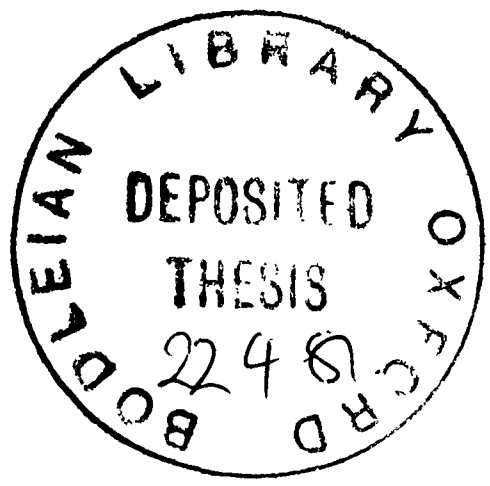
SHEAF COHOMOLOGY IN TWISTOR DIAGRAMS

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ABSTRACT

One of the earlier achievements of twistor theory was the description of free zero rest mass fields on complexified Minkowski space in terms of holomorphic functions on twistor space. Interactions between these fields are given by certain spacetime integrals (represented by Feynmann diagrams), and some of these integrals have been translated into contour integrals in products of twistor spaces (represented by twistor diagrams). The principal advantage of the twistor diagram formalism is that it is necessarily finite.

The main purpose of this thesis is to explore the uses of two mathematical techniques in twistor diagrams. The first is the "blowing up" process familiar to algebraic geometers. It arises naturally in the translation from the massless scalar ϕ^4 vertex to the corresponding twistor diagram (called the "box" diagram). A detailed study of this translation reveals that there are three contours over which the box diagram can be integrated, one for each of the channels in the ϕ^4 interaction. The second technique is sheaf cohomology theory, which was introduced to make rigorous the twistor description of zero rest mass fields by replacing twistor functions by elements of sheaf cohomology groups. We show how to interpret fragments of twistor diagrams - which normally represent twistor functions - as these sheaf cohomology elements.

Chapter 1 introduces, briefly, the basic ideas of twistor geometry, the twistor description of fields, and twistor diagrams. In chapter 2 we demonstrate the existence of contours for part of the Möller

scattering diagram using singular homology theory, while chapter 3 gives the details of the translation to the box diagram (already referred to) and compares it with the scalar product diagram. The last two chapters (4 and 5) deal with the sheaf cohomology of tree diagrams and the scalar product diagram respectively.

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Chapter 1 : Background

§1.1 Introduction

In this chapter we give a lightning review of the relevant areas of twistor theory, more to fix the notation than as a serious attempt to explain the concepts involved. Section 1.2 describes some basic twistor geometry, section 1.3 introduces the twistor treatment of zero rest mass fields, and in section 1.4 twistor diagrams and topological diagrams are defined and discussed. For a more extensive account of the material in sections 1.2 and 1.3 see, for example, (Penrose and Ward 1979). The theory of twistor diagrams is covered in (Penrose and MacCallum 1972), (Penrose 1975), and (Hodges and Huggett 1980).

§1.2 Twistor Geometry

Let $x^{AA'}$ be local coordinates on compactified complexified Minkowski space $\mathbf{CM}$. The equation

$$x^{AA'} = x_o{}^{AA'} + \lambda^A \pi^{A'} \qquad 1.2.1$$

defines an α plane if λ^A varies and $\pi^{A'}$ is fixed, and a β plane if $\pi^{A'}$ varies and λ^A is fixed. The space of α planes is a $\mathbf{CP}^3$ called projective twistor space $\mathbf{PT}$. $\mathbf{PT}$ has homogeneous coordinates $Z^\alpha = (Z^A, Z_{A'})$ and the relationship between $\mathbf{PT}$ and $\mathbf{CM}$ is determined by

$$Z^A = ix^{AA'} Z_{A'} \qquad 1.2.2.$$

We can look at this equation in two ways:

(i) if we fix Z^{α} the solutions for $x^{AA'}$ form an α plane

(ii) if we fix $x^{AA'}$ the solutions for Z^{α} form a complex projective line (CP^1) in PT.

Theorem 1.2.3

The points $x^{AA'}$ and $y^{AA'}$ in CM are null separated if and only if the lines representing them in PT intersect. □

A complex projective plane (CP^2) in PT is a dual twistor $W_{\alpha} = (W_A, W^{A'}) \in PT^*$. Dual twistors are related to CM by

$$W^{A'} = - ix^{AA'} W_A \qquad 1.2.4.$$

Again, we can fix either W_{α} and obtain a β plane in CM or $x^{AA'}$ and obtain a CP^1 in PT^*.

