String Amplitudes in AdS and their Limits

Lessons from flat space and dualities

Mathematical Institute

Green Templeton College

University of Oxford

A thesis submitted for the completion of

Doctor of Philosophy in Mathematics

Trinity Term 2025

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Alla mia famiglia,

di nascita e d'anima.

A Matteo, che è casa,

ovunque io sia.

Statement of originality

This thesis draws on several collaboratively authored papers, and I gratefully acknowledge the contributions of all my coauthors. In particular, I have selected some of my papers to form the heart of this thesis, which I list below. I will focus primarily on my works on string amplitudes in AdS:

- L. F. Alday, T. Hansen, and M. Nocchi, "High Energy String Scattering in AdS," JHEP 02 (2024) 089 , **arXiv:2312.02261**. Chapter 4
- L. F. Alday, M. Nocchi, C. Virally, and X. Zhou, "On the Regge behaviour of the AdS Virasoro–Shapiro Amplitude," JHEP 04 (2025) 064, **arXiv:2409.03695**. Chapter 4
- L. F. Alday, M. Nocchi, and A. S. Sangaré, "Stringy KLT Relations on AdS", **arXiv:2504.19973**. Chapter 5

To some extent, I will also present results about higher-point holographic correlators:

- L. F. Alday, V. Gonçalves, M. Nocchi, and X. Zhou, "Six-point AdS gluon amplitudes from flat space and factorization," Phys. Rev. Res. 6 (2024) L012041, **arXiv:2307.06884**. Chapter 3
- V. Gonçalves, M. Nocchi, and X. Zhou, "Dissecting supergraviton six-point function with lightcone limits and chiral algebra," JHEP 06 (2025) 173, **arXiv:2502.10269**. Chapter 3

Another work is cited in the final part, but not included in the main body of the thesis:

- L. F. Alday, M. Nocchi, R. Ruzziconi, and A. Y. Srikant, "Carrollian Amplitudes from Holographic Correlators," JHEP 03 (2025) 158, **arXiv:2406.19343**.

My papers on bootstrapping critical systems with impurities or defects are not included at all:

- M. Nocchi, "Random Bond perturbations of the O(2) vector model", **arXiv:2405.08072**.
- C. Behan, E. Lauria, M. Nocchi, and P. van Vliet, "Analytic and numerical bootstrap for the long-range Ising model," JHEP 03 (2024) 136, **arXiv:2311.02742**.

Acknowledgments

As I reach the end of this journey, my heart is full of gratitude for all the people who made it possible, each in their own way.

First of all, I am deeply thankful to my Master's supervisor, Alessandro, who introduced me to the world of CFTs and encouraged me to apply for a PhD abroad at a time when I was not inclined to consider it. And of course, my greatest thanks to Fernando, who believed in me during the Oxford interviews and offered me a place in the group. Thank you for welcoming me and teaching me so much, not only great Mathematica tricks, but more importantly, how to think about physics and remain curious. It has been wonderful to discuss and understand things together. I have learned so much! I am also grateful for the incredible conference trips, especially to New York and Kyoto. Discovering our shared passion for food was so much fun. Thank you for being an outstanding culinary and travel guide. Visiting such distant and special places was mind-blowing. Travelling has enriched me immensely. I am also grateful to the Mathematical Institute for awarding me a scholarship that funded all my four years and supported my conference travels. A special thank you to everyone I met at LACES in my first year. Those three weeks were absolutely unforgettable! A beautiful hotel in Florence with a grand piano in the lobby, where we played and sang every single night. The outings, the pizza at our favourite place, the long walks. Thank you especially to Giulio, Jacob, Sophie, and Will for exploring so many places together, and an extra special thanks to Andrea. Bonded by our mutual love of coccoli, we became friends, and we kept meeting at conferences until he got a postdoc here in Oxford. We had so much fun.

I have always dreamed of living in England. Still, I was scared when I first arrived. Leaving everything and everyone behind to move to a new country was not easy. I owe thanks to the amazing people who helped me make it through.

First, the Green Templeton College choir. I have always loved music and singing, and I joined as soon as I arrived. It became my safe haven for four years. Emily, with her passion and infectious energy, made me feel so welcome and loved. We shared so many laughs, rehearsals, and moments each week. A piece of my heart will always remain with them. Thanks especially to Patricia and Arnaud. They were also part of my (short-lived) rowing adventure, before tendinitis returned. Thank you to the rowing captains and crew of 2023–2024. You gave me my first taste of being part of a sports team. Thank you also to all the people I shared the passion for climbing with, a sport which taught me patience and perseverance.

A big thank you to the Mathematical Physics group, starting from Fernando's students and postdocs. Thanks to Pietro and Johan, who helped me choose Oxford and welcomed me into the group. Thanks to Tobias for being such a nice person and great collaborator. Special thanks to Connor—my go-to reference on CFTs, valued collaborator, and dear friend. Thanks to Carmen for being a point of reference in the office and the group. Thanks to Joseph and Mateo for the smiles and memes, and for continuing to share them even from afar. To every PhD student and postdoc, whether still here or gone on to new adventures. To all my collaborators.

A huge thank you to Adam, my constant friend throughout the PhD and my office mate. It began with him as my student in the String Theory I course, and over time, we became close friends. I always joke that I "decided" he would be my friend from the very first moment, but I truly believe he has a heart of gold, and I am so grateful life brought him into mine to share both hard times and beautiful ones.

A special thank you to Julius, who arrived bringing joy and energy just when I needed it most. You have become family, in Oxford and beyond. Thank you for all the dinners, from carbonara, to gnocchi with gorgonzola, salami, and much more. For the pub outings, the walks, the funny texts when we are far apart. For welcoming me into your city. Ready for whatever adventures life will lead us to next!

Thanks also to Clément and Aurélie, my academic siblings. Thanks for the collaborations we shared, where we learned so much together. Thank you, Clément, for all the Mathematica tricks and discussions. And of course, for the crêpes and the brownies, and all the trips to my favourite Thai place. Thank you, Aurélie, for being a fantastic collaborator and also a dear friend. I must thank you for introducing me to the world of Mamma Mia!—those songs got me through writing my thesis and still lift my spirit. I hope we will share many more moments of musicals and Christmas decorations. Thanks also to Rodrigo, who joined the family this year—a truly kind person and a great researcher from whom I have learned a lot. Thanks to Chris Couzens for his support with my first seminars, postdoc applications, my housing issues, for lending me his AdS/CFT books, and really, for everything.

Thanks to the happy hour crew, who brought me genuine joy. Thanks to Chris, Nico, Tim, Adam, Pieter, Marta, Tabea, and Alex for so many weekly smiles. For the Aperol Spritz, for buying my favourite snacks, for letting me play Christmas music at happy hour, for the Halloween pumpkins and decorations, and so much more. This is not our "last call". A special thank you to Pieter for helping me between my tenancies and for the wonderful moments we shared in Japan and at formal dinners.

Speaking of which, I am so grateful to everyone who shared unforgettable evenings with me at formal dinners across different colleges—what a privilege! I must especially thank Hertford College, where I taught for three years. Thanks to Fernando for proposing me as a lecturer, and to Alan and Dawid for believing in me. Thanks especially to Tom and Pat, the two principals during my time there: such extraordinary people! Thank you to all my students over the years. Thanks to the amazing people in the SCR, and the college catering team, always wonderful. Forever grateful.

Thanks to all the Italians I met in Oxford during the PhD, who made me feel at home in many moments. In particular, thanks to my friends from the Chaplaincy. Thank you also to Claudia, my friend from Abingdon, whom I met by chance one Sunday morning in town. Thank you for reminding me how to do good without expecting anything in return. And my heartfelt thanks to Nadia, from "my van" in the centre, and her lovely mum. Thanks to Rohit, for bringing Nadia and Coke Studio songs into my life, and offering support when it counted.

Thanks to everyone I met in my first-year accommodation in college. Thank you to Corina, my next-door neighbour, whose smile always lifted me up. A huge thank you to Priya, my housemate in my third year, whom I had met in that same first-year house. Life brought us back together. It was a tough year for both of us, but we supported each other and grew close. You are now a sister to me. I will never forget our random cinema trips, our love for instant ramen, our shared chocolate cakes (especially from Sainsbury's!), our beautiful home, and your sweet mum, who joined us in the last months and made me feel part of the family. You are both in my heart forever.

Finally, thank you to my current flatmates: Harry, Helen, and Sascha. We ended up in this house by chance, and I did not know what to expect. But I discovered three wonderful people with whom I formed a deep bond, and I will miss you all so much. Thank you for trying so hard to coordinate weekly plans. For the afternoon tea, the formal dinner, the BBQ, the Piña Coladas, Paddington movies, frozen pizzas, and all our moments. I am incredibly grateful and happy.

E adesso i ringraziamenti alle persone della mia famiglia italiana che sono stati *Chicchi di luce*, oltre la distanza. Grazie a mamma e papà, per avermi dato la vita e condividerne le gioie con me. Grazie per avermi trasmesso la passione per la matematica e per la musica, una parte significativa di chi sono. Grazie per aver sostenuto le mie scelte e per ricordarmi sempre di guardare avanti, anche nei momenti in cui non è per nulla facile. Grazie a Gianluca, per capirmi oltre le parole ed esserci sempre. Grazie alle zie e zio, per tutto l'amore dimostrato in questi anni. Per avere sempre un pensiero per me. Grazie ai miei nonni, per la gioia sincera che mostrano ogni volta che ci rivediamo.

Un altro grazie alla mia famiglia di anima, le persone conosciute a Pisa. Anche se ora siamo sparsi per il mondo, so che siamo legati per sempre. In questa famiglia di anima rientrano anche persone dalla mia città natale. Grazie a Claudia, Mariacristina, Riccardo, Giammarco, e tutti coloro che hanno avuto un ruolo fondamentale nella mia crescita. Grazie a Luciano, che è sempre presente ormai da dieci anni. E un grande grande grazie a Maria, la mia amica d'infanzia. Ogni volta che ci rivediamo, è come se il tempo non fosse mai passato.

Un ultimo grazie alla mia seconda famiglia di scelta. Matteo, grazie per essere stato una felicità inaspettata. Grazie per i film dello Studio Ghibli, Star Wars, Harry Potter, le puntate di Friends, e i film scelti a caso. Per i date online nel weekend. Per la nostra sfacciata fortuna con le lotterie di Broadway. Tutte le cene e le nostre colazioni. La pizza americana, i pancake, i cocktail da Turtle Bay. Le uscite per fare la spesa che tanto ci piacciono. Per le discussioni di fisica. Per avermi salvato prestandomi PC e caricatore mentre il mio stava cedendo. Per avermi salvato in tanti altri momenti. Per avermi insegnato a comunicare e costruire insieme. E per tutto ciò che verrà. Grazie anche alla tua meravigliosa famiglia, che è un po' anche la mia ora. Per essere stata una seconda casa dal primo istante. Per le partite a burraco, il sushi, Caspoggio e i pomeriggi con Filippo. Grazie di cuore!

I am deeply grateful to every single person who has crossed my path and helped shape who I am. Sempre, *in Movimento*.

Contents

Abstract

Scattering amplitudes capture the dynamics and interactions of particles and strings, serving as powerful tools to test predictions and explore conjectures, while revealing deep symmetries and mathematical structures. Recent decades have seen dramatic advances in understanding and computing flat space amplitudes, revealing unexpected patterns. This thesis explores how similar structures emerge in curved backgrounds. In particular, we use the AdS/CFT correspondence to define on-shell amplitudes in AdS from conformal correlators in the boundary CFT, referred to as *holographic correlators.*

First, we bootstrap higher-point holographic correlators, neglecting stringy corrections. This allows us to probe deeper into the dual CFT. We present our results for the six-point supergluon amplitude in AdS_5, where the answer is fixed completely by the flat space limit and factorisation, and outline the strategy for the graviton case.

We then present the AdS Virasoro–Shapiro program, which addresses the challenge of computing string amplitudes in curved spacetimes, where no direct worldsheet formulation exists, even at tree-level. We study the tree-level amplitude of four gravitons in Type IIB superstring theory on $AdS_5 \times S^5$, dual to $4d$ $\mathcal{N} = 4$ SYM. The AdS Virasoro-Shapiro amplitude is defined by a curvature expansion around flat space, organised as a series in $1/R$, where R is the radius of AdS. Motivated by the appearance of odd zeta values in the low-energy expansion in flat space, a conjectural *single-valuedness* in AdS has guided recent advances. We contribute by analysing the high-energy and Regge limits, where *exact* computations confirm a worldsheet structure involving single-valued logarithms.

To conclude, we conduct a detailed analysis of the mathematical structure governing strings in AdS backgrounds, making the deep interplay with number theory explicit. A key result is an AdS generalisation of the celebrated stringy KLT relations, expressing closed-string amplitudes as bilinears of open-string ones via an exactly computable Kernel. We furthermore show that the building blocks for open-string amplitudes are given by Aomoto-Gelfand hypergeometric functions.

Chapter 1

Introduction

The goal of Theoretical Physics is to formulate frameworks that faithfully describe nature. Mathematical Physics, in turn, develops the underlying tools—often pushing them well beyond their original scope—but history has shown that these advances can yield fresh insights into real physical theories, when applied in the appropriate limits. It is in this context that the core research for this thesis was conducted.

Gauge and gravity theories play a central role in our understanding of fundamental physics. The weak, strong and electromagnetic interactions arise from gauge symmetries, whereas gravity governs the large-scale evolution of the Universe and the geometry of spacetime. Looking back at how these frameworks have evolved to capture different aspects of nature, we recognise that each theory is only valid within a specific regime. Classical Mechanics accurately describes macroscopic motion, but breaks down at atomic scales. Non-relativistic Quantum Mechanics provides an exceptionally accurate description of atomic and subatomic phenomena, but cannot account for the Lorentz symmetry required at velocities approaching the speed of light, an issue resolved by Special Relativity.

Quantum Field Theory (QFT) unites Quantum Mechanics and Special Relativity into a self-consistent framework for particle interactions, forming the basis of our best understanding of nature and encompassing the gauge theories introduced above.[1] All particles and forces emerge from quantum fields filling space and time. What we call electrons, quarks, photons, etc., are specific vibrational modes/excitations of those fields. A paradigmatic example of a gauge QFT is the Yang–Mills (YM) theory [1], a non-abelian theory in which self-interacting gauge bosons, associated with a compact Lie group, mediate the fundamental forces such as the strong and electroweak interactions. It is renormalisable in four dimensions and becomes asymptotically free at high energies.

On the other side, Einstein's General Relativity [2] describes gravity as the geometry of a curved

[1] Relativistic Quantum Mechanics, such as the Dirac or Klein–Gordon equations, incorporates Special Relativity without full field quantisation, but cannot fully describe interacting systems with particle creation and annihilation. Notice also that QFT as a framework based on quantised fields needs not be relativistic.

spacetime, with inertial motion given by geodesics in that manifold. Famously, incorporating General Relativity into a quantum-field-theoretic description leads to uncontrollable ultraviolet (UV) divergences, revealing that naive quantisation of gravity fails, although being well-defined in the infrared (IR).

Over time, it has become clear that developing a *unified framework* for gauge theories and gravity requires moving beyond traditional methods and embracing new principles and symmetries. String theory offers one such framework. Unlike QFT, where particles are modelled as point-like objects, string theory describes all particles as different vibrational modes of a one-dimensional object. The type of particle that a string represents depends on how it vibrates. Gauge bosons (like photons, gluons, etc) arise as massless excitations of open strings. Gravitons, the quanta of the gravitational field, emerge as massless excitations of closed strings. Gauge and gravity interactions are then embedded in the same fundamental theory, emerging from the same underlying object, rather than being postulated independently. String theory is finite at the quantum level, offering a consistent theory of quantum gravity.

For the present work, string theory will serve as an elegant and unifying *mathematical* framework in which gauge and gravitational theories can be treated simultaneously. In the interest of intellectual honesty, it must be acknowledged that despite its structural elegance, there is no direct experimental evidence for it as a fundamental description of nature. Nevertheless, string theory has proven to be a powerful framework for exploring QFTs, that instead enjoy strong experimental support and emerge in the low-energy regime. In the following, we will present two compelling examples of how string theory can be used to uncover deep *dualities* between different physical theories.

The first example is the Anti-de Sitter (AdS)/Conformal Field Theory (CFT) correspondence, a duality between a gravitational theory in $(d+1)$-dimensional AdS space, and a conformal gauge theory living on its boundary, a flat d-dimensional spacetime, proposed by Maldacena in 1997 [3]. By employing holography in string theory, one can access and analyse the boundary QFTs at strong coupling, regimes that are challenging to study with conventional methods. Indeed, in a QFT with coupling constant $g \ll 1$, perturbation theory—an expansion in powers of g—provides an accurate description. However, when $g \sim \mathcal{O}(1)$ or larger, the theory enters a strongly coupled phase, and perturbative methods break down.[2] In these cases, one must resort to genuinely non-perturbative techniques, with holography in string theory being among the most powerful available. In particular, the correspondence allows for computing quantum effects in a strongly coupled field theory using a classical gravitational theory. Conversely, one can leverage powerful CFT techniques (*e.g.* bootstrap, conformal block expansions) to reconstruct and constrain the dynamics of fields propagating in the bulk. For this work, we will mainly use AdS/CFT to study *holographic correlators*, correlation

[2]The paradigmatic strong-coupling problem is the strong force, which confines quarks and gluons into protons and neutrons and binds those nucleons into atomic nuclei.

functions of boundary excitations in AdS that are dual to CFT correlators, in theories with maximal and half-maximal superconformal symmetry. In particular, while maximal superconformal symmetry (such as $4d$ $\mathcal{N}=4$) simplifies computations, half-maximal supersymmetry (like $4d$ $\mathcal{N}=2$) still provides valuable insights and supports the definition of gluon scattering in AdS.

It is always within the context of string theory that another perspective on gravity from gauge theories was proposed back in 1985 by Kawai, Lewellen and Tye (KLT) [4]. This *double-copy* structure[3] establishes a surprising map between observables in different theories, most famously relating Yang-Mills theory to gravity [5]. As is often the case, this connection is not apparent at the level of the Lagrangian but emerges instead through the study of observable quantities such as *scattering amplitudes*. Perturbatively, these objects reveal that the dynamics of two seemingly distinct classes of theories are governed by the *same* underlying kinematic structures. In this sense, the double-copy offers a potential unification of gauge theory and gravity, recasting both in terms of a single set of building blocks that obey the same algebraic relations. An open question remains whether the double-copy can be understood as a general feature of gravitational theories. This structure originally emerged in string theory: perturbative closed-string tree amplitudes can be expressed as bilinears of open-string amplitudes [4, 6]. In the low-energy/infinite tension limit (below the Planck scale), this relation becomes the statement that, for example, gravity amplitudes can be written as "squares" of Yang-Mills amplitudes. This provides a beautiful dynamic interchange between QFT and string theory, organising a web of theories in principle very different.

In this thesis, we make use of the AdS/CFT correspondence and later explore KLT relations in AdS as powerful frameworks for probing the interplay between gravity and gauge theories. We begin with presenting the AdS/CFT duality in Chapter 2, where amplitudes in AdS, naively viewed as a "box" with no asymptotic states, are given precise meaning via boundary CFT correlators. As suggested by the title of this thesis, we focus on several *limits* of the full string theory in AdS.[4] First, in the field theory limit, we compute supergravity/gauge theory higher-point holographic correlators in Chapter 3. Next, in Chapter 4, we present our findings for the high-energy limits of string amplitudes in AdS, extracting lessons that inform the *full* theory. Finally, in Chapter 5, we revisit the double-copy structures, or more precisely, the KLT relations, now in AdS, and identify the fundamental building blocks of AdS string amplitudes, compared to their flat space counterparts. Throughout, flat space remains our guiding intuition: each Chapter opens with a review of the necessary background, typically beginning with flat space results. Moreover, our curved space string amplitudes are defined via a small-curvature expansion around flat space.

We conclude by summarising the key results and proposing some avenues for future investigation.

[3] To be precise, the term double-copy is usually restricted to a field-theoretic framework.

[4] Here, we refer to a *putative* string theory in AdS, whose various limits we study formally. As will become clear, there is no fully established worldsheet description of string theory on generic curved spacetimes, so these limits effectively *define* the theory.

Chapter 2

Basics of AdS/CFT

The AdS/CFT [3] is a conjectured duality central to the analytic study of strongly coupled QFTs in the large-N limit.[1] It relates certain d-dimensional CFTs to supergravity on a spacetime of the form $AdS_{d+1} \times \mathcal{M}$, where $\mathcal{M}$ is a compact internal manifold. In the maximally supersymmetric case, $\mathcal{M}$ is a sphere. More generally, in the bulk we have superstring theory/M-theory as a quantum theory of gravity, whose low-energy limit is $10d/11d$ supergravity.

The term *holography* is used in this context to describe how the lower-dimensional theory, defined on the boundary of AdS spacetime, encodes information about the higher-dimensional gravitational bulk, much like a two-dimensional hologram encodes the appearance of a three-dimensional object. Even though an exact proof of the gauge–gravity correspondence is still lacking, a vast body of evidence supports it, including numerous non-trivial computations that yield matching results on both sides of the duality. We will cite, in due course, the extensive literature most relevant to the results in this thesis. But the AdS/CFT has been extended to make predictions for many other scenarios, such as large-N QCD-like models [7] and condensed matter systems [8–10]. Moreover, it is especially powerful for theories without a Lagrangian formulation, such as $4d$ $\mathcal{N} = 3$ SCFTs [11, 12] and $6d$ $\mathcal{N} = (2, 0)$ theories [13, 14].

The original and paradigmatic example of the AdS/CFT duality is the equivalence between $\mathcal{N} = 4$ Super Yang–Mills (SYM) theory in four dimensions, with gauge group $SU(N)$ and Yang–Mills coupling g_{YM}, and Type IIB superstring theory on $AdS_5 \times S^5$. In this setup, the string coupling g_s is related to the field theory via $g_s \sim g_{\mathrm{YM}}^2$, and both the AdS and sphere radii scale as $(g_{\mathrm{YM}}^2 N)^{1/4}$ in string units. In the large N limit with fixed 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$, the string theory becomes weakly coupled, and the supergravity approximation is valid.

The plan of this Chapter is to review the fundamentals of CFTs and the bootstrap approach, and then define (Euclidean) AdS space to lay the groundwork for understanding the AdS/CFT duality

[1] N may denote the rank of the gauge group of the boundary theory. In the stringy setup, it is the number of branes whose worldvolume theory defines the CFT.

and its associated dictionary. The main observables considered in this setup are the *holographic correlators*, namely correlators between boundary excitations in AdS, dual to correlators in the CFT. We focus on half-BPS (Bogomol'nyi-Prasad-Sommerfield) multiplets[2] under both maximal and half-maximal supersymmetry.

2.1 The conformal side

Figure 2.1: The recursive distortion in Escher's "Print Gallery" artwork evokes the behaviour of conformal maps, which preserve local angles, but not distances.

Let us start by motivating what makes CFTs so special in modern Theoretical Physics. One possible perspective is the renormalisation group (RG) [17–19], a term first introduced in [20]. This is an elegant tool to understand the behaviour of a theory under a change of the scale at which it is observed. Among all physical systems, a special role is played by those that are invariant under scale transformations and are therefore fixed points of the RG. QFTs are then trajectories in the space of theories under RG flow, typically interpolating between a UV and an IR fixed point. There are three possible IR behaviours: (i) a theory with a mass gap (such as $4d$ YM), (ii) a theory with massless particles (such as quantum electrodynamics (QED) or massless quantum chromodynamics (QCD) with a small number of fermionic flavours), or (iii) a scale-invariant theory with a continuous spectrum. It is widely believed that most physically relevant scale-invariant theories are in fact conformally invariant [21]. In such cases, the IR fixed point is described by a CFT, corresponding to the third possibility in the above RG flow classification.

Local operators of a theory include certain distinguished examples, such as the *stress-energy tensor* $T_{\mu\nu}$, the Noether current associated with spacetime translations, and *conserved currents* J_μ associated with global symmetries. Their existence is tied to the locality of the theory. In perturbation theory, these operators are protected from renormalisation.[3] At the fixed point, they retain their canonical dimensions, $\Delta_T = d$ and $\Delta_J = d - 1$, and satisfy conservation laws. Furthermore, if the theory possesses Poincaré invariance, then $T_{\mu\nu}$ can be chosen to be symmetric.

[2]As their name suggests, their superconformal primary is annihilated by half of the Poincaré supercharges [15, 16].
[3]Usually, operators acquire anomalous dimensions along the RG flow.

An RG transformation (scale transformation) rescales globally the coordinates, which in turn induces a corresponding rescaling of the background metric. The energy-momentum tensor/stress tensor can also be defined as the response of the action to variations of the background metric. If the theory is scale-invariant, the variation of the action vanishes,

$$\delta S = \int d^d x \, T_{\mu\nu} \, \delta g^{\mu\nu} \propto \int d^d x \, T^\mu_\mu = 0 \; , \tag{2.1}$$

so the trace of the energy–momentum tensor vanishes. More generally, scale invariance implies that the trace takes the form $T^\mu_\mu = \partial_\rho K^\rho$, a total divergence of a vector operator. The operator K^μ has then dimension $[K^\mu] = d - 1$, which is the canonical dimension of a conserved current in a free theory. In most cases, it is difficult to construct a consistent candidate for K^μ that is not itself conserved. Therefore, T^μ_μ must vanish. This implies that scale invariance is enhanced to full conformal invariance.[4]

This enhancement can be understood geometrically. Arbitrary local rescalings of the metric define Weyl transformations, which modify the spacetime geometry. To remain within flat spacetime, one must restrict to the subset of Weyl transformations that are also diffeomorphisms, coordinate transformations that locally rescale the metric without introducing curvature. These are the *conformal transformations*, which preserve angles between crossing curves.

More concretely, conformal transformations are invertible maps that do not affect the metric up to an overall local rescaling:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \; , \tag{2.2}$$

where $\Lambda(x)$ can be an arbitrary function of the coordinates, equal to 1 in the special case of isometries. The variation of the action is proportional to the trace of the stress-tensor, which vanishes by assumption. In summary, $T^\mu_\mu = 0$ implies invariance under (infinitesimal) Weyl transformations.

Looking at (2.2), by solving the Killing equation, one can see that if $d = 1$, each smooth transformation is conformal (trivial, since there is no notion of angle). Instead, the case $d = 2$ admits a special description [25]. Here, we will be working with CFTs with $d > 2$. The infinitesimal transformations are

$$\begin{aligned}
&x^\mu \to x^\mu + c^\mu && \text{(translations)} \; , \\
&x^\mu \to x^\mu + \omega^{\mu\nu} x_\nu && \text{(rotations)} \; , \\
&x^\mu \to x^\mu + \lambda \, x^\mu && \text{(dilatations)} \; , \\
&x^\mu \to x^\mu + 2(b \cdot x) x^\mu - x^2 b^\mu && \text{(special conformal transformations (SCTs))} \; ,
\end{aligned} \tag{2.3}$$

[4] This enhancement is automatic in $d = 2$ and can be shown in perturbation theory for $d = 4$ [22, 23]. See, however, [24] for a physical theory that is scale-invariant but not conformally invariant.

where b^μ is an arbitrary constant vector in $\mathbb{R}^d$.[5] The set of conformal transformations forms a group: it is closed under composition and each conformal transformation has an inverse. It is the largest finite-dimensional subgroup of *Diff* (diffeomorphisms of $\mathbb{R}^d$). Its generators can be written in terms of differential operators,

$$\begin{aligned} P_\mu &= -i\partial_\mu && \text{(translations)} \ , \\ M_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) && \text{(rotations)} \ , \\ D &= -ix^\mu\partial_\mu && \text{(dilatations)} \ , \\ K_\mu &= -i\left(2x_\mu x^\rho\partial_\rho - x^2\partial_\mu\right) && \text{(SCTs)} \ , \end{aligned} \tag{2.4}$$

with definite commutation relations. Beyond the usual Poincaré algebra, we emphasise that $[D, P_\mu] = iP_\mu$ and $[D, K_\mu] = -iK_\mu$, implying that P_μ and K_μ behave like ladder operators for D, building the so-called *conformal multiplet* of the operator $\mathcal{O}$. The base is called primary, with $[K_\mu, \mathcal{O}] = 0$, while the other levels of the tower, reached by application of P_μ, are named descendants. For more details, we refer, for example, to the great set of lectures [26].

Notice that, if we define

$$\begin{aligned} &J_{\mu,\nu} = M_{\mu\nu}, && J_{-1,\mu} = \tfrac{1}{2}\,(P_\mu - K_\mu), \\ &J_{-1,0} = D, && J_{0,\mu} = \tfrac{1}{2}\,(P_\mu + K_\mu), \\ &J_{a,b} = -\,J_{b,a} \quad \text{for} \quad a = -1, 0, 1, \ldots, d \ , \end{aligned} \tag{2.5}$$

the generators $J_{a,b}$ satisfy the $SO(d+1,1)$ commutation relations, or $SO(d,2)$ in Minkowski space. This implies that the conformal group is a Lie group with $(d+1)(d+2)/2$ parameters, isomorphic to $SO(d+1,1)$ (or $SO(d,2)$).

A central feature of CFTs is that the form of one-, two-, and three-point functions of primaries is entirely determined by conformal symmetry. For scalar operators:[6]

$$\begin{aligned} &\langle\mathcal{O}(x)\rangle = 0 \ , \\ &\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{\delta_{12}}{|x_{12}|^{2\Delta_1}} \ , \\ &\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}} \ , \end{aligned} \tag{2.6}$$

where Δ_i are the scaling dimensions of the operators (eigenvalues of D), and $C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$ is a constant

[5] The finite transformations can be obtained by exponentiation. However, the quadratic nature of the SCTs prevents us from passing directly to the finite form. One can use the trick of writing the infinitesimal transformation as a composition of an inversion, a translation, and again an inversion.

[6] Here, δ_{12} reflects the fact that the two-point function vanishes if the two fields have different scaling dimensions, $\Delta_1 \neq \Delta_2$. More precisely, if a theory contains several fields $\mathcal{O}_i$ with the same scaling dimension, the two-point function involves a matrix M_{ij}, which is positive definite if the theory is unitary. In this case, there always exists a field basis in which M_{ij} is the identity matrix.

but physical coefficient, which is completely symmetric in the exchange of two indices.

Starting from four points, conformal covariance allows for the extraction of a kinematical factor $\mathcal{K}_4(\Delta_i, x_i)$, but then we are left with an unknown function of the conformally invariant cross ratios,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \frac{1}{|x_{12}|^{2\Delta_1}|x_{34}|^{2\Delta_3}}\, g(U,V) \equiv \mathcal{K}_4(\Delta_i, x_i)\, g(U,V)\ , \tag{2.7}$$

where

$$U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\ , \quad V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\ , \tag{2.8}$$

and $x_{ij} \equiv x_i - x_j$. The number of independent cross ratios is related to the number of dimensions of spacetime (d) as well as the number of points (n) in the correlator, according to[7]

$$\begin{aligned} &n < d+1\ ,\ \frac{n(n-3)}{2}\ ,\\ &n \geq d+1\ ,\ nd - \frac{(d+1)(d+2)}{2}\ . \end{aligned} \tag{2.9}$$

Importantly, it is always possible to reduce the n-point correlation function to $(n-1)$-point correlation functions, so that in the end $g(U,V)$ (from four-point functions) is related to $C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$ (from three-point functions), and so on. This is motivated by the (convergent) Operator Product Expansion (OPE):

$$\mathcal{O}_i(x) \times \mathcal{O}_j(y) = \sum_{\text{primary }\mathcal{O}_k} \frac{C_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}}\Big(\mathcal{O}_k(y) + \text{descendants}\Big)\ . \tag{2.10}$$

$C_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k}$ is called *OPE coefficient* (or structure constant of the operator algebra), and it corresponds to the free parameter appearing in the three-point function (2.6). Notice that we cannot rescale it away, as we have already fixed the normalisation of the two-point function. As a consequence, if we choose a pair of operators in a correlator and expand it via its OPE, we get the nice expression

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_i \frac{C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_i}C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}_i}}{|x_{12}|^{2\Delta_1}|x_{34}|^{2\Delta_3}}\, g_{\Delta_i,l_i}(U,V)\ , \tag{2.11}$$

where, for simplicity, we have taken $\Delta_1 = \Delta_2$ and $\Delta_3 = \Delta_4$. Here, $g_{\Delta_i,l_i}(U,V)$ is called *conformal block* and it is the contribution to the four-point function of the primary operator $\mathcal{O}_i$ and of *all* its descendants. In general, each conformal block is an eigenfunction of the quadratic conformal Casimir $\mathcal{C}_2$ with eigenvalue $C_2(\Delta, l) = \Delta(\Delta - d) + l(l + d - 2)$, where l is the spin of the exchanged primary operator under the Lorentz group. Notice that $g(u,v)$ from (2.7) can be expressed as

$$g(U,V) = \sum_i C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_i}C_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}_i}\, g_{\Delta_i,l_i}(U,V)\ , \tag{2.12}$$

[7] See, for example, [27].

hence it is not an independent quantity, being related to the three-point functions in a non-trivial way.

Equation (2.11) is the result of using the OPEs in the channels $|x_{12}| \to 0$ and $|x_{34}| \to 0$, but it is clear that we could have alternatively considered $|x_{14}| \to 0$ and $|x_{23}| \to 0$, or any other permutation. Different ways of making pairs out of four operators have to be equivalent, which is mathematically captured by the so-called *crossing equation* (or OPE associativity condition). Demanding invariance under the exchange $x_1 \leftrightarrow x_4$:

$$g(U,V) = \left(\frac{U}{V}\right)^{\Delta} g(V,U) \ , \tag{2.13}$$

where $g(U,V)$ is expressed in terms of the OPE coefficients and conformal blocks as in Equation (2.12). Crossing symmetry therefore translates into non-trivial constraints on these data. In particular, given an arbitrary set of CFT data (spectrum and OPE coefficients), there is no guarantee that it defines a consistent theory: the four-point functions must satisfy the crossing equation for all values of u and v. The *bootstrap* problem is to identify spectra and OPE coefficients for which this consistency holds. This concept was first introduced by Belavin *et al.* [28] in the investigation of the $2d$ CFTs. Later, the idea has been improved numerically to the $d > 2$ case [29], with a particular interest in $3d$ systems [30]. Great lectures are [26, 31]. The general aim is to classify the space of CFTs, where each theory is completely characterised by its CFT data (scaling dimensions of the primaries and OPE coefficients). In principle, given a set of CFT data, we can compute *all* the correlation functions. Determining this set for physical systems can be very difficult. The conformal bootstrap offers the powerful perspective of focusing only on values of operator dimensions and OPE coefficients that are consistent with fundamental principles such as unitarity and crossing symmetry. Remarkably, this leads to the determination of universal bounds on key physical quantities without requiring complete knowledge of the infinite set of CFT data.

Note on superconformal symmetry

The CFTs we will be working with actually possess supersymmetry [32, 33], making them superconformal field theories (SCFTs). While we will not go into detail here, a comprehensive reference for the classification of multiplets in SCFTs can be found in [34]. Let us very briefly highlight the main new elements introduced by supersymmetry, compared to the purely bosonic case discussed above. A superconformal algebra extends the conformal algebra by including additional generators:

- supersymmetry generators Q, and their conformal counterparts S, forming a graded Lie superalgebra;

- internal symmetry generators, collectively forming the *R-symmetry* group, which rotate the supercharges.

In flat d-dimensional Minkowski space, the bosonic part of the superconformal algebra includes the Poincaré group, dilatations, special conformal transformations and the R-symmetry, whose structure depends on the spacetime dimension and number of supercharges. The fermionic part includes the Poincaré supercharges Q and the conformal supercharges S.

Superconformal algebras exist only in spacetime dimensions $d \leq 6$ [35]. A defining feature of SCFTs is the presence of the R-symmetry as part of the full symmetry algebra. As a result, operators in SCFTs will carry additional labels, corresponding to the Dynkin labels of the R-symmetry group. See [15] for the unitarity restrictions on the scaling dimensions.

2.2 The gravitational side

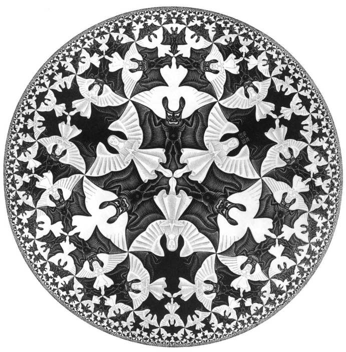


Figure 2.2: Escher's "Circle Limit IV" artwork of the Hyperbolic Plane.

We now turn to the gravitational side, and in particular to Euclidean AdS space. We will first recall how Minkowski space arises as the boundary of Lorentzian AdS, and then move to its Euclidean counterpart and the corresponding holographic framework.

Conformal transformations can send finite points of Minkowski space to infinity. To render the action of the full conformal group $SO(d, 2)$ smooth and globally well-defined, one therefore passes to the conformal compactification of Minkowski space. This is most economically realised as the projective quadric in $\mathbb{P}^{d+1}$ on which $SO(d, 2)$ acts linearly [36]. Homeomorphic to $\mathbb{R} \times S^{d-1}$, this compactified manifold matches precisely the boundary of global AdS_{d+1}, and it provides the natural stage for the holographic duality, in the sense of [37, 38].

We can still get this identification in Euclidean signature. Let us consider the Euclidean space $\mathbb{R}^{d+1}$ with some coordinate $\{y_i\}$, with $i = 0, \ldots, d$. We can define the open unit ball

$$B_{d+1} \ : \ \sum_{i=0}^{d} y_i^2 < 1 \ , \tag{2.14}$$

such that it corresponds to AdS_{d+1} with metric

$$ds^2 = \frac{4\sum_{i=0}^{d} dy_i^2}{(1-|y|^2)^2} \ . \tag{2.15}$$

The next step is to compactify the open unit ball to get the closed one, $\overline{B}_{d+1}$, whose boundary is the sphere S^d,

$$\sum_{i=0}^{d} y_i^2 = 1 \ . \tag{2.16}$$

This is the Euclidean version of the conformal compactification of Minkowski space. Schematically:

$$\partial(\overline{B}_{d+1}) = S^d \iff \partial(AdS_{d+1}) \ = \ \text{Minkowski space} \ . \tag{2.17}$$

Let us notice that the metric on the open unit ball does not extend over the closed one since it is singular at $|y| = 1$, so one has to rescale the metric by some function f on $\overline{B}_{d+1}$ which is positive on the open ball and has a first order zero on the boundary [36]. Hence, we consider $d\tilde{s}^2 = f^2 ds^2$, which is defined up to conformal transformations. By this, we mean that, strictly speaking, the Euclidean version of AdS_{d+1} has a metric invariant under $SO(d+1,1)$, while the boundary (S^d) has only a conformal structure, preserved by the action of $SO(d+1,1)$.

The Euclidean continuation of AdS_{d+1} can be viewed as the $Y_{-1} > 0$ sheet of the hyperboloid [39]

$$-(Y_{-1})^2 + (Y_0)^2 + \sum_{i=1}^{d}(Y_i)^2 = -\frac{1}{R^2} \ , \tag{2.18}$$

embedded in a $d+1$ dimensional space with metric of signature $(-++\cdots+)$. If we introduce the coordinates

$$\begin{aligned} z_i &\equiv \frac{Y_i}{R(Y_0+Y_{-1})} \ , \quad i = 1,\ldots,d \ , \\ z_0 &\equiv \frac{1}{R^2(Y_0+Y_{-1})} \ , \end{aligned} \tag{2.19}$$

they provide a complete coordinate chart $z_\mu(Y)$ such that the induced metric on the hyperboloid takes the form of the upper half-space in $z_\mu \in \mathbb{R}^{d+1}$ with $z_0 > 0$ and metric $g_{\mu\nu}$ of constant negative curvature $\mathcal{R} = -d(d+1)/R^2$:[8]

$$ds^2 = \sum_{\mu,\nu=0}^{d} g_{\mu\nu}\, dz_\mu\, dz_\nu = \frac{R^2}{z_0^2}\sum_{\mu=0}^{d} dz_\mu^2 = \frac{R^2}{z_0^2}\left(dz_0^2 + \sum_{i=1}^{d} dz_i^2\right) \ . \tag{2.20}$$

We will set $R = 1$ in the rest of the Chapter. In this representation, the boundary of AdS_{d+1} is a copy of $\mathbb{R}^d$ at $z_0 = 0$ with a point at $z_0 = \infty$, which defines the sphere S^d.

[8]The Christoffel symbols are $\Gamma^k_{\mu\nu} = \frac{1}{Rz_0}(\delta^k_0\delta_{\mu\nu} - \delta_{\mu 0}\delta^k_\nu - \delta_{\nu 0}\delta^k_\mu)$.

Notice that the inversion $z'_\mu = z_\mu/z^2$ is an isometry of the metric, where contractions as z^2 use the Euclidean metric $\delta_{\mu\nu}$, so we are indifferent to the question of raising/lowering coordinate indices in this case, $z_\mu = z^\mu$. Otherwise, we will contract indices using the AdS metric, like

$$\partial^\mu \phi \partial_\mu \phi = g^{\mu\nu} \partial_\nu \phi \partial_\mu \phi = z_0^2 \delta_{\mu\nu} \partial_\nu \phi \partial_\mu \phi = z_0^2 \partial_\mu \phi \partial_\mu \phi \ . \tag{2.21}$$

We shall focus on AdS_5, the unique maximally symmetric solution of the five-dimensional Einstein equations with negative cosmological constant.[9] Upon Wick rotation to Euclidean signature, it becomes the hyperbolic space H^5, whose isometry group is $SO(5,1)$. It can be thought of as the set of solutions of

$$x_0^2 + x_5^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = R^2 \ , \tag{2.22}$$

in flat $\mathbb{R}^{2,4}$. This matches the conformal group in four dimensions, reflecting the general fact that the conformal group in d-dimensions coincides with the isometry group of AdS_{d+1}.

Additional details can be found in the numerous good reviews available, such as [40–43].

2.3 Dictionary of the duality

The original duality [3] was made explicit in [36, 44], with a prescription for matching observables across the two theories. The precise holographic dictionary is ultimately provided by string theory in an AdS background, and the duality itself stands as one of the most remarkable achievements of string theory. For the expert reader, the key insight is that low-energy open strings ending on a stack of N coincident D3-branes give rise to a four-dimensional $U(N)$ $\mathcal{N} = 4$ SYM theory on the brane world-volume.[10] In the near-horizon limit of these D3-branes, the bulk geometry becomes $AdS_5 \times S^5$, and the spectrum of closed-string excitations matches that of single-trace operators in the boundary CFT [3].[11] Although we will not review the full string-theoretic derivation here, it is this D-brane construction that provides the concrete foundation for the map between CFT correlators and amplitudes in the bulk.

Let us explain how to map the quantities on the two sides of the duality. The fields in five dimensions will be referred to as *bulk fields* (h) and they interact according to the gravitational action with an AdS vacuum. As usual, the action is a function of the metric, gauge fields, scalars, etc. We also assume it contains a potential for the scalar field with a negative value at the minimum, such that we get the AdS vacuum. On the other side, we have the fields living in the $4d$ CFT, and

[9] The Einstein equations are the central equations of General Relativity, relating the curvature of spacetime to its matter and energy content. In the absence of matter, they reduce to $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$, where Λ is the cosmological constant and R is the Ricci scalar. For $\Lambda < 0$, the maximally symmetric vacuum solution is AdS.

[10] In string perturbation theory, a D-brane is a hypersurface on which open strings may end.

[11] We will consider only single-trace operators since multi-trace operators correspond to multi-particle states in the bulk. See [45] for interesting remarks when stringy corrections are included.

we will refer to them as the *boundary fields* $\mathcal{O}$, that are classified in (super) conformal multiplets. Importantly, h and $\mathcal{O}$ have the same quantum numbers and know about each other via boundary couplings. In this sense, for example, the stress-energy tensor in the CFT corresponds to the graviton and any conserved current will be dual to a gauge field in AdS. The general pattern is that global symmetries in the CFT correspond to gauge symmetries in AdS.

From the CFT perspective, every operator can be associated to a source. In the first instance, we think of $h(x)$ as a $4d$ background field that we use to compute correlators of the operators in the QFT. Then, we interpret it as the boundary value of a field $\hat{h}(x, x_5)$ living in five dimensions. The way that we assure it exists for any source configuration $h(x)$ is by demanding that $\hat{h}(x, x_5)$ solves the $5d$ equation of motion in AdS. The extension from the boundary to the bulk is unique once we impose the proper boundary conditions.

We can think of the duality as the identification

$$\langle e^{\int h\mathcal{O}} \rangle_{CFT} \leftrightarrow e^{S_{AdS_5}(\hat{h})} \ . \tag{2.23}$$

This allows for the computation of on-shell scattering amplitudes in AdS from correlation functions in the CFT. Indeed, in the CFT side, h is an arbitrary off-shell configuration, while it is put on-shell when extended to five dimensions. The Green's function responsible for this is the *bulk-to-boundary propagator* $G^{\Delta}_{B\partial}(z, \vec{x})$, where Δ is the conformal dimension, z is the $5d$ point in the bulk, and $\vec{x}$ is the $4d$ point on the boundary. It takes the form [36]

$$G^{\Delta}_{B\partial}(z, \vec{x}) = C_{\Delta} \left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right)^{\Delta} \ , \tag{2.24}$$

where

$$\begin{aligned} C_{\Delta} &= \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta - d/2)} \quad \text{for } \Delta > d/2 \ , \\ C_{d/2} &= \frac{\Gamma(d/2)}{2\pi^{d/2}} \ , \end{aligned} \tag{2.25}$$

and $\Delta = d/2$ corresponds to the lowest AdS mass allowed by unitarity [39]. We will always factor out the normalisation constants in the Witten diagrams and focus on the integrated vertices to show the relevant techniques.

Now, we need to consider the possible *interactions* among the bulk fields. The equations of motion derived from the AdS action include higher-order (nonlinear) terms, which in general prevent an exact solution. We can still use a perturbative approach to get the so-called *bulk-to-bulk propagator* (as in standard QFT computations [46]). For example, the scalar bulk-to-bulk propagator is the solution of

$$(-\Box_{AdS} + m^2)\, G_{BB}^{\Delta}(u) = \delta^d(z,w) \ , \tag{2.26}$$

where m^2 is the mass of the scalar field in the bulk. More precisely, a scalar field of mass m is characterised by two possible scale dimensions, namely the roots

$$\Delta_\pm = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2+4m^2} \ , \tag{2.27}$$

of the quadratic relation $\Delta(\Delta - d) = m^2$. Unless explicitly indicated, Δ will mean Δ_+. See, for example, the discussion in [47] and [41] for more details. In (2.26), z and w are points in the bulk and u is known as the chordal distance,

$$u = \frac{\delta_{\mu\nu}(z-w)_\mu (z-w)_\nu}{2 z_0 w_0} \ . \tag{2.28}$$

Invariant functions and tensors on AdS are expressed in terms of it.

The explicit expression of the scalar propagator is the hypergeometric function

$$G_{BB}^{\Delta}(u) = f(\Delta,d)\left(\frac{2}{u}\right)^{\Delta} {}_2F_1\left(\Delta, \Delta - \frac{d}{2} + \frac{1}{2}, 2\Delta - d + 1, -\frac{2}{u}\right) , \tag{2.29}$$

up to an overall function not relevant for the present discussion.

