Calabi-Yau Manifolds, Discrete Symmetries and String Theory

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to my father Jaiprakash, the most inspiring teacher I have had

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Abstract

In this thesis we explore various aspects of Calabi-Yau (CY) manifolds and compactifications of the heterotic string over them. At first we focus on classifying symmetries and computing Hodge numbers of smooth CY quotients. Being non-simply connected, these quotients are an integral part of CY compactifications of the heterotic string, aimed at producing realistic string vacua. Discrete symmetries of such spaces that are generically present in the moduli space, are phenomenologically important since they may appear as symmetries of the associated low energy theory. We classify such symmetries for the class of smooth Complete Intersection CY (CICY) quotients, resulting in a large number of regular and R-symmetry examples. Our results strongly suggest that generic, non-freely acting symmetries for CY quotients arise relatively frequently. A large number of string derived Standard Models (SM) were recently obtained over this class of CY manifolds indicating that our results could be phenomenologically important. We also specialise to certain loci in the moduli space of a quintic quotient to produce highly symmetric CY quotients. Our computations thus far are the first steps towards constructing a sizeable class of highly symmetric smooth CY quotients.

Knowledge of the topological properties of the internal space is vital in determining the suitability of the space for realistic string compactifications. Employing the tools of polynomial deformation and counting of invariant Kähler classes, we compute the Hodge numbers of a large number of smooth CICY quotients. These were later verified by independent cohomology computations. We go on to develop the machinery to understand the geometry of CY manifolds embedded as hypersurfaces in a product of del Pezzo surfaces. This led to an interesting account of the quotient space geometry, enabling the computation of Hodge numbers of such CY quotients.

Until recently only a handful of CY compactifications were known that yielded low energy theories with desirable MSSM features. The recent construction of rank 5 line bundle sums over smooth CY quotients has led to several SU(5) GUTs with the exact MSSM spectrum. We derive semi-analytic results on the finiteness of the number of such line bundle models, and study the relationship between the volume of the CY and the number of line bundle models over them. We also imply a possible correlation between the observed number of generations and the value of the gauge coupling constants of the corresponding GUTs. String compactifications with underlying $SO(10)$ GUTs are theoretically attractive especially since the discovery that neutrinos have non-zero mass. With this in mind, we construct tens of thousands of rank 4 stable line bundle sums over smooth CY quotients leading to $SO(10)$ GUTs.

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1
Introduction

The origins of string theory lie in the description of strong nuclear interactions. Over the past few decades however, it has emerged as one of the leading approaches to a quantum theory of gravity. It has also made forays into the fields of cosmology and condensed matter physics. It has led to many interesting developments in pure mathematics, particularly in enumerative geometry. One of the holy grails of string theory as applied to particle physics, is to embed the Standard Model (SM) within its framework. This attempt broadly defines the field of string phenomenology. This thesis deals with some of the questions related to this central idea.

The SM of particle physics has a very interesting mathematical structure that has successfully explained and predicted a multitude of phenomena observed in particle accelerators over the past decades. The most recent example is the observation of the long conjectured Higgs boson by the ATLAS and CMS collaborations at the LHC. The task for a string phenomenologist is to exactly reproduce the particle content of the SM, the coupling constants and the masses of the particles. In addition, one must precisely reproduce the discrete symmetries of the low energy theory that help explain unobserved couplings, the long lifetime of the proton, etc. There are several other challenges for a string phenomenologist. The fine tuning problem, moduli stabilisation and supersymmetry breaking are some examples. Another open problem not addressed by the SM is to explain the observed number of generations of matter particles.

In this thesis, we focus on the issue of the origins of discrete symmetries that are necessary to explain phenomena observed in low energy particle physics. In addition we discuss and attempt to build models in heterotic string theory, with the eventual aim of producing SM like models.

1.1 Discrete Symmetries

Discrete symmetries are an important tool in theoretical particle physics. In low energy theories, they are simply conjectured in order to accomplish tasks like avoiding undesirable couplings, reproducing the structure of mass matrices, etc. However, they are not understood at a deeper level. For internal consistency, string theory requires extra-dimensions of space, referred to as the compactification space. In the formalism of superstrings, discrete symmetries cannot merely be conjectured, but instead must be found as isometries of the compactification space.

While there are many approaches to string phenomenology, one of the earliest approaches derives from the Calabi-Yau compactifications of the 10 dimensional heterotic string, where the compactification space is a Calabi-Yau (CY) threefold. These are six dimensional compact manifolds with special geometrical features and are amongst the simplest class of solutions in certain string compactifications. A particularly interesting class of CY threefolds is that of Complete Intersection Calabi-Yau (CICY) threefolds. These manifolds are complete intersections of hypersurfaces embedded in products of complex projective spaces and are thus especially amenable to the tools of computational algebraic-geometry. A significant fraction of this thesis is devoted to the study of the symmetries of this class of manifolds, with the ultimate aim of reproducing desired discrete symmetries of low energy theories. In this process, we also wish to understand better, the geometry of CY threefolds. Discrete symmetries of CY manifolds play a dual role in string model building. Firstly, symmetries of a CY manifold can be used to construct new CY manifolds with different topological properties, e.g., a simply connected CY threefold on being quotiented by a freely acting discrete symmetry Γ becomes non-simply connected with the fundamental group Γ. The Euler characteristic of the quotient manifold is that of the original CY reduced by a factor $|\Gamma|$, although the Hodge numbers of the quotient do not scale in a similar fashion. Secondly, there could be remnant freely acting or non-freely acting symmetries in a CY quotient. Both these classes of symmetries can survive as symmetries of the low energy theory obtained by compactifying the heterotic string over a CY quotient.

1.2 Heterotic Model Building

The standard model gauge group is given by SU(3) × SU(2) × U(1), where SU(3) describes strong interactions of quantum chromodynamics, and SU(2)×U(1) describe electroweak interactions. Using the apparatus of renormalization group, one finds that the coupling constants of these three components vary with energy, and approach the same value at sufficiently high energies $\sim 10^{15}$ GeV, leading to a unification of these couplings. It is then expected that the different components of the SM gauge group then unify into one single gauge group. This leads to the phenomenon of grand unification. There are several Lie groups that do the job. Examples of Grand Unified Theory (GUT) gauge groups include $SU(5)$, $SO(10)$[1] and E_6.

All these three GUT groups are attractive from the viewpoint of string phenomenology. Within the framework of CY compactifications of the $E_8 \times E_8$ heterotic string, which is our focus in this thesis, the string gauge group is first broken into one of the GUT groups by the introduction of stable holomorphic vector bundles. Some of the simplest constructions of stable vector bundles involve the tangent bundle of the CY with the structure group $SU(n)$, or the sum of line bundles with Abelian structure groups of the kind $S(U(1)^k)$. The GUT group derived from this theory is then broken into the SM gauge group $SU(3)\times SU(2)\times U(1)$ by introducing discrete Wilson lines. To ensure this breaking, the CY manifold must be non-simply connected.

Recently a large number of string derived standard models were obtained by constructing line bundle models over CICY threefolds leading to $SU(5)$ GUTs. Until that point, only a handful of string derived standard models existed in the

[1] The actual group is Spin(10), the double cover of $SO(10)$.

literature that have the exact MSSM spectrum. In this thesis, we use the above line bundle data to derive hints about the issue of the number of generations of matter particles in the SM. In addition, we take the first important step towards the (computationally intensive) task of deriving all rank 4 line bundle models over smooth CICYs leading to $SO(10)$ GUTs.

1.3 Outline of the thesis

We begin in Chapter 2 by laying the foundation and further motivation for the work done in this thesis. We discuss several aspects of CY compactifications of the $E_8 \times E_8$ heterotic string using 'standard' and general embeddings. We also formally introduce Calabi-Yau manifolds and motivate the study of its discrete symmetries.

We devote chapter 3 to the study of one the simplest Calabi-Yau threefolds, the quintic $\mathbb{P}^4[5]$. The quintic has been one of the most well studied Calabi-Yau manifolds. In particular, the symmetries of the quintic has been the focus of a significant amount of research [1–4]. It is known to have quotients by freely acting $\mathbb{Z}_5$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$ symmetries. Some symmetries of the $\mathbb{Z}_5 \times \mathbb{Z}_5$ quotient of the quintic were studied in [1] which hinted at ways of constructing highly symmetric quintic quotients. In this chapter, we construct all quintic families with additional symmetries that descend from the automorphism group of $\mathbb{P}^4$, yielding quintic families that are highly symmetric.
Our methods could possibly be generalised to other CICYs but at the cost of being computationally much more intensive. Many of our constructions are previously unknown, smooth families of the quintic threefolds.

Taking cue from the example of the quintic, we undertake the classification of discrete global symmetries of all CICY threefolds in chapter 4. A complete list of freely acting discrete symmetries of these manifolds was obtained in the automated classification of V. Braun [3]. Some of these free CICY quotients have been shown to have residual non-freely acting discrete symmetries [1, 2, 5]. In this chapter, we classify such residual (global) symmetries of all free CICY quotients. Evidently, these are all non-freely acting symmetries and we provide an exhaustive list of such group actions on these quotients. As non-freely acting symmetries, a number of these were found to be R-symmetries. We thus produce for the first time, a comprehensive list of global discrete symmetries of all known CICY quotients and a large number of R-symmetry candidates amongst them.

Having constructed a large number of symmetric quintic quotients and classified the global symmetries of all CICY quotients, it is all but natural to devote some time to the study of the CICY quotients themselves.
Although all linearly realised freely acting groups of CICY threefolds were recently computed by Braun in [3], the Hodge numbers of the quotients were not. The freely acting groups, G_f, that arise in the classification are either $\mathbb{Z}_2$ or contain $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$ or $\mathbb{Z}_5$ as a subgroup. The Hodge numbers for the quotients for which the group G_f contains $\mathbb{Z}_3$ or $\mathbb{Z}_5$ were computed previously [2, 6]. In chapter 5 we deal with the cases, for which $G_f \supseteq \mathbb{Z}_4$ or $G_f \supseteq \mathbb{Z}_2{\times}\mathbb{Z}_2$. We also compute the Hodge numbers for 99 of the 166 CICYs which have $\mathbb{Z}_2$ quotients. We make use of the polynomial deformation method and counting of invariant Kähler classes to achieve this. In the process we populate the tip of the Hodge plot with a number of previously unoccupied points.

In chapter 6, we switch gears slightly to construct line bundle models over the class of CICYs. We follow the algorithmic model building approach espoused by the works [7–19], to construct $SO(10)$ GUT line bundle models. We present the results of a comprehensive scan over a large class of heterotic Calabi-Yau compactifications involving rank 4 line bundle sums, a promising avenue for string model building.

In chapter 7, we work on line bundle models over CICYs that lead to $SU(5)$ GUTs. Within the class of heterotic line bundle models, we argue that $\mathcal{N} = 1$ vacua which lead to a small number of low-energy chiral families are preferred. By imposing

an upper limit on the volume of the internal manifold (required in order to obtain finite values of the four-dimensional gauge couplings and for the validity of the supergravity approximation) we show that, for a given manifold, only a finite number of line bundle sums are consistent with supersymmetry. By explicitly scanning over this finite set of line bundle models on certain manifolds we show that, for a sufficiently small volume of the internal manifold, the family number distribution peaks at small values, consistent with three chiral families. The relation between the maximal number of families and the gauge coupling is discussed, which hints towards a possible explanation of the family problem.
In chapter 8, we conclude by emphasising our most important findings and claims, and lay the foundation for future work. This thesis is based on the following research articles:

1. Highly Symmetric Quintic Quotients - Philip Candelas and Challenger Mishra, accepted for publication in Fortschritte der Physik, arXiv:1709.01081.

2. Discrete Symmetries of Complete Intersection Calabi-Yau Manifolds - Andre Lukas and Challenger Mishra, accepted for publication in Communications in Mathematical Physics, arXiv:1708.08943.

3. Calabi-Yau Threefolds with Small Hodge Numbers - Philip Candelas, Andrei Constantin and Challenger Mishra, under consideration in Fortschritte der Physik, arXiv:1602.06303.

4. Hodge Numbers for CICYs with Symmetries of Order Divisible by 4 - Philip Candelas, Andrei Constantin and Challenger Mishra, Fortschritte der Physik 64 (2016) 463-509, arXiv:1511.01103.

5. The Family Problem: Hints from Heterotic Line Bundle Models - Andrei Constantin, Andre Lukas and Challenger Mishra, Journal of High Energy Physics (2016): 173, arXiv:1509.02729.

6. $SO(10)$ GUTs from Heterotic Line Bundle Models: a Comprehensive Scan - Andrei Constantin, Andre Lukas and Challenger Mishra, to appear.

2
Background

2.1 Introduction

Over the course of time, it has become increasingly clear that the SM does not easily extend to yield a complete description of nature. The primary reason for that is its inability to describe gravity. It is at best a low energy description of nature[1]. String theory has emerged as the leading candidate for a fundamental theory that can describe gravity within a quantum mechanical framework. It contains many ingredients of the SM including chiral fermions, non-Abelian gauge symmetries and Yukawa couplings. For internal consistency, string theory requires extra-dimensions of space. Since the directly observable world is four dimensional, string theories are 'compactified' over the extra-dimensions to yield a four dimensional description of nature. Examples of these extra-dimensions include the six dimensional Calabi-Yau threefolds that appear frequently in compactifications of the heterotic string, the seven dimensional G_2 manifolds appearing in M-theory compactifications and the eight dimensional Calabi-Yau fourfolds in F-theory.

A key ingredient of most compactifications of string theory is supersymmetry, which itself draws its motivation from some of the shortcomings of the SM, like the hierarchy problem and the problem of gauge coupling unification [23]. Although non-supersymmetric string theories have been explored [24–26], string compactifications

[1]There are several other shortcomings of the SM and we refer the reader to [20] for a detailed discussion on this. For an introduction to the various string theories and their generatilies, we refer to [20–22].

preserving supersymmetry are more attractive especially since they are automatically stable and devoid of tachyons.

There are five 10-dimensional superstring theories: Type I, Type IIA, Type IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$. All these theories were shown to be connected to each other and to the 11-dimensional M-theory, via S and T-dualities. It was further shown that the low-energy effective field theory descriptions of such theories are given by supergravity[2] theories. The low energy limit of heterotic string theory is the N=1 supergravity in ten dimensions [27].

Of the above, heterotic string theory [28] has been the most well studied superstring theory. In heterotic string theory the left-moving excitations are treated as a bosonic string in 26 dimensions and the right-moving excitations as a superstring in 10 dimensions. To avoid global anomalies in such a theory, modular invariance is imposed. This requires an even and 16 dimensional self-dual lattice. Even and self-dual lattices with Euclidean signature scalar product are truly rare. In fact there are only two even and self-dual lattices in 16 dimensions. These two lead to the two types of heterotic strings with different gauge groups in 10 dimensions: $SO(32)$ and $E_8 \times E_8$. These are the only choices for anomaly-free gauge groups that can couple to N=1 supergravity in 10 dimensions. The only other choices $U(1)^{496}$ and $E_8 \times U(1)^{248}$ lead to theories incompatible with heterotic string theory. In the $E_8 \times E_8$ version of the heterotic string, the SM family spectrum arises from spinor representations **16** of the $SO(10)$, an attractive GUT group. The $E_8 \times E_8$ heterotic string theory has turned out to be one of the most exciting approaches to string model building.

2.2 Compactifications of the $E_8 \times E_8$ heterotic string

Consider the 10-dimensional supergravity theory with N=1 supersymmetry that is coupled to Yang-Mills theory with the gauge symmetry $E_8 \times E_8$. This constitutes the effective field theory of the heterotic string. We assume that the spacetime is given by the direct product $M_4 \times X_6$, where M_4 is a maximally symmetric[3] space of

[2]theories which combine the principles of supersymmetry and general relativity, leading to theories with local supersymmetry.

[3]An n dimensional space with $\frac{n(n+1)}{2}$ Killing vectors is called maximally symmetric.

dimension four and X_6 is the six dimensional compactification space. X_6 should be Riemannian since we are interested in a theory of gravity which requires a metric. The existence of chiral fermions in the four-dimensional space is a requirement. This enforces that X_6 be irreducible, since a reducible compactification space will be difficult to reconcile with chiral fermions in the four-dimensional space and are thus phenomenologically not preferred [29]. These assumptions restrict the choice of X_6 to six-dimensional irreducible Riemannian manifolds with holonomy U(3), SU(3) or SO(6) owing to Berger's classification of such spaces by their holonomy [30]. Finally, if we demand that N=1 supersymmetry is preserved in four dimensions, this fixes the choice of holonomy to be SU(3).

Given a six-dimensional irreducible Riemannian manifold, it is difficult to conclude if the manifold admits a metric with SU(3) holonomy [31, 32]. However, according to the Calabi conjecture (proven by Yau), an n-dimensional complex Kähler manifold with vanishing first Chern class, admits a Ricci flat metric with $SU(n)$ holonomy. This leads us to the class of Calabi-Yau manifolds. One of the leading approaches to model building in string theory is the compactification of the heterotic string over Calabi-Yau threefolds [28, 33–35]. The ultimate goal of such a construction is to embed the SM of particle physics in all its detail. Although no such construction has actually been achieved, considerable amount of progress has been made.

Compactifications over CY manifolds often lead to models that do not satisfy cosmological constraints and often have massless scalars that are unobserved. Revisiting our assumptions above: demanding N=1 supersymmetry in 4 dimensions and assuming the factorized geometry of a maximally symmetric space M_4 and a compact space X_6, leads to M_4 and X_6 spaces being Minkowski and a CY manifold respectively. The assumption of this factorized geometry in turn amounts to demanding that the only non-trivial background that is turned on is the metric. Relaxing this assumption leads to more general compactifications. For a review of such (heterotic and other) string compactifications over non-Kähler, non-CY, generalised and exceptional complex geometries, we refer the reader to the reviews [36, 37]. Some of the alternate approaches to compactifications of the heterotic string can be found in [38–53]. We now proceed to discuss some of the generalities associated with CY manifolds and subsequently describe various CY compactifications of the heterotic string, which despite its shortcomings has been the frontrunner in realistic string model building.

2.2.1 Calabi-Yau threefolds

A Calabi-Yau manifold is a complex Kähler manifold with vanishing first Chern class. In the following we review what characterises a Calabi-Yau threefold. For a more descriptive account, we refer to [20, 54, 55]. We start by defining a complex manifold. A complex manifold of dimension n is a $2n$ dimensional real manifold that admits a complex structure J_b^a, that satisfies the following conditions:

$$\begin{aligned} J_b^a . J_a^c &= -\delta_b^c \\ 0 = N_{bc}^a &\equiv J_b^d(\partial_d J_c^a - \partial_c J_d^a) - J_c^d(\partial_d J_b^a - \partial_b J_d^a) \end{aligned} \tag{2.2.1}$$

where N_{bc}^a is the Nijenhuis tensor and is identically zero for a complex manifold. To define a Käher manifold, we first define what a Hermitian manifold is. A complex manifold is Hermitian if it is endowed with a metric g of the form

$$ds^2 = g_{a\bar{b}} dz^a d\bar{z}^{\bar{b}} \ . \tag{2.2.2}$$

A Kähler manifold is a Hermitian manifold such that the (1,1) form $\omega \equiv g_{a,\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}$ is closed i.e., $d\omega = 0$. Note that this implies that all one-dimensional complex manifolds are Kähler.

For heterotic string compactifications, we required the extra-dimensional compact manifold X_6 to have an $SU(3)$ holonomy. As previously stated, a conjecture by Calabi and proven by Yau states that an n dimensional complex Kähler manifold with vanishing first Chern class admits a metric with $SU(n)$ holonomy. However, till date there are no analytic expressions for a non-trivial metric with $SU(3)$ holonomy although these can be computed numerically.

2.2.2 Moduli space of Calabi-Yau manifolds and Hodge numbers

The total parameter space of a CY manifold consists of parameters related to its structure as a complex manifold and parameters related to the deformations of its Kähler metric. These parameter spaces are referred to as the complex structure moduli space and the Kähler structure moduli space respectively, which in turn are related to the cohomology groups of (2,1) forms and (1,1) forms respectively [56]. To determine the number of parameters needed to specify the SU(3) holonomy metric of a CY threefold, we define the Hodge numbers. The Hodge numbers ($h^{p,q}$) of a CY manifold are the dimensions of its (Dolbeault) cohomology classes $H^{p,q}$. Another topological invariant, the Euler characteristic denoted by χ, is a weighted sum of the Hodge numbers:

$$h^{p,q} = \dim\, H^{p,q} \quad \text{and} \quad \chi = \sum_{p,q=0}^{3} (-1)^{p+q} h^{p,q} \; . \tag{2.2.3}$$

For a smooth, connected, CY threefold with holonomy group $SU(3)$, $h^{0,0} = h^{3,0} = h^{0,3} = h^{3,3} = 1$, while $h^{1,1}$ and $h^{2,1}(= h^{1,2})$ remain unfixed, with all other Hodge numbers being zero. $h^{1,1}$ and $h^{2,1}$ determine the dimensions of the Kähler and complex structure moduli spaces, which together determine the complete moduli space of a CY threefold [56]. The Euler characteristic χ is given by $2(h^{1,1} - h^{2,1})$. In chapter 5, we will look at ways of computing the Hodge numbers of CY threefolds. Given a CY with the Hodge pair $(h^{1,1}, h^{2,1})$, the idea behind mirror symmetry [57–60] predicts a second CY manifold with the Hodge pair $(h^{2,1}, h^{1,1})$. Loosely speaking, the complex structure and Kähler structure moduli spaces are interchanged. This idea has had many interesting applications in the field of enumerative geometry. An equivalent way of defining a CY threefold is to require a compact Kähler manifold to have a unique and nowhere vanishing holomorphic (3,0)-form usually denoted by Ω. This holomorphic (3,0)-form will become important in differentiating between regular and R symmetries of Calabi-Yau threefolds, which we will encounter in chapter 4.

Examples of Calabi-Yau manifolds

The simplest known CY threefold is the quintic, which is a hypersurface of degree 5 embedded in a $\mathbb{P}^4$ and denoted by $\mathbb{P}^4[5]$ (throughout, by $\mathbb{P}^n$ we would mean $\mathbb{CP}^n$). We will explore this manifold in detail in the following chapter. The largest known class of CY manifolds is the class of toric CYs. With a view to constructing examples of CY threefolds, that yield a simple algebraic description, we define the class of Complete Intersection Calabi-Yau (CICY) threefolds. This class was motivated by Yau's construction of a manifold with Euler characteristic $\chi = -18$. Yau's manifold was defined over the product of two $\mathbb{P}^3$'s in terms of three polynomial constraints. The CICY threefolds form a sub-class of CY threefolds that can be expressed as the vanishing locus of a set of homogeneous polynomials with variables from a product of complex projective spaces. Let $X \subset \mathcal{A}$ denote a CICY manifold embedded in a product of m projective spaces, with $\mathcal{A} = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}$. We refer to $\mathcal{A}$ as the ambient space. The manifold X is defined as the common zero locus of K homogeneous polynomials $p_1, \ldots, p_K$. Let q_a^r denote the homogeneity degree of the a^{th} polynomial in the coordinates of the r^{th} projective space. The manifold is smooth if the hypersurfaces meet transversally, that is the K-form $\mathrm{d}p_1 \wedge \ldots \wedge \mathrm{d}p_K$ does not vanish on X. The coefficients specifying the defining polynomials can be changed, thus altering the complex structure of X. The deformation class of X is then specified by the configuration matrix, which collects the multi-degrees q_a^r of the defining polynomials.

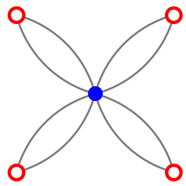
$$X = \begin{matrix} \mathbb{P}^{n_1} \\ \vdots \\ \mathbb{P}^{n_m} \end{matrix} \left[\begin{matrix} q_1^1 & \ldots & q_K^1 \\ \vdots & \ddots & \vdots \\ q_1^m & \ldots & q_K^m \end{matrix} \right]^{h^{1,1}(X),\ h^{2,1}(X)}_{\chi(X)} \tag{2.2.4}$$

where $\chi(X)$ stands for the Euler number of X. For a generic CY threefold, the only unspecified Hodge numbers are $h^{1,1}$ and $h^{2,1}$. These are listed as superscripts in (2.2.4). The complex dimension of X is given by $\sum_r n_r - K$, which for a threefold equals 3. The vanishing of the first Chern class of X (the Calabi-Yau condition), corresponds to the condition $\sum_a q_a^r = n_r + 1$, for each $r \in \{1, \ldots, m\}$. A detailed exposition of the class of CICYs, including finiteness of the class and the computation of its topological properties, can be found in [29].

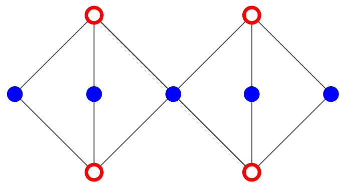
Examples of CICYs

Let $X \subset \mathcal{A} = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}$ be a CICY as defined in (2.2.4). One can associate a diagram to the configuration matrix (2.2.4) by drawing a blue disk for each polynomial, a red circle for each projective space and connecting the r-th projective space with the a-th polynomial by q_a^r lines. Some examples follow[4]:

$$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}^{4,68}_{-128} \tag{2.2.5}$$



$$\begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}^{8,44}_{-72} \tag{2.2.6}$$



It is easy to check that the sum of entries in each row in the examples above, equals the number of co-ordinates of the corresponding projective space. In addition, both the examples define a threefold because the dimension of the ambient space less the number of polynomials is 3 in both cases.

The CICY list

There are 7890 CY threefolds in the CICY list each specified by a configuration matrix, of which at least 2590 are known to be distinct as classical manifolds, but there are only 266 distinct pairs $(h^{1,1}, h^{2,1})$ of Hodge numbers [62]. The list of all CICYs can be downloaded online, see Ref. [63]. At first sight, the CICY list appears to be a menagerie of manifolds. Though true to some extent, this

[4]In a recent work [61], a generalization of Complete Intersection Calabi-Yau manifolds was presented, wherein some new manifolds, called gCICYs, were found. In this generalization, one of the columns in the CICY matrix above is allowed to assume negative integer values.

impression is mitigated by the fact that, among these manifolds, there are some very interesting spaces. The manifolds that admit symmetry groups of large order are particularly interesting.

Conifold transitions

An interesting aspect of CYs is how seemingly disconnected CYs form webs because of conifold transitions. In particular, the class of CICYs has some structure owing to the process of splitting and contraction (a detailed explanation may be found in [2] or [29]; see [64] for an illustrative account). Indeed the process of repeatedly splitting the matrices (2.2.4) that could not be further contracted was the process used to generate the list. Since this process relates the manifolds of the list it is natural that it should also relate the quotients [6]. This leads to webs of CICYs and its quotients that we explore in chapter 5. We move on to the study of further details of CY compactifications of the heterotic string.

2.2.3 Stability of vector bundles and anomaly cancellation

One of the standard approaches to $N = 1$ supersymmetric heterotic string compactifications is to compactify the theory on a CY X admitting a gauge connection A on a vector bundle $V \rightarrow X$, with field strength F, that satisfies the Hermitian Yang-Mills equations (for a more descriptive treatment including a numeric handling see [65]). These equations derive from the variations of the 10-dimensional fermionic gaugino field and are produced below:

$$F_{ab} = F_{\bar{a}\bar{b}} = 0 \quad \text{and} \quad g^{a\bar{b}} F_{a\bar{b}} = 0 \ . \tag{2.2.7}$$

The first two conditions imply that the vector bundle V is holomorphic. The final equation is relatively difficult to solve. However, a theorem due to Donaldson-Uhlenbeck-Yau [66, 67] states that a poly-stable holomorphic vector bundle V on a Kähler manifold admits a unique gauge connection A that satisfies the Hermitian Yang-Mills equations above. Therefore any poly-stable holomorphic vector bundle $V \to X$ will suffice for such compactifications preserving supersymmetry in 4 dimensions. For discussing poly-stability we introduce the slope

$$\mu(\mathcal{F}) = \frac{1}{rk(\mathcal{F})} \int_X c_1(\mathcal{F}) \wedge J^2 \ . \tag{2.2.8}$$

of a coherent sheaf $\mathcal{F}$, where J is the Kähler form of the CY X. A bundle V with a simple structure group is called stable iff $\mu(\mathcal{F}) < \mu(V)$ for all coherent sheafs $\mathcal{F} \subset V$ with $0 < rk(\mathcal{F}) < rk(V)$. Further, a bundle V is poly-stable if it can be decomposed into a sum of stable bundles with the same slope i.e., $V = V_1 \oplus \cdots \oplus V_n$ with $\mu(V_1) = \cdots = \mu(V_n) = \mu(V)$.

A second component of heterotic CY model building that we will consider now, derives from the condition of anomaly cancellation following the Green-Schwarz mechanism [68]. We merely state the anomaly cancellation condition below and refer the reader to [20, 22, 69] for a full treatment[5].

$$ch_2(TX) - ch_2(V) = W_{M5} \tag{2.2.9}$$

where ch_2 is the second Chern character, TX and V are the tangent bundle and a holomorphic vector bundle over X. $W_{M5} \in H_2(X, \mathbb{Z})$ is the class of holomorphic effective two-cycles in X that $M5$ branes can wrap.

2.2.4 Standard embedding

The scenario when V is chosen to be the holomorphic tangent bundle TX of the CY X with $SU(3)$ structure group (or its deformations), is referred to as the standard embedding and is reviewed in [70, 71]. It has many interesting features, some of which we now mention.

For starters, since $V = TX$, the Hermitian Yang-Mills equations are satisfied since the metric is Ricci flat. The tangent bundle is stable with slope 0. The breaking of the gauge group E_8 to the SM gauge group happens in two stages. First the E_8 is broken to a GUT group, which is then broken to the SM gauge

[5] The anomaly cancellation condition basically derives from modified Bianchi identities for the B field: $dH = trR \wedge R - trF \wedge F$, where H is the curvature of the B field and R is the curvature 2-form [20, 22, 69].

group. Although there are possible constructions that can lead to the SM gauge group directly without an intermediate GUT group, proceeding via the GUT group allows more freedom. It is important to note the value of the first step since E_8 is not a suitable GUT group, since it does not admit complex representations. Upon compactification, the gauge symmetry of the four dimensional theory is the commutant of the structure group of V in E_8. With the structure group being $SU(3)$ in this case, the GUT group of the corresponding low energy theory is E_6. The adjoint representation of E_8 then decomposes as [20]:

$$\begin{aligned} E_8 &\longrightarrow E_6 \times SU(3) \\ \mathbf{248} &\longrightarrow (\mathbf{78},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}) \oplus (\mathbf{27},\mathbf{3}) \oplus (\overline{\mathbf{27}},\overline{\mathbf{3}}) \ . \end{aligned} \tag{2.2.10}$$

In this thesis, although we will never be concerned with obtaining the 4-dimensional massless spectrum arising from heterotic compactifications, the chiral asymmetry will become important in chapters 6 and 7. The matter chiral multiplets in the four dimensional theory transform in the **27** and $\overline{\mathbf{27}}$ of E_6, and are given by the Hodge numbers of the Calabi-Yau[6]. The chiral asymmetry $N_{\text{gen}}(X)$ is then a function of the Euler characteristic of X alone:

$$\begin{gathered} n_{\mathbf{27}} = h^{1,1}(X) \ \text{ and } \ n_{\overline{\mathbf{27}}} = h^{2,1}(X) \\ N_{\text{gen}}(X) = |\#\text{ families} - \#\text{ anti-families}| = |h^{1,1}(X) - h^{2,1}(X)| = \frac{|\chi(X)|}{2} \ , \end{gathered} \tag{2.2.11}$$

where $n_{\mathbf{27}}$ and $n_{\overline{\mathbf{27}}}$ denote the number of chiral multiplets in the **27** and $\overline{\mathbf{27}}$ respectively. Obtaining the SM gauge group $SU(3) \times SU(2) \times U(1)$ proceeds by breaking the GUT group E_6 via discrete Wilson lines. This is contingent on the CY X having a non-trivial fundamental group Γ. For a threefold, the quotient manifold X/Γ is also a CY threefold [29]. On the quotient manifold, the chiral asymmetry is $N_{\text{gen}}(X/\Gamma) = N_{\text{gen}}(X)/|\Gamma|$.

It is noteworthy that the first heterotic string compactification using deformations of the standard embedding that yielded an exact MSSM spectrum, was only recently

[6]The low-energy fermions come from the 10-dimensional gaugino and their numbers are obtained by looking at the zero modes of the Dirac operator on the Calabi-Yau.

obtained [72]. Phenomenologically speaking, the GUT groups $SU(5)$ and $SO(10)$[7]. are more interesting [20]. These groups are closer to the SM gauge group. To break the heterotic E_8 to the SM gauge group via such GUTs, we need to consider embeddings more general than the standard embedding.

2.2.5 Non-standard embeddings

In order to obtain a GUT group that is closer to the SM gauge group, one must choose the structure group G of the vector bundle V that is larger than $SU(3)$ discussed in the previous subsection. This leads us to consider compactifications based on non-standard embeddings [18, 73–83]. Given the vector bundle V with structure group $G \subset E_8$, we noted above that the GUT group this leads to, is the commutant of G in E_8. The GUT group, often assumed simple, should contain the SM gauge group $SU(3) \times SU(2) \times U(1)$. Therefore, it should be of rank at least 4. In addition the GUT group must admit complex representations to admit chiral fermions. $SU(5)$ is the unique simple Lie group of rank 4 that admits complex representations. At rank 5, we only have the Lie groups $SU(6)$ and $SO(10)$. The GUT group $SU(6)$ is thought of as a simple extension of $SU(5)$ and may not necessarily lead to features significantly different than the $SU(5)$ case. Thus at rank 5, the $SO(10)$ GUT group is the more interesting one. At rank 6, we have $SU(7)$ and E_6. An argument similar to the one for the $SU(6)$ above, applies to the $SU(7)$. We discussed some of the features of string compactifications leading to the E_6 GUTs in the subsection above. Finally, there are also proposed GUT groups that are not simple e.g., the flipped $SU(5)$. Each of the GUT groups above have merits and demerits and we refer to [20] for an exposition to that effect. In (the second part of) this thesis we will be concerned with the GUT groups $SU(5)$ and SO(10). For a generic vector bundle V with the structure group G, the E_8 and its adjoint break as follows [20, 69]:

[7] The actual Lie group is Spin(10), the double cover of $SO(10)$, but it is common practice to use $SO(10)$.

$$\begin{aligned} E_8 &\longrightarrow \text{GUT} \times G \\ \mathbf{248} &\longrightarrow (\mathbf{adj}\ \text{GUT}, \mathbf{1}) \bigoplus \sum_k (\mathbf{R}_{\text{GUT},k}, \mathbf{R}_{G,k}) \bigoplus (\mathbf{1}, \mathbf{adj}\ \text{G}) \end{aligned} \tag{2.2.12}$$

where in the above, GUT denotes the GUT group and equals the commutant of the structure group G in E_8. **adj** denotes the adjoint representation. If we now consider the structure group G to be $SU(4)$, the resulting GUT group is then $SO(10)$ and E_8 breaks as follows:

$$\begin{aligned} E_8 &\longrightarrow SO(10) \times SU(4) \\ \mathbf{248} &\longrightarrow (\mathbf{45}, \mathbf{1}) \oplus (\mathbf{16}, \mathbf{4}) \oplus (\overline{\mathbf{16}}, \overline{\mathbf{4}}) \oplus (\mathbf{10}, \mathbf{6}) \oplus (\mathbf{1}, \mathbf{15}) \ . \end{aligned} \tag{2.2.13}$$

The full 4-dimensional massless spectrum can be obtained by computing the dimensions of the following cohomologies:

$$(n_{\mathbf{16}},\ n_{\overline{\mathbf{16}}},\ n_{\mathbf{10}},\ n_{\mathbf{1}}) = (h^1(X,V),\ h^1(X,V^*),\ h^1(X,\wedge^2 V),\ h^1(X, V \otimes V^*)) \ . \tag{2.2.14}$$

Since we are interested in computing the chiral asymmetry, we note this is given by the difference $n_{\mathbf{16}} - n_{\overline{\mathbf{16}}} = h^1(X,V) - h^1(X,V^*)$, which equals $h^1(X,V) - h^2(X,V)$. Using the fact that $h^0(X,V) = h^3(X,V) = 0$ (for stable $SU(n)$ bundles) and the conditions $c_1(TX) = c_1(V) = 0$ [69] ($c_1(V) = 0$ for special unitary bundles), we invoke the Atiyah-Singer index theorem [84–86] to cast the chiral asymmetry as the index of the bundle V:

$$\begin{aligned} N_{\text{gen}}(X) &= -\text{ind}(V) \ , \\ \text{ind}(V) &= \sum_a (-1)^a h^a(X,V) = \frac{1}{2}\int_X c_3(V) \ . \end{aligned} \tag{2.2.15}$$

For the case of $G = SU(5)$ leading to the GUT group $SU(5)$ the E_8 breaks as follows:

$$\begin{aligned} E_8 &\longrightarrow SU(5) \times SU(5) \\ \mathbf{248} &\longrightarrow (\mathbf{1}, \mathbf{24}) \oplus (\mathbf{10}, \mathbf{5}) \oplus (\overline{\mathbf{10}}, \overline{\mathbf{5}}) \oplus (\overline{\mathbf{5}}, \mathbf{10}) \oplus (\mathbf{5}, \overline{\mathbf{10}}) \oplus (\mathbf{24}, \mathbf{1}) \ . \end{aligned} \tag{2.2.16}$$

This leads to the following computation of the particle sprectrum:

$$(n_{\mathbf{10}},\ n_{\overline{\mathbf{10}}},\ n_{\overline{\mathbf{5}}},\ n_{\mathbf{5}},\ n_{\mathbf{1}}) = (h^1(X,V),\ h^1(X,V^*),\ h^1(X,\wedge^2 V), h^1(X,\wedge^2 V^*),\ h^1(X, V \otimes V^*)) \ . \tag{2.2.17}$$

The chiral asymmetry formula (2.2.15) holds in this case as well.

The requirements for Calabi-Yau compactifications of the heterotic string are truly stringent on the bundle. To construct realistic CY compactifications, one must first ensure that the structure group of the vector bundle V over the CY X can lead to a valid GUT group. The bundle V further needs to satisfy conditions of vanishing first Chern class (if V is a special unitary bundle [81]), stability (to ensure N=1 supersymmetry in 4-dimensions) and anomaly cancellation. In addition, the low energy theory must have the correct chiral asymmetry and massless spectrum. Once these conditions are met, one must derive a mechanism to break the GUT group to the SM gauge group using discrete Wilson lines, by ensuring that the CY X is not simply-connected. Computing the particle spectra and breaking of GUT to the SM gauge group are beyond the scope of this thesis. However we will focus on the remaining constraints on V to construct quasi-realistic line bundle models over smooth CY quotients in chapter 6. Line bundles have Abelian structure groups and hence different from the cases we have considered thus far.

2.2.6 Line bundle compactifications

One of the most straightforward vector bundle constructions over smooth CY threefolds is the line bundle construction[8] which can be thought of as a special case of split bundles [87,88]. The line bundle approach [7,8,81–83] is relatively new but has led to several GUT models many of which have led to SM like models. It is noteworthy that before the line bundle model building approach, only a handful of smooth heterotic standard models were known [75,78,82]. The basic idea behind these constructions is to treat the internal gauge field configurations as a set of $U(1)$ fluxes, i.e., as the sum of a number of line bundles [7]. In addition if we focus on favourable CYs X where the entire second cohomology descends from that of the ambient space, each line bundle is specified by a set of $h^{1,1}(X)$ integers $\mathbf{k}$:

$$V = \bigoplus_{a=1}^{\mathrm{n}} L_a, \quad \text{where} \quad L_a = \mathcal{O}_X(\mathbf{k_a}), \quad \text{and} \quad c_1(L_a) = \mathbf{k_a^i}.J_i \tag{2.2.18}$$

[8]A simple analog of the line bundle is the Möbius strip, a line bundle over the circle S^1.

where n is the rank of the vector bundle $rk(V)$, L_a are the component line bundles, and J_i are the $h^{1,1}(X)$ Kähler forms of the ambient space. V is then completely specified by $h^{1,1}(X) \times rk(V)$ integers. V must satisfy the condition of vanishing first Chern class: $c_1(V) = 0$. If we further assume that $c_1^r(L_a) \neq 0 \ \ \forall a$ for at least one $r \in \{1, \ldots, h^{1,1}(X)\}$ and that $\sum_{a \in S} c_1(L_a) \neq 0$ for all proper subsets $S \subset \{1, \ldots, 5\}$, it can be ensured that the structure group of V is the Abelian $S(U(1)^5)$ for $rk(V) = 5$ [9]. For the cases $rk(V) = 4$ (chapter 6) and $rk(V) = 5$ (chapter 7), if we denote the structure group of V by G, the breaking of the E_8 happens as follows:

$$
\begin{array}{lcccc}
 & E_8 & \longrightarrow & \text{GUT} \ \times & G \\
rk(V) = 4: & E_8 & \longrightarrow & SO(10) \ \times & S(U(1)^4) \\
rk(V) = 5: & E_8 & \longrightarrow & SU(5) \ \times & S(U(1)^5)\ .
\end{array}
\tag{2.2.19}
$$

There are some advantages of the line bundle construction. Given the simple nature of the construction and the Abelian nature of the structure group, they are more amenable to large computational efforts in algebraic geometry. The constructions in [7,8,81–83] and in chapter 6 clearly demonstrate this aspect. The model building requirements impose restrictions on the Chern classes (or equivalently characters) and the index of the line bundle sum $V = \bigoplus_{a=1}^{rk(V)} L_a$ which are summarised below:

$$
\text{ch}_1(V) = \sum_{a=1}^{rk(V)} k_a^i J_i, \quad \text{ch}_2(V).J_l = -\frac{1}{2} \sum_{a=1}^{rk(V)} d_{ijl}\, k_a^i\, k_a^j\ , \quad \text{ind}(V) = \frac{1}{6}\, d_{ijl} \sum_{a=1}^{rk(V)} k_a^i\, k_a^j\, k_a^l
\tag{2.2.20}
$$

where $d_{ijk} = \int_X J_i \wedge J_j \wedge J_k$ are the triple intersection numbers of the CY X. The number of resulting GUT models from a comprehensive scan of line bundle models over CY threefolds is also significantly large. This is essential since this increases the odds of embedding the SM into one of these many GUT models. Usually, line bundle constructions lead to GUT groups with additional U(1) factors. These U(1)'s are frequently broken and impose constraints on the effective field theory descriptions of these models. See [89] for example, for detailed results of line bundle constructions over smooth CY threefolds. The ingredients in heterotic CY compactification were summarised at the end of the previous subsection. In §6.2 we specialise those ingredients to the case where V is a sum of line bundles.

The class of CICY threefolds is very well suited for the construction of line bundle models for multiple reasons. First, the number of smooth CICYs, 7890, is large enough to construct plenty of line bundle models. Their freely acting symmetries have been classified [2,3] which provides numerous examples of non-simply connected CY threefolds as CICY quotients. This aids in breaking the GUT group to the SM gauge group. The global symmetries of the CICY quotients have also been classified, as part of this thesis [90, 91]. This provides examples of possible low energy discrete symmetries (regular and R), originating in the compactification space. Keeping these considerations in mind, we now move on to a discussion of discrete symmetries of CY threefolds.

2.3 Discrete symmetries

One of the key components of string model building is symmetries. In the discussion leading up to this, we sketched how the heterotic E_8 symmetry can be broken to the SM gauge group through intermediate GUT groups like $SO(10)$ and $SU(5)$. While these are all Lie symmetries, the class of discrete symmetries plays an equally important role in the overall scheme of model building. The breaking of the GUT group to the SM gauge group is facilitated by discrete Wilson lines. Once a quasi-realistic model is constructed using the means described before, known discrete symmetries of the low energy theory should agree with the surviving discrete symmetries of the string derived SM. In other words, discrete symmetries of the low energy theory are thought to descend from isometries of the compactification space in string compactifications. In addition, discrete symmetries of compactification spaces are interesting in their own right since they shed light on many algebro-geometric features of the compactification space.

2.3.1 Isometries of Calabi-Yau threefolds

It is noteworthy that a smooth CY threefold has only discrete isometries [29]. While we study in detail several aspects of discrete symmetries of CY threefolds in chapters 3-5, we take this opportunity to distinguish between the various kinds of discrete symmetries we will come across. While a CY manifold may not possess any discrete symmetry generically, in special loci of its moduli space, it may be highly symmetric. A simple example is that of the quintic threefold $\mathbb{P}^4[5]$, the most generic version of which has no symmetry. However in a special locus of its moduli space, it contains the following $\mathbb{Z}_5$ symmetry:

$$S : x_i \to \zeta^i x_i \ , \tag{2.3.1}$$

where $1+\zeta+\zeta^2+\zeta^3+\zeta^4 = 0$, and $x \in \mathbb{P}^4$. By restricting this locus even further, one obtains an even more symmetric quintic, invariant under the following two actions:

$$S : x_i \to \zeta^i x_i \quad \text{and} \quad T : x_i \to x_{i+4} \ . \tag{2.3.2}$$

The above actions generate the group $(\mathbb{Z}_5 \times \mathbb{Z}_5) \rtimes \mathbb{Z}_5$, a group of order 125, however, projectively, they simply generate the symmetry group $\mathbb{Z}_5 \times \mathbb{Z}_5$. In other words, S and T yield a projective representation of the group $\mathbb{Z}_5 \times \mathbb{Z}_5$. Both S and T act freely on the quintic, i.e., neither actions have any fixed points that lie on the corresponding quotient manifolds $\mathbb{P}^4[5]/\langle S\rangle$ and $\mathbb{P}^4[5]/\langle T\rangle$. These quotients along with $\mathbb{P}^4[5]/\langle S,\ T\rangle$ are smooth. It is impossible to proceed any further to find larger freely acting symmetries. However, quotients of CY manifolds by freely acting symmetries can still have remnant non-freely acting discrete symmetries either at a sub-locus of the moduli space of the quotient manifold or globally. To distinguish between these two we present an example for each. Consider the quintic quotient $\mathbb{P}^4[5]/\mathbb{Z}_5 \times \mathbb{Z}_5$ described above. This has following global non-freely acting $\mathbb{Z}_2$ symmetry:

$$\mathbb{Z}_2 : \quad x_0 \leftrightarrow x_4 \ , \ x_1 \leftrightarrow x_3 \ . \tag{2.3.3}$$

This symmetry has come for free in the sense that one did not have to restrict to a sub-locus of the quotient moduli space to obtain it. This has interesting implications. If one indeed restricts to a sub-locus of the quotient moduli space to find symmetries even bigger than the $\mathbb{Z}_2$ above, such symmetries must in fact contain the $\mathbb{Z}_2$ as a subgroup. The one-parameter quintic below is such an example which belongs to a sub-class of the class of quintic quotients $\mathbb{P}^4[5]/\mathbb{Z}_5\times\mathbb{Z}_5$ described above, but has a remnant non-freely acting symmetry, the dicyclic group of order 20 (Dic_5) that contains the $\mathbb{Z}_2$ above:

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + \Psi\ x_0x_1x_2x_3x_4 = 0\ . \tag{2.3.4}$$

The Dic_5 symmetry is generated by

$$\begin{aligned} &x_1 \to\ \zeta^3 x_3,\ x_2 \to\ \zeta^3 x_1,\ x_3 \to\ x_4,\ x_4 \to\ \zeta^4 x_2\ ,\ \text{and} \\ &x_1 \to\ \zeta^2 x_3,\ x_2 \to\ \zeta^2 x_1,\ x_3 \to\ x_4,\ x_4 \to\ \zeta\ x_2\ . \end{aligned} \tag{2.3.5}$$

It turns out that (2.3.4) is smooth for generic values of the parameter $\Psi \in \mathbb{C}$. However, often restricting to a sub-locus on the quotient space yields a non-smooth manifold. The following is an example of a non-smooth quintic quotient $\mathbb{P}^4[5]/\mathbb{Z}_5\times\mathbb{Z}_5$ with residual non-freely acting $\mathbb{Z}_{10}$ symmetry that contains the global $\mathbb{Z}_2$ above:

$$x_3x_4^2x_2^2 + x_0^2x_1x_2^2 + x_1^2x_3^2x_2 + x_0x_1^2x_4^2 + x_0^2x_3^2x_4 = 0\ . \tag{2.3.6}$$

The $\mathbb{Z}_{10}$ symmetry is generated by:

$$x_1 \to \zeta^4 x_1,\ x_2 \to \zeta^2 x_2,\ x_3 \to \zeta^4\ x_3,\ \text{and}\ x_0 \leftrightarrow x_4\ ,\ x_1 \leftrightarrow x_3\ . \tag{2.3.7}$$

Regular and R symmetries

Both freely acting and non-freely acting symmetries can survive the breaking of the string gauge group to the SM gauge group. However, there are subtleties. A freely acting symmetry necessarily preserves the holomorphic (3,0)-form Ω of the CY. However, a non-freely acting symmetry can transform Ω by a unitary phase. If this phase is trivial, then we classify the symmetry as a regular symmetry. Whenever this phase is non-trivial, the corresponding symmetry becomes an R symmetry in the associated 4-dimensional supersymmetric theory [1]. In fact R symmetries are those that do not commute with supersymmetry. The (3,0)-form Ω of the CY X, is related to the 4d-supersymmetry generators via the relation

$$\Omega_{ijk} = \eta^T \; \Gamma_{ijk} \; \eta \; , \tag{2.3.8}$$

where η is the internal spinor and Γ_{ijk} is the product of three gamma matrices. The superpotential W can then be seen to transform in the same way as Ω (see e.g., [92]). In order to compute the action of the discrete symmetry on the holomorphic (3,0)-form Ω on X, we introduce its ambient space counterpart $\widehat{\Omega}$. Assume that X is a complete intersection CY manifold such that $X \subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$, and is given by the vanishing locus of l polynomials $\{p^a\}$. The (3,0)-form $\widehat{\Omega}$ obeys the following relation derived from [93]:

$$\begin{gathered} \widehat{\Omega} \; \wedge dp^1 \; \wedge \cdots \wedge dp^l = \mu \; , \\ \text{where} \;\; \mu = \mu_1 \wedge \cdots \wedge \mu_k \;\; \text{and} \;\; \mu_j = \frac{1}{n_j!} \epsilon_{\beta_0 \beta_1 \ldots \beta_{n_j}} x_j^{\beta_0} dx_j^{\beta_1} \wedge \cdots \wedge dx_j^{\beta_{n_j}} \; . \end{gathered} \tag{2.3.9}$$

Given the symmetry action on the combined homogeneous co-ordinates $\{x_j^{\beta_i}\}$ of the ambient space and on the defining polynomials $\{p^a\}$, it is easy to determine the action of the symmetry on the ambient space (3,0)-form using (2.3.9). Since the symmetries of the CY threefold X descend from the linear automorphisms of the ambient space, computing the action of symmetries on $\widehat{\Omega}$ and Ω are equivalent.

A discrete R-symmetry is often invoked in the 4-dimensional theory to forbid operators that can lead to the decay of protons. We would like to understand the string theoretical origins of such and other discrete symmetries that are phenomenologically important. Our classification of the global discrete symmetries of all CICY quotients in chapter 4, results in several examples of discrete regular and R-symmetries that appear generically in the quotient moduli space. This could have significant phenomenological impact especially since many string derived SMs were recently constructed over this class of CICYs [7–9].

2.3.2 Constructing symmetric Calabi-Yau threefolds

In heterotic model building, the breaking of the GUT group to the SM gauge group usually requires a compactification manifold (X) with a non-trivial fundamental group $\pi_1(X)$. However, almost all known Calabi-Yau threefolds are simply connected. In [94, 95], the largest known class of CY threefolds was classified, of which only a handful were known to have non-trivial fundamental groups [96]. This is a serious impediment for large scale model building. Although examples of CY manifolds with non-trivial $\pi_1(X)$ were known, it was not until the classification efforts in [2, 3], that a large number of CY threefolds were obtained with non-trivial fundamental groups, in the form of CICY quotients. The basic idea behind this classification is the following. Corresponding to every non-simply connected manifold X, there exists a unique simply connected cover with a free $\pi_1(X)$ action. Thus, by looking for free actions on the list of CICYs with trivial fundamental group, one could obtain non-simply connected CICY quotients. Employing this idea, previous attempts to construct heterotic models on smooth CY quotients have been successful [70, 71, 97, 98]. Non-simply connected CY threefolds are also useful in moduli stabilisation [99]. The classification of smooth CICY quotients with freely acting groups is relatively involved and we refer the reader to the original articles [2, 3]. The automated classification of [3] relies on scanning through numerous multi-projective π-representations of all allowed groups acting on the CICY threefolds. This is followed by smoothness checks. It is noteworthy that the above classification only describes the linearly realised isometries that descend from the group of linear automorphisms of the ambient space in which the CICY is embedded. Non-linearly realised symmetries of CICYs may appear as linear symmetries in an equivalent representation of the CICY configuration matrix (2.2.4).

This classification of freely acting CICY quotients has led to 1695 distinct CY quotients by freely acting symmetries and is a valuable database to attempt to address interesting questions in string model building. What are the global symmetries of these CICY quotients? What further residual symmetries appear upon restricting the moduli space of these quotients? Which of these symmetries

survive the breaking of the heterotic gauge group to the SM gauge group? There are further geometric questions that arose as a result of this classification. The computation of Hodge numbers of CY threefolds and their quotients is an industry in itself. We review such techniques in chapter 5 and compute Hodge numbers of several CICY quotients obtained in the above classification.
The knowledge of isometries of the compactification manifold aides the construction of string derived standard models. With this motivation in mind we start by constructing highly symmetric quintic threefolds, after making a few comments on certain group theoretical aspects.

2.3.3 Group theory

Centralisers, Normalisers, and Automorphism Groups

Chapters 3–4 rely heavily on abstract group theory and projective representations of finite groups. Two group theoretical quantities that would be of considerable importance in the following two chapters are the centraliser and the normaliser. Consider the subgroup H of G. The centraliser (commutant) of H in G is defined as:

$$C_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g.h = h.g,\ \forall\ h \in H\}\ . \tag{2.3.10}$$

Similarly, the normaliser of the group H in G is defined as:

$$N_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g.H = H.g\}\ . \tag{2.3.11}$$

Further we note the result that the quotient of the normaliser by the centraliser is isomorphic to a subgroup of the automorphism group of H (denoted by Aut(H)). This is known as the NC theorem.

$$N_G(H)/C_G(H) \subseteq \text{Aut}(H)\ , \tag{2.3.12}$$

where Aut(H) is the automorphism group of H. An automorphism of a group H is a group isomorphism $\psi : H \to H$ i.e., ψ is a bijective homomorphism. The set of automorphisms of a group H, forms a group under the composition of maps, called the automorphism group of H.

Other notation

Several interesting finite groups appear in chapters 3–4. Examples include the well known quarternion group, denoted by $\mathbb{Q}_8$, the special linear group $\mathrm{SL}(2, \mathbb{Z}_5)$ which is isomorphic to the double cover of A_5.
We also encounter dicyclic groups (binary dihedral groups) denoted by Dic_n. Dic_n is a non-Abelian group of order $4n$ with the presentation

$$\langle a, x \mid a^{2n} = 1,\ x^2 = a^n,\ x^{-1}ax = a^{-1} \rangle \ . \tag{2.3.13}$$

Dic_n can be expressed as an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_{2n}$ as demonstrated by the following short exact sequence:

$$1 \to \mathbb{Z}_{2n} \to \mathrm{Dic}_n \to \mathbb{Z}_2 \to 1 \ . \tag{2.3.14}$$

3

Highly Symmetric Quintic Quotients

3.1 Introduction

Many Calabi-Yau manifolds are known but until fairly recently very few were known that have small Hodge numbers[1]. Finding such manifolds seems to require taking quotients, usually freely acting quotients of simply connected manifolds. This process of taking quotients then does double duty; on the one hand the Hodge numbers are reduced and on the other, the quotient manifold acquires a nontrivial fundamental group that permits flux breaking on the $E_8 \times E_8$ gauge group of the heterotic string, to a group that is of greater interest for the purpose of constructing a realistic heterotic string vacuum.

This much is an old story. The more recent element is that it has been possible to find freely acting quotients in a systematic way, albeit for a special class of Calabi-Yau manifolds, the so called CICYs (complete intersection Calabi-Yau manifolds) introduced in chapter 2. An important work in this direction was the systematic classification by V. Braun [3] of all freely acting quotients of manifolds on the CICY list.

Further work in [2, 6, 100, 101] calculated the Hodge numbers for each of these quotients, and was summarised in [102]. Not all of the quotient manifolds seem interesting but some are. Among these is the familiar example of the $\mathbb{Z}_5 \times \mathbb{Z}_5$

[1] By Hodge numbers $(h^{1,1}, h^{2,1})$ being small, somewhat loosely we mean $h^{1,1} + h^{2,1} \leq 20$.

quotient of the quintic:

$$\mathbb{P}^4[5]/\mathbb{Z}_5{\times}\mathbb{Z}_5, \quad (h^{1,1}, h^{2,1}) = (1, 5). \tag{3.1.1}$$

A number of other interesting quotients have a somewhat similar structure being quotients by a group $\mathbb{Z}_{p^n}{\times}\mathbb{Z}_{p^n}$ for a prime p and a positive integer n. These include:

$$\mathbb{P}^7[2,2,2,2]/\mathbb{Z}_8{\times}\mathbb{Z}_8, \quad \begin{matrix}\mathbb{P}^2\\ \mathbb{P}^2\end{matrix}\left[\begin{matrix}3\\ 3\end{matrix}\right]/\mathbb{Z}_3{\times}\mathbb{Z}_3 \tag{3.1.2}$$

and the $\mathbb{Z}_7 \times \mathbb{Z}_7$ quotient of Rodland's manifold [103]. The first manifold in (3.1.2) denotes the resolved quotients of resolutions of singular complete intersections of four quadrics in $\mathbb{P}^7$ [104]. A question that naturally arises is what symmetries do these quotient manifolds have? Such remnant symmetries may manifest themselves as global symmetries of the corresponding low energy theory and are thus potentially phenomenologically important. For the quintic quotient the general formalism for addressing this question was given by Witten [1] (in perhaps the second paper to be written on string compactification on Calabi-Yau manifolds with further results). A number of the possible symmetries, or rather the symmetric quintic quotients are given in that paper and further similar cases were provided by [105]. A number of quintics symmetric under finite simple groups were constructed in [4] employing various group and representation theory techniques and data from the ATLAS of finite groups [106].

Our interest is principally with the other quotients listed above (3.1.1) but any attempt at a complete treatment of these cases requires a certain amount of machine calculation. Developing the routines to analyse these quotients has brought us back to the quintic for which we reproduce the results of the classic papers together with many extensions. We intend to return to a parallel treatment of the quotients (3.1.2) in a future work. In this chapter, we will focus on the quintic class $\mathbb{P}^4[5]/\mathbb{Z}_5 \times \mathbb{Z}_5$, with a view to constructing sub-classes with non-trivial symmetries.

3.2 Preliminaries

The quintic is defined as the zero-locus of a homogeneous polynomial of degree 5 with variables in $\mathbb{P}^4$, denoted by $\mathbb{P}^4[5]$. It admits freely acting symmetries by the following two order 5 generators:

$$S : x_i \to \zeta^i x_i \quad \text{and} \quad T : x_i \to x_{i+4} \tag{3.2.1}$$

where x_i ($i \in \mathbb{Z}_5$) stands for the homogeneous co-ordinates of $\mathbb{P}^4$ and ζ is a nontrivial fifth root of unity. Note that $\langle S,\ T\rangle = \mathbb{Z}_5 \times \mathbb{Z}_5$. The most generic quintic, invariant under both S and T can be written as the zero locus of the following 5 parameter[2] family of degree 5 polynomials:

$$\begin{aligned}
x \in \mathbb{P}^4[5]\big/\mathbb{Z}_5\times\mathbb{Z}_5 \quad \text{satisfies} \quad p(x) \ &\stackrel{\text{def}}{=} \ \sum_{a=1}^{6} c_a \mathbf{J}_a \ = \ 0 \\
\text{where} \quad \mathbf{J}_1 &= \textstyle\prod_i x_i \\
\mathbf{J}_2 &= \frac{1}{5}\textstyle\sum_i x_{i-1}^2\, x_i\, x_{i+1}^2 \\
\mathbf{J}_3 &= \frac{1}{5}\textstyle\sum_i x_{i-2}^2\, x_i\, x_{i+2}^2 \\
\mathbf{J}_4 &= \frac{1}{5}\textstyle\sum_i x_{i-2}\, x_i^3\, x_{i+2} \\
\mathbf{J}_5 &= \frac{1}{5}\textstyle\sum_i x_{i-1}\, x_i^3\, x_{i+1} \\
\mathbf{J}_6 &= \frac{1}{5}\textstyle\sum_i x_i^5
\end{aligned} \tag{3.2.2}$$

The quintic and its quotient by the above freely acting $\mathbb{Z}_5 \times \mathbb{Z}_5$ symmetry are distinct families of manifolds evidenced by different Hodge numbers listed in Table 3.1.