Suppose an α plane Z^{α} and a β plane W_{α} intersect in CM. Then they have a point $x^{AA'}$ in common. This point is represented by a line in PT which must

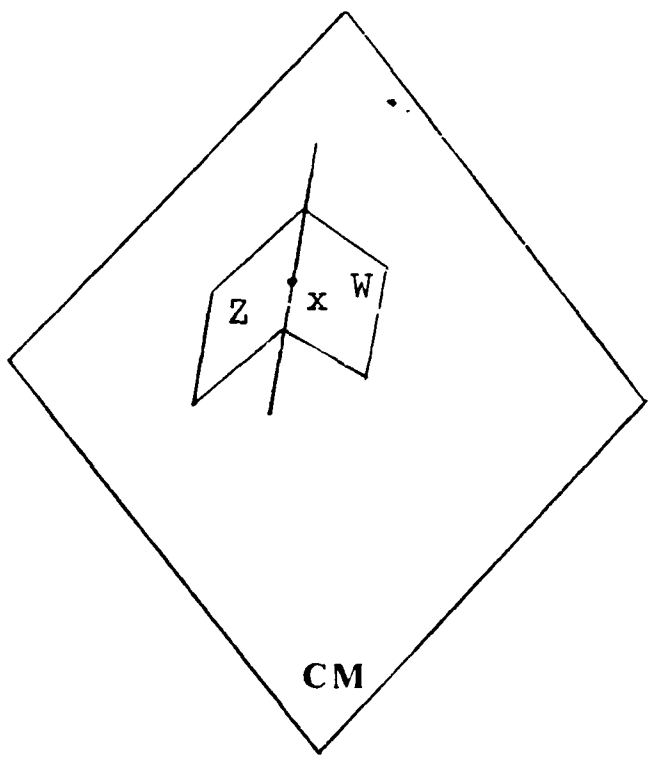
(a) go through the point in PT representing the α plane

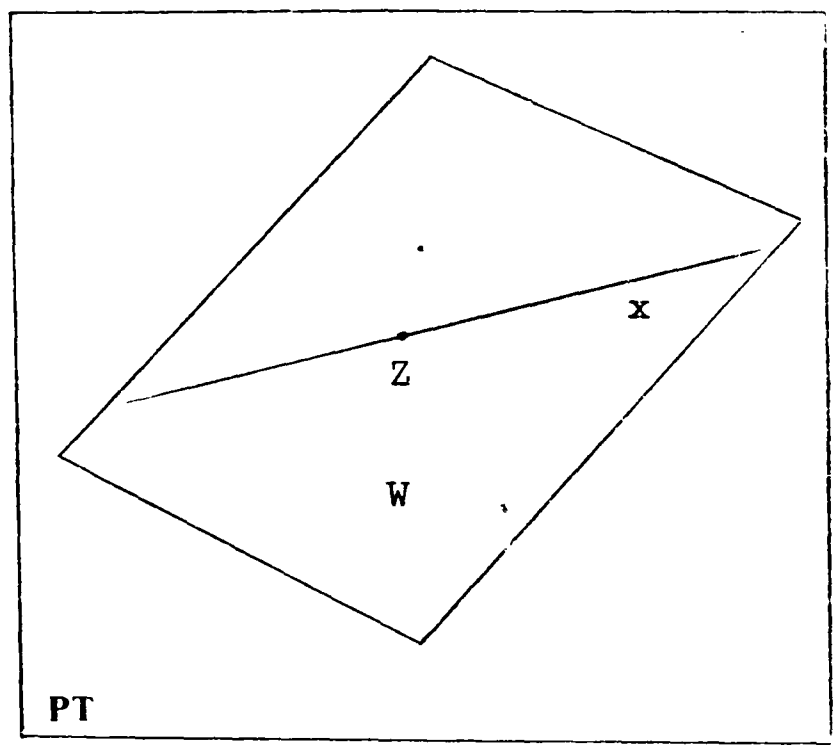
(b) lie in the plane in PT representing the β plane.

