G_2-moduli spaces: geometry, periods and degenerate limits

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A thesis submitted for the degree of
Doctor of Philosophy
Trinity Term 2025

"Le mathématicien observe cet homme primaire, et il se conforte dans l'idée désespérante qu'en additionnant des obscurités individuelles on obtient rarement une lumière collective."

Hervé Le Tellier, *L'anomalie*

Acknowledgements

As might be expected at the end of such a long project, I realise that there are more people to thank than I can name. I hope that anyone whom I could not personally acknowledge here will forgive me and rest assured that they are in my thoughts at the time of writing these lines.

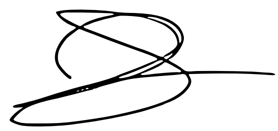
Let me first express my gratitude to my supervisor Prof. Jason Lotay, who introduced me to the beautiful field of special holonomy when I was a master student and gave me the opportunity to further explore this topic for a DPhil degree. Needless to say, this thesis could not have reached its current form without his support and advice. I would also like to thank Sir Simon Donaldson and Prof. Dominic Joyce who kindly agreed to review the present manuscript and be my DPhil examiners. Finally, I am grateful to Prof. Andrew Dancer, Prof. Spiro Karigiannis and Prof. Sakura Schäfer-Nameki for providing valuable insights at an earlier stage of this project, on the occasion of my Transfer and Confirmation of Status, and to my college tutor Prof. Stuart White for his help and advice.

My doctoral studies have been rich with mathematical encounters, and I have had the chance to make many friends along the way. It was a pleasure to plan trips around San Francisco with Izar; to talk about mathematics over endless ping-pong games with Viktor; to chase banana slugs and elephant seals in California with Dominik; and to roam around the Atlantic Forest in Brazil with Alfred, Gilles, Jacek and Nico. More generally, I would like to thank the members of the Geometry group in Oxford and the people who were staying at SLMath in Berkeley during the Fall 2024 semester for many inspiring discussions.

On a more personal note, I was lucky enough to spend four happy years in Oxford thanks to the diversity of the people that I met there. Amongst the fondest memories that I will take away from that time are blind tasting sessions with Katie, Lynn and Michał; dinners with Aida and Gabriel; and many an early morning outing on the river with Cam, Charles, Seb and the rest of St John's College Boat Club.

These acknowledgements would not be complete without a thought for the people back home. First my family, for their love and continual support. And my friends, for their unwavering humour in all circumstances: Élodie, Léa, Adrienne, Sandra, Lysander, Sarah, Hugo, Balthazar, whom I have known for longer than I dare to admit; Eliott and Maëlle and our countless late-night discussions; María and all the people that I met during my studies at ENS: Jade, Sasha, Louis, Thibault, Noé, Claire, Marc, as well as Clara, Dasha, Katarina and Léo.

And finally, my deepest gratitude goes to Connie, for everything.



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Oxford – July 4, 2025

Abstract

The overarching theme of the present dissertation is the geometry of compact G_2-manifolds and their moduli spaces. This thesis can be roughly divided into two parts: the first one is concerned with the spectral properties of twisted connected sums, whilst the second one studies the geometry of G_2-moduli spaces.

In the first part, we prove a deformation theorem allowing to construct torsion-free G_2-structures with C^k-estimates (for any $k \geq 1$), and use it to deduce improved estimates for the twisted connected sum construction. We then study the mapping properties of differential operators in 'neck-stretching' problems, where two asymptotically cylindrical manifolds are glued to form a family of compact manifolds containing a cylindrical neck region whose length is stretched to infinity. In this limit, we construct approximate Fredholm inverses for a large class of 'adapted' elliptic operators, with good control on the growth of their norm. Our results are then refined in the case of the Laplacian operator, and we derive a precise description of the asymptotic behaviour of its lower spectrum. Specified to twisted connected sum G_2-manifolds, our results give mathematical support for the so-called swampland distance conjecture in physics.

The second part of this dissertation is concerned with the geometry of the moduli space $\mathscr{M}$ of torsion-free G_2-structures on a compact G_2-manifold M, endowed with the Riemannian metric $\mathscr{G}$ induced by the volume-normalised L^2-inner product. When the first Betti number of M vanishes, this metric has the remarkable property of being Hessian and admits a global potential $\mathscr{F}$. Using this observation, we derive a formula for the energy (the integral of the squared velocity) of a path in $\mathscr{M}$. This allows us to give sufficient geometric and topological conditions for a path of torsion-free G_2-structures on M to have finite energy and length in the moduli space. By considering paths that degenerate to a singular limit, we deduce that G_2-moduli spaces may be incomplete: indeed we show that G_2-manifolds constructed by the generalised Kummer construction, by resolution of isolated conical singularities or by the Joyce–Karigiannis construction all have incomplete moduli spaces.

In the final chapters, we study the local geometry of $\mathscr{M}$ and give an alternative description of the metric $\mathscr{G}$ through the introduction of a new notion of period map for G_2-manifolds, mimicking the classical notion of period map introduced by Griffiths for Kähler manifolds. More specifically, we show using Hodge theory that there is a natural immersion $\Phi : \mathscr{M} \to \mathfrak{D}$ of the moduli space into a homogeneous space $\mathfrak{D}$ diffeomorphic to $\mathrm{GL}(n+1)/(\{\pm 1\} \times \mathrm{O}(n))$, where $n+1 = b^3(M)$, in such a way that $\mathscr{G}$ coincides with the restriction of a (degenerate) homogeneous quadratic form defined on $\mathfrak{D}$. Motivated by the question of understanding the curvature of $\mathscr{G}$, we also compute the derivatives of the potential function $\mathscr{F}$ up to order 4 and relate our formulas to the second fundamental form of $\Phi(\mathscr{M}) \subset \mathfrak{D}$. In particular, we deduce that the map Φ is a totally geodesic immersion and $\mathscr{G}$ is locally symmetric when $\mathscr{M} = T^7/\Gamma$ or $M = (T^3 \times K3)/\Gamma$. Finally, we use the theory of exterior differential systems to give some complements on the properties of the map Φ and relate it to the more classical notion of period map for G_2-manifolds, as a Lagrangian immersion of $\mathscr{M}$ into $H^3(M) \oplus H^4(M)$.

Statement of Originality

This thesis is the result of the work that I carried out between October 2021 and July 2025 during my doctoral studies at the Mathematical Institute, University of Oxford, under the supervision of Prof. Jason D. Lotay. I hereby declare that:

- No parts of this thesis have been submitted in support of another degree, diploma or qualification at the University of Oxford or any other university.
- This thesis is the product of my own work and the results presented are original, unless otherwise stated.

Most of the material in this thesis has been published in the form of three articles [82, 83, 84], with some improvements and original additions which will be signposted.

Funding. During my DPhil, my research was funded by scholarships from the Clarendon Fund (Oxford University Press, 2021–2025), the Saven European Programme (2021–2024) and a Mathematical Institute Studentship (2025).

Part of this material is also based upon work supported by the National Science Foundation under Grant No. DMS-1928930, while I was in residence at the Simons Laufer Mathematical Sciences Institute (formerly MSRI) in Berkeley, California, during the Fall 2024 semester.

Contents

Introduction

Historical perspective and motivation

Riemannian holonomy groups. One of the cornerstones of modern geometry is the concept of *holonomy*, for the holonomy group of a Riemannian manifold detects the presence of additional geometric structures. For non-symmetric spaces, the classification of the possible irreducible holonomy groups was carried out by Berger in 1955 [11], with a later amendment by Aleeksevkii [5]. Besides generic Riemannian metrics (with holonomy $\mathrm{SO}(n)$) and Kähler metrics ($\mathrm{U}(n)$), this classification contains five special cases: three infinite families corresponding to Ricci-flat Kähler/Calabi–Yau ($\mathrm{SU}(n)$), hyperkähler ($\mathrm{Sp}(n)$) and quaternionic-kähler ($\mathrm{Sp}(1).\,\mathrm{Sp}(n)$) metrics; and the two exceptional Lie groups $\mathrm{G}_2 \subset \mathrm{SO}(7)$ – which will be the main focus of this thesis – and $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$.

On the one hand, the study of Calabi–Yau and hyperkähler metrics lies at the intersection of Riemannian, complex and algebraic geometry, and since Yau's solution of the Calabi conjecture in 1978 [123] giving a necessary and sufficient condition for the existence of such metrics on a compact Kähler manifold the connections between these fields have been exploited to yield a very rich theory. In addition, quaternionic-kähler metrics can also be related to complex geometry using twistor methods.

On the other hand, the study of metrics with exceptional holonomy is only amenable to differential-geometric techniques and would turn out to be more difficult. The identification of the parallel tensors associated with the exceptional holonomy groups goes back to Bonan [14] in 1966, who also proved that metrics with holonomy G_2 or $\mathrm{Spin}(7)$ are automatically Ricci-flat. But it took another two decades before the first construction of metrics with exceptional holonomy due to Bryant in the local setting [16], shortly before the first complete examples were exhibited by Bryant and Salamon using cohomogeneity one techniques [19]. The compact case required the introduction of completely different methods, and the first examples are due to Joyce [63, 64, 65] who developed a general gluing-perturbation framework in order to resolve the singularities of certain flat G_2- or Spin(7)-orbifolds.

Over the past three decades, these construction techniques have been extended and refined, but the fundamental principles on which they rely remain unchanged. In the noncompact setting, we now know of a number of complete exceptional holonomy manifolds with various asymptotic behaviours (due to Foscolo–Haskins–Nordström for G_2 [42, 43] and more recently Cavalleri for Spin(7) [22]), which all rely on dimensional reduction methods (cohomogeneity one or higher). In the compact case, several new constructions of G_2-manifolds have been elaborated: the twisted connected sum, initially due to Kovalev [80] and then improved by Corti–Haskins–Nordström–Pacini [29, 30] and extended by Crowley and Nordström [34, 97]; and other resolution methods due to Karigiannis [72] and Joyce–Karigiannis [68], although in these cases it remains a challenge to find appropriate building blocks. All of these constructions rely on Joyce's deformation theorems [66, Ch. 11] and produce compact exceptional holonomy manifolds close to a degenerate limit. Contrary to the Calabi–Yau case, we do not know which compact manifolds admit metrics with holonomy G_2 or Spin(7): if certain necessary topological conditions are known, they are expected to be far from sufficient. More generally, our understanding of exceptional geometry so far mostly relies on specific constructions, and there is not a well-established theory comparable to what is available for the study of special metrics in complex geometry.

But despite these differences, there are deep connections between Calabi–Yau, hyperkähler and exceptional geometries [37]. This is manifest in the sequence of inclusions of holonomy groups $\mathrm{Sp}(1) = \mathrm{SU}(2) \subset \mathrm{SU}(3) \subset G_2 \subset \mathrm{Spin}(7)$, which implies that the different types of geometries can be related through dimensional reduction. For this reason, metrics with holonomy SU(2) or SU(3) often arise as building blocks in the construction of exceptional holonomy metrics, and understanding them remains a central topic in exceptional geometry. In addition, there is an expectation that certain phenomena arising in Calabi–Yau geometry should have counterparts in G_2 and Spin(7)-geometry. This general idea has been adapted in a variety of ways and is at the origin of a lot of research in the field. Besides the geometrical intuition, it is partly motivated by the fundamental role played by special holonomy manifolds for quantum gravity theories in physics, especially for Calabi–Yau and G_2-manifolds in string and M-theory.

Special holonomy and quantum gravity theories. From a historical perspective, the inception of string theory took place in the 1960s as a candidate theory of strong interactions, in an attempt to account for the growing number of particles that were regularly discovered in experiments at the time. As such it was undermined by several problems, among which the presence of a massless spin two particle in the spectrum of the closed string which did not correspond to any known particle. With the development of quantum chromodynamics in the late 1960s, string theory was discarded as a possible description of strong interactions but soon became a candidate theory for quantum gravity, interpreting the massless spin two particle as the graviton [104] – that is, the particle mediating the gravitational interaction.

The basic idea of string theory is to replace point-like particles by small 'strings', which sweep out 2-dimensional surfaces by moving through a D-dimensional space-time. Despite the fact that it naturally contains the graviton, string theory unfortunately fails to reproduce the basic observation that space-time has dimension 4. Indeed, bosonic string theory is only consistent in a space-time of dimension $D = 26$. With the addition of fermions, the critical dimension can be brought down to $D = 10$, which is arguably better than 26 but not quite as good as 4.

To solve this conundrum, physicists introduced the idea of 'compactification'. That is, one assumes that the ambient space-time is a Riemannian product $\mathbb{R}^{3,1} \times M^{D-4}$, where M^{D-4} is a compact Riemannian manifold which should be 'very small' compared with the length scale that can be reached by experiment. Consistency conditions require M to be Ricci-flat and to admit parallel spinors, and hence in string theory M^6 must be a Calabi–Yau threefold ($D = 10$). During the 1990s, Witten also made the conjecture that there should exist an 11-dimensional so-called *M-theory* (whose precise formulation is yet to be found) unifying the different types of string theories through various limits [122]. In this theory, the internal manifold M^7 must be a G_2-manifold.

Even though in quantum gravity theories the internal manifold cannot be directly detected, the idea is that its geometry and topology should govern the low-energy physics in four dimensions. One instance of this principle is the Kaluza–Klein reduction, wherein the fields in the D-dimensional action can be decomposed in a Hilbert basis and integrated out along M, thus obtaining a countable family of physical modes on $\mathbb{R}^{3,1}$. For this reason, the eigenvalues of geometric operators such as the Laplacian acting on differential forms on M have the physical interpretation of a *mass spectrum*[1]. For this prescription to be realistic, one usually only keeps the massless modes[2] to have only a finite number of particles, the other ones being deemed too heavy to be observed at low energies.

[1]More accurately, in the case of the Laplacian the eigenvalues correspond to *squared* masses.

[2]According to the Standard Model of particle physics, particles acquire a mass through the Higgs mechanism, but their 'bare mass' (the fundamental mass parameter in the Lagrangian) is zero.

The swampland programme. Due to the large number of topological types of known Calabi–Yau and G_2 manifolds, there is a huge number of possible background vacua for string and M-theory, leading to an equally huge number of possible low-energy field theories in 4 dimensions. They form what is known as the (string or M-theory) *landscape*. This raises a question as to whether *any* quantum field theory in 4 dimensions can be obtained from string/M-theory (or any other quantum gravity theory) by choosing the right internal manifold, in which case these theories would not have any predictive power for low-energy physics.

It is now widely believed by physicists that this should not be the case, and that in some sense most field theories should not occur as the low energy limit of a theory of quantum gravity: such field theories have been collectively termed the *swampland*. In order to give more substance to this idea, one would need to give precise criteria in order to distinguish the theories forming the landscape from those belonging to the swampland. This idea was originally formulated by Vafa in 2005 [115] and is at the origin of what is called the *swampland programme*.

Since then, this programme has received a lot of attention in the physics community and many criteria have been proposed, backed by arguments with various levels of mathematical rigour (see the reviews [15, 101]). An important point is that all criteria must be formulated purely in terms of the low energy field theories themselves, without reference to a particular quantum gravity theory. This is crucial for the low energy field theories can usually be given a precise definition, even though quantum gravity theories are not mathematically well-founded. Hence some of the predictions of the swampland programme yield interesting and well-defined mathematical questions, which can be studied in their own right.

Part of the work of this thesis was motivated by a bulk of conjectures known as the *swampland distance conjectures*, made by Ooguri and Vafa [98], which concern the moduli spaces of the internal manifolds (e.g. Calabi–Yau manifolds in string theory or G_2-manifolds in M-theory). Physically, they represent the moduli spaces of parameters in the low-energy theory, and they are endowed with a natural Riemannian metric (which is to be interpreted as a kinetic term in the action functional). For Calabi–Yau manifolds, this metric coincides with the *Weil–Petersson metric* on the moduli space of complex structures and with the *Hodge metric* on the Kähler cone. On the other hand, for G_2-manifolds the relevant metric to consider is the volume-normalised L^2-metric [52].

The main prediction of the swampland distance conjectures, which is often alone called the *swampland distance conjecture*, is that the low-energy effective field theory is expected to break down at infinite distance in the moduli space.

More precisely, it is conjectured that infinite-distance limits in the moduli space are related to the appearance of an 'infinite tower of light states', with masses decaying exponentially in the moduli space distance. In more geometric terms, this means that along a deformation of the internal special holonomy manifold which does not remain in a bounded region of the moduli space, an infinite number of the mass parameters determined by the internal geometry (for instance through the Kaluza–Klein reduction, but this is not the only possible source of physical states) should decay to zero at the same rate. My understanding is that physicists expect that along such deformations, the usual prescription of keeping only the massless states in the low energy limit cannot be consistent, since an arbitrarily large number of massive modes can be made lighter than any fixed 'experimental' energy scale. In such degenerate limits, the right low-energy description should therefore be a different theory, and this should explain why the different types of quantum gravity theories are related by *dualities*.

The swampland distance conjecture has been mainly studied on Calabi–Yau moduli manifolds (see for instance [31, 53]), and the eigenvalues aspects are notably backed by numerical evidence [8]. The question of studying this conjecture from a more analytical point of view was one of the motivations for my first paper [82], which in particular gave more mathematical grounding for the conjecture by studying the spectral properties of twisted connected sum G_2-manifolds. These results are exposed in Chapters 2 and 3, and a more detailed overview will be given in the next section of this introduction.

In their original article, Ooguri–Vafa also made a number of conjectures on the asymptotic behaviour of the moduli space metric and its curvature, but they seem too strong to hold in general and may be disproved in some cases (see next paragraph). Nevertheless, this provides some motivation for trying to understand the geometry of the moduli spaces of special holonomy manifolds, endowed with their natural Riemannian metric, and the properties of the associated distance. An especially interesting related question is to understand whether the moduli spaces are complete and to distinguish finite-distance limits from infinite-distance ones. These questions have been well-studied in complex geometry for Calabi–Yau moduli spaces, but comparatively the moduli spaces of manifolds with exceptional holonomy are poorly understood.

Moduli spaces. The best understood moduli space of special holonomy manifolds is the moduli space of hyperkähler metrics on the K3 surface (holonomy SU(2)). The global Torelli theorem identifies this moduli space with the complement of a countable union of codimension 3 submanifolds in the symmetric space

$SO_0(3, 19)/(SO(3) \times SO(19))$ [111] via the period map. Therefore, the natural moduli space metric is locally symmetric and nonpositively curved. Moreover, the moduli space of hyperkähler K3 surfaces is incomplete, and it is known that finite-distance limits can geometrically be interpreted as degenerations to compact hyperkähler orbifolds [76].

The situation for the moduli spaces of Calabi–Yau metrics in higher dimensions is more complicated, and one usually studies separately the Kähler and complex deformations[3]. Because of Yau's theorem, the moduli space of Ricci-flat Kähler metrics on a fixed complex Calabi–Yau manifold can be identified with the cone of Kähler classes, which can be explicitly described in terms of the intersection form and the classes of analytic cycles [36]. The Hodge metric on the Kähler cone turns out to be Hessian and is entirely determined by the intersection form: hence this is a purely topological object. It is conjectured that this metric is nonpositively curved, and this has been proved for certain classes of Calabi–Yau threefolds [121], but to the authors' knowledge the general case is still open. In terms of distances, it is not difficult to prove that cohomology classes which are big and nef correspond to finite-distance limits at the boundary of Kähler cones [92], and hence that the Hodge metric may be incomplete. Such finite-distance limits can be interpreted geometrically: the underlying Calabi–Yau metrics associated with a deformation of the Kähler class to a big class degenerate to a Kähler current which can be seen as a singular Calabi–Yau metric on a complex analytic space with *canonical* singularities [27, 113].

For the deformations of the complex structure of Calabi–Yau manifolds, the main tool used to understand the moduli spaces is the notion of *period map* originally introduced by Griffiths [49, 50], which determines the Weil–Petersson metric in a natural way [108, 112]. There is a rich theory studying the asymptotic behaviour of the period map [105], which was used by Wang to prove that finite-distance limits along one-parameter families of Calabi–Yau manifolds correspond to degenerations to varieties with canonical singularities [117]. The relation between the Weil–Petersson metric and the period map was axiomatised by Lu and Sun [90], who deduced various results about the volume and the first Chern class of the moduli spaces [91]. Regarding the local geometry, it was for a long time expected that the scalar curvature of the Weil-Petersson metric should be nonpositive at least near infinite-distance limits (see for instance [118], and this also

[3]Both types of deformations are supposed to be in some sense dual to each other, according to mirror symmetry.

motivated one of the conjectures of Ooguri and Vafa [98]), but eventually it was disproved for certain Calabi–Yau threefolds [114].

By contrast, much less is known about G_2-moduli spaces. Joyce proved that the moduli space of torsion-free G_2-structures on a compact 7-manifold M is a smooth manifold of dimension $b^3(M)$[4] (when it is nonempty), locally modelled on an open cone in $H^3(M)$ [64]; thus the moduli space is even an *affine manifold.* Moreover, it is naturally immersed as a Lagrangian submanifold of $H^3(M) \oplus H^4(M)$ [65]. But these results are purely local, and little is known about the global structure of the moduli space, partly because of the lack of an analogue of Yau's theorem [123] in G_2-geometry. Nevertheless, there have been some recent advances on the topology of the moduli spaces: it was proved by Crowley–Goette–Nordström that the quotient of the space of torsion-free G_2-structures by the full space of diffeomorphisms is disconnected in some cases [33]. Even more recently, Crowley–Goette–Hertl proved that the quotient of the space of torsion-free G_2-structures by the group of diffeomorphisms isotopic to the identity may be non-aspherical [32].

From a geometric perspective, Hitchin first noticed that the Hessian of the volume functional is non-degenerate [60], and when $b^1(M) = 0$ it defines a metric with Lorentzian signature on the moduli space. Around the same time, it was pointed out in the physics literature that by taking the logarithm of the volume one obtains a potential function with definite Hessian, which coincides (up to a constant factor) with the volume-normalised L^2-metric [10, 54, 55, 61]. This is reminiscent of the Hodge metric on Kähler cones, but the high degree of nonlinearity of the potential function makes the geometry of G_2-moduli spaces much more difficult to understand. Grigorian and Yau [52] obtained formulas for the curvatures of the moduli space metric, which are unfortunately difficult to interpret geometrically. Nevertheless, an interesting feature of these formulas is their formal similarity with the equations describing the geometry of the moduli spaces of complex structures on Calabi–Yau threefolds. Further similarities were exhibited by the work of Karigiannis and Leung [73], who developed a notion of Intermediate Jacobians for G_2-manifolds.

More recent progress was made by the author in two articles [83, 84] concerning both the distance aspects and the local geometry of the moduli spaces. These results form Chapters 4 and 5 of the present manuscript. In the next section, we shall give an overview of the results proved in this thesis and its organisation.

[4] In his article, Joyce attributes the result to Bryant and Harvey in unpublished work. Indeed, the result was announced in [16, p.561] in the case of simply connected manifolds.

Main results and organisation of the thesis

The present thesis is divided into 6 chapters. The first one contains background material and sets the notations and conventions that will be used throughout. In the first section, we briefly introduce the the fundamental concepts of holonomy, geometric structures and intrinsic torsion, and discuss the classification of Riemannian holonomy groups and Berger's list. Then in the second section we give a self-contained exposition of the basics of G_2-geometry, spanning from the notion of positive linear form on $\mathbb{R}^7$ to G_2-manifolds and their moduli spaces. Most of the material of this first chapter can be found elsewhere in the literature (for instance in Joyce's monograph [66]), with the exception of a few observations about the linear algebra of positive forms in §1.2.1 which will be clear to the experts but for which we could not find a reference.

Gluing constructions and neck-stretching. The following two chapters (2 and 3) form the most analytical part of the thesis and stem from the article [82] by the author, which studies the analysis of differential operators for a certain class of 'neck-stretching' problems. The original motivation for this came from the swampland distance conjecture mentioned in the previous section. Indeed, in the twisted connected sum construction of G_2-manifolds, two asymptotically cylindrical G_2-manifolds are glued together in order to form a family of compact manifolds M_T endowed with a closed G_2-structure, φ_T, which can be perturbed to a nearby torsion-free one, $\widetilde{\varphi}_T$, in the limit where the length of the neck region $2T \to \infty$; and this limit is at infinite-distance in the moduli space[5]. Hence it is an interesting question to study the asymptotic behaviour of the spectrum of the Laplacian operator of $(M_T, \widetilde{\varphi}_T)$ and to interpret the presence of low eigenvalues in geometrical terms.

The first technical difficulty is that $\widetilde{\varphi}_T$ is only implicitly defined, and the original analytical argument in the twisted connected sum construction only gives an estimate $\|\widetilde{\varphi}_T - \varphi_T\|_{C^0} \lesssim e^{-\delta T}$, where $\delta > 0$ is small enough and $T \to \infty$. In particular, it does not give control on the derivatives of $\widetilde{\varphi}_T$, which would be needed in order to approximate the behaviour of the Laplacian operator associated with $\widetilde{\varphi}_T$ by the Laplacian associated with φ_T[6]. This problem is tackled in Chapter 2, where we adapt Joyce's general existence theorem for torsion-free G_2-structures on compact manifolds – which gives C^0-estimates – to prove a deformation theorem

[5] This is because in this limit the volume goes to infinity; see §4.1.1 and also Lemma 4.18.

[6] For some purposes, C^1 or C^2-estimates might be enough, but to be able to apply the analytical tools developed in Chapter 3 it is more convenient to have C^k-estimates for any $k \in \mathbb{N}$.

with C^k-estimates for $k \geq 1$. This is an improvement and generalisation of an argument given by the author in the last section of [82]. Our theorem requires rather strong hypotheses, but using an additional analytical result proved in Chapter 3 we can show that they are satisfied in the case of twisted connected sums, and deduce improved estimates $\|\widetilde{\varphi}_T - \varphi_T\|_{C^k} \lesssim e^{-\delta T}$ for any $k \in \mathbb{N}$.

In the following chapter, we turn our attention to the analysis of the Laplacian operator associated with φ_T. In fact, since this problem has nothing to do with G_2, we consider the more general setting of a family of compact Riemannian manifolds (M_T, g_T) obtained from the gluing of two asymptotically cylindrical manifolds. Moreover, we will not specifically consider the Laplacian operator but a larger class of differential operators P_T adapted to the geometry of this neck-stretching problem, in a sense defined in §3.1.1. Our goal is to build an approximate Fredholm inverse for the map induced by P_T on Sobolev spaces of sections, with good control on the growth of its operator norm in the limit where $T \to \infty$.

Our first task, which is carried out in Section 3.1, is to define good notions of substitute kernels and cokernels, with a particular emphasis on the case where the model operator on the cylinder admits 0 as an indicial root. Under a certain assumption justified in §3.1.2, we develop in Sections 3.2 and 3.3 a general method to construct a Fredholm inverse of P_T on the complement of the substitute kernel and cokernel, and show that its norm (as an operator between Sobolev spaces of sections) grows at most polynomially with T. Finally, in Section 3.4 our method is refined in order to derive precise estimates for the asymptotic behaviour of the lower spectrum of the Laplacian operator acting on differential forms in the neck-stretching limit. Namely, we show that the asymptotic density of low eigenvalues of the Laplacian Δ_T of (M_T, g_T) is equivalent to the density of eigenvalues of the Laplacian acting on the product $S^1_{2T} \times X$, where X is the cross-section of the cylindrical neck which has length $2T$. Going back to our physical motivation, I have been told by physicists that for twisted connected sum G_2-manifolds this confirms the idea that M-theory should be dual to another theory compactified on X, in the limit where the length of the neck goes to infinity.

Geometry of the moduli spaces. In the second half of this thesis, from Chapter 4 to Chapter 6, we will be interested in the moduli space $\mathscr{M}$ of torsion-free G_2-structures of a compact G_2-manifold M and the properties of the volume-normalised L^2-metric, denoted by $\mathscr{G}$. A few basic facts about this metric, including the observation that it is Hessian when $b^1(M) = 0$, are explained in Section 4.1, which also gathers various results which will be useful in the following chapters.

The rest of Chapter 4 is an extended version of [84], where the author proved that G_2-moduli spaces may be incomplete (even in the case of manifolds with full holonomy). I was motivated by the corresponding statements for Calabi–Yau moduli spaces: for both the Kähler cone and the complex structure moduli space, the asymptotic behaviour of the moduli space metric near the boundary seems to detect something about the type of degenerations occurring, in so far as finite-distance limits correspond to the formation of canonical singularities as opposed to more singular degenerate limits. Moreover, in both cases there are incompleteness criteria which may be proved without knowing anything about the underlying Calabi–Yau metrics: they only take advantage of the special properties satisfied by the moduli space metric.

For G_2-manifolds, one special property is that the metric $\mathscr{G}$ is Hessian, and in Section 4.2 we exploit this observation to derive a simple formula for the energy of a path $\{\varphi_t\}_{t\in(0,1]}$ of torsion-free G_2-structures in the moduli space. This allows us to find sufficient topological and geometrical conditions for its energy and length to be finite, from which we deduce that the generalised Kummer G_2-manifolds have incomplete moduli spaces. In the last section we discuss other incompleteness criteria and prove the incompleteness of the moduli spaces for Karigiannis' resolution of isolated conical singularities and the Joyce–Karigiannis construction (cases which were not treated in [84]). We also mention some interesting open questions for future study.

In Chapter 5, which is based on the paper [83], we study the local properties of the metric $\mathscr{G}$ and define a new notion of period map for G_2-manifolds with vanishing first Betti number. In the first section, we compute the derivatives of the potential function up to order 4 and derive some consequences for the curvatures of the moduli spaces. In Section 5.2 we define a certain 'twisted version' of the Hodge decomposition of M associated with a torsion-free G_2-structure φ and show that it defines an element in the homogeneous space $\mathfrak{D} \simeq \mathrm{GL}(n+1)/(\{\pm 1\} \times \mathrm{O}(n))$, where $n+1 = b^3(M)$. In the following section we study the properties of the induced map $\Phi : \mathscr{M} \to \mathscr{D}$. We notably point out that this is an immersion, which satisfies differential constraints similar to Griffiths' notion of transversality for the period map of Calabi–Yau manifolds, and that it determines the metric on the moduli spaces in a natural way. More precisely, we show that $\mathscr{G}$ is the pull back under Φ of a degenerate homogeneous quadratic form defined on $\mathfrak{D}$. We also give a necessary and sufficient condition for the immersion $\Phi(\mathscr{M}) \subset \mathfrak{D}$ to be totally geodesic: namely, it is totally geodesic if and only if the symmetric cubic form known as the *Yukawa coupling* is a parallel tensor on $\mathscr{M}$. We then relate this condition to the formulas we obtained for the derivatives of the potential. Lastly, in Section 5.4 we study the differential constraints satisfied by the period mapping from the point of view of exterior differential systems, and relate our notion of period map to the classical notion of G_2-period map as a Lagrangian immersion of $\mathscr{M}$ into $H^3(M) \oplus H^4(M)$.

Finally, Chapter 6 is a small chapter mostly meant to complement the previous one, where we make some remarks on the moduli spaces of compact G_2-manifolds whose restricted holonomy group is a proper subgroup of G_2 and which have vanishing first Betti number. We notably prove that, in the cases of T^7/Γ and $(T^3 \times K3)/\Gamma$, the Yukawa coupling turns out to be parallel and therefore the period map is a totally geodesic immersion. We also make some brief comments on the case of $(S^1 \times CY3)/\Gamma$. In Appendix A, we also give a classification of the possible geometries for the moduli spaces in the flat case.

Chapter 1

Background

This chapter gathers some fundamental background material. The first section is a brief overview of the concepts of geometric structure and holonomy, where we also review the classification of Riemannian holonomy groups. Section 1.2 is an introduction to the geometry of G_2-manifolds, where we set the notations that will be used throughout this dissertation and emphasize a few useful algebraic properties of positive forms.

1.1 Geometric structures

1.1.1 Bundles and connections. Let M be a manifold (a second-countable Hausdorff topological space endowed with an atlas of charts with smooth transition functions) of dimension m. If E is a rank r vector bundle over M, a *connection* on E is a linear map $\nabla : \Omega^0(E) \to \Omega^1(E)$ satisfying the so-called *Leibniz rule*:

$$\nabla(fs) = df \otimes s + f\nabla s, \qquad \forall f \in C^\infty(M),\ \forall s \in \Omega^0(E).$$

There is a unique way to extend ∇ to a collection of linear maps $d_\nabla : \Omega^k(E) \to \Omega^{k+1}(E)$ such that $d_\nabla = \nabla$ on $\Omega^0(E)$ and d_∇ satisfies the following generalisation of the Leibniz rule:

$$d_\nabla(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d_\nabla \eta), \qquad \forall \omega \in \Omega^k(M),\ \forall \eta \in \Omega^l(E).$$

Using this Leibniz rule, it is easy to show that there exists a unique $\mathrm{End}(E)$-valued 2-form $F_\nabla \in \Omega^2(\mathrm{End}(E))$, called the *curvature* of ∇, such that for any $\eta \in \Omega^k(E)$ we have $d_\nabla^2 \eta = F_\nabla \wedge \eta$. When ∇ is not flat, that is $F_\nabla \not\equiv 0$, then $(\Omega^\bullet(E), d_\nabla)$ fails to be a chain complex.

More abstractly, one can think of connections in terms of the frame bundle $\mathscr{F}_E$ of E, whose fibre over a point $p \in E$ is the manifold of linear automorphisms $E_p \simeq \mathbb{R}^r$. The group $\mathrm{GL}(r)$ acts on the right on E by post-composition, and this gives $\mathscr{F}_E$ the structure of a $\mathrm{GL}(r)$-principal bundle. Then the data of a connection on E is equivalent to an equivariant splitting of the tangent space $T\mathscr{F}_E$ into vertical and horizontal components, and the curvature of the connection measures the failure of integrability of the horizontal distribution.

Given a connection ∇ on E, one may associate to any piecewise C^1 path $\gamma : [0,1] \to M$ a *parallel transport map* $P_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$, which is a linear isomorphism. In particular, when γ is a loop based at $p \in M$, P_γ is an automorphism of the fibre E_p. The set of such automorphisms forms a Lie subgroup $\mathrm{Hol}_p(\nabla) \subset \mathrm{GL}(E_p)$, called the *holonomy group* of ∇ based at p. After choosing an identification of E_p with $\mathbb{R}^r$, $\mathrm{Hol}_p(\nabla)$ can be regarded as a subgroup of $\mathrm{GL}(r)$ whose conjugacy class is independent of the choice of base point. For this reason, one often calls any representative H of this conjugacy class *the* holonomy group of ∇ and drops the subscript p. The restricted holonomy group $\mathrm{Hol}^0(\nabla)$ is by definition the identity component of $\mathrm{Hol}^0(\nabla)$. By a theorem of Ambrose and Singer [6], the curvature of a connection determines its restricted holonomy.

The importance of the notion of holonomy is best understood from the perspective of principal bundles. One can easily see that the holonomy group of a connection ∇ on E is contained (up to conjugacy) in a subgroup $G \subset \mathrm{GL}(r)$ if and only if there exists a principal subbundle $\mathscr{P} \subset \mathscr{F}_E$ with structure group G which is invariant under parallel transport; that is, such that ∇ can be reduced to a connection on $\mathscr{P}$. Hence the holonomy group of a connection is the smallest structure group to which it can be reduced.

An especially interesting case is when the subgroup $G \subset \mathrm{GL}(r)$ arises as the stabiliser of a nontrivial element Ψ in a linear representation W of $\mathrm{GL}(r)$. For such a representation, we may construct the associated bundle $W_E = (\mathscr{F}_E \times W)/\,\mathrm{GL}(r)$. Then there is a one-to-one correspondence between the principal subbundles of $\mathscr{F}_E$ with structure group G and the sections ψ of W_E such that for any $p \in M$, there is an identification $W_{E,p} \simeq W$ mapping ψ_p to Ψ. It follows that a connection ∇ restricts to a G-subbundle $\mathscr{P} \subset \mathscr{F}_E$ if and only if the associated section ψ of W_E is parallel (for the connection induced by ∇ on W_E). To summarise, one might say that the holonomy of a connection determines and is characterised by its parallel tensors.

1.1.2 *G*-structures and intrinsic torsion. Let us now consider the special case of the vector bundle $E = TM$, endowed with a connection ∇. We can define a section T_∇ of the vector bundle $\Lambda^2 T^*M \otimes TM$, called the *torsion* of ∇, by

$$T_\nabla(u,v) = \nabla_u v - \nabla_v u - [u,v], \qquad \forall u, v \in C^\infty(TM).$$

The connection ∇ is called *torsion-free* if $T_\nabla \equiv 0$. If α is a 1-form on M, one can easily calculate that $\nabla\alpha(u,v) - \nabla\alpha(v,u) = d\alpha(u,v) - \alpha(T_\nabla(u,v))$ for all $u, v \in C^\infty(TM)$. From this it follows that if ∇ is torsion-free then the antisymmetric part of $\nabla\alpha$ coincides with $d\alpha$ (up to a combinatorial coefficient). This fact readily generalises to differential forms of any degree; hence for torsion-free connections parallel forms are automatically closed.

There is also a notion of torsion for geometric structures. Recall that if G is a Lie subgroup of $\mathrm{GL}(m)$, a *G-structure* on M is a principal G-subbundle $\mathscr{P}$ of $\mathscr{F}_{TM}$. It is called *torsion-free* if it admits a torsion-free connection. Alternatively, if $\mathscr{P}$ is defined by a section ψ of an associated bundle, then it is torsion-free if and only if M admits a torsion-free connection for which ψ is parallel. The torsion-free condition can often be interpreted as an *integrability* condition. For instance, in even dimension a $\mathrm{GL}(m/2, \mathbb{C})$-structure corresponds to an almost complex structure J, and it is torsion-free if and only if it is integrable, in the sense that it is induced by an atlas of complex charts with holomorphic transition functions. Along the same lines, a non-degenerate 2-form ω is symplectic (closed) if and only if the corresponding $\mathrm{Symp}(m)$ structure is torsion-free.

There is however a very useful case where the torsion-free condition is always satisfied, which is that of $\mathrm{O}(m)$-structures. Indeed, an $\mathrm{O}(m)$ structure on M is just a Riemannian metric g and the fundamental theorem of Riemannian geometry states that it admits a unique torsion-free connection, the *Levi-Civita connection*. Unless otherwise stated, all notions of curvature, connection and holonomy for a Riemannian metric will implicitly refer to its Levi-Civita connection.

For a proper subgroup $G \subset \mathrm{O}(m)$, the torsion-free condition becomes more subtle to decipher. If $\mathscr{P}$ is a G-structure, it induces a Riemannian metric g (since $G \subset \mathrm{O}(m)$) and any connection on $\mathscr{P}$ must be compatible with g. Therefore, $\mathscr{P}$ is torsion-free if and only if the Levi-Civita connection ∇^g of g reduces to a connection on $\mathscr{P}$. In general, this condition may not be satisfied and ∇^g will be written $\nabla^g = \nabla' + a$ where ∇' is a connection on $\mathscr{P}$ and $a \in \Omega^1(\mathrm{End}(TM))$. The 1-form a depends on the choice of compatible connection ∇' only up to a 1-form taking values in $\mathrm{Ad}(\mathscr{P}) = (\mathscr{P} \times \mathfrak{g})/G$, where $\mathfrak{g}$ is the Lie algebra of G. Hence if we denote by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{so}(m)$, the projection $\tau_{\mathscr{P}}$ of a onto $\mathfrak{m}_{\mathscr{P}} = (\mathscr{P} \times \mathfrak{m})/G$ does not depend on any arbitrary choice and is called the *intrinsic torsion* of $\mathscr{P}$[1]. Moreover, we see that $\mathscr{P}$ admits a unique connection $\nabla^{\mathscr{P}}$

[1]One may also define the intrinsic torsion if G is not a subgroup of $\mathrm{O}(m)$, but the definition is somewhat more involved. See [66] for instance.

such that $\nabla^g = \nabla^{\mathscr{P}} + \tau_{\mathscr{P}}$, called the *canonical connection*[2], and $\mathscr{P}$ is torsion-free if and only if $\nabla^g = \nabla^{\mathscr{P}}$.

As previously mentioned, we will be interested in the case where G is the stabiliser of a non-trivial vector Ψ in a linear representation of $\mathrm{GL}(m)$. It follows from our discussion that if a G-structure $\mathscr{P}$ is torsion-free then the corresponding section ψ is parallel, and therefore up to conjugacy the holonomy group of g must be a subgroup of G. The converse is also true: if $\mathrm{Hol}(g) \subset G$, then the Levi-Civita connection of g admits a parallel G-structure. In this sense, holonomy detects the presence of additional compatible geometric structures on a Riemannian manifold. For this reason, the classification of Riemannian holonomy groups plays a central role in geometry, and we will outline it in the next part.

1.1.3 Berger's list of Riemannian holonomy groups. The list of possible Riemannian holonomy groups is a priori very long, so in order to classify them it is reasonable to make a few assumptions. First, up to covering maps one may restrict to the case when M is simply connected, which forces the holonomy group of any Riemannian metric g on M to be connected. Moreover, one usually assumes that the metric g is irreducible, in the sense that the holonomy group acts irreducibly on the tangent space; otherwise one can easily show that (M, g) is locally a Riemannian product, and in fact a theorem of de Rham [35] implies that if (M, g) is complete, simply connected and reducible then it is globally a Riemannian product, in which case its holonomy group is just the direct product of the holonomy groups of each irreducible component.

A Riemannian manifold (M, g) whose curvature tensor is parallel is said to be *locally symmetric*. If moreover (M, g) is complete and simply connected, this condition implies that it must be a *symmetric space* (the geodesic involution at every point of M can be extended to a global isometry). Symmetric spaces are in particular homogeneous (there is a transitive isometric action by a Lie group) and they were classified by É. Cartan using his own classification of Lie groups. From this classification, it is possible to deduce the list of holonomy groups of symmetric spaces. Since a simply connected locally symmetric space is always isometric to an open subset of a symmetric space, this settles the classification of Riemannian holonomy groups in the locally symmetric case.

In his PhD thesis [11], Berger used Cartan's theory of Lie groups in order to derive constraints on the possible holonomy groups for non-symmetric ($\nabla R \not\equiv 0$)

[2]This terminology is not widely accepted since the notion of canonical connection may depend on the context, but it is suitable for our purpose.

dimension	**group**	**name/description**
$m \geq 1$	$\mathrm{SO}(m)$	generic case
$2m,\ m \geq 2$	$\mathrm{U}(m)$	Kähler
$2m,\ m \geq 2$	$\mathrm{SU}(m)$	Calabi–Yau/Ricci-flat Kähler
$4m,\ m \geq 2$	$\mathrm{Sp}(m)$	hyperkähler
$4m,\ m \geq 2$	$\mathrm{Sp}(1).\,\mathrm{Sp}(m)$	quaternionic-kähler
7 8	G_2 $\mathrm{Spin}(7)$	exceptional cases

Table 1.1: Berger's list of Riemannian holonomy groups.

simply connected spaces[3]. For this he used the Ambrose-Singer holonomy theorem together with the symmetries of the Riemann curvature tensor and the Bianchi identity to prove that for most irreducible and simply connected subgroups H of $\mathrm{SO}(m)$, a metric with holonomy H would have to be locally symmetric. A striking fact about the list of possible candidates that he obtained is that it is actually quite short - and it became even shorter a few years later when Alekseevskii ruled out the case of $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ [5]. With this amendment, the complete list is shown in Table 1.1.

At the time of Berger, it was not known whether all the groups on the list could actually occur as the holonomy group of an irreducible non-symmetric Riemannian manifold, except for $\mathrm{SO}(m)$ (the generic case) and $\mathrm{U}(m)$ (generic Kähler metrics). It took another thirty years or so to prove that it was indeed the case. The cases of $\mathrm{SU}(m)$ (Ricci-flat Kähler metrics), $\mathrm{Sp}(m)$ (hperkähler metrics) and $\mathrm{Sp}(1).\,\mathrm{Sp}(m)$ (quaternionic-kähler metrics - which are not in fact Kähler) are related to complex geometry, and by now there is a rather rich set of tools available to study them. Complete examples of such metrics were notably given by Calabi [21] (for $\mathrm{SU}(m)$ and $\mathrm{Sp}(m)$) and Galicki and Lawson [44, 45] (for $\mathrm{Sp}(m).\,\mathrm{Sp}(1)$), and in the compact setting the existence of Kähler Ricci-flat metrics follows from Yau's solution of the Calabi conjecture [123] - hence such metrics are also called Calabi–Yau.

The question of the existence of metrics with exceptional holonomy - G_2 and $\mathrm{Spin}(7)$ - remained open for a long time, before being finally settled by Bryant [16] in 1987, using the theory of exterior differential systems[4]. A few years later, Bryant–Salamon constructed the first complete examples using cohomogeneity 1 techniques [19]. The first compact examples are due to Joyce [64, 65, 63], who developed a general gluing-perturbation method which is still some thirty years later the only available tool for constructing compact exceptional holonomy manifolds.

[3] In fact Berger considered the more general case of metrics with arbitrary signature.
[4] Which incidentally was also first developed by É. Cartan and further extended by Kähler.

1.2 G_2-geometry

This section gathers some basic notions of G_2-geometry. In §1.2.1, we recall the definition of positive forms in $\mathbb{R}^7$ and a few elements of their linear algebra. G_2-manifolds are introduced in §1.2.2, and their moduli spaces in §1.2.4.

1.2.1 Positive forms on $\mathbb{R}^7$. Let us consider $\mathbb{R}^7$ equipped with its standard orientation and denote by $\mathbb{R}^*_7$ its dual space. A 3-form $\varphi \in \Lambda^3\mathbb{R}^*_7$ is said to be *positive* if for any $v \in \mathbb{R}^7\backslash\{0\}$ we have

$$(v\lrcorner\varphi) \wedge (v\lrcorner\varphi) \wedge \varphi > 0 \tag{1.1}$$

relative to the standard orientation. Here $\cdot\lrcorner\cdot$ denotes the interior product of a vector in $\mathbb{R}^7$ and an alternating form in $\Lambda(\mathbb{R}^*_7)$. The set $\Lambda^3_+\mathbb{R}^*_7$ of positive forms is nonempty and open in $\Lambda^3\mathbb{R}^*_7$, and is acted upon transitively by the group of orientation-preserving automorphisms $\mathrm{GL}_+(7)$. The stabiliser of any positive form is conjugated to the group $\mathrm{G}_2 \subset \mathrm{SO}(7)$. This is a compact, simple Lie group of dimension 14. A positive form $\varphi \in \Lambda^3_+\mathbb{R}^*_7$ canonically determines an inner product on $\mathbb{R}^7$, which we denote by g_φ or $\langle\cdot,\cdot\rangle_\varphi$, and a 7-from $\mu_\varphi \in \Lambda^7\mathbb{R}^*_7$ characterised by

$$\begin{aligned}(v\lrcorner\varphi) \wedge (u\lrcorner\varphi) \wedge \varphi &= 6\langle u, v\rangle_\varphi \mu_\varphi, \quad \forall u, v \in \mathbb{R}^7, \quad \text{and}\\ |\varphi|^2_{g_\varphi} &= 7.\end{aligned} \tag{1.2}$$

The dual 4-form of φ with respect to the Hodge operator $*_\varphi$ associated with g_φ is commonly denoted by $\Theta(\varphi) \in \Lambda^4\mathbb{R}^*_7$. The maps $\varphi \mapsto g_\varphi$, $\varphi \mapsto \mu_\varphi$, $\varphi \mapsto *_\varphi$ and $\varphi \mapsto \Theta(\varphi)$ are non-linear and equivariant under the action of $\mathrm{GL}_+(7)$.

Example 1.1. Let $\mathbb{R}^7$ be endowed with the canonical basis $(e_1, \ldots, e_7)$ and let $(e^1, \ldots, e^7)$ be the dual basis of $\mathbb{R}^*_7$. For conciseness we shall write $e^{i_1\cdots i_k} = e^{i_1} \wedge \cdots \wedge e^{i_k}$ for any $1 \leq i_1, \ldots, i_k \leq 7$. Then we can define the following standard positive form:

$$\varphi_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}.$$

The associated inner product g_0 is the one for which the canonical basis of $\mathbb{R}^7$ is orthonormal, and the dual 4-form of φ_0 reads:

$$\Theta(\varphi_0) = e^{3456} + e^{1256} + e^{1234} - e^{2467} + e^{2357} + e^{1457} + e^{1367}.$$

Any positive 3-form φ, together with the 4-form $\Theta(\varphi)$ and the inner product $\langle\cdot,\cdot\rangle_\varphi$, satisfy the following important property. If $u, v, w, z \in \mathbb{R}^7$ have unit norm with respect to $\langle\cdot,\cdot\rangle_\varphi$, then the following inequalities hold:

$$|\varphi(u, v, w)| \leq 1 \quad \text{and} \quad |\Theta(\varphi)(u, v, w, z)| \leq 1.$$

In other words, for any oriented 3-plane $V \subset \mathbb{R}^7$ the restriction of φ to V is smaller or equal to the volume element associated with the restriction of φ. Similarly, for any oriented 4-plane $W \subset \mathbb{R}^7$ the restriction of $\Theta(\varphi)$ is smaller or equal to the volume element of the restriction of g_φ. The 3-planes (respectively 4-planes) realising the equality case are called *associatives* (respectively *co-associatives*).

Let us now fix a positive form φ on $\mathbb{R}^7$, and identify the stabiliser of φ with G_2. The exterior algebra $\Lambda(\mathbb{R}^*_7)$ can be decomposed into irreducible representations of G_2 as follows. The representation $\mathbb{R}^*_7$ is irreducible, as G_2 acts transitively on the unit sphere. The space of 2-forms can be decomposed as:

$$\Lambda^2\mathbb{R}^*_7 = \Lambda^2_{14} \oplus \Lambda^2_7$$

where Λ^2_{14} is isomorphic to the Lie algebra of G_2 and $\Lambda^2_7 \simeq \mathbb{R}^*_7$. In particular, any $\omega \in \Lambda^2\mathbb{R}^*_7$ can be written uniquely as $\omega = v \lrcorner \varphi + \chi$, where $v \in \mathbb{R}^7$ and $\chi \in \Lambda^2_{14}$. In order to decompose $\Lambda^3\mathbb{R}^*_7$, let us introduce a bilinear map $\mathrm{End}(\mathbb{R}^7) \otimes \Lambda(\mathbb{R}^*_7) \to \Lambda(\mathbb{R}^*_7)$ defined by:

$$h \cdot \eta = \left.\frac{d}{dt}\right|_{t=0} (e^{th})^*\eta = \eta(h\cdot, \cdot, \cdots) + \cdots + \eta(\cdots, \cdot, h\cdot), \quad \forall (h, \eta) \in \mathrm{End}(\mathbb{R}^7) \times \Lambda(\mathbb{R}^*_7). \tag{1.3}$$

Up to a sign, this is the derivative of the action of $\mathrm{GL}(7)$ on $\Lambda\mathbb{R}^*_7$. Since $\mathrm{GL}_+(7)$ acts transitively on $\Lambda^3_+\mathbb{R}^*_7$ which is open in $\Lambda^3\mathbb{R}^*_7$, the map $h \in \mathrm{End}(\mathbb{R}^7) \mapsto h \cdot \varphi \in \Lambda^3\mathbb{R}^*_7$ is onto. The representation $\mathrm{End}(\mathbb{R}^7)$ can be decomposed as:

$$\mathrm{End}(\mathbb{R}^7) \simeq \Lambda^2\mathbb{R}^*_7 \oplus S^2\mathbb{R}^*_7 \simeq \Lambda^2_{14} \oplus \Lambda^2_7 \oplus \mathbb{R} \oplus S^2_0\mathbb{R}^*_7$$

where Λ^2_{14} is identified with the Lie algebra of G_2 and $S^2_0\mathbb{R}^*_7$ is isomorphic to the space of trace-free self-adjoint endomorphisms with respect to g_φ. The kernel of the above map $\mathrm{End}(\mathbb{R}^7) \to \Lambda^3\mathbb{R}^*_7$ is Λ^2_{14}, and therefore we obtain the decomposition:

$$\Lambda^3\mathbb{R}^*_7 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$$

where $\Lambda^3_7 \simeq \mathbb{R}^*_7$ and $\Lambda^3_{27} \simeq S^2_0\mathbb{R}^*_7$. In particular, any 3-form $\eta \in \Lambda^3\mathbb{R}^*_7$ can be written uniquely as $\eta = \lambda\varphi + v \lrcorner \Theta(\varphi) + \nu$, where $\lambda \in \mathbb{R}$, $v \in \mathbb{R}^7$ and $\nu \in \Lambda^3_{27}$. As $\Lambda^k\mathbb{R}^*_7 \simeq \Lambda^{7-k}\mathbb{R}^*_7$ under Hodge duality, this give a full decomposition of $\Lambda\mathbb{R}^*_7$. We shall denote by π_m the projection of $\Lambda^k\mathbb{R}^*_7$ onto Λ^k_m.

We finish these generalities with a few useful formulas for the first variation of various tensors associated with an inner product or a positive form on $\mathbb{R}^7$, and some interesting consequences. First, we begin with some properties of the bilinear map $\mathrm{End}(\mathbb{R}^7) \otimes \Lambda(\mathbb{R}^*_7) \to \Lambda(\mathbb{R}^*_7)$ previously defined:

Lemma 1.2. *For $h \in \mathrm{End}(\mathbb{R}^7)$ we denote by $\delta_h : \Lambda(\mathbb{R}^*_7) \to \Lambda(\mathbb{R}^*_7)$ the linear map $\eta \mapsto h \cdot \eta$. Then for $h, h' \in \mathrm{End}(\mathbb{R}^7)$ the following properties are satisfied:*

(i) *The map δ_h is a derivation of degree 0 of $\Lambda(\mathbb{R}^*_7)$. That is, it preserves the degree of forms and $h\cdot(\omega\wedge\omega') = (h\cdot\omega)\wedge\omega' + \omega\wedge(h\cdot\omega')$ for any $\omega, \omega' \in \Lambda(\mathbb{R}^*_7)$.*

(ii) $[\delta_h, \delta_{h'}] = -\delta_{[h,h']}$.

(iii) *If h is (anti-)self-adjoint for some inner product on $\mathbb{R}^7$, then δ_h is (anti-)self-adjoint for the induced inner product on $\Lambda(\mathbb{R}^*_7)$.*

Proof. That δ_h is a derivation of degree 0 can be seen by differentiating the identity $(e^{th})^*(\omega \wedge \omega') = (e^{th})^*\omega \wedge (e^{th})^*\omega'$. Moreover, by definition $\delta : \mathrm{End}(\mathbb{R}^7) \to \mathrm{End}(\Lambda(\mathbb{R}^*_7))$ is the negative of the natural action of the Lie algebra $\mathrm{End}(\mathbb{R}^7)$ on $\Lambda(\mathbb{R}^*_7)$, and thus $[\delta_h, \delta_{h'}] = -\delta_{[h,h']}$. Last, if h is g-self-adjoint and $\omega \in \mathbb{R}^*_7$, then the dual vector of $\delta_h\omega = \omega \circ h$ is $h(v)$, where $v \in \mathbb{R}^7$ is dual to ω. From this it follows that δ_h is self-adjoint for the inner product induced by g on $\mathbb{R}^*_7$, and thus on $\Lambda(\mathbb{R}^*_7)$. We can argue similarly when h is anti-self-adjoint for g, since then the dual vector of $\omega \circ h \in \mathbb{R}^*_7$ is $-h(v)$ if $v \in \mathbb{R}^7$ is the vector dual to ω. □

The next lemma gathers a few useful identities which are easy to check.

Lemma 1.3. *Let g be an inner product on $\mathbb{R}^7$ and $h \in \mathrm{End}(\mathbb{R}^7)$, and consider a 1-parameter family of inner products g_t such that $g_0 = g$ and $\frac{dg_t}{dt}\big|_{t=0} = g(h\cdot,\cdot) + g(\cdot, h\cdot)$. Let $\omega, \omega' \in \Lambda^k\mathbb{R}^*_7$ for some $0 \le k \le 7$. Then we have the following first variation formulas:*

$$\frac{d}{dt}\Big|_{t=0} \langle\omega,\omega'\rangle_{g_t} = -\langle h\cdot\omega,\omega'\rangle_g - \langle\omega, h\cdot\omega'\rangle_g,$$
$$\frac{d}{dt}\Big|_{t=0} *_{g_t}\omega = h\cdot(*_g\omega) - *_g(h\cdot\omega),$$
$$\frac{d}{dt}\Big|_{t=0} \mu_{g_t} = \mathrm{tr}(h)\mu_g.$$

These two lemmas have a few consequences that will be useful in the rest of the article. First note that if h is self-adjoint for g, then δ_h is self-adjoint for the inner product induced by g on $\Lambda(\mathbb{R}^*_7)$ and thus with the notations of the above lemma we have

$$\frac{d}{dt}\Big|_{t=0} \langle\omega,\omega'\rangle_{g_t} = -2\langle h\cdot\omega,\omega'\rangle_g$$

for any $\omega, \omega' \in \Lambda^k\mathbb{R}^*_7$. This implies:

Corollary 1.4. *Let $h, h' \in \mathrm{End}(\mathbb{R}^7)$, and suppose that h is a trace-free endomorphism, self-adjoint with respect to an inner product g, and h' is anti-self-adjoint for g. Then for any $\omega \in \Lambda^k \mathbb{R}^*_7$ we have:*

$$h \cdot (*_g \omega) = - *_g (h \cdot \omega), \quad \text{and} \quad h' \cdot (*_g \omega) = *_g (h' \cdot \omega).$$

Proof. Consider the family of inner products $g_t = (e^{th})^* g$. Using the previous lemmas, we can differentiate the identity $\omega' \wedge *_{g_t} \omega = \langle \omega', \omega \rangle_{g_t} \mu_{g_t}$ at $t = 0$ which yields:

$$\omega' \wedge h \cdot (*_g \omega) - \omega' \wedge *_g (h \cdot \omega) = -2 \langle \omega', h \cdot \omega \rangle_g \mu_g = -2\omega' \wedge *_g (h \cdot \omega)$$

and hence $\omega' \wedge h \cdot (*_g \omega) = -\omega' \wedge *_g (h \cdot \omega)$ for any $\omega' \in \Lambda^k \mathbb{R}^*_7$, which proves the first identity.

For the second identity, we note that since h' is anti-self-adjoint for g, the linear isomorphisms $e^{th'}$ preserve g, and thus

$$*_g \omega = e^{th'} (*_g e^{-th'} \omega)$$

for any t, and differentiating at $t = 0$ it follows that $h' \cdot (*_g \omega) - *_g (h' \cdot \omega) = 0$. □

Another useful consequence to note is:

Corollary 1.5. *If φ is a positive form on $\mathbb{R}^7$, then the cubic form*

$$(h_1, h_2, h_3) \in S^2 \mathbb{R}^*_7 \times S^2 \mathbb{R}^*_7 \times S^2 \mathbb{R}^*_7 \longmapsto \langle h_3 \cdot h_1 \cdot \varphi, h_2 \cdot \varphi \rangle_\varphi \in \mathbb{R}$$

is fully symmetric.

Proof. The identity

$$\langle h_3 \cdot h_1 \cdot \varphi, h_2 \cdot \varphi \rangle_\varphi = \langle h_1 \cdot \varphi, h_3 \cdot h_2 \cdot \varphi \rangle_\varphi = \langle h_3 \cdot h_2 \cdot \varphi, h_1 \cdot \varphi \rangle_\varphi$$

holds because δ_{h_3} is self-adjoint for the inner product induced by φ in $\Lambda(\mathbb{R}^*_7)$. Thus the cubic form is symmetric under permutation of h_1 and h_2. To prove that it is also symmetric under permutation of h_1 and h_3, note that since $[\delta_{h_3}, \delta_{h_1}] = -\delta_{[h_3, h_1]}$ we have

$$\langle h_3 \cdot h_1 \cdot \varphi, h_2 \cdot \varphi \rangle_\varphi - \langle h_1 \cdot h_3 \cdot \varphi, h_2 \cdot \varphi \rangle_\varphi = \langle [h_1, h_3] \cdot \varphi, h_2 \cdot \varphi \rangle_\varphi = 0$$

where the last equality follows from the fact that $[h_1, h_3]$ is anti-self-adjoint, and thus $[h_1, h_3] \cdot \varphi \in \Lambda^3_7$ is orthogonal to $h_2 \cdot \varphi \in \Lambda^3_1 \oplus \Lambda^3_{27}$. □

Finally, we record the following well-known first variations formulas:

Lemma 1.6. *Let φ be a positive form on $\mathbb{R}^7$, $\eta \in \Lambda^3\mathbb{R}^*_7$ and let $h \in \mathrm{End}(\mathbb{R}^7)$ be the unique endomorphism orthogonal to Λ^2_{14} such that $h \cdot \varphi = \eta$. Let φ_t be a 1-parameter family of positive forms in $\mathbb{R}^7$ such that $\varphi_0 = \varphi$ and $\frac{d\varphi_t}{dt}\Big|_{t=0} = \eta$, and let $\omega, \omega' \in \Lambda^k\mathbb{R}^*_7$ for some $0 \leq k \leq 7$. Then we have the following first variation formulas:*

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \langle \omega, \omega' \rangle_{\varphi_t} &= -\langle h \cdot \omega, \omega' \rangle_{\varphi} - \langle \omega, h \cdot \omega' \rangle_{\varphi}, \\
\frac{d}{dt}\Big|_{t=0} *_{\varphi_t} \omega &= h \cdot (*_{\varphi}\omega) - *_{\varphi}(h \cdot \omega), \\
\frac{d}{dt}\Big|_{t=0} \mu_{\varphi_t} &= \mathrm{tr}(h)\mu_{\varphi}, \\
\frac{d}{dt}\Big|_{t=0} \Theta(\varphi_t) &= \frac{4}{3} *_{\varphi} \pi_1(\eta) + *_{\varphi}\pi_7(\eta) - *_{\varphi}\pi_{27}(\eta).
\end{aligned}$$

1.2.2 G_2-manifolds. Let M be an oriented 7-manifold. According to the discussion of the previous section, a G_2-*structure* corresponds to the data of a 3-form φ such that $\varphi_p \in T_pM$ is positive for every $p \in M$. Not all 7-manifolds admit G_2-structures: a necessary and sufficient existence condition is that M be orientable and spin [66]. In the remainder of this section we shall assume that these conditions are satisfied. Moreover, we will denote by $\Lambda^3_+T^*M \subset \Lambda^3T^*M$ the open subbundle of positive forms and $\Omega^3_+(M)$ the set of sections of $\Lambda^3_+T^*M$ (which is nonempty by assumption).

The properties of positive forms on $\mathbb{R}^7$ carry over to G_2-structures on manifolds. In particular, a G_2-structure $\varphi \in \Omega^3_+(M)$ determines a Riemannian metric g_φ, a volume form μ_φ and a 4-form $\Theta(\varphi) = *_\varphi\varphi$. Moreover, φ induces a splitting of the exterior bundle $\Lambda(T^*M)$ and identifications

$$\begin{aligned}
&T^*M \simeq TM, && \\
&\Lambda^2T^*M = \Lambda^2_7T^*M \oplus \Lambda^2_{14}T^*M, && \Lambda^2_7T^*M \simeq T^*M, \\
&\Lambda^3T^*M = \Lambda^3_1T^*M \oplus \Lambda^3_7T^*M \oplus \Lambda^3_{27}T^*M, && \Lambda^3_1T^*M \simeq \underline{\mathbb{R}},\ \Lambda^3_7T^*M \simeq T^*M, \\
&\Lambda^kT^*M \simeq \Lambda^{7-k}T^*M, && k = 0, \dots, 7
\end{aligned}$$

where we denote by $\underline{\mathbb{R}}$ the trivial real line bundle $M \times \mathbb{R}$. There is a corresponding splitting of the algebra of differential forms on M, and we will write $\Omega^k(M) = \oplus_m\Omega^k_m(M)$ where $\Omega^k_m(M) = C^\infty(\Lambda^k_mT^*M)$, and denote by π_m the projection of $\Omega^k(M)$ onto $\Omega^k_m(M)$. Any 2-form ω on M can be written uniquely as $\omega = \xi \lrcorner \varphi + \chi$ with $\xi \in C^\infty(TM)$ and $\chi \in \Omega^2_{14}(M)$, and any 3-form η can be written uniquely as $\eta = f\varphi + \xi \lrcorner \Theta(\varphi) + \nu$ where $f \in C^\infty(M)$, $\xi \in C^\infty(TM)$ and $\nu \in \Omega^3_{27}(M)$.

Another useful way to describe a 3-form is to decompose $\mathrm{End}(TM)$ as

$$\mathrm{End}(TM) \simeq \Lambda^2 T^*M \oplus S^2 T^*M = \Lambda^2_{14} T^*M \oplus \Lambda^2_7 T^*M \oplus \mathbb{R} g_\varphi \oplus S^2_0 T^*M$$

where $S^2_0 T^*M \simeq \Lambda^3_{27} T^*M$. Then for any 3-form $\eta \in \Omega^3(M)$, there exists a unique section $h \in C^\infty(\mathrm{End}(TM))$ orthogonal to $\Omega^2_{14}(M)$ such that $\eta = h \cdot \varphi$. Note that $\pi_7(\eta) = 0$ if and only if h is a self-adjoint endomorphism for the metric g_φ.

As we discussed in §1.1.2 in the general context of geometric structures, a G_2-structure φ has an associated (intrinsic) torsion $\tau(\varphi)$. Since the orthogonal complement of the Lie algebra $\mathfrak{g}_2$ in $\mathfrak{so}(7)$ is isomorphic to $\mathbb{R}^7$ as representation of G_2, $\tau(\varphi)$ is a section of the bundle $T^*M \otimes TM \simeq \mathrm{End}(TM)$. Hence the torsion $\tau(\varphi)$ can be decomposed into four components commonly denoted by $\tau_0 \in C^\infty(M)$, $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2_{14}(M)$ and $\tau_3 \in \Omega^3_{27}(M)$. These torsion forms are related to $d\varphi$ and $d(\Theta(\varphi))$ in the following way:

$$d\varphi = \tau_0 \Theta(\varphi) + 3\tau_1 \wedge \varphi + *_\varphi \tau_3, \qquad d(\Theta(\varphi)) = 4\tau_1 \wedge \Theta(\varphi) + \tau_2 \wedge \varphi.$$

A G_2-structure φ is called *torsion-free* if $\tau(\varphi) \equiv 0$, that is if φ is parallel for the Levi-Civita connection of g_φ. Because of the above identities, this amounts to the condition that φ be closed and co-closed [41]. If this is satisfied, then the holonomy group of g_φ is conjugated to a subgroup of G_2, and in particular the metric g_φ is *Ricci-flat* [14]. A 7-manifold M endowed with a torsion-free G_2-structure is called a G_2*-manifold*. As we mentioned in the previous section, the existence of metrics with holonomy G_2 is known in both the compact and noncompact (and complete) settings. However, the question of which 7-manifolds admit torsion-free G_2-structures is far from fully understood, even though the topological condition of existence of G_2-structures is quite simple. One reason which accounts for this is that the equations $d\varphi = 0 = d(\Theta(\varphi))$ are highly nonlinear, due to the nonlinearity of the map $\varphi \mapsto \Theta(\varphi)$. If M is compact, one can still deduce certain necessary topological conditions. For instance, since a torsion-free G_2-structure is always a non-trivial harmonic form, Hodge theory implies that $b^3(M) > 0$. Similarly, it is known that the first Pontryagin class $p_1(M) \in H^4(M; \mathbb{Z})$ of a compact nonflat G_2-manifold is always nonzero [66]. But we do not know any sufficient topological conditions for the existence of torsion-free G_2-structures on a compact manifold.

Let us now discuss further consequences of Hodge theory on a compact connected G_2-manifold (M, φ). Due to a Weitzenböck formula, the Laplacian operator $\Delta_\varphi = dd^{*_\varphi} + d^{*_\varphi} d$ associated with g_φ leaves invariant each component of the splitting $\Omega^k(M) = \oplus \Omega^k_m(M)$. Therefore, the Hodge theorem yields a decomposition of the de Rham cohomology groups $H^k(M) \simeq \oplus H^k_m(M)$, and moreover isomorphic representations lead to isomorphic components in cohomology. In particular:

$$H^1(M) \simeq H^2_7(M) \simeq H^3_7(M) \text{ and } H^3_1(M) \simeq H^0(M) \simeq \mathbb{R}.$$

This decomposition is analogous to the decomposition of the cohomology of a compact Kähler manifold into classes of type (p,q). We will denote by $\mathscr{H}^k(M,\varphi)$ the space of k-forms harmonic with respect to g_φ, by $\mathscr{H}^k_m(M,\varphi)$ the intersection of $\mathscr{H}^k(M,\varphi)$ and $\Omega^k_m(M)$, and define the refined Betti numbers $b^k_m(M) = \dim H^k_m(M)$. Since the metric g_φ is Ricci-flat, $\mathscr{H}^1(M)$ is exactly the space of parallel 1-forms on M, and is dual to the space of Killing fields. The Cheeger–Gromoll splitting theorem [24] implies that g_φ has full holonomy G_2 if and only if $\pi_1(M)$ is finite [66, Prop. 10.2.2]. When this condition (or the weaker condition $b^1(M) = 0$) is satisfied, then $H^1(M) = H^2_7(M) = H^3_7(M) = 0$ and the Hodge decomposition is reduced to $H^2(M) = H^2_{14}(M)$ and $H^3(M) = H^3_1(M) \oplus H^3_{27}(M)$. Since $b^3_1(M) = b^0(M) = 1$, the only undetermined refined Betti numbers are $b^2_{14}(M) = b^2(M)$ and $b^3_{27} = b^3(M) - 1$.

1.2.3 Dimensional reduction. For later use, it will be important to describe the geometry of G_2-manifolds which do not have full holonomy. In the compact setting, this occurs when the fundamental group is infinite. In general, if (M,φ) is a complete G_2-manifold which does not have full holonomy, then the universal cover $\widetilde{M}$ of M, endowed with the induced torsion-free G_2-structure $\widetilde{\varphi}$, must be isometric to a product $\mathbb{R}^k \times (N^{7-k}, g_N)$ where N is a simply connected manifold endowed with a complete Ricci-flat metric which does not split a Euclidean factor. If M is compact, the Cheeger–Gromoll splitting theorem implies that there is a finite cover of M isometric to $T^k \times (N, g_N)$ where T^k is a flat torus and N is compact. In any case, the holonomy group of g_N must be a proper subgroup of G_2 and also be an item of Berger's list (a priori it could also be a product thereof, but this does not occur). There are only three such subgroups: the trivial group 1, SU(2) and SU(3). Let us describe the geometry associated with each of these holonomy groups and describe their relations to G_2-geometry.

In the simplest case where the reduced holonomy group is trivial, then N is reduced to a point and the universal cover of M is isometric to a flat $\mathbb{R}^7$. Up to a linear change of coordinates, $\widetilde{\varphi}$ can be identified with the standard positive form φ_0 from Example 1.1. Hence M must be isometric to the quotient of $(\mathbb{R}^7, \varphi_0)$ by a discrete subgroup of $G_2 \ltimes \mathbb{R}^7$. Compact examples can be constructed by taking quotients of flat tori.

When the holonomy group of g_N is $\mathrm{SU}(2) \subset \mathrm{SO}(4)$, then $k = 3$ and the metric g_N is *hyperkähler*. This means that N admits a triple of symplectic (closed and non-degenerate) 2-forms $\omega_1, \omega_2, \omega_3$ satisfying the relations

$$\frac{1}{2}\omega_i \wedge \omega_j = \delta_{ij}\mu_{g_N}$$

where μ_{g_N} is the volume form of g_N. Each symplectic form is Kähler with respect to an integrable complex structure J_i defined by $\omega_i = g_N(J_i\cdot, \cdot)$. The triple of complex structures (J_1, J_2, J_3) satisfy the well-known quaternionic commutation relations; this is related to the isomorphism $\mathrm{SU}(2) \simeq \mathrm{Sp}(1)$ of Lie groups. Noncompact complete examples of hyperkähler 4-manifolds include the Eguchi–Hanson metric on T^*S^2, and ALE or ALF manifolds constructed via the Gibbons–Hawking ansatz. In the compact case, the only simply-connected examples are K3 surfaces. If one choses coordinates (t_1, t_2, t_3) on $\mathbb{R}^3$ and $(N, g_{\underline{\omega}})$ is a hyperkähler 4-manifold with hyperkähler triple $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$, then $\mathbb{R}^3 \times N$ can be endowed with a torsion-free G_2-structure defined as

$$dt_1 \wedge dt_2 \wedge dt_3 - dt_1 \wedge \omega_1 - dt_2 \wedge \omega_2 - dt_3 \wedge \omega_3$$

whose associated metric is $dt_1^2 + dt_2^2 + dt_3^2 + g_{\underline{\omega}}$.

The last possible case in this list is when $k = 1$, g_N has holonomy $\mathrm{SU}(3)$ and N has dimension 6. Like the previous case, it fits within the more general case of $2m$-manifolds with holonomy $\mathrm{SU}(m)$. Since $\mathrm{SU}(m) \subset \mathrm{U}(m)$ such manifolds are in particular Kähler: they admit an integrable complex structure J and a symplectic form $\omega = g(J\cdot, \cdot)$. The restriction from $\mathrm{U}(m)$ to $\mathrm{SU}(m)$ corresponds to the existence of a parallel holomorphic volume form Ω (a nowhere vanishing holomorphic section of the canonical bundle). The existence of a parallel holomorphic volume form is equivalent to the condition that the canonical bundle be trivial and the Kähler metric g be Ricci-flat. The fact that any compact Kähler manifold with trivial canonical bundle admits a unique Ricci-flat Kähler metric in every Kähler class is a consequence of Yau's theorem which we have already mentioned before.

It is worth pointing out that, by the Kodaira embedding theorem [77], any compact m-fold with holonomy $\mathrm{SU}(m)$ is automatically projective if $m \geq 3$ (this statement fails if $m = 2$). As a consequence of Yau's theorem, compact Calabi–Yau manifolds are easy to produce: for instance, any smooth hypersurface in $\mathbb{CP}^m$ defined by the vanishing of a homogeneous polynomial of degree $m + 1$ has trivial canonical bundle by the adjunction formula, and hence admits Ricci-flat Kähler metrics. In the noncompact setting, there is also a wealth of constructions, notably due to Calabi [21] and Tian–Yau [109, 110].

Going back to the case $m = 3$, let g_N be a Calabi–Yau metric on a complex manifold (N, J), let ω be the corresponding Kähler form and pick a holomorphic volume form Ω satisfying the normalisation condition

$$\frac{1}{6}\omega^3 = \frac{1}{4}\Omega \wedge \overline{\Omega}.$$

Then one can define an associated torsion-free G_2-structure on $\mathbb{R} \times N$ as

$$dt \wedge \omega + \operatorname{Re} \Omega$$

where t is the coordinate along $\mathbb{R}$. The associated metric is $dt^2 + g_N$.

1.2.4 Moduli spaces. Let M be a compact oriented 7-manifold which admits torsion-free G_2-structures. We denote by $\mathscr{D}$ the group of diffeomorphisms of M acting trivially on $H^3(M)$. In particular, it contains the group of diffeomorphisms isotopic to the identity, but it could be larger. The group $\mathscr{D}$ acts by pull-back on the space $\Omega^3_+(M)$ of G_2-structures on M, leaving invariant the subset of torsion-free G_2-structures. The moduli space $\mathscr{M}$ of torsion-free G_2-structures is defined as the quotient of the set of torsion-free G_2-structures by this action. It has a natural topology coming from the C^∞-topology of $\Omega^3_+(M)$, and it was proven by Joyce [64, Th. C] that it admits a compatible manifold structure of dimension $b^3(M)$. Moreover, the map $\mathscr{M} \to H^3(M)$ sending $\varphi\mathscr{D} \in \mathscr{M}$ to the cohomology class $[\varphi] \in H^3(M)$ is a local diffeomorphism. This endows $\mathscr{M}$ with a natural atlas of charts with affine transition functions, and therefore $\mathscr{M}$ even has the structure of an *affine manifold*.

Remark 1.7. In fact our definition slightly differs from the convention adopted in [64] (or in Joyce's monograph [66, §10.4]) where one takes the quotient by the group $\mathscr{D}_0$ of diffeomorphisms isotopic to the identity (the resulting space may be called the 'Teichmüller space' $\mathscr{T}$ of torsion-free G_2-structures). With our definition, we are taking a further quotient by the discrete group $\Gamma = \mathscr{D}/\mathscr{D}_0$, but since the Teichmüller space is locally diffeomorphic to $H^3(M)$ and $\mathscr{D}$ acts trivially on this space, it follows that the quotient $\mathscr{M} = \mathscr{T}/\Gamma$ is nonsingular and locally diffeomorphic to $H^3(M)$.

Hence $\mathscr{M}$ is more of a 'marked moduli space' (we fix an identification of $H^3(M)$ with $\mathbb{R}^{b_3(M)}$) of torsion-free G_2-structures on M, but we will just call it the moduli space for simplicity. This does not affect the results which we will prove in Chapters 4 and 5, since with either convention the moduli space is smooth and locally diffeomorphic to $H^3(M)$.

By means of justifying our choice of convention, let us point out the following somewhat subtle issue. Let us look at the moduli space of torsion-free G_2-structures on the torus $T^7 = \mathbb{R}^7/\mathbb{Z}^7$. It is natural to guess that it can be identified with $\Lambda^3_+\mathbb{R}^*_7$. It is not difficult to see that this is true, *with our convention*. The issue is that the structure of the mapping class group of T^7 is rather complicated, and in particular there are elements of the mapping class group which act trivially on the cohomology $H^\bullet(T^7)$ (this is true for T^n for any $n \geq 5$ [58, 62]). In particular, $\Gamma = \mathscr{D}/\mathscr{D}_0$ is a non-trivial (even infinitely generated) group, and it acts freely transitively on the fibres of the covering map $\mathscr{T} \to \mathscr{M}$.

To see this last point, let φ be a torsion-free G_2-structure on T^7, which we may assume to be induced by a linear positive form on $\mathbb{R}^7$. If $\alpha_1, \alpha_2 \in \mathscr{D}$ satisfy $\alpha_1^*\varphi = \alpha_2^*\varphi$, then $\alpha_2\alpha_1^{-1}$ is an isometry for the metric g_φ, which is induced by an inner product on $\mathbb{R}^7$. Hence $\alpha_2\alpha_1^{-1} \in \mathrm{G}_2 \ltimes \mathbb{R}^7$. The action of the linearisation $A \in \mathrm{G}_2$ of $\alpha_2\alpha_1^{-1}$ on $\mathbb{R}^*_7$ can be identified with its action on $H^1(T^7)$, and the action of $\alpha_2\alpha_1^{-1}$ on $H^3(T^7)$ can be identified with the action of A on $\Lambda^3\mathbb{R}^*_7$. But the representation of G_2 on $\Lambda^3\mathbb{R}^*_7$ is faithful and by assumption α_1, α_2 act trivially on $H^3(T^7)$, whence we deduce that $A = \mathrm{Id}$. Therefore $\alpha_2\alpha_1^{-1}$ is a translation of T^7, and as such it is isotopic to the identity. That is, α_1 and α_2 define the same element of Γ, which proves our claim.

Let us now outline the construction of the manifold structure of $\mathscr{M}$ for later use; or equivalently of the Teichmüller space $\mathscr{T}$. The first step is to find a convenient description, as a topological space, of the quotient of $\Omega^3_+(M)$ by the action of $\mathscr{D}_0$. Adapting a result of Ebin [40], this essentially boils down to finding a good slice for the action of $\mathscr{D}_0$. Near a torsion-free G_2-structure, this leads to the following local description [66, Th. 10.4.1]:

Proposition 1.8. *Let $\varphi \in \Omega^3_+(M)$ be a torsion-free G_2-structure on M, and denote by I_φ the subgroup of $\mathscr{D}_0$ fixing φ. Define $L_\varphi = \{\widetilde{\varphi} \in \Omega^3_+(M),\ \pi_7(d^*\widetilde{\varphi}) = 0\}$, where π_7 and d^* come from the G_2-structure φ and the associated metric. Then there exists an open neighbourhood S_φ of φ in L_φ, invariant under I_φ, such that the natural projection from S_φ/I_φ to $\Omega^3_+(M)/\mathscr{D}_0$ induces a homeomorphism between S_φ/I_φ and a neighbourhood of $\varphi\mathscr{D}_0$ in $\Omega^3_+(M)/\mathscr{D}_0$.*

Let us now fix a torsion-free G_2-structure on M. We seek to understand the subspace of $\Omega^3_+(M)/\mathscr{D}_0$ defined by $\mathscr{T}$ in a neighbourhood of $\varphi\mathscr{D}_0$. To that end, let us denote by L'_φ the intersection of L_φ with the set of torsion-free G_2-structures; that is, $L'_\varphi = \{\widetilde{\varphi} \in \Omega^3_+(M),\ \pi_7(d^*\widetilde{\varphi}) = 0 \text{ and } d\widetilde{\varphi} = d\Theta(\widetilde{\varphi}) = 0\}$. As in the above proposition, there exists a neighbourhood S'_φ of φ in L'_φ, invariant under I_φ, such that the natural projection from S'_φ/I_φ to $\mathscr{T}$ induces of homeomorphism from S'_φ/I_φ to a neighbourhood of $\varphi\mathscr{D}_0$ in $\mathscr{T}$. In order to obtain a characterisation of the torsion-free G_2-structures close to φ, let us write:

$$\Theta(\varphi+\eta) = \Theta(\varphi) + \frac{4}{3} *_\varphi \pi_1(\eta) + *_\varphi\pi_7(\eta) - *_\varphi\pi_{27}(\eta) + F_\varphi(\eta) \tag{1.4}$$

where the terms containing projections are the linearisation of Θ at φ and F_φ is a smooth non-linear map defined for η small enough. Let $\widetilde{\varphi}$ be a closed G_2-structure on M, written uniquely as $\widetilde{\varphi} = \varphi + \xi + d\varpi$ where ξ is harmonic and ϖ is a co-exact 2-form. Then we have the following [66, Prop. 10.4.3]:

Proposition 1.9. *There exists a universal constant $\epsilon_1 > 0$ such that, if $\widetilde{\varphi} \in L_\varphi$ and $\|\widetilde{\varphi} - \varphi\|_{C^0} \leq \epsilon_1$, then $\widetilde{\varphi}$ lies in L'_φ if and only if $(dd^* + d^*d)\varpi + *d(F_\varphi(\xi + d\varpi)) = 0$.*

With the above proposition, one may use the Implicit Function Theorem in an appropriate Banach space in order to prove that for small enough harmonic 3-forms ξ, there exists a unique torsion-free G_2-structure $\widetilde{\varphi} = \varphi + \xi + d\varpi$ lying in L'_φ and such that the norm of $\widetilde{\varphi} - \varphi$ is controlled by the norm of ξ. In particular, any small open neighbourhood of φ in L'_φ is homeomorphic to an open subset of $H^3(M)$ through the map sending a torsion-free G_2-structure to its cohomology class. Since $\mathscr{D}_0$ acts trivially on $H^3(M)$, this implies that the isotropy group I_φ acts trivially on L'_φ near φ.

