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Wasserstein stability estimates for covariance-preconditioned Fokker–Planck equations

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Abstract

We study the convergence to equilibrium of the mean field PDE associated with the derivative-free methodologies for solving inverse problems that are presented by Garbuno-Inigo *et al* (2020 *SIAM J. Appl. Dyn. Syst.* **19** 412–41), Herty and Visconti (2018 arXiv:1811.09387). We show stability estimates in the Euclidean Wasserstein distance for the mean field PDE by using optimal transport arguments. As a consequence, this recovers the convergence towards equilibrium estimates by Garbuno-Inigo *et al* (2020 *SIAM J. Appl. Dyn. Syst.* **19** 412–41) in the case of a linear forward model.

Keywords: mean-field Fokker–Planck equation, ensemble Kalman inversion, Wasserstein stability estimates

Mathematics Subject Classification numbers: 35Q70, 35Q84, 62F15, 65C35.

1. Introduction

In this paper, we are concerned with the nonlocal Fokker–Planck equation

$$\frac{\partial f}{\partial t}(\mathbf{u}, t) = \nabla \cdot \left(\mathcal{C}(f_t) (\nabla \Phi_R(\mathbf{u}; \mathbf{y}) f(\mathbf{u}, t) + \sigma \nabla f(\mathbf{u}, t)) \right), \quad \mathbf{u} \in \mathbf{R}^d, t \in \mathbf{R}_{\geq 0}, \quad (1.1)$$

where $\sigma \geq 0$, $f_t = f(\bullet, t)$, \mathcal{C} is the covariance operator defined by

$$\mathcal{C}(f) = \int_{\mathbf{R}^d} (\mathbf{u} - \mathcal{M}(f)) \otimes (\mathbf{u} - \mathcal{M}(f)) f(\mathbf{u}) d\mathbf{u}, \quad \text{with} \quad \mathcal{M}(f) = \int_{\mathbf{R}^d} \mathbf{u} f(\mathbf{u}) d\mathbf{u},$$

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and $\Phi_R(\bullet; \mathbf{y})$ is a functional of the form

$$\Phi_R(\mathbf{u}; \mathbf{y}) = \frac{1}{2}|\mathbf{y} - \mathcal{G}(\mathbf{u})|_{\Gamma}^2 + \frac{1}{2}|\mathbf{u}|_{\Gamma_0}^2 =: \Phi(\mathbf{u}, \mathbf{y}) + \frac{1}{2}|\mathbf{u}|_{\Gamma_0}^2. \quad (1.2)$$

Here $\mathcal{G} : \mathbf{R}^d \rightarrow \mathbf{R}^K$ is the so-called *forward model*, in view of the link with Bayesian inverse problems, $\mathbf{y} \in \mathbf{R}^d$ is a given vector of *observations* and $\Gamma \in \mathbf{R}^{K \times K}, \Gamma_0 \in \mathbf{R}^{d \times d}$ are symmetric, positive definite matrices. We employed the notation $|\bullet|_{\Gamma} := |\Gamma^{-\frac{1}{2}}\bullet|$, where $|\bullet|$ is the usual Euclidean norm.

Throughout this paper, we restrict our attention to the case where \mathcal{G} is a linear mapping and we write $\mathcal{G}(\mathbf{u}) = G\mathbf{u}$, with $G \in \mathbf{R}^{K \times d}$. We will assume that the matrix $\Gamma_0^{-1} + G^T \Gamma^{-1} G =: B^{-1}$ is nonsingular, so that the regularised least squares misfit Φ_R , given by (1.2), admits the unique minimiser $\mathbf{u}_0 = BG^T \Gamma^{-1} \mathbf{y}$. Our main result is that, if f^1 and f^2 are the solutions of (1.1) associated with the initial conditions f_0^1 and f_0^2 , respectively, then a stability estimate of the following form holds:

$$W_2(f_t^1, f_t^2) \leq C(f_0^1, f_0^2; G, \Gamma, \Gamma_0) \gamma(t; \sigma) W_2(f_0^1, f_0^2), \quad (1.3)$$

where $C(\bullet_1, \bullet_2; G, \Gamma, \Gamma_0)$ depends only on the first two moments of \bullet_1 and \bullet_2 and the function $\gamma(t; \sigma)$ converges to zero as $t \rightarrow \infty$ exponentially when $\sigma > 0$ and algebraically when $\sigma = 0$. Here and in the rest of the paper, we employed the notation $f_t^i = f^i(\bullet, t)$, $i = 1, 2$. If $\sigma > 0$, then by taking one solution in (1.3) to be the Gaussian equilibrium $f_{\infty} = \frac{1}{Z} e^{-\Phi_R/\sigma}$, with Z the normalisation constant, one recovers the equilibration estimate obtained in [15], but with a sharper rate of convergence. An important feature of our result is that the convergence rate, encapsulated in the function $\gamma(\bullet; \sigma)$, depends only on σ and not on the parameters, notably the Hessian, of the quadratic least-squares functional Φ_R . Roughly speaking, the reason for this universality of the convergence rate comes from the fact that the covariance preconditioner tends to accelerate equilibration in directions along which the posterior distribution has a large variance [or, equivalently, in directions associated with a small eigenvalue of $\text{Hess}(\Phi_R)$], and these are the directions which, in the absence of the covariance preconditioner, are limiting for the convergence rate. For a more precise statement on the optimality of the decay rate, we refer to proposition 3.8 and remark 3.7.

As a byproduct of our analysis, we deduce the algebraic convergence of the solution towards a Dirac delta at \mathbf{u}_0 when $\sigma = 0$, i.e. to the solution of the Bayesian inverse problem, generalising to the mean field PDE the estimates obtained for a related particle system in [21] and answering fully the equilibration open problem discussed in [18] for the linear forward model.

We now turn our attention to the connection of the PDE (1.1) to mean field descriptions of the ensemble Kalman methods for the Bayesian inverse problem. The Fokker–Planck equation (1.1) can be linked to the inverse problem of finding $\mathbf{u} \in \mathbf{R}^d$ from an *observation* $\mathbf{y} \in \mathbf{R}^K$ where

$$\mathbf{y} = \mathcal{G}(\mathbf{u}) + \boldsymbol{\eta}. \quad (1.4)$$

Here $\boldsymbol{\eta}$ is a random variable assumed to have Lebesgue density ρ . In the Bayesian approach to inverse problems [9, 12], a probability measure called the *prior* is placed on \mathbf{u} . If we assume that this measure also has a density ρ_0 and that \mathbf{u} is independent of $\boldsymbol{\eta}$, then (\mathbf{u}, \mathbf{y}) is a random variable with density $\rho(\mathbf{y} - \mathcal{G}(\mathbf{u})) \rho_0(\mathbf{u})$. The posterior density of $\mathbf{u}|\mathbf{y}$ (i.e. of \mathbf{u} given an observation \mathbf{y}) is then given by the normalised probability density

$$\rho^{\mathbf{y}}(\mathbf{u}) = \frac{\rho(\mathbf{y} - \mathcal{G}(\mathbf{u})) \rho_0(\mathbf{u})}{\int_{\mathbf{R}^d} \rho(\mathbf{y} - \mathcal{G}(\mathbf{u})) \rho_0(\mathbf{u}) d\mathbf{u}}. \quad (1.5)$$

In the particular case where ρ and ρ_0 are the densities of Gaussians $\mathcal{N}(0, \Gamma)$ and $\mathcal{N}(0, \Gamma_0)$, respectively, $\rho^y \propto e^{-\Phi_R(u; y)}$, where Φ_R is given by (1.2). We make this assumption below.

In [19], the authors proposed to solve the inverse problem (1.4) by applying a state-estimation method, or filter, to the following artificial dynamics on $\mathbf{R}^d \times \mathbf{R}^K$ and associated observational model, where we denote by \mathbf{u} the first d components of \mathbf{z} :

$$\mathbf{z}_{n+1} = \Xi(\mathbf{z}_n), \quad \Xi(\mathbf{z}) = \begin{pmatrix} \mathbf{u} \\ \mathcal{G}(\mathbf{u}) \end{pmatrix}, \quad \mathbf{y}_{n+1} = \begin{pmatrix} 0 & I \end{pmatrix} \mathbf{z}_{n+1} + \boldsymbol{\eta}_{n+1},$$

where $\{\boldsymbol{\eta}_n\}_{n \in \mathbf{N}}$ are i.i.d. $\mathcal{N}(0, h^{-1}\Gamma)$ random variables. If the observed data in the dynamics is fixed at the observation of the Bayesian inverse problem \mathbf{y} for all steps, then the \mathbf{u} -marginal of the posterior distribution at iteration n has density

$$\rho_n(\mathbf{u}) \propto \exp(-nh\Phi(\mathbf{u}; \mathbf{y})) \rho_0(\mathbf{u}),$$

which can be obtained by repeatedly applying the reasoning that led to (1.5). This iteration leads to a concentration of the mass of ρ_n at minimisers of the (non-regularized) least squares functional Φ in the limit as $n \rightarrow \infty$, and we remark that the posterior ρ_n coincides with the posterior ρ^y of the inverse problem when $nh = 1$, a fact that can be exploited to produce approximate samples of the posterior ρ^y [11].

