

Colouring proximity graphs in the plane

Colin McDiarmid^{a,*}, Bruce Reed^b

^a *Department of Statistics, University of Oxford, 1 South Parks Road, Oxford OX1 3TG, UK*

^b *CNRS, Paris, France*

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Abstract

Given a set V of points in the plane and given $d > 0$, let $G(V, d)$ denote the graph with vertex set V and with distinct vertices adjacent whenever the Euclidean distance between them is less than d . We are interested in colouring such ‘proximity’ graphs. One application where this problem arises is in the design of cellular telephone networks, where we need to assign radio channels (colours) to transmitters (points in V) to avoid interference. We investigate the case when the set V has finite positive upper density σ , and d is large. We find that, as $d \rightarrow \infty$, the chromatic number χ divided by d^2 tends to the limit $\sigma\sqrt{3}/2$, and the ratio of the chromatic number χ to the clique number ω tends to $2\sqrt{3}/\pi \sim 1.103$. © 1999 Elsevier Science B.V. All rights reserved

1. Introduction and statement of results

Given a non-empty set V of points in the plane, we may form a ‘proximity’ or ‘interference’ graph on the vertex set V by joining two distinct vertices whenever they are ‘close’ together. In particular, given $d > 0$ let $G(V, d)$ denote the graph with vertex set V and with distinct vertices u and v adjacent whenever the Euclidean distance $d(u, v)$ is less than d . This graph is a scaled version of a ‘unit circle’ graph. We are interested in colouring such graphs, that is colouring the points in V so that we do not give the same colour to any two points that are within distance less than d .

One application in which this problem arises is in the design of cellular telephone networks, where we need to assign a radio channel (colour) to each transmitter (point in V) so as to avoid interference, see for example [9] or [5] where this problem is called the ‘frequency-distance constrained cochannel assignment problem’. It is reasonable to assume that there are many points, well spread out. When the interference distance d is small, small changes in d or in V can lead to large changes in the number of colours needed. In order to gain an overview of the problem, we consider here the case

* Corresponding author. Tel.: 0186 527 2872; fax: 0186 527 2595; e-mail: cmcd@stats.ox.ac.uk.

when V is infinite and $d \rightarrow \infty$. It is hoped that our asymptotic results yield insight into finite cases with practical values for the parameters. We shall state results first for general sets; then for approximately uniformly spread sets, where we can be more precise; and finally for the special case of the triangular lattice (with hexagonal cells), where we can give an exact non-asymptotic result, and for generalised lattices. These results were first presented at the DONET meeting in Stirin near Prague in May 1996.

First, however, we recall some definitions and notation. A (*proper*) *colouring* of a graph G is a colouring of the vertices so that adjacent vertices receive distinct colours. The *chromatic number* $\chi(G)$ is the least number of colours in such a colouring. When we consider finding colourings, other graph parameters are also of interest.

The *degree* of a vertex of G is the number of vertices adjacent to it. The *maximum degree* $\Delta(G)$ is the supremum of the degrees of the vertices. The *minimum degree* $\delta(G)$ is the minimum degree of a vertex. The *maximin degree* $\delta^*(G)$ is the supremum over all finite induced subgraphs of the minimum degree. This is also called the ‘degeneracy number’ of G , and the number plus 1 is called the ‘colouring number’ of G , since it is easily seen that there must be a proper colouring using at most this last number of colours.

A *clique* is a set of pairwise adjacent vertices; the *clique number* $\omega(G)$ is the supremum of the numbers of vertices in a clique. A *stable (or independent) set* is a set of pairwise non-adjacent vertices; the *stability number* $\alpha(G)$ is the supremum of the numbers of vertices in a stable set. We let the *stability quotient* $sq(G)$ be the supremum over all finite subgraphs H of G of the ratio $|V(H)|/\alpha(H)$. Note that $\chi(G) \geq sq(G) \geq \omega(G)$. All our lower bounds on $\chi(G)$ will in fact be lower bounds on $sq(G)$.

1.1. General sets

Let V be any countable set of points in the plane. For $x > 0$ let $f(x)$ be the supremum of the ratio $|V \cap S|/x^2$ over all open $(x \times x)$ squares S with sides aligned with the axes. The *upper density* of V is $\sigma^+(V) = \inf_{x > 0} f(x)$. We shall see that $f(x) \rightarrow \sigma^+(V)$ as $x \rightarrow \infty$; and that the definition could equally well be phrased in terms of balls rather than squares, or indeed in terms of any ‘reasonable’ set with finite positive area.

Theorem 1. *Let V be a countable non-empty set of points in the plane, with upper density $\sigma^+(V) = \sigma$. For any $d > 0$, denote the clique number $\omega(G(V, d))$ by ω_d , and use χ_d, Δ_d and δ_d^* similarly for the chromatic number, maximum degree and maximin degree. Then $\omega_d/d^2 \geq \sigma\pi/4$ and $\chi_d/d^2 \geq \sigma\sqrt{3}/2$ for any $d > 0$; and, as $d \rightarrow \infty$, $\Delta_d/d^2 \rightarrow \sigma\pi$, $\delta_d^*/d^2 \rightarrow \sigma\pi/2$, $\omega_d/d^2 \rightarrow \sigma\pi/4$ and $\chi_d/d^2 \rightarrow \sigma\sqrt{3}/2$.*

It follows for example that for any countable set V of points in the plane with a finite positive upper density, the ratio of the chromatic number of $G(V, d)$ to its clique number tends to $2\sqrt{3}/\pi \sim 1.103$ as $d \rightarrow \infty$. It was suggested in [3] that such a result should hold for the triangular lattice.

