Ramsey theory

Marya Bazzi

August 13, 2012

Contents

1 Introduction

Ramsey theory is a field of mathematics dating back to approximately 100 years. It intersects with various branches of mathematics, such as combinatorics, number theory, geometry, topology and set theory [16]. Loosely speaking, Ramsey theory can be described as the study of structure which is preserved under partitions – an idea succinctly captured by the statement "complete disorder is impossible" [6, 10].

In this essay we explore Ramsey's theorems, some of the core results underpinning Ramsey theory and dealing with invariant substructures under finite set partitioning. We then discuss some extensions of these ideas in the case of infinite set partitioning.

2 Preliminaries

We begin with some notation and definitions which will be used repeatedly throughout the essay. For ease of writing, we will not consider 0 to be a natural number and take $\mathbb{N}$ to be the set of positive integers $\{1, 2, 3, \dots\}$. For any $n \in \mathbb{N}, [n]$ denotes the set $\{1, 2, \dots, n\}$. For any set X, $X^{(r)}$ denotes the set $\{Y \subseteq X : |Y| = r\}$, where $|Y|$ denotes the cardinality of the finite set Y. In other words, $X^{(r)}$ is the set of unordered subsets of X of size r, or the set of *r-tuples* of X.

Definition A *graph* G is an ordered pair of disjoint sets (V, E) such that $E \subseteq V^{(2)}$. The set V may be finite or infinite. An element of V is a *vertex*, and an element of E is an *edge*. If $E = V^{(2)}$ and V is finite, the graph is said to be *complete* and is denoted K_n where $n = |V|$. We call $G' = (V', E')$ a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$.

Definition A *hypergraph* H is a pair of disjoint sets (V, E) such that $E \subset P(V)$, where $P(V)$ is the power set of V, that is, the set of all subsets of E. Clearly, a hypergraph is more general than a graph.

Throughout the essay, we will mainly consider hypergraphs where $E \subseteq V^{(r)}$, that is, hypergraphs where all elements of E have length r. As mentioned in the introduction, Ramsey theory can be loosely described as the study of structure which is preserved under partition. The next definition provides a more intuitive way of viewing the partition of a set X.

Definition Let X be a set and $k \in \mathbb{N}$. A *k-colouring* of X is a map

$$c : X \to [k].$$

For $x \in X$, $c(x)$ is called the *colour* of x. A subset Y of X is said to be *monochromatic* (under c) if c is constant on Y.

We will be concerned with the colouring of the set of r-tuples of a set X, $X^{(r)}$ (where $(X, X^{(r)})$ forms a hypergraph). A partition of $X^{(r)}$ into k distinct classes can be visualised as a k-colouring of $X^{(r)}$, where elements belonging to the same class correspond to elements of the same colour.

With these definitions in mind, we can begin exploring some of the core results of Ramsey theory.

3 Ramsey Theorems

A common way of introducing Ramsey theory is via this simple "puzzle problem":

> *In any collection of six people, there is always a group of three who either all know each other or are all strangers to each other* [2, 14].

To see why this is true, denote the collection of six people by $X = \{X_1, X_2, X_3, X_4, X_5, X_6\}$, where each element of the set denotes one person, and where for any distinct $i, j \in [6]$, either X_i knows X_j or X_i does not know X_j[1]. The relationship is assumed to be symmetric but not transitive. Without loss of generality, consider the relation of X_1 to the five other people. The person labelled X_1 must either know three people or not know three people (otherwise the number of relationships between X_1 and the five remaining people will be strictly inferior to five). Suppose that X_1 knows three of them. If any pair of these three know each other, then we end up with a trio in which all people know each other yielding the desired result. If no pair of the three know each other, then these three constitute a trio in which all people do not know each other yielding the desired result. The argument if X_1 does not know three people is analogous to this one.

If we view the set of people as a set of vertices, it becomes clear that this example is in fact a statement about any partition of the edges of a complete[2] six vertex graph. The first class of the partition would contain all the edges linking two people who know each other and the second, all the edges linking people who do not know each other. Re-wording this result using some of the definitions mentioned in the previous section, we have shown that any 2-colouring of the edges of the complete graph K_6 contains a monochromatic triangle (i.e. a subgraph K_3). This idea of finding partition-invariant substructures within larger structures is at the heart of Ramsey theory.

3.1 Ramsey's Theorem: Finite Form

Ramsey theory derives its name from the mathematician Frank Plumpton Ramsey who proved two theorems in 1928 known as Ramsey's theorems [15]. Ramsey's theorems are considered amongst those at the origin of Ramsey theory [14, 7, 6]. A result predating Ramsey's theorems and considered by Bollòbas and the authors of [14] to be the "quintessential" and "central" result of Ramsey theory is Van der Waerden's theorem [17]. Van der Waerden's theorem states that for any k, p in $\mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that every k-colouring of $[n_0]$ contains a monochromatic arithmetic progression of length p[3]. There have also been other results in Ramsey theory predating Ramsey's theorems dealing with the colouring of $\mathbb{N}$ [14, 16, 12, 6]. However, we will focus here on Ramsey's theorems, which study the existence of partition-invariant structures in countably infinite[4] sets as well as finite sets. In this section we state and prove Ramsey's theorem for finite sets.

[1] What is meant by "know" is irrelevant, the result applies to all interpretations of the word, i.e. all partitions of the set X.

[2] Since any two people either know or do not know each other.

[3] i.e. a subsequence of $\mathbb{N}$ such that the difference between consecutive terms remains constant.

[4] In order to keep the focus on Ramsey theory and avoid subtleties in axiomatic set theory relating to the axiom of choice, we restrict our attention to countably infinite sets. However, if the axiom of choice is assumed, the infinite form of Ramsey's theorem can be generalized to uncountably infinite sets [14, 7, 3].