[2]Only 5 of the six parameters c_κ are actually independent. One of the parameters can be used to scale the others without changing the zero-locus of (3.2.2).

Γ	$\mathbb{I}$	$\mathbb{Z}_5$	$\mathbb{Z}_5 \times \mathbb{Z}_5$
$h^{1,1}(X/\Gamma)$	1	1	1
$h^{2,1}(X/\Gamma)$	101	21	5
$\chi(X/\Gamma)$	−200	−40	−8

Table 3.1: *Hodge numbers and Euler characteristics of the quintic $X = \mathbb{P}^4[5]$ and its quotients by freely acting symmetries Γ.*

The generic family of quintics is a linear combination of 126 distinct monomials with no apparent symmetry. One can think of (3.2.2) as defining the sub-class of the generic quintic that is left invariant under the symmetry actions (3.2.1). It is therefore natural to ponder whether one could obtain quintic families with even larger symmetries upon further restriction of the defining polynomial (3.2.2). Before we embark on constructing such families it is worth noting that (3.2.2) admits a $\mathbb{Z}_2$ symmetry without any further restriction of $p(\bar{x})$ (3.2.2). This global discrete symmetry is:

$$\pi^{(2)} : \quad x_0 \leftrightarrow x_4 \ , \ x_1 \leftrightarrow x_3 \ . \tag{3.2.3}$$

It is easy to verify that $p(\pi^{(2)}.\bar{x}) = p(\bar{x})$. Symmetry groups of any sub-class of the quintic quotient defined by (3.2.2) must then contain this $\mathbb{Z}_2$ as a subgroup. This provides an important clue to listing symmetric quintic families, which brings us to a summary of the results of this work listed in Table 3.2.

Symmetry Group	# parameters	# sub-classes	smooth/singular	Reference
$\mathbb{Z}_4$	3	15	smooth	Table 3.4
$\mathbb{Z}_6$	1	20	4 conifold points	Table 3.5
$\mathbb{Q}_8$	1	15	4 conifold points	Table 3.6
$\mathbb{Z}_{10}$	0	24	singular curve(s)	Table 3.7
Dic_3	1	10	smooth	Table 3.8
Dic_5	1	6	smooth	Table 3.9

Table 3.2: *An overview of the results in this chapter. The quintic quotient has sub-families with the above listed symmetries. The table lists also the number of distinct sub-families corresponding to each symmetry group. Each of the groups listed above contains the $\mathbb{Z}_2$ defined in* (3.2.3) *as a subgroup. A detailed description of the families is given in the tables in §3.4. Smoothness of our constructions was studied in the accompanying paper [91].*

The table lists the various symmetry groups and the corresponding number of families of the quintic quotient (3.2.2). In summary, we have found one parameter families of quintic quotients with remnant $\mathbb{Z}_6$, $\mathbb{Q}_8$, Dic_3 and Dic_5 symmetries and some three parameter families with $\mathbb{Z}_4$ symmetries. In addition, we have also found 24 isolated quintic quotients with $\mathbb{Z}_{10}$ symmetries. In the following we outline our methods. The co-ordinate representations and invariants of these symmetries are tabulated in §3.4.

3.2.1 Symmetries of quintic quotients from automorphisms of $\mathbb{P}^4$

The linear symmetries of the quintic and all its free quotients derive from the linear automorphisms of $\mathbb{P}^4$. These symmetries can be thought of as projective linear actions on the homogeneous co-ordinates of $\mathbb{P}^4$. It is argued in [1] that any symmetry action π on the quintic that descends to its quotient by a freely acting group G_f must satisfy the following 'normalizer condition' (see §4.3):

$$\pi.g.\pi^{-1} \in G_f \quad \forall g \in G_f \ . \tag{3.2.4}$$

This ensures that points that lie on the quintic, transform sensibly under the combined action of any $g \in G_f$ and any new symmetry π. We note that all linear symmetries of the quotient manifold descend from the group PGL(5, $\mathbb{C}$), i.e., $\pi \in$ PGL(5, $\mathbb{C}$). For $G_f \cong \mathbb{Z}_5 \times \mathbb{Z}_5$ generated by S and T defined in (3.2.1), the above normalizer condition translates to:

$$\begin{aligned} \pi.S.\pi^{-1} &= S^{\alpha}.T^{\beta} \\ \pi.T.\pi^{-1} &= S^{\gamma}.T^{\delta} \ , \end{aligned} \tag{3.2.5}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_5$. One additionally notes that $S.T.S^{-1}.T^{-1} = \zeta$. Combined with (3.2.5) above, this implies that $(\alpha\delta - \beta\gamma) \equiv 1 \bmod 5$, or alternately,

$$\sigma \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2,\ \mathbb{Z}_5)\ . \tag{3.2.6}$$

In addition, there is a one-to-one correspondence between linear projective actions π and matrices $\sigma \in$ SL(2, $\mathbb{Z}_5$), upto equivalence of actions on the quotient $\mathbb{P}^4[5]/G_f$. This is because the only π's that satisfy the normalizer condition (3.2.5) for $\sigma = \mathbb{I}_2$, are the elements of G_f itself. The importance of the normalizer condition in our context can hardly be overemphasized. We note that the global $\mathbb{Z}_2$ symmetry (3.2.3) corresponds to $\sigma = -\mathbb{I}_2$ in the current formalism, and reemphasise that $p(\pi^{(2)}.\bar{x}) = p(\bar{x})$. It is clear from the above discussion that any symmetry group of the quintic quotient is a subgroup of SL(2, $\mathbb{Z}_5$). Given the conjugating nature of the action of π on G_f, one need only solve (3.2.5) for π for each generator of each subgroup of SL(2, $\mathbb{Z}_5$). There are a total of 75 non-trivial subgroups of SL(2, $\mathbb{Z}_5$). These are listed in Table 3.3.

R	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_5$	$\mathbb{Z}_6$	$\mathbb{Q}_8$	$\mathbb{Z}_{10}$	Dic_3	Dic_5	$\text{SL}(2,\mathbb{Z}_3)$	$\text{SL}(2,\mathbb{Z}_5)$
#	1	10	15	6	10	5	6	10	6	5	1
$\lvert R \rvert$	2	3	4	5	6	8	10	12	20	24	120

Table 3.3: *The non-trivial subgroups $R \subset SL(2, \mathbb{Z}_5)$ together with the number of distinct homomorphic copies, obtained using GAP [107]. The last row records the order of the subgroups.*

While the normalizer condition (3.2.5) ensures that additional symmetry action π sensibly transforms the elements of the freely acting $\mathbb{Z}_5 \times \mathbb{Z}_5$, it is not guaranteed at this stage that π will be a symmetry of this quotient manifold. To ensure this, we should check that $p(\pi.\bar{x}) \propto p(\bar{x})$ for some non-trivial sub-space of the six dimensional vector space $\mathbb{C}^6$ spanned by the coefficients c_κ in (3.2.2). For the global $\mathbb{Z}_2$ symmetry $\pi^{(2)}$ in (3.2.3), this is the entire $\mathbb{C}^6$, since $p(\pi^{(2)}.\bar{x}) = p(\bar{x})$ $\forall x \in \mathbb{P}^4[5]/\mathbb{Z}_5{\times}\mathbb{Z}_5$. We will find soon that this is not true for all π's satisfying the normalizer condition (3.2.5).

In fact, the action of π on $\bar{x}$ induces a linear map (from the right) on the vector of coefficients $\bar{c} \equiv (c_1, c_2, c_3, c_4, c_5, c_6)$. The eigenspaces of this linear map with non-zero eigenvalues therefore define sub-classes of the quintic quotient with an additional symmetry generated by π. We now proceed to describe through examples, the process of obtaining various quintic quotient families with additional symmetries.

3.2.2 A detailed example

$\mathbb{Z}_4$ and $\mathbb{Q}_8$ Families

Consider the subgroup $\mathbb{Z}_4$ of $\mathrm{SL}(2,\mathbb{Z}_5)$ generated by $\sigma = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$. The normalizer condition (3.2.5) reads:

$$\begin{aligned} \pi.S.\pi^{-1} &= S^2.T^0 \\ \pi.T.\pi^{-1} &= S^1.T^3 \ , \end{aligned} \tag{3.2.7}$$

where S and T are as defined in (3.2.1). Solving this set of linear equations, we obtain the following action on $\mathbb{P}^4$:

$$\pi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \zeta^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta^3 & 0 \end{pmatrix} \in \mathrm{PGL}(5,\mathbb{C}) \ . \tag{3.2.8}$$

It is straightforward to check that $\pi^4 \propto \mathbb{I}_5$. The above action π on $\bar{x} \in \mathbb{P}^4$ induces the following linear action A_π on $\bar{c}$ from the right:

$$A_\pi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta^4 & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta^3 & 0 \\ 0 & 0 & 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{3.2.9}$$

Thus π preserves a sub-class of the quintic quotient iff $\bar{c}.A_\pi{=}\lambda.\bar{c}$ for some non-null vector $\bar{c}$. The eigenvalues of A_π are -1 and 1 and have multiplicities 2 and 4

respectively. The corresponding eigenspaces, denoted by $\mathrm{E}_-(A_\pi)$ and $\mathrm{E}_+(A_\pi)$ are therefore 2 and 4 dimensional respectively. $\mathrm{E}_-(A_\pi)$ is generated by the vectors $(0,0,0,-\zeta^2,1,0)$ and $(0,-\zeta,1,0,0,0)$ and thus the corresponding $\mathbb{Z}_4$ invariant combinations of $(\mathbf{J}_\kappa)$ are $-\zeta^2\mathbf{J_4}+\mathbf{J_5}$ and $-\zeta\mathbf{J_2}+\mathbf{J_3}$. On the other hand $\mathrm{E}_+(A_\pi)$ is generated by $(1,0,0,0,0,0)$, $(0,0,0,\zeta^2,1,0)$, $(0,\zeta,1,0,0,0,0)$ and $(0,0,0,0,0,1)$, which correspond to the invariants $\mathbf{J_1}$, $\zeta^2\mathbf{J_4}+\mathbf{J_5}$, $\zeta\mathbf{J_2}+\mathbf{J_3}$ and $\mathbf{J_6}$.
The eigenspace $\mathrm{E}_+(A_\pi)$ thus corresponds to a three-parameter family of quintic quotients with a remnant $\mathbb{Z}_4$ symmetry, whereas the eigenspace $\mathrm{E}_-(A_\pi)$ corresponds to a one-parameter family of quintic quotients with remnant $\mathbb{Z}_4$ symmetry. It turns out that this family coincides with the one-parameter family of quintic quotients with an even higher symmetry $\mathbb{Q}_8$ (that contains this $\mathbb{Z}_4$) and generated by

$$\sigma_1=\sigma \quad \text{and} \quad \sigma_2=\begin{pmatrix}1 & 1\\ 3 & 4\end{pmatrix} . \tag{3.2.10}$$

This can be seen as follows. The normalizer condition (3.2.5) for the group elements π_1 and π_2 corresponding to σ_1 and σ_2 above reads:

$$\begin{aligned} &\pi_1.S.\pi_1^{-1}=S^2.T^0 && \pi_2.S.\pi_2^{-1}=S^1.T^1\\ &\pi_1.T.\pi_1^{-1}=S^1.T^3 && \pi_2.T.\pi_2^{-1}=S^3.T^4 . \end{aligned} \tag{3.2.11}$$

Solving these sets of linear equations for π_1 and π_2, we obtain:

$$\pi_1=\pi \quad \text{and} \quad \pi_2=\begin{pmatrix} 1 & \zeta & \zeta & 1 & \zeta^3\\ \zeta & \zeta & 1 & \zeta^3 & 1\\ \zeta^3 & \zeta^2 & 1 & \zeta^2 & \zeta^3\\ \zeta & \zeta^4 & \zeta & \zeta^2 & \zeta^2\\ 1 & \zeta^2 & \zeta^3 & \zeta^3 & \zeta^2 \end{pmatrix} . \tag{3.2.12}$$

The corresponding linear actions on $\bar{c}$ induced by π_1 and π_2 (from the right) are respectively given by A_{π_1} and A_{π_2} where:

$$A_{\pi_1}=A_\pi \quad \text{and} \tag{3.2.13}$$

$$A_{\pi_2} = 5\begin{pmatrix} -1 & 5\zeta^3 & 5\zeta^2 & -5\zeta & -5\zeta^4 & 1\\ 4\zeta^2 & 5(\zeta+\zeta^4) & 5(\zeta+\zeta^2) & 0 & 0 & \zeta^2\\ 4\zeta^3 & 5(\zeta^3+\zeta^4) & 5(\zeta+\zeta^4) & 0 & 0 & \zeta^3\\ -6\zeta^4 & 0 & 0 & -5(\zeta^2+\zeta^3) & -5(\zeta^2+\zeta^4) & \zeta^4\\ -6\zeta & 0 & 0 & -5(\zeta+\zeta^3) & -5(\zeta^2+\zeta^3) & \zeta\\ 24 & 30\zeta^3 & 30\zeta^2 & 20\zeta & 20\zeta^4 & 1 \end{pmatrix} .$$

The eigenvalues of A_{π_1} are 1 and -1 and have multiplicities 4 and 2 respectively. A_{π_2} has eigenvalues $25\,(2\zeta^3+2\zeta^2+1)$ and $-25\,(2\zeta^3+2\zeta^2+1)$ with multiplicities 2 and 4 respectively. The corresponding four eigenspaces are denoted by $E_{\pm}(A_{\pi_1})$ and $E_{\pm}(A_{\pi_2})$. To obtain quintic quotient sub-classes that are invariant under both the actions π_1 and π_2 i.e., under the $\mathbb{Q}_8$ above, we must find those vectors $\bar{c}$ that lie at the intersection of pairs of eigenspaces that derive from A_{π_1} and A_{π_2} respectively. It is a trivial exercise in linear algebra to show that $|E_{-}(A_{\pi_1})\cap E_{+}(A_{\pi_2})| = 0$. The space $E_{-}(A_{\pi_1})\cap E_{-}(A_{\pi_2})$ is generated by the vectors $(0,0,0,-\zeta^2,1,0)$ and $(0,-\zeta,1,0,0,0)$. The corresponding invariants are $-\zeta^2\mathbf{J_4}+\mathbf{J_5}$ and $-\zeta\mathbf{J_2}+\mathbf{J_3}$, which we had previously demonstrated to be an invariant of the $\mathbb{Z}_4$ symmetry generated by $\pi=\pi_1$. Finally, $|E_{+}(A_{\pi_1})\cap E_{+}(A_{\pi_2})| = |E_{+}(A_{\pi_1})\cap E_{-}(A_{\pi_2})|$ =2. Their corresponding invariants generating one-parameter families of quintic quotients with residual $\mathbb{Q}_8$ symmetries are listed in Table 3.4.3.

In principle, we should perform an analysis analogous to that of §3.2.2, above, for all the 15 $\mathbb{Z}_4$ subgroups of $\mathrm{SL}(2,\mathbb{Z}_5)$, however, by appealing to the Galois group action $\zeta\to\zeta^{\kappa}$, for $\kappa\in\{1,2,3,4\}$ we are able to reduce the number of cases that have to be analysed, as we explain in the following section. The upshot is that we find 15 one-parameter families and 15 three-parameter families of manifolds with $\mathbb{Z}_4$ symmetry. Each of the one-parameter families coincides with a one-parameter family of manifolds with a $\mathbb{Q}_8$ symmetry. These one-parameter families are grouped together with the other $\mathbb{Q}_8$ families in Table 3.6. The three-parameter families however do not possess a symmetry larger than $\mathbb{Z}_4$ and these are listed in Table 3.4. All the quintic quotients with remnant symmetries $\mathbb{Z}_6$, Dic_3 or Dic_5 turn out to be one-parameter families. The quintic quotients with precisely $\mathbb{Z}_{10}$ symmetry have no free parameters. Without repeating the previous analysis, for all the cases, we record the results in the tables of §3.4.

3.3 Galois actions

We have been careful not to specify a value for ζ, beyond the fact that it is a nontrivial fifth root of unity. Now we enquire as to how our calculations change if we replace ζ by ζ^κ, with $\kappa \in \{1,2,3,4\}$.

Consider the effect of this transformation, which we shall refer to as a κ-transformation, on the normalizer condition (3.2.5). In this relation the matrix T is independent of ζ, while the matrix S does depend on ζ.

$$S(\zeta) \;=\; \mathrm{diag}(1,\, \zeta, \zeta^2, \zeta^3, \zeta^4) \quad \text{and so} \quad S(\zeta^\kappa) \;=\; S(\zeta)^\kappa \;. \tag{3.3.1}$$

From this observation it is easy to see that if $\pi(\zeta)$ satisfies the normalizer condition with matrix $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ then $\pi(\zeta^\kappa)$ satisfies an analogous relation with matrix

$${}^\kappa\sigma \;=\; \begin{pmatrix} \alpha & \kappa^{-1}\beta \\ \kappa\gamma & \delta \end{pmatrix} \;, \tag{3.3.2}$$

where κ^{-1} denotes the inverse in $\mathbb{Z}_5^*$. Thus a κ-transform has the effect

$$(\sigma,\, \pi(\zeta),\, A(\zeta)) \to ({}^\kappa\sigma,\, \pi(\zeta^\kappa),\, A(\zeta^\kappa)) \;, \tag{3.3.3}$$

which we may think of as induced by an action of the Galois group on ζ, or more strictly, on the roots of the irreducible polynomial, of which ζ is a root

$$1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 \;=\; 0 \;. \tag{3.3.4}$$

This observation simplifies our task significantly, since the σ-matrices fall into orbits $\{{}^\kappa\sigma\}$ as κ varies. Consider, again, the case of families with $\mathbb{Z}_4$ symmetry. We learn from GAP, or from Table 3.3, that $\mathrm{SL}(2,\mathbb{Z}_5)$ has 15 distinct $\mathbb{Z}^4$ subgroups. These fall into 3 orbits of length 4, one orbit of length 2 and one orbit of length 1, as κ varies over the Galois group. Since we understand the action of κ on the eigenvectors of A we need only calculate these for one σ taken from each orbit, so for 5 matrices rather than 15.

The orbits we are counting, in this example, are orbits of distinct $\mathbb{Z}_4$ subgroups. 'Short' orbits arise if, for example, the matrix σ is invariant under κ-transformation, as is the case for

$$\sigma \;=\; \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \;. \tag{3.3.5}$$

It can also happen that the matrices ${}^{\kappa}\sigma$, as κ varies, are distinct but they do not generate 4 distinct $\mathbb{Z}_4$ subgroups. An example is provided by

$$\sigma ~=~ \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} ~, \tag{3.3.6}$$

for which ${}^4\sigma = ({}^1\sigma)^{-1}$ and ${}^3\sigma = ({}^2\sigma)^{-1}$, so only two $\mathbb{Z}_4$ groups are generated.

We should explain a general point that turns out to relate particularly to the families with precisely $\mathbb{Z}_6$ symmetry of Table 3.5. We have seen that we have to find the eigenvectors of matrices A_π with elements in the field $\mathbb{Q}(\zeta)$. The first step in finding such eigenvectors is to solve the characteristic polynomial for the eigenvalues. The characteristic polynomial has coefficients in $\mathbb{Q}(\zeta)$. In general, we would expect this polynomial to factor over some higher field. Somewhat surprisingly, in all cases, apart from the cases corresponding to symmetry precisely $\mathbb{Z}_6$, the eigenvalues take values in $\mathbb{Q}(\zeta)$ itself. For $\mathbb{Z}_6$, however, we find that the eigenvalues take values in $\mathbb{Q}(\zeta,\omega)$, with ω a nontrivial cube root of unity. An elementary application of the primitive element theorem ensures that we can achieve this field with a single extension. In this case $\mathbb{Q}(\zeta,\omega) ~=~ \mathbb{Q}(\epsilon)$ where $\epsilon ~=~ \omega\zeta$. This quantity is a fifteenth root of unity, but it is not a general such root. There are four choices for ζ and two for ω, so eight choices for ϵ. Indeed ϵ satisfies the following irreducible polynomial equation of degree eight

$$1 - \epsilon + \epsilon^3 - \epsilon^4 + \epsilon^5 - \epsilon^7 + \epsilon^8 ~=~ 0 ~. \tag{3.3.7}$$

Given ζ, there are two choices for ϵ. These are the common roots of the above polynomial with the relation $\epsilon^6 = \zeta$. The polynomial above is $\Phi_{15}(\epsilon)$, the fifteenth cyclotomic polynomial, and the roots are ϵ^k for k coprime to 15, that is for the eight values

$$\mathcal{G} ~\stackrel{\text{def}}{=}~ \{1,\, 2,\, 4,\, 7,\, 8, 11, 13, 14\} ~.$$

The two common roots of (3.3.7) and the equation $\epsilon^6 = \zeta$ are ϵ and ϵ^{11}. This quantity ϵ is important to us because it is the quantity that appears in Table 3.5.

Now, the set $\mathcal{G}$ is in fact a group, with the group operation being multiplication mod 15. Since $\mathcal{G}$ permutes the roots of (3.3.7) via

$$\epsilon^k \mapsto \epsilon^{jk} \ ; \quad j,\, k \in \mathcal{G} \ ,$$

it is the Galois group of the polynomial. As an abstract group, $\mathcal{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$; in terms of generators, we can write $\mathcal{G} = \langle 11,\, 2 \rangle$.

For the case of families that have precisely $\mathbb{Z}_6$ symmetry, each σ-matrix gives rise to two curves of symmetric manifolds. The group $\mathcal{G}$ has the order two element that maps $\epsilon \mapsto \epsilon^{11}$, which we have noted leaves ζ invariant. In each case, this interchanges the two curves. The order 4 element that maps $\epsilon \mapsto \epsilon^2$ induces a κ-transform via the relation $\zeta = \epsilon^6$, and this acts as before.

3.4 Tables

We have noted in §3.2.1 that each symmetric polynomial corresponds to a left eigenspace of a matrix A_π. Such an eigenspace corresponds to a family of symmetric manifolds, and has a basis of eigenvectors, the general eigenvector being a linear combination of these. In the tables, the basis eigenvectors become polynomials. Basis polynomials, for a given eigenspace, are separated by commas. The generic polynomial is then a linear combination of these. Distinct eigenspaces correspond to distinct families of manifolds. These are separated, in the tables, by vertical space. Thus the first row of Table 3.4, for example, corresponds to a single, three-parameter, family of manifolds, while the first row of Table 3.6 corresponds to three, one-parameter, families of manifolds.

The $\mathbb{Z}_2$ generator $\sigma = -\mathbf{1}_2$ generating the symmetry action $\pi^{(2)}$ defined in (3.2.3) is omitted from the tables with $\mathbb{Z}_6$ and $\mathbb{Z}_{10}$ symmetries, since $\pi^{(2)}$ is a symmetry of all the quotients. Instead, the generators for the $\mathbb{Z}_3$ and $\mathbb{Z}_5$ groups are presented respectively for brevity.

3.4.1 $\mathbb{Z}_4$ Families

σ	Symm Action	$\mathbb{Z}_4$ Covariants	# Fams.
$\begin{pmatrix} 4 & 4 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \zeta & \zeta & 1 & \zeta^3 \\ 1 & \zeta^2 & \zeta^3 & \zeta^3 & \zeta^2 \\ \zeta & \zeta^4 & \zeta & \zeta^2 & \zeta^2 \\ \zeta^3 & \zeta^2 & 1 & \zeta^2 & \zeta^3 \\ \zeta & \zeta & 1 & \zeta^3 & 1 \end{pmatrix}$	$\mathbf{J_2} - \zeta^4\mathbf{J_3}$, $\mathbf{J_4} - \zeta^3\mathbf{J_5}$, $\mathbf{J_1} + (\zeta^2+\zeta^4)\mathbf{J_2} + (\zeta+\zeta^2)\mathbf{J_5}$, $4\mathbf{J_1} + 10(\zeta^2+\zeta^4)\mathbf{J_2} + \mathbf{J_6}$	4
$\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \zeta^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta^3 & 0 \end{pmatrix}$	$\mathbf{J_1}$, $\mathbf{J_2} + \zeta^4\mathbf{J_3}$, $\mathbf{J_4} + \zeta^3\mathbf{J_5}$, $\mathbf{J_6}$	4
$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & \zeta & 1 & \zeta^2 & \zeta^2 \\ \zeta^2 & \zeta^2 & 1 & \zeta & 1 \\ \zeta & 1 & \zeta^2 & \zeta^2 & 1 \\ \zeta^2 & 1 & \zeta & 1 & \zeta^2 \\ 1 & \zeta^2 & \zeta^2 & 1 & \zeta \end{pmatrix}$	$\mathbf{J_2} - \zeta^3\mathbf{J_3}$, $\mathbf{J_4} - \zeta\mathbf{J_5}$, $\mathbf{J_1} + (1+\zeta^2)\mathbf{J_2} + (1+\zeta)\mathbf{J_5}$, $4\mathbf{J_1} + 10(1+\zeta^2)\mathbf{J_2} + \mathbf{J_6}$	4
$\begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \zeta^4 & \zeta^3 & \zeta^2 & \zeta \\ 1 & \zeta^3 & \zeta & \zeta^4 & \zeta^2 \\ 1 & \zeta^2 & \zeta^4 & \zeta & \zeta^3 \\ 1 & \zeta & \zeta^2 & \zeta^3 & \zeta^4 \end{pmatrix}$	$\mathbf{J_2} - \mathbf{J_3}$, $\mathbf{J_4} - \mathbf{J_5}$, $\mathbf{J_1} + (\zeta+\zeta^4)\mathbf{J_2} + (\zeta^2+\zeta^3)\mathbf{J_5}$, $4\mathbf{J_1} + 10(\zeta+\zeta^4)\mathbf{J_2} + \mathbf{J_6}$	2
$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	$\mathbf{J_1}$, $\mathbf{J_2} + \mathbf{J_3}$, $\mathbf{J_4} + \mathbf{J_5}$, $\mathbf{J_6}$	1

Table 3.4: *The families of manifolds with $\mathbb{Z}_4$ symmetries. Each of the σ matrices of the first three rows generate four $\mathbb{Z}_4$ subgroups of $SL(2,\mathbb{Z}_5)$ under the κ-transformation (3.3.2). Each of these rows give rise, in this way, to 4 distinct three-parameter families of manifolds. The penultimate σ matrix forms an orbit of length 2 under κ-transformation and so gives rise to 2 distinct three-parameter families of manifolds. The σ matrix of the final row is fixed by κ-transformation, and so gives rise to 1 three-parameter family of manifolds.*

3.4.2 $\mathbb{Z}_6$ Families

σ	**Symm Action**	$\mathbb{Z}_6$ **Covariants**
$\begin{pmatrix} 4 & 1 \\ 4 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \zeta^4 & \zeta^3 & \zeta^2 & \zeta & 1 \\ \zeta^2 & 1 & \zeta^3 & \zeta & \zeta^4 \\ \zeta^4 & \zeta & \zeta^3 & 1 & \zeta^2 \\ 1 & \zeta & \zeta^2 & \zeta^3 & \zeta^4 \end{pmatrix}$	$4\mathbf{J_1}-5\left(2-\epsilon-\epsilon^2+\epsilon^3-2\epsilon^4+\epsilon^5-2\epsilon^7\right)\mathbf{J_2}-5(1-\epsilon)^2(1+\epsilon)\mathbf{J_3}+\mathbf{J_6}\,,$ $2\left(-1+2\epsilon+\epsilon^2-\epsilon^3+\epsilon^4-\epsilon^5+2\epsilon^7\right)\mathbf{J_1}-\left(2+\epsilon+\epsilon^3+\epsilon^5+2\epsilon^6\right)\mathbf{J_2}$ $-\left(2-2\epsilon^2+\epsilon^5+\epsilon^6-\epsilon^7\right)\mathbf{J_3}$ $+\left(5-3\epsilon-4\epsilon^2+3\epsilon^3-2\epsilon^4+4\epsilon^5+3\epsilon^6-5\epsilon^7\right)\mathbf{J_4}+\mathbf{J_5}$ $4\mathbf{J_1}+5\left(1-\epsilon+\epsilon^3-2\epsilon^4+\epsilon^5+\epsilon^6-\epsilon^7\right)\mathbf{J_2}-5\left(1+\epsilon+\epsilon^3+\epsilon^6-\epsilon^7\right)\mathbf{J_3}+\mathbf{J_6}\,,$ $2\left(1-2\epsilon+\epsilon^2-\epsilon^4+\epsilon^5-\epsilon^6\right)\mathbf{J_1}-\left(1-\epsilon+\epsilon^3-\epsilon^5+\epsilon^6\right)\mathbf{J_2}$ $-\left(1-\epsilon^2-\epsilon^5+\epsilon^6-2\epsilon^7\right)\mathbf{J_3}$ $-\left(1-3\epsilon+3\epsilon^2-\epsilon^3-2\epsilon^4+4\epsilon^5-4\epsilon^6+2\epsilon^7\right)\mathbf{J_4}+\mathbf{J_5}$
$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & \zeta^3 & \zeta^4 & \zeta^3 & 1 \\ \zeta & \zeta^3 & \zeta^3 & \zeta & \zeta^2 \\ \zeta^3 & \zeta^4 & \zeta^3 & 1 & 1 \\ \zeta & \zeta & \zeta^4 & 1 & \zeta^4 \\ 1 & \zeta^4 & \zeta & \zeta & \zeta^4 \end{pmatrix}$	$4\mathbf{J_1}-5\left(\epsilon^2+\epsilon^4+\epsilon^5+\epsilon^7\right)\mathbf{J_2}-5\left(\epsilon-\epsilon^3+\epsilon^5\right)\mathbf{J_3}+\mathbf{J_6}\,,$ $2\left(2-\epsilon-\epsilon^2+\epsilon^3-\epsilon^4+2\epsilon^5+\epsilon^6-2\epsilon^7\right)\mathbf{J_1}$ $+\left(3-\epsilon-2\epsilon^2+\epsilon^3-2\epsilon^4+\epsilon^5+\epsilon^6-3\epsilon^7\right)\mathbf{J_2}$ $-\left(1-2\epsilon-\epsilon^2+2\epsilon^3-\epsilon^4+\epsilon^5+\epsilon^6-2\epsilon^7\right)\mathbf{J_3}$ $+\left(3+\epsilon-2\epsilon^2-\epsilon^3-2\epsilon^4+\epsilon^5+3\epsilon^6\right)\mathbf{J_4}+\mathbf{J_5}$ $4\mathbf{J_1}-5(1-\epsilon)^2\epsilon^3(1+\epsilon)\mathbf{J_2}+5\left(1+\epsilon+\epsilon^3+\epsilon^5+\epsilon^6\right)\mathbf{J_3}+\mathbf{J_6}\,,$ $2\left(-1+\epsilon-\epsilon^2+\epsilon^4-2\epsilon^5+\epsilon^6\right)\mathbf{J_1}+\epsilon\left(1-\epsilon-\epsilon^2+2\epsilon^3-\epsilon^4\right)\mathbf{J_2}$ $+\left(1-2\epsilon+\epsilon^2-\epsilon^3-\epsilon^4+\epsilon^5-2\epsilon^6\right)\mathbf{J_3}-\epsilon\left(1-\epsilon+\epsilon^2\right)^2\mathbf{J_4}+\mathbf{J_5}$
$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & \zeta^2 & \zeta & \zeta^2 & 1 \\ \zeta^2 & \zeta^3 & \zeta & \zeta & \zeta^3 \\ \zeta & \zeta & \zeta^3 & \zeta^2 & \zeta^3 \\ \zeta^2 & \zeta & \zeta^2 & 1 & 1 \\ 1 & \zeta^3 & \zeta^3 & 1 & \zeta^4 \end{pmatrix}$	$4\mathbf{J_1}+5\left(2-\epsilon+\epsilon^3-\epsilon^4+\epsilon^5-2\epsilon^7\right)\mathbf{J_2}+5\epsilon\left(1-\epsilon+\epsilon^3-\epsilon^4+\epsilon^6\right)\mathbf{J_3}+\mathbf{J_6}\,,$ $2\left(1+\epsilon-\epsilon^4+\epsilon^6\right)\mathbf{J_1}+\left(2-\epsilon-\epsilon^2+2\epsilon^3-\epsilon^4+\epsilon^5-3\epsilon^7\right)\mathbf{J_2}$ $+\epsilon\left(1+\epsilon-\epsilon^2-\epsilon^3+\epsilon^5+\epsilon^6\right)\mathbf{J_3}+\left(2+3\epsilon-\epsilon^3-\epsilon^4-\epsilon^5+2\epsilon^6+2\epsilon^7\right)\mathbf{J_4}-\mathbf{J_5}$ $4\mathbf{J_1}+5\epsilon\left(1-\epsilon+\epsilon^3-\epsilon^4+\epsilon^6\right)\mathbf{J_2}+5\left(2-\epsilon+\epsilon^3-\epsilon^4+\epsilon^5-2\epsilon^7\right)\mathbf{J_3}+\mathbf{J_6}\,,$ $2\epsilon\left(1-\epsilon+\epsilon^2-\epsilon^3+\epsilon^5-\epsilon^6\right)\mathbf{J_1}-\epsilon\left(1-2\epsilon+\epsilon^2+\epsilon^3-\epsilon^4\right)\mathbf{J_2}$ $+\left(1+\epsilon-2\epsilon^2+2\epsilon^3-\epsilon^4+\epsilon^6-2\epsilon^7\right)\mathbf{J_3}$ $+\left(-2+3\epsilon-3\epsilon^2+2\epsilon^3-\epsilon^4-\epsilon^5+2\epsilon^6-\epsilon^7\right)\mathbf{J_4}+\mathbf{J_5}$

Table 3.5: *The families of $\mathbb{Z}_6$ manifolds. Together with $\sigma = -\mathbf{1}_2$, each of the σ matrices of the first two rows generate four $\mathbb{Z}_6$ subgroups under κ-transformation, and so give rise to 8 distinct one-parameter families each. The final row of σ matrix along with $\sigma = -\mathbf{1}_2$, generates two $\mathbb{Z}_6$ subgroups, under κ-transformation and gives rise to four $\mathbb{Z}_6$ families.*

3.4.3 The $\mathbb{Q}_8$ Families

σ	**Symm Action**	$\mathbb{Q}_8$ **Covariants**	**# Fams.**
$\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \zeta^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta^3 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & \zeta & \zeta & 1 & \zeta^3 \\ \zeta & \zeta & 1 & \zeta^3 & 1 \\ \zeta^3 & \zeta^2 & 1 & \zeta^2 & \zeta^3 \\ \zeta & \zeta^4 & \zeta & \zeta^2 & \zeta^2 \\ 1 & \zeta^2 & \zeta^3 & \zeta^3 & \zeta^2 \end{pmatrix}$	$\mathbf{J_2} - \zeta^4 \mathbf{J_3}$, $\mathbf{J_4} - \zeta^3 \mathbf{J_5}$ $4\mathbf{J_1} + 5\left(1+\zeta\right)\mathbf{J_2} + 5\left(1+\zeta^4\right)\mathbf{J_3} + \mathbf{J_6}$, $6\mathbf{J_1} + 5\left(1+\zeta^2\right)\mathbf{J_4} + 5\left(1+\zeta^3\right)\mathbf{J_5} - \mathbf{J_6}$ $4\mathbf{J_1} + 5\left(\zeta^2+\zeta^4\right)\mathbf{J_2} + 5\left(\zeta+\zeta^3\right)\mathbf{J_3} + \mathbf{J_6}$, $6\mathbf{J_1} + 5\left(\zeta^3+\zeta^4\right)\mathbf{J_4} + 5\left(\zeta+\zeta^2\right)\mathbf{J_5} - \mathbf{J_6}$	12
$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 & \zeta^4 \\ 1 & \zeta^2 & \zeta^4 & \zeta & \zeta^3 \\ 1 & \zeta^3 & \zeta & \zeta^4 & \zeta^2 \\ 1 & \zeta^4 & \zeta^3 & \zeta^2 & \zeta \end{pmatrix}$	$\mathbf{J_2} - \mathbf{J_3}$, $\mathbf{J_4} - \mathbf{J_5}$ $4\mathbf{J_1} + 5\left(\zeta^2+\zeta^3\right)\left(\mathbf{J_2}+\mathbf{J_3}\right) + \mathbf{J_6}$, $6\mathbf{J_1} + 5\left(\zeta+\zeta^4\right)\left(\mathbf{J_4}+\mathbf{J_5}\right) - \mathbf{J_6}$ $4\mathbf{J_1} + 5\left(\zeta+\zeta^4\right)\left(\mathbf{J_2}+\mathbf{J_3}\right) + \mathbf{J_6}$, $6\mathbf{J_1} + 5\left(\zeta^2+\zeta^3\right)\left(\mathbf{J_4}+\mathbf{J_5}\right) - \mathbf{J_6}$	3

Table 3.6: *Families of manifolds with $\mathbb{Q}_8$ symmetry. Each row corresponds, initially, to 3 one-parameter families. The first row gives rise to an orbit of length 4 under κ-transformation, so to a total of twelve one-parameter $\mathbb{Q}_8$ families. The second row is invariant under κ-transformation, so gives rise to 3 such families.*

3.4.4 The rigid $\mathbb{Z}_{10}$ manifolds

σ	**Symm Action**	$\mathbb{Z}_{10}$ **Covariants**	# **Quotients**
$\begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & \zeta^2 & \zeta & \zeta^2 \\ 1 & \zeta^4 & 1 & \zeta^3 & \zeta^3 \\ 1 & \zeta^3 & \zeta^3 & 1 & \zeta^4 \\ 1 & \zeta^2 & \zeta & \zeta^2 & 1 \\ 1 & \zeta & \zeta^4 & \zeta^4 & \zeta \end{pmatrix}$	$4\mathbf{J_1} + 5\left(1+\zeta^4\right)\mathbf{J_2} + 5\left(\zeta^2+\zeta^4\right)\mathbf{J_3} + \mathbf{J_6}$ $4\mathbf{J_1} + 5\left(\zeta+\zeta^3\right)\mathbf{J_2} + 5\left(1+\zeta\right)\mathbf{J_3} + \mathbf{J_6}$ $6\mathbf{J_1} + 5\left(\zeta+\zeta^2\right)\mathbf{J_4} + 5\left(1+\zeta^2\right)\mathbf{J_5} - \mathbf{J_6}$ $6\mathbf{J_1} + 5\left(1+\zeta^3\right)\mathbf{J_4} + 5\left(\zeta^3+\zeta^4\right)\mathbf{J_5} - \mathbf{J_6}$	16
$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \zeta^4 & 0 & 0 & 0 \\ 0 & 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & 0 & \zeta^4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\mathbf{J_2}$ $\mathbf{J_3}$ $\mathbf{J_4}$ $\mathbf{J_5}$	4
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & \zeta & \zeta^3 & \zeta \\ \zeta & 1 & 1 & \zeta & \zeta^3 \\ \zeta^3 & \zeta & 1 & 1 & \zeta \\ \zeta & \zeta^3 & \zeta & 1 & 1 \\ 1 & \zeta & \zeta^3 & \zeta & 1 \end{pmatrix}$	$4\mathbf{J_1} + 5\left(\zeta^2+\zeta^3\right)\mathbf{J_2} + 5\left(\zeta+\zeta^4\right)\mathbf{J_3} + \mathbf{J_6}$ $4\mathbf{J_1} + 5\left(\zeta+\zeta^4\right)\mathbf{J_2} + 5\left(\zeta^2+\zeta^3\right)\mathbf{J_3} + \mathbf{J_6}$ $6\mathbf{J_1} + 5\left(\zeta^2+\zeta^3\right)\mathbf{J_4} + 5\left(\zeta+\zeta^4\right)\mathbf{J_5} - \mathbf{J_6}$ $6\mathbf{J_1} + 5\left(\zeta+\zeta^4\right)\mathbf{J_4} + 5\left(\zeta^2+\zeta^3\right)\mathbf{J_5} - \mathbf{J_6}$	4

Table 3.7: *Rigid manifolds with $\mathbb{Z}_{10}$ symmetry. Each row corresponds, initially, to four rigid $\mathbb{Z}_{10}$ manifolds. The first row has an orbit of length four under κ-transformation so gives rise to a total of sixteen manifolds. The second and third rows are invariant under κ-transformation so give rise to 4 manifolds each.*

3.4.5 Dic_3 Families

σ	**Symm Action**	**Dic_3 Covariants**	**# Fams.**
$\begin{pmatrix}3&4\\0&2\end{pmatrix}$ $\begin{pmatrix}0&3\\3&4\end{pmatrix}$	$\begin{pmatrix}1&\zeta&1&\zeta^2&\zeta^2\\1&\zeta^2&\zeta^2&1&\zeta\\\zeta^2&1&\zeta&1&\zeta^2\\\zeta&1&\zeta^2&\zeta^2&1\\\zeta^2&\zeta^2&1&\zeta&1\end{pmatrix}$ $\begin{pmatrix}1&1&\zeta^3&\zeta^4&\zeta^3\\1&\zeta^3&\zeta^4&\zeta^3&1\\1&\zeta&1&\zeta^2&\zeta^2\\1&\zeta^4&\zeta&\zeta&\zeta^4\\1&\zeta^2&\zeta^2&1&\zeta\end{pmatrix}$	$4\mathbf{J_1}-10\left(1+\zeta+2\zeta^3\right)\mathbf{J_2}+10\zeta\left(1-\zeta+\zeta^2\right)\mathbf{J_3}+\mathbf{J_6}\,,$ $\mathbf{J_1}-\left(1+\zeta+2\zeta^3\right)\mathbf{J_2}+\zeta\left(1-\zeta+\zeta^2\right)\mathbf{J_3}$ $-(1+\zeta)^2\mathbf{J_4}+(\zeta+\zeta^2-\zeta^4)\mathbf{J_5}$	4
$\begin{pmatrix}3&0\\1&2\end{pmatrix}$ $\begin{pmatrix}1&1\\2&3\end{pmatrix}$	$\begin{pmatrix}1&0&0&0&0\\0&0&0&\zeta^2&0\\0&\zeta^2&0&0&0\\0&0&0&0&1\\0&0&\zeta&0&0\end{pmatrix}$ $\begin{pmatrix}1&\zeta^3&\zeta^4&\zeta^3&1\\\zeta&\zeta^3&\zeta^3&\zeta&\zeta^2\\\zeta^3&\zeta^4&\zeta^3&1&1\\\zeta&\zeta&\zeta^4&1&\zeta^4\\1&\zeta^4&\zeta&\zeta&\zeta^4\end{pmatrix}$	$4\mathbf{J_1}-10\zeta(1+\zeta)^2\mathbf{J_2}+10\left(1+\zeta-\zeta^3\right)\mathbf{J_3}+\mathbf{J_6},$ $\mathbf{J_1}-\zeta(1+\zeta)^2\mathbf{J_2}+\left(1+\zeta-\zeta^3\right)\mathbf{J_3}$ $+\left(1+\zeta^3-\zeta^4\right)\mathbf{J_4}+\left(1-\zeta+\zeta^2\right)\mathbf{J_5}$	4
$\begin{pmatrix}2&0\\0&3\end{pmatrix}$ $\begin{pmatrix}2&2\\4&2\end{pmatrix}$	$\begin{pmatrix}1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&1&0&0&0\\0&0&0&1&0\end{pmatrix}$ $\begin{pmatrix}1&1&\zeta&\zeta^3&\zeta\\\zeta^4&\zeta&\zeta^4&\zeta^3&\zeta^3\\\zeta^4&\zeta^3&\zeta^3&\zeta^4&\zeta\\1&\zeta&\zeta^3&\zeta&1\\\zeta^2&1&\zeta^4&\zeta^4&1\end{pmatrix}$	$4\mathbf{J_1}-10\left(2+\zeta^2+\zeta^3\right)(\mathbf{J_2}+\mathbf{J_3})+\mathbf{J_6}\,,$ $6(\mathbf{J_2}+\mathbf{J_3})-4\left(2-3\zeta^2-3\zeta^3\right)(\mathbf{J_4}+\mathbf{J_5})$ $-\left(1-\zeta^2-\zeta^3\right)\mathbf{J_6}$	2

Table 3.8: *The families with Dic_3 symmetry. Each row corresponds, initially to one-parameter families. The first two rows admit orbits of length 4 under κ-transformation, so each give rise to 4 one-parameter families. The third row admits an orbit of length 2 under κ-transformation, so to two distinct Dic_3 families.*

3.4.6 Dic_5 Families

σ	Symm Action	Dic_5 Covariants	# Fams.
$\begin{pmatrix}0&1\\4&0\end{pmatrix}$ $\begin{pmatrix}1&3\\1&4\end{pmatrix}$	$\begin{pmatrix}1&1&1&1&1\\1&\zeta^4&\zeta^3&\zeta^2&\zeta\\1&\zeta^3&\zeta&\zeta^4&\zeta^2\\1&\zeta^2&\zeta^4&\zeta&\zeta^3\\1&\zeta&\zeta^2&\zeta^3&\zeta^4\end{pmatrix}$ $\begin{pmatrix}1&\zeta^4&\zeta&\zeta&\zeta^4\\\zeta^4&\zeta&\zeta&\zeta^4&1\\1&1&\zeta^3&\zeta^4&\zeta^3\\\zeta^3&\zeta&\zeta^2&\zeta&\zeta^3\\\zeta^3&\zeta^4&\zeta^3&1&1\end{pmatrix}$	$4\mathbf{J_1}+10\zeta^4\mathbf{J_2}+10\zeta\mathbf{J_3}+\mathbf{J_6}$, $6\mathbf{J_1}+10\zeta^3\mathbf{J_4}+10\zeta^2\mathbf{J_5}-\mathbf{J_6}$	4
$\begin{pmatrix}3&0\\4&2\end{pmatrix}$ $\begin{pmatrix}3&0\\1&2\end{pmatrix}$	$\begin{pmatrix}1&0&0&0&0\\0&0&0&\zeta^3&0\\0&\zeta^3&0&0&0\\0&0&0&0&1\\0&0&\zeta^4&0&0\end{pmatrix}$ $\begin{pmatrix}1&0&0&0&0\\0&0&0&\zeta^2&0\\0&\zeta^2&0&0&0\\0&0&0&0&1\\0&0&\zeta&0&0\end{pmatrix}$	$\mathbf{J_1}$, $\mathbf{J_6}$	1
$\begin{pmatrix}2&2\\0&3\end{pmatrix}$ $\begin{pmatrix}2&0\\0&3\end{pmatrix}$	$\begin{pmatrix}1&\zeta^4&\zeta^2&\zeta^4&1\\\zeta^4&1&1&\zeta^4&\zeta^2\\\zeta^4&\zeta^2&\zeta^4&1&1\\1&1&\zeta^4&\zeta^2&\zeta^4\\\zeta^2&\zeta^4&1&1&\zeta^4\end{pmatrix}$ $\begin{pmatrix}1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&1&0&0&0\\0&0&0&1&0\end{pmatrix}$	$6\mathbf{J_1}+10(\mathbf{J_4}+\mathbf{J_5})-\mathbf{J_6}$, $4\mathbf{J_1}+10(\mathbf{J_2}+\mathbf{J_3})+\mathbf{J_6}$	1

Table 3.9: *The families with Dic_5 symmetry. Each row corresponds, initially, to a one-parameter family. The first row admits an orbit of length 4 under κ-transformation. The second and third rows are invariant under κ-transformation, so give rise to one family each.*

3.5 Discussion

Constructing highly symmetric Calabi-Yau threefolds has clear phenomenological advantages in the overall scheme of building realistic heterotic string vacua. As a small step towards that goal, in this chapter we constructed highly symmetric quintic quotients with various symmetry groups: $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Q}_8$, $\mathbb{Z}_{10}$, Dic_3 and Dic_5. These constructions were made by using the formalism provided by Witten in [1]. Many of our constructions are smooth families of Calabi-Yau threefolds. All $\mathbb{Z}_4$, Dic_3 and Dic_5 families of the $\mathbb{Z}_5 \times \mathbb{Z}_5$ quintic quotient that we obtained in this chapter are smooth. The $\mathbb{Z}_{10}$ isolated quotients, the one-parameter $\mathbb{Z}_6$ and $\mathbb{Q}_8$ families are singular for generic choices of the parameters. We postpone a full analysis of the nature of these singularities to a future work.

One of our original motivations in constructing highly symmetric quintic quotients was to be able to construct symmetric quotients of the manifolds described in (3.1.1) and the $\mathbb{Z}_7 \times \mathbb{Z}_7$ quotient of Rodland's manifold [103]. With the machinery developed during the course of this chapter we are now ready to address this. However, we postpone that discourse to another future work.

Since the remnant symmetries of the Calabi-Yau quotients could potentially appear as discrete symmetries of the low-energy theory, it would be interesting to classify symmetries of all known Calabi-Yau quotients. Since the CICYs are a class of Calabi-Yaus whose freely acting symmetries have been classified, one could employ the methods of this chapter to the class of all CICY quotients totalling 1695 in number. However, one quickly realises the significant computational complexity in a treatment of all CICY quotients that is akin that of the quintic quotient in this chapter. Instead, we ask a more humble question: what are the global symmetries of all known CICY quotients? We find that this problem can be addressed by an automated machine computation. In fact such a computation will turn out to be far less expensive than the classification of all free quotients of the CICYs completed by Volker in [3]. This leads us to the next chapter where we address the question of classification of the global discrete symmetries of all CICY quotients.

4

Discrete Symmetries of Complete Intersection Calabi-Yau Manifolds

4.1 Introduction

Discrete symmetries are an important tool in string model building. In the framework of superstrings, discrete symmetries that are phenomenologically important cannot simply be postulated, but rather must be found as isometries of the compactification manifold. The classification of all freely acting symmetries of CICYs led to a large number of smooth CICY quotients [2, 3], 1695 to be exact. A total of 31 distinct freely acting groups were found in this classification, with the largest group G_f being of order 32. The results of the classification were compiled in [63]. The nature of the classification meant that these CICY quotients could still have remnant non-freely acting symmetries. In this chapter we will be concerned with such non-freely acting symmetries that are generically present in the moduli space of CICY quotients. Of course one could specialise to a particular locus of the moduli space where the quotients have symmetries even larger than the global symmetry, as in the case of the quintic quotient $\mathbb{P}^4[5]/\mathbb{Z}_5{\times}\mathbb{Z}_5$, which we discussed at length in the previous chapter. We noted there that the quintic quotient $\mathbb{P}^4[5]/\mathbb{Z}_5{\times}\mathbb{Z}_5$ has a global $\mathbb{Z}_2$ symmetry that is generic and found everywhere in the quotient moduli space. However upon specialising to certain loci in the moduli space, we could find larger symmetries than contain this $\mathbb{Z}_2$. While we were able to do this for the quintic, in this chapter we will try to answer the question: Can CY manifolds obtained as quotients have additional, generic symmetries, akin to the $\mathbb{Z}_2$ of the quintic quotient above?

Both freely acting and non-freely acting symmetries of the compactification space can survive as low energy symmetries, the former if it is not quotiented out. The global (generic) symmetries of a CY quotient that we will compute in this chapter, can be either regular or R symmetries depending on their action on the holomorphic (3,0)-form. Phenomenologically, both regular and R-symmetries are useful, and thus the knowledge of global symmetries of CY quotients is important in string model building.

4.2 An overview of results

Given the pair $(X,\, G_f)$ where X is a CICY and G_f is a discrete symmetry that acts freely on X, we have computed co-ordinate presentations of the global discrete symmetries of the quotient space X/G_f, i.e., symmetries that are generically present in the quotient space. While a complete list of the projective representations of all global symmetries of all CICY quotients is too long to present here, the global symmetry groups themselves are listed in Table 4.3. We find examples of both Abelian and non-Abelian groups as symmetries of various CICY quotients. An even shorter summary of the results is given in Table 4.1.

Of the 1695 known smooth CICY quotients, a total of 381 were found to admit nontrivial global discrete symmetries i.e., symmetries that are present at all points of the quotient moduli spaces. Of these, 187 such CICY quotients were found to have a remnant non-freely acting $\mathbb{Z}_2$ symmetry. The other groups that were obtained over several distinct CICY quotients are $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{D}_8$, $\mathbb{Z}_2^4$, $\mathbb{Z}_2{\times}\mathbb{D}_8$ and $(\mathbb{Z}_3{\times}\mathbb{Z}_3){\rtimes}\mathbb{Z}_2$. A total of 113 CICY quotients were found to have global R symmetries discussed in §2.3.1.

Group	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{D}_8$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2\times\mathbb{D}_8$	$(\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_2$
Regular symms	155	35	5	31	36	11	0	2	4
R-symms	52	0	0	33	25	0	3	0	0

Table 4.1: *All non-trivial regular and R-symmetry groups of smooth CICY quotients that appear in this classification, and their counts.*

Classification Data

As mentioned in chapter 2, the list of all CICYs and their topological data is available at [63]. This also contains the data related to the freely acting symmetries of all CICYs. A particular CICY quotient is thus identified by the pair of numbers (CICY#, SYMM#), the first denoting the location of the CICY within the CICY list and the second denoting the particular quotient of the given CICY, e.g., the pair (7890, 5) stands for a $\mathbb{Z}_5\times\mathbb{Z}_5$ quotient of the quintic $\mathbb{P}^4[5]$. The data from this classification containing the projective co-ordinate representations of all global symmetry groups of CICY quotients and their corresponding actions on the holomorphic (3,0)-form, is available at [108].

4.3 Preliminaries

An obvious approach to computing the global symmetries of smooth CICY quotients would be the one adopted in the automated classification by Braun [3]. In that, the linear symmetry actions on CICYs were computed by going through all multi-projective π-representations of all groups upto a certain order, which was computationally highly intensive. This limit on the order of the group was a result of the freely acting nature of the symmetries being classified. Since the symmetries we are classifying are necessarily non-freely acting, there are no suitable bounds on the group orders of such symmetries. Thus although Braun's approach would still work in our case, it would be at the cost of introducing an artificial bound on the group order, and reproducing all his results, which we know is computationally intensive. Instead we argue that any symmetry that descends to the CY quotient must obey a certain "normalizer condition" that depends on the freely acting group of the CY. This can be stated more precisely.

Normalizer condition

Given a CY X and a freely acting symmetry G_f acting on it, any new symmetry deriving from some abstract group G (of order infinity in our classification), that descends to the CY quotient X/G_f must be contained in the normalizer $N_G(G_f)$. This can be explained by reproducing the argument from [1]. In the quotient manifold X/G_f, any point $x \in X$ is equivalent to the point $g_f.x \; \forall g_f \in G_f$. Now, if π is an element of the new symmetry of X/G_f, $\pi.x$ is equivalent to $\pi.(g_f.x)$ in the quotient manifold. For this to be so for all $x \in X$, there should exist $h_f \in G_f$ such that $h_f(\pi.x) = \pi.(g_f.x)$, i.e., $\pi.g_f.\pi^{-1} \in G_f$, or alternately, $\pi \in N_G(G_f)$. This is the criterion for a new symmetry element π to sensibly transform the fundamental group in the quotient X/G_f and is referred to as the normalizer condition.

Thus in effect we use the results from Braun's classification i.e., the pair (X, G_f) to place stringent constraints on any symmetry that can descend to the CY quotient. The normalizer condition is of paramount importance in our classification, and was used to obtain some of the symmetries of the quintic quotiented by a freely acting $\mathbb{Z}_5 \times \mathbb{Z}_5$ symmetry in [1]. We discussed this example in detail in chapter 3 but we will revisit parts of that analysis to describe it in the formalism of this chapter. We will then proceed to present a general algorithm to obtain our classification in §4.3.2.

4.3.1 Discrete symmetries of the $\mathbb{Z}_5 \times \mathbb{Z}_5$ quintic quotient

The quintic has two $\mathbb{Z}_5$ linear actions which are given by the transformations

$$S : x_i \to \zeta^i x_i \quad \text{and} \quad T : x_i \to x_{i+4}, \tag{4.3.1}$$

where x_i, $i \in \mathbb{Z}_5$ denotes the homogeneous co-ordinates of the ambient space $\mathbb{P}^4$ of the quintic and ζ is a nontrivial fifth root of unity. Any additional linear

symmetry element 'π' must descend from the group of linear automorphisms of the ambient space $\mathbb{P}^4$ which we denote by G. The new symmetry element π must also sensibly transform the group G_f acting freely on the quintic. This is expressed as: $\pi \in N_G(G_f)$, where $N_G(G_f)$ is the normalizer of the group G_f in G.
For the case of the quintic quotiented by $G_f = \langle S, T \rangle$, this boils down to the following conditions:

$$\begin{aligned} \pi.S.\pi^{-1} &= S^{\alpha}.T^{\beta} \\ \pi.T.\pi^{-1} &= S^{\gamma}.T^{\delta} \ , \end{aligned} \tag{4.3.2}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_5$. We refer to (4.3.2) and its generalization to other CICY quotients, as the 'normalizer condition'. Witten goes on to show in [1] that since S and T commute upto an overall projective factor, this should imply that $(\alpha\delta - \beta\gamma) \equiv 1 \mod 5$, or alternately,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}_5) \ . \tag{4.3.3}$$

We note that the group $\mathrm{SL}(2, \mathbb{Z}_5) \subset \mathrm{Aut}(\mathbb{Z}_5 \times \mathbb{Z}_5) \cong \mathrm{GL}(2, \mathbb{Z}_5)$. This turns out to be so, because one can view the conjugating action of $\pi \in N_G(G_f)$ viz., $\pi.G_f.\pi^{-1}$ $(\equiv G_f)$, as a group isomorphism from G_f to itself i.e., an automorphism. Note that not all automorphisms of G_f can be induced by the normalizer elements.
$\mathrm{SL}(2, \mathbb{Z}_5)$ contains 120 elements and (like $\mathrm{SL}(2, \mathbb{Z})$) is generated by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ . \tag{4.3.4}$$

Corresponding to each of the generators listed above, one can then solve the matrix equations (4.3.2) to obtain co-ordinate representations of π. Note that if $\pi \in N_G(G_f)$ i.e., π obeys (4.3.2), then so does $\pi.c$, where $c \in C_G(G_f)$. $C_G(G_f)$ is the set of solutions to (4.3.2) with

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \tag{4.3.5}$$

i.e., all the group elements of G that commute with G_f. In this example, $C_G(G_f) \cong G_f$. This means that there is a unique solution to π in (4.3.2) modulo the centralizer. Thus we have $N_G(G_f)/C_G(G_f) \cong \mathrm{SL}(2, \mathbb{Z}_5)$ in line with the NC theorem (2.3.12).

In this work, we are interested in obtaining the additional symmetry group of the manifold CICY$/G_f$ i.e., the group $N_G(G_f)/G_f$. Fortunately, $C_G(G_f) \cong G_f$ in this case and we obtain

$$N_G(G_f)/G_f \cong \mathrm{SL}(2, \mathbb{Z}_5) \ . \tag{4.3.6}$$

This is not the end of the story though. Any additional symmetry element $\pi \in N_G(G_f)$, should map the zero-locus of the defining polynomial of $\mathbb{P}^4[5]/G_f$ to itself. The subgroup of $C_G(G_f)$ ($N_G(G_f)$) that preserves this zero-locus is denoted by $C^*_G(G_f)$ ($N^*_G(G_f)$). However, it turns out that only a $\mathbb{Z}_2$ subgroup of the full $\mathrm{SL}(2, \mathbb{Z}_5)$ preserves this, which is then the global (remnant) symmetry group of this quotient space. This additional $\mathbb{Z}_2$ action is given by:

$$x_0 \leftrightarrow x_4 \ , \ x_1 \leftrightarrow x_3 \ . \tag{4.3.7}$$

This corresponds to

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = -\mathbb{I}_2 \in \mathrm{SL}(2, \mathbb{Z}_5) \ . \tag{4.3.8}$$

This shows that the quotient of a quintic by the $\mathbb{Z}_5 \times \mathbb{Z}_5$ symmetry (4.3.1), always has a global $\mathbb{Z}_2$ symmetry. In this chapter we have generalized the above procedure to list the quotient groups $N^*_G(G_f)/G_f$ for all CICY quotients by freely acting symmetries G_f.

In chapter 3, we discovered larger symmetries of the quotient $\mathbb{P}^4[5]/\ (\mathbb{Z}_5 \times \mathbb{Z}_5)$ by restricting the parameter space of coefficients of the polynomial invariants in the definition of the quotient, all the while (often) still having a non-singular manifold. The quintic quotient was eventually shown to have additional symmetry groups $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Q}_8$, $\mathbb{Z}_{10}$, Dic_3 and Dic_5. While this is possible for the $\mathbb{Z}_5 \times \mathbb{Z}_5$ quotient of the quintic, such a computation is predicated on the chance that $C_G(G_f) \cong G_f$. We find that this is not the case for a general CICY quotient, and proceeding àla Witten to restrict the CICY quotient further, becomes intractable quickly.

Thus on purely computational grounds, we restrict ourselves to computing symmetries of CICY quotients without restricting the parameter space of coefficients in the polynomials any further than it already is. This also has the added advantage of avoiding smoothness checks, since we deal with CICY quotients that are already smooth. We now proceed to describe our classification algorithm[1].