1.2. Sets with a cell structure

Next we consider sets of points that are approximately uniformly spread over the plane. For such sets we can tighten the upper bound parts of the above theorem.

Let $0 < \sigma, r < \infty$. We say that a set V of points in the plane has a *cell structure* with density σ and radius r if there is a family $(C_v: v \in V)$ of ‘cells’ indexed by V such that (a) this family partitions the plane (except perhaps for a set of measure zero); (b) each cell C_v (is measurable and) has area $1/\sigma$; and (c) $C_v \subseteq B(v, r)$ for each $v \in V$. Here $B(v, r)$ denotes the open ball with centre v and radius r (in the Euclidean metric). It is easily seen that such a set V has upper density σ . For example, the set of vertices of the square lattice with unit edge lengths has a cell structure (with square cells) with density 1 and radius $1/\sqrt{2}$.

Theorem 2. *Let the set V of points in the plane have a cell structure with density σ and radius r . Then for any $d > 0$,*

$$\sigma\pi(d-r)^2 - 1 \leq \delta(G(V, d)) \leq \Delta(G(V, d)) \leq \sigma\pi(d+r)^2 - 1, \quad (1)$$

$$(\sigma\pi/4)d^2 \leq \omega(G(V, d)) \leq (\sigma\pi/4)(d+2r)^2 \quad (2)$$

and

$$(\sigma\sqrt{3}/2)d^2 \leq \chi(G(V, d)) < ((\sigma\sqrt{3}/2)^{1/2}(d+2r) + (2/\sqrt{3}) + 1)^2. \quad (3)$$

Thus both the minimum and the maximum degree of the graph $G(V, d)$ equal $\sigma\pi d^2 + O(d)$, $\omega(G(V, d)) = (\sigma\pi/4)d^2 + O(d)$, and $\chi(G(V, d)) = (\sigma\sqrt{3}/2)d^2 + O(d)$ as $d \rightarrow \infty$.

1.3. Lattices

The key to understanding the above colouring problems is the special case of the triangular lattice (with hexagonal cells), where things work out very neatly. We assume that the lattice has the natural embedding in the plane with minimum distance 1. The origin $O = (0, 0)$ and the point $a = (1, 0)$ are lattice points, and thus so are the points $b = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $c = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let G_T denote the corresponding 6-regular graph, with vertex set the set T of lattice points, and with two vertices adjacent whenever they are at distance 1. The six neighbours of the origin O in G_T are then $\pm a, \pm b, \pm c$.

We are interested in colouring the graph $G(T, d)$. The set T has a cell structure (with hexagonal cells) with density $2/\sqrt{3}$ and radius $1/\sqrt{3}$. Thus Theorem 2 above gives good bounds on $\chi(G(T, d))$. However, we can determine the exact value.

The distance between the origin O and the lattice point $xa + yb$, where x and y are non-negative integers, is $((x+y/2)^2 + (\sqrt{3}y/2)^2)^{1/2} = (x^2 + xy + y^2)^{1/2}$. For any $d > 0$, we let \hat{d} denote the minimum value of $(x^2 + xy + y^2)^{1/2}$ such that x and y are non-negative integers and $(x^2 + xy + y^2)^{1/2} \geq d$. Then \hat{d} is the minimum Euclidean distance between

two points in T subject to that distance being at least d . Note that $d \leq \hat{d} \leq \lceil d \rceil$, and that we can compute the (rhombic) integer \hat{d}^2 quickly, in $O(d)$ arithmetic operations.

Theorem 3. *The triangular lattice T satisfies*

$$\chi(G(T, d)) = \hat{d}^2$$

for any $d > 0$.

This result appears to have been known to engineers at least since 1979 — see [9, 2]. A similar result appears as Theorem 3 in [1]. The above theorem is the result on the triangular lattice which we need in order to prove the results stated earlier. However, it is of interest to note that we can go further in this special case. We can also work with graph distance in the triangular lattice graph G_T . Given a graph G and positive integer k , let $G^{(k)}$ denote the graph with the same vertices as G , and with distinct vertices u and v adjacent whenever their distance in G is at most k . [The graph distance between u and v is the least number of edges in a path joining them.] Thus $G^{(1)}$ is just G .

Theorem 4. *The graph G_T of the triangular lattice satisfies*

$$\chi(G_T^{(k)}) = \omega(G_T^{(k)}) = \lceil \frac{3}{4}(k+1)^2 \rceil$$

for any positive integer k .

This result has been proved independently in [6], where a similar result is given for the graph of the square lattice (with $\frac{3}{4}$ replaced by $\frac{1}{2}$).

Both the above theorems in fact follow from a more general result, where we measure the closeness of two points u and v in V by both their Euclidean distance $d(u, v)$ and their distance $d_{G_T}(u, v)$ in the graph G_T . Let the ‘forbidden’ set F of ‘too small’ distance pairs be any proper subset of the non-negative vectors in \mathbb{R}^2 , such that if $x = (x_1, x_2) \in F$, and $0 \leq y_1 \leq x_1$ and $0 \leq y_2 \leq x_2$, then $y = (y_1, y_2) \in F$. Define the graph $G(T, F)$ on the vertex set T by letting distinct vertices u and v be adjacent whenever the ordered pair of distances $(d(u, v), d_{G_T}(u, v))$ is in F .