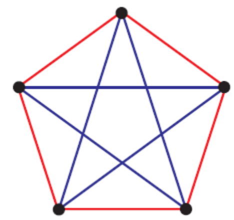

Figure 1: A red-blue colouring of K_5 that contains neither a red nor a blue K_3 .

Before stating Ramsey's theorem on the existence of partition-invariant substructures in finite sets, we start with an important definition.

Definition Let $k, r, s_1, \ldots, s_k \in \mathbb{N}$ with $1 \leq r \leq \min(s_1, s_2, \ldots, s_k)$[5]. The *Ramsey number* $R_k^{(r)}(s_1, s_2, \ldots, s_k)$ is the smallest positive integer n_0 such that for any k-colouring of a set $X^{(r)}$ with $|X| \geq n_0$, there exists $i \in [k]$ and a set $S_i \subseteq X$, $|S_i| = s_i$, so that every element of $S_i^{(r)}$ has colour i.

It is clear from the definition that the Ramsey number is symmetric in its arguments, that is, $R_k^{(r)}(s_1, s_2, \ldots, s_k) = R_k^{(r)}(s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(k)})$, where σ is any permutation of the elements of $[k]$. It is helpful to consider small values of k and r to facilitate visualising the Ramsey number. Take $k = r = 2$, so that we are considering a 2-colouring of the edges of a complete graph K_n. When $k = 2$, it is common to choose red and blue as colour assignments, rather than integers, so that we can imagine the graph as being actually coloured. For a given $s_1, s_2 \in \mathbb{N}$, the question is then whether there exists an integer $n_0 \in \mathbb{N}$ such that any red-blue colouring of K_n will contain a red K_{s_1} or a blue K_{s_2}, that is, whether $R(s_1, s_2)$ is finite, where we supress the subscript or superscript 2 whenever $k = 2$ or $r = 2$ (resp.) for ease of writing. Recall our earlier party example, in which we showed that any 2-colouring of K_n contains a monochromatic triangle. Formulating this result using the Ramsey number, we can write $R(3, 3) \leq 6$. In fact, 6 is the minimal such $n_0 \in \mathbb{N}$ for which any red-blue colouring of K_{n_0} contains either a red or a blue triangle, and thus $R(3, 3) = 6$. To show this, it suffices to find a red-blue colouring of K_5 that contains neither a red nor a blue K_3. An example of such a graph shown in Figure 1.

A natural question which arises now is whether the Ramsey number is finite for larger values of r and k, and arbitrarily large values of $s_1, s_2, \ldots, s_k$; and this is precisely the question addressed by Ramsey's theorem for finite sets.

Theorem 3.1. *(***Ramsey's Theorem: Finite Form***) For all r, k and $s_1, s_2, \ldots, s_k \in \mathbb{N}$, with $1 \leq r \leq \min(s_1, s_2, \ldots, s_k)$, the Ramsey number $R_k^{(r)}(s_1, s_2, \ldots, s_k)$ is finite.*

Basing ourselves on the proof in [2], we will prove this result by induction on $k \in \mathbb{N}$. We start by showing this is true for $k = 2$ [6] i.e.

$$R^{(r)}(s_1, s_2) \leq \infty. \tag{3.1.1}$$

[5]This inequality ensures that for all $i \in [k]$ and for any set S_i of cardinality s_i, $S_i^{(r)} \neq \emptyset$.

[6]Note that the result trivially holds for $k = 1$ with $R^{(r)}(s_1) = s_1$ since a 1-colouring is a monochromatic colouring. Showing it for $k = 1$ would usually suffice in a proof by induction; however, in the second part of the proof where we

To prove (3.1.1), we will use a double induction on r and $s_1 + s_2$, and show that

$$R^{(r)}(s_1, s_2) \leq R^{(r-1)}\big(R^{(r)}(s_1 - 1, s_2), R^{(r)}(s_1, s_2 - 1)\big) + 1, \quad 1 < r < \min(s_1, s_2), \tag{3.1.2}$$

where $R^{(r)}(s_1 - 1, s_2)$, $R^{(r)}(s_1, s_2 - 1)$ and $R^{(r-1)}\big(R^{(r)}(s_1 - 1, s_2), R^{(r)}(s_1, s_2 - 1)\big)$ are assumed to be finite by our inductive hypothesis.

Proof. We first show that $R^{(r)}(s_1, s_2) \leq \infty$ for the case $r = 1$ and the case $s_1 + s_2 = 2r$, as this is the minimal value $s_1 + s_2$ can take. For $r = 1$, a 2-colouring of $X^{(1)}$ for a set X corresponds to a red-blue colouring of all the singletons of X. Take $|X| = s_1 + s_2 - 1$, then any 2-colouring of $X^{(1)}$ contains a monochromatic subset of size s_1 or s_2 by the Pigeon-Hole principle [1]. Furthermore, for $|X| = s_1 + s_2 - 2$, the colouring of any $s_1 - 1$ singletons in red and the remaining $s_2 - 1$ singletons in blue contains neither a red nor a blue monochromatic subset of size s_1 or s_2, respectively. Thus $R^{(1)}(s_1, s_2) = s_1 + s_2 - 1$, $\forall s_1, s_2 \in \mathbb{N}, s_1, s_2 \geq r$. For the case $s_1 + s_2 = 2r$, we must have $s_1 = s_2 = r$ since $s_1, s_2 \geq r$. For these values, the inequality (3.1.1) trivially holds with $R^{(r)}(r, r) = r$.

Assume $R^{(r-1)}(s_1, s_2)$ is finite, $r > 1$, and that $R^{(r)}(s_1 - 1, s_2)$ and $R^{(r)}(s_1, s_2 - 1)$ are finite, $s_1, s_2 > r$. We will show that $R^{(r)}(s_1, s_2)$ is finite by proving the inequality (3.1.2).