4.3.2 The classification algorithm

Our goal is to compute the linearly realised global symmetries of all CICY quotients by freely acting discrete groups. The information at hand is the pair $(X \subset \mathcal{A}, G_f)$, where X denotes the CY threefold embedded in the ambient space $\mathcal{A}$. For the case of CICYs, $\mathcal{A}$ is a product of complex projective spaces. G_f is the largest group of freely acting symmetries of X. In terms of these quantities our goal is to compute the group of linear symmetries that fix the CY quotient X/G_f. Since all linearly realised symmetries must act on the combined co-ordinates of the ambient space $\mathcal{A}$, we assert that all linearly realised symmetries of X descend from the group of (linear) automorphisms of $\mathcal{A}$, denoted by $\mathrm{Aut_L}(\mathcal{A})$ and noted below:

$$X \subset \mathcal{A} = \ \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k} \ \ \text{and} \ \ G = \mathrm{Aut_L}(\mathcal{A}) \ . \tag{4.3.9}$$

Elements of G are group actions that can be represented by linear transformations of the combined homogeneous co-ordinates of $\mathcal{A}$. More precisely, these group actions are combinations of matrices in the different $\mathrm{PGL}(n_i+1,\mathbb{C})$'s and their permutations corresponding to the permutations of the factors $\mathbb{P}^{n_i}$ [3]. The most generic such group action r can thus be written as follows:

$$r \ = \ \Pi.\ \mathrm{diag}(\eta_1, \ldots, \eta_k), \ \text{where } \eta_i \in \mathrm{PGL}(n_i+1,\mathbb{C}) \tag{4.3.10}$$

and Π is a permutation action on the k blocks of $\{\eta_i\}$ such that only blocks of equal size may be permuted. Loosely speaking G can be thought of as a semi-direct product of the direct product group $\Pi_{i=1}^{k}\mathrm{PGL}(n_i+1,\mathbb{C})$ and a group that permutes projective spaces of the same dimensions. Thus the order of G is infinite. Note that $G_f \subset G$. For a new symmetry element $\pi \in G$ to descend to the quotient space X/G_f, it must obey the normalizer condition, i.e., $\pi \in N_G(G_f)$. However, to be a

[1]A more terse description of the algorithm can be found in the accompanying publication [109].

symmetry of X/G_f, π should preserve the zero locus of the defining polynomial(s) of X/G_f. The subgroup of $N_G(G_f)$ that preserves the zero-locus of these defining polynomial(s) is denoted by $N^*_G(G_f)$. Therefore the symmetry group of the quotient manifold X/G_f is the remnant group $N^*_G(G_f)/G_f$, whose computation could then proceed via the computation of $N_G(G_f)$.

To compute $N_G(G_f)$, we use the embedding of G_f in G that is available due to the classification efforts of [2, 3]. Using this representation of G_f, we look for matrices $r \in G$ that normalise G_f, i.e.,

$$r.G_f.r^{-1} = G_f \ . \tag{4.3.11}$$

Solving (4.3.11) for r involves solving a set of $|G_f|$ matrix equations, a total of $|S_{|G_f|}| = |G_f|!$ times, since the above is an equality as sets. This is computationally very intensive since $2 \leq |G_f| \leq 32$ for the CICYs. There is a simple way out. Each element of the normalizer defines an automorphism of G_f via the conjugating action expressed in (4.3.11). Given $\pi \in N_G(G_f)$, consider the map:

$$\begin{aligned} \psi_\pi : G_f \ &\rightarrow \ G_f \\ \psi_\pi(g_f) \ &= \ \pi.g_f.\pi^{-1} \ . \end{aligned} \tag{4.3.12}$$

This map is well-defined since $\pi.g_f.\pi^{-1} \in G_f$, $\forall\ \pi \in N_G(G_f)$. This essentially says that each element of the normalizer $N_G(G_f)$ induces a map ψ_π from G_f to itself. Additionally, we claim that ψ_π is an automorphism of the group G_f i.e., ψ_π is a bijective homomorphism from G_f to itself. It can be understood as follows: ψ_π is a homomorphism from G_f to itself since

$$\psi_\pi(g_f.h_f) = \pi.g_f.h_f.\pi^{-1} = (\pi.g_f.\pi^{-1}).(\pi.h_f.\pi^{-1}) = \psi_\pi(g_f).\psi_\pi(h_f) \ . \tag{4.3.13}$$

This implies that $\psi_\pi(e) = e$ and $\psi_\pi(g_f^{-1}) = \psi_\pi^{-1}(g_f)$. It remains to be shown that ψ_π is a bijection. Injectivity follows, since if $\psi_\pi(g_f) = \psi_\pi(h_f)$, then $\pi.g_f.\pi^{-1} = \pi.h_f.\pi^{-1}$, which then implies that $g_f = h_f$. In addition, the pre-image of the element $g_f \in G_f$ under ψ_π is given by $\pi^{-1}.g_f.\pi$, which clearly lies in G_f since $\pi \in N_G(G_f)$, proving surjectivity of ψ_π.

Having established that each element of the normalizer induces a group automorphism, one could then solve (4.3.14) for matrices $r \in G$ when an automorphism ψ of G_f is supplied. In our classification we used the computer algebra system GAP to compute $\mathrm{Aut}(\mathrm{G}_f)$. They are summarised in §A.1, where we also demonstrate a way of computing the group $\mathrm{Aut}(\mathbb{Z}_5 \times \mathbb{Z}_5)$.

$$r.g_f.r^{-1} = \psi(g_f), \quad \forall\, g_f \in G_f \ . \tag{4.3.14}$$

Note that one can think of solving r in (4.3.14) as obtaining the preimage of the automorphism $\psi \in \mathrm{Aut}(G_f)$ with respect to the following homomorphic map:

$$\begin{aligned} \phi : N_G(G_f) \quad &\rightarrow \quad \mathrm{Aut}(G_f) \\ \phi(\pi) \quad &= \quad \psi_\pi \end{aligned} \tag{4.3.15}$$

where ψ_π is as defined in (4.3.12). The kernel of ϕ in (4.3.15) is the centralizer $C_G(G_f)$. By the first isomorphism theorem, $\phi/\mathrm{Ker}(\phi) \cong \mathrm{Im}(\phi)$, arriving at the well known result (often referred to as the NC theorem [110]),

$$N_G(G_f)/C_G(G_f) \cong H \subseteq \mathrm{Aut}(G_f) \ . \tag{4.3.16}$$

This is an useful result esp for the cases where $C_G(G_f) = G_f$, as is true for the case of the $\mathbb{Z}_5{\times}\mathbb{Z}_5$ quotient of the quintic[2]. Note that, given $\psi \in \mathrm{Aut}(G_f)$ one obtains in general, a number of matrices r by solving the equation (4.3.14) above. The full normalizer $N_G(G_f)$ is then given by all such r's for all automorphisms ψ. The task is simplified by computing r's only for the generators of each subgroup of $\mathrm{Aut}(G_f)$, and then using the resulting r's to freely and projectively generate the normalizer $N_G(G_f)$ in case this yields a finite group. In most cases however, this is not a finite group. The r's are not fully determined by simply solving the matrix equations (4.3.14). Therefore, we impose the following restriction, where $\{p^\mu(\bar{x})\}$ denotes the set of defining polynomials for X/G_f :

$$\text{zero-locus of } \{p^\mu(r.\bar{x})\} = \text{zero-locus of } \{p^\mu(\bar{x})\} \ . \tag{4.3.17}$$

[2]For the case of the quintic manifold with the $\mathbb{Z}_5{\times}\mathbb{Z}_5$ freely acting symmetry, $H \cong \mathrm{SL}(2, \mathbb{Z}_5) \subset \mathrm{Aut}(G_f) \cong \mathrm{GL}(2, \mathbb{Z}_5)$.

Since a smooth CY threefold has no continuous symmetries, imposing (4.3.17) fixes the r's completely, completing the classification program and yielding the subgroup $N_G^*(G_f) \subset N_G(G_f)$ such that $r \in N_G^*(G_f)$ if $r \in N_G(G_f)$ and r satisfies (4.3.17). The global symmetry group of X/G_f is then the quotient group $N_G^*(G_f)/G_f$.

4.4 Discrete symmetries of CICY quotients

With the help of a couple of examples below, we will describe the broad steps in our classification program. In Table 4.3 that will follow thereafter, we will summarise the group theoretical results of the classification program. As noted before, the explicit co-ordinate presentations can be accessed online. Table 4.3 lists the quotient groups $N_G^*(G_f)/G_f$ for all CICY quotients when it is non-trivial. These include quotients for which either $C_G^*(G_f)$ or $N_G^*(G_f)$ or both are bigger than G_f. We now proceed to describe the examples.

4.4.1 Examples

A CICY quotient with a remnant non-Abelian symmetry

Consider the $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotient of the split bicubic

$$X_{14} = \begin{matrix}\mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2\end{matrix}\begin{bmatrix}1 & 1 \\ 3 & 0 \\ 0 & 3\end{bmatrix}^{19,19}_{\chi=0} . \tag{4.4.1}$$

Let the combined homogeneous co-ordinates of the ambient space $\mathcal{A} = \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ be denoted by: $\bar{w} := [\ x_0, x_1 \mid y_0, y_1, y_2 \mid z_0, z_1, z_2\ ]$. The freely acting $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry (G_f) acts on $\bar{w}$ via the following two order three actions:

$$\begin{aligned} S:\ & \bar{x} \to \bar{x}, \quad y_i \to \omega^{2i} y_i, \quad z_i \to \omega^{2i} z_i \\ T:\ & \bar{x} \to \bar{x}, \quad y_i \to\ y_{i+2}, \quad z_i \to z_{i+2}\ , \end{aligned} \tag{4.4.2}$$

where $1 + \omega + \omega^2 = 0$. X_{14}/G_f is defined as the common zero-locus of the following two polynomials invariant under the above free group action generated by S and T:

$$\begin{aligned} p_1(\bar{w}) &= (a_1 x_0 + a_2 x_1)\ z_0 z_1 z_2 + (a_3 x_0 + a_4 x_1)\ (z_0^3 + z_1^3 + z_2^3) \\ p_2(\bar{w}) &= (a_5 x_0 + a_6 x_1)\ y_0 y_1 y_2 + (a_7 x_0 + a_8 x_1)\ (y_0^3 + y_1^3 + y_2^3)\ , \end{aligned} \tag{4.4.3}$$

where a_i are arbitrary coefficients $\in \mathbb{C}$. First we note that any linear symmetry r of X_{14}/G_f must descend from the group of linear automorphisms of the ambient space $\mathcal{A}$ i.e., $\mathrm{Aut}_{\mathrm{L}}(\mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^2) \equiv G$. Requiring this new symmetry r to satisfy the normalizer condition (4.3.11), and to keep the zero-locus of the $\{p_i(\bar{w})\}$ fixed, yields the full symmetry group of X_{14}/G_f denoted by $N^*_G(G_f)$. In this case our classification algorithm yields $N^*_G(G_f) = \mathbb{Z}_3^4 \rtimes \mathbb{Z}_2$, which is a non-Abelian group of order 162. Therefore the remnant symmetry group, i.e, the global symmetry group of this CICY quotient can be obtained by taking the freely acting G_f from the full symmetry group $N^*_G(G_f)$. This is accomplished by computing the quotient group $N^*_G(G_f)/G_f$, which in this case turns out to be $\mathbb{Z}_3^2\rtimes\mathbb{Z}_2$, a group of order 18. It is generated by the following actions on the combined homogeneous-coordinates:

$$\begin{aligned}
r_1 &: \ \bar{x} \to \bar{x}, \qquad y_i \to \ y_{i+2}, \qquad z_i \to \omega^{i+2} z_{i+2} \\
r_2 &: \ \bar{x} \to \bar{x}, \qquad y_i \to y_{i+2}, \qquad z_i \to z_i \\
r_3 &: \ \bar{x} \to \bar{x}, \qquad y_i \to \ \omega^{2i} \ y_{2-i}, \qquad z_i \to \omega^{2i+1} \ z_{2-i} \ .
\end{aligned} \tag{4.4.4}$$

One notes that both the freely acting group G_f and the remnant symmetry group $N^*_G(G_f)/G_f$ act trivially on the set of polynomials $\{p_i(\bar{w})\}$. The same analysis applied to the other quotients of the split bicubic X_{14} (4.4.1) yields different global symmetry groups summarised in Table 4.2 below:

G_f	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_3\times\mathbb{Z}_3$	$\mathbb{Z}_3\times\mathbb{Z}_3$
$N^*_G(G_f)/G_f$	$\mathbb{I}$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$

Table 4.2: *Global symmetries of quotients of the split bicubic* (4.4.1) *by freely acting groups* G_f.

A CICY quotient with a remnant R symmetry

Consider the $\mathbb{Z}_2{\times}\mathbb{Z}_2{\times}\mathbb{Z}_2$ quotient of the transpose of the tetraquadric manifold:

$$X_{7861} = \mathbb{P}^7 \begin{bmatrix} 2 & 2 & 2 & 2 \end{bmatrix}^{1,65}_{\chi=-128} \ . \tag{4.4.5}$$

Let the homogeneous co-ordinates of the ambient space $\mathbb{P}^7$ be denoted by x_i, with $i \in \mathbb{Z}_7$. The freely acting $\mathbb{Z}_2{\times}\mathbb{Z}_2{\times}\mathbb{Z}_2$ symmetry (G_f) acts on $\bar{x}$ via the following three order two actions:

$$\begin{aligned} S &= \mathrm{diag}(1,-1,\ \ 1,-1,\ \ 1,-1,\ \ 1,-1)\ , \\ T &= \mathrm{diag}(1,-1,-1,\ \ 1,\ \ 1,-1,-1,\ \ 1)\ , \\ U &= \mathrm{diag}(1,-1,-1,-1,-1,\ \ 1,\ \ 1,\ \ 1)\ . \end{aligned} \tag{4.4.6}$$

X_{7861}/G_f is defined as the common zero-locus of the following polynomials invariant under the above free group action generated by S, T and U:

$$p_i \ = \ \sum_j a_{ij} x_j^2\ , \tag{4.4.7}$$

for $i \in \{1,2,3,4\}$, where $a_{ij} \in \mathbb{C}$ are arbitrary coefficients. Carrying out a procedure summarised in the example above and detailed in §4.3.2, we get the full symmetry group of the quotient of $X_{7861}/\mathbb{Z}_2^3$ to be $\mathbb{Z}_2^7$ an Abelian group of order 128.
The global symmetry group of this CICY quotient is thus the quotient group $\mathbb{Z}_2^7/G_f \cong \mathbb{Z}_2^4$, an Abelian group of order 16. It is generated by the following actions on the combined homogeneous-coordinates:

$$\begin{aligned} g_1 &= \mathrm{diag}(1,-1,-1,-1,-1,\ \ 1,\ \ 1,-1)\ , \\ g_2 &= \mathrm{diag}(1,-1,-1,-1,-1,\ \ 1,-1,\ \ 1)\ , \\ g_3 &= \mathrm{diag}(1,-1,-1,-1,-1,-1,\ \ 1,\ \ 1)\ , \\ g_4 &= \mathrm{diag}(1,-1,-1,-1,\ \ 1,\ \ 1,\ \ 1,\ \ 1)\ . \end{aligned} \tag{4.4.8}$$

Under the action of each one of these four $\mathbb{Z}_2$ actions, the holomorphic (3,0)-form transforms to the negative of itself, showing that the entire $\mathbb{Z}_2^4$ group of global symmetries of $X_{7861}/\mathbb{Z}_2^3$ is an R-symmetry. Like the previous example, one again notes that both the freely acting group G_f and the remnant symmetry group $N_G^*(G_f)/G_f$ act trivially on the set of the four polynomials $\{p_i(\bar{x})\}$. The same analysis applied to the other quotients of X_{7861} (4.4.5) yield global symmetry groups $\mathbb{Z}_2^n$, where $n \in \{1, 2, 3\}$. These appear in Table 4.3, which now follows.

Table 4.3: Global symmetry groups of smooth CICY quotients. For a CICY quotient by a freely acting group G_f and identified by the pair (CICY #, SYMM #), the centralizer $C_G^*(G_f)$, normalizer $N_G^*(G_f)$ and the quotient group $N_G^*(G_f)/G_f$ are listed. Bold (CICY #, SYMM #) pairs indicate manifolds with global R-symmetries. For such manifolds that also appear with a superscript †, only a $\mathbb{Z}_2$ subgroup of the entire global symmetry group $N_G^*(G_f)/G_f$ is an R-symmetry.

$\mathbf{G_f}$	**(CICY#, SYMM#)**	$\mathbf{C_G^*(G_f)}$	$\mathbf{N_G^*(G_f)}$	$\mathbf{N_G^*(G_f)/G_f}$
$\mathbb{Z}_2$	$(19,1), (21,3), (27,1), (28,2), (30,1)$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
	$(\mathbf{6836}, \mathbf{11})$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^3$
$\mathbb{Z}_3$	$(6,33)$	$\mathbb{Z}_3$	$\mathbb{S}_3$	$\mathbb{Z}_2$
	$(14,1), (18,1), (26,1)$	$\mathbb{Z}_3^2$	$\mathbb{Z}_3^2$	$\mathbb{Z}_3$
$\mathbb{Z}_4$	$(19,4), (20,5)$	$\mathbb{Z}_4^2$	$\mathbb{Z}_4^2$	$\mathbb{Z}_4$
	$(19, 7-8), (20, 3-6), (21, 7-8)$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_2$
	$(19,9), (21,9)$ $(30,4), (2568,8)$	$\mathbb{Z}_4$	$\mathbb{D}_8$	$\mathbb{Z}_2$
	$(21,6)$	$\mathbb{Z}_4^2$	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2$	$\mathbb{D}_8$
	$(6836,14)$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{D}_8$	$\mathbb{Z}_2^2$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(\mathbf{19}, \mathbf{10}), (\mathbf{20}, \mathbf{8}), (\mathbf{21}, \mathbf{10})$[3]	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^4$
	$(\mathbf{19}, \mathbf{11-16})$, $(\mathbf{6836}, \{\mathbf{15-17}, \mathbf{30-31}, \mathbf{38}, \mathbf{42}, \mathbf{46}, \mathbf{50}, \mathbf{52}, \mathbf{59}, \mathbf{62}, \mathbf{69}, \mathbf{71}, \mathbf{74}, \mathbf{85}, \mathbf{88}, \mathbf{92}\})$	$\mathbb{Z}_2^4 \times \mathbb{Z}_2$	$\mathbb{Z}_2^4 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3$

Continued on next page

[3]For this manifold, only a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of the entire $N_G^*(G_f)/G_f = \mathbb{Z}_2^4$ is an R-symmetry.

$\mathbf{G_f}$	(CICY#, SYMM#)	$\mathbf{C^*_G(G_f)}$	$\mathbf{N^*_G(G_f)}$	$\mathbf{N^*_G(G_f)/G_f}$
	$(\mathbf{20},\mathbf{9-14}),(\mathbf{6836},\{\mathbf{18}^\dagger-\mathbf{20},\mathbf{22},\mathbf{24},\mathbf{27},\mathbf{32}^\dagger,\mathbf{33},\mathbf{34},\mathbf{36},\mathbf{39}^\dagger,\mathbf{40}^\dagger,\mathbf{41}^\dagger,\mathbf{43},\mathbf{44}^\dagger,\mathbf{45},\mathbf{47},\mathbf{49}^\dagger,\mathbf{51},\mathbf{54},\mathbf{55},\mathbf{57}^\dagger,\mathbf{58},\mathbf{61},\mathbf{63},\mathbf{66},\mathbf{68}^\dagger,\mathbf{70},\mathbf{73},\mathbf{77},\mathbf{78},\mathbf{81},\mathbf{83}^\dagger,\mathbf{84},\mathbf{87},\mathbf{91}\})$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^2$
	$(21,11-16),(2564,4),(\mathbf{2566},\mathbf{4-10}),(2568,\{9-36,40\}),(\mathbf{5302},\mathbf{5-20}),(\mathbf{6788},\mathbf{4-6}),(6836,\{21,23,25,26,28,29,35,37,48,53,56,60,64,65,67,72,75,76,79,80,82,86,89,90\}),(\mathbf{7491},\mathbf{5-19}),(7735,\{4,5\}),(\mathbf{7823},\mathbf{2}),(7861,3)$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$
$\mathbb{Z}_6$	$(6,34-41)$	$\mathbb{Z}_6$	$\mathbb{D}_{12}$	$\mathbb{Z}_2$
$\mathbb{Z}_8$	$(19,17),(6836,93)$	$\mathbb{Z}_8$	$\mathbb{Z}_8\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2$
	$(21,17)$	$\mathbb{Z}_8$	$(\mathbb{Z}_2\times\mathbb{D}_8)\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$\mathbb{Z}_4\times\mathbb{Z}_2$	$(19,\{18,20\}),(2564,6),(6836,\{95,97,101,103,109,111\})$	$\mathbb{Z}_4\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2\times\mathbb{D}_8$	$\mathbb{Z}_2^2$
	$(19,19),(21,31)$	$\mathbb{Z}_4\times\mathbb{Z}_2^2$	$\mathbb{Z}_2\times(\mathbb{Z}_2^4\rtimes\mathbb{Z}_2)$	$\mathbb{Z}_2^3$
	$(21,\{18-20,26\}),(7861,5)$	$\mathbb{Z}_4\times\mathbb{Z}_2^2$	$\mathbb{Z}_4\times\mathbb{Z}_2^2$	$\mathbb{Z}_2$
	$(21,21)$	$\mathbb{Z}_4^2\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2\times(\mathbb{Z}_4^2\rtimes\mathbb{Z}_2)$	$\mathbb{Z}_2\times\mathbb{D}_8$

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$\mathbf{G_f}$	(CICY#, SYMM#)	$\mathbf{C^*_G(G_f)}$	$\mathbf{N^*_G(G_f)}$	$\mathbf{N^*_G(G_f)/G_f}$
	$(21, \{22, 24\})$	$\mathbb{Z}_4\times\mathbb{Z}_2^3$	$\mathbb{Z}_2^3\times\mathbb{D}_8$	$\mathbb{Z}_2^3$
	$(21, \{23, 25, 27, 28\})$	$\mathbb{Z}_4^2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times(\mathbb{Z}_4^2\rtimes\mathbb{Z}_2)$	$\mathbb{D}_8$
	$(21, 29-30), (2568, 41-42), (6836, \{96, 98, 99, 100, 102, 104-106\}), (7735, 6-7), (7861, 6)$	$\mathbb{Z}_4\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{D}_8$	$\mathbb{Z}_2$
	$(6836, \{94, 107, 108, 110\})$	$\mathbb{Z}_4\times\mathbb{Z}_2^2$	$\mathbb{Z}_2\times((\mathbb{Z}_4\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2)$	$\mathbb{Z}_2^2$
$\mathbb{Z}_2^3$	$(\mathbf{7861}, \mathbf{8})$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2^4$
$\mathbb{Q}_8$	$(19, \{21, 22, 24, 25, 27, 28\}), (21, \{33, 34\}), (2564, 7-9)$	$\mathbb{Z}_4$	$(\mathbb{Z}_4\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2$
	$(19, \{23, 26, 29\})$	$\mathbb{Z}_4$	$\mathbb{Z}_4^2\rtimes\mathbb{Z}_2$	$\mathbb{Z}_4$
	$(21, 32)$	$\mathbb{Q}_8$	$(\mathbb{Z}_4^2\rtimes\mathbb{Z}_2)\rtimes\mathbb{Z}_2$	$\mathbb{D}_8$
	$(6836, 112-113)$	$\mathbb{Q}_8$	$(\mathbb{Z}_2\times\mathbb{D}_8)\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$\mathbb{Z}_3\times\mathbb{Z}_3$	$(14, 4-7)$	$\mathbb{Z}_3^4$	$\mathbb{Z}_3^4\rtimes\mathbb{Z}_2$	$\mathbb{Z}_3^2\rtimes\mathbb{Z}_2$
	$(14, 8-39)$	$\mathbb{Z}_3\times\mathbb{Z}_3$	$\mathbb{Z}_3^3$	$\mathbb{Z}_3$
	$(7878, 2-3)$	$\mathbb{Z}_3^2$	$\mathbb{Z}_3^2\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{Z}_{10}$	$(7447, 4)$	$\mathbb{Z}_{10}$	$\mathbb{D}_{20}$	$\mathbb{Z}_2$
$\mathbb{Z}_3\rtimes\mathbb{Z}_4$	$(7246, 21-23)$	$\mathbb{Z}_2$	$(\mathbb{Z}_6\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{Z}_4\times\mathbb{Z}_4$	$(21, 35-37)$	$\mathbb{Z}_4^2$	$(\mathbb{Z}_2^3\times\mathbb{D}_8)\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^3$
	$(7861, \{9, 10\}), (7862, 7)$	$\mathbb{Z}_4^2$	$\mathbb{Z}_4^2\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2$

Continued on next page

$\mathbf{G_f}$	**(CICY#, SYMM#)**	$\mathbf{C^*_G(G_f)}$	$\mathbf{N^*_G(G_f)}$	$\mathbf{N^*_G(G_f)/G_f}$
	$(7861, 11)$	$\mathbb{Z}_4^2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times(\mathbb{Z}_4^2\rtimes\mathbb{Z}_2)$	$\mathbb{Z}_2^2$
$\mathbb{Z}_4\rtimes\mathbb{Z}_4$	$(21, 38-40)$	$\mathbb{Z}_4\times\mathbb{Z}_2$	$(\mathbb{Z}_2^3\times\mathbb{D}_8)\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^3$
	$(6836, 114-115)$	$\mathbb{Z}_2^2$	$(\mathbb{Z}_2^2\times\mathbb{D}_8)\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^2$
	$(7861, 12), (7862, 8)$	$\mathbb{Z}_2^2$	$(\mathbb{Z}_4\times\mathbb{Z}_2^2)\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{Z}_8\times\mathbb{Z}_2$	$(21, 41)$	$\mathbb{Z}_8\times\mathbb{Z}_2$	$(\mathbb{Z}_2^3\times\mathbb{D}_8)\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^3$
	$(21, 42-43)$	$\mathbb{Z}_8\times\mathbb{Z}_2$	$\mathbb{Z}_2\times((\mathbb{Z}_2\times\mathbb{D}_8)\rtimes\mathbb{Z}_2)$	$\mathbb{Z}_2^2$
	$(6836, 116-117)$, $(7861, 13)$	$\mathbb{Z}_8\times\mathbb{Z}_2$	$\mathbb{Z}_2\times(\mathbb{Z}_8\rtimes\mathbb{Z}_2)$	$\mathbb{Z}_2$
	$(7862, 9)$	$\mathbb{Z}_8\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{D}_{16}$	$\mathbb{Z}_2$
$\mathbb{Z}_8\rtimes\mathbb{Z}_2$	$(21, 44-45)$	$\mathbb{Z}_4\times\mathbb{Z}_2$	$\mathbb{Z}_2\times((\mathbb{Z}_2\times\mathbb{D}_8)\rtimes\mathbb{Z}_2)$	$\mathbb{Z}_2^2$
	$(21, 46)$	$\mathbb{Z}_4\times\mathbb{Z}_2$	$(\mathbb{Z}_2\times((\mathbb{Z}_2\times\mathbb{D}_8)\rtimes\mathbb{Z}_2))\rtimes\mathbb{Z}_2$	$\mathbb{D}_8$
$\mathbb{Z}_2\times\mathbb{Q}_8$	$(21, \{47, 48, 50\})$	$\mathbb{Z}_2\times\mathbb{Q}_8$	$\mathbb{Z}_2\times((\mathbb{Z}_4^2\rtimes\mathbb{Z}_2)\rtimes\mathbb{Z}_2)$	$\mathbb{D}_8$
	$(21, 49)$	$\mathbb{Z}_2\times\mathbb{Q}_8$	$(\mathbb{Z}_2^2\times(\mathbb{Z}_4^2\rtimes\mathbb{Z}_2))\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{D}_8$
	$(21, 51)$	$\mathbb{Z}_2\times\mathbb{Q}_8$	$\mathbb{Z}_2\times(((\mathbb{Z}_2\times\mathbb{D}_8)\rtimes\mathbb{Z}_2)\rtimes\mathbb{Z}_2)$	$\mathbb{D}_8$
	$(21, 52-53)$	$\mathbb{Z}_2\times\mathbb{Q}_8$	$(\mathbb{Z}_2^3\times\mathbb{D}_8)\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^3$
	$(7861, 17-19)$	$\mathbb{Z}_4\times\mathbb{Z}_2$	$\mathbb{Z}_2\times((\mathbb{Z}_4\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2)$	$\mathbb{Z}_2$
	$(7862, 11)$	$\mathbb{Z}_2\times\mathbb{Q}_8$	$\mathbb{Z}_2\times((\mathbb{Z}_2\times\mathbb{D}_8)\rtimes\mathbb{Z}_2)$	$\mathbb{Z}_2^2$
$\mathbb{Z}_4\times\mathbb{Z}_2^2$	$(7861, 14-16)$	$\mathbb{Z}_4\times\mathbb{Z}_2^3$	$\mathbb{Z}_2^2\times(\mathbb{Z}_2^4\rtimes\mathbb{Z}_2)$	$\mathbb{Z}_2^3$

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$\mathbf{G_f}$	**(CICY#, SYMM#)**	$\mathbf{C^*_G(G_f)}$	$\mathbf{N^*_G(G_f)}$	$\mathbf{N^*_G(G_f)/G_f}$
$\mathbb{Z}_{10}\times\mathbb{Z}_2$	$(7447, 5)$	$\mathbb{Z}_{10}\times\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{D}_{10}$	$\mathbb{Z}_2$
$\mathbb{Z}_5\times\mathbb{Z}_5$	$(7890, 2-5)$	$\mathbb{Z}_5\times\mathbb{Z}_5$	$(\mathbb{Z}_5\times\mathbb{Z}_5)\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2$
$(\mathbb{Z}_4\times\mathbb{Z}_2)\rtimes\mathbb{Z}_4$	$(7861, 20)$	$(\mathbb{Z}_4\times\mathbb{Z}_2)\rtimes\mathbb{Z}_4$	$(\mathbb{Z}_2^2\times(\mathbb{Z}_2^4\rtimes\mathbb{Z}_2))\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^3$
	$(7861, 21-23)$	$\mathbb{Z}_2^3$	$(\mathbb{Z}_2^2\times((\mathbb{Z}_2\times\mathbb{D}_8)\rtimes\mathbb{Z}_2))\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^3$
$\mathbb{Z}_8\times\mathbb{Z}_4$	$(7861, 24-25)$	$\mathbb{Z}_8\times\mathbb{Z}_4$	$(\mathbb{Z}_2\times((\mathbb{Z}_2\times\mathbb{D}_8)\rtimes\mathbb{Z}_2))\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$\mathbb{Z}_8\rtimes\mathbb{Z}_4$	$(7861, 26)$[4]	$\mathbb{Z}_4^2$	$(\mathbb{Z}_2\times((\mathbb{Z}_2\times\mathbb{D}_8)\rtimes\mathbb{Z}_2))\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$\mathbb{Z}_8\rtimes\mathbb{Z}_4$	$(7861, 28)$[4]	$\mathbb{Z}_2^2$	$(\mathbb{Z}_2\times\mathbb{D}_{16})\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2$
$(\mathbb{Z}_8\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2$	$(7861, 27)$	$\mathbb{Z}_4\times\mathbb{Z}_2$	$(((\mathbb{Z}_8\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2)\rtimes\mathbb{Z}_2)\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$\mathbb{Z}_4^2\times\mathbb{Z}_2$	$(7861, 29-36)$	$\mathbb{Z}_4^2\times\mathbb{Z}_2$	$(\mathbb{Z}_2^2\times(\mathbb{Z}_2^4\rtimes\mathbb{Z}_2))\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^3$
$\mathbb{Z}_4\rtimes\mathbb{Q}_8$	$(7861, 39)$	$\mathbb{Z}_4\times\mathbb{Z}_2$	$(\mathbb{Z}_2\times((\mathbb{Z}_2\times\mathbb{D}_8)\rtimes\mathbb{Z}_2))\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$\mathbb{Z}_2\times(\mathbb{Z}_4\rtimes\mathbb{Z}_4)$	$(7861, 37-38)$	$\mathbb{Z}_2^3$	$(\mathbb{Z}_2^2\times(\mathbb{Z}_2^4\rtimes\mathbb{Z}_2))\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^3$
$\mathbb{Z}_2^2\times\mathbb{Q}_8$	$(7861, 40-45)$	$\mathbb{Z}_2^2\times\mathbb{Q}_8$	$(\mathbb{Z}_2^2\times((\mathbb{Z}_2\times\mathbb{D}_8)\rtimes\mathbb{Z}_2))\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^3$

[4]The two distinct semi-direct products $\mathbb{Z}_8\rtimes\mathbb{Z}_4$ correspond to the presentations $\langle a, b \mid a^8 = b^4 = e, bab^{-1} = a^3\rangle$ and $\langle a, b \mid a^8 = b^4 = e, bab^{-1} = a^5\rangle$.

4.5 Discussion

In conclusion, we have classified the global discrete symmetries of all known smooth CICY quotients by freely acting symmetries. Our results include both Abelian and non-Abelian symmetries. Being isometries of the compactification manifold, these are candidates for discrete symmetries of the corresponding low energy theories. The machinery behind this work was drawn from the example of the quintic, and thus it is worth returning to it. The quotient of the quintic by the freely acting group $\mathbb{Z}_5 \times \mathbb{Z}_5$ was found to have an additional $\mathbb{Z}_2$ symmetry that does not lie in the centralizer $C^*_G(G_f)$ but crops up, only when the normalizer $N^*_G(G_f)$ is computed. The centralizer is the freely acting $\mathbb{Z}_5 \times \mathbb{Z}_5$ itself. The normalizer $N^*_G(G_f)$ in this case is the semi-direct product $(\mathbb{Z}_5 \times \mathbb{Z}_5) \rtimes \mathbb{Z}_2$, leading to a non-freely acting $\mathbb{Z}_2$ as the symmetry of the quotient. The $\mathbb{Z}_5$ quotient of the quintic was not found to have any nontrivial remnant global symmetry. This is tabulated below.

G_f	$\mathbb{Z}_5$	$\mathbb{Z}_5 \times \mathbb{Z}_5$
$N^*_G(G_f)/G_f$	$\mathbb{I}$	$\mathbb{Z}_2$

Table 4.4: *Global symmetries of quintic quotients by freely acting groups.*

We draw attention to the $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotients of the Schoen manifold (also known as the split bicubic) introduced in [111], over which three generation models have been constructed [77, 78, 112–114].

$$X_{14} = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{bmatrix}^{19,19}_{\chi=0} . \qquad (4.5.1)$$

There are 36 distinct $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotients of X_{14}. Four of these quotients yielded $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2$ as residual symmetries that act non-freely on these quotients, while the remaining 32 have remnant $\mathbb{Z}_3$ symmetries. All remnant symmetries of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotients of the split bicubic are regular symmetries. It remains to be checked whether these symmetries have phenomenological value.

Another example of a three generation CICY is the Dic_3 quotient of the following manifold:

$$X_{7246} = \begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}^{8,44}_{-72} . \tag{4.5.2}$$

This manifold was shown to be embedded in a product of two del Pezzo surfaces of degree six [5]. The residual global symmetry of the quotient manifold was found to be $\mathbb{Z}_2$, with the co-ordinate action swapping the two del Pezzo surfaces. We conclude our discussion with examples of R symmetries obtained as global symmetries of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ quotient of the following manifold:

$$X_{2566} = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^4 \end{matrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 \end{bmatrix}^{12,28}_{-32} . \tag{4.5.3}$$

This manifold was shown to be embedded in a product of two del Pezzo surfaces of degree 4 in chapter 5. While the $\mathbb{Z}_2$ quotients of the manifold do not have any residual global symmetries, all the $\mathbb{Z}_2 \times \mathbb{Z}_2$ quotients of the manifold were found to have residual $\mathbb{Z}_2$ global symmetries that are actually R symmetries. These symmetries act in a way that exchanges the second and third $\mathbb{P}^1$ spaces, except in one case, where it acts as a toric symmetry.

G_f	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$N^*_G(G_f)/G_f$	$\mathbb{I}$	$\mathbb{Z}_2$

Table 4.5: *Global symmetries of quotients of the manifold* (4.5.3) *by freely acting groups.*

While it was one of the most significant undertakings of this thesis to compute the global symmetries of all the 1695 known CICY quotients, it is worth revisiting the quotients themselves. The Hodge numbers for all such quotients were not known until recently [2, 6, 100]. The following chapter is a significant part of that effort. All such Hodge number computations were recently recomputed using an automated computation of equivariant cohomologies in [101]. The following chapter discusses the polynomial deformation method and the counting of invariant Kähler classes to compute Hodge numbers of CY quotients. We also provide a vivid algebro-geometric description of some CICYs embedded in products of del Pezzo surfaces of degree 4. CY manifolds embedded in products of del Pezzo surfaces were explored previously in [5, 115]. Subsequently we use our description of dP_4 to not only explain the Hodge numbers of the relevant CICYs but to compute the Hodge numbers of the corresponding CICY quotients.

5
Hodge Numbers of CICY Quotients

5.1 Introduction

Compactifications of the heterotic string are based on smooth Calabi-Yau threefolds [35]. This approach remains a promising avenue from string theory to realistic particle physics phenomenology [7–9, 72, 75, 76, 78, 83]. We are led to seek Calabi–Yau manifoldswith small Hodge numbers by a desire to find realistic models constructed with a minimum of complexity. While not universally the case, since the very special class of Gross-Popescu manifolds [116] yield a small number of manifolds that have small Hodge numbers and are nevertheless simply connected, the great majority of known Calabi–Yau manifolds with small Hodge numbers are realised as quotients of simply connected manifolds by a freely acting group. The process of taking a quotient by a freely acting group does double duty, by first reducing the Hodge numbers, but also by creating a multiply connected manifold. Flux lines around the irreducible paths of the manifold then allow the breaking of the gauge group to the Standard Model group.

A large number of examples of this type resulted from the work initiated in [2] and completed through the automated scan carried out by Volker Braun in [3]. Braun's scan led to a complete classification of all free linear actions of finite groups on complete intersection Calabi-Yau (CICY) manifolds embedded in products of projective spaces.[1]

[1]The embeddings used in this classification were those of the CICY list Ref. [117]. Though

For model building, the properties of the quotient manifolds X/G are of prime importance. In particular, the Hodge numbers of the quotients $h^{1,1}(X/G)$ and $h^{2,1}(X/G)$ play a central role. While Braun's scan gave a complete listing of the freely acting symmetries, the individual Hodge numbers of the quotients were not calculated, though of course the difference

$$2\Big(h^{1,1}(X/G) - h^{2,1}(X/G)\Big) = \frac{\chi(X)}{|G|} \tag{5.1.1}$$

follows immediately from the fact that the Euler number divides by the order of the group.

Braun found that there are 166 manifolds for which G is precisely $\mathbb{Z}_2$, and for all cases, where $|G| > 2$, that G is either $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$, or $\mathbb{Z}_5$, or G contains at least one of these groups as a subgroup. The computation of the Hodge numbers, for the case $G \supseteq \mathbb{Z}_5$ was given in [2], for the case $G \supseteq \mathbb{Z}_3$ the majority of the cases were studied in [2] while the remaining cases were studied in [6]. The remaining cases, for which $G \supseteq \mathbb{Z}_4$ or $G \supseteq \mathbb{Z}_2{\times}\mathbb{Z}_2$, are the subject of the present work. This completes the calculation of Hodge numbers for the CICY quotients, except for $\mathbb{Z}_2$-quotients. Though for many of the $\mathbb{Z}_2-$quotients we are able to compute Hodge numbers. These cases are discussed in the appendix A.2.

5.1.1 Webs of CICY quotients

There are 45 CICY matrices for which $G \supseteq \mathbb{Z}_4$ or $G \supseteq \mathbb{Z}_2{\times}\mathbb{Z}_2$. This somewhat overstates the number of manifolds since a number of the matrices seem to correspond to manifolds that are the same. The 45 matrices correspond to 19 distinct pairs of Hodge numbers $(h^{1,1}, h^{2,1})$. We present the corresponding web in Figure 1. From this web we obtain others. Those corresponding to $G = \mathbb{Z}_4$ are shown in Figure 4, and those corresponding to $G = \mathbb{Z}_2{\times}\mathbb{Z}_2$ are shown in Figure 5.

Although the number of manifolds is overstated there are nevertheless many cases to consider since a given manifold can admit several and, in some cases many,

the existence of a certain symmetry of a CICY does not depend on the embedding, the linearity of the action does. As such, one expects that Braun's classification is not complete, if different equivalent embeddings are considered.

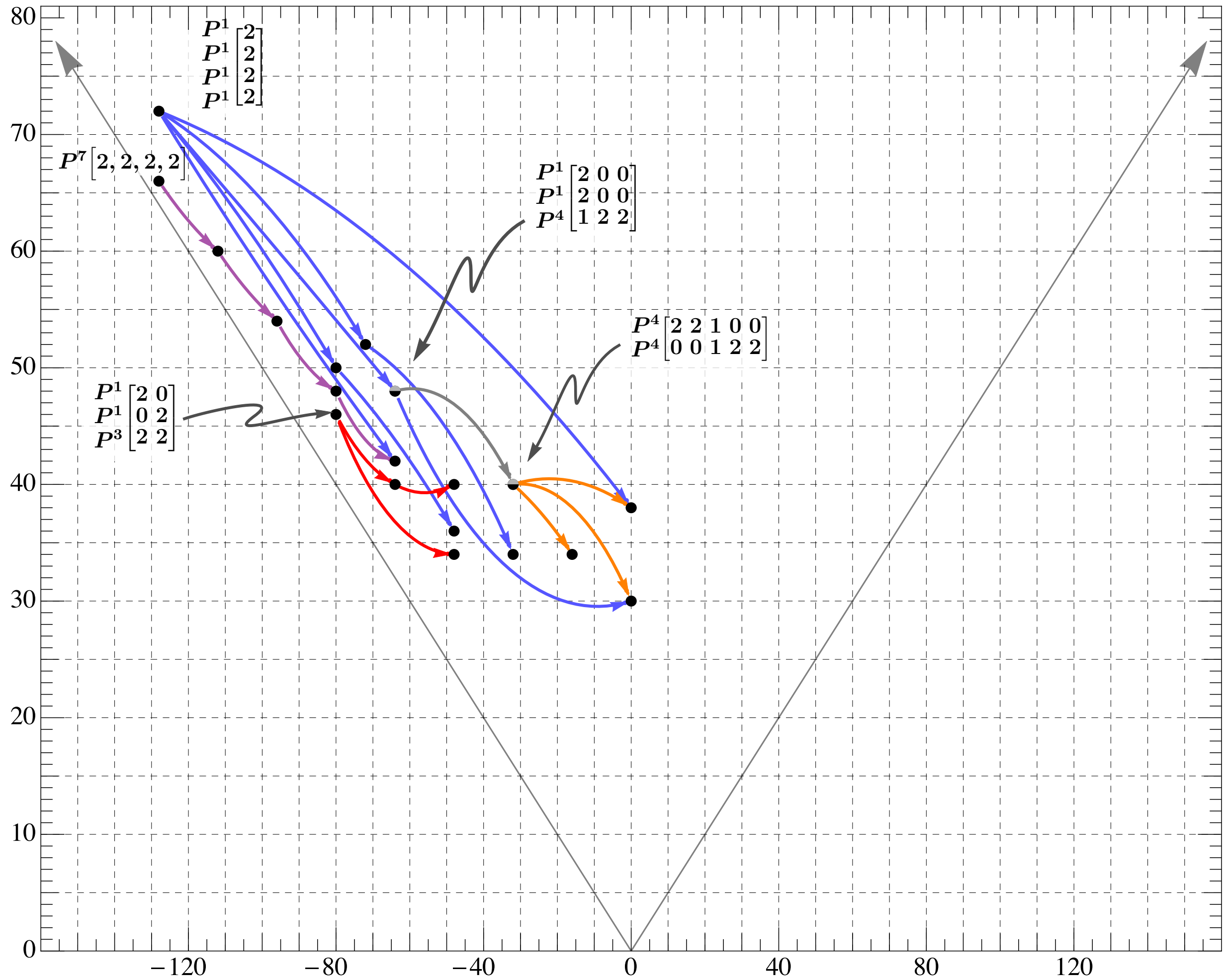


Figure 1: *The web of CICY manifolds that admit free automorphisms by $\mathbb{Z}_4$ or $\mathbb{Z}_2\times\mathbb{Z}_2$. The Euler number, $\chi = \frac{1}{2}(h^{1,1}-h^{2,1})$, is plotted horizontally while the height, $h^{1,1}+h^{2,1}$, is plotted vertically. The oblique axes correspond to the Hodge numbers $h^{1,1}$ and $h^{2,1}$. The arrows denote conifold transitions between CICYs. Conifold transitions from the 5 parent CICYs, or 'sources', are depicted in different colours.*

distinct group actions. An example is provided by $X_{7861} = \mathbb{P}^7[2,2,2,2]$. For this configuration, Table 5.1 lists 21 distinct groups and these give rise to 45 distinct group actions.

5.1.2 Redundancy of representations

While it is far from being the case that the manifolds of the list are classified by their Hodge numbers[2], nevertheless there is some redundancy in that some of the

[2]There are 7890 matrices in the CICY list, of which at least 2590 are known to be distinct as classical manifolds, but there are only 266 distinct pairs ($h^{1,1}$, $h^{2,1}$) of Hodge numbers [62].

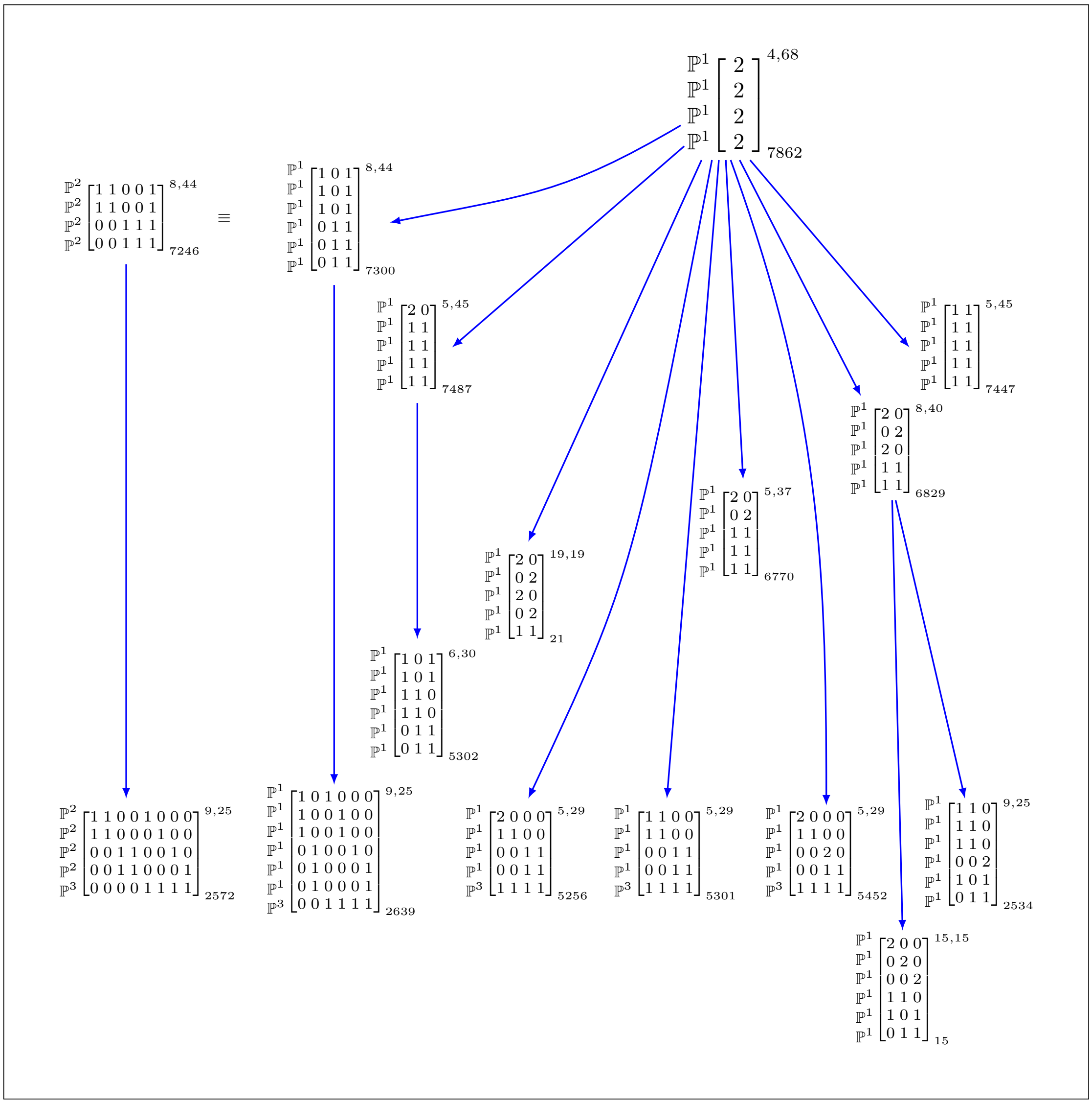


Figure 2: *The CICY Web with the tetraquadric as the source manifold. The arrows depict conifold transitions. Configuration matrices are decorated with superscripts, indicating the Hodge numbers and subscripts, indicating the position in the CICY list, which is available at [63].*

configurations with the same Hodge numbers correspond to the same manifold. The process of finding freely acting symmetries will tend to pick out identical manifolds, since if two configurations correspond to identical manifolds and one is symmetric, then so is the other. Where we have found suspected identities between manifolds we have not attempted to prove, in all cases, that some of these manifolds are in fact the same, though in some cases we do. In any event, different representations of the same manifold are often useful. Symmetries, for example, may be more evident in one representation than another. Braun's classification, as well as previous work,

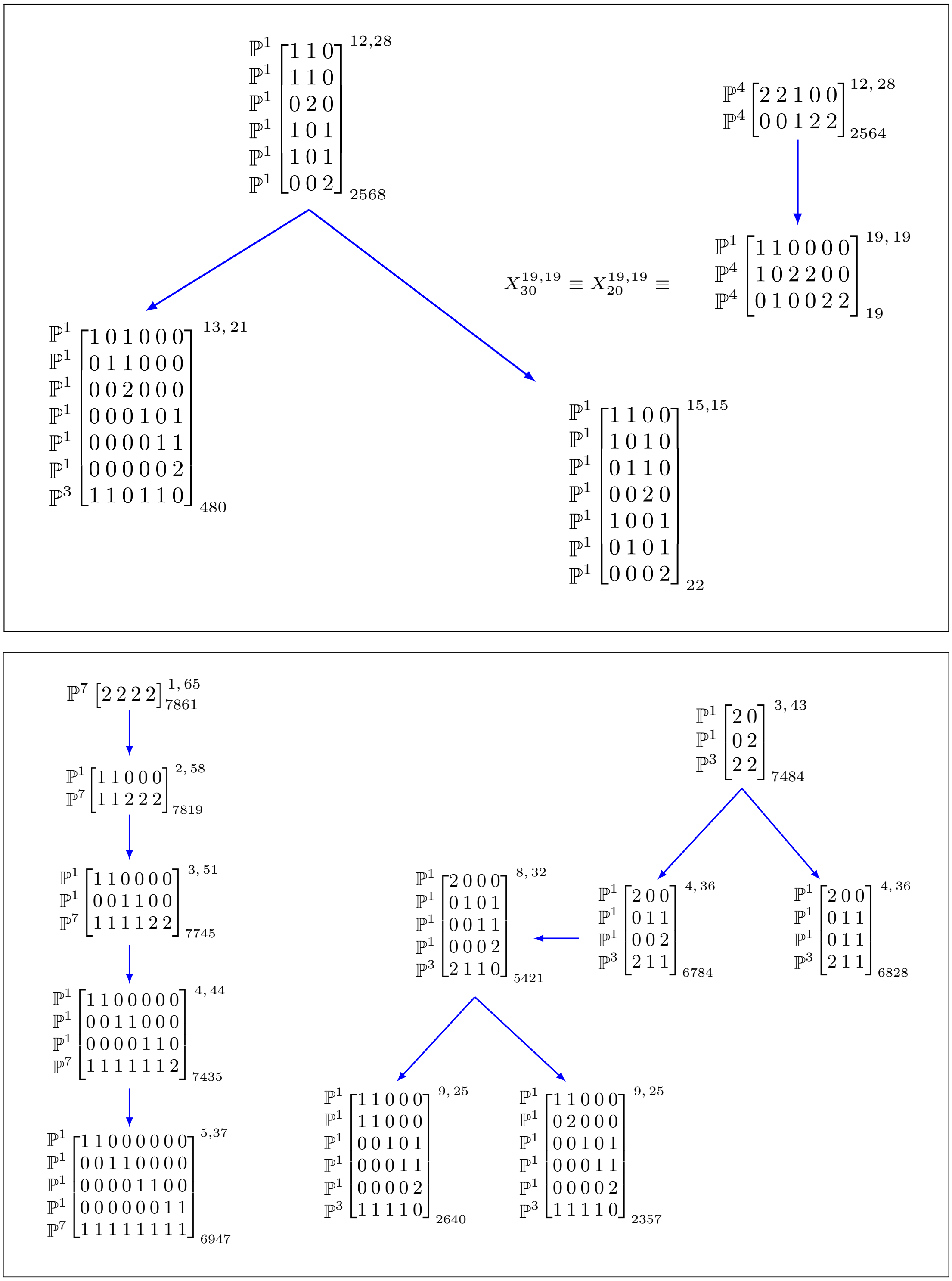


Figure 3: *The top diagram shows a CICY Web with parent manifolds* $X^{12,28}_{2564}$ *and* $X^{12,28}_{2568}$*. The diagram below shows CICY Webs with parent manifolds* $X^{1,65}_{7861}$ *and* $X^{3,43}_{7484}$*. The arrows depict conifold transitions. The conifold transitions with* $\mathbb{P}^7[2\ 2\ 2\ 2]$ *as the parent are described in detail in Figure 11.*

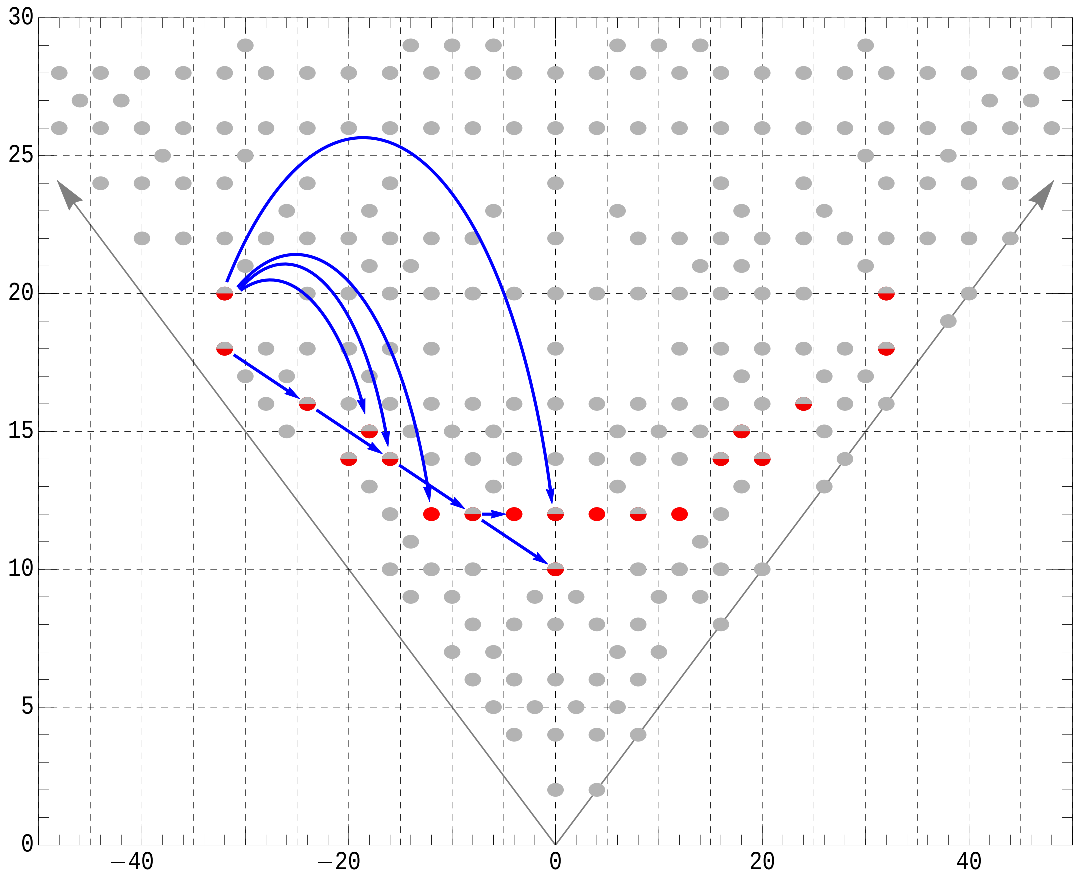


Figure 4: *The web of $\mathbb{Z}_4$ quotients of CICY manifolds and their mirrors. The red points indicate $\mathbb{Z}_4$ quotients whose Hodge numbers fall onto sites previously unoccupied, while the bicoloured points correspond to previously occupied sites.*

has identified linearly represented symmetries. It is possible, and we will identify examples in the following, for one representation of a manifold to admit a linear representation of a symmetry while another does not.

Consider a first example of the redundancy. The CICY's

$$X_{7246} = \begin{matrix}\mathbb{P}^2\\\mathbb{P}^2\\\mathbb{P}^2\\\mathbb{P}^2\end{matrix}\begin{bmatrix}1 & 1 & 0 & 0 & 1\\1 & 1 & 0 & 0 & 1\\0 & 0 & 1 & 1 & 1\\0 & 0 & 1 & 1 & 1\end{bmatrix} \qquad \text{and} \qquad X_{7300} = \begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\end{matrix}\begin{bmatrix}1 & 0 & 1\\1 & 0 & 1\\1 & 0 & 1\\0 & 1 & 1\\0 & 1 & 1\\0 & 1 & 1\end{bmatrix} \tag{5.1.2}$$

both have $(h^{1,1}, h^{2,1}) = (8, 44)$ and both admit freely acting symmetries of order 12, corresponding to the groups $\mathbb{Z}_{12}$ and Dic_3. We observe also that the del Pezzo

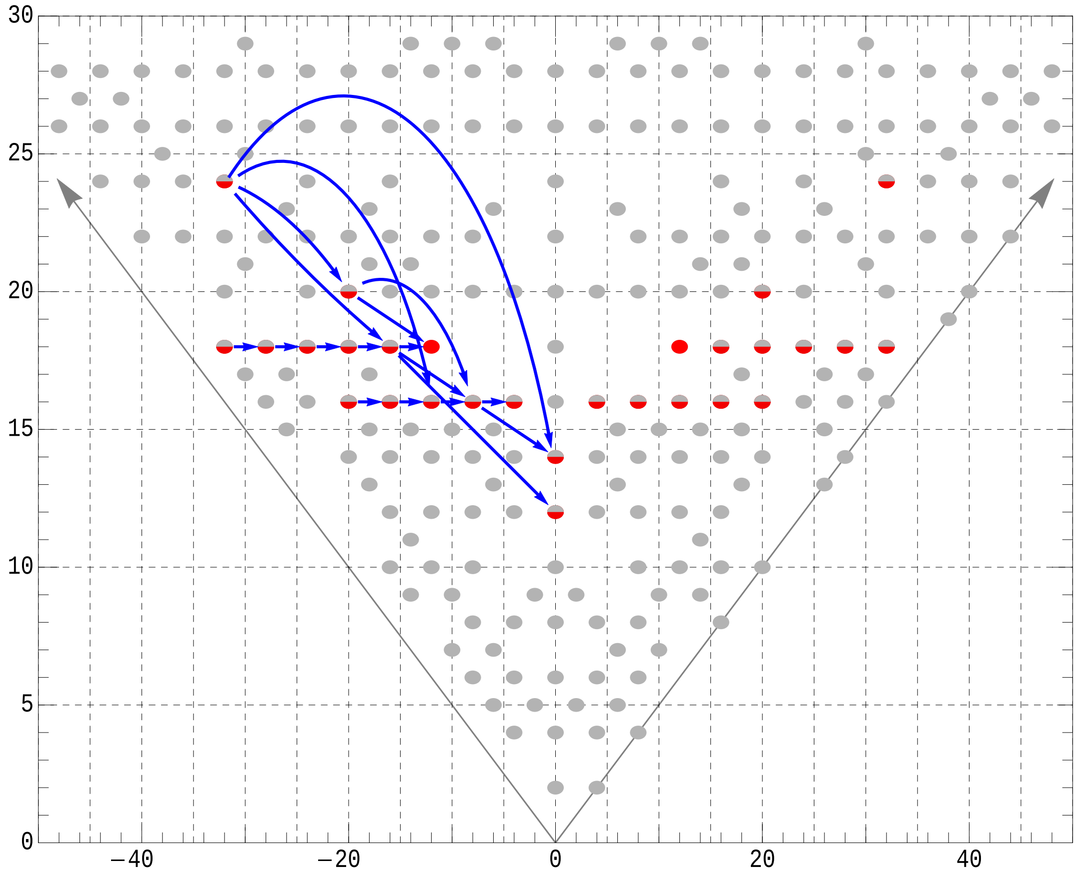


Figure 5: *The web of $\mathbb{Z}_2\times\mathbb{Z}_2$ quotients of CICY manifolds and their mirrors. The red points indicate $\mathbb{Z}_2\times\mathbb{Z}_2$ quotients whose Hodge numbers fall onto sites previously unoccupied, while the bicoloured points correspond to previously occupied sites.*

surface[3] dP_6 can be represented both as

$$dP_6 = \begin{matrix}\mathbb{P}^2\\ \mathbb{P}^2\end{matrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \qquad \text{or} \qquad dP_6 = \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} \tag{5.1.3}$$

so each of the two CICY's above corresponds to a hypersurface in $dP_6\times dP_6$. In this case the manifolds are the same[4], since in each case, we have a linear system of anti-canonical hypersurfaces in $dP_6\times dP_6$, so the only question is whether we cover the same part of the moduli space. But both constructions give the entire 44-parameter family, so they must be the same.