Although we presented the bulk-to-boundary propagator before the bulk-to-bulk one to follow the narrative flow, let us note that one can first determine the bulk-to-bulk propagator as a solution to the inhomogeneous Klein–Gordon equation in the bulk (2.26). Then, as one of the two bulk points is taken to the boundary, this defines the bulk-to-boundary propagator, being the solution of the homogeneous equation in the bulk.

We can visualise interactions in AdS/CFT by using the so-called *Witten diagrams*, the analogue of position space Feynman diagrams in flat space. In practice, we can think of Euclidean AdS_5 space as a $5d$ ball: the CFT operators live on its boundary, and the interior is the bulk. The simplest example is a contact Witten diagram, given simply by the product of n bulk-to-boundary propagators, integrated over the common bulk point (Figure 2.3). This is known in literature as D-function [48]. It is defined as the following AdS integral:

$$D_{\Delta_1,\ldots,\Delta_n} = \int \frac{dz_0\, d^d z}{z_0^{d+1}} \prod_{i=1}^{n} \left(\frac{z_0}{z_0^2 + (\vec{z}-\vec{x}_i)^2}\right)^{\Delta_i} \ . \tag{2.30}$$

One may reduce it to functions of the conformal cross ratios by extracting a kinematic factor. These are typically referred to as $\overline{D}$-functions. For $n=4$ and $\Delta_i = 1$, the $\overline{D}$-function can be expressed in

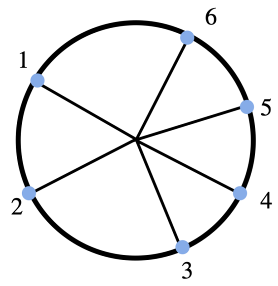


Figure 2.3: An example of contact Witten diagram. All the boundary insertions are connected to a single bulk interaction point, *without* any internal propagators connecting different bulk points.

terms of elementary functions, namely polylogarithms. More precisely, it evaluates to

$$\overline{D}_{1,1,1,1}(U,V) = \frac{1}{Z-\overline{Z}}\left[2\,\mathrm{Li}_2(Z) - 2\,\mathrm{Li}_2(\overline{Z}) + \log\big(Z\overline{Z}\big)\,\log\left(\frac{1-Z}{1-\overline{Z}}\right)\right] , \tag{2.31}$$

where we have parametrised the cross ratios as $U = Z\overline{Z}$, $V = (1-Z)(1-\overline{Z})$, with Z a complex variable. This matches the flat space scalar one-loop (box) four-point function in four dimensions [49, 50].

Instead, an exchange diagram also includes bulk-to-bulk propagators and more vertices. See Figure 2.4 for an example. Each internal line connecting two bulk vertices is a bulk-to-bulk propagator;

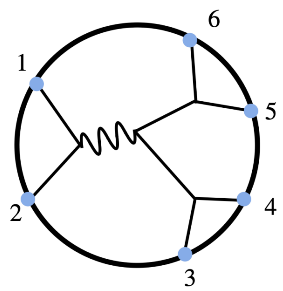


Figure 2.4: An example of exchange Witten diagram.

each external line connecting a bulk vertex and a boundary point is a bulk-to-boundary propagator. Different types of propagators can appear, as illustrated by the various lines in the example Figure. For any interacting vertex point, we have integration over the bulk AdS. The propagators are joined at interaction vertices, whose structure is typically intricate as a result of the underlying AdS geometry. In this thesis, we will rarely specify the explicit form of the interaction vertices; instead, we will employ a bootstrap strategy, leaving their coefficients undetermined and fixing them by means of consistency conditions and symmetry constraints.

The simplest kind of exchange diagram is the one with scalars only, but some exchange diagrams with vector (or tensor) propagators can be rewritten in terms of products of functions of the chordal distance on AdS (2.28) and scalar propagators. We will present an example of how to reduce the exchange of the gauge boson to the scalar exchange in Section 3.2.2. A collection of useful identities for the following:

$$
\begin{aligned}
&\Box_{AdS}\, u = D^\mu \partial_\mu u = (d+1)(1+u) \ , \\
&D^\mu u \partial_\mu u = u(2+u) \ , \\
&D_\mu \partial_\nu u = g_{\mu\nu}(1+u) \ , \\
&(D^\mu u)(D_\mu \partial_\nu \partial_{\nu'} u) = \partial_\nu u \partial_{\nu'} u \ , \\
&(D^\mu u)(\partial_\mu \partial_{\nu'} u) = (1+u)\partial_{\nu'} u \ , \\
&(D^\mu \partial_{\mu'} u)(\partial_\mu \partial_{\nu'} u) = g_{\mu'\nu'} + \partial_{\mu'} u \partial_{\nu'} u \ ,
\end{aligned}
\tag{2.32}
$$

where the primed indices refer to the bulk point w_μ.

2.4 Vertex identities in AdS

A very interesting and useful trick when dealing with position space computations of Witten diagrams is the use of vertex identities, first introduced in [47]. These identities enable the reduction of Witten diagrams to either D-functions or Feynman-like integrals.

Exchange diagrams, which involve bulk-to-bulk propagators, require at least two integrations over AdS_{d+1}, starting at the level of four-point functions. In the literature, the first bulk integration is typically referred to as the z-integral. Early studies of exchange diagrams in AdS_{d+1} employ (cumbersome) expansion and resummation techniques [51, 52]. The computation of the z-integrals is carried out by using the uniformly convergent expansion of the hypergeometric function (2.29) in powers of the parameter

$$
\xi \equiv \frac{1}{1+u} = \frac{2z_0 w_0}{(z_0^2 + w_0^2 + (\vec{z}-\vec{w})^2)} \ . \tag{2.33}
$$

The bulk-to-bulk scalar propagator then becomes

$$
G_{BB}^{\Delta}(\xi) = 2^{\Delta} C_{\Delta} \xi^{\Delta} {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta}{2} + \frac{1}{2}; \Delta - \frac{d}{2} + 1; \xi^2\right) \ . \tag{2.34}
$$

Given that by definition $|\xi| \leq 1$, the propagator has a uniformly convergent expansion in powers of ξ, implying that one can interchange the summation of the series expansion of the hypergeometric and any convergent integration of the propagator. Specifically, assuming fixed and non-coincident spacetime points, and that all conformal dimensions satisfy the unitarity bounds $\Delta, \Delta_i \geq d/2$, the convergence conditions of the integrals involved are given by

$$
\begin{aligned}
&|\Delta_1 - \Delta_2| < \Delta \ , \ |\Delta_3 - \Delta_4| < \Delta \ , \\
&\sum_{i=1}^{4} \Delta_i > 2\Delta_i \ , \ i = 1,2,3,4 \ ,
\end{aligned}
\tag{2.35}
$$

where Δ_i are the conformal dimensions of the fields in each vertex and Δ is the conformal dimension of the internal field. The first condition ensures convergence when one of the bulk points approaches a boundary point. The second condition concerns the possibility for both bulk points to approach any of the boundary points. Surprisingly, the final result for the exchange diagram is a remarkably simple function of the remaining bulk coordinate w [51, 52], leading to the suspicion that a more efficient method might exist to bypass the cumbersome intermediate computations. This insight ultimately gave rise to the so-called integrated vertex identities, which directly reduce the exchange Witten diagram to a linear combination of contact diagrams. Ideally, one would hope to obtain a finite sum of D-functions; however, in generic theories with arbitrary couplings, the result is typically an infinite series.

The strategy consists of deriving a differential equation for the z-integral, supplemented by a recursion relation that allows for its solution. Crucially, this method does not require explicit knowledge of the bulk-to-bulk propagator. The remaining bulk integrals are then evaluated using integral representations and asymptotic formulas [47].

We now present the method for integrating a cubic vertex, involving either a scalar or vector internal propagator, followed by its generalisation to the $(n+1)$-point vertex of all scalars. More attention will be devoted to the $(n+1)$-point generalisation, as it has received relatively little attention in the existing literature and, as we will see in Chapter 3, leads to interesting integrals. We will follow [47].

2.4.1 Cubic vertex: scalar exchange

We want to compute the integrated vertex

$$A(x_1, x_2, w) = \int \frac{d^{d+1}z}{z_0^{d+1}} G^{\Delta_1}_{B\partial}(z, \vec{x}_1) G^{\Delta_2}_{B\partial}(z, \vec{x}_2) G^{\Delta}_{BB}(z, w) \ , \tag{2.36}$$

where w is the other bulk point in the diagram, $G_{B\partial}$ is the bulk-to-boundary propagator, and G_{BB} is the bulk-to-bulk propagator. The scalar bulk-to-bulk propagator satisfies

$$(-\Box_{AdS} + m^2)\, G^{\Delta}_{BB} = \delta(z, w) \ , \tag{2.37}$$

where $m^2 = \Delta(\Delta - d)$. We can act on (2.36) with the differential operator $(-\Box_{AdS} + m^2)$ and use the above equation of motion to collapse the internal propagator to a delta function. This leads to a differential equation. By requiring the proper asymptotic conditions, in the end, one gets the very nice result [47]

$$A(x_1, x_2, w) = \sum_{k=k_{min}}^{k_{max}} a_k (x_{12}^2)^{k-\Delta_2} G_{B\partial}^{k+\Delta_1-\Delta_2}(w, \vec{x}_1) G_{B\partial}^{k}(w, \vec{x}_2) \ , \tag{2.38}$$

whenever the following condition holds:

$$\boxed{\Delta_1 + \Delta_2 - \Delta \in 2\,\mathbb{Z}_{\geq 0}} \tag{2.39}$$

Here,

$$\begin{aligned} k_{min} &= \frac{\Delta - \Delta_1 + \Delta_2}{2} \ , \\ k_{max} &= \Delta_2 - 1 \ , \\ a_{k-1} &= \frac{\left(k - \frac{\Delta}{2} + \frac{\Delta_1 - \Delta_2}{2}\right)\left(k - \frac{d}{2} + \frac{\Delta}{2} + \frac{\Delta_1 - \Delta_2}{2}\right)}{(k-1)(k-1-\Delta_1+\Delta_2)} a_k \ , \\ a_{\Delta_2 - 1} &= \frac{1}{4(\Delta_1 - 1)(\Delta_2 - 1)} \ . \end{aligned} \tag{2.40}$$

2.4.2 Cubic vertex: vector exchange

Similarly, one can perform the integration over a cubic vertex involving a vector internal propagator. In our case, we are primarily interested in the exchange of a massless gauge field, for which the conformal dimension is $\Delta = d - 1$. Before proceeding, let us briefly review the origin and structure of the gauge boson propagator in AdS. An abelian gauge field coupled to a conserved current source in the AdS_{d+1} is described by the action[12]

$$S_A = \int \frac{d^{d+1}z}{z_0^{d+1}} \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{\Xi}{2} (D_\mu A^\mu)^2 - A_\mu J^\mu \right) \ , \tag{2.41}$$

where Ξ is the gauge-fixing parameter and D_μ is the AdS-covariant derivative. The propagator of the massless gauge field is the solution of the AdS-covariant equation

$$D^\mu \partial_{[\mu} G_{\nu]\nu'} + \Xi\, \partial_\nu (D^\mu G_{\mu\nu'}) = -\frac{g_{\nu\nu'}}{\sqrt{g}}\ \delta^{d+1}(z, w) + \partial_{\nu'} \Lambda_\nu(z, w) \ , \tag{2.42}$$

where the primed indices refer to the bulk point w. The second term on the right-hand side (RHS) is a pure gauge term that will cancel out when integrated. The AdS gauge boson propagator can be written as [51], [53]

$$G_{\mu\nu'}(z, w) = -(\partial_\mu \partial_{\nu'} u) F(u) + \partial_\nu \partial_{\nu'} S(u) \ , \tag{2.43}$$

[12]The gauge group considered here is $U(1)$, but the generalisation to non-abelian cases is obtained by simply inserting the proper group factors.

where $F(u)$ and $S(u)$ are scalar functions of the chordal distance u, and the expression follows from the fact that any bitensor can be written as a linear combination of two independent tensor structures with scalar coefficients. In particular:

$$\begin{aligned}
\partial_\mu \partial_{\nu'} u &= -\frac{1}{z_0 w_0}\left[\delta_{\mu\nu'} + \frac{1}{w_0}(z-w)_\mu \delta_{\nu' 0} + \frac{1}{z_0}(w-z)_{\nu'}\delta_{\mu 0} - u\delta_{\mu 0}\delta_{\nu' 0}\right] , \\
(\partial_\mu u)(\partial_{\nu'} u) &= \frac{1}{z_0}\left[\frac{(z-w)_\mu}{w_0} - u\delta_{\mu 0}\right] \times \frac{1}{w_0}\left[\frac{(w-z)_{\nu'}}{z_0} - u\delta_{\nu' 0}\right] ,
\end{aligned} \tag{2.44}$$

and $S(u)$ in (2.43) is a gauge artefact that vanishes when integrated with a conserved current in a Witten diagram. $F(u)$ describes instead the propagation of the physical components of the gauge boson, and is given by

$$F(u) = \frac{\Gamma((d-1)/2)}{(4\pi)^{(d+1)/2}} \frac{1}{[u(u+2)]^{(d-1)/2}} \ . \tag{2.45}$$

The integrated vertex involving the gauge boson bulk-to-bulk propagator takes the general form[13]

$$A^\mu(x_1, x_2, w) = \int \frac{d^{d+1}z}{z_0^{d+1}} J_\nu(z)\, G^{\mu\nu}_{BB}(z,w) \ , \tag{2.46}$$

where we consider the coupling of the vector field to a conserved current. Assuming $\Delta_2 = \Delta_1$ at the vertex, the integral can be evaluated as a sum of contact vertices,

$$\begin{aligned}
A^\mu(x_1, x_2, w) &= \int \frac{d^{d+1}z}{z_0^{d+1}} \left(G^{\Delta_1}_{B\partial}(z, \vec{x}_1) \overleftrightarrow{\nabla}_\nu G^{\Delta_1}_{B\partial}(z, \vec{x}_2)\right) G^{\mu\nu}_{BB}(z,w) \\
&= -\sum_{k=k_{min}}^{k_{max}} \frac{a_k}{2k} (x_{12}^2)^{-\Delta_1+k} g^{\mu\nu}(w) \left(G^k_{B\partial}(w, \vec{x}_1) \overleftrightarrow{\nabla}_\nu G^k_{B\partial}(w, \vec{x}_2)\right) ,
\end{aligned} \tag{2.47}$$

with

$$\begin{aligned}
k_{min} &= \frac{d-2}{4} + \frac{1}{4}\sqrt{(d-2)^2 + 4(\Delta-1)(\Delta-d+1)} \ , \\
k_{max} &= \Delta_1 - 1 \ , \\
a_{k-1} &= \frac{2k(2k+2-d) - (\Delta-1)(\Delta-d+1)}{4(k-1)k} a_k \ , \\
a_{\Delta_1 - 1} &= \frac{1}{2(\Delta_1 - 1)} \ ,
\end{aligned} \tag{2.48}$$

and

$$\boxed{k_{max} - k_{min} \in \mathbb{Z}_{\geq 0}} \tag{2.49}$$

In evaluating the cubic integral, it is not necessary for the source coupled to the vector field to be conserved. Therefore, this result remains valid even when the source has non-vanishing divergence.

[13] For notational simplicity, we have omitted the primed indices distinguishing the coordinates associated with different bulk points.

2.4.3 $(n+1)$-pt vertex: scalar exchange

Since in this thesis we are interested in higher-point holographic correlators (Chapter 3), we must consider exchange Witten diagrams involving non-cubic interaction vertices. For now, let us focus on diagrams with scalars only. We shall assume that the dimension d of AdS as well as the dimensions Δ_i of the fields are integers, and satisfy the unitarity bounds $\Delta_i \geq d/2$. We would like to integrate a $(n+1)$-vertex, or equivalently solving the z-integral

$$R(w) = \int \frac{d^{d+1}z}{z_0^{d+1}}\ G^{\Delta}_{BB}(z,w) \prod_{i=1}^{n} \left(\frac{z_0}{z_0^2 + (\vec{z}-\vec{x}_i^2)} \right)^{\Delta_i} . \tag{2.50}$$

As before, given that the bulk-to-bulk propagator satisfies the equation of motion identity (2.36), $R(w)$ has to satisfy the differential equation

$$(-\Box_{AdS} + m^2) R(w) = \prod_{i=1}^{n} \left(\frac{w_0}{w_0^2 + (\vec{w}-\vec{x}_i)^2} \right)^{\Delta_i} . \tag{2.51}$$

Note that we can always write the product of bulk-to-boundary propagators in terms of Feynman integrals in the alpha-parametrisation [54], namely

$$\prod_{i=1}^{n} \left(\frac{w_0}{w_0^2 + (\vec{w}-\vec{x}_i)^2} \right)^{\Delta_i} = \frac{\Gamma(\delta)}{\prod_i \Gamma(\Delta_i)} \prod_{i=1}^{n} \int_0^1 d\alpha_i\ \alpha_i^{\Delta_i - 1}\ \delta\Big(1 - \sum_{i=1}^{n} \alpha_i\Big) \left(\frac{w_0}{w_0^2 + (\vec{w}-\vec{v})^2 + \mu^2} \right)^{\delta} , \tag{2.52}$$

with the definitions

$$\begin{aligned} \delta &= \sum_{i=1}^{n} \Delta_i \ , \\ \vec{v} &= \sum_{i=1}^{n} \alpha_i \vec{x}_i \ , \\ \mu^2 &= -\vec{v}^2 + \sum_{i=1}^{n} \alpha_i |\vec{x}_i|^2 \ . \end{aligned} \tag{2.53}$$

This agrees with the standard Schwinger parametrisation for the D-function (see Appendix C of [55]):

$$D_{\Delta_1 \ldots \Delta_i}(\vec{x_i}) = \frac{\pi^{d/2}\ \Gamma((\delta-d)/2)\Gamma(\delta/2)}{2\prod_i \Gamma(\Delta_i)} \int \frac{\prod_i d\alpha_i\ \alpha_i^{\Delta_i-1}\ \delta(\sum_i \alpha_i - 1)}{(\sum_{k,l} \alpha_k \alpha_l x_{kl}^2)^{\delta/2}} . \tag{2.54}$$

Taking all this into account, we can write

$$R(w) = -\frac{\Gamma(\delta)}{\prod_i \Gamma(\Delta_i)} \prod_{i=1}^{n} \int_0^1 d\alpha_i \alpha_i^{\Delta_i - 1} \delta\Big(1 - \sum_{i=1}^{n} \alpha_i\Big) S(w_0, \vec{w}-\vec{v}; \delta; \mu) , \tag{2.55}$$

where we have defined S as a scalar function which is a particular solution of the differential equation

$$(\Box_{AdS} - m^2)S(w;\delta;\mu) = \left(\frac{w_0}{w^2+\mu^2}\right)^{\delta} . \tag{2.56}$$

The aim is to solve for S as a function of w and plug it into (2.55), to write our initial diagram as a Feynman integral. We start from the recursion relation [47]

$$\Box_{AdS}\frac{(w_0)^l}{(w^2+\mu^2)^k} = l(l-d)\frac{(w_0)^l}{(w^2+\mu^2)^k} + 4k(k-l)\frac{(w_0)^{l+2}}{(w^2+\mu^2)^{k+1}} - 4k(k+1)\mu^2\frac{(w_0)^{l+2}}{(w^2+\mu^2)^{k+2}} , \tag{2.57}$$

and restrict to the scalar case ($k = l$),

$$\Box_{AdS}\left(\frac{w_0}{w^2+\mu^2}\right)^l = l(l-d)\left(\frac{w_0}{w^2+\mu^2}\right)^l - 4l(l+1)\mu^2\left(\frac{w_0}{w^2+\mu^2}\right)^{l+2} . \tag{2.58}$$

Since $m^2 = \Delta(\Delta - d)$, we can write

$$(\Box_{AdS} - m^2)\left(\frac{w_0}{w^2+\mu^2}\right)^l = (l-\Delta)(l+\Delta-d)\left(\frac{w_0}{w^2+\mu^2}\right)^l - 4l(l+1)\mu^2\left(\frac{w_0}{w^2+\mu^2}\right)^{l+2} . \tag{2.59}$$

Let us try to exploit this relation to find a well-defined particular solution of (2.56), that is a finite series of powers of $w_0/(w^2+\mu^2)$. We assume that

$$S(w;\delta;\mu) = \sum_{l=l_{\min}}^{l_{\max}} C_l(\mu)\left(\frac{w_0}{w^2+\mu^2}\right)^l . \tag{2.60}$$

In order to satisfy (2.56), the highest power has to be $l_{\max} = \delta - 2$. The lower powers will be given by $l = l_{\max} - 2j$, where j is a positive integer. Then, the condition for the truncation of the series is

$$\boxed{\delta - l_{\min} - 2 = 2l_0 \in \mathbb{Z}_{\geq 0}} \tag{2.61}$$

where $l_{\min}$ can be either Δ or $d - \Delta$. The second possible solution coincides with the first one in the case of $AdS_5 \times S^5$ compactification of Type IIB supergravity [47]. Let us then restrict to $l_{\min} = \Delta$.

For the boundary conditions, we look at the properties of $R(w)$. It is regular at $\vec{w} = 0$, implying that $S(w_0, \vec{w} - \vec{v}; \delta; \mu)$ has to be regular as well at $\vec{w} - \vec{v} = 0$, and this is satisfied by the solution above. Moreover, $R(w) \sim w_0^{\Delta}$ as $w_0 \to 0$, and this is automatically satisfied in AdS_5. To conclude,

$$\begin{aligned} S(w;\delta;\mu) &= \sum_{l=0}^{l_0} C_l(\mu)\frac{w_0^{\Delta+2l}}{(w^2+\mu^2)^{\Delta+2l}} , \\ C_l(\mu) &= -\frac{1}{4}\mu^{2l+\Delta-\delta}\frac{\Gamma(1/2(\delta-\Delta))\Gamma(1/2(\delta+\Delta-d))\Gamma(\Delta+2l)}{\Gamma(\delta)\Gamma(l+1)\Gamma(l+\Delta+1-d/2)} \end{aligned} \tag{2.62}$$

leads to

$$R(w) = -\frac{\Gamma(\delta)}{\prod_i \Gamma(\Delta_i)} \prod_{i=1}^{n} \int_0^1 d\alpha_i \alpha_i^{\Delta_i - 1} \delta\left(1 - \sum_{i=1}^{n} \alpha_i\right) \sum_{l=0}^{l_0} C_l(\mu) \frac{w_0^{\Delta+2l}}{(w_0^2 + (\vec{w} - \vec{v})^2 + \mu^2)^{\Delta+2l}} \,. \tag{2.63}$$

This is the solution to the z-integral if the above conditions hold, which is true for all the diagrams considered in Chapter 3 because of the R-symmetry selection rules.

As a concrete example—and as a preview of the diagrams considered later—consider the $3 \to 3$ exchange diagram Ie shown in Figure 3.4, involving a single internal propagator and three external legs on each side. If we were to compute this diagram directly in position space, we could apply the integrated vertex identity above to write it as

$$\begin{aligned} W_{3\to3} = &-\frac{\Gamma(\delta)}{\prod_i \Gamma(\Delta_i)} \int \frac{dw_0}{w_0^{d+1}} d^d\vec{w} \prod_{i=1}^{3} \int_0^1 d\alpha_i \alpha_i^{\Delta_i - 1} \delta\left(1 - \sum_{i=1}^{3} \alpha_i\right) \\ &\times \sum_{l=0}^{l_0} C_l(\mu) \left(\frac{w_0}{w_0^2 + (\vec{w} - \vec{v})^2 + \mu^2}\right)^{\Delta+2l} \prod_{j=4,5,6} \left(\frac{w_0}{w_0^2 + (\vec{w} - \vec{x}_j)^2}\right)^{\Delta_j} , \end{aligned} \tag{2.64}$$

which closely resembles Feynman-like integrals. For additional comments on position space, see Section 3.3.4.

2.5 Bootstrapping holographic correlators

Let us conclude this Chapter by introducing the bootstrap approach to holographic correlators. The general strategy is to take qualitative input from the gravitational side and use CFT methods to extract strong-coupling data while gaining insight into scattering amplitudes in AdS. We assume that we exchange only supergravity states that belong to short multiplets. Possible stringy states will be neglected at this stage, but a detailed discussion is postponed to Chapter 4.

2.5.1 Maximal supersymmetry: gravitons on AdS

In the following, d denotes the spacetime dimension of the boundary theory, i.e. the dual CFT$_d$, and equivalently the dimension of the AdS space is $d+1$, as in AdS_{d+1}. On the other hand, we will use $\mathbf{d}$ to denote the dimension of the internal sphere in the bulk geometry, which corresponds to the rank of the R-symmetry group in the dual CFT. In general, d and $\mathbf{d}$ are distinct.

In $d > 2$, we have three well-known examples of the AdS/CFT duality, which are

- Type IIB String Theory on $AdS_5 \times S^5 \leftrightarrow 4d\ \mathcal{N} = 4$ SYM with $SO(6)_R$ [3];
- M-theory on $AdS_4 \times S^7 \leftrightarrow 3d\ \mathcal{N} = 8$ ABJM with $SO(8)_R$ [56];
- M-theory on $AdS_7 \times S^4 \leftrightarrow 6d\ \mathcal{N} = (2,0)$ with $SO(5)_R$ [13].

Given the presence of a compact internal manifold, one usually performs a Kaluza-Klein (KK) reduction of the AdS action [57, 58]. For example, for the ten-dimensional action on $AdS_5 \times S^5$, this procedure reduces the higher-dimensional field theory to an effective five-dimensional theory by expanding the fields in harmonics on the compact space—in this case, the five-sphere S^5. The resulting KK modes correspond to towers of five-dimensional fields with increasing masses, organised into supermultiplets of $\mathcal{N} = 8$ gauged supergravity in AdS_5. On the CFT side, these are dual to half-BPS single-trace operators in $\mathcal{N} = 4$ SYM, labelled by an integer $k \geq 2$, where the superprimary is a Lorentz scalar in the rank-k symmetric traceless representation of the $SO(6)$ R-symmetry. The case $k = 2$ corresponds to the stress tensor multiplet,[14] which is dual to the massless graviton in five dimensions. This correspondence is what allows for the computation of graviton scattering amplitudes in AdS, and exemplifies the general principle that gauge symmetries in the bulk are dual to global symmetries in the boundary CFT. To sum up:

$$\mathbf{k = 2} \ : \ T_{\mu\nu} \leftrightarrow AdS \text{ graviton} \quad , \quad \mathbf{k > 2} \ : \ \text{KK modes of the graviton on } S^{\mathbf{d}-1}, \tag{2.65}$$

where $\mathbf{d}$ is the rank of the R-symmetry group and the scaling dimensions of the operators are $\Delta_\epsilon = \epsilon\, k$, where $\epsilon = (d-2)/2$.

The one-half BPS states considered here carry R-symmetry indices, $\mathcal{O}^{I_1 \dots I_K}(x)$, and are often contracted with auxiliary null vectors t_i, which simplifies the treatment of tensor structures:

$$\begin{aligned} &\mathcal{O}_k(x,t) \equiv \mathcal{O}^{I_1 \dots I_K}(x) t_{I_1} \dots t_{I_k} \ , \\ &I_i = 1, \dots, \mathbf{d} \ , \\ &t_i \cdot t_i = 0 \ . \end{aligned} \tag{2.66}$$

Therefore, the correlator depends on the internal coordinates t_i in addition to the spacetime coordinates x_i. In particular, it is a polynomial of $SO(\mathbf{d})$ invariants, namely $t_{ij} = t_i \cdot t_j$. Then, we can extract a kinematic factor by imposing covariance under the bosonic part of the superconformal group, that is the conformal group $SO(d+1,1)$ and the R-symmetry $SO(\mathbf{d})$.[15] For the case of four points, the correlator is reduced to a function of two pairs of cross ratios, for the conformal and R-symmetry. As a concrete example, and without loss of generality, let us fix the ordering of the external conformal dimensions as $k_1 \leq k_2 \leq k_3 \leq k_4$. We can have two cases:

$$\begin{aligned} &k_1 + k_4 \geq k_2 + k_3 \quad (\text{case I}) \\ &k_1 + k_4 < k_2 + k_3 \quad (\text{case II}) \end{aligned} \tag{2.67}$$

[14] The stress tensor itself appears as a superconformal descendant within the multiplet.
[15] See, for example, [55].

We can then extract a kinematic factor, as explained above:

$$G_{k_1k_2k_3k_4}(x_i,t_i) = \prod_{i<j}\left(\frac{t_{ij}}{x_{ij}^{2\epsilon}}\right)^{\gamma^0_{ij}}\left(\frac{t_{12}t_{34}}{x_{12}^{2\epsilon}x_{34}^{2\epsilon}}\right)^{\mathcal{E}}\mathcal{G}_{k_1k_2k_3k_4}(U,V;\sigma,\tau)\ , \tag{2.68}$$

where

$$\begin{aligned}U &= \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\ ,\ V = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}\ ,\\ \sigma &= \frac{t_{13}t_{24}}{t_{12}t_{34}}\ ,\ \tau = \frac{t_{14}^2t_{23}^2}{t_{12}^2t_{34}^2}\ ,\end{aligned} \tag{2.69}$$

while $\mathcal{E}$, called *extremality*, and γ^0_{ij} are functions of the k_i whose expressions depend on the two cases in (2.67). In particular,

$$\begin{aligned}\mathcal{E} &= \frac{k_1+k_2+k_3-k_4}{2} \qquad \text{(case I)}\\ \mathcal{E} &= k_1 \qquad \text{(case II)}\end{aligned} \tag{2.70}$$

and

$$\begin{aligned}&\gamma^0_{12}=\gamma^0_{13}=0\ ,\quad \gamma^0_{34}=\frac{\kappa_s}{2},\quad \gamma^0_{24}=\frac{\kappa_u}{2}\ ,\\ &\gamma^0_{14}=\frac{\kappa_t}{2}\ ,\quad \gamma^0_{23}=0\quad \text{(case I)}\ ,\quad \gamma^0_{14}=0\ ,\quad \gamma^0_{23}=\frac{\kappa_t}{2}\quad \text{(case II)}\ ,\end{aligned} \tag{2.71}$$

where

$$\kappa_s \equiv |k_3+k_4-k_1-k_2|\ ,\quad \kappa_t \equiv |k_1+k_4-k_2-k_3|\ ,\quad \kappa_u \equiv |k_2+k_4-k_1-k_3|\ . \tag{2.72}$$

As a result of the kinematic factor extracted in (2.68), $\mathcal{G}_{k_1k_2k_3k_4}$ becomes a polynomial of degree $\mathcal{E}$ in σ and τ. But we also have the fermionic generators! They give us some relations between conformal and R-symmetry cross-ratios in the form of Super Conformal Ward Identities (SCWIs). For any number of spacetime dimensions [59],

$$(z\partial_z - \epsilon\alpha\partial_\alpha)\ \mathcal{G}(z,\bar{z};\alpha,\bar{\alpha})\big|_{\alpha=1/z} = 0\ , \tag{2.73}$$

where we have redefined the cross ratios according to

$$\begin{aligned}U &= z\bar{z}\ ,\ V=(1-z)(1-\bar{z})\ ,\\ \sigma &= \alpha\bar{\alpha}\ ,\ \tau=(1-\alpha)(1-\bar{\alpha})\ .\end{aligned} \tag{2.74}$$

This is a very strong and interesting constraint. For $d=4$, the correlator with this special R-symmetry configuration is independent of $\bar{z}$. This holomorphicity can be understood from the chiral algebra construction [60]: a subset of protected operators restricted to live on a $2d$ plane, when subject to special position-dependent R-symmetry polarisations (that are called twists), live in the chiral algebra. This imposes non-trivial constraints in the case of higher-point functions [61].

We are interested in studying correlators of these operators. The standard approach would involve writing a diagrammatic expansion in position space, obtained by KK reducing the 10- or 11-dimensional theory on the internal sphere. The resulting infinite tower of KK modes organises into superconformal multiplets, and we focus on the superprimary operators within these multiplets. For such operators, two- and three-point functions are fixed by symmetry and therefore trivial; the first non-trivial correlator is the four-point function, as reviewed above. To compute an n-point function via this method, one would need to expand the effective AdS supergravity action to the corresponding order. For instance, three- and four-point correlators require the cubic and quartic interaction vertices, respectively. Already at this stage, the computation becomes highly involved [62]. Moreover, the brute-force diagrammatic expansion requires integrating over AdS at each interaction vertex, a generally challenging task. To obtain the full result, one would finally need to enumerate all relevant Witten diagrams, evaluate each, and sum their contributions.

However, the theories of interest typically possess a high degree of symmetry, which can be exploited to constrain the form of correlators and avoid direct diagrammatic computations. This approach is known as *bootstrapping holographic correlators*: an on-shell method that relies on symmetry and consistency conditions rather than off-shell actions. Let us emphasise that one of the main difficulties lies in the complicated structure of the exact interaction vertices in AdS. Still, the vertex identities can significantly simplify the computation by reducing the problem to a set of more tractable integrals. In particular, in the case of four-point functions of $4d$ $\mathcal{N}=4$ and $4d$ $\mathcal{N}=2$, we can apply the integrated vertex identities to each diagram. As a counterexample, the vertices on $AdS_4 \times S^7$ are not suitable for this trick, so it cannot be implemented for the dual CFT. On the other side, alternative and powerful methods have been proposed, such as working directly in Mellin space [63], which has led to tremendous progress in the computation of holographic correlators, and will also feature in our analysis in Chapter 3. Another interesting method is the Maximally R-symmetry Violating limit [64, 65].

The general bootstrap approach consists of writing an ansatz for the correlator as a linear combination of all possible exchange and contact Witten diagrams that can contribute, with undetermined coefficients. These coefficients remain unknown because the precise form of the interaction vertices is not specified; however, the allowed structures are constrained by gauge invariance and global symmetry selection rules. In our studies, we will always have a finite number of terms in our ansatz as a consequence of the R-symmetry selection rules of the theory. The coefficients are then fixed by imposing proper conditions with no need to input the details of the effective Lagrangian. In particular, we will impose the correct flat space limit, factorisation, and the constraints from the

SCWIs.

At tree-level, the correlator is given by the sum of the contributions in the three channels (s, t and u, from the kinematic invariants in (3.1)) and, possibly, some contact terms. We will assume that the latter contains at most two derivatives (contracted) to match the 2-derivative structure of the supergravity action. In the case of maximal supersymmetry, the possible exchanged fields from R-symmetry representations are

- supergraviton field, spin-0, $\Delta = 2$, representation $20'$ of $SO(6)_R$;
- graviphoton field, spin-1, $\Delta = 3$, representation 15 of $SO(6)_R$;
- graviton field, spin-2, $\Delta = 4$, singlet representation of $SO(6)_R$.

A possible (s-channel) ansatz would then be

$$A_s = \lambda_s \, Y_{20'}(\sigma,\tau) \, W_{2,0}(U,V) + \lambda_v \, Y_{15}(\sigma,\tau) \, W_{3,1}(U,V) + \lambda_g \, Y_1(\sigma,\tau) \, W_{4,2}(U,V) \ , \tag{2.75}$$

where $Y(\sigma,\tau)$ are the R-symmetry polynomials, labelled by their representation. They are functions of the R-symmetry cross ratios defined in (2.69), and can be explicitly written down as solutions of the R-symmetry 2-particle quadratic Casimir equations. This is the equivalent of the conformal blocks computation for conformal symmetry. $W_{\Delta,l}(U,V)$ is the Witten diagram related to the exchange of a field with spin l and scaling dimension Δ, as a function of the conformal cross ratios in (2.69). Finally, λ is the unknown coefficient to be bootstrapped. Notice that one can focus on one specific channel, and then the other two will be related to it by crossing symmetry.

2.5.2 Half-maximal supersymmetry: gluons on AdS

When we have half-maximal supersymmetry, the single-trace short operators in the CFT are dual to gluons and gravitons. For the expert reader, a simple stringy construction consists of starting with Type IIB string theory on $AdS_5 \times S^5$ and adding M D7-branes wrapping $AdS_5 \times S^3$, where $S^3 \subset S^5$. This configuration breaks the original $\mathcal{N} = 4$ supersymmetry of the boundary theory down to $\mathcal{N} = 2$, and introduces fundamental matter, the so-called flavour sector. The global flavour symmetry in the boundary theory gives rise to a conserved spin-1 current multiplet, which was absent in the maximally supersymmetric case. The symmetry-breaking pattern in the R-symmetry group is $SO(6) \to SO(4) \times SO(2)$, due to the presence of the D7-branes wrapping the internal S^3. In the bulk, the gluons living on AdS_5 are dual to protected scalar operators of dimension $\Delta = 2$ in the current multiplet of the boundary $\mathcal{N} = 2$ theory. The KK modes on the S^3 correspond to a tower of scalar operators with increasing conformal dimension $\Delta = k$. For more details, see [66].

We will consider holographic correlators of "supergluons", scalar operators with $\Delta = \epsilon k$, $j_R = k/2$, $j_L = (k-2)/2$, with integer $k \geq 2$, and transforming in the adjoint of a gauge group G_F. Here, j_R is the spin under R-symmetry and j_L is the spin under the $SU(2)_L$ global symmetry. Compared to the maximally supersymmetric case, such as that of the supergraviton, the supergluons also carry a flavour index (which corresponds to the colour symmetry in the bulk). Indeed, $k = 2$ corresponds to the conserved current of G_F, dual to AdS gluons with gauge group G_F, while as usual $k > 2$ describes higher KK modes of gluons on S^3. Another difference is the presence of two kinds of indices. We contract the R-symmetry indices with $SU(2)_R$ auxiliary spinors, but we also have $SU(2)_L$ indices:[16]

$$\mathcal{O}_k^I(x; v; \overline{v}) \equiv \mathcal{O}(x)^{I;\alpha_1,\ldots,\alpha_k;\overline{\alpha}_1,\ldots,\overline{\alpha}_{k-2}} v_{\alpha_1} \ldots v_{\alpha_k} \overline{v}_{\alpha_1} \ldots \overline{v}_{\alpha_1} \ldots \overline{v}_{\alpha_{k-2}} \ , \tag{2.76}$$

where I refers to the adjoint of G_F. In the correlator, the tensor product of two adjoint representations of G_F will determine the number of independent flavour structures.

In Chapter 3, we will work in the large-N limit and focus on tree-level correlators. It is important to note that at order $O(1/N)$, the coupling between two supergluons and a supergraviton is suppressed compared to the leading cubic supergluon interaction, by a power of N. As a result, graviton exchange can be neglected at this order, and the dynamics can be consistently described in terms of pure gluon scattering in AdS. The bootstrap strategy proceeds exactly as before, but the ansatz is simpler due to the reduced number of contributing diagrams, a consequence of the lower amount of supersymmetry.

[16] For $SU(2)_R$, the R-symmetry polynomials are Jacobi polynomials.

Chapter 3

Higher-point holographic correlators

In the past few decades, there has been a dramatic advance in our knowledge of and techniques for calculating flat space scattering amplitudes. The astonishing simplicity of the Parke–Taylor formula [67], especially when juxtaposed with the daunting complexity of traditional Feynman diagram expansions, hinted at a deep underlying structure. Since then, we have uncovered a wealth of remarkable phenomena, from the double-copy relationship between gauge theory and gravity amplitudes [4, 68] presented in the Introduction, to the emergence of positive-geometric frameworks [69], and our understanding of scattering processes continues to expand. Along the way, we have also learned many useful lessons. One such lesson is that gauge theory amplitudes serve as building blocks for gravity amplitudes while being *much* simpler. Therefore, one may recycle gauge theory solutions to approach gravitational problems. This can be appreciated from the stark contrast between the finite three- and four-point vertices of (renormalisable) gauge theories and the infinite tower of ever more intricate n-point vertices required in gravity. Another lesson is that to uncover the full structure behind scattering amplitudes, the computation of higher points is fundamental. In this context, the full power of modern on-shell, geometric and bootstrap techniques really gets unleashed.

Progress in curved spacetimes has been much slower, but thanks to the AdS/CFT correspondence, AdS scattering amplitudes can be extracted from conformal correlators, and we can use CFT methods as well as input from different areas of mathematics to move forward more quickly. As a result, correlation functions of $\frac{1}{2}$-BPS operators in the four-dimensional $\mathcal{N} = 4$ and $\mathcal{N} = 2$ SYM theory have received a tremendous amount of attention over the past three decades. At large N and 't Hooft coupling $\lambda = g^2_{YM}N$, these correlators can in principle be computed using Witten diagrams. In practice, however, this requires a detailed knowledge of complicated effective Lagrangians, and it becomes prohibitively hard except for the simplest four-point correlation functions. On the other side, the bootstrap strategy [27, 70] has generated many impressive results. See [55] for a review. For instance, all infinitely many tree-level four-point functions of $\frac{1}{2}$-BPS operators with arbitrary KK levels have been obtained in all maximally superconformal theories [27, 64, 65, 70], as well as in theories with half the amount of maximal superconformal symmetry [66,71,72]. These results further revealed interesting hidden structures, such as higher dimensional symmetry [66, 71, 73], dimensional reduction [66, 74], and AdS double-copy [75–79].

As in flat space, it is important to consider processes with higher particle multiplicities, where interesting features are expected to arise, allowing us to probe more deeply into the CFT structure. This study sharpens the connection between holographic correlators and flat space scattering amplitudes, through techniques like factorisation [80]. Moreover, such correlators provide access to new, unprotected CFT data at strong coupling, which is inaccessible via four-point functions. This becomes particularly evident when examining the OPE limits. For instance, taking the OPE limit once or twice for a pair of operators in a six-point function yields five-point or four-point functions containing one or two protected or unprotected double-trace operators respectively. These correlators have attracted attention recently with multi-trace operators being interpreted as bound states in AdS [81–85]. For $\mathcal{N} = 4$ SYM, the five-point function of the lowest KK mode of supergravitons, *i.e.*, the $\mathbf{20}'$ operators, was bootstrapped in [86]. In [87], the bootstrap approach was further used to obtain an infinite family of five-point functions where two operators have arbitrary KK levels.

As in flat space, gluon amplitudes in AdS are much simpler than graviton amplitudes, and progress has been more rapid for SYM on $AdS_5 \times S^3$ [66, 88–92]. In this case, tree-level correlators of supergluons have now been determined up to eight points for the lowest KK level [90, 91] and for arbitrary KK modes for five-point functions [92].

The higher-point results available in the literature have been obtained using methods that stem from two distinct strategies. The first approach, which led to the results in [86–88], is similar in spirit to the original method of [27, 70]. Superconformal symmetry plays a key role in fixing the ansatz. In contrast, in [89] we proposed a new method, which relies solely on the flat space limit (for a specific choice of polarisations), and on factorisation properties in Mellin space. Quite non trivially, this fixes the AdS amplitude completely with minimal use of supersymmetry and promises greater efficiency. However, as we will see in the following, its full implementation relies on using lower-point amplitudes as input. These may involve spinning operators other than the scalar supergluons and supergravitons, and are not always readily available. Therefore, it is still important to improve the first method in order to maximise our computational power.

This motivated the study of the six-point AdS supergraviton amplitude with another bootstrap approach, which we review in the second part of this Chapter. The key point is a novel implementation of the chiral algebra condition [93]—a central superconformal constraint—ensuring that correlators in the co-plane configuration, with suitably twisted R-symmetry polarisations, become meromorphic functions of the plane's complex coordinates. The restriction to the $2d$ kinematics makes the condition difficult to use, especially for $n \geq 6$ points. If we use this condition directly in position space, we face a gigantic ansatz, which is a complicated function of the cross ratios. In fact, it is even unclear how to write down certain parts of the ansatz explicitly when $n \geq 6$ because we do not know how to evaluate the corresponding Witten diagrams as elementary functions. In Mellin space, the ansatz becomes much simpler, but it is unclear how to use the chiral algebra condition because the Mellin representation requires the operators to be inserted at generic points. Therefore, the use of the chiral algebra constraint has been quite limited in the past works. In [61], a new method was proposed to exploit this condition by using the lightcone OPE and get differential equations among different lower-point correlation functions. These constraints can be conveniently exploited without leaving Mellin space! Moreover, this new strategy isolates individual components of the correlators so that one can analyse the impact of consistency conditions on each part separately. This leads to a more nuanced understanding of how the bootstrap constraints determine the correlators, in contrast to earlier approaches where the ansatz was fixed all at once. To sum up, these studies [61, 89] can not only advance our understanding of higher-point functions but also pave the way for innovative bootstrap techniques, showcasing their broader impact and originality.

We begin by reviewing the fundamentals of gluon scattering in flat space, emphasising the essential mathematical structures that underpin our subsequent analysis. This overview is intended simply as a concise reminder, not a comprehensive survey of flat space amplitudes, allowing us to proceed directly to the core of this thesis: our original AdS investigations. In Chapter 4, we shall revisit the mathematical structure of scattering amplitudes in greater depth when introducing stringy corrections to AdS amplitudes. While we will introduce the basic structure of gluon interactions in flat space to prepare the reader for our detailed analysis of supergluons in AdS, we will not review gravitons in flat space. This is because the bootstrap approach employed in that case requires little direct input from flat space and instead exploits its structural similarities with the gluon case, particularly at the level of the pole structure. However, we will dedicate Subsection 3.3.3 at the end of the Chapter to some comments on the flat space six-graviton amplitude. For the purposes of this thesis, we will present the full technical details of the six-supergluon analysis, whereas for the six-point supergraviton amplitude, we concentrate on the novel bootstrap strategy and defer the complete computational derivations to the original manuscript [61]. Finally, we omit the supersymmetric framework from our review, since it plays only a minor role in our AdS analysis, such as providing R-symmetry selection rules.

3.1 Field theory amplitudes in flat space

The perturbative study of scattering amplitudes in flat space is traditionally performed through a loop expansion organised via Feynman diagrams, yielding a series in a small dimensionless coupling constant that controls the strength of particle interactions. While this approach is systematic and grounded in the Lagrangian formulation of QFT, it becomes increasingly cumbersome as the number of external particles grows, due to the factorial proliferation of diagrams and the presence of unphysical, gauge-dependent redundancies in theories such as QED or QCD. The Feynman rules assign mathematical expressions (propagators, vertices, etc.) to the lines and nodes in a diagram, and, importantly, depend on the choice of the gauge. On the other side, there is an object which, at each loop order, is a gauge invariant sum of Feynman diagrams: the *on-shell* amplitude.[1] Moreover, the on-shell result for complicated processes such as multi-gluon scattering often takes a remarkably compact and simple form, despite arising from the sum of many individually complex diagrams.