Let X be a non-empty set with $|X| = R^{(r-1)}\big(R^{(r)}(s_1 - 1, s_2), R^{(r)}(s_1, s_2 - 1)\big) + 1$, which is finite by our inductive hypothesis, and let c be any red-blue colouring of $X^{(r)}$. If we show that $X^{(r)}$ contains red subset $S_1^{(r)}, |S_1| = s_1$, or a blue subset $S_2^{(r)}, |S_2| = s_2$, then we are done.

Pick $x \in X$ and let $Y = X - \{x\}$. We define a red-blue colouring c' of $Y^{(r-1)}$ such that $\forall y \in Y^{(r-1)}$, $c'(y) = c(y \cup \{x\})$. This choice of colouring is at the heart of this proof, and will be frequently encountered in the proofs of Ramsey's theorems. Noting that $|Y| = R^{(r-1)}\big(R^{(r)}(s_1 - 1, s_2), R^{(r)}(s_1, s_2 - 1)\big) < \infty$, and recalling the definition of the Ramsey number, there exists a subset $\tilde{S}_1$ of Y such that $\tilde{S}_1^{(r-1)}$ is red and $|\tilde{S}_1| = R^{(r)}(s_1 - 1, s_2)$, or a subset $\tilde{S}_2$ of Y such that $\tilde{S}_2^{(r-1)}$ is blue and $|\tilde{S}_2| = R^{(r)}(s_1, s_2 - 1)$. The proof is analogous for either case, and so we assume that there exists a red $\tilde{S}_1^{(r-1)}$. Now, consider $c|_{\tilde{S}_1^{(r)}}$ the restriction of the initial red-blue colouring of $X^{(r)}$ to the set $\tilde{S}_1^{(r)}, \tilde{S}_1 \subseteq X$, $|\tilde{S}_1| = R^{(r)}(s_1 - 1, s_2)$. Using again the definition of the Ramsey number, if $\tilde{S}_1$ contains a subset S_2 such that $S_2^{(r)}$ is blue and $|S_2| = s_2$ then we are done since $S_2^{(r)} \subseteq \tilde{S}_1^{(r)} \subseteq X^{(r)}$. If $\tilde{S}_1$ contains a subset $\hat{S}_1$ such that $\hat{S}_1^{(r)}$ is red and $|\hat{S}_1| = s_1 - 1$, then consider the set $S_1 = \hat{S}_1 \cup \{x\}$ and let's show that S_1 is also red (under c). Any element of $S_1^{(r)}$ is either in $\hat{S}_1^{(r)}$, in which case it is red (under c), or is of the form $\sigma \cup \{x\}$ with $\sigma \in \hat{S}_1^{(r-1)}$ and so $c'(\sigma)$ is red (under c') since $\hat{S}_1^{(r-1)} \subseteq \tilde{S}_1^{(r-1)}$. By definition of the red-blue colouring c', we have that $c'(\sigma) = c(\sigma \cup \{x\})$ and thus $\sigma \cup \{x\}$ is red (under c). It follows that S_1 is a subset of X such that $S_1^{(r)}$ is red and $|S_1| = s_1$. □

We have now shown that $R_k^{(r)}(s_1, s_2, \ldots, s_k)$ is finite for the case $k = 2$, that is, $R^{(r)}(s_1, s_2) < \infty$.

show that if $R_{k-1}^{(r)}(s_1, s_2, \ldots, s_k)$ is finite then $R_k^{(r)}(s_1, s_2, \ldots, s_k)$ is also finite, we will use the result $R^{(r)}(s_1, s_2) \leq \infty$, and thus need to prove it as well.

We now assume that the result of Theorem 3.1 holds for $k-1, k \geq 2$ and show that it must hold for k. To prove this, we will show the following inequality

$$R_k^{(r)}(s_1, s_2, \ldots, s_k) \leq R_{k-1}^{(r)}\big(R^{(r)}(s_1, s_2), \ldots, s_k\big), \quad k \geq 2, \quad 1 \leq r \leq \min(s_1, \ldots, s_k), \qquad (3.1.3)$$

where $k, r, s_1, \ldots, s_k$ are all positive integers, $R_{k-1}^{(r)}\big(R^{(r)}(s_1, s_2), \ldots, s_k\big) < \infty$ by the inductive hypothesis and $R^{(r)}(s_1, s_2) < \infty$ by our previous result (3.1.1).

Proof. We will prove this by an argument called the colour-grouping argument. Consider a set X with $|X| = R_{k-1}^{(r)}\big(R^{(r)}(s_1, s_2), s_3, \ldots, s_k\big)$ and let c be an arbitrary k-colouring of $X^{(r)}$. If we show that for some $i \in [k]$, there exists a set $S_i \subseteq X, |S_i| = s_i$, such that all elements of S_i have colour i, then we are done.

To be able to use the definition of the Ramsey number in cases where we know or have assumed it is finite, we need to reduce the number of colours in c. Define a $(k-1)$-colouring of $X^{(r)}$ by merging the first two colours of c into one colour that we will denote m. By the definition of the Ramsey number, there exists a set $S_m \subseteq X, |S_m| = R^{(r)}(s_1, s_2)$, such that all the elements of S_m have the colour m, or for some $i \geq 3$, there exists a set $S_i \subseteq X, |S_i| = s_i$ such that all the elements of S_i have the colour i. If the latter is true, then we are done. If the former is true, then we revert the colour m on $S_m^{(r)}$ back to what it was in c, that is, back to the 2-colouring $c_{|S_m^{(r)}}$. By definition of the Ramsey number and since $|S_m| = R^{(r)}(s_1, s_2)$, for some $i \in [2]$, there exists $S_i \subseteq S_m \subseteq X$ with cardinality s_i such that every element of S_i has colour i. □

This ends the proof of Theorem 3.1. Note that while Ramsey's theorem only makes a claim about the existence of Ramsey numbers, the method of proof presented here provides a bound on Ramsey numbers. In fact, the task of computing exactly or estimating Ramsey numbers is one of the oldest areas of Ramsey theory and has proven to be a vast and difficult area of research [2, 14, 4, 5]. We shall not discuss the results on bounds of Ramsey numbers in this essay, but it is worth noting that the only known exact values of Ramsey numbers are for the case $k = 2$ and for values of s_1, s_2 not exceeding nine, as shown in Table 1 [2]. For all other Ramsey numbers, all that is known are bounds [2].