If the prior ρ_0 is Gaussian and the forward model \mathcal{G} is linear, then the posteriors $\{\rho_n\}_{n \in \mathbf{N}}$ can in principle be captured exactly by a Kalman filter. However, when the dimension of the state space is large, which is often the case in scientific and engineering applications, the Kalman filter is computationally expensive and a particle-based method such as the ensemble Kalman filter (EnKF) becomes preferable. This approach is also more general than the Kalman filter, because it does not require that the forward model be linear. The ensemble members $U = \{\mathbf{u}^{(j)}\}_{j=1}^J$ of EnKF are evolved according to equation (4) in [21]:

$$\mathbf{u}_{n+1}^{(j)} = \mathbf{u}_n^{(j)} + hC^{\text{up}}(U_n)(hC^{\text{pp}}(U_n) + \Gamma)^{-1} \left(\mathbf{y}_{n+1}^{(j)} - \mathcal{G}(\mathbf{u}_n^{(j)}) \right), \quad j = 1, \dots, J, \quad (1.6)$$

where C^{uu} (used later), C^{up} and C^{pp} are given by

$$\begin{aligned} C^{\text{uu}}(U) &= \frac{1}{J} \sum_{j=1}^J (\mathbf{u}^{(j)} - \bar{\mathbf{u}}) \otimes (\mathbf{u}^{(j)} - \bar{\mathbf{u}}), \\ C^{\text{up}}(U) &= \frac{1}{J} \sum_{j=1}^J (\mathbf{u}^{(j)} - \bar{\mathbf{u}}) \otimes (\mathcal{G}(\mathbf{u}^{(j)}) - \bar{\mathcal{G}}), \\ C^{\text{pp}}(U) &= \frac{1}{J} \sum_{j=1}^J (\mathcal{G}(\mathbf{u}^{(j)}) - \bar{\mathcal{G}}) \otimes (\mathcal{G}(\mathbf{u}^{(j)}) - \bar{\mathcal{G}}), \\ \bar{\mathbf{u}} &= \frac{1}{J} \sum_{j=1}^J \mathbf{u}^{(j)}, \quad \bar{\mathcal{G}} = \frac{1}{J} \sum_{j=1}^J \mathcal{G}(\mathbf{u}^{(j)}), \end{aligned}$$

and $\mathbf{y}_n^{(j)} = \mathbf{y} + \boldsymbol{\eta}_n^{(j)}$, where $\{\boldsymbol{\eta}_n^{(j)}\}$ are i.i.d. vectors with $\boldsymbol{\eta}_1^{(1)} \sim \mathcal{N}(0, h^{-1}\Sigma)$. Traditionally, the distribution of the noise employed to perturb the simulated observations $\{\mathcal{G}(\mathbf{u}_n^{(j)})\}$ in the EnKF coincides with that of the noise in the observational model, which suggests taking $\Sigma = \Gamma$. It was shown in [21], however, that taking $\Sigma = 0$ also produces an efficient method for solving

inverse problems. Furthermore, the authors noticed that, when taking the limit $h \rightarrow 0$, (1.6) is a tamed Euler–Maruyama-type discretisation of the stochastic differential equation (SDE)

$$\dot{\mathbf{u}}^{(j)} = \frac{1}{J} \sum_{k=1}^J \langle \mathcal{G}(\mathbf{u}^{(k)}) - \bar{\mathcal{G}}\mathbf{y} - \mathcal{G}(\mathbf{u}^{(j)}) + \sqrt{\Sigma} \dot{\mathbf{W}}^{(j)} \rangle_{\Gamma} (\mathbf{u}^{(k)} - \bar{\mathbf{u}}), \quad j = 1, \dots, J, \quad (1.7)$$

where $\mathbf{W}^{(j)}$, $j = 1, \dots, J$, are standard independent Brownian motions. They carried out a thorough analysis of this continuous-time dynamics in the particular case where the forward model \mathcal{G} is linear and $\Sigma = 0$. Equation (1.7) can now be viewed as a derivative-free approach to inverse problems, which was recently referred in [15] as the ensemble Kalman inversion method.

More recently, in [15], a modification of (1.7) with $\Sigma = 0$ was suggested to enable sampling from the posterior distribution over an infinite time horizon; the modified dynamics read

$$\dot{\mathbf{u}}^{(j)} = \frac{1}{J} \sum_{k=1}^J \langle \mathcal{G}(\mathbf{u}^{(k)}) - \bar{\mathcal{G}}\mathbf{y} - \mathcal{G}(\mathbf{u}^{(j)}) \rangle_{\Gamma} (\mathbf{u}^{(k)} - \bar{\mathbf{u}}) - C^{\text{uu}}(U) \Gamma_0^{-1} \mathbf{u}^{(j)} + \sqrt{2C^{\text{uu}}(U)} \dot{\mathbf{W}}^{(j)}, \quad (1.8)$$

for $j = 1, \dots, J$. The second term in the right-hand side is included so as to take the prior information into account. The idea of including the covariance matrix $C^{\text{uu}}(U)$ in that term, as well as in the noise, is motivated by the fact, in the case of linear forward model, (1.8) can equivalently be written as

$$\dot{\mathbf{u}}^{(j)} = -C^{\text{uu}}(U) \nabla \Phi_{\text{R}}(\mathbf{u}^{(j)}) + \sqrt{2C^{\text{uu}}(U)} \dot{\mathbf{W}}^{(j)}, \quad j = 1, \dots, J, \quad (1.9)$$

which is expected to produce approximate samples of the posterior ρ^y for large J . Indeed, the formal mean field limit of this interacting particle system is given by the law of the process defined by the McKean-type SDE

$$\dot{\mathbf{u}} = -C(f_t) \nabla \Phi_{\text{R}}(\mathbf{u}) + \sqrt{2C(f_t)} \dot{\mathbf{W}}, \quad f_t := \text{Law}(\mathbf{u}_t), \quad (1.10)$$

which clearly admits $\frac{1}{Z} e^{-\Phi_{\text{R}}}$ as an invariant measure, where Z is the normalisation constant. The associated Fokker–Planck equation for f , which was derived formally in [15] and rigorously in [13], is precisely (1.1) with $\sigma = 1$. Two remarks are in order. First, we note that a concentration of the particles at any point of \mathbf{R}^d is a stationary solution of the dynamics (1.8) and, likewise, any Dirac delta is a stationary solution of (1.10). Second, as recently noted in [20], the J -particle distribution $(\frac{1}{Z})^J \prod_{j=1}^J e^{-\Phi_{\text{R}}(\mathbf{u}^{(j)}; y)}$ is not invariant under the dynamics (1.9).

The strategy of the proof of the stability estimate (1.3) is the following: we first realise that the moments up to second order of the equation (1.1) are governed by a closed system of ODEs. This is a common feature appearing in some of the simplest cases of homogeneous kinetic equations, such as the Fokker–Planck operator preserving the first two moments of the distribution function [22], the Maxwellian molecules case for the Landau–Fokker–Planck equation [23], and the Boltzmann equation for Maxwellian molecules; see [8, 10] and the references therein. Then, we focus on finding stability estimates for solutions that have the same covariance matrix, which is simpler because the nonlinearity of the problem does not show up and we are reduced to a kind of linear Fokker–Planck equation. Then we obtain the stability estimate for any two solutions, regardless of the values of their first two moments, by using optimal transport techniques. The strategy of our proofs follows that employed in similar results for the Boltzmann equation in the Maxwellian case as in [4–6, 10].

The paper is organised as follows. In section 2, we summarise known results and we present some equilibration estimates for the first and second moments of the solution to (1.1). In section 3, we give a simple proof of the stability estimate (1.3) in Euclidean Wasserstein distance based on analytical techniques in optimal transport.

2. Preliminaries

We remind the reader that the forward model $\mathcal{G} = G$ is assumed to be linear throughout the paper, and we recall the following result, proved in [15].

Proposition 2.1 (Closed system of ordinary differential for the first and second moments). Assume f_t is a solution of (1.1), and let $C(t) := \mathcal{C}(f_t)$ and $\delta(t) := \mathcal{M}(f_t) - \mathbf{u}_0$, where $\mathcal{M}(f_t)$ denotes the first moment of f_t . The evolution of $C(t)$ and $\delta(t)$ is governed by the system:

$$\dot{\delta}(t) = -C(t)B^{-1}\delta(t), \quad \left(\dot{\bullet} := \frac{d}{dt}\bullet \right) \quad (2.1a)$$

$$\dot{C}(t) = -2C(t)B^{-1}C(t) + 2\sigma C(t). \quad (2.1b)$$

Proof. We show this only in the case $\sigma = 0$, for simplicity. Multiplying (1.1) by \mathbf{u} , integrating over \mathbf{R}^d , and using the notation $\mathbf{m}(t) = \mathcal{M}(f_t)$, we obtain

$$\begin{aligned} \dot{\mathbf{m}}(t) &= -C(t)\nabla\Phi_{\mathbf{R}}(\mathbf{m}(t), \mathbf{y}) = -C(t)(G^T\Gamma^{-1}(G\mathbf{m}(t) - \mathbf{y}) + \Gamma_0^{-1}\mathbf{m}(t)) \\ &= -C(t)B^{-1}(\mathbf{m}(t) - \mathbf{u}_0), \end{aligned}$$

leading to (2.1a). Similarly, multiplying (1.1) by $(\mathbf{u} - \mathbf{m}(t)) \otimes (\mathbf{u} - \mathbf{m}(t))$ and noticing that

$$\int_{\mathbf{R}^d} \frac{\partial}{\partial t} ((\mathbf{u} - \mathbf{m}(t)) \otimes (\mathbf{u} - \mathbf{m}(t))) f(\mathbf{u}, t) d\mathbf{u} = \int_{\mathbf{R}^d} (\mathbf{u} - \mathbf{m}(t)) \otimes (\mathbf{u} - \mathbf{m}(t)) \frac{\partial f}{\partial t}(\mathbf{u}, t) d\mathbf{u},$$

we obtain an equation for the covariance matrix. Omitting the dependence of C and \mathbf{m} on t for convenience:

$$\begin{aligned} \frac{d}{dt}C_{ij}(t) &= - \int_{\mathbf{R}^d} C : (\nabla((u_i - m_i)(u_j - m_j)) \otimes \nabla\Phi_{\mathbf{R}}(\mathbf{u}, \mathbf{y})) f(\mathbf{u}, t) d\mathbf{u}, \\ &= - \sum_{k,\ell} \int_{\mathbf{R}^d} C_{k\ell} (\delta_{ki}(u_j - m_j) + \delta_{kj}(u_i - m_i)) (B^{-1}(\mathbf{u} - \mathbf{u}_0))_{\ell} f(\mathbf{u}, t) d\mathbf{u}. \end{aligned}$$