There is a one complex parameter family of lines having the properties (a) and (b). All these lines intersect, so the points representing them in CM are null separated and all lie on both the α plane and the β plane. Hence:

Theorem 1.2.5

If an α plane and a β plane have a point in common they must have a whole complex null geodesic in common. This geodesic is represented in PT by a point on a plane. □





The following table summarises these remarks:

PT	CM
point	α plane
line	point
plane	β plane
point on plane	complex null geodesic
intersection of lines	null separation of points

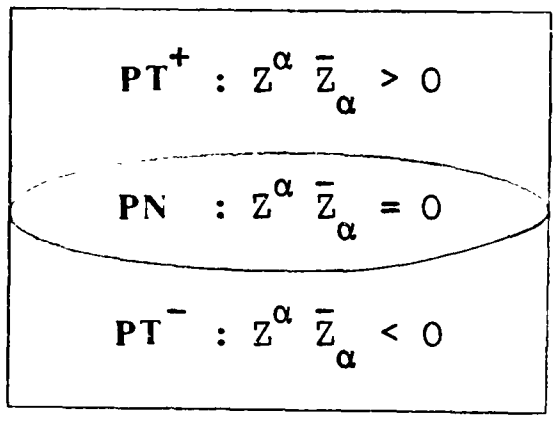
In order to be able to select lines in PT representing <u>real</u> points in CM we define the complex conjugate of the twistor Z^α to be the dual twistor

$$\bar{Z}_\alpha = (\bar{Z}_A, \bar{Z}^{A'})$$

and we define the norm of Z^α to be

$$Z^\alpha \bar{Z}_\alpha = Z^A \bar{Z}_A + Z_{A'} \bar{Z}^{A'}.$$

Using this we can divide PT into three regions:



<u>Theorem 1.2.6</u>

$x^{AA'}$ is real if and only if the line representing it lies entirely in PN. □

Now consider the null twistor $Z^\alpha \in PN$. We can choose a line lying entirely in PN and containing Z^α. If this line represents the point $x_o^{AA'} \in M$ we have

$$Z^A = ix_o^{AA'} Z_{A'}.$$

Any other line lying entirely in $\mathcal{PN}$ and containing Z^α represents a point $x^{AA'} \in \mathcal{M}$ satisfying

$$x^{AA'} = x_o{}^{AA'} + \kappa \bar{Z}^A Z^{A'}$$

for some real constant κ. Therefore the null twistor Z^α corresponds to the real null geodesic through $x_o{}^{AA'}$ in the direction $\bar{Z}^A Z^{A'}$. (In other words when Z^α is null the corresponding α plane contains a real null geodesic.)

We can define a spinor field on $\mathcal{M}$ by

$$Z^A(x^{AA'}) =: Z^A - ix^{AA'} Z_{A'} .$$

Then our real null geodesic is simply the locus

$$Z^A(x^{AA'}) = 0.$$

Moreover, we can use this spinor field to obtain a real geometrical picture of a <u>non-null</u> twistor. We construct the null vector field

$$Z^A(x^{AA'}) \bar{Z}^{A'}(x^{AA'})$$

and project it into a spacelike hypersurface S. The projected vector field is the field of tangent vectors to a <u>Robinson congruence</u> in S (Penrose 1975).

This completes our section on twistor geometry except for the following notation:

Z^{α} and W_{α} will sometimes be written $\overset{|}{Z}$ and $\overset{|}{W}$, $2X^{[\alpha}Z^{\beta]}$ will be written $\overset{++}{XZ}$, and $\sqcup\!\sqcup\!\sqcup$ and $\sqcap\!\sqcap\!\sqcap$ will denote $\epsilon^{\alpha\beta\gamma\delta}$ and $\epsilon_{\alpha\beta\gamma\delta}$ respectively.

§1.3 Zero rest mass fields

The zero rest mass (z.r.m.) free field equations for spin $\frac{n}{2}$ are

$$\nabla^{AA'} \phi_{AB\ldots L} = 0 \qquad 1.3.1$$

$$\nabla^{AA'} \psi_{A'B'\ldots L'} = 0 \qquad 1.3.2$$

where $\phi_{AB\ldots L}$ and $\psi_{A'B'\ldots L'}$ are symmetric in their n indices. A positive frequency field on $\mathbf{M}$ is one which can be extended to a field on the whole future tube

$$\mathbf{CM}^{+} = \{z^{a} \in \mathbf{CM}: z^{a} = x^{a} - iy^{a} \text{ with } x^{a} \text{ and } y^{a} \text{ real and } y^{a} \text{ timelike and future pointing}\}.$$

If the fields above are to be of positive frequency then 1.3.1 is for negative helicity ($s = -\frac{n}{2}$) while 1.3.2 is for positive helicity ($s = \frac{n}{2}$).

