

HYPERTORIC MANIFOLDS OF INFINITE TOPOLOGICAL TYPE

ANDREW DANCER

ABSTRACT. We analyse properties of hypertoric manifolds of infinite topological type, including their topology and complex structures. We show that our manifolds have the homotopy type of an infinite union of compact toric varieties. We also discuss hypertoric analogues of the periodic Ooguri-Vafa spaces.

1. INTRODUCTION

A hypertoric manifold is a hyperkähler manifold of real dimension $4n$ with a tri-Hamiltonian action of a torus T^n of dimension n . (In the literature completeness is often assumed, but in some cases we will drop this assumption). Complete examples were systematically constructed and analysed in [3], [2] in the case when they have finite topological type, meaning that all Betti numbers are finite. These particular examples generalise the four-dimensional Gibbons-Hawking spaces [6] and the higher-dimensional examples of Goto [7], and retract onto a finite union of toric varieties.

Recently, the complete four-dimensional examples, without restriction on their topology, were classified in [16]. Examples of infinite topological type had been given by Anderson, Kronheimer and LeBrun [1] and have also been studied by Hattori [11], building on work of Goto [8], that also included examples in higher dimensions. The approach of Goto was generalised in a systematic construction in higher dimensions by the author and Swann in our paper [5], using hyperKähler quotients of certain Hilbert manifolds. This was then sufficient to obtain the full classification of complete hypertoric manifolds by combining the results of [3], [16].

As in the finite case, much of the geometry and topology is encoded by a configuration of codimension 3 affine subspaces (*flats*) in \mathbb{R}^{3n} , generalising the points in \mathbb{R}^3 that are the centres of the Gibbons-Hawking metrics. In [2] there were only finitely many such flats, but we shall now consider the situation where there are infinitely many of them. These need to be spaced out suitably to make certain sums converge.

The purpose of this paper is to further analyse the geometries obtained in the case of infinite topological type. In particular, we describe their homotopy type and their structure as complex manifolds. The topology is now generated by an infinite collection of compact toric varieties. The T^n action still induces a moment map that surjects onto \mathbb{R}^{3n} . The fibres are quotient tori (T^n for generic fibres), whose dimension is controlled by the intersection properties of the flats. The Pedersen-Poon generalisation of the Gibbons-Hawking calculations enables us to write the metric in terms of a generalised monopole. The explicit formulae now involve an infinite sum involving the distances from the flats.

We also briefly discuss generalisations of the 4-dimensional (incomplete) periodic Ooguri-Vafa metric to the hypertoric set-up.

2. CONSTRUCTION OF HYPERTORIC MANIFOLDS

We begin by reviewing our construction from [5] which produces certain hypertoric manifolds $M(\beta, \lambda)$ (including examples of infinite topological type) via a hyperkähler quotient of a flat Hilbert manifold.

We let \mathbb{L} denote a finite or countably infinite set and let $\mathbb{H} = \mathbb{R}^4$ be the quaternions. Given $\Lambda = (\Lambda_k)_{k \in \mathbb{L}} \in \mathbb{H}^{\mathbb{L}}$, we define $\lambda = (\lambda_k)_{k \in \mathbb{L}} \in \text{Im } \mathbb{H}^{\mathbb{L}}$ by $\lambda_k = -\frac{1}{2} \overline{\Lambda_k} i \Lambda_k \in \text{Im } \mathbb{H}$. Put $L^2(\mathbb{H}) = \left\{ v \in \mathbb{H}^{\mathbb{L}} \mid \sum_{k \in \mathbb{L}} |v_k|^2 < \infty \right\}$ and equip the Hilbert manifold $M_\Lambda = \Lambda + L^2(\mathbb{H})$, with the flat hyperKähler structure induced by the L^2 -metric and the complex structures obtained by regarding $L^2(\mathbb{H})$ as a right \mathbb{H} -module.

For the following construction, we require λ to satisfy the growth condition

$$(1) \quad \sum_{k \in \mathbb{L}} (1 + |\lambda_k|)^{-1} < \infty.$$

Consider the Hilbert group

$$T_\lambda = \left\{ g \in T^{\mathbb{L}} = (S^1)^{\mathbb{L}} \mid \sum_{k \in \mathbb{L}} (1 + |\lambda_k|) |1 - g_k|^2 < \infty \right\}.$$

whose Lie algebra \mathfrak{t}_λ consists of those $t \in \mathbb{R}^{\mathbb{L}}$ such that $\sum_{k \in \mathbb{L}} (1 + |\lambda_k|) |t_k|^2$ is finite. Note that $|g_k h_k - 1| \leq |(g_k - 1)(h_k - 1)| + |g_k - 1| + |h_k - 1|$ and $|g_k^{-1} - 1| = |g_k|^{-1} |g_k - 1|$, so as $g_k \rightarrow 1$ as $|k| \rightarrow \infty$, the group axioms do indeed hold for T_λ .

The group T_λ acts on M_Λ via $gx = (g_k x_k)_{k \in \mathbb{L}}$. To see this, observe that for $g \in T_\lambda$ and $x = \Lambda + v \in M_\Lambda$, we have $gx = g\Lambda + gv = \Lambda - (1 - g)\Lambda + gv$, but $gv \in L^2(\mathbb{H})$, since $g_k \rightarrow 1$ as $|k| \rightarrow \infty$, and $\|(1 - g)\Lambda\|^2 = \sum_{k \in \mathbb{L}} \frac{1}{2} |\lambda_k| |1 - g_k|^2 \leq \frac{1}{2} \sum_{k \in \mathbb{L}} (1 + |\lambda_k|) |1 - g_k|^2$, which is finite by the definition of T_λ , so $(1 - g)\Lambda \in L^2(\mathbb{H})$ too. The action preserves the flat hyperKähler structure on M_Λ , and has moment map $\mu_\Lambda : M_\Lambda \rightarrow \text{Im } \mathbb{H} \otimes \mathfrak{t}_\lambda^*$ given by

$$\mu_\Lambda(x) = \sum_{k \in \mathbb{L}} (\lambda_k + \frac{1}{2} \overline{x_k} i x_k) e_k^*.$$

where $e_i = (\delta_i^k)_{k \in \mathbb{L}}$ with δ_i^k Kronecker's delta and $e_k^*(e_i) = \delta_i^k$. The choice of constant λ_k is to cancel the quadratic terms in Λ when we write $x = \nu + \Lambda$ with $\nu \in L^2(\mathbb{H})$. The cross terms that are linear in Λ give a convergent sum of terms such as $\bar{\nu}_k i t_k \Lambda_k$ when paired with $t \in \mathfrak{t}_\lambda$, as both (ν_k) and $(t_k \Lambda_k)$ are square-summable.