[3]Our convention here is that dP_n denotes the del Pezzo surface of degree n. Thus the del Pezzo surface corresponding to $\mathbb{P}^2$ blown up in k generic points is, with this convention, dP_{9-k}.

[4]We are grateful to Rhys Davies for pointing out the following easy argument.

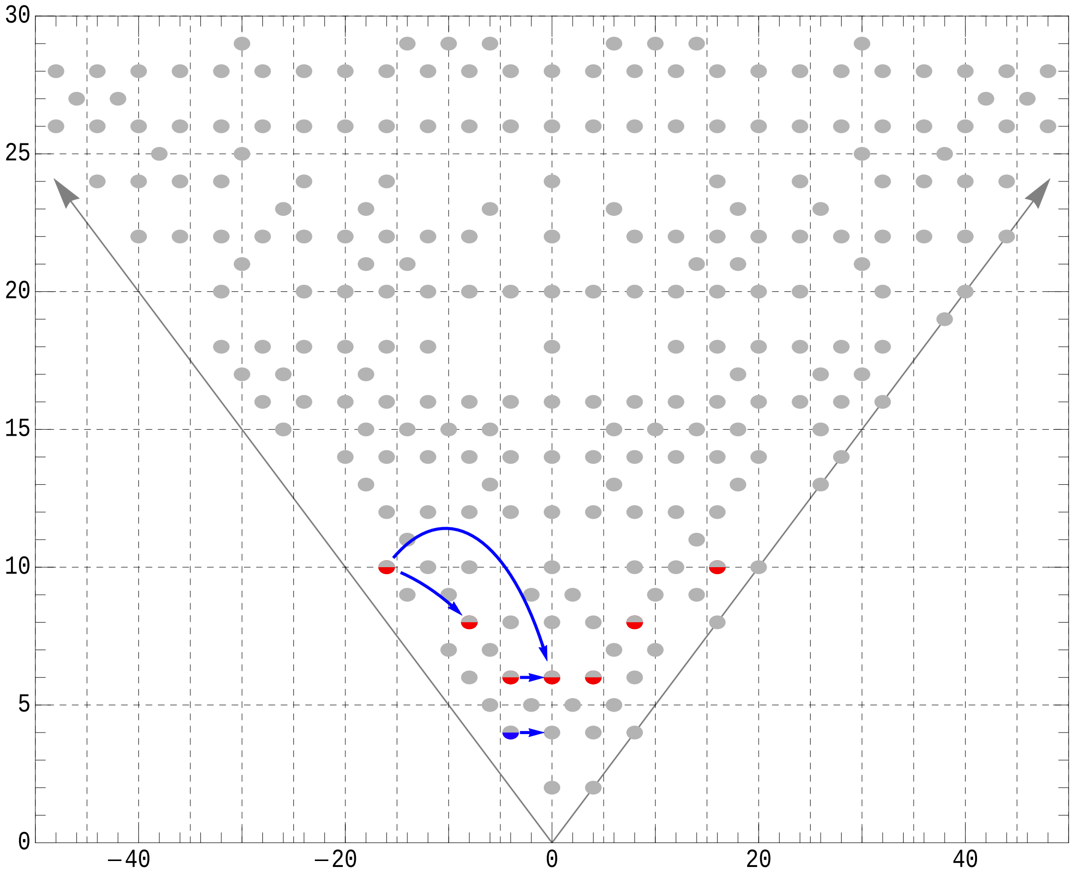


Figure 6: *The overlapping web of $\mathbb{Z}_8$ and $\mathbb{Q}_8$–quotients of CICY manifolds and their mirrors. The grey and red points indicate quotients whose Hodge numbers fall onto sites previously occupied. The grey and blue point is not part of the web and indicates the manifold $\mathbb{P}^7[2\ 2\ 2\ 2]/G$ with $|G| = 32$. The conifold transition originating there, is to a Gross-Popescu manifold with Hodge numbers $(2, 2)$ (see §5 of Ref. [2]).*

Given this identity, should there not also be a 'hybrid' matrix

$$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^2\\\mathbb{P}^2\end{matrix}\begin{bmatrix}1&0&0&1\\1&0&0&1\\1&0&0&1\\0&1&1&1\\0&1&1&1\end{bmatrix}. \tag{5.1.4}$$

Indeed there is, this is matrix 7206 of the list, it also has $(h^{1,1}, h^{2,1}) = (8, 44)$ and is assumed to be identical to the previous two representations. In Braun's classification, the hybrid appears with a maximal group $\mathbb{Z}_6$. Thus, this manifold also admits the symmetry groups of order 12, but not all the elements of the groups are represented linearly.

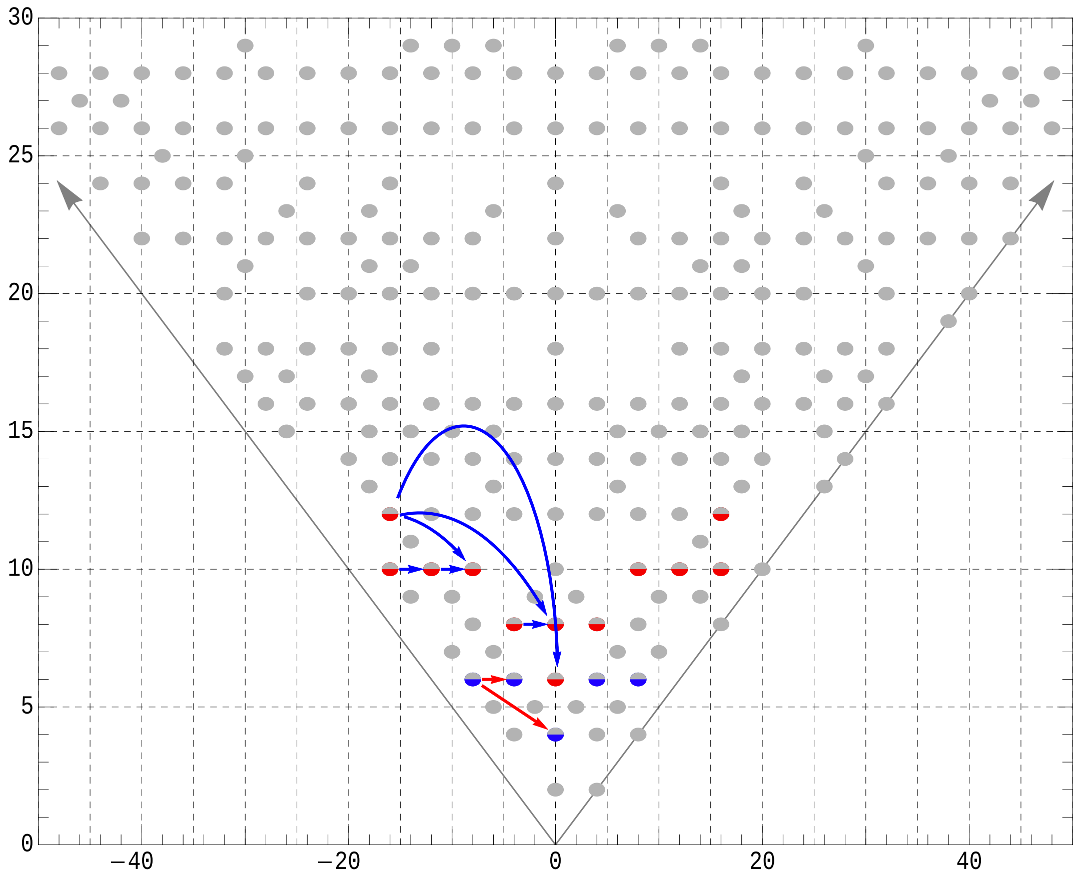


Figure 7: *The webs of* $\mathbb{Z}_2\times\mathbb{Z}_4$*,* $\mathbb{Z}_4\rtimes\mathbb{Z}_4$ *and* $\mathbb{Z}_2\times\mathbb{Z}_8$*–quotients of CICY manifolds and their mirrors. The webs of* $\mathbb{Z}_4\rtimes\mathbb{Z}_4$ *and* $\mathbb{Z}_2\times\mathbb{Z}_8$*–quotients overlap and contain the blue and grey points connected by red arrows. The* $\mathbb{Z}_2\times\mathbb{Z}_4$ *web corresponds to the red and gray points connected by blue arrows.*

Another interesting case derives from the above. This is the split of X_{7246}:

$$\begin{matrix}\mathbb{P}^1\\ \mathbb{P}^2\\ \mathbb{P}^2\\ \mathbb{P}^2\\ \mathbb{P}^2\end{matrix}\begin{bmatrix}0&0&0&0&1&1\\1&1&0&0&1&0\\1&1&0&0&1&0\\0&0&1&1&0&1\\0&0&1&1&0&1\end{bmatrix}. \tag{5.1.5}$$

A quick calculation reveals the Euler number as $\chi = 0$ and the practised reader will see that by contracting, say, the second and also the last $\mathbb{P}^2$ we arrive at the split bicubic

$$\begin{matrix}\mathbb{P}^2\\ \mathbb{P}^2\\ \mathbb{P}^2\end{matrix}\begin{bmatrix}1&1\\3&0\\0&3\end{bmatrix}^{19,19}_{\chi=0} \tag{5.1.6}$$

which also has $\chi = 0$. The significance of this is that, under a conifold transition, the Euler number changes by twice the number of nodes. In this case, since the Euler number does not change, there are no nodes. Thus the last two configurations above correspond to the same manifold. Now the configuration (5.1.5) is not in the CICY list owing to the fact that, in constructing the list, extended matrices of the type we have just seen, that are related to matrices of the list by redundant splits, were suppressed. The interesting point is that (5.1.5) inherits the linear group actions of $\mathbb{Z}_{12}$ and Dic_3 from X_{7246}. Thus the split bicubic must also admit these as freely acting symmetries, a fact that was not otherwise known. However these symmetries are not linearly realised on the configuration (8.0.1). A lesson is that there are very probably nonlinearly realised symmetries of the CICY manifolds, of which we are not aware. Some of these may correspond to linear actions on extended CICY matrices. Returning to the matter of redundancy among the matrices. We have discussed the redundancy due to the two ways of representing dP_6. It turns out that dP_4 (for us, this is $\mathbb{P}^2$ blown up in five points) also appears in our matrices in two ways. The first is the well known presentation $\mathbb{P}^4[2,2]$. Another is

$$\mathrm{dP}_4 \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\begin{bmatrix}1\\1\\2\end{bmatrix}. \tag{5.1.7}$$

To see this note that, by taking coordinates x_i, y_i, z_i, in the three $\mathbb{P}^1$'s, we can realise the space on the right as the zero locus of an equation

$$A(y,z)\,x_1 + B(y,z)\,x_2 \;=\; 0 \tag{5.1.8}$$

where A and B are polynomials of bidegree $(1,2)$ in their arguments. For generic $(y,z) \in \mathbb{P}^1 {\times} \mathbb{P}^1$, this yields a unique solution for (x_1,x_2), as a point of $\mathbb{P}^1$. However there will be four points, for sufficiently general A and B, such that $A(y,z) = B(y,z) = 0$, and for these points the solutions to (5.1.8) yield a complete $\mathbb{P}^1$'s worth of x's. In this way we see that we can think of the space on the right of (5.1.7) as $\mathbb{P}^1 {\times} \mathbb{P}^1$ blown up in four points, and this is equivalent to $\mathbb{P}^2$ blown up in five points. It remains to check that these points "are in general position", that is, no three on a line. This is most simply done in the context of an example and we do this in §5.2.7. The relevance of the remarks above is that we find among our matrices presumed identities such as

$$X_{2564} \;=\; \begin{matrix}\mathbb{P}^4\\ \mathbb{P}^4\end{matrix}\begin{bmatrix}1&2&2&0&0\\1&0&0&2&2\end{bmatrix}^{12,28}_{-32} \quad\text{and}\quad X_{2568} \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\begin{bmatrix}1&1&0\\1&1&0\\0&2&0\\1&0&1\\1&0&1\\0&0&2\end{bmatrix}^{12,28}_{-32} \tag{5.1.9}$$

as well as the corresponding splits and hybrids of these.

Now, as we have observed, the surface dP_4 can be obtained by blowing up $\mathbb{P}^2$ in five points. It follows that $h^{1,1}(\mathrm{dP}_4) = 6$. So by recognising that the configurations above are hypersurfaces in $\mathrm{dP}_4 \times \mathrm{dP}_4$ we have explained the fact that $h^{1,1} = 12$ for these configurations, a fact that was not immediately apparent from the configurations themselves. We make extensive use of this method to calculate $h^{1,1}$ for these 'difficult' cases. By working out how the groups act on the exceptional lines of the dP_4's we are also able to compute $h^{1,1}$ for the quotients.

5.1.3 Unexpected symmetries

The webs of symmetric CICY's, that we discuss here, have groups $G \supseteq \mathbb{Z}_4$ or $G \supseteq \mathbb{Z}_2{\times}\mathbb{Z}_2$. The groups that arise, by the process of splitting and contraction are, for the most part, two-groups[5], as is easily appreciated from a glance at Table 5.1 at the end of this introduction. For example, the tetraquadric has groups

$$\begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\begin{bmatrix}2\\2\\2\\2\end{bmatrix} : \qquad \begin{matrix}\mathbb{Z}_2\,,\ \mathbb{Z}_4\,,\ \mathbb{Z}_2{\times}\mathbb{Z}_2\,,\ \mathbb{Z}_8\,,\ \mathbb{Z}_4{\times}\mathbb{Z}_2\,,\ \mathbb{Q}_8\,,\\ \mathbb{Z}_4{\times}\mathbb{Z}_4\,,\ \mathbb{Z}_4{\rtimes}\mathbb{Z}_4\,,\ \mathbb{Z}_8{\times}\mathbb{Z}_2\,,\ \mathbb{Z}_8{\rtimes}\mathbb{Z}_2\,,\ \mathbb{Z}_2{\times}\mathbb{Q}_8\end{matrix}$$

'Unexpected' groups arise occasionally, however. The particularly symmetric split

$$X_{7447} \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\begin{bmatrix}1&1\\1&1\\1&1\\1&1\\1&1\end{bmatrix} \tag{5.1.11}$$

can admit a freely acting $\mathbb{Z}_5$ that corresponds to the cyclic permutation of the five $\mathbb{P}^1$ spaces, and admits in fact a freely acting $\mathbb{Z}_5{\times}\mathbb{Z}_2{\times}\mathbb{Z}_2$. A second example of this phenomenon could be the manifold X_{7300} of (5.1.2). This is a (double) split of the tetraquadric and admits the freely acting groups $\mathbb{Z}_{12} = \mathbb{Z}_3{\times}\mathbb{Z}_4$ and $\mathrm{Dic}_3 = \mathbb{Z}_3{\rtimes}\mathbb{Z}_4$.

[5] Two-groups are groups such that the order of the group, and so the order of each of its subgroups, is a power of 2.

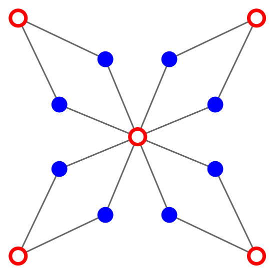
5.1.4 Layout of the chapter

In many cases, we use the polynomial deformation method to calculate the Hodge numbers. That is we compute the number of free parameters for the most general symmetric polynomials and we find $h^{2,1}$ from this. The number of invariant Kähler classes follows, since, as already remarked, the Euler number divides by the order of the group. The method and the circumstances for which it gives a reliable count are described in §5.2. We discuss here also the alternate approach of calculating the action of the group on the Kähler-forms. The cases where the manifold is a hypersurface in products of dP_6's and dP_4's leads to some very classical and beautiful algebraic geometry.

We turn next to the actual calculations for the 45 CICY's. As has been remarked, there is a partial ordering imposed by splitting. We follow Figure 1. There are five 'sources' in this diagram, corresponding to the labelled CICY's. We begin, in §5.3, with the successive splits of the tetraquadric. For each symmetric split we record the groups and the Hodge numbers corresponding to the quotients.

In §5.4 we consider the split manifolds that descend from $\mathbb{P}^7[2,2,2,2]$. These are the manifolds of Figure 11. It is interesting to remark that there is a 'sink' in Figure 1, corresponding to a manifold that descends from both $\mathbb{P}^7[2,2,2,2]$ and the tetraquadric:

$$X_{6947} \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^7\end{matrix}\begin{bmatrix}1&1&0&0&0&0&0&0\\0&0&1&1&0&0&0&0\\0&0&0&0&1&1&0&0\\0&0&0&0&0&0&1&1\\1&1&1&1&1&1&1&1\end{bmatrix}^{5,37}_{-64} \tag{5.1.12}$$



It is elementary, but interesting, to observe that, if we contract the $\mathbb{P}^1$'s we return to $\mathbb{P}^7[2,2,2,2]$. While, if we contract the $\mathbb{P}^7$, we return to the tetraquadric.

In §5.5 we follow the splits of the two remaining sources

$$X_{7484} \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^3\end{matrix}\begin{bmatrix}2&0\\0&2\\2&2\end{bmatrix}^{3,43}_{-80} \quad\text{and}\quad X_{2564} \;=\; \begin{matrix}\mathbb{P}^4\\ \mathbb{P}^4\end{matrix}\begin{bmatrix}1&2&2&0&0\\1&0&0&2&2\end{bmatrix}^{12,28}_{-32} \tag{5.1.13}$$

There are more matrices here than is apparent from Figure 1, owing to the occurrence of multiple representations, as noted previously.

Of the 166 manifolds that admit a freely acting $\mathbb{Z}_2$ symmetry, we have presented Hodge numbers of quotients of 46 in the body of the chapter. In Appendix A.2, we list a further 53 manifolds and the Hodge numbers of their $\mathbb{Z}_2$–quotients.

5.1.5 How to navigate this chapter

This chapter can be read in a linear fashion. However, the reader might also wish to look up the properties of a particular CICY. The following methods are efficient:

- If you know the CICY number: for example, you are looking up the CICY X_{6947}. The quickest way is to look up the manifold in Table 5.1, in the following subsection, this directs to the discussion of each of the manifolds. Alternatively, use the PDF search facility, on the electronic version, for 6947. This leads rapidly to all the ocurrencies of this space.

- If you have an explicit matrix, but do not know the CICY number, then draw the diagram for the configuration and then contract the matrix as far as possible. The contraction will lead to one (or more) of the 'source' manifolds of Figure 1. From there proceed to §5.3, §5.4 or §5.5, as appropriate. It is then easy to identify the relevant configuration from the diagram.

5.1.6 Table of freely acting symmetries

We now present a table of the freely acting symmetries of those CICYs that possess symmetries of order divisible by 4. There are a total of 45 such manifolds, many of which are equivalent to others in this list.

Table 5.1: Symmetry groups of CICYs with $G \supseteq \mathbb{Z}_4$ or $G \supseteq \mathbb{Z}_2{\times}\mathbb{Z}_2$.

CICY #	Symmetry Groups	References
15	$\mathbb{Z}_2$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$	§5.3.4, Table 5.19
19	$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$, $\mathbb{Z}_8$, $\mathbb{Z}_4{\times}\mathbb{Z}_2$, $\mathbb{Q}_8$	§5.5.2, Table 5.48
20	$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$	§5.5.3, Table 5.50
21	$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$, $\mathbb{Z}_8$, $\mathbb{Z}_4{\times}\mathbb{Z}_2$, $\mathbb{Q}_8$, $\mathbb{Z}_4{\times}\mathbb{Z}_4$, $\mathbb{Z}_4{\rtimes}\mathbb{Z}_4$, $\mathbb{Z}_8{\times}\mathbb{Z}_2$, $\mathbb{Z}_8{\rtimes}\mathbb{Z}_2$, $\mathbb{Z}_2{\times}\mathbb{Q}_8$	§5.3.2, Table 5.15
22	$\mathbb{Z}_2$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$	§5.5.2, Table 5.46
30	$\mathbb{Z}_2$, $\mathbb{Z}_4$	§5.5.3
480	$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$	§5.5.1, Table 5.43
2357	$\mathbb{Z}_2$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$	§5.5.1, Table 5.40
2534	$\mathbb{Z}_2$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$	§5.3.4, Table 5.20
2564	$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$, $\mathbb{Z}_8$, $\mathbb{Z}_4{\times}\mathbb{Z}_2$, $\mathbb{Q}_8$	§5.5.2, Table 5.47
2566	$\mathbb{Z}_2$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$	§5.5.2, Table 5.8
2568	$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$, $\mathbb{Z}_4{\times}\mathbb{Z}_2$	§5.5.2, Table 5.3
2572	$\mathbb{Z}_2$, $\mathbb{Z}_4$	§5.3.5, Table 5.24
2639	$\mathbb{Z}_2$, $\mathbb{Z}_4$	§5.3.5, Table 5.24
2640	$\mathbb{Z}_2$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$	§5.5.1, Table 5.40
5256	$\mathbb{Z}_2$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$	§5.3.3, Table 5.17
5301	$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$	§5.3.3, Table 5.16
5302	$\mathbb{Z}_2$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$	§5.3.4, Table 5.18
5421	$\mathbb{Z}_2$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$	§5.5.1, Table 5.35
5452	$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$	§5.3.3, Table 5.16
6715	$\mathbb{Z}_2$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$	§5.4.2, Table 5.31
6784	$\mathbb{Z}_2$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$	§5.5.1, Table 5.34

Continued on next page

CICY #	**Symmetry Groups**	**References**
6788	$\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$	§5.4.2, Table 5.31
6826	$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2$	§5.5.3, Table 5.49
6828	$\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$	§5.5.1, Table 5.34
6829	$\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$	§5.3.2, Table 5.12
6836	$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_8$, $\mathbb{Z}_4\times\mathbb{Z}_2$, $\mathbb{Q}_8$, $\mathbb{Z}_4\rtimes\mathbb{Z}_4$, $\mathbb{Z}_8\times\mathbb{Z}_2$	§5.4.2, Table 5.30
6927	$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_4\times\mathbb{Z}_2$	§5.4.2, Table 5.32
6947	$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_8$, $\mathbb{Z}_4\times\mathbb{Z}_2$, $\mathbb{Q}_8$, $\mathbb{Z}_4\rtimes\mathbb{Z}_4$, $\mathbb{Z}_8\times\mathbb{Z}_2$	§5.4.2, Table 5.30
7246	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Z}_3\rtimes\mathbb{Z}_4$, $\mathbb{Z}_{12}$	§5.3.5, Table 5.23
7300	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Z}_3\rtimes\mathbb{Z}_4$, $\mathbb{Z}_{12}$	§5.3.5, Table 5.23
7435	$\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$	§5.4.2, Table 5.29
7447	$\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_5$, $\mathbb{Z}_{10}$, $\mathbb{Z}_{10}\times\mathbb{Z}_2$	§5.3.2, Table 5.10
7462	$\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$	§5.4.2, Table 5.29
7484	$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2$	§5.5.1, Table 5.33
7487	$\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$	§5.3.2, Table 5.11
7491	$\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$	§5.4.2, Table 5.29
7522	$\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$	§5.4.2, Table 5.29
7714	$\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$	§5.4.2, Table 5.28
7735	$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_4\times\mathbb{Z}_2$	§5.4.2, Table 5.27
7745	$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_4\times\mathbb{Z}_2$	§5.4.2, Table 5.27
7819	$\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$	§5.4.2, Table 5.26
7823	$\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$	§5.4.2, Table 5.26
7861	$\mathbb{Z}_2,\mathbb{Z}_4,\mathbb{Z}_2\times\mathbb{Z}_2,\mathbb{Z}_8,\mathbb{Z}_4\times\mathbb{Z}_2,\mathbb{Q}_8,\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2,\mathbb{Z}_4\times\mathbb{Z}_4,$ $\mathbb{Z}_4\rtimes\mathbb{Z}_4,\mathbb{Z}_8\times\mathbb{Z}_2,\mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_2,\mathbb{Z}_2\times\mathbb{Q}_8,(\mathbb{Z}_4\times\mathbb{Z}_2)\rtimes\mathbb{Z}_4,$ $\mathbb{Z}_8\times\mathbb{Z}_4,\mathbb{Z}_8\rtimes\mathbb{Z}_4,\ (\mathbb{Z}_8\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2,\mathbb{Z}_8\rtimes\mathbb{Z}_4,$ $\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_2,\mathbb{Z}_2\times(\mathbb{Z}_4\rtimes\mathbb{Z}_4),\mathbb{Z}_4\rtimes\mathbb{Q}_8,\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Q}_8$	§5.4.1, Table 5.25
7862	$\mathbb{Z}_2,\ \mathbb{Z}_4,\ \mathbb{Z}_2\times\mathbb{Z}_2,\ \mathbb{Z}_8,\ \mathbb{Z}_4\times\mathbb{Z}_2,\ \mathbb{Q}_8,$ $\mathbb{Z}_4\times\mathbb{Z}_4,\ \mathbb{Z}_4\rtimes\mathbb{Z}_4,\ \mathbb{Z}_8\times\mathbb{Z}_2,\ \mathbb{Z}_8\rtimes\mathbb{Z}_2,\ \mathbb{Z}_2\times\mathbb{Q}_8$	§5.3.1, Table 5.9

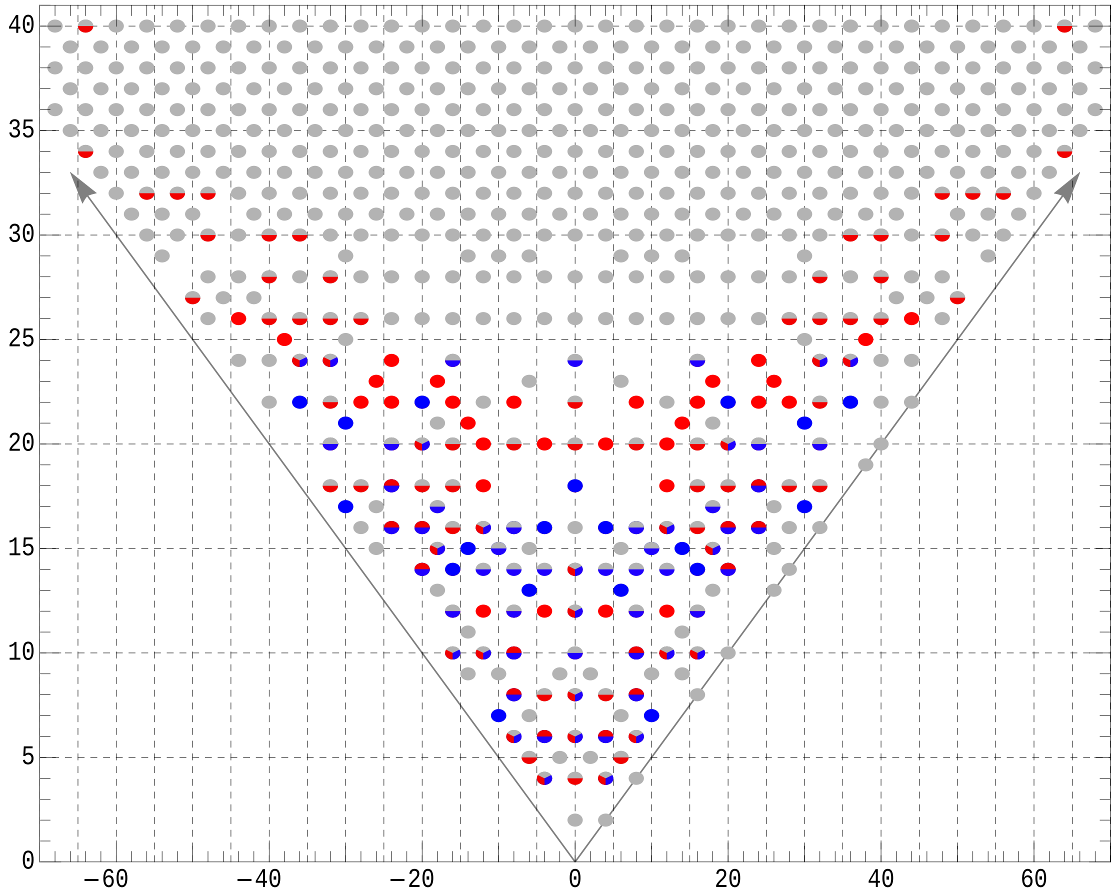


Figure 8: *The tip of the Hodge number plot for all the Calabi-Yau three-folds that we know. The grey points are the manifolds of the Kreuzer-Skarke list, CICYs, generalized CICYs, toric CICYs, resolutions of toric conifolds, Gross-Popescu manifolds, the manifold of V. Braun with Hodge numbers (1,1), manifolds obtained through hyperconifold transitions and other manifolds studied in Refs. [5, 61, 64, 94, 97, 103, 115, 116, 118–135], as well as the mirrors of the foregoing. The blue points correspond to the CICY quotients previously studied in [2, 6]. The red points correspond to CICY quotients studied in the present chapter together with their mirrors. Monochrome points indicate quotients whose Hodge numbers fall onto sites previously unoccupied, while the multicoloured points correspond to multiply occupied sites.*

5.2 Preliminaries

5.2.1 The polynomial deformation method

The polynomial deformation method, proposed in [1], has been used in the literature to compute the number of harmonic $(2,1)$-forms on Calabi-Yau manifolds defined as complete intersections of hypersurfaces given by homogeneous polynomials in products of projective spaces. The method relies on the observation that in many such cases the coefficients of the defining polynomials give a complete and non-redundant parametrisation of the complex structure moduli space. In the present section we will describe the polynomial deformation method in detail, first for CICY manifolds, and then for smooth quotients thereof.

The class of CICYs was formally introduced in §2.2.1. We reproduce below the general form of the configuration matrix of a CICY:

$$X = \begin{matrix} \mathbb{P}^{n_1} \\ \vdots \\ \mathbb{P}^{n_m} \end{matrix} \left[\begin{matrix} q_1^1 & \cdots & q_K^1 \\ \vdots & \ddots & \vdots \\ q_1^m & \cdots & q_K^m \end{matrix}\right]^{h^{1,1}(X),\ h^{2,1}(X)}_{\chi(X)} \tag{5.2.1}$$

The polynomial deformation method for computing the number of complex structure deformations of a CICY X proceeds by the following three steps:

1. Compute the number of coefficients in the defining polynomials:

$$N_{\text{coeffs}} = \sum_{a=1}^{K} \prod_{r=1}^{m} \frac{(q_a^r + n_r)!}{(q_a^r)!\, n_r!} \tag{5.2.2}$$

2. Subtract the number of parameters corresponding to the freedom to redefine the homogeneous coordinates of the m projective spaces:

$$N_{\text{c.r.}} = \sum_{r=1}^{m} \left(n_r^2 - 1\right) \tag{5.2.3}$$

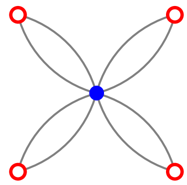
3. Subtract the number of parameters corresponding to the freedom to redefine the defining polynomials, $N_{\text{p.r.}}$. This step is not always straightforward. In the simplest case, in which there are no relations between the defining polynomials, the only available polynomial redefinition is an overall re-scaling

of each polynomial, which gives a total number of K to be subtracted from N_{coeffs}. The computation for the generic case is best illustrated through a few examples, which we pursue below.

5.2.2 Examples

We choose our examples from the class of manifolds discussed in the following sections. A simple illustration of the polynomial deformation method is afforded by the tetraquadric, a manifold defined as a hypersurface of multi-degree $(2,2,2,2)$ in a product of four $\mathbb{P}^1$ spaces:

$$X_{7862} \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\left[\begin{matrix}2\\2\\2\\2\end{matrix}\right]^{4,68}_{-128} \tag{5.2.4}$$

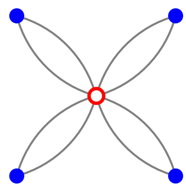


For this manifold, we have:

$$h^{2,1}(X) \;=\; \left(\frac{3!}{2!}\right)^4 - 4\cdot(2^2-1) - 1 \;=\; 68\ . \tag{5.2.5}$$

The second example involves polynomials which have the same (multi)-degrees, these polynomials, p^α, can be redefined by linear transformations $p^\alpha \to M^\alpha{}_\beta\, p^\beta$.

$$X_{7861} \;=\; \mathbb{P}^7\left[\,2\quad 2\quad 2\quad 2\right]^{1,65}_{-128} \tag{5.2.6}$$

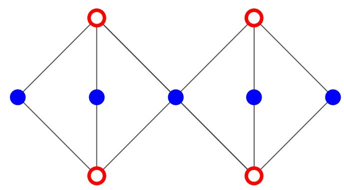


In this case, we have $N_{\text{p.r.}} = 16$, hence:

$$h^{2,1}(X) \;=\; 4\cdot\frac{9!}{7!\cdot 2!} - (8^2-1) - 16 \;=\; 65\ .$$

In the third example, the computation of $N_{\text{p.r.}}$ is somewhat more involved.

$$X_{7246} \;=\; \begin{matrix}\mathbb{P}^2\\ \mathbb{P}^2\\ \mathbb{P}^2\\ \mathbb{P}^2\end{matrix}\left[\begin{matrix}1&1&0&0&1\\1&1&0&0&1\\0&0&1&1&1\\0&0&1&1&1\end{matrix}\right]^{8,44}_{-72} \tag{5.2.7}$$



Let us denote the polynomials corresponding to the columns of the configuration, taken in order, by p^α, $\alpha = 1,\ldots,5$. In this case, there is a 4 parameter freedom to redefine p^1 and p^2 and another 4 parameter freedom to redefine p^3 and p^4. We may also redefine p^5 by

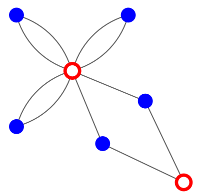
$$p^5 \;\to\; \lambda\, p^5 + \sum_{k=1}^{4} \widetilde{p}_k\, p^k \tag{5.2.8}$$

where λ is a scale, $\widetilde{p}_1$ and $\widetilde{p}_2$ have multidegree $(0,0,1,1)$ and $\widetilde{p}_3$ and $\widetilde{p}_4$ have multidegree $(1,1,0,0)$. Each $\widetilde{p}_k$ has 9 degrees of freedom, so would give a total of $4\cdot 9+1=37$ parameters. However, there is an over-counting since the 4 products $p^1\,p^3$, $p^2\,p^3$, $p^1\,p^4, p^2\,p^4$ are counted twice. This leaves $37-4=33$ parameters corresponding to redefinitions of p_5. Thus we have:

$$h^{2,1}(X) \;=\; 4\cdot\left(\frac{3!}{2!\cdot 1!}\right)^2 + \left(\frac{3!}{2!\cdot 1!}\right)^4 - 4\cdot(3^2-1) - (2\cdot 4+33) \;=\; 44\ . \tag{5.2.9}$$

The fourth example corresponds to a manifold that can be obtained from the manifold (5.2.6) through a conifold transition:

$$X_{7819} \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^7\end{matrix}\!\left[\begin{matrix}1 & 1 & 0 & 0 & 0\\ 1 & 1 & 2 & 2 & 2\end{matrix}\right]^{2,58}_{-112} \tag{5.2.10}$$



In this case, the counting of polynomial redefinitions goes as follows. There is a 2×2 matrix, so 4 parameters, of redefinitions associated with the first two polynomials and 3×3 matrix, so 9 parameters, associated with the last three polynomials. Furthermore, the first two polynomials, p_1 and p_2 are linear in the homogeneous coordinates $\{x_0, x_1\}$ of $\mathbb{P}^1$:

$$\begin{aligned} p_1 &= A_1(y)\,x_0 + B_1(y)\,x_1\ , \\ p_2 &= A_2(y)\,x_0 + B_2(y)\,x_1\ . \end{aligned} \tag{5.2.11}$$

Since the coordinates x_0, x_1 cannot both vanish, we have the determinant

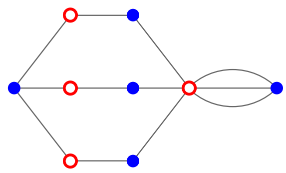
$$A_1(y)\,B_2(y) - A_2(y)\,B_1(y) \;=\; 0\ , \tag{5.2.12}$$

which corresponds to the vanishing of a polynomial of degree 2 in the coordinates of $\mathbb{P}^7$. Multiples of this determinant may be added to any of the last three polynomials, giving 3 extra parameters. Thus $N_{\text{p.r.}} = 13+3$ and we obtain in this way

$$h^{2,1}(X) \;=\; 2\cdot\frac{8!}{7!\cdot 1!}\cdot\frac{2!}{1!\cdot 1!} + 3\cdot\frac{9!}{7!\cdot 2!} - \Big((2^2-1)+(8^2-1)\Big) - 16 \;=\; 58\ . \tag{5.2.13}$$

The final example corresponds to a manifold that will not appear in the following sections, owing to the fact that it does not admit a symmetry of order 4, however we discuss it here since the computation of $N_{\text{p.r.}}$ involves an extra element. This manifold admits a smooth quotient by $\mathbb{Z}_3$ and was discussed in [2]. The manifold corresponds to the following configuration matrix:

$$X_{7664} = \begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^5\end{matrix}\left[\begin{matrix}1&0&0&1&0\\0&1&0&1&0\\0&0&1&1&0\\1&1&1&0&3\end{matrix}\right]^{5,50}_{-90} \tag{5.2.14}$$



In this case, apart from the 5 overall re-scalings of the defining polynomials, there exists an additional redefinition, which can be understood as follows. For fixed coordinates in $\mathbb{P}^5$, the equation $p_1 = 0$ gives a unique solution for $x_1\colon x_0$, and hence, the locus $p_1 = 0$ and, further, the manifold X intersects the first $\mathbb{P}^1$ in a single point. Similarly, there are unique points in the second and the third $\mathbb{P}^1$ spaces that correspond to the intersection with X, for a given set of coordinates in $\mathbb{P}^5$. As such, the fourth polynomial can be regarded as a polynomial of degree 3 in the coordinates of $\mathbb{P}^5$, and it can be used in order to redefine p_5. Consequently, we arrive to the following counting of parameters:

$$h^{2,1}(X) = 3\cdot\frac{2!}{1!\cdot 1!}\cdot\frac{6!}{5!\cdot 1!}+\left(\frac{2!}{1!\cdot 1!}\right)^3+\frac{8!}{5!\cdot 3!}-\Big(3\cdot(2^2-1)+(6^2-1)\Big)-6 = 50\,. \tag{5.2.15}$$

5.2.3 Understanding the polynomial deformation method

The validity of the polynomial deformation method, as outlined above, was studied in [136]. The procedure provides a complete and non-redundant parametrisation of the complex structure moduli space if (i) the parameter counting agrees with $h^{2,1}(X)$ and (ii) the associated diagram is not one-leg-decomposable. (A connected diagram is called one-leg-decomposable if the complement of a single leg is disconnected.) Without repeating the full discussion of [136], we would like to review the cohomology computations that underlie the polynomial deformation method. To this end, we need to relate the cohomology of X to bundle-valued cohomologies. Through Dolbeault's theorem, we have the identification:

$$H^{p,q}(X) \;\cong\; H^q(X,\, \wedge^p TX^*) \;. \tag{5.2.16}$$

Moreover, on a Calabi-Yau three-fold, the multiplication with the projectively unique and nowhere vanishing holomorphic 3-form, induces the isomorphism:

$$H^q(X, TX) \;\cong\; H^q\Big(X,\, \wedge^2 TX^*\Big) \;. \tag{5.2.17}$$

Together, (5.2.16) and (5.2.17) imply $H^{2,1}(X) \;\cong\; H^1(X, TX)$. If X is a CICY manifold, we can compute $H^1(X, TX)$ using the normal bundle sequence and the Euler sequence, as discussed below.

Let $X \subset \mathcal{A}$ be a Calabi-Yau three-fold, with $\mathcal{A} = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}$, defined as the common zero locus of the K polynomials, $p_1, \ldots, p_K$. We will assume that X is not a direct product, or equivalently, that the configuration matrix is in-decomposable (i.e. it cannot be written in block diagonal form with more than one block), or equivalently, that the associated diagram is connected. Let $\mathcal{N} \to \mathcal{A}$ denote the line bundle sum whose sections are the defining polynomials, and $N = \mathcal{N}|_X$, the restriction of $\mathcal{N}$ to X, denote the normal bundle of X. Explicitly, the normal bundle $\mathcal{N}$ can be obtained from the configuration matrix (5.2.1) as the sum of line bundles:

$$\mathcal{N} \;=\; \bigoplus_{a=1}^{K} \mathcal{O}_{\mathcal{A}}(\mathbf{q}_a) \;. \tag{5.2.18}$$

Then the normal bundle sequence reads:

$$0 \to TX \to T\mathcal{A}|_X \to N \to 0 \;. \tag{5.2.19}$$

This short exact sequence induces the long exact sequence in cohomology:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X,TX) & \longrightarrow & H^0(X,T\mathcal{A}|_X) & \longrightarrow & H^0(X,N) & \longrightarrow & \\ & \longrightarrow & H^1(X,TX) & \longrightarrow & H^1(X,T\mathcal{A}|_X) & \longrightarrow & H^1(X,N) & \longrightarrow & \\ & \longrightarrow & H^2(X,TX) & \longrightarrow & H^2(X,T\mathcal{A}|_X) & \longrightarrow & H^2(X,N) & \longrightarrow & \\ & \longrightarrow & H^3(X,TX) & \longrightarrow & H^3(X,T\mathcal{A}|_X) & \longrightarrow & H^3(X,N) & \longrightarrow & 0 \end{array} \tag{5.2.20}$$

which implies:

$$H^1(X,TX) \cong \Big(H^0(X,N)/H^0(X,T\mathcal{A}|_X)\Big) \oplus \text{Ker}\,\Big(H^1(X,T\mathcal{A}|_X) \to H^1(X,N)\Big)\ , \tag{5.2.21}$$

where we have used the fact that, for a Calabi-Yau three-fold that is not a direct product, we have $H^0(X,TX) \cong H^{3,1}(X) = 0$. Since N is a sum of line bundles, its cohomology can be relatively easily computed from line bundle cohomology on $\mathcal{A}$, using the Koszul spectral sequence. The cohomology of $T\mathcal{A}|_X$ can be obtained from the Euler sequence, restricted to X:

$$0 \to \mathcal{O}_X^{\oplus m} \to S \to T\mathcal{A}|_X \to 0\ , \tag{5.2.22}$$

where $\mathcal{O}_X$ denotes the trivial line bundle on X (or, rather, the sheaf of holomorphic functions on X, also called the structure sheaf of X), $S = \bigoplus_{r=1}^m \mathcal{O}_X(\mathbf{e}_r)^{\oplus(n_r+1)}$, and $\mathbf{e}_r$ are the standard unit vectors in m dimensions. The associated long exact sequence in cohomology,

$$\begin{array}{cccccccc} 0 & \longrightarrow & H^0(X,\mathcal{O}_X^{\oplus m}) & \longrightarrow & H^0(X,S) & \longrightarrow & H^0(X,T\mathcal{A}|_X) & \longrightarrow \\ & \longrightarrow & H^1(X,\mathcal{O}_X^{\oplus m}) & \longrightarrow & H^1(X,S) & \longrightarrow & H^1(X,T\mathcal{A}|_X) & \longrightarrow \\ & \longrightarrow & H^2(X,\mathcal{O}_X^{\oplus m}) & \longrightarrow & H^2(X,S) & \longrightarrow & H^2(X,T\mathcal{A}|_X) & \longrightarrow \\ & \longrightarrow & H^3(X,\mathcal{O}_X^{\oplus m}) & \longrightarrow & H^3(X,S) & \longrightarrow & H^3(X,T\mathcal{A}|_X) & \longrightarrow \quad 0 \end{array} \tag{5.2.23}$$

leads to the following identifications:

$$H^0(X,T\mathcal{A}|_X) \cong H^0(X,S)/\mathbb{C}^m \tag{5.2.24}$$

$$H^1(X,T\mathcal{A}|_X) \cong H^1(X,S) \tag{5.2.25}$$

where we have used the fact that, for spaces defined by in-decomposable configuration matrices, $H^0(X,\mathcal{O}_X) \cong H^3(X,\mathcal{O}_X) \cong \mathbb{C}$ and the cohomology groups $H^1(X,\mathcal{O}_X)$ and $H^2(X,\mathcal{O}_X)$ are trivial, results which follow from the application of Bott's formula for line bundles on projective spaces, the Künneth formula for cohomology of bundles over a direct product of spaces and the Koszul resolution of the restriction $\mathcal{O}_\mathcal{A} \to \mathcal{O}_X$. With the above identifications, we have:

$$H^{2,1}(X) \;\cong\; \frac{H^0(X,N)}{H^0(X,S)/\mathbb{C}^m} \oplus \text{Ker}\,\Big(H^1(X,S) \to H^1(X,N)\Big)\ . \tag{5.2.26}$$

The expression (5.2.26) is completely general, subject only to the assumption that the configuration matrix is in-decomposable. All the cohomologies involved are line bundle cohomologies, which can be computed using the Koszul spectral sequence (see e.g. [29, 137]). To relate it with the polynomial deformation method, we note the following:

1. The dimension of the cohomology group $H^0(X, N)$ corresponds to the difference:

$$N_{\text{coeffs}} - N_{\text{p.r.}} \tag{5.2.27}$$

2. The term $H^0(X, S)/\mathbb{C}^m$ corresponds to the (projective) coordinate redefinitions of the ambient space, $N_{\text{c.r.}}$.

For the case that

$$\text{Ker}\left(H^1(X, S) \to H^1(X, N)\right) \cong 0 \tag{5.2.28}$$

we obtain the simple relation:

$$h^{2,1}(X) = h^0(X, N) - \left(h^0(X, S) - m\right) . \tag{5.2.29}$$

When it fails, the polynomial deformation method gives a wrong result for $\dim(H^0(X, N))$. This has to be so, since in general the computation of $H^0(X, N)$ involves contributions from higher iterations in the Koszul spectral sequence. Some of these contributions can be accounted for by using the artifices described above in Section 5.2.2. Given a group action $G \times X \to X$, and an equivariant structure of the normal bundle N specified by the action on the defining polynomials, the cohomology groups involved in Eq. (5.2.29) split into representations of G. The part which corresponds to the trivial representation descends to the quotient manifold. Thus by counting the multiplicity of the trivial representation in a given cohomology group on X one can obtain the dimension of the cohomology group in question on X/G [101].

5.2.4 The polynomial deformation method for quotient manifolds

Let $G \times X \to X$ be a free group action of the finite group G on the CICY threefold X defined as above, as the common zero locus of K polynomials, $p_1, \ldots, p_K$, in $\mathcal{A} = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}$. We can pick $g_1, \ldots, g_k$ to be a set of generators of G. The action can be specified by two sets of matrices, $\{\gamma(g_1), \ldots, \gamma(g_k)\}$ and $\{\rho(g_1), \ldots, \rho(g_k)\}$, representing the action on the coordinates and the action on the polynomials. Note that the matrices $\{\gamma(g)\}$ and $\{\rho(g)\}$ do not form a representation in the usual sense; rather, they are obtained by multiplying a multi-projective representation of G with a permutation representation, corresponding to the part of the action of G that permutes the embedding projective spaces (see [3] for more details).

For the smooth quotient X/G, the counting of complex structure parameters using the polynomial deformation method can be systematised in the following steps:

1. Find a basis of invariant polynomial vectors $\{v_i = (p_1^i, \ldots, p_K^i)\}$. The dimension of this basis corresponds to the number of coefficients in the specialised polynomials that define manifolds with the given symmetry. For the manifolds discussed in the present chapter, a basis of invariant polynomial vectors was given in each case in [3].

2. Count the number of coordinate redefinitions consistent with the given group action. Let C_x be a block-diagonal matrix with the i-th block representing a general linear transformation of the homogeneous coordinates of the i-th projective space. Then C_x is consistent with the given group action if

$$[C_x, \gamma(g_i)] = 0 \ , \tag{5.2.30}$$

for all group generators g_i. The number of free parameters in the matrix C_x that remain after solving (5.2.30) minus the number of projective re-scalings consistent with the given symmetry corresponds to $N_{\text{c.r.}}$ for the quotient manifold. In order to find the number of allowed projective re-scalings, one starts with a matrix $\widetilde{C}_x$ of the same size and block-diagonal structure as C_x, in which each block is a multiple of the identity matrix with an arbitrary multiplication factor. Solving equations analogous to (5.2.30), leaves a certain number of free parameters, which correspond to the allowed projective re-scalings.

3. Similarly, in order to find the number of polynomial redefinitions $N_{\text{p.r.}}$, one starts with a matrix C_p representing the most general redefinition by linear transformations and then counts the number of free parameters that are left after solving the equations $[C_p, \rho(g_i)] = 0$, analogous to (5.2.30). The cases which contain "embeddings" and redefinitions with determinants are more subtle. Below, we outline the computation for a case in which determinants are present in the computation of $N_{\text{p.r.}}$.

5.2.5 Examples: Computing $h^{2,1}$ by parameter counting

The quintic

Although in this work, we will only look at CICYs with freely acting discrete symmetries whose orders are divisible by 4, and some with freely acting $\mathbb{Z}_2$ symmetries, the quintic quotient by the freely acting $\mathbb{Z}_5\times\mathbb{Z}_5$ symmetry is one of the simplest examples to illustrate the counting of complex structure parameters. The quintic and the $\mathbb{Z}_5\times\mathbb{Z}_5$ symmetry generators S and T are described below:

$$\begin{gathered} X_{7890} = \mathbb{P}^4[5]^{1,101}_{-200}; \\ S : x_i \to \zeta^i x_i \quad \text{and} \quad T : x_i \to x_{i+4} \end{gathered} \tag{5.2.31}$$

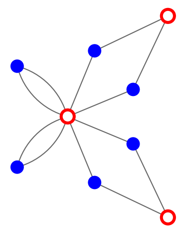
where, ζ is a non-trivial fifth root of unity. For the quintic, $h^{2,1}$ can be accounted for by first noting that there are 126 coefficients in the most generic quintic polynomial. There are $5^2 - 1 = 24$, and 1 degree of freedom in co-ordinate and polynomial redefinitions respectively. Therefore, $h^{2,1} = 126 - 24 - 1 = 101$. For the quotient we begin by noting that there are 6 quintic invariants of the $\mathbb{Z}_5\times\mathbb{Z}_5$ symmetry (3.2.2). There is one degree of freedom left in co-ordinate redefinitions of $\mathbb{P}^4$ that are consistent with the group action in (5.2.31), which is negated by an overall projective rescaling of the co-ordinates of $\mathbb{P}^4$. There is one degree of freedom for the overall rescaling of the quintic polynomial. Thus, for the quotient, $h^{2,1} = 6 - 1 = 5$.

This implies that $h^{1,1} = 1$ for the quotient, since the Euler characteristic for the quotient is $-200/25 = -8$.

A CICY embedded in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^7$

We start with the manifold given by the following configuration:

$$X_{7745} \;=\; \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^7 \end{array} \!\left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 2 \end{array}\right]^{3,51}_{-96} \qquad (5.2.32)$$



The counting of polynomial redefinitions is similar to the counting done for the manifold (5.2.10) above. There are 12 redefinitions associated with linear transformations of polynomials of the same multi-degrees. Furthermore, there is a determinant condition associated with the first and the second pairs of polynomials, which give two polynomials of multi-degree $(0, 0, 2)$ which can be added to each of the last two polynomials. This gives 4 additional redefinitions. Hence, we obtain:

$$h^{2,1}(X) \;=\; 4\cdot\frac{8!}{7!\cdot 1!}\cdot\frac{2!}{1!\cdot 1!}+2\cdot\frac{9!}{7!\cdot 2!}-\Big(2\cdot(2^2-1)+(8^2-1)\Big)-16 \;=\; 51\ . \qquad (5.2.33)$$

The manifold X admits a free action of $\mathbb{Z}_4$, which can be specified by two matrices $\gamma(g)$ and $\rho(g)$, where g is a generator of $\mathbb{Z}_4$. The matrix $\gamma(g)$ corresponds to the action on the twelve ambient space coordinates, while $\rho(g)$ gives the action on the six defining polynomials:

$$\gamma(g) \;=\; \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \oplus \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i} \end{array}\right] \qquad (5.2.34)$$

$$\rho(g) \;=\; \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \oplus \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right] \qquad (5.2.35)$$

Note that the action of $\gamma(g)$ swaps the two $\mathbb{P}^1$ spaces. More explicitly, denoting the coordinates of the $\mathbb{P}^1$ spaces by x_β and y_β respectively, with $\beta \in \mathbb{Z}_2$ and those of the $\mathbb{P}^7$ space by z_m, the polynomial equations that define the manifold can then be written as:

$$\begin{aligned} p^\alpha\left(x_\beta, z_m\right) \;=\; \sum_{m,\beta} P^\alpha_{\beta,m}\, x_\beta\, z_m \;, \qquad q^\alpha\left(y_\beta, z_m\right) \;=\; \sum_{m,\beta} Q^\alpha_{\beta,m}\, y_\beta\, z_m \;, \\ r^\alpha\left(z_m\right) \;=\; \sum_{m,n} R^\alpha_{m,n}\, z_m\, z_n \end{aligned} \tag{5.2.36}$$

where p^α, q^α and r^α denote the three pairs of polynomials, with $\alpha \in \mathbb{Z}_2$ and $m \in \mathbb{Z}_8$. With these notations, the action (5.2.34), (5.2.35) can be recast in the following form:

$$\begin{array}{lllllllll} x_\beta & \to & (-1)^\beta\, y_\beta \;, & y_\beta & \to & x_\beta \;, & z_m & \to & \mathrm{i}^m z_m \;, \\ p^\alpha & \to & q^\alpha \;, & q^\alpha & \to & (-1)^\alpha\, p^\alpha \;, & r^\alpha & \to & (-1)^\alpha\, r^\alpha \;. \end{array} \tag{5.2.37}$$

The generic form of the polynomials (5.2.36) is not consistent with this action. To achieve that, we need to impose certain relations between the coefficients $P^\alpha_{\beta,m}$, $Q^\alpha_{\beta,m}$ and $R^\alpha_{m,n}$, namely:

$$Q^\alpha_{\beta,m} \;=\; (-1)^\beta\, \mathrm{i}^m P^\alpha_{\beta,m} \;=\; (-1)^\alpha\, \mathrm{i}^{-m} P^\alpha_{\beta,m} \;, \qquad \mathrm{i}^{m+n} R^\alpha_{m,n} \;=\; (-1)^\alpha\, R^\alpha_{m,n} \;. \tag{5.2.38}$$

This implies that the coefficients $Q^\alpha_{\beta,m}$ are completely obtainable from the coefficients $P^\alpha_{\beta,m}$, subject to the condition $(-1)^\beta\, \mathrm{i}^m = (-1)^\alpha\, \mathrm{i}^{-m}$, which, in turn, implies that $\alpha + \beta - m$ must be an even integer. If $\alpha + \beta$ is odd (of which there are two cases), $m \in \{1, 3, 5, 7\}$, while if $\alpha + \beta$ is even (of which also, there are two cases), $m \in \{0, 2, 4, 6\}$, leading to a total of 16 valid tuples $\{\alpha, \beta, m\}$. This implies that there are 16 non-zero terms in the p^α polynomial, consistent with the given $\mathbb{Z}_4$ action. The third condition involving the r^α polynomials reduces to the condition $m + n \equiv 0 \mod 4$, for $\alpha = 0$, and $m + n \equiv 2 \mod 4$, for $\alpha = 1$. Together, these yield a total of 20 distinct cases, implying that there are 20 terms in the restricted r^α polynomials. Thus the total number of free coefficients in the restricted polynomials, consistent with the given $\mathbb{Z}_4$ action is $16 + 20 = 36$, which agrees with the number the number of polynomial invariants obtained in [3].

In order to find the number of coordinate redefinitions, consistent with the $\mathbb{Z}_4$ action (5.2.37), we introduce a matrix C_x, as described in the previous section. Solving the equation $[C_x, \gamma(g)] = 0$ leaves 18 free parameters in C_x, of the original 72 parameters. Out of these, we need to subtract the 2 independent projective re-scalings of the coordinates consistent with the action (5.2.37). Thus $N_{\text{c.r.}}(X/\mathbb{Z}_4) = 16$. Similarly, we introduce the matrix C_p which corresponds to linear transformations among the defining polynomials of equal multi-degrees. Solving the equation $[C_p, \rho(g)] = 0$ leaves 4 free parameters in C_p.

To account for the polynomial determinants that can be added to the last two polynomials, we note that these can be written as:

$$\Delta^1 = \text{Det}\begin{bmatrix} \sum_m P^0_{0,m}\, z_m & \sum_m P^0_{1,m}\, z_m \\ \sum_m P^1_{0,m}\, z_m & \sum_m P^1_{1,m}\, z_m \end{bmatrix} \; , \; \Delta^2 = \text{Det}\begin{bmatrix} \sum_m Q^0_{0,m}\, z_m & \sum_m Q^0_{1,m}\, z_m \\ \sum_m Q^1_{0,m}\, z_m & \sum_m Q^1_{1,m}\, z_m \end{bmatrix} \; . \tag{5.2.39}$$

For the manifold X, these lead to 4 polynomial redefinitions of the form:

$$r^\alpha(z) \to r^\alpha(z) + \sum_\beta \lambda_\beta\, \Delta^\beta \; . \tag{5.2.40}$$

However, under the action (5.2.37), $\Delta^1 \leftrightarrow -\Delta^2$. It follows that there is actually only one invariant quantity constructed out of the two determinants, $\Delta^1 - \Delta^2$, which can be added to the two polynomials r^α. Thus, we have only two polynomial redefinitions coming from the determinant constraints, as opposed to four for the covering manifold. Finally, we obtain:

$$h^{2,1}(X/\mathbb{Z}_4) = 36 - 16 - 6 = 14 \; . \tag{5.2.41}$$

This implies $h^{1,1}(X/\mathbb{Z}_4) = 2$, since the Euler characteristic of the quotient manifold is $-96/4 = -24$. This is consistent with the counting of Kähler parameters.

5.2.6 Invariant Kähler forms

We turn now to a consideration of the counting of the Kähler parameters. The Kähler parameters are most easily counted if the embedding of the CICY manifold is favourable, that is, for CICY manifolds for which the entire second cohomology descends from the ambient space, thought of as a product of projective spaces. A precise statement is given in [9], which we review below. An extension that we discuss in the following concerns the case where the $h^{1,1}$ of the manifold can be explained by virtue of it being embedded in a product of del Pezzo and projective spaces. In these cases we count separately the homology invariants of the del Pezzo part.

We begin with the case of a favourable embedding. For this case we seek to gain an explicit understanding of the cohomology group $H^{1,1}(X)$. An application of Dolbeault's theorem leads to the identification:

$$H^{1,1}(X) \;\cong\; H^1(X, TX^*) \; . \tag{5.2.42}$$

For a Calabi-Yau three-fold, Serre duality formula gives

$$H^q(X, TX^*) \;\cong\; H^{3-q}(X, TX)^* \; . \tag{5.2.43}$$

Combining these two formulas, we obtain $H^{1,1}(X) \;\cong\; H^2(X, TX)^*$. The bundle-valued cohomology $H^2(X, TX)$ can be computed using the tangent bundle and the Euler sequences, as done in Section 5.2.3. Assuming that the configuration matrix defining X is in-decomposable, we obtain:

$$\begin{aligned} H^2(X, TX) \;&\cong\; \text{Coker}\left(H^1(X, S) \to H^1(X, N)\right) \oplus \text{Ker}\left(H^2(X, T\mathcal{A}|_X) \to H^2(X, N)\right) \; , \\ H^2(X, T\mathcal{A}|_X) \;&\cong\; H^2(X, S) \oplus \text{Ker}\left(\mathbb{C}^m \to H^3(X, S)\right) \; . \end{aligned} \tag{5.2.44}$$

The part of $H^2(X, TX)$ which descends from the second cohomology of the ambient space corresponds to the $\mathbb{C}^m$ term in the second equation, where m is the number of projective spaces in the ambient space $\mathcal{A}$. Thus, a CICY three-fold X is favourable if the following conditions are satisfied:

$$\text{Coker}\left(H^1(X, S) \to H^1(X, N)\right) = 0 \; , \quad H^2(X, S) = 0 \; . \tag{5.2.45}$$

A necessary, but not sufficient condition for (5.2.45) to hold is that $h^{1,1}(X) = m$. A sufficient, however slightly too strong, condition for X to be favourable is

$$h^1(X, N) = h^2(X, S) = 0 \; . \tag{5.2.46}$$

It follows that for a favourable CICY three-fold X, $h^{1,1}(X)$ is given by the number of projective spaces in the ambient space. In this case, a basis of the second cohomology of X is given by the pullbacks of the hyperplane classes of the embedding projective spaces.