In this section we have proved the finite form of Ramsey's theorem, that is, $R_k^{(r)}(s_1, s_2, \ldots, s_k)$ is finite for all values of $k, r, s_1, s_2, \ldots, s_k$ in $\mathbb{N}$, $1 \leq r \leq \min(s_1, \ldots, s_k)$. If we consider any k-colouring of the infinite set $\mathbb{N}^{(r)}$, it follows from this result that we can find arbitrarily large monochromatic subsets of the form $S^{(r)}$ for some $S \subset \mathbb{N}$. Whether it is possible to find infinitely large monochromatic subsets of this form for any colouring of a countably infinite set is the topic of the next section.

3.2 Ramsey's theorem: Infinite Form

The infinite form of Ramsey's theorem deals with the existence of partition-invariant substructures in infinite sets. Before stating and proving the infinite form of Ramsey's theorem, we reformulate

s_1	s_2	$R(s_1, s_2)$
3	3	6
3	4	9
3	5	14
3	6	18
3	7	23
3	8	28
3	9	36
4	4	18
4	5	25

Table 1: The only known Ramsey numbers

the Pigeon-Hole principle used at the start of the proof of Theorem 3.1 for finite colourings of infinite sets: in a k-colouring of an infinite set, there exists a monochromatic infinite subset. What Ramsey showed is that not only do we have infinitely many r-tuples of the same colour, but that all the r-tuples of an infinite set have the same colour[7]. In other words, he showed that when the countably infinite set is of the form $X^{(r)}$, then the infinite monochramatic subset will also be of the form $T^{(r)}$ for some countably infinite set T.

Theorem 3.2. *(***Ramsey's Theorem: Infinite Form***) Let $r \in \mathbb{N}$ and let X be a countably infinite set. For any k-colouring $c : X^{(r)} \to [k]$, there exists an infinite subset $T \subset X$ such that $T^{(r)}$ is monochromatic.*

Proof. We will show this by induction on r. For $r = 1$, we have $X^{(r)} = X^{(1)}$. Since c is here a finite colouring of the singletons of the infinite set X, Ramsey's Theorem follows from the Pigeon-Hole principle. Now, assume that Theorem 3.2 is true for $r - 1$ and let's show that it is true for $r \in \mathbb{N}, r \geq 1$.

The idea in this proof is to construct an infinite sequence of elements of X in which all r-tuples have the same colour. We do this as follows. Put $X = X_0$, and let x_1 be an arbitrary element of X_0. As in the proof of Theorem 3.1, define a colouring $c_1 : Y_1^{(r-1)} \to [k]$, where $Y_1 = X_0 \backslash \{x_1\}$ and for any $\sigma \in Y_1$, $c_1(\sigma) = c(\{x_1\} \cup \sigma)$. By the induction hypothesis, Y_1 contains an infinite set X_1 such that all elements of $X_1^{(r-1)}$ have the same colour (under c_1), say d_1. Let x_2 be an arbitrary element of X_1 and define a colouring $c_2 : Y_2^{(r-1)} \to [k]$, where $Y_2 = X_1 \backslash \{x_2\}$ and for any $\sigma \in Y_2$, $c_2(\sigma) = c(\{x_2\} \cup \sigma)$. By the induction hypothesis, Y_2 contains an infinite set X_2 such that all elements of $X_2^{(r-1)}$ have the same colour (under c_2), say d_2. By continuing this process, we obtain an infinite sequence of elements of X_0, $\{x_1, x_2, \dots\}$ and an infinite set of nested sets $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$ (where $X_i^{(r)}$ is monochromatic $\forall i \geq 1$), such for any r-tuple $\{x_{i_1}, \dots, x_{i_r}\} \subset \{x_1, x_2, \dots\}$, $i_1 < \dots < i_r$, we have $c(\{x_{i_1}, \dots, x_{i_r}\}) = d_{i_1}$, where d_{i_1} is the colour of $X_{i_1}^{(r)}$. That is, the colour of any r-tuple only

[7]The only case where these two statement mean the same thing is for $r = 1$.

depends on the smallest i-index, $i \in \mathbb{N}$. To see why this is true note that

$$c(\{x_{i_1}, \dots, x_{i_r}\}) = c(\{x_{i_1}\} \cup \{x_{i_2}, \dots, x_{i_r}\}) = c_{i_1}(\{x_{i_2} \dots, x_{i_r}\}) = d_{i_1},$$

where the second equality follows by construction of c_i for all i, and the third equality follows from the fact that $\{x_{i_2}, \dots, x_{i_r}\} \subset X_{i_1}^{(r)}$. Now, by the Pigeon-Hole principle, since there are only finitely many colours, there must exist a colour d repeated infinitely many times, say $d = d_{n_1} = d_{n_2} = \dots$. Then the infinite set $T = \{x_{n_1}, x_{n_2}, \dots\}$ is monochromatic under c, since any r-tuple will have some have some n_i as the smallest i-index, and thus the colour $d_{n_i} = d$. □

It is worth noting that it is possible to deduce the finite form of Ramsey's theorem from Theorem 3.2, that is, the above result implies that the Ramsey number $R_k^{(r)}(s_1, s_2, \dots, s_k)$ is finite for all $k, r, s_1, ..., s_k$ in $\mathbb{N}$, $1 \leq r \leq \min(s_1, s_2, \dots, s_k)$ [2, 14, 13]. The proof is as follows.