Now suppose $f(Z^{\alpha})$ and $g(Z^{\alpha})$ are holomorphic and homogeneous of degrees $n - 2$ and $-n - 2$ respectively. These functions are not defined throughout $\mathbf{PT}$, but we postpone the discussion of where they <u>are</u> defined for the moment. We can define spinor fields

$$\phi_{AB\ldots L}(x^a) = \frac{1}{2\pi i}\oint \frac{\partial}{\partial Z^A}\,\frac{\partial}{\partial Z^B}\ldots\frac{\partial}{\partial Z^L} f(Z^\alpha)\,\Delta Z \qquad 1.3.3$$

$$\psi_{A'B'\ldots L'}(x^a) = \frac{1}{2\pi i}\oint Z_{A'}Z_{B'}\ldots Z_{L'}\, g(Z^\alpha)\,\Delta Z \qquad 1.3.4$$

where $Z^\alpha = (ix^{AA'}\, Z_{A'},\, Z_{A'})$ and $\Delta Z = \epsilon^{A'B'}\, Z_{A'}\, dZ_{B'}$. We have the remarkable result:

Theorem 1.3.5 . (Penrose)

The fields $\phi_{AB\ldots L}$ and $\psi_{A'B'\ldots L'}$ defined in 1.3.3 .and 1.3.4 satisfy 1.3.1 and 1.3.2 respectively. □

The contour integrals in 1.3.3 and 1.3.4 are performed in the projective primed spin space above x^a. Let us concentrate on 1.3.4. (Corresponding remarks will apply to 1.3.3.) For the field $\psi_{A'B'\ldots L'}$ to be non singular at a particular point x^a the line representing x^a in PT has to intersect the singularities of the function g in two disjoint regions. Then the contour γ_x for 1.3.4 can separate these regions. Because g is holomorphic the actual location of γ_x is unimportant as long as it separates the singularities of g and varies continuously with x^a. Suppose for example that we require $\psi_{A'B'\ldots L'}$ to be a positive frequency field.

Lemma 1.3.6

$x^{AA'} \in CM^+$ <=> the line in PT representing $x^{AA'}$

lies completely in PT^+. □

Therefore the singularities of g must intersect any line in PT^+ in two disjoint regions. Figure 1.3.7 illustrates the case where g is an elementary state:

$$g(Z^\alpha) = \frac{\overset{A}{(|)}\overset{a}{}\quad\overset{B}{(|)}\overset{b}{}}{\underset{Z}{}\qquad\underset{Z}{}} \Big/ \frac{\overset{C}{(|)}\overset{c+1}{}\quad\overset{D}{(|)}\overset{d+1}{}}{\underset{Z}{}\qquad\underset{Z}{}}$$

where $c + d - a - b = 2s \geq 0$.