Our hypertoric manifolds are produced as hyperkähler quotients of M_Λ by finite-codimension subtori of T_λ , defined as follows. For each $k \in \mathbb{L}$, we let $u_k \in \mathbb{Z}^n$ be a non-zero *primitive* vector, meaning that for all integers $m > 1$, we have $u_k \notin m\mathbb{Z}^n$. We also assume that $\{u_k : k \in \mathbb{L}\}$ spans \mathbb{R}^n . Now consider the linear map $\beta : \mathfrak{t}_\lambda \rightarrow \mathbb{R}^n$ given by $\beta(e_k) = u_k$,

Supposing β is continuous, then we define $\mathfrak{n}_\beta = \ker \beta \subset \mathfrak{t}_\lambda$, so \mathfrak{n}_β is a closed subspace of \mathfrak{t}_λ . (We shall shortly give sufficient conditions for continuity of β). As we are taking the u_k to be integral, we may define a Hilbert subgroup N_β of T_λ by

$$N_\beta = \ker(\exp \circ \beta \circ \exp^{-1} : T_\lambda \rightarrow T^n).$$

This gives exact sequences

$$(2) \quad 0 \longrightarrow \mathfrak{n}_\beta \xrightarrow{\iota} \mathfrak{t}_\lambda \xrightarrow{\beta} \mathbb{R}^n \longrightarrow 0,$$

$$(3) \quad 0 \longrightarrow (\mathbb{R}^n)^* \xrightarrow{\beta^*} \mathfrak{t}_\lambda^* \xrightarrow{\iota^*} \mathfrak{n}_\beta^* \longrightarrow 0,$$

and also the exact sequence of groups

$$1 \rightarrow N_\beta \rightarrow T_\lambda \rightarrow T^n = T_\lambda/N_\beta \rightarrow 1.$$

The hyperKähler moment map for the subgroup N_β is then $\mu_\beta = \iota^* \mu_\Lambda : M_\Lambda \rightarrow \text{Im } \mathbb{H} \otimes \mathfrak{n}_\beta^*$. We define the hyperkähler quotient

$$M = M(\beta, \lambda) := M_\Lambda // N_\beta = \mu_\beta^{-1}(0)/N_\beta.$$

Since (3) is exact, we have $\ker \iota^* = \text{im } \beta^*$. Now $\beta^*(s)(e_k) = s(\beta(e_k)) = s(u_k)$ for $s \in (\mathbb{R}^n)^*$, so $\beta^*(s) = \sum s(u_k) e_k^*$. Hence a point $x \in M_\Lambda$ lies in the zero set of the hyperKähler moment map μ_β for N_β if and only if there is an $a \in \text{Im } \mathbb{H} \otimes (\mathbb{R}^n)^*$ with

$$(4) \quad a(u_k) = \lambda_k + \frac{1}{2} \bar{x}_k \mathbf{i} x_k$$

for each $k \in \mathbb{L}$, where $u_k = \beta(e_k)$. □

As in [2], the codimension 3 affine subspaces $H_k \subset \text{Im } \mathbb{H} \otimes (\mathbb{R}^n)^* \cong \mathbb{R}^{3n}$

$$(5) \quad H_k = H(u_k, \lambda_k) = \{a \in \text{Im } \mathbb{H} \otimes (\mathbb{R}^n)^* \mid a(u_k) = \lambda_k\}$$

are called *flats*. Equation(4) shows that $x_k = 0$ if and only if $a(u_k) = \lambda_k$, that is, if and only if $a \in H_k$.

The following result was proved by the author and Swann in [5].

Theorem 2.1. [5] *Suppose $u_k = \beta(e_k) \in \mathbb{Z}^n$, $k \in \mathbb{L}$, are primitive and span \mathbb{R}^n . Let $\lambda_k \in \text{Im } \mathbb{H}$, $k \in \mathbb{L}$, be given such that the convergence condition (1) holds and the flats $H_k = H(u_k, \lambda_k)$, $k \in \mathbb{L}$, are distinct. Then the hyperKähler quotient $M = M(\beta, \lambda)$ is smooth and of dimension $4n$ if*

- (a) *any set of $n+1$ flats H_k has empty intersection, and*
- (b) *whenever n distinct flats $H_{k(1)}, \dots, H_{k(n)}$ have non-empty intersection the corresponding vectors $u_{k(1)}, \dots, u_{k(n)}$ form a \mathbb{Z} -basis for \mathbb{Z}^n .*

The two conditions ensure that N_β acts freely on $\mu_\beta^{-1}(0)$. Note that we have a residual hyperKähler action of $T^n = T_\Lambda/N_\beta$ on the $4n$ -manifold $M(\beta, \lambda)$, justifying the terminology *hypertoric*.

The proof involved the following result, which is useful in its own right.

Proposition 2.2. *Suppose $u_k \in \mathbb{Z}^n$, $k \in \mathbb{L}$, are primitive, span \mathbb{R}^n and satisfy condition (b) of Theorem 2.1. Then $\mathcal{U} = \{u_k \mid k \in \mathbb{L}\}$ is finite.*

Let us observe (as in the proof of Theorem 3.3 in [2]) that conditions (a), (b) imply that if J is a maximal set of indices such that $\cap_{k \in J} H_k$ is non-empty, then $\{u_k : k \in J\}$ is a \mathbb{Z} -basis for \mathbb{Z}^n . For maximality implies that every other u_i is in the span of $\{u_k : k \in J\}$, so, as we are assuming that the $u_k : k \in \mathbb{L}$ span \mathbb{R}^n , then $\{u_k : k \in J\}$ is also spanning. Condition (a) now implies that $|J| = n$, and condition (b) implies that $\{u_k : k \in J\}$ is a \mathbb{Z} -basis.

As $\mathcal{U} = \{u_k : k \in \mathbb{L}\}$ spans \mathbb{R}^n , it contains a basis, and the conditions of Proposition 2.2 imply this is a \mathbb{Z} -basis. We may change basis of \mathbb{Z}^n so that this set is $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Now for each u_k we may write $u_k = \sum_{i=1}^n a_{ki} \mathbf{e}_i$, and for each i with

$a_{ki} \neq 0$, then $\{\mathbf{e}_j : j \neq i\} \cup \{u_k\}$ is a \mathbb{Z} -basis. Thus the matrix with these vectors as columns is in $GL(n, \mathbb{Z})$ so has determinant ± 1 . But up to sign this determinant is a_{ki} so in general the coefficients of u_k lie in $\{-1, 0, 1\}$ and there are at most $3^n - 1$ distinct elements in \mathcal{U} , thus establishing Proposition 2.2.

It follows that under condition (b), the set $\{\|u_k\| \mid k \in \mathbb{L}\}$ is bounded. In particular it is now easy to show that $\beta: \mathfrak{t}_\lambda \rightarrow \mathbb{R}^n$ is indeed continuous and it follows that N_β is a Hilbert subgroup of T_λ of codimension n .

Remark 2.3. When \mathbb{L} is finite, the convergence condition (1) is vacuous, $M_\Lambda = \mathbb{H}^\mathbb{L}$, and we have the finite-dimensional construction of [2]. The construction of Hattori [11] corresponds to $n = 1$, $\mathbb{L} = \mathbb{Z}$ and $u_k = 1 \in \mathbb{R}$ for each k . For general dimension $4n$, Goto's construction [8] corresponds to $\mathbb{L} = \mathbb{N}_{>0}\mathbf{i} \cup \mathbb{N}_{>0}\mathbf{k} \cup \{1, \dots, n\}$,

$$\Lambda_k = \begin{cases} k, & \text{for } k \in \mathbb{N}_{>0}\mathbf{i}, \\ -k, & \text{for } k \in \mathbb{N}_{>0}\mathbf{k}, \\ 0, & \text{for } k \in \{1, \dots, n\}, \end{cases} \quad \text{with } u_k = \begin{cases} \mathbf{e}_1, & \text{for } k \in \mathbb{N}_{>0}\mathbf{i} \cup \mathbb{N}_{>0}\mathbf{k}, \\ \sum_{i=1}^n \mathbf{e}_i, & \text{for } k = 1, \\ -\mathbf{e}_r, & \text{for } k \in \{2, \dots, n\}. \end{cases}$$

Thus Goto's construction is for one concrete choice of $(\lambda_k)_{k \in \mathbb{L}}$ and only one of the u_k 's is repeatedly infinitely many times.

Just as in Hattori [11], one may use the T_λ action to show that different choices of $(\Lambda_k)_{k \in \mathbb{L}}$ yielding the same $(\lambda_k)_{k \in \mathbb{L}}$ result in hyperKähler structures that are isometric via a T^n -equivariant tri-holomorphic map. Choosing a complex structure and writing $\Lambda = (\Lambda_z, \Lambda_w)$ we shall often assume that either $\Lambda_z = 0$ or its terms grow like $O(k)$, and similarly for Λ_w .