If X admits a linearly represented free action of the finite group G, and X is favourably embedded in a product of projective spaces $\mathcal{A}$, one can easily count the number of Kähler parameters, $h^{1,1}(X/G)$. The action of G on $\mathcal{A}$ consists of two parts, one which permutes projective spaces of the same dimension, and one which acts internally on the coordinates of each projective space. As such, $h^{1,1}(X/G)$ equals the number of orbits of the permutation part of the action.

To give an example, consider the manifold (5.2.32) discussed in the previous section, for which $\mathcal{A}=\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^7$, and its $\mathbb{Z}_4$–quotient. The manifold is embedded favourably, hence $h^{1,1}(X) = 3$. The $\mathbb{Z}_4$ action permutes the two $\mathbb{P}^1$ spaces; consequently, $h^{1,1}(X/\mathbb{Z}_4) = 2$, in agreement with the results obtained above.

5.2.7 Manifolds embedded in products of del Pezzo surfaces

We note, on several occasions throughout this chapter, that some of our CICY manifolds can be thought of as submanifolds of products of del Pezzo surfaces. The del Pezzo surfaces are smooth algebraic surfaces with ample anticanonical bundle $-K$. These are $\mathbb{P}^1\times\mathbb{P}^1$, $\mathbb{P}^2$ and $\mathbb{P}^2$ blown up in k, $1\leq k\leq 8$ *generic* points. We here denote by dP_n the surface obtained by blowing up $\mathbb{P}^2$ in $9-n$ points, in general position. This is then the del Pezzo surface of degree n, where the degree of the surface is the self intersection number, K^2, of its anti-canonical class.

del Pezzo surfaces contain special lines and the configuration of these lines has been the subject of much study in classical algebraic geometry (for a recent text see [138]). One way to study how the symmetries of the manifold act on the Kähler classes is to study how the symmetries permute the special lines, since there is a correspondence between the lines and cohomology classes. This study was carried out for the three-generation manifold in §2 of [5] realised as a hypersurface in a product of two del Pezzo surfaces of degree 6. In the following we will work this through for manifolds that are embedded in products of del Pezzo surfaces of degree

4, which play an important role in the class of manifolds we consider here. The fact that the class of hypersurfaces in products of del Pezzo surfaces, and products of del Pezzo surfaces with projective spaces is a fruitful place to look for Calabi–Yau manifoldswith symmetries was appreciated by Bini and Favale [115].

del Pezzo surfaces of degree 4

For a thorough discussion of del Pezzo surfaces of degree 4 see also [139]. They are surfaces isomorphic to $\mathbb{P}^2$ blown up in 5 points, in general position, which in this case amounts to no 3 points on a line. Each blow-up introduces an exceptional divisor, isomorphic to $\mathbb{P}^1$, with self-intersection -1. We denote these curves by E_i, with $i \in \{1, \ldots, 5\}$. Together with the pullback of the hyperplane class from $\mathbb{P}^2$, denoted by H, these curves span $H_2(\mathrm{dP}_4, \mathbb{Z})$. Since the E_i curves are disjoint, and H can always be chosen such that it misses the blown-up points, we have the intersection numbers $H \cdot H = 1$, $H \cdot E_i = 0$ and $E_i \cdot E_j = -\,\delta_{ij}$. In terms of these classes, the anti-canonical class is given by:

$$-\,K \;=\; 3H - \textstyle\sum_i E_i \;, \tag{5.2.47}$$

and we see that $K^2 = 4$ which is the degree of the surface. In addition, dP_4 contains 11 other curves with self-intersection -1. For each pair (i, j), with $i < j$, there is an exceptional line L_{ij} that intersects both E_i and E_j, so 10 such lines. The L_{ij} line corresponds to the line in $\mathbb{P}^2$ that passes through the points blown-up to E_i and E_j. Since any two hyperplanes in $\mathbb{P}^2$ intersect at a point, we have $H \cdot L_{ij} = 1$. In homology, these intersection relations identify the L_{ij} as:

$$L_{ij} \;=\; H - E_i - E_j \;. \tag{5.2.48}$$

Finally, there is a unique quadric Q passing through the 5 blown up points of $\mathbb{P}^2$, that satisfies $Q \cdot E_i = 1$ (a smooth quadric, $\mathbb{P}^2[2]$, is a $\mathbb{P}^1$). This curve intersects a generic hyperplane H in two points. These intersections define the class Q as:

$$Q \;=\; 2H - \textstyle\sum_i E_i \;. \tag{5.2.49}$$

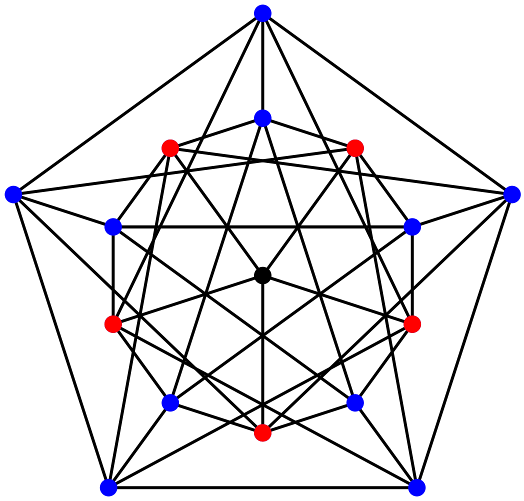


Figure 9: *The Clebsch Graph showing the intersections of the -1 curves of a del Pezzo surface of degree 4. The* 16 *vertices in the graph correspond to the curves. Two such curves intersect if the corresponding vertices are connected. From the graph, one sees that each curve meets exactly* 5 *other curves. The vertices in red and blue correspond to the lines E_i and L_{ij} respectively. The black vertex at the centre corresponds to Q.*

We summarise the intersection numbers of the lines by a table

$$\begin{array}{llll} E_i.E_j = -\delta_{ij} , & H.H = 1 , & Q.Q = -1 , & L_{ij}.L_{ij} = -1 , \\ H.Q = 2 , & H.E_i = 0 , & H.L_{ij} = 1 , & L_{ij}.L_{kl} = 1 ; \text{ if } \{i,j\} \cap \{k,l\} = \emptyset , \\ E_i.L_{ij} = 1 , & E_i.Q = 1 , & L_{ij}.Q = 0 , & E_i.L_{jk} = 0 ; \text{ if } i \notin \{j,k\} . \end{array} \tag{5.2.50}$$

The combinatorics of the intersection of the lines is summarised by the Clebsch graph of Figure 9.

It is a classical fact also that a dP_4 surface can be represented as a transverse intersection of two quadrics in $\mathbb{P}^4$, so $\mathbb{P}^4[2,2]$ [139]. At the end of §5.1.2, we came across another representation of dP_4 embedded in a product of three $\mathbb{P}^1$s:

$$\mathrm{dP}_4 = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} . \tag{5.2.51}$$

In the following we see how to identify the 16 exceptional lines for each of these two representations.

Our first task is to show that this surface (5.2.51) is indeed dP_4. Since dP_4 is $\mathbb{P}^2$ blown up in 5 points, we note:

$$c_2(\mathrm{dP}_4) = c_2(\mathbb{P}^2) + 5 = 8 , \tag{5.2.52}$$

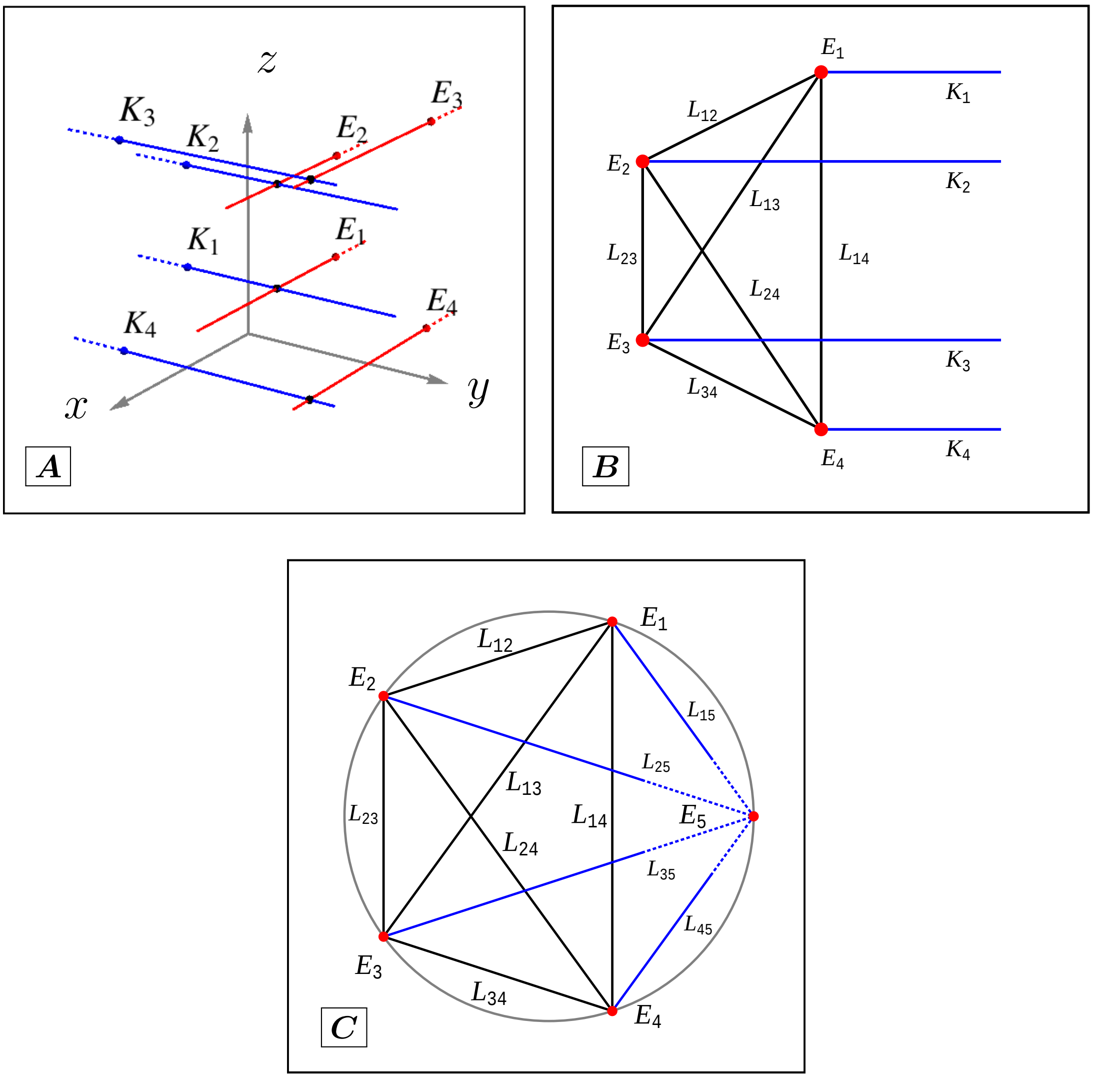


Figure 10: *Steps in obtaining the -1 lines of* dP_4*: (A) eight of the sixteen -1 lines of* dP_4 *in* $\mathbb{C}^3$*. (B) Fourteen of the sixteen -1 lines projected onto the* yz*-plane. (C) The sixteen -1 lines of* dP_4 *in the* $X_3 \neq 0$ *patch of* $\mathbb{P}^2$*. The lines* $\mathcal{K}_i$ $(= L_{i5})$ *meet in the limit* $y \to \infty$*. The circle represents the unique quadric* Q *passing through the 5 blown up points of* $\mathbb{P}^2$*.*

since each blow-up replaces a point with a $\mathbb{P}^1$. It is easy to check that the numerical invariants for c_2 and c_1^2, computed from the representation (5.2.51) match those of dP_4. We present further strong evidence of the equality in (5.2.51) by showing that the surface contains sixteen exceptional lines with the correct properties.

To begin seeing these lines, in this representation, consider the equation implicit in (5.2.51) written below:

$$A(y,z)\, x_1 + B(y,z)\, x_2 \;=\; 0 \tag{5.2.53}$$

where $x,\ y,\ z$ are coordinates of the three $\mathbb{P}^1$'s taken in order and A and B are polynomials of bidegree $(1,2)$ in their arguments. For general $(y,z) \in \mathbb{P}^1{\times}\mathbb{P}^1$, the equation above yields a unique $(x_1, x_2) \in \mathbb{P}^1$. However, there will be 4 points (y_*, z_*) such that $A(y_*, z_*){=}B(y_*, z_*){=}0$. When this happens, the solution to (5.2.53) will be an entire $\mathbb{P}^1$. These in fact correspond to the exceptional curves E_i, $i \in \{1,2,3,4\}$. In this way we see that the equation (5.2.53) yields a $\mathbb{P}^1{\times}\mathbb{P}^1$ blown up in 4 general points. It is another classical fact that blowing up $\mathbb{P}^1{\times}\mathbb{P}^1$ in 4 points is equivalent to blowing up $\mathbb{P}^2$ in 5 points. This will be subsumed in the following explicit construction. In order to refer this calculation to $\mathbb{P}^2$ and find the remaining exceptional lines, we first choose affine coordinates $x{=}x_2/x_1, y{=}y_2/y_1, z{=}z_2/z_1$. A simple choice of a specific equation, that will also be useful later, is

$$(1 + yz^2) + x(y + z^2) \;=\; 0\ . \tag{5.2.54}$$

Four lines are then associated with the simultaneous roots of the equations

$$1 + yz^2 \;=\; 0 \quad \text{and} \quad y + z^2 \;=\; 0\ . \tag{5.2.55}$$

The roots correspond to the intersections of the four lines with the (y,z) plane, see Figure 10A. One sees that (5.2.54) is symmetric under the interchange $x \leftrightarrow y$ so we obtain another set of four -1 lines by swapping x and y in the above argument. So now we have a total of eight -1 lines. These lines, shown in Figure 10A are parallel to either the x or y axis and are referred to as E_i and $\mathcal{K}_i$ respectively, $i \in \{1,2,3,4\}$. Let L_{ij} denote the exceptional line that intersects E_i and E_j. In total, there are 6 such lines. Similarly, there are 6 lines that meet the lines $\mathcal{K}_i$ and $\mathcal{K}_j$ with $i \neq j$, which we denote by $\mathcal{K}_{ij}$. The set of L_{ij} lines is the same as the set of $\mathcal{K}_{ij}$ lines. In fact $L_{ij} = \mathcal{K}_{mn}$ if $\{i,j\} \cap \{m,n\} = \emptyset$. The four intersection points, denoted as solid black points in Figure 10A can be easily obtained by solving for

$E_i \cap \mathcal{K}_j$ for $i \neq j$. If we now project to the yz-plane, the -1 lines, are as shown in Figure 10(B). We then map the lines to $\mathbb{P}^2$

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\dashrightarrow \mathbb{P}^2 \\ (1, y) \times (1, z) &\longmapsto (y,\, z,\, 1) \end{aligned} \tag{5.2.56}$$

By considering the limit $y \to \infty$, we see that the four $\mathcal{K}_i$ lines meet at $(1, 0, 0)$, thereby revealing the location of the fifth blow-up point E_5; see Figure 10C. Thus the 5 blown up points in $\mathbb{P}^2$ in our example are given by: $\{(-\zeta^2, \zeta, 1) \,|\, \zeta^4{=}1\} \cup (1, 0, 0)$. We can now see that $\mathcal{K}_i$ is in fact L_{i5}. The sixteenth line is the quadratic Q that intersects the five E_i. In our case this is given by the second of Eqs. (5.2.55). In homogeneous coordinates (X_1, X_2, X_3) this is the quadric $X_1X_3 + X_2^2 = 0$. The first of Eqs. (5.2.55) is the cubic $X_3^3 + X_1X_2^2 = 0$. This intersects Q in the five E_i with a double contact at E_5. In summary the sixteen lines are given by the five E_i, the ten L_{ij} and Q.

Many of the intersection relations of (5.2.50) between the exceptional lines are intuitively clear from Figure 10. The point of intersection of a curve with the exceptional lines E_i is determined by the tangent direction to the curve at E_i in Figure 10. This makes it clear that $Q.L_{ij} = 0$, for example.

Let us turn now to the representation of dP_4 as the surface $\mathbb{P}^4[2, 2]$. In this representation it is also easy to see the lines but harder to see the map to $\mathbb{P}^2$. Although this is well-known, we pause to indicate the map.

Note first that $\mathbb{P}^4[2, 2]$ has, by a process that is by now very familiar, $15{+}15{-}(5^2 - 1){-}2^2 = 2$ complex structure parameters. It is convenient, in order to be able to write the lines explicitly, to make a simple choice of quadrics

$$\begin{aligned} w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2 &= 0 \\ w_1^2 + 2w_2^2 + 3w_3^2 + 4w_4^2 + 5w_5^2 &= 0\ . \end{aligned} \tag{5.2.57}$$

This choice will also be useful to us in §5.2.7, since it manifests a $\mathbb{Z}_2{\times}\mathbb{Z}_2$ symmetry. Choose now 5 points, e_k, $k{=}1, .., 5$, in general position on $\mathbb{P}^2$, that is such that no three lie on a line. We may choose coordinates $(x,\, y,\, z)$ such that the first four points are $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ and $e_4 = (1, 1, 1)$. Having exhausted the coordinate freedom we write for the fifth point $e_5 = (\alpha, \beta, \gamma)$. This freedom to choose the fifth point corresponds to the two degrees of freedom computed above. Note that since e_5 does not lie on any of the lines that join the other four points, none of the coordinates (α, β, γ) vanish and no two are equal. Consider now the cubics $\psi(x)$ that pass through the five e_k. A general cubic in the coordinates (x, y, z) has 10 coefficients. Requiring that the cubic passes through the five e_k imposes five conditions so there are 5 linearly independent cubics that satisfy the condition. A specific choice is

$$\begin{aligned}
\psi_1 &= \beta(\alpha-\gamma)y(x-z)(\gamma x-\alpha z)\\
\psi_2 &= \gamma\, z\Big(\beta(\alpha-\gamma)\, x^2+(\beta\gamma-\alpha^2)\, xy+\alpha(\alpha-\beta)\, yz\Big)\\
\psi_3 &= \beta\, y\Big(\alpha(\beta-\gamma)\, z^2+(\gamma^2-\alpha\beta)\, zx+\gamma(\alpha-\gamma)\, xy\Big)\\
\psi_4 &= \beta\gamma\, yz\Big((\gamma-\beta)x+(\alpha-\gamma)y+(\beta-\alpha)z\Big)\\
\psi_5 &= \gamma\, z\Big(\gamma(\beta-\alpha)\, xy+\alpha(\gamma-\beta)\, yz+\beta(\alpha-\gamma)\, zx\Big)\ .
\end{aligned} \tag{5.2.58}$$

We think of the ψ_k as being proportional to homogenous coordinates on $\mathbb{P}^4$. It is straightforward to check that the ψ_k satisfy the two quadratic relations

$$\begin{aligned}
\gamma(\alpha-\beta)\,\psi_2\psi_3+\alpha(\beta-\gamma)\,\psi_2\psi_4+\beta(\alpha-\gamma)\,\psi_1\psi_5+(\beta\gamma-\alpha^2)\,\psi_3\psi_5+\alpha(\alpha-\beta)\,\psi_4\psi_5 &= 0\ ,\\
\beta(\gamma-\alpha)\,(\psi_2\psi_4-\psi_3\psi_5)+\gamma(\alpha-\beta)\,\psi_1\psi_4+\beta(\gamma-\beta)\,\psi_1\psi_5 &= 0\ .
\end{aligned} \tag{5.2.59}$$

We could now proceed to study the -1 lines of $\mathbb{P}^4[2,2]$ and how these relate to curves of Figure 10. Rather than do this in general, let us, with a certain prescience, choose the surface with $(\alpha, \beta, \gamma) = (-3, -2, 1)$. With this choice we can make a linear change of basis $w_i = A_{ij}\psi_j$ with

$$A = \begin{pmatrix} 1 & -1 & -\frac{3}{2} & \frac{3}{2} & -3\\ \frac{3}{2}\mathrm{i} & 0 & -2\mathrm{i} & 0 & 4\mathrm{i}\\ \sqrt{\frac{3}{2}} & \sqrt{6} & -\sqrt{\frac{3}{2}} & -\sqrt{\frac{27}{2}} & -\sqrt{6}\\ \frac{1}{2}\mathrm{i} & 4\mathrm{i} & 0 & 6\mathrm{i} & 0\\ 0 & -3 & \frac{1}{2} & -\frac{9}{2} & -1 \end{pmatrix}$$

such that the two quadrics above become equivalent to the quadrics (5.2.57). To find the lines for these quadrics we seek lines $\mathbb{P}^1 \hookrightarrow \mathbb{P}^4$ that lie in the surface defined by the quadrics. Thus we set $w_i = a_i u + b_i v$ with (u, v) coordinates on the $\mathbb{P}^1$. By choice of u and v we may take

$$w \;=\; (u,\, v,\, a_3\, u + b_3\, v,\, a_4\, u + b_4\, v,\, a_5\, u + b_5\, v) \tag{5.2.60}$$

and choose the coefficients so that w satisfies (5.2.57) identically in (u, v). This requires

$$a_5^2 \;=\; 9\;, \quad b_3^2 \;=-6\;, \quad b_4^2 \;=\; 9\;, \quad b_5^2 \;=-4\;, \quad a_4 \;=\; \frac{2}{9}\, a_5 b_4 b_5\;, \quad a_3 \;=-\frac{1}{6} a_5 b_3 b_5\;. \tag{5.2.61}$$

There is a choice of sign associated with solving for each of a_5, b_3, b_4, b_5, so there are 16 solutions. The corresponding lines are shown in Table 5.2 where the lines are also identified.

The birational map $\mathbb{P}^2 \dashrightarrow \mathrm{dP}_4$ given by $w = A\ \psi(x, y, z)$ is well defined apart from the five points e_k. The inverse map $w \to (\frac{x}{z}, \frac{y}{z})$ which is defined everywhere is now easily computable, and can be used to verify that the lines of Table 5.2 map to $\mathbb{P}^2$ as they should.

Although we have described the configuration of lines in some detail, in the following it will suffice to consider the action of the groups on the cohomology basis $\{H, E_k\}$. It is quite often the case that a group element will map an E-line to an L-line, say, and then the relation (5.2.48) is used to re-express L_{ij} as $H - E_i - E_j$.

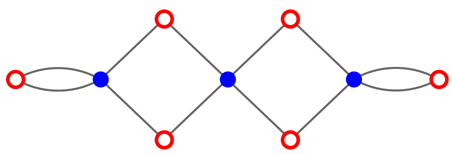
Table 5.2: *Special lines for the* dP_4 *defined by* (5.2.57).

Line	**Representation**
Q	$\left(u, v, -\sqrt{6}u - \mathrm{i}\sqrt{6}v, -3v + 4\mathrm{i}u, \ \ 3u + 2\mathrm{i}v\right)$
E_1	$\left(u, v, -\sqrt{6}u - \mathrm{i}\sqrt{6}v, \ \ 3v - 4\mathrm{i}u, \ \ 3u + 2\mathrm{i}v\right)$
E_2	$\left(u, v, -\sqrt{6}u - \mathrm{i}\sqrt{6}v, -3v + 4\mathrm{i}u, -3u - 2\mathrm{i}v\right)$
E_3	$\left(u, v, -\sqrt{6}u + \mathrm{i}\sqrt{6}v, \ \ 3v + 4\mathrm{i}u, \ \ 3u - 2\mathrm{i}v\right)$
E_4	$\left(u, v, \ \ \sqrt{6}u - \mathrm{i}\sqrt{6}v, -3v - 4\mathrm{i}u, -3u + 2\mathrm{i}v\right)$
E_5	$\left(u, v, \ \ \sqrt{6}u + \mathrm{i}\sqrt{6}v, -3v + 4\mathrm{i}u, \ \ 3u + 2\mathrm{i}v\right)$
$L_{1,2}$	$\left(u, v, -\sqrt{6}u - \mathrm{i}\sqrt{6}v, \ \ 3v - 4\mathrm{i}u, -3u - 2\mathrm{i}v\right)$
$L_{1,3}$	$\left(u, v, -\sqrt{6}u + \mathrm{i}\sqrt{6}v, -3v - 4\mathrm{i}u, \ \ 3u - 2\mathrm{i}v\right)$
$L_{1,4}$	$\left(u, v, \ \ \sqrt{6}u - \mathrm{i}\sqrt{6}v, \ \ 3v + 4\mathrm{i}u, -3u + 2\mathrm{i}v\right)$
$L_{1,5}$	$\left(u, v, \ \ \sqrt{6}u + \mathrm{i}\sqrt{6}v, \ \ 3v - 4\mathrm{i}u, \ \ 3u + 2\mathrm{i}v\right)$
$L_{2,3}$	$\left(u, v, -\sqrt{6}u + \mathrm{i}\sqrt{6}v, \ \ 3v + 4\mathrm{i}u, -3u + 2\mathrm{i}v\right)$
$L_{2,4}$	$\left(u, v, \ \ \sqrt{6}u - \mathrm{i}\sqrt{6}v, -3v - 4\mathrm{i}u, \ \ 3u - 2\mathrm{i}v\right)$
$L_{2,5}$	$\left(u, v, \ \ \sqrt{6}u + \mathrm{i}\sqrt{6}v, -3v + 4\mathrm{i}u, -3u - 2\mathrm{i}v\right)$
$L_{3,4}$	$\left(u, v, \ \ \sqrt{6}u + \mathrm{i}\sqrt{6}v, \ \ 3v - 4\mathrm{i}u, -3u - 2\mathrm{i}v\right)$
$L_{3,5}$	$\left(u, v, \ \ \sqrt{6}u - \mathrm{i}\sqrt{6}v, \ \ 3v + 4\mathrm{i}u, \ \ 3u - 2\mathrm{i}v\right)$
$L_{4,5}$	$\left(u, v, -\sqrt{6}u + \mathrm{i}\sqrt{6}v, -3v - 4\mathrm{i}u, -3u + 2\mathrm{i}v\right)$

Quotients of $X_{2568} \subset \mathbf{dP}_4 \times \mathbf{dP}_4$

In this subsection, we compute the Hodge numbers for the quotients of the manifold X_{2568} by studying the group actions on the -1 lines of dP_4, and in particular on the homology basis $\{H, E_1, E_2, E_3, E_4, E_5\}$. The number of linear invariant homology classes under this action will give the $h^{1,1}$ of the quotient. We begin with the following representation of X_{2568}:

$$X_{2568} \ = \ \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{array}\right]^{12,28}_{-32} \qquad (5.2.62)$$



The largest symmetry group acting on this manifold is $\mathbb{Z}_2{\times}\mathbb{Z}_4$. For the covering manifold, the number of complex structure parameters, seen by the polynomial deformation method is 19, which undercounts the true number. Thus the polynomial deformation method is inapplicable in this case. However, since $X_{2568} \subset \mathrm{dP}_4{\times}\mathrm{dP}_4$, we can apply the previous discussion of lines on dP_4 to compute $h^{1,1}$ for the quotients. In view of the fact that we know the Euler number for the quotient, this allows us to compute $h^{2,1}$ also.

The cohomology group $H^2(X_{2568})$ descends from that of the product of the two dP_4 spaces, that is from $H^2(\mathrm{dP}_4{\times}\mathrm{dP}_4)$. We are thus interested in the group action on the combined cohomology basis of the two dP_4 factors, $\{H,\, E_i,\, \widetilde{H},\, \widetilde{E}_i\}$, with $1 \le i \le 5$. Although Braun lists 42 (freely acting) symmetry actions on the combined set of coordinates and the polynomials for (5.2.62), only 14 of them act distinctly on the co-ordinates. For our purpose, it suffices to consider only the distinct actions on the coordinates. Table 5.4 lists these actions.

We apply our discussion of the last subsection to obtain the group action on the cohomology for each symmetry representation. All cohomology actions are linear and can be expressed in terms of the set of matrices defined in (5.2.65). We will repeat this calculation for other manifolds, in the following, and these definitions of the matrices P_i and Q_i, $i{=}1,2,3$, will hold in such cases also. For each symmetry action listed in Table 5.4, the corresponding group action on the cohomology basis is shown in Table 5.5. The table also lists the cohomology invariants corresponding to each symmetry. Since the equations describing the exceptional curves depend on the choice of coefficients in the defining polynomials of a CICY quotient, different choices will yield different cohomology actions and hence different invariants in principle. However, for all coefficient choices that yield a smooth manifold and sixteen distinct special lines, the number of invariants is fixed. This is the $h^{1,1}$ of the quotient.

Consider the first $\mathbb{Z}_2$ action listed in Table 5.4. With a generic choice of coefficients in the defining polynomials of this quotient of (5.2.62), the lines transform in the following way:

$$\begin{aligned} &H \to H\ ,\quad E_1 \leftrightarrow E_2\ ,\quad E_3 \leftrightarrow E_4\ ,\quad E_5 \to E_5\ ,\\ &\widetilde{H} \to \widetilde{H}\ ,\quad \widetilde{E}_1 \leftrightarrow \widetilde{E}_2\ ,\quad \widetilde{E}_3 \leftrightarrow \widetilde{E}_4\ ,\quad \widetilde{E}_5 \to \widetilde{E}_5\ . \end{aligned} \tag{5.2.63}$$

The transformation properties of the lines E_i and $\widetilde{E}_i$, $1 \le i \le 4$, follow from the transformation properties of the roots of the equations $A = 0$ and $B = 0$ that derive from (5.2.53). The transformation properties of H, $\widetilde{H}$, E_5 and $\widetilde{E}_5$ follow from the invariance of the canonical classes K and $\widetilde{K}$ and the preservation of the intersection numbers (5.2.50). This yields the following 8 cohomology invariants:

$$H,\ E_1+E_2,\ E_3+E_4,\ E_5;\ \ \widetilde{H},\ \widetilde{E}_1+\widetilde{E}_2,\ \widetilde{E}_3+\widetilde{E}_4,\ \widetilde{E}_5\ . \tag{5.2.64}$$

So $h^{1,1} = 8$ for this quotient of X_{2568}. The computation of $h^{1,1}$ for the remaining symmetry actions proceed in a similar way and together with $h^{2,1}$ and the Euler characteristic, they are listed in Table 5.3. The P and Q matrices, of Table 5.5, are given by

$$P_1 = \begin{bmatrix} 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{bmatrix},\ P_2 = \begin{bmatrix} 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1 \end{bmatrix},\ P_3 = \begin{bmatrix} 1&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1 \end{bmatrix};$$

$$\begin{aligned} &Q_1 = \begin{bmatrix} 3&-1&-1&-1&-1&-2\\ 1&0&-1&0&0&-1\\ 1&-1&0&0&0&-1\\ 1&0&0&0&-1&-1\\ 1&0&0&-1&0&-1\\ 2&-1&-1&-1&-1&-1 \end{bmatrix},\ Q_2 = \begin{bmatrix} 3&-1&-1&-1&-1&-2\\ 1&0&0&0&-1&-1\\ 1&0&0&-1&0&-1\\ 1&0&-1&0&0&-1\\ 1&-1&0&0&0&-1\\ 2&-1&-1&-1&-1&-1 \end{bmatrix};\\ &Q_3 = \begin{bmatrix} 3&-1&-1&-1&-1&-2\\ 1&0&0&-1&0&-1\\ 1&0&0&0&-1&-1\\ 1&-1&0&0&0&-1\\ 1&0&-1&0&0&-1\\ 2&-1&-1&-1&-1&-1 \end{bmatrix},\ Q_4 = \begin{bmatrix} 3&-1&-1&-1&-1&-2\\ 1&-1&0&0&0&-1\\ 1&0&-1&0&0&-1\\ 1&0&0&-1&0&-1\\ 1&0&0&0&-1&-1\\ 2&-1&-1&-1&-1&-1 \end{bmatrix}. \end{aligned} \tag{5.2.65}$$

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
$h^{1,1}(X/\Gamma)$	8	6	4	3
$h^{2,1}(X/\Gamma)$	16	10	8	5
$\chi(X/\Gamma)$	-16	-8	-8	-4

Table 5.3: *Hodge numbers for the quotients of the manifold* (5.2.62).

Table 5.4: *Various symmetry actions on the ambient space of the manifold* (5.2.62). *The coordinate patch of the two* dP_4*'s are chosen to be* $(1,x)\times(1,y)\times(1,z)$ *and* $(1,\tilde{x})\times(1,\tilde{y})\times(1,\tilde{z})$ *respectively.*

Index	**Group**	$(\ \mathbf{x},\ \mathbf{y},\ \mathbf{z})$	$(\ \tilde{\mathbf{x}},\ \tilde{\mathbf{y}},\ \tilde{\mathbf{z}})$
1	$\mathbb{Z}_2$	$(-x,-y,-z)$	$(-\tilde{x},-\tilde{y},-\tilde{z})$
2	$\mathbb{Z}_2$	$(\ y,\ x,-z)$	$(-\tilde{x},-\tilde{y},-\tilde{z})$
3	$\mathbb{Z}_2$	$(\ y,\ x,-z)$	$(\ \tilde{y},\ \tilde{x},-\tilde{z})$
4	$\mathbb{Z}_4$	$(\ \tilde{x},\ \tilde{y},-\tilde{z})$	$(\ y,\ x,\ z)$
5	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(-x,-y,-z)$ (x^{-1},y^{-1},z^{-1})	$(-\tilde{x},-\tilde{y},-\tilde{z})$ $(\tilde{x}^{-1},\tilde{y}^{-1},\tilde{z}^{-1})$
6	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(-x,-y,-z)$ $(\ y,\ x,z^{-1})$	$(-\tilde{x},-\tilde{y},-\tilde{z})$ $(\tilde{x}^{-1},\tilde{y}^{-1},\tilde{z}^{-1})$
7	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(\ y,\ x,-z)$ $(-y,-x,z^{-1})$	$(-\tilde{x},-\tilde{y},-\tilde{z})$ $(\tilde{x}^{-1},\tilde{y}^{-1},\tilde{z}^{-1})$
8	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(\ y,\ x,-z)$ $(-x,-y,z^{-1})$	$(-\tilde{x},-\tilde{y},-\tilde{z})$ $(\tilde{x}^{-1},\tilde{y}^{-1},\tilde{z}^{-1})$
9	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(\ y,\ x,-z)$ $(-y,-x,z^{-1})$	$(\ \tilde{y},\ \tilde{x},-\tilde{z})$ $(-\tilde{y},-\tilde{x},\tilde{z}^{-1})$
10	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(-x,-y,-z)$ $(\ y,\ x,z^{-1})$	$(-\tilde{x},-\tilde{y},-\tilde{z})$ $(\ \tilde{y},\ \tilde{x},\tilde{z}^{-1})$

Continued on next page

Index	Group	$(\mathbf{x}, \mathbf{y}, \mathbf{z})$	$(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}, \widetilde{\mathbf{z}})$
11	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(y, x, -z)$ $(-x, -y, z^{-1})$	$(\widetilde{y}, \widetilde{x}, -\widetilde{z})$ $(-\widetilde{x}, -\widetilde{y}, \widetilde{z}^{-1})$
12	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(y, x, -z)$ $(-y, -x, z^{-1})$	$(\widetilde{y}, \widetilde{x}, -\widetilde{z})$ $(-\widetilde{x}, -\widetilde{y}, \widetilde{z}^{-1})$
13	$\mathbb{Z}_2 \times \mathbb{Z}_4$	(y, x, z^{-1}) $(-\widetilde{y}, -\widetilde{x}, -\widetilde{z}^{-1})$	$(\widetilde{y}, \widetilde{x}, \widetilde{z}^{-1})$ (x, y, z)
14	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(-x, -y, -z)$ $(\widetilde{y}, \widetilde{x}, \widetilde{z}^{-1})$	$(-\widetilde{x}, -\widetilde{y}, -\widetilde{z})$ (x, y, z)

Table 5.5: *Symmetry actions on the cohomology basis and the corresponding invariants for the manifold* (5.2.62). *The matrices P_i and Q_i are defined in* (5.2.65).

Index	Group	Action on Coh Basis	Coh Invariants
1	$\mathbb{Z}_2$	$\begin{bmatrix} P_1 & 0 \\ 0 & P_1 \end{bmatrix}$	$H, E_1+E_2, E_3+E_4, E_5,$ $\widetilde{H}, \widetilde{E}_1+\widetilde{E}_2, \widetilde{E}_3+\widetilde{E}_4, \widetilde{E}_5$
2	$\mathbb{Z}_2$	$\begin{bmatrix} Q_1 & 0 \\ 0 & P_1 \end{bmatrix}$	$H-E_5, E_1+E_4-E_5, E_2+E_4-E_5,$ $E_3-E_4,\ \widetilde{H}, \widetilde{E}_1+\widetilde{E}_2, \widetilde{E}_3+\widetilde{E}_4, \widetilde{E}_5$
3	$\mathbb{Z}_2$	$\begin{bmatrix} Q_1 & 0 \\ 0 & Q_1 \end{bmatrix}$	$H-E_5, E_1+E_4-E_5, E_2+E_4-E_5,$ $E_3-E_4, \widetilde{H}-\widetilde{E}_5, \widetilde{E}_1+\widetilde{E}_4-\widetilde{E}_5,$ $\widetilde{E}_2+\widetilde{E}_4-\widetilde{E}_5, \widetilde{E}_3-\widetilde{E}_4$
4	$\mathbb{Z}_4$	$\begin{bmatrix} 0 & Q_4 \\ P_1 & 0 \end{bmatrix}$	$H-E_5+\widetilde{H}-\widetilde{E}_5, E_3-E_4-(\widetilde{E}_3-\widetilde{E}_4),$ $E_1+E_4-E_5+\widetilde{E}_2+\widetilde{E}_3-\widetilde{E}_5,$ $E_2+E_4-E_5+\widetilde{E}_1+\widetilde{E}_3-\widetilde{E}_5$
5	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} P_2 & 0 \\ 0 & P_1 \end{bmatrix}, \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$	$H, E_1+E_2+E_3+E_4, E_5,$ $\widetilde{H}, \widetilde{E}_1+\widetilde{E}_2+\widetilde{E}_3+\widetilde{E}_4, \widetilde{E}_5$
6	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} P_1 & 0 \\ 0 & P_1 \end{bmatrix}, \begin{bmatrix} Q_3 & 0 \\ 0 & P_2 \end{bmatrix}$	$H-E_5, E_1+E_2-E_5, E_3+E_4-E_5,$ $\widetilde{H}, \widetilde{E}_1+\widetilde{E}_2+\widetilde{E}_3+\widetilde{E}_4, \widetilde{E}_5$
7	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} Q_1 & 0 \\ 0 & P_1 \end{bmatrix}, \begin{bmatrix} Q_2 & 0 \\ 0 & P_2 \end{bmatrix}$	$H-E_5, E_1+E_3-E_5, E_2+E_4-E_5,$ $\widetilde{H}, \widetilde{E}_1+\widetilde{E}_2+\widetilde{E}_3+\widetilde{E}_4, \widetilde{E}_5$
8	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} Q_1 & 0 \\ 0 & P_1 \end{bmatrix}, \begin{bmatrix} P_3 & 0 \\ 0 & P_2 \end{bmatrix}$	$H-E_5, E_1+E_3-E_5, E_2+E_4-E_5,$ $\widetilde{H}, \widetilde{E}_1+\widetilde{E}_2+\widetilde{E}_3+\widetilde{E}_4, \widetilde{E}_5$

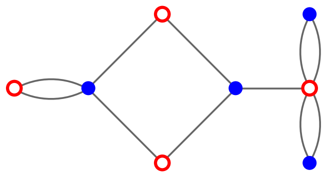
Continued on next page

Index	Group	Action on Coh Basis	Coh Invariants
9	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} Q_1 & 0 \\ 0 & Q_1 \end{bmatrix}, \begin{bmatrix} Q_2 & 0 \\ 0 & Q_3 \end{bmatrix}$	$H - E_5, E_1 + E_3 - E_5, E_2 + E_4 - E_5,$ $\widetilde{H} - \widetilde{E}_5, \widetilde{E}_1 + \widetilde{E}_4 - \widetilde{E}_5, \widetilde{E}_2 + \widetilde{E}_3 - \widetilde{E}_5$
10	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} P_2 & 0 \\ 0 & P_2 \end{bmatrix}, \begin{bmatrix} Q_3 & 0 \\ 0 & Q_1 \end{bmatrix}$	$H - E_5, E_1 + E_4 - E_5, E_2 + E_3 - E_5,$ $\widetilde{H} - \widetilde{E}_5, \widetilde{E}_1 + \widetilde{E}_4 - \widetilde{E}_5, \widetilde{E}_2 + \widetilde{E}_3 - \widetilde{E}_5$
11	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} Q_1 & 0 \\ 0 & Q_1 \end{bmatrix}, \begin{bmatrix} P_2 & 0 \\ 0 & P_2 \end{bmatrix}$	$H - E_5, E_1 + E_4 - E_5, E_2 + E_3 - E_5,$ $\widetilde{H} - \widetilde{E}_5, \widetilde{E}_1 + \widetilde{E}_4 - \widetilde{E}_5, \widetilde{E}_2 + \widetilde{E}_3 - \widetilde{E}_5$
12	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} Q_1 & 0 \\ 0 & Q_1 \end{bmatrix}, \begin{bmatrix} Q_3 & 0 \\ 0 & P_2 \end{bmatrix}$	$H - E_5, E_1 + E_4 - E_5, E_2 + E_3 - E_5,$ $\widetilde{H} - \widetilde{E}_5, \widetilde{E}_1 + \widetilde{E}_4 - \widetilde{E}_5, \widetilde{E}_2 + \widetilde{E}_3 - \widetilde{E}_5$
13	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\begin{bmatrix} Q_2 & 0 \\ 0 & Q_2 \end{bmatrix}, \begin{bmatrix} 0 & Q_3 \\ \mathbb{I}_6 & 0 \end{bmatrix}$	$H - E_5 + \widetilde{H} - \widetilde{E}_5,$ $E_1 + E_2 - E_5 + \widetilde{E}_1 + \widetilde{E}_2 - \widetilde{E}_5,$ $E_3 + E_4 - E_5 + \widetilde{E}_3 + \widetilde{E}_4 - \widetilde{E}_5$
14	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\begin{bmatrix} P_1 & 0 \\ 0 & P_1 \end{bmatrix}, \begin{bmatrix} 0 & Q_2 \\ \mathbb{I}_6 & 0 \end{bmatrix}$	$H - E_5 + \widetilde{H} - \widetilde{E}_5,$ $E_1 + E_2 - E_5 + \widetilde{E}_1 + \widetilde{E}_2 - \widetilde{E}_5,$ $E_3 + E_4 - E_5 + \widetilde{E}_3 + \widetilde{E}_4 - \widetilde{E}_5$

Quotients of $X_{2566} \subset \mathbf{dP}_4 \times \mathbf{dP}_4$

The following configuration contains a product of two dP_4's in two distinct representations.

$$X_{2566} \;=\; \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^4 \end{matrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 \end{bmatrix}^{12,\,28}_{-32} \tag{5.2.66}$$



This manifold admits quotients by free $\mathbb{Z}_2 \times \mathbb{Z}_2$ actions, one of which is given by the following generators. The computations of the Hodge numbers for the other actions proceed in a similar way.

$$R(g) = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \quad \text{and} \quad R(h) = \begin{bmatrix} A' & 0 & 0 \\ 0 & B' & 0 \\ 0 & 0 & C' \end{bmatrix} \tag{5.2.67}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};\quad A' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};\quad B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix};\quad B' = \begin{bmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{bmatrix} \tag{5.2.68}$$

and C and C' are diagonal matrices with entries $\{1, -1, -1, 1, -1\}$ and $\{1, -1, -1, -1, 1\}$ respectively.

Over $\mathbb{P}^4$, the special lines of (5.2.57) take the form shown in Table 5.2. The symmetry action on the cohomology is listed in Table 5.6 below. This action can also be written in terms of matrices acting on the cohomology basis. The advantage in doing so is that one can easily compute the dimension of the eigenspace with eigenvalue 1. This gives the number of invariants under the group action and is related to the $h^{1,1}$ of the quotient manifold under this group action.

	Q	E_1	E_2	E_3	E_4	E_5	L_{12}	L_{13}	L_{14}	L_{15}	L_{23}	L_{24}	L_{25}	L_{34}	L_{35}	L_{45}
g	L_{14}	E_4	L_{35}	L_{25}	E_1	L_{23}	L_{24}	L_{34}	Q	L_{45}	E_5	L_{12}	E_3	L_{13}	E_2	L_{15}
h	L_{24}	L_{35}	E_4	L_{15}	E_2	L_{13}	L_{14}	E_5	L_{12}	E_3	L_{34}	Q	L_{45}	L_{23}	E_1	L_{25}

Table 5.6: *Group Action on the special lines of dP_4, for the surface defined by* (5.2.57)

Group Generators	Cohomology Invariants
g	$K,\quad H-E_5,\quad E_3-E_5,\quad E_2-E_3$
h	$K,\quad H-E_5,\quad E_3-E_5,\quad E_1-E_3$
$g,\, h$	$K,\quad H-E_5,\quad E_3-E_5$

Table 5.7: *Cohomology Invariants of the group action on the dP_4 defined in* (5.2.57). *In the last row we give the invariants under both g and h.*

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	8	6
$h^{2,1}(X/\Gamma)$	16	10
$\chi(X/\Gamma)$	-16	-8

Table 5.8: *Hodge numbers for the quotients of the manifold* (5.2.66).

In order to compute the $h^{1,1}$ for the quotients of (5.2.66) though, we need to know the number of cohomology invariants of the two dP_4's. We have listed the invariants for the second dP_4 in the representation $\mathbb{P}^4[2,2]$ in Table 5.7. Using the methods in the previous subsection, the number of cohomology invariants of the first dP_4

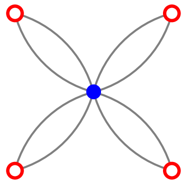
was computed to be 4 for the $\mathbb{Z}_2$ quotients and 3 for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ quotients. The corresponding group actions on the coordinates were taken from [3]. Finally, we produce the value of $h^{1,1}$ for all smooth quotients of (5.2.66) in Table 5.8. This required in summary, a knowledge of the group action on the cohomologies of two distinct representations of del Pezzo surfaces of degree 4.

5.3 The tetraquadric and its splits

5.3.1 The tetraquadric $X^{4,68}$ and its smooth quotients

We start with the class of manifolds defined by the following configuration matrix:

$$X_{7862} = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \left[\begin{matrix} 2 \\ 2 \\ 2 \\ 2 \end{matrix} \right]^{4,68}_{-128} \qquad (5.3.1)$$



This manifold has recently been studied in Refs. [11–13, 140] leading to heterotic models which come very close to the desired properties of the Standard Model. The tetraquadric manifold admits 11 different smooth quotients [3]. The finite groups in question, as well as the Hodge numbers for the corresponding quotients are listed in Table 5.9. The polynomial deformation method correctly reproduces the number of harmonic $(1, 2)$–forms on this manifold. Indeed, the defining polynomial contains 81 coefficients, of which 13 are redundant, corresponding to 12 coordinate redefinitions and an overall rescaling of the polynomial. Moreover, since the embedding (5.3.1) is favourable, we are able to count the number of linearly independent Kähler forms for each quotient. The two methods agree with the independent computation of the Euler number.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Q}_8$
$h^{1,1}(X/\Gamma)$	4	2	4	1	2	1
$h^{2,1}(X/\Gamma)$	36	18	20	9	10	9
$\chi(X/\Gamma)$	-64	-32	-32	-16	-16	-16

Γ	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_8 \rtimes \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Q}_8$
$h^{1,1}(X/\Gamma)$	1	1	1	1	1
$h^{2,1}(X/\Gamma)$	5	5	5	5	5
$\chi(X/\Gamma)$	-8	-8	-8	-8	-8

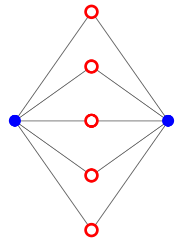
Table 5.9: *Hodge numbers for the quotients of the tetraquadric.*

5.3.2 Splits of the tetraquadric with a $\mathbb{P}^1$

Two favourable splits with Hodge numbers $(5, 45)$

The first split corresponds to the following configuration:

$$X_{7447} \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\left[\begin{matrix}1 & 1\\ 1 & 1\\ 1 & 1\\ 1 & 1\\ 1 & 1\end{matrix}\right]^{5,45}_{-80} \tag{5.3.2}$$



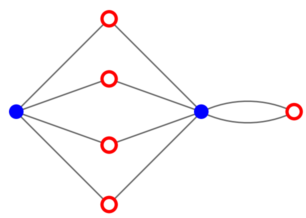
This manifold admits 5 different smooth quotients, whose Hodge numbers are presented in Table 5.10. The polynomial deformation method correctly reproduces the number of complex structure parameters. Indeed, the defining polynomials contain 65 coefficients, of which 15 can be removed by coordinate redefinitions and 4 by redefinitions of the defining polynomials. Furthermore, the configuration (5.3.2) is favourable, so are able to count the number of linearly independent Kähler forms for each quotient. The two methods agree with the independent computation of the Euler number.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_5$	$\mathbb{Z}_{10}$	$\mathbb{Z}_2\times\mathbb{Z}_{10}$
$h^{1,1}(X/\Gamma)$	5	5	1	1	1
$h^{2,1}(X/\Gamma)$	25	15	9	5	3
$\chi(X/\Gamma)$	-40	-20	-16	-8	-4

Table 5.10: *Hodge numbers for the quotients of* $X^{5,45}$.

The second split corresponds to the following configuration:

$$X_{7487} \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\left[\begin{matrix}2 & 0\\ 1 & 1\\ 1 & 1\\ 1 & 1\\ 1 & 1\end{matrix}\right]^{5,45}_{-80} \tag{5.3.3}$$



It is interesting that this manifold was used in Ref. [15] in the construction of a heterotic model with a QCD axion. It admits 6 different free group actions, which result in two pairs of Hodge numbers, as listed in Table 5.11.

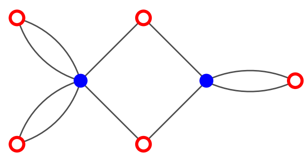
Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	5	5
$h^{2,1}(X/\Gamma)$	25	15
$\chi(X/\Gamma)$	-40	-20

Table 5.11: *Hodge numbers for the quotients of the second split of the tetraquadric,* $X^{5,45}$.

Two non-favourable splits of the tetraquadric: $X^{8,40}$ and $X^{19,19}$

The manifold $X^{8,40}$ corresponds to the following configuration:

$$X_{6829} \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\left[\begin{matrix}2 & 0\\ 0 & 2\\ 2 & 0\\ 1 & 1\\ 1 & 1\end{matrix}\right]^{8,40}_{-64} \tag{5.3.4}$$



The embedding (5.3.4) is not favourable. Moreover, the application of the polynomial deformation method does not reproduce the number of complex structure parameters. For these reasons, in order to compute the Hodge numbers for the quotients of this manifold, we make use of the fact that X_{6829} can be regarded as a hypersurface in the product $\mathrm{dP}_4 \times \mathbb{P}^1 \times \mathbb{P}^1$, where the dP_4 surface is defined by the second polynomial. This embedding explains the number of Kähler parameters for this manifold since $h^{1,1}(\mathrm{dP}_4) = 6$. We begin our computation by listing the distinct group actions on the coordinates in Table 5.13. The group action on the cohomology basis of the dP_4 and the corresponding invariants are listed in Table 5.14. Finally, the $h^{1,1}$ for each quotient equals the number of cohomology invariants plus two, since the symmetries listed in Table 5.13 do not mix the first and the third $\mathbb{P}^1$ spaces of (5.3.4).

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	6	5
$h^{2,1}(X/\Gamma)$	22	13
$\chi(X/\Gamma)$	-32	-16

Table 5.12: *Hodge numbers for the quotients of the manifolds* (5.3.4).

Table 5.13: *Various symmetry actions on the ambient space of the manifold* (5.3.4). *The coordinate patch of the* dP_4 *is chosen to be* $(1,x)\times(1,y)\times(1,z)$. (p,q) *and* (r,s) *are taken to be coordinates of the first and third* $\mathbb{P}^1$ *spaces.*

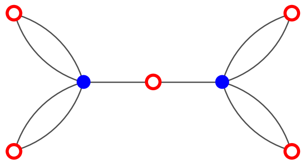
Index	**Group**	(**x**, **y**, **z**)	(**p**, **q**)	(**r**, **s**)
1	$\mathbb{Z}_2$	$(-x,-y,-z)$	$(-p,\ q)$	$(-r,\ s)$
2	$\mathbb{Z}_2$	$(\ y,\ x,-z)$	$(-p,\ q)$	$(-r,\ s)$
3	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(-x,-y,-z)$ (x^{-1},y^{-1},z^{-1})	$(\ p,-q)$ $(\ q,\ p)$	$(\ r,-s)$ $(\ s,\ r)$
4	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(-x,-y,-z)$ $(\ y,\ x,z^{-1})$	$(\ p,-q)$ $(\ q,\ p)$	$(\ r,-s)$ $(\ s,\ r)$
5	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(\ y,\ x,-z)$ $(-x,-y,z^{-1})$	$(\ p,-q)$ $(\ q,\ p)$	$(\ r,-s)$ $(\ s,\ r)$
6	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(\ y,\ x,-z)$ $(-y,-x,z^{-1})$	$(\ p,-q)$ $(\ q,\ p)$	$(\ r,-s)$ $(\ s,\ r)$

Table 5.14: *Symmetry actions on the cohomology basis and the corresponding invariants for the manifold* (5.3.4). *The matrices* P_i *and* Q_i *are defined in* (5.2.65).

Index	**Group**	**Action on Coh Basis**	**Coh Invariants**
1	$\mathbb{Z}_2$	P_1	H, E_1+E_2, E_3+E_4, E_5
2	$\mathbb{Z}_2$	Q_1	$H-E_5, E_1+E_4-E_5, E_2+E_4-E_5, E_3-E_4$
3	$\mathbb{Z}_2\times\mathbb{Z}_2$	$P_2,\ P_1$	$H, E_1+E_2+E_3+E_4, E_5$
4	$\mathbb{Z}_2\times\mathbb{Z}_2$	$P_1,\ Q_2$	$H-E_5, E_1+E_2-E_5, E_3+E_4-E_5$
5	$\mathbb{Z}_2\times\mathbb{Z}_2$	$Q_1,\ P_3$	$H-E_5, E_1+E_3-E_5, E_2+E_4-E_5$
6	$\mathbb{Z}_2\times\mathbb{Z}_2$	$Q_1,\ Q_2$	$H-E_5, E_1+E_3-E_5, E_2+E_4-E_5$

There are 15 occurrences of the Hodge numbers $(19, 19)$ in the CICY list. These can all be seen to correspond to the same manifold by a sequence of redundant splittings and contractions. One of these configurations is the following

$$X_{21} \;=\; \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \!\left[\begin{array}{cc} 2 & 0 \\ 0 & 2 \\ 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{array}\right]^{19,19}_{0} \tag{5.3.5}$$



This configuration admits 53 different free group actions by the groups

$$\mathbb{Z}_2,\ \mathbb{Z}_4,\ \mathbb{Z}_2{\times}\mathbb{Z}_2,\ \mathbb{Z}_8,\ \mathbb{Z}_2{\times}\mathbb{Z}_4,\ \mathbb{Q}_8,\ \mathbb{Z}_4{\times}\mathbb{Z}_4,\ \mathbb{Z}_4{\rtimes}\mathbb{Z}_4,\ \mathbb{Z}_8{\times}\mathbb{Z}_2,\ \mathbb{Z}_8{\rtimes}\mathbb{Z}_2 \ \text{ and } \ \mathbb{Z}_2{\times}\mathbb{Q}_8\ .$$

Many of these quotients, as well as their Hodge numbers were studied in Ref. [125]. Although the diagram (5.3.5) is one-leg decomposable, the polynomial deformation method correctly reproduces the number of complex structure parameters for the covering manifold, and we will assume that it provides a complete parametrisation of the complex structure moduli space. With this assumption, we are able to compute the Hodge numbers for the resulting quotients, as listed in Table 5.15.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_2{\times}\mathbb{Z}_2$	$\mathbb{Z}_8,\ \mathbb{Z}_2{\times}\mathbb{Z}_4,\ \mathbb{Q}_8$
$h^{1,1}(X/\Gamma)$	11	5	6	7	3
$h^{2,1}(X/\Gamma)$	11	5	6	7	3
$\chi(X/\Gamma)$	0	0	0	0	0

Γ	$\mathbb{Z}_2{\times}\mathbb{Z}_4$	$\mathbb{Z}_4{\times}\mathbb{Z}_4,\ \mathbb{Z}_4{\rtimes}\mathbb{Z}_4$ $\mathbb{Z}_8{\times}\mathbb{Z}_2,\ \mathbb{Z}_8{\rtimes}\mathbb{Z}_2,\ \mathbb{Z}_2{\times}\mathbb{Q}_8$
$h^{1,1}(X/\Gamma)$	4	2
$h^{2,1}(X/\Gamma)$	4	2
$\chi(X/\Gamma)$	0	0

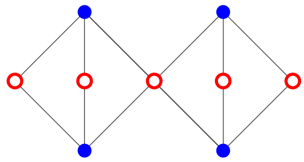
Table 5.15: *Hodge numbers for the quotients of the manifold* (5.3.5).

5.3.3 Other splits of the tetraquadric

Three favourable splits with Hodge numbers $(5, 29)$

The first split corresponds to the following configuration:

$$X_{5301} = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{5,29}_{-48} \tag{5.3.6}$$



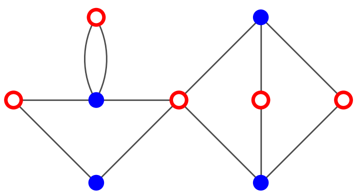
The manifold (5.3.6) admits 3 different free actions. For the Hodge number computation for the resulting quotients we are able to use both the polynomial deformation method and the counting of the invariant Kähler forms. The results are listed in Table 5.16.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	5	3	5
$h^{2,1}(X/\Gamma)$	17	9	11
$\chi(X/\Gamma)$	-24	-12	-12

Table 5.16: *Hodge numbers for the quotients of the manifolds* (5.3.6) *and* (5.3.8)*.*

The second split corresponds to the following configuration:

$$X_{5256} = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{5,29}_{-48} \tag{5.3.7}$$



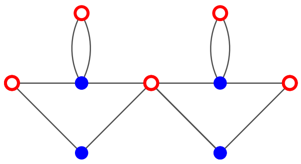
This manifold admits 6 different free actions, resulting in two pairs of Hodge numbers for the quotients. The results are listed in Table 5.17.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	5	5
$h^{2,1}(X/\Gamma)$	17	11
$\chi(X/\Gamma)$	-24	-12

Table 5.17: *Hodge numbers for the quotients of the manifold* (5.3.7)*.*

The third split corresponds to the following configuration:

$$X_{5452} \;=\; \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{array} \left[\begin{array}{cccc} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right]^{5,29}_{-48} \tag{5.3.8}$$



This manifold admits a total of 22 different free group actions by the same groups as for the manifold (5.3.6). The Hodge numbers of the quotients are identical to those in Table 5.16.

Five favourable splits with Hodge numbers $(5, 37)$

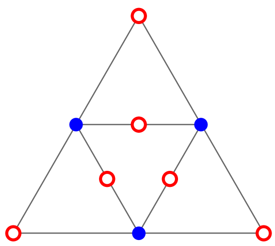
The tetraquadric also admits five different splits with Hodge numbers $(5, 37)$ whose quotients are discussed in Section 5.4.2.

5.3.4 Further splits

The favourable split $X^{5,45} \to X^{6,30}$

The manifold $X^{6,30}$ can be obtained by splitting the first column of (5.3.3), leading to the following configuration:

$$X_{5302} \;=\; \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \left[\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right]^{6,30}_{-48} \tag{5.3.9}$$



This manifold admits 20 different free group actions. We compute the Hodge numbers for the resulting quotients using the counting of Kähler parameters. In this case, the polynomial deformation method does not yield the expected number of complex structure parameters for the covering manifold. We present the Hodge numbers in Table 5.18.

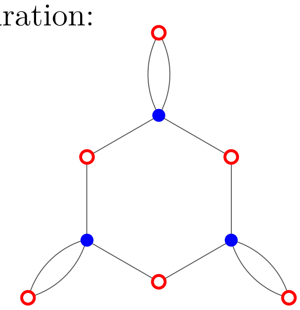
Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	6	6
$h^{2,1}(X/\Gamma)$	18	12
$\chi(X/\Gamma)$	-24	-12

Table 5.18: *Hodge numbers for the quotients of the manifold* (5.3.9).