Proof. We will prove this result by contradiction. Suppose Theorem 3.1 does not hold for some $r \in \mathbb{N}$. This implies that for all $n \in \mathbb{N}, n \geq r$, there exists a k-colouring $c_n : [n]^{(r)} \to [k]$ such that for each $i \in [k]$, there exists no subset S_i, $|S_i| = s_i$, all of whose r-tuples have colour i. The idea is to now construct a k-colouring of $N^{(r)}$ that does not contain a monochromatic subset, contradicting Theorem 3.2.

Consider the restriction of c_n to $[r]$, for all $n \geq r$. Each of $c_r|_{[r]^{(r)}}$, $c_{r+1}|_{[r]^{(r)}}$, $c_{r+2}|_{[r]^{(r)}}$,..., constitutes a k-colouring of $[r]$. Since there are only finitely many ways to colour $[r]^{(r)}$ with k colours (k, in fact), infinitely many of the k-colourings $c_i|_{[r]^{(r)}}$ must agree. We can then write $c_i|_{[r]^{(r)}} = \tilde{c}_r$ for all i in some infinite set $X_1 \subseteq \mathbb{N}$. Now, of those that agree on $[r]$, we select infinitely many that agree on $[r+1]$. Consider the restriction of c_i to $[r+1]$, for all $i \in X_1$. As argued earlier, since there are only finitely many ways to colour $[r+1]^{(r)}$ with k colours, infinitely many of these colourings must agree on $[r+1]^{(r)}$. We can then write $c_i|_{[r+1]^{(r)}} = \tilde{c}_{r+1}$ for all i in some infinite set $X_2 \subseteq X_1$. Continuing this process, we obtain an infinite sequence of colourings $\tilde{c}_n : [n]^{(r)} \to [k], n \geq r$ such that

(i) There does not exist a monochromatic set $S_i^{(r)}, i \in [k], |S_i| = s_i$ under any of the k-colourings $\tilde{c}_n$ (since $\tilde{c}_n = c_n|_{[n]^{(r)}}$ by definition, and so if $\tilde{c}$ is constant on $S_i^{(r)}$ then c_n is constant on $S_i^{(r)}$, contradicting our assumption); and

(ii) $\tilde{c}_{n'}|_{[n]^{(r)}} = \tilde{c}_n$ for all $n' \geq n$ (since $X_{n'} \subseteq X_n$ by construction).

Define a colouring $c : \mathbb{N}^{(r)} \to [k]$ by $c(\{i_1, i_2, \dots, i_r\}) = \tilde{c}_{i_r}(\{i_1, i_2, \dots, i_r\})$, where $i_1 < i_2 < \dots < i_r$. Note that this implies that for any finite subset $N \subseteq [\max(N)]$ of $\mathbb{N}$, $N \geq r$, $c|_N = \tilde{c}_{\max(N)}|_N$ by (ii). It follows from this observation and (i) that there does not exist an infinite monochramic set $T^{(r)}$, $T \subset \mathbb{N}$, under c[8]. □

[8] If there did exist an infinite monochromatic subset $T^{(r)}$ of colour i, $T \subseteq \mathbb{N}$, then take any subset S_i of T with cardinality s_i and denote by m the largest element of S_i. Then S_i would be a monochromatic set under $\tilde{c}_m$ contradicting (i).

This argument is called a compactness argument as it is a special case of Tychonoff's theorem on the product of compact spaces [18]; more details on what is called the compactness principle, which allows to deduce the finite form of Ramsey's theorem from the infinite form can be found in [2, 14, 7]. Note that this compactness argument establishes the finite form Ramsey's theorem by showing existence without giving any bounds on $R_k^{(r)}(s_1, s_2, \ldots, s_k)$.

Up to this point, we have only considered colourings of countably infinite sets with finitely many colours. Can anything be said about the colourings of countably infinite sets with infinitely many colours? In fact, the answer is yes, and in the next section we present Erdös and Rado's Canonical Theorem to illustrate this [5].

4 Erdös and Rado's Canonical Theorem

Suppose we colour $\mathbb{N}^{(r)}$ with infinitely colours[9]. We can quickly see that, in contrast to finite colourings, it is not necessary to have a infinite subset all of whose r-tuples have the same colour by considering an infinite colouring where every r-tuple of $\mathbb{N}^{(r)}$ is of a different colour. Is it then the case that there must exist an infinite subset $T \subseteq \mathbb{N}$ such that every element of $T^{(r)}$ gets a different colour or every element of $T^{(r)}$ has the same colour? This is also not true, and to see why consider the infinite colouring c such that $c(\{i_1, i_2, \ldots, i_r\}) = c(\{j_1, j_2, \ldots, j_r\})$ if and only if $i_1 = j_1$, where $\{i_1, i_2, \ldots, i_r\}$, $\{j_1, j_2, \ldots, j_r\} \in \mathbb{N}^{(r)}$, $i_1 < i_2 < \cdots < i_r$ and $j_1 < j_2 < \cdots < j_r$. Then for any infinite set T, $T^{(r)}$ would certainly contain elements with the same colour (by making the first entries coincide) and elements with distinct colours (by making the first entries distinct). It turns out that by allowing some additional types of colouring (of which the one just described is a special case), it is possible to guarantee the existence an infinite set $T^{(r)}$ of a certain form for any infinite colouring (or partition) of $\mathbb{N}^{(r)}$. This result was proved by Erdös and Rado in 1950 and is known as Erdös and Rado's Canonical Theorem. Before stating the theorem, we give the following definition.

Definition Let X be a set and $\{x_1, x_2, \ldots, x_r\}, \{y_1, y_2, ..., y_r\} \in X^{(r)}$. An *R-canonical colouring* of $X^{(r)}$ is any colouring c that verifies $c(\{x_1, x_2, ..., x_r\}) = c(\{y_1, y_2, ..., y_r\})$ if and only if $x_i = y_i \forall i \in R$ for some $R \subseteq [r]$.