Since the term in the first round brackets in the integral is mean-zero with respect to $f(\mathbf{u}, t)$, we can remove and add constants in the other factor:

$$\begin{aligned} \frac{d}{dt}C_{ij}(t) &= - \sum_{k,\ell} \int_{\mathbf{R}^d} C_{k\ell} (\delta_{ki}(u_j - m_j) + \delta_{kj}(u_i - m_i)) (B^{-1}(\mathbf{u} - \mathbf{m}))_{\ell} f(\mathbf{u}, t) d\mathbf{u}, \\ &= - \sum_{\ell,p} \int_{\mathbf{R}^d} (C_{i\ell}(u_j - m_j) + C_{j\ell}(u_i - m_i)) B_{\ell p}^{-1}(u_p - m_p) f(\mathbf{u}, t) d\mathbf{u}, \\ &= - \sum_{\ell,p} (C_{i\ell}C_{jp} + C_{j\ell}C_{ip}) B_{\ell p}^{-1} = -2 \sum_{\ell,p} C_{i\ell}C_{jp}B_{\ell p}^{-1}, \end{aligned}$$

which, in matrix form, gives (2.1b). \square

If we assume that $C_0 := C(f_0)$ is positive definite, then the solution of (2.1b) reads

$$C(t) = \begin{cases} \left(\frac{1 - e^{-2\sigma t}}{\sigma} B^{-1} + e^{-2\sigma t} C_0^{-1} \right)^{-1} & \text{if } \sigma > 0, \\ (2B^{-1}t + C_0^{-1})^{-1} & \text{if } \sigma = 0. \end{cases} \quad (2.2)$$

We notice that the solution in the case $\sigma = 0$ is the pointwise limit as $\sigma \rightarrow 0$ of that when $\sigma > 0$. For a given solution $C(t)$ of (2.1b), we will denote by $U(s, t; C)$ the fundamental matrix associated with (2.1a); this matrix solves

$$\forall s \in \mathbf{R}, t \geq s: \quad \partial_t U(s, t; C) = -C(t)B^{-1}U(s, t; C), \quad U(s, s; C) = I. \quad (2.3)$$

Lemma 2.2 (Bound for the fundamental matrix). *Let $C(t)$ be a solution of (2.1b) with initial condition $C(0)$. The matrix $U(s, t) := U(s, t; C)$ satisfies*

$$|U(s, t)|_2 \leq e^{-\sigma(t-s)} \sqrt{\frac{\alpha(s)}{\alpha(t)}} \sqrt{\max(|C(0)|_2, |B|_2)} \sqrt{\max(|C(0)^{-1}|_2, |B^{-1}|_2)}, \quad (2.4)$$

where

$$\alpha(t) = \begin{cases} 2t + 1 & \text{if } \sigma = 0, \\ \frac{1}{\sigma}(1 - e^{-2\sigma t}) + e^{-2\sigma t} & \text{if } \sigma > 0. \end{cases} \quad (2.5)$$

Proof. We notice that

$$\frac{d}{dt}(U(s, t)^T C(t)^{-1} U(s, t)) = -2\sigma(U(s, t)^T C(t)^{-1} U(s, t)),$$

which implies

$$\left(C(t)^{-1/2} U(s, t) \right)^T \left(C(t)^{-1/2} U(s, t) \right) = U(s, t)^T C(t)^{-1} U(s, t) = e^{-2\sigma(t-s)} C(s)^{-1}. \quad (2.6)$$

Let us denote the polar decomposition of $C(t)^{-1/2} U(s, t)$ by $Q(s, t)S(s, t)$, for some orthogonal matrix $Q(s, t)$ and some symmetric matrix $S(s, t)$. Substituting this decomposition in (2.6), we obtain $S(s, t) = e^{-\sigma(t-s)} C(s)^{-1/2}$ and so $U(s, t) = e^{-\sigma(t-s)} C(t)^{1/2} Q(s, t) C(s)^{-1/2}$. In particular,

$$|U(s, t)|_2 \leq e^{-\sigma(t-s)} \sqrt{|C(t)|_2 |C(s)^{-1}|_2}.$$

Rewriting $C(t)$ in a way that exhibits a convex combinations of B^{-1} and $C(0)^{-1}$,

$$C(t) = \frac{1}{\alpha(t)} \left((1 - \beta(t)) B^{-1} + \beta(t) C(0)^{-1} \right)^{-1}, \quad \beta(t) = \frac{e^{-2\sigma t}}{\alpha(t)}, \quad (2.7)$$

we deduce (2.4). \square

In the sequel, $\alpha(t)$ denotes the same function as in lemma 2.2, and we employ the notations $|\bullet|_F := \sum_{ij} \bullet_{ij}^2$ and $|\bullet|_2$ to denote the Frobenius matrix norm and the operator norm induced by the Euclidean vector norm in \mathbf{R}^d , respectively.

Lemma 2.3 (Convergence of the first and second moments). *We consider two solutions $C_1(t)$, $C_2(t)$ of (2.1b) and the corresponding solutions $\delta_1(t)$, $\delta_2(t)$ of (2.1a), and we assume that*

$$|C_1(0)|_2 \vee |C_2(0)|_2 \vee |B|_2 \leq M, \quad (2.8a)$$

$$|C_1(0)^{-1}|_2 \vee |C_2(0)^{-1}|_2 \vee |B^{-1}|_2 \leq m, \quad (2.8b)$$

$$|\delta_1(0)|_2 \vee |\delta_2(0)|_2 \leq R. \quad (2.8c)$$

Then it holds that

$$|C_1(t) - C_2(t)|_F \leq M^2 m^2 |C_1(0) - C_2(0)|_F \frac{e^{-2\sigma t}}{\alpha(t)^2}, \quad (2.9a)$$

$$|\delta_1(t) - \delta_2(t)|_2 \leq \left(\sqrt{mM} |\delta_1(0) - \delta_2(0)|_2 + \frac{1}{2} m^4 M^3 R |C_2(0) - C_1(0)|_F \right) \frac{e^{-\sigma t}}{\sqrt{\alpha(t)}}, \quad (2.9b)$$

Proof. By a sub-multiplicative property of the Frobenius norm,

$$|C_1(t) - C_2(t)|_F = |C_1(t)|_2 |C_1(t)^{-1} - C_2(t)^{-1}|_F |C_2(t)|_2,$$

We observe $C_1(t)^{-1} - C_2(t)^{-1} = e^{-2\sigma t}(C_1(0)^{-1} - C_2(0)^{-1})$ so, using the sub-multiplicative property of the norm again,

$$|C_1(t) - C_2(t)|_F = |C_1(t)|_2 |C_1(0)^{-1}|_2 |C_2(0) - C_1(0)|_F |C_2(0)^{-1}|_2 |C_2(t)|_2 e^{-2\sigma t}. \quad (2.10)$$

Since $\frac{1}{\alpha(t)} C_i(t)^{-1}$ is a convex combination of $C_i(0)^{-1}$ and B^{-1} ,

$$|C_i(t)|_2 \leq \frac{1}{\alpha(t)} \max(|C_i(0)|_2, |B|_2), \quad i = 1, 2,$$

leading to (2.9a).