This outline of proof shows that the tangent spaces $T_{\varphi\mathscr{D}_0}\mathscr{T}$ and $T_{\varphi\mathscr{D}}\mathscr{M}$ can be intrinsically identified with the space of 3-forms $\mathscr{H}^3(M,\varphi)$ which are harmonic with respect to the metric g_φ. Using this identification, we may define a natural Riemannian metric $\mathscr{G}$ on $\mathscr{M}$ by

$$\mathscr{G}_\varphi(\eta,\eta') = \frac{1}{\mathrm{Vol}(\varphi)} \int \langle \eta, \eta' \rangle_\varphi \mu_\varphi, \quad \forall \eta, \eta' \in \mathscr{H}^3(M,\varphi) \simeq T_{\varphi\mathscr{D}}\mathscr{M}.$$

That is, $\mathscr{G}$ is the volume-normalised L^2-metric. It will be our main object of study in the last three chapters of this thesis.

Remark 1.10. As we mentioned in the introduction, the metric $\mathscr{G}$ fits into the broader context of the moduli spaces of special holonomy manifolds. It is analogous to the Hodge metric on Kähler cones and to the Weil–Petersson metric on the moduli spaces of complex structures of Calabi–Yau manifolds.

Chapter 2

Gluing constructions with C^k-estimates

As of today, all the known compact G_2-manifolds are constructed by a gluing-perturbation method whose analytical foundations were laid by Joyce [65, 66]. The idea is to start from a closed G_2-structure φ with small torsion, and to deform it to a nearby torsion-free G_2-structure $\widetilde{\varphi}$ within the same cohomology class using a fixed-point argument. The general existence theorem of Joyce [66, Theorem 11.6.1] gives control on the C^0-norm of $\widetilde{\varphi} - \varphi$ and only requires bounds on certain geometric quantities (curvature, injectivity radius, and various weak norms of the torsion), making it applicable in a wide range of geometrical contexts.

For certain purposes however, it is more convenient to have control on a number of derivatives of $\widetilde{\varphi} - \varphi$, for instance in order to approximate the differential operators associated with $\widetilde{\varphi}$ by those associated with φ. In the particular case of the generalised Kummer construction, Platt was able to improve the control to $C^{1,\alpha}$-estimates for any $\alpha \in (0, \frac{1}{2})$ [103], which was crucial in order to construct associative submanifolds whose volume is shrinking to zero [39].

In this chapter we shall derive a general theorem giving sufficient conditions to be able to deform a closed G_2-structure φ to a nearby torsion-free G_2-structure $\widetilde{\varphi}$ with estimates on $\|\widetilde{\varphi} - \varphi\|_{C^k}$, for any $k \geq 1$. The precise statement is Theorem 2.14, which improves and generalises an argument which appears in the last section of the article [82] by the author. In contrast to Joyce's existence theorem, our result gives a control on any number of derivatives of $\widetilde{\varphi} - \varphi$, but the trade-off is that we need much stronger bounds on quantities that are not entirely geometrical (i.e. the operator norms of Sobolev embeddings and Green's functions) and on a high number of derivatives of the torsion.

This necessarily restricts the scope of application of our result, but in certain situations it yields an improvement on the previously known estimates. In the second section of this chapter, we will apply it to the twisted connected sum construction and show that we can well approximate any number of derivatives.

2.1 A deformation theorem with C^k-estimates

Our deformation theorem is proved in §2.1.3, and relies on a series of uniform estimates which we derive in §2.1.1 and §2.1.2.

2.1.1 Uniform estimates for compatible connections. The main ingredient from functional analysis that we need in this part is the Implicit Function Theorem for analytic maps between Banach spaces [120]. We first recall some definitions. Let $U \subseteq A$ be an open subset of a Banach space, let B be another Banach space, and let $f : U \to B$ be a map. Let us moreover denote by $L(A, B)$ the Banach space of bounded linear maps between A and B, and define inductively $L_m(A, B) = L(A, L_{m-1}(A, B))$. It is a classical fact that $L_m(A, B)$, equipped with the operator norm, is a Banach space isometric to the space of bounded m-linear maps defined on A and valued in B. The map f is said to be analytic at a point $u_0 \in U$ if there exists a family of symmetric multilinear maps $f_m \in L_m(A, B)$ such that the series $\sum \|f_m\| t^m$ has non-zero radius of convergence and $f(u) = \sum_m f_m(u - u_0)^m$ in a neighbourhood of u_0, where we use the notation $f_m u_1 \cdots u_m$ (respectively $f_m u^m$) for $f_m(u_1, \ldots, u_m)$ (respectively $f_m(u, \ldots, u)$). Moreover, f is said to be analytic if it is analytic everywhere on U, which in particular implies that f is smooth. If A and B are finite-dimensional, this definition is equivalent to the usual definition of analytic maps in terms of power series expansions, and most properties of analytic maps carry out from the finite-dimensional to the Banach space setting.

For our purpose, an especially important class of analytic maps are equivariant maps. Let (ρ, W) be a linear representation of $\mathrm{GL}_+(7)$, and let $\Upsilon : \Lambda^3_+ \mathbb{R}^*_7 \to W$ be an equivariant map. As $\mathrm{GL}_+(7)$ acts transitively on $\Lambda^3_+ \mathbb{R}^*_7$, the map Υ is determined by the image of any positive form. In the following two lemmas we state some useful properties of such maps:

Lemma 2.1. *Let $\Upsilon : \Lambda^3_+ \mathbb{R}^*_7 \to W$ be an equivariant map. Then:*

(i) The map Υ is analytic.

(ii) Let $\Upsilon_m : \Lambda^3_+ \mathbb{R}^*_7 \to L_m(\Lambda^3 \mathbb{R}^*_7, W)$ be the maps determined by the expansion $\Upsilon(\varphi + \eta) = \sum_m \Upsilon_m(\varphi)\eta^m$. Then each Υ_m is an equivariant map.

Proof. Fix a positive form φ_0 on $\mathbb{R}^7$ and identify the stabiliser of φ_0 with G_2. Let us pick a direct sum decomposition $\mathfrak{gl}_7 \simeq \mathfrak{g}_2 \oplus \mathfrak{p}$, where $\mathfrak{g}_2$ is the Lie algebra of G_2. Then there exists a neighbourhood U_0 of 0 in $\mathfrak{p}$ such that the map

$$U_0 \longrightarrow \Lambda^3_+\mathbb{R}^*_7, \quad \xi \longmapsto \exp(-\xi)^*\varphi_0$$

is a diffeomorphism onto a neighbourhood V_0 of φ_0 in $\Lambda^3_+\mathbb{R}^*_7$. The above map is polynomial and hence analytic, and so is its inverse, which we denote by $\xi(\varphi)$ for $\varphi \in V_0$. By equivariance of the map Υ, if $\varphi \in V_0$ we have:

$$\Upsilon(\varphi) = \Upsilon(\exp(-\xi(\varphi))^*\varphi_0) = \rho(\exp \xi(\varphi))\Upsilon(\varphi_0) = \exp(\rho_*(\xi(\varphi)))\Upsilon(\varphi_0)$$

which is analytic in φ, as ξ and exp are analytic. This proves point (i).

For point (ii), let $\alpha \in \mathrm{GL}_+(7)$, $\varphi \in \Lambda^3_+\mathbb{R}^*_7$ and consider a 3-form $\eta \in \Lambda^3\mathbb{R}^*_7$ such that the norm $|\eta|_\varphi$ is smaller than the radii of convergence of the series $\Upsilon(\varphi + \eta) = \sum_m \Upsilon_m(\varphi)\eta^m$ and $\Upsilon((\alpha^{-1})^*\varphi + \eta') = \sum_m \Upsilon_m((\alpha^{-1})^*\varphi)(\eta')^m$. As $|(\alpha^{-1})^*\eta|_{(\alpha^{-1})^*\varphi} = |\eta|_\varphi$, we have:

$$\Upsilon((\alpha^{-1})^*(\varphi + \eta)) = \sum_m \Upsilon_m((\alpha^{-1})^*\varphi)((\alpha^{-1})^*\eta)^m.$$

On the other hand, since the map Υ is equivariant, we also have:

$$\Upsilon((\alpha^{-1})^*(\varphi + \eta)) = \rho(\alpha)\Upsilon(\varphi + \eta) = \sum_m \rho(\alpha)(\Upsilon_m(\varphi)\eta^m).$$

By uniqueness of the expansion, we deduce that

$$\Upsilon_m((\alpha^{-1})^*\varphi)\eta^m = \rho(\alpha)\Upsilon_m(\varphi)(\alpha^*\eta)^m$$

for all $\eta \in \Lambda^3\mathbb{R}^*_7$. Hence the map $\Upsilon_m : \Lambda^3_+\mathbb{R}^*_7 \to L_m(\Lambda^3\mathbb{R}^*_7, W)$ is equivariant under the action of $\mathrm{GL}_+(7)$. $\square$

We deduce the following adaptation of [64, Lem. 3.1.1]:

Lemma 2.2. *Let $\Upsilon : \Lambda^3_+\mathbb{R}^*_7 \to W$ be an equivariant map, and assume that any positive form $\varphi \in \Lambda^3\mathbb{R}^*_7$ induces an inner product g^W_φ on W such that the map $\varphi \in \Lambda^3\mathbb{R}^*_7 \to g^W_\varphi \in S^2_+W^*$ is equivariant. We denote by $|u|_\varphi$ the norm of $u \in W$ relative to the inner product g^W_φ, and by $|\eta|_\varphi$ the norm of $\eta \in \Lambda^3\mathbb{R}^*_7$ relative to the inner product g_φ. Then there exist a sequence of nonnegative numbers $a_0, a_1, a_2, \dots$ and a constant $R > 0$ such that the following hold:*

(i) If φ is a positive form and $\eta \in \Lambda^3\mathbb{R}^*_7$ satisfies $|\eta|_\varphi \leq R$ then $\varphi + \eta$ is a positive form.

(ii) For any $\varphi \in \Lambda^3_+\mathbb{R}^*_7$ we have:

$$|\Upsilon_m(\varphi)\eta_1\cdots\eta_m|_\varphi \leq a_m|\eta_1|_\varphi\cdots|\eta_m|_\varphi, \quad \forall \eta_1,\ldots,\eta_m \in \Lambda^3\mathbb{R}^*_7.$$

(iii) The series $\sum a_m t^m$ has radius of convergence greater or equal to R.

Proof. As noted in [66, Def. 10.3.3], there exists a universal constant $\epsilon > 0$ such that for any positive form φ, if $\eta \in \Lambda^3\mathbb{R}^*_7$ satisfies $|\eta|_\varphi \leq \epsilon$ then $\varphi + \eta$ is also a positive form. Now let us fix $\varphi_0 \in \Lambda^3_+\mathbb{R}^*_7$ and denote by a_m the operator norm of $\Upsilon_m(\varphi_0) \in L_m(\Lambda^3\mathbb{R}^*_7, W)$ relative to the norms induced by φ on $\Lambda^3\mathbb{R}^*_7$ and W. That is, a_m is the smallest nonnegative number such that

$$|\Upsilon_m(\varphi_0)\eta_1\cdots\eta_m|_{\varphi_0} \leq a_m|\eta_1|_{\varphi_0}\cdots|\eta_m|_{\varphi_0}, \quad \forall \eta_1,\ldots,\eta_m \in \Lambda^3\mathbb{R}^*_7. \tag{2.1}$$

As Υ is analytic, $\sum a_m t^m$ has positive radius of convergence. If $\varphi \in \Lambda^3_+\mathbb{R}^*_7$, then there exists $\alpha \in \mathrm{GL}_+(7)$ such that $\varphi = (\alpha^{-1})^*\varphi_0$. If $\eta_1,\ldots,\eta_m \in \Lambda^3\mathbb{R}^*_7$, we have

$$|\Upsilon_m(\varphi)\eta_1\cdots\eta_m|_\varphi = |\rho(\alpha)\Upsilon_m(\varphi_0)\alpha^*\eta_1\cdots\alpha^*\eta_m|_\varphi.$$

As the map $\varphi \mapsto g^W_\varphi$ is equivariant, the corresponding norms on W satisfy

$$|\rho(\alpha)u|_\varphi = |u|_{\alpha^*\varphi} = |u|_{\varphi_0}, \quad \forall u \in W.$$

Thus we deduce from (2.1) that

$$\begin{aligned}|\Upsilon_m(\varphi)\eta_1\cdots\eta_m|_\varphi &= |\Upsilon_m(\varphi_0)(\alpha^*\eta_1)\cdots(\alpha^*\eta_m)|_{\varphi_0}\\ &\leq a_m|\alpha^*\eta_1|_{\varphi_0}\cdots|\alpha^*\eta_m|_{\varphi_0}.\end{aligned}$$

Moreover, since $\varphi \mapsto g_\varphi$ is equivariant, we have

$$|\alpha^*\eta|_{\varphi_0} = |\eta|_{(\alpha^{-1})^*\varphi_0} = |\eta|_\varphi$$

whence we finally deduce

$$|\Upsilon_m(\varphi)\eta_1\cdots\eta_m|_\varphi \leq a_m|\eta_1|_\varphi\cdots|\eta_m|_\varphi.$$

Hence the lemma holds if we take $R > 0$ to be the minimum of ϵ and the radius of convergence of the series $\sum a_m t^m$. □

Remark 2.3. Since any two norms on W are equivalent, the radius of convergence of the series $\sum a_m t^m$ does not depend on the choice of equivariant family of inner products g^W_φ on W, since the coefficients a_m only depend on this choice up to multiplication by a positive constant independent of m. Hence we can define R independently from a particular choice of family of inner products on W.

The above lemmas will be used in the following way. Let M be an oriented compact 7-manifold, equipped with its oriented frame bundle, and let E be the vector bundle associated with the representation (ρ, W). We assume moreover that W is endowed with an equivariant family of inner products $\varphi \mapsto g^W_\varphi$. Then the equivariant map $\Upsilon : \Lambda^3_+\mathbb{R}^*_7 \to W$ induces a (possibly nonlinear) bundle map $\Upsilon : \Lambda^3_+ T^*M \to E$. In the same way, each map $\Upsilon_m : \Lambda^3_+\mathbb{R}^*_7 \to L_m(\Lambda^3\mathbb{R}^*_7, W)$ induces a bundle map $\Upsilon_m : \Lambda^3_+T^*M \to L_m(\Lambda^3 T^*M, E)$. Hence if φ is a positive form on M, $\Upsilon_m(\varphi)$ is a fully symmetric m-linear map from $\Lambda^3 T^*M$ to E. Moreover, φ induces a metric h_φ on the vector bundle E. If η is a 3-form whose C^0-norm with respect to the Riemannian metric induced by φ is smaller than the radius R defined in Lemma 2.2, then we have a convergent power series expansion:

$$\Upsilon(\varphi + \eta) = \sum_{m=0}^{\infty} \Upsilon_m(\varphi)\eta^m. \tag{2.2}$$

Note that for any connection ∇ on M compatible with φ (that is, $\nabla\varphi = 0$), the tensor $\Upsilon(\varphi)$ is parallel for the induced connection on E; and in the same way $\nabla\Upsilon_m(\varphi) = 0$ for the induced connection on $L_m(\Lambda^3T^*M, E)$.

From this observation, we may deduce that Υ induces an analytic map between appropriate Sobolev spaces of sections and provide quantitative estimates for its radius of convergence. Let us first introduce some notations. Let M and E be as above, and let φ be a positive form on M, let h_φ be the induced metric on E, and let ∇ be a connection compatible with φ. From this data, we may define the Sobolev $W^{k,p}$-norm ($p \geq 1$, $k \in \mathbb{N}$) of a smooth section $u \in C^\infty(E)$ as:

$$\|u\|_{W^{k,p}} = \sum_{l=0}^{k} \|\nabla^l u\|_{L^p}$$

where $\|\cdot\|_{L^p}$ is the Lebesgue norm associated with the Riemannian metric g_φ on M and the metric h_φ on E. The Sobolev space $W^{k,p}(E)$ is defined as the completion of $C^\infty(E)$ for the $W^{k,p}$-norm. This is a Banach space. In the same way, one can define C^k-norms as:

$$\|u\|_{C^k} = \sum_{l=0}^{k} \|\nabla^l u\|_{C^0}$$

and the Banach space $C^k(E)$ is the completion of $C^\infty(E)$ for this norm. The Sobolev Embedding Theorem (see for instance [13, App.A,§C,Th.6]) states that there is a continuous embedding $L^p_k \hookrightarrow C^l$ whenever $\frac{1}{p} \leq \frac{k-l}{7}$.

Let us now fix $p \in [1, \infty)$ and integers $l \geq 0$ and $k \geq 1$, satisfying the conditions

$$l \geq \lfloor k/2 \rfloor, \quad \text{and} \quad \frac{1}{p} \leq \frac{k-l}{7}. \tag{2.3}$$

The second condition and the Sobolev Embedding Theorem imply that $W^{k,p} \subset C^l \subseteq C^0$. Hence we can talk about positive forms of regularity $W^{k,p}$, which are defined as 3-forms of regularity $W^{k,p}$ such that, as continuous sections, they are valued in the bundle of positive forms. We denote by $W^{k,p}(\Lambda^3_+T^*M)$ the set of such positive forms. Since $W^{k,p}(\Lambda^3T^*M)$ continuously embeds into $C^0(\Lambda^3T^*M)$, this is an open subset of the Banach space $W^{k,p}(\Lambda^3T^*M)$. Moreover, for any $\varphi' \in W^{k,p}(\Lambda^3_+T^*M)$, we can define $\Upsilon(\varphi')$ at least as a continuous section of E. Let E, φ, h_φ, ∇ be as before, and denote by $C_{k,p,l,\varphi,\nabla}$ the norm of the Sobolev embedding $W^{k,p}(\Lambda^3T^*M) \hookrightarrow C^l(\Lambda^3T^*M)$ (where the $W^{k,p}$- and C^l-norms of differential forms are defined with respect to ∇ and g_φ).

Lemma 2.4. *For any $m \geq 0$ and any $\eta_1, \dots, \eta_m \in W^{k,p}(\Lambda^3T^*M)$ we have*

$$\|\Upsilon_m(\varphi)\eta_1 \cdots \eta_m\|_{W^{k,p}} \leq m^{k+1} C^{m-1}_{k,p,l,\varphi,\nabla} a_m \|\eta_1\|_{W^{k,p}} \cdots \|\eta_m\|_{W^{k,p}}.$$

In particular if U is the ball of radius $R/C_{k,p,l,\varphi,\nabla}$ centred at φ in $W^{k,p}(\Lambda^3_+T^*M)$, then Υ induces an analytic map $U \to W^{k,p}(E)$ and the expansion $\Upsilon(\varphi+\eta) = \sum_{m=0}^\infty \Upsilon_m(\varphi)\eta^m$ converges in this domain.

Proof. To make notations lighter, let us write $C = C_{k,p,l,\varphi,\nabla}$, so that $\|\eta\|_{C^l} \leq C\|\eta\|_{W^{k,p}}$ for any 3-form η of regularity $W^{k,p}$. As the C^0-norm is smaller than the C^l norm, Lemma 2.2 implies that if $\|\eta\|_{W^{k,p}} \leq R/C$ then $\varphi + \eta$ is a positive form, and moreover the expansion:

$$\Upsilon(\varphi+\eta) = \sum_{m=0}^{\infty} \Upsilon_m(\varphi)\eta^m \tag{2.4}$$

is convergent with respect to the C^0-norm. To prove that $\Upsilon(\varphi+\eta) \in W^{k,p}(E)$, we want to show that the series also converges with respect to the $W^{k,p}$-norm.

Let us consider each term $\Upsilon_m(\varphi)\eta^m$ separately. As we noted before, since ∇ is compatible with φ we have $\nabla(\Upsilon_m(\varphi)) = 0$. Let $0 \leq j \leq k$ and let $\eta_1, \dots, \eta_m$ be smooth 3-forms. As $\Upsilon_m(\varphi)$ is covariantly constant we have:

$$\nabla^j(\Upsilon_m(\varphi)\eta_1 \dots \eta_m) = \sum_{j_1+\cdots j_m=j} \frac{j!}{j_1! \cdots j_m!} \Upsilon_m(\varphi)\nabla^{j_1}\eta_1 \cdots \nabla^{j_m}\eta_m$$

which yields:

$$\|\nabla^j(\Upsilon_m(\varphi)\eta_1 \dots \eta_m)\|_{L^p} \leq \sum_{j_1+\cdots j_m=j} \frac{j!}{j_1! \cdots j_m!} \|\Upsilon_m(\varphi)\nabla^{j_1}\eta_1 \cdots \nabla^{j_m}\eta_m\|_{L^p}. \tag{2.5}$$

Let us fix $j_1, \dots, j_m \geq 0$ such that $j_1 + \cdots + j_m = j$. From Lemma 2.2, we deduce the following estimate pointwise over M:

$$|\Upsilon_m(\varphi)\nabla^{j_1}\eta_1 \cdots \nabla^{j_m}\eta_m|_\varphi \leq a_m |\nabla^{j_1}\eta_1|_\varphi \cdots |\nabla^{j_m}\eta_m|_\varphi.$$

Let us pick i_0 such that $j_{i_0} = \max\{j_1, \ldots, j_m\}$. As $l \geq \lfloor k/2 \rfloor$ we have $j_i \leq l$ for all $i \neq i_0$, and therefore $|\nabla^{j_i}\eta_i|_\varphi$ is uniformly bounded by $\|\eta_i\|_{C^l}$. Hence we have:

$$\begin{aligned}\left(\int (|\nabla^{j_1}\eta_1|_\varphi \cdots |\nabla^{j_m}\eta_m|_\varphi)^p \mu_\varphi\right)^{\frac{1}{p}} &\leq \|\eta_1\|_{C^l} \cdots \|\nabla^{j_{i_0}}\eta_{i_0}\|_{L^p} \cdots \|\eta_m\|_{C^l} \\ &\leq C^{m-1}\|\eta_1\|_{W^{k,p}} \cdots \|\eta_m\|_{W^{k,p}}.\end{aligned}$$

where we use $\|\nabla^{j_{i_0}}\eta_{i_0}\|_{L^p} \leq \|\eta\|_{W^{k,p}}$ and the Sobolev embedding $W^{k,p} \subset C^l$ to obtain the second inequality. Hence we have the estimate:

$$\|\Upsilon_m(\varphi)\nabla^{j_1}\eta_1 \cdots \nabla^{j_m}\eta_m\|_{L^p} \leq C^{m-1}a_m\|\eta_1\|_{W^{k,p}} \cdots \|\eta_m\|_{W^{k,p}}.$$

Reinjecting this inequality into (2.5) and taking into account that the sum of the multinomial coefficients $\frac{j!}{j_1! \cdots j_m!}$ over all combinations of $j_1, \ldots j_m$ such that $j_1 + \cdots j_m = j$ is m^j, we finally obtain:

$$\|\nabla^j(\Upsilon_m(\varphi)\eta_1 \cdots \eta_m)\|_{L^p} \leq m^j C^{m-1}a_m\|\eta_1\|_{W^{k,p}} \cdots \|\eta_m\|_{W^{k,p}}$$

and therefore, summing over $j = 0, \ldots, k$:

$$\begin{aligned}\|\Upsilon_m(\varphi)\eta_1 \cdots \eta_m\|_{W^{k,p}} &\leq \left(\sum_{j=0}^{k} m^j\right) C^{m-1}a_m\|\eta_1\|_{W^{k,p}} \cdots \|\eta_m\|_{W^{k,p}} \\ &\leq m^{k+1}C^{m-1}a_m\|\eta_1\|_{W^{k,p}} \cdots \|\eta_m\|_{W^{k,p}}.\end{aligned}$$

The above inequality holds for any smooth 3-forms $\eta_1, \ldots, \eta_m$, and by density of $C^\infty(\Lambda^3T^*M)$ in $W^{k,p}(\Lambda^3T^*M)$ we deduce that $\Upsilon_m(\varphi)$ induces a bounded m-linear map $W^{k,p}(\Lambda^3T^*M)\times\cdots\times W^{k,p}(\Lambda^3T^*M) \to W^{k,p}(E)$ with operator norm bounded above by $m^{k+1}C^{m-1}a_m$. As the radius of convergence of the series $\sum a_m t^m$ is bounded below by $R > 0$, the radius of convergence of the series $\sum m^{k+1}C^{m-1}a_m t^m$ is greater or equal to R/C. Therefore the expansion (2.4) converges in $W^{k,p}$-norm if $\|\eta\|_{W^{k,p}} \leq R/C$, and as $W^{k,p}(E)$ is complete this implies that $\Upsilon(\varphi+\eta) \in W^{k,p}(E)$. Moreover, if U is the open ball in $W^{k,p}(\Lambda^3_+T^*M)$ of radius R/C centred at φ, then the induced map $\Upsilon : U \to W^{k,p}(E)$ is analytic. □

2.1.2 Uniform estimates for the Levi-Civita connection. In this part, we let φ be a smooth positive form on the compact manifold M, and let E and h_φ be as before. However, from now on ∇ will be the Levi–Civita connection associated with φ (which we do not assume to be compatible with φ) and we denote by ∇' the canonical connection associated with φ (which is compatible). Recall from §1.1.2 that these connections are related by $\nabla = \nabla' + \tau(\varphi)$, where $\tau(\varphi) \in \Omega^1(T^*M \otimes TM)$ is the torsion of φ. Finally, $W^{k,p}_\nabla$ and C^k_∇ will denote the norms defined with respect to the connection ∇ and the metrics g_φ and h_φ, and similarly for ∇'. The norms $W^{k,p}_\nabla$ and $W^{k,p}_{\nabla'}$ are equivalent, and so are C^k_∇ and $C^k_{\nabla'}$, but for our purpose it is important to have precise bounds. This is the object of the next lemmas.

Lemma 2.5. *Let $k \geq 1$. Then there exists a polynomial function P_k with nonnegative coefficients, depending only on the integer k and the representation (ρ, W), such that $P_k(0) = 0$ and for all $u \in C^k(E)$ we have*

$$(1 + P_k(\|\tau(\varphi)\|_{C^{k-1}_\nabla}))^{-1}\|u\|_{C^k_{\nabla'}} \leq \|u\|_{C^k_\nabla} \leq (1 + P_k(\|\tau(\varphi)\|_{C^{k-1}_\nabla}))\|u\|_{C^k_{\nabla'}}.$$

Proof. If u is a smooth section of E we have

$$\nabla u = \nabla' u + \tau^\rho(\varphi) \cdot u \tag{2.6}$$

where we wrote $\tau^\rho(\varphi) = \rho_*(\tau(\varphi)) \in \Omega^1(\mathrm{End}(E))$. By an easy induction, we deduce that for any $k \geq 1$ there is a formula of the type

$$\nabla^k u = (\nabla')^k u + \sum_{m=1}^{\infty} \sum_{j_1+\cdots j_m+l \leq k-1} A_{k,m,j_1\cdots j_m,l} \nabla^{j_1}\tau^\rho(\varphi)\cdots\nabla^{j_m}\tau^\rho(\varphi)\cdot(\nabla')^l u$$

where $A_{k,m,j_1\ldots j_m,l}$ are some combinatorial coefficients, which for a given k vanish identically when m is large enough, so that the sum is finite. Moreover we can write $\nabla^j\tau^\rho(\varphi) = \rho_*\nabla^j\tau(\varphi)$ as a section of $(T^*M)^{\otimes j} \otimes \mathrm{End}(E)$, and thus there are constants $B_{j_1\ldots j_m,l}$ depending only on the representation (ρ, W) such that

$$\begin{aligned}|\nabla^{j_1}\tau^\rho(\varphi)\cdots\nabla^{j_m}\tau^\rho(\varphi)\cdot(\nabla')^l u|_\varphi &\leq B_{j_1\ldots j_m,l}|\nabla^{j_1}\tau(\varphi)|_\varphi\cdots|\nabla^{j_m}\tau(\varphi)|_\varphi|(\nabla')^l u|_\varphi \\ &\leq B_{j_1\ldots j_m,l}\|\tau(\varphi)\|^m_{C^{k-1}_\nabla}|(\nabla')^l u|_\varphi\end{aligned}$$

everywhere on M. Taking the supremum over M we obtain an inequality

$$\|u\|_{C^k_\nabla} \leq (1 + P_k(\|\tau(\varphi)\|_{C^{k-1}_\nabla}))\|u\|_{C^k_{\nabla'}}$$

for some polynomial P_k with nonpositive coefficients depending only on the choice of representation and such that $P_k(0) = 0$. For the other inequality, we may use the identity (2.6) and an induction to find a formula of the form

$$(\nabla')^k u = \nabla^k u + \sum_{m=1}^{\infty} \sum_{j_1+\cdots j_m+l \leq k-1} A'_{k,m,j_1\cdots j_m,l} \nabla^{j_1}\tau^\rho(\varphi)\cdots\nabla^{j_m}\tau^\rho(\varphi)\cdot\nabla^l u$$

where for a given k the coefficients $A'_{k,m,j_1\cdots j_m,l}$ vanish identically when m is large enough, and apply the same reasoning. □

Remark 2.6. The point here is that P_k does not depend on the manifold M or the G_2-structure φ. In that sense, this bound is universal.

Lemma 2.7. *Let $k \geq 1$. Then there exists a polynomial function Q_k with nonnegative coefficients, depending only on the integer k and the representation (ρ, W), such that $Q_k(0) = 0$ and for all $u \in W^{k,p}(E)$ we have*

$$(1 + Q_k(\|\tau(\varphi)\|_{C^{k-1}_\nabla}))^{-1}\|u\|_{W^{k,p}_{\nabla'}} \leq \|u\|_{W^{k,p}_\nabla} \leq (1 + Q_k(\|\tau(\varphi)\|_{C^{k-1}_\nabla}))\|u\|_{W^{k,p}_{\nabla'}}.$$

Proof. The proof is exactly the same as for the previous lemma, except that we integrate instead of taking the supremum. □

Since the torsion $\tau(\varphi)$ is represented by $(d\varphi, d\Theta(\varphi))$ we immediately obtain the following consequences:

Corollary 2.8. *Let $k \geq 1$. Then there is a constant $\epsilon_k > 0$, depending only on the integer k and the representation (ρ, W), such that if $\|d\varphi\|_{C^{k-1}_\nabla} + \|d\Theta(\varphi)\|_{C^{k-1}_\nabla} \leq \epsilon_k$ then for all $u \in C^k(E)$*

$$2^{-1}\|u\|_{C^k_{\nabla'}} \leq \|u\|_{C^k_\nabla} \leq 2\|u\|_{C^k_{\nabla'}}.$$

Corollary 2.9. *Let $k \geq 1$. Then there is a constant $\epsilon'_k > 0$, depending only on the integer k and the representation (ρ, W), such that if $\|d\varphi\|_{C^{k-1}_\nabla} + \|d\Theta(\varphi)\|_{C^{k-1}_\nabla} \leq \epsilon'_k$ then for all $u \in W^{k,p}(E)$*

$$2^{-1}\|u\|_{W^{k,p}_{\nabla'}} \leq \|u\|_{W^{k,p}_\nabla} \leq 2\|u\|_{W^{k,p}_{\nabla'}}.$$

Let us now consider as before a bundle map $\Upsilon : \Lambda^3_+ T^*M \to E$ induced by an equivariant map $\Upsilon : \Lambda^3_+\mathbb{R}^*_7 \to W$. We also fix $p \in (1, \infty)$ and integers $l \geq 0$, $k \geq 1$ such that conditions (2.3) (which we recall below) hold:

$$l \geq \lfloor k/2 \rfloor, \quad \text{and} \quad \frac{1}{p} \leq \frac{k-l}{7}.$$

Then we can adapt Lemma 2.4 to Sobolev norms defined with respect to the Levi-Civita connection of g_φ for G_2-structures with small torsion:

Lemma 2.10. *There is a universal constant $\epsilon_{k,l}$, depending only on the integers k, l and the representation (ρ, W), such that the following holds. Assume that $\|d\varphi\|_{C^{k-1}_\nabla} + \|d\Theta(\varphi)\|_{C^{k-1}_\nabla} \leq \epsilon_{k,l}$ and let $C_{k,p,l,\varphi}$ be the norm of the Sobolev embedding $W^{k,p}(\Lambda^3 T^*M) \hookrightarrow C^l(\Lambda^3 T^*M)$ with respect to the norms $W^{k,p}_\nabla$ and C^l_∇. Then for any $m \geq 0$ and any $\eta_1, \ldots, \eta_m \in W^{k,p}(\Lambda^3 T^*M)$ we have*

$$\|\Upsilon_m(\varphi)\eta_1 \cdots \eta_m\|_{W^{k,p}_\nabla} \leq 8^m m^{k+1} C^{m-1}_{k,p,l,\varphi} a_m \|\eta_1\|_{W^{k,p}_\nabla} \cdots \|\eta_m\|_{W^{k,p}_\nabla}.$$

In particular the radius of convergence of the expansion $\Upsilon(\varphi+\eta) = \sum_{m=0}^\infty \Upsilon_m(\varphi)\eta^m$ with respect to the $W^{k,p}_\nabla$-norms is greater or equal to $R/(8C_{k,p,l,\varphi})$.

Proof. Let $\epsilon_{k,l} > 0$ be smaller than the constants $\epsilon'_{k,\Lambda^3}, \epsilon'_{k,W}$ provided by Corollary 2.9 for the representations $\Lambda^3\mathbb{R}^*_7$ and (ρ, W) and than the constant ϵ_{l,Λ^3} associated with the representation $\Lambda^3\mathbb{R}^*_7$. Assume that $\|d\varphi\|_{C^{k-1}_\nabla} + \|d\Theta(\varphi)\|_{C^{k-1}_\nabla} \leq \epsilon_{k,l}$. By Lemma 2.4, we know that

$$\|\Upsilon_m(\varphi)\eta_1\cdots\eta_m\|_{W^{k,p}_{\nabla'}} \leq m^{k+1}C^{m-1}_{k,p,l,\varphi,\nabla'}a_m\|\eta_1\|_{W^{k,p}_{\nabla'}}\cdots\|\eta_m\|_{W^{k,p}_{\nabla'}}.$$

Now as $\|\tau(\varphi)\|_{C^{k-1}} \leq \epsilon'_{k,\Lambda^3}, \epsilon'_{k,W}$ we have

$$\|\eta_j\|_{W^{k,p}_{\nabla'}} \leq 2\|\eta_j\|_{W^{k,p}_\nabla}$$

and

$$\|\Upsilon_m(\varphi)\eta_1\cdots\eta_m\|_{W^{k,p}_{\nabla'}} \leq 2\|\Upsilon_m(\varphi)\eta_1\cdots\eta_m\|_{W^{k,p}_\nabla}$$

and therefore we deduce

$$\|\Upsilon_m(\varphi)\eta_1\cdots\eta_m\|_{W^{k,p}_\nabla} \leq 2^{m+1}m^{k+1}C^{m-1}_{k,p,l,\varphi,\nabla'}a_m\|\eta_1\|_{W^{k,p}_\nabla}\cdots\|\eta_m\|_{W^{k,p}_\nabla}.$$

Now we need to control the Sobolev constant $C_{k,p,l,\varphi,\nabla'}$ by $C_{k,p,l,\varphi}$. Since $\|\tau(\varphi)\|_{C^{l-1}} \leq \epsilon_{l,\Lambda^3}$, for any $\eta \in W^{k,p}(\Lambda^3T^*M)$ we have

$$\|\eta\|_{C^l_{\nabla'}} \leq 2\|\eta\|_{C^l_\nabla} \leq 2C_{k,p,l,\varphi}\|\eta\|_{W^{k,p}_\nabla} \leq 4C_{k,p,l,\varphi}\|\eta\|_{W^{k,p}_{\nabla'}}$$

whence $C_{k,p,l,\varphi,\nabla'} \leq 4C_{k,p,l,\varphi}$. Thus $2^{m+1}C^{m-1}_{k,p,l,\varphi,\nabla'} \leq 2^{m+1}\cdot 2^{2m-2}C_{k,p,l,\varphi} \leq 8^mC_{k,p,l,\varphi}$ which yields the desired inequality. The rest of the lemma easily follows. □

In the next paragraph, we shall be considering the equivariant map $\Theta : \Lambda^3_+\mathbb{R}^*_7 \to \Lambda^4\mathbb{R}^*_7$, and the radius of convergence R and the uniform constants ϵ_k, ϵ'_k and $\epsilon_{k,l}$ will implicitly be the ones associated with Θ and the natural representation of $\mathrm{GL}_+(7)$ on $\Lambda^4\mathbb{R}^*_7$. If (M, φ) is a compact manifold endowed with a G_2-structure and $\eta \in \Omega^3(M)$ satisfies $\|\eta\|_{C^0} \leq R$, then $\varphi + \eta$ is a positive form and

$$\Theta(\varphi + \eta) = \sum_{m=0}^{\infty}\Theta_m(\varphi)\eta^m = \Theta(\varphi) + L_\varphi(\eta) + F_\varphi(\eta)$$

where $L_\varphi(\eta) = \Theta_1(\varphi)\eta = \frac{4}{3} * \pi_1(\eta) + *\pi_7(\eta) - \pi_{27}(\eta)$ and $F_\varphi(\eta) = \sum_{m=2}^\infty \Theta_m(\varphi)\eta^m$ (see (1.4)). For the applications of this chapter we will not need to control the full expansion; instead we will use the following consequence:

Corollary 2.11. *For any constant $A > 0$, there exists a universal constant $\delta_{k,l,A} > 0$, depending only on A and the integers k, l, such that if $\|d\varphi\|_{C^{k-1}_\nabla} + \|d\Theta(\varphi)\|_{C^{k-1}_\nabla} \leq \epsilon_{k,l}$ and $\|\eta_1\|_{W^{k,p}_\nabla}, \|\eta_2\|_{W^{k,p}_\nabla} \leq \delta_{k,l,A}/C_{k,p,l,\varphi}$ then*

$$\|F_\varphi(\eta_1) - F_\varphi(\eta_2)\|_{W^{k,p}_\nabla} \leq AC_{k,p,l,\varphi}\|\eta_1 - \eta_2\|_{W^{k,p}_\nabla}(\|\eta_1\|_{W^{k,p}_\nabla} + \|\eta_2\|_{W^{k,p}_\nabla}).$$

Proof. Let us take $\|\eta_1\|_{W^{k,p}_\nabla}, \|\eta_2\|_{W^{k,p}_\nabla} < \delta/C_{k,p,l,\varphi}$ where $\delta < R/8$ is to be determined later. Then we have

$$\begin{aligned} F_\varphi(\eta_1) - F_\varphi(\eta_2) &= \sum_{m=2}^{\infty} \Theta_m(\varphi)\eta_1^m - \Theta_m(\varphi)\eta_2^m \\ &= \sum_{m=2}^{\infty}\sum_{j=0}^{m-1} \Theta_m(\varphi)(\eta_1-\eta_2)\eta_1^j\eta_2^{m-1-j}. \end{aligned}$$

By the previous corollary we have estimates on each term:

$$\begin{aligned} \|\Theta_m(\varphi)(\eta_1-\eta_2)\eta_1^j\eta_2^{m-1-j}\|_{W^{k,p}_\nabla} &\leq 8^m m^{k+1} C^{m-1}_{k,p,l,\varphi} a_m \|\eta_1-\eta_2\|_{W^{k,p}_\nabla}\|\eta_1\|^j_{W^{k,p}_\nabla}\|\eta_2\|^{m-1-j}_{W^{k,p}_\nabla} \\ &\leq 8^m m^{k+1} C_{k,p,l,\varphi}\delta^{m-2} a_m \|\eta_1-\eta_2\|_{W^{k,p}_\nabla}(\|\eta_1\|_{W^{k,p}_\nabla}+\|\eta_2\|_{W^{k,p}_\nabla}). \end{aligned}$$

From this we deduce that

$$\|\Theta_m(\varphi)\eta_1^m - \psi_m(\varphi)\eta_2^m\|_{W^{k,p}_\nabla} \leq 8^m m^{k+2} C_{k,p,l,\varphi}\delta^{m-2} a_m\|\eta_1-\eta_2\|_{W^{k,p}_\nabla}(\|\eta_1\|_{W^{k,p}_\nabla}+\|\eta_2\|_{W^{k,p}_\nabla}).$$

As the series $\sum_m 8^m m^{k+2} a_m$ has nonzero radius of convergence, we can choose $0 < \delta_{k,l,A} < R/8$ such that

$$\sum_{m=2}^{\infty} 8^m m^{k+2} a_m \delta^{m-2}_{k,l,A} \leq A$$

which satisfies the desired property. □

2.1.3 The deformation theorem. Let us now outline the deformation argument argument of Joyce [66, §10.3] for constructing torsion-free G_2-structures. The starting point is to consider a compact manifold M^7 equipped with a closed G_2-structure φ with small torsion, and seek a nearby torsion-free G_2-structure $\tilde{\varphi} = \varphi + d\eta$ in the same cohomology class. In [66, Theorem 10.3.7], Joyce proved that there exists a universal constant $\varepsilon_0 > 0$ (which we might assume to be smaller that the radius of convergence R previously defined) which does not depend on M or φ such that:

Theorem 2.12 (Joyce). *Let (M,φ) be a compact manifold equipped with a closed G_2-structure. Suppose ϖ is a 2-form on M such that $\|d\varpi\|_{C^0} \leq \varepsilon_0$ and ψ a 4-form on M such that $d\Theta(\varphi) = d\psi$ and $\|\psi\|_{C^0} \leq \varepsilon_0$. If (ϖ,ψ) satisfy:*

$$\Delta\varpi + d^*\left(\left(1+\frac{1}{3}\langle d\varpi,\varphi\rangle\right)\psi\right) + *dF_\varphi(d\varpi) = 0$$

then $\tilde{\varphi} = \varphi + d\varpi$ is a torsion-free G_2-structure on M.

In the remainder of this part, we use this theorem together with the uniform estimates of §2.1.2 to prove that provided φ is a closed G_2-structure with sufficiently small torsion in the C^k sense and we can choose ψ with small $W^{k+1,p}$ Sobolev norm (for $p \geq 7$), then φ can be deformed to a C^k-close torsion-free G_2-structure within the same cohomology class. Note that from now on, all Sobolev norms on (M, φ) will be defined with respect to the Levi-Civita connection of g_φ and will be denoted by $W^{k,p}$ instead of $W^{k,p}_\nabla$, and similarly for C^k norms. Our argument is based on the following standard fixed-point theorem whose proof will be omitted:

Proposition 2.13. *Let $(A_1, \|\cdot\|_1), (A_2, \|\cdot\|_2)$ be Banach spaces and $f : B_\delta \subset A_1 \to A_2$ be a continuous map defined on the ball of radius δ centred at 0 which can be written as*

$$f(u) = f(0) + L(u) + F(u)$$

where $L : A_1 \to A_2$ is a bounded linear map which has a bounded inverse and $F : B_\delta \to A_2$ is a continuous map such that

$$\|F(u_1) - F(u_2)\|_2 \leq C\|u_1 - u_2\|_1(\|u_1\|_1 + \|u_2\|_1)$$

for some constant $C > 0$. Let Q be the operator norm of L^{-1} and assume that the following inequalities hold:

$$Q|f(0)| < \delta/2, \quad CQ\delta \leq 1/4.$$

Then there is a unique $u \in B_\delta$ such that $f(u) = 0$, and moreover

$$\|u\|_1 \leq 2Q\|f(0)\|_2.$$

We shall now prove the following theorem:

Theorem 2.14. *Let $k, l \geq 1$ and $p \in (1, \infty)$ satisfy the following conditions:*

$$l \geq 1 + \lfloor k/2 \rfloor \quad \text{and} \quad \frac{1}{p} \leq \frac{k+1-l}{7}.$$

Then there exist a universal constant $\kappa \geq 1$ and a constant $\varepsilon = \varepsilon_{k,l,p} > 0$ depending only on k, l and p such that the following holds.

Let (M, φ) be a compact 7-manifold endowed with a closed G_2-structure, and suppose that $\|d\Theta(\varphi)\|_{C^k} \leq \epsilon_{k+1,l}$. Assume moreover that ψ is a 4-form such that $d\Theta(\varphi) = d\psi$ and $\|\psi\|_{C^0} \leq \epsilon_0$, where ε_0 is the constant of Theorem 2.12. Let us moreover denote by:

- *$Q = Q_{k,\varphi}$ the operator norm of the Green's function G_Δ of the Laplacian, $G_\Delta : \mathscr{H}^2(M, \varphi)^\perp \cap W^{k,p}(\Lambda^2 T^*M) \to \mathscr{H}^2(M, \varphi)^\perp \cap W^{k+2,p}(\Lambda^2 T^*M)$.*

- $C = \sup\{C_{k,p,l-1,\varphi}, C_{k+1,p,l,\varphi}\}$ *the maximum of the norms of the Sobolev embeddings* $W^{k,p}(\Lambda^\bullet T^*M) \hookrightarrow C^{l-1}(\Lambda^\bullet T^*M)$ *and* $W^{k+1,p}(\Lambda^\bullet T^*M) \hookrightarrow C^l(\Lambda^\bullet T^*M)$.

Then if the inequality

$$C(1 + Q(1 + Q + C))(1 + C\|d\psi\|_{W^{k,p}})\|\psi\|_{W^{k+1,p}} \leq \varepsilon$$

holds, there exists a co-exact 2*-form* ϖ *with* $\|\varpi\|_{W^{k+2,p}} \leq 8\kappa Q\|\psi\|_{W^{k+1,p}}$ *such that* $\widetilde{\varphi} = \varphi + d\varpi$ *is torsion-free. Moreover* $\|\widetilde{\varphi} - \varphi\|_{C^l} \leq 4\kappa^2 QC\|\psi\|_{W^{k+1,p}}$.

Remark 2.15. We might in particular choose $k = l \geq 1$ and $p \geq 7$ in this theorem. Therefore, a control on the C^k-norm of the torsion and on the $W^{k+1,p}$-norm of ψ yields a C^k-estimate on $\widetilde{\varphi} - \varphi$. The main reason to allow k, p and l to satisfy the more general conditions of the theorem is that it allows us to work with $p = 2$ if $k + 1 - l$ is large enough. This sometimes simplify the analysis since L^2-spaces are in general better behaved than other L^p-spaces. In fact this will not play a role in our application to twisted connected sums in the next section since the analytical results of Chapter 3 will be valid in the L^p-range for any $p \in (1, \infty)$.

We shall make a few comments and introduce some notations before proving this theorem. Firstly, let us comment on the restrictions on k, l and p. We require $\frac{1}{p} \leq \frac{k+1-l}{7}$ in order to have the Sobolev embeddings $W^{k+1,p} \hookrightarrow C^l$ and $W^{k,p} \hookrightarrow C^{l-1}$. Moreover, the condition $l \geq 1 + \lfloor k/2 \rfloor$ implies both $l - 1 \geq \lfloor k/2 \rfloor$ and $l \geq (k+1)/2$; this allows us to use Corollary 2.11 for the triples $(k+1, l, p)$ and $(k, l-1, p)$. In fact we will only directly use this corollary for $(k+1, l, p)$, and an adaptation of its proof for $(k, l-1, p)$.

Secondly, because the Levi-Civita connection ∇ of g_φ is torsion-free, the exterior differential $d : \Omega^\bullet(M) \to \Omega^\bullet(M)$ coincides with the antisymmetrisation of $\nabla : \Omega^\bullet(TM) \to C^\infty(T^*M \otimes \Lambda^\bullet T^*M)$ (up to a combinatorial constant depending on the degree of forms). Hence there exists a universal constant κ such that $|d\eta| \leq \kappa|\nabla\eta|$ for any $\eta \in \Omega^\bullet(M)$. Me might assume $\kappa \geq 1$. Moreover, the Hodge operator $*$ is an isometry of the exterior algebra $\Lambda^\bullet T^*M$, whence $|d^*\eta| \leq \kappa|\nabla\eta|$ for the same constant. This implies uniform estimates $\|d\eta\|_{W^{k,p}} \leq \kappa\|\eta\|_{W^{k+1,p}}$, and similarly with C^l-norms and if we replace d by d^*.

Proof of Theorem 2.14. The idea is to apply Proposition 2.13 to the function

$$f(\varpi) = d^*\psi + \Delta\varpi + \frac{1}{3}d^*\left(\langle d\varpi, \varphi\rangle\psi\right) + *dF_\varphi(d\varpi)$$

defined on a ball B_{δ_1} of radius $\delta_1 = \frac{\delta_{k+1,l,1}}{\kappa C}$ centred at 0 in the Banach space $\mathscr{H}^2(M, \varphi)^\perp \cap W^{k+2,p}(\Lambda^2 T^*M)$ and taking values in $\mathscr{H}^2(M, \varphi)^\perp \cap W^{k,p}(\Lambda^2 T^*M)$.

Here $\delta_{k+1,l,1}$ is the constant of Corollary 2.11 for $A = 1$. Our choice of δ_1 is made to ensure that $\|d\varpi\|_{W^{k+1,p}} \leq \delta_{k+1,l,1}$ if $\varpi \in B_{\delta_1}$. In the notations of Proposition 2.13, $f(0) = d^*\psi$, $L(\varpi) = \Delta\varpi + \frac{1}{3}d^*\left(\langle d\varpi, \varphi\rangle\psi\right)$ and $F(\varpi) = *dF_\varphi(d\varpi)$.

Let us begin with a quadratic estimate on F. If $\|\varpi_1\|_{W^{k+2}}, \|\varpi\|_{W^{k+2}} \leq \delta_1$ then

$$\begin{aligned}\| * dF_\varphi(d\varpi_1) - *dF_\varphi(d\varpi_2)\|_{W^{k,p}} &\leq \kappa\|F_\varphi(d\varpi_1) - F_\varphi(d\varpi_2)\|_{W^{k+1,p}} \\ &\leq \kappa^2 C\|\varpi_1 - \varpi_2\|_{W^{k+2,p}}(\|\varpi_1\|_{W^{k+2,p}} + \|\varpi_2\|_{W^{k+2,p}})\end{aligned}$$

where the second inequality follows from Corollary 2.11.

Now we want to estimate the norm of the inverse of the linear map L. Since $L(\varpi) = \Delta(\varpi + \frac{1}{3}G_\Delta d^*(\langle d\varpi, \varphi\rangle\psi)$, any condition ensuring that the norm K of the linear map $\varpi \rightarrow \frac{1}{3}d^*(\langle d\varpi, \varphi\rangle\psi))$ satisfies $2KQ \leq 1$ implies that L is invertible with $\|L^{-1}\| \leq 2Q$. Now we have the inequality

$$\|d^*(\langle d\varpi, \varphi\rangle\psi))\|_{W^{k,p}} \leq \kappa\|\langle d\varpi, \varphi\rangle\psi\|_{W^{k+1,p}}.$$

In order to estimate the norm on the second line, we might use the Sobolev embeddings $W^{k+1,p} \hookrightarrow C^l$ and $W^{k,p} \hookrightarrow C^{l-1}$ as in the proof of Lemma 2.4. When calculating the covariant derivatives of $\langle d\varpi, \varphi\rangle\psi$, we encounter two types of terms. The terms of the first type are of the form $\langle\nabla^{j_1}d\varpi, \varphi\rangle\nabla^{j_3}\psi$; such terms have $|\langle\nabla^{j_1}d\varpi, \varphi\rangle\nabla^{j_3}\psi| \leq 7|\nabla^{j_1}d\varpi|_\varphi \cdot |\nabla^{j_3}\psi|$ (since $|\varphi|_\varphi = 7$) and $j_1 + j_3 \leq k + 1$. They can be estimated using the Sobolev embedding $W^{k+1,p} \hookrightarrow C^l$ as in the aforementioned lemma. The second type are terms of the form $\langle\nabla^{j_1}d\varpi, \nabla^{j_2}\varphi\rangle\nabla^{j_3}\psi$ where $j_2 \neq 0$. Since $\nabla\varphi$ is essentially the torsion of φ which is represented by $d\Theta(\varphi) = d\psi$, these terms admit the bound $|\langle\nabla^{j_1}d\varpi, \nabla^{j_2}\varphi\rangle\nabla^{j_3}\psi|_\varphi \leq \kappa'|\nabla^{j_1}\varpi|\cdot|\nabla^{j_2-1}d\psi|_\varphi\cdot|\nabla^{j_3}\psi|$ for some universal constant κ'. Here $j_1 + (j_2 - 1) + j_3 \leq k$ and we can estimate these terms using the embedding $W^{k,p} \hookrightarrow C^{l-1}$ this time. Hence we obtain

$$\|d^*(\langle d\varpi, \varphi\rangle\psi))\|_{W^{k,p}} \leq \kappa'' C\|\varpi\|_{W^{k+2,p}}\|\psi\|_{W^{k+1,p}}(1 + C\|d\psi\|_{W^{k,p}}).$$

for some universal constant κ''. Therefore if

$$2\kappa'' QC\|\psi\|_{W^{k+1,p}}(1 + C\|d\psi\|_{W^{k,p}}) \leq 1$$

then L is invertible and $\|L^{-1}\| \leq 2Q$. This condition will be satisfied provided the constant ε in the statement of the theorem is chosen to be small enough.

Assuming that this is the case, let us define $\delta = 8\kappa Q\|\psi\|_{W^{k+1,p}}$. Then

$$\|L^{-1}\|\|d^*\psi\|_{W^{k,p}} \leq 2Q\kappa\|\psi\|_{W^{k+1,p}} = \delta/4 < \delta/2 \tag{2.7}$$

and hence δ satisfies the first condition of Proposition 2.13. For δ to be smaller than δ_1, we need to have

$$8\kappa^2 QC\|\psi\|_{W^{k+1,p}} \leq \delta_{k+1,l,p}.$$

This is also satisfied if ε is small enough. Finally, the second condition in Proposition 2.13 reads

$$2Q \cdot \kappa^2 C \cdot 8\kappa Q\|\psi\|_{W^{k+1,p}} = 16\kappa^3 Q^2 C\|\psi\|_{W^{k+1,p}} \leq 1/4$$

which is also implied by the condition of the theorem provided we chose a small enough $\varepsilon > 0$.

We might therefore apply Proposition 2.13 for $\delta = 8\kappa Q\|\psi\|_{W^{k+1,p}}$ since $\kappa \geq 1$. Hence $f(\varpi) = 0$ has a unique solution such that $\|\varpi\|_{W^{k+2,p}} \leq \delta$, which must moreover satisfy $\|\varpi\|_{W^{k+2,p}} \leq 4Q\|d^*\psi\|_{W^{k,p}} \leq 4\kappa Q\|\psi\|_{W^{k+1,p}}$. Hence $\|d\varpi\|_{W^{k+1,p}} \leq 4\kappa^2 Q\|\psi\|_{W^{k+1,p}}$ and by the Sobolev embedding $W^{k+1,p} \hookrightarrow C^l$ we deduce that $\|d\varpi\|_{C^l} \leq 4\kappa^2 QC\|\psi\|_{W^{k+1,p}}$. To apply the theorem of Joyce, it remains to prove that ϖ is smooth. The main observation is that f, seen as a second-order partial differential operator, is *quasilinear*, that is, it is linear in the second-order derivatives (represented by $\nabla d\varpi$). Moreover the linearisation at 0 just the Laplacian Δ, and because this is an open property the linearisation of f at ϖ will be elliptic if the C^1-norm of ϖ is smaller than a certain universal constant. Given that $l \geq 1$ and $\|\varpi\|_{C^l} \leq 4\kappa^2 QC\|\psi\|_{W^{k+1,p}} \leq 4\kappa^2\varepsilon$ this condition will hold if ε is small enough. Thus we might use a classical bootstrap argument to prove that ϖ is smooth. □

2.2 Improved estimates for the twisted connected sum construction

We now turn our attention to the construction of compact G_2-manifolds by twisted connected sum. It was first developed by Kovalev [80], and subsequently fixed[1] by Corti–Haskins–Nordström–Pacini [29] and further extended by Crowley and Nordström [34, 97]. The main result of this section is the C^k-estimate of Proposition 2.16, which is an application of Theorem 2.14 together with estimates on the Green's function of the Laplacian that we will derive in Chapter 3, in a much more general setting.

[1]The original construction contained a number of geometric caveats which were addressed in the cited work. In fact, the analysis developed in [80], which partially relies on an adaptation of [81], is also erroneous (although one can use Joyce's general existence results to bypass it). The results of the next chapter provide a way to fix the analytical aspects of the construction, in addition to improving the known estimates.

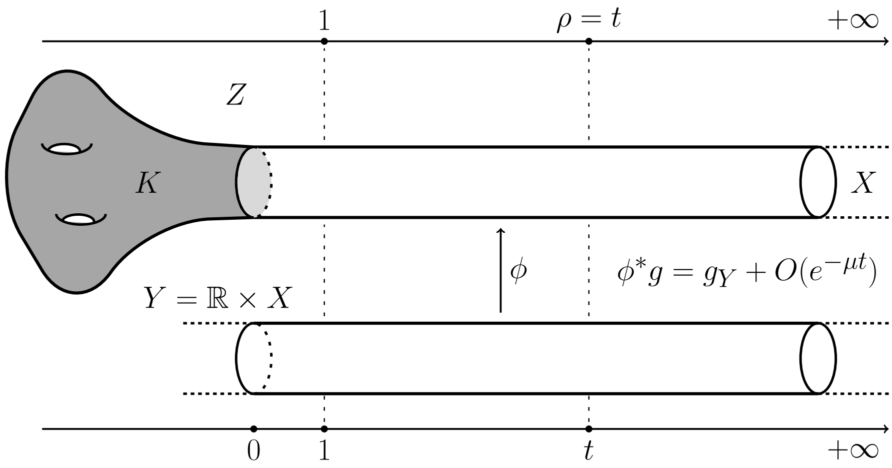

Figure 2.1: An EAC manifold.

2.2.1 Asymptotically cylindrical manifolds. Let us recall a few standard definitions, starting with the notion of *Exponentially Asymptotically Cylindrical (EAC)* manifold. Let Z be an oriented non-compact manifold of dimension n and X an oriented compact manifold of dimension $n-1$. We say that Z is asymptotic to the cylinder $Y = \mathbb{R} \times X$ at infinity if there exist a compact $K \subset Z$ and an orientation-preserving diffeomorphism $\phi : (0, \infty) \times X \to Z \backslash K$. The compact manifold X is called the *cross-section* of Z. It will often be useful to pick a positive function $\rho : Z \to \mathbb{R}$ such that $\rho(\phi(t, x)) = t$ for $(t, x) \in [1, \infty) \times X$ and $\rho < 1$ outside of $\phi([1, \infty) \times X)$. Following the terminology of [56], we will call such function a *cylindrical coordinate function*.

We say that a Riemannian metric g on Z is EAC of rate $\mu > 0$ if we have, for all integers $l \geq 0$:

$$|\nabla^l_Y (\phi^* g - g_Y)|_{g_Y} = O\left(e^{-\mu t}\right) \tag{2.8}$$

as $t \to \infty$, where $g_Y = dt^2 + g_X$ is a cylindrical metric on $Y = \mathbb{R} \times X$, ∇_Y the associated Levi-Civita connection and $|\cdot|_{g_Y}$ the associated norm on tensor bundles.

The notion of EAC manifold can be refined in the case of metrics with special holonomy. An EAC G_2-manifold (Z, φ) of rate $\mu > 0$ is required to satisfy

$$\phi^* \varphi = \varphi_0 + \eta, \qquad |\nabla^l_Y \eta|_{g_Y} = O(e^{-\mu t}) \ \ \forall l \geq 0$$

as $t \to \infty$, where φ_0 is a translation-invariant torsion-free G_2-structure on $\mathbb{R} \times X$. The cross-section X is then a Calabi–Yau threefold, and has a unique Calabi–Yau structure (ω_0, Ω_0) such that $\varphi_0 = dt \wedge \omega_0 + \operatorname{Re} \Omega_0$. In this case, we may assume that the metric g_X is induced by (ω_0, Ω_0).

2.2.2 Twisted connected sums. Let us now outline the twisted connected sum construction. The building blocks are a pair of EAC G_2-manifolds (Z_1, φ_1) and (Z_2, φ_2) of rate $\mu > 0$, asymptotic to a cylinder $\mathbb{R} \times X$. Let us denote by $\varphi_{0,i}$ the asymptotic translation-invariant model for φ_i. The G_2-structures φ_1 and φ_2 are said to be matching if there exists an isometry γ of the cross-section X such that the map

$$\underline{\gamma} : \mathbb{R} \times X \to \mathbb{R} \times X, \quad (t, x) \mapsto (-t, \gamma(x))$$

satisfies $\underline{\gamma}^* \varphi_{0,2} = \varphi_{0,1}$. If $\gamma_{0,i} = dt \wedge \omega_{0,i} + \operatorname{Re} \Omega_{0,i}$ for Calabi-Yau structures $(\omega_{0,i}, \Omega_{0,i})$ on X, the matching condition amounts to:

$$\gamma^* \omega_{0,2} = -\omega_{0,1}, \quad \gamma^* \operatorname{Re} \Omega_{0,2} = \operatorname{Re} \Omega_{0,1}. \tag{2.9}$$

In all known examples [80, 30, 97], such matching pairs are trivial circle bundles over EAC Calabi–Yau threefolds (or quotients thereof in the case of Nordström's 'extra-twisted' connected sum), and the cross-section is isometric to the product of a K3 surface with a flat 2-torus (or the corresponding quotients). Much of the subtlety of the construction lies in the choice of isometry γ, which is usually designed so that compact manifold obtained by gluing Z_1 and Z_2 along γ has finite fundamental group in order to construct manifolds with full holonomy G_2. These details go beyond the scope of the present chapter and do not affect our analysis, so we refer to the original papers for more information. In fact one could also do an 'untwisted' connected sum, resulting in a compact manifold which is globally a trivial circle bundle, and by dimensional reduction this yields a construction of compact Calabi–Yau threefolds.

Let us denote by σ the minimum of μ and of the square roots of the smallest non-trivial eigenvalues of the Laplacian acting on 2- and 3-forms on X, and pick diffeomorphisms $\phi_i : (0 \times \infty) \times X \to Z_i \backslash K_i$ of the cylindrical ends. The closed forms φ_i and $\Theta(\varphi_i)$ admit an expansion:

$$\phi_i^* \varphi_i = \varphi_{i,0} + d\eta_i, \quad \phi_i^* \Theta(\varphi_i) = \Theta(\varphi_{i,0}) + d\xi_i$$

where $\eta_i \in \Omega^2((0, \infty) \times X)$, $\xi_i \in \Omega^3((0, \infty) \times X)$ and all their covariant derivatives have exponential decay in $O(e^{-\delta t})$ as $t \to \infty$, for any $0 < \delta < \sigma$ (see §3.4.1). Let us pick a smooth cutoff function $\chi : \mathbb{R} \to [0, 1]$ such that $\chi \equiv 0$ in $(-\infty, -\frac{1}{2}]$ and $\chi \equiv 1$ in $[\frac{1}{2}, \infty)$. Then we can build 1-parameter families of closed forms:

$$\varphi_{i,T} = \varphi_i - d(\chi(\rho_i - T)(\phi_i^{-1})^* \eta_i), \quad \Theta_{i,T} = \Theta(\varphi_i) - d(\chi(\rho_i - T)(\phi_i^{-1})^* \xi_i)$$

where ρ_i are cylindrical coordinate functions on Z_i. For T large enough $\varphi_{i,T}$ is a G_2-structure on Z_i, which is closed by construction. Then we can build a family of compact manifolds $\{(M_T, \varphi_T)\}_{T\in[T_0,\infty)}$ in the following way. To construct M_T, we can glue the domains $\{\rho_i \le T+2\} \subset Z_i$ along the annuli $\{T \le \rho_i \le T+2\} \simeq [-1,1]\times X$ using the identification

$$\phi_1(T+1+t, x) \sim \phi_2(T+1-t, \gamma(x)), \qquad \forall (t,x) \in [-1,1]\times X. \tag{2.10}$$

Since the matching conditions (2.9) are satisfied, we can patch up $\varphi_{1,T}$ with $\varphi_{2,T}$ in order to obtain a closed G_2-structure φ_T on M_T. Similarly, patching up $\Theta_{1,T}$ with $\Theta_{2,T}$ we obtain a closed 4-form Θ_T. Let us write $\psi_T = \Theta(\varphi_T) - \Theta_T$, so that $d\psi_T = d\Theta(\varphi_T)$. By construction, we have estimates of the form:

$$\|\psi_T\|_{C^k} + \|d\Theta(\varphi_T)\|_{C^k} = O\left(e^{-\delta T}\right) \tag{2.11}$$

for any $k \ge 0$ and $0 < \delta < \sigma$, and since ψ_T and $d\Theta(\varphi_T)$ are supported in the gluing region which has uniformly bounded volume this induces similar estimates for any $W^{k,p}$-norms [80, Lemma 4.25]. It follows from [66, Theorem 11.6.1] that for T large enough there is a torsion-free G_2-structure $\widetilde{\varphi}_T$ cohomologous to φ_T and such that $\|\widetilde{\varphi}_T - \varphi_T\|_{C^0} = O(e^{-\delta T})$ for any $\delta > 0$ small enough.

In the next chapter, we will see that the norm of the Sobolev embeddings $W^{k,p} \hookrightarrow C^l$ (in the range where these embeddings are well-defined and continuous) are uniformly bounded (Proposition 3.5), and that the norm of the Green's function of the Laplacian is bounded above by $O(T^2)$ (Corollary 3.39). Hence Theorem 2.14 yields the following improved estimates with control on an arbitrary number of derivatives:

Proposition 2.16. *Let $k \in \mathbb{N}$ and $0 < \delta < \sigma$. Then there exists a constant $C > 0$ such that for T large enough:*

$$\|\widetilde{\varphi}_T - \varphi_T\|_{C^k} \le Ce^{-\delta T}.$$

With these estimates, we can approximate the Laplacian operator associated with $\widetilde{\varphi}_T$ by the Laplacian operator associated with φ_T. In particular, the spectral estimates which we will derive in the next chapter (Theorem 3.8), using the 'unperturbed' metric g_{φ_T}, also apply to the metric $g_{\widetilde{\varphi}_T}$. In Remark 3.10 we will mention some physical consequences of this result, in relation to the swampland distance conjecture discussed in the introduction.

Chapter 3

Spectral properties of twisted connected sums

This chapter, whose material appears in the article [82] by the author, is concerned with the analysis of differential operators for a class of 'neck-stretching' problems where two exponentially asymptotically cylindrical (EAC) manifolds are glue together in order to form a family of compact manifolds whose diameter goes to infinity. A typical example of such situation is the twisted connected sum construction of compact G_2-manifolds which we saw in the previous chapter (Section 2.2). Our original motivation came from the swampland distance conjecture in physics (see the introduction): along such deformations, physicists expect an infinite number of eigenvalues of the Laplacian acting on differential forms (which physically correspond to a mass spectrum) to decay at the same rate[1]. Hence an interesting question is to understand precisely the asymptotic behaviour of the Laplacian in the neck-stretching limit.

This type of neck-stretching problems has many applications beyond twisted connected sums in various branches of analysis, geometry, topology and mathematical physics (see for instance the review [47]). It has notably been used for proving index theorems for manifolds with boundary [9] or corners [57], and the analysis of such problems is closely related to the analysis of differential operators on noncompact manifolds (e.g. Lockhart–McOwen theory for EAC manifolds [89], or Melrose's general theory of b-calculus [93]). Concerning the spectral aspects, a very precise development of the asymptotic behaviour of the eigenvalues of the scalar Laplacian was notably obtained by Grieser [46] in the case of manifolds connected by neck regions which are exact cylinders.

[1] Physicists expect that this is the typical behaviour for infinite-distance limits in the moduli space. Twisted connected sums are an example of such infinite-distance limits for the volume diverges to infinity in the neck-stretching limit (see §4.1.1, and also Lemma 4.18).

In the present chapter, we develop a general method to analyse the mapping properties of a class of adapted differential operators in the neck-stretching limit of a connected sum of two EAC manifolds. Our method is relatively elementary in two ways: first, we only use the classical Lockhart–McOwen theory and some results on the analysis on Riemannian cylinders without appealing to more sophisticated machinery (like b-calculus), which allows us to give a mostly self-contained exposition; and second we work with *unweighted* Sobolev spaces on the family of compact manifolds. Let us briefly justify this choice. In degenerate limits, it is classical to introduce *weighted* Sobolev spaces in order to find a Fredholm inverse for the relevant differential operator with uniformly bounded norm. However, in applications it is not necessarily a problem if the operator norm of a Fredholm inverse diverges, as long as its growth rate remains under control. The real issue is to identify the right notion of substitute kernel and cokernel, and to build a Fredholm inverse which is close to being an actual inverse on the complement of those. This involves a rather delicate 'matching problem' for the obstructions coming from each of the pieces which are glued together, and this problem does not have anything to do with a particular choice of weight[2]. A second reason for working with unweighted spaces is that we will be interested in the eigenvalues of the Laplacian, which are more directly related to the estimates in the L^2-range.

Let us now outline the plan for this chapter. In Section 3.1 we describe the general gluing problem that we are interested in, define the notions of adapted operators and of substitute kernel and cokernel that we will be working with, and state our main results. Section 3.2 is concerned with the analysis of translation-invariant PDEs on cylinders and contains the main technical ingredients underlying our proofs. Section 3.3 is dedicated to the analysis of the mapping properties of adapted operators in the neck-stretching limit. Under some assumptions, we prove a theorem on the invertibility of adapted operators (Theorem 3.6), but our method is more general and we also comment on how to adapt it in different contexts. Finally, in Section 3.4 we apply our techniques to the study the asymptotic behaviour of the low eigenvalues of the Laplacian.

[2] This is notably an issue in Kovalev's analysis for the twisted connected sum construction [80]. In this case, the relevant operator is the Laplacian, and the actual obstructions are topological by Hodge theory; but by introducing weights we obtain an approximate inverse for the Laplacian where the substitute kernel and cokernel have the wrong dimension. This issue can be solved by using the analytical framework developed in this chapter.

3.1 The neck-stretching problem

In this section we explain our setup and formulate the main results of this chapter. The gluing problem under consideration is described in §3.1.1, in which we introduce the building blocks and the class of adapted operators that we are interested in. In §3.1.2 we motivate and introduce the notions of substitute kernel and cokernel for adapted operators, following ideas present for instance in [69, 71] or [81]. Our main results are discussed in §3.1.3, where we also outline our strategy of proof.

3.1.1 Model gluing problem. Let (Z, g) be an exponentially asymptotically cylindrical (EAC) manifold of rate $\mu > 0$ (see §2.2.1). By definition, there is a compact subset $K \subset Z$ and a diffeomorphism $\phi : (0, \infty) \times X \to Z \backslash K$, where the cross-section X is a compact manifold endowed with a metric g_X, and $\phi^* g = g_Y + O(e^{-\mu t})$ where $g_Y = dt^2 + g_X$ on the cylinder $Y = \mathbb{R} \times X$ (and similar estimates hold for derivatives of any degree). We can also pick a smooth cylindrical coordinate function $\rho > 0$ on Z such that $\rho(\phi(t, x)) = t$ for any $t \geq 1$ and $x \in X$.

Given the above data, we may define a notion of *adapted bundle* as follows. Any vector bundle $E_0 \to X$ equipped with a metric h_0 and a connection ∇_0 can be extended to a vector bundle $\underline{E}_0 \to Y$ with translation-invariant metric and connection $(\underline{h}_0, \underline{\nabla}_0)$ (see Section 3.2 for more details). We call such bundles on Y *translation-invariant vector bundles.* Let $E \to Z$ be a vector bundle on Z, endowed with a metric h and a connection ∇. We say that E is an adapted bundle on (Z, g) if there exist a translation-invariant vector bundle $(\underline{E}_0, \underline{h}_0, \underline{\nabla}_0)$ on Y and a bundle isomorphism $\Phi_E : \underline{E}_{0|(0,\infty)\times X} \to E_{|Z\backslash K}$ covering ϕ, such that for all integers $l \geq 0$:

$$|\underline{\nabla}_0^l(\Phi_E^* h - \underline{h}_0)|_0 = O(e^{-\mu t}), \quad \text{and} \quad |\underline{\nabla}_0^l(\Phi_E^* \nabla - \underline{\nabla}_0)|_0 = O\left(e^{-\mu t}\right) \tag{3.1}$$

as $t \to \infty$, where $|\cdot|_0$ is the norm induced by the metrics g_Y and $\underline{h}_0$.

Remark 3.1. In this chapter, we will consider *complex* vector bundles endowed with hermitian metrics for convenience, but the results of course apply to real vector bundles by taking their complexification.

We may also define the notion of *adapted differential operator* between adapted bundles. Let E, F be adapted bundles on Z and $P : C^\infty(E) \to C^\infty(F)$ be a differential operator of order $k \geq 1$. If u is a smooth section of $\underline{E}_0$ defined over the half-cylinder $(0, \infty)$, let:

$$\widetilde{P} u = \Phi_F^{-1} P \Phi_E u. \tag{3.2}$$

This defines a differential operator $\widetilde{P} : C^\infty(\underline{E}_0) \to C^\infty(\underline{F}_0)$ over the cylinder $(0,\infty)\times X$, modelling the action of P on sections supported in $Z\backslash K$. The operator $\widetilde{P}$ can be written in the form:

$$\widetilde{P} = \sum_{j=0}^{k} A_{k-j}(t)\partial_t^j \tag{3.3}$$

where ∂_t is the covariant derivative in the direction $\frac{\partial}{\partial t}$ for the connection $\underline{\nabla}_0$ on E (which coincides with the Lie derivative of $\frac{\partial}{\partial t}$ because of translation-invariance), and $A_{k-j}(t) : C^\infty(E_0) \to C^\infty(F_0)$ are differential operators depending smoothly on t. We say that P is adapted (with exponential rate $\mu > 0$) if there exists a translation-invariant differential operator $P_0 : C^\infty(E_0) \to C^\infty(F_0)$ of the form:

$$P_0 = \sum_{j=0}^{k} A_{k-j}\partial_t^j \tag{3.4}$$

such that for any smooth section u of $\underline{E}_0$ defined on $(0,\infty)\times X$ and for any $l \geq 0$ and $0 \leq j \leq k$ we have:

$$|\underline{\nabla}_0^l(A_{k-j}(t)u - A_{k-j}u)|_0 = O\left(e^{-\mu t}\sum_{i\leq l}|\underline{\nabla}_0^i u|_0\right) \tag{3.5}$$

as $t \to \infty$. That is, we essentially want the coefficients of $\widetilde{P} - P_0$ and all their derivatives to have exponential decay when $t \to \infty$. The operator P_0 is called the *indicial operator* of P. Note that the formal adjoint of an adapted P is also adapted, and its indicial operator is naturally P_0^*.

Example 3.2. The tensor bundles $TZ^{\otimes s} \otimes T^*Z^{\otimes r}$, the bundle of differential forms $\Lambda^k T^*Z$, or any direct sums or tensor products thereof are adapted (endowed with the metric induced by g and its Levi-Civita connection). Moreover, in those cases we might choose the bundle isomorphism covering ϕ to be the push-forward map. The differential operators $d + d^*$ and $\Delta = dd^* + d^*d$ are adapted.

We now describe the general gluing problem that we are interested in. Let Z_1 and Z_2 be two EAC manifolds and assume that the cross-section of Z_2 is the same as the cross-section X of Z_1, but with opposite orientation. By definition, there exist compact subsets $K_i \subset Z_i$ and diffeomorphisms $\phi_i : (0,\infty)\times X_i \to Z_i\backslash K_i$ where $X_1 = X = \overline{X}_2$, and we can pick cylindrical coordinate functions $\rho_i : Z_i \to \mathbb{R}_{>0}$. For any $T \geq 0$, we can construct an oriented compact manifold M_T by gluing the domains $\{\rho_1 \leq T+2\} \subset Z_1$ and $\{\rho_2 \leq T+2\} \subset Z_2$ along the annuli $\{T \leq \rho_i \leq T+2\} \simeq [-1,1]\times X$ with the identification:

$$\phi_1(T+1+t, x) \simeq \phi_2(T+1-t, x), \quad \forall (t,x) \in [-1,1]\times X. \tag{3.6}$$

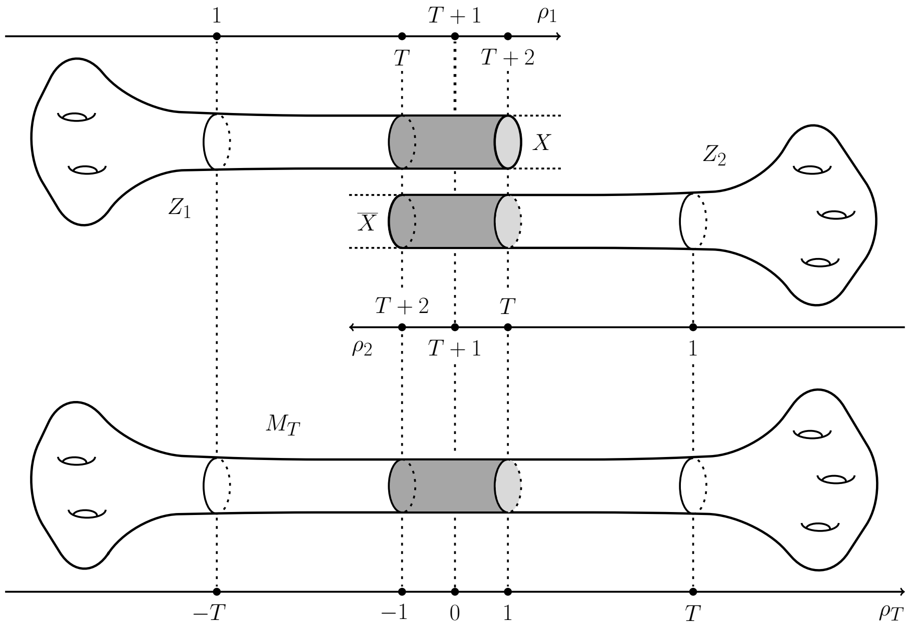

Figure 3.1: The gluing construction and neck-stretching.

Define a smooth function ρ_T on M_T by:

$$\rho_T \equiv \begin{cases} \rho_1 - T - 1 & \text{in } \{\rho_1 \leq T+2\} \\ T + 1 - \rho_2 & \text{in } \{\rho_2 \leq T+2\} \end{cases}.$$

This is well-defined as $\rho_1 - T - 1$ coincides with $T + 1 - \rho_2$ under the identification (3.6). Intuitively, the function ρ_T parametrises the neck of M_T. In particular, the domain $\{|\rho| \leq T\}$ is diffeomorphic to the finite cylinder $[-T, T] \times X$ (see Figure 3.1). Our goal is to study the mapping properties of elliptic operators defined on M_T as T becomes very large, and relate it to the corresponding properties of operators on Z_i.

Remark 3.3. In the twisted connected sum construction (see §2.2.2), we considered a variation of the above gluing construction where the EAC manifolds Z_1 and Z_2 are glued along a non-trivial isometry $\gamma : X \to X$ (see (2.10)). From the point of view of the analysis this does not change anything; in fact if we replace ϕ_2 by $\phi_2 \circ (\mathrm{id} \times \gamma)$ we can see that the twisted connected sum is a particular case of our seemingly 'untwisted' gluing problem.