The full classification of hypertoric manifolds relies on the following operation from [3] that does not change the topology: given a hyperKähler manifold M^{4n} with tri-Hamiltonian T^n action, a *Taub-NUT deformation* is any smooth hyperKähler quotient of $M \times \mathbb{H}^m$ by \mathbb{R}^m where $\mathbb{R}^m \subset \mathbb{H}^m$ acts by translation on \mathbb{H}^m and via an injective linear map $\mathbb{R}^m \rightarrow \text{Lie}(T^n)$ on M .

Theorem 2.4. [5] *Let M be a connected complete hypertoric manifold of dimension $4n$. Then M is topologically a product $M = M_2 \times (S^1 \times \mathbb{R}^3)^m$ with $M_2 = M(\beta, \lambda)$ for some β, λ . The hyperKähler metric on M is either the product hyperKähler metric or a Taub-NUT deformation of this metric.*

Remark 2.5. Several authors (see [4] for example) have explored the notion of Gale duality for hypertoric varieties. This swaps $\mathfrak{n} = \text{Lie}(N)$ and \mathfrak{n}^\perp , its orthogonal in $\mathfrak{t} = \text{Lie}(T)$. Now \mathfrak{n}^\perp can be identified with the Lie algebra of the isometry group T/N of the hyperkähler quotient. The duality also interchanges the level η at which we reduce and an element ξ of the Lie algebra of the isometry group. Thus deformation and isometry parameters are interchanged.

This duality is an example of the notion of 'symplectic duality', which ultimately comes from physics (duality between Coulomb and Higgs branches). The swapping of deformation and isometry parameters is a general feature of this picture.

In our set-up, \mathfrak{n} has finite codimension in \mathfrak{t}_λ so the 'Gale dual' would be a quotient of M_Λ by a finite-dimensional subgroup of T_λ , and hence would be an infinite-dimensional hypertoric.

3. THE T^n ACTION ON THE HYPERTORIC MANIFOLDS

We have defined M as a hyperkähler quotient of M_Λ by the finite codimension subtorus N_β of the Hilbert group T_λ . In this section we consider further the hypertoric structure of M .

We first recall that $T^n = T_\lambda/N_\beta$ acts on M preserving the induced hyperKähler structure, and with moment map $\phi: M \rightarrow \operatorname{Im} \mathbb{H} \otimes (\mathbb{R}^n)^*$ induced by μ_Λ . In more detail, $\operatorname{Lie} T^n = \mathfrak{t}_\lambda/\mathfrak{n}_\beta$ and hence $(\operatorname{Lie} T^n)^* = (\mathfrak{n}_\beta)^0$, the annihilator of \mathfrak{n}_β in \mathfrak{t}_λ^* . (We recall here that under our hypotheses \mathfrak{n}_β is a closed subspace of \mathfrak{t}_λ). Now on $\mu_\beta^{-1}(0)$ the map μ_Λ takes values in $(\mathfrak{n}_\beta)^0 = (\operatorname{Lie} T^n)^*$ and descends to M as ϕ .

Explicitly, $\phi(x) = a$ where $a \in \operatorname{Im} \mathbb{H} \otimes (\mathbb{R}^n)^* = \operatorname{Im} \mathbb{H} \otimes (\operatorname{Lie}(T^n))^*$ is defined by (4), that is

$$a(u_k) = \lambda_k + \frac{1}{2} \bar{x}_k i x_k \quad : \quad k \in \mathbb{L}$$

Note that a is uniquely determined by x if we assume, as we always do, that the u_k generate \mathbb{R}^n .

In particular, observe that the stabiliser in T_λ of x has Lie algebra spanned by the vectors e_k where $x_k = 0$, or equivalently, using (4), (5), the set of vectors e_k where $a = \phi(x)$ is in the flat H_k defined by $a(u_k) = \lambda_k$. The analogous statement for the T^n -stabiliser is that it is spanned by the u_k such that $a = \phi(x) \in H_k$.

Note also that the Lie algebra of T^n may be identified with the span of a finite collection of e_k (we can take $e_k : k \in I_1$ where $u_k : k \in I_1$ is a basis for \mathbb{R}^n).

The above discussion implies, as in [2]:

Lemma 3.1. *The hyperkähler moment map ϕ for the T^n -action on M induces a homeomorphism $M/T^n \rightarrow \operatorname{Im} \mathbb{H} \otimes (\mathbb{R}^n)^* = \mathbb{R}^{3n}$. In particular, M is connected.*

For $p \in M$, the stabiliser $\operatorname{stab}_{T^n}(p)$ is the subtorus with Lie algebra spanned by the u_k such that $\phi(p) \in H_k$.

We saw in the discussion after Proposition 2.2 that if a collection of flats H_k intersect then the corresponding u_k are contained in a \mathbb{Z} -basis. Combining this with Lemma 3.1, we see, as in the finite case:

Proposition 3.2. *Under the hypotheses of Theorem 2.1, we have that if $a \in \operatorname{Im} \mathbb{H} \otimes (\mathbb{R}^n)^* \cong \mathbb{R}^{3n}$ lies in exactly r flats then the T^n -stabiliser of a point in $\phi^{-1}(a)$ is an r -dimensional torus.*

From this result and Lemma 3.1, we see ϕ maps M onto \mathbb{R}^{3n} with generic fibre T^n , but these fibres may collapse to lower dimensional tori depending on how many flats contain the point in \mathbb{R}^{3n} . In particular the intersections of n flats correspond to fixed points of the T^n action on M .

The pre-image of the flat H_k is, from the discussion earlier, the hyperkähler subvariety S_k of M given by the quaternionic condition $x_k = 0$. So we have a collection $S_k (k \in \mathbb{L})$ of T^n -invariant hyperkähler subvarieties of M , such that the T^n action is free on the complement of $\bigcup_{k \in \mathbb{L}} S_k$ in M . The moment map ϕ expresses $M - \bigcup_{k \in \mathbb{L}} S_k$ as a T^n -bundle over $\mathbb{R}^{3n} - \bigcup_{k \in \mathbb{L}} H_k$. The real dimension of each S_k is $4(n-1)$.

We can also split the T^n -moment map into real and complex parts; equivalently, we write $a \in \operatorname{Im} \mathbb{H} \otimes (\mathbb{R}^n)^*$ as $(a_\mathbb{R}, a_\mathbb{C})$, which we view as lying in $(\mathbb{R}^n)^* \oplus (\mathbb{C} \otimes (\mathbb{R}^n)^*)$. This amounts to choosing a particular complex structure, which we refer to as l ,

in the 2-sphere of complex structures defined by the hyperkähler structure. So we split $\mathbb{H} = \mathbb{C} \oplus \mathbf{j}\mathbb{C}$ and write $x \in \mathbb{H}^{\mathbb{L}}$ as $x = z + \mathbf{j}w$ with $z, w \in \mathbb{C}^{\mathbb{L}}$.

Let us write $\lambda_k \in \text{Im } \mathbb{H}$ as $(\lambda_k^1, \lambda_k^2 + i\lambda_k^3) = (\lambda_k^{\mathbb{R}}, \lambda_k^{\mathbb{C}}) \in \mathbb{R} \oplus \mathbb{C}$. Equation (4) can now be written as

$$(6) \quad a_{\mathbb{R}}(u_k) = \lambda_k^{\mathbb{R}} + \frac{1}{2}(|z_k|^2 - |w_k|^2)$$

$$(7) \quad a_{\mathbb{C}}(u_k) = \lambda_k^{\mathbb{C}} + z_k w_k$$

(This was the form of the equations used in [2]).