A set of points in the plane is *discrete* if every bounded subset is finite. If we think of the plane as a group under co-ordinatewise addition, then a *lattice* in the plane is a subgroup which is discrete and two-dimensional. A *strict tiling* of a lattice V (see for example [6, 8]) is a colouring of V with a finite number of colours such that each colour set is a translate $v + L$ of some sublattice L , the co-channel lattice.

Theorem 5. *Define $d^* = d^*(T, F)$ to be the minimum Euclidean distance between two distinct points in the triangular lattice T which are not adjacent in the graph $G(T, F)$. Then*

$$\chi(G(T, F)) = (d^*)^2,$$

and there is an optimal colouring which is a strict tiling.

A *generalised lattice* is a finite union of cosets of a lattice. Equivalently, a generalised lattice is a discrete set with two independent translation symmetries. It is not hard to see that any generalised lattice V has a cell structure, with some density σ . Thus Theorem 2 applies, and shows in particular that

$$\chi(G(V, d)) = (\sigma\sqrt{3}/2)d^2 + O(d)$$

as $d \rightarrow \infty$. However, we may wish to focus on strict tilings as introduced above.

A *strict tiling* of the generalised lattice V is a colouring of V with a finite number of colours such that each colour set is a coset $v+L$ of some lattice L , the co-channel lattice. Many popular channel assignment methods consider only strict tilings, and indeed may consider only triangular co-channel lattices — see [8].

For the triangular lattice T and any $d > 0$, the proximity graph $G(T, d)$ always has an optimal colouring which is a strict tiling, with a triangular co-channel lattice. For any generalised lattice V in the plane, there is a similar approximate result.

Theorem 6. *Let V be a generalised lattice in the plane. Then the proximity graph $G(V, d)$ has a colouring which is a strict tiling and which uses at most $\chi(G(V, d)) + O(d)$ colours.*

2. Proofs

2.1. Upper density

Our definition of upper density could be called ‘upper density on squares’. We shall see now that it could equally well be based on balls, or indeed on more general sets in the plane.

Let C (the ‘measuring set’) be any set in the plane with (well-defined) finite positive area. For any set V of points in the plane, define $\sigma^+(V, C)$ to be the supremum of the ratio $|V \cap C'|/\text{area}(C')$ over all sets C' which are translates of C . Also, define $\sigma_C^+(V) = \inf_{x>0} \sigma^+(V, xC)$. Here xC denotes the scaled set $\{xz: z \in C\}$. Thus if we let S denote the open unit square in the plane centered at the origin, then we have $\sigma^+(V) = \sigma_S^+(V)$. It is easily seen that $\sigma_C^+(V)$ is finite if and only if $\sigma^+(V, xC)$ is finite for each $x > 0$; and in this case the supremum in the definition is always attained.

We say that C has a *small neighbourhood* if

$$\frac{\text{area}(B(C, r))}{\text{area}(C)} \rightarrow 1$$

as $r \rightarrow 0$. Here $B(C, r)$ denotes $C + B(O, r)$, that is, the set $\{x: d(x, y) < r \text{ for some } y \in C\}$. Note that such a set C must be bounded.

Lemma 1. *Let C be any set in the plane with finite positive area, and with a small neighbourhood. Then, for any countable set V of points in the plane, $\sigma_C^+(V) = \sigma^+(V)$ and $\sigma^+(V, xC) \rightarrow \sigma^+(V)$ as $x \rightarrow \infty$.*

Proof. Let D be another set in the plane like C , that is, with finite positive area, and with a small neighbourhood. We shall show that $\sigma_D^+(V) \geq \sigma_C^+(V)$, and we shall then be able to complete the proof quickly. We may assume without loss of generality that C and D both have area 1.

Suppose first that $\sigma^+(V, yC)$ is finite for each $y > 0$. Since D is bounded, we may choose $r > 0$ such that $D \subseteq B(O, r)$. Fix $x > 0$. Let $y > 0$. Consider a translate C_y of yC such that $|V \cap C_y| = \sigma^+(V, yC) y^2$. Since C has a small neighbourhood, $B(C_y, xr)$ has area $\alpha(y) = y^2 + o(y^2)$ as $y \rightarrow \infty$.

Pick a random point w uniformly from $B(C_y, xr)$, and let R be the corresponding random translate $w + xD$ of xD . For any point $v \in C_y$, $v - xD \subseteq B(C_y, xr)$, and $v \in R$ if and only if $w \in v - xD$; and so the probability that $v \in R$ equals $x^2/\alpha(y)$. Hence

$$\sigma^+(V, xD) x^2 \geq E(|V \cap R|) \geq |V \cap C_y| x^2 / \alpha(y).$$

So

$$\sigma^+(V, xD) \geq \sigma^+(V, yC) (y^2 / \alpha(y)).$$

Now let $y \rightarrow \infty$ to see that

$$\sigma^+(V, xD) \geq \limsup_{y \rightarrow \infty} \sigma^+(V, yC).$$

But this holds for all $x > 0$ and so

$$\sigma_D^+(V) \geq \limsup_{y \rightarrow \infty} \sigma^+(V, yC) \geq \liminf_{y \rightarrow \infty} \sigma^+(V, yC) \geq \sigma_C^+(V).$$

But by interchanging C and D we have also $\sigma_C^+(V) \geq \sigma_D^+(V)$. Hence $\sigma_C^+(V) = \sigma_D^+(V)$, and $\sigma^+(V, xC) \rightarrow \sigma_C^+(V)$ as $x \rightarrow \infty$.