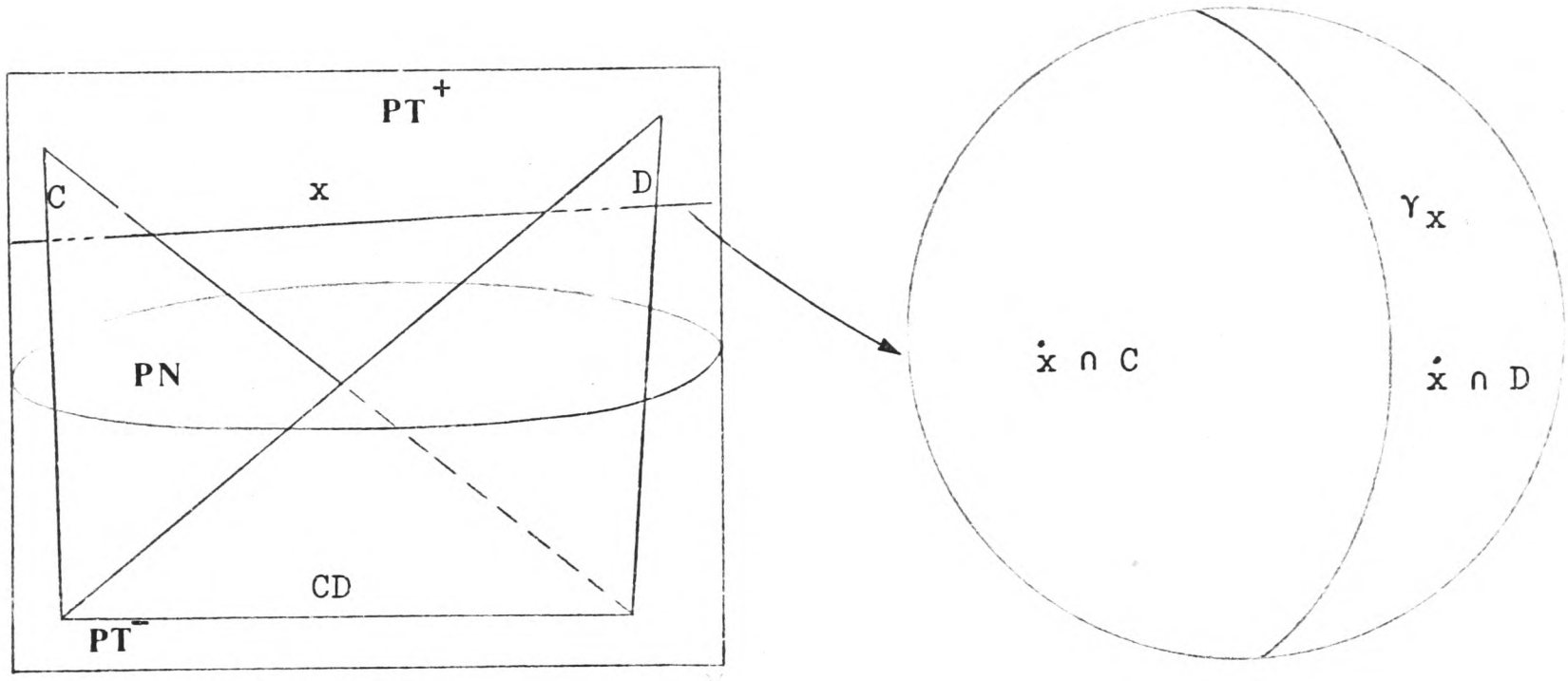


Figure 1.3.7

The relationship between twistor functions and z.r.m. fields given in Theorem 1.3.5 is not one to one. We have noted that the actual location of the contour γ_x is unimportant as long as it separates the singularities of g. However, having chosen γ_x we can

add to g any function h which is holomorphic all over one side of γ_x because the integral over γ_x of such a function h will be zero. Now we can move the contour γ_x again, and so on. Clearly there is a whole class of functions and contours all corresponding to the same z.r.m. field. It turns out that this class is a sheaf cohomology class:

Theorem 1.3.8

$$H^1(PT^+; \mathcal{O}(-n-2)) \cong \{\text{holomorphic solutions of } \nabla^{AA'} \psi_{A'B'\ldots L'} = 0 \text{ where } \psi_{A'B'\ldots L'} \text{ is symmetric in its } n \text{ indices and defined on } CM^+\}.$$

□

This important isomorphism is discussed in (Penrose 1979b) and the theorem is proved in (Eastwood, Penrose and Wells 1979). In chapter 4 we describe sheaf cohomology classes and generalise the interpretation of an elementary state as an element of H^1.

§1.4 Twistor Diagrams

We saw in the last section that a zero rest mass free field can be generated by a holomorphic twistor function or, more accurately, by a sheaf cohomology class of such functions. We now turn our attention to interacting fields and define twistor diagrams. These diagrams are to be thought of as equivalent to Feynman diagrams, in that they represent integrals defining explicit scattering amplitudes. We start by defining

twistor diagrams, and after that we discuss some important examples.