It is useful to look at just the complex moment map $\phi_{\mathbb{C}} : (z, w) \mapsto a_{\mathbb{C}}$. A fibre of $\phi_{\mathbb{C}}$ will map via $\phi_{\mathbb{R}}$ onto $(\mathbb{R}^n)^* \cong \mathbb{R}^n$ with generic fiber a copy of T^n . If $a_{\mathbb{C}}$ is in the complement of the complex flats

$$(8) \quad H_{k, \mathbb{C}} = \{b \in \mathbb{C}^n \otimes (\mathbb{R}^n)^* : b(u_k) = \lambda_k^{\mathbb{C}}\}$$

then $(a_{\mathbb{R}}, a_{\mathbb{C}})$ is in the complement of the H_k for all $a_{\mathbb{R}}$, and hence $\phi_{\mathbb{C}}^{-1}(a_{\mathbb{C}})$ is just $T^n \times \mathbb{R}^n$. If $a_{\mathbb{C}}$ does lie in some $H_{k, \mathbb{C}}$, then $\phi_{\mathbb{R}} : \phi_{\mathbb{C}}^{-1}(a_{\mathbb{C}}) \rightarrow \mathbb{R}^n$ has degenerate fibres (tori of dimension less than n) over points $a_{\mathbb{R}}$ in the hyperplanes $a_{\mathbb{R}}(u_k) = \lambda_k^{\mathbb{R}}$.

Note that $\phi_{\mathbb{C}}^{-1}(a_{\mathbb{C}})$ is a toric variety for the T^n action (the moment map is projection onto the \mathbb{R}^n factor).

Remark 3.3. In [2] the authors considered the case when all λ_k are set to zero. The space $M(\beta, \lambda)$ is now a cone with action of \mathbb{R}^+ . It was shown that the origin was the only singularity (and hence $M(\beta, \lambda)$ was a cone over a smooth 3-Sasakian manifold) if and only if every set of n vectors u_k is linearly independent and every set of less than n vectors u_k is part of a \mathbb{Z} -basis. In our situation these conditions are never satisfied because Lemma 2.2 shows that some u_i are necessarily repeated infinitely often. This is to be expected as of course the 3-Sasakian space, being compact, cannot have infinite topological type.

4. COMPLEX STRUCTURES

In this section we consider complex structures on the hypertoric quotients. For ease of notation we shall suppress the subscript β and just write N etc.

The complexification $T_{\lambda}^{\mathbb{C}}$ of T_{λ} is the group

$$T_{\lambda}^{\mathbb{C}} = \left\{ g \in (\mathbb{C}^*)^{\mathbb{L}} \mid \sum_{k \in \mathbb{L}} (1 + |\lambda_k|) |1 - g_k|^2 < \infty \right\}.$$

This acts on M_{Λ} and induces an action of the complexification $N^{\mathbb{C}}$ of N . The zero locus of the complex part of our moment map for N consists of the pairs $(z, w) \in M_{\Lambda}$ satisfying equation (7) for some $a_{\mathbb{C}} \in \mathbb{C}^n$. This locus can be identified with the set of $(a_{\mathbb{C}}, (z, w)) \in \mathbb{C}^n \times M_{\Lambda}$ satisfying equation (7), as $a_{\mathbb{C}}$ is determined by (z, w) since the u_k span \mathbb{C}^n over \mathbb{C} .

We recall the definition (8) of the complex flats $H_{k, \mathbb{C}}$.

Proposition 4.1. *Suppose that each set of $n+1$ flats $H_{k, \mathbb{C}}$ has empty intersection. Then the action of $N^{\mathbb{C}}$ on*

$$(9) \quad \{(a_{\mathbb{C}}, (z, w)) : a_{\mathbb{C}}(u_k) = \lambda_k^{\mathbb{C}} + z_k w_k\}$$

by $t.(a_{\mathbb{C}}, (z, w)) = (a_{\mathbb{C}}, (t.z, t^{-1}.w))$, has discrete stabilisers.

Proof. The stabiliser of (z, w) for the $N^{\mathbb{C}}$ action is contained in $(T_I)^{\mathbb{C}}$, where I is the set of indices k where $z_k w_k = 0$, or equivalently the set of indices where $a_{\mathbb{C}}(u_k) = \lambda_k^{\mathbb{C}}$, ie. $a_{\mathbb{C}} \in H_{k, \mathbb{C}}$. The argument after Proposition 2.2 shows that if a collection of complex flats $\{H_{k, \mathbb{C}} : k \in I\}$ intersect, then the $\{u_k : k \in I\}$ are linearly independent, so $\beta : e_i \mapsto u_i$ is injective on $\text{Lie } (T_I)^{\mathbb{C}}$, and hence $\mathfrak{n}^{\mathbb{C}} \cap \text{Lie } (T_I)^{\mathbb{C}} = 0$. We thus have that stabilisers $N^{\mathbb{C}}$ are discrete. \square

Observe also that the argument of Thm 5.1 of [2] shows that, subject to the assumptions of Proposition 4.1, $\{\alpha_i : i \in I^c\}$ spans \mathfrak{n}^* , where I^c denotes the complement of I . The key point is to show that $\mathfrak{n} \cap \text{Lie } (T_I) = 0$ implies $\mathfrak{t} = \mathfrak{n}^{\perp} + \text{Lie } (T_I)$. The space \mathfrak{t} is now an infinite-dimensional Hilbert space but we still have $(X \cap Y)^{\perp} = \overline{X^{\perp} + Y^{\perp}}$ for *closed* subspaces X and Y . Now \mathfrak{n} is closed, as remarked earlier, and $\text{Lie}(T_I)$ is closed as I is finite, so we have $\mathfrak{t} = \overline{\mathfrak{n}^{\perp} + \text{Lie } (T_I)}$. But also \mathfrak{n}^{\perp} is finite-dimensional (as \mathfrak{n} has finite codimension) so $\mathfrak{n}^{\perp} + \text{Lie } (T_I)^{\perp}$ is closed, since the sum of a closed and a finite-dimensional space is closed.

Our next aim is to identify, under certain conditions, the complex quotient $(\mu^{\mathbb{C}})^{-1}(0)/N^{\mathbb{C}}$ with the hyperkähler quotient $\mu^{-1}(\lambda^{\mathbb{R}}, 0)/N$.

We shall assume that we are in the situation of Proposition 4.1. In addition, we need to make certain assumptions about the growth rates of the components of Λ . Choosing a complex structure, let us write $\Lambda = (\Lambda_z, \Lambda_w)$. Now $|\lambda_k| = |\Lambda_k|^2 = |(\Lambda_z)_k|^2 + |(\Lambda_w)_k|^2$. The λ_k need to grow fast enough for condition (1) to hold, and we shall assume that they grow like $O(k^{\delta})$ for some $\delta > 1$. (In concrete examples, one often takes $\delta = 2$).

We shall assume in the following discussion that the growth rates of $(\Lambda_z)_k$ and $(\Lambda_w)_k$ are *commensurable* in the sense that there exist positive constants C_1 and C_2 independent of k such that

$$C_1 |(\Lambda_w)_k| \leq |(\Lambda_z)_k| \leq C_2 |(\Lambda_w)_k|$$

for sufficiently large k . This implies of course that both $|(\Lambda_z)_k|^2$ and $|(\Lambda_w)_k|^2$ have commensurable growth rates with λ_k , and in particular tend to infinity as $k \rightarrow \infty$.

Now a point $(z, w) \in M_{\Lambda}$ may be written $(z, w) = (\hat{z}, \hat{w}) + (\Lambda_z, \Lambda_w)$ where \hat{z} and \hat{w} are square-summable. So both $|z_k|^2$ and $|w_k|^2$ have commensurable growth rates with λ_k also.

We shall now turn to the study of the quotient $(\mu^{\mathbb{C}})^{-1}(0)/N^{\mathbb{C}}$. We shall make use of the theory of monotone operators, as developed for example in Showalter's book [15].

(i) Let us consider the $N^{\mathbb{C}}$ -orbit \mathcal{O} through a point $(z, w) \in \mu_{\mathbb{C}}^{-1}(0)$ as above. We want to show that the real moment map $\mu^{\mathbb{R}}$ is surjective when restricted to the orbit.