A similar argument shows that if $\sigma^+(V, yC)$ is infinite for some $y > 0$, then $\sigma^+(V, xD)$ is infinite for each $x > 0$, and the result follows. \square

Essentially the same proof shows that the above result extends to any dimension. It will also yield the more general result Lemma 4 below.

Lemma 2. Let V be a discrete set of points in the plane with upper density $\sigma > 0$. Then $\text{sq}(G(V, d)) \geq \sigma(\sqrt{3}/2)d^2$ for any $d > 0$.

Proof. A *packing* of disks is a collection of disks with pairwise disjoint interiors. Let $d > 0$ be fixed, and consider disks of diameter d . We shall use Thue's theorem (see for example [12, 13]), that the maximum density of a packing of unit disks in the plane is achieved by the hexagonal packing, and thus of course the same is true for disks with diameter d . Note that the hexagon with inner radius $d/2$ has area $6 \times \sqrt{3} \times (d/2\sqrt{3})^2 = (\sqrt{3}/2)d^2$. Given a bounded set C , let $\nu(C)$ denote the maximum number of disks with centre in C , over all packings of disks with diameter d .

Thue's theorem implies that $v(B(O, r))/\pi r^2 \rightarrow 1/((\sqrt{3}/2)d^2)$ as $r \rightarrow \infty$. Hence, for any $\varepsilon > 0$, there is an r_0 such that for any $r \geq r_0$,

$$v(B(O, r)) \leq (1 + \varepsilon)(\pi r^2) \left/ \left(\frac{\sqrt{3}}{2} d^2 \right) \right.$$

But for any ball B_r of radius r ,

$$\alpha(G(V \cap B_r, d)) = v(V \cap B_r) \leq v(B(O, r)),$$

and so

$$\alpha(G(V \cap B_r, d)) \leq (1 + \varepsilon)(\pi r^2) \left/ \left(\frac{\sqrt{3}}{2} d^2 \right) \right.$$

On the other hand, if σ is finite, then there is a ball B_r of radius r with $|V \cap B_r| \geq \sigma \pi r^2$. Hence, for any $r \geq r_0$,

$$sq(G(V, d)) \geq \frac{\sigma \pi r^2}{(1 + \varepsilon)(\pi r^2)/(\frac{\sqrt{3}}{2} d^2)} = \frac{1}{1 + \varepsilon} \sigma \frac{\sqrt{3}}{2} d^2.$$

A similar argument works for the case when σ is infinite. \square

2.2. Proofs for lattices

Proof of Theorem 5. First consider the lower bound on χ . Recall that the triangular lattice with unit edge-lengths has density $\sigma = 2/\sqrt{3}$. Hence Lemma 2 shows that

$$sq(G(T, F)) \geq sq(G(T, d^*)) \geq (d^*)^2.$$

Now consider the upper bound on χ . Recall that the points $a = (1, 0)$, $b = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $c = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ are neighbours of the origin O in the lattice graph G_T . Recall also that the distance between O and the lattice point $xa + yb$, where x and y are non-negative integers, is $(x^2 + xy + y^2)^{1/2}$.

Now suppose that the point $p = x_0a + y_0b$ is not adjacent in $G(T, F)$ to O , where x_0 and y_0 are non-negative integers such that $(d^*)^2 = x_0^2 + x_0y_0 + y_0^2$. Let $q = x_0b + y_0c$. Note that the set $U = \{xp + yq : x, y \text{ integers}\}$ is the set of lattice points of a triangular sublattice of the original triangular lattice, and the 6 points in U closest to O are $\pm p, \pm q, \pm r$ where $r = x_0c - y_0a$. Also, if v is any of these six points then O and v are not adjacent in $G(T, F)$. Thus U is a stable set in this graph.

Now let $R = \{xp + yq : 0 \leq x, y < 1\}$. Then $\text{area}(R) = (\sqrt{3}/2)(d^*)^2$, and the family $(R + v : v \in U)$ partitions (tiles) the plane. So R contains exactly $\text{area}(R) \sigma = (d^*)^2$ lattice points. We colour the lattice points in R with $(d^*)^2$ distinct colours, and use the tiling of the plane by R to extend this colouring to the whole of U . Thus we give the lattice point $xp + yq$ the colour of the point $x'p + y'q$, where $x' = x - \lfloor x \rfloor$, $y' = y - \lfloor y \rfloor$. The set of points with a given colour is then a translate of the stable

set U , the co-channel lattice, and so we have defined a proper colouring of $G(T, F)$, which is a strict tiling as required. \square

Proof of Theorem 3. This is immediate from Theorem 5 and its proof, with F as the set of non-negative pairs (x, y) with $x < d$, since then $d^+ = d^*(T, F)$. \square

Proof of Theorem 4. Let F be the set of non-negative pairs (x, y) with $y \leq k$. In the notation of the proof of Theorem 5, a closest lattice point to O at graph distance k is $(k/2)a + (k/2)b$ if k is even, and is $((k-1)/2)a + ((k+1)/2)b$ if k is odd. It follows easily that $(d^*(T, F))^2$ equals $3(k+1)^2/4$ if k is odd, and $3((k+1)^2 + 1)/4$ if k is even. The theorem now follows from Theorem 5, except for the reference to $\omega(L^{(k)})$.