Splits of the manifold $X^{8,40}$: $X^{15,15}$ and $X^{9,25}$

The manifold $X^{15,15}$ can be obtained by splitting the first column of (5.3.4) with a $\mathbb{P}^1$, which leads to the following configuration:

$$X_{15} \; = \; \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \left[\begin{array}{ccc} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]^{15,15}_{0} \tag{5.3.10}$$



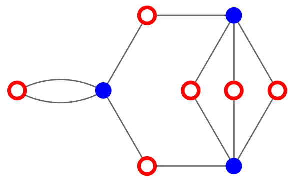
The manifold admits 20 different free group actions by $\mathbb{Z}_2$ and $\mathbb{Z}_2\times\mathbb{Z}_2$. We can compute the Hodge numbers for the corresponding quotients by using the polynomial deformation method. The polynomials defining the covering manifold contain 36 parameters, of which 18 account for coordinate redefinitions and 3 for rescalings of the polynomials. For the $\mathbb{Z}_2$-actions, the restricted polynomials contain 18 coefficients, of which 6 can be removed by coordinate redefinitions and 3 by overall scalings of the polynomials. Similar considerations apply to the $\mathbb{Z}_2\times\mathbb{Z}_2$ actions. The results are summarised in Table 5.19.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	9	6
$h^{2,1}(X/\Gamma)$	9	6
$\chi(X/\Gamma)$	0	0

Table 5.19: *Hodge numbers for the quotients of the manifold* (5.3.10).

The manifold $X^{9,25}$ can be obtained through a different splitting the first column of (5.3.4), which corresponds to the following configuration:

$$X_{2534} \;=\; \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]^{9,25}_{-32} \qquad (5.3.11)$$



The above embedding (5.3.11) is not favourable and the polynomial deformation method does not reproduce the number of complex structure parameters. However, X_{2534} is embedded in the product $\text{dP}_4 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. This will allow us to compute the $h^{1,1}$ of the quotients by computing the group action on the cohomology basis directly, using methods of section §5.2.7. The $h^{1,1}$ of this manifold is 9 since $h^{1,1}(\text{dP}_4) = 6$ and the three $\mathbb{P}^1$'s contribute 1 each. The distinct group actions on the coordinates are listed in Table 5.21.

The various group actions on the cohomology basis $\{H, E_1, E_2, E_3, E_4, E_5\}$ and corresponding invariants are listed in Table 5.22. Since $X_{2534} \subset \text{dP}_4 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and none of the symmetries mix the three additional $\mathbb{P}^1$'s, the $h^{1,1}$ for the quotients is equal to three plus the number of corresponding cohomology invariants. They are listed in Table 5.20.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	7	6
$h^{2,1}(X/\Gamma)$	15	10
$\chi(X/\Gamma)$	-16	-8

Table 5.20: *Hodge numbers for the quotients of the manifold* (5.3.11).

A still different split of the first column of (5.3.4) leads to a manifold with Hodge numbers $(12, 28)$, which can be further split to yield a $(15, 15)$ manifold. We discuss these in Section 5.5.2.

Table 5.21: *Various symmetry actions on the ambient space of the manifold* (5.3.11). *The coordinate patch of the* dP_4 *is chosen to be* $(1,x)\times(1,y)\times(1,z)$. (p,q), (r,s) *and* (u,v) *are taken to be coordinates of the first three* $\mathbb{P}^1$ *spaces.*

Index	Group	(**x**, **y**, **z**)	(**p**, **q**)	(**r**, **s**)	(**u**, **v**)
1	$\mathbb{Z}_2$	$(-x,-y,-z)$	$(-p,\ q)$	$(-r,\ s)$	$(-u,\ v)$
2	$\mathbb{Z}_2$	$(\ y,\ x,-z)$	$(-p,\ q)$	$(-r,\ s)$	$(-u,\ v)$
3	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(-x,-y,-z)$ (x^{-1},y^{-1},z^{-1})	$(\ p,-q)$ $(\ q,\ p)$	$(\ r,-s)$ $(\ s,\ r)$	$(\ u,-v)$ $(\ v,\ u)$
4	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(\ y,\ x,-z)$ $(-y,-x,z^{-1})$	$(\ p,-q)$ $(\ q,\ p)$	$(\ r,-s)$ $(\ s,\ r)$	$(\ u,-v)$ $(\ v,\ u)$
5	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(\ y,\ x,-z)$ $(-x,-y,z^{-1})$	$(\ p,-q)$ $(\ q,\ p)$	$(\ r,-s)$ $(\ s,\ r)$	$(\ u,-v)$ $(\ v,\ u)$
6	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(-x,-y,-z)$ $(\ y,\ x,z^{-1})$	$(\ p,-q)$ $(\ q,\ p)$	$(\ r,-s)$ $(\ s,\ r)$	$(\ u,-v)$ $(\ v,\ u)$

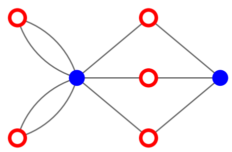
Table 5.22: *Symmetry actions on the cohomology basis and the corresponding invariants for the manifold* (5.3.11). *The matrices* P_i *and* Q_i *are defined in* (5.2.65).

Index	Group	Action on Coh Basis	Coh Invariants
1	$\mathbb{Z}_2$	P_1	H, E_1+E_2, E_3+E_4, E_5
2	$\mathbb{Z}_2$	Q_1	$H-E_5, E_1+E_4-E_5, E_2+E_4-E_5, E_3-E_4$
3	$\mathbb{Z}_2\times\mathbb{Z}_2$	$P_2,\ P_1$	$H, E_1+E_2+E_3+E_4, E_5$
4	$\mathbb{Z}_2\times\mathbb{Z}_2$	$Q_1,\ Q_2$	$H-E_5, E_1+E_3-E_5, E_2+E_4-E_5$
5	$\mathbb{Z}_2\times\mathbb{Z}_2$	$Q_1,\ P_2$	$H-E_5, E_1+E_4-E_5, E_2+E_3-E_5$
6	$\mathbb{Z}_2\times\mathbb{Z}_2$	$P_2,\ Q_1$	$H-E_5, E_1+E_4-E_5, E_2+E_3-E_5$

5.3.5 The manifold $X^{8,44}$ and its split $X^{9,25}$

This manifold can be obtained from the tetraquadric through a sequence of two splits. The first split leads to the configuration:

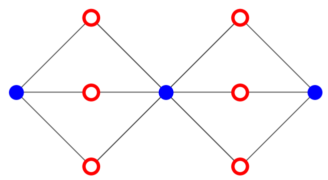
$$X_{7709} = \begin{array}{c}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\end{array}\left[\begin{array}{cc}2&0\\2&0\\1&1\\1&1\\1&1\end{array}\right]^{6,54}_{-96} \qquad (5.3.12)$$



This manifold admits two free actions of $\mathbb{Z}_2$. The polynomial deformation method can be applied in this case. For the covering manifold, we have a number of 80 parameters in the defining polynomials, 15 of which can be removed by coordinate redefinitions and 11 by redefinitions of the polynomials. For the quotient manifold, there are 40 coefficients in the specialised polynomials, 5 of which can be eliminated by projective linear coordinate transformations, and 6 by redefinitions of the polynomials. This leads to $h^{2,1}(X/\mathbb{Z}_2) = 29$ and hence $h^{1,1}(X/\mathbb{Z}_2) = 5$.

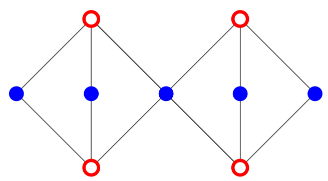
Splitting the first column of the above configuration matrix with a $\mathbb{P}^1$ space leads us to the manifold $X^{8,44}$ defined by the following configuration:

$$X_{7300} = \begin{array}{c}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\end{array}\left[\begin{array}{ccc}1&0&1\\1&0&1\\1&0&1\\0&1&1\\0&1&1\\0&1&1\end{array}\right]^{8,44}_{-72} \qquad (5.3.13)$$



This manifold has been used in Refs. [2, 5] in order to construct a class of heterotic models with three generations and the MSSM spectrum. As shown in §5.1.2 the manifold $X^{8,44}$ can be represented by the following equivalent configuration:

$$X_{7246} = \begin{array}{c}\mathbb{P}^2\\\mathbb{P}^2\\\mathbb{P}^2\\\mathbb{P}^2\end{array}\left[\begin{array}{ccccc}1&1&0&0&1\\1&1&0&0&1\\0&0&1&1&1\\0&0&1&1&1\end{array}\right]^{8,44}_{-72} \qquad (5.3.14)$$



In the embedding (5.3.13), the manifold admits 15 free actions of the following groups:

$$\mathbb{Z}_2,\ \mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{Z}_6,\ \mathbb{Z}_3 \rtimes \mathbb{Z}_4,\ \mathbb{Z}_{12}, \qquad (5.3.15)$$

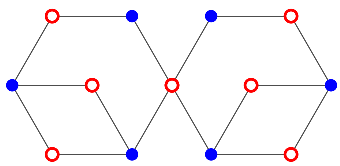
while in the embedding (5.3.14) it admits 26 free actions of the same groups. Using either of the two embeddings, one can compute the Hodge numbers for the resulting quotients through the polynomial deformation method. Alternatively, one can analyse the action of the above groups on the second cohomology of dP_6, as done in [2, 5]. By either method, we obtain the Hodge numbers listed in Table 5.23.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_6$	$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_{12}$
$h^{1,1}(X/\Gamma)$	6	4	3	2	1	1
$h^{2,1}(X/\Gamma)$	24	16	12	8	4	4
$\chi(X/\Gamma)$	-36	-24	-18	-12	-6	-6

Table 5.23: *Hodge numbers for the quotients of the manifolds* (5.3.13) *and* (5.3.14).

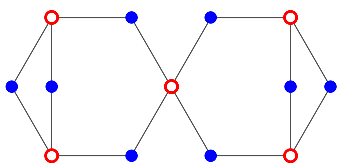
The last column of the manifold (5.3.13) can be split with a $\mathbb{P}^3$, giving the following configuration for the manifold $X^{9,25}$:

$$X_{2639} \;=\; \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{array} \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]^{9,25}_{-32} \qquad (5.3.16)$$



Similarly, the last column of the manifold (5.3.14) can be split with a $\mathbb{P}^3$, an operation which leads to the following configuration matrix:

$$X_{2572} \;=\; \begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^3 \end{array} \left[\begin{array}{cccccccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]^{9,25}_{-32} \qquad (5.3.17)$$



We do not have a good method for computing the Hodge numbers for the quotients of (5.3.16) directly. However, given the identity of (5.3.13) and (5.3.14) we will assume that (5.3.16) and (5.3.17) are also the same and have equivalent group actions. The manifold (5.3.17) admits 2 different free group actions by $\mathbb{Z}_2$ and $\mathbb{Z}_4$. We compute the Hodge numbers for the resulting quotients using the polynomial deformation method.

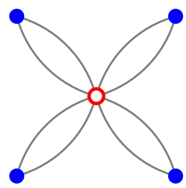
Γ	$\mathbb{Z}_2$	$\mathbb{Z}_4$
$h^{1,1}(X/\Gamma)$	7	4
$h^{2,1}(X/\Gamma)$	15	8
$\chi(X/\Gamma)$	-16	-8

Table 5.24: *Hodge numbers for the quotients of the manifolds* (5.3.16) *and* (5.3.17).

5.4 The manifold $\mathbb{P}^7[2,2,2,2]$ and its splits

The parent manifold in this sequence of splits is defined by the following configuration:

$$X_{7861} = \mathbb{P}^7 \left[2 \;\; 2 \;\; 2 \;\; 2 \right]^{1,65}_{-128}$$



(5.4.1)

In this section we will describe smooth quotients of not only this manifold but also of manifolds that descend from this via a sequence of conifold transitions: $X^{1,65} \to X^{2,58} \to X^{3,51} \to X^{4,44} \to X^{5,37}$. These manifolds appear in multiple guises so we end up considering all the configurations with the 'flower' diagrams of Figure 11.

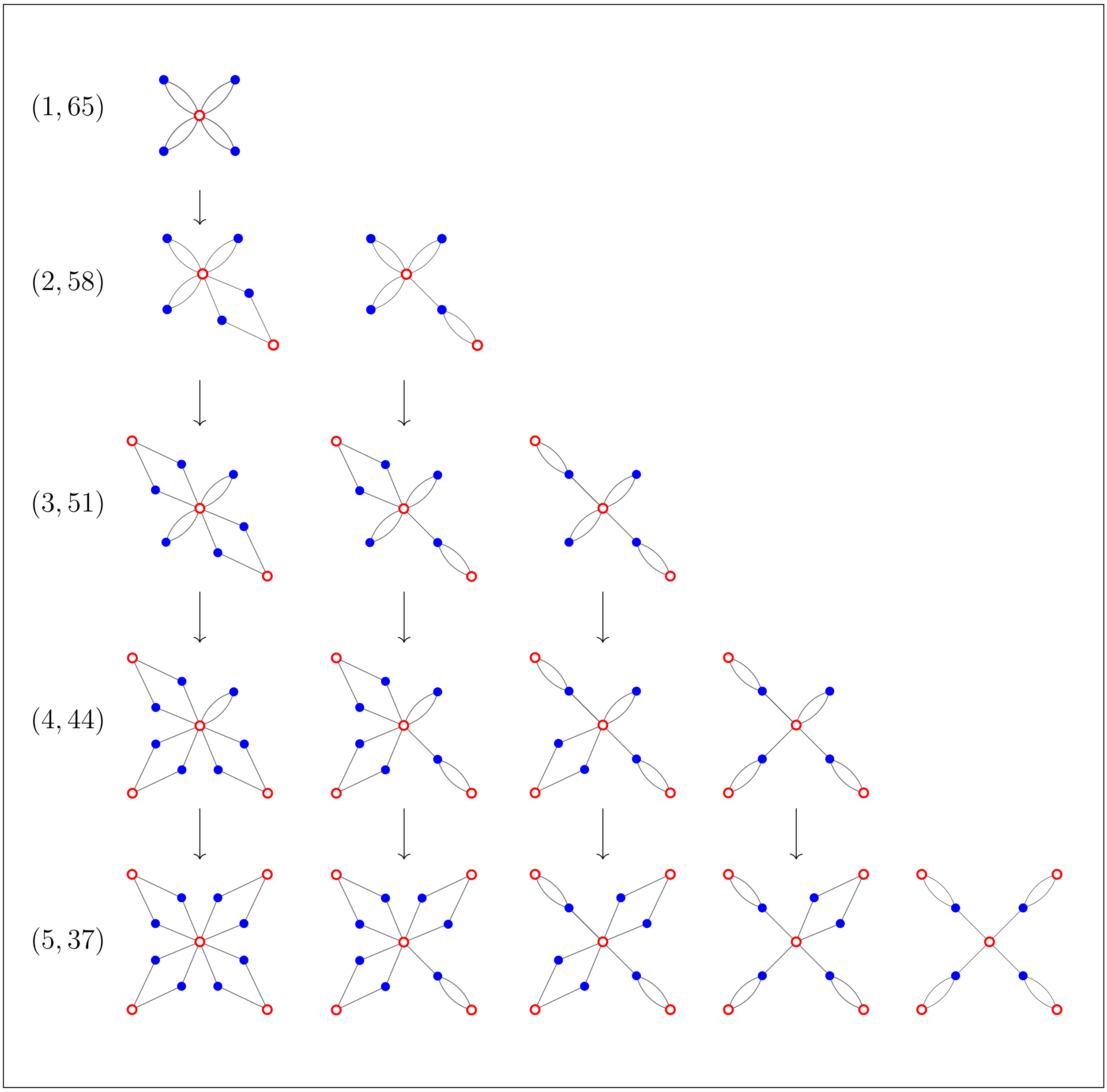


Figure 11: *Conifold transitions depicting the splits:* $X^{1,65} \to X^{2,58} \to X^{3,51} \to X^{4,44} \to X^{5,37}$. $\Delta(h^{1,1}, h^{2,1}) = (1, -7)$ *for each transition depicted above. The CICYs in each row are equivalent.*

5.4.1 Transpose of the tetraquadric $X^{1,65}$ and its smooth quotients

The manifold $X^{1,65}$ admits 45 different smooth quotients by the finite groups listed in Table 5.25. The counting of invariant Kähler forms in this case is trivial, since the embedding space is a single projective space. The polynomial deformation method also yields consistent results. For this manifold the classification of free group actions was first performed by Hua *et al.* [123].

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_4$ $\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_8$ $\mathbb{Z}_2\times\mathbb{Z}_4$, $\mathbb{Q}_8$ $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_4\times\mathbb{Z}_4$, $\mathbb{Z}_4\rtimes\mathbb{Z}_4$ $\mathbb{Z}_2\times\mathbb{Z}_8$, $\mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_2$ $\mathbb{Z}_2\times\mathbb{Q}_8$	$(\mathbb{Z}_4\times\mathbb{Z}_2)\rtimes\mathbb{Z}_4$, $\mathbb{Z}_8\times\mathbb{Z}_4$, $\mathbb{Z}_8\rtimes\mathbb{Z}_4$ $(\mathbb{Z}_8\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2$, $\mathbb{Z}_8\rtimes\mathbb{Z}_4$, $\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_2$ $\mathbb{Z}_2\times(\mathbb{Z}_4\rtimes\mathbb{Z}_4)$, $\mathbb{Z}_4\rtimes\mathbb{Q}_8$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Q}_8$
$h^{1,1}(X/\Gamma)$	1	1	1	1	1
$h^{2,1}(X/\Gamma)$	33	17	9	5	3
$\chi(X/\Gamma)$	−64	−32	−16	−8	−4

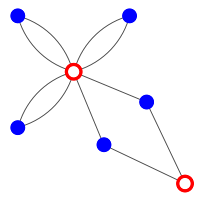
Table 5.25: *Hodge numbers for the quotients of the manifold* $X^{1,65}$.

5.4.2 The sequence of splits: $X^{1,65} \to X^{2,58} \to X^{3,51} \to X^{4,44} \to X^{5,37}$

Two favourable splits with Hodge numbers $(2, 58)$

The first split corresponds to the following configuration:

$$X_{7819} = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^7 \end{array}\left[\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 & 2 \end{array}\right]^{2,\,58}_{-112} \tag{5.4.2}$$



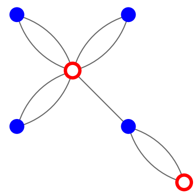
The manifold (5.4.2) admits two different smooth quotients. The corresponding Hodge numbers are listed in Table 5.26. These can be obtained by either the polynomial deformation method or counting the Kähler parameters.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	2	2
$h^{2,1}(X/\Gamma)$	30	16
$\chi(X/\Gamma)$	-56	-28

Table 5.26: *Hodge numbers for the quotients of the manifolds* (5.4.2) *and* (5.4.3).

Another configuration that we shall show presently corresponds to the same manifold as the configuration above is

$$X_{7823} \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^6\end{matrix}\left[\begin{matrix}2 & 0 & 0 & 0\\ 1 & 2 & 2 & 2\end{matrix}\right]^{2,\,58}_{-112} \qquad (5.4.3)$$



This manifold also admits two different smooth quotients, whose Hodge numbers were computed by counting of Kähler parameters. The polynomial deformation method does not lead to a correct count of the complex structure parameters in this case. It is interesting to note that the Hodge numbers of the quotient manifolds are the same as in the case of the manifold (5.4.2). We will show next that these manifolds are in fact the same and that this identity extends to an identity between the members of each row of Figure 5.25. Although the representations (5.4.2) and (5.4.3) define equivalent CICYs, a symmetry linearly realised in one representation, may not be linearly realised in the other. In the discussion that follows, we will find such examples.

We wish now to show that the two configurations above, (5.4.2) and (5.4.3) define identical manifolds. Consider first (5.4.3) and denote the first polynomial by p and the remaining three by q^i, i=1, 2, 3. The polynomial p has the form

$$\begin{aligned} p &= \sum_{k=1}^{7}(A_k\, t_1^2 + 2B_k\, t_1 t_2 + C_k\, t_2^2)\, x_k \\ &= t_1\sum_{k=1}^{7}(A_k t_1 + B_k t_2)\, x_k + t_2\sum_{k=1}^{7}(B_k t_1 + C_k t_2)\, x_k\ , \end{aligned} \qquad (5.4.4)$$

where t and x are coordinates on the $\mathbb{P}^1$ and $\mathbb{P}^6$, respectively. Define x_8 by the equations

$$\begin{aligned} t_1\, x_8 - \sum_{k=1}^{7}(B_k t_1 + C_k t_2)\, x_k &= 0 \\ t_2\, x_8 + \sum_{k=1}^{7}(A_k t_1 + B_k t_2)\, x_k &= 0\ . \end{aligned} \tag{5.4.5}$$

The quantity x_8 is uniquely defined since the coordinates (t_1, t_2) cannot vanish simultaneously and the equations are consistent by virtue of the equation $p=0$. Note also that if all the x_k vanish, k=1, . . . , 7, then so does x_8. Denoting the two equations above by (p^1, p^2), we see that there is a one-to-one relation between the zero loci of the polynomials $\{p,\, q^i\}$ and $\{p^1,\, p^2,\, q^i\}$, so taking $(x_k,\, x_8)$ as the coordinates of a $\mathbb{P}^7$ we have established the equality.

The following generalisation is immediate and relates the configurations of the rows of Figure 11:

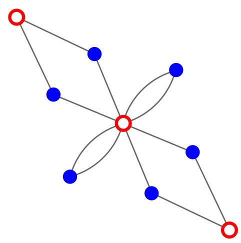
$$\begin{matrix} \mathcal{A}_{n+3-k} \\ \mathbb{P}^1 \\ \mathbb{P}^{k+1} \end{matrix} \begin{bmatrix} 0 & 0 & M \\ 1 & 1 & \mathbf{0} \\ 1 & 1 & \mathbf{d} \end{bmatrix} \;=\; \begin{matrix} \mathcal{A}_{n+3-k} \\ \mathbb{P}^1 \\ \mathbb{P}^{k} \end{matrix} \begin{bmatrix} 0 & M \\ 2 & \mathbf{0} \\ 1 & \mathbf{d} \end{bmatrix}\ , \tag{5.4.6}$$

here $\mathcal{A}_{n+3-k}$ is an ambient space of the indicated dimension, M is a matrix and $\mathbf{d}$ is a degree vector with n components.

Three favourable splits with Hodge numbers $(3, 51)$

The second split corresponds to the following configuration:

$$X_{7745} \;=\; \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^7 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 2 \end{bmatrix}^{3,51}_{-96} \tag{5.4.7}$$



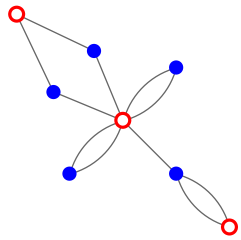
The manifold (5.4.7) admits five different free group actions. The Hodge numbers for the resulting quotients are listed in Table 5.27, the computation of which are amenable to both the polynomial deformation method and the counting of Kähler parameters.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_4$
$h^{1,1}(X/\Gamma)$	3	2	3	2
$h^{2,1}(X/\Gamma)$	27	14	15	8
$\chi(X/\Gamma)$	−48	−24	−24	−12

Table 5.27: *Hodge numbers for the quotients of the manifolds* (5.4.7) *and* (5.4.9).

There are two other favourable embeddings of the manifold described by (5.4.7) that we discuss now. The first one corresponds to the following configuration:

$$X_{7714} \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^6\end{matrix}\left[\begin{matrix}0 & 0 & 0 & 0 & 2\\ 0 & 0 & 1 & 1 & 0\\ 2 & 2 & 1 & 1 & 1\end{matrix}\right]^{3,51}_{-96} \qquad (5.4.8)$$



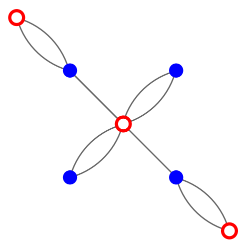
This manifold admits two different smooth quotients, whose Hodge numbers are listed in Table 5.28. We can make use of only the counting of Kähler parameters for this favourable embedding.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	3	3
$h^{2,1}(X/\Gamma)$	27	15
$\chi(X/\Gamma)$	−48	−24

Table 5.28: *Hodge numbers for the quotients of the manifold* (5.4.8).

The second equivalent representation of the manifold (5.4.7) is given by the embedding:

$$X_{7735} \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^5\end{matrix}\left[\begin{matrix}2 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 1 & 1 & 2 & 2\end{matrix}\right]^{3,51}_{-96} \qquad (5.4.9)$$

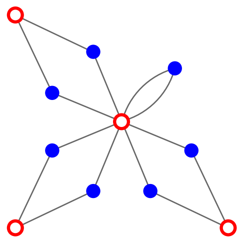


The manifold (5.4.9) admits 8 different free group actions by $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$. The Hodge numbers for the resulting quotients are identical to those in Table 5.27, obtained for the manifold (5.4.7). In order to compute the Hodge numbers, we count the Kähler parameters, since the polynomial deformation cannot be expected to give correct answers, in this case, owing to the fact that it does not count correctly the complex structure parameters of the covering space and, moreover, the diagram for the configuration is 1-leg decomposable.

Four favourable splits with Hodge numbers $(4, 44)$

The third split corresponds to the transition from the third to the fourth row in Figure 11. The first CICY in this set of four equivalent CICYs are given by the following configuration:

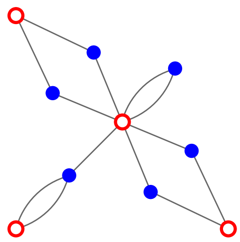
$$X_{7435} = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^7 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}^{4,44}_{-80} \tag{5.4.10}$$



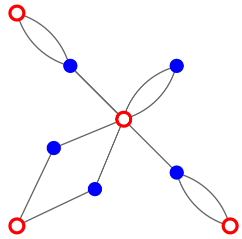
The four CICYs appearing in this class admit smooth quotients by $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. The Hodge numbers of the quotients are the same for each manifold and are shown in Table 5.29. Both the polynomial deformation method and the counting of Kähler parameters can be employed to compute the Hodge numbers listed in Table 5.29 for the manifold (5.4.10).

The remaining three equivalent descriptions of (5.4.10) are given by the configuration matrices and diagrams below. These manifolds admit respectively 2, 4 and 19 different group actions by $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. Their Hodge numbers are listed in Table 5.29.

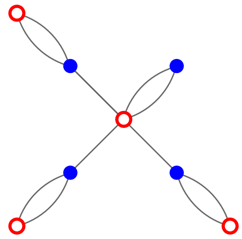
$$X_{7522} = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^6 \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^{4,44}_{-80} \tag{5.4.11}$$



$$X_{7462} = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^5 \end{array} \left[\begin{array}{ccccc} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 \end{array}\right]^{4,44}_{-80} \qquad (5.4.12)$$



$$X_{7491} = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^4 \end{array} \left[\begin{array}{cccc} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 2 \end{array}\right]^{4,44}_{-80} \qquad (5.4.13)$$



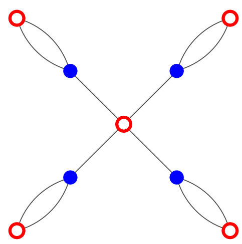
Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	4	4
$h^{2,1}(X/\Gamma)$	24	14
$\chi(X/\Gamma)$	−40	−20

Table 5.29: *Hodge numbers for the quotients of the manifolds* (5.4.10), (5.4.11), (5.4.12) *and* (5.4.13).

Five favourable splits with Hodge numbers $(5, 37)$

The fourth split of the manifold $X^{1,65}$ is the manifold (5.4.18) with Hodge numbers $(5, 37)$. This CICY has four other equivalent embeddings. The first CICY corresponds to the following configuration:

$$X_{6836} = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{array} \left[\begin{array}{cccc} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right]^{5,37}_{-64} \qquad (5.4.14)$$



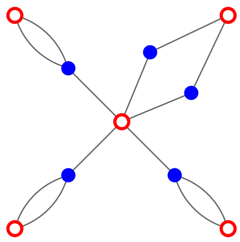
The manifold (5.4.14) admits an impressive number of 117 different free group actions [3]. However, the Hodge numbers for the quotients corresponding to a fixed group are all the same. We summarise this information in Table 5.30. The polynomial deformation method does not reproduce correctly the number of complex structure parameters for the covering manifold $X^{5,37}$, and we cannot use it for computing $h^{2,1}(X/\Gamma)$. On the other hand, the embedding (5.4.14) is favourable; as such, we are able to count the number of linearly independent Kähler forms for each quotient and then infer $h^{2,1}(X/\Gamma)$ from the Euler number.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_2\times\mathbb{Z}_4$	$\mathbb{Q}_8$	$\mathbb{Z}_4\rtimes\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_8$
$h^{1,1}(X/\Gamma)$	5	3	5	2	3	2	2	2
$h^{2,1}(X/\Gamma)$	21	11	13	6	7	6	4	4
$\chi(X/\Gamma)$	-32	-16	-16	-8	-8	-8	-4	-4

Table 5.30: *Hodge numbers for the quotients of the manifolds* (5.4.14) *and* (5.4.18).

The second CICY in this class of $(5, 37)$ manifolds corresponds to the following configuration:

$$X_{6788} = \begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^4\end{matrix}\begin{bmatrix}2&0&0&0&0\\0&2&0&0&0\\0&0&2&0&0\\0&0&0&1&1\\1&1&1&1&1\end{bmatrix}^{5,37}_{-64} \qquad (5.4.15)$$



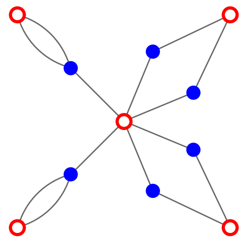
The manifold (5.4.15) admits 12 different free actions. Similar comments to the previous manifold apply here as well. The Hodge numbers for the quotients are listed in Table 5.31.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	5	5
$h^{2,1}(X/\Gamma)$	21	13
$\chi(X/\Gamma)$	-32	-16

Table 5.31: *Hodge numbers for the quotients of the manifolds* (5.4.15) *and* (5.4.17).

The third CICY in this class of $(5, 37)$ manifolds corresponds to the following configuration:

$$X_{6927} = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^5 \end{matrix} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^{5,37}_{-64} \tag{5.4.16}$$



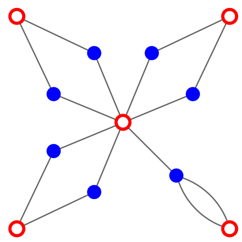
The manifold (5.4.16) admits 8 different free actions. As before, we compute the Hodge number for the different quotients by making use of the favourable embedding, as listed in Table 5.32.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_4$
$h^{1,1}(X/\Gamma)$	5	3	5	3
$h^{2,1}(X/\Gamma)$	21	11	13	7
$\chi(X/\Gamma)$	-32	-16	-16	-8

Table 5.32: *Hodge numbers for the quotients of the manifold* (5.4.16).

The fourth CICY corresponds to the following configuration:

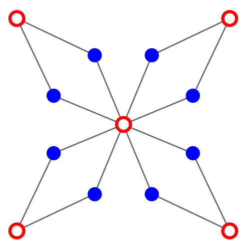
$$X_{6715} = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^6 \end{matrix} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^{5,37}_{-64} \tag{5.4.17}$$



The manifold (5.4.17) admits only 2 different free actions. The Hodge numbers for the resulting quotients are identical to those of the manifold (5.4.15), see Table 5.31.

The final CICY in this class of $(5, 37)$ manifolds corresponds to the following configuration:

$$X_{6947} = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^7 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^{5,37}_{-64} \tag{5.4.18}$$



The manifold (5.4.18) admits 9 different free actions by the same groups as for the manifold (5.4.14). The resulting quotients have Hodge numbers that are identical to those obtained in the previous discussion, see Table 5.30. Note, however, that unlike for the manifold (5.4.14), in this case the polynomial deformation method can be applied and agrees with the counting of Kähler parameters.

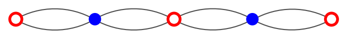
5.5 The remaining sources, and their descendants

5.5.1 The manifold $X^{3,43}$ and its splits

The manifold $X^{3,43}$ and its smooth quotients

A third sequence of splits starts with the manifold defined by the following configuration:

$$X_{7484} \quad = \quad \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{matrix} \left[\begin{matrix} 2 & 0 \\ 0 & 2 \\ 2 & 2 \end{matrix} \right]^{3,\,43}_{-80} \tag{5.5.1}$$



The manifold $X^{3,43}$ admits three different smooth quotients, the details of which are listed in Table 5.33. For this manifold, the polynomial deformation method does not reproduce correctly the number of complex structure parameters, and as such, it cannot be used in order to compute $h^{2,1}(X/\Gamma)$. However, we can exploit the fact that the embedding (5.5.1) is favourable and hence compute the number of Kähler parameters invariant under each group action.

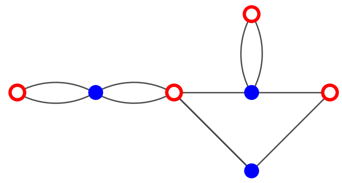
Γ	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	3	2	3
$h^{2,1}(X/\Gamma)$	23	12	13
$\chi(X/\Gamma)$	-40	-20	-20

Table 5.33: *Hodge numbers for the quotients of the manifold* (5.5.1).

Two favourable splits with Hodge numbers $(4, 36)$

The first split corresponds to the following configuration:

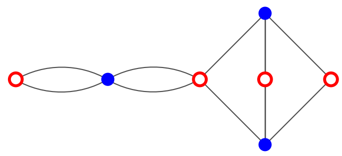
$$X_{6784} \;=\; \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{array} \left[\begin{array}{ccc} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 1 & 1 \end{array}\right]^{4,36}_{-64} \qquad (5.5.2)$$



The manifold (5.5.2) admits six different free group actions. Also for this case, the polynomial deformation method does not yield the correct number of complex structure parameters. We use instead the counting of Kähler parameters, which leads to the Hodge numbers presented in Table 5.34.

The second split corresponds to the following configuration:

$$X_{6828} \;=\; \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{array} \left[\begin{array}{ccc} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{array}\right]^{4,36}_{-64} \qquad (5.5.3)$$



The manifold (5.5.3) admits two smooth quotients, having the same Hodge numbers as the quotients of the manifold (5.5.2).

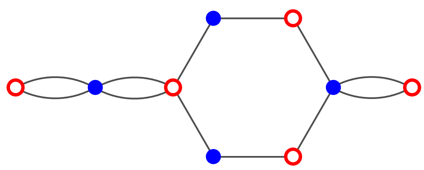
Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	4	4
$h^{2,1}(X/\Gamma)$	20	12
$\chi(X/\Gamma)$	-32	-16

Table 5.34: *Hodge numbers for the quotients of the manifolds* (5.5.2) *and* (5.5.3).

Further non-favourable splits: $X^{4,36} \to X^{8,32} \to X^{9,25} \to X^{13,21}$

The manifold $X^{8,32}$ can be obtained by splitting the second column of the configuration matrix (5.5.2), and corresponds to the following configuration:

$$X_{5421} \;=\; \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{array} \left[\begin{array}{cccc} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 1 & 0 \end{array}\right]^{8,32}_{-48} \qquad (5.5.4)$$



This manifold admits a total number of 27 different free group actions, by $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. In this case, the polynomial deformation method does not reproduce correctly the number of complex structure parameters. Moreover, the embedding (5.5.4) is not favourable in the sense of Section 5.2.6. However, we note that X_{5421} is embedded in the product $\mathrm{dP}_4 \times \mathbb{P}^1 \times \mathbb{P}^3$. Using our knowledge of del Pezzo surfaces of degree 4 described in §5.2.7, we compute the Hodge numbers of the quotients. In Table 5.36, we list the distinct symmetry actions on the ambient space coordinates.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	6	5
$h^{2,1}(X/\Gamma)$	18	11
$\chi(X/\Gamma)$	-24	-12

Table 5.35: *Hodge numbers for the quotients of the manifold* (5.5.4).

Table 5.36: *Various symmetry actions on the ambient space of the manifold* (5.5.4)*. The coordinate patch of the* dP_4 *is chosen to be* $(1,x)\times(1,y)\times(1,z)$*.* (p,q) *and* (a,b,c,d) *are taken to be coordinates of the first* $\mathbb{P}^1$ *space and the* $\mathbb{P}^3$ *respectively.*

Index	**Group**	(**x**, **y**, **z**)	(**p**, **q**)	(**a**, **b**, **c**, **d**)
1	$\mathbb{Z}_2$	$(-x,-y,-z)$	$(-p, q)$	$(-a,-b, c, d)$
2	$\mathbb{Z}_2$	$(y, x,-z)$	$(-p, q)$	$(-a,-b, c, d)$
3	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(-x,-y,-z)$ (x^{-1},y^{-1},z^{-1})	$(p,-q)$ (q, p)	$(a,-b, c,-d)$ (b, a, d, c)
4	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(y, x,-z)$ $(-y,-x,z^{-1})$	$(p,-q)$ (q, p)	$(a,-b, c,-d)$ (b, a, d, c)
5	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(y, x,-z)$ $(-x,-y,z^{-1})$	$(p,-q)$ (q, p)	$(a,-b, c,-d)$ (b, a, d, c)
6	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(-x,-y,-z)$ (y, x,z^{-1})	$(p,-q)$ (q, p)	$(a,-b, c,-d)$ (b, a, d, c)

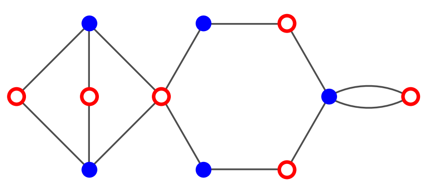
Table 5.37: *Symmetry actions on the cohomology basis and the corresponding invariants for the manifold* (5.5.4)*. The matrices P_i and Q_i are defined in* (5.2.65)*.*

Index	Group	Action on Coh Basis	Coh Invariants
1	$\mathbb{Z}_2$	P_1	$H, E_1 + E_2, E_3 + E_4, E_5$
2	$\mathbb{Z}_2$	Q_1	$H - E_5, E_1 + E_4 - E_5, E_2 + E_4 - E_5, E_3 - E_4$
3	$\mathbb{Z}_2{\times}\mathbb{Z}_2$	$P_1,\ P_2$	$H, E_1 + E_2 + E_3 + E_4, E_5$
4	$\mathbb{Z}_2{\times}\mathbb{Z}_2$	$Q_1,\ Q_2$	$H - E_5, E_1 + E_3 - E_5, E_2 + E_4 - E_5$
5	$\mathbb{Z}_2{\times}\mathbb{Z}_2$	$Q_1,\ P_2$	$H - E_5, E_1 + E_4 - E_5, E_2 + E_3 - E_5$
6	$\mathbb{Z}_2{\times}\mathbb{Z}_2$	$P_1,\ Q_2$	$H - E_5, E_1 + E_2 - E_5, E_3 + E_4 - E_5$

The group action on the cohomology and the invariants are listed in Table 5.37. For the quotient manifolds, the counting of Kähler classes includes the two hyperplane classes corresponding to the $\mathbb{P}^1$ and $\mathbb{P}^3$ spaces, plus the number of cohomology invariants listed in Table 5.37. The resulting Hodge numbers are shown in Table 5.35.

Now we turn to the manifold $X^{9,25}$, which can be obtained by splitting the first column of the configuration matrix (5.5.4), and corresponds to the following configuration:

$$X_{2640} = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{array} \left[\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 0 \end{array} \right]^{9,25}_{-32} \qquad (5.5.5)$$



This manifold is embedded in the product $\mathrm{dP}_4 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$ and admits a total of 27 different free group actions, by $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. Because of this embedding, $h^{1,1} = 6 + 3 = 9$ for the manifold. In order to compute the Hodge numbers of the quotients, we resort to computing the group action on the cohomology basis of the dP_4, since this manifold is neither favourable nor amenable to polynomial deformation methods. In Table 5.38, we list the distinct symmetry actions on the ambient space coordinates.

The group action on the cohomology and corresponding invariants are listed in Table 5.39. The Hodge numbers of the quotients of (5.5.5) are listed in Table 5.40.

Table 5.38: *Various symmetry actions on the ambient space of the manifold* (5.5.5)*. The coordinate patch of the* dP_4 *is chosen to be* $(1, x) \times (1, y) \times (1, z)$*.* (p, q)*,* (r, s) *and* (a, b, c, d) *are taken to be coordinates of the first two* $\mathbb{P}^1$ *spaces and the* $\mathbb{P}^3$ *respectively.*

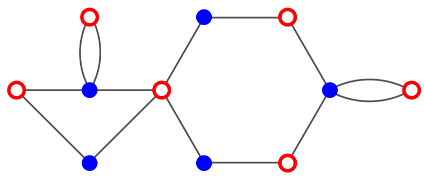
Index	Group	(x, y, z)	(p, q)	(r, s)	(a, b, c, d)
1	$\mathbb{Z}_2$	$(-x, -y, -z)$	$(-p, q)$	$(-r, s)$	$(-a, -b, c, d)$
2	$\mathbb{Z}_2$	$(y, x, -z)$	$(-p, q)$	$(-r, s)$	$(-a, -b, c, d)$
3	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(-x, -y, -z)$ (x^{-1}, y^{-1}, z^{-1})	$(p, -q)$ (q, p)	$(r, -s)$ (s, r)	$(a, -b, c, -d)$ (b, a, d, c)
4	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(-x, -y, -z)$ (y, x, z^{-1})	$(p, -q)$ (q, p)	$(r, -s)$ (s, r)	$(a, -b, c, -d)$ (b, a, d, c)
5	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(y, x, -z)$ $(-x, -y, z^{-1})$	$(p, -q)$ (q, p)	$(r, -s)$ (s, r)	$(a, -b, c, -d)$ (b, a, d, c)
6	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(y, x, -z)$ $(-y, -x, z^{-1})$	$(p, -q)$ (q, p)	$(r, -s)$ (s, r)	$(a, -b, c, -d)$ (b, a, d, c)

Table 5.39: *Symmetry actions on the cohomology basis and the corresponding invariants for the manifold* (5.5.5)*. The matrices* P_i *and* Q_i *are defined in* (5.2.65)*.*

Index	Group	Action on Coh Basis	Coh Invariants
1	$\mathbb{Z}_2$	P_1	$H, E_1 + E_2, E_3 + E_4, E_5$
2	$\mathbb{Z}_2$	Q_2	$H - E_5, E_1 + E_4, E_2 + E_3, E_5$
3	$\mathbb{Z}_2\times\mathbb{Z}_2$	$P_1,\ P_3$	$H, E_1 + E_2 + E_3 + E_4, E_5$
4	$\mathbb{Z}_2\times\mathbb{Z}_2$	$P_1,\ Q_2$	$H - E_5, E_1 + E_2 - E_5, E_3 + E_4 - E_5$
5	$\mathbb{Z}_2\times\mathbb{Z}_2$	$Q_1,\ P_2$	$H - E_5, E_1 + E_4 - E_5, E_2 + E_3 - E_5$
6	$\mathbb{Z}_2\times\mathbb{Z}_2$	$Q_1,\ Q_2$	$H - E_5, E_1 + E_3 - E_5, E_2 + E_4 - E_5$

Another manifold $X^{9,25}$ can be obtained through a different splitting of the first column of the configuration matrix (5.5.4), which corresponds to the following configuration:

$$X_{2357} \;=\; \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{array} \left[\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right]^{9,25}_{-32} \tag{5.5.6}$$



Like (5.5.5), this manifold is embedded in the product $\mathrm{dP}_4\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^3$ and admits a total of 27 different free group actions, by $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. $h^{1,1} = 6 + 3 = 9$ for the manifold because of contributions from the dP_4 and the three other $\mathbb{P}^n$'s respectively. We compute the Hodge numbers of the quotients by computing the group action on the cohomology basis of the dP_4, since this manifold is neither favourable nor amenable to polynomial deformation methods. In Table 5.41, we list the distinct symmetry actions on the ambient space coordinates. The group action on the cohomology and corresponding invariants are listed in Table 5.42. Finally, the Hodge numbers of the quotients of (5.5.6) are listed in Table 5.40.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	7	6
$h^{2,1}(X/\Gamma)$	15	10
$\chi(X/\Gamma)$	-16	-8

Table 5.40: *Hodge numbers for the quotients of the manifolds* (5.5.5) *and* (5.5.6).

Table 5.41: *Various symmetry actions on the ambient space of the manifold* (5.5.6)*. The coordinate patch of the* dP_4 *is chosen to be* $(1,x)\times(1,y)\times(1,z)$*.* (p,q)*,* (r,s) *and* (a,b,c,d) *are the coordinates of the first two* $\mathbb{P}^1$ *spaces and the* $\mathbb{P}^3$ *respectively.*

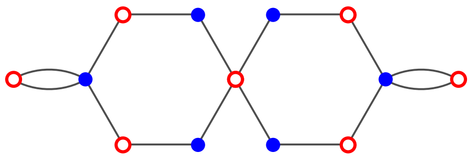
Index	Group	(**x**, **y**, **z**)	(**p**, **q**)	(**u**, **v**)	(**a**, **b**, **c**, **d**)
1	$\mathbb{Z}_2$	$(-x,-y,-z)$	$(-p, q)$	$(-r, s)$	$(-a,-b, c, d)$
2	$\mathbb{Z}_2$	$(y, x,-z)$	$(-p, q)$	$(-r, s)$	$(-a,-b, c, d)$
3	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(-x,-y,-z)$ (x^{-1},y^{-1},z^{-1})	$(p,-q)$ (q, p)	$(r,-s)$ (s, r)	$(a,-b, c,-d)$ (b, a, d, c)
4	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(y, x,-z)$ $(-y,-x,z^{-1})$	$(p,-q)$ (q, p)	$(r,-s)$ (s, r)	$(a,-b, c,-d)$ (b, a, d, c)
5	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(-x,-y,-z)$ (y, x,z^{-1})	$(p,-q)$ (q, p)	$(r,-s)$ (s, r)	$(a,-b, c,-d)$ (b, a, d, c)
6	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(y, x,-z)$ $(-x,-y,z^{-1})$	$(p,-q)$ (q, p)	$(r,-s)$ (s, r)	$(a,-b, c,-d)$ (b, a, d, c)

Table 5.42: *Symmetry actions on the cohomology basis and the corresponding invariants for the manifold* (5.5.6)*. The matrices* P_i *and* Q_i *are defined in* (5.2.65)*.*

Index	Group	Action on Coh Basis	Coh Invariants
1	$\mathbb{Z}_2$	P_1	H, E_1+E_2, E_3+E_4, E_5
2	$\mathbb{Z}_2$	Q_1	$H-E_5, E_1+E_4-E_5, E_2+E_4-E_5, E_3-E_4$
3	$\mathbb{Z}_2\times\mathbb{Z}_2$	P_2, P_1	$H, E_1+E_2+E_3+E_4, E_5$
4	$\mathbb{Z}_2\times\mathbb{Z}_2$	Q_1, Q_2	$H-E_5, E_1+E_3-E_5, E_2+E_4-E_5$
5	$\mathbb{Z}_2\times\mathbb{Z}_2$	P_1, Q_2	$H-E_5, E_1+E_2-E_5, E_3+E_4-E_5$
6	$\mathbb{Z}_2\times\mathbb{Z}_2$	Q_1, P_2	$H-E_5, E_1+E_4-E_5, E_2+E_3-E_5$

A further splitting of the second column of (5.5.6) leads us to the manifold $X^{13,21}$, specified by the following configuration:

$$X_{480} \;=\; \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{array} \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{array}\right]^{13,\,21}_{-16} \qquad (5.5.7)$$



The above manifold, embedded in a product of $\mathrm{dP}_4{\times}\mathrm{dP}_4{\times}\mathbb{P}^3$ has a large set of freely acting symmetries (394 in number). The value of $h^{1,1}$ for this manifold can be understood as the sum of $h^{1,1}$'s for the two dP_4's and the $\mathbb{P}^3$. Thus $h^{1,1}(X_{480}) = 6 + 6 + 1 = 13$. The symmetries although large in number, act on the coordinates in a grand total of 13 distinct ways. The number 394 is due to the large number of different actions on the polynomials. None of these symmetries mix the $\mathbb{P}^3$ with the $\mathbb{P}^1$'s, although 5 of them do interchange the two dP_4's. These symmetries are listed in Table 5.44.

The action of the symmetries on the combined 32 special lines of the two dP_4's are listed in Table 5.45. This also contains all the cohomology invariants. Note that the Z_4 actions interchange the two del Pezzo surfaces. Finally, the $h^{1,1}$ of each quotient is the number of such invariants plus 1 for the $\mathbb{P}^3$ space, since $X_{480} \subset \mathrm{dP}_4 \times \mathrm{dP}_4 \times \mathbb{P}^3$. These values are listed in Table 5.43.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	9	5	7
$h^{2,1}(X/\Gamma)$	13	7	9
$\chi(X/\Gamma)$	-8	-4	-4

Table 5.43: *Hodge numbers for the quotients of the manifold* (5.5.7).

Table 5.44: *Various symmetry actions on the ambient space of the manifold* (5.5.7). *The coordinate patch of the two dP_4's are chosen to be* $(1,x)\times(1,y)\times(1,z)$ *and* $(1,\tilde{x})\times(1,\tilde{y})\times(1,\tilde{z})$ *respectively.* (a,b,c,d) *is taken to be coordinates of the* $\mathbb{P}^3$.

Index	**Group**	$(\mathbf{x},\ \mathbf{y},\ \mathbf{z})$	$(\tilde{\mathbf{x}},\ \tilde{\mathbf{y}},\ \tilde{\mathbf{z}})$	$(\mathbf{a},\ \mathbf{b},\ \mathbf{c},\ \mathbf{d})$
1	$\mathbb{Z}_2$	$(-x,-y,-z)$	$(-\tilde{x},-\tilde{y},-\tilde{z})$	$(-a,-b,\ c,\ d)$
2	$\mathbb{Z}_2$	$(\ y,\ x,-z)$	$(-\tilde{x},-\tilde{y},-\tilde{z})$	$(-a,-b,\ c,\ d)$
3	$\mathbb{Z}_2$	$(\ y,\ x,-z)$	$(\ \tilde{y},\ \tilde{x},-\tilde{z})$	$(-a,-b,\ c,\ d)$
4	$\mathbb{Z}_4$	$(-\tilde{y},-\tilde{x},-\tilde{z})$	$(\ y,\ x,\ z)$	$(\ a,-b,\ \mathrm{i}c,-\mathrm{i}d)$
5	$\mathbb{Z}_4$	$(\ \tilde{x},\ \tilde{y},-\tilde{z})$	$(\ y,\ x,\ z)$	$(\ a,-b,\ \mathrm{i}c,-\mathrm{i}d)$
6	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(-x,-y,-z)$ (x^{-1},y^{-1},z^{-1})	$(-\tilde{x},-\tilde{y},-\tilde{z})$ $(\tilde{x}^{-1},\tilde{y}^{-1},\tilde{z}^{-1})$	$(\ a,-b,\ c,-d)$ $(\ b,\ a,\ d,\ c)$
7	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(\ y,\ x,-z)$ $(-y,-x,z^{-1})$	$(-\tilde{x},-\tilde{y},-\tilde{z})$ $(\tilde{x}^{-1},\tilde{y}^{-1},\tilde{z}^{-1})$	$(\ a,-b,\ c,-d)$ $(\ b,\ a,\ d,\ c)$
8	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(-x,-y,-z)$ $(\ y,\ x,z^{-1})$	$(-\tilde{x},-\tilde{y},-\tilde{z})$ $(\tilde{x}^{-1},\tilde{y}^{-1},\tilde{z}^{-1})$	$(\ a,-b,\ c,-d)$ $(\ b,\ a,\ d,\ c)$
9	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(\ y,\ x,-z)$ $(-x,-y,z^{-1})$	$(-\tilde{x},-\tilde{y},-\tilde{z})$ $(\tilde{x}^{-1},\tilde{y}^{-1},\tilde{z}^{-1})$	$(\ a,-b,\ c,-d)$ $(\ b,\ a,\ d,\ c)$
10	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(\ y,\ x,-z)$ $(-x,-y,z^{-1})$	$(\ \tilde{y},\ \tilde{x},-\tilde{z})$ $(-\tilde{x},-\tilde{y},\tilde{z}^{-1})$	$(\ a,-b,\ c,-d)$ $(\ b,\ a,\ d,\ c)$
11	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(-x,-y,-z)$ $(\ y,\ x,z^{-1})$	$(-\tilde{x},-\tilde{y},-\tilde{z})$ $(\ \tilde{y},\ \tilde{x},\tilde{z}^{-1})$	$(\ a,-b,\ c,-d)$ $(\ b,\ a,\ d,\ c)$
12	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(\ y,\ x,-z)$ $(-y,-x,z^{-1})$	$(\ \tilde{y},\ \tilde{x},-\tilde{z})$ $(-\tilde{y},-\tilde{x},\tilde{z}^{-1})$	$(\ a,-b,\ c,-d)$ $(\ b,\ a,\ d,\ c)$
13	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(\ y,\ x,-z)$ $(-y,-x,z^{-1})$	$(\ \tilde{y},\ \tilde{x},-\tilde{z})$ $(-\tilde{x},-\tilde{y},\tilde{z}^{-1})$	$(\ a,-b,\ c,-d)$ $(\ b,\ a,\ d,\ c)$

Table 5.45: *Symmetry actions on the cohomology basis and the corresponding invariants for the manifold* (5.5.7). *The matrices* P_i *and* Q_i *are defined in* (5.2.65).

Index	Group	Action on Coh Basis	Coh Invariants
1	$\mathbb{Z}_2$	$\begin{bmatrix} P_2 & 0 \\ 0 & P_2 \end{bmatrix}$	$H, E_1+E_4, E_2+E_3, E_5,$ $\widetilde{H}, \widetilde{E}_1+\widetilde{E}_4, \widetilde{E}_2+\widetilde{E}_3, \widetilde{E}_5$
2	$\mathbb{Z}_2$	$\begin{bmatrix} Q_1 & 0 \\ 0 & P_2 \end{bmatrix}$	$H-E_5, E_1+E_4-E_5, E_2+E_4-E_5,$ $E_3-E_4,\ \widetilde{H}, \widetilde{E}_1+\widetilde{E}_4, \widetilde{E}_2+\widetilde{E}_3, \widetilde{E}_5$
3	$\mathbb{Z}_2$	$\begin{bmatrix} Q_1 & 0 \\ 0 & Q_1 \end{bmatrix}$	$H-E_5, E_1+E_4-E_5, E_2+E_4-E_5, E_3-E_4,$ $\widetilde{H}-\widetilde{E}_5, \widetilde{E}_1+\widetilde{E}_4-\widetilde{E}_5, \widetilde{E}_2+\widetilde{E}_4-\widetilde{E}_5, \widetilde{E}_3-\widetilde{E}_4$
4	$\mathbb{Z}_4$	$\begin{bmatrix} 0 & Q_1 \\ Q_4 & 0 \end{bmatrix}$	$H+3\widetilde{H}-\widetilde{E}_1-\widetilde{E}_2-\widetilde{E}_3-\widetilde{E}_4-2\widetilde{E}_5,$ $E_1+E_2+2\widetilde{H}-\widetilde{E}_1-\widetilde{E}_2-2\widetilde{E}_5,$ $E_3+E_4+2\widetilde{H}-\widetilde{E}_3-\widetilde{E}_4-2\widetilde{E}_5,$ $E_5+2\widetilde{H}-\widetilde{E}_1-\widetilde{E}_2-\widetilde{E}_3-\widetilde{E}_4-\widetilde{E}_5$
5	$\mathbb{Z}_4$	$\begin{bmatrix} 0 & P_1 \\ Q_4 & 0 \end{bmatrix}$	$H-E_5+\widetilde{H}-\widetilde{E}_5, E_3-E_4-(\widetilde{E}_3-\widetilde{E}_4),$ $E_1+E_4-E_5+\widetilde{E}_2+\widetilde{E}_3-\widetilde{E}_5,$ $E_2+E_4-E_5+\widetilde{E}_1+\widetilde{E}_3-\widetilde{E}_5$
6	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\begin{bmatrix} P_1 & 0 \\ 0 & P_1 \end{bmatrix}, \begin{bmatrix} P_2 & 0 \\ 0 & P_2 \end{bmatrix}$	$H, E_1+E_2+E_3+E_4, E_5,$ $\widetilde{H}, \widetilde{E}_1+\widetilde{E}_2+\widetilde{E}_3+\widetilde{E}_4, \widetilde{E}_5$
7	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\begin{bmatrix} Q_1 & 0 \\ 0 & P_2 \end{bmatrix}, \begin{bmatrix} Q_2 & 0 \\ 0 & P_1 \end{bmatrix}$	$H-E_5, E_1+E_3-E_5, E_2+E_4-E_5,$ $\widetilde{H}, \widetilde{E}_1+\widetilde{E}_2+\widetilde{E}_3+\widetilde{E}_4, \widetilde{E}_5$
8	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \begin{bmatrix} Q_2 & 0 \\ 0 & P_1 \end{bmatrix}$	$H-E_5, E_1+E_2-E_5, E_3+E_4-E_5,$ $\widetilde{H}, \widetilde{E}_1+\widetilde{E}_2+\widetilde{E}_3+\widetilde{E}_4, \widetilde{E}_5$
9	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\begin{bmatrix} Q_1 & 0 \\ 0 & P_2 \end{bmatrix}, \begin{bmatrix} P_2 & 0 \\ 0 & P_1 \end{bmatrix}$	$H-E_5, E_1+E_4-E_5, E_2+E_3-E_5,$ $\widetilde{H}, \widetilde{E}_1+\widetilde{E}_2+\widetilde{E}_3+\widetilde{E}_4, \widetilde{E}_5$
10	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\begin{bmatrix} Q_1 & 0 \\ 0 & Q_1 \end{bmatrix}, \begin{bmatrix} P_2 & 0 \\ 0 & P_2 \end{bmatrix}$	$H-E_5, E_1+E_4-E_5, E_2+E_3-E_5,$ $\widetilde{H}-\widetilde{E}_5, \widetilde{E}_1+\widetilde{E}_4-\widetilde{E}_5, \widetilde{E}_2+\widetilde{E}_3-\widetilde{E}_5$
11	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \begin{bmatrix} Q_2 & 0 \\ 0 & Q_1 \end{bmatrix}$	$H-E_5, E_1+E_2-E_5, E_3+E_4-E_5,$ $\widetilde{H}-\widetilde{E}_5, \widetilde{E}_1+\widetilde{E}_4-\widetilde{E}_5, \widetilde{E}_2+\widetilde{E}_3-\widetilde{E}_5$
12	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \begin{bmatrix} Q_2 & 0 \\ 0 & Q_1 \end{bmatrix}$	$H-E_5, E_1+E_3-E_5, E_2+E_4-E_5,$ $\widetilde{H}-\widetilde{E}_5, \widetilde{E}_1+\widetilde{E}_3-\widetilde{E}_5, \widetilde{E}_2+\widetilde{E}_4-\widetilde{E}_5$
13	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\begin{bmatrix} Q_1 & 0 \\ 0 & Q_1 \end{bmatrix}, \begin{bmatrix} Q_2 & 0 \\ 0 & Q_2 \end{bmatrix}$	$H-E_5, E_1+E_3-E_5, E_2+E_4-E_5,$ $\widetilde{H}-\widetilde{E}_5, \widetilde{E}_1+\widetilde{E}_4-\widetilde{E}_5, \widetilde{E}_2+\widetilde{E}_3-\widetilde{E}_5$

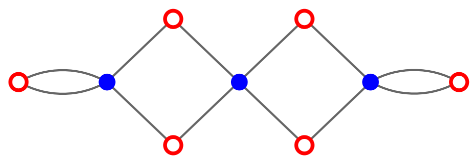
5.5.2 The manifold $X^{12,28}$ and its splits

In our sub-class of CICYs admitting freely acting symmetries that are either $\mathbb{Z}_2$ or of order divisible by 4, there are three manifolds with Hodge numbers $(12, 28)$. All three of them can be argued to be embedded in a product of two del Pezzo surfaces of degree 4.

$X^{12,28}$ and its split $X^{15,15}$

The first of the $(12, 28)$ manifolds can be obtained by splitting the first column of the $X^{8,40}$ manifold (5.3.4). It is defined by the following configuration:

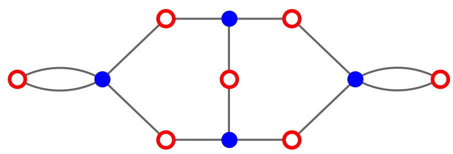
$$X_{2568} \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\left[\begin{matrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 2 & 0\\ 1 & 0 & 1\\ 1 & 0 & 1\\ 0 & 0 & 2\end{matrix}\right]^{12,28}_{-32}$$



(5.5.8)

This manifold as well as its quotients have been discussed in detail in Section 5.2.7. A splitting of the first column of (5.5.8), leads to the manifold $X^{15,15}$, defined by the following configuration:

$$X_{22} \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\left[\begin{matrix}1 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 2 & 0\\ 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 2\end{matrix}\right]^{15,15}_{0}$$



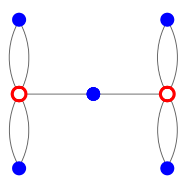
(5.5.9)

The manifold (5.5.9) admits 15 different free group actions by $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. We compute the Hodge numbers for the resulting quotients using the polynomial deformation method. The results are listed in Table 5.46.