As N is abelian, we have $\mathfrak{n}^{\mathbb{C}} = \mathfrak{n} + \mathfrak{a}$ with $\mathfrak{a} = i\mathfrak{n} \cong \mathfrak{n}$ Abelian. Writing $A = \exp(\mathfrak{a})$ we have $N^{\mathbb{C}} = NA$. Now $\mu^{\mathbb{R}}$ is N -invariant, as it is the moment map for an Abelian action, so we can view $\mu^{\mathbb{R}}$, restricted to the orbit \mathcal{O} , as a map from \mathfrak{a} to \mathfrak{n}^* (identifying A with \mathfrak{a} via the exponential map). Explicitly

$$\mu^{\mathbb{R}}(e^y \cdot (z, w)) = \frac{1}{2} \sum_{i \in \mathbb{L}} (|z_i|^2 e^{2\alpha_i \cdot y} - |w_i|^2 e^{-2\alpha_i \cdot y}) \alpha_i + c_1$$

where $y \in \mathfrak{a}$. (Compare with the expressions in [2] and [10]). Also $\alpha_i = \iota^* e_i^* \in \mathfrak{n}^*$ and $c_1 = \sum \lambda_i^1 \alpha_i$ is the constant term.

Writing $\tilde{\mu}(y) = \mu^{\mathbb{R}}(e^y \cdot (z, w))$, we get a map $\tilde{\mu} : \mathfrak{n} \rightarrow \mathfrak{n}^*$, with the same image as $\mu^{\mathbb{R}}$ on the orbit:

$$\tilde{\mu}(y) = \frac{1}{2} \sum_{i \in \mathbb{L}} (|z_i|^2 e^{2\alpha_i \cdot y} - |w_i|^2 e^{-2\alpha_i \cdot y} + 2\lambda_i^1) \alpha_i$$

This is the Legendre transform DF of the map $F : \mathfrak{n} \rightarrow \mathbb{R}$

$$F = \frac{1}{4} \sum_{i \in \mathbb{L}} (|z_i|^2 e^{2\alpha_i \cdot y} + |w_i|^2 e^{-2\alpha_i \cdot y} - (|z_i|^2 + |w_i|^2) + 4\lambda_i^1 \alpha_i \cdot y)$$

The constant term $(|z_i|^2 + |w_i|^2)$ is subtracted to ensure convergence. In more detail, let us write $t_i = 2\alpha_i \cdot y$, so for $y \in \mathfrak{n}$ we have $t_i = 2e_i^*(y) = 2y_i$. We now see that F is $\frac{1}{4}$ times a sum of terms

$$(|z_i|^2 - |w_i|^2 + 2\lambda_i^1)t_i + |z_i|^2(e^{t_i} - 1 - t_i) + |w_i|^2(e^{-t_i} - 1 + t_i).$$

The sum of the first terms just gives $\tilde{\mu}(0) = \mu^{\mathbb{R}}(z, w)$ evaluated on $y \in \mathfrak{n}$. The second and third terms are $\sim |z_i|^2 t_i^2$ (resp. $\sim |w_i|^2 t_i^2$) for large i , so the sums converge, since $\sum (1 + |\lambda_i|) t_i^2 < \infty$ by our definition of the Lie algebra \mathfrak{t}_λ of which \mathfrak{n} is a subalgebra.

Now $D^2F = \sum_{i \in \mathbb{L}} (|z_i|^2 e^{2\alpha_i \cdot y} + |w_i|^2 e^{-2\alpha_i \cdot y}) \alpha_i \otimes \alpha_i$. From above, $\{\alpha_i : i \in I^c\}$ spans \mathfrak{n}^* where I^c is the set of indices i for which z_i and w_i are both nonzero, so F is convex, and $\tilde{\mu} = DF$ is a *monotone operator*, in the sense that $\langle \tilde{\mu}(u) - \tilde{\mu}(v), u - v \rangle \geq 0$ for all u, v . (This follows by applying the one-variable mean value theorem to the function $t \mapsto \langle \tilde{\mu}(tu + (1-t)v), u - v \rangle$).

(ii) To show surjectivity of the monotone operator $\tilde{\mu} : \mathfrak{n} \rightarrow \mathfrak{n}^*$ it remains (see [15] section II.2) to show that $\tilde{\mu}$ maps bounded sets to bounded sets, and to prove a coercivity condition

$$\frac{\tilde{\mu}(y) \cdot y}{\|y\|} \rightarrow \infty \text{ as } \|y\| \rightarrow \infty.$$

We first show coercivity. Writing as above $y_i = \alpha_i \cdot y$, so

$$\tilde{\mu}(y) \cdot y = \frac{1}{2} \sum_{i \in \mathbb{L}} (|z_i|^2 e^{2y_i} - |w_i|^2 e^{-2y_i} + 2\lambda_i^1) y_i$$

Let us look at the i th term, which we can rewrite (suppressing the overall factor of $\frac{1}{2}$) as

$$(10) \quad y_i(|z_i|^2 - |w_i|^2 + 2\lambda_i^1) + f(y_i)|z_i|^2 + f(-y_i)|w_i|^2$$

where $f(t) = t(e^{2t} - 1) = 2t \sinh(t)e^t$. Note $f(t)$ is always nonnegative and we have $f(0) = f'(0) = 0$ and $f''(t) = 4(1+t)e^{2t}$, so $f(t) \geq 2t^2$ for all $t \geq 0$.

Moreover the first term in (10) is just the i th term in the linear functional $\tilde{\mu}(0)$ evaluated on y . Thus

$$2\tilde{\mu}(y)(y) \geq \tilde{\mu}(0)(y) + \sum_{i \in \mathbb{L}} 2 \min(|z_i|^2, |w_i|^2) y_i^2.$$

Recall $(z, w) = (\hat{z}, \hat{w}) + (\Lambda_z, \Lambda_w)$ where \hat{z} and \hat{w} are square-summable. We see there exists a constant B_z depending on z but independent of i such that $|z_i|^2 \leq B_z(1 + |\lambda_i|)$ for all i , and similarly for w_i . Moreover the discussion on comensurability above shows that there exists a *positive* constant D_z such that

$|z_i|^2 \geq D_z(1 + |\lambda_i|)$ for all i such that $z_i \neq 0$. Again we have an analogous statement for w .

Recall now that $z_i w_i = 0$ if and only if $i \in I$. We deduce that, subject to our assumptions, $\min(|z_i|^2, |w_i|^2)$ is bounded below by a positive constant times $(1 + |\lambda_i|)$ for $i \in I^c$.

Now $\tilde{\mu}(y)(y)$ is the sum of an expression that grows at most linearly, and a term bounded below by a positive constant times $\sum_{i \in I^c} (1 + |\lambda_i|) t_i^2$, which is greater than or equal to a positive constant times $\|y\|^2$ as, from above, the α_i span \mathfrak{n}^* .

(iii) To show $\tilde{\mu}$ maps bounded sets to bounded sets, we need to check that $\tilde{\mu}(y)(\xi)$ can be bounded in terms of y and linearly in $\|\xi\|$ for $y, \xi \in \mathfrak{n}$. A similar calculation to that above gives that

$$(11) \quad \tilde{\mu}(y)(\xi) = \tilde{\mu}(0)(\xi) + \frac{1}{2} \sum_{i \in \mathbb{L}} (|z_i|^2(e^{2y_i} - 1) - |w_i|^2(e^{-2y_i} - 1)) \alpha_i(\xi)$$

The first term is bounded linearly in terms of $\|\xi\|$.