Suppose that the manifolds Z_i are endowed with EAC metrics g_i asymptotic to the same translation-invariant metric $g_Y = dt^2 + g_0$ on $Y = \mathbb{R} \times X$. It will also be useful to fix a cutoff function $\chi : \mathbb{R} \to [0, 1]$ such that $\chi \equiv 0$ on $(-\infty, -\frac{1}{2}]$ and $\chi \equiv 1$ on $[\frac{1}{2}, +\infty)$. If $T \in \mathbb{R}$ we let $\chi_T(t) = \chi(t - T)$. Then, for T large enough

$$g_{i,T} = (1 - \chi_T(\rho_i))g_i + \chi_T(\rho_i)g_Y$$

is a Riemannian metric on Z_i which coincides with g_i on $\{\rho_i \leq T - \frac{1}{2}\}$ and with g_Y on $\{\rho_i \geq T + \frac{1}{2}\}$. Moreover, the difference $g_i - g_{i,T}$ and all their derivatives are uniformly bounded by $O(e^{-\mu T})$. Note that here we implicitly identify $Z_i \backslash K_i$ with the half cylinder $(0, \infty) \times X_i$ to make notations lighter. We can patch $g_{1,T}$ and $g_{2,T}$ to form a Riemannian metric g_T on M_T, defining:

$$g_T \equiv \begin{cases} g_{1,T} & \text{if } \rho_T \leq 0 \\ g_{2,T} & \text{if } \rho_T \geq 0 \end{cases}.$$

Similarly, if we have adapted bundles (E_i, h_i, ∇_i) on Z_i such that their asymptotic models are both isomorphic to the same translation-invariant vector bundle $(\underline{E}_0, \underline{h}_0, \underline{\nabla}_0)$ on $\mathbb{R} \times X$, we can use the same cutoff procedure to patch them up on M_T and form a vector bundle E_T with metric h_T and connection ∇_T.

Consider matching adapted bundles E_i, F_i on Z_i $(i = 1, 2)$ asymptotic to the same translation-invariant bundles $\underline{E}_0, \underline{F}_0$, and adapted elliptic operators $P_i : C^\infty(E_i) \to C^\infty(F_i)$ of order k. Denote by $P_{i,0}(x, \partial_x, \partial_t) : C^\infty(\underline{E}_0) \to C^\infty(\underline{F}_0)$ the indicial operator of P_i, where we use ∂_x as a loose notation for the derivatives along the cross-section X. In order to patch up these operators we need to assume the following compatibility condition [81]:

$$P_{2,0}(x, \partial_x, \partial_t) = P_{1,0}(x, \partial_x, -\partial_t). \tag{3.7}$$

Assuming that it is satisfied, define:

$$P_{i,T} = (1 - \chi_T(\rho_i))P_i + \chi_T(\rho_i)P_{i,0}$$

which coincides with P_i for $\rho_i \leq T - \frac{1}{2}$ and with $P_{i,0}$ for $\rho_i \geq T + \frac{1}{2}$. For large enough T, the operators $P_{i,T}$ are elliptic, and moreover the coefficients of $P_i - P_{i,T}$ and all their derivatives are uniformly bounded by $O(e^{-\mu T})$. Patching $P_{1,T}$ and $P_{2,T}$ together in the same way as for the metrics $g_{i,T}$, we obtain a family of operators $P_T : C^\infty(E_T) \to C^\infty(F_T)$ which are elliptic for large enough T (see Figure 3.2). Elliptic regularity on compact manifolds implies that the action of P_T on Sobolev spaces of sections induce Fredholm maps. Our goal is to construct Fredholm inverses for these maps, with a good control on their norm as $T \to \infty$.

Before explaining our results in more detail in the next part, let us make our conventions for Sobolev and C^l-norms explicit. For $p \geq 1$ and $l \in \mathbb{N}$, the $W^{l,p}$-norm of a section $u \in C^\infty(E_T)$ can be defined as:

$$\|u\|_{W^{l,p}} = \sum_{j \leq l} \|\nabla_T^j u\|_{L^p} \tag{3.8}$$

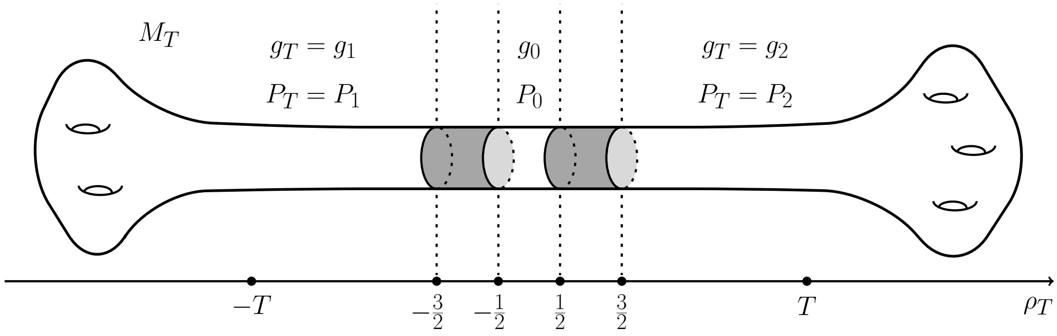

Figure 3.2: The metric g_T and the operator P_T.

where the fibrewise norm of $\nabla_T^j u$ is computed with respect to the metrics h_T, g_T and we integrate over the volume form of g_T. The Sobolev space $W^{l,p}(E_T)$ is defined as the completion of $C^\infty(E_T)$ for the $W^{l,p}$-norm. In the same way, the C^l norm of a smooth section $u \in C^\infty(E_T)$ will be defined as

$$\|u\|_{C^l} = \sum_{j \le l} \|\nabla_T^j u\|_{C^0} \tag{3.9}$$

and $C^l(E_T)$ is the completion of $C^\infty(E_T)$ for this norm.

In Section 3.3, we will see that the adapted operators P_i on the EAC manifolds Z_i induce bounded maps $W^{k+l,p}(E_i) \to W^{l,p}(E_i)$ and satisfy a priori estimates

$$\|u\|_{W^{k+l,p}} \le C(\|Pu\|_{W^{l,p}} + \|u\|_{L^p}).$$

These estimates for P_i induce a priori estimates for the family of operators P_T on M_T which are uniform in T:

Proposition 3.4. *With the above setup, let $p > 1$ and $l \in \mathbb{N}$. Then the map*

$$P_T : W^{k+l,p}(E_T) \to W^{l,p}(F_T)$$

is uniformly bounded as $T \to \infty$. Moreover there exist constants $C, C' > 0$ such that for T large enough and for any $u \in W^{k+l,p}(E_T)$ we have:

$$\|u\|_{W^{k+l,p}} \le C\left(\|P_T u\|_{W^{l,p}} + \|u\|_{L^p}\right).$$

In the same way, there are continuous embeddings $W^{r,p}(E_i) \hookrightarrow W^{s,q}(E_i)$ and $W^{r,p}(E_i) \hookrightarrow C^l(E_i)$ whenever $\frac{1}{q} \le \frac{1}{q} + \frac{r-s}{n}$ and $\frac{1}{p} \le \frac{r-l}{n}$, for the spaces of sections over the EAC manifolds Z_i [88]. Hence on M_T we deduce the uniform boundedness of the Sobolev embeddings:

Proposition 3.5. *Let $p, q > 1$ and $r, s, l \in \mathbb{N}$ such that $\frac{1}{q} \le \frac{1}{q} + \frac{r-s}{n}$ and $\frac{1}{p} \le \frac{r-l}{n}$. Then there exist constants $C, C' > 0$ which do not depend on T such that*

$$\|u\|_{W^{s,q}} \le C\|u\|_{W^{r,p}} \quad \textit{and} \quad \|u\|_{C^l} \le C'\|u\|_{W^{r,p}}, \qquad \forall\, u \in W^{r,p}(E_T).$$

By elliptic regularity, the P_T can be inverted in the L^2-orthogonal complements of the kernels of P_T and of its adjoint. However, since the dimensions of these spaces are not deformation-invariant (only the index is), they do depend on the precise way we take cutoffs to define our gluing, and so will the norm of the inverse of P_T. In order to make general statements, we would like to define notions of substitute kernel and cokernel in the fashion of [69] (see also [71, §18]), determined by the gluing data and in the complement of which we have a good control on the norm of the inverse of P_T. Under the restricting assumption that the map induced by the indicial operator $P_0 = P_{1,0}$ on Sobolev spaces of sections on Y is an isomorphism, these have been defined and studied in [81]. However in many cases of interest this assumption is not satisfied, as the indicial operator may have *real roots* (see next part). Thus we need to define notions of substitute kernel and cokernel adapted to that case.

3.1.2 Substitute kernel and cokernel. In order to define the substitute kernel and cokernel, a good understanding of the mapping properties of translation-invariant operators on cylinders and of adapted operators on EAC manifolds is crucial. For completeness, the results that we need are gathered in §3.2.1 and §3.3.1. Some original references are [3], [89] and [93].

In the situation described in the previous part, let $P_0 = P_{1,0} : C^\infty(\underline{E}_0) \to C^\infty(\underline{F}_0)$ be the indicial operator of P_1, acting on the cylinder $Y = \mathbb{R} \times X$. Points in Y will be denoted by $y = (t, x)$. A particularly important role in our analysis is played by solutions of the homogeneous equation $P_0 u = 0$ of the form:

$$u(t,x) = \sum_{j=1}^{m} e^{i\lambda_j t} u_j(t,x)$$

where $\lambda_1, ..., \lambda_m$ are real numbers and the sections u_j are polynomial in the variable t. Such solutions are called *polyhomogeneous solutions* of rate 0, and we denote by $\mathscr{E}$ the vector space they span. As a matter of general theory, this is a finite-dimensional space, and in particular there are only finitely many values $\lambda \in \mathbb{R}$ such that the homogeneous equation $P_0 u = 0$ admits a non-trivial solution of the form $u(t,x) = e^{i\lambda t} u_\lambda(t,x)$, where u_λ is polynomial in t. These values are called the *real roots* of P_0 (see Section 3.2 for a detailed discussion). In Section 3.2 we will see that each root λ_j has a certain order $d_j \in \mathbb{N}^*$ such that the sections u_j in a polyhomogeneous solution of rate 0 are polynomials of order at most $d_j - 1$ in the variable t. We will usually denote by d the maximum of the orders d_j. Let us point out here that the real roots of the formal adjoint P_0^* are the same as the real roots of P_0, and denote by $\mathscr{E}^*$ the space of polyhomogeneous solutions of rate 0 of $P_0^* u = 0$. It follows from the compatibility condition (3.7) that the space of polyhomogeneous solutions of $P_{2,0}u = 0$ of rate 0 is $\{u(-t,x), u \in \mathscr{E}\}$, and similarly for the adjoint operators.

Let us denote by $\mathscr{K}_i$ the space of solutions of $P_i u = 0$ with sub-exponential growth and $\mathscr{K}_{i,0}$ the subspace of decaying solutions, for $i = 1, 2$. By Lockhart–McOwen theory ([89], see also §3.3.1 for more details), $\mathscr{K}_i$ has finite dimension and each of its elements is asymptotic to a polyhomogeneous solution of $P_{i,0}u = 0$ with rate 0, up to an exponentially decaying term. More precisely, for any $u \in \mathscr{K}_1$, there exists a polyhomogeneous solution $u_0 \in \mathscr{E}$ such that for any $l \in \mathbb{N}$:

$$|\underline{\nabla}_0^l (u(t,x) - u_0(t,x))|_0 = O\left(e^{-\delta t}\right)$$

when $t \to \infty$, for any sufficiently small $\delta > 0$. Here we implicitly identify u over $Z_1 \backslash K_1$ with a section of $\underline{E}_0$ over $(0, \infty) \times X$. Therefore, we can define a linear map $\kappa_1 : \mathscr{K}_1 \to \mathscr{E}$, such that for any $u \in \mathscr{K}_1$, the difference $u - \kappa_1[u]$ and all its derivatives have exponential decay at infinity. Taking care of the fact that we need to change the sign of the variable t, we can similarly define a map $\kappa_2 : \mathscr{K}_2 \to \mathscr{E}$ such that $|u(x,t) - \kappa_2[u](x,-t)|_0 = O(e^{-\delta t})$ as $t \to \infty$ for all $u \in \mathscr{K}_2$, with the usual identifications. For $i = 1, 2$, the kernel of the map κ_i in $\mathscr{K}_i$ is $\mathscr{K}_{i,0}$. Considering adjoint operators, we may also define $\mathscr{K}_i^*$, $\mathscr{K}_{i,0}^*$ and linear maps $\kappa_i^* : \mathscr{K}_i^* \to \mathscr{E}^*$.

With these notations in hand, let $u_1 \in \mathscr{K}_1$, $u_2 \in \mathscr{K}_2$ and fix $T > 0$. We say that u_1 and u_2 are matching at T if the following condition is satisfied:

$$\kappa_1[u_1](t + T + 1, x) = \kappa_2[u_2](t - T - 1, x), \quad \forall (t,x) \in \mathbb{R} \times X. \tag{3.10}$$

Given a matching pair of solutions (u_1, u_2), we can define a section of the bundle $E_T \to M_T$ as follows:

$$u_T = (1 - \chi_{T+1}(\rho_1))u_1 + (1 - \chi_{T+1}(\rho_2))u_2$$

where we consider $\chi(\rho_i)u_i$ as a section of E_T supported in the domain $\{\rho_i \leq T + 2\} \subset M_T$. In particular, $u_T \equiv u_1$ in the domain $\{\rho_1 \leq T + \frac{1}{2}\}$, $u_T \equiv u_2$ in $\{\rho_2 \leq T + \frac{1}{2}\}$ and it smoothly interpolates between the two in $\{|\rho_T| \leq \frac{1}{2}\}$. It is easy to see that $P_T u_T \equiv 0$ outside of the annulus $\{|\rho_T| \leq \frac{3}{2}\}$. The matching condition (3.10) ensures that for any $l \in \mathbb{N}$, small enough $\delta > 0$ and arbitrary norms on $\mathscr{K}_1$, $\mathscr{K}_2$, there exists a constant $C > 0$ independent of $T \geq 1$ such that:

$$\|P_T u_T\|_{C^l} \leq C e^{-\delta T}(\|u_1\| + \|u_2\|) \tag{3.11}$$

for any matching pair of solutions (u_1, u_2). In that sense, u_T is an approximate solution of $P_T u = 0$. The *substitute kernel* $\mathscr{K}_T$ of P_T is defined as the finite-dimensional subspace of $C^\infty(E_T)$ of approximate solutions constructed in this way, from a matching pair (u_1, u_2) of solutions of $P_i u = 0$. Similarly we define the *substitute cokernel* $\mathscr{K}_T^*$ as the substitute kernel of P_T^*. These definitions depend on the arbitrary choice of cutoff function χ, but since the difference for two choices of cutoff function would decay exponentially with T this will not be an issue.

For these notions of substitute kernel and cokernel to be convenient to handle in practice, it is simpler to assume that for T large enough the dimensions of $\mathscr{K}_T$ and $\mathscr{K}_T^*$ are independent of T. This is automatically satisfied if the indicial operator P_0 has only one root. Indeed, in this case we can express the matching condition at T as a finite-dimensional linear system depending polynomially on T by choosing convenient bases for $\operatorname{im}\kappa_1$, $\operatorname{im}\kappa_2$ and $\mathscr{E}$. The minors of this system are polynomial in T, and therefore are either identically 0 or do not vanish for T large enough. Hence the rank of the system does not depend on T for T large enough, and neither does the dimension of its kernel. We can argue similarly for the substitute cokernel $\mathscr{K}_T^*$.

For more general operators, the matching condition will be expressed as a finite-dimensional linear system with coefficients depending analytically on T, and although the non-trivial minors of the system only have isolated zeroes we cannot always ensure that there are only finitely many of them. This is the situation that we want to avoid. Therefore we will assume that P has only one real root to state our main result, about the existence of a Fredholm inverse for P in the complement of the substitute kernel and cokernel. This is sufficient for our applications in Section 3.4. However the method we develop is more general, and for most of this chapter we do not need to take any restricting assumptions on the roots of the indicial operator.

Assuming that the spaces $\mathscr{K}_T$ and $\mathscr{K}_T^*$ have constant dimension for T large enough, it follows from (3.11) that for any Sobolev norm $W^{k,p}$ and any small enough $\delta > 0$, there exists a constant $C > 0$ such that for T large enough and for any $u \in \mathscr{K}_T$ we have:

$$\|P_T u\|_{W^{k,p}} \leq C e^{-\delta T} \|u\|_{L^p}.$$

Similar bounds hold for P_T^*. Hence there is no hope to have a control on the norm of the inverse of P_T better than $O(e^{\delta T})$ if we do not work on the complement of $\mathscr{K}_T$ and $\mathscr{K}_T^*$.

3.1.3 Results and strategy. Our first main result is the following theorem, which says that under the limiting assumption described above we can find a Fredholm inverse for P_T in the complement of the substitute kernel and cokernel, with norm bounded by a power of T:

Theorem 3.6. *Let $p > 1$ and $l \in \mathbb{N}$, and assume that P_0 has only one real root. Then there exist constants $C, C' > 0$ and an exponent $\beta \geq 0$ such that for T large enough the following holds.*

For any $f \in W^{l,p}(F_T)$, there exist a unique $u \in W^{k+l,p}(E_T)$ orthogonal to $\mathscr{K}_T$ and a unique $w \in \mathscr{K}_T^*$ such that $f = P_T u + w$. Moreover, u satisfies the bound:

$$\|u\|_{W^{k+l,p}} \leq C\|f\|_{W^{l,p}} + C'T^{\beta}\|f\|_{L^p}.$$

Remark 3.7. The case when the indicial operator P_0 has no real roots has been studied by Kovalev–Singer in [81], who show that under this assumption one can build an approximate inverse for P_T with uniform bounds (independent of T).

In some cases, we are also able to determine the optimal exponent β. This is for instance the case of the Laplacian operator Δ_T of the metric g_T. The Laplacian acting on q-forms on $\mathbb{R} \times X$ has no real roots when $b^{q-1}(X) + b^q(X) = 0$ and admits 0 as unique real root when $b^{q-1}(X) + b^q(X) > 0$. If $b^{q-1}(X) + b^q(X) = 0$, it follows from the results of [81] (see Remark 3.7) that the norm of the inverse of Δ_T orthogonally to the space of harmonic q-forms is bounded independently of T. When $b^{q-1}(X) + b^q(X) > 0$, we will see that $\beta = 2$ is optimal in Theorem 3.6 and the substitute kernel gives a good approximation of the space of harmonic forms (see Corollary 3.39).

If we consider the L^2-range, this means that the behaviour of the low eigenvalues of Δ_T depends on the topology of the cross-section X. When $b^{q-1}(X)+b^q(X) = 0$, the lowest non-zero eigenvalue of Δ_T acting on q-forms is uniformly bounded below as $T \to \infty$. On the other hand, if $b^{q-1}(X) + b^q(X) > 0$ then the first eigenvalue satisfies a bound of the type $\lambda_1(T) \geq \frac{C}{T^2}$ for some constant $C > 0$. It is an interesting problem to determine the distribution the eigenvalues that have the fastest decay rate. Let us define the densities of low eigenvalues as:

$$\Lambda_{q,\inf}(s) = \liminf_{T\to\infty} \# \left\{\text{eigenvalues of } \Delta_T \text{ acting on } q\text{-forms in } \left(0, \pi^2 s T^{-2}\right]\right\}$$
$$\Lambda_{q,\sup}(s) = \limsup_{T\to\infty} \# \left\{\text{eigenvalues of } \Delta_T \text{ acting on } q\text{-forms in } \left(0, \pi^2 s T^{-2}\right]\right\}$$

where we count eigenvalues with multiplicity. The normalisation by T^{-2} comes from the fact that we expect the lowest eigenvalues to be decaying at precisely this rate, whilst the factor π^2 is just a matter of convenience. We can similarly define the densities $\Lambda^*_{q,\inf}(s)$ and $\Lambda^*_{q,\sup}(s)$ of low eigenvalues of the Laplacian acting on co-exact q-forms. We are interested in understanding the asymptotic behaviour of these densities as $s \to \infty$. In §3.4.2 we prove the following:

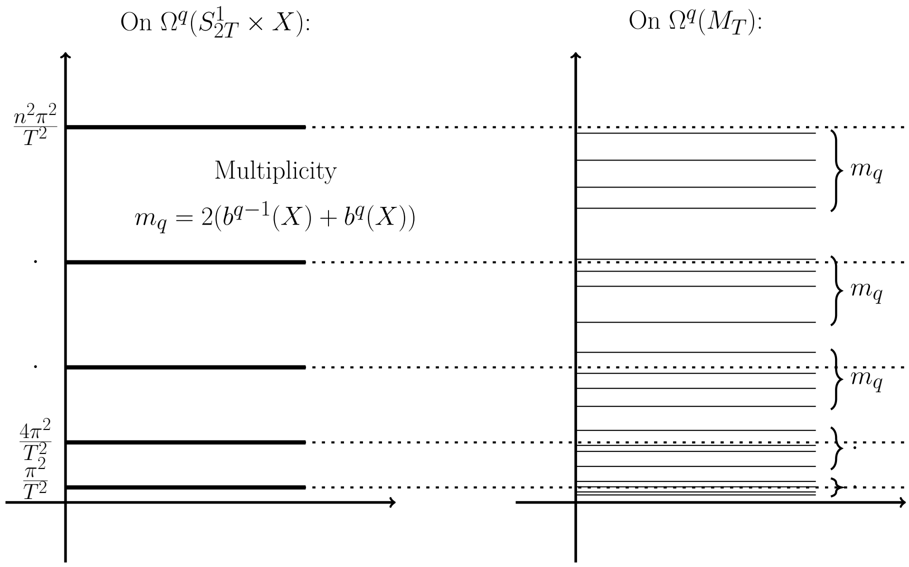

Figure 3.3: Asymptotic behaviour of the lower spectrum.

Theorem 3.8. *If $b^{q-1}(X) + b^q(X) = 0$, then the lowest eigenvalue of Δ_T acting on $\Omega^q(M_T)$ is uniformly bounded below as $T \to 0$, and in particular $\Lambda_{q,\sup}(s) = \Lambda_{q,\inf}(s) = 0$.*

If on the other hand $b^{q-1}(X) + b^q(X) > 0$, then the densities of low eigenvalues satisfy:

$$\Lambda_{q,\sup}(s) = \Lambda_{q,\inf}(s) + O(1) = 2(b^{q-1}(X) + b^q(X))\sqrt{s} + O(1)$$

as $s \to \infty$. Moreover, we also have

$$\Lambda^*_{q,\sup}(s) = \Lambda^*_{q,\inf}(s) + O(1) = 2b^q(X)\sqrt{s} + O(1).$$

Remark 3.9. As discussed above, the work of [81] implies this theorem when $b^{q-1}(X)+b^q(X) = 0$, so we will focus on the case where $b^{q-1}(X)+b^q(X) > 0$. Moreover, by Hodge duality it is enough to prove the estimates on $\Lambda^*_{q,\sup}(s), \Lambda^*_{q,\inf}(s)$ when $b^q(X) > 0$.

This theorem essentially says that the lowest eigenvalues of Δ_T are asymptotically distributed as the low eigenvalues of the Laplacian acting on the product $S^1_{2T} \times X$, where the first factor is a circle of length $2T$ (see Figure 3.3 above). We can only express our result as an asymptotic statement on the distribution of the eigenvalues and cannot obtain a more precise asymptotic development (unlike in the simpler setting of [46]). This is because the interaction between the building blocks of the construction creates a shift in the spectrum of Δ_T compared with the spectrum of the Laplacian on the product in a way which we cannot explicitly describe, because we do not have exactly cylindrical ends.

Remark 3.10. In the case of the twisted connected sum of G_2-manifolds, the cross-section is $T^2 \times K3$ (or a quotient thereof). Using the C^k-estimates of the previous chapter, Theorem 3.8 can be applied. Hence we find an infinite number of eigenvalues decaying at the same rate. This is consistent with the predictions of the swampland distance conjecture, where they correspond to the infinite towers of asymptotically light states. I have been told by physicists that this asymptotic density of eigenvalues is related to the concept of dualities, that is, in this limit the correct low-energy physics is described by a different theory compactified on the cross-section X instead of M_T.

Let us finish this section with an overview of our strategy. We prove Theorem 3.6 by an explicit construction method, similar to the constructions by Kapouleas of minimal surfaces in Euclidean space [69, 70], by which we were inspired. The idea is to use cutoffs to separate the analysis in three different domains: the neck region, which is close to a finite cylinder $[-T, T] \times X$, and two regions isometric to the domains $\{\rho_i \leq T + 1\} \subset Z_i$. One challenge is that when the indicial operator P_0 acting on the cylinder has real roots, it is not invertible nor even Fredholm in the Sobolev range that we would like to consider. However, this failure is due to the asymptotic behaviour of solutions, and we only need to work on a compact region of the cylinder. To deal with this issue, let us denote by $W^{l,p}_c$ the subspace of $W^{l,p}$ constituted by sections with compact essential support. The main analytical ingredient of our construction is the following theorem, proved in Section 3.2:

Theorem 3.11. *Let $P_0 : C^\infty(\underline{E}_0) \to C^\infty(\underline{F}_0)$ be an elliptic translation-invariant operator of order k acting on the cylinder $\mathbb{R} \times X$. Let d be the maximal order of a real root of P_0.*

For any $p > 1$, there exists a map $Q_0 : L^p_c(\underline{F}_0) \to W^{k,p}_{loc}(\underline{E}_0)$ such that for any $f \in L^p(\underline{F}_0)$ with compact essential support, $P_0 Q_0 f = f$. Moreover, there exists a constant $C > 0$ such that for any $T \geq 1$ and any $f \in L^p_c(\underline{F}_0)$ with essential support contained in $(-T, T) \times X$:

$$\|Q_0 f\|_{W^{k,p}((-T,T)\times X)} \leq C T^d \|f\|_{L^p}.$$

Remark 3.12. The existence of the map Q_0 can be deduced from standard results as [89] or [93] for instance. However, the explicit expression that we give for Q_0 will be important for our purpose, since a precise understanding of the asymptotic behaviour of $Q_0 f$ will play a key role in the construction of Section 3.3.

Using this theorem, we can try to build approximate solutions of the equation $P_T u = f$ by first taking a cutoff f_0 of f in the neck region, and considering an equation of the type $P_0 u_0 = f_0$. We can consider f_0 as a section of $\underline{F}_0$ supported in $[-T, T] \times X$. This equation can be solved using the above theorem. Taking a cutoff u_0 of the solution, the equation $P_T u = f$ can be replaced with an equation of the form $P_T(u - u_0) = f'$, where f' is appropriately small in the neck region. Thus f' can be written as a sum $f_1 + f_2$, where each f_i is a section of F_i defined in the domain $\{\rho_i \leq T + 1\} \subset Z_i$ and satisfies good decay properties. This allows us to use weighted analysis to study the equations $P_i u_i = f_i$ in a range where the operators P_i satisfy the Fredholm property. Similar ideas can be found for instance in [99].

Unfortunately, there are obstructions to solving $P_i u_i = f_i$ in weighted spaces, and the main difficulty is to understand how these obstructions interact. Using a pairing defined in §3.2.2, we can keep track of the obstructions and express their vanishing (up to an exponentially decaying error term) as a finite-dimensional linear system, which we call the *characteristic system* of our gluing problem. The unknown of this system is an element $v \in \mathscr{E}$ which represents our degrees of freedom in solving the equation $P_0 u = f_0$. The coefficients of the system are linearly determined by f. In §3.3.3, we prove that in full generality the characteristic system admits a solution if and only if f is orthogonal to the substitute cokernel. With the extra assumption that P has only one root, this allows us to build an approximate solution of the equation $P_T u = f$, and when T is large enough we can prove Theorem 3.6 using an iterative process. However, our method could apply more generally, as long as one can ensure that the characteristic system admits a solution with reasonable bounds.

3.2 Translation-invariant differential operators

Throughout this section, we fix a compact oriented manifold X and let $Y = \mathbb{R} \times X$. If $E \to X$ is a vector bundle, we denote by $\underline{E} \to Y$ the pull-back of E by the projection $Y \to X$ on the second factor. Given any connection ∇ on E, we can endow $\underline{E}$ with the pull-back connection $\underline{\nabla}$. Parallel transport along the vector field $\frac{\partial}{\partial t}$ naturally defines a translation operator on $\underline{E}$. A section of $\underline{E}$ is translation-invariant if and only if it is the pull-back of a section of E. We also equip Y with a cylindrical metric $g_Y = dt^2 + g_X$ and endow $\underline{E}$ with a translation-invariant metric. Sobolev and C^k norms on Y are defined with respect to this data.

In §3.2.1, we introduce some background about analysis on cylinders. In §3.2.2 we study the action of a general elliptic translation-invariant operator P on polyhomogeneous sections and define a pairing between the spaces of polyhomogeneous solutions of $Pu = 0$ and of $P^*v = 0$. Last, we prove Theorem 3.11 in §3.2.3 by constructing explicit solutions of the equation $Pu = f$. Using the above pairing, we can precisely analyse the asymptotic behaviour of these solutions, which will play a key role in Section 3.3.

3.2.1 Analysis on cylinders by separation of variables. On the cylinder $Y = \mathbb{R} \times X$, we have natural isomorphisms identifying $L^p(\underline{E})$ with $L^p(\mathbb{R}, L^p(E))$ for any $p \geq 1$, which follow from Fubini's theorem. Therefore we can think of sections of translation-invariant vector bundles over Y as maps from $\mathbb{R}$ to an appropriate Banach space of sections over X. Moreover, for $p \geq 2$ there is a continuous embedding $L^p(E) \hookrightarrow L^2(E)$, which therefore induces a continuous embedding $L^p(\underline{E}) \to L^p(\mathbb{R}, L^2(E))$. Hence there exists a constant $C > 0$ such that for any $u \in L^p(\underline{E})$:

$$\|u\|_{L^p(\mathbb{R}, L^2(E))} = \left(\int_{\mathbb{R}} \|u_t\|_{L^2}^p dt \right)^{\frac{1}{p}} \leq C \|u\|_{L^p}$$

where $u_t = u|_{\{t\} \times X}$. The main tools that we will need for the analysis of PDEs on a cylinder $\mathbb{R} \times X$ are the Fourier transform and the convolution along the variable $t \in \mathbb{R}$. Below we recall some definitions.

Let H be a complex Hilbert space, and consider functions $f : \mathbb{R} \to H$. Later, we will take H to be the space $L^2(E)$. We denote by $\mathscr{S}(\mathbb{R}, H)$ the space of H-valued Schwartz functions, that is, the space of smooth functions taking values in H that have all derivatives rapidly decaying at infinity. If $f \in \mathscr{S}(\mathbb{R}, H)$, we can define its Fourier transform $\hat{f} : \mathbb{R} \to H$ by:

$$\hat{f}(\lambda) = \int_{\mathbb{R}} e^{-i\lambda t} f(t) dt \tag{3.12}$$

for any $\lambda \in \mathbb{R}$. As in the case of scalar-valued functions, $\hat{f}$ also belongs to the space $\mathscr{S}(\mathbb{R}, H)$. This defines an invertible map $\mathscr{S}(\mathbb{R}, H) \to \mathscr{S}(\mathbb{R}, H)$, and the inverse Fourier transform takes the usual expression. As H is a Hilbert space, the Plancherel theorem holds and the Fourier transform extends to a bounded linear map $L^2(\mathbb{R}, H) \to L^2(\mathbb{R}, H)$, which is, up to constant, an isometry [1, Th. 2.47].

Just as for scalar-valued maps, the Plancherel theorem also implies that the Fourier transform can be extended to the dual space $\mathscr{S}'(\mathbb{R}, H)$ of $\mathscr{S}(\mathbb{R}, H)$. In particular, for any $p \geq 1$ we can define the Fourier transform of an L^p-function through the embedding $L^p(\mathbb{R}, H) \to \mathscr{S}'(\mathbb{R}, H)$. On $\mathscr{S}'(\mathbb{R}, H)$ we can also define weak derivatives by duality, and moreover the relation $\widehat{f'}(\lambda) = i\lambda \hat{f}(\lambda)$ can also be proved by evaluating against test functions.

Let us consider two complex Hilbert spaces H_1, H_2 and let $B(H_1, H_2)$ be the Banach space of bounded linear operators from H_1 to H_2. Let $R : \mathbb{R} \to B(H_1, H_2)$ be a smooth map, such that R and all its derivatives have at most polynomial growth. Then R induces a linear operator $A_R : \mathscr{S}(\mathbb{R}, H_1) \to \mathscr{S}(\mathbb{R}, H_2)$ acting on a Schwartz function f by:

$$A_R[f](t) = \frac{1}{2\pi} \int e^{i\lambda t} R(\lambda) \hat{f}(\lambda) d\lambda, \quad \forall t \in \mathbb{R}. \tag{3.13}$$

The Plancherel theorem implies that if R is bounded, then A_R extends to a bounded linear operator $L^2(\mathbb{R}, H_1) \to L^2(\mathbb{R}, H_2)$. The Hilbert-space-valued *Mikhlin multiplier theorem* gives a sufficient condition for A_R to extend as a bounded linear map for other L^p-spaces (see [1, Th. 5.8] or [12, Th. 6.1.6]):

Theorem 3.13. *Assume that there exists a constant C such that $\|R(\lambda)\| + \|\lambda R'(\lambda)\| \leq C$ for all $\lambda \in \mathbb{R}$, where $\|\cdot\|$ denotes the norm of $B(H_1, H_2)$. Then A_R extends as a bounded linear map $L^p(\mathbb{R}, H_1) \to L^p(\mathbb{R}, H_2)$ for all $1 < p < \infty$.*

In §3.2.3, we will also need to consider functions $F : \mathbb{R} \to H$ defined by integrals of the form

$$F(t) = \int_{-\infty}^{t} \frac{(t-\tau)^{l-1}}{(l-1)!} f(\tau) d\tau$$

where $l \geq 1$ and $f \in L^1_c(\mathbb{R}, H)$ is a compactly supported integrable function. The function F is continuous, and since the support of f is compact F has at most polynomial growth at infinity, and therefore it defines an element of $\mathscr{S}'(\mathbb{R}, H)$. If we denote the Heaviside step function by H and define $H_l(t) = \frac{t^{l-1}}{(l-1)!} H(t)$, then F can be written more compactly as the convolution $H_l * f$. Note that the n-th order weak derivative of H_l is H_{l-n} if $n < l$ and the Dirac mass δ if $l = n$. The weak derivatives of $F \in \mathscr{S}'(\mathbb{R}, H)$ are naturally given by:

$$F^{(n)} = \begin{cases} H_{l-n} * f & \text{if } l < n, \\ f & \text{if } l = n, \\ f^{(n-l)} & \text{if } l > n. \end{cases} \tag{3.14}$$

This can be proved by integrating against a test function $g \in \mathscr{S}(\mathbb{R}, H)$, as in the case of scalar-valued functions.

Let us now turn to the study of PDEs on cylinders. Let $\underline{E}$ and $\underline{F}$ be translation-invariant vector bundles over $Y = \mathbb{R} \times X$, equipped with translation-invariant metrics and connections. We will denote by $y = (t, x)$ the points in Y. Moreover let ∂_t be the covariant derivative along $\frac{\partial}{\partial_t}$, and define $D_t = -i\partial_t$.

A differential operator $P : C^\infty(\underline{E}) \to C^\infty(\underline{F})$ of order k is translation-invariant if it takes the form:

$$P(x, \partial_x, D_t) = \sum_{l=0}^{k} A_{k-l}(x, \partial_x) D_t^l$$

where $A_{k-l}(x, \partial_x)$ are differential operators $C^\infty(E) \to C^\infty(F)$. If P has order k, it is a standard fact that it induces continuous maps $P : W^{k+l,p}(\underline{E}) \to W^{l,p}(\underline{F})$ on Sobolev spaces of sections.

From now on we assume that P is elliptic, and for any $T > 0$ we denote by $\underline{E}_T$ the restriction of $\underline{E}$ to the finite cylinder $(-T, T) \times X$. If $u \in L^p(\underline{E}_2)$ and $Pu \in W^{l,p}(\underline{E}_2)$, then by elliptic regularity the restriction of u to $(-1, 1) \times X$ is in $W^{k+l,p}(\underline{E}_1)$, and moreover we have interior estimates:

$$\|u\|_{W^{k+l,p}(\underline{E}_1)} \leq C \left(\|Pu\|_{W^{l,p}(\underline{E}_2)} + \|u\|_{L^p(\underline{E}_2)} \right).$$

Combined with the translation-invariance of P, we get interior estimates for sections of $\underline{E}_T$ that are independent of T:

Proposition 3.14. *Let $P : C^\infty(E) \to C^\infty(F)$ be a translation-invariant elliptic operator of order k, and let $p > 1$ and $l \in \mathbb{N}$. Then there exists $C > 0$ such that for any $T \geq 1$ the following holds. If $f \in W^{l,p}(\underline{F}_{T+1})$ and $u \in L^p(\underline{E}_{T+1})$ is a solution of $Pu = f$, then $u|_{(-T,T)\times X} \in W^{k+l,p}(\underline{E}_T)$ with the bound:*

$$\|u\|_{W^{k+l,p}(\underline{E}_T)} \leq C \left(\|f\|_{W^{l,p}(\underline{E}_{T+1})} + \|u\|_{L^p(\underline{E}_{T+1})} \right).$$

When $p \geq 2$, we can make a stronger statement. If $u \in L^2(\underline{E}_2)$ and $Pu \in W^{l,p}(\underline{E}_2)$, then the restriction of u to $(-1, 1) \times X$ is in $W^{k+l,p}(\underline{E}_1)$ and moreover we have an estimate:

$$\|u\|_{W^{k+l,p}(\underline{E}_1)} \leq C \left(\|Pu\|_{W^{l,p}(\underline{E}_2)} + \|u\|_{L^2(\underline{E}_2)} \right).$$

Since $L^p((-2, 2), L^2(E))$ continuously embeds into $L^2((-2, 2), L^2(E)) \simeq L^2(\underline{E}_2)$, there exists a constant $C' > 0$ such that if $u \in L^p((-2, 2), L^2(E))$ then we have the following interior estimate:

$$\|u\|_{W^{k+l,p}(\underline{E}_1)} \leq C \left(\|Pu\|_{W^{l,p}(\underline{E}_2)} + \|u\|_{L^p((-2,2),L^2(E))} \right).$$

Using translation-invariance this implies:

Proposition 3.15. *Let $P : C^\infty(E) \to C^\infty(F)$ be a translation-invariant elliptic operator of order k, and let $p \geq 2$ and $l \in \mathbb{N}$. Then there exists $C > 0$ such that for any $T \geq 1$ the following holds. If $f \in W^{l,p}(\underline{F}_{T+1})$ and $u \in L^p((-T-1, T+1), L^2(E))$ is a solution of $Pu = f$, then $u|_{(-T,T)\times X} \in W^{k+l,p}(\underline{E}_T)$ with the bound::*

$$\|u\|_{W^{k+l,p}(\underline{E}_T)} \leq C \left(\|f\|_{W^{l,p}(\underline{E}_{T+1})} + \|u\|_{L^p((-T-1,T+1),L^2(E))} \right).$$

In the remainder of this part, we will be concerned with equations of the type

$$P(x, \partial_x, D_t)u(t, x) = f(t, x) \tag{3.15}$$

where P is a translation-invariant elliptic operator. It is usually studied by taking its Fourier transform in the variable t, which takes the form

$$P(x, \partial_x, \lambda)\hat{u}(x, \lambda) = \hat{f}(x, \lambda). \tag{3.16}$$

For any fixed $\lambda \in \mathbb{C}$, the operator $P(x, \partial_x, \lambda) : C^\infty(E) \to C^\infty(F)$ is an elliptic operator of order k, and hence defines Fredholm maps $W^{k+l,p}(E) \to W^{l,p}(F)$ for $p > 1$ and $l \geq 0$. By the results of [3] these maps are analytic in the variable λ, and there exists a discrete set $\mathscr{C}_P \subset \mathbb{C}$ such that the homogeneous equation

$$P(x, \partial_x, \lambda)\hat{u}(x, \lambda) = 0 \tag{3.17}$$

has a non-trivial solution if and only if $\lambda \in \mathscr{C}_P$. Moreover, the intersection of $\mathscr{C}_P$ with any strip $\{\delta_1 < \operatorname{im} \lambda < \delta_2\}$ of $\mathbb{C}$ is finite. The elements of $\mathscr{C}_P$ are called the *roots* of P. The discrete set $\mathscr{D}_P = \{\operatorname{im} \lambda, \lambda \in \mathscr{C}_P\}$ is called the set of *indicial roots*.

Example 3.16. Consider the translation-invariant bundle $\Lambda_{\mathbb{C}}T^*Y$ of complex-valued differential forms. It splits as a direct sum:

$$\Lambda_{\mathbb{C}}T^*Y = \underline{\Lambda_{\mathbb{C}}T^*X} \oplus dt \wedge \underline{\Lambda_{\mathbb{C}}T^*X}$$

where $\underline{\Lambda_{\mathbb{C}}T^*X}$ is the pull-back of the bundle of differential forms on X. The operators d_Y and d_Y^* take the form:

$$\begin{cases} d_Y(\alpha + dt \wedge \beta) = d_X\alpha + dt \wedge (\partial_t\alpha - d_X\beta) \\ d_Y^*(\alpha + dt \wedge \beta) = d_X^*\alpha - \partial_t\beta - dt \wedge d_X^*\beta \end{cases}$$

Thus if we define $J \in \operatorname{End}(\Lambda_{\mathbb{C}}T^*Y)$ by $J\eta = dt \wedge \eta - \iota_{\frac{\partial}{\partial t}}\eta$, where ι denotes the interior product, we can write the Fourier transform of the operator $d_Y + d_Y^*$ as

$$(d_Y + d_Y^*)(\lambda)\eta = (d_X + d_X^*)\alpha - dt \wedge (d_X + d_X^*)\beta + i\lambda J\eta.$$

with $\eta = \alpha + dt \wedge \beta$. The Laplacian $\Delta_Y = d_Y d_Y^* + d_Y^* d_Y$ can be written as $\Delta_Y = -\partial_t^2 + \Delta_X$, so that its Fourier transform is $\Delta_X + \lambda^2$. For both operators, the roots are exactly the values $\pm i\sqrt{\lambda_n}$, where $\lambda_n \geq 0$ are the eigenvalues of the Laplacian Δ_X. In particular the only real root is $\lambda_0 = 0$, and the corresponding translation-invariant solutions are of the form $\alpha + dt \wedge \beta$, where α and β are harmonic forms on X.

For $\lambda \in \mathbb{C}$, we will write $P(\lambda)$ as a short-hand for $P(x, \partial_x, \lambda)$. It can be seen as a Fredholm map $W^{k+l,p}(E) \to W^{l,p}(F)$, analytic in the variable λ. This implies that $P(\lambda)$ is invertible for $\lambda \notin \mathscr{C}_P$ [3, 4]. Its inverse $R(\lambda)$ is called the *resolvent* of $P(\lambda)$; for any $m \leq k+l$ it can be considered as a bounded operator form $W^{l,p}(F)$ to $W^{m,p}(E)$ (which is compact when $m < k+l$). We will denote by $\|R(\lambda)\|_{l,m}$ the operator norm of the resolvent seen as a map $W^{l,p}(F) \to W^{m,p}(E)$. By the results of [3] the resolvent is meromorphic in $\lambda \in \mathbb{C}$, with poles exactly at the roots of P. That is, around any $\lambda_0 \in \mathscr{C}_P$ we can write:

$$R(\lambda) = \frac{R_{-d}(\lambda_0)}{(\lambda - \lambda_0)^d} + \cdots + \frac{R_{-1}(\lambda_0)}{\lambda - \lambda_0} + \sum_{n=0}^{\infty} R_n(\lambda_0)(\lambda - \lambda_0)^n$$

where $R_l(\lambda_0)$ are bounded operators $W^{l,p}(F) \to W^{m,p}(E)$ and the series has positive radius of convergence. The largest positive integer d such that $R_{-d}(\lambda_0) \neq 0$ is called the order of λ_0. The notions of root, pole and order do not depend on the Sobolev spaces we choose to work with.

The following bounds on the resolvent $R(\lambda)$ and its derivative $R'(\lambda) = \frac{dR}{d\lambda}(\lambda)$ are crucial for our purpose, and follow from the more general [3, Theorem 5.4]:

Theorem 3.17. *Let $p > 1$, $l \in \mathbb{N}$ and P be a translation-invariant elliptic operator. Then the following holds:*

(i) The resolvent $R(\lambda)$ has no poles in a double sector $\{\arg(\pm\lambda) \leq \delta, \quad |\lambda| \geq N\}$ and in this domain there exists a constant $C > 0$ such that:

$$\sum_{j=0}^{k} \left\| \lambda^{k-j} R(\lambda) \right\|_{l,l+j} \leq C.$$

(ii) Furthermore, as $|\lambda| \to \infty$ along the real axis:

$$\sum_{j=0}^{k} \left\| \lambda^{k-j} R'(\lambda) \right\|_{l,l+j} = O\left(\frac{1}{\lambda}\right).$$

The last result that we want to mention here is the following well-known proposition (see [79] for an original reference), which can be seen as a particular case of Theorem 3.11. When P has no roots along the real axis the following holds.

Proposition 3.18. *Let $p > 1$, $l \in \mathbb{N}$, and assume that P has no real roots. Then the map $W^{k+l,p}(\underline{E}) \to W^{l,p}(\underline{F})$ induced by P admits a bounded inverse.*

A sketch proof of this proposition is as follows. If f is a smooth, compactly supported section of $\underline{F}$, then equation (3.16) admits a solution $\hat{u}(\lambda) = R(\lambda)\hat{f}(\lambda)$, where we can consider $\hat{f}$ as a Schwartz function valued in $L^2(F)$ and the resolvent

$R(\lambda)$ as a bounded map $L^2(F) \to L^2(E)$. Hence we have a solution $u = Q[f] \in \mathscr{S}(\mathbb{R}, L^2(E))$ of $Pu = f$ defined as:

$$Q[f](t) = \frac{1}{2\pi} \int e^{i\lambda t} R(\lambda) \hat{f}(\lambda) d\lambda, \quad \forall t \in \mathbb{R}. \tag{3.18}$$

It follows from Theorem 3.13 and the above bounds on the resolvent that Q extends to a bounded linear map $L^p(\mathbb{R}, L^2(F)) \to L^p(\mathbb{R}, L^2(E))$ for any $1 < p < \infty$. If $p \geq 2$, the fact that P admits a bounded inverse in the L^p-Sobolev range can therefore be deduced from Proposition 3.15 and the continuous embedding $L^p(\underline{E}) \hookrightarrow L^p(\mathbb{R}, L^2(E))$, and the case $1 < p < 2$ can be treated by duality.

When P has real roots the statement of Proposition 3.18 no longer holds and the map induced by P on Sobolev spaces is not even Fredholm. It still has finite-dimensional kernel but the cokernel has infinite dimension. In order to understand the mapping properties of P in more detail, we want to make sense of the inverse Fourier transform of the singular part of the resolvent.

3.2.2 Polyhomogeneous sections. In this part, we prove that the action of P on polyhomogeneous sections admits a right inverse and introduce a pairing which will play an important role in Section 3.3. A section of $\underline{E} \to Y$ is called *exponential* if it is of the form $u(x,t) = e^{i\lambda t} p(x,t)$, where $\lambda \in \mathbb{C}$ is called the rate of u and p is polynomial in the variable t. A *polyhomogeneous* section is a finite sum of exponential sections.

To understand the action of P on polyhomogeneous sections, we fix $\lambda_0 \in \mathbb{C}$ and define:

$$P_{\lambda_0}(x, \partial_x, D_t) = e^{-i\lambda_0 t} P(x, \partial_x, D_t) e^{i\lambda_0 t} \tag{3.19}$$

which is a translation-invariant operator on Y. More explicitly,

$$P_{\lambda_0}(D_t) = \sum_{n \geq 0} \frac{1}{n!} \frac{\partial^n P}{\partial \lambda^n}(\lambda_0) D_t^n.$$

We consider P_{λ_0} as an operator mapping the space $W^{k,p}(E)[t]$ into $L^p(F)[t]$, that is, we consider the action on sections of $\underline{E} \to Y$ that are polynomials in t and have $W^{k,p}$ coefficients. Our goal is to show that P_{λ_0} admits a right inverse Q_{λ_0}.

Consider the resolvent $R(\lambda)$ as an operator $L^p(F) \to W^{k,p}(E)$. If λ_0 is a root of P, it is a pole of R and we denote by $d(\lambda_0)$ its degree. By convention we set $d(\lambda_0) = 0$ if λ_0 is not a root of P. In general we may expand $R(\lambda)$ near λ_0 as:

$$R(\lambda) = \frac{R_{-d(\lambda_0)}(\lambda_0)}{(\lambda - \lambda_0)^{d(\lambda_0)}} + \cdots + \frac{R_{-1}(\lambda_0)}{\lambda - \lambda_0} + R_0(\lambda_0) + \sum_{m \geq 1} R_m(\lambda_0)(\lambda - \lambda_0)^m.$$

where for $m \geq -d(\lambda_0)$, $R_m(\lambda_0) : L^p(X,F) \to W^{k,p}(X,E)$ are bounded operators. The relations $R(\lambda)P(\lambda) = \mathrm{Id}_{W^{k,p}(E)}$ and $P(\lambda)R(\lambda) = \mathrm{Id}_{L^p(F)}$ that hold away form the roots of P imply:

$$\sum_{m+n=0} \frac{1}{n!} R_m(\lambda_0) \frac{\partial^n P}{\partial \lambda^n}(\lambda_0) = \mathrm{Id}_{W^{k,p}(E)}, \quad \sum_{m+n=0} \frac{1}{n!} \frac{\partial^n P}{\partial \lambda^n}(\lambda_0) R_m(\lambda_0) = \mathrm{Id}_{L^p(F)} \tag{3.20}$$

and for any non-zero $l \in \mathbb{Z}$:

$$\sum_{m+n=l} \frac{1}{n!} R_m(\lambda_0) \frac{\partial^n P}{\partial \lambda^n}(\lambda_0) = 0 = \sum_{m+n=l} \frac{1}{n!} \frac{\partial^n P}{\partial \lambda^n}(\lambda_0) R_m(\lambda_0). \tag{3.21}$$

Example 3.19. One can easily see from Example 3.16 that $\lambda_0 = 0$ is a root of order 1 of the operator $d_Y + d_Y^*$, and of order 2 for the operator Δ_Y. The singular parts of their resolvent can be computed with the above relations. For the operator $d_Y + d_Y^*$, relations (3.21) for $l = -1$ imply that $R_{-1}^{d+d^*}(0)$ vanishes on the orthogonal space to harmonic forms and maps into the space of harmonic forms. Relations (3.20) imply that:

$$iR_{-1}^{d+d^*}(0) J\eta = \eta = iJR_{-1}^{d+d^*}(0)\eta$$

for any translation-invariant harmonic form on Y. As $J^2 = -1$ we obtain:

$$R_{-1}^{d+d^*}(0) = iJ \circ p_h = ip_h \circ J$$

where p_h is the L^2-orthogonal projection onto the space of harmonic forms. As the Laplacian Δ_Y is the square of the operator $d_Y + d^*$ this implies that:

$$R_{-2}^{\Delta}(0) = (R_{-1}^{d+d^*}(0))^2 = (iJ)^2 p_h^2 = p_h.$$

On the other hand, as the Fourier transform of Δ_Y is an analytic function of the variable λ^2 it is easy to see that $R_{-1}^{\Delta}(0) = 0$.

With these notations in hand, let D_t^{-1} be the endomorphism of $L^p(F)[t]$ mapping $\frac{(it)^j}{j!}v$ to $\frac{(it)^{j+1}}{(j+1)!}v$ for any $v \in L^p(F)$. This is a right inverse of D_t. Let us define the operator $Q_{\lambda_0} : L^p(F)[t] \to W^{k,p}(E)[t]$ by:

$$Q_{\lambda_0}(D_t, D_t^{-1}) = \sum_{m \geq -d(\lambda_0)} R_m(\lambda_0) D_t^m.$$

It maps polynomials of order m to polynomials of order at most $m + d(\lambda_0)$. Moreover relations (3.20) and (3.21) imply the following:

Lemma 3.20. *The map $Q_{\lambda_0} : L^p(F)[t] \to W^{k,p}(E)[t]$ is a right inverse of P_{λ_0}.*

Let us now turn our attention to the kernel of P_{λ_0}. It is non-trivial if and only if λ_0 is a root of P, which amounts to saying that the homogeneous equation $P_{\lambda_0} u = 0$ admits a non-trivial translation-invariant solution. Moreover, the kernel of P_{λ_0} acting on polynomial sections in the variable t is always finite-dimensional, and the degree of its elements is bounded above by the order of the root λ_0 minus one [3]. In particular, if λ_0 has order 1 the only polynomial solutions of $P_{\lambda_0} u = 0$ are translation-invariant.

For any root λ_0 of P, let us denote by $\mathscr{E}_{\lambda_0}$ the (finite-dimensional) space of exponential solutions of $Pu = 0$ of rate λ_0, and $\mathscr{E}^*_{\overline{\lambda}_0}$ the space of exponential solutions of $P^* v = 0$ of rate $\overline{\lambda}_0$ (note that $P(\lambda)^* = P^*(\overline{\lambda})$ for any $\lambda \in \mathbb{C}$). As we are mainly interested in the real roots of P, we denote by $\lambda_1, ..., \lambda_m$ the real roots and define:

$$\mathscr{E} = \bigoplus_{j=1}^{m} \mathscr{E}_{\lambda_j}, \quad \mathscr{E}^* = \bigoplus_{j=1}^{m} \mathscr{E}^*_{\lambda_j}.$$

We shall now define a pairing $\mathscr{E} \times \mathscr{E}^* \to \mathbb{C}$ and derive its basic properties. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\chi \equiv 0$ in a neighbourhood of $-\infty$ and $\chi \equiv 1$ in a neighbourhood of $+\infty$. We define a sesquilinear pairing $(\cdot, \cdot) : \mathscr{E} \times \mathscr{E}^* \to \mathbb{C}$ by the integral:

$$(u, v) = \int_{\mathbb{R}} \langle P(D_t) \left[\chi(t)u(t)\right], v(t)\rangle \, dt. \tag{3.22}$$

Here, we denote by $\langle \cdot, \cdot \rangle$ the L^2-product on the compact manifold X. This is well-defined as $P(D_t) \left[\chi(t)u(t))\right]$ is compactly supported for any $u \in \mathscr{E}$. Further, it does not depend on the choice of function χ. Indeed, if $\tilde{\chi}$ is another smooth function that satisfies the same assumptions, define $\chi_\tau = (1 - \tau)\chi + \tau\tilde{\chi}$ for $\tau \in [0, 1]$. As $\frac{\partial \chi_\tau}{\partial \tau}(t)u(t)$ is compactly supported, we can integrate by parts to obtain:

$$\frac{d}{d\tau} \int_{\mathbb{R}} \langle P(D_t) \left[\chi_\tau(t)u(t)\right], v(t)\rangle \, dt = \int_{\mathbb{R}} \left\langle P(D_t) \left[\frac{\partial \chi_\tau}{\partial \tau}(t)u(t)\right], v(t)\right\rangle dt = 0$$

as $P^*(D_T)v(t) = 0$. Therefore the pairing does not depend on the choice of χ.

An important consequence of this observation is that $\mathscr{E}_{\lambda_i}$ is orthogonal to $\mathscr{E}^*_{\lambda_j}$ for the pairing $(\cdot, \cdot)$ unless $i = j$. Indeed, let $u \in \mathscr{E}_{\lambda_i}$ and $v \in \mathscr{E}^*_{\lambda_j}$, and replace $\chi(t)$ by $\chi(t - \tau)$ in the definition of the pairing, for $\tau \in \mathbb{R}$. Then we can compute by a change of variables:

$$\int_{\mathbb{R}} \langle P(D_t) \left[\chi(t - \tau)u(t)\right], v(t)\rangle \, dt = e^{i(\lambda_i - \lambda_j)\tau} \left[(u, v) + \sum_{l \geq 1} a_l(u, v)\tau^l\right]$$

where the coefficients $a_l(u, v)$ are independent of τ, and only finitely many of them are non-zero. As this has to be equal to (u, v) for all $\tau \in \mathbb{R}$, this implies $a_l = 0$ for $l \geq 1$ and $(u, v) = 0$ when $i \neq j$.

The key property of the pairing $(\cdot,\cdot)$ is the following:

Lemma 3.21. *The pairing $(\cdot,\cdot)$ is non-degenerate.*

Proof. By the above remarks it suffices to show that the restriction of $(\cdot,\cdot)$ to $\mathscr{E}_{\lambda_j} \times \mathscr{E}^*_{\lambda_j}$ is non-degenerate. Consider first $v \in \ker P^*(\lambda_j)$, so that $\tilde{v}(t,x) = e^{i\lambda_j t}v(x)$ is an element of $\mathscr{E}^*_{\lambda_j}$. Considering v as an element of $L^2(F)[t]$, we define $u(t,x) = Q_{\lambda_j}v$. This is a polynomial of order at most $d(\lambda_j)$ in the variable t, and it satisfies:

$$P_{\lambda_j}(D_t)u(t) = v.$$

Differentiating this expression in the variable t, it follows that:

$$P_{\lambda_j}(D_t)[D_t u(t)] = 0$$

so that $\tilde{u}(t,x) = e^{i\lambda_j t}D_t u(t,x)$ is in $\mathscr{E}_{\lambda_j}$. Let us now pick a function χ as above and compute:

$$\begin{aligned}\int_{\mathbb{R}} \left\langle P(D_t)\left[\chi(t)\tilde{u}(t)\right], e^{i\lambda_j t}v\right\rangle dt &= \int_{\mathbb{R}} \left\langle P_{\lambda_j}(D_t)\left[\chi(t)D_t u(t)\right], v\right\rangle dt \\ &= \frac{1}{i}\int \frac{d}{dt}\left\langle P_{\lambda_j}(D_t)\left[\chi(t)u(t)\right], v\right\rangle dt - \frac{1}{i}\int \left\langle P_{\lambda_j}(D_t)\left[\chi'(t)u(t)\right], v\right\rangle dt \\ &= \frac{1}{i}\langle v, v\rangle\end{aligned}$$

which holds because $P(D_t)[\chi(t)u(t)] \equiv v$ as t goes to $+\infty$ and $P(D_t)[\chi(t)u(t)] = 0$ as t goes to $-\infty$. Thus we have $(\tilde{u},\tilde{v}) = -i\|v\|^2_{L^2}$ which is non-zero when $v \neq 0$.

In general, let $v(t,x)$ be an element of $\mathscr{E}^*_{\lambda_j}$ of degree m. Then $e^{i\lambda_j t}D_t^m e^{-i\lambda_j t}v(t,x)$ is a non-zero element of $\mathscr{E}^*_{\lambda_j}$ of degree zero. By the above argument there exists $u(t,x)$ in $\mathscr{E}_{\lambda_j}$ such that $(u, e^{i\lambda_j t}D_t^m e^{-i\lambda_j t}v) \neq 0$. Moreover one can easily check that:

$$(u, e^{i\lambda_j t}D_t^m e^{-i\lambda_j t}v) = (e^{i\lambda_j t}D_t^m e^{-i\lambda_j t}u, v)$$

and $e^{i\lambda_j t}D_t^m e^{-i\lambda_j t}u \in \mathscr{E}_{\lambda_j}$. Hence the pairing $(\cdot,\cdot)$ is non-degenerate. $\square$

Example 3.22. The space of translation-invariant solutions of the operator $d_Y + d_Y^*$ acting on $\Lambda_{\mathbb{C}}T^*Y$ is:

$$\mathscr{E}_{d+d^*} = \mathscr{E}^*_{d+d^*} = \{\alpha + dt\wedge\beta,\ \alpha,\beta \in C^\infty(\Lambda_{\mathbb{C}}T^*X),\ \Delta_X\alpha = \Delta_X\beta = 0\}$$

If $\alpha + dt\wedge\beta,\ \alpha' + dt\wedge\beta' \in \mathscr{E}$ we can compute their pairing:

$$\begin{aligned}(\alpha + dt\wedge\beta, \alpha' + dt\wedge\beta') &= \int \langle (d_Y + d_Y^*)(\chi(\tau)\alpha + dt\wedge\beta), \alpha' + dt\wedge\beta'\rangle d\tau \\ &= \int \chi'(\tau)\langle dt\wedge\alpha - \beta, \alpha' + dt\wedge\beta'\rangle d\tau \\ &= \langle\alpha,\beta'\rangle - \langle\beta,\alpha'\rangle\end{aligned}$$

which is clearly non-degenerate.

For the Laplacian Δ_Y acting on q-forms, the spaces $\mathscr{E}_q$ and $\mathscr{E}_q^*$ are both isomorphic to the space q-forms that can be written as $\eta_0 + t\eta_1$, where $\eta_i = \alpha_i + dt \wedge \beta_i$ with $\alpha_i \in \Omega^q_{\mathbb{C}}(X)$ and $\beta_i \in \Omega^{q-1}_{\mathbb{C}}(X)$ harmonic. In the same way one can easily derive:

$$(\eta_0 + t\eta_1, \eta_0' + t\eta_1') = \langle \alpha_0, \alpha_1' \rangle + \langle \beta_0, \beta_1' \rangle - \langle \alpha_1, \alpha_0' \rangle - \langle \beta_1, \beta_0' \rangle.$$

3.2.3 Existence of solutions. In this part we prove Theorem 3.11, beginning by the case $p \geq 2$. Let us consider a translation-invariant elliptic differential operator $P : C^\infty(\underline{E}) \to C^\infty(\underline{F})$ of order k with real roots $\lambda_1, \ldots, \lambda_m$. For $1 \leq j \leq m$, let $d(\lambda_j)$ be the order of the root λ_j. Considering the resolvent as a family of (compact) operators from $L^2(F)$ to $L^2(E)$, we have a decomposition of the form:

$$R(\lambda) = R_r(\lambda) + \sum_{j=1}^{m} \sum_{l=1}^{d(\lambda_j)} \frac{R_{-l}(\lambda_j)}{(\lambda - \lambda_j)^l} \tag{3.23}$$

where the regular part of the resolvent $R_r(\lambda)$ is an analytic function from a neighbourhood of the real line in $\mathbb{C}$ to $B(L^2(F), L^2(E))$. We will denote the second term of the right-hand side of equation (3.23) by $R_s(\lambda)$; this is the singular part of the resolvent.

Since $p \geq 2$, a section $f \in L^p_c(\underline{F})$ can be considered as an element of $L^p(\mathbb{R}, L^2(F))$, which has compact essential support. We want to find a solution of equation (3.15) through the study of the Fourier transformed equation (3.16). As the resolvent has poles we need to make sense of the expression $\hat{u}(\lambda) = R(\lambda)\hat{f}(\lambda)$, or rather of its inverse Fourier transform.

Differentiating the identity $P(\lambda)R(\lambda) = \mathrm{Id}_{L^2(F)}$ and using the bounds of Theorem 3.17, we see that the resolvent and all its derivatives have at most polynomial growth at infinity. Since this also true of the singular part of the resolvent, which is bounded at infinity as well as all of its derivatives, then the same holds for the regular part of the resolvent. On the other hand, from Theorem 3.17 we have a bound:

$$\|R(\lambda)\| + \|\lambda R'(\lambda)\| = O\left(\frac{1}{\lambda}\right)$$

as $|\lambda| \to \infty$. Further this bound clearly also holds for the singular part of the resolvent. Therefore there exists a constant $C > 0$ such that for all $\lambda \in \mathbb{R}$ we have:

$$\|R_r(\lambda)\| + \|\lambda R_r'(\lambda)\| \leq C$$

Thus Theorem 3.13 implies that $R_r(\lambda)$ induces a bounded map $Q_r : L^p(\mathbb{R}, L^2(F)) \to L^p(\mathbb{R}, L^2(E))$ defined as:

$$Q_r[v](t) = \frac{1}{2\pi} \int e^{i\lambda t} R_r(\lambda)\hat{v}(\lambda)d\lambda, \quad \forall v \in L^p(\mathbb{R}, L^2(E)). \tag{3.24}$$

Define $u_r = Q_r[f] \in L^p(\mathbb{R}, L^2(E))$. By definition we have $\hat{u}_r(\lambda) = R_r(\lambda)\hat{f}(\lambda)$ for $\lambda \in \mathbb{R}$. As f has compact support, its Fourier transform $\hat{f}(\lambda)$ can be continued as an analytic $L^2(F)$-valued function of the variable $\lambda \in \mathbb{C}$. Moreover, $R_r(\lambda)$ has no poles in a complex strip of the form $\{|\operatorname{im}\lambda| < \delta\}$ for some $\delta > 0$, and therefore $\hat{u}_r(\lambda)$ can be extended as an analytic $L^2(E)$-valued function for λ varying in a neighbourhood of the real line in $\mathbb{C}$.

We now deal with the singular part of the resolvent. Our main problem is that we cannot directly make sense of the inverse Fourier transform of $R_s(\lambda)\hat{f}(\lambda)$. Nevertheless, it is natural to define the following:

$$u_s(t) = Q_s[f](t) = \sum_{j=1}^{m} \sum_{l=1}^{d(\lambda_j)} i^l e^{i\lambda_j t} \int_{-\infty}^{t} \frac{(t-\tau)^{l-1}}{(l-1)!} e^{-i\lambda_j \tau} R_{-l}(\lambda_j) f(\tau) d\tau. \tag{3.25}$$

Note that the integrals are well-defined because f has compact essential support, and therefore u_s is a map $\mathbb{R} \to L^2(E)$. If we define $H_{l,\lambda}(t) = e^{i\lambda_j t} \frac{i^l t^{l-1}}{(l-1)!} H(t)$ where H is the Heaviside step function, then we can write u_s more compactly as a convolution:

$$u_s = \sum_{j=1}^{m} \sum_{l=1}^{d(\lambda_j)} H_{l,\lambda_j} * (R_{-l}(\lambda_j) f). \tag{3.26}$$

In general u_s is not in $L^p(\mathbb{R}, L^2(E))$, but its restriction to any finite interval $(-T, T)$ is L^p. We will shortly provide more precise estimates, but we first want to prove that $u = u_r + u_s$ satisfies $Pu = f$.

In order to do this, let us first compute $P(D_t)u_r(t)$, considering D_t as a weak derivative wherever appropriate. Taking the Fourier transform, we may compute $P(\lambda)u_r(\lambda)$ for $\lambda \in \mathbb{R}\backslash\{\lambda_1, ..., \lambda_m\}$ as follows. As $P(\lambda)R(\lambda) = \mathrm{Id}_{L^2(F)}$, we have:

$$P(\lambda)\hat{u}_r(\lambda) = \hat{f}(\lambda) - P(\lambda)R_s(\lambda)\hat{f}(\lambda).$$

For each root λ_j, we can expand $P(\lambda)$ in Taylor series around λ_j to compute:

$$P(\lambda) \sum_{l=1}^{d(\lambda_j)} \frac{R_{-l}(\lambda_j)}{(\lambda - \lambda_j)^l} = \sum_{n} \sum_{l=1}^{d(\lambda_j)} \frac{(\lambda - \lambda_j)^{n-l}}{n!} \frac{\partial^n P}{\partial \lambda^n}(\lambda_j) R_{-l}(\lambda_j).$$

By relations (3.21), the expansion of the sum in powers of $\lambda - \lambda_j$ is polynomial, that is the sum of the terms containing negative powers of $\lambda - \lambda_j$ vanishes. This yields:

$$P(\lambda)\hat{u}_r(\lambda) = \hat{f}(\lambda) - \sum_{j=1}^{m} \sum_{l\geq 1,\ n-l\geq 0} \frac{(\lambda - \lambda_j)^{n-l}}{n!} \frac{\partial^n P}{\partial \lambda^n}(\lambda_j) R_{-l}(\lambda_j)\hat{f}(\lambda)$$

which holds for $\lambda \neq \lambda_j$. As both sides of the equality are analytic in the variable λ, this is in fact true for all λ contained in a neighbourhood of the real line in $\mathbb{C}$. We can therefore take the inverse Fourier transform to obtain:

$$P(D_t)u_r(t) = f(t) - \sum_{j=1}^{m} \sum_{l\geq 1,\ n-l\geq 0} e^{i\lambda_j t} D_t^{n-l} \left[\frac{1}{n!} \frac{\partial^n P}{\partial \lambda^n}(\lambda_j) R_{-l}(\lambda_j) e^{-i\lambda_j t} f(t) \right]. \quad (3.27)$$

Next, we compute $P(D_t)u_s$. Let us remark that for $n < l$ we have the following identity:

$$e^{i\lambda t} D_t^n e^{-i\lambda t} H_{l,\lambda} = (D_t - \lambda)^n H_{l,\lambda} = H_{l-n,\lambda} \quad (3.28)$$

and for $n = l$, we have:

$$e^{i\lambda t} D_t^l e^{-i\lambda t} H_{l,\lambda} = \delta \quad (3.29)$$

where δ here is a Dirac mass centred at $t = 0$. Writing $P(D_t) = e^{i\lambda_j t} P_{\lambda_j}(D_t) e^{-i\lambda_j t}$ where $P_{\lambda_j}(D_t)$ is the operator defined in §3.2.2, we have

$$P \sum_{l=1}^{d(\lambda_j)} H_{l,\lambda_j} * (R_{-l}(\lambda_j) f) = \sum_{n\geq 0} \sum_{l=1}^{d(\lambda_j)} e^{i\lambda_j t} D_t^n e^{-i\lambda_j t} H_{l,\lambda_j} * \left[\frac{1}{n!} \frac{\partial^n P}{\partial \lambda^n}(\lambda_j) R_{-l}(\lambda_j) f \right].$$

If we split the sum into two parts, we see using (3.14) and (3.29) that the sum of the terms for which $n \geq l$ is equal to:

$$\sum_{n\geq l} \sum_{l=1}^{d(\lambda_j)} e^{i\lambda_j t} D_t^{n-l} \left[\frac{1}{n!} \frac{\partial^n P}{\partial \lambda^n}(\lambda_j) R_{-l}(\lambda_j) e^{-i\lambda_j t} f(t) \right]. \quad (3.30)$$

On the other hand, the sum of the terms for which $n < l$ can be computed using (3.14) and (3.28), and in fact this sum vanishes by (3.21):

$$\sum_{n<l} \sum_{l=1}^{d(\lambda_j)} H_{l-n,\lambda_j} * \left[\frac{1}{n!} \frac{\partial^n P}{\partial \lambda^n}(\lambda_j) R_l(\lambda_j) f \right] = 0. \quad (3.31)$$

Comparing with (3.27), this proves that $Pu = f$. Once we prove that u_s is in $L^p(I, L^2(E))$ for any finite interval $I \subset \mathbb{R}$, it will follow from Proposition 3.15 that u is $W^{k,p}_{loc}$. Thus we have a well-defined map $Q = Q_r + Q_s : L^p_c(\underline{F}) \to W^{k,p}_{loc}(\underline{E})$ which is a right inverse for P.

It remains to prove the estimates of Theorem 3.11. Let $T \geq 1$ and $f \in L^p_c(\underline{F})$ with essential support contained in $(-T, T) \times X$. For $-T-1 \leq t \leq T+1$, we can write:

$$\begin{aligned} u_s(t) &= \sum_{j=1}^{m} \sum_{l=1}^{d(\lambda_j)} \int_0^\infty \frac{i^l \tau^{l-1}}{(l-1)!} e^{i\lambda_j \tau} R_{-l}(\lambda_j) f(t-\tau) d\tau \\ &= \sum_{j=1}^{m} \sum_{l=1}^{d(\lambda_j)} \int_0^{2T+1} \frac{i^l \tau^{l-1}}{(l-1)!} e^{i\lambda_j \tau} R_{-l}(\lambda_j) f(t-\tau) d\tau \end{aligned}$$

In particular the L^2-norm of $u_s(t)$ is bounded by:

$$
\begin{aligned}
\|u_s(t)\|_{L^2(E)} &\leq \sum_{j=1}^{m} \sum_{l=1}^{d(\lambda_j)} \int_0^{2T+1} \frac{\tau^{l-1}}{(l-1)!} \|R_{-l}(\lambda_j) f(t-\tau)\|_{L^2(F)} d\tau \\
&\leq C \int_{-\infty}^{+\infty} \chi_{(0,2T+1)}(\tau) \left(\sum_{j=1}^{m} \sum_{l=1}^{d(\lambda_j)} \frac{\tau^{l-1}}{(l-1)!} \right) \|f(t-\tau)\|_{L^2(F)} d\tau
\end{aligned}
$$

where C is a constant depending only on the maps $R_{-l}(\lambda_j)$ and $\chi_{(0,2T+1)}$ is the characteristic function of the interval $(0, 2T+1)$. The function $\chi_{(0,2T+1)}(\tau) \left(\sum_{j=1}^{m} \sum_{l=1}^{d(\lambda_j)} \frac{\tau^{l-1}}{(l-1)!} \right)$ is L^1, and since d is the maximum of the $d(\lambda_j)$ it has L^1-norm bounded by CT^d for some constant $C > 0$ that does not depend on $T \geq 1$. Thus, as a function of the variable $t \in (-T-1, T+1)$, the function $\|u_s(t)\|_{L^2(E)}$ is bounded above by the convolution of the L^1-function $\chi_{(0,2T+1)}(\tau) \left(\sum_{j=1}^{m} \sum_{l=1}^{d(\lambda_j)} \frac{\tau^{l-1}}{(l-1)!} \right)$ and the L^p-function $\|f(\tau)\|_{L^2}$, and therefore Young's inequality yields:

$$
\|u_s\|_{L^p((-T-1,T+1),L^2(E))} \leq CT^d \|f\|_{L^p(\mathbb{R},L^2(E))}.
$$

Consequently, the restriction of $u = u_s + u_r$ to $(-T-1, T+1) \times X$ is in $L^p((-T-1, T+1), L^2(E))$ for any $T \geq 1$. By Proposition 3.15, $u|_{(-T,T)\times X} \in W^{k,p}(\underline{E}_T)$ and:

$$
\begin{aligned}
\|u\|_{W^{k,p}(\underline{E}_T)} &\leq C(\|f\|_{L^p(\underline{F}_{T+1})} + \|u\|_{L^p((-T-1,T+1),L^2(E))}) \\
&\leq C(\|f\|_{L^p(\underline{F})} + \|u_r\|_{L^p(\mathbb{R},L^2(E))} + \|u\|_{L^p((-T-1,T+1),L^2(E))}) \\
&\leq C(\|f\|_{L^p(\underline{F})} + \|f\|_{L^p(\mathbb{R},L^2(F))}) + CT^d \|f\|_{L^p(\mathbb{R},L^2(F))}) \\
&\leq CT^d \|f\|_{L^p(\underline{F})}
\end{aligned}
$$

which holds since the $L^p(\underline{E})$-norm of f controls its $L^p(\mathbb{R}, L^2(F))$-norm. Moreover, we can apply the same argument for any arbitrarily large $T' \geq T$ to deduce that $u \in W^{k,p}_{loc}(\underline{E})$. This finishes the proof of Theorem 3.11 in the case where $p \geq 2$. The case $1 < p < 2$ can be treated by duality, since the formal adjoint P^* is also a translation-invariant elliptic operator, and the maximal order of the real roots of P^* is also d.

In the remainder of this part, we shall comment on the asymptotic behaviour of the solutions constructed above. Let $f \in L^p_c(\underline{F})$ and let u, u_s and u_r be defined as above. Assume that the essential support of f is contained in $(-T, T) \times X$. Then, outside of this compact set we have:

$$
Pu_s = 0 = Pu_r.
$$

As $Pu = 0$ in this domain it suffices to show that $Pu_s = 0$. This is a consequence of (3.30) which yields:

$$
Pu_s = \sum_{n \geq l} \sum_{l=1}^{d(\lambda_j)} e^{i\lambda_j t} D_t^{n-l} \left[\frac{1}{n!} \frac{\partial^n P}{\partial \lambda^n} (\lambda_j) R_{-l}(\lambda_j) e^{-i\lambda_j t} f(t) \right].
$$

For $|t| > T$ the expression under brackets vanishes identically, which proves our claim.

An important consequence of this fact is that u_r has exponential decay as $|t| \to \infty$, in the sense that the $W^{k,p}$-norm of $e^{\delta\rho}u_r$ is finite for some $\delta > 0$, where ρ denotes an arbitrary smooth function on Y equal to $|t|$ when $|t| \geq 1$. This can be seen as a particular case of Lockhart–McOwen theory (see §3.3.1). On the other hand, it is easy to see from its definition that u_s vanishes identically in the domain $\{t < -T\}$, and more interestingly, u_s is equal to the restriction of a polyhomogeneous solution of $Pu = 0$ in the domain $\{t > T\}$. Indeed for $t > T$ (3.25) reads:

$$u_s(t) = \sum_{j=1}^{m} \sum_{l=1}^{d(\lambda_j)} i^l e^{i\lambda_j t} \int_{-T}^{T} \frac{(t-\tau)^{l-1}}{(l-1)!} e^{-i\lambda_j \tau} R_{-l}(\lambda_j) f(\tau) d\tau$$

which is manifestly polyhomogeneous. Let us denote the right-hand-side $u_f \in \mathscr{E}$. We may use the pairing $(\cdot,\cdot)$ introduced in §3.2.2 to characterise u_f by duality:

Lemma 3.23. *With the above notations, $(u_f, v) = \langle f, v\rangle$ for any $v \in \mathscr{E}^*$.*

Proof. Let χ be a smooth function such that $\chi \equiv 1$ in $(-\infty, 0]$ and $\chi \equiv 0$ in $[1,\infty)$, and let $\chi_\tau(t) = \chi(t-\tau)$. For any $\tau > T$ we have the equality $\langle \chi_\tau P u, v\rangle = \langle f, v\rangle$. On the other hand, let us prove that $\langle P\chi_\tau u, v\rangle = 0$ for any $\tau \in \mathbb{R}$. If $\tau, \tau' \in \mathbb{R}$, $\chi_\tau - \chi_{\tau'}$ has compact support and hence:

$$\langle P\chi_\tau u, v\rangle - \langle P\chi_{\tau'} u, v\rangle = \langle P(\chi_\tau - \chi_{\tau'})u, v\rangle = \langle (\chi_\tau - \chi_{\tau'})u, P^* v\rangle = 0$$

Therefore the value of $\langle P\chi_\tau u, v\rangle = 0$ does not depend on τ. Hence we may send τ to $-\infty$, and as $u(t)$ has exponential decay as $t \to -\infty$ we obtain $\langle P\chi_\tau u, v\rangle = 0$.

It follows that $\langle f, v\rangle = -\lim_{\tau\to\infty}\langle [P, \chi_\tau]u, v\rangle$. Given the exponential decay of $u_r(t)$ and its k first derivatives as $t \to \infty$, this yields:

$$\langle f, v\rangle = -\lim_{\tau\to\infty}\langle [P, \chi_\tau]u_s, v\rangle = \lim_{\tau\to\infty}\langle [P, 1-\chi_\tau]u_s, v\rangle = (u_f, v)$$

as claimed. □

Example 3.24. Consider the case of the Laplacian Δ_Y acting on q-forms. The singular part of the resolvent is $\lambda^{-2}p_h$, where p_h is the projection on the space of harmonic forms. Thus if η is a q-form on Y supported in $[-T, T] \times X$ and we denote by $\xi(\tau)$ the L^2-projection of η_τ onto the space of harmonic forms, and write $\xi(\tau) = \alpha(\tau) + dt \wedge \beta(\tau)$, we have by definition:

$$u_\eta(t) = i^2 \int_{-T}^{T} (t-\tau)\xi(\tau)d\tau = \int_{-T}^{T} \tau\alpha(\tau) + dt \wedge \tau\beta(\tau)d\tau - t\int_{-T}^{T} \alpha(\tau) + dt \wedge \beta(\tau)d\tau.$$

For any $v(t) = \alpha_0 + dt\wedge\beta_0 + t(\alpha_1 + dt\wedge\beta_1) \in \mathscr{E}_q$ we can use the formula of Example 3.22 to check:

$$(u_\eta, v) = \int_{-T}^{T} \tau(\langle\alpha(\tau),\alpha_1\rangle + \langle\beta(\tau),\beta_1\rangle) + \langle\alpha(\tau),\alpha_0\rangle + \langle\beta(\tau),\beta_0\rangle d\tau = \langle\eta, v\rangle.$$

3.3 The matching problem

In this section we explain the main construction of this chapter. In §3.3.1 we review the mapping properties of adapted operators on EAC manifolds. In §3.3.2 we explain our method for constructing approximate solutions of the equation $P_T u = f$ and show that it can be reduced to a finite-dimensional linear system. In §3.3.3 we prove that this system admits a solution if and only if f is orthogonal to the substitute cokernel defined in §3.1.2. Under the restricting assumption that the indicial operator of the gluing problem has only one real root, this enables us to prove Theorem 3.6. We also discuss other possible conditions which would yield the same result.

3.3.1 Analysis on EAC manifolds. The mapping properties of adapted operators on EAC manifolds have been studied by Lockhart–McOwen in [89], and we will give a brief review of their theory. The right function spaces to consider in this situation are weighted Sobolev spaces. Let (Z, g) be an EAC manifold asymptotic to a cylinder $Y = \mathbb{R}\times X$ at infinity, (E, h, ∇) an adapted bundle, and pick a cylindrical coordinate function $\rho : Z \to \mathbb{R}_{>0}$. If u is a smooth compactly supported section of E, we can define its $W^{l,p}_\nu$-norm ($p \geq 1$, $l\in\mathbb{N}$, $\nu\in\mathbb{R}$) as follows:

$$\|u\|_{W^{l,p}_\nu} = \sum_{j=0}^{l} \|e^{\nu\rho}\nabla^j u\|_{L^p}.$$

Note that for $\nu = 0$ this is just the usual $W^{l,p}$ norm. The weighted Sobolev space $W^{l,p}_\nu(E)$ can be defined as the completion of $C^\infty_c(E)$ with respect to the $W^{l,p}_\nu$-norm. We also denote by $C^\infty_\nu(E)$ the space of smooth sections of E that have all derivatives bounded by $O(e^{-\nu\rho})$.