$X^{12,28}$ and its split $X^{19,19}$

The second $(12, 28)$ manifold corresponds to the following configuration:

$$X_{2564} \;=\; \begin{matrix}\mathbb{P}^4\\ \mathbb{P}^4\end{matrix}\left[\begin{matrix}2 & 2 & 1 & 0 & 0\\ 0 & 0 & 1 & 2 & 2\end{matrix}\right]^{12,\,28}_{-32}$$



(5.5.10)

The manifold (5.5.10) admits 9 different free group actions by $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_2\times\mathbb{Z}_2, \mathbb{Z}_8, \mathbb{Z}_2\times \mathbb{Z}_4$ and $\mathbb{Q}_8$. The polynomial deformation method correctly reproduces the number of

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	9	10	7
$h^{2,1}(X/\Gamma)$	9	10	7
$\chi(X/\Gamma)$	0	0	0

Table 5.46: *Hodge numbers for the quotients of the manifold* (5.5.9).

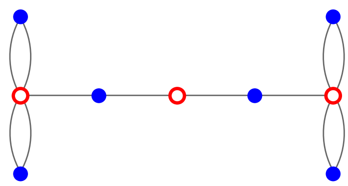
complex structure parameters for the covering manifold, and, although the diagram is one-leg decomposable, we will assume that it provides a complete parametrisation of the complex structure moduli space. With this assumption, we are able to compute the Hodge numbers listed in Table 5.47.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Q}_8$
$h^{1,1}(X/\Gamma)$	8	4	6	2	3	2
$h^{2,1}(X/\Gamma)$	16	8	10	4	5	4
$\chi(X/\Gamma)$	-16	-8	-8	-4	-4	-4

Table 5.47: *Hodge numbers for the quotients of the manifold* (5.5.10).

The configuration matrix (5.5.10) can be split with a $\mathbb{P}^1$ to give the manifold $X^{19,19}$: This manifold corresponds to the following configuration:

$$X_{19} \;=\; \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^4 \\ \mathbb{P}^4 \end{array} \left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 2 \end{array}\right]^{19,\,19}_{0} \qquad (5.5.11)$$



The manifold (5.5.11) admits 29 different free group actions by the same groups as the previous manifold. We compute the Hodge numbers for the quotient manifolds using the same assumption as above. The results are listed in Table 5.48.

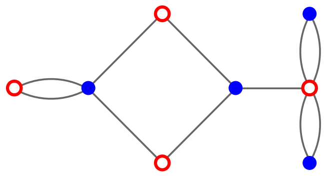
Γ	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_2\times\mathbb{Z}_4$	$\mathbb{Q}_8$
$h^{1,1}(X/\Gamma)$	11	5	6	7	3	4	3
$h^{2,1}(X/\Gamma)$	11	5	6	7	3	4	3
$\chi(X/\Gamma)$	0	0	0	0	0	0	0

Table 5.48: *Hodge numbers for the quotients of the manifold* (5.5.11).

Another $X^{12,28}$

An $X^{12,28}$ manifold can be obtained through a splitting of the first column of a $(8,40)$ manifold (5.5.13). This corresponds to the following configuration:

$$X_{2566} \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^4\end{matrix}\left[\begin{matrix}2 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 1 & 2 & 2\end{matrix}\right]^{12,\,28}_{-32} \qquad (5.5.12)$$



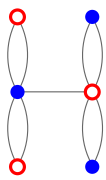
This manifold, embedded in a product of two dP_4's, admits 10 different free group actions by $\mathbb{Z}_2$ and $\mathbb{Z}_2\times\mathbb{Z}_2$. The polynomial deformation method does not correctly reproduce the number of complex structure parameters in this case. We have listed the Hodge numbers of the quotient manifolds in Table 5.8.

5.5.3 Other non-favourable manifolds

The manifold $X^{8,40}$ and its split $X^{19,19}$

The manifold $X^{8,40}$ is defined by the following configuration:

$$X_{6826} \;=\; \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^4\end{matrix}\left[\begin{matrix}2 & 0 & 0\\ 2 & 0 & 0\\ 1 & 2 & 2\end{matrix}\right]^{8,\,40}_{-64} \qquad (5.5.13)$$



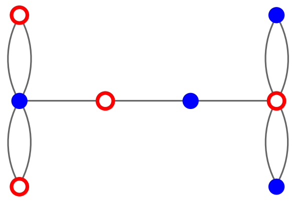
The manifold (5.5.13) admits 3 different free group actions by $\mathbb{Z}_2, \mathbb{Z}_4$ and $\mathbb{Z}_2\times\mathbb{Z}_2$. Although the diagram is one-leg decomposable, the polynomial deformation method correctly reproduces the number of complex structure parameters for the covering manifold. Assuming that the method provides a complete parametrisation of the complex structure moduli space, we compute in this way the Hodge numbers $h^{2,1}(X/\Gamma)$.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	6	3	5
$h^{2,1}(X/\Gamma)$	22	11	13
$\chi(X/\Gamma)$	-32	-16	-16

Table 5.49: *Hodge numbers for the quotients of the manifold* (5.5.13).

The manifold $X^{19,19}$ can be obtained by splitting the first column of (5.5.13), leading to the following favourable embedding:

$$X_{20} = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^4 \end{array} \left[\begin{array}{cccc} 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 \end{array}\right]^{19,19}_{0} \tag{5.5.14}$$



The manifold (5.5.14) admits 14 free group actions by $\mathbb{Z}_2, \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. For the computation of the Hodge numbers for the quotient manifolds, the same comments as for the previous manifold apply. We present the results in Table 5.50.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$h^{1,1}(X/\Gamma)$	11	5	7
$h^{2,1}(X/\Gamma)$	11	5	7
$\chi(X/\Gamma)$	0	0	0

Table 5.50: *Hodge numbers for the quotients of the manifold* (5.5.14).

A different splitting of the first column of (5.5.13) yields the $X^{12,28}$ that was discussed in 5.5.2.

The manifold $X^{19,19}$ once again

There is another configuration matrix which leads to the manifold $X^{19,19}$:

$$X_{30} = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}^{19,\,19}_{0}$$

(5.5.15)

In this embedding, $X^{19,19}$ admits only 4 different, linearly represented, free group actions, by $\mathbb{Z}_2$ and $\mathbb{Z}_4$. The Hodge numbers for the $\mathbb{Z}_2$–quotients are $(11, 11)$, while for the $\mathbb{Z}_4$–quotient $(6, 6)$. We have obtained these through the application of the polynomial deformation method.

5.6 Conclusions

Since we are, in a sense, summarising a series of papers, it seems worthwhile to take stock of the manifolds that have been found. The following has some overlap with [2] and [130], which give more detail and contain references to the original literature. A graphical depiction of our present knowledge of the existence of Calabi-Yau manifolds of small height ($= h^{1,1} + h^{1,2}$) is provided by Figure 8. For the purposes of the present discussion we follow [130] in making the subjective choice of considering Hodge numbers to be small if $h^{1,1}+h^{1,2} \leq 24$. Most of the manifolds of the plot have been found as quotients of CICYs. Although many of these manifolds are interesting in themselves, it is also evident that the resolution of singular limits and singular quotients of these yields other important manifolds. Examples of this will be indicated below.

Any list of interesting manifolds must include Yau's manifold

$$\begin{array}{c} \mathbb{P}^3 \\ \mathbb{P}^3 \end{array} \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}^{6,9}_{/\mathbb{Z}_3} \tag{5.6.1}$$

that led to the first of the three generation models and which inspired the CICY class [70, 71]. The three generation UPenn models [77, 78, 112–114] are based on a $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotient of the split bicubic.

$$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{bmatrix}^{19,19} \tag{5.6.2}$$

The split bicubic itself appears in many guises in the CICY list. As remarked previously, all 15 occurrences of the Hodge numbers $(19, 19)$ can be seen to correspond to the same manifold by a series of ineffective splittings and contractions. By considering these different representations and drawing also on [125] we have the following quotients, all of which, of course, have $\chi = 0$.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_5$	$\mathbb{Z}_6$	$\mathbb{Z}_8$	$\mathbb{Q}_8$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
$(h^{1,1}, h^{2,1})$	(11,11)	$(7, 7)$	$(5, 5)$	$(6, 6)$	$(7, 7)$	$(3, 3)$	$(3, 3)$	$(3, 3)$	$(3, 3)$	$(4, 4)$
Ref.	[2, 125]	[2, 125]	[125]		[125]	[125]	[2, 125]		[2]	

Γ	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_{12}$	Dic_3	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_8 \rtimes \mathbb{Z}_2$	$\mathbb{Q}_8 \times \mathbb{Z}_2$	
$(h^{1,1}, h^{2,1})$	$(3, 3)$	$(3, 3)$	$(2, 2)$	$(2, 2)$	$(2, 2)$	$(2, 2)$	$(2, 2)$	$(2, 2)$	$(2, 2)$	
Ref.	[125]	[125]	[5]	[5]						

Table 5.51: *Hodge numbers for the quotients of the manifold $X^{19,19}$. Whenever a reference is not given, the corresponding quotients have not, to our knowledge, been previously discussed.*

The manifold with Hodge numbers $(15, 15)$, which we have met here as the configuration (5.3.10), also occurs 15 times in the CICY list in different guises. There are also extended representations that are not in the list, but which might manifest additional symmetries, including as a 15×18 matrix (see [2, Table 15]), which is the maximum size for a CICY matrix. One way to think of this manifold is as a codimension 3 submanifold of $\text{dP}_6 \times \text{dP}_6 \times \text{dP}_6$. This manifold is special because it can be highly symmetric and, like the split bicubic, it has many quotients. A somewhat more detailed discussion of the manifold may be found in [2]. We give here the quotients that we know.

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2{\times}\mathbb{Z}_2$	$\mathbb{Z}_2{\times}\mathbb{Z}_2$	$\mathbb{Z}_3{\times}\mathbb{Z}_2$
$(h^{1,1},\, h^{2,1})$	$(10,10)$	$(9,9)$	$(7,7)$	$(7,7)$	$(6,6)$	$(3,3)$
Ref.	[2]		[2]			[2]

Table 5.52: *Hodge numbers for the quotients of the manifold* $X^{15,15}$.

It is interesting that there is a conifold transition [64] from the Yau manifold to $X^{19,19}/\mathbb{Z}_3$, a quotient with Hodge numbers $(7,7)$. Particularly intriguing is the role of $X^{19,19}/\mathbb{Z}_3$ as one of the two superimposed sinks in the $\mathbb{Z}_3$ web of CICYs, see Figure 12 which we reproduce in a slightly updated version from [6]. This is a single web with two endpoints. These are the $\mathbb{Z}_3$ quotients of $X^{19,19}$ and of $X^{15,15}$, both of which have Hodge numbers $(7,7)$.

Notable also are the manifolds of §5.3.5, with Hodge numbers (8,44), that admit smooth quotients by $\mathbb{Z}_{12}$ and Dic_3 which also lead directly (i.e. via the standard embedding or deformations thereof) to three generation models. It is interesting to note that these manifolds also have conifold transitions to quotients of the split bicubic. For the $\mathbb{Z}_3$ quotient we have a transition [2] to the sink just mentioned

$$\left(X^{8,44}/\mathbb{Z}_3\right)^{4,16} \;\rightarrow\; \left(X^{19,19}/\mathbb{Z}_3\right)^{7,7}~. \tag{5.6.3}$$

For quotients by groups $G = \mathbb{Z}_{12}$ or $G = \text{Dic}_3$ we have conifold transitions to quotients of the split bicubic that have Hodge numbers $(2,2)$.

$$\left(X^{8,44}/G\right)^{1,4} \;\rightarrow\; \left(X^{19,19}/G\right)^{2,2}~. \tag{5.6.4}$$

We have hitherto concentrated on conifold transitions that preserve the fundamental group of the manifold. Davies [131, 141, 142] has exploited hyperconifold transitions, in which a quotient of a node is blown up into a divisor. These transitions are more drastic in that they do not preserve the fundamental group. In [130] Davies constructs two explicit examples that relate to our results. One is a hyperconifold transition from $X^{8,44}/\text{Dic}_3$ that yields a manifold with Hodge numbers $(2,3)$ and fundamental group S_3. This is the origin of the gray points at height 5 in Figure 8. The second example relates to a $\mathbb{Z}_{10}$ quotient of the $\mathbb{P}^4$-split of the quintic. A

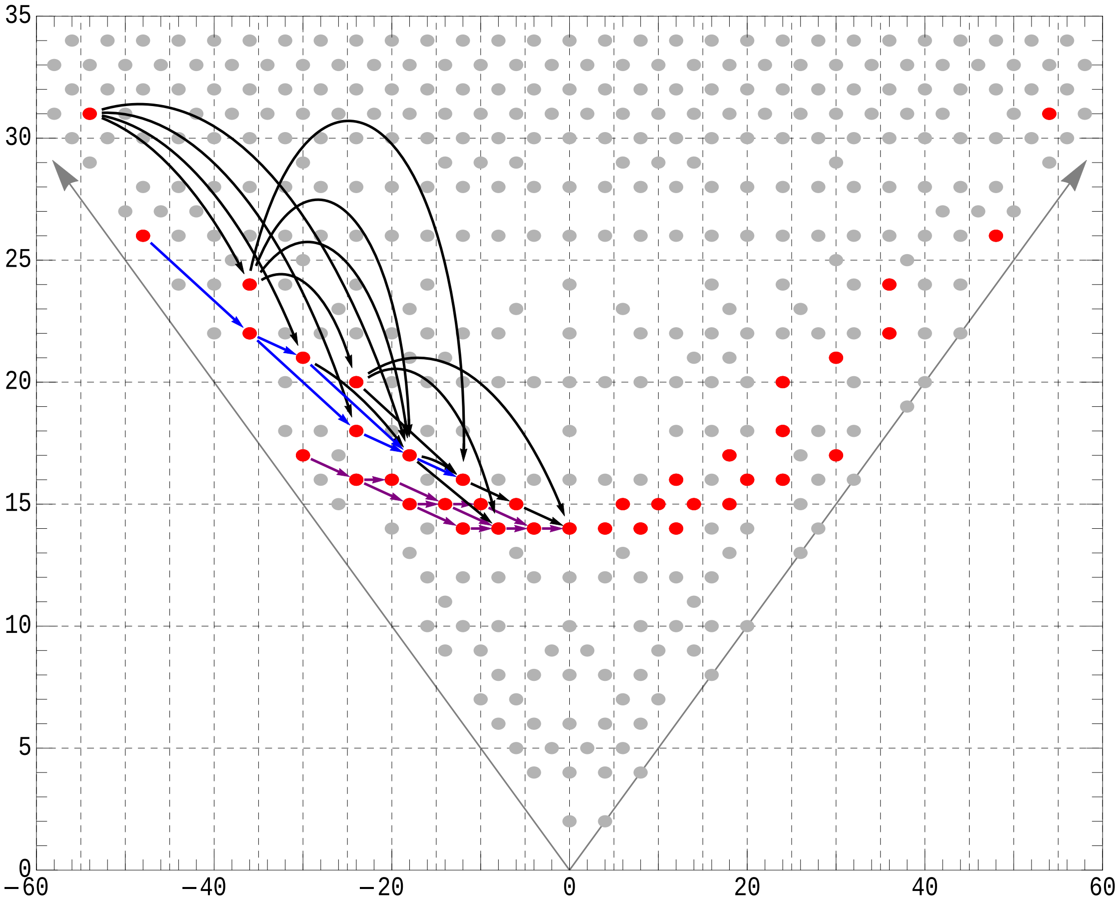


Figure 12: *The web of $\mathbb{Z}_3$ quotients of CICY manifolds and their mirrors [6]. The red points indicate $\mathbb{Z}_3$ quotients. The arrows correspond to conifold transitions, and the different colours indicate three different webs of $\mathbb{Z}_3$ quotients. The manifold $X^{19,19}/\mathbb{Z}_3$ with Hodge numbers $(7,7)$ is the endpoint of two such sequences.*

hyperconifold transition creates, in this case, a manifold with Hodge numbers $(2,5)$ and with fundamental group $\mathbb{Z}_5$. This manifold has $\chi = -6$ but does not appear promising for the purposes of model building.

It was realised very early [143] that the $\mathbb{P}^7[2,2,2,2]$ family admits smooth quotients by freely acting groups of order 32 and has manifolds with 64 nodes that admit free quotients by groups of order 64. The nodes are identified under the group action so the quotients have a single node. The smooth quotients by groups of order 32 have Hodge numbers $(1,3)$, so are very near the tip of the distribution. The nodal quotient by a group of order 64 can be resolved to give one of the remarkable Gross-Popescu manifolds with Hodge numbers $(2,2)$. The Gross-Popescu manifolds are fibered by abelian varieties and occupy some of the sites with $\chi = 0$ in Figure 8. For a slightly more detailed discussion of the Gross-Popescu manifolds see [2], which makes reference to the original papers. The tetraquadric, together with its quotients by groups of order 16, which all have Hodge numbers $(1,5)$, also deserves special mention.

Some of the CICY's can be represented in different ways. The Yau manifold can be thought of as a quotient of a hypersurface in $\mathrm{dP}_3{\times}\mathrm{dP}_3$ and the three-generation manifolds with Hodge numbers $(1,4)$ can be thought of as quotients of hypersurfaces in $\mathrm{dP}_6{\times}\mathrm{dP}_6$. More generally many of the special CICY's can be thought of as hypersurfaces in an embedding space $\mathcal{S}{\times}\mathcal{S}'$ where $\mathcal{S}$ and $\mathcal{S}'$ are del Pezzo surfaces. The importance of this class of Calabi-Yau manifolds was recognised by Bini and Favale [115]. Since $\mathbb{P}^2$ and $\mathbb{P}^1{\times}\mathbb{P}^1$ are also del Pezzo surfaces this class contains also the configuration that follows, as well as the tetraquadric.

$$X_{7884} \;=\; \begin{matrix}\mathbb{P}^2\\ \mathbb{P}^2\end{matrix}\!\begin{bmatrix}3\\3\end{bmatrix} \tag{5.6.5}$$

Another of these spaces is the manifold

$$X_{2564} \;=\; \begin{matrix}\mathbb{P}^4\\ \mathbb{P}^4\end{matrix}\!\begin{bmatrix}2&2&1&0&0\\0&0&1&2&2\end{bmatrix}^{12,28}_{-32}\;, \tag{5.6.6}$$

being a hypersurface in $\mathrm{dP}_4{\times}\mathrm{dP}_4$, is, in many ways, an analogue of the covering space of Yau's manifold and is notable since it admits $\mathbb{Q}_8$ as a freely acting symmetry.

Returning to the embedding space $\mathrm{dP}_6{\times}\mathrm{dP}_6$, we note that this embedding space is a toric variety and has moreover a high degree of symmetry since each dP_6 has 6 exceptional lines that form a hexagon, so each dP_6 has Dih_6, the group of the hexagon, as a symmetry group. The product $\mathrm{dP}_6{\times}\mathrm{dP}_6$ has then a symmetry group $(\mathrm{Dih}_6{\times}\mathrm{Dih}_6)\ltimes\mathbb{Z}_2$ and leads to the unexplored topic of symmetric reflexive polyhedra.

Volker Braun's manifolds, with Hodge numbers $(2,2)$ are not the quotient of a CICY, but were found by an extension of the techniques used to analyse the $(1,4)$ three-generation manifold. These manifolds have Hodge numbers $(1,1)$ and are clearly very remarkable [132]. They are free quotients, by groups of order 24, of manifolds specified by a reflexive polyhedron, where the polyhedron in question is the 24-cell, a four dimensional regular and self-dual polyhedron with 24 vertices and 24 faces. The role of transposition of configurations remains an ill understood though intriguing phenomenon. It is known, and easy to see, that the transpose of a configuration matrix is again a configuration matrix of a Calabi–Yau threefold. The intriguing fact is that the transposes of interesting configurations are themselves interesting. The reader will already have noted that the covering space for the Yau manifold corresponds to the transpose of the split bicubic. The transpose of X_{2564}, above, is

$$X_{21} \;=\; \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix}\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 2 & 0 \\ 0 & 2 \\ 0 & 2 \end{bmatrix}^{19,19} \tag{5.6.7}$$

which is another of the avatars of the split bicubic and which admits freely acting groups of order 16 including $\mathbb{Z}_2{\times}\mathbb{Q}_8$.

Consider the split

$$\mathbb{P}^4[5]^{1,101}_{\chi=-200} \;\to\; \begin{matrix} \mathbb{P}^4 \\ \mathbb{P}^4 \end{matrix}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}^{2,52}_{\chi=-100}\,, \tag{5.6.8}$$

which has not been considered in the present chapter owing to the fact that the freely acting symmetries have $\mathbb{Z}_5$ as a subgroup rather than $\mathbb{Z}_4$ or $\mathbb{Z}_2{\times}\mathbb{Z}_2$. There are quintics that have $\mathbb{Z}_5{\times}\mathbb{Z}_5$ as a freely acting symmetry and members of the family, shown on the right that have freely acting symmetry $\mathbb{Z}_5{\times}\mathbb{Z}_2$. The quintic is notable, in part for the group $\mathbb{Z}_5{\times}\mathbb{Z}_5$ which is large. The largest free groups that act on smooth CICYs have order 32 and these act on just one space, $\mathbb{P}^7[2,2,2,2]$. Now the configuration on the right of the above split is a resolution of a quintic with 50 nodes. We may go to a singular limit in which the configuration on the right has a freely acting symmetry $\mathbb{Z}_5{\times}\mathbb{Z}_5$ and has 50 nodes. The singular variety can be resolved to give the Gross Popescu manifold $\text{GP}^{4,4}$, which as the notation suggests, has Hodge numbers $(4,4)$ and vanishing Euler number. The manifold $\text{GP}^{4,4}$ inherits the freely acting symmetry $\mathbb{Z}_5{\times}\mathbb{Z}_5$. The point being made here is that if the quintic and the manifold $GP^{4,4}$ are notable, then so is the half way house corresponding to the right hand side of the above split. The transpose of this manifold is the configuration

$$X_{7447} = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}^{5,45}_{\chi=-80} \qquad (5.6.9)$$

which is notable because it admits the large group $\mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and so leads to a manifold with the small Hodge numbers $(1, 3)$.

The tetraquadric and its transpose, the configuration $\mathbb{P}^7[2, 2, 2, 2]$, have free quotients of order up to 16 and 32 respectively. By allowing nodal limits of $\mathbb{P}^7[2, 2, 2, 2]$ it is possible to find freely acting symmetries of order 64. These nodal varieties can be resolved to give Gross-Popescu manifolds $GP^{2,2}$ that inherit the symmetry.

One last example: consider the transpose of one of the avatars of the manifold $X^{8,44}$.

$$X_{7240} = \begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^5 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^{3,39}_{\chi=-72} . \qquad (5.6.10)$$

Here one can find smooth manifolds that admit a freely acting $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry. However one can also find singular varieties with 36 nodes that admit a free action by $\mathbb{Z}_6 \times \mathbb{Z}_6$. These nodes may be resolved to give the Gross Popescu manifold $GP^{6,6}$, which inherits the free action by $\mathbb{Z}_6 \times \mathbb{Z}_6$.

The real test for these manifolds is whether they admit interesting holomorphic vector bundles. This question has been partly addressed in a recent ongoing program (see Refs. [7–9, 81, 83]) aiming to construct in a systematic manner large classes of holomorphic and poly-stable vector bundles, realised as monad bundles or sums of line bundles over CICY quotients, in a search for phenomenologically viable models. So far, these searches were limited to favourable CICY quotients with a small number (< 7) of Kähler parameters, but have already revealed hundreds of models with the correct gauge group, an exact MSSM spectrum and one or several pairs of Higgs doublets. Interestingly, the study undertaken in Ref. [14] reinforced the role played by manifolds with non-trivial fundamental group, and hence by CICY quotients, revealing a significant conflict between direct symmetry breaking approaches in heterotic compactifications and a realistic particle spectrum.

A parallel search for interesting vector bundles over the 16 manifolds from the Kreuzer-Skarke list exhibiting a non-trivial fundamental group was undertaken in Refs. [10, 144, 145]. The full extension of this work to hypersurfaces in toric varieties would require a systematic study of discrete quotients thereof, analogous to Braun's classification of CICY quotients.

6

$SO(10)$ GUTs from Heterotic Line Bundle Models

6.1 Introduction

Until this point of the thesis we have focused on mainly two aspects of heterotic string compactifications. Firstly, we have developed a way of constructing highly symmetric CY threefolds (chapter 3) and classified all global symmetries of a class of CY quotients (chapter 4). Secondly, we have developed tools to compute the topological properties of Hodge numbers of many of these manifolds (chapter 5). In the process we have developed a deeper understanding of the geometry and symmetries of CY threefolds. In fact understanding the embedding of certain CICYs in products of del Pezzo surfaces of degree 4 was one of the highlights of the previous chapter. Although the above are interesting in their own right, when seen from the broader context of heterotic model building, symmetries can play a decisive role in phenomenologically restricting string models. An apt example of such symmetries are the R symmetries of chapter 4. These can determine whether the proton is stable in a string derived MSSM. Understanding whether a string model has the right symmetries is crucial to its validity.

Heterotic Calabi-Yau models with $\mathcal{N} = 1$ supersymmetry are specified by a Calabi-Yau three-fold, X, and a vector bundle $V \to X$ with structure group contained in $E_8 \times E_8$. CY compactifications based on holomorphic deformations of the tangent bundle has been the subject of much research [70, 71] and has led to an exact

MSSM [72]. CY compactifications over non-standard embeddings are more recent. In fact, one of the most fruitful endeavours in heterotic model building in recent times has been the construction of line bundle models over smooth CY threefolds. A comprehensive classification of line bundle models leading to $SU(5)$ GUTs was completed in the papers [7–9]. This scan over all favorable non-simply connected CICYs with $h^{1,1} < 7$ led to many phenomenologically viable $SU(5)$ GUT models, the data for which is available at [89]. In this chapter we will summarise the model building tools that we use in our constructions of line bundle models that lead to the interesting GUT group $SO(10)$[1]. Low energy theories based on spontaneously broken $SO(10)$ gauge groups received tremendous boost upon the discovery that neutrinos are not massless [146]. In such theories not only does a right handed neutrino exist naturally, but each family of matter particles is unified under a single spin representation of $SO(10)$. Similar to the efforts in [7–9], we chose our base manifolds from the class of favourable CICYs with a non-trivial fundamental group and $h^{1,1} < 6$. We perform an automated scan of a large number of line bundle sums in an effort to determine viable bundles. We will return to the determination of the $SO(10)$ GUT spectrum and whether these GUT models lead to viable SM like models in a future publication. We will now proceed to describe our model building set-up generalising it sufficiently to include the case of rank 5 line bundle sums leading to $SU(5)$ GUTs that we will come across in the following chapter.

6.2 Model building setup

6.2.1 The Base Space: CICYs

Our choice of the base CY manifolds is the class of CICYs in keeping with the theme of this thesis. However, there are three more significant factors dictating this choice. Firstly, for building realistic string derived SMs, the GUT group (in our case $SO(10)$) needs to be broken to the SM gauge group. This process requires

[1]Although the actual GUT group is the Lie group Spin(10), the double cover of $SO(10)$, in literature such GUTs are referred to as $SO(10)$ GUTs.

non-simply connected CYs. The class of CICYs consists of simply connected CYs embedded in a product of projective spaces. However, their symmetries have been classified, and a huge repository of 1695 CICY quotients by freely acting discrete symmetries is available at [63]. The CICY quotients are non-simply connected and thus aid in the process of breaking the GUT group to the SM gauge group. Secondly, the global discrete symmetries of these CICY quotients were classified in chapter 4. If these symmetries survive in the low energy theory, they can test the validity of our line bundle models by imposing further phenomenological restrictions. Finally, since the CICYs are embedded in products of projective spaces, the cohomology of line bundles over them can be computed with relative ease and is very amenable to automation. In fact many of the CICYs are 'favourably' embedded, by which we mean that their entire second cohomology class descends from that of the ambient space. The cohomology computations are even simpler for these cases. In addition, a line bundle on such a CY threefold is the restriction of the line bundle on the corresponding ambient space.

Each rank 4 line bundle sum over a CY manifold X is characterised by 4 $h^{1,1}(X)$ number of integers, or alternately a matrix of dimension $h^{1,1}(X)\times 4$. An automated scan of such line bundle sums thus quickly becomes intractable with increasing $h^{1,1}(X)$. Therefore we restrict ourselves to CYs with small $h^{1,1}$. Therefore, in summary we choose to build our line bundle models over the class of favourable non-simply connected CICYs with small $h^{1,1}$. More precisely, we restrict ourselves to $h^{1,1} < 6$.

6.2.2 Heterotic Line Bundle $SO(10)$ and $SU(5)$ GUTs

As mentioned before, the structure group of V must be contained in $E_8\times E_8$. For a consistent heterotic string vacuum, the bundle V needs to satisfy two further conditions: it needs to obey the heterotic anomaly cancellation condition, and in order to preserve $\mathcal{N}=1$ supersymmetry, it needs to be poly-stable with slope zero. Since both conditions are crucial for the subsequent discussion we would now like to briefly review them in turn.

The slope of a coherent sheaf $\mathcal{F}$ is defined by the rational number:

$$\mu(\mathcal{F}) = \frac{1}{rk(\mathcal{F})} \int_X c_1(\mathcal{F}) \wedge J^2 \ , \tag{6.2.1}$$

where J is the Kähler form of the CY X. A direct sum bundle $V = V_1 \oplus \cdots \oplus V_n$ is called poly-stable if all summands V_i are slope-stable (see §2.2.3) and if they have the same slope, that is, $\mu(V_1) = \cdots = \mu(V_n) = \mu(V)$. In the present context we are interested in rank n line bundle sums

$$V = \bigoplus_{a=1}^{n} L_a \quad \text{satisfying} \quad c_1(V) = \sum_{a=1}^{n} c_1(L_a) \stackrel{!}{=} 0 \ , \tag{6.2.2}$$

which have a typical structure group $S(U(1)^n)$ [9]. In this chapter and the next, we will focus on rank four and rank five bundles respectively, so $n = 4$ or 5. For the case of $n = 5$, i.e., rank 5 line bundle sums, the structure group is indeed $S(U(1)^5)$ if one ensures in addition to (6.2.2), that each component of the line bundle sum is non-trivial and that the first Chern class of any sum of a proper subset of the 5 line bundles is non-zero, i.e., the line bundles are sufficiently generic and such accidental cancellations do not take place. In case they do, the structure group can be different e.g., $SO(5)$ or $Sp(5)$ (see Appendix A of [9] for a detailed analysis). Since we will impose similar conditions on the Chern classes of rank 4 line bundle sums, it is reasonable to assume that the structure group in that case will be $S(U(1)^4)$ when the line bundle sum is generic enough. We will postpone a detailed analysis of what other structure groups are possible for now, since the accidental cancellations, rare as they are, will not affect the qualitative nature of the rest of our analysis.

For $n = 4$, the structure group can be embedded into E_8 via the subgroup chain $S(U(1)^4) \subset SU(4) \subset E_8$ which leads to a low-energy GUT group $\mathrm{SO}(10) \times S(U(1)^4)$ and the discrete symmetry Γ of X. Similarly for $n = 5$, the structure group can be embedded into E_8 via the subgroup chain $S(U(1)^5) \subset SU(5) \subset E_8$ which leads to a low-energy GUT group $SU(5) \times S(U(1)^5)$ and Γ. The extra $U(1)$ factors are generically Green-Schwarz anomalous and hence the associated gauge bosons often acquire Stückelberg masses which are close to the compactification scale in magnitude. Quasi-realistic standard models can be obtained from these GUT models after dividing by a freely-acting discrete symmetry (in cases when X is simply-connected) and including a suitable Wilson line.

Due to the constraint on the rank of the sub-sheaf $\mathcal{F}$, line bundles are automatically slope-stable. All we have to require for a supersymmetric line bundle sum is, therefore, the vanishing of all slopes, that is

$$\mu(L_a) = c_1(L_a) \cdot J^2 \stackrel{!}{=} 0 \tag{6.2.3}$$

for all $a = 1, \ldots, n$. Note that, for fixed line bundles L_a, these are conditions on the Kähler form, J, of the Calabi-Yau manifold X. In practice, we have to check if the slopes of all line bundles can simultaneously vanish somewhere in (the interior of) the Kähler cone of X.

A slope poly-stable bundle V automatically satisfies a positivity condition on the second Chern class [147], which for $c_1(V) = 0$, is given by

$$\int_X c_2(V) \wedge J \geq 0 \,, \tag{6.2.4}$$

and is known as the Bogomolov bound. Here, J is any Kähler form for which V is poly-stable. For a line bundle sum (6.2.2) the second Chern class is given by

$$c_2(V) = -\frac{1}{2} \sum_{a=1}^{n} c_1(L_a)^2 \,. \tag{6.2.5}$$

In order to be able to satisfy the anomaly cancellation condition we require that

$$c_2(TX) - c_2(V) \in \text{Mori cone of } X \,. \tag{6.2.6}$$

Provided this condition is satisfied, we can always saturate the anomaly condition by adding a suitable hidden sector of five-branes (or a suitable bundle in the other, hidden E_8 factor or a combination of five branes and hidden bundle). In practice, it will be useful to introduce an integral basis C_i, where $i = 1, \ldots, h^{1,1}(X)$, of holomorphic curves for the second homology of X and a corresponding dual basis, J_i, of $H^{1,1}(X)$. Then, the Kähler form can be expanded as

$$J = \sum_{i=1}^{h^{1,1}(X)} t^i J_i \,, \tag{6.2.7}$$

where t^i are the Kähler moduli. For our examples, the Kähler cone of X is simply characterised by all $t^i > 0$ and the holomorphic curves C_i generate the Mori cone. The latter property means that the anomaly condition (6.2.6) can be re-written as

$$(c_2(TX) - c_2(V)) \cdot J_i \geq 0 \ , \tag{6.2.8}$$

for all $i = 1, \ldots, h^{1,1}(X)$. Another crucial quantity for our discussion is the number of chiral families, given by the index

$$N_{\text{gen}}(X) = -\text{ind}(V) = \frac{1}{2} \int_X c_3(V) \ , \tag{6.2.9}$$

where we have used $c_1(V) = 0$ in the last step. Equations (6.2.2), (6.2.4) and (6.2.8) constrain the first and the second Chern classes of V, while Eq. (6.2.3) guarantees the existence of points in the Kähler moduli space of X for which V is poly-stable with zero slope.

6.2.3 Equivariance

To ensure that the bundles we construct descend to bundles on the quotient CY X/Γ, one has to impose equivariance. Although we do not impose equivariance on our bundle constructions we intend to return to it in the future. Thus here, we simply define what imposing equivariance entails. For equivariance of a bundle $V \xrightarrow{\pi} X$, it is necessary that the automorphisms of X i.e., Γ should lift to the automorphisms of the bundle $V \xrightarrow{\pi} X$, i.e., $\forall g \in \Gamma$ there should exist bundle morphism $\phi_g : U \to U$ such that the following diagram commutes:

$$\begin{array}{ccccc} & V & \xrightarrow{\phi_g} & V & \\ \pi & \downarrow & & \downarrow & \pi \\ & X & \xrightarrow{g} & X & \end{array} \tag{6.2.10}$$

If in addition, the morphisms ϕ_g obey the 'cocycle' condition:

$$\phi_g \circ \phi_h = \phi_{gh} \ , \tag{6.2.11}$$

then V descends to a bundle $\widetilde{V}$ on X/Γ (see Appendix A of [8] for a detailed account).

6.3 Computer Scan Results

We have classified all rank 4 line bundle sums over non-simply connected smooth CICY quotients with $h^{1,1} < 6$ leading to $SO(10)$ GUT models, subject to the conditions of vanishing first Chern class (6.2.2), poly-stability (6.2.3), anomaly cancellation (6.2.8) and the correct chiral asymmetry, i.e., the N_{gen} in (6.2.9) to be $3|\Gamma|$, where $|\Gamma|$ is the order of the discrete symmetry group acting on the CY X. This is so since,

$$N_{\text{gen}}(X/\Gamma) = \frac{N_{\text{gen}}(X)}{|\Gamma|} \ . \tag{6.3.1}$$

Our automated scan was performed over a few months on the Hydra computing cluster. The results are tabulated in Tables 6.1, 6.2 and 6.5, interspersed with examples from our constructions. Due to our ongoing efforts in this classification, we are able to produce an example of a line bundle sum over a CICY quotient with $h^{1,1} = 6$.

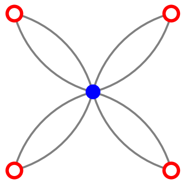
Table 6.1: *Number of line bundle models as a function of the maximum entry k_m of the line bundle sum on CICYs with $h^{1,1} = 2$ or 3. The CICY X is denoted by its index in the CICY list available at [63]. Total number of models: 48*

X, $\|\Gamma\|$	$k_{\rm m} = 1$	$k_{\rm m} = 2$	$k_{\rm m} = 3$	$k_{\rm m} = 4$	$k_{\rm m} = 5$	$k_{\rm m} = 6$	$k_{\rm m} = 7$	$k_{\rm m} = 8,\ 9$
7240, 3	0	0	2	3	3	3		
7484, 4	0	1	2	5	7	7	7	
7636, 2	0	1	1	1	1			
7647, 2	0	0	2	2	2			
7669, 3	0	3	7	8	8	8		
7669, 9	0	0	1	1	1			
7714, 4	0	0	0	0	2	2	2	
7735, 4	0	0	0	0	1	1	1	
7735, 8	0	0	0	1	4	4	5	5
7745, 4	0	0	0	0	1	1	1	
7745, 8	0	0	0	1	4	4	5	5
7788, 2	0	0	2	5	6	6	6	
7792, 2	0	0	2	5	6	6	6	

Example 1: A CICY $\subset (\mathbb{P}^1)^{\times 4}$ with $h^{1,1} = 4$: Tetraquadric

Consider the tetraquadric X_{7862} with an order 4 freely acting discrete symmetry $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\Gamma = \mathbb{Z}_4$ and the rank four line bundle V over it:

$$X_{7862} \;=\; \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \left[\begin{matrix} 2 \\ 2 \\ 2 \\ 2 \end{matrix} \right]^{4,68}_{-128}$$



$$V = \begin{bmatrix} -1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 3 & -6 & 4 & -1 \\ 4 & 2 & -1 & -5 \end{bmatrix} .$$

The triple intersection numbers of this manifold can be written as

$$d_{ijk} = \int_X J_i \wedge J_j \wedge J_k = \begin{cases} 2 & \text{if } i \neq j, j \neq k, i \neq k \\ 0 & \text{otherwise} \end{cases} . \tag{6.3.2}$$

It will be interesting to verify that the line bundle sum V satisfies the anomaly cancellation conditions and yields the correct number of generations:

$$\begin{aligned} c_2(V).J_i &= (12, 10, 22, 12) \; , \\ c_2(\text{TX}).J_i &= (24, 24, 24, 24) \; . \end{aligned} \tag{6.3.3}$$

V indeed satisfies the anomaly cancellation condition since $c_{2i}(V) \leq c_{2i}(TX)$ from above. The number of generations is given by the index of the bundle:

$$N_{\text{gen}}(X_{7862}/\Gamma) = \frac{N_{\text{gen}}(X_{7862})}{|\Gamma|} = \frac{-\text{ind}(V)}{|\Gamma|} = \frac{12}{4} = 3 \; . \tag{6.3.4}$$

The above is consistent with our expectation. Finally we should verify that V is slope stable, i.e., the $rk(V)$ number of equations $d_{ijk}\, k^i_a\, t^j\, t^k = 0$ are satisfied simultaneously in the interior of the Kähler cone, i.e.,

$$\begin{aligned} t^i > 0 \;\; \forall\, i \; , \\ 28t^1t^2 + 12t^3t^2 + 8t^4t^2 + 16t^1t^3 + 12t^1t^4 - 4t^3t^4 = 0 \; , \\ -16t^1t^2 + 4t^3t^2 - 28t^4t^2 + 12t^1t^3 - 20t^1t^4 = 0 \; , \\ 12t^1t^2 + 20t^4t^2 - 8t^1t^3 + 12t^1t^4 = 0 \; , \\ -24t^1t^2 - 16t^3t^2 - 20t^1t^3 - 4t^1t^4 + 4t^3t^4 = 0 \; . \end{aligned} \tag{6.3.5}$$

It is straightforward to verify (computaionally) that the above set of equations does yield a solution.

Table 6.2: *Number of line bundle models as a function of the maximum entry k_m of the line bundle sum on CICYs with $h^{1,1} = 4$. The CICY X is denoted by its index in the CICY list available at [63]. Total number of models: 44,415*

X, $\|\Gamma\|$	$k_m = 1$	$k_m = 4$	$k_m = 6$	$k_m = 7$	$k_m = 8$	$k_m = 9$	$k_m = 10$	$k_m = 11,\ 12$
6831, 2	0	143	158	161	163	164	164	164
7204, 2	0	1222	1584	1616	1618	1618	1618	
7218, 2	0	613	794	810	811	811	811	
7241, 2	0	613	794	810	811	811	811	
7245, 2	0	598	678	696	703	706	706	706
7247, 3	0	223	227	227	227			
7270, 2	0	1222	1584	1616	1618	1618	1618	
7403, 2	0	156	170	172	174	176	177	177
7435, 2	0	210	338	360	361	361	361	
7435, 4	0	424	609	624	633	635	635	635
7462, 2	0	629	1010	1076	1079	1079	1079	
7462, 4	0	1265	1819	1864	1891	1897	1897	1897
7468, 2	0	469	553	569	581	586	588	588
7491, 2	0	210	338	360	361	361	361	
7491, 4	0	424	609	624	633	635	635	635
7522, 2	0	629	1010	1076	1079	1079	1079	
7522, 4	0	1265	1819	1864	1891	1897	1897	1897

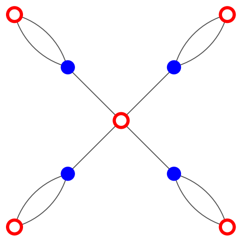
X, $\|\Gamma\|$	$k_m = 1$	$k_m = 6$	$k_m = 11$	$k_m = 15$	$k_m = 16$	$k_m = 17$	$k_m = 18$	$k_m = 19,\ 20$
6784, 2	0	2629	2903	2929	2929	2929		
6784, 4	0	4500	5300	5318	5318	5318		
6828, 2	0	1317	1454	1467	1467	1467		
6828, 4	0	2253	2653	2662	2662	2662		
7719, 2	0	4154	5014	5076	5076	5076		
7736, 2	0	2082	2512	2543	2543	2543		
7742, 2	0	2082	2512	2543	2543	2543		
7862, 2	0	1221	1425	1447	1448	1449	1450	1450
7862, 4	0	1920	2356	2408	2410	2410	2410	

X, $\|\Gamma\|$	$k_m = 1$	$k_m = 10$	$k_m = 20$	$k_m = 21$	$k_m = 22$	$k_m = 23$	$k_m = 32$	$k_m = 33,\ 34$
7862, 8	0	2172	2268	2270	2270	2270		
7862, 16	0	853	1067	1069	1071	1073	1082	1083

Example 2: A CICY $\subset (\mathbb{P}^1)^{\times 4}\times\mathbb{P}^3$ with $h^{1,1}=5$

Consider the CICY X_{6836} with an order 16 freely acting discrete symmetry $\Gamma = \mathbb{Z}_4\rtimes\mathbb{Z}_4$ or, $\Gamma=\mathbb{Z}_8\times\mathbb{Z}_2$ and the rank four line bundle V over it:

$$X_{6836} = \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^3\end{matrix}\left[\begin{matrix}2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&2\\1&1&1&1\end{matrix}\right]^{5,37}_{-64}$$



$$V = \left[\begin{matrix}-36&-1&9&28\\1&-5&4&0\\1&-1&-1&1\\1&0&0&-1\\-1&1&0&0\end{matrix}\right] .$$

The triple intersection numbers of this manifold can be expressed in terms of the following $h^{1,1}(X_{6836})$ matrices of dimension $h^{1,1}(X_{6836})\times$ $h^{1,1}(X_{6836})$ each:

$$d_{ijk} = \left\{ \left[\begin{matrix}0&0&0&0&0\\0&0&2&2&4\\0&2&0&2&4\\0&2&2&0&4\\0&4&4&4&8\end{matrix}\right], \left[\begin{matrix}0&0&2&2&4\\0&0&0&0&0\\2&0&0&2&4\\2&0&2&0&4\\4&0&4&4&8\end{matrix}\right], \left[\begin{matrix}0&2&0&2&4\\2&0&0&2&4\\0&0&0&0&0\\2&2&0&0&4\\4&4&0&4&8\end{matrix}\right], \left[\begin{matrix}0&2&2&0&4\\2&0&2&0&4\\2&2&0&0&4\\0&0&0&0&0\\4&4&4&0&8\end{matrix}\right], \left[\begin{matrix}0&4&4&4&8\\4&0&4&4&8\\4&4&0&4&8\\4&4&4&0&8\\8&8&8&8&16\end{matrix}\right] \right\} . \tag{6.3.6}$$

The second Chern classes of the bundle V and the tangent bundle TX are:

$$\begin{aligned} c_2(V).J_i &= (22,\ 24, -4, -98,\ 64)\ , \\ c_2(\mathrm{TX}).J_i &= (24,\ 24,\ 24,\ \ 24,\ 64)\ . \end{aligned} \tag{6.3.7}$$

V indeed satisfies the anomaly cancellation condition since $c_{2i}(V) \leq c_{2i}(TX)$. The number of generations is a function of the index of the bundle:

$$N_{\mathrm{gen}}(X_{6836}/\Gamma) = \frac{N_{\mathrm{gen}}(X_{6836})}{|\Gamma|} = \frac{-\mathrm{ind}(V)}{|\Gamma|} = \frac{48}{16} = 3\ . \tag{6.3.8}$$

The slope stability conditions for V i.e., $d_{ijk}\, k_a^i\, t^j\, t^k = 0$ and $t^i > 0$, turn out to be:

$$
\begin{aligned}
&t^i > 0 \;\forall\; i \;, \\
-280(t^5)^2 + 8t^1t^5 - 288t^2t^5 - 288t^3t^5 - 288t^4t^5 - 148t^2t^3 - 148t^2t^4 - 148t^3t^4 &= 0 \;, \\
-40(t^5)^2 - 32t^1t^5 - 32t^3t^5 - 40t^4t^5 + 4t^1t^2 - 12t^1t^3 + 4t^2t^3 - 16t^1t^4 - 16t^3t^4 &= 0 \;, \\
96(t^5)^2 + 24t^1t^5 + 64t^2t^5 + 104t^3t^5 + 96t^4t^5 - 4t^1t^2 + & \\
16t^1t^3 + 36t^2t^3 + 12t^1t^4 + 32t^2t^4 + 52t^3t^4 &= 0 \;, \\
224(t^5)^2 + 224t^2t^5 + 216t^3t^5 + 232t^4t^5 - 4t^1t^3 + 108t^2t^3 + & \\
4t^1t^4 + 116t^2t^4 + 112t^3t^4 &= 0 \;.
\end{aligned}
\tag{6.3.9}
$$

As in the previous example, it is easy to verify computationally that the above set of equations does yield a solution. This example also illustrates the intensive nature of our machine computation. Two of the entries in the line bundle sum are relatively large integers and thus convergence of our classification algorithm would take long.

Table 6.5: *Number of line bundle models as a function of the maximum entry k_m of the line bundle sum on CICYs with $h^{1,1} = 5$. The CICY X is denoted by its index in the CICY list available at [63]. Total number of models: 2,071*

X, $\|\Gamma\|$	$k_{\rm m} = 21$	$k_{\rm m} = 22$	$k_{\rm m} = 23$	$k_{\rm m} = 24$	$k_{\rm m} = 25$	$k_{\rm m} = 26,\ 27,\ 28$
5256, 2	4	8	8	8		
5301, 2	1	2	2	2		
6204, 2	24	41	50	55	56	57
6715, 4	4	8	12	12	12	
6732, 2	16	28	38	44	48	48
6770, 2	1	1	1			
6777, 2	16	28	38	44	48	48
6788, 4	4	8	12	12	12	
6802, 2	16	28	38	44	48	48
6834, 2	8	14	19	22	24	24
6836, 4	1	2	3	3	3	
6836, 8	9	12	15	18	18	18
6890, 2	32	56	76	88	96	96
6896, 2	8	14	19	22	24	24

Continued on next page

Table 6.5 – *Continued from previous page*

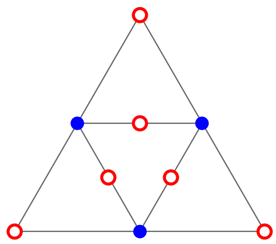
6927, 4	6	12	18	18	18	
6927, 8	54	72	90	108	108	108
6947, 4	1	2	3	3	3	
6947, 8	9	12	15	18	18	18

X, $\|\Gamma\|$	$k_{\rm m} = 28$	$k_{\rm m} = 29$	$k_{\rm m} = 30$	$k_{\rm m} = 31$	$k_{\rm m} = 32$	$k_{\rm m} = 33$	$k_{\rm m} = 34$	$k_{\rm m} = 35$
5256, 4	160	164	164	164				
5301, 4	40	41	41	41				
6225, 2	216	224	230	234	237	238	239	240
6724, 2	99	101	102	102	102			
6804, 2	240	252	260	268	273	277	279	280
6836, 16	69	73	77	81	85	89	90	91
6947, 16	69	73	77	81	85	89	90	91
7279, 2	197	202	204	206	207	208	208	208
7447, 4	9	9	9					
7447, 10	89	93	95	97	99	101	102	103
7447, 20	112	119	127	134	139	143	147	149
7487, 4	45	45	45					

Example 3: A CICY $\subset (\mathbb{P}^1)^{\times 6}$ with $h^{1,1} = 6$

Consider the CICY X_{5302} with an order 4 freely acting discrete symmetry $\Gamma = \mathbb{Z}_4$ or, $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ and the rank four line bundle V over it:

$$X_{5302} = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \left[\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right]^{6,30}_{-48}$$



$$V = \left[\begin{array}{cccc} -15 & 2 & 6 & 7 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \\ 1 & -1 & 0 & 0 \end{array} \right] .$$

The triple intersection numbers of this manifold is given by:

$$d_{ijk} = \int_X J_i \wedge J_j \wedge J_k = \begin{cases} 2 & \text{if } i \neq j, j \neq k, i \neq k \\ 0 & \text{otherwise} \end{cases} . \qquad (6.3.10)$$

V indeed satisfies the anomaly cancellation condition since $c_{2i}(V) \leq c_{2i}(TX)$ as can be deciphered from below:

$$c_2(V).J_i = (22, -4,\ 18,\ 12,\ 24, -24)\ ,$$
$$c_2(\mathrm{TX}).J_i = (24,\ 24,\ 24,\ 24,\ 24,\ \ 24)\ . \tag{6.3.11}$$

The number of generations on the quotient space is:

$$N_{\mathrm{gen}}(X_{5302}/\Gamma) = \frac{N_{\mathrm{gen}}(X_{5302})}{|\Gamma|} = \frac{-\mathrm{ind}(V)}{|\Gamma|} = \frac{12}{4} = 3\ . \tag{6.3.12}$$

The formulation of the slope stability conditions for V is as follows:

$$\begin{aligned}
&t^i > 0\ \forall\ i\ ,\\
&4t^1t^2 - 56t^3t^2 - 56t^4t^2 - 56t^5t^2 - 60t^6t^2 - 60t^3t^4 - 60t^3t^5 - 60t^4t^5 -\\
&\quad 4t^1t^6 - 64t^3t^6 - 64t^4t^6 - 64t^5t^6 = 0\ ,\\
&-16t^1t^2 - 4t^3t^2 - 8t^4t^2 - 4t^6t^2 - 4t^1t^4 + 8t^3t^4 + 4t^1t^5 + 16t^3t^5 + 12t^4t^5 +\\
&\quad 12t^3t^6 + 8t^4t^6 + 16t^5t^6 = 0\ ,\\
&4t^1t^2 + 24t^3t^2 + 28t^4t^2 + 28t^5t^2 + 28t^6t^2 - 4t^1t^3 + 20t^3t^4 + 20t^3t^5 + 24t^4t^5 +\\
&\quad 20t^3t^6 + 24t^4t^6 + 24t^5t^6 = 0\ ,\\
&8t^1t^2 + 36t^3t^2 + 36t^4t^2 + 28t^5t^2 + 36t^6t^2 + 4t^1t^3 + 4t^1t^4 + t^3t^4 - 4t^1t^5 +\\
&\quad 24t^3t^5 + 24t^4t^5 + 4t^1t^6 + 32t^3t^6 + 32t^4t^6 + 24t^5t^6 = 0\ .
\end{aligned} \tag{6.3.13}$$

As in the previous examples, one can computationally verify that the above set of equations does yield a solution.

6.4 Conclusions: The low energy theory

Consider a CY X with a known discrete symmetry group Γ and a vector bundle $V \to X$ defined over it. The possible SO(10) multiplets of the GUT theory over X are **16**, $\overline{\mathbf{16}}$, **10** and **1**. If the structure group of V i.e., $S(U(1)^4)$ embeds into E_8 with a Γ-equivariant structure, then the cohomologies corresponding to the matter multiplets would be $H^1(X,V)$, $H^1(X,V^*)$, $H^1(X,\wedge^2 V)$ and $H^1(X, V\bigotimes V^*)$ respectively [148]. Chiral asymmetry can be ensured by demanding $\mathrm{ind}(V) = -3|\Gamma|$. The absence of anti-families is ensured by imposing $h^1(X,V) = 0$. Finally the presence of a Higgs multiplet can be safeguarded by imposing $h^1(\wedge^2 V) > 0$ so that at least one **10** multiplet remains.

For the sake of completeness, and in preparation for the next chapter, the possible $SU(5)$ GUT multiplets (arising from taking the structure group of V to be $S(U(1)^5)$) are $\mathbf{10}$, $\overline{\mathbf{10}}$, $\overline{\mathbf{5}}$, $\mathbf{5}$ and $\mathbf{1}$ with the corresponding cohomologies $H^1(X,V)$, $H^1(X,V^*)$, $H^1(X,\wedge^2 V)$, $H^1(X,\wedge^2 V^*)$ and $H^1(X, V \bigotimes V^*)$. In this case the absence of anti-families is ensured by imposing $h^1(X,V^*) = 0$ and the presence of Higgs is safeguarded by the condition $h^1(X, V \bigotimes V^*) > 0$.

Returning to the case at hand i.e., the heterotic $SO(10)$ GUT: the GUT underlying the CY quotient X/Γ is determined as the quotient of the GUT underlying the CY X. Wilson line then breaks the GUT group to the SM gauge group with additional factors.

All of the above is well established. It would thus be natural to compute the matter spectrum corresponding to each triplet (X,Γ,V) of our classification. However, we first pause to note how the breaking of the $SO(10)$ GUT group to the SM gauge group could proceed. It was shown in [149], that the simplest Wilson line that can break the $SO(10)$ GUT group to the SM gauge group with an additional $\mathrm{U}(1)_{B-L}$ factor requires that $\pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$. It turns out that there is only a sole model in our classification that fits this description, namely,

$$(X_{7669},\Gamma,V) = \left(\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right]^{3,48}_{-90}, \quad \mathbb{Z}_3 \times \mathbb{Z}_3, \quad \left[\begin{array}{cccc} -1 & -1 & 0 & 2 \\ 0 & 0 & -2 & 2 \\ 1 & 1 & 1 & -3 \end{array} \right] \right). \quad (6.4.1)$$

Cohomology computations indicate that this yields an undesirable spectrum. Therefore before proceeding further, it is imperative to work out mechanisms for the breaking of the $SO(10)$ GUT group to the SM gauge group via discrete Wilson lines, using symmetries more general than the $\mathbb{Z}_3 \times \mathbb{Z}_3$ mentioned above. Note that there are more than 500 distinct group actions that act freely on the CICYs, many of them being non-Abelian, that one could explore for the above enterprise.

We will end this chapter noting that the blind automated scan of heterotic $SU(5)$ GUT models on CICYs yielded several quasi-realistic standard models. This motivated our classification efforts in this chapter. We have successfully constructed tens of thousands of potentially viable heterotic line bundle $SO(10)$ GUT models. Further work is now necessary to construct more models on CICYs with more suitable symmetries to carry this large computational project forward. In the following chapter we will focus on some heterotic $SU(5)$ GUT models, and derive hints about chiral asymmetry, i.e., the family problem.

7

A Finiteness Problem and Family Number in Heterotic Line Bundle Models

7.1 Introduction

String theory provides a natural explanation for the family replication observed in nature: Kaluza-Klein compactification from ten to four dimensions generically leads to low-energy multiplets appearing with a certain multiplicity which is governed by a topological number associated to the compactification [35, 150]. Despite the mathematical beauty of this mechanism and the qualitative solution of the family problem it provides, there is no immediate prediction for the number of families. Different compactifications lead to different family numbers and, although three families can be achieved by appropriate model building choices, a wide range of values can be obtained.

In this chapter, we would like to study the question of family number in string theory in the context of heterotic line bundle models, a class of models introduced in Refs. [7–9] and further developed in Refs. [10–17]; see also the earlier studies [18,19]. These models are based on Calabi-Yau compactifications of the heterotic string and vector bundles with Abelian structure group, that is, line bundle sums. As has been shown in Refs. [7–9], these models are very well under control from a model building point of view and large numbers of quasi-realistic models with a standard model spectrum can be obtained. For these reasons, heterotic line bundle models provide a useful setting to study the family number problem. One observation, made in Ref. [9], is of particular importance for this discussion. Based on extensive computer scans, it was noted that for a given Calabi-Yau threefold X, the class of Abelian bundles which correspond to $\mathcal{N} = 1$ supersymmetric vacua in the interior of the Kähler cone and lead to three chiral families is finite.

In Ref. [11], this finiteness result was shown analytically, based on two assumptions on the Kähler moduli of the theory: the Kähler moduli have to be sufficiently away from the boundary of the Kähler cone and the Calabi-Yau volume is finite[1]. The first of these assumptions stems from the requirement that supergravity remains valid at the chosen locus in moduli space and the second one is motivated by the finiteness of the four-dimensional couplings constants. Interestingly, the proof of this statement does not require imposing a fixed number of families - supersymmetry and anomaly cancellation together with the aforementioned constraints on the Kähler moduli space are sufficient. This suggests that the class of bundles that lead to consistent $\mathcal{N} = 1$ heterotic vacua with an arbitrary number of chiral families is also finite, in line with the conjecture made in Ref. [151].

Provided this finiteness result holds, it is clear that there exists an upper bound on the number of families for a given Calabi-Yau manifold. What is more, this upper bound depends on and monotonically increases with the Calabi-Yau volume. Roughly, the more stringent the bound on the Calabi-Yau volume the fewer line bundle models can be supersymmetric in the so-prescribed portion of Kähler moduli space and the lower the bound on the number of families. This relation between the maximal number of families and the Calabi-Yau volume is rather surprising and hints at a possible explanation of the family problem: The number of families is small because the four-dimensional coupling constants have sizeable, finite values which require a relatively small Calabi-Yau volume.

The aim of the present chapter is to establish this connection between the Calabi-Yau volume and the number of families for heterotic line bundle models in detail.

[1] An infinite class of heterotic line bundle models has been constructed in a recent paper [17], by including the boundary of the Kähler cone. In the present chapter, the Kähler moduli are always constrained to the interior of the Kähler cone, so we do not encounter such infinite families of models.

We reviewed the basic model-building set-up in Section 6.2. We will discuss the coupling constants in §7.2. In §7.3, we revisit the proof presented in Ref. [11] and derive a semi-analytical formula for the maximal number of families. §7.4 presents an explicit construction of heterotic line bundle models with low-energy gauge group $SU(5)$ and a variable number of families and determines the maximal number of generations as a function of the Calabi-Yau volume. We conclude in Section 7.5. In the following we would like to provide evidence that the class of line bundle models on a given Calabi-Yau manifold, subject to these conditions on the first and second Chern classes is finite. Of course, this implies the existence of an upper bound on N_{gen} for a given Calabi-Yau manifold. We will prove finiteness of the class after excluding a finite neighbourhood at the boundary of the Kähler moduli space and requiring the Calabi-Yau volume to be finite. These restrictions are motivated by the validity of the supergravity approximation and the finiteness of the four-dimensional couplings, as mentioned earlier. On the other hand, the results of our automated scans indicate that the class remains finite even without these restrictions on the Kähler moduli space.

7.2 Kähler moduli space and low-energy coupling constants

For a supersymmetric line bundle sum, the slope zero conditions (6.2.3) must have a common solution in the Kähler cone of the Calabi-Yau manifold X. However, from a physical point of view, the acceptable locus in the Kähler cone is further restricted by the values of low-energy coupling constants and the requirement that the supergravity approximation be consistent. We would like to discuss the interplay between those physical restrictions in Kähler moduli space and the slope zero conditions. Unification of gauge couplings, including the gravitational coupling, in the heterotic string is most naturally realised in the strong-coupling limit [152], described by 11-dimensional Horava-Witten theory [153]. For this reason, we will be working in the 11-dimensional theory and measure all internal volumes using the relevant part of the 11-dimensional metric.