To complete the proof we observe that there are cliques in $L^{(k)}$ of this size: if k is even then take the set of vertices of L at graph distance at most $k/2$ from a given point, and if k is odd then take the set of vertices at graph distance at most $(k-1)/2$ from a given triangle. [It is straightforward to check that these cliques correspond to a tiling of the plane as in the proof of Theorem 5: this gives an alternative proof of the theorem.] \square

Theorem 6 will follow easily from the next lemma.

Lemma 3. *Let L be a lattice in the plane. Then there are constants c and d_0 such that for any $d \geq d_0$, there is a sublattice \tilde{L} of L which has minimum distance at least d and has density at least $(2/\sqrt{3})/(d^2 + cd)$.*

Proof. Consider the lattice points $a = (1, 0)$ and $b = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ in the triangular lattice. Let $0 < \delta \leq \frac{1}{2}$. Let $\tilde{a} \in B(a, \delta)$ and $\tilde{b} \in B(b, \delta)$. We claim that the lattice \tilde{L} generated by \tilde{a} and \tilde{b} has minimum distance greater than $1 - 2\delta$, and density $2/\sqrt{3} + O(\delta)$.

To prove the claim, note first that each of the four points $\pm \tilde{a}, \pm \tilde{b}$ has (Euclidean) norm greater than $1 - \delta$. Now for any integers m and n ,

$$\begin{aligned} \|ma + nb\|^2 &= m^2 + mn + n^2 \\ &= \frac{1}{4}(|m| + |n|)^2 + \frac{1}{4}(|m| - |n|)^2 + \frac{1}{2}(m^2 + 2mn + n^2) \\ &\geq \frac{1}{4}(|m| + |n|)^2. \end{aligned}$$

But any point of the lattice \tilde{L} other than the four listed above is of the form $m\tilde{a} + n\tilde{b}$ for some integers m and n with $|m| + |n| \geq 2$, and then

$$\begin{aligned} \|m\tilde{a} + n\tilde{b}\| &> \|ma + nb\| - (|m| + |n|)\delta \\ &\geq (|m| + |n|) \left(\frac{1}{2} - \delta\right) \\ &\geq 1 - 2\delta. \end{aligned}$$

This shows that the minimum distance in \tilde{L} is greater than $1 - 2\delta$. The final part of the claim concerning the density of \tilde{L} comes from the observation that the

fundamental region of \tilde{L} has area $\sqrt{3}/2 + O(\delta)$. This completes the proof of the claim.

Now let $\delta > 0$ be such that every open ball of radius δ contains a point of L . Let $d > 0$, and let $D = d + 2\delta$. Let \hat{a} be a point of L in $B(Da, \delta)$ and let \hat{b} be a point of L in $B(Db, \delta)$. Then $\tilde{a} = (1/D)\hat{a}$ is in $B(a, \delta/D)$, and similarly $\tilde{b} = (1/D)\hat{b}$ is in $B(b, \delta/D)$. Now the minimum distance in the lattice \hat{L} generated by \hat{a} and \hat{b} is D times the minimum distance in the lattice \tilde{L} , and hence, by the claim, it is greater than $D(1 - 2\delta/D) = d$.

Finally, the density of \hat{L} equals the density of \tilde{L} divided by D^2 , and so it equals

$$(2/\sqrt{3} + O(\delta/D))/D^2 = (2/\sqrt{3})/(d^2 + O(d)),$$

as required. \square

Proof of Theorem 6. Let the generalised lattice V consist of k cosets of the lattice L . If V has density σ_V and L has density σ_L then $k = \sigma_V/\sigma_L$. By the above lemma, there are constants c and d_0 such that for any $d \geq d_0$, there is a sublattice \hat{L} of L which has minimum distance at least d and has density $\hat{\sigma}$ at least $(2/\sqrt{3})/(d^2 + cd)$. Then V may be partitioned into $(\sigma_V/\sigma_L)(\sigma_L/\hat{\sigma}) = \sigma_V/\hat{\sigma}$ cosets of \hat{L} . Thus we have a colouring of V which is a strict tiling and the number of colours used is

$$\sigma_V/\hat{\sigma} = \sigma_V(\sqrt{3}/2)d^2 + O(d). \quad \square$$

2.3. Proofs for general sets

We need an easy extension of Lemma 1. Let \mathcal{A} be a family of sets in the plane, each with finite positive area. For any set V of points in the plane, define $\sigma^+(V, \mathcal{A})$ to be the supremum of the quantities $\sigma^+(V, A)$ over the sets $A \in \mathcal{A}$. Also, define $\sigma_{\mathcal{A}}^+(V) = \inf_{x>0} \sigma^+(V, x\mathcal{A})$. Here $x\mathcal{A}$ denotes the family of scaled sets $\{xA: A \in \mathcal{A}\}$. Thus if \mathcal{A} contains only the set A , then we have $\sigma_{\mathcal{A}}^+(V) = \sigma_A^+(V)$. We say that \mathcal{A} has *small neighbourhoods* if the sets in \mathcal{A} have uniformly small neighbourhoods, that is, if

$$\sup_{A \in \mathcal{A}} \frac{\text{area}(B(A, r))}{\text{area}(A)} \rightarrow 1$$

as $r \rightarrow 0$. The same proof method as for Lemma 1 will yield the following extension of that result.