For the first moments, we have

$$\frac{d}{dt}(\delta_1(t) - \delta_2(t)) = -C_1(t)B^{-1}(\delta_1(t) - \delta_2(t)) - (C_2(t) - C_1(t))B^{-1}\delta_2(t).$$

By the variation-of-constants formula, and with the shorthand notation $U_i(s, t) := U(s, t; C_i)$, we deduce that

$$\delta_1(t) - \delta_2(t) = -U_1(s, t)(\delta_1(s) - \delta_2(s)) - \int_s^t U_1(u, t)(C_2(u) - C_1(u))B^{-1}\delta_2(u) du.$$

Employing (2.4) and (2.9a), and using the fact that $\delta_2(u) = U_2(s, u)\delta_2(s)$, we obtain

$$\begin{aligned} |\delta_1(t) - \delta_2(t)|_2 &\leq \sqrt{mM} \sqrt{\frac{\alpha(s)}{\alpha(t)}} e^{-\sigma(t-s)} |\delta_1(s) - \delta_2(s)|_2 \\ &\quad + m^3 M^3 |\delta_2(s)| \sqrt{\frac{\alpha(s)}{\alpha(t)}} e^{-\sigma(t-s)} |C_2(0) - C_1(0)|_F \int_s^t \frac{e^{-2\sigma u}}{\alpha(u)^2} |B^{-1}|_2 du. \end{aligned} \quad (2.11)$$

We calculate that

$$\begin{aligned} \forall \sigma \neq 0, 1, \quad I(s, t) &:= \int_s^t \frac{e^{-2\sigma u}}{\alpha(u)^2} du = \frac{1}{2(\sigma - 1)} \left(\frac{1}{\alpha(t)} - \frac{1}{\alpha(s)} \right) = \frac{e^{-2\sigma s} - e^{-2\sigma t}}{2\sigma \alpha(s) \alpha(t)}, \\ &\leq \lim_{t \rightarrow \infty} I(s, t) = \frac{e^{-2\sigma s} - e^{-2\sigma \infty}}{2\sigma \alpha(s) \alpha(\infty)} = \frac{e^{-2\sigma s}}{2\alpha(s)}. \end{aligned} \quad (2.12)$$

(This calculation fails for $\sigma = 0$ and $\sigma = 1$, but it is easy to check that the conclusion holds for any $\sigma \geq 0$.) This leads to (2.9b) after taking $s = 0$ (the case $s > 0$ will be useful in remark 2.1 below) and rearranging. \square

We note that, in the case $\sigma = 0$, equation (2.9a) cannot be employed with $C_2(0) \rightarrow 0$ in order to deduce the rate of convergence of $C_1(t)$ to 0, because the bound m in the assumptions grows to $+\infty$ as $C_2(0) \rightarrow 0$. It can, however, be employed (setting $\delta_2(0) = 0$ and $C_2(0) = C_1(0)$) to deduce that $\delta_1(t)$ converges to zero with rate $e^{-\sigma t}/\sqrt{\alpha(t)}$, which is consistent with (2.4).

Remark 2.1. Since $\delta_i(t) = U_i(s, t)\delta_i(s)$, for $i = 1, 2$, by definition of $U_i(s, t)$, it follows from (2.11) that

$$\forall s \leq t, \quad |U_2(s, t) - U_1(s, t)|_2 \leq m^4 M^3 |C_2(0) - C_1(0)|_F \frac{e^{-\sigma(s+t)}}{\sqrt{\alpha(s)\alpha(t)}}, \quad (2.13)$$

where the constants m and M are defined as before.

In the rest of this paper, we denote by $g(\bullet; \mu, \Sigma)$ the density of the Gaussian $\mathcal{N}(\mu, \Sigma)$.

Lemma 2.4 (Propagation of Gaussians for the linear equation). *Let $C(t)$ be the solution of*

$$\dot{C}(t) = -2C(t)B^{-1}C(t) + 2\sigma C(t), \quad C(0) = C_0,$$

for a given matrix C_0 . Then the solution of the linear Fokker–Planck equation

$$\frac{\partial f}{\partial t}(\mathbf{u}, t) = \nabla \cdot (C(t)B^{-1}(\mathbf{u} - \mathbf{u}_0)f(\mathbf{u}, t)) + \sigma \nabla \cdot (C(t)\nabla f(\mathbf{u}, t)), \quad (2.14a)$$

$$f(\mathbf{u}, 0) = g(\mathbf{u}; \mu_0, \Sigma_0), \quad (2.14b)$$

is given by the Gaussian density $f(\mathbf{u}, t) = g(\mathbf{u}; \mu(t), \Sigma(t))$ where

$$\mu(t) = \mathbf{u}_0 + U(0, t)(\mu_0 - \mathbf{u}_0), \quad (2.15a)$$

$$\Sigma(t) = U(0, t)\Sigma_0 U(0, t)^T + 2\sigma \int_0^t U(s, t)C(s)U(s, t)^T ds. \quad (2.15b)$$

Here $U(\bullet, \bullet) := U(\bullet, \bullet; C)$ is given by (2.3). If $\Sigma_0 = 0$, then the matrix $\Sigma(t)$ admits the following explicit expression in terms of $C(t)$:

$$\Sigma(t) = (1 - e^{-2\sigma t})C(t). \quad (2.16)$$

Proof. Proceeding as in proposition 2.1, we deduce that the first and second moments of any solution to (2.14a), which we denote μ and Σ , satisfy

$$\dot{\mu}(t) = -C(t)B^{-1}(\mu(t) - \mathbf{u}_0), \quad (2.17a)$$

$$\dot{\Sigma}(t) = -C(t)B^{-1}\Sigma(t) - \Sigma(t)B^{-1}C(t) + 2\sigma C(t). \quad (2.17b)$$

We then verify, proceeding similarly to [2, 14], that the Gaussian ansatz

$$f(\mathbf{u}, t) = \frac{1}{(2\pi)^{d/2} \sqrt{\det \Sigma(t)}} e^{-\frac{1}{2}(\mathbf{u} - \mu(t))^T \Sigma(t)^{-1}(\mathbf{u} - \mu(t))}$$

is indeed a solution. Omitting the dependence of C , μ and Σ on t for notational convenience, we calculate that the left-hand side of (2.14a) reads

$$\frac{\text{LHS}}{f(\mathbf{u}, t)} = \dot{\mu}^T \Sigma^{-1}(\mathbf{u} - \mu) + \frac{1}{2}(\mathbf{u} - \mu)^T \Sigma^{-1} \dot{\Sigma} \Sigma^{-1}(\mathbf{u} - \mu) + \frac{1}{2} \frac{1}{\det \Sigma} \frac{d}{dt} (\det \Sigma)$$

and the right hand is

$$\frac{\text{RHS}}{f(\mathbf{u}, t)} = -(\mathbf{u} - \mathbf{u}_0)^T B^{-1} C \Sigma^{-1}(\mathbf{u} - \mu) + \sigma(\mathbf{u} - \mu) \Sigma^{-1} C \Sigma^{-1}(\mathbf{u} - \mu) + \text{tr}(CB^{-1} - \sigma C \Sigma^{-1}).$$

Both sides of the equation are quadratic polynomials in \mathbf{u} . Equating the Hessians w.r.t. \mathbf{u} of the coefficients of both sides, and multiplying left and right by Σ , we obtain (2.17b). Taking this equation into account and equating the gradients, we obtain (2.17a). It remains to check that the constant terms (w.r.t. \mathbf{u}) coincide, which can be seen from the fact that $\text{tr}(CB^{-1} - \sigma C \Sigma^{-1}) = -\frac{1}{2} \text{tr}(\dot{\Sigma} \Sigma^{-1})$, by (2.17b), and the formula for the derivative of the determinant function:

$$0 = \frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2} \sqrt{\det \Sigma}} e^{-\frac{1}{2} \mathbf{u}^T \Sigma^{-1} \mathbf{u}} d\mathbf{u} = \frac{1}{2} \left(\text{tr}(\dot{\Sigma} \Sigma^{-1}) - \frac{1}{\det \Sigma} \frac{d}{dt} (\det \Sigma) \right).$$

For general initial conditions μ_0 and Σ_0 , we can check that the solution to the system of equation (2.17) is given by (2.15a) and (2.15b). Equation (2.16) can be checked by simple substitution in (2.17b). \square

Remark 2.2. We remark that, for $\sigma > 0$, $U(t-s, t) \rightarrow e^{-\sigma s} I$ as $t \rightarrow \infty$ because $C(t) \rightarrow \sigma B$. Therefore, employing lemmas 2.2 and 2.4 and dominated convergence, we deduce

$$\begin{aligned} \lim_{t \rightarrow \infty} \Sigma(t) &= 2\sigma \lim_{t \rightarrow \infty} \int_0^t U(s, t) C(s) U(s, t)^T ds, \\ &= 2\sigma \lim_{t \rightarrow \infty} \int_0^t U(t-s, t) C(t-s) U(t-s, t)^T ds, \\ &= 2\sigma \lim_{t \rightarrow \infty} \int_0^\infty U(t-s, t) C(t-s) U(t-s, t)^T \mathbf{1}_{\{\bullet \leq t\}}(s) ds, \\ &= 2\sigma \lim_{t \rightarrow \infty} \int_0^\infty e^{-\sigma s} \sigma B e^{-\sigma s} ds = \sigma B \left(= \lim_{t \rightarrow \infty} C(t) \right), \end{aligned}$$

which holds for any initial condition Σ_0 .

Remark 2.3. A byproduct of lemma 2.4 is that the mean field equation (1.1) too propagates Gaussians when the forward model G is linear, as was proved in [15, proposition 4].

3. Stability in Wasserstein distance

The aim of this section is to derive a stability property for both the linear Fokker–Planck equation (2.14a) (where $C(t)$ is a given parameter) and the nonlocal mean field equation (1.1) (where $\mathcal{C}(f_t)$ depends on the solution), which we undertake in sections 3.1 and section 3.2, respectively.

3.1. Stability for the linear Fokker–Planck equation (2.14a)

Throughout this subsection we consider that $C(t)$ is a given solution of (2.1b) and $U(0, t) = U(0, t; C)$. For some probability measure f over \mathbf{R}^d and a mapping $T : \mathbf{R}^d \rightarrow \mathbf{R}^d$, we will denote the pushforward measure by $T_{\#}f$. We remind the reader that, if f admits a density \hat{f} with respect to the Lebesgue measure and $A \in \mathbf{R}^{d \times d}$ is a nonsingular matrix, then $A_{\#}f$ (identifying A with the associated linear mapping) has density $\frac{1}{\det(A)} \hat{f}(A^{-1} \bullet)$. We show the following result.