A bound on y gives a bound, uniform in i , on the y_i (since $|\lambda_i| \rightarrow \infty$ as $i \rightarrow \infty$). Writing $e^t - 1 = th(t)$ where $h(t) = e^t \sinh(t)/t$ is continuous, we see that, for bounded $\|y\|$, we have $|z_i|^2|e^{2y_i} - 1| \leq C|z_i|^2|y_i|$ for a constant C independent of i , and we similarly bound $|w_i|^2(e^{-2y_i} - 1)$.

Now $\sum_{i \in \mathbb{L}} |z_i|^2(e^{2y_i} - 1) \alpha_i(\xi)$ is bounded by

$$C \sum |z_i|^2 |y_i| |\alpha_i(\xi)| \leq C \left(\sum |z_i|^2 y_i^2 \right)^{\frac{1}{2}} \left(\sum |z_i|^2 (\alpha_i(\xi))^2 \right)^{\frac{1}{2}}$$

As we have a constant B independent of i such that $|z_i|^2, |w_i|^2 \leq B(1 + |\lambda_i|)$, we see finally the second and third terms in (11) are bounded by a constant depending on (z, w) and the bound on $\|y\|$, times $\|\xi\|$, thus concluding the argument.

(iv) So, subject to our hypotheses, we have that the $N^{\mathbb{C}}$ -orbit through (z, w) meets each level set of $\mu^{\mathbb{R}}$. Standard arguments (see eg Prop 5.1 of [8]) show that in fact the $N^{\mathbb{C}}$ -orbit meets the level set in an N -orbit.

So, we can identify the hyperkähler quotient with the $N^{\mathbb{C}}$ -quotient of the zero locus of the complex equation (7).

We now deduce that $M(\lambda^1, \lambda^2, \lambda^3)$ is diffeomorphic to $M(\lambda^1, \lambda^2, 0)$ and (rotating complex structures) with $M(\lambda^1, 0, 0)$. We shall analyse the topology of this space in the next section.

Remark 4.2. Considering $M(\lambda^1, 0, 0)$ means we take Λ_z or $\Lambda_w = 0$, so the λ_k are pure multiples of i . We now have a holomorphic (but not triholomorphic) circle action on the hyperkahler quotient, induced by rotation in the z (resp. w) factor. For, as this rotation fixes Λ it induces an action on $M_{\Lambda} = \Lambda + L^2(\mathbb{H})$ and hence on the hyperkähler quotient, since $\lambda_k^{\mathbb{C}} = \lambda_k^2 + i\lambda_k^3 = 0$. In terms of equations (6) and (7), $a_{\mathbb{R}}$ remains unchanged while $a_{\mathbb{C}}$ is rotated. The fixed point set of this action is therefore contained in $\phi_{\mathbb{C}}^{-1}(0)$, and is a union of toric varieties contained in the locus $z_k w_k = 0$.

5. TOPOLOGY

We can analyse the topology using similar methods to the finite case. In view of the above discussion, let us take $\lambda_k = (\lambda_k^{\mathbb{R}}, 0, 0)$, that is, to have zero complex part.

First recall the homeomorphism $\tau : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R} \times \mathbb{R}_{\geq 0}$ defined by

$$(12) \quad (x, y) \mapsto \left(\frac{1}{2}(x^2 - y^2), xy\right)$$

whose inverse is

$$\tau^{-1} : (p, q) \mapsto \left(+\sqrt{p + \sqrt{p^2 + q^2}}, +\sqrt{-p + \sqrt{p^2 + q^2}}\right)$$

The deformation $j_t : (p, q) \mapsto (p, tq)$ of $\mathbb{R} \times \mathbb{R}_{\geq 0}$ now induces a deformation $j_t = \tau^{-1} \circ j_t \circ \tau$ of $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. We shall take the deformation parameter t to lie in $[0, 1]$.

Explicitly,

$$(13) \quad j_t^1(x, y) = \sqrt{\frac{1}{2}(x^2 - y^2) + \frac{1}{2}\sqrt{(x^2 - y^2)^2 + 4t^2x^2y^2}}$$

$$(14) \quad j_t^2(x, y) = \sqrt{\frac{1}{2}(y^2 - x^2) + \frac{1}{2}\sqrt{(x^2 - y^2)^2 + 4t^2x^2y^2}}$$

Note that j_1 is the identity, while j_0 sends $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ into the union of the non-negative x and y axes.

Writing $j_t(x, y) = (j_t^1(x, y), j_t^2(x, y))$, we obtain a deformation h_t of $\mathbb{H} = \mathbb{C} \times \mathbb{C}$ given by

$$h_t(z, w) = \left(j_t^1(|z|, |w|)\frac{z}{|z|}, j_t^2(|z|, |w|)\frac{w}{|w|}\right)$$

This can now be extended diagonally to a $T^{\mathbb{L}}$ -equivariant deformation h_t of $\mathbb{H}^{\mathbb{L}}$ with h_1 equal to the identity.

Note that $(j_t^1(x, y))^2 + (j_t^2(x, y))^2 \leq x^2 + y^2$ for $0 \leq t \leq 1$ so this actually induces a deformation of $\mathbb{L}^2(\mathbb{H})$. Moreover, as $(j_t^1(x, y))^2 - (j_t^2(x, y))^2 = x^2 - y^2$, we deduce that $(j_t^1(x, y))^2 \leq x^2$ and $(j_t^2(x, y))^2 \leq y^2$.

Now $|h_t(z, w) - (z, w)|^2 = (j_t^1(x, y) - x)^2 + (j_t^2(x, y) - y)^2$, where $x = |z|$ and $y = |w|$.

The second term is bounded by $4y^2$. The first term is $O(1/x^2)$ for x large in comparison with y .

In our situation (as we are taking $\lambda_k = (\lambda_k^{\mathbb{R}}, 0, 0)$, we can take $\Lambda = (\Lambda^{(1)}, 0)$. Now let us consider $(z_k, w_k)_{k \in \mathbb{L}} \in M_{\Lambda}$, so $(z_k, w_k) = (\Lambda_k + \hat{z}_k, w_k)$ where (\hat{z}_k) and (w_k) are square summable sequences. Moreover $|z_k|^2 = O(\lambda_k)$, so, since $\sum_k \frac{1}{\lambda_k} < \infty$ the above calculations show $(h_t(z_k, w_k) - (z_k, w_k))$ is square-summable. So if $(z, w) - \Lambda \in L^2(\mathbb{H})$ then $h_t(z, w) - \Lambda \in L^2(\mathbb{H})$ also.

So h_t in fact gives a deformation of $M_{\Lambda} = \Lambda + L^2(\mathbb{H})$. The equivariance implies that this induces a deformation of the hyperkähler quotient $M = M(\beta, \lambda)$.

As

$$\tau \circ j_t(x, y) = j_t \circ \tau(x, y) = \left(\frac{1}{2}(x^2 - y^2), txy\right)$$

we see that the real component $\mu^{\mathbb{R}}$ of our moment map is invariant under the deformation, while the complex component $\mu^{\mathbb{C}} = \mu_2 + i\mu_3$ scales by t . Passing to the hyperkähler quotient M , similar statements apply to the moment map ϕ for the T^n -action on M .