Lemma 4. *Let \mathcal{A} be a family of sets in the plane, each with finite positive area, and suppose that \mathcal{A} has small neighbourhoods. Then, for any discrete set V of points in the plane, $\sigma_{\mathcal{A}}^+(V) = \sigma^+(V)$ and $\sigma^+(V, x\mathcal{A}) \rightarrow \sigma^+(V)$ as $x \rightarrow \infty$.*

Lemma 4 yields immediately

Lemma 5. *Let the set V of points in the plane have finite upper density $\sigma > 0$. Let $\eta > 0$. Then there exists r_0 such that for all $r \geq r_0$, for all balls B_r of radius r , for all points $w \in B_r$, and for all $0 < s \leq r/2$,*

$$|V \cap (B_r \setminus B(w, s))| \leq (1 + \eta)\sigma \text{ area}(B_r \setminus B(w, s)).$$

Lemma 6. *Let the non-empty set V of points in the plane have finite upper density σ . Then $\Delta(G(V, d))/d^2 \rightarrow \sigma\pi$ as $d \rightarrow \infty$.*

Proof. Let B denote the unit ball $B(O, 1)$. Let $\varepsilon > 0$. Clearly $\Delta(G(V, d)) \leq \sigma^+(V, dB)\pi d^2 - 1$. Hence $\Delta(G(V, d)) \leq (\sigma + \varepsilon)\pi d^2$ for d sufficiently large.

To prove a corresponding lower bound on the maximum degree, we may assume that $\sigma > 0$. Again let $\varepsilon > 0$. Let A be the annulus $B \setminus \varepsilon B$. Since A has area strictly less than π , by Lemma 5, there is an $x_0 > 0$ such that for $x \geq x_0$ and all points z , we have $|V \cap (z + xA)| < \sigma\pi x^2$. Let $y_0 = (1 + \varepsilon)x_0$. Let $y \geq y_0$, and let $x = y/(1 + \varepsilon)$. Let z_0 be such that $|V \cap B(z_0, x)| \geq \sigma\pi x^2$. Let $A' = z_0 + xA = B(z_0, x) \setminus B(z_0, \varepsilon x)$. Then $|V \cap A'| < \sigma\pi x^2$, so there is a point v_0 in $V \cap B(z_0, \varepsilon x)$. But now

$$\Delta(G(V, y)) + 1 \geq |V \cap B(v_0, y)| \geq |V \cap B(z_0, x)| \geq \sigma\pi x^2 = \left(\frac{1}{1 + \varepsilon}\right)^2 \sigma\pi y^2. \quad \square$$

Lemma 7. *Let the set V of points in the plane have finite upper density σ , and let $\varepsilon > 0$. Then there exists r_0 such that for all $r \geq r_0$, there exists a ball B_r of radius r such that for all points $w \in B_r$ and all $\varepsilon r \leq s \leq r/2$,*

$$|V \cap B_r \cap B(w, s)| \geq (1 - \varepsilon)\sigma \text{ area}(B_r \cap B(w, s)).$$

Proof. We may assume that $\sigma > 0$. Let $0 < \eta \leq \varepsilon^3/3$. Let r_0 be as in Lemma 5, and let $r \geq r_0$. Let B_r be a ball of radius r with $|V \cap B_r| \geq \sigma\pi r^2$. Let $w \in B_r$ and let $s \geq \varepsilon r$.

$$\begin{aligned} |V \cap B_r \cap B(w, s)| &= |V \cap B_r| - |V \cap (B_r \setminus B(w, s))| \\ &\geq \sigma\pi r^2 - (1 + \eta)\sigma(\text{area}(B_r) - \text{area}(B_r \cap B(w, s))) \\ &\geq \sigma \text{ area}(B_r \cap B(w, s)) - \eta\sigma\pi r^2. \end{aligned}$$

But

$$\text{area}(B_r \cap B(w, s)) \geq \frac{1}{3}\pi s^2 \geq \frac{1}{3}\pi \varepsilon^2 r^2.$$

Hence $\eta\pi r^2 \leq \varepsilon \text{ area}(B_r \cap B(w, s))$, and the result follows. \square

Lemma 8. *Let the non-empty set V of points in the plane have finite upper density σ . Then $\delta^*(G(V, d))/d^2 \rightarrow \sigma\pi/2$ as $d \rightarrow \infty$.*

Proof. Again let B denote the unit ball $B(O, 1)$. Let $0 < \varepsilon < 1$. Let B' denote the set of points $(x, y) \in B$ with $x > 0$. Then $\delta^*(G(V, d))$ is at most the number of points of V in some translate of dB' ; and hence $\delta^*(G(V, d)) \leq (\sigma + \varepsilon)\pi d^2/2$ for d sufficiently large.

Now we prove a corresponding lower bound for $\delta^*(G(V, d))$. Note first that there exists $0 < \rho \leq \frac{1}{2}$ such that for all points $w \in B$, the area of $B \cap B(w, \rho)$ is at least $(1 - \varepsilon)\pi\rho^2/2$. Consider the value r_0 from Lemma 7 corresponding to $\varepsilon' = \min(\varepsilon, \rho)$. Let $s = d$ be sufficiently large that $r = d/\rho \geq r_0$; let B_r be a ball as in that lemma; and let H be the subgraph of $G(V, d)$ induced by $V' = V \cap B_r$, that is, H is $G(V', d)$. Consider any point $v \in V'$. The degree of v in H plus 1 equals $|V' \cap B(v, d)|$. But by Lemma 7, this is at least

$$\begin{aligned} (1 - \varepsilon)\sigma \text{ area}(B_r \cap B(v, d)) &\geq (1 - \varepsilon)\sigma r^2 (1 - \varepsilon)\pi\rho^2/2 \\ &= (1 - \varepsilon)^2\sigma\pi d^2/2. \quad \square \end{aligned}$$

We shall need the two-dimensional case of the Bieberbach inequality (1915), see for example [4, p. 65] or [14, (6.2.6), p. 318], which is as follows.