Proposition 3.1 (Convergence of solutions when the covariance is given). *Let f^1 and f^2 be two solutions of (2.14a) associated with initial conditions f_0^1 and f_0^2 , respectively. Then*

$$W_2(f_t^1, f_t^2) \leq \sqrt{|U(0, t)^T U(0, t)|_2} W_2(f_0^1, f_0^2). \quad (3.1)$$

Under the same assumptions as in lemma 2.3, it therefore holds, in view of (2.4),

$$W_2(f_t^1, f_t^2) \leq \sqrt{mM} \left(\frac{e^{-\sigma t}}{\sqrt{\alpha(t)}} \right) W_2(f_0^1, f_0^2).$$

To prove proposition 3.1 we will need the following lemma.

Lemma 3.2 (Influence of linear transformations on the Wasserstein distance).

Let $A \in \mathbf{R}^{d \times d}$ be nonsingular and let us consider two probability measures with finite second moment, $f, g \in \mathcal{P}_2(\mathbf{R}^d)$. Then also $A_{\#}f, A_{\#}g \in \mathcal{P}_2(\mathbf{R}^d)$ and

$$W_2(A_{\#}f, A_{\#}g) \leq \sqrt{|A^T A|_2} W_2(f, g).$$

Proof. Let γ_o be an optimal transference plan (by [10, proposition 2.1], the infimum in the definition of the Wasserstein distance is achieved) such that

$$\int \int_{\mathbf{R}^d \times \mathbf{R}^d} |x - y|^2 \gamma_o(dx \, dy) = W_2(f, g)^2,$$

and consider the map $r : (x, y) \mapsto (Ax, Ay)$. The pushforward plan $r_{\#}\gamma_o$ has the correct marginals: looking for example at the x marginal, we calculate that for all $\varphi \in C_b(\mathbf{R}^d)$,

$$\begin{aligned} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} \varphi(x) r_{\#}\gamma_o(dx \, dy) &= \int \int_{\mathbf{R}^d \times \mathbf{R}^d} \varphi(Ax) \gamma_o(dx \, dy) \\ &= \int_{\mathbf{R}^d} \varphi(Ax) f(dx) = \int_{\mathbf{R}^d} \varphi(x) A_{\#}f(dx). \end{aligned}$$

Furthermore, by a change of variable,

$$\begin{aligned} \int \int_{\mathbf{R}^d \times \mathbf{R}^d} |x - y|^2 r_{\#}\gamma_o(dx \, dy) &= \int \int_{\mathbf{R}^d \times \mathbf{R}^d} |Ax - Ay|^2 \gamma_o(dx \, dy) \\ &\leq |A^T A|_2 \int \int_{\mathbf{R}^d \times \mathbf{R}^d} |x - y|^2 \gamma_o(dx \, dy) \\ &= |A^T A|_2 W_2(f, g)^2. \end{aligned}$$

We notice that orthogonal transformations do not influence the Wasserstein distance. \square

Proof of theorem 3.1. Let us denote by $\zeta(\mathbf{u}, t; \mathbf{v})$ the fundamental solution provided by lemma 2.4. By linearity, the solution of (2.14a) associated with initial condition f_0 can be expressed as follows:

$$\begin{aligned} f(\mathbf{u}, t) &= \int_{\mathbf{R}^d} f_0(\mathbf{v}) \zeta(\mathbf{u}, t; \mathbf{v}) d\mathbf{v} \\ &= \int_{\mathbf{R}^d} f_0(\mathbf{v}) g(\mathbf{u}; \mathbf{u}_0 + U(0, t)(\mathbf{v} - \mathbf{u}_0), \Sigma(t)) d\mathbf{v}. \end{aligned}$$

By the change of variables $\mathbf{v} \mapsto U(0, t)(\mathbf{v} - \mathbf{u}_0) =: \mathbf{w}(\mathbf{v})$, we can rewrite this integral as

$$\begin{aligned} f(\mathbf{u}, t) &= \int_{\mathbf{R}^d} \frac{f_0(U(0, t)^{-1}\mathbf{w} + \mathbf{u}_0)}{\det(U(0, t))} g(\mathbf{u}; \mathbf{u}_0 + \mathbf{w}, \Sigma(t)) d\mathbf{w} \\ &= \left(\frac{f_0(U(0, t)^{-1}\bullet + \mathbf{u}_0)}{\det(U(0, t))} \star g(\bullet; \mathbf{u}_0, \Sigma(t)) \right)(\mathbf{u}). \end{aligned} \quad (3.2)$$

By the convexity property of the Wasserstein distance [10, proposition 2.1], its invariance under translation and lemma 3.2, we obtain

$$\begin{aligned} W_2(f_t^1, f_t^2) &\leq W_2\left(\frac{f_0^1(U(0, t)^{-1}\mathbf{w} + \mathbf{u}_0)}{\det(U(0, t))}, \frac{f_0^2(U(0, t)^{-1}\mathbf{w} + \mathbf{u}_0)}{\det(U(0, t))}\right) \\ &\leq \sqrt{|U(0, t)^T U(0, t)|_2} W_2(f_0^1, f_0^2), \end{aligned}$$

which is the desired inequality. \square

Remark 3.1. Proposition 3.1 can also be proved via a purely probabilistic approach, employing the approach presented e.g. in [7, 24]. Indeed a solution of (2.14a) with initial condition f_0 can be viewed, by Itô's formula, as the law of the process $(X_t)_{t \geq 0}$ that solves the stochastic SDE

$$dX_t = -C(t)B^{-1}(X_t - \mathbf{u}_0)dt + \sqrt{2\sigma C(t)}dW_t, \quad X_0 \sim f_0,$$

where W is a standard Wiener process on \mathbf{R}^d . Considering two solutions X_t and Y_t associated with the initial conditions $X_0 \sim f_0^1$ and $Y_0 \sim f_0^2$ (and with the same Wiener process), we calculate

$$dX_t - dY_t = -C(t)B^{-1}(X_t - Y_t)dt,$$

and therefore

$$X_t - Y_t = U(0, t)(X_0 - Y_0),$$

which implies

$$|X_t - Y_t|^2 \leq |U(0, t)^T U(0, t)|_2 |X_0 - Y_0|^2. \quad (3.3)$$

Recalling that the Wasserstein distance can equivalently be defined as

$$W_2(\rho_1, \rho_2) = \left(\inf_{X, Y} \mathbf{E}|X - Y|^2 \right)^{1/2},$$

where the infimum is over all \mathbf{X} and \mathbf{Y} with laws ρ_1 and ρ_2 , respectively, and taking the expectation of both sides of (3.3), we obtain

$$W_2(f_t^1, f_t^2) \leq |U(0, t)^T U(0, t)|_2 \mathbf{E} |\mathbf{X}_0 - \mathbf{Y}_0|^2.$$

Infimizing over all \mathbf{X}_0 and \mathbf{Y}_0 with laws f_0^1 and f_0^2 , respectively, we obtain precisely (3.1).

We remark that the first moment of f^1 and f^2 need not coincide for proposition 3.1 to hold.

3.2. Stability for the nonlocal mean field equation

To prepare the terrain for the derivation of our main result, we begin by showing a stability property on the set of Gaussian solutions. To this end, we will employ the following bound for the distance between the square root of the covariant matrices associated with two solutions.

Lemma 3.3 (Convergence of the square root of the covariance matrix). *Under the assumptions of lemma 2.3, it holds that*

$$\left| C_1(t)^{1/2} - C_2(t)^{1/2} \right|_F \leq C_R M m \left| C_1(0)^{1/2} - C_2(0)^{1/2} \right|_F \frac{e^{-\sigma t}}{\alpha(t)}, \quad (3.4)$$

where C_R is a constant that depends only on the dimension of the problem.

Proof. We restrict ourselves in the proof to the case $\sigma > 0$ for simplicity. Employing the same reasoning as in the first part of the proof of lemma 2.3, we write

$$\left| C_1(t)^{1/2} - C_2(t)^{1/2} \right|_F = \left| C_1(t)^{1/2} \right|_2 \left| C_1(t)^{-1/2} - C_2(t)^{-1/2} \right|_F \left| C_2(t)^{1/2} \right|_2.$$

The middle term can be written as

$$\left| C_1(t)^{-1/2} - C_2(t)^{-1/2} \right|_F = \left| (M + M_1)^{1/2} - (M + M_2)^{1/2} \right|_F,$$

where $M = (1 - e^{-2\sigma t})B^{-1}/\sigma$ and $M_i = e^{-2\sigma t}C_i(0)^{-1}$, for $i = 1, 2$. Therefore, using the technical bound presented in lemma A.1 in the appendix A,

$$\begin{aligned} \left| C_1(t)^{-1/2} - C_2(t)^{-1/2} \right|_F &\leq C_R \left| M_1^{1/2} - M_2^{1/2} \right|_F = C_R e^{-\sigma t} \left| C_1(0)^{-1/2} - C_2(0)^{-1/2} \right|_F \\ &\leq C_R m e^{-\sigma t} \left| C_1(0)^{1/2} - C_2(0)^{1/2} \right|_F, \end{aligned}$$

which leads to (3.4) after employing the convex decomposition (2.7) to bound $\left| C_i^{1/2} \right|_2$. \square

The Wasserstein distance between two Gaussian measures admits an explicit expression, which we recall in the following lemma.