Let P be an adapted elliptic differential operator $P : C^\infty(E) \to C^\infty(F)$ of order k, and let $P_0 : C^\infty(\underline{E}_0) \to C^\infty(\underline{F}_0)$ be its indicial operator. The maps $W^{k+l,p}_\nu(E) \to W^{l,p}_\nu(F)$ induced by P are bounded linear operators. Moreover, combining the estimates of Proposition 3.14 with standard interior elliptic estimates, we obtain a priori estimates of the form:

$$\|u\|_{W^{k+l,p}_\nu} \leq C\left(\|Pu\|_{W^{l,p}_\nu} + \|u\|_{L^p_\nu}\right).$$

Contrary to the compact case, these estimates are not enough to ensure that the maps induced by P on weighted spaces are Fredholm, essentially because of the failure of compactness in the Sobolev embedding theorem between spaces with the same weight. Nevertheless, Lockhart–McOwen showed that the Fredholm property holds if and only if ν is not an indicial root of P_0. When $\nu \notin \mathscr{D}_{P_0}$ we denote by $\operatorname{ind}_\nu(P)$ the index of the maps induced by P on spaces of weight ν. The following theorem summarises the mapping properties of P in weighted spaces and the index change formula as proved in [89].

Theorem 3.25. *Let $p > 1$ and $l \in \mathbb{N}$. Then the following holds.*

(i) The maps $W^{k+l,p}_\nu(E) \to W^{l,p}_\nu(F)$ induced by P are Fredholm if and only if $\nu \notin \mathscr{D}_{P_0}$. In that case, the image of P is the L^2-orthogonal complement of $\ker P^* \cap C^\infty_{-\nu}(F)$.

(ii) If $\nu < \nu'$ are not indicial roots of P, then the index change is given by

$$\operatorname{ind}_\nu(P) - \operatorname{ind}_{\nu'}(P) = \sum_{\nu < \operatorname{im}\lambda < \nu'} \dim \mathscr{E}_\lambda.$$

By elliptic regularity, the solutions of the homogeneous equation $Pu = 0$ are smooth. An important property of solutions with sub-exponential growth is that they have a polyhomogeneous expansion at infinity. More precisely, if $0 < \nu' - \nu < \mu$ and $\nu, \nu' \notin \mathscr{D}_P$, then the following holds. For any $u \in C^\infty_\nu$ such that $Pu = 0$, there exists $u' \in C^\infty_{\nu'}$ such that when $\rho \to \infty$, the difference $u - u'$ is an element of $\bigoplus_{\nu < \operatorname{im}\lambda < \nu'} \mathscr{E}_\lambda$ under the usual identification of the domain $\{\rho > 1\}$ with the cylinder $(1, \infty) \times X$.

From now on, let us assume that 0 is an indicial root of P_0, and let:

$$\sigma = \min\{\mu, \min_{\nu \in \mathscr{D}_{P_0} \setminus \{0\}} |\nu|\} \tag{3.32}$$

Take any $\delta \in (0, \sigma)$. Recall that we defined $\mathscr{K}$ as the kernel of P acting on sections with sub-exponential growth, and $\mathscr{K}_0$ the kernel of P acting on decaying sections. In particular, $\mathscr{K}$ is the kernel of P acting on $W^{k,p}_{-\delta}(E)$ and $\mathscr{K}_0$ the kernel of the action of P on $W^{k,p}_\delta(E)$. In §3.1.2 we defined a map $\kappa : \mathscr{K} \to \mathscr{E}$ such that any element $v \in \mathscr{K}$ is asymptotic to $\kappa(v)$. Hence $\mathscr{K}_0$ is the kernel of κ. Similarly we defined $\mathscr{K}^*$, $\mathscr{K}^*_0$ and $\kappa^* : \mathscr{K}^* \to \mathscr{E}^*$. Let us point out that the index change formula in Theorem 3.25 implies:

$$\dim \operatorname{im} \kappa + \dim \operatorname{im} \kappa^* = \dim \mathscr{E}. \tag{3.33}$$

We want to study equations of the type $Pu = f$ when f has exponential decay, say $f \in L^p_\delta$. By Theorem 3.25, the obstructions to solve this equation for $u \in W^{k,p}_\delta$ lie in $\mathscr{K}^*$, whereas the obstructions to solve it in $W^{k,p}_{-\delta}$ lie in $\mathscr{K}_0^*$. Here, we want to use the pairing defined in §3.2.2 to give a precise description of these obstructions and of the asymptotic behaviour of solutions in $W^{k,p}_{-\delta}$.

Let $v \in \mathscr{K}^*$ be asymptotic to $\kappa^*(v) = v_0 \in \mathscr{E}^*$ and consider $u \in C^\infty(E)$ asymptotic to $u_0 \in \mathscr{E}$, such that $u - u_0$ and all their derivatives are exponentially decaying as $\rho \to \infty$. The L^2 product $\langle Pu, v\rangle$ is well-defined as Pu decays exponentially. It turns out that its value only depends on the asymptotic data. More precisely we claim that:

Lemma 3.26. *With the above notations,* $\langle Pu, v\rangle = (u_0, v_0)$.

Proof. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\chi \equiv 0$ in $(-\infty, 0]$ and $\chi \equiv 1$ in $[1, \infty)$, and let $\chi_\tau(t) = \chi(t - \tau)$ for $\tau \in \mathbb{R}$. Then for any $\tau \geq 1$ we have:

$$\begin{aligned}\langle Pu, v\rangle &= \langle P\chi_\tau(\rho)u, v\rangle + \langle P(1 - \chi_\tau(\rho))u, v\rangle \\ &= \langle P\chi_\tau(\rho)u, v\rangle + \langle (1 - \chi_\tau(\rho))u, P^*v\rangle \\ &= \langle P\chi_\tau(\rho)u, v\rangle\end{aligned}$$

since $P^*v = 0$. Thus $\langle Pu, v\rangle = \lim_{\tau\to\infty}\langle P\chi_\tau(\rho)u, v\rangle$. As $u - u_0$, $v - v_0$ and the coefficients of $P - P_0$ decay exponentially as $\rho \to \infty$, as well as all derivatives, this implies:

$$\langle Pu, v\rangle = \lim_{\tau\to\infty}\langle P_0\chi_\tau u_0, v_0\rangle = (u_0, v_0)$$

since $\langle P_0\chi_\tau u_0, v_0\rangle = (u_0, v_0)$ for any $\tau \in \mathbb{R}$. □

As a consequence of this lemma, $\operatorname{im}\kappa$ and $\operatorname{im}\kappa^*$ are orthogonal for the pairing $(\cdot,\cdot)$. Together with equality (3.33), this implies that $\operatorname{im}\kappa$ is exactly the orthogonal space of $\operatorname{im}\kappa^*$ for the pairing $(\cdot,\cdot)$.

Let us denote by $\mathscr{K}_+^*$ the subspace of $\mathscr{K}$ orthogonal to $\mathscr{K}_0^*$ for the L^2-product, so that κ^* induces an isomorphism between $\mathscr{K}_+^*$ and $\operatorname{im}\kappa^*$. We also choose an arbitrary complement $\mathscr{A}_0$ of $\operatorname{im}\kappa$ in $\mathscr{E}$. Let $m = \dim\operatorname{im}\kappa^*$. Pick smooth sections $h_1, \ldots, h_m$ which are asymptotic to a basis of $\mathscr{A}_0$ at infinity, with the difference and all their derivatives exponentially decaying, and denote by $\mathscr{A} \subset C^\infty(E)$ the vector space they span. By Lemma 3.26 we may choose a basis $g_1, \ldots, g_m$ of $\mathscr{K}_+^*$ such that $\langle Ph_i, g_j\rangle = \delta_{ij}$ for all $1 \leq i, j \leq m$.

Let $f \in L^p_\delta$ be a section of F, and w be the L^2-projection of f onto $\mathscr{K}_0^*$. Let us write:

$$f = f' + \sum_{j=1}^{m}\langle f, g_j\rangle Ph_j + w.$$

where $f' \in L^p_\delta$ is by construction orthogonal to the obstruction space $\mathscr{K}^*$. As $|\langle f, g\rangle| \leq C\|f\|_{L^p_\delta}\|g\|_{L^q_{-\delta}}$ for any $g \in \mathscr{K}^*$, where q is the conjugate exponent of p and $C > 0$ is some constant, we have $\|f'\|_{L^p_\delta} \leq C'\|f\|_{L^p_\delta}$ for some universal constant $C' > 0$. By Theorem 3.25 there exists u' such that $Pu' = f'$ and $\|u'\|_{W^{k,p}_\delta} \leq C''\|f'\|_{L^p_\delta}$. This proves the following:

Proposition 3.27. *For any $p > 1$ and $0 < \delta < \sigma$, there exists a constant $C > 0$, depending only on p and δ, such that the following holds. Let $f \in L^p_\delta(F)$, and let w by its L^2-projection onto $\mathscr{K}_0$. Then there exists a section $u' \in W^{k,p}_\delta(E)$ with $\|u'\|_{W^{k,p}_\delta} \leq C\|f\|_{L^p_\delta}$ and such that*

$$P\left(u' + \sum_{j=1}^{m}\langle f, g_j\rangle h_j\right) = f - w.$$

3.3.2 Characteristic system. In the same setup as Section 3.1, we now consider the gluing problem of two adapted operators P_1, P_2 of order k on EAC manifolds Z_1, Z_2. For the present discussion there are no restrictions on the real roots of the indicial operator P_0. By definition, there is a compact $K_1 \subset Z_1$ and an orientation-preserving diffeomorphism $\phi_1 : (0,\infty) \times X \to Z_1 \backslash K_1$, and we picked a positive cylindrical coordinate function ρ_1 on Z_1 such that $\rho_1(\phi_1(t,x)) = t$ when $t \geq 1$ and $\rho_1 < 1$ everywhere else in Z_1. As in Section 3.1 we fix a cutoff function $\chi : \mathbb{R} \to [0,1]$ such that $\chi \equiv 0$ in $(-\infty, -\frac{1}{2}]$ and $\chi \equiv 1$ in $[\frac{1}{2}, \infty)$. For $\tau \in \mathbb{R}$ we keep our usual notation $\chi_\tau(t) = \chi(t - \tau)$. It will be convenient to introduce a family $\zeta^1_\tau : Z_1 \to [0,1]$ of cutoff functions for the construction. For $\tau \geq 0$ define:

$$\zeta^1_\tau(z) = \begin{cases} 0 & \text{if } \ z \in K_1 \\ \chi(t - \tau - \frac{1}{2}) & \text{if } \ z = \phi_1(t,x), \ (t,x) \in (0,\infty) \times X \end{cases}.$$

We similarly define a family of cutoff functions on Z_2, denoted by ζ^2_τ for $\tau \geq 0$. Consider now the compact manifold M_T obtained by gluing the compact domains $\{\rho_1 \leq T + 2\} \subset Z_1$ and $\{\rho_2 \leq T + 2\} \subset Z_2$ along the annulus $\{T \leq \rho_i \leq T + 2\}$. We can define a family of cutoff functions $\zeta_\tau : M_T \to [0,1]$ for $0 \leq \tau \leq T$ by patching together ζ^1_τ with ζ^2_τ in the following way:

$$\zeta_\tau \equiv \begin{cases} \zeta^1_\tau & \text{if } \rho_T \leq 0 \\ \zeta^2_\tau & \text{if } \rho_T \geq 0 \end{cases}.$$

Note that the support of ζ_τ is diffeomorphic to the finite cylinder $[-T-1+\tau, T+1-\tau] \times X$.

We now turn to the gluing problem of two adapted operators $P_i : C^\infty(E_i) \to C^\infty(F_i)$ as described in §3.1.1. Our goal is to prove that we can construct solutions of the equation $P_T u = f$ for f taking values in a complement of the substitute cokernel introduced in §3.1.2. We shall do this by considering three regions in M_T: the neck region $\{|\rho_T| \leq T\}$ for which our main tool is Theorem 3.11, and the two compact regions $\{\rho_T \leq 0\}$ and $\{\rho_T \geq 0\}$, for which we will use weighted analysis on Z_1 and Z_2 in the form of Proposition 3.27. The crucial point of the construction is to understand the interactions between these three regions, especially in terms of the obstructions to solving the equation $P_i u = f$ on each Z_i. Using the pairing $(\cdot,\cdot)$ defined in §3.2.2 in order to implicitly keep track of these obstructions, we will be able to essentially reduce this problem to a finite-dimensional linear system.

From now on we fix $p > 1$ and work with Sobolev spaces $W^{l,p}$. Let $f \in L^p(F_T)$ be an arbitrary section. We may identify the section $\zeta_1 f$ with a section of the translation-invariant vector bundle $\underline{F}_0$ over the cylinder $Y = \mathbb{R} \times X$, which we denote by f_0. Moreover, the essential support of f_0 is contained in the finite cylinder $[-T, T] \times X$. Note that the Sobolev norm of sections supported in the neck region of M_T and the Sobolev norm of sections supported in the finite cylinder $[-T, T] \times X$ are equivalent. Hence, we have a bound:

$$\|f_0\|_{L^p} \leq C\|f\|_{L^p}.$$

By Theorem 3.11, the operator P_0 admits a right inverse $Q_0 : L^p_c(\underline{F}_0) \to W^{k,p}_{loc}(\underline{E}_0)$. Thus we can define $u_0 = Q_0 f_0$, which satisfies $P_0 u_0 = f_0$. Using the cutoff function ζ_0 to identify $\zeta_0 u_0$ with a section of $E_T \to M_T$, one has:

$$\begin{aligned} f - P_T\zeta_0 u_0 &= f - [P_T, \zeta_0]u_0 - \zeta_0 P_T u_0 \\ &= (1-\zeta_1)f - [P_T, \zeta_0]u_0 - \zeta_0(P_T - P_0)u_0. \end{aligned}$$

Note that the section $(1-\zeta_1)f - [P_T, \zeta_0]u_0$ is supported in the compact region $\{|\rho_T| \geq T\}$. Moreover the operator $\zeta_0(P_T - P_0)$ vanishes in the region $\{|\rho_T| \leq \frac{1}{2}\}$ so that we may write:

$$f - P_T\zeta_0 u_0 = f_1 + f_2 \tag{3.34}$$

where $f_1 = \chi(\rho_T)(f - P_T\zeta_0 u_0)$ can be identified with a section of F_1 supported in $\{\rho_1 \leq T+1\} \subset Z_1$, and $f_2 = (1-\chi(\rho_T))(f - P_T\zeta_0 u_0)$ can be identified with a section of F_2 over $\{\rho_2 \leq T+1\} \subset Z_2$. Both sections are L^p-bounded.

From now on we fix some $\delta \in (0, \sigma)$. As the coefficients of $P_i - P_0$ and all their derivatives have exponential decay as $\rho_i \to \infty$, the L^p-norm of f controls the L^p_δ-norms of f_1 and f_2. More precisely, the following estimates hold:

Lemma 3.28. *Let d be the maximal order of the real roots of P_0. Then there exists a constant $C > 0$ such that:*

$$\|f_i\|_{L^p_\delta} \leq CT^d\|f\|_{L^p}, \qquad i = 1, 2.$$

Proof. Let us prove the estimate for f_1, which can be written as:

$$f_1 = (1-\chi(\rho_T))((1-\zeta_1)f - [P_T,\zeta_0]u_0 - \zeta_0(P_T - P_0)u_0).$$

The term $(1-\chi(\rho_T))(1-\zeta_1)f$ is supported in the compact region $\{\rho_1 \leq 2\} \subset Z_1$ and therefore satisfies:

$$\|(1-\chi(\rho_T))(1-\zeta_1)f\|_{L^p_\delta} \leq e^{2\delta}\|f\|_{L^p}$$

since the function $(1-\chi(\rho_T))(1-\zeta_1)$ is bounded by 1. On the other hand, the second term $(1-\chi(\rho_T))[P_T,\zeta_0]u_0$ is supported in $\{\rho_1 \leq 1\}$, and the $W^{k,p}$-norm of u_0 in the cylinder $[-T-1, T+1]\times X$ is bounded by $CT^d\|f\|$ for some constant C. As ζ_0 and all its derivatives are uniformly bounded independently from T, this yields an estimate:

$$\|(1-\chi(\rho_T))[P_T,\zeta_0]u_0\|_{L^p} \leq C'T^d\|f\|_{L^p}.$$

For the last term $(1-\chi(\rho_T))\zeta_0(P_T-P_0)u_0$, we can use the bound on the $W^{k,p}$-norm of u_0 and the exponential decay of the coefficients of $P_1 - P_0$ and all their derivatives to obtain a similar bound:

$$\|(1-\chi(\rho_T))\zeta_0(P_T-P_0)u_0\|_{L^p} \leq C''T^d\|f\|_{L^p}.$$

These three bounds prove the lemma. □

Next we want to understand the obstructions to solving $P_iu_i = f_i$ with $u_i \in W^{k,p}_\delta(E_i)$. Let us denote by $\langle\cdot,\cdot\rangle_0$ the L^2-product on the cylinder $\mathbb{R}\times X$ equipped with its translation-invariant metric. The key result is the following:

Lemma 3.29. *Choose arbitrary norms on $\mathscr{K}_1^*$ and $\mathscr{K}_2^*$. For $T\to\infty$ the following holds. If $g_1 \in \mathscr{K}_1^*$ and $g_{1,T}(t) = \kappa_1^*[g_1](t+T+1)$, then:*

$$\langle f_1, g_1\rangle = \langle f, (1-\chi_{T+1}(\rho_1))g_1\rangle - \langle(1-\chi)f_0, g_{1,T}\rangle_0 + O\left(e^{-\delta T}\|f\|_{L^p}\|g_1\|\right).$$

If $g_2 \in \mathscr{K}_2^*$ and $g_{2,T} = \kappa_2^*[g_2](t-T-1)$ then:

$$\langle f_2, g_2\rangle = \langle f, (1-\chi_{T+1}(\rho_2))g_2\rangle + \langle(1-\chi)f_0, g_{2,T}\rangle_0 + O\left(e^{-\delta T}\|f\|_{L^p}\|g_2\|\right).$$

Remark 3.30. In the statement of the lemma and in the following proof, the notation $O(e^{-\delta T}\|f\|_{L^p}\|g_i\|)$ means that there is a constant $C > 0$, depending on $p > 1$, $\delta \in (0,\sigma)$ and possibly on the choice of norms on $\mathscr{K}_i^*$ but independent of T, f and g_i, such that

$$|O(e^{-\delta T}\|f\|_{L^p}\|g_i\|)| \leq Ce^{-\delta T}\|f\|_{L^p}\|g_i\|.$$

Proof. Notice first that for any $\tau \le T-2$ we have:

$$\begin{aligned}\langle f_1, g_1\rangle &= \langle (1-\chi(\rho_T))f, g_1\rangle - \langle (1-\chi(\rho_T))P_T\zeta_0 u_0, g_1\rangle \\ &= \langle (1-\chi(\rho_T))f, g_1\rangle - \langle (1-\chi(\rho_T))P_T\zeta_\tau u_0, g_1\rangle\end{aligned}$$

since $(\zeta_\tau - \zeta_0)u_0$ has support in $\{\rho_1 \le T-1\}$ and $P_1^* g_1 = 0$. Given the decay of the coefficients of $P_1 - P_0$ we have:

$$\langle (1-\chi(\rho_T))P_T\zeta_{T-2}u_0, g_1\rangle = \langle (1-\chi)P_0\chi_{-2}u_0, g_{1,T}\rangle_0 + O\left(e^{-\delta T}\|f\|_{L^p}\|g_1\|\right). \quad (3.35)$$

Moreover $1-\chi(\rho_T) = (1-\chi_{T+1}(\rho_1))$ with the usual identifications. Thus the equality $\langle (1-\chi(\rho_T))f, g_1\rangle = \langle f, (1-\chi_{T+1}(\rho_1))g_1\rangle$ clearly holds.

It remains to compute the value of $\langle (1-\chi)P_0\chi_{-2}u_0, g_{1,T}\rangle_0$. By integration by parts, for any $\tau \le -2$ we have:

$$\langle (1-\chi)P_0\chi_{-2}u_0, g_{1,T}\rangle_0 = \langle (1-\chi)P_0\chi_\tau u_0, g_{1,T}\rangle_0. \quad (3.36)$$

But now we can write $P\chi_\tau u_0 = \chi_\tau f_0 + [P_0, \chi_\tau]u_0$. As u_0 and its derivatives of order less than k have exponential decay at infinity and the differential operator $[P_0, \chi_\tau]$ has uniformly bounded coefficients and is supported in $[\tau - \frac{1}{2}, \tau + \frac{1}{2}] \times X$, it follows that

$$\lim_{\tau\to-\infty} \langle (1-\chi)[P_0, \chi_\tau]u_0, g_{1,T}\rangle_0 = 0$$

and therefore we can send $\tau \to -\infty$ in (3.36) and obtain:

$$\begin{aligned}\langle (1-\chi)P_0\chi_{-2}u_0, g_{1,T}\rangle_0 &= \lim_{\tau\to-\infty} \langle (1-\chi)\chi_\tau f_0, g_{1,T}\rangle_0 \\ &= \langle (1-\chi)f_0, g_{1,T}\rangle_0\end{aligned}$$

This proves the first equality of Lemma 3.29.

For the second equality we can prove as above that:

$$\langle f_2, g_2\rangle = \langle f, \chi_{T+1}(\rho_2)g_2\rangle - \lim_{\tau\to\infty} \langle \chi P_0(1-\chi_\tau)u_0, g_{2,T}\rangle_0 + O\left(e^{-\delta T}\|f\|_{L^p}\|g_2\|\right). \quad (3.37)$$

Then for τ large enough we have:

$$\begin{aligned}\langle \chi P_0(1-\chi_\tau)u_0, g_{2,T}\rangle_0 &= \langle \chi f_0, g_{2,T}\rangle_0 + \langle \chi[P_0, 1-\chi_\tau]u_0, g_{2,T}\rangle_0 \\ &\longrightarrow \langle \chi f_0, g_{2,T}\rangle_0 - (u_{f_0}, g_{2,T})\end{aligned}$$

as $\tau \to \infty$, where $u_{f_0} \in \mathscr{E}$ is the polyhomogeneous solution defined in §3.2.3. By Lemma 3.23, the last term is equal to:

$$(u_{f_0}, g_{2,T}) = \langle f_0, g_{2,T}\rangle_0.$$

The second equality follows. □

In the next section, it will be useful to use a variation of the above lemma for arbitrary solutions of the equation $P_0 u = f_0$. Thus let $v \in \mathscr{E}$ and define $u_0' = Q_0 f_0 + v$, and as above write:

$$f - P_T \zeta_0 u_0' = f_1' + f_2' \tag{3.38}$$

where $f_i' \in L^p_\delta(F_i)$. As a corollary of Lemma 3.29, we can describe the obstructions to solving $P_i u = f_i'$ as follows.

Corollary 3.31. *Choose arbitrary norms on $\mathscr{E}$, $\mathscr{K}_1^*$ and $\mathscr{K}_2^*$. Then, if $g_1 \in \mathscr{K}_1^*$ and $g_{1,T}(t) = \kappa_1^*[g_1](t + T + 1)$ it holds:*

$$\langle f_1', g_1\rangle = \langle f, (1-\chi_{T+1}(\rho_1))g_1\rangle - \langle (1-\chi)f_0, g_{1,T}\rangle_0 - (v, g_{1,T}) + O\left(e^{-\delta T}(\|f\|_{L^p} + \|v\|)\|g_1\|\right).$$

If $g_2 \in \mathscr{K}_2^*$ and $g_{2,T} = \kappa_2^*[g_2](t - T - 1)$ then:

$$\langle f_2', g_2\rangle = \langle f, (1-\chi_{T+1}(\rho_2))g_2\rangle + \langle (1-\chi)f_0, g_{2,T}\rangle_0 + (v, g_{2,T}) + O\left(e^{-\delta T}(\|f\|_{L^p} + \|v\|)\|g_2\|\right).$$

Remark 3.32. As in the previous lemma, the notation $O(e^{-\delta T}(\|f\|_{L^p} + \|v\|)\|g_i\|)$ means that there is a constant $C > 0$, depending on $p > 1$, $\delta \in (0, \sigma)$, and possibly on the choice of norms on $\mathscr{E}$, $\mathscr{K}_1^*$ and $\mathscr{K}_2^*$ but independent of T, f, g_i and v such that

$$|O(e^{-\delta T}(\|f\|_{L^p} + \|v\|)\|g_i\|)| \leq Ce^{-\delta T}(\|f\|_{L^p} + \|v\|)\|g_i\|.$$

So far, everything we did works without need to impose any conditions on the real roots of the indicial operator P_0. We now outline the construction which we will perform in the next part and emphasise where the restricting assumption of Theorem 3.6 come from. The general idea of our construction is to identify a subspace of $L^p(F_T)$ on which we can find approximate solutions of the equation $P_T u = f$ with good estimates and a control on the error of the form $\|f - P_T u\|_{L^p} \leq Ce^{-\delta T}\|f\|_{L^p}$ for T large enough. Once we can achieve this, we will simply use an iterative process to build exact solutions, by taking successive projections onto this good subspace.

By taking cutoffs as above, we can solve the equation $P_0 u = f_0$ on the cylinder, with a general solution of the form $u = Q_0 f_0 + v$ for some arbitrary $v \in \mathscr{E}$. With the above notations, it remains to consider the equations $P_i u_i = f_i'$ on the EAC manifolds Z_1 and Z_2. The idea is to choose v appropriately so that all the obstructions to finding decaying solutions $u_i \in W^{k,p}_\delta(E_i)$ vanish, up to exponentially decaying errors. If this can be done, we just need to take cutoffs of these solutions to build an approximate solution, up to an exponentially decaying term. Using Corollary 3.31 we have essentially reduced our linear PDE problem to the following finite-dimensional system, where the unknown is $v \in \mathscr{E}$:

$$\begin{cases} (v, g_{1,T}) = \langle f, (1-\chi_{T+1}(\rho_1))g_1\rangle - \langle (1-\chi)f_0, g_{1,T}\rangle_0, & \forall g_1 \in \mathscr{K}_1^* \\ (v, g_{2,T}) = -\langle f, (1-\chi_{T+1}(\rho_2))g_2\rangle + \langle (1-\chi)f_0, g_{2,T}\rangle_0, & \forall g_2 \in \mathscr{K}_2^* \end{cases} \tag{3.39}$$

where we use the notations:

$$g_{1,T}(t) = \kappa_1^*[g_1](t+T+1), \quad \text{and} \quad g_{2,T} = \kappa_2^*[g_2](t-T-1).$$

We call this system the *characteristic system* of our gluing problem. There are obvious obstructions to finding a solution to this system. We need at least to impose f to be orthogonal to all the sections of the form $(1-\chi_{T+1}(\rho_i))g_i$ with $g_i \in \mathscr{K}_{i,0}$, as in this case $g_{i,T} = 0$. Actually, a more careful examination of the characteristic system shows that we need f to be orthogonal to the full substitute cokernel. Indeed, a pair $(g_1, g_2) \in \mathscr{K}_1^* \times \mathscr{K}_2^*$ is matching at T if and only if $g_{1,T} = g_{2,T}$ with the above notations. Thus, if there exists $v \in \mathscr{E}$ solving the system we must have

$$\langle f, (1-\chi_{T+1}(\rho_1))g_1 + (1-\chi_{T+1}(\rho_2))g_2\rangle = 0$$

for any pair (g_1, g_2) matching at T.

As a consequence, the substitute cokernel $\mathscr{K}_T^*$ naturally arises as a space of obstructions to constructing approximate solutions of $P_T u = f$ by our method. In fact, we will see that this is also a sufficient condition (Lemma 3.37). Unfortunately, the coefficients of this system vary analytically with T, and therefore the rank of the system might drop at some points. Furthermore, the system is generally underdetermined, with an obvious kernel formed by the subspace of $\mathscr{E}$ orthogonal to all $g_{1,T}$ and $g_{2,T}$, for g_i varying in $\mathscr{K}_i^*$. Hence, even if the characteristic system admits a solution v whenever f is orthogonal to the substitute cokernel we might not be able to obtain reasonable estimates on the norm of v, especially near the values of T at which the rank of the system drops. We shall prove that these difficulties can be avoided in the case where the indicial operator P_0 has only one root, which will be sufficient for our applications.

3.3.3 Main construction. Let us first consider the case where P_0 has a single root λ_0 of order 1, before generalising to any order. In that case, the elements of $\mathscr{E}$ are of the form $e^{i\lambda_0 t}u(x)$ with u translation-invariant section of $\underline{E}_0$, and similarly for $\mathscr{E}^*$. As a consequence, the matching condition (3.10) does not really depend on T, up to overall factors of $e^{\pm i\lambda_0(T+1)}$. In particular, the dimensions of $\mathscr{K}_T$ and $\mathscr{K}_T^*$ are independent of T:

$$\dim \mathscr{K}_T = \dim \mathscr{K}_{0,1} + \dim \mathscr{K}_{0,2} + \dim(\operatorname{im} \kappa_1 \cap \operatorname{im} \kappa_2)$$

and similarly for the substitute cokernel $\mathscr{K}_T^*$. This implies that we can uniformly bound the L^2-orthogonal projections onto $\mathscr{K}_T$ and $\mathscr{K}_T^*$:

Lemma 3.33. *Let $p > 1$ and $l \in \mathbb{N}$. Then for T large enough the norm of the L^2-orthogonal projection of $W^{l,p}(F_T)$ onto $\mathscr{K}_T^*$ is bounded from above by a uniform constant $C_1 > 0$, depending on l and p but independent of T. Similarly, the norm of the L^2-orthogonal projection of $W^{l,p}(E_T)$ onto $\mathscr{K}_T$ is bounded from above by a uniform constant $C_1' > 0$.*

Proof. This can be proved by fixing basis for $\mathscr{K}_{0,1}$, $\mathscr{K}_{0,2}$ and $\operatorname{im} \kappa_1 \cap \operatorname{im} \kappa_2$ and considering the corresponding basis of $\mathscr{K}_T$. By Gram–Schmidt orthonormalisation, one can deduce an explicit expression for the L^2-projection, from which the lemma easily follows. □

Let us now choose an arbitrary complement $\mathscr{E}_1^*$ of $\operatorname{im} \kappa_1^* \cap \operatorname{im} \kappa_2^*$ in $\operatorname{im} \kappa_1^*$, and a complement $\mathscr{E}_2^*$ of $\operatorname{im} \kappa_1^* \cap \operatorname{im} \kappa_2^*$ in $\operatorname{im} \kappa_2^*$. Thus we have a direct sum decomposition:

$$\operatorname{im} \kappa_1^* + \operatorname{im} \kappa_2^* = \operatorname{im} \kappa_1^* \cap \operatorname{im} \kappa_2^* \oplus \mathscr{E}_1^* \oplus \mathscr{E}_2^* \subset \mathscr{E}^*.$$

Pick a complement $\mathscr{E}'$ of $\operatorname{im} \kappa_1 \cap \operatorname{im} \kappa_2$ in $\mathscr{E}$, so that the pairing

$$\mathscr{E}' \times (\operatorname{im} \kappa_1^* + \operatorname{im} \kappa_2^*) \to \mathbb{C}$$

induced by $(\cdot, \cdot)$ is non-degenerate. For $i = 1, 2$ define $\mathscr{E}_i = \operatorname{im} \kappa_i \cap \mathscr{E}'$. Then the pairings:

$$\mathscr{E}_1 \times \mathscr{E}_2^* \to \mathbb{C} \quad \text{and} \quad \mathscr{E}_2 \times \mathscr{E}_1^* \to \mathbb{C} \tag{3.40}$$

induced by $(\cdot, \cdot)$ are non-degenerate. Indeed, if $u \in \mathscr{E}_1$ is orthogonal to $\mathscr{E}_2^*$ then it is orthogonal to $\operatorname{im} \kappa_1^* + \operatorname{im} \kappa_2^*$ and therefore belongs to $\operatorname{im} \kappa_1 \cap \operatorname{im} \kappa_2$, which means that $u = 0$ as it is an element of $\mathscr{E}'$. On the other hand if $v \in \mathscr{E}_2^*$ is orthogonal to $\mathscr{E}_1$, then it is orthogonal to $\operatorname{im} \kappa_1$ and to $\operatorname{im} \kappa_2$, which means that it belongs to $\operatorname{im} \kappa_1^* \cap \operatorname{im} \kappa_2^*$ and thus $v = 0$ by definition of $\mathscr{E}_2^*$. Therefore, if we define $\mathscr{E}_0$ as the orthogonal space of $\mathscr{E}_1^* \oplus \mathscr{E}_2^*$ in $\mathscr{E}'$ for the above pairing, we have a direct sum decomposition:

$$\mathscr{E}' = \mathscr{E}_0 \oplus \mathscr{E}_1 \oplus \mathscr{E}_2.$$

This implies that the pairing

$$\mathscr{E}_0 \times (\operatorname{im} \kappa_1^* \cap \operatorname{im} \kappa_2^*) \to \mathbb{C} \tag{3.41}$$

is non-degenerate. These conventions will be useful to put the system (3.39) in a more tractable form. For definiteness we prefer to work in a complement of $\operatorname{im}\kappa_1 \cap \operatorname{im}\kappa_2$, as this is the kernel of the system. This allows us to prove:

Proposition 3.34. *Let $p > 1$ and $\delta \in (0, \sigma)$. Then there exist constants $C_2, C_3 > 0$, depending on p and δ but not on T, such that for T large enough the following holds. If $f \in L^p(F_T)$ is L^2-orthogonal to $\mathscr{K}_T^*$, then there exists $u \in W^{k,p}(E_T)$ L^2-orthogonal to $\mathscr{K}_T$ such that:*

$$\|f - P_T u\|_{L^p} \le C_2 e^{-\delta T}\|f\|_{L^p} \quad \text{and} \quad \|u\|_{W^{k,p}} \le C_3 T \|f\|_{L^p}.$$

Remark 3.35. The constants C_2 and C_3 may depend on the geometric data of the gluing problem (the manifolds Z_i, the adapted bundles E_i and the matching operators P_i); in this sense they are not universal. But once this gluing data is fixed they only depend on p and δ, the point being that they do not depend on T.

Proof. For $i = 1, 2$, let us fix subspaces $\mathscr{A}_i \subset C^\infty(E_i)$ as in the discussion preceding Proposition 3.27. The orthogonal space $\mathscr{K}_{i,+}^*$ of $\mathscr{K}_{i,0}$ in $\mathscr{K}_i$ is isomorphic to $\operatorname{im}\kappa_i^*$, so that the decomposition $\operatorname{im}\kappa_i^* = \operatorname{im}\kappa_1^* \cap \operatorname{im}\kappa_2^* \oplus \mathscr{E}_i^*$ induce a corresponding decomposition $\mathscr{K}_{i,+}^* = \mathscr{K}_{i,m}^* \oplus \mathscr{K}_{i,\perp}^*$ (the subscript m stands for matching).

If $f \in L^p(F_T)$ is orthogonal to the substitute cokernel, we can use the above decompositions of $\mathscr{E}$ amd $\mathscr{E}^*$ to put the system (3.39) in the form:

$$\begin{cases} (v_0, \kappa_1^*[g_0]) = e^{-i\lambda_0(T+1)}\langle f, (1-\chi_{T+1}(\rho_1))g_0\rangle - \langle (1-\chi)f_0, \kappa_1^*[g_0]\rangle_0, & \forall g_0 \in \mathscr{K}_{1,m}^* \\ (v_1, \kappa_1^*[g_1]) = e^{-i\lambda_0(T+1)}\langle f, (1-\chi_{T+1}(\rho_1))g_1\rangle - \langle (1-\chi)f_0, \kappa_1^*[g_1]\rangle_0, & \forall g_1 \in \mathscr{K}_{1,\perp}^* \\ (v_2, \kappa_2^*[g_2]) = -e^{i\lambda_0(T+1)}\langle f, (1-\chi_{T+1}(\rho_2))g_2\rangle + \langle (1-\chi)f_0, \kappa_2^*[g_2]\rangle_0, & \forall g_2 \in \mathscr{K}_{2,\perp}^* \end{cases}$$

where we decompose any element $v \in \mathscr{E}'$ as $v = v_0 + v_1 + v_2 \in \mathscr{E}_0 \oplus \mathscr{E}_1 \oplus \mathscr{E}_2$ and the factors $e^{\pm i\lambda_0(T+1)}$ come from $\kappa_{1,T}^* = e^{i\lambda_0(T+1)}\kappa_1^*$ and $\kappa_{2,T} = e^{-i\lambda_0(T+1)}\kappa_2^*$. Non-degeneracy of the pairings (3.40) and (3.41) implies that this is of the form:

$$Av = b_T(f) \in \mathbb{R}^N$$

where $N = \dim \mathscr{E}' = \dim\operatorname{im}\kappa_1^* + \dim\operatorname{im}\kappa_2^*$ and $A : \mathscr{E}' \to \mathbb{R}^N$ is an invertible linear map which does not depend on T. Thus there is a unique solution $v = A^{-1}b(f)$, and if we fix norms on $\mathscr{E}'$ and $\mathbb{R}^N$ we have a uniform bound:

$$\|v\| \le C\|b_T(f)\|.$$

As the elements of $\mathscr{K}_1^*$ and $\mathscr{K}_2^*$ are bounded in C^0 norm, each of the sections $(1-\chi_{T+1})g_i$ have L^q-norm bounded by $CT^{\frac{1}{q}}\|g_i\|$, where q is the conjugate exponent of p. Thus, we can deduce that the norm of $b_T(f)$ satisfies a bound of the form:

$$\|b_T(f)\| \le C' T^{\frac{1}{q}}\|f\|_{L^p}.$$

Hence $\|v\| \leq C''T^{\frac{1}{q}}\|f\|_{L^p}$ for a constant independent of T.

Following the idea outlined in the previous part, let us write:

$$f - P_T(\zeta_0 Q_0 f_0 + \zeta_0 v) = f_1 + f_2$$

with $f_1 \in L^p(F_1)$ and $f_2 \in L^p(F_2)$, each of the sections f_i being supported in the domain $\{\rho_i \leq T+2\} \subset Z_i$. By Theorem 3.11, we have estimates:

$$\|\zeta_0 Q_0 f_0\|_{W^{k,p}} \leq CT\|f_0\|_{L^p} \leq C'T\|f\|_{L^p}.$$

Further, as v is uniformly bounded and $\zeta_0 v$ has support in a domain equivalent to a finite cylinder $[-T-1, T+1] \times X$ we have a bound:

$$\|\zeta_0 v\|_{W^{k,p}} \leq CT^{\frac{1}{p}}\|v\| \leq C'T\|f\|_{L^p}.$$

As in the proof of Lemma 3.28, we can use the uniform bound on v to prove that the weighted norms of f_1 and f_2 satisfy bounds:

$$\|f_i\|_{L^p_\delta} \leq CT\|f\|_{L^p}.$$

We now consider the equations $P_1 u_1 = f_1$ on Z_1 and $P_2 u_2 = f_2$ on Z_2. By Proposition 3.27, there exist $w_i \in \mathscr{K}^*_{i,0}$, $h_i \in \mathscr{A}_i$ and $u_i \in W^{k,p}_\delta(E_i)$ such that:

$$P_i(u_i + h_i) = f_i - w_i.$$

Moreover, our choice of v implies uniform bounds of the form:

$$\|u_i\|_{W^{k,p}_\delta} \leq C\|f_i\|_{L^p_\delta} \leq C'T\|f\|_{L^p}, \quad \|h_i\| + \|w_i\| \leq C''e^{-\delta T}\|f\|_{L^p}$$

for some uniform constants C' and C''. Taking cutoffs we can write:

$$f_i - P_T(\chi_{T+1}(\rho_i)u_i + \chi_{T+1}(\rho_i)h_i) = \chi_{T+1}(\rho_i)w_i + r_i$$

where r_i is an error term of the form

$$r_i = (P_i - P_T)(\chi_{T+1}(\rho_i)u_i + \chi_{T+1}(\rho_i)h_i) + [P_i, \chi_{T+1}(\rho_i)](u_i + h_i).$$

As the coefficients of $P_i - P_T$ and their derivatives have exponential decay with T, u_i has exponential decay at infinity. Given the bound on h_i, it follows that for any $0 < \epsilon < \delta$ we can bound the errors terms by:

$$\|r_i\|_{L^p} \leq Ce^{-(\delta-\epsilon)T}\|f\|_{L^p} \tag{3.42}$$

for some uniform constant. Let us define:

$$u = \zeta_0 u_0 + \chi_{T+1}(\rho_1)(u_1 + h_1) + \chi_{T+1}(\rho_2)(u_2 + h_2).$$

Then $f - P_T u = \chi_{T+1}(\rho_1)w_1 + \chi_{T+2}(\rho_2)w_2 + r_1 + r_2$ satisfies $\|f - P_T u\|_{L^p} \leq Ce^{-(\delta-\epsilon)T}\|f\|_{L^p}$, and $\|u\|_{W^{l,p}} \leq CT\|f\|_{L^p}$, for some constant C. By Lemma 3.33, we can decompose $u = u' + w$ where $w \in \mathscr{K}_T$ and u' is orthogonal to the substitute kernel. Moreover we have bounds:

$$\|u'\|_{W^{k,p}} \leq (1 + C_1')\|u\|_{W^{k,p}} \leq C'T\|f\|_{L^p}, \quad \|w\|_{W^{k,p}} \leq C_1\|u\|_{W^{k,p}} \leq C''T\|f\|_{L^p}$$

and u' satisfies

$$f - P_T u' = f - P_T u + P_T w \tag{3.43}$$

and as $w \in \mathscr{K}_T$, then $\|P_T w\|_{L^p} \leq Ce^{-\delta T}\|w\|_{W^{k,p}} \leq Ce^{-(\delta-\epsilon)T}\|f\|_{L^p}$. □

Now we have all the tools to prove Theorem 3.6, in the case where λ_0 is a root of order 1. Let $f \in L^p(F_T)$ be an arbitrary section. By Lemma 3.33, there exist $\tilde{f} \in L^p(F_T)$ and $w_0 \in \mathscr{K}_T^*$ such that $f = \tilde{f} + w_0$, $\tilde{f}$ is orthogonal to $\mathscr{K}_T^*$ and:

$$\|\tilde{f}\|_{L^p} \leq (1 + C_1)\|f\|_{L^p}, \quad \|w_0\|_{L^p} \leq C_1\|f\|_{L^p}.$$

Moreover, by Proposition 3.34 there exist $u_0 \in W^{k,p}(E_T)$ orthogonal to $\mathscr{K}_T^*$ and $f_1 \in L^p(F_T)$ such that:

$$\tilde{f} = P_T u_0 + f_1$$

with bounds:

$$\|u_0\|_{W^{k,p}} \leq C_3 T\|\tilde{f}\|_{L^p} \leq (1 + C_1)C_3 T\|f\|_{L^p}$$

and

$$\|f_1\|_{L^p} \leq C_2 e^{-\delta T}\|\tilde{f}\|_{L^p} \leq (1 + C_1)C_2 e^{-\delta T}\|f\|_{L^p}.$$

Choose T large enough such that $\eta = (1 + C_1)C_2 e^{-\delta T} < 1$, and define $f_0 = f$. Inductively, we can construct sequences $\{f_n,\ n \geq 0\}$ in $L^p(F_T)$, $\{u_n, n \geq 0\}$ in the L^2-orthogonal complement of $\mathscr{K}_T$ in $W^{k,p}(E_T)$ and $\{w_n,\ n \geq 0\}$ in $\mathscr{K}_T^*$ such that for all $n \geq 0$ we have:

$$f_n - f_{n+1} = P_T u_n + w_n \tag{3.44}$$

with the bounds:

$$\|f_n\|_{L^p} \leq \eta^n\|f\|, \quad \|u_n\|_{W^{k,p}} \leq \eta^n(1 + C_1)C_3 T\|f\|_{L^p}, \quad \text{and} \quad \|w_n\|_{L^p} \leq \eta^n C_1\|f\|_{L^p}. \tag{3.45}$$

As $W^{k,p}(E_T)$ is complete and $\eta < 1$, the series $\sum u_n$ converges. Let $u = \sum_{n=0}^{\infty} u_n$. As each term of the series is orthogonal to $\mathscr{K}_T$, u belongs to the orthogonal space to $\mathscr{K}_T$ in $W^{k,p}(E_T)$. In the same way, the series $\sum w_n$ converges to an element $w \in \mathscr{K}_T^*$. It follows from the bounds (3.45) that we have:

$$\|u\|_{W^{k,p}} \leq \frac{(1 + C_1)C_2}{1 - \eta}T\|f\|_{L^p}, \quad \|w\|_{L^p} \leq \frac{C_1}{1 - \eta}\|f\|_{L^p}$$

Further the map $W^{k,p}(E_T) \to L^p(F_T)$ is continuous, and therefore we can sum over n in equality (3.44) to obtain $f = P_T u + w$.

This proves the existence part in Theorem 3.6 for f in the L^p range. For the uniqueness, remark that the index of P_T satisfies the inequality:

$$\operatorname{ind}(P_T) \geq \dim \mathscr{K}_T - \dim \mathscr{K}_T^*. \tag{3.46}$$

As the map $W^{k,p}(E_T) \to L^p(F_T)$ induced by P_T is Fredholm, the uniqueness of $u \in W^{k,p}(E_T)$ orthogonal to $\mathscr{K}_T$ and $w \in \mathscr{K}_T^*$ satisfying $f = P_T u + w$ is equivalent to proving that inequality (3.46) is in fact an equality. But the same reasoning applied to P_T^* yields:

$$\operatorname{ind}(P_T^*) \geq \dim \mathscr{K}_T^* - \dim \mathscr{K}_T.$$

Since $\operatorname{ind}(P_T^*) = -\operatorname{ind}(P_T)$, uniqueness in Theorem 3.6 follows.

To complete the proof of the theorem in the Sobolev range, it remains to remark that if one further assumes that $f \in W^{l,p}(F_T)$, Proposition 3.4 implies that

$$\|u\|_{W^{k+l,p}} \leq C(\|f\|_{W^{l,p}} + \|u\|_{L^p}) \leq C\|f\|_{W^{l,p}} + C'T\|f\|_{L^p}$$

for some constant $C' > 0$.

Remark 3.36. One of the advantages of treating the case of a root of order 1 first is that we proved that the Sobolev constant does not grow more than linearly with T, whereas in the general case it is more complicated to find the optimal rate of growth of the constant. This will be useful in our applications in Section 3.4 to derive the rate of decay of the low eigenvalues of the Laplacian.

Let us go back to the general case, before indicating how to modify our construction to treat the case of a single root of any order. We first prove our previous claim, that without any restrictions on the number of real roots of P_0 the characteristic system admits a solution if and only if f is orthogonal to the substitute cokernel:

Lemma 3.37. *For any $T \geq 1$ and any $f \in L^p(F_T)$ orthogonal to $\mathscr{K}_T^*$, the characteristic system (3.39) admits a solution $v \in \mathscr{E}$.*

Proof. Let us use the following notations for $u_1 \in \mathscr{K}_1$ and $g_1 \in \mathscr{K}_1^*$:

$$\kappa_{1,T}[u_1](t,x) = \kappa_1[u_1](t+T+1,x), \quad \kappa_{1,T}^*[u_1](t,x) = \kappa_1^*[u_1](t+T+1,x)$$

and for $u_2 \in \mathscr{K}_2$ and $g_2 \in \mathscr{K}_2^*$:

$$\kappa_{2,T}[u_2](t,x) = \kappa_2[u_2](t-T-1,x), \quad \kappa_{2,T}^*[u_2](t,x) = \kappa_2^*[u_2](t-T-1,x).$$

As the pairing $(\cdot,\cdot)$ is invariant by translation, it is still true that $\operatorname{im}\kappa_{i,T}$ is the orthogonal space to $\operatorname{im}\kappa_i^*$. Thus we may proceed exactly as above, choosing a complement $\mathscr{E}'_T$ of $\operatorname{im}\kappa_{1,T}\cap\operatorname{im}\kappa_{2,T}$ in $\mathscr{E}$, and complements $\mathscr{E}^*_{i,T}$ of $\operatorname{im}\kappa^*_{1,T}\cap\operatorname{im}\kappa^*_{2,T}$ in $\kappa^*_{i,T}$. Once these arbitrary choices are made we can decompose:

$$\mathscr{E}'_T = \mathscr{E}_{0,T} \oplus \mathscr{E}_{1,T} \oplus \mathscr{E}_{2,T}$$

where $\mathscr{E}_{i,T} = \operatorname{im}\kappa_{i,T} \cap \mathscr{E}'_T$ and $\mathscr{E}_{0,T}$ is the orthogonal space of $\mathscr{E}_{1,T} \oplus \mathscr{E}_{2,T}$ in $\mathscr{E}'_T$. Hence the non-degenerate pairing $\mathscr{E}'_T \times (\operatorname{im}\kappa_{1,T} + \operatorname{im}\kappa_{2,T}) \to \mathbb{C}$ induced by $(\cdot,\cdot)$ decomposes as the orthogonal sum of the non-degenerate pairings:

$$\mathscr{E}_{0,T} \times (\operatorname{im}\kappa_{1,T} \cap \operatorname{im}\kappa_{2,T}) \to \mathbb{C}, \quad \mathscr{E}_{1,T} \times \mathscr{E}^*_{2,T} \to \mathbb{C} \quad \text{and} \quad \mathscr{E}_{2,T} \times \mathscr{E}^*_{1,T} \to \mathbb{C}.$$

As in the proof of Proposition 3.34, for $i = 1,2$ the orthogonal space $\mathscr{K}^*_{i,+}$ of $\mathscr{K}_{i,0}$ in $\mathscr{K}_i$ is isomorphic to $\operatorname{im}\kappa^*_{i,T}$, so that the decomposition $\operatorname{im}\kappa^*_{i,T} = \operatorname{im}\kappa^*_{1,T} \cap \operatorname{im}\kappa^*_{2,T} \oplus \mathscr{E}^*_{i,T}$ induces a corresponding decomposition $\mathscr{K}^*_{i,+} = \mathscr{K}^*_{i,T,m} \oplus \mathscr{K}^*_{i,T,\perp}$. Thus if $f \in L^p(M_T)$ is orthogonal to the substitute cokernel $\mathscr{K}^*_T$ the characteristic system can be written as:

$$\begin{cases}(v_0, \kappa^*_{1,T}[g_0]) = \langle f, (1-\chi_{T+1}(\rho_1))g_0\rangle - \langle (1-\chi)f_0, \kappa^*_1[g_0]\rangle_0, & \forall g_0 \in \mathscr{K}^*_{1,T,m}\\ (v_1, \kappa^*_{1,T}[g_1]) = \langle f, (1-\chi_{T+1}(\rho_1))g_1\rangle - \langle (1-\chi)f_0, \kappa^*_1[g_1]\rangle_0, & \forall g_1 \in \mathscr{K}^*_{1,T,\perp}\\ (v_2, \kappa^*_{2,T}[g_2]) = -\langle f, (1-\chi_{T+1}(\rho_2))g_2\rangle + \langle (1-\chi)f_0, \kappa^*_2[g_2]\rangle_0, & \forall g_2 \in \mathscr{K}^*_{2,T,\perp}\end{cases}$$

where $v = v_0 + v_1 + v_2 \in \mathscr{E}_{0,T} \oplus \mathscr{E}_{1,T} \oplus \mathscr{E}_{2,T}$. Given the non-degeneracy of the above pairings this system is manifestly invertible. □

Despite the fact that we can solve the characteristic system whenever f is orthogonal to the substitute cokernel $\mathscr{K}^*_T$, this does not imply that we can find a solution $v \in \mathscr{E}$ with bounds of the form $\|v\| \le C(T)\|f\|_{L^p}$ with a good control on $C(T)$, which was a key argument in the previous construction. This is due to the fact that the characteristic system is in general underdetermined, and only becomes determined after a choice of arbitrary complements $\mathscr{E}'_T$ of $\operatorname{im}\kappa_{1,T} \cap \operatorname{im}\kappa_{2,T}$ in $\mathscr{E}$ and $\mathscr{E}^*_{i,T}$ of $\operatorname{im}\kappa^*_{1,T} \cap \operatorname{im}\kappa^*_{2,T}$ in $\operatorname{im}\kappa^*_{i,T}$, using the notations introduced in the above proof. In the case where P_0 has a single root of order 1, this was not problematic as we could simply make any arbitrary choice independently of T, but in general we cannot make such a consistent choice. This is especially true at values of T where the rank of the characteristic system drops.

As we discussed in §3.1.2, if P_0 has only one real root, then in a good basis the coefficients of the characteristic system are polynomial in T. Therefore, the rank of the system is constant whenever T is large enough and we can fix a complement of its kernel independent of T. On this complement, the system can be inverted with polynomial control on the norm. Thus if $f \in L^p(F_T)$ is orthogonal to the substitute cokernel we can find solutions of the characteristic system with $\|v\| \leq CT^\beta \|f\|_{L^p}$ for some exponent $\beta > 0$. In the same way, all the matching conditions can be expressed as linear equations with coefficients depending polynomially on T. Therefore, the norm of the L^2-orthogonal projections onto $\mathscr{K}_T$ and $\mathscr{K}_T^*$ do not grow more than polynomially. Then we can use the same argument as above to prove Theorem 3.6 in the general case.

In fact, any assumptions ensuring that we can invert the characteristic system with less than exponential growth on the norm after fixing complements of its image and kernel, and that the norm of the projections onto $\mathscr{K}_T$ and $\mathscr{K}_T^*$ do not grow too quickly, would yield the same result.

3.4 Spectral aspects

In this section, we want to interpret our results from a spectral perspective. Indeed, for self-adjoint operators the approximate kernel can be regarded as a finite-dimensional space associated with very low eigenvalues of the operator P_T. For the Laplacian, we shall see in §3.4.1 that the substitute kernel is a good approximation of the space of harmonic forms. Orthogonally to the space of harmonic forms, the results of the previous section imply a bound in $O(T^2)$ on the L^2-norm of the inverse of Δ_T. In particular if 0 is a root of the Laplacian acting on q-forms on $\mathbb{R} \times X$, then the lowest non-zero eigenvalues of Δ_T acting on q-forms admit a lower bound of the form $\frac{C}{T^2}$. In §3.4.2 we study the density of eigenvalues with fastest decay rate and prove Theorem 3.8.

3.4.1 Approximate harmonic forms. We begin with a review of standard properties of the Laplacian on EAC manifolds (see for instance [93, 6.4]). Let (Z, g) be an oriented EAC Riemannian manifold of rate $\mu > 0$, let $Y = \mathbb{R} \times X$ be its asymptotic cylinder and ρ be a cylindrical coordinate function. The space $\mathscr{H}^q$ of bounded closed and co-closed q-forms is equal to the space of bounded harmonic q-forms. Moreover, there is a direct sum decomposition:

$$\mathscr{H}^q = \mathscr{H}_0^q \oplus \mathscr{H}_d^q \oplus \mathscr{H}_{d^*}^q$$

where $\mathscr{H}_0^q$ is the space of decaying harmonic q-forms, $\mathscr{H}_d^q$ is the space of bounded *exact* harmonic q-forms and $\mathscr{H}_{d^*}^q$ the space of bounded *co-exact* harmonic q-forms. On the other hand, the map κ_q mapping a bounded harmonic q-form to its translation-invariant expansion at infinity induces two maps

$$\alpha_q : \mathscr{H}^q \to H^q(X), \quad \beta_q : \mathscr{H}^q \to H^{q-1}(X)$$

such that $\kappa_q(\eta) = \alpha_0 + dt \wedge \beta_0$ where α_0 and β_0 are the harmonic representatives of $\alpha_q(\eta)$ and $\beta_q(\eta)$ respectively. The map α_q can be factorised as $\mathscr{H}^q \to H^q(Z) \to H^q(X)$, where any q-form in $\mathscr{H}^q$ is mapped to its de Rham cohomology class in $H^q(Z)$ and the map $H^q(Z) \to H^q(X)$ comes from the long exact sequence of the pair (Z, X). By [9, Proposition 4.9], $\mathscr{H}_0^q$ is mapped isomorphically to the image of the map $H_c^q(Z) \to H^q(Z)$ coming from the same exact sequence. In particular this implies that $\mathscr{H}_0^q \oplus \mathscr{H}_d^q \subset \ker \alpha_q$, and by considering Hodge duals it follows that $\mathscr{H}_0^q \oplus \mathscr{H}_{d^*}^q \subset \ker \beta_q$. As the kernel of the map κ_q is $\mathscr{H}_0^q$ this implies that

$$\ker \alpha_q = \mathscr{H}_0^q \oplus \mathscr{H}_d^q, \quad \ker \beta_q = \mathscr{H}_0^q \oplus \mathscr{H}_{d^*}^q.$$

By [93, Proposition 6.18], the map $\mathscr{H}_0^q \oplus \mathscr{H}_{d^*}^q \to H^q(Z)$ is an isomorphism, and α_q maps $\mathscr{H}_{d^*}^q$ isomorphically onto the image of the map $H^q(Z) \to H^q(X)$ coming from the long exact sequence of (Z, X).

Let $0 \leq q \leq \dim Z$ and denote by σ_q the minimum of μ and of the square roots of the lowest eigenvalues of the Laplacian acting on $(q-1)$- and q-forms on X. Any bounded closed and co-closed q-form η on Z is asymptotic to a translation-invariant form $\eta_0 = \alpha_0 + dt \wedge \beta_0$, up to terms in $O(e^{-\delta\rho})$ for any $\delta < \sigma_q$. With the above notations (α_0, β_0) are the harmonic representatives of $(\alpha_q(\eta), \beta_q(\eta))$. It is a standard fact that there exists a $(q-1)$-from ξ on Z such that $\eta - \eta_0 = d\xi$ in the domain $\{\rho > 1\}$, with $|\nabla^l \xi| = O(e^{-\delta\rho})$ for any $l \geq 0$ and $\delta < \sigma_q$. A suitable ξ can be constructed as follows. Identify the region $\{\rho > 1\}$ with the cylinder $(1, \infty) \times X$, and write $\eta - \eta_0 = \alpha(t) + dt \wedge \beta(t)$ where α, β and all their derivatives have the usual exponential decay. As η and η_0 are closed this implies:

$$d_X \alpha(t) = 0 = \partial_t \alpha(t) - d_X \beta(t)$$

for all $t > 1$, where d_X denotes the exterior differential on X. Hence we can define ξ in the domain $\{\rho > 1\}$ by

$$\xi(t, x) = \int_{+\infty}^{t} \beta(\tau, x) d\tau, \quad \forall (t, x) \in (1, \infty) \times X.$$

This $(q-1)$-form ξ allows us to build a 1-parameter family of closed q-forms

$$\eta_T = \eta - d(\chi_T(\rho)\xi)$$

interpolating between η when $\rho < T - \frac{1}{2}$ and η_0 when $\rho > T + \frac{1}{2}$, which all represent the cohomology class of η in $H^q(Z)$. Moreover, the difference $\eta_T - \eta$ and all its derivatives satisfy uniform bounds in $O(e^{-\delta T})$ for any $0 < \delta < \sigma_q$.

Let (Z_1, g_1) and (Z_2, g_2) be two matching EAC manifolds, and consider the 1-parameter family of compact Riemannian manifolds (M_T, g_T) obtained by the gluing procedure explained in §3.1.1. We want to control the mapping properties of the associated operators $d + d_T^*$ and Δ_T as $T \to \infty$. Strictly speaking, these operators differ from the operators obtained by gluing $d + d_1^*$ with $d + d_2^*$ and Δ_1 with Δ_2 in the gluing region $\{|\rho_T| \leq \frac{3}{2}\}$. Nevertheless, the results of the previous parts still apply as the coefficients of the difference and and all their derivatives have exponential decay with T. It is convenient to slightly modify our definition of approximate kernel. If (η_1, η_2) is a matching pair of harmonic forms, define a closed form on M_T by:

$$\eta_T = \begin{cases} \eta_{1,T} & \text{if } \rho_T \leq -\frac{1}{2} \\ \eta_{2,T} & \text{if } \rho_T \geq \frac{1}{2} \\ \eta_0 & \text{if } |\rho_T| \leq \frac{1}{2} \end{cases}$$

where both η_1 and η_2 are asymptotic to η_0 and $\eta_{1,T}$ and $\eta_{2,T}$ are closed forms constructed as above. It follows that η_T is closed. We denote by $\mathscr{H}_T^q$ the finite-dimensional space of q-forms constructed as above from a pair of matching q-forms. Again, this differs from our previous definition of substitute kernel only up to terms that are bounded in $O(e^{-\delta T})$ as well as all their derivatives for any $\delta < \sigma_q$. Hence our results, and in particular Theorem 3.6, still apply. As the elements of $\mathscr{H}_T^q$ are closed there is a well-defined map:

$$\mathscr{H}_T^q \to H^q(M_T)$$

sending every element to its de Rham cohomology class. The key point is the following theorem [96, Theorem 3.1]:

Theorem 3.38. *For T large enough, the map $\mathscr{H}_T^q \to H^q(M_T)$ is an isomorphism.*

Let us briefly sketch the proof of this theorem. It relies on a close examination of the Mayer-Vietoris sequence:

$$\ldots \to H^{q-1}(Z_1) \oplus H^{q-1}(Z_2) \to H^{q-1}(X) \to H^q(M_T) \to H^q(Z_1) \oplus H^q(Z_2) \to \ldots$$

As the space of approximate harmonic q-forms $\mathscr{H}_T^q$ is isomorphic to the space

$$H_c^q(Z_1) \oplus H_c^q(Z_2) \oplus \operatorname{im} \alpha_{1,q} \cap \operatorname{im} \alpha_{2,q} \oplus \operatorname{im} \beta_{1,q} \cap \operatorname{im} \beta_{2,q}$$

and since $\ker \beta_{i,q} \simeq H^q(Z_i)$, it is clear that the restriction of $\mathscr{H}_T^q \to H^q(M_T)$ to the space obtained by matching pairs in $\mathscr{H}_{i,0}^q \oplus \mathscr{H}_{i,d^*}^q$ yields an isomorphism:

$$H_c^q(Z_1) \oplus H_c^q(Z_2) \oplus \operatorname{im} \alpha_{1,q} \cap \operatorname{im} \alpha_{2,q} \simeq \operatorname{im}(H^q(M_T) \to H^q(Z_1) \oplus H^q(Z_2)).$$

Moreover, the subspace of $\mathscr{H}_T^q$ obtained by gluing matching pairs of bounded exact harmonic q-forms, which is isomorphic to $\operatorname{im} \beta_{1,q} \cap \operatorname{im} \beta_{2,q}$, maps into the image of

$H^{q-1}(X) \to H^q(M_T)$. By Lemma 3.26, $\operatorname{im}\beta_{1,q} \cap \operatorname{im}\beta_{2,q}$ is the orthogonal space of $\operatorname{im}\alpha_{1,q-1} \oplus \operatorname{im}\alpha_{2,q-1}$ for the inner product induced by the L^2-product on harmonic representatives, and therefore $\operatorname{im}\beta_{1,q} \cap \operatorname{im}\beta_{2,q}$ has the same dimension as the kernel of $H^q(M_T) \to H^q(Z_1) \oplus H^q(Z_2)$. Thus it only remains to prove that the subspace of $\mathscr{H}^q_T$ obtained by gluing matching pairs of exact q-forms maps isomorphically onto the kernel of the map $H^q(M_T) \to H^q(Z_1) \oplus H^q(Z_2)$ coming from the exact sequence. In [94, Theorem 3.1] it is proven that this is the case for T large enough.

Alternatively, one could also argue using Theorem 3.6. The spaces $\mathscr{H}^q_T$ and $H^q(M_T)$ have same dimension by the above argument. Moreover, the Laplacian Δ_T maps the orthogonal space of $\mathscr{H}^q_T$ in $W^{2,2}(\Lambda^q T^* M_T)$ isomorphically onto a complement of $\mathscr{H}^q_T$ in $L^2(\Lambda^q T^* M_T)$, for T large enough. Hence the map $\mathscr{H}^q_T \to H^q(M_T)$ must be an isomorphism for large T, since otherwise there would be a non-trivial exact form in $\mathscr{H}^q_T$ and the image of the Laplacian would have codimension strictly less than $b^q(M_T)$ in $L^2(\Lambda^q T^* M_T)$.

As a consequence, the L^2-projection of the space $\mathscr{H}^q_T$ of approximate harmonic q-forms onto the space $\mathscr{H}^q(M_T)$ of genuine harmonic q-forms is an isomorphism for T large enough. It is natural to ask how close to their harmonic part the elements of $\mathscr{H}^q_T$ are. If $\eta \in \mathscr{H}^q_T$ is decomposed in harmonic and exact parts $\eta = \xi + d\nu$, then:

$$\|\Delta_T d\nu\|_{L^2} = \|\Delta_T(\eta - \xi)\|_{L^2} = \|\Delta_T \eta\|_{L^2} = O\left(e^{-\delta T}\|\eta\|_{L^2}\right)$$

for any $\delta < \sigma_q$. By Theorem 3.6, there exists a $(q-1)$-form η' with $\Delta_T \eta' = \Delta_T d\nu$ and satisfying a bound of the form

$$\|\eta'\|_{W^{2,2}} \leq C T^\beta \|\Delta_T d\nu\|_{L^2} \leq C' e^{-\delta T}\|\eta\|_{L^2}$$

for some constant C'. As $\eta' - d\nu$ is harmonic it follows that $\|d\nu\|_{L^2} \leq \|\eta'\|_{L^2}$, which yields:

$$\|\eta - \xi\|_{L^2} = O\left(e^{-\delta T}\|\eta\|_{L^2}\right)$$

for any $\delta < \sigma_q$. Thus not only is the L^2-projection of $\mathscr{H}^q_T$ onto $\mathscr{H}^q(M_T)$ an isomorphism, but the norm of the projection is close to 1, up to $O(e^{-\delta T})$ terms. Once this inequality is established in L^2, the a priori estimates of Proposition 3.4 imply that

$$\|\eta - \xi\|_{W^{l,2}} = O\left(e^{-\delta T}\|\eta\|_{L^2}\right)$$

for any $l \geq 0$. Then, the Sobolev embedding theorem (see Proposition 3.5) yields estimates:

$$\|\eta - \xi\|_{W^{l,p}} = O\left(e^{-\delta T}\|\eta\|_{L^p}\right), \quad \|\eta - \xi\|_{C^l} = O\left(e^{-\delta T}\|\eta\|_{C^0}\right)$$

for any $l \geq 0$, $p > 1$ and $0 < \delta < \sigma_q$.

By the same bootstrapping argument, we can prove that if $\nu \in W^{l,p}(\Lambda^q T^* M_T)$ is orthogonal to $\mathscr{H}_T^q$ and ν' is the unique q-form orthogonal to $\mathscr{H}^q(M_T)$ such that $\Delta_T \nu = \Delta_T \nu'$ (or equivalently $(d + d_T^*)\nu = (d + d_T^*)\nu'$) then

$$\|\nu - \nu'\|_{W^{l,p}} = O(e^{-\delta T}\|\nu\|_{W^{l,p}}).$$

These remarks, the fact that the Laplacian Δ_T is the square of the operator $d + d_T^*$ whose only root has order 1 and Theorem 3.6 imply the following:

Corollary 3.39. *Let $p > 1$ and $l \in \mathbb{N}$, and assume that 0 is an indicial root of the Laplacian action on q-forms on Y, that is $b^{q-1}(X) + b^q(X) > 0$. Then there exist constants $C, C' > 0$ such that, for large enough T and any $\eta \in W^{l,p}(\Lambda^q T^* M_T)$ orthogonal to $\mathscr{H}^q(M_T)$, the unique solution $\eta' \in W^{2+l,p}(\Lambda^q T^* M_T)$ of $\Delta \eta' = \eta$ orthogonal to $\mathscr{H}^q(M_T)$ satisfies:*

$$\|\eta'\|_{W^{l+2,p}} \leq C\|\eta\|_{W^{l,p}} + C' T^2 \|\eta\|_{L^p}.$$

Remark 3.40. If instead we assume that $b^{q-1}(X) + b^q(X) = 0$, the results of [81] imply uniform bounds (independent of T) for the Green's function of the Laplacian (see Remark 3.7). Hence in that case the result would hold with $C' = 0$.

Proof. Let us consider the operator $d + d_T^*$ acting on $\Lambda^\bullet T^* M_T$. Since the only real root of $d + d_T^*$ is 0 and this is a root of order 1, we saw in §3.3.3 that in this case Theorem 3.6 holds with $\beta = 1$. Moreover, for T large enough the L^2-projection of $\mathscr{H}_T$ on the space of harmonic forms $\mathscr{H}$ is an isomorphism, and therefore $\mathscr{H}_T$ is a linear complement of the image of $d + d_T^*$ in $W^{l,p}$. Thus by Theorem 3.6 any differential form η of regularity $W^{l,p}$ which is orthogonal to the space of harmonic forms can be written as

$$\eta = (d + d_T^*)\nu$$

for a unique differential form ν of regularity $W^{l+1,p}$ orthogonal to the space of approximate harmonic forms $\mathscr{H}_T$, which satisfies bounds of the form

$$\|\nu\|_{W^{l+1,p}} \leq C(\|\eta\|_{W^{l,p}} + T\|\eta\|_{L^p}), \quad \|\nu\|_{L^p} \leq \|\nu\|_{W^{1,p}} \leq C' T \|\eta\|_{L^p}$$

where in the second equality we use the case $l = 0$ of the theorem. By the previous remarks, the unique differential form ν' orthogonal to $\mathscr{H}(M_T)$ satisfying $(d + d_T^*)\nu' = \eta$ satisfies bounds of the form $\|\nu - \nu'\|_{W^{k,p}} \leq C_k e^{-\delta T}\|\nu\|_{W^{k,p}}$ for any $k \geq 0$ and small enough $\delta > 0$, and thus for large enough T we also have

$$\|\nu'\|_{W^{l+1,p}} \leq C(\|\eta\|_{W^{l,p}} + T\|\eta\|_{L^p}), \quad \|\nu'\|_{L^p} \leq \|\nu'\|_{W^{1,p}} \leq C' T \|\eta\|_{L^p}$$

for some constants C, C', possibly different from the previous ones but independent of T. Iterating this argument, the unique differential form η' orthogonal to $\mathscr{H}(M_T)$ such that $(d + d_T^*)\eta' = \nu'$, which is a solution of $\Delta_T \eta' = \eta$, satisfies the bounds

$$\|\eta'\|_{W^{l+2,p}} \leq C(\|\nu'\|_{W^{l+1,p}} + T\|\nu'\|_{L^p}) \leq C'\|\eta\|_{W^{l,p}} + C''T^2\|\eta\|_{L^p}$$

for some constants C, C', C'' which do not depend on η and T large enough. □

Consequently, if $b^{q-1}(X) + b^q(X) > 0$ the lowest eigenvalue of Δ_T acting on q-forms satisfies a bound of the form $\lambda_1(T) \geq \frac{C}{T^2}$ as $T \to \infty$. In the next part we study the distribution of the eigenvalues that have the fastest decay rate, that is of order T^{-2}.

3.4.2 Density of low eigenvalues. We want bounds on the densities $\Lambda_{q,\inf}(s)$, $\Lambda_{q,\sup}(s)$ of low eigenvalues of the Laplacian Δ_T acting on q-forms defined in §3.1.3. When $b^{q-1}(X) + b^q(X) = 0$, the Laplacian acting on q-forms does not admit any real root, and thus it has no decaying eigenvalues. From now on we assume that $b^{q-1}(X) + b^q(X) > 0$. We shall prove Theorem 3.8 using a min-max principle.

The easiest part, which does not require the results of Section 3.3, is to find a lower bound for $\Lambda_{q,\inf}(s)$. Let us denote by $0 < \lambda_1(T) \leq \ldots \leq \lambda_n(T) \leq \ldots$ the non-decreasing sequence of positive eigenvalues of the Laplacian, counted with multiplicity. The n-th eigenvalue (counted with multiplicity) is determined by:

$$\lambda_n(T) = \min\left\{\max\left\{\frac{\|\Delta_T\eta\|_{L^2}}{\|\eta\|_{L^2}}, \eta \in V\backslash\{0\}\right\},\ V \subset W^{2,2}(\Lambda^q T^* M_T),\ \dim V = n\right\}$$

where V ranges over spaces orthogonal to harmonic forms. Using this we claim:

Lemma 3.41. *Let $V \subset C^2([-1,1], \mathbb{C})$ be an n-dimensional space of functions such that $f(-1) = f(1) = f'(-1) = f'(1) = 0$ for all $f \in V$. Let $A > 0$ such that for all non-zero $f \in V$ we have:*

$$\int_{-1}^{1} |f''(t)|^2 dt < A^2 \int_{-1}^{1} |f(t)|^2 dt.$$

Then for T large enough $\lambda_{(b^{q-1}(X)+b^q(X))n-b^q(M_T)}(T) \leq \frac{A}{T^2}$.

Proof. Any $f \in V$ can be extended as a C^1 function to $\mathbb{R}$ by setting $f(t) = 0$ for any $|t| \geq 1$. With this extension, $f \in W^{2,2}(\mathbb{R})$ and $f'' \in L^2(\mathbb{R})$ vanishes outside of $[-1,1]$ and is equal to the usual second derivative inside this interval. Let us choose $0 < \tau < 1$ small enough so that:

$$\int_{-1}^{1} |f''(t)|^2 dt < A^2(1-\tau)^4 \int_{-1}^{1} |f(t)|^2 dt \tag{3.47}$$

for any $f \in V$. For $T \geq 1$, let V_T be the subspace of $W^{2,2}(\Lambda^q T^*_{\mathbb{C}} Y)$ spanned by sections of the form:

$$\eta(t,x) = f\left(\frac{t}{(1-\tau)T}\right)\nu(x)$$

where $f \in V$ and ν is a translation-invariant harmonic form on Y. In particular, it has dimension $\dim V_T = (b^{q-1}(X) + b^q(X))n$ over $\mathbb{C}$. As the elements of V_T vanish outside of the finite cylinder $[-(t-\tau)T, (1-\tau)T] \times X$, it can be identified with a subspace of $W^{2,2}(\Lambda^q T^*_{\mathbb{C}} M_T)$. On the support of the elements of V_T, the Laplacian Δ_T and the metric g_T approach $\Delta_0 = \Delta_X - \partial_t^2$ and $g_0 = g_X + dt^2$ up to terms in $O(e^{-\delta\tau T})$ and similarly for all derivatives, for some $\delta > 0$ appropriately small. Therefore, there exist constants $C, C' > 0$ such that:

$$\sup_{\eta \in V_T \backslash \{0\}} \frac{\|\Delta_T \eta\|_{L^2}}{\|\eta\|_{L^2}} \leq \frac{(1 + Ce^{-\delta\tau T})}{(1-\tau)^2 T^2} \sup_{f \in V \backslash \{0\}} \frac{\|f''\|_{L^2}}{\|f\|_{L^2}} + C' e^{-\delta\tau T} \sup_{\eta \in V_T \backslash \{0\}} \frac{\|\eta\|_{W^{2,2}}}{\|\eta\|_{L^2}}.$$

As the ratio between the $W^{2,2}$-norm and the L^2-norm on V_T does not grow more than polynomially with T, the second term in the right-hand-side has exponential decay. On the other hand, by (3.47) the first term is less than $\frac{(A-\epsilon)}{T^2}$ for large enough T and small enough $\epsilon > 0$. This proves the lemma. □

We can apply the above lemma to the spaces:

$$V_n = \left\{ \sum_{1 \leq |k| \leq n} a_k e^{ik\pi t}, \sum_{1 \leq |k| \leq n} (-1)^k a_k = \sum_{1 \leq |k| \leq n} (-1)^k k a_k = 0 \right\} \tag{3.48}$$

For $n \geq 2$, the space V_n has dimension $2n-2$ and for any non-zero $f \in V_n$ we have:

$$\int_{-1}^{1} |f''(t)|^2 dt < (n\pi)^2 \int_{-1}^{1} |f(t)|^2 dt$$

The above lemma yields the inequality:

$$\Lambda_{q,\inf}(s) \geq (2\lfloor\sqrt{s}\rfloor - 2)(b^{q-1}(X) + b^q(X)) - b^q(M_T), \quad \forall s \geq 1. \tag{3.49}$$

We now want an upper bound on $\Lambda_{q,\sup}(s)$. Let us denote by

$$G_T : L^2(\Lambda^q T^* M_T) \cap \mathscr{H}^q(M_T)^\perp \to L^2(\Lambda^q T^* M_T) \cap \mathscr{H}^q(M_T)^\perp$$

the composition of the inverse of the Laplacian acting on $W^{2,2}(\Lambda^q T^* M_T) \cap \mathscr{H}^q(M_T)^\perp$ with the compact embedding $W^{2,2} \hookrightarrow L^2$. The eigenvalues $\lambda_{n+1}(T)$ can be characterised by:

$$\lambda_{n+1}^{-1}(T) = \min\left\{ \max\left\{ \frac{\|G_T \eta\|_{L^2}}{\|\eta\|_{L^2}}, \eta \in V \backslash \{0\} \right\}, V \subset L^2(\Lambda^q T^* M_T), \operatorname{codim} V = n \right\}$$

where moreover V ranges over closed subspaces orthogonal to $\mathscr{H}^q(M_T)$. Since we developed an explicit construction of solutions to the equation $\Delta_T\nu = \eta$, the idea is to show that if we impose enough orthogonality conditions to $\eta \in L^2(\Lambda^q T^* M_T)$, and not only orthogonality to the space of harmonic forms (or to the substitute kernel), we can give explicit bounds for the norm of $G_T\eta$.

Let us denote by N the sum of the dimensions of the spaces of harmonic forms with at most polynomial growth on Z_1 and Z_2. Moreover, denote by $E \subset L^2([-1,1],\mathbb{C})$ the closed subspace of functions $f(t) = \sum a_k e^{ik\pi t}$ which satisfy:

$$a_0 = 0, \quad \sum_{|k|\geq 1} (-1)^k \frac{a_k}{k} = \sum_{|k|\geq 1} (-1)^k \frac{a_k}{k^2} = 0.$$

Thus E is the intersection of the kernels of 3 independent continuous linear forms on $L^2([-1,1],\mathbb{C})$, and therefore has codimension 3. For $f \in L^2(\mathbb{R},\mathbb{C})$ with compact essential support let us define:

$$Hf(t) = \int_{-\infty}^{t} (\tau - t) f(\tau) d\tau.$$

The first two conditions in the definition of E imply that for any $f \in E$ on has:

$$\int_{-1}^{1} f(\tau) d\tau = \int_{-1}^{1} \tau f(\tau) d\tau = 0. \tag{3.50}$$

On the other hand, the last condition is a matter of scaling under change of variables. Since we have

$$\int_{-T}^{t} (\tau - t) e^{\frac{ik\pi\tau}{T}} dt = \frac{T^2}{(k\pi)^2} e^{\frac{ik\pi t}{T}} + (-1)^k \left(\frac{T(T+t)}{ik\pi} - \frac{T^2}{(k\pi)^2} \right)$$

if follows that for any $f(t) = \sum a_k e^{ik\pi t} \in E$, the function $f_T(t) = f\left(\frac{t}{T}\right)$ satisfies:

$$Hf_T(t) = \frac{T^2}{\pi^2} \sum_{|k|\geq 1} \frac{a_k}{k^2} e^{\frac{ik\pi t}{T}} \tag{3.51}$$

for any $-T \leq t \leq T$.

Bearing this in mind, we shall find an upper bound on $\Lambda_{q,\sup}(s)$ with the help of the following technical lemma:

Lemma 3.42. *Let $V \subset E$ be a closed subspace of codimension n, and let $B, \epsilon > 0$ such that for all $f \in V$ we have:*

$$\int_{-1}^{1} |Hf(t)|^2 dt \leq \frac{1}{(B+\epsilon)^2} \int_{-1}^{1} |f(t)|^2 dt.$$

Then for T large enough $\lambda_{(b^{q-1}(X)+b^q(X))(n+3)+N}(T) \geq \frac{B}{T^2}$.

Proof. The idea is to follow the construction of §3.3.3 to build a solution of $\Delta_T \nu = \eta$, where η is a complex q-form orthogonal to the space of harmonic forms, and showing that if we assume sufficiently many orthogonality conditions we can give a precise bound on the constant C such that $\|\nu\|_{L^2} \leq CT^2\|\eta\|_{L^2}$. To do this we need to introduce a parameter $\tau > 0$ and replace the cutoffs ζ_0 and ζ_1 (see §3.3.2) by ζ_τ and $\zeta_{\tau+1}$.