Note that for a given $r \in \mathbb{N}$, the number of canonical colourings is the number of subsets of $[r]$, that is, $\sum_{k=0}^{r} \binom{k}{r}$ or simply 2^r by the binomial formula. Note also that for $R = \emptyset$, the canonical colouring corresponds to a monochromatic colouring and for $R = [r]$, to an all distinct or injective colouring.

Theorem 4.1. *(***Erdös and Rado's Canonical Theorem***) Let $r \in \mathbb{N}$. Then for any colouring $c : \mathbb{N}^{(r)} \to \mathbb{N}$, there exists an infinite subset T of $\mathbb{N}$ such that the restriction of c to $T^{(r)}$ is canonical.*

Note that this result is stronger than the infinite form of Ramsey's theorem, since the only possible canonical colouring of an infinite set with finitely many colours is the monochromatic colouring (i.e.

[9] Any countably infinite set is in bijection with $\mathbb{N}$, and so without loss of generality we use the set of natural numbers to prove our results for ease of writing.

$R = \emptyset$). It is easy to see that the above result holds for $r = 1$. As shown earlier, if the colouring is finite, Theorem 4.1 holds with $R = \emptyset$ by the Pigeon-Hole principle; and if the colouring is infinite then clearly there are infinitely many singletons with distinct colour and thus Theorem 4.1 holds with $R = \{1\}$. We now give a proof of this result for $r = 2$, where there are four possible canonical colourings: $R = \emptyset, R = \{1, 2\}, R = \{1\}$ and $R = \{2\}$. In particular, we will show the following:

Theorem 4.2. *For any colouring $c : \mathbb{N}^{(2)} \to \mathbb{N}$, there exists an infinite set T such that*

(i) c is constant on $T^{(2)}$; or

(ii) c is injective on $T^{(2)}$; or

(iii) $c(\{i, j\} = c(\{k, l\})$ if and only if $i = k$, for all i, j, l, k in T, with $i < j$ and $k < l$; or

(iv) $c(\{i, j\} = c(\{k, l\})$ if and only if $j = l$, for all i, j, l, k in T, with $i < j$ and $k < l$.

Proof. Basing ourselves on the proof in [13], we will prove this result by considering successive 2-colourings of $N^{(4)}$ and $N^{(3)}$ induced by c. As it involves many steps, we present this proof in the form of a dicussion to make it easier to follow. Throughout the proof, it will be useful to think of a 2-colouring as the partition of a set into two classes, C_1 and C_2, where elememts of C_1 verify a condition p and elements of C_2 verify $\bar{p}$, where $\bar{p}$ denotes the negation of p. For this reason, a colour is denoted by a word that reflects the condition p (for ease of writing, when definining the successive 2-colourings, we will sometimes use the same word although we are referring to different conditions). When writing an r-tuple of $\mathbb{N}^{(r)}$ as $\{x_1, x_2, \dots, x_r\}$, we assume without loss of generality that $x_1 < x_2 < \dots < x_r$.

To give some intuition about the method of proof, note that in all but the first canonical colouring of Theorem 4.2, for any 2-tuples $\{i, j\}, \{k, l\}$ where i, j, k, l are all distinct, we must have $c(\{i, j\}) \neq c(\{k, l\})$. We start by constructing a set in which this holds.

Define a 2-colouring of $\mathbb{N}^{(4)}$ by giving $\{i, j, k, l\}$ colour SAME if $c(\{i, j\}) = c(\{k, l\})$ and colour DIFF if $c(\{i, j\}) \neq c(\{k, l\})$. By Theorem 3.2, there exists an infinite subset T_1 of $\mathbb{N}$ such that $T_1^{(4)}$ is monochromatic under this colouring, that is, either $c(\{i, j\}) = c(\{k, l\})$ holds for all $\{i, j, k, l\} \in \mathbb{N}^{(4)}$ or $c(\{i, j\}) \neq c(\{k, l\})$ holds for all $\{i, j, k, l\} \in \mathbb{N}^{(4)}$. If the former is true, i.e. if $T_1^{(4)}$ has colour SAME, then we will show that the canonical colouring of case (i) holds. Let $\{i, j\}$ and $\{k, l\}$ be arbitrary elements of $T_1^{(2)}$. Choose any m and n in T_1 such that $m < n$ and $m > i, j, k, l$[10]. Then $c(\{i, j\}) = c(\{m, n\})$ since $\{i, j, m, n\} \in T_1^{(4)}$ and $c(\{k, l\}) = c(\{m, n\})$ since $\{k, l, m, n\} \in T_1^{(4)}$; and thus $c(\{i, j\}) = c(\{k, l\})$ and we are done.

Suppose that $T_1^{(4)}$ has colour DIFF, i.e.

$$c(\{i, j\}) \neq c(\{k, l\}) \quad \text{for all } i < j < k < l \in T_1. \tag{4.0.1}$$

[10]Such an m and n always exist since T_1 is infinite.

Define a 2-colouring of $T_1^{(4)}$ by giving $\{i,j,k,l\}$ colour SAME if $c(\{i,l\}) = c(\{j,k\})$ and DIFF if $c(\{i,l\}) \neq c(\{j,k\})$. By Theorem 3.2, there exists an infinite subset T_2 of T_1 such that $T_2^{(4)}$ is monochromatic under this colouring. Let's show that by virtue of being a subset of T_1, $T_2^{(4)}$ cannot have colour SAME. Suppose that $T_2^{(4)}$ has colour SAME and choose i,j,m,n,k,l in T_2 such that $i < j < m < n < k < l$. Recalling the condition verified by tuples of colour SAME under this colouring and applying it to $\{i,j,m,l\}$ and $\{i,n,k,l\}$ in $T_2^{(4)}$, we get that $c(\{j,m\}) = c(\{i,l\}) = c(\{n,k\})$, which is a contradiction since by condition (4.0.1), we must have $c(\{j,m\}) \neq c(\{n,k\})$ as $T_2 \subseteq T_1$. It follows that $T_2^{(4)}$ has colour DIFF, and the following conditions hold in T_2

$$\begin{aligned} c(\{i,j\}) &\neq c(\{k,l\}) \quad \text{for all } i<j<k<l \in T_2 \\ c(\{i,l\}) &\neq c(\{j,k\}) \quad \text{for all } i<j<k<l \in T_2, \end{aligned} \tag{4.0.2}$$

where the first line follows from the fact that $T_2 \subseteq T_1$ and condition (4.0.1).