This motivated the development of an alternative approach, namely the on-shell formalism (see, *e.g.*, [94, 95]), which focuses directly on physical, gauge-invariant observables. A key insight here is that the analytic structure of scattering amplitudes is highly constrained. For instance, locality restricts the allowed singularities in momentum space (k_i^μ): at tree-level, amplitudes can only develop poles when internal particles go on-shell. This implies that a four-gluon amplitude may contain simple poles in the Mandelstam variables, the Lorentz-invariant combinations of external momenta,

$$\begin{aligned} &s = (k_1+k_2)^2 \;, \quad t = (k_1+k_4)^2 \;, \quad u = (k_1+k_3)^2 \;, \\ &s+t+u = \sum_i m_i^2 = 0 \;, \quad \text{massless particles} \;, \end{aligned} \tag{3.1}$$

but must not contain higher-order (*e.g.* double) poles, which would be incompatible with locality. Tree amplitudes are rational functions of the kinematic invariants, while loop amplitudes are integrals of rational functions, leading to more complicated structures. Some of them are reviewed in Section 4.1.2. For the present Chapter, we just need to know that useful information about a scattering amplitude can be extracted by analysing its singularities. By tuning the external kinematics so that an internal propagator becomes on-shell, one reaches the *factorisation* limit. Under the assumption that the amplitude vanishes at large momenta (*i.e.* it has no poles at infinity), this limit leads to the factorisation of the amplitude into a product of lower-point amplitudes. For instance, in the four-gluon case, a simple pole in s corresponds to the exchange of an intermediate particle between the pair $(1, 2)$ and $(3, 4)$, and the factorisation behaviour reads schematically

$$\lim_{s\to 0} s\, A_4 = A_3\, A_3 \,. \tag{3.2}$$

[1]This refers to configurations that satisfy classical equations of motion.

In general, one must sum over all factorisation channels and all on-shell internal states compatible with the theory.[2] This structure not only reflects the analytic constraints imposed by locality and unitarity, but also highlights how on-shell methods render symmetries more manifest, which aligns naturally with the bootstrap philosophy. Rather than being imposed diagram by diagram, these symmetries are intrinsically built into the structure of the amplitude through tools such as recursion relations (*e.g.* BCFW [96]).

To sum up, Feynman diagrams remain a valuable and instructive tool, as they offer direct access to the interaction structure and help identify the physical channels contributing to a given process. This is complemented by on-shell techniques, which make the analytic structure and underlying symmetries of the amplitude more transparent. In what follows, we will keep both perspectives in mind.

3.1.1 Pure Yang-Mills

The kinematic data required to describe scattering amplitudes of massless particles consist of the momentum vectors $k_\mu = (k_0, \vec{k})$ and, when relevant, the polarisation vectors.[3] In four dimensions, any amplitude involving massless particles can be written entirely in terms of these quantities. Since we want to study scattering amplitudes of massless spin-1 particles in $4d$, the two physical polarisations are in the fundamental representation of $SO(2)$. In the usual covariant notation, they are encoded into 4-dimensional vectors $e_\mu(k)$ and required to satisfy the transversality condition $k \cdot e = 0$, as dictated by gauge invariance, as well as the equivalence relation $e_\mu \sim e_\mu + c\, k_\mu$, where c is a scalar constant. Scattering amplitudes are multilinear in the external polarisations and encode physical information about the interactions between the external states.

Gluons, transversely polarised massless spin-1 particles, are described by the Yang-Mills theory with gauge group $G = SU(N)$,[4] where N is the number of colours, and Lagrangian

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} \;, \tag{3.3}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\sqrt{2}}[A_\mu, A_\nu]$, and $A_\mu = A^a_\mu\, T^a$, with T^a the generators. The associated fields A^a_μ live in the adjoint representation of G and are thus labelled by a colour index $a = 1, \ldots, N^2 - 1$. The gluons are self-interacting, in contrast to the abelian case (for example, photons). After gauge fixing, one obtains three- and four-gluon interaction vertices. We will work in Feynman gauge. The

[2] For gluons, this means summing over the allowed helicity states of the exchanged particle.

[3] For massless particles, helicity is a Lorentz-invariant and physically meaningful quantum number. In contrast, for massive particles, helicity is frame-dependent and can flip under boosts. The spinor-helicity formalism offers a particularly elegant description of massless states in $4d$ [97].

[4] QCD corresponds to $N = 3$.

cubic vertex reads

$$\mathbf{V}^{abc}_{\mu\nu\rho}(k_1,k_2,k_3) = if^{abc}\left[(k_3-k_2)_\mu g_{\nu\rho} + (k_1-k_3)_\nu g_{\mu\rho} + (k_2-k_1)_\rho g_{\mu\nu}\right], \tag{3.4}$$

where f^{abc} are the structure constants, and we have assumed all momenta to be in-going. The quartic vertex is instead

$$\mathbf{V}^{abcd}_{\mu\nu\rho\sigma} = -\left[f^{abe}f^{cde}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) + f^{ace}f^{dbe}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\rho\sigma}) + f^{ade}f^{bce}(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma})\right] , \tag{3.5}$$

where we have set the coupling to 1. When some of the legs in the vertices are external, we must contract (3.4) and (3.5) with the corresponding external propagators and polarisation vectors. The gluon propagator in the Feynman gauge is

$$\Delta^{ab}_{\mu\nu} = \frac{\delta_{ab}g_{\mu\nu}}{k^2} . \tag{3.6}$$

3.1.2 Four gluons

Let us first present the 4-gluon case. The amplitudes can be organised according to distinct group-theoretical structures, each multiplied by a corresponding kinematic factor. For the colour structures, we can define

$$\begin{aligned} &c_s \equiv f^{a_1a_2e}f^{a_3a_4e} \ , \ c_t \equiv f^{a_1a_4e}f^{a_2a_3e} \ , \ c_u \equiv f^{a_1a_3e}f^{a_4a_2e} \ , \\ &c_s + c_t + c_u = 0 \ , \quad \text{Jacobi identity} \ . \end{aligned} \tag{3.7}$$

In addition to the exchange diagrams (s-, t-, and u-channels), the four-point amplitude also receives contributions from the contact interaction. This term can be treated using a splitting procedure that involves multiplying and dividing by appropriate propagators, effectively reconstructing the structure of exchange diagrams (using only cubic vertices). In the end, the tree-level amplitude takes the compact form

$$A_4 = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right) , \tag{3.8}$$

where the c-factors are the colour factors associated with the s-, t-, and u-channel diagrams defined above, and the n-factors are the corresponding kinematic numerators, which encode the dependence on momenta and polarisations. For example:

$$n_s = (e_1\cdot e_2)(e_3\cdot e_4)(k_2-k_1)\cdot(k_4-k_3) - ((e_1\cdot e_3)(e_2\cdot e_4) - (e_1\cdot e_4)(e_2\cdot e_3))\,(k_1+k_2)^2 \ . \tag{3.9}$$

These numerators play a central role in the colour-kinematics duality [98], where, under certain conditions, they satisfy algebraic relations analogous to the Jacobi identity obeyed by the colour factors:

$$n_s + n_t + n_u = 0 \ . \tag{3.10}$$

We can also express the colour structure constants in terms of traces of the generators T^a to write the four-point amplitude as

$$A_4 = g^2 (A_4[1234] \, \mathrm{Tr}\,(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) + \text{perms of } (234)) \ , \tag{3.11}$$

where $A_4[1234]$ is gauge-invariant. More specifically, one can pass from the f^{abc}-based decomposition introduced above to the trace-based decomposition through

$$f^{abc} = -i \, \mathrm{Tr}(T^a T^b T^c - T^b T^a T^c) \ , \tag{3.12}$$

which follows from the Lie algebra identity for the generators $[T^a, T^b] = i f^{abc} T^c$. For example, the colour structure (3.7) can be equivalently written as

$$c_s = \mathrm{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) - \mathrm{Tr}(T^{a_1} T^{a_2} T^{a_4} T^{a_3}) - \mathrm{Tr}(T^{a_1} T^{a_3} T^{a_4} T^{a_2}) + \mathrm{Tr}(T^{a_1} T^{a_4} T^{a_3} T^{a_2}) \ . \tag{3.13}$$

In the representation (3.11), the partial amplitudes are gauge invariant, although some of their properties are not manifest. Let us examine this more generally and in greater detail at higher multiplicity, which is the primary regime of interest in this Chapter.

3.1.3 Higher points

Extending to n gluons, we can disentangle the colour and kinematic algebra by writing the tree-level amplitude as[5]

$$A_n = g^{n-2} \sum_{\text{perms } \sigma} A_n[1, \sigma(2), \ldots, \sigma(n)] \ \mathrm{Tr}(T^{a_1} T^{\sigma(a_2)} \ldots T^{\sigma(a_n)}) \ , \tag{3.14}$$

where $\sigma \in S_{n-1} \equiv S_n / \mathbb{Z}_n$, the group of permutations of $n-1$ elements. The first entry has been fixed by the cyclic invariance of the trace. Then, the trace factor encodes the colours of the external legs while the remaining partial amplitudes are the so-called *colour-ordered* amplitudes, since they have a fixed ordering of the external particles. These colour-ordered amplitudes are:

- invariant under cyclic permutations,

[5]This is usually referred to as colour decomposition in terms of Chan-Paton factors, or traces of $SU(N_c)$ matrices T^a in the fundamental representation. See, for example, [99] and references therein, for more details and the original formulations.

- antisymmetric under reflection,

- subject to the photon decoupling (or $U(1)$ decoupling) identity [100, 101].

In summary:

$$\begin{aligned} &A_n[1,2,\ldots,n] = A_n[2,\ldots n,1]\,, \\ &A_n[1,2,\ldots,n] = (-1)^n A_n[n,\ldots,2,1]\,, \\ &\sum_\sigma A_n[1,\sigma(2,3,\ldots,n)] = 0\,. \end{aligned} \tag{3.15}$$

The last sum runs over cyclic permutations of legs $2,3,\ldots,n$ and enforces the vanishing of the full amplitude when a gluon is replaced by a colour-neutral photon ($T^{a_1} \to 1$ in (3.14)).

The trace decomposition in (3.14) is overcomplete. Naively, there are $(n-1)!$ single-trace structures, and hence at most that many colour-ordered amplitudes. The above identities reduce the number of independent partial amplitudes. In particular, the Kleiss-Kuijf (KK) relations [102] allow to express all colour-ordered amplitudes in terms of $(n-2)!$ amplitudes with fixed legs at positions 1 and n (*i.e.* $\sigma \in S_{n-2}$). Further reductions arise from the BCJ relations [98], which follow from the duality between colour and kinematic numerators in gauge theory amplitudes discussed above, and ultimately yield a minimal basis of $(n-3)!$ independent partial amplitudes. Notice that, if we return to the conventional colour factors for gluonic Feynman diagrams (in terms of f^{abc}), certain properties, such as the $U(1)$ decoupling identity, become manifest [99]. Motivated by this, one can further consider an alternative colour decomposition for n gluons at tree level, using structure constants instead of traces, but with the kinematic coefficients being already the $(n-2)!$ independent colour-ordered amplitudes ([99, 103] (DDM)), namely the ones with legs 1 and n adjacent. We will return to this shortly.

In pure Yang-Mills theory, interactions arise only from the cubic and quartic vertices (3.4), (3.5). This implies, for example, that for five external gluons, the allowed tree-level diagrams include single- and double-exchange topologies, but no contact term. By contact term, we mean a diagram where all external legs are attached to a single interaction vertex, with no internal/exchanged propagators.[6] Moreover, for our purposes, let us restrict to a special configuration, where *all* polarisations are perpendicular to *all* momenta. This is the relevant setup for amplitudes coming from AdS, as explained later. Clearly, this simplifies the computation quite drastically, and it can be easily checked that the five-point amplitude in this configuration becomes trivial, as all the allowed diagrams involve contractions of polarisations and momenta.

In the case of $n = 6$ external gluons, the full colour-dressed tree-level amplitude can be expressed

[6]For five external gluons, if we want to work with diagrams constructed purely from cubic interactions, they necessarily contain two internal propagators and therefore exhibit two kinematic poles. There are 15 such diagrams, corresponding to the different ways of connecting five external legs via trivalent graphs. For more details and the specific expressions of the amplitude, we refer the interested reader to Section IV of [98].

as a sum over $(n-1)! = 120$ colour-ordered amplitudes multiplying corresponding trace structures. Using the Kleiss-Kuijf and the BCJ relations, the number of independent colour-ordered amplitudes reduces to $(n-3)! = 6$. Each of these six BCJ-basis amplitudes receives contributions from a specific set of cubic (trivalent) diagrams. In this representation, each partial amplitude is written as a sum over 14 cubic diagrams, each with distinct propagator structures and fixed external ordering. These diagrams fall into two basic topologies: the linear (comb) and the star (snowflake) topology, as illustrated in Figure 3.1. Note that via the Jacobi identity, the snowflake diagrams

$$\mathbb{S}_{[a_1^\sigma a_2^\sigma a_3^\sigma a_4^\sigma a_5^\sigma a_6^\sigma]} = f^{a_1^\sigma a_2^\sigma b_1} f^{a_3^\sigma a_4^\sigma b_2} f^{a_5^\sigma a_6^\sigma b_3} f^{b_1 b_2 b_3} \ , \tag{3.16}$$

can be expressed in terms of the comb diagrams

$$\mathbb{T}_{[a_1^\sigma a_2^\sigma a_3^\sigma a_4^\sigma a_5^\sigma a_6^\sigma]} = f^{a_1^\sigma a_2^\sigma b_1} f^{b_1 a_3^\sigma b_2} f^{b_2 a_4^\sigma b_3} f^{b_3 a_5^\sigma a_6^\sigma} \ . \tag{3.17}$$

In fact, $\mathbb{T}_{[a_1 a_2^\sigma a_3^\sigma a_4^\sigma a_5^\sigma a_6]}$ with σ being a permutation of $\{a_2, a_3, a_4, a_5\}$ form the so-called DDM basis [99], already mentioned above. When we talk about a basis here, we mean a basis for the colour structures, namely a minimal set of independent colour structures for tree-level gluon amplitudes. Of course, every chain of f^{abc}'s in the DDM basis can be expanded into a sum of traces, giving the trace decomposition (3.14), as seen for the case of four gluons.

The relevant diagrams for the colour-ordered amplitude are shown in Figure 3.2.

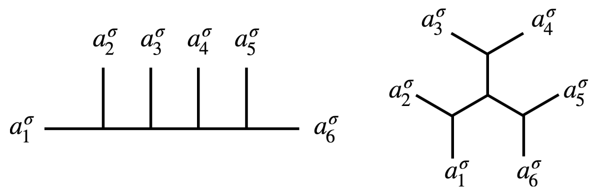


Figure 3.1: Two topologies of cubic tree colour diagrams, where σ denotes a permutation of $\{a_1, a_2, a_3, a_4, a_5, a_6\}$.

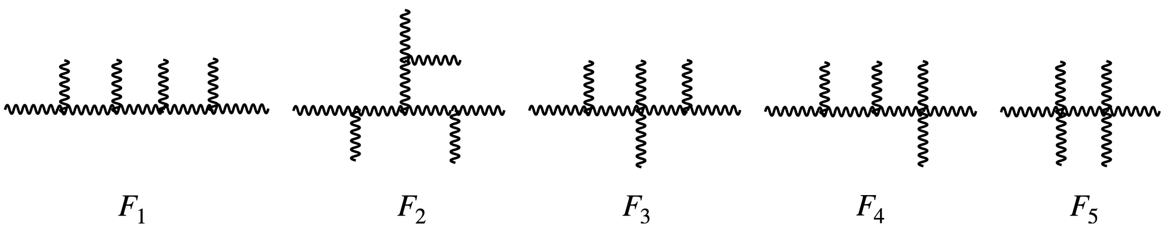


Figure 3.2: All six-point Feynman diagrams in flat space.

It is easy to get their expressions when all polarisation vectors are orthogonal to all momenta:

$$
\begin{aligned}
F_1 &= \frac{1}{2} f^{a_1a_2b_1} f^{b_1a_3b_2} f^{b_2a_4b_3} f^{b_3a_5a_6} \frac{(e_1\cdot e_2)(e_3\cdot e_4)(e_5\cdot e_6)}{s_{12}(s_{12}+s_{13}+s_{23})s_{56}} (s_{23}-s_{13})(s_{45}-s_{46}) \ , \\
F_2 &= \frac{1}{4} f^{a_1a_2b_2} f^{a_3a_4b_1} f^{a_5a_6b_3} f^{b_1b_2b_3} \frac{(e_1\cdot e_2)(e_3\cdot e_4)(e_5\cdot e_6)}{s_{12}s_{34}s_{56}} \\
&\quad \times \Big((s_{13}-s_{14})(s_{25}-s_{26}) - (s_{23}-s_{24})(s_{15}-s_{16}) + (s_{14}-s_{24})(s_{35}-s_{36}) \\
&\quad - (s_{13}-s_{23})(s_{45}-s_{46}) + (s_{15}-s_{25})(s_{36}-s_{46}) - (s_{16}-s_{26})(s_{35}-s_{45}) \Big) \ , \\
F_3 &= -\frac{1}{4} f^{a_1a_2b_1} f^{b_1a_3b_2} f^{b_2a_4b_3} f^{b_3a_5a_6} \frac{(e_1\cdot e_2)(e_3\cdot e_4)(e_5\cdot e_6)}{s_{12}(s_{12}+s_{13}+s_{23})s_{56}} \\
&\quad \times (s_{12}+s_{13}+s_{23})(s_{26}+s_{15}-s_{25}-s_{16}) + \cdots \ , \\
F_4 &= -\frac{1}{2} f^{a_1a_2b_1} f^{b_1a_3b_2} f^{b_2a_4b_3} f^{b_3a_5a_6} \frac{(e_1\cdot e_2)(e_3\cdot e_5)(e_4\cdot e_6) - (e_1\cdot e_2)(e_3\cdot e_6)(e_4\cdot e_5)}{s_{12}(s_{12}+s_{13}+s_{23})s_{56}} \\
&\quad \times (s_{23}-s_{13})s_{56} + \cdots \ , \\
F_5 &= \frac{1}{2} f^{a_1a_2b_1} f^{b_1a_3b_2} f^{b_2a_4b_3} f^{b_3a_5a_6} \frac{(e_1\cdot e_3)(e_2\cdot e_5)(e_4\cdot e_6) + \cdots}{s_{12}s_{56}(s_{12}+s_{13}+s_{23})s_{56}} + \cdots \ ,
\end{aligned} \tag{3.18}
$$

where $s_{ij} \equiv k_i \cdot k_j$ and the dots refer to additional terms from permutations of the indices.

The full flat space amplitude is

$$
\mathcal{A}^\flat = \sum_{i=1}^{5} C_i F_i + \text{perms} \ , \tag{3.19}
$$

where C_i are symmetry factors determined by combinatorics. However, one can also show that by imposing colour-kinematic duality, all the C_i coefficients are fixed up to an overall factor. The result fully agrees with the field theory limit of the six-point open-string amplitude at tree-level obtained from using pure spinor techniques [104],[7] by further imposing the orthogonality condition (3.25). However, it is worth emphasising again that the simplification from this configuration is drastic, allowing even for a diagrammatic evaluation by hand.

3.2 Six-point AdS gluon amplitudes

Now, we consider SYM on AdS_5 with $4d$ $\mathcal{N}=2$ superconformal symmetry in the boundary parlance. While pure Yang-Mills theory may be more physically relevant, the supersymmetric version provides a more suitable testing ground for developing new techniques, due to its more tractable kinematics. As anticipated in 2.5.2, the theory contains a scalar field s^a (supergluon) and a spin-1 gauge field v^a_μ as bosonic fields, transforming in the adjoint representation of a gauge group G_F ($a = 1, \ldots, \dim(G_F)$),

[7] The explicit expression is available at `https://www.southampton.ac.uk/ ∼crm1n16/6pt-SYM_bbbbbb.h`.

as well as fermionic super partners.[8] On the boundary, the supergluon is dual to a scalar field $\mathcal{O}^{a;\alpha_1\alpha_2}$, $\alpha_i = 1, 2$, with dimension $\Delta = 2$ and transforms in the spin-1 representation of $SU(2)_R$. The gluon is dual to a conserved flavor current $\mathcal{J}^a_\mu$ which has $\Delta = 3$ and is an R-symmetry singlet. From the boundary perspective, G_F is a global flavor symmetry.

Our goal is to compute the six-point function of supergluons:

$$G_6(x_i; v_i) = \langle \mathcal{O}^{a_1}(x_1; v_1) \ldots \mathcal{O}^{a_6}(x_6; v_6) \rangle \ , \tag{3.20}$$

where we have absorbed the $SU(2)_R$ indices by contracting them with R-symmetry polarisation spinors

$$\mathcal{O}^a(x; v) = \mathcal{O}^{a;\alpha_1\alpha_2}(x) v^{\beta_1} v^{\beta_2} \epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2} \ . \tag{3.21}$$

Compared to spinning correlators, the scalar six-point function (3.20) is kinematically simpler, as it avoids complications related to polarisation tensors and the index structure associated with higher-spin fields. However, it still encodes key features of AdS scattering, such as the analytic structure of Witten diagrams, the emergence of flat space limits, and the interplay between conformal and bulk factorisation. In particular, it allows us to explore the underlying dynamics in a setting where the complexity is reduced.

We will focus on tree-level scattering in AdS. Then fermionic fields will not be exchanged and therefore are irrelevant. As on the left-hand-side (LHS) of (3.20), we will often suppress the colour indices to lighten the notation. But the colour structures are described in the same way as in flat space gluon amplitudes and are given by cubic tree colour diagrams in Figure 3.1. While our perspective will be entirely from the bulk, the operators $\mathcal{O}^a$ are mesons in the dual gauge theory.[9] The correlator (3.20) computes the leading $1/N$ contribution to the connected six-point meson correlator in a large N gauge theory at infinite 't Hooft coupling.

The best way to describe these holographic correlators is to use the Mellin space formalism [48, 108], which defines a scattering amplitude in AdS. In this formalism, we write[10]

$$G_6(x_i; v_i) = \int [d\delta_{ij}] \bigg(\prod_{i<j} x_{ij}^{-2\delta_{ij}} \Gamma[\delta_{ij}] \bigg) M(\delta_{ij}; v_i) \ , \tag{3.22}$$

where the integration contour runs parallel to the imaginary axis with $\mathrm{Re}(\delta_{ij}) > 0$. See [48] for more details about the integration.

Here, $M(\delta_{ij}; v_i)$ is the Mellin amplitude and the Mellin variables satisfy the constraints $\delta_{ij} = \delta_{ji}$, $\delta_{ii} = -\Delta_i$, $\sum_{j\neq i} \delta_{ij} = 0$, ensuring that the integrand is conformally covariant with scaling dimension

[8]It can also be viewed as a consistent truncation of 8d $\mathcal{N} = 1$ SYM on $AdS_5 \times S^3$, which can be obtained by D3 branes probing an F-theory 7-brane singularity [105, 106] or D3 branes with D7 probes [107].

[9]The D3-D7 setup corresponds to $4d$ $\mathcal{N} = 4$ SYM coupled to a small number of $\mathcal{N} = 2$ fundamental hypermultiplets.

[10]The Gamma functions in the Mellin space representation of CFT correlators provide the double-trace poles.

Δ_i at the point x_i. It is most convenient to think of δ_{ij} as the Mandelstam variables formed from a set of fictitious flat space momenta, $\delta_{ij} = \vec{k}_i \cdot \vec{k}_j$, satisfying momentum conservation and the on-shell condition $\vec{k}_i^2 = -\Delta_i$. Then the constraints are automatically solved. Moreover, much of the flat space intuition also extends to the Mellin amplitude. For example, Mellin amplitudes also enjoy factorisation properties similar to flat space amplitudes.

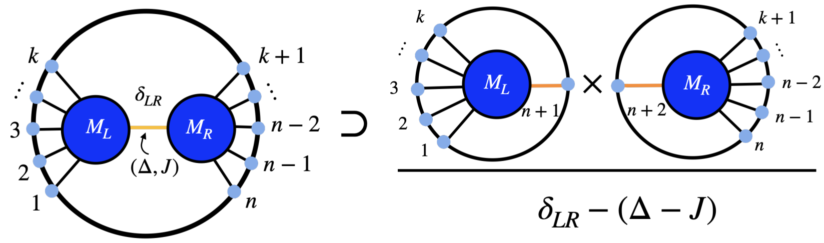


Figure 3.3: The Mellin amplitude factorises into the product of two lower-point amplitudes at a pole.

More precisely, Mellin amplitudes are meromorphic functions with simple poles at $\delta_{LR} = \Delta - J + 2\,m$, with $m = 0, 1, 2, \ldots$, that are associated with the exchange of an operator with dimension Δ and spin J. Here the propagator divides the n-point function into a $(k+1)$- and a $(n-k+1)$-point function as in Figure 3.3,[11] and $\delta_{LR} = \sum_{a=1}^{k} \sum_{i=k+1}^{n} \delta_{ai}$. The residues are controlled by lower-point amplitudes involving both the external and the exchanged operators [63, 80]. This is the CFT analogue of the well-known factorisation property of flat space amplitudes.

The simplest example is the exchange of a supergluon between two operators, say $\mathcal{O}_1$, $\mathcal{O}_2$, and the remaining ones. For this case, the Mellin amplitude has a pole at $\delta_{12} = 1$ given by

$$M(\delta_{ij}; v_i) \supset \frac{M_3 M_5}{\delta_{12} - 1} \,, \tag{3.23}$$

where M_3, M_5 are three- and five-point Mellin amplitudes of supergluons, respectively. Similar formulas also exist for the exchange of spinning operators, as well as for residues at satellite poles ($m > 0$) [80]. On the RHS of (3.23), we have suppressed the R-symmetry dependence, which itself factorises and can be reinstated at the end. The R-symmetry polarisations must be chosen consistently for the new legs associated with the exchanged particle. In practice, this can be achieved by inserting a projector that restricts the exchange channel to a given irreducible representation. A worked-out example of this procedure is provided, for instance, in Section 3.4 of [61].

The Mellin amplitude contains information about flat space in the high-energy limit. More precisely, the six-point scattering amplitude $\mathcal{A}^\flat$ of spin-1 gluons in flat space is related to the Mellin amplitude by

$$\lim_{\beta\to\infty} \beta M(\beta s_{ij}; v_i) \sim \mathcal{A}^\flat(s_{ij}; e_i) \,, \tag{3.24}$$

[11] In Figure 3.3 and in the following, $\supset$ means that the Mellin amplitude contains (among other terms) a contribution of the displayed form.

where $s_{ij} = k_i \cdot k_j$ are the flat space Mandelstam variables. This relation was first conjectured by Penedones in [48] and verified there for scalar theories at tree and one-loop level. It was later confirmed in [63] through diagrammatic Mellin rules reducing to standard Feynman rules, and more generally derived in [109] using wavepackets localised to flat regions of AdS.

Since the amplitude originates from AdS, the polarisation vectors e_i in (3.24) are restricted to the special configuration where they are orthogonal to *all* momenta [66]:

$$\boxed{e_i \cdot k_j = 0} \tag{3.25}$$

In fact, they lie within a four-dimensional subspace and are related to the $SU(2)_R$ polarisation spinors by

$$e_i^A = \frac{i}{\sqrt{2}} \sigma^A_{\alpha\beta} v_i^\alpha v_i^\beta \ , \quad A = 0, 1, 2, 3 \ , \tag{3.26}$$

where $\sigma^A_{\alpha\beta}$ are the Pauli matrices. This subspace is orthogonal to the subspace where k_i live, which ensures the condition (3.25). Note that the scalar supergluons and the spin-1 gluons in AdS lose their difference in the flat space limit. They both become spin-1 gluons but with different polarisations. The flat space amplitude $\mathcal{A}^\flat$ with polarisations obeying (3.25) can be viewed as the dimensional reduction of the eight dimensional gluon amplitude into a scalar amplitude, in agreement with the picture of consistent truncation into AdS_5 SYM before taking the limit.

3.2.1 Taxonomy of Witten diagrams

The AdS amplitude is essentially a collection of Witten diagrams. Equation (3.24) tells us that the Mellin amplitude reduces to the flat space amplitude in the high-energy limit. But this mapping is in fact more refined and holds at the level of individual diagrams. It is not difficult to check that the five flat space Feynman diagrams in Figure 3.2 correspond to the Witten diagrams in the first row of Figure 3.4 respectively.[12] For example, the Mellin amplitude of the diagram Ia is

$$M_{\mathrm{Ia}} = \sum_{m=0}^{1} \frac{\mathbb{T}_{[a_1 a_2 a_3 a_4 a_5 a_6]}\, e_{12}\, e_{34}\, e_{56}\, (\delta_{13} - \delta_{23})(\delta_{45} - \delta_{46})}{(\delta_{12} - 1)(\delta_{12} + \delta_{13} + \delta_{23} + m - 2)(\delta_{56} - 1)} \ , \tag{3.27}$$

which reduces to the first diagram in 3.18 in the flat space limit.[13] Here $e_{ij} \equiv e_i \cdot e_j$.

Note that the Mellin amplitudes of the Witten diagrams Ia to Ie all scale as β^{-1}, as required by (3.24). Diagrams with a slower decaying behaviour are disallowed as they violate the flat space limit. For example, D1 and D2 in Figure 3.5 decay as β^0 and are therefore excluded. The exclusion of D1 implies the non-existence of quartic vertices with three scalars and one vector. This, in turn, implies that diagrams D3 and D4 do not contribute and also decay as β^{-1}, but already from matching the

[12] See, *e.g.*, [48, 55, 63] for computation of Witten diagrams in Mellin space.
[13] The finite sum over m can be understood in terms of the truncation properties observed in [87].

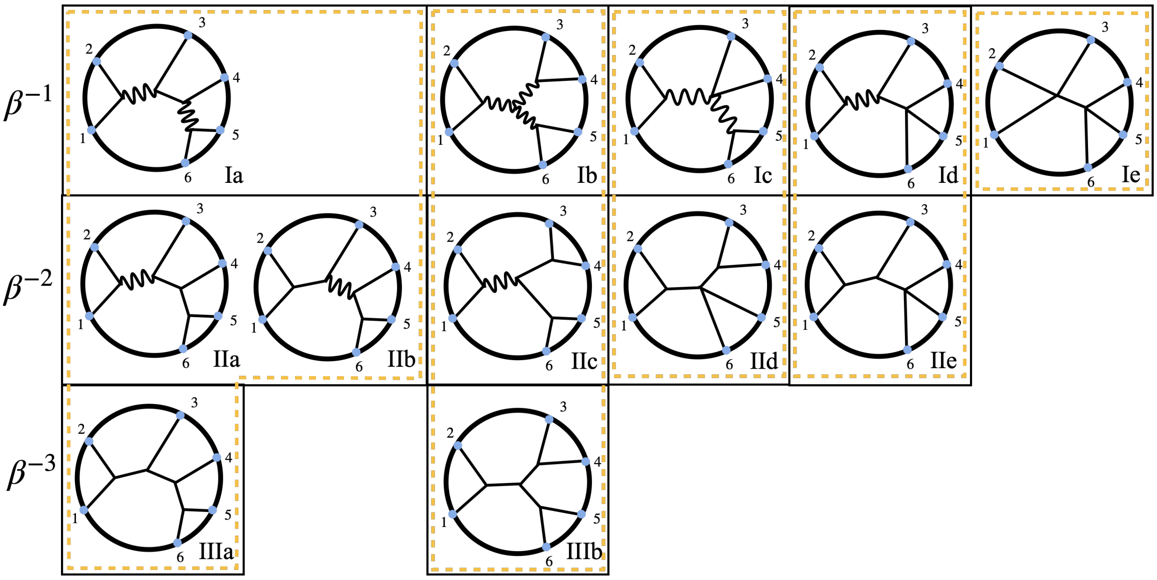


Figure 3.4: Tree-level Witten diagrams. Straight lines represent scalar supergluons (s) and wavy lines are spin-1 gluons (v). Diagrams in the i-th row decay as β^{-i} in the large β limit. Diagrams grouped together by the yellow dashed line are related by replacing v with s.

flat space amplitude, we can conclude that there are no such diagrams. However, in the opposite direction, diagrams with faster decaying rates are not detected by the flat space limit (3.24) and are not prohibited. These diagrams are catalogued in the second and third rows of Figure 3.4 and are obtained from the first row by changing some vector internal lines into scalar lines. This requires some care, as certain replacements may violate R-symmetry (*e.g.* replacing only one v by s in diagram Ib) and are therefore not listed. To summarise, Figure 3.4 contains all possible six-point Witten diagrams up to permutations.

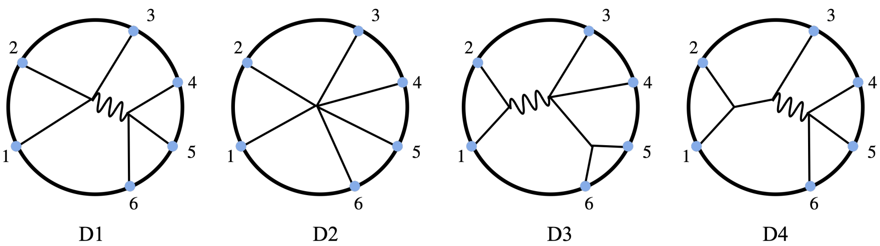


Figure 3.5: Examples of disallowed Witten diagrams.

While in this Section we focus on the computation of Witten diagrams in Mellin space, we would like to emphasise that these diagrams can also be evaluated directly in position space. In fact, this is how we initially approached the problem. As reviewed in Section 2.4, certain exchange diagrams can be reduced to a finite sum of contact diagrams using vertex identities. Implementing this trick for the $3 \to 3$ exchange diagram (Ie in Figure 3.4) is more involved, as previewed in (2.64). In the next Section, we shall present an example of a computation in position space, which offers valuable insights into the utility of vertex identities and is of interest beyond the specific case under consideration.

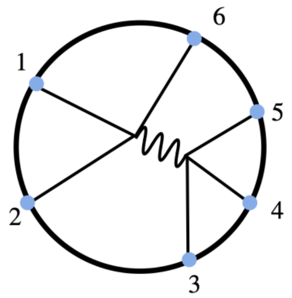


Figure 3.6: Single exchange diagram with internal gluon.

3.2.2 A note on position space computations

In the case of n scalars and a vector in the vertex, there is no such trick as 2.4.3 available in the literature, but we can find another approach to simplify the integration even in this case.

Let us study the diagram in Figure 3.6. We will show how to reduce it to a scalar exchange, such that we can implement the trick for the $(n+1)$-vertex in the case of all scalars. We assume that the vertex involving one vector and an arbitrary number of scalar fields contains only a single derivative. Since it is not possible to construct a conserved current from three scalars using only one derivative, we choose the derivative to act on the full product of bulk-to-boundary propagators within the vertex. As a consequence, the diagram under consideration is not gauge invariant, since the current is not conserved due to the vertex being totally symmetric under the exchange of the involved external legs. We will return to this point shortly.

We want to simplify the single-exchange integral[14]

$$W_v = \int \frac{d^{d+1}z}{z_0^{d+1}} \frac{d^{d+1}w}{w_0^{d+1}} \left[D_\mu^z \left(G_{B\partial}^{\Delta=2}(z,\vec{x}_1) G_{B\partial}^{\Delta=2}(z,\vec{x}_2) G_{B\partial}^{\Delta=2}(z,\vec{x}_3) \right) \right] \\ \times G_{BB}^{\mu\nu}(z,w) \left[D_\nu^w \left(G_{B\partial}^{\Delta=2}(w,\vec{x}_4) G_{B\partial}^{\Delta=2}(w,\vec{x}_5) G_{B\partial}^{\Delta=2}(w,\vec{x}_6) \right) \right] . \tag{3.28}$$

This structure is well-suited for the integration by parts technique, allowing the derivatives to be transferred from the vertices to the bulk-to-bulk propagator. Notice that, in performing the integration by parts, we neglect surface terms. However, one can verify at the end that these terms do not contribute once the inequivalent permutations of the external legs are summed. We end up with the structure

$$D_z^\mu D_w^\nu G_{\mu\nu}(z,w) , \tag{3.29}$$

where we can use the covariant expression (2.43) for the bulk-to-bulk-propagator (only the contri-

[14]The subscript v of the Witten diagram stands for *vector exchange*. Similary, s will stand for *scalar exchange* in (3.37).

bution of the physical term). We can write

$$D_z^\mu D_w^\nu[(\partial_\mu^z \partial_\nu^w u)F(u)] = D_z^\mu[(\partial_\mu^z \Box_{AdS}\, u)F(u) + (\partial_\mu^z \partial_\nu^w u)(F'(u)\partial_w^\nu u)] \ , \tag{3.30}$$

where $F'(u) = \partial_u F(u)$. By applying the identities (2.32) and massaging the resulting expression to use (2.61), we find the simple expression[15]

$$4(1+u)F + \delta(z,w) \ . \tag{3.31}$$

In $d = 4$,

$$(1+u)F = \frac{\Gamma(3/2)}{(4\pi)^{5/2}}[u(u+2)]^{-3/2} = \frac{1}{64\pi^2 u^2}\, {}_2F_1\left(2, \frac{1}{2}, 1, -\frac{2}{u}\right) \ , \tag{3.32}$$

which is the scalar bulk-to-bulk propagator with $\Delta = 2$. We report below the general expression for completeness:

$$G_{BB}^{\Delta}(u) = \frac{\Gamma(\Delta)\Gamma(\Delta - d/2 + 1/2)}{(4\pi)^{(d+1)/2}\Gamma(2\Delta - d + 1)} \left(\frac{2}{u}\right)^{\Delta} {}_2F_1\left(\Delta, \Delta - \frac{d}{2} + \frac{1}{2}, 2\Delta - d + 1, -\frac{2}{u}\right) \ . \tag{3.33}$$

In our case,

$$G_{BB}^{2}(u) = \frac{1}{8\pi^2 u^2}\ {}_2F_1\left(2, \frac{1}{2}, 1, -\frac{2}{u}\right) \ . \tag{3.34}$$

In the end

$$4(1+u)F = \frac{1}{2} G_{BB}^{2}(u) \ . \tag{3.35}$$

We conclude that the vector exchange (3.28) is the same as

$$\begin{aligned} W_v = &-\frac{1}{2}\int \frac{d^{d+1}z}{z_0^{d+1}}\frac{d^{d+1}w}{w_0^{d+1}}\left(G_{B\partial}^{\Delta=2}(z,\vec{x}_1)G_{B\partial}^{\Delta=2}(z,\vec{x}_2)G_{B\partial}^{\Delta=2}(z,\vec{x}_3)\right) \\ &\times G_{BB}^{\Delta=2}(z,w)\left(G_{B\partial}^{\Delta=2}(w,\vec{x}_4)G_{B\partial}^{\Delta=2}(w,\vec{x}_5)G_{B\partial}^{\Delta=2}(w,\vec{x}_6)\right) - D_{222222} \ , \end{aligned} \tag{3.36}$$

namely, a scalar exchange and a contact term.

Let us make a final remark. The diagram in Figure 3.6 is not directly part of our ansatz, and instead we need the diagram in Figure 3.7. In this case, the expression is

$$\begin{aligned} W_{s,v,s} = &\int \frac{d^5z}{z_0^5}\frac{d^5y}{y_0^5}\frac{d^5w}{w_0^5}\frac{d^5\rho}{\rho_0^5} G_{B\partial}^{\Delta=2}(z,\vec{x}_1)G_{B\partial}^{\Delta=2}(z,\vec{x}_2)\left(G_{BB}^{\Delta=2}(z,y)\overleftrightarrow{\nabla}_{y,\mu}G_{B\partial}^{\Delta=2}(y,\vec{x}_6)\right)G_{BB}^{\mu\nu}(y,w) \\ &\times\left(G_{BB}^{\Delta=2}(w,\rho)\overleftrightarrow{\nabla}_{w,\nu}G_{B\partial}^{\Delta=2}(w,\vec{x}_5)\right)G_{B\partial}^{\Delta=2}(\rho,\vec{x}_3)G_{B\partial}^{\Delta=2}(\rho,\vec{x}_4) \ . \end{aligned} \tag{3.37}$$

[15] We drop terms like $u\,\delta(z,w)$, because they vanish when integrated.

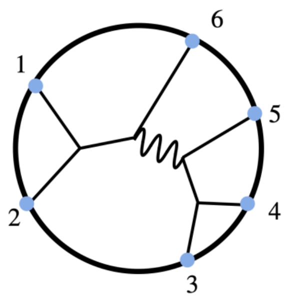


Figure 3.7: Triple exchange diagram.

By integration over two of the cubic vertices, we get

$$\begin{aligned}W_{s,v,s} =& \frac{1}{16x_{12}^2x_{34}^2}\int\frac{d^5y}{y_0^5}\frac{d^5w}{w_0^5}\left[G_{B\partial}^{\Delta=2}(y,\vec{x}_6)\overset{\leftrightarrow}{\nabla}_{y,\mu}\left(G_{B\partial}^{\Delta=1}(y,\vec{x}_1)G_{B\partial}^{\Delta=1}(y,\vec{x}_2)\right)\right]G_{BB}^{\mu\nu}(y,w)\\ &\times\left[G_{B\partial}^{\Delta=2}(w,\vec{x}_5)\overset{\leftrightarrow}{\nabla}_{w,\nu}\left(G_{B\partial}^{\Delta=1}(w,\vec{x}_3)G_{B\partial}^{\Delta=1}(w,\vec{x}_4)\right)\right].\end{aligned} \tag{3.38}$$

The result is symmetric under the exchanges $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$. However, to directly apply Equation (3.36), we would also require symmetry under $1 \leftrightarrow 6$, $2 \leftrightarrow 6$, $3 \leftrightarrow 5$, and $4 \leftrightarrow 5$. Nonetheless, when we include all possible permutations of the external legs at the cubic vertices, we end up with nine distinct diagrams which, when summed, manifest the required symmetry. Hence, the combined contribution of our ansatz (with all its permutations) is precisely equivalent to that of the fully symmetric $3 \to 3$ vector exchange diagram.

3.2.3 Bootstrapping the six-point amplitude

Using all these ingredients (input from position space and Mellin space techniques), we can now formulate an efficient algorithm to compute the six-point amplitude. This comes in two steps. The flat space limit $\mathcal{A}^\flat$ clearly allows us to fix the coefficient of each diagram in the first row of Figure 3.2. For the rest of the Witten diagrams, we observe that they have at least one internal scalar line which separates the diagram into a five-point diagram and a three-point diagram. Although these Witten diagrams are not captured by the flat space limit, they are detected by the AdS amplitude factorisation. We can therefore fix all their contributions in terms of the five-point and three-point Mellin amplitudes of supergluons.

To see how the strategy works in detail, it is instructive to first look at an explicit example where we reproduce the four-point function [66]. By different large β scalings, we have

$$M_{\text{ansatz}}^{\text{4pt}} = M_{\text{L}} + M_{\text{SL}}\,. \tag{3.39}$$

The leading part consists of the gluon exchange diagrams and the contact diagrams

$$M_{\mathrm{L}} = \mathtt{c}_s \left[\lambda_v \frac{e_{12}\, e_{34}(t-u)}{s-2} + \lambda_{c,1}\, e_{13}\, e_{24} + \lambda_{c,2}\, e_{14}\, e_{23} \right] + (t\text{- and } u\text{-channels}) \ ,$$

where $\mathtt{c}_s = \mathbb{T}_{[a_1 a_2 a_3 a_4]}$ and

$$\delta_{12} = \delta_{34} = \frac{4-s}{2} \ , \quad \delta_{14} = \delta_{23} = \frac{4-t}{2} \ , \quad \delta_{13} = \delta_{24} = \frac{4-u}{2} \ . \tag{3.40}$$

The relative coefficient between the exchange and contact diagrams is fixed by the flat space limit to be

$$\frac{\lambda_{c,1}}{\lambda_v} = -1 \ , \quad \frac{\lambda_{c,2}}{\lambda_v} = 1 \ . \tag{3.41}$$

The subleading part is proportional to the scalar exchange

$$M_{\mathrm{SL}} = \mathtt{c}_s \lambda_s \frac{e_{12}\, e_{34} - 2 v_{13}\, v_{24}\, v_{12}\, v_{34}}{s-2} + (t\text{- and } u\text{-channels}) \ ,$$

where $v_{ij} = v_i^\alpha v_j^\beta \epsilon_{\alpha\beta}$. By factorising the ansatz on the scalar internal line, we get a product of two supergluon three-point functions, and this gives

$$\frac{\lambda_s}{\lambda_v} = 2 \ . \tag{3.42}$$

For the six-point case, instead of starting from a sum of all the Witten diagrams in Figure 3.4 with unfixed coefficients, it is more convenient to write an ansatz with the same analytic structure:

$$\begin{aligned} M_{\text{ansatz}} =& \frac{P^{(2)}_{12;34;56}(\delta_{kl}, v)}{(\delta_{12}-1)(\delta_{34}-1)(\delta_{56}-1)} + \frac{P^{(1)}_{12;34}(\delta_{kl}, v)}{(\delta_{12}-1)(\delta_{34}-1)} + \frac{P^{(0)}_{12}(\delta_{kl}, v)}{(\delta_{12}-1)} \\ &+ \sum_{m=0}^{1} \frac{C^{(2)}_{123,m}}{(\delta_{12}+\delta_{13}+\delta_{23})+m-2} + \text{perms} \ . \end{aligned} \tag{3.43}$$

Here $P^{(n)}_{\cdots}$ are polynomials of degree n in the Mellin variables and $C^{(2)}_{\cdots}$ are rational functions with the same analytic structure as the four-point Mellin amplitude. The highest degree terms in the polynomials $P^{(n)}$ are easily fixed by matching with the flat space scattering amplitude. In fact, the flat space limit fixes completely $P^{(0)}_{12}$ and leaves 8 undetermined coefficients inside the polynomials $P^{(2)}_{12;34;56}$ and $P^{(1)}_{12;34}$. These remaining coefficients, as well as those in $C^{(2)}_{123,m}$, are determined by the factorisation of Mellin amplitudes. In addition to the factorisation into M_5 and M_3, we can also consider the factorisation of the amplitude on a scalar line into two four-point supergluon amplitudes. We used this to fix the relative coefficient of the contributions corresponding to the first row of Figure 3.4 and the last two rows.

The explicit result is provided in the notebook accompanying the publication [87]. It passes several non-trivial checks. First, it factorises correctly into a spinning five-point amplitude and a spinning three-point amplitude on an internal gluon line. Note that this condition was not part of the constraints imposed in our algorithm. More precisely, this factorisation requires the six-point amplitude to contain a contribution

$$M(\delta_{ij}; v_i) \supset \frac{\delta_{ai}\, M_3^a\, M_5^i}{\delta_{12} - 1} \ , \tag{3.44}$$

where the lower-point spinning amplitudes carry additional indices $a = 1, 2,\ i = 3, 4, 5, 6$. Let us briefly review the framework of spinning Mellin amplitudes.