Let us define $\eta_\tau = \zeta_{\tau+1}\eta$, considered as a q-form on the cylinder $Y = \mathbb{R} \times X$ supported in the cylinder $[-T+\tau, T-\tau] \times X$. We pick a basis $\eta_1, \ldots, \eta_m$ of the space of translation-invariant harmonic q-forms on Y, orthonormal for the L^2-product on X. Then we may write:

$$\eta_\tau(t,x) = \widetilde{\eta}_\tau(t,x) + \sum_{j=1}^{m} f_{\tau,j}(t)\eta_j(x)$$

with $\widetilde{\eta}_\tau$ orthogonal to any function of the form $f(t)\eta_j(x)$ with f compactly supported smooth function, and $f_{\tau,j} \in L^2([-T+\tau, T-\tau], \mathbb{C})$. Moreover the solution $\nu_\tau = Q\eta_\tau$ of $\Delta_0 \nu = \eta_\tau$ provided by Theorem 3.11 can be written as (see Ex. 3.24):

$$\nu_\tau = Q_r[\eta_\tau] + \sum_{j=1}^{m} Hf_{j,\tau} \cdot \eta_j.$$

with Q_r defined as in §3.2.3. Let us assume that each of the functions

$$t \in [-1,1] \mapsto f_{j,\tau}((T-\tau)t)$$

belongs to $V \subset E$. This imposes $(b^{q-1}(X)+b^q(X))(n+3)$ orthogonality conditions on η. Given (3.50), the L^2-functions $Hf_{j,\tau}$ vanish outside of $[-T+\tau, T-\tau]$, and therefore $\nu_\tau \in L^2(\Lambda^q T^*_{\mathbb{C}} Y)$ and from (3.51) it satisfies:

$$\|\nu_\tau\|_{L^2} \leq C\|\eta_\tau\|_{L^2} + \frac{(T-\tau)^2}{B+\epsilon}\|\eta_\tau\|_{L^2} \leq \frac{T^2}{B+\epsilon}\|\eta_\tau\|_{L^2} \tag{3.52}$$

for large enough T. Let us consider $\zeta_\tau \nu_\tau$ as a section of $\Lambda^q T^*_C M_T$ supported in the neck region. As such, there exists a constant $C > 0$ independent of τ such that:

$$\|\zeta_\tau \nu_\tau\|_{L^2} \leq \frac{1+Ce^{-\delta\tau}}{B+\epsilon} T^2 \|\eta\|_{L^2}$$

Following the method of §3.3.2, we can write:

$$\eta - \Delta_T(\zeta_\tau \nu\tau) = \eta_1 + \eta_2$$

with $\eta_i \in L^2_{\delta'}(\Lambda^q T^*_C Z_i)$, for some small $\delta' > 0$ that we fix. Moreover, we can argue as in the proof of Lemma 3.28 to show the bounds:

$$\|\eta_i\|_{L^2_{\delta'}} \leq Ce^{\delta'\tau}\|\eta\|_{L^2} + C'T^2 e^{-\delta\tau}\|\eta\|_{L^2} \leq C''T^2 e^{-\delta\tau}\|\eta\|_{L^2} \tag{3.53}$$

for T large enough, where δ, δ' and τ are fixed, and C'' does not depend on any of these choices. Up to $O(e^{-\delta T})$ terms, the vanishing of the obstructions to solving $f_i = \Delta_i \nu_i$ with $\nu_i \in W^{2,2}_{\delta'}$ can be expressed as the vanishing of N linear forms (this is to say that the coefficients of the characteristic system are linear in $\eta \in L^2$). Thus imposing N additional orthogonality conditions on η, we can use the same argument as in Proposition 3.34 to show that there exists $\nu' \in W^{2,2}$, $\eta' \in L^2$ such that $\eta - \Delta_T \nu' = \eta'$ with $\|\eta'\|_{L^2} \leq CT^2 e^{-\delta' T}\|\eta\|_{L^2}$ for some constant C' possibly depending on δ' but not on τ or T. From (3.52) and (3.53) we can deduce:

$$\|\nu'\|_{L^2} \leq \left(\frac{1 + Ce^{-\delta\tau}}{B+\epsilon} + C'e^{-\delta\tau}\right) T^2 \|\eta\|_{L^2}$$

for large enough T. On the other hand, as η is by assumption orthogonal to the space of harmonic forms, so is η' and by Corollary 3.39 there exists ν'' such that $\Delta_T \nu'' = \eta'$ with a bound:

$$\|\nu''\|_{L^2} \leq CT^2 \|\eta'\|_{L^2} \leq C'' T^4 e^{-\delta' T} \|\eta\|_{L^2}$$

for some constant C'' which does not depend on τ or on T large enough. Thus if $\nu = \nu' + \nu''$ we have $\Delta_T \nu = \eta$ with a universal bound:

$$\|\nu\|_{L^2} \leq \left(\frac{1 + Ce^{-\delta\tau}}{B+\epsilon} + C'e^{-\delta\tau} + C''T^2 e^{-\delta' T}\right) T^2$$

for some constants C, C', C'' that may depend on the choices of δ, δ' but not on τ and T. As $\|G_T \eta\|_{L^2} \leq \|\nu\|_{L^2}$ it follows that if τ and T are large enough then

$$\|G_T \eta\|_{L^2} \leq \frac{T^2}{B} \|\eta\|_{L^2}.$$

This inequality holds true provided η satisfies all the orthogonality conditions described above, which define a closed subspace of codimension no more than $(b^{q-1}(X) + b^q(X))(n+3) + N$ in the orthogonal space to harmonic forms in $L^2(\Lambda^q T^*_{\mathbb{C}} M_T)$. The lemma follows. □

To use this lemma we consider the subspaces $V'_n \subset E$ defined by:

$$V'_n = \left\{ f(t) = \sum a_k e^{ik\pi t} \in E,\ a_k = 0\ \forall |k| \leq n \right\}. \tag{3.54}$$

The space V'_n has codimension $2n$ in E, and for any $f \in V'_n$, (3.51) implies:

$$\int_{-1}^{1} |Hf(t)|^2 dt^2 \leq \frac{1}{(n+1)^4 \pi^4} \int_{-1}^{1} |f(t)|^2 dt$$

Hence we have an upper bound:

$$\Lambda_{q,\sup}(s) \leq 2(\lfloor \sqrt{s} \rfloor + 3)(b^{q-1}(X) + b^q(X)) + N.$$

Together with the bound on $\Lambda_{q,\inf}(s)$ this proves the first part of Theorem 3.8.

In order to prove the second assertion in Theorem 3.8, let us consider the subset $W_n \subset V_n$ defined by:

$$W_n = \left\{ \sum_{1 \le |k| \le n} a_k e^{ik\pi t} \in V_n, \sum_{1 \le |k| \le n} (-1)^k k^2 a_k = 0 \right\} \tag{3.55}$$

seen as a subspace of $C^3([-1,1],\mathbb{C})$. Any $f \in W_n$ can be extended as a C^2-function on $\mathbb{R}$, with $f' \in V_n$. Moreover if β is a harmonic $(q-1)$-form then $d(f\beta) = f' dt \wedge \beta$. Using this, we can deduce that the density of low eigenvalues of the Laplacian acting on exact q-forms, which we define as the density of low eigenvalues, satisfies:

$$\Lambda^e_{q,\inf}(s) \ge 2b^{q-1}(X)\sqrt{s} - N_q$$

for some constant $N_q \ge 0$. By Hodge duality, this implies the lower bound:

$$\Lambda^*_{q,\inf}(s) \ge 2b^q(X)\sqrt{s} - N_{\dim M_T - q}.$$

As we know that $\Lambda^*_{q,\sup}(s) + \Lambda^e_{q,\sup}(s) \le 2(b^{q-1}(X) + b^q(X))\sqrt{s} + O(1)$, this means that when $b^q(X) \ne 0$ we have:

$$\Lambda^*_{q,\sup}(s) = \Lambda^*_{q,\inf}(s) + O(1) = 2b^q(X)\sqrt{s} + O(1).$$

Chapter 4

Geometry and incompleteness of the moduli spaces

In this chapter, we begin our study of the geometry of G_2-moduli spaces, which will be the main theme for the remainder of this dissertation. Throughout this chapter, we will denote by M a compact 7-manifold admitting torsion-free G_2-structures, by $\mathscr{M}$ the moduli space of torsion-free G_2-structures on M and by $\mathscr{G}$ the volume-normalised L^2-metric on $\mathscr{M}$. Our main interest is the case of manifolds with full holonomy, which necessarily have finite fundamental group, but we will only make the weaker assumption $b^1(M) = 0$ unless otherwise stated.

Under this assumption, the metric $\mathscr{G}$ has the remarkable property of being the Hessian of a global potential function $\mathscr{F}$, with respect to the affine structure induced by the natural map $\mathscr{M} \to H^3(M)$. To the author's knowledge, this was first noticed in the physics literature [10, 54, 55, 61] following an observation of Hitchin [60] that the volume functional has non-degenerate (but indefinite) Hessian on the moduli spaces.

We shall introduce the potential function $\mathscr{F}$ in Section 4.1, where we present its basic properties and derive a few technical results which will be useful in the next chapters. The goal of the following two sections will be to prove that G_2-moduli spaces may be incomplete, even in the case of manifolds with full holonomy. This is based on the article [84] by the author, with some additions and slight improvements. In Section 4.2, we give sufficient conditions for a path in the moduli space to have finite length, and apply it to the generalised Kummer construction [64, 65]. In Section 4.3, we make further observations on the incompleteness question, and tackle the cases of the resolution of isolated conical singularities [72] and the Joyce–Karigiannis construction [68]. We also briefly discuss the general resolution of flat G_2-orbifolds using R-data from Joyce's monograph [66, Ch. 11], which we would also expect to yield finite-distance degenerate limits. These cases were not treated in the article [84] and only appear in this thesis.

4.1 The moduli spaces as Riemannian manifolds

The aim of this section is to present the basic properties of the metric $\mathscr{G}$ and to set the notations which will be used in the next three chapters. In §4.1.1, we introduce the potential function $\mathscr{F}$ and make some comments as to why $\mathscr{G}$ is the natural metric to consider on $\mathscr{M}$. In §4.1.2 we introduce the concept of adapted sections of the moduli space, and explain how to use them in order to compute the infinitesimal variations of geometric quantities defined on $\mathscr{M}$. For the sake of completeness, we give a self-contained proof of the existence of adapted sections in §4.1.3, where we also show that the potential is a real-analytic function.

4.1.1 The metric. In Chapter 1, we gave a brief description (following [66]) of the manifold structure of $\mathscr{M}$, and noted that for a torsion-free G_2-structure φ on M, the tangent space $T_{\varphi\mathscr{D}}\mathscr{M}$ can be identified with the space $\mathscr{H}^3(M,\varphi)$ of 3-forms which are harmonic with respect to the metric g_φ. This identification allows us to define a natural Riemannian metric $\mathscr{G}$ on $\mathscr{M}$ as follows:

$$\mathscr{G}(\eta,\eta') = \frac{1}{\mathrm{Vol}(\varphi)}\int \langle \eta,\eta'\rangle_\varphi \mu_\varphi, \quad \forall \eta,\eta' \in \mathscr{H}^3(M,\varphi) \simeq T_{\varphi\mathscr{D}}\mathscr{M}, \tag{4.1}$$

where $\mathrm{Vol}(\varphi)$ is the volume of (M, g_φ) which can be written as

$$\mathrm{Vol}(\varphi) = \int \mu_\varphi = \frac{1}{7}\int \varphi \wedge \Theta(\varphi). \tag{4.2}$$

That is, the metric $\mathscr{G}$ is just the volume-normalised L^2-metric on $\mathscr{M}$. It is perhaps worth commenting on the volume normalisation in this definition. If we denote by $\mathscr{M}_1 \subset \mathscr{M}$ the moduli space of torsion-free G_2-structures with unit volume, then the metric $\mathscr{G}$ restricts to the usual L^2-metric on $\mathscr{M}_1$, denoted by $\mathscr{G}_1$. Moreover, there is a diffeomorphism $\mathbb{R}\times\mathscr{M}_1 \to \mathscr{M}$ mapping $(t,\varphi\mathscr{D})$ to $e^t\varphi\mathscr{D}$. Note that the tangent space of $\mathscr{M}_1$ can be identified with the space of harmonic 3-forms in Ω^3_{27}. In the following easy lemma we show that $(\mathscr{M},\mathscr{G})$ splits a line and is isometric to $\mathbb{R}\times(\mathscr{M}_1,\mathscr{G}_1)$:

Lemma 4.1. $\mathscr{G} = 7dt^2 + \mathscr{G}_1$.

Proof. The vector field $\frac{\partial}{\partial t} = e^t\varphi \in \mathscr{H}^3(M, e^t\varphi)$ is orthogonal to $\mathscr{H}^3_{27}(M, e^t\varphi)$ and we easily see that:

$$\mathscr{G}_{(t,\varphi)}(\partial_t,\partial_t) = \frac{\int_M e^t\varphi\wedge\Theta(e^t\varphi)}{\mathrm{Vol}(e^t\varphi)} = 7$$

whence $\mathscr{G} = 7dt^2 + \mathscr{G}_t$ for a family of metrics $\mathscr{G}_t$ on $\mathscr{M}_1$. It remains to see that $\mathscr{G}_t$ does not depend on t. Under the identification $T_{\varphi\mathscr{D}}\mathscr{M}_1 \simeq \mathscr{H}^2_{27}(M,\varphi)$, we see that:

$$\mathscr{G}_t(\eta,\eta') = \frac{\int \langle e^t\eta, e^t\eta'\rangle_{e^t\varphi}\mu_{e^t\varphi}}{\mathrm{Vol}(e^t\varphi)} = \frac{\int e^{2t}e^{-3\cdot\frac{2t}{3}}\langle \eta,\eta'\rangle_\varphi e^{\frac{7t}{3}}\mu_\varphi}{e^{\frac{7t}{3}}} = \mathscr{G}_1(\eta,\eta')$$

which proves the lemma. □

Because of this lemma, there is no essential difference between studying the Riemannian properties of $(\mathscr{M},\mathscr{G})$ and those of $(\mathscr{M}_1,\mathscr{G}_1)$. This would not be the case without the volume normalisation: the unnormalised L^2-metric would instead be written $7dt^2 + e^{7t/3}\mathscr{G}_1$ and the splitting would be lost.

Another motivation for this choice of normalisation is that, when the first Betti number of M vanishes, the metric $\mathscr{G}$ is Hessian. To see this, recall that there is a local diffeomorphism $\mathscr{M} \to H^3(M)$. By pulling back the natural flat connection of $H^3(M)$, we obtain a flat connection D on $\mathscr{M}$. If $(u_0,\ldots,u_n)$ is a basis of $H^3(M)$, where $n = b^3(M) - 1$, we will denote by $(x^0,\ldots,x^n)$ the associated local coordinates on $\mathscr{M}$ and call them *affine coordinates*. Then the connection D is just the usual differentiation in these coordinates. Since the volume functional is invariant under diffeomorphisms, it descends to a function on the moduli space, and we can define $\mathscr{F} : \mathscr{M} \to \mathbb{R}$ by:

$$\mathscr{F}(\varphi\mathscr{D}) = -3\log \mathrm{Vol}(\varphi). \tag{4.3}$$

This defines a smooth (in fact, real analytic as we will show in a moment) function on the moduli space, which we refer to as the *potential*. If $(x^0,\ldots x^n)$ are local affine coordinates, we denote by $\mathscr{F}_a = \frac{\partial\mathscr{F}}{\partial x^a}$, $\mathscr{F}_{ab} = \frac{\partial^2\mathscr{F}}{\partial x^a\partial x^b}$, and so on the derivatives of $\mathscr{F}$. If φ is a torsion-free G_2-structure on M, we will also denote by $\eta_a \in \mathscr{H}^3(M,\varphi)$ the harmonic representative of the cohomology class $\frac{\partial}{\partial x^a} \in H^3(M)$. Then the first and second derivatives of $\mathscr{F}$ admit the following expressions [52, 73]:

Proposition 4.2. *Let $x = (x^0,\ldots,x^n)$ be affine coordinates on $\mathscr{M}$ and let φ be a torsion-free G_2-structure. Then at $\varphi\mathscr{D} \in \mathscr{M}$ we have:*

$$\mathscr{F}_a = -\frac{1}{\mathrm{Vol}(\varphi)}\int \eta_a \wedge \Theta(\varphi), \quad \textit{and} \quad \mathscr{F}_{ab} = \frac{1}{\mathrm{Vol}(\varphi)}\int \langle \eta_a, \pi_{1\oplus 27}(\eta_b) - \pi_7(\eta_b)\rangle_\varphi \mu_\varphi.$$

If $b^1(M) = 0$, the harmonic 3-forms with respect to a torsion-free G_2-structure on M have no Ω^3_7-component. In this case, the second derivative of $\mathscr{F}$ takes the simpler form:

$$\mathscr{F}_{ab} = \frac{1}{\mathrm{Vol}(\varphi)}\int \langle \eta_a, \eta_b\rangle_\varphi \mu_\varphi. \tag{4.4}$$

Thus the Hessian $\mathscr{F}_{ab}$ is non-degenerate and positive, and in affine coordinates

$$\mathscr{G} = \mathscr{G}_{ab}dx^a dx^b = \mathscr{F}_{ab}dx^a dx^b$$

where we write $dx^k dx^l$ as a short-hand for the (unsymmetrised) tensor product $dx^k \otimes dx^l \in T^*\mathscr{M} \otimes T^*\mathscr{M}$, and we will use similar notations for tensor products of higher degree. Thus the metric $\mathscr{G}$ is the Hessian of the potential $\mathscr{F}$ for the flat connection D. In general, if the first Betti number of M is non-zero, the Hessian of $\mathscr{F}$ is still non-degenerate and defines a metric of signature $(b^3(M)-b^1(M), b^1(M))$ on $\mathscr{M}$. Even in the case $b^1(M) = 0$, one could take the volume functional Vol instead of $\mathscr{F}$ as a potential, which has non-degenerate Hessian and defines a metric on $\mathscr{M}$ with Lorentzian signature [60, 73]. In the present work we prefer to use $\mathscr{F}$ as a potential, which is the convention usually adopted by physicists. In fact, both conventions agree when restricted to the moduli space $\mathscr{M}_1$ of torsion-free G_2-structures with unit volume, but we prefer to use $\mathscr{F}$ since it is more convenient to work with a Riemannian metric instead of a Lorentzian one. Moreover, since $(\mathscr{M}, \mathscr{G})$ is isometric to $\mathbb{R} \times (\mathscr{M}_1, \mathscr{G}_1)$ all geometric invariants of interest can be computed in $\mathscr{M}$, which has a natural affine structure, and directly restricted to $\mathscr{M}_1$, whereas it would be more difficult to do computations directly in $\mathscr{M}_1$ for lack of natural coordinates.

Remark 4.3. The moduli space $\mathscr{M}$ also has a second natural affine structure coming from the local diffeomorphism $\mathscr{M} \to H^4(M)$, $\varphi\mathscr{D} \to [\Theta(\varphi)]$. The metric $\mathscr{G}$ is also Hessian for this affine structure, and in fact the potentials are related by a Legendre transform [73]. Formally, the properties of these two Hessian structures are entirely similar, and the results of this and the next chapters could easily be adapted. In practice however, it is more convenient to consider the affine structure coming from the cohomology class of the 3-form since it can be explicitly computed, for in the construction of compact G_2-manifolds we always perturb a closed 3-form within a fixed cohomology class. This will play an important role in Section 4.2.

4.1.2 Adapted sections. For the purpose of computing higher derivatives of the potential, it will be convenient to adopt the following definition:

Definition 4.4. Let $\mathscr{U} \subseteq \mathscr{M}$ be an open subset of the moduli space. A *local section* of the moduli space defined on $\mathscr{U}$ is a smooth map $\underline{\varphi} : \mathscr{U} \times M \to \Lambda^3 T^* M$, such that for any $u \in \mathscr{U}$ the restriction $\varphi_u = \underline{\varphi}_{\{u\}\times M}$ is a torsion-free G_2-structure on M and $u = \varphi_u\mathscr{D}$ in $\mathscr{M}$. A section $\underline{\varphi}$ is said to be *adapted* at $u_0 \in \mathscr{U}$ if the tangent map $T_{u_0}\mathscr{U} \to \Omega^3(M)$ of the induced map $\mathscr{U} \to \Omega^3(M)$ takes values in the space $\mathscr{H}^3(M, \varphi_{u_0})$ of harmonic 3-forms for the metric induced by φ_{u_0}.

In affine coordinates $x = (x^0, \dots, x^n)$, where $n + 1 = b^3(M)$, a local section $\underline{\varphi} = (\varphi_x)_x$ of the moduli space is adapted at a point u_0 with coordinates x_0 if and only if for any $0 \leq a \leq n$, the 3-form $\frac{\partial \varphi_x}{\partial x^a}\big|_{x=x_0}$ is harmonic for the metric induced by φ_{x_0}. In §4.1.3 we will show that there exist adapted sections through any point of the moduli space. The interest of working with sections that are adapted at a point is the following lemma, which will simplify many computations:

Lemma 4.5. *Let $\mathscr{U}$ be an open subset of $\mathscr{M}$, $x = (x^0, \dots, x^n)$ be affine coordinates on $\mathscr{U}$ and $u_0 \in \mathscr{U}$ with coordinates x_0. Let $\underline{\varphi} = (\varphi_x)_x$ be a local section of the moduli space adapted at u_0, and $f : \mathscr{U} \times M \to \mathbb{R}$ be a smooth function. Then:*

$$\frac{\partial}{\partial x^a}\bigg|_{x=x_0} \left(\frac{1}{\mathrm{Vol}(\varphi_x)} \int f_x \mu_{\varphi_x} \right) = \frac{1}{\mathrm{Vol}(\varphi_{x_0})} \int \frac{\partial f_x}{\partial x^a}\bigg|_{x=x_0} \mu_{\varphi_{x_0}}, \quad \forall a = 0, \dots, n.$$

Proof. After a linear change of coordinates, we may choose a basis $\eta_0, \dots, \eta_n$ of $\mathscr{H}^3(M, \varphi_{x_0})$ such that $\eta_0 \in \mathscr{H}^3_1(M, \varphi_{x_0})$ and $\eta_a \in \mathscr{H}^3_{27}(M, \varphi_{x_0})$ for $a = 1, \dots, n$ and assume that $(x^0, \dots, x^n)$ are the associated local coordinates (that is, $\frac{\partial}{\partial x^a} = [\eta_a] \in H^3(M)$). In these coordinates we have:

$$\begin{aligned} \frac{\partial}{\partial x^a}\bigg|_{x=x_0} \left(\frac{1}{\mathrm{Vol}(\varphi_x)} \int f_x \mu_{\varphi_x} \right) &= \frac{1}{\mathrm{Vol}(\varphi_{x_0})} \int \frac{\partial f_x}{\partial x^a}\bigg|_{x=x_0} \mu_{\varphi_{x_0}} \\ &+ \frac{1}{\mathrm{Vol}(\varphi_{x_0})} \int f_{x_0} \frac{\partial \mu_{\varphi_{x_0}}}{\partial x^a}\bigg|_{x=x_0} + \frac{\partial}{\partial x^a} \left(\frac{1}{\mathrm{Vol}(\varphi_x)} \right)\bigg|_{x=x_0} \int f_{x_0} \mu_{\varphi_{x_0}}. \end{aligned} \tag{4.5}$$

Since the section is adapted at the point x_0, $\frac{\partial \varphi_x}{\partial x^0}$ is a harmonic section of $\Omega^3_1(M)$ and $\frac{\partial \varphi_x}{\partial x^a}$ are harmonic sections of $\Omega^3_{27}(M)$ for $a = 1, \dots, n$ at $x = x_0$. Hence, if $a \geq 1$ then $\frac{\partial \mu_{\varphi_x}}{\partial x^a} = 0$ at $x = x_0$, which also implies $\frac{\partial \,\mathrm{Vol}(\varphi_x)}{\partial x^a} = 0$. Therefore, both terms in the second line of (4.5) vanish. For the derivative along the coordinate x^0, there exists $\lambda \neq 0$ such that $\frac{\partial \varphi_x}{\partial x^0} = \lambda \varphi_{x_0}$ at $x = x_0$, and by Lemma 1.6 this implies:

$$\frac{\partial \mu_{\varphi_x}}{\partial x^0} = \frac{7\lambda}{3} \mu_{\varphi_x}, \quad \frac{\partial}{\partial x^0} \left(\frac{1}{\mathrm{Vol}(\varphi_x)} \right) = -\frac{7\lambda}{3} \frac{1}{\mathrm{Vol}(\varphi_x)}$$

at $x = x_0$. Therefore the lemma also holds for $a = 0$ since the two terms in the second line of (4.5) cancel each other. □

4.1.3 Regularity results. In this part, we prove our previous claims that there exist adapted sections through every point of the moduli space $\mathscr{M}$ and that the potential function $\mathscr{F}$ is real analytic in local affine coordinates. The main ingredient from PDE theory is that an *elliptic solution* of class C^2 of a nonlinear PDE of order 2 is smooth [2, Th. 12.1] (see also [13, App. A, Th. 41]). Here a solution is called *elliptic* if the linearisation of the PDE at this point is an elliptic linear differential operator. In practice, we will use the fact that this condition is open in the C^2-topology.

Let φ be a torsion-free G_2-structure on M and let (ξ_a) be a basis of the space of harmonic three-forms with respect to g_φ, and let x^a be the corresponding affine coordinates centred at $\varphi\mathscr{D}$. Using Theorem 2.12, we seek a family of torsion-free G_2-structures $\varphi_x = \varphi + x^a\xi_a + d\varpi_x$ where the 2-form ϖ_x satisfies:

$$\Delta\varpi_x + *d(F_\varphi(x^a\xi_a + d\varpi_x)) = 0$$

where $F_\varphi(\eta) = \Theta(\varphi+\eta) - \Theta(\varphi) - \frac{4}{3} * \pi_1(\eta) - *\pi_7(\eta) + *\pi_{27}(\eta)$. For $\epsilon > 0$ small enough, we can consider this equation for x in an open ball $B_\epsilon \subset \mathbb{R}^{n+1}$ and η contained in an open ball U_ϵ of the space of 2-forms of regularity $W^{k+2,p}$ that are L^2-orthogonal to harmonic forms, where we choose for instance $p \geq 7$ and $k \geq 1$ so that the triple $(p, k+1, k)$ satisfies conditions (2.3) in the previous chapter.

With these choices we can apply Lemma 2.4, and thus the map induced by Θ on sections of regularity $W^{k+1,p}$ is analytic at φ. If we write $\Theta(\varphi+\eta) = \sum \Theta_m(\varphi)\eta^m$, then we have

$$F_\varphi(\eta) = \sum_{m=2}^{\infty} \Theta_m(\varphi)\eta^m \tag{4.6}$$

and therefore if ϵ is small enough the map:

$$B_\epsilon \oplus U_\epsilon \to W^{k,p}(\Lambda^2T^*M) \cap \mathscr{H}^2(M,\varphi)^\perp,\ (x,\varpi) \mapsto \Delta\varpi + *dF_\varphi(x^a\xi_a + d\varpi) \tag{4.7}$$

is analytic. Moreover, by Corollary 2.11 F_φ satisfies a quadratic bound of the form $\|F(\eta)\|_{W^{k+1,p}} \leq O(\|\eta\|^2_{W^{k+1,p}})$ near $\eta = 0$. Hence (4.7) maps $(0,0)$ to 0 and its derivative at $(0,0)$ in the direction of ϖ is the Laplacian

$$\Delta : W^{k+2,p}(\Lambda^2T^*M) \cap \mathscr{H}^2(M,\varphi)^\perp \to W^{k,p}(\Lambda^2T^*M) \cap \mathscr{H}^2(M,\varphi)^\perp$$

which is bounded and admits a bounded inverse by elliptic regularity. By the Implicit Function Theorem for analytic maps between Banach spaces [120], for $\delta > 0$ small enough there exists an analytic map $x \in B_\delta \mapsto \varpi_x \in U_\epsilon$ such that $\eta_0 = 0$ and for (x,ϖ) near $(0,0)$ the equation

$$\Delta\varpi + *dF_\varphi(x^a\xi_a + d\varpi) = 0 \tag{4.8}$$

is satisfied if and only if $\varpi = \varpi_x$. As $p \geq 7$ and $k \geq 1$, $W^{k+2,p}(\Lambda^2T^*M)$ embeds continuously into $C^2(\Lambda^2T^*M)$, and thus we can see ϖ_x as a family of 2-forms of class C^2 depending smoothly (even analytically) on x. Moreover, for x small enough, the linearisation of (4.8) at ϖ_x is elliptic, since the linearisation at ϖ is the Laplacian. It follows that ϖ_x is a smooth 2-form for x close enough 0. Hence if $\delta > 0$ is small enough, for all $x \in B_\delta$ the 3-form $\varphi_x = \varphi + x^a \xi_a + d\varpi_x$ is a smooth torsion-free G_2-structure on M, with affine coordinates (x^a). Moreover, the map $x \in B_\delta \mapsto \varphi_x \in W^{k+1,p}(\Lambda^3_+ T^*M)$ is analytic, and by Lemma 2.4 it follows that the map $x \in B_\delta \mapsto \mu_{\varphi_x} \in W^{k+1,p}(\Lambda^7 T^*M)$ is analytic. As integration defines a continuous linear form on $W^{k+1,p}(\Lambda^7 T^*M)$, we deduce that the map $x \mapsto \mathrm{Vol}(\varphi_x) = \int \mu_{\varphi_x}$ is analytic. Taking the logarithm we finally obtain:

Theorem 4.6. *The potential $\mathscr{F}$ is real-analytic in affine coordinates.*

From the proof of the analyticity of the potential, we may also deduce our second claim about the existence of smooth adapted sections through every point of the moduli space. In order to do this, we want to use again the fact that an elliptic solution of class C^2 of a nonlinear PDE of order 2 is smooth. One problem is that the sections $x \mapsto \varpi_x$ previously constructed do not satisfy any particular elliptic equation jointly in the variables $x \in B_\delta$ and $p \in M$. To solve this issue, we can take advantage of the analyticity of the equations to replace the real variable x by a complex variable z and use the fact that complex-analytic maps are harmonic (this argument is similar to that of [78, §6]).

As in the proof of the analyticity of the potential we let $p \geq 7$ and $k \geq 1$. Using the expansion (4.6), we extend F to a function acting on complex 3-forms by setting:

$$F_\varphi(\eta_1 + i\eta_2) = \sum_{m=2}^{\infty} \sum_{l=0}^{m} \binom{m}{l} i^{m-l} \Theta_m(\varphi) \eta_1^l \eta_2^{m-l}.$$

This expression makes sense for complex 3-forms that are small enough in C^0-norm, and it defines an analytic map in a neighbourhood of 0 in $W^{k+1,p}(\Lambda^3 T^*M \otimes \mathbb{C})$. As the map which associates to any $x \in B_\delta$ the unique solution $\varpi_x \in U_\epsilon \subset W^{k+2,p}(\Lambda^2 T^*M)$ of (4.8) is analytic, there exists an expansion $\varpi_x = \sum_\alpha x^\alpha \varpi_\alpha$ that converges in $W^{k+2,p}$-norm. We can therefore extend it to an analytic map in the complex coordinates $z^a = x^a + iy^a$ by $\varpi_z = \sum_\alpha z^\alpha \varpi_\alpha$, which converges in $W^{k+2,p}(\Lambda^3 T^*M \otimes \mathbb{C})$ for z belonging to a small polydisc B'_δ centred at 0 in $\mathbb{C}^{n+1}$. For all $x \in B_\delta$, ϖ_x satisfies equation (4.8), and by analyticity we may deduce that for all $z \in B'_\delta$ we have:

$$\Delta \varpi_z + *dF_\varphi(z^a \xi_a + d\varpi_z) = 0.$$

As $p \geq 7$ and $k \geq 1$ we have $W^{k+2,p} \subset C^2$, and thus the above equations holds strongly in the C^2-sense. Moreover, by composition the map $z \in B'_\delta \mapsto \varpi_z \in C^2(\Lambda^2 T^*M \otimes \mathbb{C})$ is complex-analytic, and since the evaluation map at a point $p \in M$ defines a continuous linear map from $C^2(\Lambda^2 T^*M \otimes \mathbb{C})$ to $\Lambda^2 T^*_p M \otimes \mathbb{C}$ the map $z \in B'_\delta \mapsto \varpi_z(p) \in \Lambda^2 T^*_p M \otimes \mathbb{C}$ is also complex analytic. In particular it is harmonic, in the sense that it satisfies:

$$\sum_{a=0}^{n} \frac{\partial^2 \varpi_z(p)}{(\partial x^a)^2} + \frac{\partial^2 \varpi_z(p)}{(\partial y^a)^2} = 0, \quad \forall p \in M,\ \forall z \in B'_\delta.$$

Thus, if we consider ϖ as depending jointly on the variables $z = (z^0, \ldots, z^n) \in B'_\delta$ and $p \in M$ and define the elliptic differential operator

$$\Box = \Delta - \sum_{a=0}^{n} \frac{\partial^2}{(\partial x^a)^2} + \frac{\partial^2}{(\partial y^a)^2}$$

then ϖ is a C^2 solution of the following nonlinear PDE:

$$\Box \varpi_z + *dF_\varphi(z^a \xi_a + d\varpi_z) = 0. \tag{4.9}$$

The linearisation at ϖ of the above equation at any point $(0, p) \in B'_\delta \times M$ is the operator $\Box$, and thus if δ is chosen small enough then ϖ is an elliptic solution of (4.9) on $B'_\delta \times M$. Applying again [2, Th.12.1], we deduce that ϖ is smooth jointly in the variables (z, p). Restricting to real values of z, this proves that $x \mapsto \varphi_x = \varphi_0 + x^a \xi_a + d\varpi_x$ defines a smooth section of the moduli space through φ_0. The fact that this section is adapted at φ_0, i.e. that the derivatives with respect to x at $x = 0$ are harmonic 3-forms, follows from the quadratic bound on F_φ near $\eta = 0$.

4.2 Incompleteness of the moduli spaces

In this section, whose results appear in the paper [84] by the author, we prove that G_2-moduli spaces may be incomplete. We begin with some motivation.

4.2.1 Motivation and strategy. One of the most basic questions that one may ask about the geometry of $\mathscr{M}$[1] is whether it is complete or not. As we mentioned in the introduction, the analogous question has been extensively studied in the complex geometry literature in various contexts. A well-understood case is that of the Kähler cone of a compact Kähler manifold, where the natural metric also admits a global Hessian potential. The Kähler cone can be described in terms of the intersection form and the classes of analytic cycles [36], and there is a simple necessary and sufficient condition for a cohomology class at the boundary of the cone to be a finite-distance limit [92]. Examples where the Kähler cone is

[1]Note that there is no essential difference between considering the full moduli space $\mathscr{M}$ or the unit-volume moduli space $\mathscr{M}_1$ for the question of completeness since $\mathscr{M}$ is isometric to $\mathbb{R} \times \mathscr{M}_1$. We use $\mathscr{M}$ as a matter of convenience in order to take advantage of its affine structure.

incomplete include Kummer K3 surfaces, where a sequence of hyperkähler metrics degenerating to $T^4/\mathbb{Z}_2$ occurs at finite distance. In the case of moduli spaces of polarised Calabi–Yau manifolds, the Weil–Petersson metric can be studied using Hodge-theoretic methods, and there are known examples where the moduli spaces are incomplete as well as some characterisations of finite-distance degenerations [117]. See the introduction for a more detailed discussion of these results.

Based on these results, one would expect that it is a general feature of special holonomy manifolds that certain degenerate limits occur at finite distance, causing the moduli spaces to be incomplete. Therefore, a more refined question is to derive geometric conditions to characterise finite-distance limits and to distinguish them from infinite-distance limits. This question is also very relevant to physics and the swampland programme which we already discussed at various points: at least naively, the hope might be that finite-distance limits correspond to the formation of singularities which are relatively mild, as opposed to infinite-distance limits where the low-energy effective description of physics would break down.

A difficulty in trying to adapt the known results about the incompleteness of Kähler cones or Calabi–Yau moduli spaces is that they ultimately rely on techniques of complex algebraic geometry together with the link to Riemannian geometry provided by Yau's solution of the Calabi conjecture [123]. By contrast, G_2-manifolds are only amenable to differential-geometric methods and little is known about the global properties of $\mathscr{M}$. In this section, we shall obtain sufficient conditions for a degenerate family of G_2-manifolds to represent a finite-distance limit in the moduli space. These conditions have the advantage of being easy to check for the known constructions of compact G_2-manifolds, and we shall deduce an elementary proof that G_2-moduli spaces can indeed be incomplete. However, we will not attempt to address the question of whether these conditions are optimal or necessary; in fact, the author tends to think that they are not, but improving them might require new tools and ideas.

To prove that the moduli space of a certain G_2-manifold is incomplete, the idea is to find a path of torsion-free G_2-structures degenerating towards a singular limit and prove that it has finite length in the moduli space. The natural paths to consider are those constructed by gluing-perturbation methods, which are typically indexed by a parameter representing the size of the gluing region. To compute the length of a such a path, we a priori need to differentiate the family of torsion-free G_2-structures with respect to the gluing parameter to deduce the velocity vector along the corresponding path in the moduli space and estimate its L^2-norm. There is ongoing work by J. Li [86] using this approach, but there are many analytical difficulties related to the fact that the torsion-free G_2-structures obtained after perturbation are only implicitly defined, making this method difficult to implement in detail.

Here we adopt a much simpler approach, in which the analytical difficulties disappear. To circumvent the analysis, the idea is to consider not the length but the energy (the integral of the squared velocity) of a path and make use of the special properties of the metric $\mathscr{G}$. We derive an expression for the energy of a curve that involves derivatives only of the cohomology class of the 3-form and no derivatives of the cohomology class of its dual 4-form. Interpreting this expression in geometrical terms allows us to give simple sufficient conditions for a path of torsion-free G_2-structures to have finite energy and length, which notably applies to Joyce's generalised Kummer construction [64, 65].

4.2.2 Length and energy of paths in the moduli space. Let M be a compact oriented 7-manifold with $b^1(M) = 0$ admitting torsion-free G_2-structures. We aim to compute the energy of a path in the moduli space $\mathscr{M}$ for the metric $\mathscr{G}$. The idea is to use the fact that the metric $\mathscr{G}$ is the Hessian of a global potential. Indeed, for such metrics we have the following result:

Lemma 4.7. *Let P be a manifold equipped with a flat connection D and let g be a Riemannian metric on P which can be written as the Hessian of a global smooth potential $F : P \to \mathbb{R}$, that is, $g = D^2F$. Then if $\gamma : [0,1] \to P$ is a path of class C^2 in P we have*

$$\int_0^1 g(\dot\gamma(t), \dot\gamma(t))dt = d_{\gamma(1)}F(\dot\gamma(1)) - d_{\gamma(0)}F(\dot\gamma(0)) - \int_0^1 d_{\gamma(t)}F(\ddot\gamma(t))dt$$

where $\dot\gamma(t) = \frac{\partial\gamma}{\partial t}(t) \in T_{\gamma(t)}P$ and $\ddot\gamma(t) = \frac{D}{dt}\dot\gamma(t) \in T_{\gamma(t)}P$.

Proof. Let us consider the function $h(t) = (d\mathscr{F})_{\gamma(t)}(\dot\gamma(t))$. Since F is smooth and γ is a path of class C^2, the function h is C^1. Moreover the first derivative of h satisfies

$$h'(t) = D^2_{\gamma(t)}F(\dot\gamma(t), \dot\gamma(t)) + d_{\gamma(t)}F(\ddot\gamma(t)) = g(\dot\gamma(t), \gamma(t)) + d_{\gamma(t)}F(\ddot\gamma(t)).$$

The lemma follows by integration by parts. □

This lemma allows us to derive a simple energy formula for a path in $\mathscr{M}$. First, we make a few remarks about notations and the regularity of paths. We will say that a family of torsion-free G_2-structures $\{\varphi_t\}_{t\in(0,T]}$ on M *induces a path of class C^k in $\mathscr{M}$* ($k \in \mathbb{N} \cup \{\infty\}$) if the path $\{\varphi_t\mathscr{D}\}_{t\in(0,T]}$ is of regularity C^k in $\mathscr{M}$. We emphasize that we do not require the 3-forms φ_t to be of class C^k jointly in the variable t and in local coordinates on M, since at no point will we need to consider partial derivatives of φ_t with respect to the variable t. In practice, if φ_t is a continuous family of G_2-structures on M and the cohomology class $[\varphi_t]$ defines a path of class C^k in $H^3(M)$, then $\{\varphi_t\}_{t\in(0,T]}$ induces a path of class C^k in $\mathscr{M}$, since the moduli space is locally diffeomorphic to $H^3(M)$. Although the energy is defined for a path in $\mathscr{M}$ which is merely C^1, we will need to consider paths of class at least C^2 so as to differentiate and integrate by parts some expressions.

If $\{\varphi_t\}_{t\in(0,T]}$ induces a path of class C^2 in $\mathscr{M}$, we denote by $\dot\varphi_t \in T_{\varphi_t}\mathscr{M}$ the velocity vector along the induced path in $\mathscr{M}$ and by $\ddot\varphi_t = \frac{D}{dt}\dot\varphi_t \in T_{\varphi_t}\mathscr{M}$ the covariant derivative of $\dot\varphi_t$ along the path for the flat connection D associated with the local diffeomorphism $\mathscr{M} \to H^3(M)$. In particular $[\dot\varphi_t] = \frac{d[\varphi_t]}{dt}$ and $[\ddot\varphi_t] = \frac{d^2[\varphi_t]}{dt^2}$. From the previous lemma we may deduce:

Proposition 4.8. *Let $\{\varphi_t\}_{t\in(0,T]}$ be a family of torsion-free G_2-structures on M, inducing a path of class C^2 in $\mathscr{M}$. Then for any $\tau \in (0,T]$ the energy of the path $\{\varphi_t\}_{t\in[\tau,T]}$ reads:*

$$\begin{aligned} E_\tau^T(\varphi_t) = \int_\tau^T \mathscr{G}_{\varphi_t}(\dot\varphi_t,\dot\varphi_t)dt = &\frac{1}{\mathrm{Vol}(\varphi_\tau)}\left\langle \left.\frac{d[\varphi_t]}{dt}\right|_{t=\tau} \cup [\Theta(\varphi_\tau)], [M]\right\rangle \\ &-\frac{1}{\mathrm{Vol}(\varphi_T)}\left\langle \left.\frac{d[\varphi_t]}{dt}\right|_{t=T} \cup [\Theta(\varphi_T)], [M]\right\rangle \\ &+\int_\tau^T \frac{1}{\mathrm{Vol}(\varphi_t)}\left\langle \frac{d^2[\varphi_t]}{dt^2} \cup [\Theta(\varphi_t)], [M]\right\rangle dt. \end{aligned}$$

Proof. This follows from the previous lemma and from Proposition 4.2 with implies that

$$d\mathscr{F}_{\varphi_t}(\dot\varphi_t) = -\frac{1}{\mathrm{Vol}(\varphi_t)}\int_M \dot\varphi_t \wedge \Theta(\varphi_t) = -\frac{1}{\mathrm{Vol}(\varphi_t)}\left\langle \frac{d[\varphi_t]}{dt} \cup [\Theta(\varphi_t)], [M]\right\rangle$$

and similarly

$$d\mathscr{F}_{\varphi_t}(\ddot\varphi_t) = -\frac{1}{\mathrm{Vol}(\varphi_t)}\left\langle \frac{d[\varphi_t]}{dt} \cup [\Theta(\varphi_t)], [M]\right\rangle.$$

These expressions together with Lemma 4.7 yield the desired formula. □

By the Cauchy–Schwarz inequality, the length $L_0^T(\varphi_t) = \int_0^T \sqrt{\mathscr{G}_{\varphi_t}(\dot\varphi_t,\dot\varphi_t)}dt$ and the energy of the curve $\{\varphi_t\}_{t\in(0,T]}$ satisfy the inequality $L_0^T(\varphi_t)^2 \leq T E_0^T(\varphi_t)$ and therefore we immediately deduce:

Corollary 4.9. *With the same assumptions as in the previous proposition, assume that there exist a constant $C > 0$ and a nonnegative integrable function $A : (0,T] \to \mathbb{R}$ such that for all $t \in (0,T]$ we have*

$$\left|\left\langle \frac{d[\varphi_t]}{dt} \cup [\Theta(\varphi_t)], [M]\right\rangle\right| \leq C\,\mathrm{Vol}(\varphi_t)$$

and

$$\left|\left\langle \frac{d^2[\varphi_t]}{dt^2} \cup [\Theta(\varphi_t)], [M] \right\rangle\right| \leq A(t)\,\mathrm{Vol}(\varphi_t).$$

Then the energy and the length of $\{\varphi_t\}_{t\in(0,T]}$ *are finite.*

We finish this part with a few remarks about (co)homology groups. If M is a smooth manifold, we denote by $H^p(M)$ the de Rham cohomology groups, which are isomorphic to the singular cohomology groups with real coefficients. The singular homology groups with real coefficients are denoted by $H_p(M)$ and $\langle\cdot,\cdot\rangle$ denotes the natural pairing between $H^p(M)$ and $H_p(M)$. Any homology class $[C] \in H_p(M)$ can be represented by a smooth singular p-cycle $C = \sum a_i\sigma_i$, where $\sigma : \Delta_p \to M$ are smooth p-simplices and Δ_p denotes the standard oriented p-simplex. If η is a closed p-form on M, we have

$$\langle[\eta],[C]\rangle = \sum a_i \int_{\Delta_p} \sigma_i^*\eta.$$

In particular the right-hand side does not depend on a particular choice of representative for $[C]$, and for this reason we will denote (with a slight abuse of notation) $\int_{[C]}\eta = \langle[\eta],[C]\rangle$. In the remainder of the chapter, all singular chains are assumed to be smooth.

If g is a Riemannian metric on M, the volume of a p-chain $C = \sum_i a_i\sigma_i$ is defined by

$$\mathrm{Vol}(C,g) = \sum_i |a_i| \int_{\Delta_p} \mathrm{Vol}_{\sigma_i^* g}\,.$$

If M has dimension 7 and is endowed with a co-closed G_2-structure φ, the dual 4-form $\Theta(\varphi)$ is a calibration [67]. Hence for any 4-simplex σ we have $\pm\sigma^*\Theta(\varphi) \leq \mathrm{Vol}_{\sigma^* g_\varphi}$, and therefore for any 4-cycle D we have a bound

$$\left|\int_{[D]} \Theta(\varphi)\right| \leq \mathrm{Vol}(D, g_\varphi). \tag{4.10}$$

Note that the left-hand side is topological and independent of the choice of representative of $[D]$, whilst the right-hand side is geometric and depends on the choice of 4-cycle D.

4.2.3 Incompleteness for generalised Kummer G_2-manifolds. In this section, we consider a simple model of gluing construction of compact G_2-manifolds. Topologically, it can be described as follows. Let $\overline{U}$ be a compact oriented 7-manifold with boundary and denote $U = \overline{U}\backslash\partial\overline{U}$. We denote by $\Sigma_1,\dots,\Sigma_m$ the connected components of $\partial\overline{U}$, each of which is a compact oriented 6-manifold. We may assume that there are disjoint neighbourhoods W_i of Σ_i, diffeomorphic

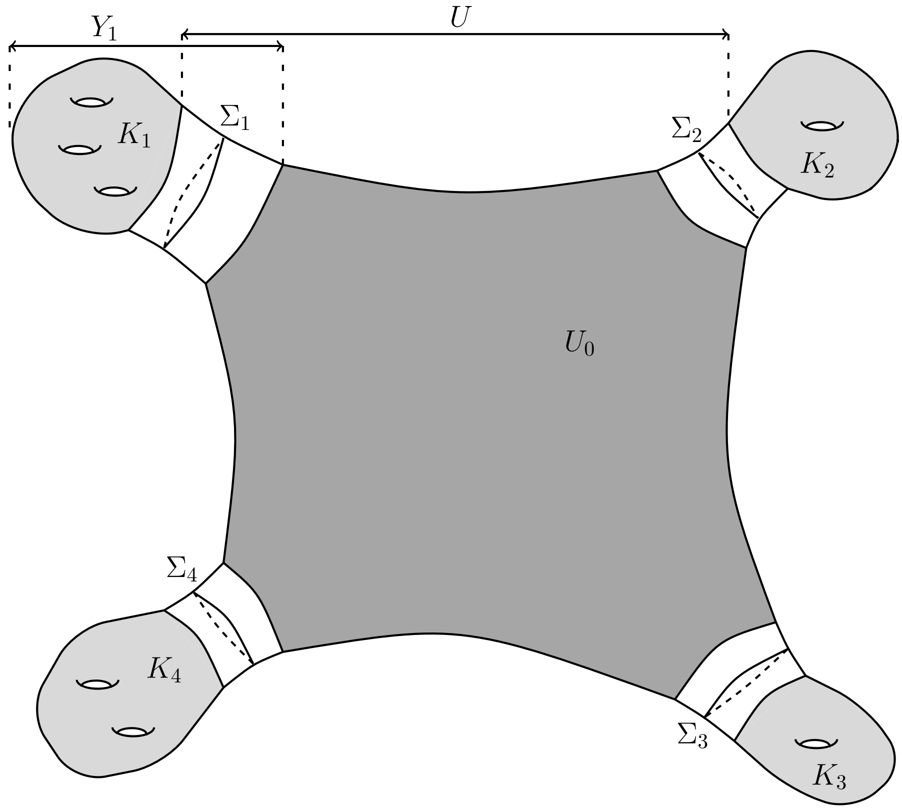

Figure 4.1: Gluing construction.

to $(0,1] \times \Sigma_i$, such that $\overline{U}_0 = \overline{U} \backslash \coprod_i W_i$ is a manifold with boundary diffeomorphic to $\overline{U}$ and a deformation retract of $\overline{U}$. We denote by U_0 its interior, which is diffeomorphic to U. For each i, we consider a compact oriented 7-manifold $\overline{Y}_i$ with boundary $\partial\overline{Y}_i = \Sigma_i$, and let K_i be a compact subset of $Y_i = \overline{Y}_i \backslash \Sigma_i$ such that $Y_i \backslash K_i \simeq (-1, 0) \times \Sigma_i$. We may assume that the orientation induced by $\overline{Y}_i$ on Σ_i is the opposite of the one induced by $\overline{U}$, and that K_i is a compact manifold with boundary diffeomorphic to $\overline{Y}_i$ and a deformation retract of $\overline{Y}_i$. Given this data we construct a compact oriented 7-manifold $M = (U \coprod_i Y_i)/\sim$ by identifying the i-th end of U with the end of Y_i; that is, the equivalence relation $\sim$ identifies $(s, x_i) \in (0,1) \times \Sigma_i \subset U$ with $(-s, x_i) \in (-1, 0) \times \Sigma_i \subset Y_i$. Thus U and each Y_i can be seen as open subsets of M, and moreover M can be decomposed as the disjoint union $\overline{U}_0 \coprod (\amalg_i (0,1) \times \Sigma_i) \coprod (\amalg_i K_i)$.

The real homology groups of M can be deduced from the Mayer–Vietoris exact sequence of the decomposition $M = U \cup (\coprod_i Y_i)$, which reads:

$$\cdots \to \oplus_i H_p(\Sigma_i) \to H_p(U) \oplus (\oplus_i H_p(Y_i)) \to H_p(M) \to \oplus_i H_{p-1}(\Sigma_i) \to \cdots \quad (4.11)$$

We are mainly interested in $p = 3, 4$. As $H_p(U) \simeq H_p(\overline{U})$ and $H_p(Y_i) \simeq H_p(\overline{Y}_i)$, the maps $\oplus_i H_p(\Sigma_i) \to H_p(U) \oplus (\oplus_i H_p(Y_i))$ in (4.11) are determined by the long exact sequences of the pairs $(\overline{U}, \partial\overline{U})$ and $(\overline{Y}_i, \partial\overline{Y}_i)$. A particular role is played by the boundary maps

$$\delta_i : H_3(\overline{Y}_i, \partial\overline{Y}_i) \to H_2(\partial\overline{Y}_i)$$

coming from the exact sequences of $(\overline{Y}_i, \partial\overline{Y}_i)$. In the next lemma, we show that if all the boundary maps δ_i are trivial, then $H_3(M)$ has a basis represented by cycles supported away from the gluing region.

Lemma 4.10. *Assume that $\delta_i = 0$ for all i. Then there are 3-cycles $C_1, \dots, C_n$ supported in U_0 and $C_{i,1}, \dots, C_{i,n_i}$ supported in K_i such that:*

(i) The homology classes $[C_k], [C_{ij}]$ form a basis of $H_3(M)$.

(ii) If $[D_k], [D_{ij}] \in H_4(M)$ is the dual basis for the intersection product, then the classes $[D_{ij}]$ can be represented by cycles supported in K_i.

Proof. Since $\delta_i = 0$, the maps $H_3(\overline{Y}_i) \to H_3(\overline{Y}_i, \partial\overline{Y}_i)$ are surjective. Thus we deduce that $H_3(Y_i) \simeq H_3(\overline{Y}_i) \simeq \mathrm{im}(H_3(\partial\overline{Y}_i) \to H_3(\overline{Y}_i)) \oplus E_i$, where $E_i \subset H_3(Y_i)$ is isomorphic to $H_3(\overline{Y}_i, \partial\overline{Y}_i)$. As the maps $H_2(\partial\overline{Y}_i) \to H_2(\overline{Y}_i)$ are injective, so is the map $\oplus_i H_2(\Sigma_i) \to H_2(U) \oplus (\oplus_i H_2(Y_i))$ in the exact sequence (4.11). Thus we obtain an exact sequence

$$\cdots \to \oplus_i H_3(\Sigma_i) \to H_3(U) \oplus (\oplus_i H_3(Y_i)) \to H_3(M) \to 0.$$

Hence there exists a subspace $E \subset H_3(U)$ such that $E \oplus E_1 \oplus \cdots E_m$ is a complement of the image of $\oplus_i H_3(\Sigma_i)$ in $H_3(U) \oplus (\oplus_i H_3(Y_i))$, and therefore $H_3(M) \simeq E \oplus E_1 \oplus \cdots \oplus E_m$. It follows that there are homology classes $[C_k] \in E \subset H_3(U)$, $k = 1, \dots, n$, and $[C_{ij}] \in E_i \subset H_3(Y_i)$, $i = 1, \dots, m$, $j = 1, \dots, n_i$, which form a basis of $H_3(M)$, where $n = \dim E$ and $n_i = \dim E_i$. Moreover, K_i is a deformation retract of $\overline{Y}_i$ and hence the classes $[C_{ij}]$ can be represented by cycles supported in K_i, and similarly the classes $[C_k]$ can be represented by cycles supported in U_0 since $H_3(U_0) \simeq H_3(U)$.

Now we prove that the homology classes $[D_{ij}] \in H_4(M)$ can be represented by cycles supported in K_i. As each $\overline{Y}_i$ is an oriented compact manifold with boundary, there are non-degenerate intersection pairings $H_4(\overline{Y}_i) \times H_3(\overline{Y}_i, \partial\overline{Y}_i) \to \mathbb{R}$. By construction, the basis $[C_{ij}]$ of $E_i \subset H_3(Y_i)$ induces a basis of $H_3(\overline{Y}_i, \partial\overline{Y}_i)$, and we denote by $[D'_{ij}] \in H_4(\overline{Y}_i)$ its dual basis for the intersection product of $\overline{Y}_i$. We can assume that the cycles D'_{ij} are supported in K_i since $H_4(K_i) \simeq H_4(\overline{Y}_i)$. As the classes $[C_1], \dots, [C_n] \in H_3(M)$ are represented by cycles supported in U_0 and the classes induced by $[D'_{ij}]$ in $H_4(M)$ are represented by cycles supported in K_i, the intersection of $[D'_{ij}]$ and $[C_k]$ is trivial in M, and the intersection of $[D'_{ij}]$ and $[C_{i'j'}]$ is 1 if $(i,j) = (i',j')$ and 0 otherwise. Thus $[D_{ij}] = [D'_{ij}] \in H_4(M)$. □

From the G_2-perspective, we typically think of U as the smooth locus of a singular G_2-manifold, and in particular U comes equipped with a torsion-free G_2-structure φ_0. The noncompact manifolds Y_i are endowed with families $\varphi_{i,t}$ of torsion-free G_2-structures with prescribed asymptotic behaviour, which should match the behaviour of φ_0 near the i-th end of U. One then uses some interpolation procedure to construct a family of closed G_2-structures φ_t on M, such that outside of the gluing region $\varphi_t|_{U_0} = \varphi_0|_{U_0}$ and $\varphi_t|_{K_i} = \varphi_{i,t}|_{K_i}$. Much of the subtlety of the construction lies in the choice of interpolation in the gluing region, but for our purpose these details are irrelevant. Provided the torsion of φ_t is small enough and there is some control on other geometric quantities (notably the injectivity radius and the norm of the curvature tensor), the general result of Joyce [66, Th. 11.6.1] ensures the existence of a torsion-free G_2-structure $\widetilde{\varphi}_t$ on M such that $[\widetilde{\varphi}_t] = [\varphi_t] \in H^3(M)$ and $\|\widetilde{\varphi}_t - \varphi_t\|_{C^0} \leq \epsilon_1$, where $\epsilon_1 > 0$ is some fixed small constant. By taking ϵ_1 small enough we can assume that $\|\widetilde{\varphi} - \varphi\|_{C^0} \leq \epsilon_1$ implies $2^{-1}g_\varphi \leq g_{\widetilde{\varphi}} \leq 2g_\varphi$ for any G_2-structures $\widetilde{\varphi}, \varphi$. The following theorem gives sufficient conditions for the path $\{\widetilde{\varphi}_t\}_{t\in(0,T]}$ to have finite energy and length in the moduli space:

Theorem 4.11. *Let $\{\widetilde{\varphi}_t\}_{t\in(0,T]}$ be a continuous family of torsion-free G_2-structures and $\{\varphi_t\}_{t\in(0,T]}$ be a family of closed G_2-structures on M, such that $[\widetilde{\varphi}_t] = [\varphi_t] \in H^3(M)$ and $\|\widetilde{\varphi}_t - \varphi_t\|_{C^0} \leq \epsilon_1$ for all $t \in (0,T]$. We assume that $\varphi_0 = \varphi_t|_{U_0}$ is independent of t, that each Y_i is endowed with a family of closed G_2-structures $\{\varphi_{i,t}\}_{t\in(0,T]}$ such that $\varphi_t|_{K_i} = \varphi_{i,t}|_{K_i}$ for all $t \in (0,T]$, and that the following assumptions are satisfied:*

- *(i) $b^1(M) = 0$, and each boundary map $\delta_i : H_3(\overline{Y}_i, \partial\overline{Y}_i) \to H_2(\partial\overline{Y}_i)$ is trivial.*
- *(ii) For all i and all $[C] \in H_3(Y_i)$, the function $f_{i,[C]}(t) = \int_{[C]} \varphi_{i,t}$ is of class C^2 and $f''_{i,[C]} \in L^1((0,T])$.*
- *(iii) There exists a metric g_i on each Y_i such that $g_{\varphi_{i,t}}\big|_{K_i} \leq g_i|_{K_i}$ for all $t \in (0,T]$.*

Then $\{\widetilde{\varphi}_t\}_{t\in(0,T]}$ induces a path of class C^2 in $\mathscr{M}$ with finite energy and length.

Proof. Let us consider the bases $[C_k], [C_{ij}] \in H_3(M)$ and $[D_k], [D_{ij}] \in H_4(M)$ provided by Lemma 4.10, and denote by $[C_k^*], [C_{ij}^*] \in H^3(M)$ and $[D_k^*], [D_{ij}^*] \in H^4(M)$ their respective dual bases. The cohomology class $[\widetilde{\varphi}_t] \in H^3(M)$ reads

$$[\widetilde{\varphi}_t] = [\varphi_t] = \sum_{k=1}^{n} a_k(t)[C_k^*] + \sum_{i=1}^{m}\sum_{j=1}^{n_i} a_{ij}(t)[C_{ij}^*]$$

where

$$a_k(t) = \int_{[C_k]} \varphi_t, \quad \text{and} \quad a_{ij}(t) = \int_{[C_{ij}]} \varphi_t.$$

Since the restriction $\varphi_t|_{U_0}$ is constant and the homology classes $[C_1], \ldots, [C_n]$ are represented by cycles supported in U_0, the functions $a_1, \ldots, a_n$ are constant. Moreover, as the homology classes $[C_{ij}]$ are represented by cycles supported in K_i and $\varphi_t|_{K_i} = \varphi_{i,t}|_{K_i}$, we deduce that $a_{ij} = f_{i,[C_{ij}]}$, and thus a_{ij} is of class C^2, a''_{ij} is L^1 and therefore a'_{ij} is uniformly bounded. This implies in particular that $\{\widetilde{\varphi}_t\}_{t\in(0,T]}$ induces a path of class C^2 in $\mathscr{M}$.

Similarly, the cohomology class of the 4-form can be written

$$[\Theta(\widetilde{\varphi}_t)] = \sum_{k=1}^{n} b_k(t)[D_k^*] + \sum_{i=1}^{m}\sum_{j=1}^{n_i} b_{ij}(t)[D_{ij}^*]$$

where

$$b_k(t) = \int_{[D_k]} \Theta(\widetilde{\varphi}_t) \quad \text{and} \quad b_{ij}(t) = \int_{[D_{ij}]} \Theta(\widetilde{\varphi}_t).$$

As the bases $[C_k^*], [C_{ij}^*] \in H^3(M)$ and $[D_k^*], [D_{ij}^*] \in H^4(M)$ are dual for the cup-product and the functions a_k are constant, it follows that:

$$\left\langle \frac{d[\widetilde{\varphi}_t]}{dt} \cup [\Theta(\widetilde{\varphi}_t)], [M] \right\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n_i} b_{ij}(t) a'_{ij}(t), \text{ and}$$
$$\left\langle \frac{d^2[\widetilde{\varphi}_t]}{dt^2} \cup [\Theta(\widetilde{\varphi}_t)], [M] \right\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n_i} b_{ij}(t) a''_{ij}(t).$$

We remark that $\mathrm{Vol}(\varphi_t) \geq \int_{U_0} \varphi_0 > 0$ for all $t \in (0, T]$, and as $\|\widetilde{\varphi}_t - \varphi_t\|_{C^0} \leq \epsilon_1$ it follows that $\mathrm{Vol}(\widetilde{\varphi}_t)$ is uniformly bounded below away from zero. Moreover, since the functions a'_{ij} are uniformly bounded and the functions a''_{ij} are L^1, it is enough to show that the functions b_{ij} are uniformly bounded to apply Corollary 4.9.

Since $\widetilde{\varphi}_t$ is co-closed the 4-form $\Theta(\widetilde{\varphi}_t)$ is a calibration, and by (4.10) we have

$$|b_{ij}(t)| = \left| \int_{[D_{ij}]} \Theta(\widetilde{\varphi}_t) \right| \leq \mathrm{Vol}(D_{ij}, g_{\widetilde{\varphi}_t}).$$

As $\|\widetilde{\varphi}_t - \varphi_t\|_{C^0} \leq \epsilon_1$ we have $g_{\widetilde{\varphi}_t} \leq 2 g_{\varphi_t}$, and thus $g_{\widetilde{\varphi}_t}\big|_{K_i} \leq 2\, g_{\varphi_{i,t}}\big|_{K_i} \leq 2\, g_i|_{K_i}$. By Lemma 4.10, we can assume that the 4-cycle D_{ij} is supported in K_i and thus

$$|b_{ij}(t)| \leq \mathrm{Vol}(D_{ij}, g_{\widetilde{\varphi}_t}) \leq 4\,\mathrm{Vol}(D_{ij}, g_i).$$

Hence the functions b_{ij} are uniformly bounded. Therefore the path $\{\widetilde{\varphi}_t\}_{t\in(0,T]}$ satisfies the assumptions of Corollary 4.9 and the theorem follows. □

In the case of the generalised Kummer construction [64, 65], U is the complement of the singular set of a G_2-orbifold T^7/Γ, where Γ is a finite subgroup of G_2, and thus U carries a flat G_2-structure φ_0. Each connected component of the singular set of T^7/Γ is assumed to have a neighbourhood isometric to either

1. $(T^3 \times B^4/G_i)/F_i$, where T^3 is a flat 3-torus, $B^4 \subset \mathbb{C}^2$ a Euclidean ball, G_i a finite subgroup of $\mathrm{SU}(2)$ acting freely on $\mathbb{C}^2$ except at the origin, and F_i a group of isometries of $T^3 \times \mathbb{C}^2/G_i$ acting freely on T^3; or
2. $(S^1 \times B^6/G_i)/F_i$, where S^1 is a flat circle, $B^6 \subset \mathbb{C}^3$ a Euclidean ball, G_i a finite subgroup of $\mathrm{SU}(3)$ acting freely on $\mathbb{C}^3$ except at the origin, and F_i a group of isometries of $S^1 \times \mathbb{C}^3/G_i$ acting freely on S^1.

The noncompact manifold Y_i used to resolve a singularity of the first type is $Y_i = (T^3 \times X_i)/F_i$ where X_i is an Asymptotically Locally Euclidean (ALE) space with holonomy $\mathrm{SU}(2)$ asymptotic to $\mathbb{C}^2/G_i$, equipped with an F_i-action such that $(T^3 \times X_i)/F_i$ is asymptotic to $(T^3 \times \mathbb{C}^2/G_i)/F_i$. It has boundary $\Sigma_i = (T^3 \times S^3/G_i)/F_i$, and in particular $H_2(\Sigma_i) \simeq H_2(T^3)^{F_i}$. In addition, it follows from the Künneth theorem – taking into account that $H_1(X_i) = 0$ since hyperkähler ALE spaces are simply connected – that $H_2(Y_i) \simeq H_2(T^3)^{F_i} \oplus H_2(X_i)^{F_i}$. Thus the map $H_2(\partial\overline{Y}_i) \to H_2(\overline{Y}_i)$ is injective, which implies that the boundary map $\delta_i : H_3(\overline{Y}_i, \partial\overline{Y}_i) \to H_2(\partial\overline{Y}_i)$ is trivial. The manifold Y_i is endowed with a family of torsion-free G_2-structures lifting to $T^3 \times X_i$ as

$$\varphi_{i,t} = \theta_1 \wedge \theta_2 \wedge \theta_3 - t^2(\theta_1 \wedge \omega_{i,1} + \theta_2 \wedge \omega_{i,2} + \theta_3 \wedge \omega_{i,3})$$

where $(\theta_1, \theta_2, \theta_3)$ is a basis of harmonic 1-forms on T^3 and $(\omega_{i,1}, \omega_{i,2}, \omega_{i,3})$ is an ALE hyperkähler triple on X_i. It follows that for any homology class $[C] \in H_3(Y_i)$ we have $f_{i,[C]}(t) = a_{i,[C]} + b_{i,[C]}t^2$ for some constants $a_{i,[C]}, b_{i,[C]} \in \mathbb{R}$. Moreover, the associated metric on $T^3 \times X_i$ is a product $g_{T^3} + t^2 g_{X_i}$, where g_{T^3} is a flat metric on T^3 and g_{X_i} is an ALE metric on X_i. In particular, if we fix an F_i-invariant radius function r on X_i adapted to the ALE asymptotics of g_{X_i} and $R > 0$ large enough such that the domain $\{r < R\} \subset Y_i$ is a deformation retract of X_i, then the compact subset $K_i = (T^3 \times \{r \leq R\})/F_i \subset Y_i$ can be identified with a compact subset of M which does not intersect the gluing region, so that for $t > 0$ small enough the closed G_2-structure φ_t obtained by gluing satisfies $g_{\varphi_t}|_{K_i} = g_{\varphi_{i,t}}\big|_{K_i} \leq g_{\varphi_{i,1}}\big|_{K_i}$ when $t \leq 1$.

For the second type of singularities, the manifold Y_i is of the form $Y_i = (S^1 \times Z_i)/F_i$ where Z_i is an ALE manifold with holonomy $\mathrm{SU}(3)$ asymptotic to $\mathbb{C}^3/G_i$, equipped with an F_i-action such that $(S^1 \times Z_i)/F_i$ is asymptotic to $(S^1 \times \mathbb{C}^3/G_i)/F_i$.

Here the boundary of $\overline{Y}_i$ is $\Sigma_i = (S^1 \times S^5/G_i)/F_i$ so that $H_2(\Sigma_i) = 0$, which in particular implies that the boundary map $\delta_i : H_3(\overline{Y}_i, \partial\overline{Y}_i) \to H_2(\partial\overline{Y}_i)$ is trivial. There is a family of torsion-free G_2-structures on Y_i which lifts to $S^1 \times Z_i$ as

$$\varphi_{i,t} = t^2\theta \wedge \omega_i + t^3 \operatorname{Re}\Omega_i$$

on Y_i, where θ is a nontrivial harmonic form on S^1, ω_i a Kähler form and Ω_i a holomorphic volume form on Z_i such that (ω_i, Ω_i) is an ALE torsion-free SU(3)-structure. Thus for any homology class $[C] \in H_3(Y_i)$ there are constants $a_{i,[C]}$ and $b_{i,[C]}$ such that $f_{i,[C]}(t) = a_{i,[C]}t^2 + b_{i,[C]}t^3$. As in the previous case the metric $g_{\varphi_{i,t}}$ lifts to the product $g_{S^1} + t^2 g_{Z_i}$ on $S^1 \times Z_i$, and hence we can fix a compact region $K_i \subset Y_i$, identified with a compact subset of M which does not intersect the gluing region, such that for any small enough $t > 0$ the glued closed G_2-structure φ_t satisfies $g_{\varphi_t}|_{K_i} = g_{\varphi_{i,t}}\big|_{K_i} \leq g_{\varphi_{i,1}}\big|_{K_i}$ for $t \leq 1$.

When the gluing data of the generalised Kummer construction is chosen so that $b^1(M) = 0$, all of the assumptions of our theorem are satisfied and thus the degeneration to T^7/Γ corresponds to a finite-distance limit in the moduli space.

Corollary 4.12. *The generalised Kummer* G_2*-manifolds constructed in [64, 65] have incomplete moduli spaces.*

4.3 Further observations

In the last section of this chapter we make some additional comments on the incompleteness question. In §4.3.1, we give a necessary condition for the limit of a path whose cohomology classes form a line segment in $H^3(M)$ to be at infinite distance (this appears in [84]). In the next part, which is original material and does not appear in the article, we use the contrapositive to prove that the other known resolution methods also yield G_2-manifolds with incomplete moduli spaces. In 4.3.3 we finish with some open questions.

4.3.1 Infinite-distance limits and the volume of cycles. Let us consider a path of torsion-free G_2-structures $\{\varphi_t\}_{t\in(0,T]}$ whose cohomology classes form a line segment in $H^3(M)$ (see Figure 4.2). We saw in the previous section that this occurs in the generalised Kummer construction when all the singularities of T^7/Γ are resolved by gluing quotients of products of a 3-torus and a hyperkähler ALE space (then the cohomology class $[\varphi_t]$ is an affine function of t^2). We will see that it is also satisfied for the Joyce–Karigiannis construction and the resolution of isolated conical singularities. In this general situation, we can give a very simple

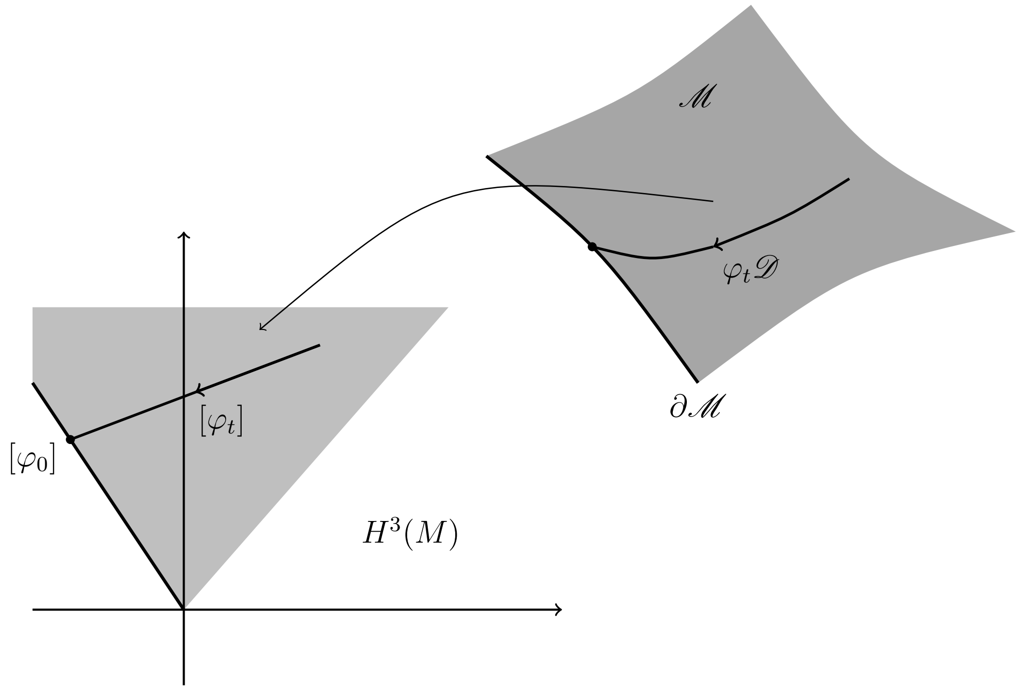

Figure 4.2: A path in $\mathscr{M}$ forming a line segment in $H^3(M)$.

sufficient condition for the length of the path to be finite; or by contrapositive, a necessary condition for the limit of this path to be at infinite distance.

To further motivate this question, let us compare with the case of Kähler cones, which we mentioned in introduction. Let X be a compact Kähler manifold of (complex) dimension $n \geq 2$ and let $\mathscr{K}$ be the cone of Kähler classes on X. This is an open convex cone of the space of real $(1,1)$-forms. Let us denote by $\overline{\mathscr{K}}$ the closure of $\mathscr{K}$ in $H^{1,1}(X;\mathbb{R})$, and $\partial\mathscr{K} = \overline{\mathscr{K}} \backslash \mathscr{K}$. The Kähler cone has a natural metric (the Hodge metric), which comes from the Hessian potential $-\log\int_X \omega^n/n! = -\log\operatorname{Vol}(\omega)$. A natural question to ask is which classes $\alpha \in \partial\mathscr{K}$ (called numerically effective, or *nef* classes in complex geometry) correspond to infinite-distance limits for this metric. The answer is simple: either $\int_X \alpha^n = 0$, in which case this is an infinite-distance limit; or $\int_X \alpha^n > 0$ (such a class is called *big*) and the limit is at finite distance [92]. Let us give a brief sketch of the argument. If $\int_X \alpha^n = 0$, then any path $\omega_t \in \mathscr{K}$ converging to $\alpha \in H^{1,1}(X;\mathbb{R})$ has $\int_X \omega_t^n \to 0$, and such limit is at infinite distance. Otherwise $\int_X \alpha^n > 0$, and if $\omega \in \mathscr{K}$ we can consider the path $\omega_t = \alpha + t\omega$. For $t \in (0,\infty)$ we have $\omega_t \in \mathscr{K}$, and moreover $\frac{d\omega_t}{dt} = \omega$ and $\frac{d^2\omega_t}{dt^2} = 0$. Arguing along the same lines as in the proof of Proposition 4.8, one can see that for any $\tau \in (0,1]$ the energy of the path $\{\omega_t\}_{t\in(\tau,1]}$ is

$$E^1_\tau(\omega_t) = \frac{\int_X \omega \wedge (\alpha + \tau\omega)^{n-1}}{(n-1)!\operatorname{Vol}(\omega_\tau)} - \frac{\int_X \omega \wedge (\alpha + \omega)^{n-1}}{(n-1)!\operatorname{Vol}(\omega_1)}.$$

Since $\int_X \alpha^n > 0$, $\mathrm{Vol}(\omega_\tau)$ is uniformly bounded below away from zero as $\tau \to 0$, and as the numerator of the first term is a polynomial function of τ it remains bounded as $\tau \to 0$. It follows that the path $\{\omega_t\}_{t\in(0,1]}$ has finite energy and length in $\mathscr{K}$, and thus α is a finite-distance limit.

Let us now go back to the G_2-case, and consider family of torsion-free G_2-structures $\{\varphi_t\}_{t\in(0,T]}$ on a compact 7-manifold M with $b^1(M) = 0$, inducing a smooth path in the moduli space $\mathscr{M}$. We assume that $[\varphi_t] = [\varphi_0] + t[\dot\varphi]$, where $[\varphi_0], [\dot\varphi] \in H^3(M)$ are fixed cohomology classes. Heuristically, we want to think of $[\varphi_0]$ as lying on the boundary of the image of $\mathscr{M}$ in $H^3(M)$. As $\ddot\varphi_t = 0$ we have $\frac{d^2\mathscr{F}(\varphi_t)}{dt^2} = D^2_{\varphi_t}\mathscr{F}(\dot\varphi_t, \dot\varphi_t) \geq 0$, and thus the function $\mathscr{F}(\varphi_t) = -3\log \mathrm{Vol}(\varphi_t)$ is convex on $(0,T]$. Hence $\mathrm{Vol}(\varphi_t)$ is bounded above, and either $\mathrm{Vol}(\varphi_t) \to 0$ as $t \to 0$ or it is uniformly bounded below away from zero. If the volume shrinks to zero, then the limit of φ_t as $t \to 0$ is at infinite distance in the moduli space. The remaining interesting case is when $\mathrm{Vol}(\varphi_t)$ is uniformly bounded below away from zero. As $\frac{d^2[\varphi_t]}{dt^2} = 0$, the energy takes a particularly simple form:

$$E^T_\tau(\varphi_t) = \frac{\langle [\dot\varphi] \cup [\Theta(\varphi_\tau)], [M]\rangle}{\mathrm{Vol}(\varphi_\tau)} - \frac{\langle [\dot\varphi] \cup [\Theta(\varphi_T)], [M]\rangle}{\mathrm{Vol}(\varphi_T)}. \tag{4.12}$$

So far the situation is very similar to that of Kähler cones, except for one crucial difference, which is that $[\Theta(\varphi_\tau)]$ has no reason to be a merely polynomial function of τ. Thus we do not know whether the numerator of the first term remains bounded as $\tau \to 0$, and it may be that the limit of φ_t as $t \to 0$ is at infinite distance in the moduli space even though the volume is bounded below. Such a phenomenon, if it occurs, would be a feature of G_2-moduli spaces that has no analogy in the geometry of Kähler cones.

To gain more insight into the geometry of such a situation, let us denote by $\mathrm{PD}[\dot\varphi] \in H_4(M)$ the Poincaré-dual class of $[\dot\varphi]$. Then we can bound the numerators in (4.12) by

$$|\langle [\dot\varphi] \cup [\Theta(\varphi_t)], [M]\rangle| = \left|\int_{\mathrm{PD}[\dot\varphi]} \Theta(\varphi_t)\right| \leq \mathrm{Vol}(\mathrm{PD}[\dot\varphi], g_{\varphi_t}) \tag{4.13}$$

where we define $\mathrm{Vol}(\mathrm{PD}[\dot\varphi], g_{\varphi_t})$ as the infimum of $\mathrm{Vol}(D, g_{\varphi_t})$ taken over the 4-cycles D representing $\mathrm{PD}[\dot\varphi] \in H_4(M)$. If $\mathrm{Vol}(\mathrm{PD}[\dot\varphi], g_{\varphi_t})$ is bounded, then we easily deduce that the energy of $\{\varphi_t\}_{t\in(0,T]}$ is finite. As the energy $E^T_\tau(\varphi_t)$ is a decreasing function of τ, it is in fact enough to assume that $\mathrm{Vol}(\mathrm{PD}[\dot\varphi], g_{\varphi_{t_i}})$ is uniformly bounded for some sequence $t_i \to 0$. By contrapositive, we obtain:

Proposition 4.13. *Let $\{\varphi_t\}_{t\in(0,T]}$ be a family of torsion-free G_2-structures on M inducing a smooth path in $\mathscr{M}$, and suppose that the cohomology class $\frac{d[\varphi_t]}{dt} = [\dot\varphi]$*

is constant in $H^3(M)$ and that the volume of (M, φ_t) is uniformly bounded below away from zero. If the limit of φ_t as $t \to 0$ is at infinite distance in the moduli space, then

$$\mathrm{Vol}(\mathrm{PD}[\dot{\varphi}], g_{\varphi_t}) \to \infty \quad \textit{as} \quad t \to 0.$$

Heuristically, this result says that if there is a point of the boundary of $\mathscr{M}$ that can be approached by a path of torsion-free G_2-structures whose cohomology classes form a line segment in $H^3(M)$, then there is the following trichotomy. Either the volume is shrinking to zero along the path, in which case the limit is at infinite distance; or the volume is bounded below away from zero and the length of the path is infinite, and in that case there must be a homology class in degree 4 whose volume is going to infinity; or this is a finite-distance limit. Some of the generalised Kummer G_2-manifolds provide examples of the third case, and the first case occurs for instance by scaling any torsion-free G_2-structure. However, we do not know if the second case can happen, and as previously noted this would have to be a phenomenon specific to G_2-moduli spaces, by contrast with what can occur at the boundary of Kähler cones.

Proposition 4.13 can be generalised to families of torsion-free G_2-structures whose cohomology classes define a path in $H^3(M)$ which is regular enough. For instance, it is straightforward to deduce from Corollary 4.9 the following:

Proposition 4.14. *Let $\{\varphi_t\}_{t\in(0,T]}$ be a family of torsion-free G_2-structures on M inducing a path of class C^2 in $\mathscr{M}$, and assume that the following conditions are satisfied:*

(i) The volume $\mathrm{Vol}(\varphi_t)$ is uniformly bounded below away from zero.

(ii) The function $t \in (0, T] \to \frac{d^2}{dt^2}[\varphi_t] \in H^3(M)$ is L^1.

(iii) The volume of any homology class $[D] \in H_4(M)$ with respect to g_{φ_t} is uniformly bounded.

Then $\{\varphi_t\}_{t\in(0,T]}$ has finite energy and length in the moduli space.

In the next part we shall use this proposition to prove that G_2-manifolds constructed by the Joyce–Karigiannis construction [68] or by resolution of isolated conical singularities [72] also correspond to finite-distance degenerations in the moduli space. In these cases it is clear that the total volume is bounded below away from zero and that all homology classes have bounded volume, and the only assumption that is left to check is the one concerning the derivatives of the path of cohomology classes. In fact in the first two cases the path of cohomology classes form a line segment in H^3 so this condition is easy to verify; and for the resolutions of T^7/Γ the path of cohomology classes is a polynomial function of the gluing parameter.