Lemma 3.4 (Wasserstein distance between Gaussians). *Consider two Gaussians probability measures $\mathcal{N}(\boldsymbol{\mu}_1, \Sigma_1)$ and $\mathcal{N}(\boldsymbol{\mu}_2, \Sigma_2)$ on \mathbf{R}^d . The Wasserstein distance between them is given by*

$$|W_2(\mathcal{N}(\boldsymbol{\mu}_1, \Sigma_2), \mathcal{N}(\boldsymbol{\mu}_2, \Sigma_2))|^2 = |\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2|^2 + \text{tr} \left(\Sigma_1 + \Sigma_2 - 2(\Sigma_1^{1/2} \Sigma_2 \Sigma_1^{1/2})^{1/2} \right). \quad (3.5)$$

Proof. Equation (3.5) is proved in [17], but we will include a sketch of the proof in the simpler case where $\Sigma_1, \Sigma_2 \succ 0$ (the proof of the general case requires an additional step) for the reader's convenience and because we will employ the intermediate inequality (3.6) below in

remark 3.7. We will see that, by taking an appropriate singular value decomposition, the proof presented in the aforementioned paper can be slightly simplified. The key idea is to notice that the covariance matrix of the optimal transference plan (a probability measure on $\mathbf{R}^d \times \mathbf{R}^d$) must have the form

$$\Sigma = \begin{pmatrix} \Sigma_1 & X \\ X^T & \Sigma_2 \end{pmatrix},$$

and that the Wasserstein distance is given by $|\mu_1 - \mu_2|^2 + \text{tr}(\Sigma_1 + \Sigma_2 - 2X)$. Using Schur's complement, and denoting the squared Wasserstein distance on the left-hand side of (3.5) by W^2 for short, we deduce

$$W^2 \geq |\mu_2 - \mu_1|^2 + \min_X \text{tr}(\Sigma_1 + \Sigma_2 - 2X) \quad \text{subject to} \quad \Sigma_2 - X^T \Sigma_1^{-1} X \succeq 0.$$

(The infimum is attained because the admissible set is compact.) By polar decomposition of $\Sigma_1^{-1/2}X$, it is possible to write $X = \Sigma_1^{1/2}QS^{1/2}$, for an orthogonal matrix Q and a symmetric positive-semidefinite matrix $S^{1/2}$. Since Q does not appear in the constraint, and since $\text{tr}(X) = \text{tr}(QS^{1/2}\Sigma_1^{1/2}) = \text{tr}(QV_1DV_2^T) = \text{tr}(V_2^TQV_1D)$, where $V_1^TDV_2$ is the singular value decomposition of $S^{1/2}\Sigma_1^{1/2}$, is clearly maximised when $V_2^TQV_1 = I$ with maximal value $\text{tr}(D)$, we deduce

$$W^2 \geq |\mu_2 - \mu_1|^2 + \text{tr}(\Sigma_1 + \Sigma_2) - 2 \max_S \text{tr} \left((\Sigma_1^{1/2}S\Sigma_1^{1/2})^{1/2} \right) \quad \text{subject to} \quad \Sigma_2 - S \succeq 0,$$

where the maximum is taken over all symmetric positive-semidefinite matrices. Here we employed that $\text{tr}(D) = \text{tr}((V_2^TD^2V_2)^{1/2}) = \text{tr}((\Sigma_1^{1/2}S\Sigma_1^{1/2})^{1/2})$. Since the matrix square root is monotonous over the cone of positive semi-definite matrices, and since clearly $\Sigma_1^{1/2}S\Sigma_1^{1/2} \preceq \Sigma_1^{1/2}\Sigma_2\Sigma_1^{1/2}$ on the set of admissible S (that is, congruence preserves the order \preceq), we conclude that the optimum is reached when $S = \Sigma_2$, which leads to

$$W^2 \geq |\mu_2 - \mu_1|^2 + \text{tr} \left(\Sigma_1 + \Sigma_2 - 2(\Sigma_1^{1/2}\Sigma_2\Sigma_1^{1/2})^{1/2} \right). \quad (3.6)$$

Considering the following transportation map,

$$T : x \mapsto \mu_2 + \Sigma_1^{-1/2}(\Sigma_1^{1/2}\Sigma_2\Sigma_1^{1/2})^{1/2}\Sigma_1^{-1/2}(x - \mu_1),$$

we notice that the lower bound is in fact attained for Gaussian densities. Indeed, it is simple to check that $T_{\#}\mathcal{N}(\mu_1, \Sigma_1) = \mathcal{N}(\mu_2, \Sigma_2)$ and, by a change of variable,

$$\begin{aligned} \int |x - Tx|^2 g_{\mu_1, \Sigma_1}(x) dx &= \int \left| \mu_1 - \mu_2 + x - \Sigma_1^{-1/2}(\Sigma_1^{1/2}\Sigma_2\Sigma_1^{1/2})^{1/2}\Sigma_1^{-1/2}x \right|^2 g_{\mathbf{0}, \Sigma_1}(x) dx, \\ &= |\mu_1 - \mu_2|^2 + \text{tr}(\Sigma_1) + \text{tr}(\Sigma_2) - 2\text{tr} \left((\Sigma_1^{1/2}\Sigma_2\Sigma_1^{1/2})^{1/2} \right), \end{aligned}$$

where we employed the notation $g_{\mu, \Sigma} = g(\bullet, \mu, \Sigma)$ for short. \square

Lemma 3.5 (Bounds on the Wasserstein distance between Gaussians). *Consider two Gaussians $\mathcal{N}(\mu_1, \Sigma_1)$ and $\mathcal{N}(\mu_2, \Sigma_2)$. Denoting the Wasserstein distance between them by W for convenience, it holds*

$$\frac{1}{2} \left| \Sigma_1^{1/2} - \Sigma_2^{1/2} \right|_F^2 \leq W^2 - |\mu_2 - \mu_1|^2 \leq \left| \Sigma_1^{1/2} - \Sigma_2^{1/2} \right|_F^2. \quad (3.7)$$

Proof. The first inequality in (3.7) can be rewritten as

$$\mathrm{tr} \left(\Sigma_1^{1/2} \Sigma_2 \Sigma_1^{1/2} \right)^{1/2} \leq \frac{1}{4} \mathrm{tr} \left(\Sigma_1 + \Sigma_2 + 2 \Sigma_1^{1/2} \Sigma_2^{1/2} \right) = \frac{1}{4} \left| \Sigma_1^{1/2} + \Sigma_2^{1/2} \right|_F^2$$

or, equivalently,

$$\left| \Sigma_1^{1/2} \Sigma_2^{1/2} \right|_{s_1} = \sum_j s_j (\Sigma_1^{1/2} \Sigma_2^{1/2}) \leq \frac{1}{4} s_j \left((\Sigma_1^{1/2} + \Sigma_2^{1/2})^2 \right) = \frac{1}{4} \left| (\Sigma_1^{1/2} + \Sigma_2^{1/2})^2 \right|_{s_1},$$

where $s_j(\bullet)$ is the j th singular value and $|\bullet|_{s_1}$ denotes the Schatten matrix norm with $p = 1$, defined as the sum of the (all positive) singular values of its argument. This inequality follows from the general arithmetic mean/geometric mean inequality, valid for any unitarily invariant matrix norm and any positive matrices in place of $\Sigma_1^{1/2}$ and $\Sigma_2^{1/2}$, that is the subject of [3]. To obtain the second inequality in (3.7), we employ the standard Araki–Lieb–Thirring inequality with $r = 1/2$ and $q = 1$,

$$\mathrm{tr} \left((\Sigma_1^{1/2} \Sigma_2 \Sigma_1^{1/2})^{1/2} \right) \geq \mathrm{tr} \left(\Sigma_1^{1/4} \Sigma_2^{1/2} \Sigma_1^{1/4} \right) = \mathrm{tr} \left(\Sigma_1^{1/2} \Sigma_2^{1/2} \right),$$

which concludes the proof. \square

Remark 3.2. Equation (3.6) is in fact true for any probability measures with positive-definite covariance matrices, as Gaussianity had not entered the proof at that point. It is possible to show that this inequality holds also for degenerate covariant matrices, see [16, theorem 2.1]. Consequently, the first inequality in (3.7) is true for non-Gaussian probability measures too.

Remark 3.3. It is in fact possible to recover the second inequality in the bound (3.7) without having recourse to the Araki–Lieb–Thirring inequality, by simply using the (nonsymmetric) transportation map $x \mapsto \mu_2 + \Sigma_2^{1/2} \Sigma_1^{-1/2} (x - \mu_1)$ to obtain an upper-bound for the Wasserstein distance.

Proposition 3.6. Let f^1 and f^2 be two Gaussian solutions of (1.1), associated with (Gaussian) initial conditions f_0^1 and f_0^2 , respectively. Under the assumptions of lemma 2.3, it holds that

$$W_2(f_t^1, f_t^2) \leq C(1 + mM + m^4 M^{7/2} R) \frac{e^{-\sigma t}}{\sqrt{\alpha(t)^{1+[1 \wedge \sigma]}}} W_2(f_0^1, f_0^2), \quad (3.8)$$

where C is a constant that depends only on the dimension d and $\alpha(t)$ is given by (2.5).