They are most conveniently defined using the embedding space formalism, where the action of the conformal group is linearised. Each point $x^\mu \in \mathbb{R}^d$ is lifted to a null ray

$$P^A \in \mathbb{R}^{1,d+1} \ , \quad \text{with } P \cdot P = 0 \ , \ P \sim \lambda P \ . \tag{3.45}$$

Using the rescaling symmetry, we can gauge fix P to be

$$P = \left(\frac{1+x^2}{2}, \frac{1-x^2}{2}, x^\mu \right) \ . \tag{3.46}$$

Operators of dimension Δ, spin J are homogeneous functions of P and Z, where the polarisation $Z^A \in \mathbb{R}^{1,d+1}$ encodes tensor structures and satisfies $P \cdot Z = 0$. The homogeneity condition is

$$\mathcal{O}(\lambda\, P, \alpha\, Z) = \lambda^{-\Delta}\, \alpha^J\, \mathcal{O}(P, Z) \ , \tag{3.47}$$

while the transversality condition reads

$$\mathcal{O}(P, Z + \beta P) = \mathcal{O}(P, Z) \ . \tag{3.48}$$

For a correlator with n scalars and 1 spinning (focusing on $J = 1$) operator, we can generalise 3.22 as [80]

$$G_{n,1}(P_1, \ldots, P_n; P_0, Z) = \sum_{a=1}^{n} (Z \cdot P_a) \int [d\delta] M_n^a \prod_{i,j=1, i<j}^{n} \frac{\Gamma(\delta_{ij})}{(-2P_i \cdot P_j)^{\delta_{ij}}} \prod_{i=1}^{n} \frac{\Gamma(\delta_{0i} + \boldsymbol{\delta}_i^a)}{(-2P_i \cdot P_0)^{\delta_{0i} + \boldsymbol{\delta}_i^a}} \ , \tag{3.49}$$

where here we denote the Kronecker delta by $\boldsymbol{\delta}_i^a$ to avoid confusion with the Mellin-Mandelstam variables, and

$$\delta_{i0} = -\sum_{j=1}^{n} \delta_{ij} \ , \quad \delta_{ij} = \delta_{ji} \ , \quad \delta_{ii} = -\Delta_i \ , \quad \sum_{i=1}^{n} \delta_{i0} = \Delta - 1 \ .$$

The spinning Mellin amplitude is now a collection of partial amplitudes labelled by an index $a = 1, \ldots, n$ because of the different structures $Z \cdot P_a$. When the current is conserved, we further have

$$\frac{\partial}{\partial P_0} \cdot \frac{\partial}{\partial Z} G_{n,1}(P_1, \ldots, P_n; P_0, Z) = 0 \ . \tag{3.50}$$

In Mellin space, transversality and conservation translate to the constraints

$$\sum_{a=1}^{n} \delta_{a0} M_n^a = 0 \ , \quad \sum_{p,q=1, p\neq q}^{n} \delta_{pq} \left[M_n^a\right]^{pq} = 0 \ , \tag{3.51}$$

where

$$\left[f(\delta_{ij})\right]^{pq} = f(\delta_{ij} + \boldsymbol{\delta}_i^p \boldsymbol{\delta}_j^q + \boldsymbol{\delta}_i^q \boldsymbol{\delta}_j^p) \ . \tag{3.52}$$

To perform the consistency check mentioned above, we need to consider the five-point spinning amplitude M_5^a. This amplitude has poles at $\delta_{ij} = 1, \ \delta_{0i} = 0, 1$, which correspond to the exchange of *AdS* fields in the OPE. Note that the appearance of two poles in δ_{0i} is a distinguishing feature compared to the scalar five-point amplitude, where there is only one. However, this can be inferred from the structure of the OPE between the scalar and the current. Alternatively, the appearance of the additional pole can be expected from the consistency with the pole structure of the supergluon six-point amplitude.

The five-point spinning Mellin amplitude is then just a rational function with finitely many poles. Moreover, this rational function should be consistent with the flat space limit, which in turn constrains the degree of the numerator. We can make an ansatz where we also restore the R-symmetry polarisations and colour structures which have been suppressed in (3.49). Notice also that the Mellin amplitude should have permutation symmetry among the four supergluon operators. Taking this into account, we arrive at an ansatz with 67 unfixed coefficients. Imposing transversality leaves 27 coefficients, while conservation further reduces the number to 9. By using the fact that the residues at the δ_{0i} poles are related to the known four-point spinning amplitude, we eventually have just two unfixed coefficients.[16]

To sum up, the five-point spinning amplitudes can be fixed up to two undetermined coefficients by imposing only basic consistency conditions, which include permutation symmetry, transversality, conservation, and factorisation. The three-point amplitude is determined by these conditions up to an overall constant. The compatibility with our six-point amplitude provides a strong consistency check of our results.

Another independent check could be provided by the chiral-algebra condition [60, 110]. This condition is highly non-trivial and has been essential in previous approaches [27, 64–66, 70, 86, 88].

[16]The spinning four-point and three-point amplitudes can also be bootstrapped in the same way. The difference is that they are fully determined by permutation symmetry, transversality and conservation, up to an overall constant.

At the same time, the chiral algebra condition is practically cumbersome to implement, especially at higher points. However, our algorithm does not require this constraint, which is one of the key reasons it is powerful and efficient. This greatly extends the range of holographic correlators which we can compute, including, for instance, SYM on $AdS_7 \times S^3$, where a chiral algebra structure is not available.

3.3 Six-point AdS graviton amplitudes

In the remainder, we outline the analysis of the six-point supergraviton correlator on $AdS_5 \times S^5$, equivalently the tree-level six-point function of $20'$ operators in $4d\,\mathcal{N}=4$ SYM at large N, working in the supergravity approximation (i.e. $\alpha' \to 0$, neglecting stringy corrections). We will briefly explain the ansatz and how to fix its free coefficients. The main goal of this Section is to highlight the bootstrap strategy developed for this case, emphasising its key strengths.

The $20'$ operator, $\mathcal{O}_{20'}^{IJ}$, is the bottom component of the super multiplet that contains the R-symmetry current $J_\mu^{[IJ]}$ and the stress tensor $\mathcal{T}^{\mu\nu}$ among other operators. It has protected conformal dimension $\Delta = 2$ and transforms in the rank-2 symmetric traceless representation of $SO(6)_R$. Via the AdS/CFT correspondence, this operator is dual to the supergraviton, which is a scalar field, while the spinning graviton is dual to $\mathcal{T}^{\mu\nu}$. We are interested in computing

$$\langle \mathcal{O}_{20'}^{I_1 J_1}(x_1) \dots \mathcal{O}_{20'}^{I_6 J_6}(x_6) \rangle \, . \tag{3.53}$$

Again, we keep track of the R-symmetry indices by contracting them with six-dimensional null polarisation vectors y_I:[17]

$$\begin{aligned} &\mathcal{O}_{20'}(x,y) \equiv \mathcal{O}_{20'}^{IJ}(x) y_I y_J \, , \qquad y^2 = 0 \, . \\ &G_6(x_1, y_1, \dots, x_6, y_6) = \langle \mathcal{O}_{20'}(x_1,y_1) \dots \mathcal{O}_{20'}(x_6,y_6) \rangle \, . \end{aligned} \tag{3.54}$$

As usual, the bosonic part of superconformal symmetry allows us to write the six-point function in terms of invariant cross ratios, and the fermionic generators impose further constraints which relate the spacetime and R-symmetry dependence. While the full implications are not clear, two weaker conditions are known in the literature. The first condition comes from the *chiral algebra* construction [93]. When all operators are inserted on a $2d$ plane with complex coordinates $(z_i, \bar{z}_i)$ and R-symmetry polarisations are restricted to the special configuration $y_{ij} = (z_i - z_j)(v_i - v_j)$, with

[17] Invariance under R-symmetry requires G_6 to depend polynomially on $y_{ij} = y_i \cdot y_j$ with degree 2 for each vector y_i [61].

arbitrary v_i, the correlator becomes independent of the meromorphic coordinates z_i:

$$G_6(z_i, \bar{z}_i, y_i)\big|_{y_{ij}=(z_i-z_j)(v_i-v_j)} = g(\bar{z}_i, v_i) \ . \tag{3.55}$$

The second condition comes from the topological *Drukker-Plefka twist* [111]: when the R-symmetry polarisations are restricted to $y_{ij} = x_{ij}^2$ and the locations of the operator insertions are unconstrained, the correlation function becomes topological:

$$G_6(x_i, y_i)\big|_{y_{ij}=x_{ij}^2} = \text{constant} \ . \tag{3.56}$$

We note here that for four-point functions, the Drukker-Plefka twist is implied by the chiral algebra twist by further setting $v_i = \bar{z}_i$ because we can use conformal symmetry to put four points on a plane. This is, however, no longer the case when we have more than four points, and these two conditions are complementary.

3.3.1 Mellin space

We have seen that Mellin amplitudes are analytic functions whose poles in the variables δ_{ij} correspond to the exchange of single-trace operators appearing in the OPE. By partitioning the external operators into two subsets, L and R, we can study the exchange of an operator with dimension Δ and spin J between these two groups. Schematically:

$$M(\delta_{ij}, y_{ij}) \approx \sum_m \frac{Q_m(\delta_{ij}, y_{ij})}{\delta_{LR} - (\Delta - J + 2m)} \ , \quad \delta_{LR} = \sum_{i\in L}\sum_{j\in R} \delta_{ij} \ . \tag{3.57}$$

The residues at these poles are related to lower-point amplitudes depending on the spin of the exchanged operator. In the following, we suppress the R-symmetry dependence, which obeys its own "factorisation" and can be multiplied back in the end. The residue of the amplitude (3.57), for $m = 0$, can be expressed as [80]

$$Q_0(\delta_{ij}) = k_{\Delta,J} \sum_{a\in L}\sum_{i\in R} M_L^{\{a\}} M_R^{\{i\}} \prod_{\ell=1}^{J} (\delta_{a_\ell i_\ell} + \boldsymbol{\delta}_{a_\ell}^{i_{\ell+1}} \boldsymbol{\delta}_{i_\ell}^{i_{\ell+1}} + \cdots + \boldsymbol{\delta}_{a_\ell}^{i_{\ell+J}} \boldsymbol{\delta}_{i_\ell}^{i_{\ell+J}}) \ , \tag{3.58}$$

where $k_{\Delta,J}$ is a normalization constant, $M_L^{\{a\}}$, $M_R^{\{i\}}$ are lower-point Mellin amplitudes and $\{a\} = a_1 \dots a_J$, $\{i\} = i_1 \dots i_J$. For $m > 0$, so far, there is no general formula valid for any spin. However, residue formulas for spins up to two have been obtained in [80] and are sufficient for the purpose of this computation. The residues with $m > 0$ are determined by conformal symmetry.

A convenient way to implement conformal symmetry is via the conformal Casimir, which was first introduced in [112] to compute four-point conformal blocks. Here, the Casimir operator is a multi-particle construct built from the sum of conformal generators acting on the operators in group L. By simple arguments based on conformal invariance (see, *e.g.*, [113]), the Casimir operator is mapped to the Laplacian operator in AdS, which collapses the propagator of the exchanged field to a delta function. In position space, the Casimir operator is a differential operator acting on the coordinates. Acting on the definition (3.22) and shifting δ_{ij}, it becomes a difference operator, which will remove the poles in δ_{LR} in (3.57) and relate the residues for different m via recursion relations. For the six-point supergraviton amplitude, there are two non-trivial factorisations where the six-point amplitude splits into the product of three-point and five-point amplitudes or the product of two four-point amplitudes:

$$M(\delta_{ij}) \approx \frac{Q_m^{(4-4)}(\delta_{ij})}{(\delta_{l_1r_1}+\delta_{l_1r_2}+\delta_{l_1r_3}+\delta_{l_2r_1}+\delta_{l_2r_2}+\delta_{l_2r_3}+\delta_{l_3r_1}+\delta_{l_3r_2}+\delta_{l_3r_3})-(\tau+2m)}\,, \tag{3.59}$$

$$M(\delta_{ij}) \approx \frac{Q_m^{(3-5)}(\delta_{ij})}{(\delta_{l_1r_1}+\delta_{l_1r_2}+\delta_{l_1r_3}+\delta_{l_1r_4}+\delta_{l_2r_1}+\delta_{l_2r_2}+\delta_{l_2r_3}+\delta_{l_2r_4})-(\tau+2m)}\,. \tag{3.60}$$

Here, l_i and r_i belong to different groups of external operators, and we have added superscripts to the residues to indicate their factorisations into lower-point amplitudes. In the factorisation channel, only three operators can appear in our setup. These are the $20'$ operator itself $\mathcal{O}_{20'}$, the R-symmetry current $\mathcal{J}$ and the stress tensor operator $\mathcal{T}$. Note that all these operators have the same conformal twist $\tau = \Delta - J = 2$, which provides simplification to the pole structure of the Mellin amplitude.

To determine the residues in (3.59), one would need the following four-point functions

$$\langle \mathcal{O}_{20'}(x_1,y_1)\mathcal{O}_{20'}(x_2,y_2)\mathcal{O}_{20'}(x_3,y_3)\mathcal{O}_{20'}(x_0,y_0)\rangle\,, \tag{3.61}$$

$$\begin{aligned}&\langle \mathcal{O}_{20'}(x_1,y_1)\mathcal{O}_{20'}(x_2,y_2)\mathcal{O}_{20'}(x_3,y_3)\mathcal{J}(x_0,z_0)\rangle\,,\\ &\langle \mathcal{O}_{20'}(x_1,y_1)\mathcal{O}_{20'}(x_2,y_2)\mathcal{O}_{20'}(x_3,y_3)\mathcal{T}(x_0,z_0)\rangle\,.\end{aligned} \tag{3.62}$$

Fortunately, these have already been computed in [86, 114]. By contrast, to compute the residues in (3.60) the following five-point functions are needed

$$\langle \mathcal{O}_{20'}(x_1,y_1)\mathcal{O}_{20'}(x_2,y_2)\mathcal{O}_{20'}(x_3,y_3)\mathcal{O}_{20'}(x_4,y_4)\mathcal{O}_{20'}(x_0,y_0)\rangle\,, \tag{3.63}$$

$$\begin{aligned}&\langle \mathcal{O}_{20'}(x_1,y_1)\mathcal{O}_{20'}(x_2,y_2)\mathcal{O}_{20'}(x_3,y_3)\mathcal{O}_{20'}(x_4,y_4)\mathcal{J}(x_0,z_0)\rangle\,,\\ &\langle \mathcal{O}_{20'}(x_1,y_1)\mathcal{O}_{20'}(x_2,y_2)\mathcal{O}_{20'}(x_3,y_3)\mathcal{O}_{20'}(x_4,y_4)\mathcal{T}(x_0,z_0)\rangle\,.\end{aligned} \tag{3.64}$$

While the supergraviton five-point function has been computed in [86], the other two spinning correlators are not known. Moreover, unlike the four-point function case, where the spinning correlators are related to the scalar correlator by superconformal symmetry, this is not true for five-point functions [115].[18]

The lesson to learn is that applying factorisation across one or more compatible[19] channels of the supergraviton Mellin amplitude inevitably leads to lower-point spinning correlators whose explicit forms are still unknown. On top of that, remember once again that it is not known how to further extend the Mellin representation for a general number of spinning operators. We will not present a general solution to this problem here. However, there is a way to implement multi-factorisation within the scope of the case of study. One can use the (position space) lightcone OPE between two scalar identical operators [116]:[20]

$$\mathcal{O}(x_1)\,\mathcal{O}(x_2) \approx \sum_J C_{12J} \int_0^1 [dt]\, \frac{\mathcal{O}_J(x_1 + t x_{21}, x_{12})}{(x_{12}^2)^{\frac{2\Delta_{\mathcal{O}}-\tau}{2}}} + \dots \,. \tag{3.65}$$

Here τ is the twist of the exchanged operator and C_{12J} is the OPE coefficient. In the lightcone limit, $x_{12}^2 \to 0$, and the polarisation vector of the spinning operator is taken to be the null vector x_{12}. Applying the lightcone OPE multiple times, it is possible to reconstruct multiple factorisations. For example, if we apply it twice to reduce the six-point function into four-point functions, we get

$$\begin{aligned}\langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_6)\rangle\big|_{\substack{x_{12}^2\to 0\\ x_{34}^2\to 0}} &= \sum_{J_1,J_2} \frac{C_{12J_1}C_{34J_2}}{(x_{12}^2 x_{34}^2)^{\Delta_{\mathcal{O}}}} \\ &\times \int [dt_1][dt_2](x_{12}^2x_{34}^2)^{\frac{\tau}{2}} \langle \mathcal{O}_{J_1}(x_1+t_1x_{21},x_{12})\mathcal{O}_{J_2}(x_3+t_2x_{43},x_{34})\mathcal{O}(x_5)\mathcal{O}(x_6)\rangle\,,\end{aligned} \tag{3.66}$$

where we have assumed that the leading contribution comes from a family of operators with twist τ. The resulting spinning Mellin amplitudes can be decomposed into tensor structures such that the Mellin transform can be applied directly to the accompanying scalar functions, bypassing the problem of dealing with spinning Mellin amplitudes [61].

3.3.2 Ansatz and strategy

As for the supergluons, the ansatz follows from the properties of the underlying Witten diagrams. In particular, the Mellin amplitude of the supergraviton six-point function is a rational function which has poles in δ_{ij} and is a polynomial in y_{ij}. We can parametrise the ansatz as follows:

$$\begin{aligned}M(\delta_{ij},y_{ij}) &= \left(\frac{P_4(\delta_{ij},y_{ij})}{(\delta_{12}-1)(\delta_{34}-1)(\delta_{56}-1)} + \text{perms}\right) + \left(\frac{P_3(\delta_{ij},y_{ij})}{(\delta_{12}-1)(\delta_{34}-1)} + \text{perms}\right) \\ &+ \left(\frac{P_2(\delta_{ij},y_{ij})}{(\delta_{12}-1)} + \text{perms}\right) + \left(\sum_{m=0}^{m_{\max=2}} \frac{B_m(\delta_{ij},y_{ij})}{(\delta_{12}+\delta_{13}+\delta_{23}+m-2)} + \text{perms}\right) + P_1(\delta_{ij},y_{ij})\,.\end{aligned} \tag{3.67}$$

[18] Notice that, however, these five-point functions can be obtained once the six-point function is completely fixed.

[19] Here "compatible channels" refers to multiple factorisation channels that can be applied without conflicting with each other's kinematic constraints or analytic structures.

[20] The integration measure is given by $[dt] = dt\,(t(1-t))^{\frac{\Delta+J-2}{2}}$.

This ansatz has a similar structure as the one for supergluons in AdS (3.43). The main differences are that here m runs up to 2 instead of 1 and the degrees of the polynomials in the residues are higher. Moreover, there is a regular term P_1 which is absent in the supergluon case. Similarly to the gluon case, from the pole structures of the ansatz, it is easy to associate P_i and B_m with various exchange (or contact) processes which are enumerated in Figure 3.8. The numerators P_i are polynomials in δ_{ij}, while B_m are rational functions and can have poles in the compatible channels at $\delta_{12}, \delta_{13}, \delta_{23}, \delta_{45}, \delta_{45}, \delta_{46}, \delta_{56} = 1$.

Recall that the high-energy limit of the Mellin amplitude is related to the flat space amplitude via (3.24), and the flat space amplitude grows linearly with energy. This implies that the polynomials P_4, P_3, P_2, P_1 should respectively have degrees 4, 3, 2 and 1. In the ansatz, they are written as general polynomials of the corresponding degrees with unfixed coefficients and include all possible R-symmetry structures. For B_m, since it is multiplied by a simple pole, it should grow quadratically at large energies. Additional comments on the flat space limit are presented in Section 3.3.3 below.

The strategy for computing the six-point amplitude is *to divide and conquer* and is illustrated in Figure 3.8. We proceed in three steps. First, we take three simultaneous lightcone limits for (12), (34) and (56). In Mellin space, this corresponds to taking the residue at $\delta_{12} = \delta_{34} = \delta_{56} = 1$ and singles out P_4 in the ansatz. This channel is named the *snowflake channel*, and we use superconformal symmetry to constrain P_4. Importantly, the snowflake channel analysis forms a closed sector under the chiral algebra condition, because in (3.55) we are free to choose $\bar{z}_i$. This freedom allows us to focus on the triple lightcone limit by taking $\bar{z}_{12} = \bar{z}_{34} = \bar{z}_{56} = 0$ and study the consequence of chiral algebra with all other contributions in the ansatz turned off. The upshot is that there are only two constants unfixed in P_4. Next, we similarly take two simultaneous lightcone limits by setting $\bar{z}_{12} = \bar{z}_{34} = 0$ and impose the chiral algebra condition. This weaker condition also allows us to probe P_3 and B_m. We find that all but a few coefficients are left unfixed. Finally, we impose the Drukker-Plefka twist, which completely fixes the ansatz up to an overall constant [61].

3.3.3 Comparing with the flat space limit

For the flat space six-graviton amplitude, we exploit the fact that gauge theory amplitudes serve as building blocks for gravity amplitudes via the KLT relations [4]. Denoting the tree-level colour ordered gluon six-point amplitude as $A_{\text{gluon}}[123456]$, then the graviton amplitude is given by

$$A_{\text{graviton}} = -\delta_{12}\,\delta_{45}\,A_{\text{gluon}}[123456]\,(\delta_{35}A_{\text{gluon}}[153462] + (\delta_{34}+\delta_{35})A_{\text{gluon}}[154362]) + \text{perms}\ . \tag{3.68}$$

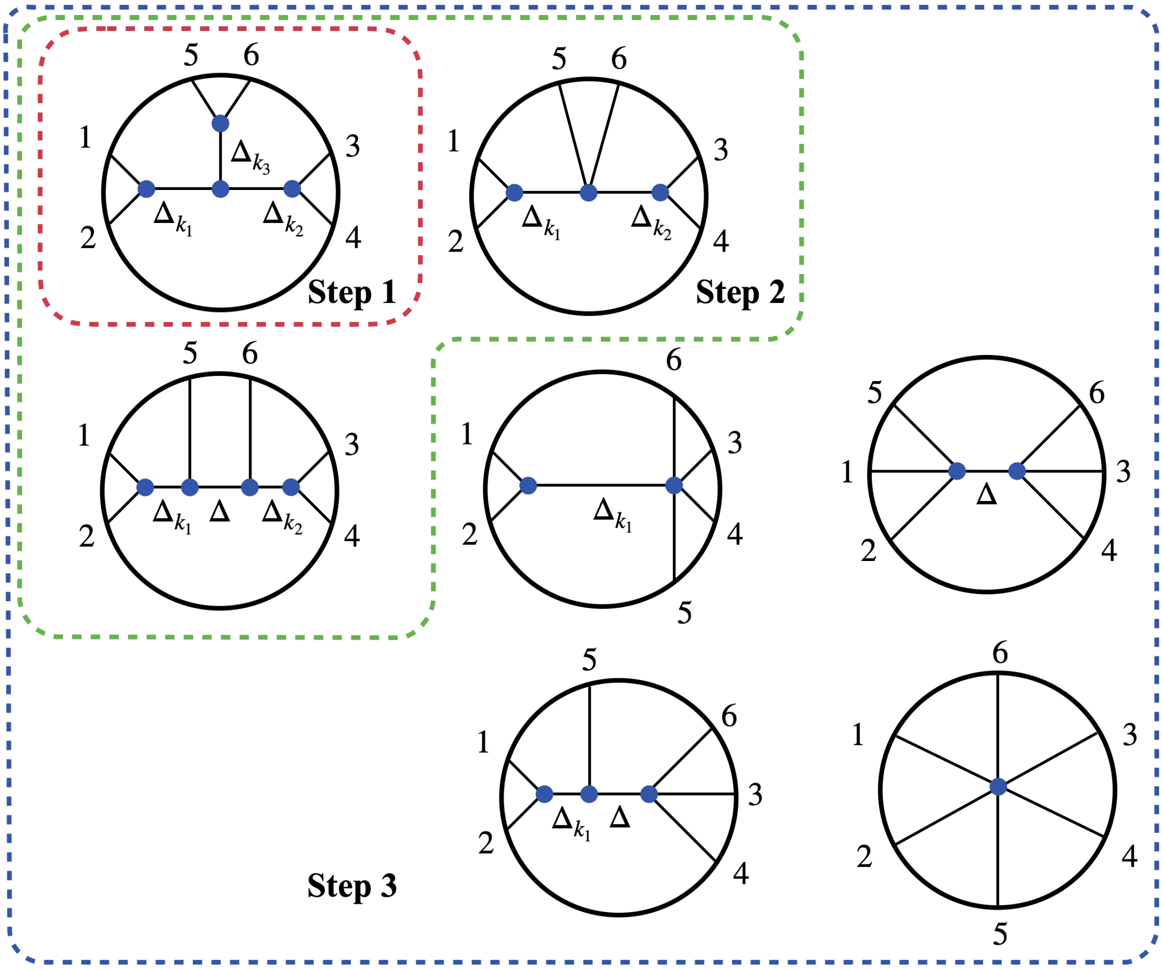


Figure 3.8: Illustration of the strategy for computing the supergraviton six-point amplitude. Here we enumerated Witten diagrams of different topology up to permutations, and the internal lines can be supergraviton, R-symmetry current and stress tensor. These diagrams are used to represent different parts of the ansatz. Our algorithm is a three-step procedure where, at each step, an increasingly larger part of the ansatz is solved.

The flat space amplitude is obtained from AdS via

$$\lim_{\beta\to\infty} M\big(\beta\,\delta_{ij},\,y_{ij}\big) \;\propto\; A_{\rm graviton}\big(\delta_{ij},\,y_{ij}\big)\ , \tag{3.69}$$

and is in the special kinematic configuration where all polarisations are orthogonal to all momenta. This offers considerable simplification as all terms with $p_i \cdot y_j$ drop out. To obtain the flat space graviton amplitude in this configuration, it is sufficient to use in (3.68) the gluon amplitude we already computed in the orthogonal configuration [89]. Although it may be too lengthy to write down, certain parts appear quite simple and suggestive. For example, the triple pole part of the snowflake channel is

$$A_{\rm graviton}\bigg|_{\delta_{12}=\delta_{34}=\delta_{56}=0} = \frac{y_{12}^2\,y_{34}^2\,y_{56}^2}{\delta_{12}\,\delta_{34}\,\delta_{56}}\,\left(\delta_{16}\,\delta_{24}+\left(\delta_{26}+\delta_{46}\right)\delta_{24}+\delta_{26}\,\delta_{45}+\delta_{23}\,\delta_{46}+\delta_{26}\,\delta_{46}\right)^2\ . \tag{3.70}$$

The terms with just poles in δ_{12} and δ_{34} are

$$\begin{aligned} A_{\rm graviton}\bigg|_{\delta_{12}=\delta_{34}=0} =& A_{\rm graviton}\bigg|_{\delta_{12}=\delta_{34}=\delta_{56}=0} + \frac{y_{12}^2\,y_{34}^2\,y_{56}^2}{\delta_{12}\,\delta_{34}}\left(\frac{\left(\delta_{16}\,\delta_{24}+\delta_{26}\,\delta_{24}+\delta_{26}\,\delta_{45}\right)^2}{\delta_{16}+\delta_{26}}\right.\\ &\left. - \frac{\left(\delta_{23}+\delta_{24}+\delta_{26}\right)^2\delta_{46}^2}{\delta_{16}+\delta_{26}+\delta_{56}}\right)\ . \end{aligned} \tag{3.71}$$

In the computation of the supergraviton six-point Mellin amplitude, we have only made minimal use of the flat space amplitude. The only input from flat space is that the Mellin amplitude should grow linearly at large energies, and the contribution from R-symmetry current needs to be subleading. Therefore, comparing the flat space limit of the Mellin amplitude against (5.10) provides a non-trivial check of the correctness of the final result. But it is also useful to look into the details of this comparison. Recall that in the single pole part P_2 in the ansatz (3.67), we were still left with four free coefficients after imposing the Casimir equation and the conservation conditions. While these can be fixed using the Drukker-Plefka twist, they are also sensitive to the flat space limit. That they give rise to the same coefficients shows the extent to which the flat space limit check is non-trivial. Similarly, in the contact part P_1, the linear term coefficients also survive the flat space limit and can be fixed either by the flat space amplitude or the Drukker-Plefka twist.

Although here the flat space limit serves only as a consistency check, it can provide a powerful constraint in future applications. Once the bootstrap approach is established, the flat space amplitude will be invaluable when extending the analysis to Mellin amplitudes with more than six points. The same advantage applies once stringy corrections are included, as we will show in detail in Chapter 4.

3.3.4 A note on position space computations

The six-point function of $20'$ operators in the supergravity approximation has been computed in [61] without computing explicitly any Witten diagram. Moreover, the algorithm is entirely within Mellin space. Nevertheless, it might still be useful to write the result in terms of position space functions.

By using the integrated vertex identities from Section 2.4, one can integrate out particle exchanges and reduce *all* Witten diagrams to two basic types. These are six-point contact diagrams and $3 \to 3$ exchange Witten diagrams. We reproduce here the integral representation of the contact diagram for reference:

$$D_{\Delta_1,\ldots,\Delta_n} = \int \frac{dz_0\, d^d z}{z_0^{d+1}} \prod_{i=1}^{n} \left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x}_i)^2} \right)^{\Delta_i} . \tag{3.72}$$

The D-function has a constant Mellin amplitude [48, 63]. When dressed with factors of x_{ij}^2, we can obtain either polynomials or poles in the Mellin amplitude, depending on the power of the factors. More precisely,

$$\prod_{i<j} (x_{ij}^2)^{-\alpha_{ij}} D_{\Delta_1 \ldots \Delta_n} = \int [d\delta] \prod_{i<j} (x_{ij}^2)^{-\delta_{ij}} \Gamma(\delta_{ij}) \left(\frac{\pi^{\frac{d}{2}} \Gamma\left(\frac{\sum_k \Delta_k - d}{2} \right) \prod_i \Gamma(\delta_{ij} - \alpha_{ij})}{\prod_i \Gamma(\Delta_i) \Gamma(\delta_{ij})} \right) , \tag{3.73}$$

which gives the Mellin amplitude

$$M(\delta_{ij}) = \frac{\pi^{\frac{d}{2}} \Gamma\left(\frac{\sum_k \Delta_k - d}{2} \right)}{\prod_i \Gamma(\Delta_i)} \prod_{i<j} \frac{\Gamma(\delta_{ij} - \alpha_{ij})}{\Gamma(\delta_{ij})} . \tag{3.74}$$

It is then clear that these dressed D-functions can provide the singularities for all terms involving $P_{1,2,3,4}$ in (3.67). A useful property of these D-functions are the differential recursion relations

$$D_{\Delta_1,\ldots,\Delta_i+1,\ldots,\Delta_j+1,\ldots,\Delta_n} = \frac{d - \sum_k \Delta_k}{2\Delta_i \Delta_j} \frac{\partial^2}{\partial x_{ij}^2} D_{\Delta_1,\ldots,\Delta_n} , \tag{3.75}$$

which relate D-functions with different weights.[21] In our case, all the D-functions can be reduced by these relations to D_{111111} only. This basic D-function can also be represented as a conformal one-loop integral in six dimensions [48, 117]

$$D_{111111} = \int \frac{d^d x_0}{x_{10}^2\, x_{20}^2\, x_{30}^2\, x_{40}^2\, x_{50}^2\, x_{60}^2} . \tag{3.76}$$

This integral has recently been computed and expressed in terms of classical polylogarithms with weight 3 [118] and obtained around three non-consecutive lightcones in [119].

[21] We used these relations also in the supergluon case, when working in position space.

The remaining Witten diagrams have poles separating the external points into two groups of three and three, like the one analysed in Section 3.2.2. These poles involve the sum of three δ_{ij} and cannot be generated from D-functions. However, they can be expressed in terms of the following two-loop six-point integral in four-dimensional flat space

$$I_{123,456} = \int \frac{d^4x_7\, d^4x_8}{x_{17}^2\, x_{27}^2\, x_{37}^2\, x_{78}^2\, x_{48}^2\, x_{58}^2\, x_{68}^2} \,. \tag{3.77}$$

The analytic structure of this integral is quite well understood. It evaluates to elliptic multiple polylogarithms and its symbol is known [120, 121]. However, for specific configurations, the integral can be expressed in terms of Goncharov polylogarithms. One such case is when all points are on a common line [122].

The Mellin representation of this Feynman integral is given by [117]

$$I_{123,456} = \int [d\delta] \frac{1}{\delta_{12}+\delta_{13}+\delta_{23}-1} \prod_{i<j} \frac{\Gamma(\delta_{ij})}{(x_{ij}^2)^{\delta_{ij}}} \,, \tag{3.78}$$

where the Mellin variables satisfy $\sum_j \delta_{ij} = 0$ with $\delta_{ii} = -1$. Similar to the D-functions, any Mellin amplitude of the comb type present in our result can be expressed in terms of the derivatives of $I_{123,456}$ with respect to x_{ij}^2 and multiplication by their powers. To see this more explicitly, let us define

$$A(x_{ij}^2) = \int [d\delta_{ij}] M(\delta_{ij}) \prod_{1\leq i<j\leq n} \Gamma(\delta_{ij})(x_{ij}^2)^{-\delta_{ij}} \,, \tag{3.79}$$

as a generic conformal function of n points written in Mellin space. Then the Mellin amplitude of the derivative of (3.79) with respect to x_{mn}^2 is given by

$$\frac{\partial}{\partial x_{mn}^2} A(x_{ij}) = \int [d\delta] M(\delta'_{ij} - \boldsymbol{\delta}_i^m \boldsymbol{\delta}_j^n - \boldsymbol{\delta}_i^n \boldsymbol{\delta}_j^m)\, \Gamma(\delta'_{ij})(x_{ij}^2)^{-\delta'_{ij}} \,, \tag{3.80}$$

where we have defined the shifted variables $\delta'_{ij} \equiv \delta_{ij} + \boldsymbol{\delta}_i^m \boldsymbol{\delta}_j^n + \boldsymbol{\delta}_i^n \boldsymbol{\delta}_j^m$, and they satisfy $\sum_j \delta'_{ij} = \Delta_i + \boldsymbol{\delta}_i^m + \boldsymbol{\delta}_i^n$. On the other hand, the Mellin amplitude obtained by multiplying a power of x_{ij}^2 is given by

$$(x_{mn}^2)^{-\alpha_{mn}} A(x_{ij}^2) = \int [d\delta] M(\delta'_{ij} - \alpha_{mn} \delta_i^m \delta_j^n)(\delta'_{ij})_{-\alpha_{mn}} \Gamma(\delta'_{ij}) \,. \tag{3.81}$$

For example, using these formulas, we can obtain the position space representations for the following

Mellin amplitudes:

$$\frac{2\,x_{16}^2 x_{35}^2}{x_{12}^2 x_{34}^2}\partial_{x_{13}^2}\partial_{x_{56}^2}^2 I_{126,345} \to -\frac{\delta_{16}\,\delta_{35}}{(\delta_{12}-1)\,(\delta_{34}-1)\,(\delta_{35}+\delta_{45})}\;, \tag{3.82}$$

$$\frac{x_{35}^2}{x_{12}^2 x_{34}^2}\partial_{x_{35}^2}\partial_{x_{56}^2} I_{126,345} \to \frac{\delta_{35}}{2\,(\delta_{12}-1)\,(\delta_{34}-1)\,(\delta_{35}+\delta_{45}-1)}\;. \tag{3.83}$$

Chapter 4

The AdS Virasoro–Shapiro program

The main contribution of this thesis is framed within the *AdS Virasoro–Shapiro program*, which addresses the long-standing challenge of computing string scattering amplitudes in curved spacetimes. Accordingly, we are committed to presenting our work in this domain in comprehensive detail. In this Chapter, we will review recent developments that extend our understanding of string amplitudes in AdS space, drawing inspiration from various limits—particularly the *flat space limit* and *high-energy limits*—where exact computations are possible and still offer valuable insight into the *full* theory.

We begin by defining perturbative string theory in flat backgrounds, whose structure is by now relatively well-understood [123], emphasising its key distinctions from point-particle interactions. We then present the essential ingredients that expose its elegant mathematical structure, whose hidden symmetries point the way to far more efficient amplitude computations. With this foundation, we will transition into the AdS setting. After a gentle introduction to the AdS Virasoro–Shapiro program, we present our results for the high-energy limits of the amplitude and motivate why these findings constitute a natural step toward a worldsheet theory on this curved background, enabling us to probe the theory's mathematical structure across different regimes.

4.1 String amplitudes in flat space

Scattering amplitudes encode the differential probability for a certain process to happen. This predictive power makes them an essential object in particle physics, mathematics, string theory, and more. By capturing the interactions and dynamics of particles and strings, they offer deep insights into the fundamental structure of physical theories. In this Section, we will try to explain how to make sense of the concept of scattering amplitudes of strings and give some intuition for how to compute them. The standard references are [5, 124–127], but for the purposes of this thesis, we will mainly follow [128].

In string theory, instead of point particles interacting at a vertex, we describe fundamental objects as one-dimensional strings whose interactions are smooth processes, splitting and joining of strings. Scattering amplitudes describe the probability of such interactions, just like in QFT, but now with a richer structure due to the extended nature of strings. While a point particle traces out a one-dimensional *worldline* in spacetime, parametrised by the particle's proper time τ, a string sweeps out a two-dimensional surface known as the *worldsheet*. We then need two parameters to study the string evolution: the timelike coordinate τ and the spacelike coordinate σ. We will package them together as $\sigma^a = (\sigma, \tau)$, with $a = 0, 1$. The worldsheet is embedded into a d-dimensional spacetime, called the *target space*, with metric $G_{\mu\nu}(X)$, by the embedding functions $X^\mu(\sigma, \tau)$, with $\mu = 0, \dots, d-1$.

From QFT, we know that the *action* functional $S[\phi]$ fully encodes the *dynamics* of a theory [46]. Its stationary condition, $\delta S[\phi] = 0$, yields the classical equations of motion, while in the quantum path integral,

$$\mathcal{Z} = \int D\phi \, e^{iS[\phi]/\hbar} \ , \tag{4.1}$$

each field configuration ϕ is weighted by $e^{iS[\phi]/\hbar}$, governing all quantum fluctuations [129].

To describe the dynamics of the string, we start from the Polyakov action[1]

$$S_P[X, g] = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g}\, g^{ab} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} G_{\mu\nu}(X) \ , \tag{4.2}$$

where g on the RHS is the determinant of the worldsheet metric. Importantly, it is a dynamical metric in this formulation, implying that the action describes scalar fields X^μ coupled to $2d$ gravity. We will review the main results in the case of flat spacetime target space, where $G_{\mu\nu} = \eta_{\mu\nu}$, the flat Minkowski metric with signature $\text{diag}(-1, +1, +1, \dots, +1)$. Our focus will be predominantly on closed strings, with the worldsheet coordinate σ taken to be periodic and $X^\mu(\sigma, \tau) = X^\mu(\sigma + 2\pi, \tau)$. This choice not only sidesteps boundary complications but is also what we need to describe the graviton scattering in our program. We will return to open strings in Chapter 5.

The Polyakov action has Poincaré global symmetry of spacetime and two gauge symmetries. The first one is the reparametrisation invariance (diffeomorphisms), which tells us that X^μ are worldsheet scalars and g_{ab} is a $2d$ metric:

$$\begin{aligned} X^\mu(\sigma) &\to X^\mu(\tilde{\sigma}) = X^\mu(\sigma) \ , \\ g_{ab}(\sigma) &\to \tilde{g}_{ab}(\tilde{\sigma}) = \frac{\partial \sigma^c}{\partial \tilde{\sigma}^a} \frac{\partial \sigma^d}{\partial \tilde{\sigma}^b} g_{cd}(\sigma) \ . \end{aligned} \tag{4.3}$$

[1] Actually, S. Deser and B. Zumino first proposed the worldsheet action for the spinning string in [130], and L. Brink, P. Di Vecchia, and P. S. Howe arrived at the same formulation independently in [131]. In 1981, A. Polyakov applied the formalism to the bosonic string's quantisation [132], and it has since become known as the Polyakov action.

Then we have Weyl invariance,

$$\begin{aligned} X^\mu(\sigma) &\to X^\mu(\sigma) \ , \\ g_{ab}(\sigma) &\to \Omega^2(\sigma) g_{ab}(\sigma) \ , \end{aligned} \tag{4.4}$$

which tells us that two metrics related by a Weyl transformation describe the same physical state. Infinitesimally, we can write $\Omega^2(\sigma) = e^{2\phi(\sigma)}$, where ϕ is a function of the worldsheet coordinates. By exploiting both diffeomorphism and Weyl invariance, we can gauge-fix the worldsheet metric to be flat:

$$g_{ab} = \eta_{ab} \quad \textbf{(conformal gauge)} \ , \tag{4.5}$$

where η_{ab} is the flat metric on the worldsheet in Minkowski coordinates. With this choice, the Polyakov action describes d free scalars propagating on a flat worldsheet:

$$S_P[X, g] = -\frac{1}{4\pi\alpha'} \int d^2\sigma \, \partial_a X \cdot \partial^a X \ , \tag{4.6}$$

where the $\cdot$ product contracts μ-indices. The remnant of the $2d$ gravity is the equation of motion for g_{ab}, which leads to the vanishing of the worldsheet stress-tensor. Equivalently, the worldsheet theory is conformally invariant. Let us emphasise that, unlike in standard applications (like Statistical Physics [133]) where conformal symmetry is a global symmetry on a fixed background, in string theory, conformal transformations are residual gauge symmetries. We shall return to this point shortly.

Before moving to the path integral, let us make some comments on important differences between string amplitudes and particle amplitudes. First, in QFT, interactions are introduced through non-linear terms added to the action. In contrast, string interactions are determined by the *free* worldsheet theory, and the Polyakov action is all we need. Interactions are indeed naturally embedded in the topology of the worldsheet. As an example, in Figure 4.1 the cylinder represents a closed string propagating through spacetime, while the other diagram depicts two strings merging into a single one. In this sense, the worldsheet *encodes* the interactions, which are no longer localised at a specific point in spacetime, unlike in QFT. Locally, every part of the worldsheet looks like a free propagating string. Only when viewed as a whole, it describes interactions.

In practice, what we ultimately want to calculate is the string S-matrix, which captures the probability amplitudes for a set of incoming string states to scatter into a set of outgoing ones. We are instructed to take all external legs in the worldsheet diagrams to infinity. Each external string state is associated with a state of the free string theory and carries a definite spacetime momentum p_i. Let us then express the S-matrix as a path integral, as in (4.1).

Thanks to the state–operator correspondence in two-dimensional CFT [28], any asymptotic string

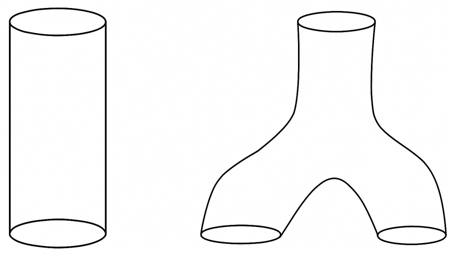


Figure 4.1: Possible worldsheets for closed strings.

state with momentum p_i can be equivalently represented by inserting a vertex operator $e^{ip_i \cdot X(z,\bar{z})}$ at a point on the worldsheet. For the (Euclidean) worldsheet coordinates, we use the complex variables ($z = \sigma^1 + i\,\sigma^2$, $\bar{z} = \sigma^1 - i\,\sigma^2$), where $\sigma^2 = i\,\sigma^0$. A conformal transformation[2] brings the infinite ends of the worldsheet diagram to finite positions, which we will refer to as *insertion points* or *punctures.* For tree-level scattering, the resulting worldsheet has the topology of a Riemann sphere, punctured by insertions of vertex operators, one for each external string state (Figure 4.2).

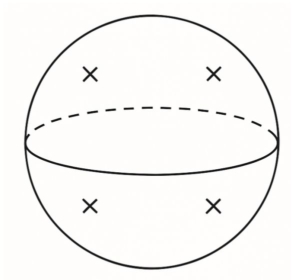


Figure 4.2: Tree-level worldsheet topology.

However, not all operator insertions are physically allowed. In string theory, consistency under Weyl invariance (the local rescaling of the worldsheet metric) imposes strong constraints: only on-shell string states ($m_i^2 = -k^i_\mu k^{i,\mu}$, $\forall i$) are admissible. String theory only computes on-shell scattering amplitudes [124, 128].[3]

For the path integral, we need to integrate over all fields (embedding coordinates) X^μ and all worldsheet metrics g_{ab}:

$$\mathcal{Z} = \frac{1}{\mathrm{Vol}} \int DX Dg\, e^{-S_P[X,g]} \,, \tag{4.7}$$

where we are working in Euclidean space, and we set $\hbar \equiv 1$. The presence of the prefactor reminds us we are dealing with a gauge theory: we should not integrate over all the fields, but rather over configurations of fields not related by gauge symmetries. Then, Vol is the volume of the gauge group. We will not review the details here; it suffices to remind that one can employ the usual Faddeev–Popov gauge-fixing procedure [46, 134]. This decouples the physical fields from the gauge

[2]Remember that conformal symmetry is one of the symmetries of the Polyakov action.
[3]The vertex operator is required to be a primary of conformal weight $(1, 1)$, in the notation of $2d$ CFTs [25].

orbits and solves the integral over the metrics, cancelling the prefactor. This introduces additional dynamical fields (ghosts) which themselves form a CFT. However, since we restrict our attention to tree-level amplitudes, the ghosts will not enter the game.

The n-point amplitude is then a sum over worldsheet topologies, weighted by the string coupling, with insertions of vertex operators for the external states:

$$A_{(n)}(\Lambda_i, p_i) = \sum_{\text{topologies}} g_s^{-\chi} \frac{1}{\text{Vol}} \int DX Dg \, e^{-S_P} \prod_{i=1}^{n} V_{\Lambda_i}(p_i) \ , \tag{4.8}$$

where Λ_i labels the string state, V_{Λ_i} is the associated vertex operator, Vol is the volume of the worldsheet gauge group introduced above, and χ is the Euler number of the worldsheet (number of handles for closed strings). Remember that there is also another parameter, α', hidden in the Polyakov action, which can be tuned to 0, leading to supergravity (for closed strings) or gauge theory (for open strings) amplitudes. These are the regimes we investigated in AdS in Chapter 3. The α'-corrections correspond to an infinite series of higher-derivative corrections to field theory amplitudes.

To sum up, in *perturbative* string theory the amplitude is a sum over topologies, namely punctured Riemann surfaces of increasing genus with insertions of vertex operators. A tree-level amplitude is computed by a genus-zero worldsheet. Higher-genus surfaces correspond to loop corrections. Unlike in field theory, where a large number of distinct Feynman diagrams appear at a given perturbative order, in string theory, at fixed genus and number of punctures, there is a single worldsheet topology contributing to the amplitude, with all interaction channels encoded in its moduli-space integration, the set of inequivalent complex structures on the Riemann surfaces.[4]

4.1.1 Illustrative computation example

We conclude by reviewing the computation of the tree-level amplitude of n closed-string tachyons, whose vertex operators are

$$V(p_i) = \int d^2 z_i \; e^{ip_i \cdot X(z_i, \bar{z}_i)} \ . \tag{4.9}$$

We learnt that the path integral over worldsheet metrics can be traded for a gauge-fixing of diffeomorphism and Weyl symmetries, leaving only a residual invariance, namely $SL(2,\mathbb{C})$ (conformal Killing vectors of the sphere). All together:

$$A_{(n)}(p_1, \ldots, p_n) = g_s^{n-2} \frac{1}{\text{Vol}(SL(2,\mathbb{C}))} \int DX \, e^{-S_P} \prod_{i=1}^{n} V(p_i) \ . \tag{4.10}$$

[4]For example, for the four-point case at tree-level, the punctured Riemann sphere is the only possible topology, with its one-dimensional moduli-space encoded by the cross ratio z we integrate over, as in (4.44). Let us also stress that here we consider oriented strings (for Type II superstring theories), so non-orientable surfaces, such as the Klein bottle for closed strings and the Möbius strip for open strings at tree-level, do not appear.