Lemma 9. *Any convex set in the plane with diameter at most 1 has area at most $\pi/4$.*

Lemma 10. *Let the non-empty set V of points in the plane have finite upper density σ . Then $\omega(G(V, d))/d^2 \geq \sigma\pi/4$ for any $d > 0$, and $\omega(G(V, d))/d^2 \rightarrow \sigma\pi/4$ as $d \rightarrow \infty$.*

Proof. The lower bound is easy, since

$$\omega(G(V, d)) \geq \sigma^+(V, (d/2)B)\pi(d/2)^2 \geq (\sigma\pi/4)d^2.$$

Here B denotes the unit ball $B(O, 1)$.

Now consider an upper bound. Let S denote the unit square $\{(x, y): 0 \leq x, y < 1\}$. Let $\varepsilon > 0$. For $x > 0$ let us denote $\sigma^+(V, xS)$ by σ_x . There is an $x > 0$ such that $\sigma_x < \sigma + \varepsilon/2$. Fix a partition of the plane into a grid of aligned $x \times x$ squares which are copies of xS . Let d be sufficiently large that $\sigma_x(\hat{d}/d)^2 < \sigma + \varepsilon$, where $\hat{d} = d + 2\sqrt{2}x$.

Let K be any clique in $G(V, d)$, and let C be the convex hull of the points of V in K . Then C has diameter less than d . Any grid square meeting C is contained in $\hat{C} = C + \hat{B}$, where \hat{B} is the closed ball centered at the origin with radius $\sqrt{2}x$. Now \hat{C} has diameter less than \hat{d} , and so by Lemma 9, \hat{C} has area less than $(\pi/4)\hat{d}^2$. Thus, the number of grid squares meeting C is less than $(\pi/4)\hat{d}^2/x^2$. Hence

$$|K|/d^2 = |V \cap C|/d^2 < (\pi/4)(\hat{d}/d)^2 \sigma_x^+ < (\pi/4)(\sigma + \varepsilon).$$

It follows that $\omega(G(V, d))/d^2 < (\pi/4)\sigma + \varepsilon$. \square

Given two sets A and B of points in the plane, and $w > 0$, we say that a function $\phi: A \rightarrow B$ is w -wobbling if the Euclidean distance $d(a, \phi(a)) \leq w$ for each $a \in A$. We shall use the observation that if there is a w -wobbling injection from A into B , then for any $d > 0$, $\chi(G(A, d)) \leq \chi(G(B, d + 2w))$.

Lemma 11. *Let the non-empty set V of points in the plane have finite upper density σ , and let $\varepsilon > 0$. Then*

$$\chi(G(V, d))/d^2 \leq (\sigma + \varepsilon)\sqrt{3}/2$$

for d sufficiently large.

Proof. Recall that T denotes the set of lattice points of the triangular lattice with unit edge lengths. Let $\gamma = ((\sigma + \varepsilon/2)\sqrt{3}/2)^{1/2}$. Let $T' = \gamma^{-1}T$, so that T' is T scaled so that the density is $(\sigma + \varepsilon/2)$.

Let S denote the half-open unit square $S = \{(x, y): 0 \leq x, y < 1\}$. For any $x > 0$ sufficiently large, every translate of the $(x \times x)$ square xS contains at least $(\sigma + \varepsilon/4)x^2$ points of T' , and contains at most this number of points of V . Partition the plane into a grid of $x \times x$ squares as above. For each grid square X there is a w -wobbling injection from $V \cap X$ into $T' \cap X$, where $w = \sqrt{2}x$. We may patch together these injections, to give a w -wobbling injection $\phi: V \rightarrow T'$.

Hence, by Theorem 3,

$$\begin{aligned} \chi(G(V, d)) &\leq \chi(G(T', d + 2w)) \\ &= \chi(G(T, \gamma(d + 2w))) \\ &< (\gamma(d + 2w) + 1)^2 \\ &< (\sigma + \varepsilon)\sqrt{3}/2 d^2 \end{aligned}$$

if d is sufficiently large. \square

Lemma 12. *Let the discrete set V of points in the plane have infinite upper density. Then $\omega(G(V, d))$ is infinite for any $d > 0$.*

Proof. As we noted in the proof of Lemma 10,

$$\omega(G(V, d)) \geq \sigma^+(V, (d/2)B)\pi(d/2)^2,$$

where B denotes the unit ball; and now the right hand side is infinite. \square

Proof of Theorem 1. This now follows from Lemmas 2, 6, 8, 10, 11 and 12. \square

2.4. Proofs for sets with a cell structure

Proof of (1) in Theorem 2. Consider first the lower bound on $\delta(G(V, d))$. Let $w \in V$. The number of points $v \in V$ such that the cell C_v meets the ball $B(w, d - r)$ is at least $\sigma\pi(r - d)^2$, and for each such point v we have $d(v, w) < r + (d - r) = d$. Hence w has degree at least $\sigma\pi(r - d)^2 - 1$ in $G(V, d)$.