Now, define a 2 colouring of $T_2^{(4)}$ by giving $\{i,j,k,l\}$ colour SAME if $c(\{i,k\}) = c(\{j,l\})$ and DIFF if $c(\{i,k\}) \neq c(\{j,l\})$. By Theorem 3.2, there exists an infinite subset T_3 of T_2 such that $T_3^{(4)}$ is monochromatic under this colouring. In the same that we proved this in the last colouring, we show that $T_3^{(4)}$ must have colour DIFF. Suppose $T_3^{(4)}$ has colour SAME and choose i,j,m,n,k,l in T_3 such that $i<j<m<n<k<l$. Recalling the condition expressed by the colour SAME under this colouring, and applying it to $\{i,j,m,k\}$ and $\{j,n,k,l\}$, we get that $c(\{i,m\}) = c(\{j,k\}) = c(\{n,l\})$, which is a contradiction since $T_3 \subseteq T_2 \subseteq T_1$ and by condition (4.0.1) we must have $c(\{i,m\}) \neq c(\{n,l\})$. It follows that $T_3^{(4)}$ has colour DIFF, and the following conditions hold in T_3

$$\begin{aligned} c(\{i,j\}) &\neq c(\{k,l\}) \quad \text{for all } i<j<k<l \in T_3 \\ c(\{i,l\}) &\neq c(\{j,k\}) \quad \text{for all } i<j<k<l \in T_3, \\ c(\{i,k\}) &\neq c(\{j,l\}) \quad \text{for all } i<j<k<l \in T_3, \end{aligned} \tag{4.0.3}$$

where the first two lines follow from the fact that $T_3 \subseteq T_2 \subseteq T_1$ and the conditions in (4.0.2).

Note that for tuples $\{i,j\}$ and $\{k,l\}$ in $T_3^{(r)}$ where i,j,k,l are all distinct, the above three conditions are sufficient to ensure that $c(\{i,j\}) \neq c(\{k,l\})$ (showing this reduces to enumerating all possible cases of distinct tuples and we do this later in the proof). This means that to complete the proof of the theorem, we need find an infinite subset of T_3 in which for any tuples $\{i,j\}$ and $\{k,l\}$ that have one element in common, the equality between $c(\{i,j\})$ and $c(\{k,l\})$ corresponds to one of the equalities in cases (ii), (iii) or (iv) (this part is clear since these cover all possible equalities) and is the same for all of them. This is what we do next.

Define a 2-colouring of $T_3^{(3)}$ by giving $\{i,j,k\}$ colour SAME if $c(\{i,j\}) = c(\{j,k\})$ and DIFF if $c(\{i,j\}) \neq c(\{j,k\})$. By Theorem 3.2, there exists an infinite subset T_4 of T_3 such that $T_4^{(3)}$ is monochromatic. We now show that $T_4^{(3)}$ must have colour DIFF. Suppose $T_4^{(3)}$ has colour SAME and choose i,j,m,n,k,l in T_4 such that $i<j<k<l$. We have $c(\{i,j\}) = c(\{j,k\}) = c(\{k,l\})$, contradicting condition (4.0.1) since $T_4 \subseteq T_1$. It follows that $T_4^{(3)}$ has colour DIFF and the following

conditions hold in T_4

$$\begin{aligned} c(\{i,j\}) &\neq c(\{k,l\}) \quad \text{for all } i<j<k<l \in M_3 \\ c(\{i,l\}) &\neq c(\{j,k\}) \quad \text{for all } i<j<k<l \in M_3, \\ c(\{i,k\}) &\neq c(\{j,l\}) \quad \text{for all } i<j<k<l \in M_3 \\ c(\{i,j\}) &\neq c(\{j,k\}) \quad \text{for all } i<j<k \in M_3, \end{aligned} \tag{4.0.4}$$

where the first three lines follow from the fact that $T_4 \subseteq T_3 \subseteq T_2 \subseteq T_1$ and the conditions in (4.0.3).

Define a 2-colouring of $T_4^{(3)}$ by giving $\{i,j,k\}$ colour LEFT-SAME if $c(\{i,j\}) = c(\{i,k\})$ and LEFT-DIFF if $c(\{i,j\}) \neq c(\{i,k\})$. By Theorem 3.2, there exists an infinite subset T_5 of T_4 such that $T_5^{(3)}$ is monochromatic. Finally, define a 2-colouring of $T_5^{(3)}$ by giving $\{i,j,k\}$ colour RIGHT-SAME if $c(\{i,k\}) = c(\{j,k\})$ and RIGHT-DIFF if $c(\{i,k\}) \neq c(\{j,k\})$. By Theorem 3.2, there exists an infinite subset T_6 of T_5 such that $T_6^{(3)}$ is monochromatic. We now consider all the possible combinations of colours T_6 can have and show that these are either impossible or reduce to one of the canonical colouring in (ii), (iii) or (iv).

If T_6 is LEFT-SAME and RIGHT-SAME, then choose $i,j,k \in T_6 \subseteq T_4$ such that $i<j<k$. We get that $c(\{i,j\}) = c(\{i,k\}) = c(\{j,k\})$, contradicting the last condition in 4.0.4.