Now the family h_t ($0 \leq t \leq 1$) deforms M to $\phi_{\mathbb{C}}^{-1}(0)$, i.e. the locus where $a_{\mathbb{C}} = 0$. As $\lambda_k^{\mathbb{C}} = 0$, points (z, w) of this locus satisfy

$$z_k w_k = 0.$$

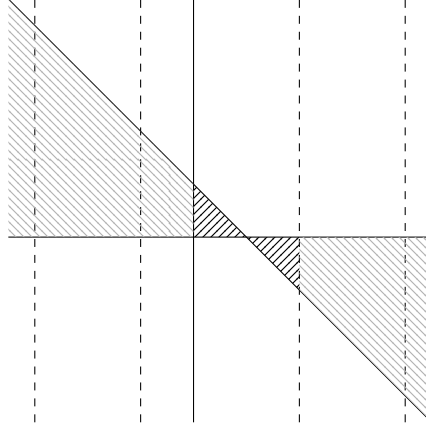


FIGURE 1. Singly infinite configuration in two dimensions

Moreover, from the moment map equation (6), we have

$$a_{\mathbb{R}}(u_k) = \lambda_k^{\mathbb{R}} + \frac{1}{2}(|z_k|^2 - |w_k|^2)$$

So if $a_{\mathbb{R}}$ is on the positive side of the hyperplane $H_{k,\mathbb{R}} = \{x : x(u_k) = \lambda_k^{\mathbb{R}}\}$ in \mathbb{R}^n , then $w_k = 0$ and $|z_k|^2 = 2(a_{\mathbb{R}}(u_k) - \lambda_k^{\mathbb{R}})$. If $a_{\mathbb{R}}$ is on the negative side of the hyperplane then $z_k = 0$ and $|w_k|^2 = -2(a_{\mathbb{R}}(u_k) - \lambda_k^{\mathbb{R}})$. So $\phi_{\mathbb{C}}^{-1}(0)$ is a union of the (in general compact and non-compact) toric varieties corresponding to the (in general bounded and unbounded) polyhedra in \mathbb{R}^n defined by the arrangement of hyperplanes $H_{k,\mathbb{R}}$. Now as in [2] we can retract $\phi_{\mathbb{C}}^{-1}(0)$ onto the union of compact toric varieties corresponding to the bounded polytopes.

In summary, the hypertoric variety has the homotopy type of a union of (in general infinitely many) compact toric varieties intersecting in toric subvarieties.

These toric varieties are Kähler submanifolds of M with respect to the complex structure \mathbb{I} , because the condition $\phi_{\mathbb{C}} = 0$ (i.e. $z_k w_k = 0$ for all k) is \mathbb{I} -holomorphic. As the \mathbb{I} -holomorphic complex symplectic form $\omega_{\mathbb{C}} = \omega_{\mathbb{J}} + i\omega_{\mathbb{K}}$ is induced from the form $dz \wedge dw$ on \mathbb{H} , we also see that these toric varieties are complex Lagrangian with respect to $\omega_{\mathbb{C}}$.

Example 5.1. Let us revisit the example due to Goto [7]. Now the array of hyperplanes is obtained from that defining a simplex by adding infinitely many translates of the hyperplane defining one face. We obtain a picture where the bounded chambers are a pair of simplices and an infinite collection of simplices truncated at one vertex. The corresponding compact toric varieties are a pair of complex projective spaces \mathbb{CP}^n and an infinite collection of the blow-ups of \mathbb{CP}^n at one point. Each toric variety intersects the next one in a \mathbb{CP}^{n-1} (corresponding to the $(n-1)$ -simplex where the associated chambers meet), except that the two projective spaces meet in a point, because the associated chambers just meet in a point. We illustrate this for $n = 2$ in Figure 1.

In the two-dimensional case, we can also have doubly or triply infinite configurations, see Figure 2.

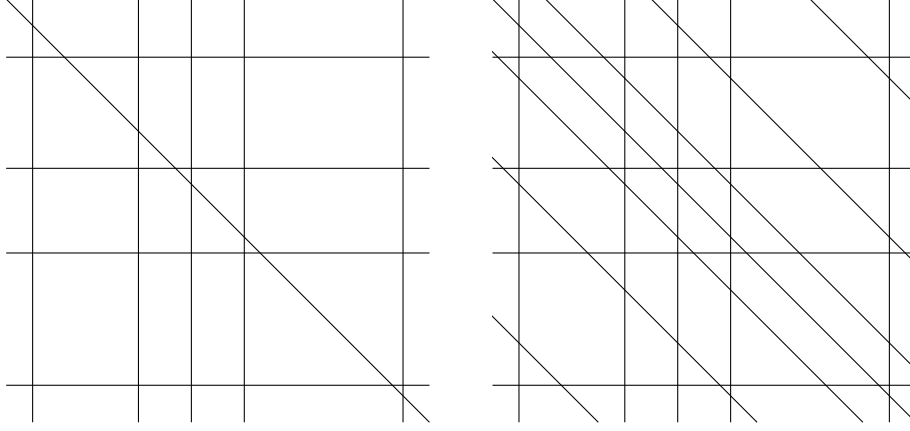


FIGURE 2. Doubly and triply infinite configurations in two dimensions

6. THE HYPERTORIC METRIC

Let us now describe the metric on M , using the notation of [2]. Let $\phi : M \rightarrow \text{Im } \mathbb{H} \otimes (\mathbb{R}^n)^* \cong \mathbb{R}^{3n}$ denote the moment map and let U be an open subset of $\text{Im } \mathbb{H} \otimes (\mathbb{R}^n)^*$ such that T^n acts freely on $\phi^{-1}(U)$. The hyperkähler structure on $\phi^{-1}(U)$ is uniquely determined via a *polyharmonic* function F on $U \subset \text{Im } \mathbb{H} \otimes (\mathbb{R}^n)^*$. (A function is polyharmonic if it is harmonic on each affine subspace $c + \text{Im } \mathbb{H} \otimes \mathbb{R}\alpha$, $\alpha \in (\mathbb{R}^n)^* \setminus \{0\}$).

Explicitly, we write the base \mathbb{R}^{3n} as $\mathbb{R}^n \times \mathbb{C}^n$ and use (a, b) as coordinates on $\mathbb{R}^n \times \mathbb{C}^n$. As we are identifying \mathbb{R}^n with its dual, expressions like $a(u_k)$ will be written $\langle a, u_k \rangle$ etc.

The hyperKähler metric is of the form

$$(15) \quad g = \sum_{i,j} [\Phi_{ij}(da_i da_j + db_i d\bar{b}_j) + (\Phi^{-1})_{ij}(dy_i - A_i)(dy_j - A_j)]$$

for suitable fibre coordinates y_i on the T^n fibres, and where A is the connexion form on the T^n -bundle. The pair (A, Φ) satisfy the generalised monopole equations of Pedersen and Poon [14]. Φ is often referred to as the *potential* for the metric (if $n = 1$ of course our set-up reduces to the Gibbons-Hawking ansatz and Φ satisfies the equation $\nabla \Phi = \text{curl } A$).

Moreover the monopole may be described in terms of a *prepotential* function F via the equations

$$(16) \quad (\Phi_{ij}, A_j) = \left(F_{a_i a_j}, \frac{1}{2} \sum_k \sqrt{-1} (F_{a_j b_k} db_k - F_{a_j \bar{b}_k} d\bar{b}_k) \right).$$

where F satisfies the polyharmonic equations $F_{a_i a_j} + F_{b_i \bar{b}_j} = 0$. The fiber coordinates y_i may also be related to F via the formula $dy_i = \partial_1 F_{a_i} - \partial_1 F_{\bar{a}_i}$ where ∂_1 is the Dolbeault operator corresponding to the choice of complex structure given by the splitting of \mathbb{R}^{3n} as $\mathbb{R}^n \times \mathbb{C}^n$.

The prepotential F may be written explicitly as a formal sum as follows. We let

$$(17) \quad s_k = 2(\langle a, u_k \rangle - \lambda_k^{\mathbb{R}}),$$

$$(18) \quad v_k = \langle b, u_k \rangle - \lambda_k^{\mathbb{C}}$$

and define $r_k \geq 0$ by $r_k^2 = s_k^2 + 4v_k^2$. So the distance from (a, b) to the flat H_k is $r_k/2|u_k|$. Now we have

$$F = \frac{1}{2} \sum_{k \in \mathbb{L}} (s_k \log(s_k + r_k) - r_k).$$

and hence

$$(19) \quad \Phi_{ij} = F_{a_i a_j} = \sum_{k \in \mathbb{L}} \frac{(u_k)_i (u_k)_j}{r_k}$$

Note that our convergence condition (1) guarantees convergence of the sum (19). We also remark that Φ may be rewritten as

$$\sum_{k \in \mathbb{L}} \frac{u_k \otimes u_k}{r_k}$$

Each individual term in the sum is positive semidefinite with kernel u_k^\perp , and as the u_k span \mathbb{R}^n , the sum is positive definite as required.