A twistor diagram is a connected finite graph whose vertices are either of degree 4 or of degree 1. Those of degree 4 are called <u>variables</u> and those of degree 1 are called <u>parameters</u>. The variables are either drawn ● or ○ , and the parameters are omitted. The edges of the graph, which may be multiple, never join two variables of the same type, and an edge joining a variable to a parameter is called a <u>twig</u>. The edges are numbered in such a way that the sum of the numbers on all edges having a common vertex is zero and the vertices are labelled with capital letters. The graph defines a differential form for a contour integral. The vertex

● Z represents $\mathcal{D}Z = :$ $\overbrace{Z\ dZ \wedge dZ \wedge dZ}$, the vertex

○ W represents $\mathcal{D}W = :$ $\underbrace{W\ dW \wedge dW \wedge dW}$, and a line

numbered r and of multiplicity m joining vertices labelled P and Q represents

$$\frac{1}{(P_\alpha Q^\alpha)^{m+r}} \qquad 1.4.1$$

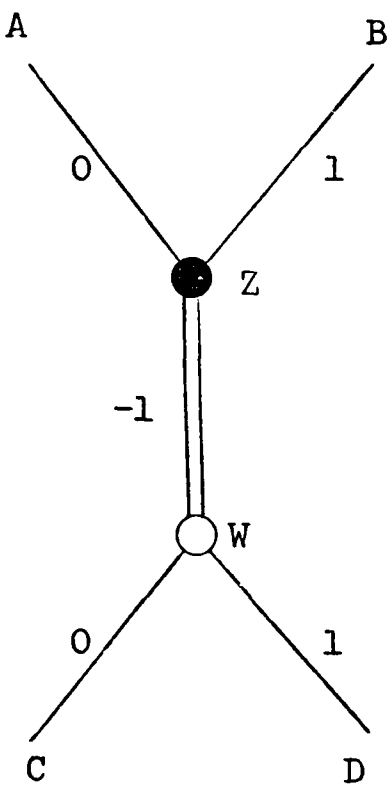
(whether P and Q are parameters or variables).

Strictly speaking we should use the bracket factor - defined in (Penrose 1975) - instead of 1.4.1 but it will make no difference here.

Consider, for example, the twistor diagram in figure 1.4.2 for the <u>scalar product</u> between the two spin $\frac{1}{2}$ z.r.m. fields generated by

the elementary states

$$\frac{1}{\begin{array}{cccc} A & B & & 2 \\ | & (& | &) \\ Z & Z & & \end{array}} \qquad \text{and} \qquad \frac{1}{\begin{array}{cccc} W & W & & 2 \\ | & (& | &) \\ C & D & & \end{array}} \quad .$$



$$= \oint \frac{\mathcal{D}Z \wedge \mathcal{D}W}{\begin{array}{ccccccccc} A & B & & 2 & W & W & W & & 2 \\ | & (& | &) & & | & | & (& | &) \\ Z & Z & & & Z & C & D & & \end{array}}$$

Figure 1.4.2

We postpone until chapter 3 the explanation of <u>why</u> this twistor diagram represents the scalar product, and instead focus on some of its mathematical properties. The first question concerns the contour over which we perform the integral in 1.4.2. The multiple lines in the diagrams are designed to determine - to some extent at least - the topology of the contour to be used (Penrose and MacCallum 1972):



contributes CP^1 ,



contributes CP^2 , and



contributes CP^3 to the contour.

These rules do not uniquely define a contour, however, so that given a twistor diagram we must

(a) find out how many contours there are, and

(b) discover which of them is a "physical" contour.

To answer (a) we first of all use the twistor diagram to define a topological diagram:

(i) replace each (possibly multiple) edge for which $m + r > 0$ by a single unnumbered straight line,

(ii) replace each (possibly multiple) edge for which $m + r \leq 0$ by a single unnumbered wavy line.

The twistor diagram in figure 1.4.2 becomes

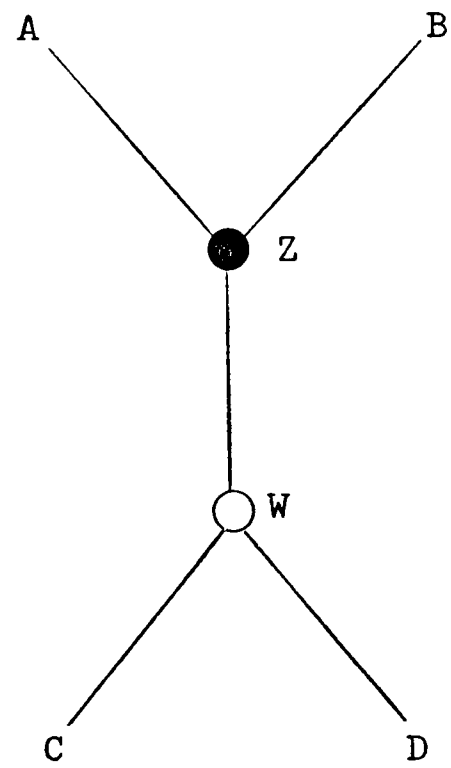


Figure 1.4.3

Now we use the topological diagram to define a pair of topological spaces (X,A). Instead of defining these spaces formally we give a couple of examples. For the diagram in figure 1.4.3 the space A is empty - because there are no wavy lines - and X is the space

$$X = \{(\,|_{Z}\,,\,{}^{W}|\,) \in \; PT \times PT^{*} \; : \; {}^{A}|_{Z} \; {}^{B}|_{Z} \; {}^{W}|_{Z} \; {}^{W}|_{C} \; {}^{W}|_{D} \neq 0\} \quad .$$