We now show that all the other combinations of colours for T_6 coupled with the conditions in (4.0.4) (which are verified by T_6 since $T_6 \subseteq T_5$) will constitute one of the canonical colourings in (ii), (iii), or (iv). We go through the details in the first case only, as the idea is the same for the rest.

Suppose that T_6 is LEFT-DIFF and RIGHT-DIFF, and let $\{i,j\}$, $\{k,l\}$ be two arbitrary elements of T_6. We will show this corresponds to an injective colouring of T_6 under c. Assume that $\{i,j\} \neq \{k,l\}$. We need to check that $c(\{i,j\}) \neq c(\{k,l\})$ and we do this by enumerating all possible cases of $\{i,j\}$ and $\{k,l\}$. If i,j,k,l are all distinct we can assume without lost of generality that $i<k$ (this just comes down to relabelling). The integers in the tuples $\{i,j\}$ and $\{k,l\}$ are then in one of the following orders: $i<j<k<l$, $i<k<j<l$ or $i<k<l<j$. By relabelling, we see that all these cases correspond to one of the first three in (4.0.4), and thus that $c(\{i,j\}) \neq c(\{k,l\})$ in all of these cases. Now, if $i=l$ or $j=k$, then we have by the last condition in (4.0.4) that $c(\{i,j\}) \neq c(\{k,l\})$. If $i=k$ then since T_6 is LEFT-DIFF we have that $c(\{i,j\}) \neq c(\{k,l\})$. If $j=l$ then since T_6 is RIGHT-DIFF we have that $c(\{i,j\}) \neq c(\{k,l\})$. In all cases, $c(\{i,j\}) \neq c(\{k,l\})$ and thus c is injective on T_6 for this combination of colours.

Similarly, if T_6 is LEFT-SAME and RIGHT-DIFF then case (iii) holds; and if T_6 is LEFT-DIFF and RIGHT-SAME then case (iv) holds. We have thus found an infinite set T_6 such that the restriction of c to T_6 is canonical. □

It is also possible to prove this result by considering a single colouring of $\mathbb{N}^{(4)}$ where the colour of each 4-tuple $\{i,j,k,l\}$ is the set of equalities among the 2-tuples in $\{i,j,k,l\}$ under c [2, 14]. The number of colours here would be the number of partitions of a six-element set. On this note, we end our discussion on partition-invariant structures for infinite partitions.

5 Conclusion

We started by introducing the general idea behind Ramsey theory, that of finding partition-invariant substructures in larger structures by giving the famous party of six example. In the case where the set is partitioned into finitely many subsets, Ramsey's theorems provide answers to this question for finite and infinite sets of the form $X^{(r)}$ for some set X. Namely, any finite partition of a finite (resp. infinite) set of this form will contain a finite (resp. infinite) subset of this form in one of its partition classes. Erdös and Rado extended the study of partition-invariant substructures to infinite partitioning of countably infinite sets. Namely, they showed that any partitioning (finite or infinite) of $\mathbb{N}^{(r)}$ set will induce a canonical colouring on $T^{(r)}$ for some countably infinite subset T of $\mathbb{N}$.

Ramsey theory is a very vast area of mathematics, with many more areas to explore[16, 14, 12]. Two mathematicians that have significantly contributed to the body of research in this field are Ronald Graham (1935-) and Paul Erdös (1913-1996). In fact, Graham stated in [9] that although "Ramsey's theorems were not discovered by Paul Erdös, Ramsey theory was largely created by him". In addition to many established results, there are also many conjectures and open problems in Ramsey theory, with some of the oldest dating back to Erdös, and where Graham has personnally offered cash rewards for solutions of some of the most prominent outstanding problems [8, 11].

References

[1] N. Biggs. *Discrete Mathematics.* Oxford University Press, 2nd edition, 2002.

[2] B. Bollòbas. *Modern Graph Theory.* Springer New York, 1998.

[3] F. Drake and D. Singh. *Intermediate Set Theory.* John Wiley and Sons, 1996.

[4] P. Erdös. Some remarks on the theory of graphs. *American Mathematical Society*, 2:264–286, 1947.

[5] P. Erdös and R. Rado. A combinatorial theorem. *Journal of the London Mathematical Society*, 5:249–255, 1950.

[6] R. Graham. Roots of ramsey theory. *AT&T Bell Laboratories. Available on http://www.math.ucsd.edu/ ronspubs/92_08_ramsey_roots.pdf.*

[7] R. Graham. *Rudiments of Ramsey Theory*, volume 45. 1980.

[8] R. Graham. *Some of My Favorite Problems in Ramsey Theory*, volume 7(2). 2007.

[9] R. Graham and J. Nešetřil. Ramsey theory in the work of paul erdös. 95:Charles Univ., Dept. of Applied Mathematics, 1995.

[10] R. Graham and V. Rödl. Numbers in ramsey theory. *AT&T Bell Laboratories. Available on http://www.math.ucsd.edu/ ronspubs/87_04_ramsey_numbers.pdf.*

[11] R. Graham and E. Tressler. *Open Problems in Euclidean Ramsey Geometry*, volume 13. 2004.

[12] B. Landman and A. Robertson. *Ramsey Theory on the Integers*, volume 24. 2004.

[13] I. Leader. *Lecture notes on Ramsey Theory of Cambridge University. 2000-2001. Available at http://www.dpmms.cam.ac.uk/ par31/notes/ramsey.pdf.*

[14] B. Rothschild R. Graham and J. Spencer. *Ramsey Theory*. Wiley Interscience, 2nd edition, 1990.

[15] F. Ramsey. On a problem of formal logic. *Proc. London Math. Soc*, 2:264–286, 1930.

[16] A. Soifer. *Ramsey Theory Yesterday, Today, and Tomorrow*. Birkhäuser, 1998.

[17] B. van der Waerden. Beweis einer baudetschen vermutung. *Nieuw Arch. Wiskd.*, 15:212–216, 1927.

[18] S. Willard. *General Topology*. Dover Publications, 2004.