4.3.2 Other degenerate limits. Let us begin with the Joyce–Karigiannis construction. Although the geometrical aspects of this construction are substantially more complicated than the generalised Kummer construction that we treated earlier, its topology is quite simple. Specifically, we will see that we are again in the very favourable situation where we can choose a basis of 3-cycles supported away from the gluing region, which makes it very easy to compute the path of cohomology classes.

First, let us briefly outline the topology of the construction. It starts with a G_2-manifold (N, φ) endowed with an involution ι preserving φ, such that the fixed locus is an associative submanifold $L \subset N$, and a nowhere-vanishing harmonic 1-form $\lambda \in \Omega^1(M)$. Let ν be the normal bundle of L in N. The G_2-structure φ induces an identification of T^*L with $\Lambda^2_+\nu^*$ by inserting a 1-form into φ. Consequently, there is a unique family of complex structures $J : \nu \to \nu$ defined on the fibres of the normal bundle such that λ can be identified with $h(J\cdot, \cdot)$, where h is the restriction of g_φ to ν. One can use this data to construct a fibre bundle $P \to L$ by blowing-up each fibre $\nu_p/\langle\iota\rangle \simeq \mathbb{R}^4/\{\pm 1\}$ with respect to the complex structure J_p. The exceptional divisors fit into a bundle $Q \to L$ with fibres diffeomorphic to S^2, and there is a natural map $P\backslash Q \to (\nu/\langle\iota\rangle)\backslash L$. Using a ι-invariant tubular neighbourhood of L in N, one can resolve the orbifold $N/\langle\iota\rangle$ by excising a small neighbourhood of L and gluing in a neighbourhood of Q in P so as to obtain a compact manifold M.

In [68, Prop. 6.1], the authors prove that for any $0 \leq k \leq 7$ the cohomology group $H^k(M)$ is isomorphic to $H^k(N/\langle\iota\rangle) \oplus H^{k-2}(L)$. The proof makes use of the important property that the normal bundle ν of L is trivial [68, Rem. 2.14]. In particular, the bundle P retracts onto $Q \simeq L \times S^2$ and therefore

$$H^k(P) \simeq H^k(L \times S^2) \simeq H^k(L) \oplus H^{k-2}(L).$$

Similarly, the gluing region of the construction, homeomorphic to $(\nu/\langle\iota\rangle)\backslash L$, is homotopy-equivalent to a trivial $\mathrm{SO}(3)$-bundle over L, and since $\mathrm{SO}(3) \simeq S^3/\{\pm 1\}$ is a rational homology sphere it follows that

$$H^k((\nu/\langle\iota\rangle)\backslash L) \simeq H^k(L \times \mathrm{SO}(3)) \simeq H^k(L) \oplus H^{k-3}(L).$$

By duality, if follows that $H_k(M) \simeq H_k(N/\langle\iota\rangle) \oplus H_{k-2}(L)$, $H_k(P) \simeq H_k(L) \oplus H_{k-2}(L)$ and $H_k((\nu/\langle\iota\rangle)\backslash L) \simeq H_k(L) \oplus H_{k-3}(L)$.

Let us now prove that we are in the situation described in Lemma 4.10. First, we claim that $H_3(N/\langle\iota\rangle) \simeq H_3((N/\langle\iota\rangle)\backslash L)$. It suffices to prove that the relative homology group $H_3(N/\langle\iota\rangle, (N/\langle\iota\rangle)\backslash L)$ vanishes. By excision, it is isomorphic to $H_3(\nu/\langle\iota\rangle, (\nu/\langle\iota\rangle)\backslash L)$. Now $\nu/\langle\iota\rangle$ retracts onto L and we see that the map $H_3(\nu/\langle\iota\rangle) \to H_3((\nu/\langle\iota\rangle)\backslash L)$ is surjective and the map $H_2(\nu/\langle\iota\rangle) \to H_2((\nu/\langle\iota\rangle)\backslash L)$ is an isomorphism. Therefore $H_3(N/\langle\iota\rangle, (N/\langle\iota\rangle)\backslash L) \simeq H_3(\nu/\langle\iota\rangle, (\nu/\langle\iota\rangle)\backslash L) = 0$. In particular $H_3(N/\langle\iota\rangle)$ has a basis of represented by cycles supported away from L. On the other hand, we have $H_2(P\backslash Q) \simeq H_2((\nu/\langle\iota\rangle)\backslash L) \simeq H_2(L)$ and $H_2(P) \simeq H_2(L) \oplus H_0(L)$ and therefore the map $H_2(P\backslash Q) \to H_2(P)$ is injective, which in turn implies that the boundary map $H_3(P, P\backslash Q) \to H_2(P\backslash Q)$ must be trivial. Therefore there is a basis of $H_3(M)$ represented by 3-cycles C_j supported away from the gluing region, and if D_j are cycles representing the dual basis of $H_4(M)$ we can ensure that D_j is supported in a compact region of P if C_j is.

The construction of a family of closed G_2-structures φ_t on M with sufficiently small torsion to be perturbed to a family of nearby torsion-free G_2-structures $\widetilde{\varphi}_t$ is quite involved, and we will not attempt to summarise it. Rather, we will just point out the two facts that are relevant to us. The first one is that, as for the generalised Kummer construction, the restriction of φ_t to the interior of $(N/\langle\iota\rangle)\backslash L$ away from the gluing region coincides with the original torsion-free G_2-structure φ. Secondly, in a neighbourhood of Q in P the 3-form φ_t is affine in the variable t^2. Hence the path of cohomology classes $[\widetilde{\varphi}_t] \in H^3(M)$ forms a line segment. In the limit where $t \to \infty$, $(M, g_{\widetilde{\varphi}_t})$ converges to the orbifold $(N/\langle\iota\rangle, g_\varphi)$, and we can apply either Theorem 4.11 or Proposition 4.13 to deduce that this occurs at finite distance in the moduli space. Hence we deduce:

Corollary 4.15. *Let $(M, \widetilde{\varphi}_t)$ be a 1-parameter family of torsion-free G_2-structures obtained from the Joyce–Karigiannis construction [68]. Then the limit $t \to 0$ lies at finite distance in the moduli space.*

Let us now move on to the resolution of G_2-manifolds with isolated conical singularities. This construction was carried out by Karigiannis in [72], and takes as building blocks a singular compact G_2-manifold (N, φ_0) with isolated singularities $p_1, \dots, p_m$ asymptotic (with rates $\mu_1, \dots, \mu_m > 0$) to a G_2-cone $(C_i \simeq (0,\infty) \times \Sigma_i, \varphi_{C_i})$[2] and a finite collection of noncompact asymptotically conical G_2-manifolds $(Y_1, \varphi_1), \dots, (Y_m, \varphi_m)$ asymptotic to $C_1, \dots, C_m$ (with rates $\nu_1, \dots, \nu_m \leq -3$). We

[2]As of today, there are no examples of compact G_2-manifolds with isolated conical singularities, and only a handful of G_2-cones. Hence this construction does not produce new examples of compact G_2-manifolds yet. However, it is generally expected that compact conically singular G_2-manifolds should exist in large numbers.

do not need to know much about the technical details of the construction, but an important fact is that it is generally obstructed and we need to describe the obstructions in order to understand the topological aspects of the construction and compute the path of cohomology classes.

In a nutshell, each singularity p_i has a pointed neighbourhood diffeomorphic to $(0, \epsilon) \times \Sigma_i$ where φ reads

$$\varphi_0 = \varphi_{C_i} + d\alpha_i$$

where φ_{C_i} is a conical torsion-free G_2-structure on the cone C_i and $d\alpha_i = O(r^{\mu_i})$ (r is the radial coordinate function). Likewise, the end of Y_i is diffeomorphic to $(R, \infty) \times \Sigma_i$ and φ_i can be written

$$\varphi_i = \varphi_{C_i} + \xi_i + d\zeta_i$$

where ξ_i is a harmonic 3-form on the link Σ_i and ζ_i a 2-form with $|\zeta_i| = O(r^{-3})$ and $|d\zeta_i| = O(r^{-4})$. Note that for the cone metric we have $|\xi_i| = O(r^{-3})$. In addition, the leading order term of $d\zeta_i$ can be written $-\nu_i/r$ for some harmonic 2-form ν_i on σ_i, so that $d\zeta_i = r^{-2} dr \wedge \nu_i$. Let us write η_i the dual 4-form of ν_i on Σ_i.

From this we see that in order to interpolate between φ_0 and φ_i, one needs a harmonic 3-form ξ on N asymptotic to ξ_i near each singular point p_i. This gives a first obstruction which can be shown to be topological using weighted Hodge theory (see [72, Th. 3.10]): the obstruction vanishes if and only if the tuple of cohomology classes $([\xi_1], \ldots, [\xi_m]) \in \oplus_i H^3(\Sigma_i)$ lies in the image of $H^3(N')$, where N' is the compact manifold with boundary obtained by replacing each p_i with a boundary component homeomorphic to Σ_i. There is also a second topological obstruction coming from the interpolation of the dual 4-forms: namely, in order to do the construction one needs to assume that the tuple of cohomology classes $([\eta_1], \ldots, [\eta_m]) \in \oplus_i H^4(\Sigma_i)$ lies in the image of $H^4(N')$ in the long exact sequence of the pair $(N', \amalg_i \Sigma_i)$.

Remark 4.16. The harmonic form ξ_i necessarily vanishes if $\nu_i < -3$, and so does η_i if $\nu_i < -4$. Hence the obstructions only occur if the AC manifolds Y_i have a (relatively) slow convergence rate to their asymptotic cone. One cannot hope that the convergence could be faster in general since the three asymptotically conical Bryant-Salamon manifolds have convergence rate -3 or -4 [75].

Assuming that the first obstruction vanishes, then one may find a harmonic 3-form on N such that $\xi = \xi_i + dA_i$ on $(0, \epsilon) \times \Sigma_i$ near p_i, where dA_i decays faster than $O(r^{-3})$. Let us then define, for some small $t > 0$, the closed G_2-structures $\varphi_{0,t} = \varphi_0 + t^3 \xi$ on N and $\varphi_{i,t} = t^3 \varphi_t$ on Y_i. Then $\varphi_{0,t}$ is equal to

$\varphi_{C_i}+t^3\xi_i+d(\alpha_i+t^3A_i)$ on $(0,\epsilon)\times\Sigma_i$, and $\varphi_{i,t}=t^3\varphi_{C_i}+t^3\xi_i+d(t^3\zeta_i)$ on $(R,\infty)\times\Sigma_i$. In particular, the domain $(R,t^{-1}\epsilon)\times\Sigma_i$ of Y_i is almost isometric to the domain (tR,ϵ) of N, and we can form a compact manifold M_t by identifying the two regions in order to resolve each conical singularity p_i by gluing in a rescaled copy of Y_i. Clearly, all the compact manifolds M_t are diffeomorphic to some fixed manifold M. Moreover M_t is endowed with a closed G_2-structure φ_t which is equal to $\varphi_{0,t}$ away from the singularities in the bulk of N, to $\varphi_{i,t}$ away from the ends in the bulk of Y_i, and to

$$\varphi_{C_i}+t^3\xi_i+d(\chi_t(\alpha_i+t^3A_i)+(1-\chi_t)t^3T_t^*\zeta_i) \tag{4.14}$$

in the gluing region $(tR,\epsilon)\times\Sigma$. In this expression, T_t is the homothety of factor t^{-1} of the cone and χ_t is a cutoff function[3] of r equal to 0 if $r\leq tR$ and 1 if $r\geq\epsilon$. Note that $T_t^*\varphi_i=t^{-3}\varphi_{C_i}$ and $T_t^*\xi_i=\xi_i$ which explains why T_t only acts on ζ_i.

Using Joyce's theorem, Karigiannis proved that provided the second obstruction also vanishes, then for t small enough φ_t can be perturbed to a nearby torsion-free G_2-structure $\widetilde{\varphi}_t$ within the same cohomology class. Thus we obtain a continuous 1-parameter family of torsion-free G_2-structures on M, which can be shown to converge to the original conically singular manifold (N,φ_0) in the Gromov-Hausdorff sense as $t\to 0$. In particular, there is a definite volume lower bound, and moreover for any $0\leq k\leq 7$ it is easy to construct a basis of $H_k(M)$ represented by cycles whose volume is uniformly bounded (in a moment we will do it for $k=3$). In turns out that the path of cohomology classes $[\widetilde{\varphi}_t]=[\varphi_t]\in H^3(M)$ is an affine function of the variable t^3, and therefore:

Corollary 4.17. *Let $(M,\widetilde{\varphi}_t)$ be the 1-parameter family of torsion-free G_2-structures constructed by resolving a compact G_2-manifold with isolated conical singularities as in [72]. Then the limit $t\to 0$ lies at finite distance in the moduli space.*

Let us justify our claim about the path of cohomology classes. If suffices to prove that there is a basis of homology classes $[C_j]\in H_3(M)$ such that $\int_{[C_j]}\varphi_t = a_i+t^3b_i$ for some $a_i,b_i\in\mathbb{R}$. Now there are two sources of nontrivial homology classes in $H_3(M)$. The first source comes from the cycles supported away from the gluing region, for which the claim is obvious. Those cycles represent the image of $H_3(U_t)\oplus(\oplus_i H_3(Y_{i,t}))\to H_3(M)$ in the exact sequence, where U_t is obtained from N by excising the neighbourhood $(0,tR)\times\Sigma_i$ of each singularity and $Y_{i,t}$ is obtained from Y_i by excising the end $(t^{-1}\epsilon,\infty)\times\Sigma_i$. A complement

[3] For the analysis it is important to make a good choice of cutoff function, but this does not affect the cohomology class of $\widetilde{\varphi}_t$ since a different choice would only add a globally exact form.

of this image is spanned by cycles which can be constructed as follows. Let B_i be a collection of 2-cycles in Σ_i such that $[B_i] \in H_2(\Sigma_i)$ lies in the image of the boundary map $\delta_i : H_3(Y_i, \Sigma_i) \to H_2(\Sigma_2)$ and $\oplus_i[B_i]$ belongs to the image the map $H_3(U_\epsilon) \to \oplus_i H_2(\Sigma_i)$. Then there is a 3-cycle C supported in $\overline{U}_\epsilon$ such that $\partial C_0 = -(B_1 + \cdots + B_m) \times \{\epsilon\}$ and for each i there is a 3-cycle C_i in $\overline{Y}_{i,R}$ such that $\partial C_i = B_i \times \{R\}$. In the gluing region $(t^{-1}R, \epsilon) \times \Sigma_i$ we consider $C' = \oplus_i B_i \times [t^{-1}R, \epsilon]$ (appropriately subdivided into 3-simplices), and adding them all we obtain a cycle $C = C_0 + C_1 + \ldots + C_m + C'$ in M.

Let us now calculate $\int_C \varphi_s$. Clearly each term $\int_{C_j} \varphi_t = \int_{C_j} \varphi_{j,t}$ is an affine function of t^3 for $0 \leq j \leq m$, so the only thing that requires some care is the computation of $\int_{C'} \varphi_t$. In the gluing region $(tR, \epsilon) \times \Sigma_i$, φ_t is given by (4.14). The integral $\int_{C'} \xi_i$ vanishes since ξ_i is a 3-form on Σ_i and therefore $\partial_r \lrcorner \xi_i = 0$. On the other hand, the conical torsion-free G_2-structure can be written $\varphi_{C_i} = d(r^3\omega_i)$ for some 2-form on Σ_i [72, Prop. 2.4] and hence Stockes' theorem yields

$$\int_{C'} \varphi_t = \sum_{i=1}^{m} (\epsilon^3 - t^3R^3) \int_{[B_i]} \omega_i + \int_{B_i \times \{\epsilon\}} \alpha_i + t^3 \int_{B_i \times \{\epsilon\}} A_i - \int_{B_i \times \{R\}} t^3 \zeta_i$$

which is again manifestly an affine function of the variable t^3. It is interesting to note that we are again in the situation where the path of cohomology classes forms a line segment in $H^3(M)$.

Let us make a few comments on the general method of resolution of flat G_2-orbifolds T^7/Γ described in [66, Ch. 11], which we will not attempt to treat in detail. The idea is to resolve the singularities by gluing in various rescaled manifolds with appropriate asymptotics (coined *QALE* G_2*-manifolds* in the monograph). Describing the topology of these resolutions would be fastidious in general since various components of the singular set might intersect, but since we are rescaling the QALE manifolds by powers of a gluing parameter seems possible that one could obtain polynomial bounds for the derivatives of the path of cohomology classes of the glued G_2-structure (by choosing an appropriate parametrisation, if necessary). In the same way, we can always construct a collection of cycles representing a basis of the homology of M by patching up various cycles on each piece of the construction, and these will have uniformly bounded volume. Hence we would expect that these more complicated resolutions of T^7/Γ also yield finite-distance degenerate limits in the moduli space, although one would need to write things more carefully in order to properly justify it.

4.3.3 Follow-up questions. Let us finish this chapter with a few remarks and open questions. An interesting problem would be to give sufficient conditions for the limit of a path (or a sequence) in the moduli space to be at infinite distance. Since this is always the case when the volume diverges to 0 or ∞, we need to fix the volume for this question to be interesting. In the case of twisted connected sums for instance, the volume grows linearly with the length of the neck region and therefore this represents an infinite-distance limit in the moduli space – even though we could deduce from Proposition 4.8 that the energy of the obvious path is bounded. However, if we normalise the volume it is no longer clear that the distance is unbounded. The main challenge is that in order to find a lower bound on the distance between two points in the moduli space, we need to control the length of *all* paths connecting them, whereas the length of one particular path is enough to give an upper bound. It is nevertheless interesting to remark that the length of the volume-normalised neck-stretching path is indeed infinite.

Let us briefly sketch the argument. First, let us recall the notations of Section 2.2: we consider a twisted connected sum of two asymptotically cylindrical G_2-manifolds (Z_1, φ_1) and (Z_2, φ_2), resulting in a compact manifold M_T endowed with a glued G_2-structure φ_T, which can be deformed to a nearby torsion-free G_2-structure $\widetilde{\varphi}_T$ for T large enough, with $\|\widetilde{\varphi}_T - \varphi_T\|_{C^k} = O(e^{-\delta T})$ as $T \to \infty$ (for any $k \geq 0$ and small enough $\delta > 0$). Moreover, $[\widetilde{\varphi}_T] = [\varphi_T] \in H^3(M_T)$, and with the estimates of Proposition 2.16 we can well approximate the harmonic forms with respect to $g_{\widetilde{\varphi}_T}$ by the harmonic forms with respect to g_{φ_T}, which themselves can be approximated by matching pairs of harmonic forms as in §3.4.1.

The matching G_2-structures φ_i are asymptotic to $\mathrm{Re}(\Omega) + dt \wedge \omega$, where (ω, Ω) is a torsion-free SU(3)-structure on the common cross-section X of Z_i. Moreover, using the decomposition of the space of bounded harmonic forms described in §3.4.1, there is a unique *exact* harmonic form η_i on Z_i asymptotic to $dt \wedge \omega$ on each Z_i. By [95, Prop. 3.2], we have $\frac{d}{dt}[\varphi_T] = 2\delta([\omega])$ in $H^3(M_T)$, where $[\omega] \in H^2(X)$ and the map $\delta : H^2(X) \to H^3(M_T)$ comes from the Mayer-Vietoris exact sequence associated with the gluing. Moreover, the class $\delta([\omega]) \in H^3(M_T)$ can be represented by the closed form $\eta_T/T \in \Omega^3(M_T)$, where η_T is the approximate harmonic form obtained from the matching pair η_1, η_2 as in §3.4.1 (this can be deduced from the proof of [95, Prop. 3.2]).

Let $\mathscr{M}$ be the moduli space of torsion-free G_2-structures on $M = M_T$. Since the metric $\mathscr{G}$ splits a line in the volume direction (Lemma 4.1), the length of the *volume-normalised* path induced by $\{\widetilde{\varphi}_T / \mathrm{Vol}(\widetilde{\varphi}_T)^{3/7}\}_{T \in [T_0, \infty)} \subset \mathscr{M}$ is equal to $\int_{T_0}^{\infty} \sqrt{\mathscr{G}_{\widetilde{\varphi}_T}(\pi_{27}(\dot{\widetilde{\varphi}}_T), \pi_{27}(\dot{\widetilde{\varphi}}_T))} dt$. As $T \to \infty$, the harmonic form $\pi_{27}(\dot{\widetilde{\varphi}}_T)$ can be approximated by

$$\nu_T = \frac{2}{T}\eta_T - \frac{2}{7T}\langle \eta_T, \varphi_T \rangle_{\varphi_T} \varphi_T = \frac{2}{T}(\eta_T - f\varphi_T)$$

where $f = \langle \eta_T, \varphi_T \rangle_{\varphi_T}/7$. In the neck region, $f \sim c$ is almost a constant, and hence

$$\eta_T \sim \frac{2}{T}((1-c)dt \wedge \omega - c\,\mathrm{Re}(\Omega))$$

in the neck region, whence in this region

$$|\eta_T|^2_{\varphi_T} \sim \frac{(1-c)^2|dt \wedge \omega|^2_{g_X} + c^2|\,\mathrm{Re}(\Omega)|^2_{g_X}}{T^2} \sim \frac{C}{T^2}$$

where $C > 0$ is a *non-trivial* constant. Since the length of the neck region and the volume of $\tilde{\varphi}_T$ both grow linearly with T, it follows that

$$\sqrt{\mathscr{G}_{\widetilde{\varphi}_T}(\pi_{27}(\dot{\tilde{\varphi}}_T), \pi_{27}(\dot{\tilde{\varphi}}_T))} = \frac{C'}{T} + o(1/T)$$

as $T \to \infty$, for some other non-trivial constant $C' > 0$. Hence we deduce:

Lemma 4.18. *The volume-normalised path* $\{\widetilde{\varphi}_T / \operatorname{Vol}(\widetilde{\varphi}_T)^{3/7}\}_{T_{0,\infty}} \subset \mathscr{M}$ *has infinite length with respect to the metric* $\mathscr{G}$*. More precisely, the length of the path grows logarithmically with* T*.*

Remark 4.19. The fact that the speed of the path is proportional to $\frac{1}{T}$ is consistent with the fact that this path has bounded energy, even though the length diverges.

Remark 4.20. From the physics perspective, the logarithmic behaviour of the length of the path would be in line with the conjectures mentioned in the introduction. Indeed, if this path turned out to be close to distance-minimising, it would mean that the parameter T is roughly some power of the exponential of the moduli space distance, and by the spectral estimates of the previous chapter this means that the eigenvalues of the Laplacian decay exponentially with the distance. This is precisely what the swampland distance conjecture predicts.

Coming back to the general case, perhaps (if we assume that the volume is normalised) one could argue that the diameter remains bounded on bounded subsets of the moduli space; that is, finite-distance limits are noncollapsed. This probably has more to do with Ricci-flatness than G_2 holonomy, and might hold more generally for manifolds with holonomy $SU(m)$ and Spin(7). Note that it is true for hyperkähler K3 surfaces (holonomy SU(2)) and that this does not seem to contradict the known results about finite-distance limits in Calabi–Yau moduli spaces[4]. In the case of G_2, all the degenerations of compact manifolds with G_2-holonomy which we proved to occur at finite distance in this chapter are noncollapsed. But to the author's knowledge there is no general statement of this sort[5].

If this turned out to be true, one could even fantasize about studying the completion of the moduli spaces (in the sense of the completion with respect to the distance associated with $\mathscr{G}$). Indeed, if any bounded sequence in the moduli space has bounded diameter, then (if the volume is normalised) Bishop–Gromov volume monotonicity implies that the sequence is noncollapsed, and the theory developed by Cheeger and Colding [23] implies that a subsequence must converge in the Gromov–Hausdorff sense to a compact, singular space isometric to the completion of an open Ricci-flat manifold. In fact the limit is a singular space with holonomy contained in G_2 (or whichever Ricci-flat holonomy group we started with) [25]. If the sequence we started with is in fact Cauchy, maybe one could hope that it actually converges (and not just subconverges) in the Gromov–Hausdorff sense. But even assuming that we can prove appropriate diameter bounds this would be far from obvious because the moduli space distance is a priori much weaker than the Gromov–Hausdorff distance.

[4]Remark that the Calabi–Yau case seems to indicate that the converse statement is unlikely to hold, that is, not all noncollapsed limits are at finite distance in the moduli space. Indeed, the result of Wang [117] states that one-parameter degenerations of polarised projective Calabi–Yau manifolds which occur at finite-distance correspond to varieties with *canonical singularities*, whilst Donaldson–Sun proved that in general non-collapsed limits of projective Calabi–Yau manifolds have *log terminal singularities* [38, §4].

[5]There is a statement along those lines in [7, Th. II], which concerns the case of Ricci-flat 4-manifolds (not necessarily hyperkähler) although the article claims that it would be straightforward to adapt it to all dimensions. Unfortunately, the proof appears to be erroneous (even in dimension 4). At first I thought that there might be an easy fix for it and spent some time trying to find one, but now I tend to believe that the argument cannot be saved.

Chapter 5

A period mapping

The material presented in this chapter originally grew out of an attempt to better understand the curvature of the metric $\mathscr{G}$ on G_2-moduli spaces. I was interested in knowing if it had any special properties or if there would be some universal bounds, in relation with certain conjectures in Kähler geometry [121] and the swampland distance conjectures in physics [98]. Since the metric $\mathscr{G}$ is Hessian, these properties are determined by the derivatives of the potential $\mathscr{F}$, which are quite difficult to understand due to the high degree of nonlinearity of this function. The third-order derivative is also of particular interest in physics: it determines a symmetric cubic form called the *Yukawa coupling*, by analogy with the Yukawa coupling of Calabi–Yau moduli spaces which plays an important role in mirror symmetry [116].

Up to order 3, one may obtain compact formulas for the derivatives, but this becomes substantially more difficult at higher order. At order 4, the author managed to obtain a formula depending on the lower order derivatives together with some 'extra terms' depending on the Green's function of the Laplacian. Unfortunately, we could not find a way to compute or estimate these terms and go further in our understanding of the metric $\mathscr{G}$ using only local coordinate computations[1], except in some easy cases described in Chapter 6.

This difficulty motivated us to introduce a new perspective on the geometry of G_2-moduli spaces. For this we drew inspiration from the notion of Weil–Petersson geometry developed by Lu and Sun [90, 91] which axiomatises the geometric properties of Calabi–Yau moduli spaces, relying on the theory of period maps introduced by Griffiths. Our starting observation is that by 'twisting' the Hodge decomposition of a compact G_2-manifold M, we can define a natural immersion of the moduli space $\mathscr{M}$ into a homogeneous space $\mathfrak{D}$ diffeomorphic to

[1] Although we are aware of ongoing work in this direction by Karigiannis and Loftin [74], whom we thank for pointing it out.

$\mathrm{GL}(b^3(M))/(\{\pm 1\} \times \mathrm{O}(b^3(M)-1))$, which satisfies properties analogous to Griffiths' transversality and naturally determines the metric $\mathscr{G}$.

The chapter is organised as follows. In Section 5.1, we compute the derivatives of the potential $\mathscr{F}$ and present a few geometric consequences of our formulas. Then in Section 5.2 we introduce the period domain $\mathfrak{D}$ and explain how to define a natural immersion $\Phi : \mathscr{M} \to \mathfrak{D}$. The geometric properties of this immersion and its relation to the metric $\mathscr{G}$ are studied in Section 5.3, where we also prove that $\Phi(\mathscr{M}) \subset \mathfrak{D}$ is a totally geodesic immersion if and only if the Yukawa coupling is parallel with respect to the Levi-Civita connection of $\mathscr{G}$. The material of these first three sections are from the article [83] by the author. In Section 5.4, we make some further comments (which do not appear in detail in the paper) and explain how to relate this immersion to the more classical notion of G_2-period map as a Lagrangian immersion of $\mathscr{M}$ into $H^3(M) \oplus H^4(M)$.

5.1 Higher derivatives of the potential

In this section as in the rest of the chapter, M^7 will be a compact manifold admitting torsion-free G_2-structures, and we assume that $b^1(M) = 0$. Recall from the previous chapter that the natural map $\mathscr{M} \to H^3(M)$ endows $\mathscr{M}$ with the structure of an affine manifold, and if $(u_0, \dots, u_n)$ is a basis of $H^3(M)$ we have associated local affine coordinates $(x^0, \dots, x^n)$ on $\mathscr{M}$, where $n = b^3(M) - 1$. In these coordinates, the Riemannian metric $\mathscr{G}$ is the Hessian of the potential function $\mathscr{F}(\varphi) = -3 \log \mathrm{Vol}(\varphi)$. We denote by $\mathscr{F}_a$, $\mathscr{F}_{ab}$, etc. the partial derivatives of $\mathscr{F}$; the expression of the first two derivatives was given in Proposition 4.2.

Remark 5.1. For later purpose, we note a couple of useful identities:

$$x^k \mathscr{G}_{ak} = x^k \mathscr{F}_{ak} = -\mathscr{F}_a, \quad \text{and} \quad x^k \mathscr{F}_k = -7.$$

They just follow from the fact that x^k are by definition the coordinates of the cohomology class $[\varphi] \in H^3(M)$ and

$$\mathscr{G}_\varphi(\varphi, \eta_a) = \frac{1}{\mathrm{Vol}(\varphi)} \int \langle \varphi, \eta_a \rangle_\varphi \mu_\varphi = -\mathscr{F}_a,$$
$$d_\varphi \mathscr{F}(\varphi) = -\frac{1}{\mathrm{Vol}(\varphi)} \int \varphi \wedge \Theta(\varphi) = -7.$$

In this section, we present a new derivation of the derivatives of the potential $\mathscr{F}$ up to order 4, and deduce a few consequences for the geometry of the moduli space. As a key part of our computations, we first we study in §5.1.1 the infinitesimal deformations of harmonic forms along a family of Riemannian metrics. The derivations of the third and fourth derivatives of the potential are carried out in §5.1.2. In §5.1.3 we relate them to the curvature of $\mathscr{G}$.

5.1.1 Deformations of harmonic forms along a family of metrics. In this part, we let (M^7, g) be an oriented compact Riemannian 7-manifold and $h \in \mathrm{End}(TM)$ be a *trace-free* endomorphism, self-adjoint for the metric g. Moreover, let $\{g_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth family of metrics such that $g_0 = g$ and $\frac{\partial g_t}{\partial t}\Big|_{t=0} = 2g(h,\cdot)$. For all $|t| < \epsilon$, we denote by h_t the unique g_t-self-adjoint endomorphism of TM such that $\frac{\partial g_t}{\partial t} = 2g_t(h_t\cdot,\cdot)$. In particular, $h_0 = h$, but we do not require h_t to be trace-free with respect to g_t for $t \neq 0$. We also denote by $*$ the Hodge operator associated with g, and by d^* and $\Delta = (dd^* + d^*d)$ the corresponding operators; similarly for $t \in (-\epsilon, \epsilon)$ we denote by $*_t$, d^{*_t} and Δ_t the operators associated with g_t. We want to understand the infinitesimal variations of the harmonic representative of a fixed cohomology class along the path $\{g_t\}_{t\in(-\epsilon,\epsilon)}$. We start by describing the deformations of the operator d^{*_t}.

Lemma 5.2. *If $\eta \in \Omega^k(M)$ is a k-form, we have*

$$\frac{\partial d^{*_t}\eta}{\partial t}\bigg|_{t=0} = 2h\cdot(d^*\eta) - 2d^*(h\cdot\eta).$$

Proof. By definition, $d^{*_t}\eta = (-1)^k *_t d *_t \eta$. Using Lemma 1.3, we know that

$$\frac{\partial *_t}{\partial t}\bigg|_{t=0}\eta = h\cdot(*\eta) - *(h\cdot\eta) = 2h\cdot(*\eta) = -2*(h\cdot\eta)$$

where the last two inequalities follow from Corollary 1.4, since h is trace-free and self-adjoint for the metric g. The lemma follows. ☐

Lemma 5.3. *Let $\{\eta_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth family of k-forms on M, such that η_t is harmonic for the metric g_t for all $|t| < \epsilon$, and let $\eta = \eta_0$. Then we have:*

$$\Delta\,\frac{\partial \eta_t}{\partial t}\bigg|_{t=0} = 2dd^*(h\cdot\eta).$$

Proof. The k-form η_t is closed for all $t \in (-\epsilon, \epsilon)$, and thus if we differentiate the equality

$$(d^{*_t}d + dd^{*_t})\eta_t = 0$$

with respect to t we obtain

$$d\frac{\partial d^{*_t}}{\partial t}\eta_t + \Delta_t\frac{\partial \eta_t}{\partial t} = 0.$$

At $t=0$, $h_0 = h$ is trace-free, $\eta_0 = \eta$ satisfies $d^*\eta = 0$, and thus the previous lemma yields

$$\left.\frac{\partial d^{*_t}}{\partial t}\right|_{t=0} \eta = 2h \cdot (d^*\eta) - 2d^*(h\cdot\eta) = -2d^*(h\cdot\eta).$$

From this it follows that

$$-2dd^*(h\cdot\eta) + \Delta \left.\frac{\partial \eta_t}{\partial t}\right|_{t=0} = 0$$

which proves our claim. □

In the next part we will need the following consequence of the previous lemmas:

Corollary 5.4. *Let η be harmonic k-form with respect to the metric g. For $t \in (-\epsilon, \epsilon)$, we denote by η_t the harmonic representative of $[\eta] \in H^k(M)$ for the metric g_t and by ν_t the harmonic representative of the cohomology class $[*\eta] \in H^{7-k}(M)$. Then the decomposition of $h\cdot\eta$ into harmonic, exact and co-exact parts reads:*

$$h\cdot\eta = \mathscr{H}(h\cdot\eta) + \frac{1}{2}\left.\frac{\partial\eta_t}{\partial t}\right|_{t=0} - \frac{1}{2} * \left.\frac{\partial \nu_t}{\partial t}\right|_{t=0}.$$

Proof. By the previous lemma, $h\cdot\eta$ satisfies the equation

$$\Delta \left.\frac{\partial\eta_t}{\partial t}\right|_{t=0} = 2dd^*(h\cdot\eta).$$

Moreover, as η_t represents a fixed cohomology class, the k-forms $\frac{\partial\eta_t}{\partial t}$ are exact. Therefore, the exact part of $h\cdot\eta$ is $\frac{1}{2}\left.\frac{\partial\eta_t}{\partial t}\right|_{t=0}$.

The co-exact part of $h\cdot\eta$ can be deduced by symmetry. Indeed, as $*^2 = (-1)^{k(7-k)} = 1$ on k-forms, the co-exact part of $h\cdot\eta$ is the Hodge dual of the exact part of $*(h\cdot\eta)$. As h is trace-free, Corollary 1.4 implies that $*(h\cdot\eta) = -h\cdot(*\eta)$. Using the above characterisation of the exact part, we deduce that the exact part of $h\cdot(*\eta)$ is precisely $\frac{1}{2}\left.\frac{\partial\nu_t}{\partial t}\right|_{t=0}$. Thus the co-exact part of $h\cdot\eta$ is $-\frac{1}{2} * \left.\frac{\partial\nu_t}{\partial t}\right|_{t=0}$. □

5.1.2 The third and fourth derivatives. In this part, M is a compact oriented 7-manifold with $b^1(M) = 0$ admitting torsion-free G_2-structures, and we aim to compute the third and fourth derivative of the potential $\mathscr{F}$. Using a basis $u_0, \ldots, u_n$ of $H^3(M)$, $n = b^3_{27}(M) = b^3(M) - 1$, we define affine coordinates $x = (x^0, \ldots, x^n)$ on $\mathscr{M}$. If φ is a torsion-free G_2-structures on M, we denote by $\eta_a \in \Omega^3(M)$ the unique g_φ-harmonic representative of the cohomology class $u_a \in H^3(M)$, and by $h_a \in C^\infty(\mathrm{End}(TM))$ the unique endomorphism orthogonal to $\Omega^2_{14}(M)$ such that $h_a \cdot \varphi = \eta_a$. Since $b^1(M) = 0$, the 3-form η_a has no Ω^3_7-component, and thus h_a is self-adjoint with respect to the metric g_φ. Similarly, if $\{\varphi_x\}$ is a local section of the moduli space, we denote by $\eta_{a,x} \in \Omega^3(M)$ and by $h_{a,x} \in C^\infty(\mathrm{End}(TM))$ the tensors associated with φ_x.

Various formulas for the third derivative of the potential have already appeared in the literature [52, 51, 73, 85]. Here we give an independent derivation:

Proposition 5.5. *Let φ be a torsion-free* G_2*-structure on M. Then the third derivative of the potential satisfies:*

$$\mathscr{F}_{abc}(\varphi\mathscr{D}) = -\frac{2}{\mathrm{Vol}(\varphi)} \int \langle h_c \cdot \eta_a, \eta_b \rangle_\varphi \mu_\varphi.$$

Proof. Let $x = (x^0, \ldots, x^n)$ be local affine coordinates on $\mathscr{M}$, let x_0 be the coordinates of φ, and let $\{\varphi_x\}$ be a local adapted section of the moduli space through φ (see §4.1.2). Differentiating the identity

$$\mathscr{F}_{ab}(\varphi_x\mathscr{D}) = \frac{1}{\mathrm{Vol}(\varphi_x)} \int \langle \eta_{a,x}, \eta_{b,x} \rangle_{\varphi_x} \mu_{\varphi_x}$$

and using Lemma 4.5 from the previous chapter, we obtain at $x = x_0$:

$$\begin{aligned}\mathscr{F}_{abc}(\varphi\mathscr{D}) = \frac{1}{\mathrm{Vol}(\varphi)} \int \left.\frac{\partial g_{\varphi_x}}{\partial x^c}\right|_{x=x_0} (\eta_a, \eta_b) \mu_\varphi \\ + \frac{1}{\mathrm{Vol}(\varphi)} \int \langle \left.\frac{\partial \eta_{a,x}}{\partial x^c}\right|_{x=x_0}, \eta_b \rangle_\varphi \mu_\varphi + \frac{1}{\mathrm{Vol}(\varphi)} \int \langle \eta_a, \left.\frac{\partial \eta_{b,x}}{\partial x^c}\right|_{x=x_0} \rangle_\varphi \mu_\varphi.\end{aligned}$$

The 3-forms $\frac{\partial \eta_{a,x}}{\partial x^c}$ and $\frac{\partial \eta_{b,x}}{\partial x^c}$ are exact since $\eta_{a,x}$ and $\eta_{b,x}$ represent constant cohomology classes, and therefore the second and third terms above vanish. On the other hand, as the section $\{\varphi_x\}$ is adapted at $x = x_0$, we have $\left.\frac{\partial \varphi_x}{\partial x^c}\right|_{x=x_0} = \eta_c = h_c \cdot \varphi$. Thus we can compute the first term using Lemma 1.6 and the fact that h_c is self-adjoint with respect to g_φ:

$$\mathscr{F}_{abc}(\varphi\mathscr{D}) = -\frac{1}{\mathrm{Vol}(\varphi)} \int (\langle h_c \cdot \eta_a, \eta_b \rangle_\varphi + \langle \eta_a, h_c \cdot \eta_b \rangle_\varphi) \mu_\varphi = -\frac{2}{\mathrm{Vol}(\varphi)} \int \langle h_c \cdot \eta_a, \eta_b \rangle_\varphi \mu_\varphi$$

at $x = x_0$. □

We now proceed with the derivation of the fourth derivative. As a first step, we prove a formula which depends on a particular choice of local section of the moduli space:

Proposition 5.6. *Let φ be a torsion-free* G_2*-structure, let $\{\varphi_x\}$ be a local adapted section of the moduli space through φ and denote by $x = x_0$ the coordinates of $\varphi\mathscr{D}$.*

Then the fourth derivative of the potential satisfies:

$$\begin{aligned}\mathscr{F}_{abcd}(\varphi\mathscr{D}) = \ & \frac{2}{\mathrm{Vol}(\varphi)}\int \langle h_d\cdot\eta_a - \frac{\partial\eta_{a,x}}{\partial x^d}\Big|_{x=x_0}, h_c\cdot\eta_b\rangle_\varphi\mu_\varphi \\ & + \frac{2}{\mathrm{Vol}(\varphi)}\int \langle h_d\cdot\eta_b - \frac{\partial\eta_{b,x}}{\partial x^d}\Big|_{x=x_0}, h_c\cdot\eta_a\rangle_\varphi\mu_\varphi \\ & + \frac{2}{\mathrm{Vol}(\varphi)}\int \langle h_d\cdot\eta_c - \frac{\partial\eta_{c,x}}{\partial x^d}\Big|_{x=x_0}, h_a\cdot\eta_b\rangle_\varphi\mu_\varphi.\end{aligned}$$

Proof. To lighten notations, we will keep the x-dependence implicit and write η_a and h_a instead of $\eta_{a,x}$ and $h_{a,x}$ when this does not create any confusion. Also, unless otherwise noted we differentiate at $x = x_0$. By the previous proposition, the third derivative of the potential can be written:

$$\mathscr{F}_{abc}(\varphi_x\mathscr{D}) = -\frac{1}{\mathrm{Vol}(\varphi_x)}\int\langle h_c\cdot\eta_a, \eta_b\rangle_{\varphi_x}\mu_{\varphi_x} - \frac{1}{\mathrm{Vol}(\varphi_x)}\int\langle \eta_a, h_c\cdot\eta_b\rangle_{\varphi_x}\mu_{\varphi_x}.$$

Differentiating with respect to x^d at $x = x_0$ and using Lemma 4.5 we obtain:

$$\begin{aligned}\mathscr{F}_{abcd}(\varphi\mathscr{D}) = & -\frac{1}{\mathrm{Vol}(\varphi)}\int\frac{\partial g_{\varphi_x}}{\partial x^d}(h_c\cdot\eta_a,\eta_b)\mu_\varphi - \frac{1}{\mathrm{Vol}(\varphi)}\int\frac{\partial g_{\varphi_x}}{\partial x^d}(\eta_a, h_c\cdot\eta_b)\mu_\varphi \\ & -\frac{1}{\mathrm{Vol}(\varphi)}\int\langle h_c\cdot\eta_a, \frac{\partial\eta_{b,x}}{\partial x^d}\rangle_\varphi\mu_\varphi - \frac{1}{\mathrm{Vol}(\varphi)}\int\langle\frac{\partial\eta_{a,x}}{\partial x^d}, h_c\cdot\eta_b\rangle_\varphi\mu_\varphi \\ & -\frac{1}{\mathrm{Vol}(\varphi)}\int\langle h_c\cdot\frac{\partial\eta_{a,x}}{\partial x^d}, \eta_b\rangle_\varphi\mu_\varphi - \frac{1}{\mathrm{Vol}(\varphi)}\int\langle\eta_a, h_c\cdot\frac{\partial\eta_{b,x}}{\partial x^d}\rangle_\varphi\mu_\varphi \\ & -\frac{1}{\mathrm{Vol}(\varphi)}\int\langle\frac{\partial h_{c,x}}{\partial x^d}\cdot\eta_a, \eta_b\rangle_\varphi\mu_\varphi - \frac{1}{\mathrm{Vol}(\varphi)}\int\langle\eta_a, \frac{\partial h_{c,x}}{\partial x^d}\cdot\eta_b\rangle_\varphi\mu_\varphi.\end{aligned} \tag{5.1}$$

Since the section $\{\varphi_x\}$ is adapted, at $x = x_0$ we have $\frac{\partial\varphi_x}{\partial x^d} = \eta_d = h_d\cdot\varphi$ and by Lemma 1.6 we have the identities:

$$\frac{\partial g_{\varphi_x}}{\partial x^d}(h_c\cdot\eta_a,\eta_b) = -2\langle h_c\cdot\eta_a, h_d\cdot\eta_b\rangle_\varphi, \quad \frac{\partial g_{\varphi_x}}{\partial x^d}(\eta_a, h_c\cdot\eta_b) = -2\langle h_d\cdot\eta_a, h_c\cdot\eta_b\rangle_\varphi.$$

Moreover, since the section h_c of $\mathrm{End}(TM)$ is self-adjoint for the metric induced by φ, the second and third lines in (5.1) are equal. These observations yield:

$$\begin{aligned}\mathscr{F}_{abcd}(\varphi\mathscr{D}) = \ & \frac{2}{\mathrm{Vol}(\varphi)}\int\langle h_d\cdot\eta_a - \frac{\partial\eta_{a,x}}{\partial x^d}, h_c\cdot\eta_b\rangle_\varphi\mu_\varphi \\ & + \frac{2}{\mathrm{Vol}(\varphi)}\int\langle h_d\cdot\eta_b - \frac{\partial\eta_{b,x}}{\partial x^d}, h_c\cdot\eta_a\rangle_\varphi\mu_\varphi \\ & - \frac{1}{\mathrm{Vol}(\varphi)}\int\langle\frac{\partial h_{c,x}}{\partial x^d}\cdot\eta_a,\eta_b\rangle_\varphi + \langle\eta_a, \frac{\partial h_{c,x}}{\partial x^d}\cdot\eta_b\rangle_\varphi\mu_\varphi.\end{aligned} \tag{5.2}$$

It remains to show that the last line in (5.2) can be put in a form similar to the first two lines. Decomposing $\frac{\partial h_{c,x}}{\partial x^d}$ into g_φ-self-adjoint and g_φ-anti-self-adjoint parts, we can further write:

$$\begin{aligned}\langle\frac{\partial h_{c,x}}{\partial x^d}\cdot\eta_a,\eta_b\rangle_\varphi+\langle\eta_a,\frac{\partial h_{c,x}}{\partial x^d}\cdot\eta_b\rangle_\varphi &= \langle\left(\frac{\partial h_{c,x}}{\partial x^d}+\left(\frac{\partial h_{c,x}}{\partial x^d}\right)^{\dagger_\varphi}\right)\cdot\eta_a,\eta_b\rangle_\varphi \\ &= \langle\left(\frac{\partial h_{c,x}}{\partial x^d}+\left(\frac{\partial h_{c,x}}{\partial x^d}\right)^{\dagger_\varphi}\right)\cdot\varphi,h_a\cdot\eta_b\rangle_\varphi\end{aligned}$$

where the second equality follows from Corollary 1.5 and $\left(\frac{\partial h_{c,x}}{\partial x^d}\right)^{\dagger_\varphi}$ denotes the adjoint of $\frac{\partial h_{c,x}}{\partial x^d}$ with respect to the metric g_φ. Taking the self-adjoint part of a section h of $\mathrm{End}(TM)$ corresponds to projecting $h\cdot\varphi$ onto the $\Omega^3_1\oplus\Omega^3_{27}$-components, and hence we obtain:

$$\langle\frac{\partial h_{c,x}}{\partial x^d}\cdot\eta_a,\eta_b\rangle_\varphi+\langle\eta_a,\frac{\partial h_{c,x}}{\partial x^d}\cdot\eta_b\rangle_\varphi=2\langle\frac{\partial h_{c,x}}{\partial x^d}\cdot\varphi,\pi_{1\oplus 27}(h_a\cdot\eta_b)\rangle_\varphi.$$

Differentiating the relation $h_{c,x}\cdot\varphi_x=\eta_{c,x}$ at $x=x_0$ gives $\frac{\partial h_{c,x}}{\partial x^d}\cdot\varphi_x=\frac{\partial\eta_{c,x}}{\partial x^d}-h_c\cdot\eta_d$ and thus:

$$\begin{aligned}2\langle\frac{\partial h_{c,x}}{\partial x^d}\cdot\varphi,\pi_{1\oplus 27}(h_a\cdot\eta_b)\rangle_\varphi &= -2\langle h_c\cdot\eta_d-\frac{\partial\eta_{c,x}}{\partial x^d},\pi_{1\oplus 27}(h_a\cdot\eta_b)\rangle_\varphi \\ &= -2\langle h_d\cdot\eta_c-\frac{\partial\eta_{c,x}}{\partial x^d},\pi_{1\oplus 27}(h_a\cdot\eta_b)\rangle_\varphi\end{aligned}$$

where the second equality also holds because this expression is invariant under permutation of h_c and h_d. It remains to prove that the component $\pi_7(h_d\cdot\eta_c-\frac{\partial\eta_{c,x}}{\partial x^d})$ vanishes. This component can be singled out by wedging with φ. On the one hand, we have:

$$(h_d\cdot\eta_c)\wedge\varphi=h_d\cdot(\eta_c\wedge\varphi)-\eta_c\wedge(h_d\cdot\varphi)=-\eta_c\wedge\eta_d$$

as $\eta_c\wedge\varphi=0$ since $\pi_7(\eta_c)=0$. On the other hand, at $x=x_0$ we can write

$$\frac{\partial\eta_{c,x}}{\partial x^d}\wedge\varphi_x=\frac{\partial}{\partial x^d}(\eta_{c,x}\wedge\varphi_x)-\eta_c\wedge\frac{\partial\varphi_x}{\partial x^d}=-\eta_c\wedge\eta_d \tag{5.3}$$

since $\frac{\partial\varphi_x}{\partial x^d}=\eta_d$ at $x=x_0$. Therefore $\pi_7(h_d\cdot\eta_c-\frac{\partial\eta_c}{\partial x^d})=0$. Putting everything together this implies that, at $x=x_0$:

$$\langle\frac{\partial h_{c,x}}{\partial x^d}\cdot\eta_a,\eta_b\rangle_\varphi+\langle\eta_a,\frac{\partial h_{c,x}}{\partial x^d}\cdot\eta_b\rangle_\varphi=-2\langle h_d\cdot\eta_c-\frac{\partial\eta_{c,x}}{\partial x^d},h_a\cdot\eta_b\rangle_\varphi$$

which yields the claimed expression for $\mathscr{F}_{abcd}(\varphi\mathscr{D})$. □

The above expression for $\mathscr{F}_{abcd}$ is unsatisfactory, as it involves choosing an adapted section at a point of $\mathscr{M}$. In order to rewrite it in a more intrinsic way, we need to decompose the 3-forms $h_d\cdot\eta_a$, $h_d\cdot\eta_b$ and $h_d\cdot\eta_c$ using the results of the previous section:

Lemma 5.7. *With the notations of the previous proposition, the decomposition of $h_d \cdot \eta_c$ into harmonic, exact and co-exact parts reads:*

$$h_d \cdot \eta_c = \mathscr{H}(h_d \cdot \eta_c) + \frac{1}{2}\left.\frac{\partial \eta_c}{\partial x^d}\right|_{x=x_0} - \frac{1}{2} *_\varphi \left.\frac{\partial \nu_c}{\partial x^d}\right|_{x=x_0}$$

where $\nu_{c,x}$ is the harmonic representative of the cohomology class $[*_\varphi \eta_c] \in H^4(M)$ for the metric induced by φ_x.

Proof. After applying a linear change of coordinates if necessary, we may assume that at $x = x_0$ the harmonic form η_0 is proportional to φ and $\eta_1, \dots, \eta_n$ are in $\mathscr{H}^3_{27}(M, \varphi)$. Thus if $d = 0$, $h_d \in C^\infty(\mathrm{End}(TM))$ is a constant multiple of the identity, and therefore $h_d \cdot \eta_c$ is harmonic. Moreover, variations of φ in the direction η_0 correspond to scaling the G_2-structure, and the harmonic representatives of a fixed cohomology class are constant under scaling of the metric. Therefore the proposition holds if $d = 0$. On the other hand, if $d = 1, \dots, n$ then the result follows from Corollary 5.4. □

As a consequence of this lemma, we can write with the notations of Proposition 5.6

$$h_d \cdot \eta_c - \left.\frac{\partial \eta_{c,x}}{\partial x^d}\right|_{x=x_0} = \mathscr{H}(h_d \cdot \eta_c) + G_\Delta((d^*d - dd^*)(h_d \cdot \eta_c))$$

where G_Δ denotes the Green's function of the Laplacian (acting on the orthogonal component of the space of harmonic forms) associated with g_φ. Moreover, we can use Proposition 5.5 to decompose the harmonic 3-form $\mathscr{H}(h_d \cdot \eta_c)$ in the basis $\eta_0, \cdots, \eta_n$ as:

$$\mathscr{H}(h_d \cdot \eta_c) = \frac{\mathscr{G}^{kl}}{\mathrm{Vol}(\varphi)} \int \langle h_d \cdot \eta_c, \eta_k \rangle_\varphi \mu_\varphi \cdot \eta_l = -\frac{1}{2}\mathscr{G}^{kl}\mathscr{F}_{cdk}\eta_l$$

and thus

$$\frac{2}{\mathrm{Vol}(\varphi)} \int \langle \mathscr{H}(h_d \cdot \eta_c), \mathscr{H}(h_a \cdot \eta_b) \rangle_\varphi \mu_\varphi = \frac{1}{2}\mathscr{G}^{kl}\mathscr{F}_{abk}\mathscr{F}_{cdl}.$$

Therefore, we obtain a formula which does not depend on any choice of local section:

Theorem 5.8. *The fourth derivative of the potential is given by*

$$\mathscr{F}_{abcd} = \frac{1}{2}\mathscr{G}^{kl}\left(\mathscr{F}_{abk}\mathscr{F}_{cdl} + \mathscr{F}_{ack}\mathscr{F}_{bdl} + \mathscr{F}_{adk}\mathscr{F}_{bcl}\right) + \mathscr{E}_{abcd} + \mathscr{E}_{cabd} + \mathscr{E}_{cbad}$$

where for any torsion-free G_2-structure φ on M we have

$$\mathscr{E}_{abcd}(\varphi\mathscr{D}) = \frac{2}{\mathrm{Vol}(\varphi)} \int \langle G_\Delta((d^*d - dd^*)h_d \cdot \eta_c), h_a \cdot \eta_b \rangle_\varphi \mu_\varphi.$$

Remark 5.9. Said in words, the integral $\int \langle G_\Delta((d^*d - dd^*)h_d \cdot \eta_c), h_a \cdot \eta_b\rangle_\varphi \mu_\varphi$ is the L^2-inner product of the co-exact parts of $h_a \cdot \eta_b$ and $h_d \cdot \eta_c$ minus the L^2-inner product of the exact parts of $h_a \cdot \eta_b$ and $h_d \cdot \eta_c$. Hence the term $\mathscr{E}_{abcd}$ vanishes exactly when these inner products are equal.

Remark 5.10. Since the operator $G_\Delta(d^*d - dd^*)$ is self-adjoint we have $\mathscr{E}_{abcd} = \mathscr{E}_{dcba}$. A slightly less obvious symmetry is the fact that $\mathscr{E}_{abcd} = \mathscr{E}_{bacd}$, which we can prove in two ways. The first comes from the symmetry of the partial derivatives of $\mathscr{F}$, which implies that $\mathscr{E}_{abcd} + \mathscr{E}_{cabd} + \mathscr{E}_{cbad}$ is fully symmetric in its indices. The sum of the last two terms is symmetric under permutations of a and b, and hence the first term $\mathscr{E}_{abcd}$ must be symmetric in the indices a and b. As a sanity check, we can also recover this symmetry property from the expression given in Theorem 5.8. Indeed we can deduce from Lemma 1.2 the expression

$$\mathscr{E}_{abcd} - \mathscr{E}_{bacd} = \frac{2}{\mathrm{Vol}(\varphi)} \int \langle G_\Delta((d^*d - dd^*)h_d \cdot \eta_c), [h_b, h_a] \cdot \varphi\rangle_\varphi \mu_\varphi.$$

Now $[h_b, h_a]$ is anti-self-adjoint for the metric g_φ and hence the 3-form $[h_a, h_b]$ is of type Ω^3_7. In particular, it is orthogonal to the space of harmonic 3-forms, and hence Lemma 5.7 implies that if we choose a section φ_x of the moduli space adapted at $x = x_0$ we have

$$\mathscr{E}_{abcd}(x_0) - \mathscr{E}_{bacd}(x_0) = \frac{2}{\mathrm{Vol}(\varphi)} \int \langle h_d \cdot \eta_c - \left.\frac{\partial \eta_{c,x}}{\partial x^d}\right|_{x=x_0}, [h_b, h_a] \cdot \varphi\rangle_\varphi \mu_\varphi.$$

In the proof of Proposition 5.6, we showed that $\pi_7(h_d \cdot \eta_c - \left.\frac{\partial \eta_{c,x}}{\partial x^d}\right|_{x=x_0}) = 0$ which means that the expression under the integral vanishes identically for type reasons. Hence we recover the fact that $\mathscr{E}_{abcd} = \mathscr{E}_{bacd}$.

Besides the above symmetries (and the ones we can deduce from them), there is no reason to think that $\mathscr{E}_{abcd}$ is fully symmetric in its indices; only the combination of the terms $\mathscr{E}_{abcd} + \mathscr{E}_{cabd} + \mathscr{E}_{cbad}$ is.

5.1.3 Yukawa coupling and curvatures. In this part, we want to interpret the expressions of the third and fourth derivatives of the potential in geometric terms and relate them to the curvatures of the moduli spaces. As in the previous chapter, let us denote by D the flat connection coming from the local diffeomorphism $\pi : \mathscr{M} \to H^3(M)$ and $\nabla^{\mathscr{G}}$ the Levi-Civita of the metric $\mathscr{G}$. Then there is a unique matrix-valued 1-form γ on $\mathscr{M}$, called the *difference tensor* of the Hessian structure $(D, \mathscr{G})$, such that $\nabla^{\mathscr{G}} = D + \gamma$. In local affine coordinates $x = (x^0, \ldots, x^n)$, the difference tensor can be written as

$$\gamma = \Gamma^k_{ab} dx^a dx^b \otimes \frac{\partial}{\partial x^k}$$

where Γ^k_{ab} are the Christoffel symbols of the metric $\mathscr{G}$ [106]. As the metric is the Hessian of $\mathscr{F}$ in affine coordinates, the Christoffel symbols read:

$$\Gamma^k_{ab} = \frac{1}{2}\mathscr{G}^{kl}\mathscr{F}_{abl}. \tag{5.4}$$

In particular, the difference tensor γ is dual to the symmetric cubic form

$$\Xi = \frac{1}{2}\mathscr{F}_{abc}dx^a dx^b dx^c.$$

The cubic form Ξ is often called the *Yukawa coupling* of $\mathscr{M}$ [52, 73, 85]. The covariant derivative of the Yukawa coupling is given by:

$$\begin{aligned} \nabla^{\mathscr{G}}_d \Xi_{abc} &= \partial_d \Xi_{abc} - \Gamma^k_{da}\Xi_{kbc} - \Gamma^k_{db}\Xi_{akc} - \Gamma^k_{dc}\Xi_{abk} \\ &= \frac{1}{2}\mathscr{F}_{abcd} - \frac{1}{4}\mathscr{G}^{kl}\left(\mathscr{F}_{abk}\mathscr{F}_{cdl} + \mathscr{F}_{ack}\mathscr{F}_{bdl} + \mathscr{F}_{adk}\mathscr{F}_{bcl}\right). \end{aligned} \tag{5.5}$$

Hence, Theorem 5.8 implies that:

$$\nabla^{\mathscr{G}}_d \Xi_{abc} = \frac{1}{2}(\mathscr{E}_{abcd} + \mathscr{E}_{cabd} + \mathscr{E}_{cbad}). \tag{5.6}$$

Therefore, $\mathscr{E}_{abcd} + \mathscr{E}_{cabd} + \mathscr{E}_{cbad} = 0$ at a point for any a, b, c, d if and only if the covariant derivative (with respect to the Levi-Civita connection of $\mathscr{G}$) of the Yukawa coupling Ξ, or equivalently of the difference tensor γ, vanishes at this point. For later use, we gather a few properties of the Yukawa coupling and its covariant derivative:

Lemma 5.11. *The Yukawa coupling satisfies the following properties:*

(i) Under the identification $\mathscr{M} \simeq \mathbb{R} \times \mathscr{M}_1$ of 4.1.1, $\Xi = -dt \otimes \mathscr{G} + \Xi_1$ where Ξ_1 is the restriction of Ξ to $\mathscr{M}_1$.

(ii) In local affine coordinates, $x^k\mathscr{F}_{abk} = -2\mathscr{G}_{ab}$.

(iii) $\nabla^{\mathscr{G}}\Xi$ is a fully symmetric quartic form on $T\mathscr{M}$.

(iv) In local affine coordinates, $x^k\nabla^{\mathscr{G}}_a\Xi_{bck} = 0$.

Proof. Properties (i) and (ii) are essentially equivalent since $\mathscr{F}_{abc} = 2\Xi_{abc}$. Moreover (ii) can be seen from the observation that x^k are the coordinates of the cohomology class $[\varphi] \in H^3(M)$, and thus

$$x^k\mathscr{F}_{abk} = -\frac{2}{\mathrm{Vol}(\varphi)}\int \langle h_a \cdot \eta_b, \varphi\rangle_\varphi \mu_\varphi = -\frac{2}{\mathrm{Vol}(\varphi)}\int \langle \eta_a, \eta_b\rangle_\varphi \mu_\varphi = -2\mathscr{G}_{ab}$$

using the symmetry of h_a and the fact that $h_a \cdot \varphi = \eta_a$ by definition.

For point (iii), the symmetry of $\nabla^{\mathscr{G}}\Xi$ follows from the symmetry of the partial derivatives of $\mathscr{F}$ and (5.5). Finally, because point (iv) will be a key argument in the next section (in the proof of Theorem 5.22), we shall give it two proofs.

The first proof follows from the observation that

$$x^k\mathscr{E}_{kbcd} = x^k\mathscr{E}_{akcd} = x^k\mathscr{E}_{abkd} = x^k\mathscr{E}_{abck} = 0 \tag{5.7}$$

for any a, b, c, d. Indeed, from the expression given in Theorem 5.8, we have

$$x^k\mathscr{E}_{kbcd} = \int \langle G_\Delta((d^*d - dd^*)h_d \cdot \eta_c), (x^k h_k \cdot \eta_b)\rangle_\varphi \mu_\varphi.$$

Notice that $x^k h_k$ is a self-adjoint endomorphism of TM for the metric g_φ, and moreover $x^k h_k \cdot \varphi = x^k\eta_k = \varphi$. It follows that $x^k h_k = \frac{1}{3}\operatorname{Id}$. Hence $x^k h_k \cdot \eta_b = \frac{1}{3}\operatorname{Id}\cdot\eta_b = \eta_b$ is harmonic, whence it is L^2-orthogonal to $G_\Delta((d^*d - dd^*)h_d \cdot \eta_c)$ and therefore $x^k\mathscr{E}_{kbcd} = 0$. The other identities of (5.7) are proved in the same way, since $x^k h_a \cdot \eta_k = h_a \cdot \varphi = \eta_a$, $x^k h_d \cdot \eta_k = \eta_d$ and $x^k h_k \cdot \eta_c = \eta_c$ are all harmonic forms. Since a, b, c, d are arbitrary point (iv) now follows from (5.6).

We can also give point (iv) a second proof which does not rely on the particular expression of the terms $\mathscr{E}_{abcd}$ given in Theorem 5.8 but only on the properties of the potential $\mathscr{F}$. The idea is to differentiate the expression of point (ii) with respect to the variable x^c, which yields the identity $x^k\mathscr{F}_{abck} + \mathscr{F}_{abc} = -2\mathscr{F}_{abc}$, that is:

$$x^k\mathscr{F}_{abck} = -3\mathscr{F}_{abc}. \tag{5.8}$$

On the other hand, using (5.5) we have

$$x^k\nabla^{\mathscr{G}}_a\Xi_{bck} = \frac{1}{2}x^k\mathscr{F}_{abck} - \frac{1}{4}\mathscr{G}^{rs}\left(\mathscr{F}_{abr}\cdot x^k\mathscr{F}_{cks} + \mathscr{F}_{acr}\cdot x^k\mathscr{F}_{bks} + x^k\mathscr{F}_{akr}\cdot\mathscr{F}_{bcs}\right)$$

and using point (ii) again we see that

$$\mathscr{G}^{rs}x^k\mathscr{F}_{cks} = -2\mathscr{G}^{rs}\mathscr{G}_{cs} = -2\delta^r_c, \quad \mathscr{G}^{rs}x^k\mathscr{F}_{bks} = -2\delta^r_b, \quad \mathscr{G}^{rs}x^k\mathscr{F}_{akr} = -2\delta^r_a.$$

Substituting this into the previous expression, we obtain

$$x^k\nabla^{\mathscr{G}}_a\Xi_{bck} = \frac{1}{2}(x^k\mathscr{F}_{abck} + 3\mathscr{F}_{abc}) = 0$$

because of (5.8). □

It is interesting to relate the previous observations to the curvature of $\mathscr{G}$. By convention, we define the Riemann curvature tensor of $\mathscr{G}$ as

$$\mathscr{R}\left(\frac{\partial}{\partial x^c}, \frac{\partial}{\partial x^d}\right)\frac{\partial}{\partial x^b} = \mathscr{R}^a{}_{bcd}\frac{\partial}{\partial x^a} = \nabla^{\mathscr{G}}_{\partial_c}\nabla^{\mathscr{G}}_{\partial_d}\frac{\partial}{\partial x^b} - \nabla^{\mathscr{G}}_{\partial_d}\nabla^{\mathscr{G}}_{\partial_c}\frac{\partial}{\partial x^b}.$$

Lowering the first index, we also denote

$$\mathscr{R}_{abcd} = \mathscr{G}_{ak}\mathscr{R}^{k}{}_{bcd}.$$

For Hessian metrics, the Riemann curvature tensor has a particularly simple expression [106, Prop. 2.3]:

$$\mathscr{R}_{abcd} = \frac{1}{4}\mathscr{G}^{kl}(\mathscr{F}_{adk}\mathscr{F}_{bcl} - \mathscr{F}_{ack}\mathscr{F}_{bdl}) = \mathscr{G}^{kl}\Xi_{adk}\Xi_{bcl} - \mathscr{G}^{kl}\Xi_{ack}\Xi_{bdl}. \tag{5.9}$$

Since the Yukawa coupling determines the curvature, we deduce the following:

Proposition 5.12. *If $\mathscr{E}_{abcd} + \mathscr{E}_{cabd} + \mathscr{E}_{cbad} = 0$ for any $0 \leq a, b, c, d \leq n$ at a point of the moduli space, then the covariant derivative of $\mathscr{R}$ vanishes at this point. In particular, if $\mathscr{E}_{abcd} + \mathscr{E}_{cabd} + \mathscr{E}_{cbad}$ vanishes identically on the moduli space, then $(\mathscr{M}, \mathscr{G})$ is locally symmetric.*

In the next chapter, we will see that this condition is satisfied in certain simple cases, e.g. when $M = T^7/\Gamma$ or $M = (T^3 \times K3)/\Gamma$. Hence for these very simple examples the moduli spaces are locally symmetric, a fact which can be easily proved independently and constitutes a good sanity check for the results which we derived in this section.

Beyond these cases, there is no reason to think that the Yukawa coupling will be a parallel tensor, because the constraints it imposes on $\mathscr{G}$ are too strong. Therefore, much of the difficulty in further analysing the geometric properties of the moduli spaces lies in the fact that the terms $\mathscr{E}_{abcd}$ cannot be computed more explicitly in local coordinates. In the next sections of this chapter, we will propose a more geometric interpretation for the presence of these terms, and in §5.3.3 we prove a stronger version of Proposition 5.12 which shows that if they vanish then the sectional curvature of $\mathscr{G}$ is nonpositive.

5.2 Period domains

In the remainder of this chapter, we shall introduce an immersion Φ of the moduli space $\mathscr{M}$ into the homogeneous space $\mathrm{GL}(n+1)/(\{\pm 1\}\times \mathrm{O}(n))$, study its properties and relate it to the metric $\mathscr{G}$. The idea is inspired by the period map introduced by Griffiths on Calabi–Yau moduli spaces [49, 50], and the related notion of Weil–Petersson geometry of Lu and Sun [90].

By means of motivation, let us recall a few facts. If Y is a compact Calabi–Yau threefold, the cohomology group $H^3(Y; \mathbb{C})$ admits a Hodge decomposition

$$H^3(Y; \mathbb{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}.$$

The cup-product induces a symplectic structure Q on $H^3(Y;\mathbb{C})$ and the Hodge decomposition is subject to the following conditions:

(A) $\overline{H^{p,3-p}} = H^{3-p,p}$, for all $p = 0, 1, 2, 3$.

(B) $iQ(H^{p,3-p}, \overline{H^{q,3-q}}) = 0$ if $p \neq q$, and $(-1)^{p+1} iQ(H^{p,3-p}, \overline{H^{p,3-p}}) > 0$ for all p.

(C) $\dim H^{3,0} = 1$.

By considering the Hodge filtration $F^p = H^{3,0} \oplus \cdots \oplus H^{3-p,p}$, it can be shown that the domain parametrising such decompositions (called Hodge structures of weight $(1, h^{2,1})$) is a complex homogeneous space diffeomorphic to $\mathrm{Sp}(Q)/(\mathrm{U}(1) \times \mathrm{U}(n))$, where $\mathrm{Sp}(Q)$ is the group of real endomorphisms of H preserving the symplectic form Q and $n = h^{2,1}(Y)$. Griffiths proved that the Hodge filtration varies holomorphically along an analytic deformation of the complex structure of Y, and that these variations satisfy the *transversality condition* $dF^p \subset F^{p-1}$ [49, 50]. This condition in particular implies that the Weil–Peterson metric can be seen as the pull-back of a homogeneous *indefinite* hermitian form defined on the period domain [108, 112, 90].

The goal of the present section is to explain how to 'twist' the Hodge decomposition of a G_2-manifold (M, φ) in order to obtain a flag in $H^3(M) \oplus H^4(M)$ satisfying analogous axioms, and to describe the geometry of the corresponding 'period domains'.

5.2.1 Some observations on the Hodge decomposition. Let (M, φ) be a compact G_2-manifold, with the usual assumption $b^1(M) = 0$. For simplicity, we denote $H^3 = H^3(M)$, $H^4 = H^4(M)$ and $H = H^3 \oplus H^4$, and $n = b^3_{27}(M) = b^3(M) - 1$. We can define an involution $\iota = \mathrm{Id}_{H^3} - \mathrm{Id}_{H^4}$ on H. As the cup-product identifies H^4 with the dual space of H^3, H is endowed with a natural symplectic form Q. Explicitly, if η, η' are closed 3-forms and ν, ν' closed 4-forms we have

$$Q([\eta] + [\nu], [\eta'] + [\nu']) = \int_M \eta \wedge \nu' - \int_M \eta' \wedge \nu.$$

Let us consider the decomposition $H = H^{(3)}_\varphi \oplus H^{(2)}_\varphi \oplus H^{(1)}_\varphi \oplus H^{(0)}_\varphi$ defined by

$$\begin{aligned} H^{(3)}_\varphi &= \{[\eta] + [*_\varphi \eta],\ \eta \in \mathscr{H}^3_1(M, \varphi)\}, \\ H^{(2)}_\varphi &= \{[\eta] - [*_\varphi \eta],\ \eta \in \mathscr{H}^3_{27}(M, \varphi)\}, \\ H^{(1)}_\varphi &= \{[\eta] + [*_\varphi \eta],\ \eta \in \mathscr{H}^3_{27}(M, \varphi)\}, \text{ and} \\ H^{(0)}_\varphi &= \{[\eta] - [*_\varphi \eta],\ \eta \in \mathscr{H}^3_1(M, \varphi)\}. \end{aligned} \tag{5.10}$$

It satisfies the following properties, analogous to **(A)**, **(B)** and **(C)** above:

(1) $H_\varphi^{(3-p)} = \iota(H_\varphi^{(p)})$ for $p = 0, \dots, 3$.

(2) $Q(\iota(H_\varphi^{(p)}), H_\varphi^{(q)}) = 0$ if $p \neq q$, and $(-1)^{p+1}Q(\iota(H_\varphi^{(p)}), H_\varphi^{(p)}) > 0$, for any $0 \leq p, q \leq 3$; that is, $(-1)^{p+1}Q(\iota(w), w) > 0$ for any $w \in H^{(p)} \backslash \{0\}$.

(3) $\dim H_\varphi^{(3)} = 1$ and $\dim H_\varphi^{(2)} = n$.

The first and third properties are clear, and the second one follows from the fact that if $w = [\eta] + (-1)^{p+1}[*_\varphi \eta] \in H_\varphi^{(p)}$ and $w' = [\eta'] + (-1)^{q+1}[*_\varphi \eta'] \in H_\varphi^{(q)}$ then

$$Q(\iota(w), w') = ((-1)^{p+1} + (-1)^{q+1}) \int \langle \eta, \eta' \rangle_\varphi \mu_\varphi.$$

Let us denote by $\mathfrak{D} \subset \mathbb{P}(H) \times \mathrm{Gr}(n, H) \times \mathrm{Gr}(n, H) \times \mathbb{P}(H)$ the set of decompositions $\mathbf{H} = (H^{(3)}, H^{(2)}, H^{(1)}, H^{(0)})$ of H satisfying the above properties, where $\mathrm{Gr}(n, H)$ is the Grassmannian of n-planes in H. The subgroup of $\mathrm{GL}(H)$ of automorphisms fixing (Q, ι) naturally acts on $\mathfrak{D}$. This group can be identified with $\mathrm{GL}(H^3) \simeq \mathrm{GL}(n+1)$. Explicitly, if we fix bases $(u_0, \dots, u_n)$ of H^3 and $(v_0, \dots, v_n)$ of H^4 such that

$$Q(u_i, v_j) = \delta_{ij}, \quad \forall 0 \leq i, j \leq n \tag{5.11}$$

then any matrix $A \in \mathrm{GL}(n+1)$ acts on H via

$$A(u_j) = \sum_{i=0}^{N} A_{ij} u_i, \quad A(v_j) = \sum_{i=0}^{N} (A^{-1})_{ji} v_i. \tag{5.12}$$

This action has the following properties:

Lemma 5.13. *There is an equivariant diffeomorphism $\mathfrak{D} \to \mathbb{P}(H^3) \times S_+^2(H^3)^*$. In particular the action of $\mathrm{GL}(H^3)$ on $\mathfrak{D}$ is transitive, and $\mathfrak{D}$ is diffeomorphic to the homogeneous space $\mathrm{GL}(n+1)/(\{\pm 1\} \times \mathrm{O}(n))$.*

Proof. If $\mathbf{H} \in \mathfrak{D}$, we can define a line $\ell_\mathbf{H} \in \mathbb{P}(H^3)$ by

$$\ell_\mathbf{H} = \{w + \iota(w),\ w \in H^{(3)}\}.$$

There is also a quadratic form $q_\mathbf{H}$ on H^3 defined as

$$q_\mathbf{H}(u) = 2Q(\pi_\mathbf{H}^{(0)} u, \pi_\mathbf{H}^{(3)} u) - 2Q(\pi_\mathbf{H}^{(1)} u, \pi_\mathbf{H}^{(2)} u), \quad \forall u \in H^3,$$

where $\pi_\mathbf{H}^{(p)}$ denotes the projection of H onto $H^{(p)}$ in the decomposition $H = H^{(3)} \oplus H^{(2)} \oplus H^{(1)} \oplus H^{(0)}$. Properties **(1)** and **(2)** imply that $q_\mathbf{H}$ is positive definite on H^3, and thus $q_\mathbf{H} \in S_+^2(H^3)^*$. This way we have defined a map $\mathfrak{D} \to \mathbb{P}(H^3) \times S_+^2(H^3)^*$, and it is clear that it is equivariant under the action of $\mathrm{GL}(H^3)$. This map is invertible, and its inverse can be constructed as follows. Let $(\ell, q) \in \mathbb{P}(H^3) \times$

$S^2_+(H^3)^*$, and let $(u_0,\dots,u_n)$ be an orthonormal basis of H^3 such that u_0 spans ℓ. Then there exists a unique basis $(v_0,\dots,v_n)$ of H^4 such that $Q(u_i,v_j)=\delta_{ij}$, and we can define

$$H^{(3)}_{(\ell,q)}=\operatorname{span}\{u_0+v_0\},\quad H^{(2)}_{(\ell,q)}=\operatorname{span}\{u_j-v_j, 1\leq j\leq n\}$$

as well as $H^{(p)}_{(\ell,q)}=\iota(H^{(3-p)}_{(\ell,q)})$ for $p=0,1$. It is easy to see that this decomposition $\mathbf{H}_{(\ell,q)}$ is an element of $\mathfrak{D}$, and that the map $\mathbb{P}(H^3)\times S^2_+(H^3)^*$ defined in this way is an inverse for the map $\mathbf{H}\mapsto(\ell_{\mathbf{H}},q_{\mathbf{H}})$. The rest of the lemma follows. □

Remark 5.14. Under the diffeomorphism $\mathfrak{D}\simeq\mathbb{P}(H^3)\times S^2_+(H^3)^*$, we can easily see that for any torsion-free G_2-structure φ on M we have $\ell(\mathbf{H}_\varphi)=H^3_1(M,\varphi)$, and $q(\mathbf{H}_\varphi)$ is the inner product on H^3 induced by the L^2-inner product on $\mathscr{H}^3(M,g_\varphi)$.

Throughout this section, it will be convenient to adopt the following definition. If $\mathbf{H}\in\mathfrak{D}$, a basis $(u_0,\dots,u_n,v_0,\dots,v_n)$ of H will be called a *standard basis* for $\mathbf{H}$ if it satisfies the following properties:

(i) $(u_0,\dots,u_n)$ is a basis of H^3, $(v_0,\dots,v_n)$ is a basis of H^4, and relations (5.11) are satisfied.

(ii) The basis $(u_0,\dots,u_n)$ is orthonormal for the inner product $q_{\mathbf{H}}$.

(iii) $H^{(3)}=\operatorname{span}\{u_0+v_0\}$ and $H^{(2)}=\operatorname{span}\{u_i-v_i,\ 1\leq i\leq n\}$.

Standard bases always exist, and are uniquely determined by a $q_{\mathbf{H}}$-orthonormal basis $(u_0,\dots,u_N)$ of H^3 such that $u_0\in\ell_{\mathbf{H}}$.

5.2.2 The horizontal and transverse distributions. Let us denote by $G_{\mathbf{H}}\subset\mathrm{GL}(H^3)$ the stabiliser of an element $\mathbf{H}\in\mathfrak{D}$, and by $\mathfrak{g}_{\mathbf{H}}\subset\mathfrak{gl}(H^3)$ its Lie algebra. In a standard basis $(u_0,\dots,u_n,v_0,\dots,v_n)$ of H associated with $\mathbf{H}$, $\mathfrak{g}_{\mathbf{H}}$ corresponds to the space of matrices

$$\mathfrak{g}_{\mathbf{H}}=\{(a_{ij})_{0\leq i,j\leq n},\ a_{0i}=a_{i0}=0\ \forall 0\leq i\leq n,\ a_{ij}=-a_{ij}\ \forall 1\leq i,j\leq n\}.$$

The quadratic form $q_{\mathbf{H}}$ induces an inner product on $\mathfrak{gl}(H^3)$: if $a\in\mathfrak{gl}(H^3)$ corresponds to the matrix $(a_{ij})_{0\leq i,j\leq n}$ in the basis $(u_0,\dots,u_n)$, we have:

$$|a|^2_{\mathbf{H}}=\sum_{i,j=0}^n a_{ij}^2.$$

We denote by $\mathfrak{p}_{\mathbf{H}}$ the orthogonal complement of $\mathfrak{g}_{\mathbf{H}}$ for this inner product. That is, in a standard basis,

$$\mathfrak{p}_{\mathbf{H}}=\{(a_{ij})_{0\leq i,j\leq n},\ a_{ij}=a_{ij}\ \forall 1\leq i,j\leq n\}.$$

The tangent space $T_{\mathbf{H}}\mathfrak{D}$ can be identified with $\mathfrak{p}_{\mathbf{H}}$, which is endowed with the inner product induced by $q_{\mathbf{H}}$. This defines a Riemannian metric $g_{\mathfrak{D}}$ on $\mathfrak{D}$, homogeneous with respect to $\mathrm{GL}(H^3)$. Let us denote by $T^v\mathfrak{D}$ the distribution tangent to the fibres of $q : \mathfrak{D} \to S^2_+(H^3)^*$ and call it the *vertical distribution* of $\mathfrak{D}$. The *horizontal distribution* of $\mathfrak{D}$ is defined as the orthogonal complement of the vertical distribution, and will be denoted by $T^h\mathfrak{D}$. If $\mathbf{H} \in \mathfrak{D}$ and $T_{\mathbf{H}}\mathfrak{D}$ is identified with $\mathfrak{p}_{\mathbf{H}} \subset \mathfrak{gl}(H^3)$, then the splitting $T_{\mathbf{H}}\mathfrak{D} = T^v_{\mathbf{H}}\mathfrak{D} \oplus T^h_{\mathbf{H}}\mathfrak{D}$ corresponds to the decomposition $\mathfrak{p}_{\mathbf{H}} = \mathfrak{v}_{\mathbf{H}} \oplus \mathfrak{h}_{\mathbf{H}}$, where $\mathfrak{h}_{\mathbf{H}}$ is the space of endomorphisms of H^3 that are self-adjoint with respect to the inner product $q_{\mathbf{H}}$ and $\mathfrak{v}_{\mathbf{H}}$ its orthogonal complement in $\mathfrak{p}_{\mathbf{H}}$. In particular, the map $q : \mathfrak{D} \to S^2_+(H^3)^*$ is a Riemannian fibration for the natural symmetric metric $g_{S^2_+}$ of $S^2_+(H^3)^*$. Recall that this metric can be defined as follows: if $q \in S^2_+(H^3)^*$ is an inner product and $\dot{q} \in S^2(H^3)^* \simeq T_q S^2_+(H^3)^*$, there is a unique q-self-adjoint endomorphism a of H^3 such that $\dot{q} = \frac{d}{dt}\Big|_{t=0} (e^{ta})^* q = q(a\cdot, \cdot) + q(\cdot, a\cdot) = 2q(a\cdot, \cdot)$, and then we define $|\dot{q}|^2_q = \mathrm{tr}(a^2) = \sum a^2_{ij}$ in a q-orthonormal basis.