Now consider the upper bound on $\Delta(G(V, d))$. Let $w \in V$. If $d(v, w) < d$ then $C_v \subseteq B(w, d + r)$, and the number of such points (including w) is at most $\sigma\pi(r + d)^2$. Hence w has degree at most $\sigma\pi(r + d)^2 - 1$. \square

Proof of (2) in Theorem 2. The lower bound follows from Theorem 1. To prove the upper bound, consider a fixed cell structure for V with density σ and radius r , and let K be any clique in $G(V, d)$. Let C be the convex hull of the points in K , and let $\hat{C} = B(C, r)$. Then \hat{C} has diameter at most $d + 2r$. Hence, by Lemma 9, \hat{C} has area at most $(\pi/4)(d + 2r)^2$. But all the cells corresponding to the points in K are contained in \hat{C} , and so

$$|K| \leq (\pi/4)(d + 2r)^2 \sigma.$$

The upper bound on $\omega(G(V, d))$ follows. \square

Let A and B be sets of points in the plane, and let $w > 0$. If there are w -wobbling injections from A to B and from B to A , then by the Mendelsohn-Dulmage theorem — see for example [11] — there is a w -wobbling bijection between A and B . In this case let us call the sets A and B w -close.

Lemma 13. *Let the sets A and B have cell structures with the same density σ and with radius r_A, r_B respectively. Then A and B are $(r_A + r_B)$ -close.*

Proof. Fix appropriate cell structures $(C_a: a \in A)$ and $(D_b: b \in B)$ for A and B respectively. Let B' be a copy of B disjoint from A . Form a bipartite graph $G = (A, B', E)$, by letting vertices $a \in A$ and $b \in B'$ be adjacent whenever C_a and D_b meet. Note that if a and b are adjacent then $d(a, b) < r_A + r_B = w$ say.

Let \hat{A} be a finite subset of A , and let \hat{B} be the set of points $b \in B'$ which are adjacent in G to some point in \hat{A} . Let C be the union of the cells C_a for $a \in \hat{A}$, and note that \hat{B} is the set of points $b \in B'$ such that D_b meets C . But since C has area $|\hat{A}|/\sigma$, clearly $|\hat{B}| \geq |\hat{A}|$. Hence, by Hall's theorem applied to the locally finite graph G , there is a matching in G covering all of \hat{A} . Thus there is a w -wobbling injection from A to B . In exactly the same way we see that there is a w -wobbling injection from B to A , and so there is w -wobbling bijection as required. \square

Proof of (3) in Theorem 2. The lower bound follows directly from Lemma 2, so let us consider the upper bound. Recall that the set T of lattice points of the triangular lattice with unit edge lengths has a cell structure with density $2/\sqrt{3}$, and radius $1/\sqrt{3}$. Now $(\frac{3}{4})^{1/4} \sigma^{1/2} V$ has the same density $2/\sqrt{3}$, and has radius $(\frac{3}{4})^{1/4} \sigma^{1/2} r$. Hence, by Lemma 13, $(\frac{3}{4})^{1/4} \sigma^{1/2} V$ and T are w -close, where $w = 1/\sqrt{3} + (\frac{3}{4})^{1/4} \sigma^{1/2} r$. It follows that, for any $d > 0$,

$$\chi(G(V, d)) = \chi\left(G\left(\left(\frac{3}{4}\right)^{1/4} \sigma^{1/2} V, \left(\frac{3}{4}\right)^{1/4} \sigma^{1/2} d\right)\right) \leq \chi(G(T, D)),$$

where

$$D = \left(\frac{3}{4}\right)^{1/4} \sigma^{1/2} d + 2w = \left(\frac{3}{4}\right)^{1/4} \sigma^{1/2} (d + 2r) + \frac{2}{\sqrt{3}}.$$

Hence

$$\chi(G(V, d)) \leq \chi(G(T, D)) = \hat{D}^2 < (D + 1)^2. \quad \square$$

Let us call two sets in the plane *close* if for some finite w they are w -close, that is there is a w -wobbling bijection between them. It follows from Lemma 13 above that a set of points in the plane has a cell structure with density 1 if and only if it is close to the set Z^2 of lattice points. On the way to solving Tarski's circle-squaring problem, Laczkovicz [7] gave a characterization of those sets which are close to Z^2 in terms of 'discrepancy'. For a discrete set S and a bounded measurable set H , the *discrepancy* of S with respect to H is

$$\Delta(S; H) = | |S \cap H| - \lambda_2(H) |,$$

where λ_2 denotes Lebesgue measure on the plane. The set S is *uniformly spread* if there are positive constants C and a such that for every Jordan domain A with perimeter $p(A) \geq a$ we have $\Delta(S; A) \leq Cp(A)$. Then S is uniformly spread if and only if it is close to Z^2 .

3. Concluding remarks

We have told a fairly full story about the basic problem of colouring proximity graphs in the plane, in the asymptotic case when the distance d tends to ∞ . We have considered general sets of points in the plane; sets which have a cell structure or are close to lattices where the error term is better; generalised lattices where strict tilings give similar good approximations; and the triangular lattice (with hexagonal cells) where we have exact results. There are many areas for further investigation, including the following.

- To what extent do the asymptotic results yield insights into finite cases with interesting values of the parameters? When is it possible to improve the various error terms, in particular the $o(d^2)$ error term for colouring in general?
- To what extent do the results extend when there are more general conditions on what assignments are allowable, in particular when there are 'frequency-distance constraints'? Initial investigations on this question appear in [10].
- The Euclidean plane seems to be the natural setting for most problems arising in channel assignment (perhaps not in Manhattan). But, to what extent do the results above carry over for different norms (we briefly considered different distance measures for the triangular lattice) and for different dimensions?

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