On the flat H_k we have $s_k = v_k = 0$ and hence $r_k = 0$, giving a singularity of Φ_{ij} , but these are just coordinate singularities corresponding to the fact that the T^n fibres collapse to lower-dimensional tori over the flats. The metric on M remains smooth at these points.

If $n = 1$, of course, the flats are points in \mathbb{R}^3 , and our expression gives the Gibbons-Hawking form of the Anderson-Kronheimer-Lebrun metrics [1].

Remark 6.1. The expression $\Phi_{ij}(da_i da_j + db_i d\bar{b}_j)$ gives the projected metric on the base \mathbb{R}^{3n} and projection of the hypertoric metric to this base is a Riemannian submersion. Similar arguments to those in [1] and [8] show that our hypertoric metrics are complete. For a Cauchy sequence in the hypertoric metric projects to a Cauchy sequence in the metric on the base, and comparison with the analogous metric with potential Φ given by a finite sum shows that the projected sequence gets trapped in a compact region of \mathbb{R}^{3n} and hence converges. As the projection onto \mathbb{R}^{3n} is proper, this means that the original sequence converges also.

Remark 6.2. Theorem 2.4 stated that a connected hypertoric manifold of dimension $4n$ is topologically (up to a product with $(S^1 \times \mathbb{R}^3)$ factors) a hypertoric manifold of the type constructed in this paper, ie. the hyperKähler quotient of a flat Hilbert hyperKähler manifold by an Abelian Hilbert Lie group. The hyperKähler metric is either the one induced by the hyperKähler quotient construction or a Taub-NUT deformation of this metric.

In the case of the Taub-NUT deformations the potential is modified by the addition of extra terms thus:

$$F = \sum_{k \in \mathbb{L}} a_k (s_k \log(s_k + r_k) - r_k) + \sum_{i,j=1}^n c_{ij} (4x_i x_j - z_i \bar{z}_j - z_j \bar{z}_i)$$

where the c_{ij} terms give the deformation of a metric determined by the first sum.

7. PERIODIC EXAMPLES

In order to make the hyperkähler quotient construction work, we assumed that the λ_k grow fast enough with k so that, as in equation (1), the sum $\sum_k (1 + |\lambda_k|)^{-1}$ is finite. For example, in Goto's examples, λ_k is taken to grow like k^2 . The λ_k of course essentially give the distances of the flats from the origin.

In the four dimensional case, when the flats are just points in \mathbb{R}^3 , Ooguri-Vafa [13] considered another viewpoint involving points spaced out at distances growing linearly in k —in fact the points are arranged periodically so the set-up is invariant under \mathbb{Z} translations. Of course this does not fit into the hyperkähler quotient picture, but it can be interpreted using the Gibbons-Hawking form of the metric (15). The points are located at $(k, 0, 0)$ and the Ooguri-Vafa potential is now defined to be

$$\Phi = \sum_k \frac{1}{\sqrt{(a-k)^2 + b^2}} - \frac{1}{|k|}.$$

so the $1/r_k$ terms are modified by the subtraction of $1/|k|$ to ensure convergence. We refer to [9] for further background on the Ooguri-Vafa metric. Note that this metric is incomplete so not included in the classification result Thm. 2.4.

We can generalise this idea to the situation in our paper. Our potential was $\Phi_{ij} = \sum_{k=1}^d \frac{(u_k)_i (u_k)_j}{r_k}$ where $r_k = +\sqrt{s_k^2 + 4v_k \bar{v}_k}$.

We can now modify this expression in the spirit of Ooguri-Vafa by changing the $1/r_k$ terms to $1/r_k - 1/|k|$. We choose the λ_k to have complex part zero. (Note that the nature of the singularity of Φ_{ij} on the flats remains the same, so the metric extends smoothly over these points).

Recall that there are only finitely many distinct u_k , subject to the assumptions of Proposition 2.2. We can now arrange that for each u_k the associated λ 's just take all possible values in a translate of \mathbb{Z} . So our diagram of hyperplanes has periodicity—invariant under a group of translations isomorphic to \mathbb{Z}^n .

This is most clearly seen in terms of the prepotential function F introduced in 6, satisfying $\Phi_{ij} = F_{a_i a_j} = -F_{b_i \bar{b}_j}$. The basic four-dimensional example is $F_0(x, w) = x \log(x + r) - r$ where $r^2 = x^2 + 4w\bar{w}$ giving $\Phi = \frac{1}{r}$. One can obtain higher dimensional examples by considering $F(a, b) = F_0(s_k, v_k)$ where s_k, v_k are given by (17), (18). Note that the second derivatives for F now scale by $(u_k)_i (u_k)_j$ so F is still polyharmonic. The linearity of the equations for F means that we can superpose solutions to get

$$F = \sum_k F_0(s_k, v_k) = \sum_k s_k \log(s_k + r_k) - r_k$$

In the finite case this is just formula (8.2) in [2], and on taking second derivatives yields the formula of Theorem 9.1 in that paper. In the infinite case this gives the formula (19) of the preceding section of the current paper.

But we can also replace F_0 by the corresponding prepotential F_{OV} for the Ooguri-Vafa metric (which need not be known explicitly). This prepotential is now periodic in the x variable— $F_{OV}(x + 1, w) = F_{OV}(x, w)$. We can now consider

$$\sum_k F_{OV}(s_k, v_k)$$

where k ranges over a finite set of indices in one-to-one correspondence with the set of *distinct* vectors u_k . This prepotential is invariant under $(a, b) \mapsto (a + u_j, b)$, as the

u_k are all integral, so the metric we obtain is invariant under the lattice isomorphic to \mathbb{Z}^n spanned by the vectors u_j . An example of this construction occurs in the physics literature in [12].

Note that in the original Ooguri-Vafa metric, the complex coordinate b has to lie in the unit disc for the correct convergence properties to hold. In our more general set up, we will need $|v_k| < 1$, for all k , which defines a non-empty neighbourhood U of the origin in \mathbb{C}^n as there are only finitely many distinct u_k .

As in §3 we can consider projection onto the complex coordinate b (this is $a_{\mathbb{C}}$ in the notation of §3). If b is in the complement of the complex flats then the fibre is a copy of $T^n \times \mathbb{R}^n$. We can then quotient by \mathbb{Z}^n and get a picture where the quotient space fibres over $U \subset \mathbb{C}^n$ with generic fibres tori T^{2n} . If b lies in a complex flat, the fibre is obtained from a $2n$ -torus by collapsing some T^n fibres to lower-dimensional tori over the subtori of the base T^n that are obtained by quotienting the real hyperplanes $\langle a_{\mathbb{R}}, u_k \rangle = \lambda_k^1$ in \mathbb{R}^n .

This generalises the $n = 1$ Ooguri-Vafa case where one can quotient by \mathbb{Z} and get an elliptic fibration over the complex disc with singular fibre a nodal cubic over the origin. The nodal cubic is of course obtained by pinching a single S^1 in the elliptic curve to a point.

Acknowledgement. It is a pleasure to thank Andrew Swann for many useful discussions, Melanie Rupflin for advice on monotone operators, Greg Sankaran for discussions on toric geometry, and Michael Thaddeus for suggesting that there should be hypertoric analogues of the Ooguri-Vafa metric.

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(Dancer) JESUS COLLEGE, OXFORD, OX1 3DW, UNITED KINGDOM
Email address: **dancer@maths.ox.ac.uk**