Let us focus on the X-integral:[5]

$$\begin{aligned}\left\langle \prod_{i=1}^{n} e^{ip_i\cdot X(z_i,\overline{z}_i)} \right\rangle_{S^2} &= \int DX\ \prod_{i=1}^{n} e^{-S_P} e^{ip_i\cdot X(z_i,\overline{z}_i)} \\ &= \int DX\ e^{-\frac{1}{2\pi\alpha'}\int d^2z\ \partial X_\mu \overline{\partial} X^\mu + i\sum_{i=1}^{n} p_i^\mu X_\mu(z_i,\overline{z}_i)} \\ &\equiv \int DX\ e^{\frac{1}{2\pi\alpha'}\int d^2z\ X_\mu \partial\overline{\partial} X^\mu + iJ^\mu X_\mu}\ ,\end{aligned} \tag{4.11}$$

where the formal source is $J^\mu(z,\overline{z}) = \sum_{i=1}^{n} p_i^\mu\,\delta(z-z_i,\overline{z}-\overline{z}_i)$. We clearly have a Gaussian integral over X^μ, which is easily solved to get

$$\begin{aligned}\left\langle \prod_{i=1}^{n} e^{ip_i\cdot X(z_i,\overline{z}_i)} \right\rangle_{S^2} &\sim \prod_{i\neq j}^{n} e^{\frac{\alpha'}{2}(p_i\cdot p_j)\log|z_i-z_j|} \\ &= \prod_{i<j}^{n} |z_i-z_j|^{\alpha' p_i\cdot p_j}\ .\end{aligned} \tag{4.12}$$

All together:[6]

$$A_{(n)}(p_1,\ldots,p_n) \sim \frac{g_s^{n-2}}{\mathrm{Vol}(SL(2;\mathbb{C}))}\int \prod_{i=1}^{n} d^2z_i \prod_{j<l} |z_j-z_l|^{\alpha' p_j\cdot p_l}\ . \tag{4.14}$$

Finally, we can remove the remnant gauge symmetry by fixing any three points on the complex plane:

$$z_1=\infty\ ,\ z_2=0\ ,\ z_3=z\ ,\ z_4=1\ , \tag{4.15}$$

such that, for example, for four points (the Virasoro-Shapiro amplitude), we just have one integral left. More generally, the building blocks of a tree-level closed-string amplitude are the worldsheet integrals [6, 135]

$$\int \prod_{i=1}^{n} d^2z_i \prod_{0\leq j<l\leq n+1} |z_j-z_l|^{2s_{jl}}\,(z_j-z_l)^{n_{jl}}(\overline{z}_j-\overline{z}_l)^{\tilde{n}_{jl}}\ , \tag{4.16}$$

with $z_0 := 0$, $z_{n+1} := 1$, $n\in\mathbb{N}$, $n_{ij},\tilde{n}_{ij}\in\mathbb{Z}$, $s_{ij}\in\mathbb{C}$ (Mandelstam variables). We will return to this point in greater detail in the AdS setting, but it is worth noting already at this level that both the global and local behaviour of these integrals are intimately connected with the theory of single-valued periods. In particular, the global behaviour corresponds to the famous KLT formula [4]: closed-string integrals factorise into products of pairs of open-string integrals.[7] On the other side, the

[5] We will defer the integrals over the vertex-operator insertions in (4.9) until the end.

[6] The contribution of the zero modes of $X^\mu(z,\overline{z})$ is the usual delta function for the conservation of momentum:

$$\int d^dX_0\ e^{i\sum_{i=1}^{n} p_i\cdot X_0} \sim \delta^d(\sum_i p_i)\ . \tag{4.13}$$

[7] How to extend this to AdS will be one of the principal outcomes of the Chapter 5.

local structure is governed by a rich number-theoretic framework. Each coefficient in the low-energy (asymptotic) expansion can be written as an iterated integral over the complex plane of single-valued polylogarithms, leading to single-valued multiple zeta values [135]. This fundamental observation paves the way for the next Section, where we will review the number-theoretic structures at the heart of the scattering amplitudes considered in this work.

4.1.2 Interplay with number theory

The goal of this Section is to present the basic objects and properties that underlie scattering amplitudes, as a stepping stone toward developing and applying techniques that go *beyond* the standard flat space tools, which become incredibly complicated, if not impossible to implement, in general curved spacetime.[8]

Let us take a step back to perturbative QFT. We describe a scattering process as a sum over Feynman diagrams, in a loop expansion. An important property is that complicated functions with branch cuts appear when intermediate virtual particles go on-shell. Indeed, in loop computations, one needs to go beyond rational functions to express the result. If we work in dimensional regularisation [138–140] in $D = D_0 - 2\epsilon$ dimensions, where D_0 is a positive integer that will be 4 here, a general loop scalar Feynman integral takes the form[9]

$$I = \int \left(\prod_{j=1}^{L} e^{\gamma_E \epsilon} \frac{d^D k_j}{i\pi^{D/2}} \right) \frac{\mathcal{N}(\{p_i, k_j\})}{(q_1^2 - m_1^2 + i\varepsilon)^{\nu_1} \dots (q_N^2 - m_N^2 + i\varepsilon)^{\nu_N}} \ , \tag{4.17}$$

where $\nu_i \in \mathbb{Z}$ and $m_i \geq 0$, with $1 \leq i \leq N$, are the masses of the propagators. The loop momenta k_i and the external momenta p_i (assumed to be all in-going) linearly combine to give the momenta q_i flowing through the propagators. Momentum is conserved. I is a meromorphic function of ϵ, with Laurent expansion

$$I = \sum_k I_k \epsilon^k \ . \tag{4.18}$$

Special numbers and functions appear in the analytic expressions for the coefficients of the Laurent expansion in dimensional regularisation. This can be appreciated from the simple one-loop two- and three-point functions:

$$\begin{aligned} B(p^2) &\sim e^{\gamma_E \epsilon} \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^2 (k+p)^2} = \frac{1}{\epsilon} + 2 - \log(-p^2) + \epsilon \left[\frac{1}{2} \log^2(-p^2) - 2 \log(-p^2) - \frac{1}{2} \zeta_2 + 4 \right] + O(\epsilon^2) \ , \\ T(p_1^2, p_2^2, p_3^2) &\sim e^{\gamma_E \epsilon} \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^2 (k+p_1)^2 (k+p_1+p_2)^2} = \frac{2}{\sqrt{\lambda}} \left[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) - \log(z\bar{z}) \log \frac{1-z}{1-\bar{z}} \right] + O(\epsilon) \ , \end{aligned} \tag{4.19}$$

[8]For further reading, consult, for example, the lecture notes [136]. See also [137] for a systematic, algorithmic approach to Feynman integrals within the realm of multiple polylogarithms.

[9]The choice of the normalisation with the Euler constant follows [136].

where $\lambda \equiv \lambda(p_1^2, p_2^2, p_3^2)$ denotes the Källen function, and

$$z\bar{z} = \frac{p_1^2}{p_3^2} \ , \quad (1-z)(1-\bar{z}) = \frac{p_2^2}{p_3^2} \ , \quad \lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \ . \tag{4.20}$$

Compared to the general form (4.17), the numerator factor of the integrand is $\mathcal{N} = 1$, for a scalar Feynman integral, and the powers of propagators are $\nu_i = 1$.

We notice, for example, the Riemann zeta function at integer values,

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n} \ , \tag{4.21}$$

and the logarithms as well as their generalisations. It has long been understood that, in four dimensions, every one-loop integral admits an expression solely in terms of logarithms and the Euler dilogarithm [141].

We will now introduce (multiple) polylogarithms and summarise their key properties. We follow the conventions of [6], but restrict ourselves to the basic results relevant for our analysis. For a more detailed treatment of these functions, see the reference above, as well as [137, 142], and references therein.

Polylogarithms

Let us start from the classical polylogarithms, defined by the convergent series on the unit disk, $|z| < 1$:

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \ . \tag{4.22}$$

They can be continued to the cut plane $\mathbb{C}\backslash[1,\infty]$ by the iterated integral representation

$$\mathrm{Li}_n(z) = \int_0^z dz' \frac{\mathrm{Li}_{n-1}(z')}{z'} \ . \tag{4.23}$$

When $z = 1$, the polylogarithms reduce to zeta values, $\mathrm{Li}_n(1) = \zeta_n$, for $n > 1$.

Multiple polylogarithms (MPLs) are instead functions $L_w(z)$ of one variable, labelled by a word drawn from some chosen alphabet. We will restrict to the special case with letters $\{0, 1\}$.[10] They are recursively defined by the relations

$$\frac{d}{dz} L_{aw}(z) = \frac{1}{z-a} L_w(z) \ , \quad a = 0, 1 \ , \tag{4.24}$$

together with

$$L_e(z) = 1 \tag{4.25}$$

[10]Sometimes, the resulting family of MPLs is referred to as (restricted) harmonic polylogarithms (originally defined with alphabet $\{0, 1, -1\}$ in [143]), as in [144], or as multiple polylogarithms in one variable.

for the empty word, and

$$\lim_{z\to 0} L_w(z) = 0 \ , \tag{4.26}$$

unless $w = 0^p$ in which case

$$L_{0^p}(z) = \frac{1}{p!}\log^p z \ . \tag{4.27}$$

We define multiple zeta values as

$$\zeta(s_1, \cdots, s_d) = \sum_{\ell_1>\ell_2>\cdots>\ell_d>0}^{\infty} \frac{1}{\ell_1^{s_1}\ell_2^{s_2}\cdots\ell_d^{s_d}} \ , \tag{4.28}$$

where d is called the depth and $s_1 + \cdots + s_d$ is called the weight [145, 146].

Multiple zeta values are intimately connected to MPLs [143]. As before, they can be defined as MPLs evaluated at $z = 1$. We will use the conventions

$$\zeta(s_1, \cdots, s_d) = (-1)^d L_{0^{s_1-1}10^{s_2-1}\cdots 0^{s_d-1}1}(1) \ , \tag{4.29}$$

with the regularisation condition $L_{1^p}(1) = 0$. Note that also $L_{0^p}(1) = 0$.

Polylogarithms satisfy shuffle relations:

$$L_w(z)L_{w'}(z) = \sum_{W\in w\sqcup\!\sqcup w'} L_W(z) \ , \tag{4.30}$$

where the shuffle product $w \sqcup\!\sqcup w'$ of two words $w = a\,u$ and $w' = b\,v$ (with letters a, b) is defined recursively by

$$\emptyset \sqcup\!\sqcup w' = \{\, w'\}, \quad w \sqcup\!\sqcup \emptyset = \{\, w\},$$
$$(a\,u) \sqcup\!\sqcup (b\,v) = a\,(u \sqcup\!\sqcup b\,v) \ \cup \ b\,(a\,u \sqcup\!\sqcup v)\,,$$

i.e. all interleavings of the letters of w and w' that preserve the internal order of each. Evaluating (4.30) at $z = 1$, we obtain shuffle relations for the multiple zeta values.[11]

Polylogarithms can also be written as iterated integrals [149, 150], which make manifest their differential relations:

$$L_{a_1a_2\cdots a_r}(z) = \int_0^z \frac{dz_1}{z_1 - a_1}\int_0^{z_1}\frac{dz_2}{z_2 - a_2}\cdots \ . \tag{4.31}$$

The integrals are convergent provided the last letter is $a_r = 1$. Note that the shuffle relations can be used to write all MPLs in terms of these and powers of $L_0(z) = \log z$. In Chapter 5, we will often restrict to this case, without loss of generality. We will occasionally use the condensed notation:

$$L_{s_1,\cdots,s_d}(z) \equiv (-1)^d L_{0^{s_1-1}10^{s_2-1}\cdots 0^{s_d-1}1}(z) \ , \tag{4.32}$$

[11] Let us recall also the stuffle (or quasi-shuffle) algebra, the natural product structure on MPLs. It describes how the product of two polylogarithms expands as a $\mathbb{Q}$-linear combination of higher-depth polylogarithms [147, 148].

with $s_i > 0$, so that $\zeta(s_1, \cdots, s_d) = L_{s_1,\cdots,s_d}(1)$. Here and throughout, we make this notation unambiguous by inserting commas between the letters in the word labelling the polylogarithm. In compact notation:

$$\begin{aligned} \frac{d}{dz} L_{s_1,\cdots,s_d}(z) &= \frac{1}{z} L_{s_1-1,\cdots,s_d}(z) \ , \quad \text{for } s_1 > 1 \ , \\ \frac{d}{dz} L_{1,\cdots,s_d}(z) &= \frac{1}{1-z} L_{s_2,\cdots,s_d}(z) \ . \end{aligned} \tag{4.33}$$

For all $s_i > 0$, we can write down a regular series expansion of the MPL around zero:

$$L_{s_1,\cdots,s_d}(z) = \sum_{\ell_1 > \ell_2 > \cdots > \ell_d > 0}^{\infty} \frac{z^{\ell_1}}{\ell_1^{s_1} \ell_2^{s_2} \cdots \ell_d^{s_d}} \ . \tag{4.34}$$

MPLs are *multi-valued* functions in the complex plane. They have branch cuts starting at $z = 0$ and/or $z = 1$. It is possible to define *single-valued* functions $\mathcal{L}_w(z)$, denoted single-valued multiple polylogarithms (SVMPLs), from a combination of polylogarithms such that all the branch cuts cancel and we get single-valued functions in the $(z, \overline{z})$ plane.[12] There are many possible choices of bases of SVMPLs, but we will restrict to the construction from holomorphic and anti-holomorphic MPLS only. They have been completely characterised by F.Brown in [151].

These functions are labelled by a word w in the alphabet with letters $\{0, 1\}$ such that they satisfy the same derivative relations as MPLs, under holomorphic derivatives [151],

$$\frac{\partial}{\partial z} \mathcal{L}_{aw}(z) = \frac{1}{z-a} \mathcal{L}_w(z) \ , \quad a = 0, 1 \ , \tag{4.35}$$

together with

$$\mathcal{L}_e(z) = 1 \tag{4.36}$$

for the empty word, and

$$\lim_{z \to 0} \mathcal{L}_w(z) = 0 \ , \tag{4.37}$$

unless $w = 0^p$ in which case

$$\mathcal{L}_{0^p}(z) = \frac{1}{p!} \log^p |z|^2 \ . \tag{4.38}$$

At any given weight, there is a finite-dimensional vector space of SVMPLs [152].

SVMPLs also satisfy the shuffle identities

$$\mathcal{L}_w(z) \mathcal{L}_{w'}(z) = \sum_{W \in w \sqcup\!\sqcup w'} \mathcal{L}_W(z) \ . \tag{4.39}$$

These relations can be used to isolate the logarithmic singularities of SVMPLs as the argument

[12] We will often write only z as the argument of the SVMPLs, to streamline the notation.

approaches the first label. For instance,

$$\lim_{z\to\sigma_i} \mathcal{L}_{\sigma_i\sigma_j w}(z) = \mathcal{L}_{\sigma_j w}(z_i)\mathcal{L}_{\sigma_i}(z) + \text{finite}\,, \qquad i \neq j\,, \tag{4.40}$$

with the divergence arising from

$$\mathcal{L}_{\sigma_i}(z) = \log\left|1 - \frac{z}{\sigma_i}\right|^2 \ . \tag{4.41}$$

Finally, evaluated at one, SVMPLs generate single-valued multiple zeta values

$$\zeta^{sv}(s_1, \cdots, s_d) = \mathcal{L}_{s_1,\cdots,s_d}(1) \ . \tag{4.42}$$

Each single-valued multiple zeta value can be written as a linear combination of ordinary multiple zeta values. More generally, we can define a single-valued map such that

$$\mathcal{L}_w(z) = sv\left(L_w(z)\right) \ . \tag{4.43}$$

When working with SVMPLs, we found the tools [153, 154] very useful.

4.2 Foundation of the AdS Virasoro-Shapiro program

We are now ready to summarise the key concepts of the AdS Virasoro-Shapiro program.

In the introductory part of this Chapter, we first reviewed the flat space technology to compute string amplitudes. Once we include supersymmetry and move to curved backgrounds, it all becomes far more subtle. While a genus expansion still exists in principle, there is currently no direct world-sheet formulation available, even at tree-level. This difficulty is particularly evident in the presence of Ramond–Ramond (RR) fluxes: the traditional Ramond-Neveu–Schwarz (RNS) formalism [125] does not apply.[13] Other formalisms, such as the Green-Schwarz [155] and pure spinor [156], are not yet developed to the point of providing scattering amplitudes in this background.

A significant success beyond the supergravity approximation has occurred in the special case of AdS_3 with a pure NS–NS background, where NS stands for Neveu–Schwarz. Notably, it admits an exact worldsheet description, owing to its underlying Wess–Zumino–Witten structure [157]. In this case, the AdS_3/CFT_2 correspondence has been checked beyond the supergravity approximation by comparing correlators. This stands in contrast to more general backgrounds with finite RR flux, such as the paradigmatic case of $AdS_5 \times S^5$, where α'-corrections to the supergravity approximation remain difficult to compute, despite the presence of a maximally supersymmetric dual CFT. We do not know how to quantise perturbative string theory on this background.

[13] In principle, it is possible to use the RNS formulation to compute amplitudes in a curvature expansion around flat space. We expect this computation to become very cumbersome very quickly.

The AdS Virasoro–Shapiro program emerges as a response to this challenge. The strategy is to combine intuition from flat space with alternative tools, such as AdS/CFT, Mellin and Borel space, integrability, and number theory. The overarching goals of the program are twofold:
- to analytically capture and systematically incorporate curvature corrections,
- to explore and highlight the rich interplay between string theory and number theory.

The main object of study is the tree-level scattering amplitude of four graviton states. In flat space, this is given by the well-known Virasoro–Shapiro amplitude [158], which is one of the earliest and most iconic results in string theory:[14]

$$A^{(0)}(S,T) = -\frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)} \ . \tag{4.44}$$

It is crossing symmetric in the three Mandelstam variables,

$$S = -\frac{\alpha'}{4}(p_1+p_2)^2 \ , \quad T = -\frac{\alpha'}{4}(p_1+p_3)^2 \ , \quad U = -\frac{\alpha'}{4}(p_1+p_4)^2 \ , \tag{4.45}$$

with $S+T+U=0$, as a consequence of momentum conservation for the massless supergravity states. The worldsheet origin of this amplitude can be seen from the integral representation

$$A^{(0)}(S,T) = \frac{1}{U^2}\int d^2z|z|^{-2S-2}|1-z|^{-2T-2} \ , \tag{4.46}$$

which is a special case of (4.16). We have stripped off a prefactor encoding the dependence on the polarisation tensors, as we want to focus on the *reduced* amplitude. Let us make a few comments on the symmetries of this amplitude. Clearly,

$$\begin{aligned} A(S,T) &= A(T,S) \ , \\ A(S,T) &= A(S,-S-T) \ , \\ A(S,T) &= A(-S-T,T) \ . \end{aligned} \tag{4.47}$$

This is crossing symmetry, reflecting the equivalence of different channels. Equivalently, we can understand it from the worldsheet integral. By sending $z \to 1-z$ in (4.46), we exchange S and T, recovering the first symmetry in (4.47). More generally, once we have fixed the conformal gauge symmetry on the worldsheet by sending three points to $0,1,\infty$, the change of variables/transformations

$$\begin{aligned} &z \to 1-z \ , \\ &z \to 1/z \ , \\ &z \to z/(1-z) \ , \end{aligned} \tag{4.48}$$

[14] It was actually derived long before people knew it had anything to do with strings.

exchange the points $0, 1, \infty, z$, where z is the remaining cross ratio on the Riemann sphere. The other two transformations then indicate the symmetries of the amplitude

$$
\begin{aligned}
A(S,T) &= \frac{(S+1)^2}{(S+T)^2} A(-S-T-1,T) \ , \\
A(S,T) &= \frac{(T+1)^2}{(S+T)^2} A(S,-S-T-1) \ .
\end{aligned}
\tag{4.49}
$$

These further imply that, for example,

$$
A(S,T) = \frac{(T+1)^2}{(S+T)^2} A(S,T+1) \ . \tag{4.50}
$$

This symmetry relates the residues on the poles of the amplitude at different values of T. We will get back to these interesting symmetries in Chapter 5, where we will also show how to extend them to AdS.

A maybe less appreciated representation of the Virasoro-Shapiro amplitude is

$$
A^{(0)}(S,T) = \frac{\exp\left(\sum_{n=1}^{\infty} \frac{\zeta^{\rm sv}(2n+1)\left(S^{2n+1}+T^{2n+1}+U^{2n+1}\right)}{2n+1}\right)}{STU} \ , \tag{4.51}
$$

where only odd zeta values appear. This is deeply significant: odd zeta values are single-valued! More precisely, $\zeta^{sv}(2n+1) = 2\zeta(2n+1)$, while $\zeta^{sv}(2n) = 0$, $\forall n \geq 1$. This structure directly reflects the single-valued nature of the integral representation (4.46), as we move around the punctures.

On the other hand, in $AdS_5 \times S^5$, we do not yet have a complete understanding of how to quantise perturbative Type IIB string theory. Fortunately, the AdS/CFT correspondence allows us to relate the string amplitude to a four-point correlator of stress-tensor multiplets in $\mathcal{N} = 4$ SYM at large central charge. Now, how do the two sides of the gauge-gravity correspondence talk to each other? In Chapter 2, we learnt that the fundamental parameters are:

String theory on $AdS_5 \times S^5$	$\mathcal{N} = 4$ **SYM in 4d**
AdS_5 and S^5 radius: R	Gauge group: $SU(N)$
String length: ℓ_s (or $\alpha' = \ell_s^2$)	't Hooft coupling: $\sqrt{\lambda} = \sqrt{g_{\rm YM}^2 N}$
String coupling: g_s	

In particular, $g_s \sim \frac{1}{N}$. We consider weakly coupled strings (in the genus expansion in powers of g_s) mapped to the planar limit of the CFT (in the large-N limit). Moreover,

$$
\frac{\alpha'}{R^2} = \frac{1}{\sqrt{\lambda}} \ , \tag{4.52}
$$

which means that the regime where stringy corrections to supergravity become small ($\alpha' \to 0$) corresponds to the strongly coupled limit of the CFT ($\lambda \to \infty$).

The analytic structure of the amplitude in *AdS*, namely its poles and residues, is then determined by the CFT data of the intermediate operators in planar $\mathcal{N} = 4$ SYM [159], a primary target of integrability (see *e.g.* [160–162]).[15] This analytic structure becomes highly constraining when combined with a conjectural *single-valuedness* condition: the low-energy expansion of the *AdS* Virasoro–Shapiro amplitude contains only single-valued zeta values, as in flat space.

4.2.1 A chain of integral transforms

The *AdS* Virasoro-Shapiro amplitude is defined in terms of the four-point position space correlator of chiral primary operators in $\mathcal{N} = 4$ SYM at leading non-trivial order in the large central charge expansion:

$$\langle \mathcal{O}^{I_1 J_1}(x_1)\mathcal{O}^{I_2 J_2}(x_2)\mathcal{O}^{I_3 J_3}(x_3)\mathcal{O}^{I_4 J_4}(x_4)\rangle\big|_{\frac{1}{c}} = G^{I_i J_i}(x_i)_{tree} \; . \tag{4.53}$$

Here $\mathcal{O}^{I_1 J_1}(x)$ is the superconformal primary operator of the stress tensor multiplet. It is a scalar and has protected dimension $\Delta = 2$. The indices I, J are in the fundamental representation of the $SO(6)$ R-symmetry group and the operator transforms in the symmetric traceless representation of rank 2. The superconformal Ward identities [168] fix the dependence on the R-symmetry indices through a computable prefactor:[16]

$$G^{I_i J_i}(x_i)_{tree} = f(I_i, J_i, x_i) \times \mathcal{T}(U, V) \; . \tag{4.54}$$

The non-trivial dynamics is encoded in the reduced correlator $\mathcal{T}(U, V)$, which is a function of the two conformally invariant cross-ratios

$$U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \, , \qquad V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \; . \tag{4.55}$$

The reduced correlator $\mathcal{T}(U, V)$ is related to the *AdS* Virasoro–Shapiro amplitude by a chain of integral transforms. First, we implement the Mellin transform, in the conventions of [169]:

$$\mathcal{T}(U, V) = \int_{-i\infty}^{i\infty} \frac{ds dt}{(4\pi i)^2} U^{\frac{s}{2}+\frac{2}{3}} V^{\frac{t}{2}-\frac{4}{3}} \Gamma\left(\frac{4}{3} - \frac{s}{2}\right)^2 \Gamma\left(\frac{4}{3} - \frac{t}{2}\right)^2 \Gamma\left(\frac{4}{3} - \frac{u}{2}\right)^2 M(s, t) \, , \tag{4.56}$$

where the Mellin variables are defined so that $s + t + u = 0$. Defined this way, $M(s, t)$ is usually denoted as the *reduced* Mellin amplitude, since it is the transform of the reduced correlator. Mellin

[15] See also [163–167] for very interesting developments on the interplay between integrability and the *AdS* Virasoro–Shapiro amplitude.

[16] The details of the prefactor $f(I_i, J_i, x_i)$ will not be relevant for our purposes, as we focus on the structure and properties of the reduced correlator $\mathcal{T}(U, V)$.

space provides the natural language for holographic correlators [48], since it recasts them in a form directly analogous to scattering amplitudes. Accordingly, we shall begin our analysis in Mellin space. From the Mellin amplitude $M(s,t)$, we then define the Borel transform $A(S,T)$, via

$$M(s,t) = \frac{1}{2R^6}\int_0^\infty d\beta e^{-\beta}\beta^5 A\left(\frac{s\beta}{2R^2}, \frac{t\beta}{2R^2}\right). \tag{4.57}$$

When we talk about the *AdS* amplitude, we always mean the Borel amplitude, in a small-curvature expansion:

$$\boxed{A(S,T) = A^{(0)}(S,T) + \frac{\alpha'}{R^2}A^{(1)}(S,T) + \dots} \tag{4.58}$$

After this chain of integral transforms, the leading-order result in the flat space limit, $A^{(0)}(S,T)$, reproduces the Virasoro–Shapiro amplitude *on the nose*. Therefore, even at leading order, and in order to obtain the Virasoro–Shapiro amplitude, one is *required* to perform the Borel transform. We then define the functions $A^{(k)}(S,T)$ as our working definition of the *AdS* corrections. As a working definition, this requires no further justification. Nevertheless, if one wishes to motivate this more concretely, at leading order in $1/R$, Equation (4.57) precisely implements the flat space limit as discussed in [48], with S and T interpreted as Mandelstam variables. Here, however, we retain the subleading corrections to the flat space formula. Another motivation for the use of Borel space comes from a more technical observation: attempting to resum the low-energy expansion of the amplitude in Mellin space leads to a divergent series, as explained for example in [169,170]. It is, however, Borel summable. Understanding how to resum the low-energy expansion is essential for making contact with the string worldsheet description and its number-theoretic structure. The low-energy expansion takes the form

$$A^{(0)}(S,T) = \frac{1}{STU} + 2\sum_{a,b=0}^{\infty}\alpha^{(0)}_{a,b}\sigma_2^a\sigma_3^b\ , \quad \sigma_2 = \frac{1}{2}(S^2+T^2+U^2)\ ,\ \sigma_3 = STU\ , \tag{4.59}$$

where the first contribution $\frac{1}{STU}$ is the famous supergravity result [5]. Then, there is an infinite tower of stringy corrections, where the Wilson coefficients $\alpha^{(0)}_{a,b}$ live in the ring of single-valued multiple zeta values.

The advances of the last few years in the *AdS* Virasoro-Shapiro program may be distilled into three key points; the reader is referred to the original articles for full details [159, 169–171]:

- Structure of **poles**: while the flat space amplitude (4.44) contains simple poles only (from the Gamma functions), the *AdS* corrections feature poles of increasing order, jumping by 3:

$$A^{(k)}(S,T) = \frac{R^{(k)}_{3k+1}(T,\delta)}{(S-\delta)^{3k+1}} + \frac{R^{(k)}_{3k}(T,\delta)}{(S-\delta)^{3k}} + \dots + \frac{R^{(k)}_1(T,\delta)}{S-\delta} + \text{regular}\ , \quad \delta = 1,2,\dots\ . \tag{4.60}$$

The R_κ-numerators are polynomials in T of order κ. For the highest order pole, they take the general form [170]

$$R^{(k)}_{3k+1}(T,\delta) = -\frac{(3k)!\delta^{2k-2}}{6^k k!}\frac{\Gamma(T+\delta)^2}{\Gamma(\delta)^2\Gamma(T+1)^2} \ . \tag{4.61}$$

- Low-energy expansion: the (unknown) coefficients are *assumed* to be **single-valued** multiple zetas, as in flat space.
- Intuition from the **worldsheet**.

These insights led to the proposal of the AdS amplitude as the genus-0 integral

$$\boxed{A(S,T) = \int d^2z\, |z|^{-2S-2}|1-z|^{-2T-2}\, G(S,T,z)} \tag{4.62}$$

where $G(S,T,z)$ admits a small-curvature expansion,

$$G(S,T,z) = \sum \frac{1}{R^{2k}} G^{(k)}(S,T,z) \ , \tag{4.63}$$

and

$$G^{(0)}(S,T,z) = \frac{1}{(S+T)^2} = \frac{1}{U^2} \ . \tag{4.64}$$

More generally, $G^{(k)}(S,T,z)$ is a rational function in S,T and a transcendental single-valued function in z of weight $3k$ (matching the jump by 3 in the poles), more precisely a SVMPL. This structure also holds for scattering amplitudes in type IIB string theory on AdS_3 [172, 173] (with different choices of fluxes), as well as type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ [174].

Combining this structure with the dimension of the Konishi operator in planar $\mathcal{N} = 4$ SYM, $G^{(1)}(S,T,z)$ and $G^{(2)}(S,T,z)$ have been fully determined, modulo certain ambiguities that integrate to zero. These results pass several highly non-trivial checks, such as reproducing all localisation results, see *e.g.* [175], and all CFT data available from integrability [160–162]. Therefore, it is a precise proposal for the structure of the tree-level amplitude on $AdS_5 \times S^5$, and provides an algorithm to compute *each AdS* correction, modulo input from integrability. Let us stress once more that this is **not** the result of a direct worldsheet computation, currently unavailable. The lesson to learn is that single-valuedness plays a fundamental role in the construction of AdS scattering amplitudes, as in flat space.

The big goal of the program is to compute this amplitude to all orders in the curvature expansion. While refining the methods to achieve the full result, attention can be placed on more tractable limits, which provide greater insight into the structure of the theory and teach us valuable lessons on its physics. One such limit is the high-energy regime of large S and T, with fixed scattering angle

S/T, where we keep the leading large energy contribution at each order in $1/R$, with R the radius of AdS. In this setup, the amplitude can be computed to *all orders* and curvature corrections exponentiate [176]. This is captured and reproduced by a classical scattering computation in AdS.

We can also explore a more intricate limit, the Regge limit, characterised by large T but finite S. In this regime, the amplitude provides comprehensive information about the intermediate operators on the leading Regge trajectory. In the AdS/CFT context, these operators are stringy or short operators, such as the Konishi operator [177]. We already mentioned that their CFT data is a central focus of integrability; this makes the Regge limit particularly compelling, with strong potential for synergy between integrability techniques and the computation of the AdS Virasoro–Shapiro amplitude. Even in this regime, to *all orders* in $1/R$, the AdS Virasoro-Shapiro amplitude can be expressed as derivatives of the flat space result in the Regge limit [178]. Interestingly, this leads to families of possible worldsheet integrands in the Regge limit involving single-valued logarithms, fully consistent with the proposed *full AdS* Virasoro-Shapiro amplitude! In the next two Sections, we will present a detailed analysis of these high-energy limits.

4.3 The high-energy limit

In our approach of looking at QFT in flat space to draw inspiration for strings in AdS, we first recall that in QFT the short-distance behaviour of the theory plays a crucial role (via the OPE, RG flow, etc.), uncovering *universal* structures. It is therefore interesting to investigate these features in string theory as well. String amplitudes in flat space display remarkable properties in the high-energy limit $|S|, |T| \gg 1$ with fixed scattering angle S/T [179–181]. In particular, the Virasoro–Shapiro amplitude $A^{(0)}(S,T)$ falls off exponentially in this regime,[17]

$$A^{(0)}_{HE}(S,T) \sim e^{-2S\log|S|-2T\log|T|-2U\log|U|} \ , \tag{4.65}$$

in clear contrast to field theory amplitudes, which either diverge or fall off as a power. Let us now derive this result from two complementary viewpoints, the worldsheet and the spacetime perspectives, and explain how to extend them to AdS.

From the worldsheet

The soft exponential behaviour can be directly understood from the integral representation (4.46), which we rewrite, for the purposes of this Section, as

$$A^{(0)}(S,T) = \int d^2z|z|^{-2S}|1-z|^{-2T}W_0(z,\overline{z}) \ , \tag{4.66}$$

[17]Stirling's formula can be used to get the answer directly from the ratio of Gamma functions (4.44).

where $W_0(z,\bar{z}) = (U^2|z|^2|1-z|^2)^{-1}$. At high energies $|S|,|T| \gg 1$, the integral can be computed by saddle point approximation around the saddle[18]

$$z = \bar{z} = \frac{S}{S+T} \equiv z_0 \ , \tag{4.67}$$

which leads to

$$A^{(0)}_{HE}(S,T) \sim W_0(z_0) e^{-2S\log|S| - 2T\log|T| - 2U\log|U|} \ . \tag{4.68}$$

It is well-known that this soft exponential behaviour is universal [179–181], meaning that it is independent of the particular string theory and quantum numbers of scattered states.

Let us now make the following interesting remark for the case of AdS: the proposed worldsheet representation (4.62) has the structure

$$A(S,T) = \int d^2z |z|^{-2S}|1-z|^{-2T} W_0(z,\bar{z}) \left(1 + \frac{S^2}{R^2} W_3(z,\bar{z}) + \frac{S^4}{R^4} W_6(z,\bar{z}) + \cdots \right) \ , \tag{4.69}$$

where W_3, W_6 (explicitly known) are specific single-valued transcendental functions, of weight three and six in the worldsheet coordinates, and we have set $\alpha' = 1$. Importantly, the $W_n(z,\bar{z})$ depend on S,T in a polynomial way: the location of the saddle is not modified in a $1/R$ expansion! Therefore, we can readily compute $A(S,T)$ in the high-energy limit by evaluating the worldsheet integral representation on the saddle point:

$$A_{HE}(S,T) \sim e^{-2S\log|S| - 2T\log|T| - 2U\log|U|}\ W_0(z_0) \left(1 + \frac{S^2}{R^2} W_3(z_0) + \frac{S^4}{R^4} W_6(z_0) + \cdots \right) \ , \tag{4.70}$$

where at each order in $1/R$ we keep the leading large energy contribution. Effectively, we are looking into a regime with large R, S and S^2/R^2 finite.[19]

From spacetime

The high-energy regime in flat space can also be understood from the point of view of spacetime, in terms of classical solutions. To do that, we consider the path integral representation for the amplitude,

$$A^{(0)}(S,T) \sim \int DX\ Dg \exp\left(-\frac{1}{4\pi}\int d\zeta_1 d\zeta_2 \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X_\mu \right) \prod_{i=1}^{4} V_i(p_i) \ , \tag{4.71}$$

[18] Further details about the evaluation of leading and sub-leading terms by saddle point approximation can be found in Appendix B of [176].

[19] One might instead keep S/R^2 fixed, to probe greater curvature. The regime presented here, of S^2/R^2 fixed, is more tractable for computation.

where the vertex operators are of the form $V_i(p_i) \sim \int d^2 z_i e^{ip_i \cdot X(z_i)}$ with $p_i^2 = 0$[20] and the locations z_i are assumed to be real for simplicity. We use the letter ζ for worldsheet (complex) coordinates. The Mandelstam variables are related to the momenta by $p_1 \cdot p_2 = -2S$, $p_1 \cdot p_3 = -2T$, $p_1 \cdot p_4 = -2U$. At high energies, the path integral is dominated by the classical solution

$$X^\mu_{\text{classical}}(\zeta) = -i \sum_k p_k^\mu \log \left| 1 - \frac{\zeta}{z_k} \right| \ , \tag{4.72}$$

together with the condition

$$z_0 = \frac{S}{S+T} = -\frac{(z_2 - z_1)(z_4 - z_3)}{(z_3 - z_2)(z_4 - z_1)} \ , \tag{4.73}$$

for the location of the punctures. This condition ensures that the induced metric on the worldsheet is conformal. Plugging the classical solution into the path integral reproduces the correct high-energy result (4.68). Transferring these insights from flat spacetime to the more intricate AdS background requires additional care; we shall devote the next Subsection to this task.

4.3.1 A classical scattering problem in AdS

The obvious generalisation to AdS of the path integral (4.71) is

$$A(S,T) \sim \int DX \ Dg \exp\left(-\frac{1}{4\pi} \int d\zeta_1 d\zeta_2 \sqrt{g} g^{ab} \partial_a X^M \partial_b X_M \right) \prod_{i=1}^4 V_i(P_i) \ , \tag{4.74}$$

where now X^M denote embedding coordinates with

$$X^M X_M = -(X^0)^2 + X^\mu X_\mu = -R^2 \ , \tag{4.75}$$

with R being the AdS radius. The vertex operators are of the form $V_i(P_i) \sim \int d^2 z_i e^{iP_i^M X_M(z_i)}$.

We expect that at high energies the behaviour of the amplitude is captured by classical solutions, now in AdS. The relevant Lagrangian to study the classical string scattering problem on AdS_d is

$$\mathcal{L} = \frac{1}{2\pi} \partial X^M \overline{\partial} X_M + \Lambda(X^M X_M + R^2) - i \sum_{k=1}^4 P_k^M X_M \delta^{(2)}(\zeta - z_k) \ , \tag{4.76}$$

where we inserted a Lagrangian multiplier Λ to select the AdS_d subspace.

Away from the punctures $\zeta = z_k$, the equations of motion are

$$\partial \overline{\partial} X^M = \frac{\partial X^N \overline{\partial} X_N}{R^2} X^M \ , \tag{4.77}$$

[20] In the high-energy limit, all operators with a finite number of excitations are effectively massless.

with the following boundary conditions as we approach each of the punctures[21]

$$X^M = -iP_k^M \log\left|1 - \frac{\zeta}{z_k}\right| + Q_k^M + \cdots \ . \tag{4.78}$$

The quadratic constraint $X^M X_M = -R^2$ then implies that for each puncture

$$P_k^M P_{k,M} = P_k^M Q_{k,M} = 0 \ . \tag{4.79}$$

The scattering problem in flat space considered in [179–181] arises as a limit of the AdS problem as follows. We split the indices into $M = (0, \mu)$ with $\mu = 1, \cdots, d$ and such that

$$X^M X_M = -X^0 X^0 + X^\mu X_\mu \equiv -X^0 X^0 + X \cdot X \ , \tag{4.80}$$

where we reserve the notation $X \cdot X = X^\mu X_\mu$ for the μ-indices. We then consider solutions with a large R expansion,

$$\begin{aligned} X^0 &= R + \frac{1}{R} X_1^0 + \cdots \ , \\ X^\mu &= X_0^\mu + \frac{1}{R^2} X_1^\mu + \cdots \ , \end{aligned} \tag{4.81}$$

where X_0^μ is the flat space solution. The corresponding momentum at each puncture k also admits a $1/R$ expansion:

$$\begin{aligned} P_k^0 &= \frac{1}{R} p_{k,1}^0 + \cdots \ , \\ P_k^\mu &= p_{k,0}^\mu + \frac{1}{R^2} p_{k,1}^\mu + \cdots \ . \end{aligned} \tag{4.82}$$

As a result of the relations (4.79), $p_{k,0}^\mu$ is null from the d-dimensional flat space point of view, which is precisely the condition for massless scattering in flat space.

The (conformal gauge) equations of motion arising from the Lagrangian are supplemented by the Virasoro constraints,

$$\partial X^N \partial X_N = \bar{\partial} X^N \bar{\partial} X_N = 0 \, . \tag{4.83}$$

Solving for $X^0 = \sqrt{R^2 + X \cdot X}$, the equations of motion away from the punctures and Virasoro constraints for the flat space coordinates X_0^μ are simply

$$\partial \bar{\partial} X_0^\mu = 0, \qquad \partial X_0 \cdot \partial X_0 = \bar{\partial} X_0 \cdot \bar{\partial} X_0 = 0 \ . \tag{4.84}$$

Together with the boundary conditions, the equations of motion imply

$$X_0^\mu = -i \sum_k p_{k,0}^\mu \log\left|1 - \frac{\zeta}{z_k}\right| + q_0^\mu \ , \tag{4.85}$$

[21]The dots denote terms which vanish as $\zeta \to z_k$.

for any constant q_0^μ. Plugging this into the Virasoro constraints at leading order, we learn that the locations of the punctures are related to the Mandelstam variables of the scattering process via

$$\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_2)(z_3-z_4)} = -\frac{T}{S} \ , \tag{4.86}$$

where $p_{1,0}\cdot p_{2,0} = -2S$, $p_{1,0}\cdot p_{3,0} = -2T$, $p_{1,0}\cdot p_{4,0} = -2U$ with $S+T+U=0$. This of course agrees with the solution found in flat space [179].

Now, we would like to solve the equations of motion and Virasoro constraints in a $1/R$ expansion. To understand the systematics of this expansion, remember that the flat space solution X_0^μ is single-valued as we move around each puncture $\zeta = z_k$ in the $\zeta-$plane (4.72). Indeed, the solution can be written in terms of SVMPLs, whose letters are the locations of the punctures:

$$X_0^\mu = -\frac{i}{2}\sum_k p_{k,0}^\mu \mathcal{L}_{z_k}(\zeta) \ . \tag{4.87}$$

Let us consider the equations of motion at the next order:

$$\partial\bar\partial X_1^\mu = \partial X_0\cdot\bar\partial X_0\, X_0^\mu = \frac{i}{8}\sum_{i,j,k}\frac{p_{i,0}\cdot p_{j,0}}{(\zeta-z_i)(\bar\zeta-z_j)}p_{k,0}^\mu\mathcal{L}_{z_k}(\zeta) \ . \tag{4.88}$$

There is a systematic procedure to "integrate" the RHS in terms of SVMPLs. For the holomorphic derivative, this is very easily done:

$$\int d\zeta\,\frac{\mathcal{L}_w(\zeta)}{(\zeta-z_i)} \to \mathcal{L}_{z_i w}(\zeta) \ . \tag{4.89}$$

For the anti-holomorphic derivative, it is slightly more complicated, but a recursive algorithm has been worked out in [6, 151, 152]. The result always takes the form

$$\int d\bar\zeta\,\frac{\mathcal{L}_w(\zeta)}{(\bar\zeta-z_j)} \to \mathcal{L}_{w z_j}(\zeta) + \dots \ , \tag{4.90}$$

where the ellipsis denotes a sum of terms of uniform weight $|w|+1$, consisting of products of lower-weight SVMPLs evaluated at ζ and at the letters z_i. For a detailed discussion, see Appendix A of [176]. For instance, for $|w|=1$, we obtain

$$\int d\bar\zeta\,\frac{\mathcal{L}_{z_k}(\zeta)}{(\bar\zeta-z_j)} \to \mathcal{L}_{z_k z_j}(\zeta) + \mathcal{L}_{z_k}(z_j)\mathcal{L}_{z_j}(\zeta) - \mathcal{L}_{z_j}(z_k)\mathcal{L}_{z_k}(\zeta) \ . \tag{4.91}$$

This allows us to write

$$X_1^\mu = \frac{i}{8}\sum_{i,j,k=1}^{4} p_{i,0}\cdot p_{j,0}\; p_{k,0}^\mu\left(\mathcal{L}_{z_i z_k z_j}(\zeta) + \mathcal{L}_{z_k}(z_j)\mathcal{L}_{z_i z_j}(\zeta) - \mathcal{L}_{z_j}(z_k)\mathcal{L}_{z_i z_k}(\zeta)\right) \ , \tag{4.92}$$

so that the equations of motion are satisfied. Note that at each order we have the freedom to add weight one functions $\mathcal{L}_{z_i}(\zeta)$ in ζ. At each step, we use this freedom to remove all weight one functions from the solution. This procedure can be repeated to higher orders and the solution finally has the following schematic form

$$\boxed{X^\mu = \mathcal{L}_1(\zeta) + \frac{1}{R^2}\mathcal{L}_3(\zeta) + \frac{1}{R^4}\mathcal{L}_5(\zeta) + \cdots} \tag{4.93}$$

where $\mathcal{L}_n(\zeta)$ are linear combinations of pure SVMPLs of weight n, with either ζ or z_i as their arguments, and letters from the alphabet $\{z_1, z_2, z_3, z_4\}$. More precisely, by pure we mean that the entire ζ-dependence is through the SVMPLs, and not, for instance, through rational functions multiplying those. The relation $X^0 = \sqrt{R^2 + X \cdot X}$ implies a very similar structure for X^0:

$$X^0 = R\,\mathcal{L}_0(\zeta) + \frac{1}{R}\mathcal{L}_2(\zeta) + \frac{1}{R^3}\mathcal{L}_4(\zeta) + \cdots \;, \tag{4.94}$$

with $\mathcal{L}_0(\zeta) = 1$.

A salient feature of our method is that once the "seed" (flat space) solution X_0^μ is given, the *whole* tower in $1/R$ is fixed by the equations of motion and the integration procedure described above:

$$X_0^\mu \quad \to \quad X^\mu = X_0^\mu + \frac{1}{R^2}X_1^\mu + \cdots \;. \tag{4.95}$$

Looking at the behaviour around each puncture, this also determines the momenta in a $1/R$ expansion from the seed/flat space momenta $p_{k,0}^\mu$:

$$p_{k,0}^\mu \quad \to \quad P_k^\mu = p_{k,0}^\mu + \frac{1}{R^2}p_{k,1}^\mu + \cdots \;. \tag{4.96}$$

Let us conclude by emphasising two features of our solution. First, note that SVMPLs of weight higher than zero are defined so that they vanish at the base point $\zeta = 0$. This, in particular, means that the solution we just constructed satisfies

$$X^0(0) = R \;, \quad X^\mu(0) = 0 \;. \tag{4.97}$$

The second condition is the statement that $\partial X^M(0)$ and $\overline{\partial} X^M(0)$ do not receive $1/R$ corrections. For $\partial X^M(0)$, this is easy to see, since at each order, weight one SVMPLs with argument ζ are removed, consistently with the equations of motion. For $\overline{\partial} X^M(0)$, this is also true, but one needs to use the fact that z_i are real (which was an assumption in constructing our solutions). These conditions imply that our solution satisfies the Virasoro constraints. Let us show why it is the case.