Written in a standard basis, the horizontal and vertical spaces are given by

$$\begin{aligned}
\mathfrak{v}_{\mathbf{H}} &= \{(a_{ij})_{0 \le i,j \le n},\ a_{0i} = -a_{i0}\ \forall 0 \le i \le n,\ a_{ij} = 0\ \forall 1 \le i,j \le n\},\\
\mathfrak{h}_{\mathbf{H}} &= \{(a_{ij})_{0 \le i,j \le n},\ a_{ij} = a_{ij}\ \forall 0 \le i,j \le n\}.
\end{aligned}$$

The horizontal distribution admits a further equivariant splitting. By the previous lemma, $\mathbf{H}$ determines a line $\ell_{\mathbf{H}} \in \mathbb{P}(H^3)$ which is fixed by $G_{\mathbf{H}}$, and therefore there is a 1-dimensional subspace $\mathfrak{l}_{\mathbf{H}} \subset \mathfrak{h}_{\mathbf{H}}$ consisting of those self-adjoint endomorphisms that send $\ell_{\mathbf{H}}$ to itself and act trivially on its orthogonal complement. We denote by $\mathfrak{t}_{\mathbf{H}}$ the orthogonal complement of $\mathfrak{l}_{\mathbf{H}}$ in $\mathfrak{h}_{\mathbf{H}}$ and by $T^t_{\mathbf{H}}\mathfrak{D}$ the corresponding subspace of $T_{\mathbf{H}}\mathfrak{D}$. This defines an equivariant distribution $T^t\mathfrak{D} \subset T\mathfrak{D}$, which we call the *transverse distribution* of $\mathfrak{D}$. Again, in a standard basis we have

$$\begin{aligned}
\mathfrak{l}_{\mathbf{H}} &= \{(a_{ij})_{0 \le i,j \le n},\ a_{ij} = 0 \text{ if } (i,j) \neq (0,0)\},\\
\mathfrak{t}_{\mathbf{H}} &= \{(a_{ij})_{0 \le i,j \le n},\ a_{ij} = a_{ij}\ \forall 0 \le i,j \le N,\ a_{00} = 0\}.
\end{aligned}$$

Another convenient description of the horizontal and transverse distributions can be given by introducing the filtration $F^{(3)} \subset F^{(2)} \subset F^{(1)} \subset F^{(0)} = H$ associated with $\mathbf{H} \in \mathfrak{D}$:

$$F^{(p)} = H^{(3)} \oplus \cdots \oplus H^{(p)}.$$

Clearly this filtration determines $\mathbf{H}$, and therefore this defines an equivariant embedding of $\mathfrak{D}$ in a manifold of flags in H. Via this embedding, any tangent vector $\xi \in T_{\mathbf{H}}\mathfrak{D}$ can be represented by a triple of linear maps $F^{(p)} \to H/F^{(p)}$ for

$p = 1, 2, 3$. Since $F^{(p)} \subset F^{(p-1)}$ and $H^{(p-1)} \oplus \cdots \oplus H^{(0)}$ is a complement of $F^{(p)}$, we can in fact represent ξ by $(\phi_\xi^{(3)}, \phi_\xi^{(2)}, \phi_\xi^{(1)})$ where

$$\phi_\xi^{(p)} : H^{(p)} \to H^{(p-1)} \oplus \cdots \oplus H^{(0)}.$$

Lemma 5.15. *Let $\mathbf{H} \in \mathfrak{D}$ and $\xi \in T_{\mathbf{H}}\mathfrak{D}$ be represented by the triple of linear maps $(\phi_\xi^{(3)}, \phi_\xi^{(2)}, \phi_\xi^{(1)})$. Then ξ is a horizontal vector if and only if*

$$\phi_\xi^{(3)}(H^{(3)}) \subseteq H^{(2)} \oplus H^{(0)}$$

and in this case

$$\phi_\xi^{(2)}(H^{(2)}) \subseteq H^{(1)}.$$

Moreover, ξ is transverse if and only if

$$\phi_\xi^{(3)}(H^{(3)}) \subseteq H^{(2)}.$$

In particular if ξ is transverse then $\phi_\xi^{(p)} \in \mathrm{Hom}(H^{(p)}, H^{(p-1)})$.

Proof. Let $(u_0, \ldots, u_n, v_0, \ldots, v_n)$ be a standard basis of H associated with $\mathbf{H}$. In this basis, $\mathfrak{D} \simeq \mathrm{GL}(n+1)/(\{\pm 1\} \times \mathrm{O}(n))$ and the vector $\xi \in T_{\mathbf{H}}\mathfrak{D}$ is uniquely represented by a matrix $a_\xi = (a_{ij})_{0 \le i,j \le n}$ satisfying

$$a_{ji} = a_{ij}, \quad \forall 1 \le i, j \le n.$$

Now a_ξ acts on H^3 by $a(u_j) = a_{ij}u_i$ and on H^4 by $a(v_j) = -a_{ji}v_i$, and therefore the linear map $\phi_\xi^{(3)}$ is characterised by:

$$\begin{aligned}\phi_\xi^{(3)}(u_0 + v_0) &= \sum_{i=0}^{n} a_{i0}u_i - a_{0i}v_i \\ &= a_{00}(u_0 - v_0) + \sum_{i=1}^{n} \left\{ \frac{a_{i0} + a_{0i}}{2}(u_i - v_i) + \frac{a_{i0} - a_{0i}}{2}(u_i + v_i) \right\} ,\end{aligned}$$

where the first term belongs to $H^{(0)}$, the second term to $H^{(2)}$ and the third to $H^{(1)}$. Hence $\phi_\xi^{(3)}$ maps into $H^{(2)} \oplus H^{(0)}$ if and only if $a_{0i} = a_{i0}$, that is if a_ξ is symmetric. This is exactly the condition for ξ to define a horizontal vector in $T_{\mathbf{H}}\mathfrak{D}$. Moreover $\phi_\xi^{(3)}$ maps into $H^{(2)}$ if and only if a_ξ is symmetric and $a_{00} = 0$, that is, if $a_\xi \in \mathfrak{t}_{\mathbf{H}}$.

Now assume that ξ is a horizontal vector, that is, a_ξ is symmetric. The only nontrivial inclusion left to check is $\phi_\xi^{(2)}(H^{(2)}) \subset H^{(1)}$. On $H^{(2)}$, a_ξ acts by

$$a_\xi(u_j - v_j) = \sum_{i=0}^{n} a_{ij}u_i + a_{ji}v_i = a_{0j}(u_0 + v_0) + \sum_{i=1}^{n} a_{ij}(u_i + v_i)$$

where the first term $a_{0j}(u_0 + v_0) \in H^{(3)}$ and the second term is an element of $H^{(1)}$. Only the projection of $a_\xi(u_j - v_j)$ onto $H^{(1)} \oplus H^{(0)}$ contributes to $\phi_\xi^{(2)}(u_j - v_j)$ and therefore $\phi_\xi^{(2)}(H^{(2)}) \subseteq H^{(1)}$. □

Another useful description uses the identification $\mathfrak{D} \simeq \mathbb{P}(H^3) \times S^2_+(H^3)^*$. Let $\mathbf{H} \in \mathfrak{D}$ correspond to $(\ell, q) \in \mathbb{P}(H^3) \times S^2_+(H^3)^*$. Then any tangent vector is characterised by a couple (ϕ, κ) where $\phi : \ell \to \ell^{\perp_q}$ is a linear map and $\kappa \in S^2(H^3)^*$. Here $\ell^{\perp_q}$ denotes the orthogonal complement of ℓ in H^3 for the inner product induced by q. The conditions of horizontality and transversality are:

Lemma 5.16. *The couple (ϕ, κ) defines a horizontal vector in $T_{\mathbf{H}}\mathfrak{D}$ if and only if*

$$\kappa(u, u') + 2q(\phi(u), u') = 0, \quad \forall u \in \ell, \ \forall u' \in \ell^{\perp_q}.$$

Moreover it defines a transverse vector if and only if the above holds for all $u \in \ell$ and $u' \in H^3$, that is, if $\kappa(u, u) = 0$ for $u \in \ell$.

Proof. Let $(u_0, \ldots, u_n)$ be a q-orthonormal basis of H^3 such that $u \in \ell$, and let $a = (a_{ij})_{0 \leq i,j \leq n}$ representing the tangent vector characterised by (ϕ, κ) in this basis. That is, a satisfies $a_{ij} = a_{ji}$ for $1 \leq i, j \leq n$, and

$$\kappa = -q(a\cdot, \cdot) - q(\cdot, a\cdot) \quad \text{and} \quad \phi(u_0) \equiv a \cdot u_0 \mod \ell.$$

For $1 \leq j \leq n$ we have:

$$\begin{aligned} \kappa(u_0, u_j) + 2q(\phi(u_0), u_j) &= -q(a \cdot u_0, u_j) - q(u_0, a \cdot u_j) + 2q(a \cdot u_0, u_j) \\ &= a_{j0} - a_{0j} \end{aligned}$$

and thus its vanishing is equivalent to $a_{ij} = a_{ji}$ for all $0 \leq i, j \leq n$, that is, a defines a horizontal vector. Moreover,

$$\kappa(u_0, u_0) = -2q(a \cdot u_0, u_0) = -2a_{00}$$

and hence a defines a transverse vector if and only if it defines a horizontal vector and the above vanishes. □

5.3 Properties of the period mapping

As the decomposition $H = H^{(3)}_\varphi \oplus H^{(2)}_\varphi \oplus H^{(1)}_\varphi \oplus H^{(0)}_\varphi$ associated with a torsion-free G_2-structure φ only depends on the class of φ modulo $\mathscr{D}$, there is a well-defined map $\Phi : \mathscr{M} \to \mathfrak{D}$. Under the identification of $\mathfrak{D}$ with $\mathbb{P}(H^3) \times S^2_+(H^3)^*$, $\varphi\mathscr{D} \in \mathscr{M}$ is mapped to $(H^3_1(M, \varphi), e^{-\mathscr{F}(\varphi)/3}\mathscr{G}_\varphi)$ (see Remark 5.14) and thus we see that Φ is a smooth, even real-analytic map (by Theorem 4.6). In this section we study its local properties and relate it more intrinsically to the metric $\mathscr{G}$. In §5.3.1, we show that it satisfies a property analogous to Griffith's transversality, in §5.3.2 we prove that $\mathscr{G}$ is induced by a homogeneous quadratic form on $\mathfrak{D}$ and in §5.3.3 we prove that Φ is a totally geodesic immersion if and only if the Yukawa coupling is a parallel tensor on $\mathscr{M}$.

5.3.1 Infinitesimal variations. The properties of the tangent map of Φ are summarised in the following theorem:

Theorem 5.17. *The map $\Phi : \mathscr{M} \to \mathfrak{D}$ is a horizontal immersion, and the restriction of Φ to $\mathscr{M}_1$ is transverse.*

Moreover, if $\varphi \in \mathscr{M}_1$ and $\eta \in \mathscr{H}^3_{27}(M,\varphi) \simeq T_{\varphi\mathscr{D}}\mathscr{M}_1$, then $T_{\varphi\mathscr{D}}\Phi(\eta)$ is determined by the triple of linear maps $\phi^{(p)}_\eta \in \mathrm{Hom}(H^{(p)}_\varphi, H^{(p-1)}_\varphi)$, $p = 1,2,3$, defined as follows. Let h be the unique trace-free self-adjoint endomorphism such that $h \cdot \varphi = \eta$ and let $\eta' \in \mathscr{H}^3_{27}(M,\varphi)$. Then we have:

(i) $\phi^{(3)}_\eta([\varphi] + [\Theta(\varphi)]) = [\eta] - [*_\varphi \eta]$,

(ii) $\phi^{(2)}_\eta([\eta'] - [*\eta']) = [\pi_{27}\mathscr{H}(h\cdot\eta')] + [*_\varphi \pi_{27}\mathscr{H}(h\cdot\eta'))]$,

(iii) $\phi^{(1)}_\eta([\eta'] + [*\eta']) = \frac{1}{7}\int \langle \eta', \eta\rangle_\varphi \mu_\varphi \cdot ([\varphi] - [\Theta(\varphi)])$.

Proof. Let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a family of torsion-free G_2-structures on M such that $\varphi_0 = \varphi$ and assume that $\frac{\partial\varphi_t}{\partial t}\big|_{t=0} = \eta$ is a harmonic 3-form. Let $\mathbf{H}_t = \Phi(\varphi_t)$ and $(\phi^{(3)}_\eta, \phi^{(2)}_\eta, \phi^{(1)}_\eta)$ be the triple of linear map representing $T_{\varphi\mathscr{D}}\Phi(\eta)$. For all $t \in (-\epsilon,\epsilon)$, $H^{(3)}_t \subset H$ is spanned by $[\varphi_t] + [\Theta(\varphi_t)]$. Differentiating at $t=0$ we have

$$\begin{aligned}\left.\frac{\partial\varphi_t}{\partial t}\right|_{t=0} + \left.\frac{\partial\Theta(\varphi_t)}{\partial t}\right|_{t=0} &= \eta + \frac{4}{3} *_\varphi \pi_1(\eta) - *_\varphi\pi_{27}(\eta) \\ &= \pi_1(\eta) + \frac{4}{3} *_\varphi \pi_1(\eta) + \pi_{27}(\eta) - *_\varphi\pi_{27}(\eta).\end{aligned} \tag{5.13}$$

Since η is harmonic with respect to g_φ, the first two terms term represent an element of $H^{(3)}_\varphi \oplus H^{(0)}_\varphi$, and the last two terms an element of $H^{(2)}_\varphi$, and hence $\phi^{(3)}_\eta(H^{(3)}_\varphi) \subseteq H^{(2)}_\varphi \oplus H^{(0)}_\varphi$. If moreover all φ_t have unit volume then $\pi_1(\eta) = 0$, and thus $\phi^{(3)}_\eta(H^{(3)}_\varphi) \subset H^{(2)}_\varphi$. Hence the first part of the theorem follows from the previous lemma.

Let us now assume that $\mathrm{Vol}(\varphi_t) = 1$ for all t and let us compute the differential of Φ. The expression for $\phi^{(3)}_\eta$ follows from (5.13) since the first two terms vanish. Now let $\eta_1,\dots,\eta_n$ be a basis of $\mathscr{H}^3_{27}(M,\varphi)$, and denote by $\eta_{a,t}$ the element of $\mathscr{H}^3(M,\varphi_t)$ such that $[\eta_{a,t}] = [\eta_a] \in H^3(M)$. For small enough t, the differential forms $\eta'_{a,t}$ defined by

$$\eta'_{a,t} = \eta_{a,t} - \frac{1}{7}\int (\eta_{a,t} \wedge \Theta(\varphi_t)) \cdot \varphi_t$$

form a basis of $\mathscr{H}^3_{27}(M,\varphi_t)$. Thus $H^{(2)}_t$ is spanned by the cohomology classes $[\eta'_{a,t}] - [*_t\eta'_{a,t}]$, $a = 1,\dots,n$, for small t. At $t=0$, each η_a is orthogonal to φ and thus by Lemma 1.6 we obtain

$$\left.\frac{\partial\eta'_{a,t}}{\partial t}\right|_{t=0} = \left.\frac{\partial\eta_{a,t}}{\partial t}\right|_{t=0} + \left(\frac{1}{7}\int\langle\eta_a,\eta\rangle_\varphi\mu_\varphi\right)\cdot\varphi$$

where $\eta = \left.\frac{\partial \varphi_t}{\partial t}\right|_{t=0}$. In particular since $\left.\frac{\partial \eta_{a,t}}{\partial t}\right|_{t=0}$ is exact the harmonic part of $\left.\frac{\partial \eta'_{a,t}}{\partial t}\right|_{t=0}$ is $(\frac{1}{7}\int \langle \eta_a, \eta\rangle_\varphi \mu_\varphi)\varphi$. On the other hand, if we write $\eta = h \cdot \varphi$ where h is traceless and self-adjoint, then by Lemma 1.3 and Corollary 1.4 we have

$$\left.\frac{\partial *_t \eta'_{a,t}}{\partial t}\right|_{t=0} = h \cdot *_\varphi \eta_a - *_\varphi (h \cdot \eta_a) + *_\varphi \left.\frac{\partial \eta'_{a,t}}{\partial t}\right|_{t=0}$$

and since h anticommutes with $*_\varphi$, the harmonic part of $\left.\frac{\partial *_t \eta'_{a,t}}{\partial t}\right|_{t=0}$ is

$$-2 *_\varphi \mathscr{H}(h \cdot \eta_a) + \left(\frac{1}{7}\int \langle \eta_a, \eta\rangle_\varphi \mu_\varphi\right) \cdot \Theta(\varphi).$$

Moreover, we have

$$\pi_1 \mathscr{H}(h \cdot \eta_a) = \left(\frac{1}{7}\int \langle h \cdot \eta, \varphi\rangle_\varphi \mu_\varphi\right) \cdot \varphi = \left(\frac{1}{7}\int \langle \eta_a, \eta\rangle_\varphi \mu_\varphi\right) \cdot \varphi$$

and thus gathering all the results we obtain

$$\begin{aligned}\left.\frac{\partial([\eta'_{a,t}] - [*_t \eta'_{a,t}])}{\partial t}\right|_{t=0} &= 2[*_\varphi \pi_{27}(h \cdot \eta_a)] + \left(\frac{1}{7}\int \langle \eta, \eta_a\rangle_\varphi \mu_\varphi\right) \cdot ([\varphi] + [\Theta(\varphi)]) \\ &\equiv [\pi_{27}\mathscr{H}(h \cdot \eta_a)] + [*_\varphi \pi_{27}\mathscr{H}(h \cdot \eta_a)] \mod F_t^{(2)}.\end{aligned}$$

This yields the claimed expression for $\phi_\eta^{(2)}$. By a mere change of sign, the expression for $\phi_\eta^{(1)}$ follows from the fact that

$$\left.\frac{\partial([\eta'_{a,t}] - [*_t \eta'_{a,t}])}{\partial t}\right|_{t=0} \equiv \left(\frac{1}{7}\int \langle \eta, \eta_a\rangle_\varphi \mu_\varphi\right) \cdot ([\varphi] - [\Theta(\varphi)]) \mod F_t^{(1)}.$$

This finishes the proof of the theorem. □

5.3.2 Riemannian aspects. The map $\Phi : \mathscr{M} \to \mathfrak{D}$ is not a local isometry for the metrics $\mathscr{G}$ on $\mathscr{M}$ and $g_{\mathfrak{D}}$ on $\mathfrak{D}$. Nonetheless, it naturally determines the metric $\mathscr{G}$. Since $\mathscr{G} = 7dt^2 + \mathscr{G}_1$ under the splitting $\mathscr{M} \simeq \mathbb{R} \times \mathscr{M}_1$, it is enough to prove that the restriction of Φ to $\mathscr{M}_1$ determines the metric $\mathscr{G}_1$. Because the map $\Phi : \mathscr{M}_1 \to \mathfrak{D}$ is transverse, it turns out that $\mathscr{G}_1$ can be seen as the pull-back of an *indefinite* quadratic form $h_{\mathfrak{D}}$ on the transverse distribution. To define $h_{\mathfrak{D}}$, let $\mathbf{H} \in \mathfrak{D}$, and consider a transverse tangent vector $\xi \in T^t_{\mathbf{H}}\mathfrak{D}$. By Lemma 5.15, it can be represented by a triple of linear maps $(\phi_\xi^{(3)}, \phi_\xi^{(2)}, \phi_\xi^{(1)})$ where $\phi_\xi^{(p)} \in \mathrm{Hom}(H^{(p)}, H^{(p-1)})$. If $w \in H^{(3)}\backslash\{0\}$, define

$$h_{\mathfrak{D}}(\xi, \xi) = -\frac{Q(\iota(\phi_\xi^{(3)}(w)), \phi_\xi^{(3)}(w))}{Q(\iota(w), w)}$$

This does not depend on the choice of w, and since $Q(\iota(H^{(2)}), H^{(2)}) < 0$ this defines a nonnegative, equivariant quadratic form on the transverse distribution $T^t\mathfrak{D}$.

Proposition 5.18. $\mathscr{G}_1 = 7\Phi^* h_{\mathfrak{D}}$.

Proof. Let φ be a unit volume torsion-free G_2-structure on M and take $w = [\varphi] + [\Theta(\varphi)] \in H^{(3)}_{\varphi}$, so that

$$Q(\iota(w), w) = Q([\varphi] - [\Theta(\varphi)], [\varphi] + [\Theta(\varphi)]) = 14.$$

Let $\eta \in \mathscr{H}^3_{27}(M, \varphi)$, identified with an element of $T_{\varphi\mathscr{D}}\mathscr{M}_1$, and let $(\phi^{(3)}_\eta, \phi^{(2)}_\eta, \phi^{(1)}_\eta)$ be the triple of linear maps representing $T\Phi(\eta) \in T_{\mathbf{H}_\varphi}\mathfrak{D}$. By Theorem 5.17 we have

$$Q(\iota(\phi^{(3)}_\eta(w)), \phi^{(3)}_\eta(w)) = Q([\eta] + [*_\varphi \eta], [\eta] - [*_\varphi \eta]) = -2 \int |\eta|^2_\varphi \mu_\varphi.$$

Thus $\Phi^* h_{\mathfrak{D}}(\eta, \eta) = \mathscr{G}_1(\eta, \eta)/7$. □

In the same way, Φ determines the Yukawa coupling Ξ on $\mathscr{M}$; by Lemma 5.11, $\Xi = -dt \otimes \mathscr{G} + \Xi_1$ and thus we just need to show that Ξ_1 is the pull-back of an equivariant cubic form defined on the transverse distribution in $\mathfrak{D}$. If $\mathbf{H} \in \mathfrak{D}$ and $\xi, \xi', \xi'' \in T^t_{\mathbf{H}}\mathfrak{D}$, each transverse vector is represented by a triple of linear maps $(\phi^{(3)}_\xi, \phi^{(2)}_\xi, \phi^{(1)}_\xi)$ and similarly for ξ' and ξ''. Since each $\phi^{(p)}_\xi$ maps $H^{(p)}$ to $H^{(p-1)}$, the composition $\phi^{(1)}_\xi \circ \phi^{(2)}_{\xi'} \circ \phi^{(3)}_{\xi''}$ defines a linear map from $H^{(3)}$ to $H^{(0)}$. Both are 1-dimensional spaces, and thus there exists a unique $\Xi_{\mathfrak{D}}(\xi, \xi', \xi'')$ such that

$$\phi^{(1)}_\xi \circ \phi^{(2)}_{\xi'} \circ \phi^{(3)}_{\xi''}(w) = -\Xi_{\mathfrak{D}}(\xi, \xi', \xi'') \cdot \iota(w), \quad \forall w \in H^{(3)}.$$

This defines equivariantly a cubic form $\Xi_{\mathfrak{D}}$ on $T^t\mathfrak{D}$.

Proposition 5.19. $\Xi_1 = 7\Phi^* \Xi_{\mathfrak{D}}$.

Proof. Let φ be a unit-volume torsion-free G_2-structure on M and $\eta, \eta', \eta'' \in \mathscr{H}^3_{27}(M, \varphi)$. Theorem 5.17 yields:

$$\begin{aligned} \phi^{(1)}_\eta \circ \phi^{(2)}_{\eta'} \circ \phi^{(3)}_{\eta''}([\varphi] + [\Theta(\varphi)]) &= \frac{1}{7} \int \langle \pi_{27}\mathscr{H}(h' \cdot \eta''), \eta \rangle_\varphi \mu_\varphi \cdot ([\varphi] - [\Theta(\varphi)]) \\ &= \frac{1}{7} \int \langle h' \cdot \eta'', \eta \rangle_\varphi \mu_\varphi \cdot ([\varphi] - [\Theta(\varphi)]) \end{aligned}$$

which proves the proposition. □

Remark 5.20. This is similar to the way the Yukawa coupling is defined on the moduli spaces of Calabi–Yau threefolds, as described by Bryant and Griffiths [18].

Remark 5.21. The way we defined it, $\Xi_{\mathfrak{D}}$ is actually not a symmetric cubic form on $T^t\mathfrak{D}$. However, we will in Section 5.4 consider the transversality condition as an exterior differential system on $\mathfrak{D}$, and show that the restriction of $\Xi_{\mathfrak{D}}$ to any integral element is fully symmetric (see Remark 5.28). Hence $\Xi_{\mathfrak{D}}$ will be symmetric along any integral submanifold of the transverse distribution.

5.3.3 A condition for Φ to be totally geodesic. In this part, we relate the geometry of the immersion $\Phi : \mathscr{M} \to \mathfrak{D}$ with the computations of the Section 5.1 and refine the observations of §5.1.3. Our main result is that the covariant derivative of the Yukawa coupling Ξ, or equivalently the extra term $\mathscr{E}_{abcd} + \mathscr{E}_{cabd} + \mathscr{E}_{cbad}$, essentially characterises the second fundamental form of $\Phi(\mathscr{M})$ inside the domain $\mathfrak{D}$. More precisely, we have

Theorem 5.22. *The Yukawa coupling is a parallel tensor if and only if $\Phi : \mathscr{M} \to \mathfrak{D}$ is a totally geodesic immersion. Moreover, if these conditions are satisfied then the Levi-Civita connections of $\mathscr{G}$ and $\Phi^* g_{\mathfrak{D}}$ coincide and $(\mathscr{M}, \mathscr{G})$ is a locally symmetric space with nonpositive sectional curvature.*

Remark 5.23. This result is a G_2-counterpart for theorems of Liu–Yin [87] and Wei [119] for moduli spaces of Calabi–Yau 3- and 4-folds.

For the proof of the theorem, first remark that since Φ is a horizontal immersion, $\Phi(\mathscr{M})$ is totally geodesic in $\mathfrak{D}$ if and only if the composition $q \circ \Phi : \mathscr{M} \to S^2_+(H^3)^*$ is a totally geodesic immersion. Moreover, the metrics $\Phi^* g_{\mathfrak{D}}$ and $\Phi^* q^* g_{S^2_+}$ coincide, and therefore it is enough to prove that the results hold for the map $q \circ \Phi$ instead of Φ. The advantage of working in $S^2_+(H^3)^*$ instead of $\mathfrak{D}$ is that we can work directly in coordinates which are compatible with affine coordinates on $\mathscr{M}$.

First we need to introduce some notations. For the remainder of this part we will fix a basis $(u_0, \dots, u_n)$ of H^3 and denote $(x^0, \dots, x^n)$ the associated system of coordinates, considered as local coordinates on $\mathscr{M}$. Any symmetric bilinear form $q \in S^2(H^3)^*$ can be written uniquely $q = q_{kl} dx^k dx^l$ where $q_{kl} = q_{lk}$ and as before we write $dx^k dx^l$ as a short-hand for the tensor product $dx^k \otimes dx^l$. Then $(q_{kl})_{1 \le k \le l \le n}$ define global coordinates on the open cone $S^2_+(H^3)^*$ of inner products on H^3. Let us write the canonical symmetric metric of $g_{S^2_+}$ in these coordinates. Let us pick $q \in S^2_+(H^3)^*$ and $\dot{q} \in S^2(H^3)^* \simeq T_q S^2_+(H^3)^*$ written as

$$q = q_{kl} dx^k dx^l \in S^2_+(H^3)^*, \qquad \dot{q} = \dot{q}_{kl} dx^k dx^l \in T_q S^2_+(H^3)^*.$$

There exists a unique q-self-adjoint endomorphism a of H^3 such that $\dot{q} = \frac{d}{dt}\Big|_{t=0} (e^{ta})^* q = q(a\cdot, \cdot) + q(\cdot, a\cdot) = 2q(a\cdot, \cdot)$ and by definition $|\dot{q}|^2_q = \operatorname{tr}(a^2) = a^k_l a^l_k$. In coordinates we have $a^k_l = \frac{1}{2} q^{kr} \dot{q}_{rl}$ and hence it follows that

$$g_{S^2_+}(\dot{q}, \dot{q}) = \frac{1}{4} q^{kl} q^{rs} \dot{q}_{kr} \dot{q}_{sl}.$$

One can use this expression to compute the Christoffel symbols of $g_{S^2_+}$ and deduce that its Levi-Civita connection $\overline{\nabla}$ can be characterised as follows:

Lemma 5.24. *Let $\dot q = \dot q_{kl} dx^k dx^l$ and $\dot q' = \dot q'_{kl} dx^k dx^l$ be vector fields with constant coefficients on $S^2_+(H^3)^*$. Then the covariant derivative $\overline{\nabla}_{\dot q}\dot q'$ is given by*

$$\overline{\nabla}_{\dot q}\dot q' = -\frac{1}{2} q^{rs}(\dot q_{kr}\dot q'_{ls} + \dot q_{lr}\dot q'_{ks}) dx^k dx^l.$$

If φ is a torsion-free G_2-structure on M then $q \circ \Phi(\varphi\mathscr{D})$ is the L^2-inner product induced by φ on $H^3 \simeq \mathscr{H}^3(M, g_\varphi)$, and therefore in the coordinates x^a we have

$$q(x) = e^{-\mathscr{F}/3}\mathscr{G}_{kl} dx^k dx^l \tag{5.14}$$

where the factor $e^{-\mathscr{F}/3} = \mathrm{Vol}$ compensates the volume normalisation in the definition of the metric $\mathscr{G}$. Thus as a subspace of $S^2_+(H^3)^*$, $(q\circ\Phi)_* T\mathscr{M}$ is spanned by the vectors

$$\frac{\partial q}{\partial x^a} = e^{-\mathscr{F}/3}\left(\mathscr{F}_{akl} - \frac{1}{3}\mathscr{F}_a\mathscr{G}_{kl}\right) dx^k dx^l, \qquad a = 0, \dots, n. \tag{5.15}$$

With a small abuse, we still denote $\overline{\nabla}$ the pull-back connection $(q\circ\Phi)^*\overline{\nabla}$, considered as a connection on the trivial vector bundle $\mathscr{M} \times S^2(H^3)^*$. Using Lemma 5.24, we have

$$\begin{aligned}
\overline{\nabla}_{\partial_a}\frac{\partial}{\partial x^b} =& e^{-\mathscr{F}/3}\left(\mathscr{F}_{abkl} - \frac{1}{3}\mathscr{G}_{ab}\mathscr{G}_{kl} - \frac{1}{3}\mathscr{F}_a\mathscr{F}_{bkl} - \frac{1}{3}\mathscr{F}_b\mathscr{F}_{akl} + \frac{1}{9}\mathscr{F}_a\mathscr{F}_b\mathscr{G}_{kl}\right) dx^k dx^l \\
&- \frac{1}{2}e^{-\mathscr{F}/3}\mathscr{G}^{rs}\left(\mathscr{F}_{akr} - \frac{1}{3}\mathscr{F}_a\mathscr{G}_{kr}\right)\left(\mathscr{F}_{bls} - \frac{1}{3}\mathscr{F}_b\mathscr{G}_{ls}\right) dx^k dx^l \\
&- \frac{1}{2}e^{-\mathscr{F}/3}\mathscr{G}^{rs}\left(\mathscr{F}_{aks} - \frac{1}{3}\mathscr{F}_a\mathscr{G}_{ks}\right)\left(\mathscr{F}_{blr} - \frac{1}{3}\mathscr{F}_b\mathscr{G}_{lr}\right) dx^k dx^l \\
=& e^{-\mathscr{F}/3}\left(\mathscr{F}_{abkl} - \frac{1}{2}\mathscr{G}^{rs}\mathscr{F}_{akr}\mathscr{F}_{bls} - \frac{1}{2}\mathscr{G}^{rs}\mathscr{F}_{aks}\mathscr{F}_{blr} - \frac{1}{3}\mathscr{G}_{ab}\mathscr{G}_{kl}\right) dx^k dx^l.
\end{aligned}$$

In the next proposition, we rewrite this expression in a more intrinsic way:

Proposition 5.25. *The connections $\overline{\nabla}$, $\nabla^{\mathscr{G}}$ and the covariant derivative of the Yukawa coupling are related by*

$$\overline{\nabla}_{\partial_a}\frac{\partial}{\partial x^b} = \nabla^{\mathscr{G}}_{\partial_a}\frac{\partial}{\partial x^b} + 2e^{-\mathscr{F}/3}\nabla^{\mathscr{G}}_a \Xi_{bkl} dx^k dx^l$$

where we see $\nabla^{\mathscr{G}}_{\partial_a}\frac{\partial}{\partial x^b}$ as an element of $S^2(H^3)^*$ via the inclusion $(q\circ\Phi)_*T\mathscr{M} \subset S^2(H^3)^*$.

Proof. By our previous computation we have

$$\begin{aligned}
\overline{\nabla}_{\partial_a}\frac{\partial}{\partial x^b} =& e^{-\mathscr{F}/3}\left(\mathscr{F}_{abkl} - \frac{1}{2}\mathscr{G}^{rs}(\mathscr{F}_{akr}\mathscr{F}_{bls} + \mathscr{F}_{aks}\mathscr{F}_{blr} + \mathscr{F}_{abr}\mathscr{F}_{kls})\right) dx^k dx^l \\
&+ e^{-\mathscr{F}/3}\left(\frac{1}{2}\mathscr{G}^{rs}\mathscr{F}_{abr}\mathscr{F}_{kls} - \frac{1}{3}\mathscr{G}_{ab}\mathscr{G}_{kl}\right) dx^k dx^l.
\end{aligned} \tag{5.16}$$

Comparing with the expression of the covariant derivative of the Yukawa coupling given in §5.1.3, the term on the first line is $2e^{-\mathscr{F}/3}\nabla^{\mathscr{G}}_a\Xi_{bkl}$. We need to rewrite the second term using the special properties of the function $\mathscr{F}$ and its derivatives. By Remark 5.1 and Lemma 5.11 we have the identities

$$x^m\mathscr{G}_{sm} = -\mathscr{F}_s, \quad x^m\mathscr{F}_{mab} = -2\mathscr{G}_{ab}. \tag{5.17}$$

Now let us compute:

$$\frac{1}{2}\mathscr{G}^{rs}\mathscr{F}_{abr}\mathscr{F}_s = -\frac{1}{2}\mathscr{G}^{rs}\mathscr{G}_{sm}x^m\mathscr{F}_{abr} = -\frac{1}{2}x^r\mathscr{F}_{abr} = \mathscr{G}_{ab}$$

and thus the term on the second line of (5.16) can be written as

$$e^{-\mathscr{F}/3}\left(\frac{1}{2}\mathscr{G}^{rs}\mathscr{F}_{abr}\mathscr{F}_{kls} - \frac{1}{3}\mathscr{G}_{ab}\mathscr{G}_{kl}\right) = \frac{1}{2}\mathscr{G}^{rs}\mathscr{F}_{abr}\cdot e^{-\mathscr{F}/3}\left(\mathscr{F}_{kls} - \frac{1}{3}\mathscr{F}_s\mathscr{G}_{kl}\right).$$

By (5.4), $\frac{1}{2}\mathscr{G}^{rs}\mathscr{F}_{abs}$ are the Christoffel symbols of the metric $\mathscr{G}$ in the affine coordinates x^k, whilst $e^{-\mathscr{F}/3}(\mathscr{F}_{kls} - \frac{1}{3}\mathscr{F}_s\mathscr{G}_{kl})dx^kdx^l$ is just $\frac{\partial q}{\partial x^s}$ by (5.15). Hence

$$\frac{1}{2}\mathscr{G}^{rs}\mathscr{F}_{abr}\cdot e^{-\mathscr{F}/3}\left(\mathscr{F}_{kls} - \frac{1}{3}\mathscr{F}_s\mathscr{G}_{kl}\right) = \nabla^{\mathscr{G}}_{\partial_a}\frac{\partial}{\partial x^b}$$

which finishes the proof of the lemma. □

After these preliminary computations, let us now prove the theorem:

Proof of Theorem 5.22. From the previous proposition it follows that if $\nabla^{\mathscr{G}}\Xi \equiv 0$ then the connections $\overline{\nabla}$ and $\nabla^{\mathscr{G}}$ coincide. In that case, $\overline{\nabla}$ has no component along the normal space of $q\circ\Phi(\mathscr{M})$ in $S^2_+(H^3)^*$ and thus $q\circ\Phi$ is a totally geodesic immersion. This also implies that $\overline{\nabla}$ is equal to its projection on the tangent space of $\mathscr{M}$, which is exactly the Levi-Civita connection of $\Phi^*g_{\mathfrak{D}}$, and therefore $\mathscr{G}$ and $\Phi^*g_{\mathfrak{D}}$ have the same Levi-Civita connection. Moreover, since $S^2_+(H^3)^*$ is a symmetric space with nonpositive sectional curvature [59], the metric $\Phi^*g_{\mathfrak{D}} = \Phi^*q^*g_{S^2_+}$ is locally symmetric and has nonpositive sectional curvatures, and as these properties only depend on the Levi-Civita connection the metric $\mathscr{G}$ must also satisfy this property.

It remains to prove that if Φ is a totally geodesic immersion then the Yukawa coupling is parallel. Thus let us assume that Φ is totally geodesic. Since the map $q:\mathfrak{D}\to S^2_+(H^3)^*$ is a Riemannian fibration and Φ is a horizontal map, it follows that $q\circ\Phi$ is also a totally geodesic immersion. Therefore, the connection $\overline{\nabla}$ of the bundle $\mathscr{M}\times S^2(H^3)^*$ must preserve the tangent space $T\mathscr{M}$ (seen as a subbundle). Given the expression of the connection $\overline{\nabla}$ given in Proposition 5.25, we deduce that for all $0\leq a\leq b\leq n$, the quadratic form $\nabla_a\Xi_{bkl}dx^kdx^l$ is a section of $T\mathscr{M}$ when Φ

is a totally geodesic immersion. We shall now prove that there exists a subbundle E of $\mathscr{M} \times S^2(H^3)^*$ such that $T\mathscr{M} \oplus E = \mathscr{M} \times S^2(H^3)^*$ and $\nabla_a \Xi_{bkl} dx^k dx^l$ is a section of E for any $0 \leq a \leq b \leq n$. Once we have shown this, then when Φ is totally geodesic each $\nabla_a \Xi_{bkl} dx^k dx^l$ is a section of both $T\mathscr{M}$ and its complement E and therefore it must vanish, whence $\nabla_a \Xi_{bkl} = 0$ for all $0 \leq a, b, k, l \leq n$ and the theorem is proved.

Using local affine coordinates, let us define E by

$$E_x = \{q \in S^2(H^3)^*,\ q(\cdot, x) = 0\} \subset S^2(H^3)^*.$$

The subspace E_x has codimension $n+1$ in $S^2(H^3)^*$, that is $\operatorname{codim}(E_x) = \dim(T_x\mathscr{M})$. By Lemma 5.11, for any $0 \leq a, b \leq n$ the covariant derivative of the Yukawa coupling satisfies

$$x^r \nabla^{\mathscr{G}}_a \Xi_{bkr} = 0$$

and therefore the quadratic form $\nabla^{\mathscr{G}}_a \Xi_{bkl} dx^k dx^l$ takes values in E_x.

In order to prove that E_x is a complement of $T_x\mathscr{M}$ in $S^2(H^3)^*$, we need to prove that the $n+1$ linear forms $\frac{\partial q}{\partial x^a}(\cdot, x) \in (H^3)^*$ are linearly independent. By (5.15), we have

$$\begin{aligned} e^{\mathscr{F}/3} \frac{\partial q}{\partial x^a}(\cdot, x) &= x^r \mathscr{F}_{akr} dx^k - \frac{1}{3} x^r \mathscr{F}_a \mathscr{G}_{kr} dx^k \\ &= -2\mathscr{G}_{ak} dx^k + \frac{1}{3} \mathscr{F}_a \mathscr{F}_k dx^k \\ &= -2\mathscr{G}\left(\frac{\partial}{\partial x^a}, \cdot\right) + \frac{1}{3} \frac{\partial \mathscr{F}}{\partial x^a} \cdot d\mathscr{F}, \end{aligned}$$

where we used the identities (5.17) to pass from the first to the second line. After a linear change of coordinates, we may assume that $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ are tangent to the level set of $\mathscr{F}$ at the point x. Hence we just have $\frac{\partial q}{\partial x^a}(\cdot, x) = -2e^{-\mathscr{F}/3}\mathscr{G}(\partial_a, \cdot)$ for $1 \leq a \leq n$, and this gives n linearly independent linear forms. Moreover, using Remark 5.1 we can compute that

$$e^{\mathscr{F}/3} x^r \frac{\partial q}{\partial x^r}(\cdot, x) = -2x^a \mathscr{G}\left(\frac{\partial}{\partial x^a}, \cdot\right) + \frac{1}{3} x^a \frac{\partial \mathscr{F}}{\partial x^a} \cdot d\mathscr{F} = 2d\mathscr{F} - \frac{7}{3} d\mathscr{F} = \frac{1}{3}\mathscr{G}(x, \cdot)$$

and this gives another linear form independent from the previous ones, since $x^r \frac{\partial}{\partial x^r}$ is linearly independent of $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^r}$ (as $x^r \mathscr{F}_r \neq 0$). Thus the $n+1$ linear forms $\frac{\partial q}{\partial x^a}(\cdot, x)$, $a = 0, \dots, n$, are independent. Hence $T_x\mathscr{M}$ is a complement of E_x in $S^2(H^3)^*$, which completes the proof of the theorem. □

5.4 Transversality as an exterior differential system

By means of closing this chapter, this section gathers a few general observations about the transversality conditions for immersions into the domain $\mathfrak{D}$. This notably allows us to relate the map Φ which we defined on G_2-moduli spaces to the classical notion of period map defined by Joyce [66, §10.4] in a natural way.

Our general setup is the following. We will denote by H a $2(n+1)$-dimensional vector space ($n \geq 1$) endowed with a symplectic structure Q and an involution ι of H such that $\iota^* Q = -Q$. We denote by $H_\pm \subset H$ the ± 1 eigenspaces of ι, which must be Lagrangian. In particular, Q identifies H_- with H_+^*. This structure is exactly what we need in order to define abstractly the domain $\mathfrak{D}$ as the space of decompositions $H = H^{(3)} \oplus H^{(2)} \oplus H^{(1)} \oplus H^{(0)}$ satisfying the axioms **(1)**, **(2)** and **(3)**. Of course if $H = H^3(M) \oplus H^4(M)$ for a compact G_2-manifold M, then $H_+ = H^3(M)$ and $H_- = H^4(M)$. All the notions defined in Section 5.2 carry on to our abstract setting in a straightforward way, including the notion of adapted basis of an element $\mathbf{H} \in \mathfrak{D}$, and the definitions of $\ell_{\mathbf{H}} \in \mathbb{P}(H)$ and $q_{\mathbf{H}} \in S^2_+ H^*_+$.

5.4.1 Dimension and generality of transverse submanifolds. In this part we study general properties of transverse immersions. For this purpose, it will be useful to express the condition of transversality as an *exterior differential system*. For generalitites about exterior differential systems, we refer to the lectures [17]. Let us denote by $I \subset T^*\mathfrak{D}$ the annihilator of $T^t\mathfrak{D}$. Then a map $\Phi : P \to \mathfrak{D}$ is transverse if and only if

$$\Phi^* \alpha = 0, \qquad \forall \alpha \in C^\infty(I)$$

where $C^\infty(I)$ is the space of smooth sections of I over $\mathfrak{D}$. We denote by $\{I\} \subset \Lambda(T^*\mathfrak{D})$ the ideal algebraically generated by I, seen as a subbundle of the exterior algebra $\Lambda(T^*\mathfrak{D})$. Alternatively, $\{I\}$ can be defined as the space of differential forms vanishing on the transverse distribution. Therefore, the pull-back of any section of $\{I\}$ vanishes along a transverse map, and so does the pull-back of the exterior differential of such a section. This leads us to considering the bundle map $\delta : I \to \Lambda^2(T^*\mathfrak{D})/\{I\}$ defined as

$$\delta\alpha \equiv d\alpha \mod \{I\}$$

for any section of I. The differential ideal $\mathscr{I} \subset \Omega^*(\mathfrak{D})$ generated by I is the space of sections of the ideal $\{I, \delta I\} \subset \Lambda(T^*\mathfrak{D})$, where we think of $\Lambda^2(T^*\mathfrak{D})/\{I\}$ as

$\Lambda^2(T^t\mathfrak{D})^* \subset \Lambda^2(T^*\mathfrak{D})$. With these definitions, a map $\Phi : P \to \mathfrak{D}$ is transverse if and only if

$$\Phi^*\eta = 0, \qquad \forall \eta \in \mathscr{I}.$$

We will be interested in transverse maps satisfying certain *independence conditions*. Namely, we want to study n-dimensional transverse immersions $\Phi : P \to \mathfrak{D}$ such that the composition $\ell \circ \Phi : P \to \mathbb{P}(H_+)$ is a local diffeomorphism. This condition is equivalent to requiring that the pull-back ϖ of a local volume form on $\mathbb{P}(H_+)$ does not vanish on P. Such an immersion locally defines a submanifold of $\mathfrak{D} \simeq \mathbb{P}(H_+)$ which is graphical over an open subset of $\mathbb{P}(H_+)$, and is said to be an integral submanifold of $(\mathscr{I}, \varpi)$ (with the understanding that ϖ is only locally defined if $\mathbb{P}(H_+)$ is not orientable).

Since the transverse distribution is equivariant, the properties of the differential ideal $\mathscr{I}$ are determined by the fibre $I_{\mathbf{H}}$ of I and the map $\delta : I_{\mathbf{H}} \to \Lambda^2(T^*_{\mathbf{H}}\mathfrak{D})/I_{\mathbf{H}}$ over any point of $\mathfrak{D}$. Thus we shall fix $\mathbf{H} \in \mathfrak{D}$ and a standard basis $(u_0, \dots, u_n, v_0, \dots, v_n)$ of H for $\mathbf{H}$. Recall from §5.2.2 that the tangent space $T_{\mathbf{H}}\mathfrak{D}$ can be identified with the space of matrices

$$\mathfrak{p}_{\mathbf{H}} = \{(a_{ij})_{0\le i,j\le n},\ a_{ij} = a_{ji}\ \forall 1 \le i, j \le n\}.$$

Moreover, the distributions $T^v\mathfrak{D}$, $T^h\mathfrak{D}$, $T^l\mathfrak{D}$ and $T^t\mathfrak{D}$ correspond to the subspaces

$$\begin{aligned}
\mathfrak{v}_{\mathbf{H}} &= \{a \in \mathfrak{p}_{\mathbf{H}},\ a_{ij} = 0\ \forall 1 \le i, j \le n \ \text{ and } \ a_{0j} = -a_{j0}\ \forall 0 \le j \le n\},\\
\mathfrak{h}_{\mathbf{H}} &= \{a \in \mathfrak{p}_{\mathbf{H}},\ a_{ij} = a_{ji}\ \forall 0 \le i, j \le n\},\\
\mathfrak{l}_{\mathbf{H}} &= \{a \in \mathfrak{p}_{\mathbf{H}},\ a_{ij} = 0 \text{ if } (i,j) \neq (0,0)\}, \text{and}\\
\mathfrak{t}_{\mathbf{H}} &= \{a \in \mathfrak{p}_{\mathbf{H}},\ a_{00} = 0 \text{ and } a_{ij} = a_{ji}\ \forall 0 \le i, j \le n\}.
\end{aligned}$$

Let us consider the linear forms on $\mathfrak{p}_{\mathbf{H}}$ defined as:

$$\begin{aligned}
\alpha_j(a) &= (a_{j0} - a_{0j})/2, & 1 \le j \le n,\\
\beta_j(a) &= (a_{j0} + a_{0j})/2, & 0 \le j \le n, \text{ and}\\
\beta_{ij}(a) &= (a_{ij} + a_{ji})/2 = \beta_{ji}, & 1 \le i, j \le n.
\end{aligned}$$

Then $I_{\mathbf{H}}$ is spanned by $\beta_0, \alpha_1, \dots, \alpha_n$, and moreover we can choose $\varpi = \beta^1 \wedge \cdots \wedge \beta_n$ as independence condition at $\mathbf{H}$. Let us denote by $\{e_j\}_{1\le j\le n} \cup \{f_j\}_{0\le j\le n} \cup \{f_{ij} = f_{ji}\}_{1\le i\le j\le n}$ the dual basis of $\mathfrak{p}_{\mathbf{H}}$.

The structure equations for the transverse distribution are the following:

Proposition 5.26. *The map $\delta : I_{\mathbf{H}} \to \Lambda^2(\mathfrak{p}^*_{\mathbf{H}})/\{I_{\mathbf{H}}\}$ is determined by:*

$$\delta\beta_0 \equiv 0 \mod \{I_{\mathbf{H}}\}, \quad \textit{and} \quad \delta\alpha_j \equiv \sum_{i=1}^n \beta_i \wedge \beta_{ij} \mod \{I_{\mathbf{H}}\},\ 1 \le j \le n.$$

Proof. The dual space $\mathfrak{p}_{\mathbf{H}}^*$ is naturally identified with the annihilator of $\mathfrak{g}_{\mathbf{H}}$ in $\mathfrak{gl}(H_+)$, and $\Lambda^2(\mathfrak{p}_{\mathbf{H}}^*)/\{I_{\mathbf{H}}\}$ can be identified with $\Lambda^2(\mathfrak{t}_{\mathbf{H}}^*)$. Under this identification, the Maurer-Cartan formula yields:

$$\delta\alpha(a,b) = -\alpha([a,b]), \qquad \forall \alpha \in I_{\mathbf{H}}, \forall a, b \in \mathfrak{t}_{\mathbf{H}}$$

where $[\cdot,\cdot]$ denotes the Lie bracket on $\mathfrak{gl}(H_+)$. Therefore, if we write

$$\delta\alpha \equiv \frac{1}{2}\sum B_{ij}\beta_i \wedge \beta_j + \frac{1}{2}\sum C_{ijk}\beta_i \wedge \beta_{jk} + \frac{1}{4}\sum D_{ijkl}\beta_{ij} \wedge \beta_{kl} \mod \{I_{\mathbf{H}}\}$$

where all indices range over integers in between 1 and n, the coefficients are defined by:

$$\begin{aligned} B_{ij} &= -\alpha([f_i, f_j]) = -B_{ji}, \\ C_{ijk} &= -2^{\delta_{jk}}\alpha([f_i, f_{jk}]) = C_{ikj}, \text{ and} \\ D_{ijkl} &= -2^{\delta_{ij}+\delta_{jk}}\alpha([f_{ij}, f_{jk}]) = D_{jikl} = D_{ijlk}. \end{aligned}$$

Note that for any $[a,b] \in \mathfrak{t}_{\mathbf{H}}$, the bracket $[a,b]$ is $q_{\mathbf{H}}$-antisymmetric, and since β_0 vanishes on $q_{\mathbf{H}}$-antisymmetic endomorphisms we deduce

$$\delta\beta_0 \equiv 0 \mod \{I_{\mathbf{H}}\}.$$

As endomorphisms of H_+, the f_{ij}'s act trivially on $\ell_{\mathbf{H}}$ and leave invariant its orthogonal space, and thus so do the brackets $[f_{ij}, f_{kl}]$. Hence we deduce that $\alpha([f_{ij}, f_{kl}]) = 0$ for any $\alpha \in I_{\mathbf{H}}$. On the other hand, as the e_j's are $q_{\mathbf{H}}$-antisymmetric, the brackets $[e_i, e_j]$ are $q_{\mathbf{H}}$-symmetric, and thus $\alpha_k([e_i, e_j]) = 0$ for all $1 \leq i, j, k \leq n$. These observations yield

$$\delta\alpha_l \equiv -\frac{1}{2}\sum \alpha_l([f_j, f_{jk}])\beta_i \wedge \beta_{jk} \mod \{I_{\mathbf{H}}\}$$

and it only remains to compute the coefficients $\alpha_l([f_i, f_{jk}])$. Let us introduce the basis $\{E_{ij}\}_{0\leq i,j\leq n}$ of $\mathfrak{gl}(H_+)$ defined by $E_{ij}(u_k) = \delta_{jk}u_i$, for $0 \leq i, j, k \leq n$, so that:

$$\begin{aligned} e_j &= E_{j0} - E_{0j}, & 1 \leq j \leq n, \\ f_j &= 2^{-\delta_{0j}}(E_{j0} + E_{0j}), & 0 \leq j \leq n, \\ f_{ij} &= 2^{-\delta_{jk}}(E_{ij} + E_{ji}), & 1 \leq i, j \leq n. \end{aligned}$$

From the commutators $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}$ we deduce:

$$2^{\delta_{jk}}[f_i, f_{jk}] = [E_{i0} + E_{0i}, E_{jk} + E_{kj}] = -\delta_{ij}e_k - \delta_{ik}e_j$$

and hence

$$\delta\alpha_l \equiv \frac{1}{2}\sum_{1\leq i,j,k\leq n}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl})\beta_i \wedge \beta_{jk} \equiv \sum_{i=1}^{n}\beta_i \wedge \beta_{il} \mod \{I_{\mathbf{H}}\}$$

for any $1 \leq l \leq n$. □

Let $E \subset T_{\mathbf{H}}\mathfrak{D}$ be a linear subspace. It is said to be an *integral element* of $\mathscr{I}$ if the restriction of any differential form in $\mathscr{I}$ vanishes on E. If moreover E has dimension n and ϖ does not vanish on E then it is called an integral element of $(\mathscr{I}, \varpi)$. The integral elements of $(\mathscr{I}, \varpi)$ based at $\mathbf{H}$ are easily determined using the structure equations of the transverse distribution. Indeed, the independence condition $\varpi \neq 0$ is equivalent to the linear independence of the linear forms $\beta_1, \ldots, \beta_n$, and any n-dimensional subspace $E \subset \mathfrak{t}_{\mathbf{H}}$ satisfying this condition is uniquely characterised by the equations:

$$\beta_0 = \alpha_1 = \cdots = \alpha_n = 0,$$
$$\beta_{ij} = \sum_{k=1}^{n} C_{ijk}\beta_k, \quad \forall 1 \leq i, j \leq n,$$

where C_{ijk} are arbitrary coefficients symmetric in the indices i and j. The necessary and sufficient condition for E to be an integral element of $\mathscr{I}$ is that the 2-forms $\delta\alpha_1, \ldots \delta\alpha_n \in \Lambda^2(\mathfrak{t}^*_{\mathbf{H}})$ vanish on E:

$$0 = \sum_{i,k=1}^{n} C_{ijk}\beta_i \wedge \beta_k = \frac{1}{2}\sum_{i,k=1}^{n} (C_{ijk} - C_{kji})\beta_i \wedge \beta_k, \quad \forall 1 \leq j \leq n.$$

Hence E is an integral element of $\mathscr{I}$ if and only if the coefficients C_{ijk} are fully symmetric in the indices i, j, k.

Remark 5.27. From the definition of the quadratic form $h_{\mathfrak{D}}$ on $T^t\mathfrak{D}$ and the expression of the linear forms $\beta_1, \ldots, \beta_n$, one can easily see that $h_{\mathfrak{D}} = \sum_i \beta_i^2$. Hence an integral element E of $\mathscr{I}$, of rank n, satisfies the independence condition $\varpi \neq 0$ if and only if the restriction of the quadratic form $h_{\mathfrak{D}}$ to E is positive-definite.

Remark 5.28. Similarly, the coefficients C_{ijk} correspond (up to a combinatorial factor) to the coefficients of the cubic form $\Xi_{\mathfrak{D}}$, in the basis of E dual to $\beta_1, \ldots, \beta_n$. This explains why the restriction of $\Xi_{\mathfrak{D}}$ to any integral element of $(\mathscr{I}, \varpi)$ is symmetric, even though $\Xi_{\mathfrak{D}}$ itself is not a symmetric cubic form on $T^t\mathfrak{D}$.

Remark 5.29. Let E_1, E_2 be integral elements of $(\mathscr{I}, \varpi)$ (not necessarily based at the same point), and denote by (h_i, Ξ_i) the restriction of $(h_{\mathfrak{D}}, \Xi_{\mathfrak{D}})$ to E_i. Then it is not difficult to prove that if there is a linear isomorphism $\phi : E_1 \to E_2$ such that $\phi^*(h_2, \Xi_2) = (h_1, \Xi_1)$, then there exists an element $A \in \mathrm{Aut}(H, Q, \iota)$, unique up to multiplication by $\pm\,\mathrm{Id}$, such that ϕ is the restriction to E_1 of the action of A on the tangent space $T\mathfrak{D}$.

This algebraic fact has an interesting counterpart (which we will not prove for lack of space) for real-analytic integral submanifolds of $(\mathscr{I}, \varpi)$. Namely, let P

be a connected real-analytic manifold and Φ_1, Φ_2 be two real-analytic transverse immersions of P into $\mathfrak{D}$, such that $\Phi_1^*(h_{\mathfrak{D}}, \Xi_{\mathfrak{D}}) = \Phi_2^*(h_{\mathfrak{D}}, \Xi_{\mathfrak{D}})$ and $\Phi_1^* h_{\mathfrak{D}}$ is non-degenerate. Then there exists an automorphism $A \in \mathrm{Aut}(H, Q, \iota)$, unique up to multiplication by $\pm\,\mathrm{Id}$, such that $\Phi_2 = A \circ \Phi_1$. This can be proved by first considering the tangent map at one point, using the previous algebraic fact, and then extending it to the rest of the manifold.

This might explain why, in the literature, the metric $\mathscr{G}$ and the Yukawa coupling Ξ are the only natural tensors considered on G_2-moduli spaces (at least to the author's knowledge); in some sense they completely characterise the geometry of $\mathscr{M}$ and how it is immersed into $\mathfrak{D}$.

An integral element of $\mathscr{I}$ is called *maximal* if it is not strictly contained in another integral element of $\mathscr{I}$. Any integral element of $\mathscr{I}$ is contained in the transverse distribution, but due to the non-triviality of the map δ the codimension of a maximal integral element of $\mathscr{I}$ is in general much larger than $n+1$. As a consequence of the previous proposition we prove:

Corollary 5.30. *Any integral element of $(\mathscr{I}, \varpi)$ is a maximal integral element of $\mathscr{I}$. Thus any integral submanifold of $(\mathscr{I}, \varpi)$ is a maximal transverse submanifold of $\mathfrak{D}$.*

Proof. It suffices to prove that any integral element of $\mathscr{I}$ on which $\beta_1, \dots, \beta_n$ are linearly independent has dimension exactly n. Let E be such an integral element. As $\delta\alpha_j$ vanishes on E, then so does the $n+1$-form

$$\beta_1 \wedge \cdots \wedge \beta_{i-1} \wedge \beta_{i+1} \wedge \cdots \wedge \beta_n \wedge \delta\alpha_j = (-1)^{n-i+1} \varpi \wedge \beta_{ij}$$

for $1 \le i, j \le n$. Hence $\beta_1, \dots, \beta_n, \beta_{ij}$ are linearly dependent on E, and therefore there exist coefficients C_{ijk} symmetric in i and j such that E is contained in the kernel of the $\frac{n(n+1)}{2}$ linearly independent linear forms

$$\beta_{ij} = \sum_{k=1}^{n} C_{ijk} \beta_k.$$

Since moreover $\beta_0, \alpha_1, \dots, \alpha_n$ vanish on E, E is contained in the kernel of $n + 1 + \frac{n(n+1)}{2}$ linearly independent linear forms on $T_{\mathbf{H}}\mathfrak{D}$, and thus it has dimension at most n. Since $\beta_1, \dots, \beta_n$ are linearly independent on E we deduce that E has dimension n and is an integral element of $(\mathscr{I}, \varpi)$. □

Remark 5.31. Using the structure equations of the transverse distribution, one can prove that any integral element of $(\mathscr{I}, \varpi)$ is a regular integral element of $\mathscr{I}$ in the sense of Cartan–Kähler theory. Moreover, the general real-analytic integral submanifold of $(\mathscr{I}, \varpi)$ depends on $n+1$ constants, n functions of one variable, ..., 2 functions of $n-1$ variables and 1 function of n variables.

5.4.2 The canonical contact system. Let us denote by $\Omega \subset \mathbb{P}(H)$ the open subset defined by

$$\Omega = \{\langle w \rangle \in \mathbb{P}(H),\ Q(\iota(w), w) > 0\}. \tag{5.18}$$

The automorphism group of (H, Q, ι), which can be identified with $\mathrm{GL}(H_+)$, acts transitively on Ω, and there is an equivariant fibration $\pi : \mathfrak{D} \to \Omega$ mapping any $\mathbf{H} \in \mathfrak{D}$ to $\pi(\mathbf{H}) = H^{(3)} \in \Omega$. The domain Ω carries a differential ideal obtained by restriction of the *canonical contact system* of $\mathbb{P}(H)$, which can be described as follows. On $H\backslash\{0\}$ we can consider the one-form

$$\Gamma = Q(dw, w). \tag{5.19}$$

Since Q is alternated, Γ_w vanishes on $\langle w \rangle$ for any $w \neq 0$, and moreover $\Gamma_{fw} = f^2\Gamma_w$ for any smooth nowhere vanishing function f defined on $H\backslash\{0\}$. Hence Γ induces a well-defined 1-dimensional subbundle of $T^*\mathbb{P}(H)$, and the associated exterior differential system $\mathscr{J}$ is the canonical contact system. The name contact system corresponds to the fact that

$$\gamma \wedge (d\gamma)^n \neq 0$$

for any nonvanishing one-form in $\mathscr{J}$. It is a classical fact that the maximal integral submanifolds of such a system have dimension n. Using coordinates $(w^0, \ldots, w^n, w_0, \ldots, w_n)$ on H such that

$$Q = \sum_{j=0}^{n} dw_j \wedge dw^j \tag{5.20}$$

then the contact system, in homogeneous coordinates $[w^0 = 1 : w^1 : \ldots : w^n : w_0 : \ldots : w_n]$, is generated by:

$$\gamma = dw_0 + \sum_{j=1}^{n} w^j dw_j - w_j dw^j \tag{5.21}$$

and in particular:

$$d\gamma = -2 \sum_{j=1}^{n} dw_j \wedge dw^j. \tag{5.22}$$

In this part, we point out that there is a one-to-one correspondence between integral submanifolds of $(\mathscr{I}, \varpi)$ and maximal solutions of the canonical contact system on Ω, together with some open condition. In fact, the exterior differential system $(\mathscr{I}, \varpi)$ appears as the *first prolongation* of $\mathscr{J}$. Since our construction is a straightforward adaptation of an argument of Bryant and Griffiths [18] who prove a similar result for variations of Hodge structures, we will only briefly explain the correspondence and refer to their paper for more details on contact systems and the process of prolongation.

Let us denote by $V_n(\Omega, \mathscr{J})$ the space of n-dimensional integral element of $\mathscr{J}$ over Ω. We consider the open subset $V_\Omega \subset V_n(\Omega, \mathscr{J})$ defined as:

$$V_\Omega = \{E \in V_n(\Omega, \mathscr{J}),\ Q(\iota(\phi(w)), \phi(w)) < 0,\ \forall \phi \in E\}$$

where we identify the tangent space $T_\pi\Omega$ with the space of linear maps from π to its $Q(\iota\cdot, \cdot)$-orthogonal complement in H. The domain V_Ω is acted upon by the automorphism group of (H, Q, ι), identified with $\mathrm{GL}(H_+)$. The key observation is:

Lemma 5.32. *There is an equivariant diffeomorphism $\mathfrak{D} \to V_\Omega$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{D} & \longrightarrow & V_\Omega \\ & \searrow \qquad \swarrow & \\ & \Omega & \end{array}$$

Proof. It is enough to show that we can identify the fibres of $\mathfrak{D}$ and V_Ω over an element $\pi \in \Omega$. Let us use coordinates $(w^0, \dots, w^n, w_0, \dots, w_n)$ on H such that Q takes the form (5.20) and ι reads:

$$\iota(w^j, w_j) = (w_j, w^j).$$

We can moreover assume that $\frac{\partial}{\partial w^0} \in \pi$ since $\pi \in \Omega$. Hence the integral elements of the contact system lying over π are subject to the equations:

$$dw_0 = 0, \quad \text{and} \quad \sum_{j=1}^{n} dw_j \wedge dw^j = 0.$$

Denoting $H^{(3)} = \pi$, the space of integral elements lying over π can therefore be described as the set of n-dimensional subspaces $H^{(2)} \subset H$ satisfying:

$$Q(\iota(H^{(3)}), H^{(2)}) = Q(H^{(3)}, H^{(2)}) = Q(H^{(2)}, H^{(2)}) = 0,$$

and moreover such an integral element belongs to V_Ω if and only if

$$Q(\iota(H^{(2)}), H^{(2)}) < 0.$$

If we let $H^{(0)} = \iota(H^{(3)})$ and $H^{(1)} = \iota(H^{(2)})$, the above conditions are equivalent to requiring that $H = H^{(3)} \oplus H^{(2)} \oplus H^{(1)} \oplus H^{(0)}$ be an element of $\mathfrak{D}$. $\square$

Let us denote by $\mathrm{Gr}(n, T\Omega)$ the Grassmannian of n-planes in $T\Omega$, and consider V_Ω as a submanifold of $\mathrm{Gr}(n, T\Omega)$. Let us pick $E \in V_\Omega$, and denote by $\mathbf{H} \in \mathfrak{D}$ the corresponding base point. We may use coordinates $(w^0, \dots, w^n, w_0, \dots, w_n)$ such that Q takes the form (5.20), $\iota(w^j, w_j) = (w_j, w^j)$, and moreover the decomposition $H = H^{(3)} \oplus H^{(2)} \oplus H^{(1)} \oplus H^{(0)}$ satisfies:

$$H^{(3)} = \operatorname{span}\left\{\frac{\partial}{\partial w^0}\right\}, \quad H^{(2)} = \operatorname{span}\left\{\frac{\partial}{\partial w_j},\ 1 \leq j \leq n\right\}.$$

We use homogeneous coordinates $[w^0 = 1 : w^1 : \ldots : w^n : w_0 : \ldots : w_n]$ on $\mathbb{P}(H)$. In a neighbourhood of E in $\mathrm{Gr}(n, T\Omega)$, the 1-forms $dw_1, \ldots, dw_n$ are linearly independent, and therefore $\mathrm{Gr}(n, T\Omega)$ has a system of coordinates $w^1, \ldots, w^n, w_0, \ldots, w_n$, q^i, p^{ij} for $1 \leq i, j \leq n$ so that:

$$dw_0 = \sum_{i=1}^n q^i dw_i, \quad \text{and} \quad dw^j = \sum_{i=1}^n p^{ij} dw_i. \tag{5.23}$$

In homogeneous coordinates, the canonical contact system of $\mathbb{P}(H)$ is generated by a 1-form γ satisfying (5.21) and (5.22), and thus the submanifold $V_\Omega \subset \mathrm{Gr}(n, T\Omega)$ is cut out by the equations

$$q^i = \sum_{j=1}^n w_j p^{ij} - w^i, \quad 1 \leq i \leq n,$$

and

$$p^{ij} = p^{ji}, \quad 1 \leq i, j \leq n.$$

In particular, the variables $w^1, \ldots, w^n, w_0, \ldots, w_n, p^{ij}$ for $1 \leq i \leq j \leq n$ are independent of V_Ω near E.

In the domain where $dw_1, \ldots, dw_n$ are linearly independent, (5.23) defines an exterior differential system on $\mathrm{Gr}(n, T\Omega)$, called the *canonical system*. Its restriction to V_Ω, often denoted $\mathscr{J}^{(1)}$, is called the *first prolongation* of the exterior differential system $\mathscr{J}$ on Ω, and its n-dimensional integral submanifolds are in one-to-one correspondence with the integral submanifolds of $\mathscr{J}$ whose tangent spaces lie in V_Ω. At the point $E \in V_\Omega$, we have $q^i = p^{ij} = 0$ and thus:

$$d(dw_0 - \sum_{i=1}^n q^i dw_i) \equiv 0 \mod \{dw_0, dw^1, \ldots, dw^n\},$$

and

$$d(dw^j - \sum_{i=1}^n p^{ij} dw_i) \equiv \sum_{i=1}^n dw_i \wedge dp^{ij} \mod \{dw_0, dw^1, \ldots, dw^n\}.$$

Examining the proof of Lemma 5.15, we see that we have the identifications $\alpha_i = dw^i$ for $1 \leq i \leq n$, $\beta_j = dw_j$ for $0 \leq j \leq n$ and $\beta_{ij} = dp^{ij}$ for $1 \leq i, j \leq n$, in the notations of the previous part. Comparing with the structure equations of the transverse distribution in $\mathfrak{D}$, we deduce:

Proposition 5.33. *Under the identification $\mathfrak{D} \simeq V_\Omega$, $(\mathscr{I}, \varpi)$ is the restriction to V_Ω of the first prolongation $\mathscr{J}^{(1)}$ of the canonical contact system of Ω. Consequently, there is a one-to-one correspondence between the immersed transverse submanifolds of $\mathfrak{D}$ on which $h_\mathfrak{D}$ is a Riemannian metric and the maximal integral submanifolds of the canonical contact system whose tangent spaces lie in V_Ω.*

For the moduli spaces of G_2-manifolds, this result has a clear interpretation. Indeed, it was proved by Joyce that the map $\varphi\mathscr{D} \in \mathscr{M} \mapsto [\varphi] + [\Theta(\varphi)] \in H^3(M) \oplus H^4(M)$ is a Lagrangian immersion. In particular, the restriction of this map to $\mathscr{M}_1$ composed with the quotient map $H\backslash\{0\} \to \mathbb{P}(H)$ is a Legendrian immersion. Hence $\mathscr{M}_1$ can be seen as a maximal immersed integral submanifold of the canonical contact system of $\mathbb{P}(H)$, and it is easy to see that its tangent spaces lie in V_Ω. The transverse map $\Phi : \mathscr{M}_1 \to \mathfrak{D}$ which we constructed is exactly the associated integral submanifold of $(\mathscr{I}, \varpi)$; and moreover, up to a factor the restriction of $h_\mathfrak{D}$ coincides with the metric $\mathscr{G}_1$ (by Proposition 5.18).

5.4.3 Transverse submanifolds and local potentials. To finish this chapter, we show that the potential $\mathscr{F}$ can be recovered (at least locally, and up to some choices of normalisation) from the map $\Phi : \mathscr{M}_1 \to \mathfrak{D}$. In fact, we can locally associate a convex function to any integral submanifold of $(\mathscr{I}, \varpi)$. In particular this shows that the result of Theorem 5.22 is not specific to G_2-moduli spaces, but holds for any integral submanifold P of $(\mathscr{I}, \varpi)$: the restriction of $\Xi_\mathfrak{D}$ is parallel for the Levi-Civita connection of the metric defined by restriction of $h_\mathfrak{D}$ if and only if P is a totally geodesic submanifold of $\mathfrak{D}$.

In the remainder of this part, let us fix a basis $u_0, \dots, u_n$ of H_+ and consider the corresponding coordinates $x^0, \dots, x^n$ on H_+. For any $x = (x^0, \dots, x^n) \in H_+\backslash\{0\}$, we denote by $\langle x \rangle \in \mathbb{P}(H_+)$ the line generated by x and $[x^0 : \dots : x^n]$ its homogeneous coordinates. For any open subset $U \subseteq \mathbb{P}(H_+)$ we denote by $C(U) \subseteq H_+\backslash\{0\}$ the open cone over U, that is:

$$C(U) = \{x \in H_+\backslash\{0\},\ \langle x \rangle \in U\}.$$

Locally, an integral submanifold of $(\mathscr{I}, \varpi)$ can be described by a map

$$U \longrightarrow \mathbb{P}(H_+) \times S^2_+(H_+)^*, \quad \ell \longmapsto (\ell, q_\ell)$$

where U is an open subset of $\mathbb{P}(H_+)$. On $C(U)$, let us consider the function

$$W(x) = q_{\langle x \rangle}(x, x).$$

This is a positive function, homogeneous of degree 2.

Lemma 5.34. *The first derivative of W is given by:*

$$\frac{\partial W}{\partial x^a} = 2q_{\langle x\rangle}(x, u_a)\cdot$$

Proof. The partial derivatives of W satisfy:

$$\frac{\partial W}{\partial x^a} = (\partial_a q_{\langle x\rangle})(x, x) + 2q_{\langle x\rangle}(x, u_a)\cdot$$

Since the map $\langle x\rangle \to (\langle x\rangle, q_{\langle x\rangle})$ is transverse, Lemma 5.16 yields $\partial_a q_{\langle x\rangle}(x, x) = 0$, from which the proposition follows. □

Proposition 5.35. *The function $F = -\frac{1}{2}\log W$ is convex on $C(U)$, and moreover its Hessian matrix is given by:*

$$\frac{\partial^2 F}{\partial x^a \partial x^b} = \frac{q_{\langle x\rangle}(u_a, u_b)}{q_{\langle x\rangle}(x, x)}\cdot$$

Proof. We first compute the Hessian matrix of W. The precedent lemma yields:

$$\frac{\partial^2 W}{\partial x^a \partial x^b} = 2\partial_b(q_{\langle x\rangle}(x, u_a)) = 2(\partial_b q_{\langle x\rangle})(x, u_a) + 2q_{\langle x\rangle}(u_a, u_b).$$

Let $\phi_b : \langle x\rangle \to \langle x\rangle^{\perp_{q_{\langle x\rangle}}}$ be the linear map representing the vector $\partial_a\langle x\rangle \in T_{\langle x\rangle}\mathbb{P}(H_+)$. Then it has for expression:

$$\phi_b(x) = u_b - \frac{q_{\langle x\rangle}(x, u_b)}{q_{\langle x\rangle}(x, x)}x.$$

By transversality, Lemma 5.16 gives:

$$(\partial_b q_{\langle x\rangle})(x, u_a) = -2q_{\langle x\rangle}(\phi_b(x), u_a) = \frac{2q_{\langle x\rangle}(u_a, x)q_{\langle x\rangle}(u_b, x)}{q_{\langle x\rangle}(x, x)} - 2q_{\langle x\rangle}(u_a, u_b)$$

so that

$$\frac{\partial^2 W}{\partial x^a \partial x^b} = \frac{4q_{\langle x\rangle}(u_a, x)q_{\langle x\rangle}(u_b, x)}{q_{\langle x\rangle}(x, x)} - 2q_{\langle x\rangle}(u_a, u_b)\cdot$$

We may now deduce:

$$\begin{aligned}\frac{\partial^2 \log W}{\partial x^a \partial x^b} &= \frac{4q_{\langle x\rangle}(u_a, x)q_{\langle x\rangle}(u_b, x)}{q_{\langle x\rangle}(x, x)^2} - 2\frac{q_{\langle x\rangle}(u_a, u_b)}{q_{\langle x\rangle}(x, x)} - \frac{2q_{\langle x\rangle}(u_a, x)\cdot 2q_{\langle x\rangle}(u_b, x)}{q_{\langle x\rangle}(x, x)^2}\\ &= -2\frac{q_{\langle x\rangle}(u_a, u_b)}{q_{\langle x\rangle}(x, x)}\end{aligned}$$

which gives the claimed expression for the Hessian matrix of F. □

We shall refer to F as the *local potential* associated with the transverse map $\ell \mapsto (\ell, q_\ell)$. The above proposition shows that $q_{\langle x\rangle}$ can be reconstructed from F, since it coincides with the Hessian of F at the point x of the line such that $W(x) = 1$. In fact, there is a converse to this statement:

Proposition 5.36. *Let $C \subset H_+$ be an open cone, $U = S_+ \cap C$ be its cross-section and let $W : C \to \mathbb{R}$ be a positive function, homogeneous of degree 2, and assume that the function $F = -\frac{1}{2}\log W$ is strictly convex. For any $u \in U$, let us denote by $x(u) \in H_+$ the unique element of $\langle u \rangle$ such that $W(x(u)) = 1$. Then the map:*

$$u \in U \longmapsto \left(\langle u \rangle, \left. \frac{\partial^2 F}{\partial x^a \partial x^b} \right|_{x=x(u)} \right) \in \mathbb{P}(H_+) \times S^2_+(H_+)^*$$

is transverse.

Proof. Let us first determine the tangent space to the level sets of F. By homogeneity, we have:

$$F(tx) = F(x) - \log t, \quad \forall t > 0$$

and differentiating this expression with respect to t and then setting $t = 1$ yields:

$$x^a \frac{\partial F}{\partial x^a} = -1, \quad \forall x \in C.$$

Since the left-hand side is constant, the partial derivatives of the expression on the right-hand-side vanish:

$$x^a \frac{\partial^2 F}{\partial x^a \partial x^b} + \frac{\partial F}{\partial x^b} = 0, \quad \forall x \in C, \ \forall b = 0, \ldots, n. \tag{5.24}$$

Let us denote by $q_x \in S^2_+(H_+)^*$ the quadratic form associated with the Hessian matrix of F. With these notations, the above equation can be written as:

$$\frac{\partial F}{\partial x^b} + q_x(x, u_b) = 0, \quad \forall x \in C, \ \forall b = 0, \ldots, n.$$

Thus the tangent space to the level set of F at a point $x \in C$, or equivalently the tangent space to the level sets of W, is the orthogonal complement of x for the quadratic form q_x. Let $X_1(x), \ldots, X_n(x)$ be any local frame of the tangent space to the level sets. By Lemma 5.16, transversality is equivalent to the fact that the partial derivatives of q_x in the directions of $X_1, \ldots, X_n$ along the level set $W = 1$ satisfy:

$$(\partial_{X_b} q_x)(x, u_a) + 2q_x(X_b, u_a) = 0, \quad \forall a = 0, \ldots, n.$$

Since q_x is the Hessian of a function we can write:

$$(\partial_{X_b} q_x)(x, u_a) = \sum_{c=0}^{n} x^c (\partial_c q_x)(X_b, u_a)$$

and taking partial derivatives in (5.24) we obtain:

$$\sum_{c=0}^{n} x^c \frac{\partial^3 F}{\partial x^a \partial x^b \partial x^c} + 2 \frac{\partial F}{\partial x^a \partial x^b} = 0$$

that is, $\sum x^c \partial_c q_x + 2q_x = 0$, which implies the desired identity. □

Remark 5.37. In fact, the proof shows that there is nothing special about the level set $W = 1$, or about the normalisation factor $\frac{1}{2}$, since we only used the fact that the function $\sum x^a \frac{\partial F}{\partial x^a}$ is constant. Hence we could replace W by a positive function, homogeneous of degree $d > 0$ and the result would still hold. In the case of G_2-moduli spaces, the function F does not coincide with the the potential $\mathscr{F}$, but they are related by an affine transformation. This does not really affect any interesting geometric property.

Proposition 5.38. *Let $\lambda \in \mathbb{R}\backslash\{2\}$. Then in the setup of the previous proposition,*

$$x \in C \longmapsto \left(\langle x \rangle, e^{\lambda F} \frac{\partial^2 F}{\partial x^a \partial x^b}(x) \right) \in \mathbb{P}(H_+) \times S^2_+(H_+)^*$$

is a horizontal immersion.

Proof. We have already proved that the differential of this map is transverse along the level sets of W. Thus it only remains to prove that the map is horizontal in the radial direction. Let use denote by q_x the quadratic form defined by the Hessian of F. By Lemma 5.16, this amounts to proving that

$$\sum_c x^c \partial_c (e^{\lambda F} q_x)(x, X) = e^{\lambda F}(x^c \lambda \partial_c F q(x, X) + x^c \partial_c q_x(x, X))0$$

for any vector X orthogonal to x, which is satisfied since $\sum_c x^c(\partial_c q_x) + 2q_x = 0$. To prove that the map is an immersion, it suffices to remark the the differential of the map in the radial direction is not transverse, since $\sum_c x^c \partial_c (e^{\lambda F} q_x)(x, x) = (\lambda - 2) q_x(x, x) \neq 0$, whilst the differential of the map $x \mapsto \langle x \rangle$ vanishes in the radial direction. □

Using this proposition, it is easy see that the result of Theorem 5.22 applies to any integral submanifold of $(\mathscr{I}, \varpi)$, or rather to the extension to a horizontal submanifold constructed as above, for any choice of $\lambda \neq 2$: the Hessian of the potential F and its third derivative will define a natural Riemannian metric and a symmetric cubic form on this extension, and their restrictions to the transverse leaves coincide (up to some factor depending on the choice of λ) with the geometric structures induced by $h_{\mathfrak{D}}$ and $\Xi_{\mathfrak{D}}$. Since the proof of the theorem only depended on certain properties of the potential $\mathscr{F}$ which have straightforward counterparts for the function F, it is readily extended to this more abstract setting.

Chapter 6

Manifolds with holonomy strictly contained in G_2

In this short chapter we study examples of moduli spaces of compact G_2-manifolds with vanishing first Betti number and infinite fundamental group. They correspond to G_2-manifolds whose restricted holonomy (the identity component of the holonomy group) is a proper subgroup of G_2. The material contained in this section is not elsewhere published, and it is meant to be a complement to the previous chapter in order to exemplify and give some perspective on our results.

Let (M, φ) be a G_2-manifold with $b^1(M) = 0$ and $\pi_1(M)$ infinite. As we discussed in §1.2.3, M has a finite cover $\pi : M' \to M$, where M' is isometric to the product of a flat torus T^k and a compact simply connected Ricci-flat manifold N^{7-k}. In particular, the identity component of the holonomy group of M is isomorphic to the holonomy group of N. Since the only proper subgroups of G_2 appearing in the Berger's list of holonomy groups are $\{1\}$, $SU(2)$ and $SU(3)$, there are only three possible cases. Either M is flat, and is covered by a flat torus T^7; or M has a cover isometric to $T^3 \times X$, where T^3 is a flat 3-torus and X a hyperkähler K3-surface; or M has a cover isometric to $S^1 \times Y^3$, where S^1 is a circle and Y a compact simply-connected Calabi–Yau threefold. We seek to describe the moduli space $\mathscr{M}$ of torsion-free G_2-structures on M, and to push further the computations of Chapter 5 by finding an expression for the terms $\mathscr{E}_{abcd}$ in order to understand the properties of the metric $\mathscr{G}$.