Proof. Combining the moment bounds (3.4) and (2.9b) with (3.7), and denoting the Wasserstein distance on the left-hand side of (3.8) by W for short, we obtain

$$\begin{aligned} W^2 &\leq \left| C_1(t)^{1/2} - C_2(t)^{1/2} \right|_F^2 + |\delta_2(t) - \delta_1(t)|^2 \\ &\leq C_R^2 M^2 m^2 \left| C_1(0)^{1/2} - C_2(0)^{1/2} \right|_F^2 \left(\frac{e^{-2\sigma t}}{\alpha(t)^2} \right) \\ &\quad + \left(2mM |\delta_1(0) - \delta_2(0)|_2^2 + m^8 M^6 R^2 |C_2(0) - C_1(0)|_F^2 \right) \left(\frac{e^{-2\sigma t}}{\alpha(t)} \right) \\ &\leq \left(2(C_R m^2 M^2 + mM) W_2(f_0^1, f_0^2)^2 + m^8 M^6 R^2 |C_2(0) - C_1(0)|_F^2 \right) \frac{e^{-2\sigma t}}{\alpha(t) \wedge \alpha(t)^2}. \end{aligned}$$

Employing lemma A.2, which generalises the inequality

$$\forall a, b \geq 0: \quad |a - b| = |\sqrt{a} + \sqrt{b}| |\sqrt{a} - \sqrt{b}| \leq 2 \max(\sqrt{a}, \sqrt{b}) |\sqrt{a} - \sqrt{b}|$$

to symmetric positive semi-definite matrices, and using (3.7) again, we finally obtain

$$W_2(f_t^1, f_t^2)^2 \leq (2C_R m^2 M^2 + 2mM + C m^8 M^7 R^2) W_2(f_0^1, f_0^2)^2 \frac{e^{-2\sigma t}}{\alpha(t) \wedge \alpha(t)^2},$$

which leads to our claim. \square

To prove a more general stability result, we will combine the ideas of propositions 3.1 and 3.6. Additionally, we will need the following lemma.

Lemma 3.7 (Wasserstein distance between linearly transformed densities). *Let $A, B \in \mathbf{R}^{d \times d}$ be nonsingular, possibly nonsymmetric matrices, and let f be a probability measure with finite second moment, $f \in \mathcal{P}_2(\mathbf{R}^d)$. Then it holds that*

$$W_2(A_{\#}f, B_{\#}f) \leq |A - B|_2 \sqrt{\text{tr}(\mathcal{C}(f)) + |\mathcal{M}(f)|^2}, \quad (3.9)$$

where $\mathcal{M}(f)$ and $\mathcal{C}(f)$ are the first and second moments of f :

$$\mathcal{M}(f) = \int x f(\mathrm{d}x), \quad \mathcal{C}(f) = \int (x - \mathcal{M}(f)) \otimes (x - \mathcal{M}(f)) f(\mathrm{d}x).$$

Proof. Let us consider the transference plan $\gamma = (A \times B)_{\#}f$, which clearly has the required marginals. (Here $A \times B$ is the operator $x \mapsto (Ax, Bx)$.) We calculate, by a change of variable,

$$\int \int_{\mathbf{R}^d \times \mathbf{R}^d} |x - y|^2 \gamma(\mathrm{d}x \, \mathrm{d}y) = \int_{\mathbf{R}^d} |Ax - Bx|^2 f(\mathrm{d}x) \leq \int_{\mathbf{R}^d} |A - B|_2^2 |x|^2 f(\mathrm{d}x),$$

which directly leads to the conclusion. \square

Proposition 3.8. *Let f^1 and f^2 be two solutions of the nonlinear nonlocal mean field equation (1.1) with linear forward model G . Under the assumptions of lemma 2.3, it holds that*

$$W_2(f_t^1, f_t^2) \leq C(1 + m^4 M^4 + m^4 M^{7/2} R) \frac{e^{-\sigma t}}{\sqrt{\alpha(t)}^{1+[1 \wedge \sigma]}} W_2(f_0^1, f_0^2), \quad (3.10)$$

where $\alpha(t)$ is given by (2.5).

Proof. Let us denote the fundamental matrices associated with the two solutions by $U_i(s, t)$, $i = 1, 2$. Our starting point will be (3.2), rewritten in a such a way that the Gaussian densities are centered at zero:

$$f^i(\mathbf{u}, t) = \int_{\mathbf{R}^d} \frac{f_0^i(U_i(0, t)^{-1}(\mathbf{w} + \mathbf{u} - \mathbf{u}_0) + \mathbf{u}_0)}{\det(U_i(0, t))} g(\mathbf{w}; 0, \Sigma_i(t)) \mathrm{d}\mathbf{w}, \quad i = 1, 2. \quad (3.11)$$

Introducing new functions $\hat{f}^i(\mathbf{u}, t) := f^i(\mathbf{u} + \mathbf{u}_0, t)$ and $\hat{f}_0^i(\mathbf{u}) = f_0^i(\mathbf{u} + \mathbf{u}_0)$ for convenience, we obtain the simpler expression

$$\hat{f}^i(\mathbf{u}, t) = \int_{\mathbf{R}^d} \frac{\hat{f}_0^i(U_i(0, t)^{-1}(\mathbf{w} + \mathbf{u}))}{\det(U_i(0, t))} g(\mathbf{w}; 0, \Sigma_i(t)) \mathrm{d}\mathbf{w}, \quad i = 1, 2.$$

Since the Wasserstein distance is invariant under translation of its arguments, it holds that

$$W_2(f_t^1, f_t^2) = W_2(\hat{f}_t^1, \hat{f}_t^2), \quad W_2(f_0^1, f_0^2) = W_2(\hat{f}_0^1, \hat{f}_0^2).$$

In other words, we can assume without loss of generality that $\mathbf{u}_0 = 0$. From here on, we will drop the hats in \hat{f}^i and \hat{f}_0^i for notational convenience. Let us now introduce

$$f^{1,2}(\mathbf{u}, t) = \int_{\mathbf{R}^d} \frac{f_0^1(U_1(0, t)^{-1}(\mathbf{w} + \mathbf{u}))}{\det(U_1(0, t))} g(\mathbf{w}; 0, \Sigma_2(t)) d\mathbf{w}.$$

Then, using the triangle inequality, we have

$$W_2(f_t^1, f_t^2) \leq W_2(f_t^1, f_t^{1,2}) + W_2(f_t^{1,2}, f_t^2).$$

Both terms can be simplified by using the convexity property of the Wasserstein metric, leading to the inequality

$$\begin{aligned} W_2(f_t^1, f_t^2) &\leq W_2(g(\bullet; 0, \Sigma_1(t)), g(\bullet; 0, \Sigma_2(t))) \\ &\quad + W_2\left(\frac{f_0^1(U_1(0, t)^{-1}\bullet)}{\det(U_1(0, t))}, \frac{f_0^2(U_2(0, t)^{-1}\bullet)}{\det(U_2(0, t))}\right). \end{aligned} \quad (3.12)$$

Using (2.16) and employing the triangle inequality again for the second term, we obtain

$$\begin{aligned} W_2(f_t^1, f_t^2) &\leq (1 - e^{-2\sigma t}) W_2(g(\bullet; 0, C_1(t)), g(\bullet; 0, C_2(t))) \\ &\quad + W_2\left(\frac{f_0^1(U_1(0, t)^{-1}\bullet)}{\det(U_1(0, t))}, \frac{f_0^1(U_2(0, t)^{-1}\bullet)}{\det(U_2(0, t))}\right) \\ &\quad + W_2\left(\frac{f_0^1(U_2(0, t)^{-1}\bullet)}{\det(U_2(0, t))}, \frac{f_0^2(U_2(0, t)^{-1}\bullet)}{\det(U_2(0, t))}\right). \end{aligned}$$

Employing (3.7) for the first term, lemma 3.7 for the second, and lemma 3.2 for the third, we deduce

$$\begin{aligned} W_2(f_t^1, f_t^2) &\leq (1 - e^{-2\sigma t}) |C_1(t)^{1/2} - C_2(t)^{1/2}|_{\mathbb{F}} \\ &\quad + |U_1(0, t) - U_2(0, t)|_2 \sqrt{\text{tr } C_1(0) + |\delta_1(0)|^2} \\ &\quad + |U_2(0, t) U_2(0, t)^T|_2 W_2(f_0^1, f_0^2). \end{aligned}$$

Employing (3.4) for the first term, (2.13) and (A.2) for the second, (2.4) for the third, and remark 3.2 to bound $|C_1(t)^{1/2} - C_2(t)^{1/2}|_{\mathbb{F}}$ from above by the Wasserstein distance, we finally obtain

$$W_2(f_t^1, f_t^2) \leq C \left(mM + m^4 M^4 + m^4 M^{7/2} R + \sqrt{mM} \right) \frac{e^{-\sigma t}}{\sqrt{\alpha(t)} \wedge \alpha(t)} W_2(f_0^1, f_0^2),$$

which concludes the proof. \square

Remark 3.4. In the case $\sigma = 0$, assuming without loss of generality that $\mathbf{u}_0 = 0$, we have the following simpler expression instead of (3.11):

$$f^i(\mathbf{u}, t) = \frac{f_0^i(U_i(0, t)^{-1}(\mathbf{u}))}{\det(U_i(0, t))}, \quad i = 1, 2,$$

so we directly obtain (3.12) without the first term on the right-hand side.

Remark 3.5. Proposition 3.8 can be proved with a probabilistic approach too, although with slightly different constants on the right-hand side. Since the probabilistic proof is very similar in spirit to the one given above, we will not present it here.

Remark 3.6. Strictly speaking, proposition 3.8 is not a generalisation of proposition 3.6 because the constant on the right-hand side of (3.10) contains the term $m^4 M^4$, which was not present in (3.8).