The structure of the solutions in a $1/R$ expansion implies that

$$\partial X^M = \sum_{k=1}^{4} \frac{H_k^M}{\zeta - z_k} \ , \quad \overline{\partial} X^M = \sum_{k=1}^{4} \frac{\overline{H}_k^M}{\overline{\zeta} - z_k} \ , \tag{4.98}$$

where H_k^M, $\overline{H}_k^M$ for $k = 1, 2, 3, 4$ are pure SVMPLs in ζ, of higher and higher weight in a $1/R$ expansion. Let us study these objects in more detail. First, they satisfy

$$\sum_{k=1}^{4} H_k^M = \sum_{k=1}^{4} \overline{H}_k^M = 0 \ , \tag{4.99}$$

which is simply the statement that ∂X^M and $\overline{\partial} X^M$ do not have a pole at infinity. Furthermore, $X^M X_M = -R^2$ implies $X_M \partial X^M = X_M \overline{\partial} X^M = 0$, so that

$$X_M H_k^M = X_M \overline{H}_k^M = 0 \ . \tag{4.100}$$

We can write the equations of motion in terms of the quantities H_k^M, $\overline{H}_k^M$. There are two equivalent ways of writing them, depending on the order in which we take the derivatives. For instance, one can derive

$$\sum_k \frac{\overline{\partial} H_k^M}{\zeta - z_k} = \frac{1}{R^2} \sum_{i,j} \frac{H_i^N \overline{H}_{j\,N}}{(\zeta - z_i)(\overline{\zeta} - z_j)} X^M \ . \tag{4.101}$$

Focusing on a given pole $\zeta = z_k$, we obtain

$$\overline{\partial} H_k^M = \frac{1}{R^2} \sum_j \frac{H_k^N \overline{H}_{j\,N}}{(\overline{\zeta} - z_j)} X^M \ . \tag{4.102}$$

We can now consider the contractions

$$S_{k,k'} = H_k^M H_{k'\,M} \ . \tag{4.103}$$

As a consequence of the equations of motions, $S_{k,k'}$ is holomorphic ($\overline{\partial} S_{k,k'} = 0$). On the other hand, our procedure implies that $S_{k,k'}$ is given by SVMPLs in ζ. The only SVMPL which is also holomorphic is the constant function, so $S_{k,k'}$ is independent of $\zeta, \overline{\zeta}$. This result is valid to all orders in $1/R$. For the solution computed above, we can actually do better and compute these products exactly. In our procedure, H_k^M, $\overline{H}_k^M$ are given in terms of SVMPLs, where the point $\zeta = 0$ has been chosen as the base point. This implies that we can compute H_k^M, $\overline{H}_k^M$ exactly at that point. Indeed,

$$H_k^\mu(0) = \overline{H}_k^\mu(0) = -\frac{i}{2} p_{k,0}^\mu \ , \quad H_k^0(0) = \overline{H}_k^0(0) = 0 \ . \tag{4.104}$$

Since $S_{k,k'}$ is constant, we can evaluate it at any point, and in particular at $\zeta = 0$, where we obtain

$S_{k,k'} = -\frac{p_{k,0}\cdot p_{k',0}}{4}$. Note in particular $S_{k,k} = 0$. Furthermore, we also have $S_{1,2} + S_{1,3} + S_{1,4} = 0$, which follows from (4.99) together with $S_{k,k} = 0$. The same considerations follow for the contractions $\overline{S}_{k,k'} = \overline{H}_k^M \overline{H}_{k'\,M}$. With this structure, the Virasoro constraints take exactly the same form as in flat space. In particular, we obtain a single constraint on the location of the punctures, which is now

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)} = -\frac{p_{1,0} \cdot p_{3,0}}{p_{1,0} \cdot p_{2,0}} , \tag{4.105}$$

exactly as in flat space. This proves that our construction in terms of SVMPLs satisfies both the equations of motion and the Virasoro constraints.

4.3.2 Evaluation of the action

Now, we need to evaluate the action at the classical solution we found. To make contact with flat space ($R \to \infty$) while still having a finite answer, we need to subtract a constant,[22] and the Lagrangian to consider is

$$\mathcal{L} = \frac{1}{2\pi}\partial X^M \overline{\partial} X_M - i\sum_{k=1}^{4} \left(P_k^M X_M + P_k^0\, R\right) \delta^{(2)}(\zeta - z_k) = \frac{1}{2\pi}\partial X^M \overline{\partial} X_M - iR\sum_{k=1}^{4} P_k^0\, \delta^{(2)}(\zeta - z_k) , \tag{4.106}$$

where in our signature $X_0 = -X^0 = -R + \cdots$. In the second expression, we took into account that $P_k^M X_M \to 0$ as we approach the puncture, so its contribution vanishes.

In other words, the total contribution to the action is given by the bulk action plus the sum over the "masses" $R m = i\, R\, P_k^0$. The contribution from both terms is finite. Indeed, $R\, P_k^0$ is clearly finite, while the bulk contribution is integrable. The non-integrable divergence as we approach the puncture,

$$\partial X^M \overline{\partial} X_M \sim \frac{P_k^M P_{k\,M}}{|\zeta - z_k|^2} , \tag{4.107}$$

vanishes since $P_k^M P_{k\,M} = 0$.

We are then ready to recover flat space from our AdS model. We solve for $X^0 = \sqrt{R^2 + X \cdot X}$ and expand in $1/R$:

$$\mathcal{L} = \frac{\partial X \cdot \overline{\partial} X}{2\pi} - i\sum_{k=1}^{4} P_k \cdot X \delta^{(2)}(\zeta - z_k) - \frac{1}{R^2}\left(\frac{X \cdot \partial X X \cdot \overline{\partial} X}{2\pi} - \frac{i}{2}\sum_{k=1}^{4} p_{k,1}^0 X \cdot X \delta^{(2)}(\zeta - z_k) \right) + \dots . \tag{4.108}$$

At leading order, we obtain the usual Lagrangian in flat space. Then, we write the action on the classical solution as

$$\mathcal{S} = \mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{vertex}} , \tag{4.109}$$

[22]This is analogous to the vacuum-energy subtraction in QFT.

with

$$
\begin{aligned}
\mathcal{S}_{\text{bulk}} &= \frac{1}{2\pi}\int d^2\zeta\left(\partial X\cdot\bar{\partial}X - \frac{1}{R^2}X\cdot\partial X X\cdot\bar{\partial}X + \dots\right), \\
\mathcal{S}_{\text{vertex}} &= -i\sum_{k=1}^{4}\left(p^0_{k,1} + \frac{1}{R^2}p^0_{k,2} + \dots\right) .
\end{aligned}
\tag{4.110}
$$

We denote different terms in the expansion by $\mathcal{S} = \mathcal{S}^{(0)} + \frac{1}{R^2}\mathcal{S}^{(1)} + \cdots$. The first two terms in the bulk action are given by

$$
\begin{aligned}
\mathcal{S}^{(0)}_{\text{bulk}} &= \frac{1}{2\pi}\int d^2\zeta\,\partial X_0\cdot\bar{\partial}X_0 = \int d^2\zeta\sum_{i,j=1}^{4}\frac{c_{ij}}{(\zeta-z_i)(\bar{\zeta}-z_j)} , \\
\mathcal{S}^{(1)}_{\text{bulk}} &= \frac{1}{2\pi}\int d^2\zeta\left(\partial X_0\cdot\bar{\partial}X_1 + \partial X_1\cdot\bar{\partial}X_0 - X_0\cdot\partial X_0 X_0\cdot\bar{\partial}X_0\right) = \int d^2\zeta\sum_{i,j=1}^{4}\frac{f_{ij}(\zeta)}{(\zeta-z_i)(\bar{\zeta}-z_j)} ,
\end{aligned}
\tag{4.111}
$$

where c_{ij} are rational numbers and $f_{ij}(\zeta)$ is of transcendental weight two and can be written in terms of SVMPLs. The integral can be done using the formula [6]

$$
\frac{1}{2\pi}\int d^2z\frac{\mathcal{L}_w(z)}{(z-z_i)(\bar{z}-z_j)} = -\mathcal{L}_{\sigma_i w}(z_j) + \operatorname*{Res}_{\bar{z}=\infty}\frac{\mathcal{L}_{\sigma_i w}(z)}{\bar{z}-z_j} ,
\tag{4.112}
$$

where the total contribution at $\bar{z}=\infty$ vanishes. The results, expressed in terms of the cross ratio $z_0 = -\frac{z_{12}z_{34}}{z_{23}z_{14}}$, read

$$
\begin{aligned}
\mathcal{S}^{(0)}_{\text{bulk}} &= -S\left(\mathcal{L}_0(z_0) + \frac{1-z_0}{z_0}\mathcal{L}_1(z_0)\right) , \\
\mathcal{S}^{(1)}_{\text{bulk}} &= S^2\left(\frac{2-z_0}{z_0}\mathcal{L}_{001}(z_0) + \frac{z_0-2}{z_0}\mathcal{L}_{010}(z_0) + \frac{1-z_0^2}{z_0^2}\mathcal{L}_{011}(z_0) + \frac{z_0^2-1}{z_0^2}\mathcal{L}_{101}(z_0) + 6\zeta(3)\right) .
\end{aligned}
\tag{4.113}
$$

In order to evaluate the source terms, we can simply use the relation $P_k^M Q_{k,M} = 0$ to write P_k^0 in terms of P_k^μ and Q_k^μ, which can be read off from X^μ using (4.78) and (4.40). We have

$$
p^0_{k,1} = p_{k,0}\cdot q_{k,0} , \qquad p^0_{k,2} = p_{k,0}\cdot q_{k,1} + p_{k,1}\cdot q_{k,0} - \frac{1}{2}q_{k,0}\cdot q_{k,0}p_{k,0}\cdot q_{k,0} ,
\tag{4.114}
$$

leading to the expressions

$$
\begin{aligned}
\mathcal{S}^{(0)}_{\text{vertex}} &= 2S\left(\mathcal{L}_0(z_0) + \frac{1-z_0}{z_0}\mathcal{L}_1(z_0)\right) , \\
\mathcal{S}^{(1)}_{\text{vertex}} &= S^2\Bigg(\frac{1}{2}\mathcal{L}_{000}(z_0) + \frac{(z_0-3)}{z_0}\mathcal{L}_{001}(z_0) - \frac{(3z_0-5)}{2z_0}\mathcal{L}_{010}(z_0) \\
&\quad + \frac{(z_0^2+z_0-2)}{z_0^2}\mathcal{L}_{011}(z_0) - \frac{(3z_0^2-z_0-2)}{2z_0^2}\mathcal{L}_{101}(z_0) + \frac{(z_0-1)^2}{2z_0^2}\mathcal{L}_{111}(z_0) - 8\zeta(3)\Bigg) .
\end{aligned}
\tag{4.115}
$$

In summary, this classical solution predicts the following behaviour of the amplitude in the high-energy limit:

$$A_{HE}(S,T) \sim e^{-\mathcal{S}} = e^{SV_1(z_0)+\frac{S^2}{R^2}V_3(z_0)+\frac{S^3}{R^4}V_5(z_0)+\cdots} \,, \tag{4.116}$$

where $V_i(z_0)$ are combinations of transcendental functions of weight i. The first two terms are given by

$$\begin{aligned} V_1(z_0) =& \; -\mathcal{L}_0(z_0) + \frac{z_0-1}{z_0}\mathcal{L}_1(z_0) \,, \\ V_3(z_0) =& -\frac{1}{2}\mathcal{L}_{000}(z_0) - \frac{(z_0-1)^2}{2z_0^2}\mathcal{L}_{111}(z_0) + \frac{1}{z_0}\mathcal{L}_{001}(z_0) + \frac{z_0-1}{2z_0}\mathcal{L}_{010}(z_0) + \frac{z_0-1}{2z_0}\mathcal{L}_{101}(z_0) \\ &+ \frac{1-z_0}{z_0^2}\mathcal{L}_{011}(z_0) + 2\zeta(3) \,. \end{aligned} \tag{4.117}$$

4.3.3 Deformations of the solution and exponentiation

Let us now comment on possible deformations of our classical solution, consistent with the equations of motion and Virasoro constraints, such that the flat space momenta $p^\mu_{k,0}$ are invariant. The first deformation is labelled by a constant vector W^μ and reduces to translations in the flat space limit. It acts on coordinates as $X^\mu \to \hat{X}^\mu$ and $X^0 \to \hat{X}^0$ with

$$\hat{X}^\mu = X^\mu + W^\mu \frac{X^0}{R} + W^\mu W \cdot X \frac{\sqrt{1+W^2/R^2}-1}{W^2} \,, \tag{4.118}$$

$$\hat{X}^0 = \sqrt{1+W^2/R^2}\left(X^0 + \frac{W\cdot X}{\sqrt{R^2+W^2}}\right) \,. \tag{4.119}$$

In other words, it is a Lorentz transformation on the embedding coordinates

$$X^M \to \hat{X}^M = \Lambda^M_N X^N \,, \tag{4.120}$$

with $\Lambda^M_N \eta_{MM'} \Lambda^{M'}_{N'} = \eta_{NN'}$. As such, it preserves the inner products $S_{k,k'}$, as well as the equations of motion. As already mentioned, in the flat space limit, this transformation reduces to shifts by $w^\mu = W^\mu(R=\infty)$, and hence the flat space momenta are invariant. On the other hand, it acts on the momenta at higher orders in $1/R$. We will fix the freedom implied by this deformation as follows. While the flat space/seed momenta are conserved, $\sum_k p^\mu_{k,0} = 0$, this is not true at higher orders in $1/R$. Since we are considering a scattering problem around flat space, where the μ-coordinates parametrise this space, it seems reasonable to impose momentum conservation on the μ-plane. We can use the W-deformation to achieve that, so that the total momentum in the μ-coordinates is conserved: $\sum_k \hat{P}^\mu_k = 0$. Imagine we carry out the procedure described previously. This will lead to

$P_T^\mu = \sum_k P_k^\mu$, which is non-zero from order $1/R^2$. Perform now a transformation such that

$$\hat{P}_T^\mu = P_T^\mu + W^\mu \frac{P_T^0}{R} + W^\mu W \cdot P_T \frac{\sqrt{1+W^2/R^2}-1}{W^2} = 0 \ . \tag{4.121}$$

This can be achieved by choosing (for $P_T^0 > 0, (P_T^0)^2 - P_T^\mu P_T^\mu > 0$)

$$W^\mu = -\frac{R P_T^\mu}{\sqrt{(P_T^0)^2 - P_T^\mu P_T^\mu}} \ . \tag{4.122}$$

To leading order, this gives

$$W^\mu = \frac{i}{16} \sum_{\substack{i,j,k=1 \\ i\neq j, i\neq k, j\neq k}}^{4} p_{i,0}^\mu \left(\mathcal{L}_{z_i}(z_j) - \mathcal{L}_{z_j}(z_k)\right) + O\left(\frac{1}{R^2}\right) \ . \tag{4.123}$$

On the other hand, note that the momentum will not be conserved in the 0-direction. Indeed, this cannot be achieved, as translations are not a symmetry of AdS. Note also the following. While the embedding momentum at each puncture is null, $P_k^M P_{k\,M} = 0$, from order $1/R^2$ there is an "induced" mass $(P_k^0)^2 = P_k \cdot P_k$. It turns out that for the choice in which the momentum along the $\mu-$coordinates is conserved, this induced mass does not depend on the puncture: $P_k \cdot P_k = -m^2$. When we talk about the solution to the classical scattering problem in AdS, we always mean the transformed solution (4.118), with W^μ given in (4.122).

Let us discuss another type of deformation: rescaling the seed momenta by a factor λ, $p_0^\mu \to \lambda\, p_0^\mu$. This acts in a very simple way at higher orders in $1/R$. More specifically, at each puncture,

$$p_0^\mu + \frac{1}{R^2} p_1^\mu + \frac{1}{R^4} p_2^\mu + \cdots \to \lambda\, p_0^\mu + \frac{\lambda^3}{R^2} p_1^\mu + \frac{\lambda^5}{R^4} p_2^\mu + \cdots \ . \tag{4.124}$$

At the level of the Mandelstam invariants, this rescales $S \to \lambda^2 S$. In particular, if

$$\lambda = 1 + \frac{\alpha}{R^2} + \cdots \ , \tag{4.125}$$

it modifies the Mandelstam invariants as

$$S \to \left(1 + 2\frac{\alpha}{R^2} + \cdots\right) S \ . \tag{4.126}$$

Note that ratios of Mandelstam invariants are invariant. On the solutions $X^\mu(R, p_0)$, where p_0 denotes the seed flat space momenta, this acts as

$$X^\mu(R, \lambda\, p_0) = \lambda\, X^\mu\left(\frac{R}{\lambda}, p_0\right) \quad \Leftrightarrow \quad X^\mu(\lambda\, R, \lambda\, p_0) = \lambda\, X^\mu(R, p_0) \ , \tag{4.127}$$

as expected.

Now, let us go back to (4.116) and use the freedom discussed around (4.124) to rescale the momenta such that

$$S \to S\left(1+\frac{SF_2(z_0)}{R^2}+\dots\right) , \tag{4.128}$$

while keeping ratios S/T invariant. Although these corrections are suppressed in the regime we are studying (large S, R with fixed S^2/R^2), the first correction does enter the leading high-energy limit of the amplitude via the term $S\,V_1(z_0)$ in the exponential. After performing this rescaling, we have

$$S\,V_1(z_0) = -\mathcal{S}^{(0)} , \qquad S^2 V_3(z_0) = -\mathcal{S}^{(1)} - 2\,S\,F_2(z_0)\,\mathcal{S}^{(0)} , \tag{4.129}$$

while $V_5(z_0)$ and further terms do not contribute to the leading high-energy limit. While $F_2(z_0)$ cannot be determined purely within the context of this classical bosonic model (being a subleading quantity), it will be fixed by the comparison below.

The final step is indeed the comparison to the expected high-energy limit of the AdS Virasoro–Shapiro amplitude computed in [169, 171]. We reviewed that the AdS amplitude is given in terms of a worldsheet integral representation, identical to that for flat space, with the extra insertion of single-valued functions $W_n(z)$ (4.69). Then, its high-energy limit, in a $1/R$ expansion, is obtained by simply evaluating the worldsheet integral representation on the saddle point from flat space $z = z_0$ and takes the form (4.70), with

$$\begin{aligned} W_3(z) &= \frac{U^2}{S^2} G^{(1)}_{\text{tot}}(S,T,z) , \\ W_6(z) &= \frac{U^2}{S^4} G^{(2)}_{\text{tot}}(S,T,z) , \end{aligned} \tag{4.130}$$

with $G^{(1,2)}_{\text{tot}}(S,T,z)$ being the AdS corrections to the flat space integrand, computed in [169]. At the saddle point, we find

$$\begin{aligned} W_3(z_0) = {}& \mathcal{L}_{000}(z_0) - \mathcal{L}_{001}(z_0) - \frac{1}{z_0}\mathcal{L}_{010}(z_0) - \frac{(z_0-1)^2}{z_0^2}\mathcal{L}_{011}(z_0) \\ & + \frac{(z_0-1)}{z_0^2}\mathcal{L}_{101}(z_0) + \frac{(z_0-1)^2}{z_0^2}\mathcal{L}_{111}(z_0) + 2\zeta(3) , \end{aligned} \tag{4.131}$$

and

$$W_6(z_0) = \frac{1}{2} W_3(z_0)^2 . \tag{4.132}$$

This is quite non-trivial! We are now ready to compare (4.116) with (4.70). The leading term of course reproduces the flat space result:

$$e^{SV_1(z_0)} = e^{-2S\log|S| - 2T\log|T| - 2U\log|U|} . \tag{4.133}$$

More interestingly, the *full* high-energy limit of the AdS Virasoro–Shapiro amplitude to *all* orders in S^2/R^2 is determined by the subleading exponent (first AdS correction, suppressed by $1/R^2$), and it precisely matches the expected result,

$$e^{\frac{S^2}{R^2}V_3(z_0)} = \left(1 + \frac{S^2}{R^2}W_3(z_0) + \frac{S^4}{R^4}W_6(z_0) + \cdots\right), \tag{4.134}$$

provided we choose

$$F_2(z_0) = \frac{1}{4}\left(-\mathcal{L}_{00}\left(z_0\right) + \frac{2}{z_0}\mathcal{L}_{01}\left(z_0\right) + \frac{z_0-1}{z_0}\mathcal{L}_{11}\left(z_0\right)\right). \tag{4.135}$$

The existence of such $F_2(z_0)$ comes from the fact that the combination $S^2W_3(z_0) + \mathcal{S}^{(1)}$ neatly factorises into something of weight two times $\mathcal{S}^{(0)}$. Furthermore, the bosonic model predicts exponentiation to all orders in S^2/R^2, via (4.134). The highly non-trivial relation (4.132) is a test of exponentiation to quadratic order.

To sum up, the AdS Virasoro–Shapiro amplitude in the high-energy limit, namely large R, S, T with fixed S/R and S/T, takes the form

$$\boxed{A_{HE}(S,T) = A^{(0)}_{HE}(S,T) \times e^{\frac{S^2}{R^2}W_3(S/T)}} \tag{4.136}$$

Curvature corrections exponentiate! The high-energy limit truly is a regime where the amplitude can be computed to all orders in the curvature expansion. The leading behaviour is captured by the bosonic model describing the scattering of classical strings on AdS, $A(S,T)_{\text{HE}} \sim e^{-\mathcal{S}}$, evaluated on the classical solution

$$-\mathcal{S} = S\,V_1(z_0) + \frac{S^2}{R^2}V_3(z_0) + \frac{S^3}{R^4}V_5(z_0) + \cdots, \tag{4.137}$$

Let us emphasise the universal scope of these findings. First, our result does not depend on the dimensionality of the AdS space. Notice also that, at the level of the action, the contribution we are computing is much smaller than the flat space contribution, indeed $S\,V_1(z_0) \gg \frac{S^2}{R^2}V_3(z_0)$, but we keep it because it is in the exponential. Higher-order curvature corrections, on the other hand, can be safely ignored. In other words, in the regime we are considering, only the first-order curvature corrections around flat space are important.

In closing, we note that our analysis omits quantum corrections, including contributions from fermionic fields. These corrections can indeed affect our results and are encapsulated in the "non-universal" function $F_2(z_0)$, given in (4.135). It would be very interesting to derive $F_2(z_0)$ directly from a worldsheet computation. At present, this is unavailable.

4.4 The Regge limit

In this Section, we consider a much richer limit, namely the large T, finite S Regge regime. Notoriously, in the Regge limit, amplitudes simplify dramatically. At the heart of Regge theory lies the connection between the analytic structure of a scattering amplitude and its high-energy asymptotics. For example, the flat space Virasoro–Shapiro amplitude (4.44) has an infinite sequence of simple poles, each corresponding to the exchange of a spin J string state. These poles arrange themselves into linear Regge trajectories, which control the $T \gg -S > 0$, S finite, behaviour. A remarkable achievement of Regge theory is that the amplitude in this regime can be fixed without knowledge of the full scattering result: one needs only the spectrum of the leading Regge trajectory and the corresponding cubic couplings.[23]

More precisely, any amplitude admits a partial–wave expansion, namely a decomposition into contributions of definite angular momentum J (called "spin" in this context):

$$A(S,T) = \sum_{J=0}^{\infty} (2J+1)\, a_J(S)\, G_J\left(z_S\right) \ , \tag{4.138}$$

where $a_J(S)$ are S-channel partial wave amplitudes and $G_J(z_S) = C_J^{(D-3)/2}$ are Gegenbauer polynomials, which go as T^J in the Regge limit. D is the bulk spacetime dimension. These polynomials are evaluated at the cosine of the scattering angle for massless external states,

$$z_S = \cos\theta_S = 1 + \frac{2T}{S} \ . \tag{4.139}$$

The main idea is that the behaviour of the amplitude in the Regge limit is dominated by an exchange of states in the S-channel. Under mild assumptions, the analyticity in the spin of the partial wave decomposition implies that any amplitude behaves in this limit as [183, 184]

$$A(S,T) \sim \beta(S)\ T^{\alpha(S)} \ . \tag{4.140}$$

The exponent encodes key spectral data. At tree-level in string theory, the leading trajectory is linear, $\alpha(S) = J + \alpha' S$, where α' is the Regge slope setting the spacing of masses, and the intercept $J = \alpha(0)$ equals the highest spin of any massless state: $J = 1$ for open strings and $J = 2$ for closed strings. These parameters fix both the mass spectrum and the asymptotic behaviour of the amplitude.

However, this prediction, based solely on the S-channel partial-wave expansion, is not entirely accurate because the series (4.138) does not converge in the physical T-channel region. More specif-

[23] See [182] for the differences between the Regge limit in QFT as opposed to string theory from a one-loop analysis.

ically, it converges in $-1 < z_S < 1$, but the T-channel has $z_S < -1$. The usual remedy is to extend the spin J into the complex plane. We will review and implement this procedure in what follows, to study the Regge limit of the AdS Virasoro-Shapiro amplitude.

Before doing so, we should mention that in the interesting work [185], the authors developed a Regge theory generalisation for AdS scattering in Mellin space. Our results may be viewed as the Borel transform of theirs, although translating between the two is technically involved due to the intricate analytic structure of $A(S,T)$. On our side, we also supply a complementary new description from the worldsheet perspective.

4.4.1 Regge limit of the AdS Virasoro-Shapiro amplitude

We are interested in the Regge behaviour of the AdS Virasoro–Shapiro amplitude. This limit is dominated by the exchange of operators in the leading Regge trajectory. As usual, the starting point is the Mellin amplitude for the exchange of an operator of spin J and twist τ:

$$M_{\tau,J}(s,t) = \sum_{m=0}^{\infty} \frac{\mathcal{Q}^{\tau+4,d=4}_{J,m}(t)}{s+\frac{4}{3}-\tau-2m} + P_{J-1}(s,t)\ , \tag{4.141}$$

where

$$\mathcal{Q}^{\tau,d}_{J,m}(t) \equiv -\frac{2^{3J+2\tau-2}\Gamma\left(J+\frac{\tau}{2}-\frac{1}{2}\right)\Gamma\left(J+\frac{\tau}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{d}{2}+J+\tau+1\right)Q^{\tau,d}_{J,m}(t)}{\pi\Gamma(m+1)\Gamma\left(J+\frac{\tau}{2}\right)^2\Gamma(J+\tau-1)\Gamma\left(-m-\frac{\tau}{2}+4\right)^2\Gamma\left(-\frac{d}{2}+J+m+\tau+1\right)}\ , \tag{4.142}$$

with $Q^{\tau,d}_{J,m}(t)$ a Mack polynomial [108], and $P_{J-1}(s,t)$ is a regular contribution. Mack polynomials are in general very complicated, but here we are only interested in the leading large t behaviour for which

$$Q^{\tau,d}_{J,m}(t) = t^J + \cdots\ . \tag{4.143}$$

Furthermore, in this limit, we can also ignore regular terms present in the exchange, given by a polynomial $P_{J-1}(s,t)$ of degree $J-1$, since this grows at most as t^{J-1} for t large. In this limit, we can compute explicitly the exchange amplitude $M_{\tau,J}(s,t) = M_{\tau,J}(s)t^J + \cdots$, but we are more interested in the difference equation it satisfies:

$$\left(s-\frac{8}{3}\right)^2 M_{\tau,J}(s-2) - \left(s^2+2Js+\frac{20}{3}s-R^2m^2\right)M_{\tau,J}(s) = M_c\ , \tag{4.144}$$

where

$$\left(\tau-\frac{4}{3}\right)\left(2J+\tau+\frac{16}{3}\right) = m^2R^2\ . \tag{4.145}$$

This relation follows from the equation of motion identity satisfied by the exchange Witten diagram. When expanding around the flat space limit, m^2 will be kept fixed as we take R large.[24] M_c is a complicated but explicit quantity, independent of s:

$$M_c(\tau,J) = \frac{2^{3J+2\tau+7}\Gamma(J+6)\Gamma\left(J+\frac{\tau}{2}+\frac{3}{2}\right)\Gamma\left(J+\frac{\tau}{2}+\frac{5}{2}\right)}{\pi\Gamma\left(2-\frac{\tau}{2}\right)^2\Gamma\left(J+\frac{\tau}{2}+2\right)^2\Gamma\left(J+\frac{\tau}{2}+4\right)^2} \,. \tag{4.146}$$

All together, we can write

$$M(s,t) \simeq \sum_{J=0} C^2(J) M_{\tau,J}(s) t^J \,, \tag{4.147}$$

where the sum involves only operators in the leading Regge trajectory, and for each exchange we have only kept the leading contribution at large t. Here, τ and J denotes the twist and the spin of the exchanged operator, while $C(J)$ denotes the OPE coefficient. More precisely, one should first perform the sum over the spin, and then take the large t limit. Since the leading contribution arises from the Regge spin $J=J^*$, it is a valid approximation to consider the leading large t asymptotics of each exchange.

The expansion (4.147) has a corresponding expression after Borel transform:

$$A(S,T) \simeq \sum_{J=0} C^2(J) A_{\tau,J}(S) T^J \,, \tag{4.148}$$

where

$$M_{\tau,J}(s) = \frac{1}{2R^6}\int_0^\infty d\beta e^{-\beta}\beta^5 A_{\tau,J}\left(\frac{s\beta}{2R^2}\right)\left(\frac{\beta}{2R^2}\right)^J \,. \tag{4.149}$$

As usual, we are interested in expansions around the flat space limit. It is convenient to introduce the rescaled Mellin variable $\hat{s} = \frac{s}{2R^2}$, which is kept finite as we take the flat space limit.

We can write the difference equation satisfied by $M_{\tau,J}(\hat{s})$ in the form

$$\mathcal{D} M_{\tau,J}(\hat{s}) = M_c(\tau,J) \,, \tag{4.150}$$

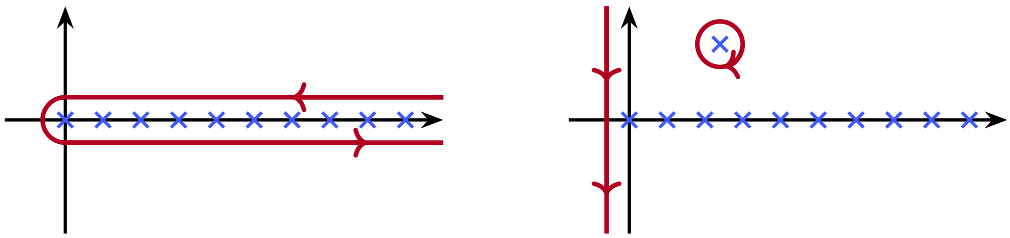
with $M_c(\tau,J)$ independent of $\hat{s}$, and the difference operator can be written as

$$\mathcal{D} = (-4U_+ + m^2) + \frac{1}{R^2}\sum_{n=0}^{\infty}\frac{4(-1)^n}{R^{2n}\Gamma(n+1)}\left(\frac{U_0^2 + \left(\frac{8(n+2)}{3}-1\right)U_0}{(n+1)(n+2)} + \frac{16}{9}\right)U_-^n \,, \tag{4.151}$$

where

$$U_0 = \hat{s}\partial_{\hat{s}} \,, \quad U_+ = -q\hat{s} + \hat{s}^2\partial_{\hat{s}} \,, \quad U_- = \partial_{\hat{s}} \,. \tag{4.152}$$

[24] Recall that at strong coupling $\tau \sim \lambda^{1/4} \sim R$.



(a) Original contour (b) Deformed contour

Figure 4.3: Integration contours in the complex-J plane. The original contour reproduces the sum over spins. The deformed contour, closed to the right, picks up the original poles, as well as Regge poles, of which the residues are subtracted to keep the same result. Picture from [178].

The relation (4.151) in turns implies an analogous relation for $A_{\tau,J}(S)$:

$$\mathcal{D}A_{\tau,J}(S) = A_c(\tau, J) \ , \tag{4.153}$$

where $\mathcal{D}$ takes exactly the same form as in (4.151) but now the operators $U_0, U_\pm$ act on functions of S.

We will now apply the ideas of Regge theory to $A(S,T)$. The first step is to write the sum over the spin J as a contour integral:

$$A(S,T) = \int \frac{dJ}{2i} \frac{1+(-1)^J}{2\sin(\pi J)} C^2(J) A_{\tau,J}(S) T^J \ . \tag{4.154}$$

In principle, we should sum over all trajectories, but as already mentioned, only the leading Regge trajectory, with $\tau = \tau(J)$, will contribute in the Regge limit. The contour integral picks the poles at $J = 0, 2, \ldots$. Note that we think of $\tau(J)$ and $C^2(J)$ as analytic functions of J. Then, we deform the contour, as in Figure 4.3. The deformed contour, parallel to the imaginary axis, picks up residues from Regge poles, which must be subtracted from this new integral to give $A(S,T)$. In the large T limit, the factor of T^J causes the pole with the largest real value of J—which corresponds to the leading Regge trajectory—to provide the dominant contribution to the Borel amplitude.

The analytic structure of $A_{\tau,J}(S)$ is more complicated than in flat space. In particular, in a $1/R$ expansion, it contains poles of higher and higher order, as reviewed above. The locations and residues of these poles are determined, in terms of $m(J)^2$, entirely by the relation (4.153) that $A_{\tau,J}(S)$ satisfies. In a $1/R$ expansion, this relation becomes a recursion relation and can be solved order by order. To any given order in $1/R$, $A_{\tau,J}(S)$ contains poles of the form

$$A^{(k)}(S) = \frac{r_k(J)}{x^k} \ , \quad x = -4S + m(J)^2 \ , \tag{4.155}$$

where k is a positive integer. In the $J-$plane, this leads to a k-th order pole at $J = J^*(S)$, where

J^* is implicitly given by the solution of

$$m(J^*)^2 = 4S \ . \tag{4.156}$$

Which contribution does this give in the Regge limit? In the J-plane, $-4S + m(J)^2$ has a simple zero at $J = J^*$, so that we can write

$$-4S + m(J)^2 \equiv (J - J^*)\beta(J) \ . \tag{4.157}$$

Deforming the contour and using Cauchy's theorem, we see that the contribution from the pole $A^{(k)}(S)$ to the Regge limit is

$$\int \frac{dJ}{2i} \frac{1 + (-1)^J}{2\sin(\pi J)} C^2(J) \frac{r_k(m^2)}{(J - J^*)^k \beta(J)^k} T^J = -\pi \frac{\partial_J^{k-1}}{\Gamma(k)} \frac{1 + (-1)^J}{2\sin(\pi J)} C^2(J) \frac{r_k(J)}{\beta(J)^k} T^J \bigg|_{J=J^*} \ . \tag{4.158}$$

In other words, for every pole of order k in $A_{\tau,J}(S)$ we can write its contribution as an operator

$$A_{\tau,J}(S) \sim \frac{r_k(J)}{x^k} \quad \to \quad \frac{\partial_J^{k-1}}{\Gamma(k)} \frac{r_k(J)}{\beta(J)^k} \ , \tag{4.159}$$

acting on functions of J. The derivatives act on everything on the right. We can then formally define the operators

$$y = \partial_J \frac{1}{\beta(J)} \ , \quad :y^n: = \partial_J^n \frac{1}{\beta(J)^n} \ . \tag{4.160}$$

We introduced the "normal ordered" $:y^2: \neq yy$ so that all the derivative operators ∂_J are placed on the left, and then they act on everything on their right. The result in the Regge limit is then given by

$$A_{Regge}(S,T) = -\pi :\mathcal{R}(y): \frac{1 + (-1)^J}{2\sin(\pi J)} C^2(J) \frac{1}{\beta(J)} T^J \bigg|_{J=J^*(S)} \ , \tag{4.161}$$

where $\mathcal{R}(y)$ is the following integral transform of $A(x)$:

$$A(x) = \int_0^\infty e^{-yx} \mathcal{R}(y) dy \ . \tag{4.162}$$

This integral transform basically implements Cauchy's theorem. We can write down the final answer as follows. We define the normalised function $\hat{\mathcal{R}}(y)$ such that it satisfies the relation

$$\mathcal{D}\hat{\mathcal{R}}(y) = \delta(y) \ , \tag{4.163}$$

where $\mathcal{D}$ has the same formal expression as before (4.151), but now the operators $U_\pm, U_0$ are defined as operators acting on functions of y.[25] In a large R expansion, $\hat{\mathcal{R}}(y) = \theta(y) + \cdots$, with $\theta(y)$ the Heaviside step function. As an operator, $\theta(y)$ acts as the identity operator (since $y \geq 0$ is the range the integral transform is defined). Our final formula for the AdS Virasoro–Shapiro amplitude in the Regge limit is then

$$\boxed{A_{Regge}(S,T) = : \hat{\mathcal{R}}(y) : \frac{1+(-1)^J}{2\sin(\pi J)} \mathcal{C}(J) \frac{1}{\beta(J)} T^J \Big|_{J=J^*(S)}} \tag{4.164}$$

with $\mathcal{C}(J)$ a rescaled OPE coefficient,

$$\mathcal{C}(J) = -\frac{\pi\, 2^{J+1} R^{2J+4}}{\Gamma(J+6)} M_c(\tau, J)\, C^2(J)\ , \tag{4.165}$$

where $\tau = \tau(J)$ is the twist of the operators in the leading Regge trajectory. We have written the AdS Virasoro-Shapiro amplitude in the Regge limit in terms of the CFT data of the exchanged operators (planar $4d$ $\mathcal{N} = 4$ SYM at strong coupling).

4.4.2 Small-curvature expansion

Our final expression (4.164) depends on three ingredients: the dimensions of the operators in the leading Regge trajectory, through $m^2(J)$, which determine the trajectory $J^*(S)$ and the Regge slope $\beta(J)$; the OPE coefficients $\mathcal{C}(J)$; and the operator $\hat{\mathcal{R}}(y)$ encoding background-curvature effects. Let us report these ingredients in a $1/R$ expansion, for the first few orders.

We start with the CFT data for the operators in the leading Regge trajectory. Their twist is given by [162]

$$\tau(J) = \sqrt{2(J+2)}R - J - 2 + \frac{1}{R}\frac{3J^2 + 10J + 16}{4\sqrt{2(J+2)}} + \cdots\ . \tag{4.166}$$

From this, we can determine m^2 and

$$\beta(J) = 2 + \frac{3J + 6S - 16}{6R^2} + \cdots\ , \tag{4.167}$$

$$J^*(S) = 2(S-1) + \frac{-9S^2 + 33S - 10}{9R^2} + \cdots\ . \tag{4.168}$$

The OPE coefficients can be read off from [169]. We have

$$\mathcal{C}(J) = -\frac{4\pi}{\Gamma\left(\frac{J+4}{2}\right)^2} + \frac{1}{R^2}\frac{2\pi(-4(J+2)\zeta(3) + 7J + 4)}{(J+2)\Gamma\left(\frac{J}{2}+1\right)^2} + \cdots\ . \tag{4.169}$$

[25]See [178] for more details on these operators.

Finally, solving (4.163) in a $1/R$ expansion, and for $y > 0$, we find

$$\hat{\mathcal{R}}(y) = 1 - \frac{1}{R^2}\frac{2}{9}(3m^4y^3 + 6m^2y^2 + 2y) + \cdots \ . \tag{4.170}$$

With these ingredients, we can write

$$\begin{aligned} A_{Regge}(S,T) &= A^{(0)}_{Regge}(S,T) + \frac{1}{R^2}A^{(1)}_{Regge}(S,T) + \cdots \\ &= A^{(0)}_{Regge}(S,T)\left(1 + \frac{1}{R^2}\hat{A}^{(1)}_{Regge}(S,T) + \cdots\right) \ , \end{aligned} \tag{4.171}$$

with

$$A^{(0)}_{Regge}(S,T) = e^{i\pi S}\frac{\Gamma(-S)}{\Gamma(S+1)}T^{2S-2} \ , \tag{4.172}$$

the flat space result in the Regge limit, where we have assumed T has a small negative imaginary part.

The first curvature correction reads

$$\begin{aligned} \hat{A}^{(1)}_{Regge}(S,T) \ = \ & -\frac{4S^2}{3}\log^3 T + \frac{2}{3}S\left(-3i\pi S + 3\pi S\cot(\pi S) + 6S\psi^{(0)}(S) - 2\right)\log^2 T + \\ & S^2\Big(2i\pi^2\cot(\pi S) + 4i\pi\psi^{(0)}(S) - 1 - 2\pi^2\cot^2(\pi S) - 4\psi^{(0)}(S)^2 + 2\psi^{(1)}(S) \\ & -4\pi\cot(\pi S)\psi^{(0)}(S) - \frac{4i\pi}{3S} + \frac{11}{3S} + \frac{4}{3S}\pi\cot(\pi S) + \frac{8\psi^{(0)}(S)}{3S}\Big)\log T + \\ & \frac{1}{6}i\pi S^2\Big(\frac{11}{S} + \frac{4\pi\cot(\pi S)}{S} - 6\pi^2\csc^2(\pi S) - 12\psi^{(0)}(S)^2 + \frac{8\psi^{(0)}(S)}{S} + 6\psi^{(1)}(S) \\ & -12\pi\cot(\pi S)\psi^{(0)}(S) + 4\pi^2 - 3\Big) + 2S^2\zeta(3) - \frac{7S^2}{2} - \frac{2}{3}\pi^3S^2\cot(\pi S) + \frac{1}{2}\pi S^2\cot(\pi S) \\ & +\frac{4}{3}S^2\psi^{(0)}(S)^3 - 2\pi^2S^2\psi^{(0)}(S) + S^2\psi^{(0)}(S) - 2S^2\psi^{(1)}(S)\psi^{(0)}(S) + \frac{S^2\psi^{(2)}(S)}{3} \\ & +\pi^3S^2\cot(\pi S)\csc^2(\pi S) + 2\pi S^2\cot(\pi S)\psi^{(0)}(S)^2 - \pi S^2\cot(\pi S)\psi^{(1)}(S) \\ & +2\pi^2S^2\csc^2(\pi S)\psi^{(0)}(S) + \frac{2\pi^2S}{3} + \frac{5S}{2} + \frac{2}{3S} - \frac{11}{6}\pi S\cot(\pi S) - \frac{2}{3}\pi^2S\csc^2(\pi S) \\ & -\frac{4S\psi^{(0)}(S)^2}{3} - \frac{11S\psi^{(0)}(S)}{3} + \frac{2S\psi^{(1)}(S)}{3} - \frac{4}{3}\pi S\cot(\pi S)\psi^{(0)}(S) - \frac{11}{6} \ , \end{aligned}$$

where $\psi^{(0)}(S) = \Gamma'(S)/\Gamma(S)$ is the digamma function. The above result is quite complicated, but fully explicit! Note that the terms proportional to $\log^3 T$, $\log^2 T$ are relatively simple. Then, $A^{(2)}_{Regge}(S)$ will contain terms proportional to $\log^6 T$, $\log^5 T, \ldots$, and so on, which can also be explicitly computed, but are very involved. In general, higher-order terms can be computed in terms of the CFT data of operators in the leading Regge trajectory. To each order in $1/R$, the result is complicated but completely explicit.

Actually, to any order in $1/R$ we can write the result in terms of derivatives acting on the Regge limit of the flat space Virasoro–Shapiro amplitude. For instance,

$$A^{(1)}_{Regge}(S,T) = \left(P^{(2)}_3(S)\partial_S^3 + P^{(2)}_2(S)\partial_S^2 + P^{(2)}_1(S)\partial_S + P^{(2)}_0(S)\right) A^{(0)}_{Regge}(S,T)\ , \tag{4.173}$$

with the second order polynomials given by

$$P^{(2)}_3(S) = -\frac{S^2}{6},\ \ P^{(2)}_2(S) = -\frac{4}{3}S,\ \ P^{(2)}_1(S) = \frac{11}{6}S - \frac{S^2}{2} - \frac{7}{3},\ \ P^{(2)}_0(S) = 2S^2\zeta(3) - \frac{7}{2}S^2 + \frac{3}{2}S + \frac{11}{6} \tag{4.174}$$

More generally, we obtain

$$A^{(k)}_{Regge}(S,T) = \left(P^{(2k)}_{3k}(S)\partial_S^{3k} + \cdots + P^{(2k)}_0(S)\right) A^{(0)}_{Regge}(S,T)\ . \tag{4.175}$$

From the expression (4.164) we can also infer some results to all orders in $1/R$. More precisely, to order $1/R^{2k}$ we get terms proportional to $\log^{3k} T$, $\log^{3k-1} T, \ldots$. We can then resum all leading logs, all subleading logs, and so on. To compute this explicitly, we need to compute $\hat{\mathcal{R}}(y)$ in a limit of large y, R^2, with y^3/R^2 fixed. For the first orders, we obtain

$$\hat{\mathcal{R}}(y) = \sum_{k=0} \frac{(-1)^k}{R^{2k}} \left(\frac{2m^4}{3}\right)^k \frac{1}{\Gamma(k+1)} \left(y^{3k} + \frac{2k(11-6k)}{5}\frac{y^{3k-1}}{m^2} + \cdots\right)\ . \tag{4.176}$$

With this at hand, we obtain, for the resummation of the leading logs,

$$A^{LL}_{Regge}(S,T) = A^{(0)}_{Regge}(S,T) \times e^{-\frac{4}{3}\frac{S^2}{R^2}\log^3 T}, \tag{4.177}$$

so that the leading logs exponentiate. This makes contact with our high-energy results (4.136). In particular, in the high-energy limit, the leading logs in T come from the SVMPL $\mathcal{L}_{000}(z_0)$ in (4.131). The limit considered in this Section, however, is much richer. For the subleading poles, we obtain

$$\begin{aligned} A^{SL}_{Regge}(S,T) =& A^{(0)}_{Regge}(S,T) \times \frac{1}{R^2} e^{-\frac{4}{3}\frac{S^2}{R^2}\log^3 T} \\ & \times \log^2 T \left(\frac{16S^3\log^3 T}{5R^2} + 2\pi S^2\cot(\pi S) + 4S^2\psi^{(0)}(S) - \frac{4S}{3} - 2i\pi S^2\right). \end{aligned} \tag{4.178}$$

To order $1/R^4$, this agrees with the explicit results obtained above. This expression, however, provides *all* order results in the limit of large R, T with $\log^3 T/R^2$ fixed!

4.4.3 Worldsheet perspective

The Regge limit of the AdS Virasoro-Shapiro amplitude can also be computed from the worldsheet representation found in [169]. In a large R expansion, the amplitude takes the form (4.62) with

$$G^{(0)}(S,T,z) = \frac{1}{3}\left(\frac{1}{U^2} + \frac{|z|^2}{S^2} + \frac{|1-z|^2}{T^2}\right) \tag{4.179}$$

reproducing the usual Virasoro-Shapiro amplitude in flat space.[26] At an arbitrary order $1/R^{2k}$, $G^{(k)}(S,T,z)$ is built from SVMPLs of weight $3k$ and rational functions of S,T of the form $P^{(2k)}(S,T)/U^2$ and crossing symmetric combinations, where $P^{(2k)}(S,T)$ are homogenous polynomials of degree $2k$.