The 11-dimensional Newton constant κ_{11} and the 11-dimensional Planck length l are related by $4\pi\kappa_{11} = (2\pi l)^9$. We also introduce the six-dimensional coordinate volume $v = (2\pi l)^6$. The dimensionless Kähler moduli t^i and the triple intersection numbers are defined by

$$t^i = \frac{1}{(2\pi l)^2}\int_{C_i} J \ , \qquad d_{ijk} = \frac{1}{v}\int_X J_i \wedge J_j \wedge J_k \ , \tag{7.2.1}$$

where J is the Kähler form associated to the 11-dimensional metric. Hence, the Kähler moduli t^i measure the volume of the holomorphic cycles C_i in units of the 11-dimensional Planck length. As usual, we introduce the pre-potential

$$\kappa = 6\mathcal{V} = d_{ijk}t^i t^j t^k \ , \tag{7.2.2}$$

where $\mathcal{V}$ is the volume in units of the coordinate volume v, so that the physical volume is given by

$$V_{\text{phys}} = \frac{1}{3!}\int_X J \wedge J \wedge J = v\mathcal{V} \ . \tag{7.2.3}$$

The four-dimensional GUT coupling constant can then be written as [152]

$$\alpha_{\text{GUT}} = \frac{(2\pi l)^6}{2V_{\text{phys}}} = \frac{1}{2\mathcal{V}} \ . \tag{7.2.4}$$

We are now ready to discuss the relevant physical restrictions on the Kähler moduli space. Validity of the supergravity approximation requires the volume of all cycles C_i to be larger than one in 11-dimensional Planck units which implies restricting the Kähler moduli as

$$t^i \overset{!}{>} 1 \quad \text{for all} \quad i = 1, \ldots, h^{1,1}(X) \ . \tag{7.2.5}$$

Further, in order to match the GUT coupling constant we should require that the value

$$\mathcal{V} = \frac{1}{6}d_{ijk}\, t^i\, t^j\, t^k \overset{!}{\simeq} \frac{1}{2\alpha_{\text{GUT}}} \simeq 12 \tag{7.2.6}$$

can be realised in the restricted Kähler moduli space defined by Eqs. (7.2.5) intersected with the locus obtained by imposing the slope-zero conditions (6.2.3). Note that in (7.2.6) we have used $\alpha_{\text{GUT}} \simeq 1/25$ (see e.g., [154]). If this intersection region is non-empty, it is in fact unbounded, since the slope-zero conditions (6.2.3) are homogeneous in the Kähler moduli t^i and are, therefore, satisfied on rays (indicated by the red lines in Fig. 1) in Kähler moduli space. Accordingly, the Calabi-Yau volume assumes arbitrarily large values over this region, while being bounded from below. Thus, if the minimal value attained by the Calabi-Yau volume is larger than the unification value, the model is ruled out.

In practice, this means that we can search for models with realistic gauge couplings by restricting the Kähler moduli such that $\mathcal{V} \leq \mathcal{V}_{\rm max}$. From Eq. (7.2.6), the physically relevant value of $V_{\rm max}$ should be approximately 12. However, later on we will allow a wider range for $V_{\rm max}$ values in order to study the dependence of the maximal number of generations on the value of the gauge coupling. The situation in Kähler moduli space for two Kähler moduli is schematically indicated in Fig. 1.

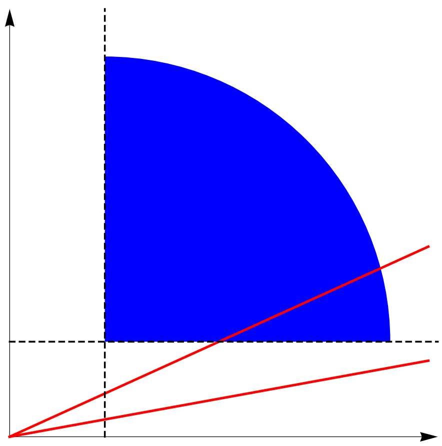


Figure 1: *Sketch of the allowed region in Kähler moduli space. The slope-zero equations are invariant under a change $t^i \to a\, t^i$, hence a line bundle sum which is consistent at some point, is consistent along the entire ray containing that point.*

The Kähler cone corresponds to the positive quadrant and the blue region indicates the physically allowed region, which is detached from the boundaries of the Kähler cone due to the Eqs. (7.2.5) and is bounded[2] due to the condition $V \leq V_{\rm max}$. The

[2]A finite volume may not necessarily bound the Kähler moduli space if some intersection numbers are negative. Here we assume that the region in Kähler moduli space is indeed bounded by the finite volume requirement. This is certainly the case if all intersection numbers are positive as will be the case for our examples.

rays defined by the slope-zero conditions (6.2.3) are indicated by the red lines in Fig. 1. A given line bundle model is supersymmetric and consistent with the physical restrictions on the Kähler moduli space if the corresponding ray intersects the blue region in Fig. 1. In the next section we will show that this class of supersymmetric, physical line bundle sums is finite.

7.3 A semi-analytic bound

We write the line bundles in Eq. (6.2.2) as $L_a = \mathcal{O}_X(\mathbf{k}_a)$, so that their first Chern class is given by $c_1(L_a) = k_a^i J_i$. The integers k_a^i are constrained by the conditions

$$\sum_{a=1}^{n} k_a^i = 0 \tag{7.3.1}$$

for all $i = 1, \ldots, h^{1,1}(X)$ which are equivalent to $c_1(V) = 0$. Further, from Eq. (6.2.5), the second Chern class is given by

$$c_2(V) = -\frac{1}{2} \sum_{a=1}^{n} d_{ijk}\, k_a^i\, k_a^j \ , \tag{7.3.2}$$

and in order to be able to satisfy the anomaly cancellation condition we require that

$$c_{2i}(V) \le c_{2i}(TX) \ , \tag{7.3.3}$$

in accordance with Eq. (6.2.8). The slope zero conditions (6.2.3) take the form

$$d_{ijk}\, k_a^i\, t^j\, t^k = 0 \ , \tag{7.3.4}$$

for $i = 1, \ldots, h^{1,1}(X)$, and these equations have to be simultaneously satisfied in the interior of the Kähler cone (here taken to be characterised by $t^i > 0$ for all i). The question we would like to address is whether line bundle sums V on a given Calabi-Yau manifold X, subject to the $c_1(V) = 0$ constraint (7.3.1), the anomaly constraint (7.3.3) and the slope zero conditions (7.3.4) constitute a finite class. If they do, the number of generations

$$N_{\rm gen} = -{\rm ind}(V) = -\frac{1}{6}\, d_{ijl} \sum_{a=1}^{n} k_a^i\, k_a^j\, k_a^l \tag{7.3.5}$$

for this class will also be finite and we are more specifically interested in any bounds on this number. The automated scans described in the next section indicate that the answer is in the affirmative, although it seems difficult to provide a general proof and derive a precise expression for the bound on N_{gen}.

However, we would like to provide an analytical finiteness argument under the assumption that the slope conditions (7.3.4) are satisfied in the *physical* part of Kähler moduli space (as discussed in the previous section). First, we recall that the Kähler moduli space is equipped with a positive-definite metric [56]

$$G_{ij} = \frac{1}{2v\mathcal{V}} \int_X J_i \wedge \star J_j = -3 \left(\frac{\kappa_{ij}}{\kappa} - \frac{2\kappa_i \kappa_j}{3\kappa^2} \right) , \tag{7.3.6}$$

where $\kappa_i = d_{ijk}\, t^j\, t^k$ and $\kappa_{ij} = d_{ijk}\, t^k$. Due to the slope zero conditions (7.3.4), which can also be written as $\kappa_i k_a^i = 0$, we obtain

$$0 < \sum_a \mathbf{k}_a^T G\, \mathbf{k}_a = -\frac{3}{\kappa} d_{ijk} \sum_a k_a^i\, k_a^j\, t^k = \frac{6}{\kappa}\, t^i\, c_{2i}(V) \leq \frac{6}{\kappa} t^i\, c_{2i}(TX) \leq \frac{6}{\kappa} |\mathbf{t}||c_2(TX)| \, . \tag{7.3.7}$$

Introducing the modified moduli space metric $\widetilde{G} = \kappa\, G/(6|\mathbf{t}|)$ (which is homogeneous of degree zero in t^i) this means that

$$\sum_a \mathbf{k}_a^T \widetilde{G}\, \mathbf{k}_a \leq |c_{2i}(TX)| \, . \tag{7.3.8}$$

Since $\widetilde{G}$ is positive definite in the (interior of the) Kähler cone, Eq. (7.3.8) seems to imply the existence of a bound on $|\mathbf{k}_a|$, and hence a bound on N_{gen}. However, the Kähler metric can degenerate on the boundary of the Kähler cone which means that, in the interior of the Kähler cone, the eigenvalues of $\widetilde{G}$ cannot be bounded from below by a strictly positive number. Hence, if we allow solutions in the entire interior of the Kähler cone, Eq. (7.3.8) does not provide an argument for finiteness. The situation improves when we strengthen our assumptions and demand a solution in the *physical* region of Kähler moduli space, as defined in the previous section and schematically indicated in Fig. 1. In this case, the eigenvalues of $\widetilde{G}$ are bounded from below by the minimal (but strictly positive) eigenvalue $\lambda_{\text{min}} > 0$ of $\widetilde{G}$ over this physical region and the line bundle integers are bounded by

$$\sum_a |\mathbf{k}_a|^2 \leq \frac{|c_2(TX)|}{\lambda_{\text{min}}} \, . \tag{7.3.9}$$

Note that the value of $\lambda_{\rm min}$ depends on the topology of the Calabi-Yau manifold (notable on the triple-intersection numbers) and the maximal volume $\mathcal{V}_{\rm max}$ used to define the physical region. For a given Calabi-Yau manifold with an upper bound on the volume, $\lambda_{\rm min}$ can be determined, although we do not have an explicit analytic formula. Taking into account the constraint $c_1(V) = 0$ and assuming that the k_a^i do not vanish for all a and a given i, Eq. (7.3.9) ensures that the line bundle entries are bounded by a value $|k_{\rm max}|$ that can be determined. Since $t^i > 1$, we have $-k_a^i < |k_{\rm max}|\, t^i$ and

$$\sum_a d_{ijk}\,(k_a^i + |k_{\rm max}|t^i)\,(k_a^j + |k_{\rm max}|t^j)\,(k_a^k + |k_{\rm max}|t^k) > 0 \tag{7.3.10}$$

which, given the slope zero conditions (7.3.4), becomes

$$N_{\rm gen} < rk(V)|k_{\rm max}|^3\,\mathcal{V} - |k_{\rm max}|c_{2i}(V)t^i\ . \tag{7.3.11}$$

Given a finite volume $\mathcal{V} < \mathcal{V}_{\rm max}$, strict positivity of $\lambda_{\rm min}$ over the physical part of Kähler moduli space and the positivity condition (6.2.4), this does indeed provide an upper bound for the number of generations on a given Calabi-Yau manifold.

7.4 Computer scan results

In this section we present the results of an automated scan performed on several different Calabi-Yau three-folds for $SU(5)-$models with a variable number of generations.[3] As we will see, these results provide evidence for the finiteness of line bundle models with vanishing slope in the interior of the Kähler cone and lead to an upper bound for the number of generations. We impose the constraints (7.3.1), (7.3.3), and (7.3.4) and, in addition, we will require that, for each pair of indices $a < b$, $\text{ind}(L_a \otimes L_b) \leq 0$, a condition necessary in order to project out all Higgs triplets after the inclusion of a Wilson line.

[3]Frequency distributions of models with various numbers of generations have been performed within other string contexts, such as orientifolds of Gepner models [155, 156], intersecting D-brane models [157], heterotic constructions with free-fermions [158, 159], heterotic Calabi-Yau compactifications with monad bundles [82] and heterotic orbifold constructions [160]

Moreover, it would be both interesting and important to use these results in order to find all line bundle models with vanishing slope in the physical region of Kähler moduli space, for varying Calabi-Yau volume $\mathcal{V}$, and to determine an upper bound for the number of generations as a function of $\mathcal{V}$. However, in general this is not an easy task, as explained below.

Finding the locus where the line bundle sum is poly-stable corresponds to simultaneously solving the quadratic equations (7.3.4) in the t^i variables. This makes the process described above difficult to implement. However, things greatly simplify in the case in which the locus defined by the slope zero conditions is a ray, which corresponds to having the number of linearly independent $\mathbf{k}_a$ vectors equal to $h^{1,1}(X) - 1$, i.e.,

$$\text{rank}(k_a^i) = h^{1,1}(X) - 1 \ . \tag{7.4.1}$$

In this case, after finding an arbitrary non-trivial solution to the slope zero equations (e.g. using the routine `FindInstance` in Mathematica), the minimal value assumed by the Calabi-Yau volume in the region of interest is obtained by rescaling the solution such that all the t_i parameters are greater than or equal to 1.

For the tetra-quadric manifold discussed below most of the line bundle models are such that the poly-stability locus is a single ray. Excluding the small number of models which do not satisfy the condition (7.4.1) does not influence the qualitative conclusions of our discussion.

7.4.1 The tetra-quadric manifold

Tetra-quadric manifolds are simply connected hypersurfaces in a product of four $\mathbb{P}^1$ spaces, defined as the zero locus of a homogeneous polynomial that is quadratic in the homogeneous coordinates of each $\mathbb{P}^1$ space. Manifolds in this class have Euler number $\chi = -128$ and Hodge numbers $h^{1,1}(X) = 4$ and $h^{2,1}(X) = 68$. This information is summarised by the following configuration matrix:

$$X \ = \ \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \left[\begin{matrix} 2 \\ 2 \\ 2 \\ 2 \end{matrix} \right]^{4,68}_{-128} \tag{7.4.2}$$

The second cohomology is spanned by the four Kähler forms, J_i of the $\mathbb{P}^1$ factors, pulled-back to the tetra-quadric, and we also introduce a dual basis, ν^i of the fourth cohomology. The triple intersection numbers have the following simple form

$$d_{ijk} = \int_X J_i \wedge J_j \wedge J_k = \begin{cases} 2 & \text{if } i \neq j, j \neq k, i \neq k \\ 0 & \text{otherwise} \end{cases} \tag{7.4.3}$$

and they lead to the pre-potential

$$\kappa = 6\mathcal{V} = 12\,(t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4)\ . \tag{7.4.4}$$

The Kähler cone is characterised by all $t^i > 0$ while the Mori cone corresponds to positive linear combinations of the ν^i. The second Chern class of the tangent bundle of the tetra-quadric, in the basis ν^i, is given by

$$c_2(TX) = (24, 24, 24, 24)\ . \tag{7.4.5}$$

The tetraquadric manifold admits smooth quotients by finite groups Γ of orders $|\Gamma| = 2, 4, 8$ and 16 (see [3]):
$\Gamma = \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_8,\ \mathbb{Q}_8,\ \mathbb{Z}_4 \times \mathbb{Z}_4,\ \mathbb{Z}_4 \rtimes \mathbb{Z}_4,\ \mathbb{Z}_8 \times \mathbb{Z}_2,\ \mathbb{Z}_8 \rtimes \mathbb{Z}_2,\ \mathbb{Q}_8 \times \mathbb{Z}_2\ .$

The physical model with standard model group and three chiral families is defined on the quotient manifold X/Γ (provided the bundle $V \to X$ descends to the quotient) while, in practice, we calculate the chiral asymmetry for the model on X. The two chiral asymmetries of upstairs and downstairs model are related by

$$N_{\text{gen}}(X/\Gamma) = \frac{N_{\text{gen}}(X)}{|\Gamma|} \stackrel{!}{=} 3\ . \tag{7.4.6}$$

Hence, three chiral families in the downstairs model require

$$N_{\text{gen}}(X) = 6, 12, 24, 48 \tag{7.4.7}$$

families in the upstairs model, depending on which symmetry Γ is involved. Also, the value imposed on the volume from Eq. (7.2.6) should be considered in the downstairs model, hence for the upstairs model we require $\mathcal{V} \simeq 12\,|\Gamma|$.
A total of $\sim 10^{40}$ rank five line bundle sums were analysed in [9] and all models which satisfy the anomaly condition (7.3.3) and the slope zero conditions (7.3.4) in the

interior of the Kähler cone were extracted. This has been done for increasing sizes of the line bundle integers $|k_a^i|$, until no further models could be found. In practice, this means that all consistent rank five line bundle models on the tetra-quadric were found. Within this set, the generation number does not exceed the value $N_{\rm gen} = 126$ and the distribution of the generation number is shown in Figure 2. We can further focus on the sub-sets of these models which satisfy the slope zero condition in the physical region of Kähler moduli space, corresponding to a certain maximal Calabi-Yau volume $\mathcal{V}_{\rm max}$. The distribution of the generation number for these sub-sets is indicated by the colour-coding in Figure 2. In particular, for the unification value $\mathcal{V} \simeq 12\,|\Gamma|$ from Eq. (7.2.6), it can be seen from Figure 3 that $N_{\rm gen} \lesssim 40$ which is well in line with the required upstairs values in Eq. (7.4.7).

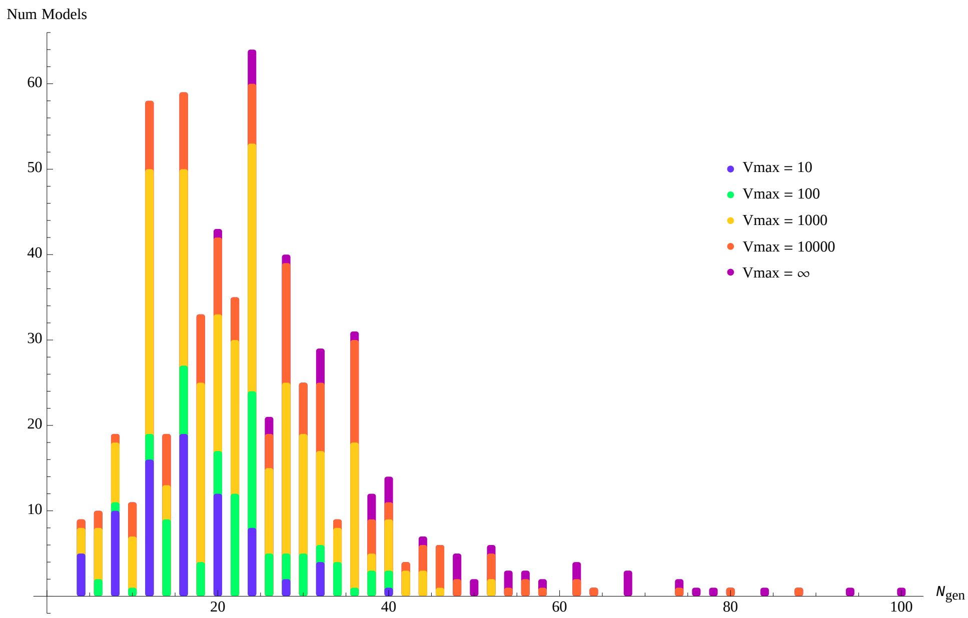


Figure 2: *Plot of the number of $SU(5)$ line bundle models on the tetra-quadric manifold as a function of the number of generations, $N_{gen}(X)$. Different colours correspond to different values of $\mathcal{V}_{\rm max}$. Note that the manifold does not admit any models with an odd number of generations, due to the fact that all intersection numbers d_{ijk} are divisible by 2. However, smooth quotients of X do admit models with an odd number of generations, in particular 3.*

The condition that the bundle $V \to X$ descends to a bundle on the quotient manifold X/Γ can be checked by ensuring that V admits an equivariant structure with respect to the Γ-action in question. In general, this check will reduce the

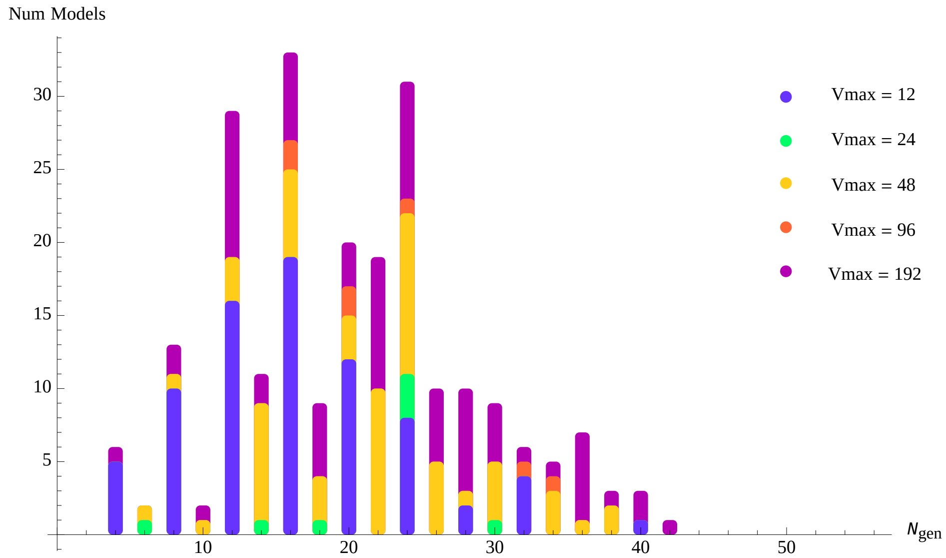


Figure 3: *Plot of the number of $SU(5)$ line bundle models on the tetra-quadric manifold as a function of the number of generations, $N_{gen}(X)$. Different colours correspond to different values of $\mathcal{V}_{\max}$.*

number of viable models. It would be interesting to see how equivariance affects the distribution of models according to the number of generations. For the case $\Gamma = \mathbb{Z}_m$, there are no obstructions to equivariance, and hence all line bundle sums will descend to the quotient manifold [83].

A non-trivial situation arises when, e.g. $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$, see Figure 4. As the figure shows, equivariance rules out a number of models. It is interesting to note that the peak of the distribution shifts to a lower value of N_{gen}, incidentally 3 for the case in consideration.

7.4.2 Manifolds with $h^{1,1}(X) = 5$

It is interesting to study the distribution of generation numbers for manifolds with different values for $h^{1,1}(X)$. In this section we present the results for two complete intersection Calabi-Yau manifolds with $h^{1,1}(X) = 5$. However, we will not be able to carry out the dependence of the upper bound for the number of generations on the Calabi-Yau volume, due to the technical difficulty of the problem, as explained above.

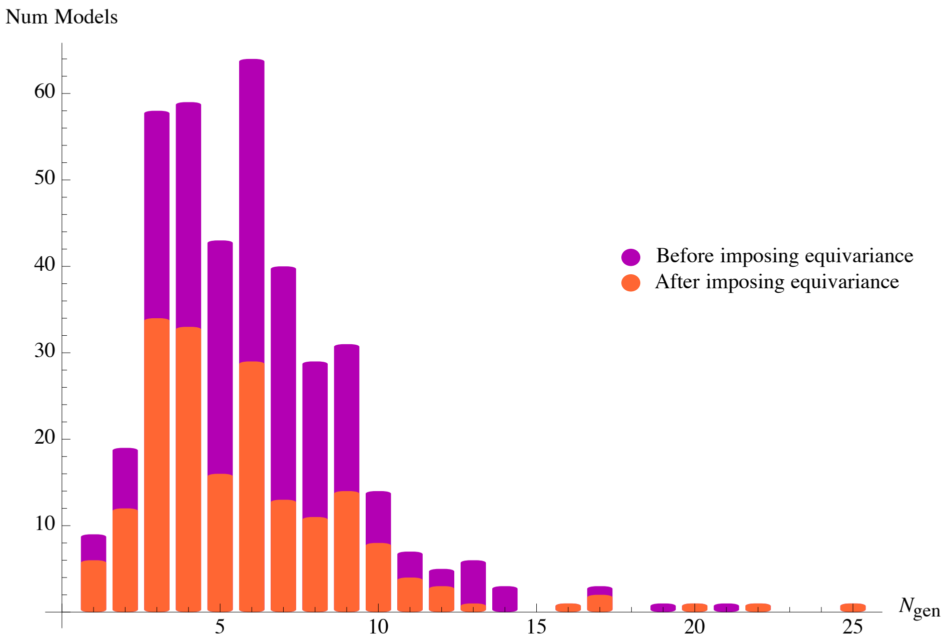


Figure 4: *Plot of the number of $SU(5)$ line bundle models on the $\mathbb{Z}_2 \times \mathbb{Z}_2$-quotient of the tetra-quadric manifold as a function of the number of generations, $N_{gen}(X)$, before and after imposing equivariance.*

The first manifold is defined by the following configuration matrix:

$$X = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{array} \left[\begin{array}{cccc} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{array} \right]^{5,37}_{-64} . \tag{7.4.8}$$

For this manifold the upper bound on the number of generations is $N_{\text{gen}} \lesssim 90$, with the distribution peaking in the range $5 - 25$ generations. Since the manifold admits quotients by finite groups of orders $2, 4, 8$ and 16, the distributions for the corresponding quotients peak at considerably lower values, consistent with $N_{\text{gen}} = 3$.

The second manifold is defined by the following configuration matrix:

$$X = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^4 \\ \mathbb{P}^4 \end{array} \left[\begin{array}{cccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{array} \right]^{5,37}_{-64} . \tag{7.4.9}$$

The distribution of models as a function of the number of generations is presented in

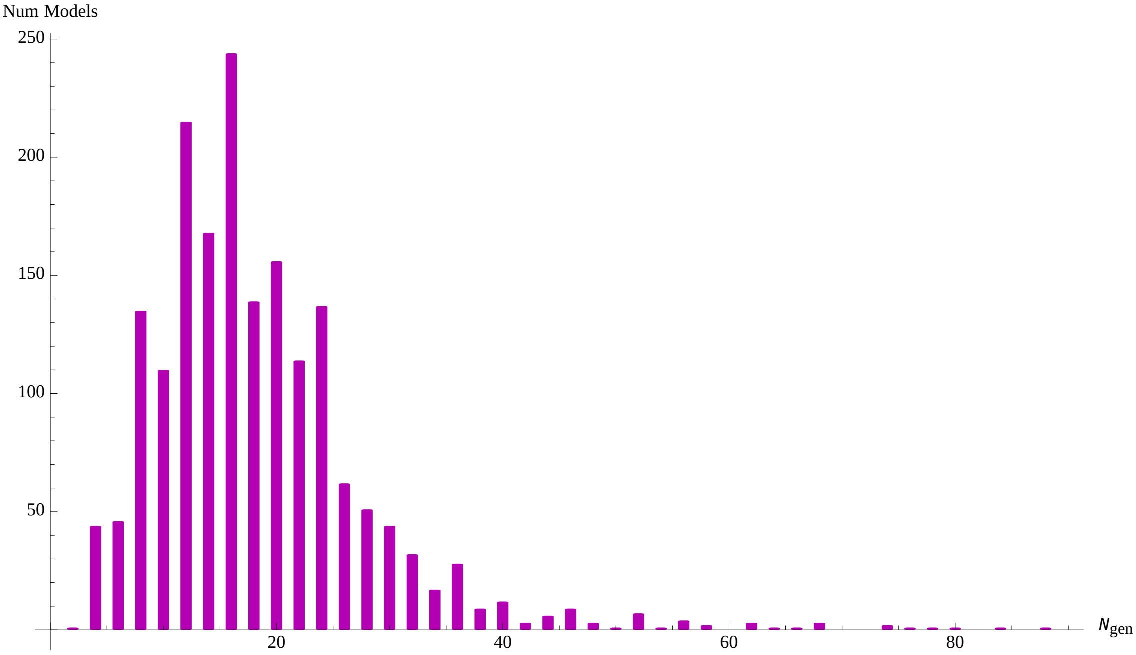


Figure 5: *Plot of the number of $SU(5)$ line bundle models on the manifold (7.4.8) as a function of the number of generations, $N_{gen}(X)$. The manifold does not admit line bundle models with an odd number of generations, since all its intersection numbers are divisible by 2.*

Figure 6. The manifold admits a quotient by $\mathbb{Z}_2$, with Hodge numbers $h^{1,1}(X) = 4$ and $h^{2,1}(X) = 20$ (see [100] for the computation of the Hodge numbers).

7.5 Conclusions

In this note, we have provided evidence for the existence of an upper bound on the number of low-energy fermion generations resulting from Calabi-Yau compactifications of the heterotic string with Abelian bundles. Line bundle sums V lead to consistent models provided $c_1(V) = 0$ (to allow for an embedding of the structure group into E_8), the anomaly condition $c_2(V) \leq c_2(TX)$ is satisfied and the slope zero conditions for all line bundles in V have a common solution in Kähler moduli space. We have formulated two versions of the last condition. In the first, mathematical version all slope zero conditions have to be satisfied in the interior of the Kähler cone. In the second, physical, version all slope zero conditions have to be satisfied in the "physical" region of Kähler moduli space. By this we mean

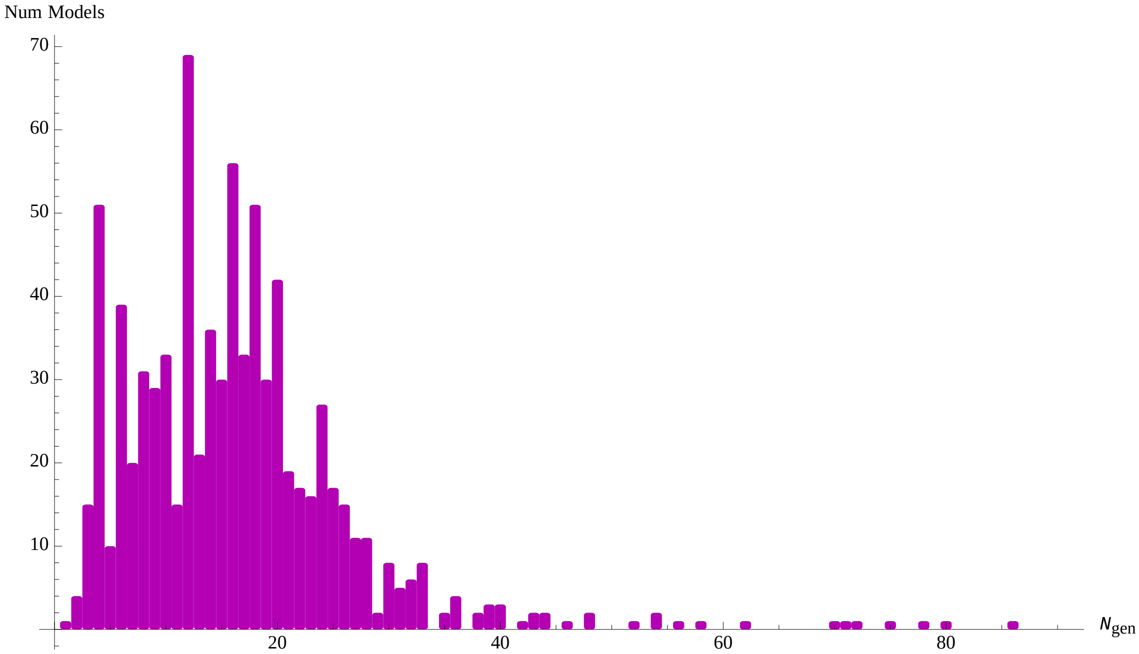


Figure 6: *Plot of the number of* $SU(5)$ *line bundle models on the manifold* (7.4.9) *as a function of the number of generations,* $N_{gen}(X)$.

a region away from the boundaries of the Kähler cone (so that the supergravity approximation is valid) and consistent with a finite Calabi-Yau volume $\mathcal{V}$, in order to account for the value of the low-energy gauge couplings.

For the second, physical, version of the slope zero condition, we have shown semi-analytically that the number of consistent line-bundle models on a given Calabi-Yau manifold must be finite. In particular, this means that the number of generations, N_{gen}, is finite and is subject to an upper bound which depends on the Calabi-Yau manifold and the maximum value of the volume $\mathcal{V}$. This suggests a possible correlation between the observed number of generations and the value of the gauge coupling constants: The sizeable value of the GUT coupling constant implies a relatively small Calabi-Yau volume which, in turn, leads to a stringent constraint from the zero-slope conditions and, hence, to a tight upper bound on the number of families.

Using computer scans, we have explicitly analysed this relation for two Calabi-Yau manifolds: the tetra-quadric Calabi-Yau manifold with $h^{1,1}(X) = 4$ and a couple of other complete intersection Calabi-Yau manifolds with $h^{1,1}(X) = 5$. In both cases we have found that the number of models is finite even when the weaker, mathematical version of the slope zero conditions is used. In the context of the stronger, physical version of the slope zero conditions, we have also determined the maximal number of generations as a function of the maximal volume. In agreement with general expectations, a small Calabi-Yau volume, $\mathcal{V} \simeq 12$, as required to account for the physical value of the GUT gauge coupling, implies a stringent bound of $N_{\text{gen}} \lesssim 40$ for the tetra-quadric, with peaks of the relevant distributions at significantly lower values. These numbers still have to be divided by the order of freely-acting symmetries to obtain the physical number of generations and are, therefore, easily consistent with three generations.

8
Concluding Remarks

In this thesis, we have addressed a number of issues related to CY manifolds and heterotic string compactifications over them. In the first part of the thesis, we primarily dealt with isometries of CY threefolds. A classification of the discrete symmetries of the class of CICYs was completed recently. However, a number of open questions cropped up as a result. One of the most important questions from a phenomenological point of view, was whether the CICY quotients by freely acting symmetries, had any remnant global discrete symmetries? Are any such symmetries R symmetries? Is it possible to construct highly symmetric CY quotients by restricting to special loci in the moduli space?

We addressed these questions in chapters 3 and 4 starting with the quintic. The quintic quotient $\mathbb{P}^4[5]/\mathbb{Z}_5 \times \mathbb{Z}_5$ is known to have a global $\mathbb{Z}_2$ symmetry. We constructed highly symmetric quintic quotients $\mathbb{P}^4[5]/\mathbb{Z}_5 \times \mathbb{Z}_5$ with symmetry groups $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Q}_8$, $\mathbb{Z}_{10}$, Dic_3 and Dic_5 each containing the global $\mathbb{Z}_2$ above. To our knowledge, quintic quotient families with these symmetries were constructed for the first time. Three of these families were found to be smooth, and the remaining were found to be singular. It would be interesting to assess the nature of singularities found in these classes in a future work, although preliminary investigations suggest singularities of the conifold type exist in two of the symmetric quintic families above. During the process of constructing highly symmetric quintic quotients, we realised that our group theoretical approach has tremendous potential in understanding the symmetries of other CY quotients. We thus undertook the classification of the global discrete symmetries of all CICY quotients.

In particle phenomenology, both regular and R discrete symmetries play an important role. In superstrings, such discrete symmetries cannot simply be conjectured, but must be discovered as isometries of the compactification space. In chapter 4, we modified our approach of the previous chapter, to compute the discrete global symmetries of all CICY quotients by freely acting groups. This led to a plethora of results and was one of the most significant undertakings of this thesis. We considered CICY quotients $Y = X/G_{\rm f}$, obtained as quotients of CICY manifolds X by freely-acting symmetries $G_{\rm f}$, and studied the symmetries $G_{\rm Y}$, freely and non-freely acting, of these quotients. Such symmetries $G_{\rm Y}$ can lead to discrete gauge symmetries in low-energy theories obtained by string compactification on Y and are, therefore, of phenomenological relevance. More specifically, we focused on those $G_{\rm Y}$ which are symmetries everywhere in the moduli space of the quotient Y. Only those "generic" symmetries can lead to low-energy symmetries which are manifest for all values of the moduli fields, rather than just for special values.
The experience so far, particularly from the classification of freely-acting symmetries, suggests that such generic symmetries of CY manifolds typically do not exist. Put simply, CY manifolds are too complicated to display symmetries at a generic point in moduli space - only at lower-dimensional sub-loci in moduli space do symmetries appear. However, this expectation is derived from the study of (freely-acting) symmetries for CY manifolds with a non-trivial first fundamental group. The main result of chapter 4 is that the situation is quite different for CY quotient manifolds and non-freely acting symmetries. Our classification strongly suggests that generic, non-freely acting symmetries for CY quotients arise relatively frequently. For the 1695 CICY quotients $Y = X/G_{\rm f}$ which can be constructed from the CICY manifolds X in the standard list [117] and freely-acting symmetries $G_{\rm f}$ as classified in [3] we find such generic, non-freely acting symmetries on about 23% of these quotient manifolds. This figure should, for example, be compared with the frequency

of freely-acting symmetries for CICY manifolds which, from the classification of [3], stands at about 2.5%, but with each of these symmetries appearing only at non-generic points in moduli space.

CY quotient manifolds are the preferred compactification manifolds for realistic model building in the context of the heterotic string. Hence, our results suggest that low-energy discrete symmetries which originate from the compactification space are a common occurrence for heterotic string models.

On the 381 CICY quotients Y with non-trivial generic symmetry group, we find that 9 different symmetry groups $G_{\rm Y}$ can arise, namely

$$G_{\rm Y} \in \left\{\mathbb{Z}_2,\ \mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{Z}_2^2,\ \mathbb{Z}_2^3,\ \mathbb{D}_8,\ \mathbb{Z}_2^4,\ \mathbb{Z}_2\times\mathbb{D}_8,\ (\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_2\right\}\ .$$

The largest symmetry group that we obtained was over the well-studied manifold of Yau (X_{14} below) over which three generation models have been constructed [77, 78, 112–114].

$$X_{14} = \begin{matrix}\mathbb{P}^1\\ \mathbb{P}^2\\ \mathbb{P}^2\end{matrix}\begin{bmatrix}1 & 1\\ 3 & 0\\ 0 & 3\end{bmatrix}^{19,19}_{\chi=0}\ . \tag{8.0.1}$$

On 4 of the 36 free $\mathbb{Z}_3\times\mathbb{Z}_3$ quotients of this manifold, we found $\mathbb{Z}_3^2\rtimes\mathbb{Z}_2$ global symmetries.

For 113 of the 1695 CICY quotients, all or part of $G_{\rm Y}$ corresponds to an R-symmetry, for the others $G_{\rm Y}$ is a regular symmetry. Although both regular and R-symmetries are phenomenologically important, not a lot of discrete R-symmetry candidates are known. These symmetries are often conjectured in the low energy supersymmetric theory to stabilise the proton. Thus our results of global R symmetries can prove to be phenomenologically impactful.

There are several obvious extensions of this work. We classified symmetries $G_{\rm Y}$ which leave the CICY quotients Y invariant for each choice of complex structure. This means that resulting low-energy discrete symmetries will act trivially on the complex structure moduli. It is also possible to consider symmetries which map between manifolds Y corresponding to different choices of complex structure, leading to low-energy symmetries with a non-trivial action on the complex structure moduli. We expect that such symmetries can be found by methods similar to the ones described in chapter 4, subject to a suitable modification of the invariance condition (4.3.17) for the defining polynomials. Another possible extension would be to find non-generic symmetries which only arise at a sub-locus in the complex structure moduli space of a CICY quotient. Since the present method heavily relies on the invariance of the *entire* family of polynomials describing the quotient CICY, finding such non-generic symmetries will likely require a different set of methods, possibly a modification of the approach taken in [3]. For the specific case of the quintic CY, this was achieved in chapter 3. It would also be interesting to know if results similar to the present ones arise for free quotients of CY manifolds constructed as hyper-surfaces in toric four-folds. However, this requires a classification of freely-acting symmetries for these CY manifolds which, to date, has been achieved only partially [161, 162]. Finally, the symmetries found in this chapter may be of direct relevance for the heterotic line bundle standard models on CICY quotients found in [7, 8]. It would be interesting to analyse this in more detail and, in particular, check if some of the present symmetries lift to the gauge bundle.

In chapter 5, we computed the Hodge numbers of a large number of CICY quotients. Our efforts completed the task of computing Hodge numbers of such manifolds apart from some $\mathbb{Z}_2$ quotients, which were later completed in [101]. This task involved employing the polynomial deformation method and the counting of invariant Kähler classes. In many cases, we discovered that the CICY threefolds were embedded in a product of del Pezzo surfaces of various degrees. We focussed our attention on those embedded in dP_4 surfaces or its product with either $\mathbb{P}^n$'s or additional dP_4 surfaces. In those examples we studied the exceptional lines of the dP_4 in vivid detail. In addition we worked out how the different isometries of the threefold acted on such lines, enabling us to compute the invariant Kähler classes. This entire process yielded results that were verified by the equivariant cohomology computations later undertaken in [101], leading to a completion of the program of computing Hodge numbers of all known smooth CICY quotients.

In the later half of the thesis, we took to the business of model building over the class of CICYs with known symmetries. $SO(10)$ GUTs have gained traction since the discovery that neutrinos have mass. Therefore string derived Standard Models via $SO(10)$ GUT breaking are extremely promising. In chapter 6, we took an algorithmic approach to constructing $SO(10)$ GUT line bundle models over CICYs with small Picard numbers, and found a very large number of stable rank 4 Abelian line bundle models. Our efforts thus far simply mark the beginning of the program of finding large classes of string derived $SO(10)$ GUTs over smooth CYs. Further work is needed to construct quasi-realistic models over CYs with suitable isometries, that lead to $SO(10)$ GUTs.

In chapter 7, we used the existing $SU(5)$ GUT line bundle data on CICY quotients to understand the generation problem: why are there three generations of matter particles found in nature? We also investigated the relationship between the volume of a CY and the number of generations of particles observed in nature. This exercise can be repeated over a large class of CY manifolds admitting suitable $SU(5)$ and $SO(10)$ GUT line bundles to extract further insights.

Our forays into different aspects of the geometry of string compactifications over smooth CY manifolds has been exciting, addressing several open questions and throwing up even more. By exploring CY geometries and their symmetries, we have strengthened string theory's attempts at reproducing low energy descriptions of nature. We remain hopeful that it will be possible to shed light on the question of whether string theory is the correct description of nature at all length scales, by exploring further string geometries.

Appendices

A
Appendix

A.1 Automorphism Groups

Below we compute $\text{Aut}(\mathbb{Z}_5 \times \mathbb{Z}_5)$ which exemplifies a key component in our classification efforts in §4.3.2 of chapter 4.

A.1.1 Automorphism group of $\mathbb{Z}_5 \times \mathbb{Z}_5$

Consider the group $G{=}\mathbb{Z}_5 \times \mathbb{Z}_5$ with the presentation $\langle S, T \mid S^5{=}T^5{=}S.T.S^{-1}.T^{-1}{=}e \rangle$. Being Abelian, any element of G can be expressed as $S^k T^l$ with $k, l \in Z_5$. Any homomorphism ψ from G to itself can be defined uniquely by specifying its action on the generators S and T. Consider the following map:

$$\begin{aligned} &\psi : G \to G \\ \psi(S) = S^\alpha.T^\beta \;\text{ and }\; \psi(T) &= S^\gamma.T^\delta, \text{ where } \alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{Z}_5 \end{aligned} \tag{A.1.1}$$

It is easy to see that this map is homomorphic, since G is Abelian. For this map to be bijective however, the integers α, β, γ and δ cannot be arbitrary.

Claim: The map defined in (A.1.1) is bijective iff $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \in \text{GL}(2, \mathbb{Z}_5)$

Proof: For injectivity we demand that $\psi(g) = \psi(h)$ implies $g = h$. Taking

$g = S^k.T^l$ and $h = S^m.T^n$ and using (A.1.1)

$$\begin{aligned}
\psi(S^k.T^l) &= \psi(S^m.T^n) \\
\text{i.e.,} \quad \psi(S^{k-m}.T^{l-n}) &= e \\
\text{i.e.,} \quad S^{\alpha(k-m)+\gamma(l-n)}.T^{\beta(k-m)+\delta(l-n)} &= e \\
\text{i.e.,} \quad \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}_{2\times 2} . \begin{bmatrix} k-m \\ l-n \end{bmatrix}_{2\times 1} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned} \tag{A.1.2}$$

Thus ψ is injective iff $k = m$ and $l = n$ is the only solution to the above. This is true iff the determinant $\begin{vmatrix} \alpha & \gamma \\ \beta & \delta \end{vmatrix} \neq 0$, i.e., iff $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Z}_5)$. We will see now that surjectivity implies the same condition. The homomorphism ψ is surjective iff $\psi^{-1}(S^k.T^l) \in G$ for any $k, l \in \mathbb{Z}_5$. Assume that $S^k.T^l$ has a preimage $S^m.T^n$ under ψ. This implies

$$\begin{aligned}
\psi(S^m.T^n) &= S^k.T^l \\
\text{i.e.,} \quad S^{\alpha m+\gamma n}.T^{\beta m+\delta n} &= S^k.T^l \\
\text{i.e.,} \quad \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}_{2\times 2} . \begin{bmatrix} m \\ n \end{bmatrix}_{2\times 1} &= \begin{bmatrix} k \\ l \end{bmatrix}
\end{aligned} \tag{A.1.3}$$

Therefore $\begin{bmatrix} m \\ n \end{bmatrix}$ exists for general $\begin{bmatrix} k \\ l \end{bmatrix}$ iff the determinant $\begin{vmatrix} \alpha & \gamma \\ \beta & \delta \end{vmatrix} \neq 0$, i.e., iff $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Z}_5)$, yielding the result:

$$\mathrm{Aut}(\mathbb{Z}_5 \times \mathbb{Z}_5) = \mathrm{GL}(2, \mathbb{Z}_5) \tag{A.1.4}$$

□

We tabulate below, the automorphism groups of the freely acting groups G_f that act on CICYs to yield smooth quotients. These automorphism groups $\mathrm{Aut}(G_f)$, play a central role in the cause of our classification of global discrete symmetries of CY quotients in chapter 4.

Table A.1: Automorphism groups $\text{Aut}(G_\text{f})$ of groups G_f acting freely on CICY manifolds, computed using GAP. For convenience, we also list the GAP identifiers for all groups, a pair of two numbers, the first of which represents the group order. For some groups with large order, the complete identifier or the structure description of the automorphism group was not available.

#	$\mathbf{G_f}$	GAP ID	$\mathbf{Aut(G_f)}$	GAP ID
1	$\mathbb{Z}_2$	[2, 1]	$\mathbb{I}$	[1, 1]
2	$\mathbb{Z}_3$	[3, 1]	$\mathbb{Z}_2$	[2, 1]
3	$\mathbb{Z}_4$	[4, 1]	$\mathbb{Z}_2$	[2, 1]
4	$\mathbb{Z}_2\times\mathbb{Z}_2$	[4, 2]	$\mathbb{S}_3$	[6, 1]
5	$\mathbb{Z}_5$	[5, 1]	$\mathbb{Z}_4$	[4, 1]
6	$\mathbb{Z}_6$	[6, 2]	$\mathbb{Z}_2$	[2, 1]
7	$\mathbb{Z}_8$	[8, 1]	$\mathbb{Z}_2\times\mathbb{Z}_2$	[4, 2]
8	$\mathbb{Z}_4\times\mathbb{Z}_2$	[8, 2]	$\mathbb{D}_8$	[8, 3]
9	$\mathbb{Z}_2^3$	[8, 5]	PSL(3,2)	[168, 42]
10	$\mathbb{Q}_8$	[8,4]	S_4	[24, 12]
11	$\mathbb{Z}_3\times\mathbb{Z}_3$	[9, 2]	GL(2,3)	[48, 29]
12	$\mathbb{Z}_{10}$	[10, 2]	$\mathbb{Z}_4$	[4, 1]
13	Dic_3	[12, 1]	$\mathbb{D}_{12}$	[12, 4]
14	$\mathbb{Z}_{12}$	[12, 2]	$\mathbb{Z}_2\times\mathbb{Z}_2$	[4, 2]
15	$\mathbb{Z}_4\times\mathbb{Z}_4$	[16, 2]	$(\mathbb{Z}_2^2\times\mathbb{A}_4)\rtimes\mathbb{Z}_2$	[96, 195]
16	$\mathbb{Z}_8\times\mathbb{Z}_2$	[16, 5]	$\mathbb{Z}_2\times\mathbb{D}_8$	[16, 11]
17	$\mathbb{Z}_4\times\mathbb{Z}_2^2$	[16, 10]	$[((\mathbb{Z}_2\times\mathbb{D}_8)\rtimes\mathbb{Z}_2)\rtimes\mathbb{Z}_3]\rtimes\mathbb{Z}_2$	[192, 1493]
18	$\mathbb{Z}_4\rtimes\mathbb{Z}_4$	[16, 4]	$\mathbb{Z}_2^4\rtimes\mathbb{Z}_2$	[32, 27]
19	$\mathbb{Z}_8\rtimes\mathbb{Z}_2$	[16, 6]	$\mathbb{Z}_2\times\mathbb{D}_8$	[16, 11]
20	$\mathbb{Z}_2\times\mathbb{Q}_8$	[16, 12]	$((\mathbb{Z}_2^4\rtimes\mathbb{Z}_3)\rtimes\mathbb{Z}_2)\rtimes\mathbb{Z}_2$	[192, 955]
21	$\mathbb{Z}_{10}\times\mathbb{Z}_2$	[20, 5]	$\mathbb{Z}_4\times\mathbb{S}_3$	[24, 5]
22	$\mathbb{Z}_5\times\mathbb{Z}_5$	[25, 2]	GL(2,5)	[480, 218]
23	$(\mathbb{Z}_4\times\mathbb{Z}_2)\rtimes\mathbb{Z}_4$	[32, 2]	$(\mathbb{Z}_2\times\mathbb{Z}_2\times(\mathbb{Z}_2^4\rtimes\mathbb{Z}_3))\rtimes\mathbb{Z}_2$	[384, 20100]
24	$\mathbb{Z}_8\rtimes\mathbb{Z}_4$	[32, 4]	$[\mathbb{Z}_2\times(((\mathbb{Z}_4\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2)\rtimes\mathbb{Z}_2)]\rtimes\mathbb{Z}_2$	[128, 753]
25	$(\mathbb{Z}_8\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2$	[32, 5]	$\mathbb{Z}_2\times(\mathbb{Z}_2^4\rtimes\mathbb{Z}_2)$	[64, 202]

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#	$\mathbf{G_f}$	GAP ID	$\mathbf{Aut(G_f)}$	GAP ID
26	$\mathbb{Z}_8 \rtimes \mathbb{Z}_4$	[32, 13]	$(\mathbb{Z}_2^3 \times \mathbb{D}_8) \rtimes \mathbb{Z}_2$	[128, 1735]
27	$\mathbb{Z}_2 \times (\mathbb{Z}_4 \rtimes \mathbb{Z}_4)$	[32, 23]	$[((\mathbb{Z}_2 \times \mathbb{Z}_2 \times ((\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2] \rtimes \mathbb{Z}_2$	[512, *]
28	$\mathbb{Z}_4 \rtimes \mathbb{Q}_8$	[32, 35]	$[(\mathbb{Z}_2 \times ((((\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2)) \rtimes \mathbb{Z}_2] \rtimes \mathbb{Z}_2$	[512, *]
29	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Q}_8$	[32, 47]	$\|G\| = 9216$	[9216, *]
30	$\mathbb{Z}_8 \times \mathbb{Z}_4$	[32, 3]	$[\mathbb{Z}_2 \times (((\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2)] \rtimes \mathbb{Z}_2$	[128, 753]
31	$\mathbb{Z}_4^2 \times \mathbb{Z}_2$	[32, 21]	$[((\mathbb{Z}_2^2 \times (\mathbb{Z}_2^4 \rtimes \mathbb{Z}_2)) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3] \rtimes \mathbb{Z}_2$	[1536, *]

A.2 Other $\mathbb{Z}_2$-quotients

In this appendix we list the Hodge numbers for the $\mathbb{Z}_2-$quotients of manifolds that have not been discussed in chapter 5. According to [3], the CICY list contains 166 manifolds that admit smooth quotients by $\mathbb{Z}_2$. Out of these, 46 have been presented in the main part of the text in chapter 5. From the remaining 120, 36 correspond to favourable embeddings, and hence one can compute the Hodge numbers for the resulting quotients by counting Kähler parameters.

A.2.1 Favourable embeddings

For each of the 36 manifolds mentioned above, we list below the position in the CICY list, the configuration matrix and the Hodge numbers for the $\mathbb{Z}_2-$quotients.

CICY #	Configuration Matrix	$(h^{1,1}, h^{2,1})$
4109	$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \left[\begin{array}{ccccc} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]^{8,40}_{-64}$	$(5, 15)$

Continued on next page

CICY #	Configuration Matrix	$(h^{1,1},\, h^{2,1})$
5273	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}^{6,30}_{-48}$	$(5, 17)$
5425	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}^{6,30}_{-48}$	$(5, 17)$
5958	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^4 \\ \mathbb{P}^4 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}^{6,32}_{-52}$	$(5, 18)$
6204	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}^{5,33}_{-56}$	$(4, 18)$
6225	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}^{5,33}_{-56}$	$(4, 18)$
6724	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^4 \\ \mathbb{P}^4 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}^{5,37}_{-64}$	$(4, 20)$
6732	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^5 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^{5,37}_{-64}$	$(5, 21)$

Continued on next page

CICY #	Configuration Matrix	$(h^{1,1}, h^{2,1})$
6738	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}^{6,38}_{-64}$	$(5, 21)$
6770	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}^{5,37}_{-64}$	$(5, 21)$
6777	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{5,37}_{-64}$	$(5, 21)$
6802	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^4 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}^{5,37}_{-64}$	$(5, 21)$
6804	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}^{5,37}_{-64}$	$(4, 20)$
6831	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}^{4,36}_{-64}$	$(3, 19)$
6834	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{5,37}_{-64}$	$(5, 21)$
6890	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^4 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}^{5,37}_{-64}$	$(5, 21)$

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CICY #	Configuration Matrix	$(h^{1,1}, h^{2,1})$
6896	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^5\end{matrix}\begin{bmatrix}1&1&0&0&0&0\\0&0&1&1&0&0\\0&0&0&0&1&1\\0&0&0&0&1&1\\1&1&1&1&1&1\end{bmatrix}^{5,37}_{-64}$	$(5, 21)$
7204	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^4\end{matrix}\begin{bmatrix}1&1&0&0\\0&0&2&0\\2&0&0&0\\1&1&1&2\end{bmatrix}^{4,40}_{-72}$	$(4, 22)$
7218	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^5\end{matrix}\begin{bmatrix}1&1&0&0&0\\0&0&1&1&0\\0&0&1&1&0\\1&1&1&1&2\end{bmatrix}^{4,40}_{-72}$	$(4, 22)$
7241	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^4\end{matrix}\begin{bmatrix}2&0&0&0\\0&1&1&0\\0&1&1&0\\1&1&1&2\end{bmatrix}^{4,40}_{-72}$	$(4, 22)$
7245	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^2\\\mathbb{P}^2\end{matrix}\begin{bmatrix}2&0&0\\0&0&2\\1&1&1\\1&1&1\end{bmatrix}^{4,40}_{-72}$	$(3, 21)$
7270	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^5\end{matrix}\begin{bmatrix}1&1&0&0&0\\0&0&1&1&0\\0&0&2&0&0\\1&1&1&1&2\end{bmatrix}^{4,40}_{-72}$	$(4, 22)$
7279	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^3\\\mathbb{P}^3\end{matrix}\begin{bmatrix}1&1&0&0&0&0\\0&0&1&1&0&0\\0&0&0&0&2&0\\1&0&1&0&1&1\\0&1&0&1&1&1\end{bmatrix}^{5,41}_{-72}$	$(4, 22)$
7403	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^4\\\mathbb{P}^4\end{matrix}\begin{bmatrix}1&1&0&0&0&0&0\\0&0&1&1&0&0&0\\1&0&1&0&1&1&1\\0&1&0&1&1&1&1\end{bmatrix}^{4,42}_{-76}$	$(3, 22)$
7450	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^5\end{matrix}\begin{bmatrix}0&0&1&1\\0&0&0&2\\2&2&1&1\end{bmatrix}^{3,43}_{-80}$	$(3, 23)$
7468	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^3\\\mathbb{P}^3\end{matrix}\begin{bmatrix}1&1&0&0&0\\0&0&2&0&0\\1&0&1&1&1\\0&1&1&1&1\end{bmatrix}^{4,44}_{-80}$	$(3, 23)$

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CICY #	Configuration Matrix	$(h^{1,1}, h^{2,1})$
7481	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^5\end{matrix}\begin{bmatrix}0&0&1&1\\0&0&1&1\\2&2&1&1\end{bmatrix}^{3,43}_{-80}$	$(3,23)$
7636	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^4\\\mathbb{P}^4\end{matrix}\begin{bmatrix}1&1&0&0&0&0\\1&0&1&1&1&1\\0&1&1&1&1&1\end{bmatrix}^{3,47}_{-88}$	$(2,24)$
7647	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^3\\\mathbb{P}^3\end{matrix}\begin{bmatrix}0&0&0&2\\1&1&1&1\\1&1&1&1\end{bmatrix}^{3,47}_{-88}$	$(2,24)$
7719	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^4\end{matrix}\begin{bmatrix}1&1&0&0\\0&0&0&2\\0&0&2&0\\1&1&2&1\end{bmatrix}^{4,52}_{-96}$	$(4,28)$
7736	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^3\end{matrix}\begin{bmatrix}0&0&2\\2&0&0\\0&2&0\\1&2&1\end{bmatrix}^{4,52}_{-96}$	$(4,28)$
7742	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^5\end{matrix}\begin{bmatrix}1&1&0&0&0\\0&0&1&1&0\\0&0&0&0&2\\1&1&1&1&2\end{bmatrix}^{4,52}_{-96}$	$(4,28)$
7761	$\begin{matrix}\mathbb{P}^4\\\mathbb{P}^4\end{matrix}\begin{bmatrix}1&1&1&1&1\\1&1&1&1&1\end{bmatrix}^{2,52}_{-100}$	$(1,26)$
7788	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^5\end{matrix}\begin{bmatrix}1&1&0&0\\0&0&2&0\\1&1&2&2\end{bmatrix}^{3,55}_{-104}$	$(3,29)$
7792	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^4\end{matrix}\begin{bmatrix}0&2&0\\2&0&0\\2&1&2\end{bmatrix}^{3,55}_{-104}$	$(3,29)$
7822	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^5\end{matrix}\begin{bmatrix}0&0&2\\2&2&2\end{bmatrix}^{2,58}_{-112}$	$(2,30)$

A.2.2 Hodge numbers obtained by counting complex structure parameters

Apart from the manifolds discussed in the main body of chapter 5, the CICY list contains 17 manifolds that admit smooth $\mathbb{Z}_2-$quotients, and for which the polynomial deformation method is applicable.

CICY #	Configuration Matrix	$(h^{1,1}, h^{2,1})$
4	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\end{matrix}\left[\begin{matrix}1&1&0&0&0\\0&0&1&0&1\\1&0&0&1&0\\0&1&0&0&1\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&1\\2&0&0&0&0\end{matrix}\right]^{15,15}_{0}$	$(9,9)$
5	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\end{matrix}\left[\begin{matrix}1&1&0&0&0&0\\0&0&1&0&0&1\\0&0&0&1&1&0\\1&0&0&0&0&1\\0&1&1&0&0&0\\0&0&1&0&1&0\\0&0&0&1&0&1\\1&0&0&1&0&0\\0&1&0&0&1&0\end{matrix}\right]^{15,15}_{0}$	$(9,9)$
6	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\end{matrix}\left[\begin{matrix}1&1&0&0&0&0\\0&0&1&0&0&1\\0&0&0&1&1&0\\1&0&0&0&0&1\\0&1&1&0&0&0\\0&0&1&0&1&0\\0&0&0&1&0&1\\0&1&0&1&0&0\\1&0&0&0&1&0\end{matrix}\right]^{15,15}_{0}$	$(9,9)$
90	$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^2\\\mathbb{P}^2\\\mathbb{P}^2\\\mathbb{P}^2\end{matrix}\left[\begin{matrix}1&1&0&0&0&0&0&0&0\\0&0&1&0&1&0&0&0&0\\0&0&0&1&1&0&0&0&0\\0&0&0&0&2&0&0&0&0\\0&0&0&0&0&1&1&0&1\\0&0&0&0&0&1&0&1&1\\1&0&0&1&0&0&1&0&0\\0&1&1&0&0&0&0&1&0\end{matrix}\right]^{13,17}_{-8}$	$(9,11)$

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CICY #	**Configuration Matrix**	$(h^{1,1},\, h^{2,1})$
261	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}^{11,19}_{-16}$	$(8, 12)$
343	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}^{11,19}_{-16}$	$(8, 12)$
376	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}^{11,19}_{-16}$	$(8, 12)$
379	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}^{11,19}_{-16}$	$(8, 12)$
1262	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}^{9,21}_{-24}$	$(7, 13)$
1701	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^3 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}^{9,23}_{-28}$	$(7, 14)$

Continued on next page

CICY #	Configuration Matrix	$(h^{1,1}, h^{2,1})$
2544	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}^{7,23}_{-32}$	$(6, 14)$
3381	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^{9,27}_{-36}$	$(7, 16)$
3929	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \\ \mathbb{P}^3 \end{matrix}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}^{7,27}_{-40}$	$(6, 16)$
4108	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^3 \end{matrix}\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}^{7,27}_{-40}$	$(6, 16)$
4335	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^4 \\ \mathbb{P}^4 \end{matrix}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}^{7,27}_{-40}$	$(6, 16)$
5423	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^5 \end{matrix}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}^{7,31}_{-48}$	$(6, 18)$
6173	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}^{7,35}_{-56}$	$(6, 20)$

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