As mentioned in the introduction, when $\sigma > 0$ proposition 3.8 implies an equilibration estimate: taking f_0^2 to be the Gaussian equilibrium f_∞ in (3.10), we obtain

$$W_2(f_t^1, f_\infty) \leq C e^{-\sigma t} W_2(f_0^1, f_\infty), \quad (3.13)$$

for a constant depending only on σ and on the first and second moments of f_0^1 and f_∞ . In the case $\sigma = 0$, however, proposition 3.8 does not yield a rate of convergence to the equilibrium Dirac at \mathbf{u}_0 , because the constant prefactor in (3.10) diverges to $+\infty$ as $|C_1(0)|_2 \rightarrow 0$ or $|C_2(0)|_2 \rightarrow 0$. In that case, an estimate can be obtained more directly from the equation

$$W_2(f_t, \delta_{\mathbf{u}_0})^2 = \int_{\mathbf{R}^d} |\mathbf{u} - \mathbf{u}_0|^2 f_t(\mathbf{u}) d\mathbf{u} = \text{tr}(\mathcal{C}(f_t)) + |\mathcal{M}(f_t) - \mathbf{u}_0|^2.$$

Combining this with (2.2) and (2.4), we obtain the equilibration estimate

$$W_2(f_t, \delta_{\mathbf{u}_0})^2 \leq \frac{1}{\alpha(t)} (dM + mMR^2) = \frac{1}{2t+1} (dM + mMR^2). \quad (3.14)$$

Remark 3.7. Notice that, in contrast to [15, proposition 2], the rate of decay (3.13) does not depend on B , i.e., on the Hessian of Φ_R . More importantly, the rate of decay we obtain is sharp. In order to check this, note that by (2.6) the mean $\delta(t)$ satisfies

$$|\delta(t)|_{C(t)} = \sqrt{\delta(0)^T U(0, t)^T C(t)^{-1} U(0, t) \delta(0)} = e^{-\sigma t} \sqrt{\delta(0)^T C(0)^{-1} \delta(0)} = e^{-\sigma t} |\delta(0)|_{C(0)},$$

for any $\sigma \geq 0$. Therefore

$$|\delta(t)| \geq \sqrt{\frac{\lambda_{\min}(C(t))}{\lambda_{\max}(C(0))}} e^{-\sigma t} |\delta(0)|,$$

where $\lambda_{\min}(\bullet)$ and $\lambda_{\max}(\bullet)$ denote the minimum and maximum eigenvalues, respectively. By the assumptions (2.8a) and (2.8b) and by (2.7), it holds

$$\lambda_{\max}(C(0)) \leq M \quad \text{and} \quad \lambda_{\min}(C(t)) \geq \left(\frac{1}{m}\right) \frac{1}{\alpha(t)},$$

and thus,

$$|\delta(t)| \geq \frac{1}{\sqrt{mM}} \frac{e^{-\sigma t}}{\sqrt{\alpha(t)}} |\delta(0)|.$$

On the other hand, since the first inequality of (3.7) in lemma 3.5 holds for general probability measures, it holds

$$W_2(f_t, f_\infty) \geq |\delta(t)|$$

for any solution f_t of (1.1), with f_∞ being the Gaussian equilibrium. Consequently,

$$W_2(f_t, f_\infty) \geq \frac{1}{\sqrt{mM}} \frac{e^{-\sigma t}}{\sqrt{\alpha(t)}} |\delta(0)|.$$

This shows that (3.13) and (3.14) are optimal, in the sense that it is not possible to obtain a better rate of convergence with respect to time on the right-hand side of these equations.

Remark 3.8. Finally, we remark that, in the case $\sigma = 0$, the convergence to the equilibrium Dirac at u_0 could also have been derived using an approach similar to that in [15], but this does not provide an optimal bound. Defining

$$E(f) = \int_{\mathbb{R}^d} \Phi_R(u; y) f(u) du = \frac{1}{2} \int_{\mathbb{R}^d} (u - u_0)^T B^{-1} (u - u_0) f_t(u) du$$

we calculate

$$\begin{aligned} \frac{d}{dt} E(f_t) &= - \int_{\mathbb{R}^d} \nabla_u \Phi_R(u; y)^T \mathcal{C}(f_t) \nabla_u \Phi_R(u; y) f_t(u) du \\ &= - \int_{\mathbb{R}^d} (u - u_0)^T B^{-1} \mathcal{C}(f_t) B^{-1} (u - u_0) f_t(u) du. \end{aligned}$$

By (2.2), this leads to

$$\begin{aligned} \frac{d}{dt} E(f_t) &= - \frac{1}{2t+1} \int_{\mathbb{R}^d} (u - u_0)^T B^{-1} \left(\frac{2t}{2t+1} B^{-1} + \frac{1}{2t+1} C_0^{-1} \right)^{-1} B^{-1} (u - u_0) f_t(u) du \\ &\leq - \left(\frac{1}{mM} \right) \frac{1}{2t+1} \int_{\mathbb{R}^d} (u - u_0)^T B^{-1} (u - u_0) f_t(u) du = - \left(\frac{2}{mM} \right) \frac{1}{2t+1} E(f_t). \end{aligned}$$

Consequently, we obtain

$$E(f_t) \leq \frac{1}{(2t+1)^{\frac{1}{mM}}} E(f_0).$$

Since $W_2(f_t, \delta_{u_0})^2 = \int_{\mathbb{R}^d} |u - u_0|^2 f_t(u) du$, we deduce from this an equilibration estimate in Wasserstein distance:

$$W_2(f_t, \delta_{u_0})^2 \leq \frac{mM}{(2t+1)^{\frac{1}{mM}}} W_2(f_0, \delta_{u_0})^2.$$

Since $mM \geq 1$, the convergence rate obtained is not as sharp as the one of (3.14).

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Appendix A. Auxiliary technical results

Lemma A.1 (A concavity inequality). *Let M_1 , M_2 and M be symmetric, positive-semidefinite matrices in $\mathbf{R}^{d \times d}$. Then it holds that*

$$\left| (M + M_1)^{1/2} - (M + M_2)^{1/2} \right|_F \leq C_R(d) \left| M_1^{1/2} - M_2^{1/2} \right|_F, \quad (\text{A.1})$$

for a constant C_R that depends only on d .

Proof. The statement is obvious in one dimension. For the general case, we start by showing the statement for the metric

$$d(M_1, M_2) = \sup_{|x|=1} ||M_1 x| - |M_2 x|| = \sup_{|x|=1} \left| \sqrt{x^T M_1^2 x} - \sqrt{x^T M_2^2 x} \right|, \quad (\text{A.2})$$

and then we show that this metric is equivalent to the that induced by the Frobenius norm (or any other matrix norm) on the space of symmetric positive-semidefinite matrices. To complete the first part, we expand (A.2) and use the one-dimensional version of this lemma:

$$\begin{aligned} d((M + M_1)^{1/2}, (M + M_2)^{1/2}) &= \sup_{|x|=1} \left| \sqrt{x^T (M + M_1) x} - \sqrt{x^T (M + M_2) x} \right| \\ &= \sup_{|x|=1} \left| \sqrt{x^T M x + x^T M_1 x} - \sqrt{x^T M x + x^T M_2 x} \right| \\ &\leq \sup_{|x|=1} \left| \sqrt{x^T M_1 x} - \sqrt{x^T M_2 x} \right| = d(M_1^{1/2}, M_2^{1/2}). \end{aligned}$$

To complete the second part, we must show that there exist constants C_1 and C_2 such that

$$\forall M_1, M_2 \succeq 0, \quad C_1 |M_1 - M_2|_F \leq d(M_1, M_2) \leq C_2 |M_1 - M_2|_F.$$

The first inequality is proved in [1, lemma C.1]. The second inequality follows after taking the supremum (over the sphere $|x| = 1$) in the following equation, where we employ the triangle inequality:

$$||M_1 x| - |M_2 x|| \leq |M_1 x - M_2 x| \leq |M_1 - M_2|_2 |x|.$$

This completes the proof. \square

Using the same trick, of passing to the equivalent distance $d(\bullet, \bullet)$, we can show the following.

Lemma A.2. *Let M_1, M_2 be symmetric, positive-semidefinite matrices in $\mathbf{R}^{d \times d}$. It holds that*

$$|M_1 - M_2|_F \leq C(d) \max \left(\left| M_1^{1/2} \right|_F, \left| M_2^{1/2} \right|_F \right) \left| M_1^{1/2} - M_2^{1/2} \right|_F, \quad (\text{A.3})$$

for a constant C that depends only on d .

Proof. In one dimension, the statement follows from the equation

$$\forall m_1, m_2 \geq 0: \quad |m_1 - m_2| = |\sqrt{m_1} - \sqrt{m_2}| |\sqrt{m_1} + \sqrt{m_2}|.$$

We can then show pass to $d(\bullet, \bullet)$ as follows:

$$\begin{aligned} \left| M_1^{1/2} - M_2^{1/2} \right|_F &\geq C \sup_{|x|=1} \left| \sqrt{x^T M_1 x} - \sqrt{x^T M_2 x} \right| \\ &= C \sup_{x \in S} \frac{|x^T M_1 x - x^T M_2 x|}{\sqrt{x^T M_1 x} + \sqrt{x^T M_2 x}} \geq C \left(\left| M_1^{1/2} \right|_2 + \left| M_2^{1/2} \right|_2 \right)^{-1} |M_1 - M_2|_2, \end{aligned}$$

where $S := \{x : |x| = 1, x^T(M_1 + M_2)x > 0\}$. This leads to the statement after rearranging. \square

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