We are interested in the Regge, large T, finite S, limit. At the level of the worldsheet integrand, the relevant limit is that of small $z,\bar{z}$ and large T, with $T\,z, T\,\bar{z}$ fixed. We can then define

$$\hat{G}(S,T,z) = \lim_{\epsilon\to 0} \frac{1}{\epsilon^2} G\left(S, \frac{T}{\epsilon}, \epsilon\, z\right) \ . \tag{4.180}$$

We assume the integrand remains finite as we take $\epsilon \to 0$. Let us make an important observation. In the AdS Virasoro–Shapiro program, the ansatz for the worldsheet theory is formulated directly at the level of the integrand (4.62). This approach carries an inherent ambiguity: one may always add terms that vanish upon integration without affecting the final amplitude. For a detailed analysis of these integrand-level degeneracies, see Section 4.2 of [169]. For the specific study of the Regge limit, for example, the finiteness of the integrand as ϵ goes to 0 is true for the first two curvature corrections explicitly obtained in [169,171], provided that these ambiguities are chosen appropriately. Of course, the final result does not depend on the choice of such ambiguities, but for this particular choice, one can take the Regge limit at the level of the integrand.

The Regge amplitude can then be computed as

$$A_{Regge}(S,T) = \int d^2z |z|^{-2S-2} |1-z|^{-2T-2} \hat{G}(S,T,z) \ . \tag{4.181}$$

The structure of $\hat{G}(S,T,z)$ in a $1/R$ expansion is relatively simple. At each order in $1/R$, $\hat{G}(S,T,z)$ is given by a polynomial in z, $\bar{z}$ and $\log|z|$. More precisely,

$$\hat{G}^{(k)}(S,T,z) = \frac{S^{2k}}{T^2} h_0^{(k)}(\log|z|) + \frac{S^{2k-1}}{T} h_1^{(k)}(z,\bar{z},\log|z|) + \cdots + \frac{T^{2k}}{S^2} h_{2k+2}^{(k)}(z,\bar{z},\log|z|) \ , \tag{4.182}$$

where $h_q^{(k)}(z,\bar{z},\log|z|)$ is a polynomial of degree $3k$ in $\log|z|$ and a homogenous symmetric polynomial of degree q in $z,\bar{z}$. For example, we can explicitly compute $\hat{G}(S,T,z)$ from the results in [169]. To

[26] This choice is equivalent to the one in (4.64), but more symmetric-looking.

leading order we find

$$\hat{G}^{(0)}(S,T,z) = \frac{2}{3}\frac{1}{T^2} + \frac{1}{3}\frac{|z|^2}{S^2} \ , \tag{4.183}$$

while for the first curvature correction

$$\begin{aligned}\hat{G}^{(1)}(S,T,z) =& \frac{2S^2 - 2ST(z+\bar{z}) + 5T^2 z\bar{z}}{18T^2}\log^3|z|^2 + \frac{(z+\bar{z})(T(z+\bar{z})-2S)}{6T}\log^2|z|^2 \\ &+ \frac{1}{4}\left(z^2+\bar{z}^2\right)\log|z|^2 - \frac{1}{4}\left(5z^2+5\bar{z}^2+z\bar{z}(4-8\zeta(3))\right) \ .\end{aligned}$$

The second order integrand in the Regge limit $\hat{G}^{(2)}(S,T,z)$ can also be computed from the results in [169], but the final answer is very lengthy and not particularly illuminating.

In general, in order to compute the integral from a given integrand, we can use the result [6]

$$I^{mn}(S,T) = \int d^2z|z|^{-2S-2}|1-z|^{-2T-2}z^m\bar{z}^n = \frac{\Gamma(m-S)\Gamma(-T)\Gamma(1-U-n)}{\Gamma(S+1-n)\Gamma(T+1)\Gamma(U+m)} \ , \tag{4.184}$$

with the following Regge behaviour

$$I^{mn}_{Regge}(S,T) = e^{i\pi S}\frac{\Gamma(-S)}{\Gamma(S+1)}T^{2S} \times \frac{\Gamma(S+1)^2}{\Gamma(-m+S+1)\Gamma(-n+S+1)}T^{-m-n} \ , \tag{4.185}$$

where we have assumed T is large with a small, negative imaginary part and we have pulled out the large T result for $m=n=0$, relevant for Virasoro–Shapiro in flat space. Note furthermore that the Virasoro–Shapiro amplitude in flat space is given by $A^{(0)}(S,T) = U^{-2}I^{00}(S,T)$. Insertions of $\log|z|^2$ in the integrand can be computed in terms of derivatives:

$$\int d^2z|z|^{-2S-2}|1-z|^{-2T-2}z^m\bar{z}^n\log^p|z|^2 = (-1)^p\ \partial_S^p\ I^{mn}(S,T) \ . \tag{4.186}$$

With these results at hand, we can compute $A_{Regge}(S,T)$ to order $1/R^4$ from the worldsheet perspective. The results are in full agreement with what we found in the previous section.

Let us now reverse our logic and look for integrands that lead to the correct result in the Regge limit, to all orders in $1/R$. For a given solution, at order $1/R^{2k}$, we always have the freedom to add insertions $K^{(k)}(S,T,z)$ in the "kernel" such that they integrate to zero:

$$\int d^2z|z|^{-2S-2}|1-z|^{-2T-2}K^{(k)}(S,T,z) = 0 \ . \tag{4.187}$$

Given the structure (4.175), we look for integrands such that

$$\frac{1}{T^2}\int d^2z|z|^{-2S-2}|1-z|^{-2T-2}F^{(k)}(S,T,z) = \left(P^{(2k)}_{3k}(S)\partial_S^{3k} + \cdots + P^{(2k)}_0(S)\right)A^{(0)}_{Regge}(S) \ . \tag{4.188}$$

It turns out that it is always possible to find a solution with the structure (4.182), namely[27]

$$F^{(k)}(S,T,z) = S^{2k} f_0^{(k)}(\log|z|) + S^{2k-1} T f_1^{(k)}(z,\overline{z},\log|z|) + \cdots + \frac{T^{2k+2}}{S^2} f_{2k+2}^{(k)}(z,\overline{z},\log|z|) \ , \quad (4.189)$$

where the functions $f_q^{(k)}(z,\overline{z},\log|z|)$ are homogenous polynomials of degree $3k$ in $\log|z|$, the single-valued logarithm, as a consequence of the number of derivatives in (4.175). This structure is fully consistent, and provides independent evidence, for the structure proposed in [169–171]!

However, these polynomials are not fully fixed, due to the possibility of adding kernels, as explained above. In order to find an integrand systematically, let us instead fix the kernel ambiguity by considering the following family of integrands:

$$\frac{1}{T^2}\int d^2z |z|^{-J-4}|1-z|^{-2T-2} F\left(S,-\log|z|\right) \simeq -F(S,\partial_J) e^{i\pi\frac{J}{2}} \frac{\Gamma(-1-J/2)}{\Gamma(2+J/2)} T^J \ , \quad (4.190)$$

where $F\left(S,-\log|z|\right)$ is a generic insertion that depends only on $\log|z|$ (*i.e.* it has no powers of $z,\overline{z}$ in a small $z,\overline{z}$ expansion), and the above equality is true in the Regge limit. In this expression, J is simply a parameter and S a spectator. Note that the factor $|z|^{-J-4}$ coincides with the usual factor $|z|^{-2S-2}$ at leading order in $1/R$ upon setting $J=J^*(S)$, which we will do momentarily. With this family of insertions, we get the following functional equation in order to reproduce the correct result in the Regge limit, namely Equation (4.164):

$$\boxed{-F(S,\partial_J) e^{i\pi\frac{J}{2}} \frac{\Gamma(-1-J/2)}{\Gamma(2+J/2)} T^J \Big|_{J=J^*(S)} =: \hat{\mathcal{R}}\left(\partial_J \frac{1}{\beta(J)}\right) : \frac{1+(-1)^J}{2\sin(\pi J)} \mathcal{C}(J) \frac{1}{\beta(J)} T^J \Big|_{J=J^*(S)}} \quad (4.191)$$

On both sides of this equation, one should first take derivatives with respect to J and then set $J=J^*(S)$. This should be seen as an equation for the insertion $F(S,\partial_J)$ in terms of $\hat{\mathcal{R}}(y)$, which can be explicitly computed to any order, and the CFT data for the operators in the leading Regge trajectory. To any order in $1/R$, this equation admits a unique solution. Indeed, to order $1/R^{2k}$, $\hat{\mathcal{R}}(y)$ is a polynomial of degree $3k$. Hence, $F(S,\partial_J)$ is a polynomial of degree $3k$ in ∂_J:

$$F(S,\partial_J)|_{1/R^{2k}} = f_{3k}(S)\partial_J^{3k} + \cdots + f_1(S)\partial_J + f_0(S) \ . \quad (4.192)$$

The coefficients of this polynomial are uniquely fixed by matching the powers of $\log T$ on both sides of the functional equation (4.191).

[27] Indeed, the insertion of positive powers of $z,\overline{z}$ produces polynomials in S times the flat space Virasoro–Shapiro amplitude in the Regge limit, see (4.184), so that the choice below leads to an overcomplete basis of polynomials of degree $2k$ in S, upon integration, multiplying each power of $\log T$.

For clarity of notation, let us define

$$G(\partial_J) \equiv : \hat{\mathcal{R}}\left(\partial_J \frac{1}{\beta(J)}\right) : , \quad -e^{i\pi\frac{J}{2}}\frac{\Gamma(-1-J/2)}{\Gamma(2+J/2)} \equiv h(J) \ , \quad \frac{1+(-1)^J}{2\sin(\pi J)}\mathcal{C}(J)\frac{1}{\beta(J)} \equiv g(J) \ . \quad (4.193)$$

We are looking for $F(S,\partial_J)$ such that

$$F(S,\partial_J)h(J)T^J\big|_{J=J^*(S)} = G(\partial_J)g(J)T^J\big|_{J=J^*(S)} \ , \quad (4.194)$$

but because this equation is valid for all T, we can write an operator solution:

$$F(S,\partial_J) = G(\partial_J)\frac{g(J)}{h(J)}\bigg|_{J=J^*(S)} \ . \quad (4.195)$$

On the RHS, all the derivatives ∂_J are on the left (due to the normal ordering in $:\hat{\mathcal{R}}(\partial_J\frac{1}{\beta(J)}):$). At any order in $1/R$, there is a finite number of such derivatives. We then move all derivatives to the right (as operators), and then set $J = J^*(S)$. This gives $F(S,\partial_J)$.

As an example, we can compute $F(S,\partial_J)$ to order $1/R^2$. We can do this by plugging all the ingredients, given Section 4.4.2, to this order. We find

$$F(S,\partial_J) = 1 + \frac{1}{R^2}\left(-\frac{4}{3}S^2\partial_J^3 - \frac{16}{3}S\partial_J^2 - \frac{32}{9}\partial_J + 2\zeta(3)S^2 - \frac{7}{2}S^2 + \frac{3}{2}S + \frac{11}{6}\right) + \cdots \ . \quad (4.196)$$

The resulting insertion does not have the structure (4.182). The reason for this is that we fixed the kernel ambiguity by forbidding positive powers of $z,\bar{z}$ in a small $z,\bar{z}$ expansion. Relaxing this condition, we can make the insertion consistent with (4.182), as seen above. Both solutions differ by a kernel that integrates to zero, of course.

As another example, let us find the insertion $F(S,-\log|z|)$ that reproduces the leading $\log^{3k} T$ to each order $1/R^{2k}$. In this approximation, we can set $h(J)$ and $g(J)$ to their flat space values,

$$\frac{g(J)}{h(J)} = 1 + O\left(\frac{1}{R^2}\right) \ , \quad (4.197)$$

together with $\beta(J) = 2 + \cdots$, $m^2(J^*) = 4S + \cdots$. From (4.176), keeping only the leading order terms at each order in $1/R$, we then find

$$F(\partial_J) = : \hat{\mathcal{R}}\left(\frac{1}{2}\partial_J\right) : \simeq e^{-\frac{4S^2}{3R^2}\partial_J^3} \ . \quad (4.198)$$

The corresponding worldsheet integral is

$$\frac{1}{T^2}\int d^2z|z|^{-J^*(S)-4}|1-z|^{-2T-2}e^{\frac{4S^2}{3R^2}\log^3|z|} \ , \quad (4.199)$$

where in this approximation we can take $J^*(S) = 2S - 2$. To this solution, we can add an integrand that integrates to zero. More precisely, combining this result with the proposal in [169] leads to[28]

$$\begin{aligned}
A^{LL}_{Regge}(S,T) = \quad & \frac{1}{T^2}\int d^2z|z|^{-2S-2}|1-z|^{-2T-2}e^{\frac{4S^2}{3R^2}\log^3|z|} \times \\
& \times\left(f_0\left(\frac{S^2}{R^2}\log^3|z|\right) + (z+\bar{z})\frac{T}{S}f_1\left(\frac{S^2}{R^2}\log^3|z|\right) + z\bar{z}\frac{T^2}{S^2}f_2\left(\frac{S^2}{R^2}\log^3|z|\right)\right) \ ,
\end{aligned}$$

where requiring the correct leading order logs to all orders in $1/R$, in the Regge limit, implies

$$f_0\left(\frac{S^2}{R^2}\log^3|z|\right) + 2f_1\left(\frac{S^2}{R^2}\log^3|z|\right) + f_2\left(\frac{S^2}{R^2}\log^3|z|\right) = 1 \ . \tag{4.200}$$

This is a prediction for the integrand, to *all* orders in $1/R$, in the limit of large R, small z with $\log^3|z|/R^2$ fixed.

[28]In particular, note that for the leading logs at each order in $1/R$ the series in $z, \bar{z}$ truncates to second order and does not contain $z^2 + \bar{z}^2$. This is a consequence of the specific structure of the proposal in [169].

Chapter 5

Stringy KLT Relations on AdS

After deriving higher-point holographic correlators in both maximally and half-maximally supersymmetric field theories, we turned our attention to the α'-corrections for four-graviton scattering on $AdS_5 \times S^5$. In this final Chapter, we broaden our scope to the investigation of the mathematical structure governing string scattering in AdS backgrounds, including the open-string sector. More precisely, the open-string building blocks will teach us about closed-string amplitudes, now in AdS, in the spirit of [4]. Indeed, our flagship result is an AdS analogue of the KLT relations. Concretely, we will show that the elementary worldsheet integrals, $J_w(s,t)$ (open strings) and $I_w(s,t)$ (closed strings), are related by

$$I_w(s,t) = \sum_{w_1,w_2} J_{w_1}(s,t)\, K_w^{w_1,w_2}(s,t)\, J_{w_2}(s,t) \ , \tag{5.1}$$

where the Kernel can be computed exactly.

Before presenting our results, one last time, let us review some relevant results in flat space. We begin by specialising to the field-theoretic formulation of the double-copy construction presented in the Introduction, whose structure —in unifying gauge and gravity theories at the level of perturbative amplitudes— is essentially the same as in string theory, which will ultimately be our main framework. The prescription is to take colour-ordered tree-level amplitudes from two "single-copy" theories and assemble them into the full non-colour-ordered tree-level amplitude of the "double-copy" theory. Schematically:[1]

$$\underbrace{A_n^{L\otimes R}}_{\text{double-copy}} = \sum_{\alpha,\beta} \underbrace{A_n^L[\alpha]}_{\text{single copy}} \quad K_n[\alpha|\beta] \quad \underbrace{A_n^R[\beta]}_{\text{single copy}} \quad . \tag{5.2}$$

Here, α and β each label a particular cyclic ordering of the external legs. By working with colour-ordered amplitudes $A_n[\alpha]$, one separates the purely kinematic dependence from the non-abelian colour algebra, as we did in (3.14). Each $A_n[\alpha]$ is computed by summing only over those planar

[1] We assume all the states to be massless and in the adjoint representation of colour groups.

diagrams whose external states appear in the fixed cyclic sequence α. The double-copy sum then recombines these kinematic building blocks to produce the full colour-blind tree-level amplitude. $K_n[\alpha|\beta]$ is a universal function of the Mandelstam variables, called the *Kernel*. It plays two crucial roles: first, its zeros cancel the double poles that arise in the naive product of two gauge theory amplitudes; second, it supplies the missing simple poles required for the correct factorisation of the gravity amplitude. This non-trivial cancellation of spurious singularities makes the very existence of the Kernel remarkable. Moreover, it can be defined for any number of external legs n, even when considering the full string theory [4].

Notice that the resulting double-copy amplitude is independent of the particular colour orderings α, β. Accordingly, the double-copy defines a bilinear (multiplicative) map between field theories,

$$(\text{theory})_L \otimes (\text{theory})_R = (\text{theory})_{L\otimes R} \ , \tag{5.3}$$

where the multiplication rule is given by the Kernel. Although a full treatment lies beyond this Chapter, one can define the so-called "KLT algebra" that characterises which single-copy theories admit a consistent double-copy construction, as well as the most general form of the Kernel [186]. It turns out that, in the "field-theory KLT" (or double-copy) map, the entries of the *inverse* of the Kernel are the colour-ordered amplitudes of the cubic bi-adjoint scalar (BAS) theory [187]. This surprising dual role makes the Kernel itself a highly non-trivial and illuminating object. In [188, 189], Mizera showed that the inverse of the stringy KLT Kernel can be understood as the set of α'–deformed amplitudes of BAS. Equivalently, the simplest stringy extension of the BAS amplitudes is precisely the inverse KLT Kernel itself, which encodes the scattering of cubic scalars with all order α'-corrections.[2]

To summarise, the Kernel encodes how disparate "single-copy" theories are knitted together into a unified double-copy framework, and, through its role as the inverse of a BAS amplitude, carries its own rich structure. The physical significance of the connections it reveals between different models remains an open area of research, as does the generalisation of this algebraic machinery to curved backgrounds. In this Chapter, we shall propose a concrete prescription for lifting the stringy KLT construction into a full *string-theoretic* setting on *AdS*. Indeed, so far, double-copy relations in *AdS* had been restricted to the supergravity regime only. See for instance [75–79].

We will first study the building blocks of string amplitudes in *AdS* to gently introduce the reader to the right formalism for the KLT relations. Let us just preview that, even in our case, the *inverse* of the Kernel will have a special role, and take a very elegant and simple form. Our results are naturally reminiscent of what was found in flat space [188, 189], where it was shown that intersection

[2] Actually, it has been recently shown how all the stringy amplitudes for scalars and pions come from the inverse of the KLT Kernel on α'-shifted kinematics [190]. See also references therein for different perspectives on this universality of amplitudes.

theory provides a geometric interpretation of the KLT relations. In particular, the inverse of the KLT Kernel in flat space can be interpreted as intersection numbers of twisted cycles. This aspect will not be covered in this thesis.

5.1 Preliminary mathematical aspects of string amplitudes

5.1.1 Flat space

The Veneziano amplitude [191] for the scattering of four open-string tachyons is given by the Euler beta function[3]

$$\beta(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}dx = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \ . \tag{5.4}$$

While the integral converges for $Re(s) > 0$, $Re(t) > 0$, the RHS can be continued beyond this region. The closed-string amplitude for the scattering of four tachyons in flat space, the Virasoro–Shapiro amplitude, is given by the complex beta function

$$\beta_{\mathbb{C}}(s,t) = \int |z|^{2s-2}|1-z|^{2t-2}d^2z = \frac{\Gamma(s)\Gamma(t)\Gamma(1-s-t)}{\Gamma(s+t)\Gamma(1-s)\Gamma(1-t)} \ . \tag{5.5}$$

Notice that we do not have the prefactor $1/(s+t)^2$ as in (4.46). This is because here we are considering tachyons ($s+t+u=1$), instead of gravitons ($s+t+u=0$). While the integral converges for $Re(s) > 0$, $Re(t) > 0$, $Re(1-s-t) > 0$, the RHS can be continued beyond this region.

These two functions satisfy various functional equations and relations. For instance, under shifts in the Mandelstam variables,

$$\beta(s+1,t) = \frac{s}{s+t}\beta(s,t), \quad \beta(s,t+1) = \frac{t}{s+t}\beta(s,t) \ . \tag{5.6}$$

Another beautiful functional equation is

$$\beta(s,t)\beta(-s,-t) = -\frac{\pi(s+t)(\cot(\pi s)+\cot(\pi t))}{st} \ , \tag{5.7}$$

which can be shown to be related to Poincaré duality [192].

The Euler and complex beta functions are intimately connected. In a small s,t expansion, they are related by a single-valued map

$$\beta_{\mathbb{C}}(s,t) = sv(\beta(s,t)) \ , \tag{5.8}$$

[3]Notice that there is a small change of convention compared to Chapter 4: we map the Mandelstam variables S and T to $-s$ and $-t$ to align with the number-theoretic literature.

where sv acts term by term and is such that

$$sv(\zeta(2n)) = \zeta_{sv}(2n) = 0 \ , \quad sv(\zeta(2n+1)) = \zeta_{sv}(2n+1) = 2\zeta(2n+1) \ , \quad n = 1, 2, \dots \ . \tag{5.9}$$

The implications of this relation away from $s = t = 0$ are not clear, but it can be shown [192, 193] that they are equivalent to

$$\boxed{\beta_{\mathbb{C}}(s,t) = \frac{\sin(\pi s)\sin(\pi t)}{\pi \sin(\pi(s+t))}\beta(s,t)^2} \tag{5.10}$$

which are the KLT relations [4]. This interplay between open- and closed-tree-level string amplitudes in flat space goes beyond four points. It is now understood that closed-string amplitudes, of any multiplicity, are the single-valued projection of open-string amplitudes. This was initially conjectured in [194], and then shown in [6, 135, 192, 195] using different methods. See also [196–198].

5.1.2 Going into AdS

We have seen that different techniques can be combined to advance the computation of massless tree-level amplitudes for closed strings in AdS [159, 169–171, 173, 174, 199]. Recently, progress has also been made for open strings in AdS [200, 201]. In both cases, the ansatz for the amplitude involves polylogarithmic functions: multiple polylogarithms (MPLs) for open strings, and their single-valued counterparts (SVMPLs) for closed strings. This leads us to consider the natural generalisations of the Euler and complex beta functions:

$$\begin{aligned} J_w(s,t) &= \int_0^1 x^{s-1}(1-x)^{t-1} L_w(x) dx \ , \\ I_w(s,t) &= \int |z|^{2s-2}|1-z|^{2t-2}\mathcal{L}_w(z) d^2z \ , \end{aligned} \tag{5.11}$$

where $L_w(x)$ is the MPL labelled by the word w and $\mathcal{L}_w(z)$ is the SVMPL labelled by the word w. For the empty word $w = e$ we obtain back the Euler and complex beta functions

$$J_e(s,t) = \beta(s,t) \ , \quad I_e(s,t) = \beta_{\mathbb{C}}(s,t) \ . \tag{5.12}$$

Before showing that these building blocks satisfy several properties and relations, let us review some additional details about (SV)MPLs, which will be relevant in the following. For further details, see for instance [6].

5.1.3 Generating functionals for polylogarithms

Let us introduce two non-commutative variables $\{e_0, e_1\}$, associated to the letters $\{0, 1\}$. We define the generating function of multiple polylogarithms:

$$L(e_0, e_1; x) = L_e(x) + L_0(x)e_0 + L_1(x)e_1 + L_{00}(x)e_0^2 + L_{01}(x)e_0e_1 + L_{10}(x)e_1e_0 + \dots \ , \tag{5.13}$$

where the order of the variables is important. The derivative relations for MPLs are then equivalent to the Knizhnik–Zamolodchikov (KZ) equation,

$$\frac{\partial}{\partial x} L(e_0, e_1; x) = \left(\frac{e_0}{x} + \frac{e_1}{x-1}\right) L(e_0, e_1; x) \ , \tag{5.14}$$

with the boundary condition $L(e_0, e_1; x) = e^{e_0 \log x}$ as $x \to 0$. The generating function for multiple zeta values, or in other words, the regularised generating function $L(e_0, e_1; x)$ at $x = 1$, is the Drinfeld associator

$$Z(e_0, e_1) = L(e_0, e_1; 1) \ , \tag{5.15}$$

which satisfies the duality relation

$$Z(e_0, e_1)Z(e_1, e_0) = 1 \ . \tag{5.16}$$

As a series expansion in e_0, e_1, it is given by

$$Z(e_0, e_1) = 1 - \zeta(2)[e_0, e_1] + \zeta(3)[[e_0, e_1], e_0 + e_1] + \cdots \ , \tag{5.17}$$

where $[e_0, e_1] = e_0e_1 - e_1e_0$ denotes the commutator. The Drinfeld associator can also be defined by

$$Z(e_0, e_1) = L(e_1, e_0; 1 - x)^{-1} L(e_0, e_1; x) \ . \tag{5.18}$$

Moving on to the single-valued case, we can define a generating function of SVMPLs as

$$\mathcal{L}(e_0, e_1; z) = \mathcal{L}_e(z) + \mathcal{L}_0(z)e_0 + \mathcal{L}_1(z)e_1 + \mathcal{L}_{00}(z)e_0^2 + \mathcal{L}_{01}(z)e_0e_1 + \mathcal{L}_{10}(z)e_1e_0 + \dots \ , \tag{5.19}$$

which is a formal solution to the holomorphic and anti-holomorphic KZ equations,

$$\begin{aligned} \frac{\partial}{\partial z} \mathcal{L}(e_0, e_1; z) &= \left(\frac{e_0}{z} + \frac{e_1}{z-1}\right) \mathcal{L}(e_0, e_1; z) \ , \\ \frac{\partial}{\partial \bar{z}} \mathcal{L}(e_0, e_1; z) &= \mathcal{L}(e_0, e_1; z) \left(\frac{e_0}{\bar{z}} + \frac{e_1'}{\bar{z}-1}\right) \ , \end{aligned} \tag{5.20}$$

where e_1' is a deformed variable which can be obtained from e_0, e_1 by solving

$$Z^R(e_0, e_1')e_1'(Z^R(e_0, e_1'))^{-1} = Z(e_0, e_1)^{-1}e_1 Z(e_0, e_1) \tag{5.21}$$

recursively. In an expansion, we obtain

$$e_1' = e_1 - 2\zeta(3)[e_0 + e_1, [e_1, [e_0, e_1]]] + \cdots \ . \tag{5.22}$$

The boundary conditions is $\mathcal{L}(e_0, e_1; z) = e^{e_0 \log |z|^2}$ as $z \to 0$. Its regularised value at $z = 1$ defines the Deligne associator

$$W(e_0, e_1) = \mathcal{L}(e_0, e_1; 1) \ , \tag{5.23}$$

which satisfies the duality relation

$$W(e_0, e_1)W(e_1, e_0) = 1 \ . \tag{5.24}$$

In an expansion, the Deligne associator is given by

$$W(e_0, e_1) = 1 + 2\zeta(3)[[e_0, e_1], e_0 + e_1] + \cdots \ . \tag{5.25}$$

The Deligne associator can also be defined by

$$W(e_0, e_1) = \mathcal{L}(e_1, e_0; 1 - z)^{-1}\mathcal{L}(e_0, e_1; z) \ . \tag{5.26}$$

SVMPLs can be constructed from MPLs in a recursive way [152]. In particular, we can write the generating functional for SVMPLs in terms of the one for MPLS as

$$\mathcal{L}(e_0, e_1; z) = L(e_0, e_1; z)L^R(e_0, e_1'; \overline{z}) \ , \tag{5.27}$$

where the Reverse operator R acts by reversing the order of the words labelling the components of $\mathcal{L}(e_0, e_1'; \overline{z})$, namely

$$L^R(e_0, e_1'; z) = 1 + \cdots + L_{0011}(z)e_1'e_1'e_0e_0 + \cdots \ . \tag{5.28}$$

The above construction implies that the Drinfeld and Deligne associators are related by

$$W(e_0, e_1) = Z(e_0, e_1)Z(e_0, e_1')^R \ . \tag{5.29}$$

5.2 Open-string amplitudes on AdS - building blocks

We start with the $J_w(s,t)$ integrals

$$J_w(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1} L_w(x) dx \ . \tag{5.30}$$

$$J_w(s,t) = \text{poles} + \sum_{p,q=0} s^p t^q \sum_{W \in 0^p \sqcup 1^q \sqcup w} \left(L_{0W}(1) - L_{1W}(1)\right) \ , \tag{5.31}$$

where notice that the evaluation of the multiple polylogarithms at unity gives the multiple zeta values. However, we are after analytic expressions valid for generic s,t. As explained around (5.13), it is convenient to introduce non-commutative variables e_0, e_1 - associated with the letters $0,1$ - and formally define the generating function of MPLs $L(e_0,e_1;x)$. Correspondingly, we introduce a generating function for the $J_w(s,t)$ integrals:

$$\begin{aligned} \mathcal{J}(s,t;e_0,e_1) &= \int_0^1 x^{s-1}(1-x)^{t-1} L(e_0,e_1;x) dx \\ &= J_e(s,t) + J_0(s,t)e_0 + J_1(s,t)e_1 + J_{00}(s,t)e_0^2 + J_{01}(s,t)e_0e_1 + \cdots \ . \end{aligned} \tag{5.32}$$

This generating function satisfies interesting properties. The shuffle identities, together with

$$\begin{aligned} \frac{\partial}{\partial s} x^s &= L_0(x) x^s \ , \\ \frac{\partial}{\partial t}(1-x)^t &= L_1(x)(1-x)^t \ , \end{aligned} \tag{5.33}$$

imply that

$$\partial_s J_w(s,t) = \sum_{w' \in 0 \sqcup w} J_{w'}(s,t) \ , \quad \partial_t J_w(s,t) = \sum_{w' \in 1 \sqcup w} J_{w'}(s,t) \ . \tag{5.34}$$

At the level of the generating function, we can write

$$\frac{\partial}{\partial s}\mathcal{J}(s,t;e_0,e_1) = \frac{\partial}{\partial e_0}\mathcal{J}(s,t;e_0,e_1) \ , \quad \frac{\partial}{\partial t}\mathcal{J}(s,t;e_0,e_1) = \frac{\partial}{\partial e_1}\mathcal{J}(s,t;e_0,e_1) \ , \tag{5.35}$$

where the non-commutative derivative is defined such that

$$\frac{\partial e_i}{\partial e_j} = \delta_{ij} \ , \tag{5.36}$$

together with the product rule

$$\frac{\partial}{\partial e_i}\left(f(e_0,e_1)g(e_0,e_1)\right) = \frac{\partial f(e_0,e_1)}{\partial e_i} g(e_0,e_1) + f(e_0,e_1)\frac{\partial g(e_0,e_1)}{\partial e_i} \ , \tag{5.37}$$

where the order of products needs to be respected.[4] Furthermore, from its integral definition, it follows that

$$\mathcal{J}(s,t;e_0,e_1) = \mathcal{J}(s+1,t;e_0,e_1) + \mathcal{J}(s,t+1;e_0,e_1)\ . \tag{5.38}$$

The transformation property under shifts in only one of the two Mandelstam variables can be found by combining the KZ equation (5.14) with integration by parts. In particular,

$$\begin{aligned} \mathcal{J}(s+1,t;e_0,e_1) &= \frac{1}{s+t+e_0+e_1}(s+e_0)\mathcal{J}(s,t;e_0,e_1)\ , \\ \mathcal{J}(s,t+1;e_0,e_1) &= \frac{1}{s+t+e_0+e_1}(t+e_1)\mathcal{J}(s,t;e_0,e_1)\ , \end{aligned} \tag{5.39}$$

where e_0 and e_1 are non-commutative variables, so expressions such as $\frac{1}{s+t+e_0+e_1}$ are defined via geometric series in the Mandelstam variables. The shift relations imply a hierarchy for the functions $J_w(s,t)$ as the weight w increases:

$$\begin{aligned} (s+t)\mathcal{J}(s+1,t;e_0,e_1) - s\mathcal{J}(s,t;e_0,e_1) &= \frac{1}{s+t+e_0+e_1}(e_0\,t - e_1\,s)\mathcal{J}(s,t;e_0,e_1)\ , \\ (s+t)\mathcal{J}(s,t+1;e_0,e_1) - t\mathcal{J}(s,t;e_0,e_1) &= \frac{1}{s+t+e_0+e_1}(e_1\,s - e_0\,t)\mathcal{J}(s,t;e_0,e_1)\ . \end{aligned} \tag{5.40}$$

These relations fix, in principle, $J_w(s,t)$ in terms of lower weight integrals, modulo the Euler beta function, which is the solution of the homogenous shift equations. The solution can be fixed by the behaviour as $s\to 0$. For $w \neq 0^p$, $J_w(s,t)$ is regular as $s \to 0$, while $J_{0^p}(s,t) = \frac{(-1)^p}{s^{p+1}} + \text{reg}$. In terms of generating functions:

$$\mathcal{J}(s,t;e_0,e_1) = \frac{1}{s+e_0} + \text{reg}, \quad \text{as } s\to 0\ . \tag{5.41}$$

As before, since e_0 is a non-commutative variable, $\frac{1}{s+e_0}$ is to be interpreted as an expansion $\frac{1}{s+e_0} = \frac{1}{s} - \frac{e_0}{s^2} + \cdots$, so that $s\to 0$ is a singular limit.

MPLs $L_w(x)$ are closed under the argument transformation $x \to 1-x$, and their transformation properties are governed by the Drinfeld associator (5.18). Since the region of integration defining the $J_w(s,t)$ integrals is invariant under such transformation, the integrals have a symmetry under the exchange $s \leftrightarrow t$, also governed by the Drinfeld associator. In terms of the generating functions:

$$\mathcal{J}(t,s;e_1,e_0) = \mathcal{J}(s,t;e_0,e_1)Z(e_1,e_0)\ . \tag{5.42}$$

[4] *e.g.* note that $\frac{\partial}{\partial e_0}(e_0e_1 - e_1e_0) = 0$, so that $\frac{\partial}{\partial e_0}f(e_0,e_1) = 0$ does not mean that $f(e_0,e_1)$ is independent of e_0.

This is consistent with the derivative relations (5.35) since

$$\frac{\partial}{\partial e_0} Z(e_0, e_1) = \frac{\partial}{\partial e_1} Z(e_0, e_1) = 0 \ . \tag{5.43}$$

Finally, let us mention that $\mathcal{J}(s, t; e_0, e_1)$ also satisfies a generalisation of Poincaré duality (5.7). This relation involves the *AdS* Kernel, and is discussed at the end of the Chapter.

5.2.1 Explicit results

Up to weight four, the $J_w(s,t)$ integrals can be computed analytically, generally in terms of generalised hypergeometric functions. At weight zero, we have

$$J_e(s,t) = \beta(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \ . \tag{5.44}$$

The relations (5.35) then fix the integrals at weight one:

$$J_0(s,t) = \partial_s J_e(s,t) \ , \quad J_1(s,t) = \partial_t J_e(s,t) \ . \tag{5.45}$$

At weight two, three combinations are fixed by the derivative relations (5.35):

$$J_{00}(s,t) = \frac{1}{2}\partial_s^2 J_e(s,t) \ , \quad J_{11}(s,t) = \frac{1}{2}\partial_t^2 J_e(s,t) \ , \quad J_{01}(s,t) + J_{10}(s,t) = \partial_s \partial_t J_e(s,t) \ . \tag{5.46}$$

In addition, we have

$$J_{01}(s,t) = -\frac{\Gamma(1+s)\Gamma(t)}{\Gamma(1+s+t)} {}_4F_3 \left(\begin{matrix} 1, 1, 1, \quad 1+s \\ 2, \quad 2, \ 1+s+t \end{matrix} ; 1 \right) \ , \tag{5.47}$$

which can be computed by direct integration. Here, we have used the general result

$$L_{0^{n-1}1}(x) = -\mathrm{Li}_n(x). \tag{5.48}$$

At weight three, there are 8 independent integrals. Two of them are given by

$$J_{001}(s,t) = -\frac{\Gamma(1+s)\Gamma(t)}{\Gamma(1+s+t)} {}_5F_4 \left(\begin{matrix} 1, 1, 1, 1, \quad 1+s \\ 2, \quad 2, \quad 2, 1+s+t \end{matrix} ; 1 \right) \ , \tag{5.49}$$

and

$$J_{110}(s,t) = J_{001}(t,s) + \zeta(2) J_0(t,s) + \zeta(3) J_e(t,s) \ . \tag{5.50}$$

The other six combinations can be written in terms of derivatives:

$$
\begin{aligned}
J_{000}(s,t) &= \tfrac{1}{6}\partial_s^3 J_e(s,t)\ , \quad & J_{111}(s,t) &= \tfrac{1}{6}\partial_t^3 J_e(s,t)\ , \\
\partial_s J_{11}(s,t) &= J_{011}(s,t)+J_{101}(s,t)+J_{110}(s,t)\ , \quad & \partial_t J_{00}(s,t) &= J_{100}(s,t)+J_{010}(s,t)+J_{001}(s,t)\ , \\
\partial_s J_{01}(s,t) &= 2J_{001}(s,t)+J_{010}(s,t)\ , \quad & \partial_t J_{10}(s,t) &= 2J_{110}(s,t)+J_{101}(s,t)\ .
\end{aligned}
\tag{5.51}
$$

As we increase the weight, the integrals become more and more complicated. We will show below that they are particular cases of Aomoto-Gelfand generalised hypergeometric functions. At weight four, they can still be written in terms of hypergeometric functions and their derivatives. Let us start with words with two zeros and two ones, which are the most complicated examples. There are six of them. The following combinations can be written in terms of derivatives of lower order integrals, already computed:

$$
\begin{aligned}
\partial_s J_{011}(s,t) &= 2J_{0011}(s,t)+J_{0101}(s,t)+J_{0110}(s,t)\ , \\
\partial_s J_{101}(s,t) &= J_{0101}(s,t)+2J_{1001}(s,t)+J_{1010}(s,t)\ , \\
\partial_s J_{110}(s,t) &= J_{0110}(s,t)+J_{1010}(s,t)+2J_{1100}(s,t)\ , \\
\partial_t J_{001}(s,t) &= J_{1001}(s,t)+J_{0101}(s,t)+2J_{0011}(s,t)\ , \\
\partial_t J_{010}(s,t) &= J_{1010}(s,t)+2J_{0110}(s,t)+J_{0101}(s,t)\ , \\
\partial_t J_{100}(s,t) &= 2J_{1100}(s,t)+J_{1010}(s,t)+J_{1001}(s,t)\ .
\end{aligned}
\tag{5.52}
$$

These relations are not all independent, and fix only four integrals. Two extra integrals can be computed as follows. Let us start with $J_{0011}(s,t)$. Plugging the series representation around zero for $L_{0011}(x)$ from (4.34), and integrating term by term, we obtain

$$
J_{0011}(s,t)=\sum_{\ell=1}^{\infty}\frac{\Gamma(s+\ell)\Gamma(t)}{\Gamma(\ell+s+t)}\frac{H(\ell-1)}{\ell^3}\ , \tag{5.53}
$$

where $H(\ell-1)$ is the harmonic number. To perform the sum we notice that

$$
H(\ell-1)=\gamma_e+\partial_\epsilon\left.\frac{\Gamma(\ell+\epsilon)}{\Gamma(\ell)}\right|_{\epsilon=0}\ . \tag{5.54}
$$

The sum over ℓ can now be performed and we obtain

$$
J_{0011}(s,t)=\frac{\Gamma(1+s)\Gamma(t)}{\Gamma(1+s+t)}\partial_\epsilon\, {}_5F_4\left.\left(\begin{matrix}1,1,1,1+\epsilon, & 1+s\\ 2,\quad 2, & 2,1+s+t\end{matrix};1\right)\right|_{\epsilon=0}\ . \tag{5.55}
$$

Using similar tricks we can compute all other integrals, for instance

$$J_{0101}(s,t) = \frac{\Gamma(1+s)\Gamma(t)}{2\Gamma(1+s+t)} \left(\partial_{\epsilon_1}^2 - \partial_{\epsilon_2}^2\right) {}_5F_4 \left(\begin{matrix} 1, 1, 1, 1+\epsilon_2, & 1+s \\ 2, \quad 2, & 1+\epsilon_1, 1+s+t \end{matrix} ; 1 \right) \Bigg|_{\epsilon_1=\epsilon_2=0} . \tag{5.56}$$

Together with the relations given above, this fixes all integrals at weight four with two zeros and two ones. Let us now focus on the integrals with three zeros and one one. Integrals with three ones and one zero are related to them by $s \leftrightarrow t$ symmetry (5.42). By direct integration, or by applying the trick above, we obtain

$$J_{0001}(s,t) = -\frac{\Gamma(1+s)\Gamma(t)}{\Gamma(1+s+t)} {}_6F_5 \left(\begin{matrix} 1, \cdots 1, & 1+s \\ 2, \cdots 2, & 1+s+t \end{matrix} ; 1 \right) . \tag{5.57}$$

The other three independent combinations can be fixed by the derivative relations, as can $J_{0000}(s,t)$ and $J_{1111}(s,t)$.

Before proceeding, let us mention that the $J_w(s,t)$ integrals have an interesting structure of poles. We can always use the shuffle relations to write polylogarithms in terms of $L_0(x)$ and polylogarithms whose label ends in the letter 1. For instance,

$$L_{10}(x) = -L_{01}(x) + L_1(x)L_0(x) . \tag{5.58}$$

In terms of the $J_w(s,t)$ integrals, this means that we can always focus on words ending in 1 and derivatives of those functions with respect to s. In compact notation, we have

$$J_{s_1,\cdots,s_d}(s,t) = \sum_{\ell_1>\ell_2>\cdots>\ell_d>0}^{\infty} \frac{\beta(s+\ell_1,t)}{\ell_1^{s_1}\ell_2^{s_2}\cdots\ell_d^{s_d}} . \tag{5.59}$$

This has poles at $s=-n$ and $t=-n$. We can write

$$\beta(s,t) = \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(t)}{\Gamma(n+1)\Gamma(t-n)} \frac{1}{s+n} = \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(s)}{\Gamma(n+1)\Gamma(s-n)} \frac{1}{t+n} . \tag{5.60}$$

For the poles in t, we find

$$J_{s_1,\cdots,s_d}(s,t) = \sum_{n=0}^{\infty} \frac{1}{t+n} \sum_{\ell_1>\ell_2>\cdots>\ell_d>0}^{\infty} \frac{(-1)^n\Gamma(s+\ell_1)}{\Gamma(n+1)\Gamma(s+\ell_1-n)\ell_1^{s_1}\ell_2^{s_2}\cdots\ell_d^{s_d}} . \tag{5.61}$$

For the poles in s, we find

$$J_{s_1,\cdots,s_d}(s,t) = \sum_{n=0}^{\infty} \sum_{\ell_1>\ell_2>\cdots>\ell_d>0}^{\infty} \frac{(-1)^n\Gamma(t)}{\Gamma(n+1)\Gamma(t-n)} \frac{1}{\ell_1^{s_1}\ell_2^{s_2}\cdots\ell_d^{s_d}} \frac{1}{s+n+\ell_1} . \tag{5.62}$$

Renaming $n+\ell_1=n'$ so that $\ell_1\leq n'$, we find

$$J_{s_1,\cdots,s_d}(s,t)=\sum_{n'=0}^{\infty}\frac{1}{s+n'}\sum_{\ell_1>\ell_2>\cdots>\ell_d>0}^{n'}\frac{(-1)^{n'-\ell_1}\Gamma(t)}{\Gamma(n'-\ell_1+1)\Gamma(t+\ell_1-n')}\frac{1}{\ell_1^{s_1}\ell_2^{s_2}\cdots\ell_d^{s_d}}\ . \tag{5.63}$$

These remarks are useful for applications to dispersive sum rules and in order to compute the spectrum of intermediate operators from the amplitude.

Low-energy expansion

In Equation (5.31) we have presented the low-energy expansion of the J-integrals, which mirrors the low-energy expansion of the Veneziano amplitude, uplifting it to AdS. Once the explicit results for the J-integrals are given, one can also derive the low-energy expansion from the hypergeometric functions that appear, which brings to light some interesting structures. Concretely, we first expand the hypergeometric function in a power series. Next, we expand the general term for small S and T, and finally, resum. As usual, the last step is non-trivial, especially if one does not already know the relevant sums involving (generalised) harmonic numbers. On the one hand, one can determine (potentially) previously unknown Euler sums by comparing with the result from (5.31), which is far more immediate. On the other hand, we can guess these sums in Mathematica by making an ansatz in terms of zeta values at fixed weight. In this approach, it is fundamental to include only the independent zeta values.

More specifically, we work with the Riemann zeta function,

$$\zeta_s=\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}\ . \tag{5.64}$$

Thanks to Euler, we know that $\zeta_2=\pi^2/6$. We need to consider the algebraic relations among the numbers $\pi,\zeta_3,\ldots,\zeta_{2n+1}$. It is widely believed that there are no relations: π and all the zetas at odd integers are believed to be algebraically independent. It is known that π is transcendental over $\mathbb{Q}$. Therefore, ζ_{2n} is transcendental for each $n\geq 1$, since

$$\zeta_{2n}=(-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!}\ , \tag{5.65}$$

where B_{2n} denote the $2n$-th Bernoulli number. In 1978, Apéry showed that ζ_3 is irrational [202].[5] Actually, infinitely many odd zetas are irrational. The picture changes upon the inclusion of *multiple* zeta values,

$$\zeta_{s_1,\ldots,s_k}=\zeta(s_1,\ldots,s_k)=\sum_{n_1>n_2>\cdots>n_k\geq 1}\frac{1}{n_1^{s_1}\cdots n_k^{s_k}}\ . \tag{5.66}$$

[5]The question of whether $\zeta(3)$ is transcendental is still open.

Very generally, the $\mathbb{Q}$-vector space spanned by multiple zeta values forms an algebra: the product of linear combinations of numbers of the form $\zeta_{s_1,\ldots,s_n}$ is again a linear combination of such numbers. Below we list a few examples of (multiple) zeta values of low weights, along with some of their linear relations:

$$\begin{aligned} &w=2\ , \quad \zeta_2 \\ &w=3\ , \quad \zeta_3,\, \zeta_{2,1}=\zeta_3 \\ &w=4\ , \quad \zeta_4,\, \zeta_{3,1}=\tfrac{1}{4}\zeta_4,\, \zeta_{2,2}=\tfrac{3}{4}\zeta_4,\, \zeta_{2,1,1}=\zeta_4=\tfrac{2}{5}\zeta_2^2\ . \end{aligned} \tag{5.67}$$

In our computation, we need to solve standard sums such as

$$\sum_{n=0}^{\infty}\left(\frac{H_n^{(2)}}{(1+n)^3}+\frac{H_n^{(3)}}{(1+n)^2}\right)=-\zeta_5+\zeta_2\,\zeta_3\ , \tag{5.68}$$

and more complicated ones. Starting from weight 8, the computation is more subtle as, for the first time, we have an independent multiple zeta value, namely $\zeta_{3,5}$, which satisfies the stuffle (or quasi-shuffle) product,

$$\zeta_{3,5}+\zeta_{5,3}=\zeta_3\,\zeta_5-\zeta_8\ . \tag{5.69}$$

5.2.2 Relation to Aomoto-Gelfand hypergeometric functions

Following the previous Section, without loss of generality, let us then focus on words ending in the letter 1. The representation of MPLs in terms of iterated integrals (4.31) leads to

$$J_{a_1a_2\cdots a_{r-1}1}(s,t)=\int_0^1 dx x^{s-1}(1-x)^{t-1}\int_0^x\frac{dx_1}{x_1-a_1}\int_0^{x_1}\frac{dx_2}{x_2-a_2}\cdots\frac{dx_r}{x_r-1}\ . \tag{5.70}$$

We can make a change of coordinates

$$u_0=x\ ,\quad u_n=\frac{x_n}{x_{n-1}}\ ,\quad \text{for } n=1,2,\cdots\ , \tag{5.71}$$

so that $u_n\in[0,1]$. The integral then becomes

$$J_{a_1a_2\cdots a_{r-1}1}(s,t)=\int_0^1\prod_{i=0}^{r}du_i u_0^{s-1}(1-u_0)^{t-1}\frac{u_0^r u_1^{r-1}u_2^{r-2}\cdots u_{r-2}^2u_{r-1}}{(u_0u_1-a_1)(u_0u_1u_2-a_2)\cdots(u_0\cdots u_r-1)}\ . \tag{5.72}$$

This is a particular case of Aomoto-Gelfand hypergeometric function of type $(r+2,2r+d+2)$ (see [203–205]),

$$\int_0^1\prod_{i=0}^{r}du_i u_i^{\alpha_i-1}(1-u_i)^{\beta_i-\alpha_i-1}(1-y_ru_0\cdots u_r)^{-\gamma_r}\cdots(1-y_1u_0u_1)^{-\gamma_1}\ , \tag{5.73}$$

where to make contact with (5.72) we choose $\gamma_i \neq 0$ only for $a_i = 1$. Here r is the weight, or length of the word, and d is the depth, or number of ones. The integrals we consider in this Chapter are obtained in the limit $y_i \to 1$. Let us consider for example $J_{0^{r-1}1}(s,t)$. In this case, $J_{0^{r-1}1}(s,t) = J_{0^{r-1}1}(s,t;1)$, where

$$J_{0^{r-1}1}(s,t;y) = -\int_0^1 \prod_{i=0}^{r} du_i \frac{u_0^s(1-u_0)^{t-1}}{1-yu_0u_1\cdots u_r} = -\frac{\Gamma(1+s)\Gamma(t)}{\Gamma(1+s+t)}\,{}_{r+2}F_{r+1}\left(\begin{matrix} 1 \cdots 1 & 1+s \\ 2 \cdots 2 & 1+s+t \end{matrix}; y\right) \tag{5.74}$$

is a standard generalised hypergeometric function.