Compared with the case of manifolds with full holonomy G_2, we are in an easier situation since the deformations the torsion-free G_2-structures on M correspond to a combination of variations of the flat metric on T^k and of the variations of the Ricci-flat metric on N – and of the associated geometric structures – which are much better understood. Therefore, we may compute the term $\mathscr{E}_{abcd}$ by lifting everything to $M' = T^k \times N$ via the covering map $\pi : M' \to M$. That is, let

$(\varphi_x)_{x\in(-1,1)}$ be a family of torsion-free G_2-structures on M and $(\varphi'_x)_{x\in(-1,1)}$ be a family of torsion-free G_2-structures on M' such that

$$\varphi'_0 = \pi^*\varphi_0, \quad \text{and} \quad \left.\frac{\partial\varphi'_x}{\partial x}\right|_{x=0} = \pi^* \left.\frac{\partial\varphi_x}{\partial x}\right|_{x=0}.$$

Note that we *do not* assume $\varphi'_x = \pi^*\varphi_x$ for all $x \in (-1,1)$. Then we have the following easy consequence of Corollary 5.4:

Lemma 6.1. *Let $\alpha \in H^k(M)$ be a cohomology class and let $\alpha' = \pi^*\alpha \in H^k(M')$. For $x \in I$, let $\eta(x)$ be the harmonic representative of α for the metric induced by φ_x, and $\eta'(x)$ the harmonic representative of α' for the metric induced by φ'_x. Then we have:*

$$\left.\frac{\partial\eta'}{\partial x}\right|_{x=0} = \pi^* \left.\frac{\partial\eta}{\partial x}\right|_{x=0}.$$

6.1 Flat G_2-manifolds

In this section, we consider the simplest type of compact G_2-manifolds M with $b^1(M) = 0$. Such manifolds are quotients of a flat torus $T^7 = \mathbb{R}^7/F$ by a finite subgroup F of G_2[1] fixing no line in $\mathbb{R}^7$, and the moduli space of torsion-free G_2-structures on M can be identified with $(\Lambda^3_+\mathbb{R}^*_7)^F$ (see Remark 1.7). Hence in those cases understanding the moduli spaces becomes merely an exercise in linear algebra, and in Appendix A we will give an explicit classification of all the possibilities.

6.1.1 Geometry of the space of positive forms. In this part, we describe the geometry of $\Lambda^3_+\mathbb{R}^*_7$, which we can see as an open cone in $\Lambda^3\mathbb{R}^*_7$ or as the homogeneous space $\mathrm{GL}_+(7)/\mathrm{G}_2$. Either way, it is endowed with a homogeneous metric which can be described as follows. If φ is a positive form and $\eta \in T_\varphi\Lambda^3_+\mathbb{R}^*_7 \simeq \Lambda^3\mathbb{R}^*_7$, then the squared norm $|\eta|^2_\varphi$ of η as a tangent vector is just the squared norm of $\eta \in \Lambda^3\mathbb{R}^*_7$ for the inner product induced by g_φ. If we write $\eta = h\cdot\varphi$ where $h \in \mathrm{End}(\mathbb{R}^7)$ is orthogonal to the Lie algebra of the stabiliser of φ, then there are constants $c_1, c_7, c_{27} > 0$ such that

$$|\eta|^2_\varphi = c_1|\pi_1(h)|^2_\varphi + c_7|\pi_7(h)|^2_\varphi + c_{27}|\pi_{27}(h)|^2_\varphi$$

[1] The action of F on T^7 also contains a translation part if we want the quotient to be smooth, but by moving to an appropriate finite cover we can arrange that F be isomorphic to its linearisation as an abstract group, which we always implicitly assume.

where $|\pi_k(h)|^2_\varphi$ denotes the squared norm of the components of h for the inner product induced by g_φ on $\mathrm{End}(\mathbb{R}^7)$. After some computations, we obtain

$$c_1 = 9, \quad c_7 = 2, \quad c_{27} = 2.$$

It is interesting to consider the homogeneous fibration $\varphi \in \Lambda^3_+\mathbb{R}^*_7 \mapsto g_\varphi \in S^2_+\mathbb{R}^*_7$. The space $S^2_+\mathbb{R}^*_7 \simeq \mathrm{GL}_+(7)/\mathrm{SO}(7)$ of inner products on $\mathbb{R}^7$ can be given a symmetric space structure, where for any $g \in S^2_+\mathbb{R}^*_7$ and any g-self-dual endomorphism h, the norm of $h \cdot g \in S^2\mathbb{R}^*_7 \simeq T_g S^2_+\mathbb{R}^*_7$ is given by $|h|^2_g = \mathrm{tr}(h^2)$. This space is a Riemannian product $\mathbb{R} \times (\mathrm{SL}(7)/\mathrm{SO}(7))$, which corresponds to writing $|h|^2_g = (\mathrm{tr}(h))^2/7 + |h_0|^2_g$ where h_0 denotes the traceless part of h. If $g = g_\varphi$ for some $\varphi \in \Lambda^3_+\mathbb{R}^*_7$ we have $\pi_1(h) = \mathrm{tr}(h)g/7$ and $\pi_{27}(h) = h_0$; in particular the map $g : \Lambda^3\mathbb{R}^*_7 \to S^2_+\mathbb{R}^*_7$ is a Riemannian fibration for a symmetric metric on $S^2_+\mathbb{R}^*_7$ which is not the standard one, since $c_1, c_{27} \neq 1$. If $\varphi \in \Lambda^3_+\mathbb{R}^*_7$, the vertical space of the fibration is $\Lambda^3_{7,\varphi}$ and the horizontal space is $\Lambda^3_{1,\varphi} \oplus \Lambda^3_{27,\varphi}$.

There is at least another natural homogeneous metric that we can consider on $\Lambda^3_+\mathbb{R}^*_7$, which has signature $(28, 7)$. Let us fix an element $\mu \in \Lambda^7_+\mathbb{R}^*_7$, and let $f_\mu : \Lambda^3_+\mathbb{R}^*_7 \to \mathbb{R}$ be the function defined by

$$\mu_\varphi = f_\mu(\varphi)\mu. \tag{6.1}$$

This function is positive and homogeneous of degree $\frac{7}{3}$. In fact, if we take any lattice Γ_μ in $\mathbb{R}^7$ such that $\int_{\mathbb{R}^7/\Gamma_\mu} \mu = 1$, we see that we have

$$f_\mu(\varphi) = \int_{\mathbb{R}^7/\Gamma_\mu} \mu_\varphi = \mathrm{Vol}(\mathbb{R}^7/\Gamma_\mu, \varphi).$$

Thus we deduce from Lemma 4.2 that the function $F_\mu = -3\log f_\mu$ has non-degenerate Hessian, and if we denote by D the natural flat connection of $\Lambda^3_+\mathbb{R}^*_7$ we have

$$\begin{aligned} D^2_\varphi F_\mu(\eta, \eta) &= \frac{1}{f_\mu(\varphi)} \int_{\mathbb{R}^7/\Gamma_\mu} (|\pi_1(\eta)|^2_\varphi + |\pi_{27}(\eta)|^2_\varphi - |\pi_7(\eta)|^2_\varphi)\mu_\varphi \\ &= |\pi_1(\eta)|^2_\varphi + |\pi_{27}(\eta)|^2_\varphi - |\pi_7(\eta)|^2_\varphi. \end{aligned}$$

Thus $(\Lambda^3_+\mathbb{R}^*_7, D, D^2F_\mu)$ is a pseudo-Hessian manifold. Moreover, D^2F_μ coincides with the natural homogeneous metric of $\Lambda^3_+\mathbb{R}^*_7$ on the horizontal space of the fibration $\Lambda^3_+\mathbb{R}^*_7 \to S^2_+\mathbb{R}^*_7$.

6.1.2 Positive forms invariant under the action of a finite group. Using the remarks made in the previous part, we prove the following proposition which describes $(\Lambda^3_+\mathbb{R}^*_7)^F$ when $F \subset \mathrm{G}_2$ is a finite subgroup such that $(\mathbb{R}^7)^F = 0$:

Proposition 6.2. *Let F be a finite subgroup of $\mathrm{GL}_+(7)$ that fixes no line in $\mathbb{R}^7$. Then the following properties hold:*

(i) $(\Lambda^3_+\mathbb{R}^*_7)^F$ *is a complete, horizontal and totally geodesic submanifold of* $\Lambda^3_+\mathbb{R}^*_7$ *for the natural homogeneous Riemannian structure.*

(ii) $(\Lambda^3_+\mathbb{R}^*_7)^F$ *has finitely many connected components.*

(iii) Each connected component is isometric to $(S^2_+\mathbb{R}^*_7)^F$ *endowed with the symmetric metric for which* $\Lambda^3_+\mathbb{R}^*_7 \to S^2_+\mathbb{R}^*_7$ *is a Riemannian fibration.*

Proof. To prove (i), assume that $(\Lambda^3_+\mathbb{R}^*_7)^F$ is nonempty and let φ be a positive form that is fixed under the action of G. As F fixes no line in $\mathbb{R}^7$, it follows that $(\Lambda^3_{7,\varphi})^F \simeq (\mathbb{R}^*_7)^F = 0$, and therefore $(\Lambda^3\mathbb{R}^*_7)^F \subset \Lambda^3_{1,\varphi} \oplus \Lambda^3_{27,\varphi}$. Thus $(\Lambda^3_+\mathbb{R}^*_7)^F$ is a horizontal submanifold of $\Lambda^3_+\mathbb{R}^*_7$. Hence any $\eta \in (\Lambda^3\mathbb{R}^*_7)^F$ can be written as $\eta = h \cdot \varphi$ where $h \in \mathrm{End}(\mathbb{R}^7)$ is an F-invariant endomorphism, symmetric for the inner product g_φ. For any $t \in \mathbb{R}$, the positive form $\varphi_t = (e^{th})^*\varphi$ is also F-invariant. Note that φ_t is a horizontal curve in $\Lambda^3_+\mathbb{R}^*_7$, and moreover $g_{\varphi_t} = (e^{th})^* g_\varphi$ is a geodesic in $S^2_+\mathbb{R}^*_7$; thus φ_t is a geodesic in $\Lambda^3_+\mathbb{R}^*_7$ [100]. This proves that $(\Lambda^3_+\mathbb{R}^*_7)^F$ is totally geodesic in $\Lambda^3_+\mathbb{R}^*_7$, and also a complete Riemannian manifold.

For part (iii), the above argument also shows that $T_\varphi(\Lambda^3\mathbb{R}^*_7)^F$ can be identified with the space of F-invariant endomorphisms of $\mathbb{R}^7$ that are symmetric for the metric g_φ, which is also the tangent space of $(S^2_+\mathbb{R}^*_7)^F$ at g_φ. Thus the restriction of $g : \Lambda^3_+\mathbb{R}^*_7 \to S^2_+\mathbb{R}^*_7$ to $(\Lambda^3_+\mathbb{R}^*_7)^F$ induces a local isometry on $(S^2_+\mathbb{R}^*_7)^F$ (for the symmetric metric on $S^2_+\mathbb{R}^*_7$ such that g is a Riemannian fibration). Now $(S^2_+\mathbb{R}^*_7)^F$ is a symmetric space totally geodesically embedded into $S^2_+\mathbb{R}^*_7$, and the exponential map at every point is a global diffeomorphism. We proved above that the geodesics of $(S^2_+\mathbb{R}^*_7)^F$ lift to geodesics in $(\Lambda^3_+\mathbb{R}^*_7)^F$, and thus we deduce that g is surjective, and moreover the restriction of g to each connected component of $(\Lambda^3\mathbb{R}^*_7)^F$ is a global isometry onto $(S^2_+\mathbb{R}^*_7)^F$.

It remains to prove that $(\Lambda^3_+\mathbb{R}^*_7)^F$ has finitely many connected components. By contradiction, assume that there are infinitely many. Then if $g \in (S^2_+\mathbb{R}^*_7)^F$, there exists a sequence $\{\varphi_i\}_{i\in\mathbb{N}}$ of elements of $(\Lambda^3_+\mathbb{R}^*_7)^F$ such that $\varphi_i \neq \varphi_j$ for $i \neq j$ but $g_{\varphi_i} = g$ for all $i \in N$. Any φ_i can be written $\alpha_i^*\varphi_0$ where α_i preserves the orientation and the inner product g. As $\mathrm{SO}(n)$ is compact, up to a subsequence we can assume that α_i converges to an automorphism α_∞ of $\mathbb{R}^7$ preserving g and the orientation of $\mathbb{R}^7$, and thus $\varphi_i \to \varphi_\infty = \alpha_\infty^*\varphi_0$ as $i \to \infty$ in $\Lambda^3\mathbb{R}^*_7$. As φ_i is F-invariant for all i the limit $\varphi_\infty \in (\Lambda^3_+\mathbb{R}^*_7)^F$, and moreover by continuity $g_{\varphi_\infty} = g$. Hence for i large enough, φ belongs to the connected component of φ_∞

in $(\Lambda^3_+\mathbb{R}^*_7)^F$, and as $g_{\varphi_i} = g_{\varphi_\infty}$ it follows that $\varphi_i = \varphi_\infty$ for i large enough, which gives a contradiction. □

Remark 6.3. It is not difficult to classify all the possible geometries for $(S^2_+\mathbb{R}^*_7)^F$ where F is a finite subgroup of G_2 such that $(\mathbb{R}^7)^F = 0$; this is essentially an exercise about the representation theory of finite subgroups of G_2. Out of curiosity, we carried it out in Appendix A; in most cases we just obtain a flat space, except for certain representations of the dihedral groups D_4, D_6 and D_8 (see Proposition A.1 and Proposition A.3).

6.1.3 Moduli spaces of flat G_2-manifolds. Let us now come back to the case where $M = T^7/F$ where (the linearisation of) F is a finite subgroup of G_2. Then the moduli space $\mathscr{M}$ of torsion-free G_2-structures on M can be identified with $(\Lambda^3_+\mathbb{R}^*_7)^F$, and under this identification $\mathscr{G}$ can be identified with the homogeneous metric since

$$\frac{1}{\mathrm{Vol}(\varphi)}\int_M |\eta|^2_\varphi \mu_\varphi = |\eta|^2_\varphi$$

for any $\varphi \in (\Lambda^3_+\mathbb{R}^*_7)^F$ and $\eta \in (\Lambda^3\mathbb{R}^*_7)^F$, where we see φ as a torsion-free G_2-structure on M and η has a harmonic 3-form. Hence each connected component of $\mathscr{M}$ is a symmetric space isometric to $(S^2_+\mathbb{R}^*_7)^F$; in particular, it has nonpositive sectional curvature.

This easy case is a good sanity check for the results of the previous chapter. Indeed, from Theorem 5.8 it is easy to see that the terms $\mathscr{E}_{abcd}$ vanish, since the space of harmonic forms is fixed along families of torsion-free G_2-structures induced by positive forms $\varphi \in (\Lambda^3_+\mathbb{R}^*_7)^F$: the harmonic 3-forms are induced by the constant alternating forms $\eta \in \Lambda^3\mathbb{R}^*_7$. Hence the fact that $\mathscr{M}$ is locally symmetric and has nonpositive sectional curvature can also be seen as a consequence of Theorem 5.22. This allows us to check the consistency of the results in that case.

Below we give a couple of examples of flat compact G_2-manifolds M with $b^1(M) = 0$. We also describe the geometry of the moduli spaces (see Appendix A for proofs). To describe these examples, it will be convenient to identify $\mathbb{R}^7$ with $\mathbb{C}^3 \oplus \mathbb{R}$ and use coordinates (z_1, z_2, z_3, θ) where $z_k = x_k + iy_k$, and to consider the positive form

$$\varphi = \mathrm{Re}(dz_1 \wedge dz_2 \wedge dz_3) + \frac{i}{2}\sum_{k=1}^{3} dz_k \wedge d\overline{z}_k \wedge d\theta.$$

Example 6.4. Let $T = \mathbb{R}^7/\mathbb{Z}^7$ and consider the action of $\mathbb{Z}^3_2$ on T generated by the isometries α, β, γ defined by

$$\alpha(z_1, z_2, z_3, \theta) = (-z_1, -z_2, z_3, \theta + 1/2),$$
$$\beta(z_1, z_2, z_3, \theta) = (-z_1, z_2 + 1/2, -z_3, \theta),$$
$$\sigma(z_1, z_2, z_3, \theta) = (\overline{z}_1 + 1/2, \overline{z}_2, \overline{z}_3 + 1/2, -\theta).$$

We can check that any elements in the group acts as a translation on one of the circle factors, and thus the $\mathbb{Z}_2^3$ acts freely, properly discontinuously on T. Moreover, this action preserves φ, and hence the quotient is a compact G_2-manifold M with $b^1(M) = 0$, $b^2(M) = 0$ and $b^3(M) = 7$. It turns out that each connected component of the moduli space is isomorphic to a flat $\mathbb{R}^7$ (see Appendix A for a proof).

Example 6.5. Again, let $T = \mathbb{R}^7/\mathbb{Z}^7$, and consider the isometries α, σ defined as

$$\alpha(z_1, z_2, z_3, \theta) = (iz_1, iz_2, -z_3, \theta + 1/4),$$
$$\sigma(z_1, z_2, z_3, \theta) = (\overline{z}_1, \overline{z}_2, \overline{z}_3 + (1+i)/2, -\theta).$$

These isometries preserve φ and satisfy the relations $\alpha^4 = \sigma^2 = 1$ and $\sigma\alpha\sigma = \alpha^{-1}$, and hence they generate a finite group isomorphic to the dihedral group D_8. The action of D_8 on T is free and properly discontinuous: clearly the subgroup isomorphic to C_4 generated by α acts without fixed points since all the elements act by translation on the coordinate θ, and the elements of $D_4 \backslash C_4$ act by translation either on the coordinate x_3 or on the coordinate y_3, so they also have no fixed points. The resulting compact G_2-manifold $M = T/D_8$ has $b^1(M) = 0$, $b^2(M) = 0$ and $b^3(M) = 6$. Each connected component of the moduli space $\mathscr{M}$ is isometric to $\mathbb{R}^4 \times \mathbb{H}(-1/8)$, where $\mathbb{H}(-1/8)$ is a hyperbolic space with sectional curvature $-1/8$ (again, see Appendix A for a proof).

6.2 Manifolds with restricted holonomy $\mathrm{SU}(2)$

When the restricted holonomy group of M is $SU(2)$, that is, if M is a finite quotient $(T^3 \times K3)/F$, it turns out that the Yukawa coupling is also a parallel tensor on the moduli space $\mathscr{M}$, and hence the moduli space is locally symmetric and the period map defined in the previous chapter is a totally geodesic immersion.

6.2.1 From K3 surfaces to compact G_2-manifolds. By definition, a *K3 surface* X is a smooth, compact, connected and simply connected complex surface with trivial canonical bundle. By a theorem of Siu [107], every K3 surface is Kähler, and therefore Yau's solution of the Calabi conjecture implies that K3 surfaces admit a unique Ricci-flat Kähler metric in each Kähler class [123]. Such a metric has holonomy SU(2), and since SU(2) $\simeq$ Sp(1) the Ricci-flat Kähler metrics on X are hyperkähler. Geometrically, this correspondence between Ricci-flat Kähler metrics and hyperkähler metrics can be realised as follows. Let g be a Ricci-flat Kähler metric on X and $\omega \in \Omega^2(X)$ be associated Kähler form. As the canonical bundle of X is trivial, $H^{2,0}(X) \simeq \mathbb{C}$ is spanned by a holomorphic volume form $\Omega \in \Omega^{2,0}(X)$, and after multiplication by an element of $\mathbb{C}^*$ we can assume that

$$\omega^2 = \frac{1}{2}\Omega \wedge \overline{\Omega}. \tag{6.2}$$

Moreover, considering the type of forms we have the following relations

$$\omega \wedge \Omega = 0 = \Omega^2. \tag{6.3}$$

Thus if we denote $\omega = \omega_1$, $\Omega = \omega_2 + i\omega_3$ and μ_g the volume form associated with g, relations (6.2) and (6.3) imply that the triple of forms $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ satisfies

$$\frac{1}{2}\omega_i \wedge \omega_j = \delta_{ij}\mu_g, \quad \forall i, j = 1, 2, 3. \tag{6.4}$$

Moreover, the real 2-forms ω_i are self-dual with respect to the metric g. Hence $\underline{\omega}$ is a *hyperkähler triple*, that is, a triple self-dual 2-forms for the metric g satisfying (6.4) and parallel for the Levi–Civita connection of g. Conversely, given a hyperkähler triple $\underline{\omega}$ with associated metric g on X, each symplectic form in the triple can be written $\omega_i = g(J_i\cdot, \cdot)$ where J_i is an integrable complex structure on X, and after possibly permuting the indices we may assume that the complex structures (J_1, J_2, J_3) satisfy quaternionic permutation relations. Then if we denote $\omega = \omega_1$, $J = J_1$ and $\Omega = \omega_2 + i\omega_3$, ω is a Kähler metric on the complex K3 surface (X, J) and Ω is a holomorphic volume form. By the work of various authors, it is known that all K3 surfaces have the same underlying smooth manifold X and the moduli space of hyperkähler metrics on X can be described by means of the *period map* [20, 102, 107, 111].

If (M, φ) is a compact torsion-free G$_2$-manifold whose universal cover is diffeomorphic to $\mathbb{R}^3 \times X$, we noted before that there exists a hyperkäler triple $\underline{\omega}$ and a discrete group action $\Gamma \simeq \pi_1(M) \to I(\mathbb{R}^3) \times I(X, g_{\underline{\omega}})$, where $I(\mathbb{R}^3)$ is the group of affine isometries of $\mathbb{R}^3$ for the standard Euclidean inner product and $I(X, g_{\underline{\omega}})$ is the group of isometries of X with respect to the metric induced by $\underline{\omega}$, such that Γ leaves invariant the G$_2$-form

$$\varphi_{\underline{\omega}} = d\theta_1 \wedge d\theta_2 \wedge d\theta_3 - d\theta_1 \wedge \omega_1 - d\theta_2 \wedge \omega_2 - d\theta_3 \wedge \omega_3 \tag{6.5}$$

and (M, φ) is isometric to $(\mathbb{R}^3 \times X)/\Gamma$ endowed with the G$_2$-structure induced by $\varphi_{\underline{\omega}}$. The associated metric reads

$$g_{\varphi_{\underline{\omega}}} = d\theta_1^2 + d\theta_2^2 + d\theta_3^2 + g_{\underline{\omega}}.$$

As Γ acts as a product, the 3-forms $d\theta_1 \wedge d\theta_2 \wedge d\theta_3$ and $\sum d\theta_i \wedge \omega_i$ are both invariant under Γ, and hence Γ preserves the orientation on $\mathbb{R}^3$ and on X. Thus Γ can be identified with a discrete subgroup of $I_+(\mathbb{R}^3) \times I_+(X, g_{\underline{\omega}})$. Moreover, there is a normal subgroup Γ_0 of finite index in Γ acting trivially on X and as a lattice of translations on $\mathbb{R}^7$. If we denote by F the finite quotient group Γ/Γ_0 and $T^3 = \mathbb{R}^3/\Gamma_0$, then F acts by isometries preserving $\varphi_{\underline{\omega}}$ on $T^3 \times X$ and $M \simeq (T^3 \times X)/F$.

Before describing the moduli spaces in the next part, let us give a few examples. One idea to construct such compact G_2-manifolds is to start with a finite group F acting freely on a flat torus T^3, which preserves the orientation of T^3 and such that there are no nontrivial fixed cohomology class in $H^1(T^3)$. An example of such group would be $F = \mathbb{Z}_2^2$ generated by the isometries α, β defined as follows:

$$\alpha(\theta_1, \theta_2, \theta_3) = (-\theta_1, -\theta_2, \theta_3 + 1/2),$$
$$\beta(\theta_1, \theta_2, \theta_3) = (-\theta_1 + 1/2, \theta_2 + 1/2, -\theta_3).$$

Notice that if we let $\gamma = \alpha\beta$ then

$$\gamma(\theta_1, \theta_2, \theta_3) = (\theta_1 + 1/2, -\theta_2 + 1/2, -\theta_3 + 1/2)$$

so that $\mathbb{Z}_2^2$ indeed acts without fixed points on T^3.

Given such a group action, we may look for a hyperkähler triple $\underline{\omega}$ on X and an isometric action ρ of F onto $(X, g_{\underline{\omega}})$, which preserves $\varphi_{\underline{\omega}}$ for the product action of F onto $T^3 \times K3$. Using the Torelli theorem, it is enough to find a right action ρ^* of F onto the lattice $H^2(X;\mathbb{Z})$, preserving the intersection form, and a triple of cohomology classes $[\omega_1], [\omega_2], [\omega_3]$ representing the periods of some hyperkähler structure and such that $\sum [d\theta_i] \otimes [\omega_i]$ is preserved by the induced (right) action on $H^1(T^3) \otimes H^2(X)$.

Let us describe a few such actions, for $F = \mathbb{Z}_2^2$ as above. The K3 lattice $H^2(X;\mathbb{Z})$ is isomorphic to $E_8(-1)^2 \oplus H^3$, and under this identification we shall write its elements $x = (w_1, w_2, u_1, v_1, u_2, v_2, u_3, v_3)$, where $w_i \in E_8(-1)$ and $(u_i, v_i) \in U$. In particular, the intersection form reads

$$x \bullet x = w_1^2 + w_2^2 + 2(u_1 v_1 + u_2 v_2 + u_3 v_3)$$

where w_i^2 is the square of w_i for the $E_8(-1)$-quadratic form.

Example 6.6. With the previous notations, let us consider the following action of $\mathbb{Z}_2^2$ on $H^3(X;\mathbb{Z})$:

$$\rho^*(\alpha)(w_1, w_2, u_1, v_1, u_2, v_2, u_3, v_3) = (-w_1, -w_2, -u_1, -v_1, -u_2, -v_2, u_3, v_3),$$
$$\rho^*(\beta)(w_1, w_2, u_1, v_1, u_2, v_2, u_3, v_3) = (-w_1, -w_2, -u_1, -v_1, u_2, v_2, -u_3, -v_3),$$
$$\rho^*(\gamma)(w_1, w_2, u_1, v_1, u_2, v_2, u_3, v_3) = (w_1, w_2, u_1, v_1, -u_2, -v_2, -u_3, -v_3).$$

This action clearly preserves the intersection form. The elements $[d\theta_i] \otimes x_i \in H^1(T^3) \otimes H^2(X)$ invariant under the action of $\mathbb{Z}_2^3$ are exactly of the form

$$\begin{aligned} x_1 &= (w_1, w_2, u_1, v_1, 0, 0, 0, 0), \\ x_2 &= (0, 0, 0, 0, u_2, v_2, 0, 0), \\ x_3 &= (0, 0, 0, 0, 0, 0, u_3, v_3). \end{aligned}$$

In particular x_1, x_2, x_3 are always orthogonal. Moreover $\underline{x} = (x_1, x_2, x_3)$ span a positive 3-plane in $H^2(X)$ if and only if

$$2u_1v_2 + w_1^2 + w_2^2 > 0, \quad u_2v_2 > 0, \quad u_3v_3 > 0.$$

Let us prove that there are hyperkähler structures on X whose periods satisfy these conditions. It boils down to finding an orthogonal set of classes $x_1, x_2, x_3 \in H^2(X)$ satisfying the above conditions and such that the vector space they span in $H^2(X)$ is not orthogonal to any root of the lattice $H^2(X;\mathbb{Z})$. For instance, this condition holds if u_i, v_i as well as the coefficients of w_1 and w_2 in an integral basis of $E_8(-1)$ are all linearly independent over $\mathbb{Q}$, since for any root δ the orthogonality conditions $\delta \bullet x_i = 0$ can be expressed as a set of $\mathbb{Q}$-linear equations.

Hence the quotient $(T^3 \times X)/Z_2^3$ is a compact G_2-manifold with $b^1(M) = b^2(M) = 0$, and we can easily compute that $b^3(M) = 1 + 18 + 2 + 2 = 23$.

Example 6.7. Replace ρ^* with

$$\begin{aligned} \rho^*(\alpha)(w_1, w_2, u_1, v_1, u_2, v_2, u_3, v_3) &= (w_2, w_1, -u_1, -v_1, -u_2, -v_2, u_3, v_3), \\ \rho^*(\beta)(w_1, w_2, u_1, v_1, u_2, v_2, u_3, v_3) &= (w_2, w_1, -u_1, -v_1, u_2, v_2, -u_3, -v_3), \\ \rho^*(\gamma)(w_1, w_2, u_1, v_1, u_2, v_2, u_3, v_3) &= (w_1, w_2, u_1, v_1, -u_2, -v_2, -u_3, -v_3). \end{aligned}$$

This time the admissible classes $[d\theta_i] \otimes x_i \in H^1(T^3) \otimes H^2(X)$ have

$$\begin{aligned} x_1 &= (w, -w, u_1, v_1, 0, 0, 0, 0), \\ x_2 &= (0, 0, 0, 0, 0, u_2, v_2, 0, 0), \\ x_3 &= (0, 0, 0, 0, 0, 0, u_3, v_3). \end{aligned}$$

and

$$2w^2 + u_1v_1 > 0, \quad u_2v_2 > 0 \text{ and } u_3v_3 > 0.$$

Again, if we assume that the coefficients of w, u_i, v_i in an integral basis of $H^2(X;\mathbb{Z})$ are rationally independent, then (after normalisation if necessary) x_1, x_2, x_3 will be represent the periods of a hyperkähler structure. Indeed if δ is a root orthogonal to such x_1, x_2, x_3, then we may deduce first that $\delta \in E_8(-1) \oplus E_8(-1)$, and second that $\delta = (w', w', 0, \ldots, 0)$ for some $w' \in E_8(-1)$. But since the lattice $E_8(-1)$ is even, the norm of (w', w') is divisible by 4, and we obtain a contradiction with the fact that $\delta^2 = -2$.

This time, the quotient would be a compact G_2-manifold with $b^1(M) = 0$, $b^2(M) = 8$ and $b^3(M) = 15$.

Example 6.8. For a more explicit description, we can for instance use the example of $\mathbb{Z}_2^2$ action on a K3 surface X given in [68, Ex. 7.2]. Here the K3 surface is constructed as a double cover of $\mathbb{CP}^2$ branched over the sextic curve $C = \{[z_1 : z_2 : z_3] | z_1^6 + z_2^6 + z_3^6 = 0\}$. The $\mathbb{Z}_2^2$ action is generated by the involution swapping the sheets of the double cover and the lift of the anti-holomorphic involution $[z_1 : z_2 : z_3] \mapsto [\overline{z}_1, \overline{z}_2, \overline{z}_3]$. The resulting manifold $M = (T^3 \times X)/\mathbb{Z}_2^2$ has $b^1(M) = b^2(M) = 0$ and $b^3(M) = 23$.

6.2.2 Structure theorem for the moduli spaces. Let us describe the deformations of (M, φ). First, we need to start with the deformations of $(T^3 \times X, \varphi_{\underline{\omega}})$. The space of harmonic 3-forms on $T^3 \times X$ decomposes as:

$$\mathscr{H}^3(T^3 \times X, \varphi_{\underline{\omega}}) = \Lambda^3\mathbb{R}_3^* \oplus (\mathbb{R}_3^* \otimes \mathscr{H}_{\underline{\omega}}^+(X)) \oplus (\mathbb{R}_3^* \otimes \mathscr{H}_{\underline{\omega}}^-(X))$$

where $\mathscr{H}_{\underline{\omega}}^{\pm}(X)$ are the spaces of harmonic (anti-)self-dual 2-forms on $(X, \underline{\omega})$ and $\mathbb{R}_3^*$ is the dual space of $\mathbb{R}^3$. Using this decomposition, we can describe the deformations of $\varphi_{\underline{\omega}}$ by analysing separately each component.

- The first component $\Lambda^3\mathbb{R}_3^*$ is spanned by $d\theta_1 \wedge d\theta_2 \wedge d\theta_3$. Deforming of $\varphi_{\underline{\omega}}$ along this direction corresponds to rescaling the inner product on T^3 by some factor $\lambda > 0$, together with a rescaling of the hyperkähler triple $\underline{\omega}$ by a factor $\lambda^{-\frac{1}{2}}$.

- $\mathbb{R}_3^* \otimes \mathscr{H}_{\underline{\omega}}^+(X)$ has dimension 9 and contains $\mathscr{H}_1^3(T^3 \times X, \varphi_{\underline{\omega}})$ as a 3-dimensional subspace spanned by $\frac{\partial}{\partial \theta_k} \lrcorner \Theta(\varphi_{\underline{\omega}}) = d\theta_j \wedge \omega_i - d\theta_i \wedge \omega_j$ for cyclic permutations (ijk) of $\{1, 2, 3\}$, corresponding to the isometric deformations of the G_2-structure $\varphi_{\underline{\omega}}$. Its orthogonal complement has dimension 6, and decomposes as the direct sum of the 5-dimensional space $\{\sum a_{ij} d\theta_i \wedge \omega_j, a_{ij} = a_{ji}, \sum a_{ii} = 0\}$ corresponding to the infinitesimal deformations of the inner product on T^3 with fixed volume element, and a 1-dimensional space spanned by $d\theta_1 \wedge \omega_1 + d\theta_2 \wedge \omega_2 + d\theta_3 \wedge \omega_3$ corresponding to an infinitesimal rescaling of the hyperkähler triple.

- The third component $\mathbb{R}_3^* \otimes \mathscr{H}^-_{\underline{\omega}}(X)$ corresponds to the deformations of the hyperkähler metric $g_{\underline{\omega}}$ on X with fixed volume, where the inner product on $\mathbb{R}^3$ is fixed.

Remark that F preserves this decomposition, since F can be identified with a finite subgroup of $I_+(T^3) \times I_+(X, g_{\underline{\omega}})$. Hence the quotient map $T^3 \times X \to M$ induces an identification of $\mathscr{H}^3(M, \varphi) \simeq \mathscr{H}^3(T^3 \times X, \varphi_{\underline{\omega}})^F$, and we obtain a decomposition:

$$\mathscr{H}^3(M, \varphi) = \Lambda^3\mathbb{R}_3^* \oplus \left(\mathbb{R}_3^* \otimes \mathscr{H}^+_{\underline{\omega}_0}(X)\right)^F \oplus \left(\mathbb{R}_3^* \otimes \mathscr{H}^-_{\underline{\omega}_0}(X)\right)^F. \tag{6.6}$$

Note that as $\varphi_{\underline{\omega}}$ is fixed by Γ, $d\theta_1 \wedge d\theta_2 \wedge d\theta_3$ must also be fixed by Γ. This decomposition induces a splitting $T\mathscr{M} = T^0\mathscr{M} \oplus T^+\mathscr{M} \oplus T^-\mathscr{M}$ of the tangent bundle of $\mathscr{M}$. Using this splitting and Theorem 5.8, we can prove:

Proposition 6.9. *Let (M, φ) be a compact G_2-manifold with $b^1(M) = 0$ whose universal cover is $\mathbb{R}^3 \times K3$. Then the Yukawa coupling on $\mathscr{M}$ is parallel for the Levi-Civita connection of $\mathscr{G}$, and in particular $(\mathscr{M}, \mathscr{G})$ is locally symmetric.*

Proof. Let us choose affine coordinates $(x^0, \ldots, x^n)$ near $\varphi\mathscr{D}$ in $\mathscr{M}$ and prove that the extra term $\mathscr{E}_{abcd} + \mathscr{E}_{cabd} + \mathscr{E}_{cbad}$ in Theorem 5.8 vanishes. Let us write $n = n_+ + n_-$ where $n_\pm$ is the dimension of $(\mathbb{R}_3^* \otimes \mathscr{H}^\pm_{\underline{\omega}_0}(X))^F$. Up to a linear change of coordinates, we can assume that we chose coordinates adapted to the decomposition (6.6), in the sense that the harmonic representative of $\frac{\partial}{\partial x^0} \in H^3(M)$ for the metric g_φ lies in $\Lambda^3\mathbb{R}_3^*$, the harmonic representatives of $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{n_+}}$ lie in $(\mathbb{R}_3^* \otimes \mathscr{H}^+_{\underline{\omega}_0}(X))^F$, and the harmonic representatives of $\frac{\partial}{\partial x^{n_+ + 1}}, \ldots, \frac{\partial}{\partial x^n}$ lie in $(\mathbb{R}_3^* \otimes \mathscr{H}^-_{\underline{\omega}_0}(X))^F$. Note that this can only be imposed at the point $\varphi\mathscr{D} \in \mathscr{M}$, not locally near this point. Throughout the proof our computations will be local (in M), and therefore we can lift everything to $T^3 \times X$, where the variations of the space of harmonic forms are easier to understand (using the result of Lemma 6.1).

First we prove that if one of the indices a, b, c or d is between 0 and n_+ then $\mathscr{E}_{abcd}(\varphi\mathscr{D}) = 0$, and similarly for $\mathscr{E}_{cabd}$ and $\mathscr{E}_{cbad}$. Since $\mathscr{F}_{abcd}$ is fully symmetric in its indices, we may assume that $0 \leq d \leq n_+$, and seek to prove that $h_d \cdot \eta$ is harmonic for any $\eta \in \mathscr{H}^3(M, \varphi)$. As a consequence of our discussion of the deformations of $\varphi_{\underline{\omega}}$ on $T^3 \times X$, there is a deformation $\{\varphi_{\underline{\omega}_t}\}_{t \in (-\epsilon, \epsilon)}$ of $\varphi_{\underline{\omega}}$ on $T^3 \times X$ which consists in a variation of the inner product on T^3 combined with a rotation and a dilation of the hyperkähler triple on X, and such that $\left.\frac{\partial \varphi_{\underline{\omega}_t}}{\partial t}\right|_{t=0}$ is the lift of η_d. In particular the space of harmonic forms on $T^3 \times X$ with respect to $g_{\varphi_{\underline{\omega}_t}}$ is fixed along this deformation of $\varphi_{\underline{\omega}}$. Hence Lemma 5.7 implies that the lift of $h_d \cdot \eta$ to $T^3 \times X$ is harmonic whenever η is a harmonic form on M, and thus $h_d \cdot \eta$ is harmonic on M. Hence $\mathscr{E}_{abcd}(\varphi\mathscr{D}) = \mathscr{E}_{cabd}(\varphi\mathscr{D}) = \mathscr{E}_{cbad}(\varphi\mathscr{D}) = 0$.

Now let us assume that $n_+ + 1 \leq a, b, c, d \leq n$. In this case, it is no longer true that $h_d \cdot \eta_c$ is harmonic, but we want to prove that the contribution to $\mathscr{E}_{abcd}$ of its exact part cancels with the contribution of the co-exact part. This time, our discussion of the deformations of torsion-free G_2-structures on $T^3 \times X$ implies that there is a deformation $\{\varphi_{\underline{\omega}_t}\}_{t\in(-\epsilon,\epsilon)}$ of $\varphi_{\underline{\omega}}$ such that $\frac{\partial \varphi_{\underline{\omega}_t}}{\partial t}\Big|_{t=0} = \pi^*\eta_d$ is the lift of η_d and $\varphi_{\underline{\omega}_t}$ can be written

$$\varphi_{\underline{\omega}_t} = d\theta_1 \wedge d\theta_2 \wedge d\theta_3 - \sum_{j=1}^{3} d\theta_j \wedge \omega_{j,t}$$

where $\underline{\omega}_t = (\omega_{1,t}, \omega_{2,t}, \omega_{3,t})$ is a family of hyperkähler triples on X. Now let $\eta \in \mathscr{H}^3(M,\varphi)$, representing a vector in $T^-_\varphi\mathscr{M}$. Its lift $\widetilde{\eta} = \pi^*\eta$ on $T^3 \times X$ can be written

$$\widetilde{\eta} = d\theta_1 \wedge \alpha_1 + d\theta_2 \wedge \alpha_2 + d\theta_3 \wedge \alpha_3$$

where $\alpha_1, \alpha_2, \alpha_3$ are anti-self-dual harmonic 2-forms on $X, g_{\underline{\omega}}$. In particular the dual 4-form of $\widetilde{\eta}$, which we denote by $\widetilde{\nu} = *_{\varphi_{\underline{\omega}}}\widetilde{\eta} = \pi^*(*_\varphi\eta)$ is

$$\widetilde{\nu} = -d\theta_2 \wedge d\theta_3 \wedge \alpha_1 - d\theta_3 \wedge d\theta_1 \wedge \alpha_2 - d\theta_1 \wedge d\theta_2 \wedge \alpha_3.$$

If we now denote by $\widetilde{\eta}_t$ the harmonic representative of $[\widetilde{\eta}] = \pi^*[\eta] \in H^3(T^3 \times X)$ for the metric $g_{\varphi_{\underline{\omega}_t}}$ and $\widetilde{\nu}_t$ the harmonic representative of $[\widetilde{\nu}] = \pi^*[*_\varphi\eta] \in H^4(T^3 \times X)$, we see that

$$\begin{aligned}
\widetilde{\eta}_t &= d\theta_1 \wedge \alpha_{1,t} + d\theta_2 \wedge \alpha_{2,t} + d\theta_3 \wedge \alpha_{3,t}, \\
\widetilde{\nu}_t &= -d\theta_2 \wedge d\theta_3 \wedge \alpha_{1,t} - d\theta_3 \wedge d\theta_1 \wedge \alpha_{2,t} - d\theta_1 \wedge d\theta_2 \wedge \alpha_{3,t}
\end{aligned}$$

where $\alpha_{j,t}$ is the harmonic representative of $[\alpha_j] \in H^2(X)$ for the hyperkähler metric associated with $\underline{\omega}_t$. In particular, the lift of the exact part of $2(h_d \cdot \eta)$ to $T^3 \times X$ is

$$\frac{\partial \widetilde{\eta}_t}{\partial t}\bigg|_{t=0} = d\theta_1 \wedge \frac{\partial \alpha_{1,t}}{\partial t}\bigg|_{t=0} + d\theta_2 \wedge \frac{\alpha_{2,t}}{\partial t}\bigg|_{t=0} + d\theta_3 \wedge \frac{\alpha_{3,t}}{\partial t}\bigg|_{t=0}$$

and the lift of its co-exact part is

$$\frac{\partial \widetilde{\nu}_t}{\partial t}\bigg|_{t=0} = -d\theta_2 \wedge d\theta_3 \wedge \frac{\partial \alpha_{1,t}}{\partial t}\bigg|_{t=0} - d\theta_3 \wedge d\theta_1 \wedge \frac{\alpha_{2,t}}{\partial t}\bigg|_{t=0} - d\theta_1 \wedge d\theta_2 \wedge \frac{\alpha_{3,t}}{\partial t}\bigg|_{t=0}.$$

If we now let $\eta = \eta_c$ and describe in a similar way the exact and co-exact parts of $h_a \cdot \eta_b$, we see that the inner product of the exact parts of $h_d \cdot \eta_c$ and $h_a \cdot \eta_b$ is equal to the inner product of their co-exact parts, and thus $\mathscr{E}_{abcd}(\varphi\mathscr{D}) = 0$ (see Remark 5.9). Similarly $\mathscr{E}_{cabd}(\varphi\mathscr{D}) = \mathscr{E}_{cbad}(\varphi\mathscr{D}) = 0$, which completes the proof of the proposition. ☐

Similarly to the case of T^7/F, one may prove in a more direct way that the moduli space of torsion-free G_2-structures on $(T^3 \times X)/F$ is a locally symmetric space with nonpositive sectional curvature, using the period map for hyperkähler surfaces. For lack of space we will not justify everything in detail but we want to outline the idea.

The basic observation is that the moduli space of torsion-free G_2-structures on $T^3 \times X$ can be seen as an open subspace of a homogeneous space. To introduce it, let use denote by $\mathrm{Hom}_+(\mathbb{R}^3, H^2(X))$ the space of linear maps $\mathbb{R}^3 \to H^2(X)$ whose image is a positive 3-plane in $H^2(X) \simeq \mathbb{R}^{3,19}$, and $E_+ = \Lambda^3_+\mathbb{R}^*_3 \times \mathrm{Hom}_+(\mathbb{R}^3, H^2(X))$. The idea behind this is that if we look at (6.5), $\varphi_{\underline{\omega}}$ should be identified with $(d\theta_1 \wedge d\theta_2 \wedge d\theta_3, \sum d\theta_i \otimes [\omega_i]) \in E_+$. E_+ is a homogeneous space under the action of $\mathrm{GL}_+(3) \times \mathrm{SO}_0(3, 19)$, and there is a homogeneous fibration to the symmetric space $(\mathrm{GL}_+(3)/SO(3)) \times (SO_0(3, 19)/SO(3) \times SO(19))$. Geometrically, the interpretation of this fibration is clear: the first factor parametrises the flat metric on T^3, and the second factor is the Grassmannian of positive 3-planes in $\mathbb{R}^{3,19} \simeq H^2(X)$, corresponding to the period of the hyperkähler metric $g_{\underline{\omega}}$.

It is not very difficult to prove that, under the identification of the moduli space of torsion-free G_2-structures on $T^3 \times X$ with an open subset of E_+, the metric G_2 is isometric to a homogeneous metric on E_+; and that there is a symmetric metric on $S^2_+\mathbb{R}^*_3 \times \mathrm{Gr}_+(3, \mathbb{R}^{3,19})$ which makes the map $E_+ \to S^2_+\mathbb{R}^*_3 \times G_+(3, \mathbb{R}^{3,19})$ a homogeneous fibration. Now if F is a finite group of isometries of $T^3 \times X$ such that the quotient $(T^3 \times X)^F = M$ has $b^1(M) = 0$ (that is, if F acts without fixing a non-zero vector in $\mathbb{R}^3$), the moduli space $\mathscr{M}$ of torsion-free G_2-structures on M will be identified with an open subset of E^F_+. The point is that E^F_+ is a horizontal submanifold of E_+, and hence each connected component of the moduli space is isometric to an open subset of $(S^2_+\mathbb{R}^*_3 \times \mathrm{Gr}_+(3, \mathbb{R}^{3,19}))^F$, which is totally geodesic inside $S^2_+\mathbb{R}^*_3 \times \mathrm{Gr}_+(3, \mathbb{R}^{3,19})$. We therefore recover the fact that $\mathscr{M}$ is in this case a locally symmetric space of nonpositive sectional curvature.

6.3 Manifolds with restricted holonomy $\mathrm{SU}(3)$

The case of $(S^1 \times CY3)/F$ was less conclusive and we could not find formulas with a clear geometrical interpretation, so we will just make a few very basic observations. Not that in this case we must have $F = \mathbb{Z}_2$ since (up to a translation) the only nontrivial isometry of the S^1 factor acts by the antipodal map, that is $\theta \mapsto -\theta$.

6.3.1 Calabi–Yau threefolds. Let Y^6 be a compact simply connected manifold, with a torsion-free SU(3)-structure (J, ω, Ω). Recall that the almost complex structure J is integrable and the metric $g_\omega = \omega(\cdot, J\cdot)$ is Kähler, Ricci-flat, and Ω is a holomorphic volume form. Moreover, there is an induced torsion-free G_2-structure

$$\varphi = d\theta \wedge \Omega + \mathrm{Re}(\Omega)$$

on $S^1 \times Y$. The space of harmonic 3-forms on $S^1 \times Y$ decomposes as:

$$\mathscr{H}^3(S^1 \times Y, \varphi) = d\theta \wedge \mathscr{H}^2(Y, g_\omega) \oplus \mathscr{H}^3(Y, g_\omega)$$

where we have further splittings:

$$\begin{aligned} \mathscr{H}^2(Y, g_\omega) &= \mathbb{R}\omega \oplus \mathscr{H}^{(1,1)_0}_{\mathbb{R}}(Y, g_\omega), \\ \mathscr{H}^3(Y, g_\omega) &= \mathbb{R}\,\mathrm{Re}\,\Omega \oplus \mathbb{R}\,\mathrm{Im}\,\Omega \oplus \mathscr{H}^{(2,1)}_{\mathbb{R}}(Y, g_\omega) \end{aligned}$$

where $\mathscr{H}^{(1,1)_0}_{\mathbb{R}}(Y, g_\omega)$ is the space of harmonic real primitive $(1,1)$-forms and similarly $\mathscr{H}^{(2,1)}_{\mathbb{R}}(Y, g_\omega)$ is the space of real harmonic forms of type $(2,1)+(1,2)$. This decomposition of the space of harmonic 3-forms on $S^1 \times Y$ has the following interpretation in terms of deformations of the product G_2-structure:

- Deformations along the space $\mathbb{R} d\theta \wedge \omega \oplus \mathbb{R}\,\mathrm{Re}\,\Omega \oplus \mathbb{R}\,\mathrm{Im}\,\Omega$ correspond to a variation of the length of the circle factor, or a rescaling and rotation of the holomorphic volume form. Hence this space corresponds to infinitesimal deformations of the G_2-metric by a mere rescaling of each factor of the Riemannian product, which does not affect the space of harmonic forms.

- Deformations along $d\theta \wedge \mathscr{H}^{(1,1)_0}_{\mathbb{R}}(Y, g_\omega)$ correspond to a variation of the Kähler class, with fixed complex structure J on Y. The space of harmonic forms varies along such deformations, but not the Hodge decomposition of the cohomology in classes of type (p, q).

- Deformations along $\mathscr{H}^{(2,1)}_{\mathbb{R}}(Y, g_\omega)$ correspond to changes of the real part of the holomorphic volume form orthogonal to those changes corresponding to multiplying Ω by a complex scalar. This amounts to deforming the complex structure of Y with fixed Kähler class (see for instance [60]). Such deformations modify both the space of harmonic forms and the Hodge decomposition into classes of type (p, q) of the cohomology of Y.

6.3.2 Some remarks on the moduli spaces. Let us now consider a G_2-manifold (M, φ) with vanishing first Betti number and restricted holonomy SU(3). Then there exists a simply-connected Calabi–Yau threefold (Y, J, ω, Ω) such that M has a double cover $\pi : M \to S^1 \times Y$, where the G_2-structure $\pi^*\varphi$ is a product as described above. Moreover, $M \simeq (S^1 \times Y)/\mathbb{Z}_2$, where without loss of generality we can assume that $\mathbb{Z}_2$ acts by $\theta \in \mathbb{R}/L\mathbb{Z} \mapsto -\theta \in \mathbb{R}/L\mathbb{Z}$ on the S^1 factor and by an isometry σ on Y. For $\mathbb{Z}_2$ to preserve φ, we therefore need $\sigma^*\omega = -\omega$ and $\sigma^* \operatorname{Re}(\Omega) = \operatorname{Re}(\Omega)$. Thus σ must be a anti-holomorphic isometry of Y (and in particular $\sigma^* \operatorname{Im}(\Omega) = -\operatorname{Im}(\Omega)$); moreover, it needs to be without fixed points for the quotient $M = (S^1 \times Y)/\mathbb{Z}_2$ to be a smooth manifold. Conversely, if Y is a simply connected compact Calabi-Yau threefold and σ a fixed-point-free anti-holomorphic involution, then for any Kähler Ricci-flat 2-form ω the 2-form $-\sigma^*\omega$ is Kähler Ricci-flat, and thus the cohomology class $[\omega] - \sigma^*[\omega] \in H^2(Y)$ is Kähler and therefore must contain a Kähler Ricci-flat metric ω' with $\sigma^*\omega' = -\omega'$. Before proceeding further, let us give an example of a suitable pair (Y, σ).

Example 6.10. By the adjunction formula and the Lefschetz Hyperplane Theorem, we can construct simply connected Calabi–Yau threefolds by intersecting transversely a smooth quadric and a smooth quartic in $\mathbb{CP}^5$.

Let us consider the two hypersurfaces $V_1, V_2 \subset \mathbb{CP}^5$ defined as $V_1 = \{[z_1 : \cdots : z_6] | \sum_i z_i^2 = 0\}$ and $V_2 = \{[z_1 : \cdots : z_6] | \sum_i z_i^4 = 0\}$. These hypersurfaces are smooth, and their intersection is transverse. To see this, define $f_k(\underline{z}) = \sum_i z_i^{2k}$ for $\underline{z} = (z_1, \ldots, z_6) \in \mathbb{C}^6$, and let $\underline{z} \neq \underline{0}$ such that $f_1(\underline{z}) = f_2(\underline{z}) = 0$. We want to prove that $\ker d_{\underline{z}} f_1 \oplus \ker d_{\underline{z}} f_2 = \mathbb{C}^6$. By contradiction, if this is not the case then $(z_1, \ldots, z_6)$ must be colinear to $(z_1^3, \ldots, z_6^3)$ in $\mathbb{C}^6$. Since these vectors are non-trivial, there exists $\lambda \neq 0$ such that for any $0 \leq i \leq 6$, either $z_i^2 = 0$ or $z_i^2 = \lambda$. But since $f_1(\underline{z}) = 0$ it follows that $d\lambda = 0$, where d is the number of indices such that $z_i \neq 0$, whence either $d = 0$ or $\lambda = 0$, contradicting our assumptions.

Let $Y \subset \mathbb{CP}^5$ be the intersection $V_1 \cap V_2$: this is a simply connected Calabi–Yau threefold. Moreover, it is endowed with an involution σ induced by complex conjugation of the homogeneous complex coordinates of $\mathbb{CP}^5$. Since the real locus of Y is empty, this involution has no fixed points. Hence (Y, σ) is a suitable pair.

Assuming that these conditions are satisfied, the space of harmonic 3-forms $\mathscr{H}^3(M, \varphi)$ is isomorphic to $\mathscr{H}^3(S^1 \times Y, \pi^*\varphi)^{\mathbb{Z}_2}$, and we deduce the following decomposition of the space of harmonic 3-forms on M:

$$\mathscr{H}^3(M, \varphi) \simeq \mathbb{R} d\theta \wedge \omega \oplus \mathbb{R} \operatorname{Re} \Omega \oplus d\theta \wedge \mathscr{H}^{(1,1)_0}_{\mathbb{R},-}(Y, g_\omega) \oplus \mathscr{H}^{(2,1)}_{\mathbb{R}}(Y, g_\omega)^\sigma$$

where $\mathscr{H}^{(2,1)}_{\mathbb{R}}(Y, g_\omega)^\sigma \subset \mathscr{H}^{(2,1)}_{\mathbb{R}}(Y, g_\omega)$ is the subspace fixed by σ and and the subspace $\mathscr{H}^{(1,1)_0}_{\mathbb{R},-}(Y, g_\omega) \subset \mathscr{H}^{(1,1)_0}_{\mathbb{R}}(Y, g_\omega)$ is the -1 eigenspace of σ.

We can consider affine coordinates (x^a) on a neighbourhood of $\varphi\mathscr{D}$ in the moduli space $\mathscr{M}$ of torsion-free G_2-structures on M, and denote by x_0 the coordinates of $\varphi\mathscr{D}$. Although we could not give a fully explicit computation of the fourth derivative of $\mathscr{F}$, we can easily see that for certain choices of indices the terms $\nabla^{\mathscr{G}}_d \Xi_{abc}$ will automatically vanish:

Lemma 6.11. *Let $0 \leq a, b, c, d \leq n$ be indices and assume that either one of the following conditions is satisfied:*

(i) At least one of the indices corresponds to a direction of deformation which lies in $\mathbb{R}d\theta \wedge \omega \oplus \mathbb{R}\,\mathrm{Re}\,\Omega$, or

(ii) Not all indices correspond to directions of deformations lying in the same space $d\theta \wedge \mathscr{H}^{(1,1)_0}_{\mathbb{R},-}(Y, g_\omega)$ or $\mathscr{H}^{(2,1)}_{\mathbb{R}}(Y, g_\omega)^\sigma$.

Then $\mathscr{E}_{abcd} + \mathscr{E}_{cabd} + \mathscr{E}_{cbad} = 0$ at $x = x_0$.

Proof. If condition (i) is satisfied, then the result is immediate since one of the indices corresponds to deforming $\pi^*\varphi$ in a direction which does not modify the space of harmonic forms. For condition (ii), there are only two cases to consider: either three of the indices correspond to deformations lying in one of the spaces $d\theta \wedge \mathscr{H}^{(1,1)_0}_{\mathbb{R},-}(Y, g_\omega)$ or $\mathscr{H}^{(2,1)}_{\mathbb{R}}(Y, g_\omega)^\sigma$ and the other index corresponds to a direction of deformation lying in the other space, or two of the indices correspond to deformations lying in $d\theta \wedge \mathscr{H}^{(1,1)_0}_{\mathbb{R},-}(Y, g_\omega)^\sigma$ and two of the indices correspond to deformations lying in $\mathscr{H}^{(2,1)}_{\mathbb{R}}(Y, g_\omega)^\sigma$. A close examination of the expression of $\mathscr{E}_{abcd}$ in terms of inner products of variations of harmonic forms shows that in all cases, we can find a permutation of the indices a, b, c, d such that $\mathscr{E}_{abcd}$, $\mathscr{E}_{cabd}$ and $\mathscr{E}_{cbad}$ are computed by taking the L^2-inner product of a 3-form of the type $d\theta \wedge \kappa$, where $\kappa \in \Omega^2(Y)$, with a 3-form of the type $\rho \in \Omega^3(Y)$. Such inner products vanish, which yields the lemma. □

Appendix A

Finite subgroups of G_2

In Chapter 6, we have shown that when F is a finite subgroup of G_2 fixing no line in $\mathbb{R}^7$, each connected component of $(\Lambda^3_+\mathbb{R}^*_7)^F$ is a symmetric space. In this appendix we want to classify the possible geometries. There is a classification of the conjugacy classes of finite subgroups of G_2 [26, 48]; however, we will not need the details of the full classification. As each connected component of $(\Lambda^3_+\mathbb{R}^*_7)^F$ is isometric (up to a factor of 2) to $(S^2_+\mathbb{R}^*_7)^F$, it is enough to understand the geometry of the latter space. Moreover, if $F \subset O(n)$ the description of $(S^2_+\mathbb{R}^*_n)^F$ is determined by the decomposition of $\mathbb{R}^7$ as a direct sum of irreducible representations of F, due to the following consequences of Schur's lemma:

(i) Assume that there is an orthogonal decomposition $\mathbb{R}^n = W_1 \oplus W_2$, where W_1, W_2 are subrepresentations of F, such that W_1 and W_2 contain no common irreducible subrepresentations. Then $(S^2_+\mathbb{R}^*_n)^F$ is isometric to the product $(S^2_+W_1^*)^F \times (S^2_+W_2^*)^F$.

(ii) Assume that there is an orthogonal decomposition $\mathbb{R}^n = W_1 \oplus \cdots \oplus W_m$ such that for $i \neq j$ the representations W_i and W_j are non-isomorphic irreducible representations of F. Then $(S^2_+\mathbb{R}^*_n)^F$ is isometric to $\mathbb{R}^m$ equipped with a Euclidean metric.

(iii) Assume that $n = dk$ and that there is an orthogonal decomposition $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k$, where all the V_j are isomorphic irreducible representations of dimension d, and there is (up to multiplication by a scalar) a unique isomorphism between any two of them. Then $(S^2_+\mathbb{R}^*_n)^F$ is isometric to $\mathbb{R} \times d \cdot \mathrm{SL}(k)/\mathrm{SO}(k)$ (i.e. the standard symmetric metric of $\mathrm{SL}(k)/\mathrm{SO}(k)$ is multiplied by a factor d).

To classify the finite subgroups of G_2 fixing no line in $\mathbb{R}^7$, it will be convenient to adopt various points of view on positive forms. A first viewpoint, which is well-adapted when a group leaves invariant a line in $\mathbb{R}^7$, is to write $\mathbb{R}^7 = \mathbb{C}^3 \oplus \mathbb{R}$, with linear coordinates (z_1, z_2, z_3, θ), where $z_j = x_j + iy_j$. In this decomposition, we can take the canonical positive form to be

$$\varphi = \mathrm{Re}(dz_1 \wedge dz_2 \wedge dz_3) + \frac{i}{2} \sum_{j=1}^{3} dz_j \wedge d\overline{z}_j \wedge d\theta \tag{A.1}$$

In particular, any linear transformation which leaves invariant φ and the decomposition $\mathbb{R}^7 = \mathbb{C}^3 \oplus \mathbb{R}$ is either an element of SU(3) (if it fixes the θ-line) or the composition of an element of SU(3) and the real endomorphism acting by complex conjugation on $\mathbb{C}^3$ and by multiplication by -1 on the θ-line.

Another point of view, which is useful when we consider a subgroup of G_2 leaving invariant a coassociative subspace in $\mathbb{R}^7$, is to write $\mathbb{R}^7 = \mathbb{R}^4 \oplus \Lambda^2_+$, where Λ^2_+ is the space of self-dual 2-forms for the standard orientation and inner product of $\mathbb{R}^4$. Any linear transformation fixing φ and leaving the decomposition $\mathbb{R}^7 = \mathbb{R}^4 \oplus \Lambda^2_+$ invariant can be identified with an element of SO(4), acting with the natural representation on $\mathbb{R}^4$ and with the induced representation on Λ^2_+. In particular, any subgroup of G_2 leaving invariant a coassociative subspace can be identified with a subgroup of SO(4), and its action on $\mathbb{R}^7$ is determined by its action on the coassociative subspace.

We first prove that there is, up to conjugacy, only one abelian subgroup of G_2 fixing no line in $\mathbb{R}^7$:

Proposition A.1. *Let $F \subset \mathrm{G}_2$ be an abelian subgroup such that $(\mathbb{R}^7)^F = 0$. Then, up to conjugation in G_2, $F \simeq \mathbb{Z}_2^3$ is generated by α, β, σ acting by*

$$\alpha(z_1, z_2, z_3, \theta) = (-z_1, -z_2, z_3, \theta),$$
$$\beta(z_1, z_2, z_3, \theta) = (-z_1, z_2, -z_3, \theta),$$
$$\sigma(z_1, z_2, z_3, \theta) = (\overline{z}_1, \overline{z}_2, \overline{z}_3, -\theta).$$

Hence $(S^2_+\mathbb{R}^*_7)^F$ is isometric to a flat $\mathbb{R}^7$.

Proof. Since F is abelian and any orthogonal transformation in $\mathbb{R}^7$ has a non-trivial eigenvector, the elements of F have a common eigenvector. Up to conjugacy, we can write $\mathbb{R}^7 = \mathbb{C}^3 \oplus \mathbb{R}$ with coordinates (z_1, z_2, z_3, θ), and assume that F leaves invariant the θ-line. Let $F_0 \lhd F$ be the subgroup of index 2 fixing this line. Then F_0 can be identified with an abelian subgroup of SU(3), and hence up to conjugacy in SU(3) we can assume that there are homomorphisms $\zeta_1, \zeta_2, \zeta_3 : F_0 \to \mathbb{U}$, where $\mathbb{U}$ is the group complex numbers with unit modulus, such that $\zeta_1\zeta_2\zeta_3 = 1$ and any $\alpha \in F_0$ acts by

$$\alpha(z_1, z_2, z_3, \theta) = (\zeta_1(\alpha)z_1, \zeta_2(\alpha)z_2, \zeta_3(\alpha)z_3, \theta).$$

Let $\sigma \in F \backslash F_0$; since F_0 has index 2 in F, the group F is generated by F_0 and σ. There exists a matrix $A = (a_{ij})_{1\leq i,j\leq 3}$ in SU(3) such that

$$\sigma(z_1, z_2, z_3, \theta) = (\sum a_{1j}\overline{z}_j, \sum a_{2j}\overline{z}_j, \sum a_{3j}\overline{z}_j, -\theta).$$

As F is abelian, $\alpha\sigma\alpha^{-1} = \sigma$ for any $\alpha \in F_0$, which is equivalent to the condition

$$\zeta_i(\alpha)\zeta_j(\alpha)a_{ij} = a_{ij}, \quad \forall 1 \leq i,j \leq 3. \tag{A.2}$$

Let us prove that A must be a diagonal matrix. By contradiction, assume that this is not the case, so that after an appropriate cyclic permutation of z_1, z_2, z_3 we can assume $a_{12} \neq 0$. Then condition (A.2) imposes $\zeta_1\zeta_2 = 1$, and as $\zeta_1\zeta_2\zeta_3 = 1$ we have $\zeta_3 = 1$. Since F fixes no line in $\mathbb{R}^7$, one of the coefficients $a_{13}, a_{31}, a_{23}, a_{32}$ must be nonzero; otherwise F_0 fixes the z_3-plane and σ acts as a reflection on this plane so F would fix a line in the z_3-plane. By (A.2), we deduce that $\zeta_1 = \zeta_2 = \zeta_3 = 1$ so that $F = \mathbb{Z}_2$ is generated by σ. But σ is a rotation in $\mathbb{R}^7$ and therefore fixes a line, and we have a contradiction.

Hence A is a diagonal matrix in SU(3). Let $\lambda_1.\lambda_2, \lambda_3 \in \mathbb{U}$ be its eigenvalues; in particular $\lambda_1\lambda_2\lambda_3 = 1$. Thus there is a choice of square roots $\lambda_j^{1/2}$ such that $\lambda_1^{1/2}\lambda_2^{1/2}\lambda_3^{1/2} = 1$. Hence the linear change of coordinates

$$(z_1, z_2, z_3, \theta) \to (\lambda_1^{1/2}z_1, \lambda_2^{1/2}z_2, \lambda_3^{1/2}z_3, \theta)$$

is in G_2, and in these coordinates σ acts as

$$\sigma(z_1, z_2, z_3, \theta) = (\overline{z}_1, \overline{z}_2, \overline{z}_3, -\theta).$$

Now condition (A.2) implies that $\zeta_i^2 = 1$ for all i, and together with the fact that $\zeta_1\zeta_2\zeta_3 = 1$ and the fact that F fixes no line in $\mathbb{R}^7$, we deduce that $F_0 \simeq \mathbb{Z}_2^2$ is generated by α, β acting as

$$\alpha(z_1, z_2, z_3, \theta) = (-z_1, -z_2, z_3, \theta),$$
$$\beta(z_1, z_2, z_3, \theta) = (-z_1, z_2, -z_3, \theta).$$

Thus $F \simeq \mathbb{Z}_2^3$ is generated by α, β, σ. In particular, $\mathbb{R}^7$ decomposes as the direct orthogonal sum of 7 irreducible representations of dimension 1, and it is easy to see that they are all non-isomorphic. Thus $(S^2_+\mathbb{R}^*7)^F$ is the maximal flat totally geodesic submanifold of $S^2_+\mathbb{R}^*_7$ formed by the inner products that are diagonalisable in this decomposition. $\square$

Next we classify the possible types of decomposition of $\mathbb{R}^7$ into irreducible representations for the action of a finite nonabelian subgroup of G_2. If we identify $\mathbb{R}^7$ with the imaginary part of the space of octonions, then a positive form is essentially dual to the cross-product. In particular, if W is a subrepresentation of F then the space generated by the cross products of elements of W is also F-invariant. Thus if $\mathbb{R}^7$ contains a subrepresentation W of F of dimension 2, then the line generated by the cross-product of the elements of a basis of W is also invariant under F. The only possibilities compatible with these constraints are the following:

Proposition A.2. *Let F be a finite nonabelian subgroup of G_2 fixing no line in $\mathbb{R}^7$. Denote by $L_1, L_2, \dots$ the non-isomorphic 1-dimensional representations of F, by $P_1, P_2, \dots,$ the 2-dimensional non-isomorphic irreducible representations of F, and by $V_1^k, V_2^k, \dots$ the $(k \geq 3)$-dimensional irreducible representations. Then the orthogonal decomposition of $\mathbb{R}^7$ into irreducible subrepresentations is either one of:*

1. $\mathbb{R}^7 = V_1^7$.
2. $\mathbb{R}^7 = V_1^6 \oplus L_1$.
3. $\mathbb{R}^7 = V_1^4 \oplus V_1^3$.
4. $\mathbb{R}^7 = V_1^4 \oplus P_1 \oplus L_1$.
5. $\mathbb{R}^7 = V_1^4 \oplus L_1 \oplus L_2 \oplus L_3$.
6. $\mathbb{R}^7 = V_1^3 \oplus V_2^3 \oplus L_1$.
7. $\mathbb{R}^7 = P_1 \oplus P_2 \oplus L_1 \oplus L_2 \oplus L_3$.
8. $\mathbb{R}^7 = P_1 \oplus P_1 \oplus L_1 \oplus L_2 \oplus L_3$.
9. $\mathbb{R}^7 = P_1 \oplus P_2 \oplus P_3 \oplus L_1$.
10. $\mathbb{R}^7 = P_1 \oplus P_1 \oplus P_2 \oplus L_1$.
11. $\mathbb{R}^7 = P_1 \oplus P_1 \oplus P_1 \oplus L_1$.

Moreover, in cases 8, 10 and 11 the only automorphisms of the representation P_1 are (real) homotheties.

Proof. If there is a subrepresentation W of dimension 5, then its orthogonal space $W^\perp$ is an invariant subspace of dimension 2, and looking at the cross-product W must contain a subrepresentation of dimension 1. Hence $\mathbb{R}^7$ has no irreducible subrepresentation of dimension 5. Moreover, any two one-dimensional subrepresentations must be non-isomorphic, otherwise the cross-product would yield a fixed line in $\mathbb{R}^7$. Therefore, cases 1 to 5 enumerate the possibilities when there is an irreducible subrepresentation of dimension at least 4. Since we assumed F to be nonabelian, $\mathbb{R}^7$ cannot be direct sum of representations of dimension 1, and thus the cases left are when the dimension of the largest irreducible subrepresentation is 2 or 3.

Suppose it is 3. If there is a subrepresentation of dimension 2, then there must be one of dimension 1 as well by cross-product. For the same reason, if there are two one-dimensional subrepresentations then there must be at least a third one. Finally, if there are two 3-dimensional irreducible subrepresentations they cannot be isomorphic for determinant reasons. Hence the possibilities are case 6 in the list and $\mathbb{R}^7 = V_1^3 \oplus L_1 \oplus L_2 \oplus L_3 \oplus L_4$. We want to rule out the second possibility. In this case, we can assume after changes the indices that $V_1^3 \oplus L_1$ is a coassociative space; thus we can write $\mathbb{R}^7 = V_1^3 \oplus L_1 \oplus \Lambda^2_+(V_1^3 \oplus L_1)^*$. We want to prove that F acts irreducibly on $\Lambda^2_+(V_1^3 \oplus L_1)$, thus reaching a contradiction. Let us take linear coordinates x_1 on L_1 and (y_1, x_2, y_2) on V_1^3. Then there is a homomorphism $A : F \to \mathrm{O}(3)$ such F acts by A on $V_1^3 \simeq \mathbb{R}^3$ and by $\det A$ on $L_1 \simeq \mathbb{R}$. Using the linear coordinates $(\theta_1, \theta_2, \theta_3)$ on $\Lambda^2_+(V_1^3 \oplus L)^*$ associated with its standard basis, we see that F acts on $\Lambda^2_+(V_1^3 \oplus L)^*$ by $\det(A) \cdot A$, which is an irreducible action. Hence only case 6 can occur.

Now suppose all the irreducible subrepresentations of F contained in $\mathbb{R}^7$ have dimension 1 or 2, and that at least one has dimension 2. Then, besides cases 7–8, there is the possibility that $\mathbb{R}^7$ decomposes as $P_1 \oplus L_1 \oplus L_2 \oplus L_3 \oplus L_4 \oplus L_5$. But after changing the indices, we could assume that $P_1 \oplus L_1$ is associative, and hence its orthogonal $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ would be coassociative. As F is determined by its action on the coassociative subspace, this would force F to be abelian, so in fact this case cannot occur.

It remains to prove our claim about the automorphisms of P_1 in cases 8, 10 and 11. In each of these cases, the direct sum of two copies of $P_1 \oplus P_1$ is a coassociative subspace of $\mathbb{R}^7$, and hence F can be identified with a finite subgroup of $\mathrm{O}(2)$ acting in the same way on each component $P_1 \simeq \mathbb{R}^2$. Since we assumed that F is not abelian it must contain a reflection, and therefore F is isometric to a dihedral group D_{2n} for some $n \geq 2$. Now the claim follows from the fact that the only automorphisms of the natural representation of the dihedral group D_{2n} on $\mathbb{R}^2$ are precisely the real homotheties. □

We now give examples and more details on each case in the list:

Cases 1–5: In cases 1 to 5, no two irreducible subrepresentations are isomorphic and thus $(S^2_+\mathbb{R}^*_7)^F$ is isometric to $\mathbb{R}^m$, where m is the number of irreducible subrepresentations. All of these cases do occur for some finite subgroup of G_2. For case 1, there are finite subgroups of G_2 acting irreducibly on $\mathbb{R}^7$; for instance there is one of order 168 which is isomorphic to $PSL(2,7)$ [26, 48]. The second case can be realised by taking a group of the form $F = G \rtimes \mathbb{Z}_2$, generated by a finite subgroup of SU(3) acting irreducibly on $\mathbb{C}^3$ and the real automorphism of complex conjugation (at least for a good choice of G; but a finite group of SO(3) acting irreducibly on $\mathbb{R}^3$, seen as a subgroup of SU(3), would do the trick with $F = G \times \mathbb{Z}_2$ in that case).

It remains to see that cases 3, 4 and 5 can occur as well. Thus we seek a finite subgroup of SO(4), acting irreducibly on $\mathbb{R}^4$, such that the induced action on Λ^2_+ is either irreducible (for case 3), or has one irreducible subrepresentation of dimension 2 (for case 4), or has three one-dimensional subrepresentations of dimension 1 (for case 5). It is more convenient to consider the double cover Spin(4) of SO(4), which is isomorphic to $\mathrm{SU}(2)_- \times \mathrm{SU}(2)_+$, where $\mathrm{SU}(2)_-$ fixes Λ^3_+ and $\mathrm{SU}(2)_+$ acts on Λ^2_+ in a way that realises the double cover of SO(3). Now take any finite subgroup G_- of $\mathrm{SU}(2)_-$ acting irreducibly on $\mathbb{R}^4$ (for instance the group of order 8 generated by i, j, k seen as unit quaternions), and G_+ a finite subgroup of $\mathrm{SU}(2)_+$. Then let $G_- \times G_+ \subset \mathrm{Spin}(4)$ and let F be its image in SO(4); it acts irreducibly on $\mathbb{R}^4$ since G_- does, and its action on Λ^2_+ is determined by the choice of G_+. For case 3, one can take G_+ to be the lift in SU(2) of a finite group of SO(3) acting irreducibly on $\mathbb{R}^3$. Up to conjugacy, there are three such groups: the chiro-tetrahedral, chiro-octahedral and chiro-icosahedral groups [28], respectively corresponding to the group of rotations leaving invariant a regular tetrahedron, a regular octahedron or a regular icosahedron in $\mathbb{R}^3$. For case 4, we can take G_+ to be the lift in SU(2) of a dihedral group D_{2n} for some $n \geq 3$, and for case 5 we can choose G_+ as the lift in SU(2) of the dihedral group $D_4 \simeq \mathbb{Z}_2^2$.

Case 6: Let us assume that $V^3_1 \oplus L_1$ is coassociative. In the proof of the previous proposition, we have seen that there is a homomorphism $A : F \to \mathrm{O}(3)$ such that F acts by A on $V^3_1 \simeq \mathbb{R}^3$, by $\det(A)$ on the invariant line and by $\det(A) \cdot A$ on $V^3_2 \simeq \mathbb{R}^3$. Hence $(S^2_+\mathbb{R}^*_7)^F$ is isometric to a flat $\mathbb{R}^3$, and F can be identified with a subgroup of O(3) acting irreducible on $\mathbb{R}^3$ without preserving the orientation. As in the case of the finite subgroups of SO(3), there are (up to conjugacy) three such groups: the holo-tetrahedral, holo-octahedral and holo-icosahedral groups. They respectively correspond to the full group of orthogonal transformations leaving invariant a regular tetrahedron, a regular octahedron or a regular icosahedron [28].

Cases 7–8: The case where the two 2-dimensional irreducible subrepresentations are non-isomorphic occurs for instance when considering the action of the dihedral group D_{4n}, for some $n \geq 3$, acting on $\mathbb{R}^7 = \mathbb{C}^3 \oplus \mathbb{R}$ by two generators α, β:

$$\begin{aligned} \alpha(z_1, z_2, z_3, \theta) &= (e^{\frac{i\pi}{n}} z_1, -e^{-\frac{i\pi}{n}} z_2, -z_3, \theta), \\ \sigma(z_1, z_2, z_3, \theta) &= (\overline{z}_1, \overline{z}_2, \overline{z}_3, -\theta). \end{aligned}$$

In case 7, $(S^2_+\mathbb{R}^*_7)^F$ is isometric to a flat $\mathbb{R}^5$.

When the two 2-dimensional representations are isomorphic, we have seen that F is isometric to a dihedral group and that the only automorphisms of P_1 are homotheties. Thus we deduce that $(S^2_+\mathbb{R}^*_7)^F$ is isometric to $\mathbb{R}^4 \times 2 \cdot (\mathrm{SL}(2)/\,\mathrm{O}(2)) \simeq \mathbb{R}^4 \times \mathbb{H}(-1/4)$, where the metric on the hyperbolic plane $\mathbb{H}$ is normalised to have constant sectional curvature $-1/4$. It is easy to see that up to conjugacy the only possibility is D_8, generated by α, σ acting on $\mathbb{R}^7$ in the following way:

$$\begin{aligned} \alpha(z_1, z_2, z_3, \theta) &= (iz_1, iz_2, -z_3, \theta), \\ \sigma(z_1, z_2, z_3, \theta) &= (\overline{z}_1, \overline{z}_2, \overline{z}_3, -\theta). \end{aligned} \tag{A.3}$$

Cases 9–11: If all the 2-dimensional representations are mutually non-isomorphic, then $(S^2_+\mathbb{R}^*_7)^F$ is isometric to a flat $\mathbb{R}^4$. For any pair of integers $n_1, n_2 \geq 3$, an example of such action for a finite group is $(C_{n_1} \times C_{n_2}) \rtimes \mathbb{Z}_2$, generated by three elements α, β, γ acting by:

$$\begin{aligned} \alpha(z_1, z_2, z_3, \theta) &= (e^{\frac{2i\pi}{n_1}} z_1, z_2, e^{-\frac{2i\pi}{n_1}} z_3, \theta), \\ \beta(z_1, z_2, z_3, \theta) &= (z_1, e^{\frac{2i\pi}{n_2}} z_2, e^{-\frac{2i\pi}{n_2}} z_3, \theta), \\ \sigma(z_1, z_2, z_3, \theta) &= (\overline{z}_1, \overline{z}_2, \overline{z}_3, -\theta). \end{aligned}$$

Another possibility is when exactly two representations of dimension 2 are isomorphic, in which case $(S^2_+\mathbb{R}^*_7)^F$ is isometric to $\mathbb{R}^3 \times 2 \cdot (\mathrm{SL}(2)/\,\mathrm{SO}(2)) \simeq \mathbb{R}^3 \times \mathbb{H}(-1/4)$. One can easily see that, up to conjugacy, the only possibility is $F \simeq D_{2n}$, where $n \geq 5$, acting with two generators α, σ as

$$\begin{aligned} \alpha(z_1, z_2, z_3, \theta) &= (e^{\frac{2i\pi}{n}} z_1, e^{\frac{2i\pi}{n}} z_2, e^{-\frac{4i\pi}{n}} z_3, \theta), \\ \sigma(z_1, z_2, z_3, \theta) &= (\overline{z}_1, \overline{z}_2, \overline{z}_3, -\theta). \end{aligned} \tag{A.4}$$

The last possibility is when all of the dimension 2 subrepresentations are isomorphic. In that case, $(S^2_+\mathbb{R}^*_7)^F$ is isometric to $\mathbb{R}^2 \times 2 \cdot (\mathrm{SL}(3)/\mathrm{SO}(3))$. Up to conjugacy in G_2, the only possibility for a finite group is D_6 acting with two generators α, σ as

$$\begin{aligned} \alpha(z_1, z_2, z_3, \theta) &= (e^{\frac{2i\pi}{3}} z_1, e^{\frac{2i\pi}{3}} z_2, e^{\frac{2i\pi}{3}} z_3, \theta), \\ \sigma(z_1, z_2, z_3, \theta) &= (\overline{z}_1, \overline{z}_2, \overline{z}_3, -\theta). \end{aligned} \tag{A.5}$$

Gathering the previous results we finally obtain:

Proposition A.3. *Let F be a finite nonabelian subgroup of G_2 fixing no line in $\mathbb{R}^7$. Then $(S^2_+\mathbb{R}^*_7)^F$ is isometric to one of the following:*

1. *A flat $\mathbb{R}^m$, where $m = 1, 2, 3, 4$ or 5.*
2. *$\mathbb{R}^3 \times \mathbb{H}(-1/4)$, in which case $F \simeq D_{2n}$ for some integer $n \geq 5$, and up to conjugacy F has two generators α, σ acting as (A.4).*
3. *$\mathbb{R}^4 \times \mathbb{H}(-1/4)$, in which case $F \simeq D_8$, and up to conjugacy F has two generators α, σ acting as (A.3).*
4. *$\mathbb{R}^2 \times 2 \cdot (\mathrm{SL}(3)/\mathrm{SO}(3))$, in which case $F \simeq D_6$, and up to conjugacy F has two generators α, σ acting as (A.5).*

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