The relation to Aomoto-Gelfand hypergeometric functions suggests the introduction of extra variables y_i such that at $\mathrm{y}_i = 1$ the integrals reduce to the expressions that appear in the open-string scattering problem. It is convenient to do this in the compact notation where

$$J_{s_1,s_2,\cdots,s_d}(s,t) = (-1)^d J_{0^{s_1-1}10^{s_2-1}\ldots 1}(s,t), \quad s_i \geq 1 \ . \tag{5.75}$$

By writing $L_{s_1,\cdots,s_d}(x)$ as a nested sum and integrating term by term, we obtain

$$J_{s_1,\cdots,s_d}(s,t) = \sum_{\ell_1>\ell_2>\cdots>\ell_d>0}^{\infty} \frac{\beta(s+\ell_1,t)}{\ell_1^{s_1}\ell_2^{s_2}\cdots\ell_d^{s_d}} \ , \tag{5.76}$$

where $\beta(s+\ell,t) = \frac{\Gamma(s+\ell)\Gamma(t)}{\Gamma(s+t+\ell)}$. We can now introduce extra variables y_i,

$$J_{s_1,\cdots,s_d}(s,t;\mathrm{y}_1,\cdots,\mathrm{y}_d) = \sum_{\ell_1>\ell_2>\cdots>\ell_d>0}^{\infty} \frac{\beta(s+\ell_1,t)\mathrm{y}_1^{\ell_1}\cdots\mathrm{y}_d^{\ell_d}}{\ell_1^{s_1}\ell_2^{s_2}\cdots\ell_d^{s_d}} \ , \tag{5.77}$$

so that

$$J_{s_1,s_2,\cdots,s_d}(s,t) = J_{s_1,s_2,\cdots,s_d}(s,t;1,\cdots,1) \ . \tag{5.78}$$

The functions $J_{s_1,s_2,\cdots,s_d}(s,t;\mathrm{y}_1,\cdots,\mathrm{y}_d)$ are generalised hypergeometric functions.[6] As such, they satisfy differential relations in the variables y_i. Indeed,

$$\mathrm{y}_q\frac{\partial}{\partial \mathrm{y}_q} J_{s_1,\cdots,s_d}(s,t;\mathrm{y}_1,\cdots,\mathrm{y}_d) = J_{s_1,\cdots,s_q-1,\cdots,s_d}(s,t;\mathrm{y}_1,\cdots,\mathrm{y}_d) \ , \tag{5.79}$$

so that the operator $\mathrm{y}_q\frac{\partial}{\partial \mathrm{y}_q}$ lowers the index s_q, and the total weight, by one. By repeated action of

[6] More precisely, they fall into the category of Horn's hypergeometric series, in general given by

$$\sum_{n_1,n_2,\cdots=0} c(n_1,n_2,\cdots) z_1^{n_1} z_2^{n_2}\cdots$$

with $c(n_1+1,n_2,\cdots)/c(n_1,n_2,\cdots), c(n_1,n_2+1,\cdots)/c(n_1,n_2,\cdots)$ and so on, rational functions of the n_i.

such operators, we reach

$$J_{0,\cdots,0}(s,t;\mathrm{y}_1,\cdots,\mathrm{y}_d) \equiv \sum_{\ell_1>\ell_2>\cdots>\ell_d>0}^{\infty} \beta(s+\ell_1,t)\mathrm{y}_1^{\ell_1}\cdots\mathrm{y}_d^{\ell_d} \ . \tag{5.80}$$

These are particular cases of Lauricella hypergeometric functions [206], and for the present case, they can be written as linear combinations of hypergeometric functions.

5.3 Closed-string amplitudes on AdS - building blocks

We now turn our attention to the building blocks of closed-string amplitudes on AdS,

$$I_w(s,t) = \int_{\mathbb{CP}_1} |z|^{2s-2}|1-z|^{2t-2}\mathcal{L}_w(z)d^2z \ , \tag{5.81}$$

where $\mathcal{L}_w(z)$ is the SVMPL labelled by the word w in the alphabet with letters $\{0,1\}$. The integral converges for $Re(s)>0$, $Re(t)>0$, $Re(s+t)<1$, but the explicit results we will find can be continued beyond that region. The integrals can be computed in a low-energy expansion [171],

$$I_w(s,t) = \text{poles} + \sum_{p,q=0} s^p t^q \sum_{W\in 0^p \sqcup\!\sqcup 1^q \sqcup\!\sqcup w} \left(\mathcal{L}_{0W}(1) - \mathcal{L}_{1W}(1)\right) \ , \tag{5.82}$$

where we now get single-valued multiple zeta values, matching what was discussed around (4.59). From (5.31) and (5.82) we can see that

$$\boxed{I_w(s,t) = sv\left(J_w(s,t)\right)} \tag{5.83}$$

since $\mathcal{L}_w(1) = sv\left(L_w(1)\right)$, so that the closed-string amplitude building blocks are the single-valued version of the open-string amplitude building blocks, as expected.

We are, however, after a quadratic relation à la KLT (5.10). We introduce a generating function for the SVMPLs, $\mathcal{L}(e_0,e_1;z)$, as in (5.19). Integrating it term by term leads to the generating function for the $I_w(s,t)$ integrals:

$$\mathcal{I}(s,t;e_0,e_1) = \int_{\mathbb{CP}_1} |z|^{2s-2}|1-z|^{2t-2}\mathcal{L}(e_0,e_1;z)d^2z \ . \tag{5.84}$$

We will start by discussing the general properties of $\mathcal{I}(e_0,e_1;z)$. Then, we will turn to its explicit computation and relation to the generating function $\mathcal{J}(e_0,e_1;z)$ appearing in the problem of open-string scattering on AdS.

The shuffle identities, together with

$$\frac{\partial}{\partial s}|z|^{2s} = \mathcal{L}_0(z)|z|^{2s} \,, \qquad \frac{\partial}{\partial t}|1-z|^{2t} = \mathcal{L}_1(z)|1-z|^{2t} \,, \tag{5.85}$$

imply that

$$\frac{\partial}{\partial s}\mathcal{I}(s,t;e_0,e_1) = \frac{\partial}{\partial e_0}\mathcal{I}(s,t;e_0,e_1) \,, \quad \frac{\partial}{\partial t}\mathcal{I}(s,t;e_0,e_1) = \frac{\partial}{\partial e_1}\mathcal{I}(s,t;e_0,e_1) \,. \tag{5.86}$$

Furthermore, the holomorphic and anti-holomorphic KZ equations for $\mathcal{L}(e_0,e_1;z)$ together with integration by parts lead to[7]

$$\begin{aligned} \mathcal{I}(s+1,t;e_0,e_1) &= \frac{1}{s+t+e_0+e_1}(s+e_0)\mathcal{I}(s,t;e_0,e_1)(s+e_0)\frac{1}{s+t+e_0+e_1'} \,, \\ \mathcal{I}(s,t+1;e_0,e_1) &= \frac{1}{s+t+e_0+e_1}(t+e_1)\mathcal{I}(s,t;e_0,e_1)(t+e_1')\frac{1}{s+t+e_0+e_1'} \,. \end{aligned} \tag{5.87}$$

SVMPLs are closed under $z \to 1-z$, and their transformation is governed by the Deligne associator. Performing a change of coordinates $z \to 1-z$ in (5.81) implies that

$$\mathcal{I}(t,s;e_1,e_0) = \mathcal{I}(s,t;e_0,e_1)W(e_1,e_0) \,. \tag{5.88}$$

The properties obtained so far mimic the properties for the building blocks of open-string amplitudes in AdS. In addition, it follows from (5.81) that the result for $I_w(s,t)$ is only sensitive to the symmetric part of $\mathcal{L}_w(z)$ under the exchange of $z \leftrightarrow \bar{z}$. Together with the explicit construction of SVMPLs, this leads to the relation

$$\mathcal{I}(s,t;e_0,e_1) = \mathcal{I}^R(s,t;e_0,e_1') \,, \tag{5.89}$$

where

$$\mathcal{I}^R(s,t;e_0,e_1') = I_e(s,t) + \cdots + I_{00010}e_0e_1'e_0e_0e_0 + \cdots \,, \tag{5.90}$$

i.e. the order of the letters in the word labelling $I_w(s,t)$ is reversed.

5.3.1 Explicit results

Let us now compute the generating function $\mathcal{I}(s,t;e_0,e_1)$ in a series expansion in the non-commutative variables e_0, e_1:

$$\mathcal{I}(s,t;e_0,e_1) = I_e(s,t) + \cdots \,, \tag{5.91}$$

[7] To be precise, we have obtained these relations by doing formal manipulations that take us outside the convergence region of the relevant integrals. We assume that these relations hold for the appropriate analytic continuation.

with $I_e(s,t) = \beta_{\mathbb{C}}(s,t)$ the complex beta function. The condition (5.89) imposes relations among different components. Up to weight four:

$$\begin{gathered}
I_{10}(s,t) = I_{01}(s,t) \ , \quad I_{100}(s,t) = I_{001}(s,t), \quad I_{110}(s,t) = I_{011}(s,t) \ , \\
I_{1110}(s,t) = I_{0111}(s,t) + 2\zeta(3) I_1(s,t) \ , \quad I_{1100}(s,t) = I_{0011}(s,t) - 2\zeta(3) I_1(s,t) \ , \\
I_{1101}(s,t) = I_{1011}(s,t) - 6\zeta(3) I_1(s,t) \ , \quad I_{1010}(s,t) = I_{0101}(s,t) + 4\zeta(3) I_1(s,t) \ , \\
I_{1000}(s,t) = I_{0001}(s,t) \ , \quad I_{0100}(s,t) = I_{0010}(s,t) \ ,
\end{gathered} \tag{5.92}$$

leaving three independent integrals at weight two, six at weight three and ten at weight four. Next, we impose the derivative relations (5.86). Up to weight two, this fixes the remaining integrals, in terms of the complex beta function, equivalently $I_e(s,t)$:

$$\begin{gathered}
I_0(s,t) = \partial_s I_e(s,t) \ , \quad I_1(s,t) = \partial_t I_e(s,t) \ , \\
I_{00}(s,t) = \frac{1}{2}\partial_s^2 I_e(s,t) \ , \quad I_{11}(s,t) = \frac{1}{2}\partial_t^2 I_e(s,t) \ , \quad I_{01}(s,t) = \frac{1}{2}\partial_s\partial_t I_e(s,t) \ .
\end{gathered} \tag{5.93}$$

At weight three, it leaves only two independent integrals, which we take to be $I_{001}(s,t)$ and $I_{011}(s,t)$:

$$\begin{gathered}
I_{000}(s,t) = \frac{1}{6}\partial_s^3 I_e(s,t) \ , \quad I_{111}(s,t) = \frac{1}{6}\partial_t^3 I_e(s,t) \ , \\
I_{101}(s,t) = -2 I_{011}(s,t) + \frac{1}{2}\partial_s\partial_t^2 I_e(s,t) \ , \quad I_{010}(s,t) = -2 I_{001}(s,t) + \frac{1}{2}\partial_s^2\partial_t I_e(s,t) \ .
\end{gathered} \tag{5.94}$$

At weight four, it turns out the relations (5.86) are quite powerful, and only one independent integral remains, which we take to be $I_{0101}(s,t)$:

$$\begin{aligned}
I_{0000}(s,t) &= \frac{1}{24}\partial_s^4 I_e(s,t) \ , \\
I_{1111}(s,t) &= \frac{1}{24}\partial_t^4 I_e(s,t) \ , \\
I_{1011}(s,t) &= -\frac{1}{2}\partial_t I_{011}(s,t) + \frac{1}{8}\partial_s\partial_t^3 I_e(s,t) + 3\zeta(3)\partial_t I_e(s,t) \ , \\
I_{0010}(s,t) &= -\frac{1}{2}\partial_s I_{001}(s,t) + \frac{1}{8}\partial_s^3\partial_t I_e(s,t) \ , \\
I_{0001}(s,t) &= \frac{1}{2}\partial_s I_{001}(s,t) - \frac{1}{24}\partial_s^3\partial_t I_e(s,t) \ , \\
I_{0110}(s,t) &= -I_{0101}(s,t) - \partial_t I_{001}(s,t) + \frac{1}{4}\partial_s^2\partial_t^2 I_e(s,t) - 2\zeta(3)\partial_t I_e(s,t) \ , \\
I_{1001}(s,t) &= -I_{0101}(s,t) - \partial_s I_{011}(s,t) + \frac{1}{4}\partial_s^2\partial_t^2 I_e(s,t) - 2\zeta(3)\partial_t I_e(s,t) \ , \\
I_{0011}(s,t) &= \frac{1}{2}\partial_s I_{011}(s,t) + \frac{1}{2}\partial_t I_{001}(s,t) - \frac{1}{8}\partial_s^2\partial_t^2 I_e(s,t) + \zeta(3)\partial_t I_e(s,t) \ , \\
I_{0111}(s,t) &= \frac{1}{2}\partial_t I_{011}(s,t) - \frac{1}{24}\partial_s\partial_t^3 I_e(s,t) - \zeta(3)\partial_t I_e(s,t) \ .
\end{aligned} \tag{5.95}$$

Our next task is to compute explicitly the independent integrals. We do so by extending the ideas of KLT [4] to the integration of SVMPLs, following [200]. In the Appendix to that paper, they considered integrals of the form

$$A_{\text{closed}}(s,t) = \int |z|^{2s-2}|1-z|^{2t-2} \sum_i^n F_i(z) G_i(\bar{z}) d^2 z \ , \tag{5.96}$$

where the inserted sum is finite and single-valued, but each independent term is not necessarily single-valued. By deforming the contours as in [4], it was shown that these integrals can be factorised into one-dimensional integrals:

$$A_{\text{closed}}(s,t) = \frac{1}{2\pi i} \sum_i^n \int_0^1 x^{s-1}(1-x)^{t-1} G_i(x) dx \int_1^\infty y^{s-1} \text{Disc}_1 \left[(1-y)^{t-1} F_i(y) \right] dy \ , \tag{5.97}$$

where the discontinuity across the real axis for $y > 1$ is defined as

$$\text{Disc}_1 \left[f(y) \right] = f(y+i\epsilon) - f(y-i\epsilon) \ , \quad y > 1 \ . \tag{5.98}$$

For $\sum_i^n F_i(z) G_i(\bar{z}) = 1$, we get

$$\text{Disc}_1 \left[(1-y)^{t-1} \right] = 2i \sin(\pi t)(y-1)^{t-1} \ , \tag{5.99}$$

and

$$A_{\text{closed}}(s,t) = \frac{1}{\pi} \sin(\pi t) \int_0^1 x^{s-1}(1-x)^{t-1} dx \int_1^\infty y^{s-1}(y-1)^{t-1} dy \ , \tag{5.100}$$

which is the KLT formula [4].

When we insert the SVMPLs $\mathcal{L}_w(z)$, we can still factorise the integrals into products of $1d$ integrals involving MPLs $L_w(x)$. These are almost identical to the integrals considered above, except that the second integral is in the range $y \in [1,\infty]$ and one needs to keep track of discontinuities. The resulting expressions are very lengthy, but the result can always be written as bilinears of the $1d$ $J_w(s,t)$ integrals. For example,

$$\begin{aligned}
I_{001}(s,t) \quad = \quad & \kappa(s,t) \left(J_{001}(s,t) J_e(s,t) + J_{100}(s,t) J_e(s,t) + J_{00}(s,t) J_1(s,t) + J_0(s,t) J_{10}(s,t) \right) \\
& + \partial_t \kappa(s,t) J_{00}(s,t) J_e(s,t) + \partial_s \kappa(s,t) (J_{10}(s,t) J_e(s,t) + J_0(s,t) J_1(s,t)) \\
& + \frac{1}{2} \partial_s \partial_t \kappa(s,t) J_0(s,t) J_e(s,t) + \frac{1}{2} \partial_s^2 \kappa(s,t) J_1(s,t) J_e(s,t) \\
& + \pi^2 \frac{2\cos^2(\pi(s+t)) - \cos(2\pi s) - \cos(2\pi t)}{4 \sin(\pi(s+t))^4} J_e(s,t) J_e(s,t) \ ,
\end{aligned} \tag{5.101}$$

where $\kappa(s,t) = \frac{\sin(\pi s)\sin(\pi t)}{\pi \sin(\pi(s+t))}$ is the KLT Kernel. This suggests writing our results in an illuminating way.

5.4 *AdS* KLT relations

Motivated by the KLT relations, the construction of SVMPLs from MPLs, and the results above, we propose the following relation between the building blocks of open- and closed-string amplitudes on *AdS*:

$$\boxed{\mathcal{I}(s,t;e_0,e_1) = \mathcal{J}(s,t;e_0,e_1)\,\mathcal{K}(s,t;e_0,e_1)\,\mathcal{J}^R(s,t;e_0,e_1')} \tag{5.102}$$

where $\mathcal{K}(s,t;e_0,e_1)$ admits an expansion in the non-commutative variables,

$$\mathcal{K}(s,t;e_0,e_1) = \kappa(s,t) + \kappa_0(s,t)e_0 + \kappa_1(s,t)e_1 + \cdots \ . \tag{5.103}$$

It starts with the KLT Kernel $\kappa(s,t)$, and contains only trigonometric functions in s,t. Indeed, because of the shift relations (5.39) and (5.87), the *AdS* KLT Kernel is periodic in both s and t,

$$\mathcal{K}(s+1,t;e_0,e_1) = \mathcal{K}(s,t+1;e_0,e_1) = \mathcal{K}(s,t;e_0,e_1) \ . \tag{5.104}$$

As a consequence of the derivative relations (5.35) and (5.86), the Kernel also satisfies

$$\partial_s\,\mathcal{K}(s,t;e_0,e_1) = \frac{\partial}{\partial e_0}\mathcal{K}(s,t;e_0,e_1) \ , \quad \partial_t\,\mathcal{K}(s,t;e_0,e_1) = \frac{\partial}{\partial e_1}\mathcal{K}(s,t;e_0,e_1) \ , \tag{5.105}$$

where we have used the fact that (recall $\frac{\partial}{\partial e_1}$ is a non-commutative derivative)[8]

$$\frac{\partial}{\partial e_1}e_1' = 1 \ . \tag{5.106}$$

In addition, (5.89) leads to

$$\mathcal{K}(s,t;e_0,e_1) = \mathcal{K}^R(s,t;e_0,e_1') \ . \tag{5.107}$$

Finally, the *AdS* Kernel also possesses a symmetry under the exchange of s and t, inherited from the respective symmetries for the $1d$ and $2d$ integrals:

$$\mathcal{K}(t,s;e_1,e_0) = Z(e_0,e_1)\mathcal{K}(s,t;e_0,e_1)Z(e_1,e_0) \ . \tag{5.108}$$

To a given order, one can compute the *AdS* Kernel by computing the generating functions $\mathcal{J}(s,t;e_0,e_1)$ and $\mathcal{I}(s,t;e_0,e_1)$ explicitly. The relation (5.102) hence has a unique solution for $\mathcal{K}(s,t;e_0,e_1)$, which can be computed order by order in e_0,e_1.

However, let us present another way, which is much more instructive and allows us to compute

[8]This follows from the fixed-point equation for e_1' (5.21) together with (5.43).

the AdS KLT Kernel to *all* orders. Consider the generating function

$$\begin{aligned}\mathcal{I}(s,t;e_0,e_1) &= \int |z|^{2s-2}|1-z|^{2t-2}\mathcal{L}(e_0,e_1;z)d^2z \\ &= \int |z|^{2s-2}|1-z|^{2t-2}L(e_0,e_1;z)L^R(e_0,e_1';\bar{z})d^2z \ .\end{aligned} \tag{5.109}$$

The holomorphic factorisation formula (5.97) leads to

$$\mathcal{I}(s,t;e_0,e_1) = \frac{1}{2\pi i}\int_1^\infty y^{s-1}\text{Disc}_1\left[(1-y)^{t-1}L(e_0,e_1;y)\right]\int_0^1 dx\, x^{s-1}(1-x)^{t-1}L^R(e_0,e_1';x) \ , \tag{5.110}$$

where the order of the two factors is important and recall that $\text{Disc}_1[f(y)] = f(y+i\epsilon) - f(y-i\epsilon)$. Note that the second factor is nothing but the generating functional $\mathcal{J}^R(s,t;e_0,e_1')$. Let us now focus on the first factor. In order to convert the integral to an integral in the correct range $[1,\infty] \to [0,1]$, consider

$$\left(\int_{-\infty}^0 dz + \int_0^1 dz + \int_1^\infty dz\right) z^{s-1}(1-z)^{t-1}F(z)\Big|_{z=x\pm i\epsilon} = 0 \ , \tag{5.111}$$

where in our case $F(z) = L(e_0,e_1;z)$ and the contour is just above or just below the real line. Furthermore, we have

$$\begin{aligned} z^{s-1}\big|_{z=x\pm i\epsilon} &= -e^{\pm i\pi s}(-x)^{s-1}, \quad x<0 \ , \\ (1-z)^{t-1}\big|_{z=x\pm i\epsilon} &= -e^{\mp i\pi t}(x-1)^{t-1}, \quad x>1 \ . \end{aligned} \tag{5.112}$$

The discontinuities of MPLs as we cross the real line, either for $x<0$ or $x>1$, are governed by monodromy matrices:

$$\begin{aligned} L(e_0,e_1;x+i\epsilon) &= L(e_0,e_1;x-i\epsilon)M_0 \ , \quad x<0 \ , \\ L(e_0,e_1;x-i\epsilon) &= L(e_0,e_1;x+i\epsilon)M_1 \ , \quad x>1 \ , \end{aligned} \tag{5.113}$$

where the monodromy matrices have been computed in [152] and are given by

$$M_0 = e^{2\pi i e_0} \ , \ M_1 = Z(e_1,e_0)e^{2\pi i e_1}Z(e_0,e_1) \ . \tag{5.114}$$

For clarity of notation, let us introduce three objects:

$$\begin{aligned} J^{(-)} &= \int_{-\infty}^0 dx(-x)^{s-1}(1-x)^{t-1}L(e_0,e_1;x-i\epsilon) \ , \\ J &= \int_0^1 dx x^{s-1}(1-x)^{t-1}L(e_0,e_1;x) \ , \\ J^{(+)} &= \int_1^\infty dx x^{s-1}(x-1)^{t-1}L(e_0,e_1;x+i\epsilon) \ , \end{aligned}$$

where we have suppressed the $(s,t;e_0,e_1)$ dependence. As a consequence of (5.111), they satisfy

$$\begin{aligned} -e^{i\pi s}J^{(-)}M_0 + J - e^{-i\pi t}J^{(+)} &= 0 \ , \\ -e^{-i\pi s}J^{(-)} + J - e^{i\pi t}J^{(+)}M_1 &= 0 \ , \end{aligned} \tag{5.115}$$

where the monodromy matrices do not commute with $J, J^{(-)}, J^{(+)}$. We can always write

$$e^{-i\pi t}J^{(+)} - e^{i\pi t}J^{(+)}M_1 = JK \ , \tag{5.116}$$

where K is invertible[9] and related to the AdS KLT Kernel by $\mathcal{K} = -\frac{1}{2\pi i}K$. Plugging this in the previous two relations, we find

$$\begin{aligned} -e^{i\pi s}J^{(-)}M_0 + J^{(+)}\left(-e^{-i\pi t} + e^{-i\pi t}K^{-1} - e^{i\pi t}M_1K^{-1}\right) &= 0 \ , \\ -e^{-i\pi s}J^{(-)} + J^{(+)}\left(-e^{i\pi t}M_1 + e^{-i\pi t}K^{-1} - e^{i\pi t}M_1K^{-1}\right) &= 0 \ . \end{aligned} \tag{5.117}$$

Since J^- and J^+ are invertible, K^{-1} should satisfy

$$e^{-i\pi s}\left(-e^{-i\pi t} + e^{-i\pi t}K^{-1} - e^{i\pi t}M_1K^{-1}\right) = e^{i\pi s}\left(-e^{i\pi t}M_1 + e^{-i\pi t}K^{-1} - e^{i\pi t}M_1K^{-1}\right)M_0 \ .$$

We can solve for K^{-1} and find

$$K^{-1} = 1 + \frac{e^{2i\pi s}M_0}{1-e^{2i\pi s}M_0} + \frac{e^{2i\pi t}M_1}{1-e^{2i\pi t}M_1} \ . \tag{5.118}$$

At zeroth order in the non-commutative variables, $M_0 = M_1 = 1$ and we get the inverse of the usual KLT Kernel ($\mathcal{K}^{-1} = -2\pi i K^{-1}$),

$$\mathcal{K}^{-1} = \pi(\cot(\pi s) + \cot(\pi t)) + \cdots \ . \tag{5.119}$$

At higher orders, and up to weight four, it precisely links the explicit integrals we computed. Plugging the explicit expressions for the monodromy matrices, we find

$$\boxed{\mathcal{K}^{-1} = \pi\left(\cot\left(\pi(s+e_0)\right) + Z(e_1,e_0)\cot\left(\pi(t+e_1)\right)Z(e_0,e_1)\right)} \tag{5.120}$$

valid to *all* orders.

[9] A function $f(e_0,e_1) = \alpha + \alpha_0 e_0 + \alpha_1 e_1 + \cdots$ with $\alpha \neq 0$ is always invertible.

5.5 Poincaré duality

We conclude this Chapter by generalising the relation (5.7) due to Poincaré duality. We consider

$$P(s,t;e_0,e_1) \equiv \mathcal{J}(s,t;e_0,e_1)\mathcal{K}(s,t;e_0,e_1)\mathcal{J}^R(-s,-t;-e_0,-e_1)\ , \tag{5.121}$$

and deduce properties of the function $P(s,t;e_0,e_1)$. In particular, let us consider shifts in s:

$$P(s+1,t;e_0,e_1) = \mathcal{J}(s+1,t;e_0,e_1)\mathcal{K}(s+1,t;e_0,e_1)\mathcal{J}^R(-s-1,-t;-e_0,-e_1)\ . \tag{5.122}$$

Then, let us also recall two key points: the shift relations (5.39),

$$\mathcal{J}(s+1,t;e_0,e_1) = \frac{1}{s+t+e_0+e_1}(s+e_0)\mathcal{J}(s,t;e_0,e_1)\ , \tag{5.123}$$

and the fact that $\mathcal{K}(s,t;e_0,e_1)$ is periodic. This leads to

$$P(s+1,t;e_0,e_1) = \frac{1}{s+t+e_0+e_1}(s+e_0)P(s,t;e_0,e_1)(s+t+1+e_0+e_1)\frac{1}{s+1+e_0}\ . \tag{5.124}$$

We can write both shifts in s and t as

$$\begin{aligned}(s+t+e_0+e_1)P(s+1,t;e_0,e_1)(s+1+e_0) &= (s+e_0)P(s,t;e_0,e_1)(s+t+1+e_0+e_1)\ ,\\ (s+t+e_0+e_1)P(s,t+1;e_0,e_1)(t+1+e_1) &= (t+e_1)P(s,t;e_0,e_1)(s+t+1+e_0+e_1)\ .\end{aligned}$$

We also have symmetry under the exchange of s and t, see (5.42) and (5.108). It follows that

$$P(s,t;e_0,e_1) = P(t,s;e_1,e_0)\ . \tag{5.125}$$

We have used the fact that $L(e_0,e_1,x)^{-1} = L^R(-e_0,-e_1,x)$. See [207]. In particular, evaluating this at $x=1$ implies a relation between the Drinfeld associator with *reversed* non-commutative variables and $e_i \to -e_i$. Using the symmetry (5.125) together with the recursion relations, and the result at zeroth order, we obtain

$$\boxed{\mathcal{J}(s,t;e_0,e_1)\mathcal{K}(s,t;e_0,e_1)\mathcal{J}^R(-s,-t;-e_0,-e_1) = -\frac{1}{s+e_0} - \frac{1}{t+e_1}} \tag{5.126}$$

which generalise the relation in flat space (5.7), that we copy below to appreciate the comparison:

$$\beta(s,t)\mathcal{K}_0(s,t)\beta(-s,-t) = -\frac{1}{s} - \frac{1}{t}\ , \tag{5.127}$$

where we denote the flat space Kernel by $\mathcal{K}_0(s,t)$. Notice that the order of the terms in the LHS of (5.126) is crucial, contrary to (5.127), making our result highly non-trivial.

Chapter 6

Discussion and outlook

This dissertation has been guided by a sustained drive to integrate classic CFT methods with advanced mathematical frameworks, from special functions to number–theoretic techniques, which are deeply embedded in Theoretical Physics. By bringing these different methods together, we aimed to uncover the mathematical structure of scattering amplitudes in AdS. This relates to the question of extracting CFT data from strongly coupled theories. Below, we offer a summary of our principal findings, placing particular emphasis on the string amplitude sector in AdS, which forms the heart of this thesis. We then highlight cross-cutting techniques and observations, and propose several avenues for future research, some of which are already underway.

6.1 Main results

In Chapter 3, we presented two bootstrap strategies for computing the six-point correlation functions of supergluons and supergravitons in AdS. The first method relies exclusively on the flat space limit and factorisation of the Mellin amplitude, with only minimal input from supersymmetry. This completely fixed the answer in AdS. The second approach decomposes the correlators into distinct components and fixes each in turn by applying the chiral algebra twist and the light-cone OPE limits. Our results hold in the field-theory limit of the AdS/CFT correspondence.

In Chapter 4, we then incorporated stringy corrections to the four-point graviton amplitude in $AdS_5 \times S^5$, with particular emphasis on its high-energy regimes, where exact results yield deep insight into the full theory. For the limit of large S, T, we showed that the complete answer to *all* orders in S/R is fixed by the subleading correction, and curvature effects exponentiate (4.136). This is reproduced to all orders by a classical scattering computation in AdS. In the Regge limit of large T and finite S, we expressed the AdS Virasoro–Shapiro amplitude in terms of CFT data for the leading Regge trajectory (4.164), echoing what happens in flat space. To *all* orders in $1/R$, the amplitude can be written explicitly as derivatives acting on the flat space Virasoro–Shapiro amplitude in the Regge limit, with leading logarithms exponentiating (4.177) and matching the high-energy result. We could furthermore resum subleading contributions in (4.178) to *all* orders in the limit of large R, T with $\log^3 T/R^2$ held fixed. These two complementary high-energy limits illuminate the worldsheet: they open a pathway towards a direct worldsheet description of strings on $AdS_5 \times S^5$. In particular, the richer Regge regime yields specific constraints. Our functional equation (4.191) determines the worldsheet integrand in the Regge limit to *all* orders in $1/R$, providing compelling evidence for the full proposal. This equation selects families of integrands that reproduce the correct Regge behaviour and manifestly ensure single-valuedness around the worldsheet point $z = 1$.

Finally, in Chapter 5, we undertook a detailed study of the mathematical structure of string amplitudes in AdS, addressing both closed strings (such as the Virasoro–Shapiro amplitude) and open strings. This investigation was originally driven by the curiosity to carry out the worldsheet integrals in (4.69), and obtain the AdS amplitude as an explicit function of the kinematic variables S and T. Indeed, before our work, all (finite-energy) expressions were known only at the level of the integrand. While Appendix F of [200] introduced preliminary guidance on integrating certain pieces of the worldsheet integrand, our work [208] presents the first complete theory. Our results allow us to compute in an analytic/closed form the first AdS curvature corrections for all results available in the literature. Notice also that an advantage of our expressions, as opposed to their integral form, is that they can be analytically continued away from the region of convergence of the integrals. This opens up the possibility of studying the amplitudes in the complex s, t plane more directly. The explicit expression for the AdS curvature corrections $A^{(k)}(S, T)$ in (4.58) involves a variety of special functions. The building blocks are laid out in detail in Chapter 5 as infinite towers of worldsheet integrals generalising the Euler and complex Beta functions. This aims to provide a more schematic and rigorous mathematical language. In particular, we have developed a new machinery to compute certain classes of integrals over the Riemann sphere. Exact examples in the literature include single-valued hypergeometric functions and generalisations of those. See for example [135, 157, 205, 209]. Our results complement those and can be useful in other contexts involving worldsheet computations. In the language of generating functions, we could further demonstrate that the open- and closed-string building blocks in AdS obey KLT relations, whose Kernel can be computed *exactly*, as shown in (5.120).

6.1.1 General observations

A recurring theme throughout this thesis is the special significance of the *flat space limit*: it provides both the starting point for novel methods and the guiding template for extending amplitude properties to curved backgrounds. Our two-stage algorithm presented in [89] begins with the flat space result, crucially using the transverse polarisation choice of (3.25), and then promotes it to AdS by enforcing the correct factorisation into lower-point amplitudes. This very same two-step procedure also underpins our study of string amplitudes in AdS: we take the Borel transform of the Mellin amplitude (4.57), thereby implementing Penedones' flat space limit [48] while keeping all $1/R$ corrections, which define the AdS corrections. The full power and significance of Borel space are still not completely understood. It would be worthwhile to develop a deeper understanding of the single-valuedness property along with the special role of the Borel transform. At leading order, it realises the flat space limit, but what further insights might it unveil? How can we forge a clearer connection with the physical CFT correlator? These questions are still open. The subsequent Section introduces other possible trajectories.

6.2 Future directions

6.2.1 Field-theory regime

Higher-point holographic correlators

Drawing on our analysis of the six-gluon correlator in AdS, several compelling directions for future investigation present themselves. First, it would be interesting to further explore colour-kinematics duality and double-copy relation in Mellin space, extending previous observations at lower points [66, 76, 88]. The appearance of subleading poles in (3.43) suggests a generalised version of the duality in this case.

More generally, the remarkably simple structure of the results makes it plausible that a recursive method can be developed to compute higher-point correlators. The simplification of the flat space amplitude in the orthogonal configuration certainly warrants more attention. It should be possible to obtain all multiplicity results by adapting on-shell techniques. On the other hand, it can be checked that the five-point amplitude of supergluons [88] can also be fixed from factorisation if the four-point function with one spinning leg is used. Therefore, it would be important to systematically extend our analysis to spinning amplitudes after further developing the spinning Mellin formalism [80]. In particular, this would allow us to apply the strategy to spinning gluon amplitudes in pure YM in AdS. Combining these elements, we can hope to generate higher-point AdS amplitudes directly from lower-point amplitudes.

Moving to the six-graviton case, a key ingredient of our approach was the use of the chiral algebra twist [60], which relates correlation functions of different operators belonging to the same supermultiplet. We focused on the $\frac{1}{2}$-BPS sector, which defines the supergraviton multiplet. However, this technique can be used more generally, and other multiplets can be similarly analysed. It would be interesting to use this approach to derive relations for correlation functions of other operators, in particular, operators residing in unprotected long multiplets. The bootstrap method may also be useful for deriving superconformal blocks for higher-point functions, either in the co-plane configuration or general kinematics. Finally, a natural extension would be to include stringy corrections to these correlators. Importantly, in that case, we could take full advantage of the tree-level string amplitude known in flat space, as we did in the AdS Virasoro-Shapiro program presented in Chapter 4.

A change of perspective

While it falls outside our current sphere of expertise and interest, we nevertheless offer a brief remark on the possibility of exploring alternative ways of implementing the flat space limit in holographic setups. In this thesis, we used the holographic principle for a spacetime with negative cosmological constant, AdS. Extending this understanding to asymptotically flat spacetimes remains challenging. Typically, the flat space limit is implemented by making specific kinematic choices and "zooming in" on the centre of AdS, allowing flat space physics to emerge. See Figure 6.1.

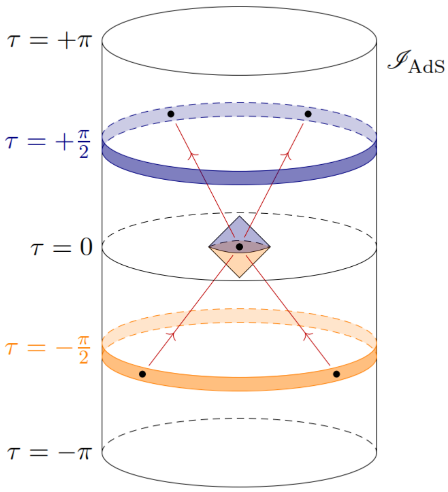


Figure 6.1: Usual setup to take the flat space limit of holographic correlators in a global patch of AdS. Here τ represents the global time coordinate. The boundary operators are inserted in little strips (blue strip for outgoing operators and orange strip for incoming operators), such that the scattering occurs in a small region in the centre of AdS, which can be approximated by flat space. Figure from [210].

However, this raises a key question: What are the implications of taking such a bulk flat space limit for the dual boundary theory? While AdS/CFT relates gravity in asymptotically AdS_{d+1} spacetime to a CFT_d, our understanding of a holographic description of flat space is still in its early stages. Two main frameworks have been proposed: Celestial [211] and Carrollian Holography [212]. The latter applies to gravity in $d+1$-dimensional asymptotically flat spacetime and establishes a correspondence with a Carrollian CFT_d, obtained from a standard CFT by formally taking the speed of light to zero. The transition from the usual holographic setup for AdS is achieved through specific limits. Taking the $R \to \infty$ limit in AdS leads to flat spacetime. Taking the $c \to 0$ (speed of light) limit in the dual CFT gives rise to a Carrollian structure. In [210], we explored this setup and analysed the flat space limit of AdS *Lorentzian* boundary correlators. A key point is that the use of Bondi coordinates allows for a flat space limit implemented directly in position space. This induces the Carrollian limit in the boundary theory by identifying $\frac{1}{R} \leftrightarrow c$. Familiar features, such as the emergence of bulk singularities [213, 214][1] follow from enforcing a finite and non-trivial Carrollian limit.

6.2.2 String theory regime

Within the AdS Virasoro–Shapiro program [169], many broad avenues for future work remain. In Section 4.2, we presented its main ingredients: guided by the principle of single-valuedness [170], the ansatz at each order k in the small-curvature expansion involves worldsheet integrals of weight-$3k$ SVMPLs together with a finite set of rational coefficients. These unknown coefficients are then fixed by imposing the correct supergravity limit, enforcing the expected pole structure (for example, via dispersive sum rules [159]), and matching the dimensions of the lowest-lying Konishi-like operators provided by integrability. The ultimate goal is to compute the amplitude to every order in $1/R$, and ultimately at finite R. It would be very interesting to reproduce these results for the AdS Virasoro–Shapiro amplitude by a direct string theory computation. Among the other possible approaches, the pure-spinor formalism seems to be the most promising framework for a direct worldsheet derivation. In recent years, there has been some progress in constructing vertex operators within this approach [216, 217], yet the precise integration measure required for amplitude computations is still unknown. It would therefore be particularly interesting to leverage the results of this program to reconstruct that measure in a $1/R$ expansion. We should also mention the recent progress in developing string field theory in the presence of RR backgrounds, see [218, 219].

High-energy limit in AdS

Our results in the high-energy limit put strong constraints on curvature corrections at higher orders, in particular fixing the answer at the saddle point. It would be instructive to see if this can be combined with results from integrability and the structure of the solution to go to higher orders. It would also be worthwhile to see if subleading corrections to the high-energy limit follow a simple pattern.

As a short aside, let us mention that there is another beautiful infinite series representation of the Virasoro-Shapiro amplitude (as opposed to (4.51)). From Appendix C of [220], we learn that the

[1] See also the interesting recent work [215].

amplitude can be written as

$$A^{(0)}(S,T) = \frac{i}{STU} A^{(0)}(S,T)_{HE} \left[\sum_{n\geq 1} \frac{B_{2n}}{n(2n-1)} \left(\frac{1}{S^{2n-1}} + \frac{1}{T^{2n-1}} + \frac{1}{U^{2n-1}} \right) \right] , \tag{6.1}$$

where B_n are Bernoulli numbers, which encode the quantum fluctuations about the saddle point's classical configuration. Understanding the interpolation between the low-energy and high-energy regimes is at the heart of [220], and it would be super insightful to explore similar patterns for our results.[2]

From another perspective, the worldsheet saddle point prediction is reproduced by a classical scattering computation in AdS. First, we could study the bosonic model exactly, for finite R, perhaps by using a Pohlmeyer-type reduction [221], as was done for the problem of scattering amplitudes in MSYM at strong coupling [222]. Moreover, we could extend the computation of this bosonic model to higher-point functions. The location of the punctures would serve as the letters of the SVMPLs that form the solution. Notice that, again, we expect that, in a $1/R$ expansion, the location of the saddle point is the same as for flat space. Another common avenue is to extend to higher-genus surfaces. A fascinating feature of the worldsheet computation is that one can go to higher genus, and the genus expansion is Borel summable [223]. It would be very interesting to explore this question in AdS.

To wrap up, we present a selection of conceptual open questions. We reviewed Gross and Mende's result in flat space for the high-energy string scattering amplitude, with fixed angle, with the notable exponentiation (4.68). This can be interpreted as the saddle point action for a minimal area surface in flat space, contributing to the four-point amplitude. In AdS, we also found an exponentiation, in the limit of large S, T, R, with S/T and S^2/R^2 fixed. More specifically:

$$A_{HE}(S,T) = A^{(0)}_{HE}(S,T)\, e^{-\frac{\alpha'}{R^2}(\mathcal{S}^{(1)}+2SF_2(z_0)\mathcal{S}^{(0)})} , \tag{6.2}$$

where $\mathcal{S}^{(0)}$ and $\mathcal{S}^{(1)}$ are the leading and subleading contributions of the action evaluated on the classical solution. $\mathcal{S}^{(0)}$ is of weight 1, $\mathcal{S}^{(1)}$ of weight 3, and $F_2(z_0)$ of weight 2. The exponentiation phenomenon usually conceals profound physics waiting to be uncovered. Therefore, it would be very instructive to deepen our understanding of (6.2). This is fundamental and subtle for various reasons. First, what is the interpretation of $F_2(z_0)$ as defined in equation (4.135)? We found it by comparison with the high-energy limit of the AdS Virasoro-Shapiro amplitude in $AdS_5 \times S^5$, and it gives the contribution from the first correction to our classical computation. One could argue that a full one-loop fluctuation calculation would fix it. Of course, this would depend on more details, such

[2] I thank Pedro Vieira, one of the authors of the paper cited above, for raising this point during my visit to the Perimeter Institute in October 2024.

as the prefactors of the exact vertex operators. But on the other side, and remarkably, the *same* $F_2(z_0)$ has been recently found in $AdS_3 \times S^3 \times M_4$ [173] and $AdS_4 \times \mathbb{CP}^3$ [174]. Therefore, it might be a universal quantity. This requires further investigation. More generally, it would be illuminating to inspect the interplay between the two branches of our computation in the high-energy regime. From the worldsheet perspective, the AdS amplitude is defined as the Borel transform of the Mellin amplitude. Then, it is a function of the Borel variables S and T. However, from the spacetime perspective, the AdS amplitude is defined in terms of the flat space momenta, which are ambiguous at subleading order $1/R^2$.[3] It is necessary to further develop this interplay to understand the role of $F_2(z_0)$ as a result of the conversion between the two sets of variables. This links to the fact that there is yet no proof for the Penedones' formula [48] beyond flat space. For us, it is a working definition, which gives reasonable results.

Regge limit in AdS

Regarding our AdS Virasoro–Shapiro amplitude results in the Regge regime, we begin by noting that our integrand solutions admit large families of ambiguities, namely, those integrands whose total integral vanishes. Similar ambiguities were also present in [169, 171]. While at present it is not clear how to fully fix or understand these ambiguities, we noted that for a specific choice (that fixes them partially), one can take the Regge limit at the level of the integrand first (4.180).

An open question is how to impose the correct Regge limit together with single-valuedness at $z = 1$. This should settle the question of how constraining the CFT data of the leading twist operators is in constructing the AdS Virasoro–Shapiro amplitude. The results presented here are the first steps in this direction. Let us stress once more that our findings connect very directly the AdS Virasoro–Shapiro amplitude to the impressive developments in integrability [160–162], via the CFT data of leading twist operators in planar $\mathcal{N} = 4$ SYM. It would be fascinating to develop a *quantum spectral curve* for the AdS Virasoro–Shapiro amplitude!

Finally, building on the discussion above, it would also be interesting to understand whether/how the pure spinor formalism simplifies at high energies, and whether it can be used to compute subleading corrections.

KLT relations in AdS

Our results in Chapter 5 open many avenues for future research. Most importantly, while we have focused on the relation between *building blocks* for open and closed string amplitudes on AdS, the natural next step is to focus on specific *amplitudes* in specific theories, and look for KLT/double-copy relations, extending more concretely the beautiful work in flat space [4, 68, 224], to AdS.

[3] I thank Tobias Hansen for discussions about this point.

Speaking of extensions of flat space structures, it is well-known that the Drinfeld and Deligne associators provide the generating functions for the Euler and complex Beta functions, more generally capturing open and closed string amplitudes [194, 225, 226]. This construction has recently been extended to AdS [227], to which we refer for additional possible future directions within the same framework. Another interesting direction is the generalisation of our results to higher multiplicity amplitudes. The first step is to identify the corresponding building blocks. Presumably, these involve the insertions of multiple polylogarithms in two or more variables for open strings and their single-valued analogues for closed strings. See [142, 228, 229], where such single-valued functions are introduced and studied.

On a more conceptual level, we have shown that much of the beautiful mathematical structure found in string theory amplitudes in flat space persists when AdS curvature corrections are taken into account. It would be compelling to fully understand the reasons and implications of this for string theory on curved backgrounds.

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