If we next consider the example of the twistor diagram (figure 1.4.4) for Möller scattering between massless electrons we find that its topological diagram (figure 1.4.5) does have wavy lines.

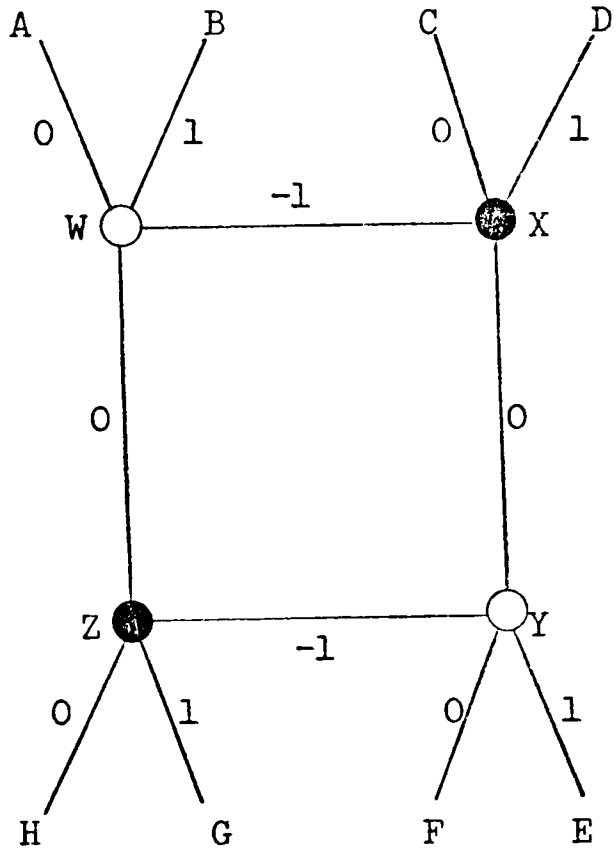


Figure 1.4.4

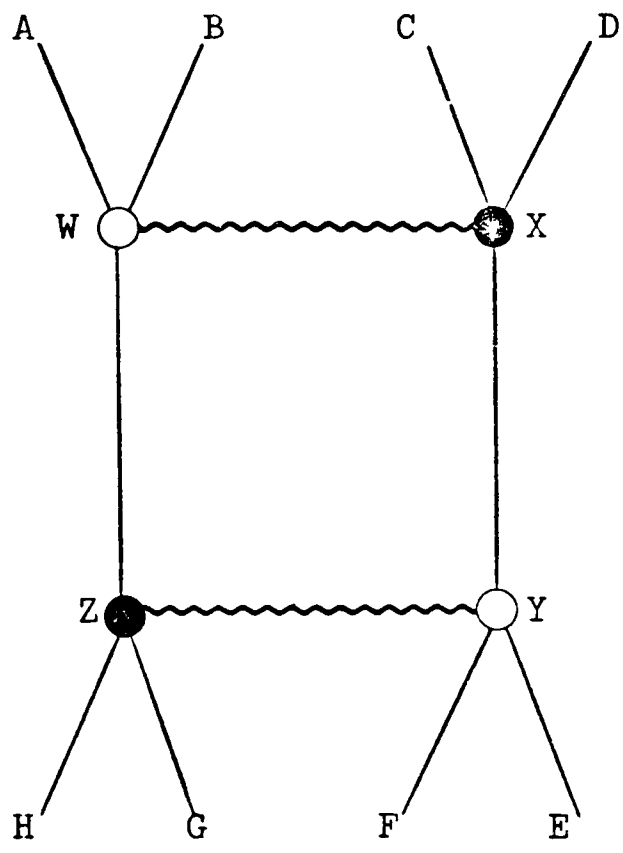


Figure 1.4.5

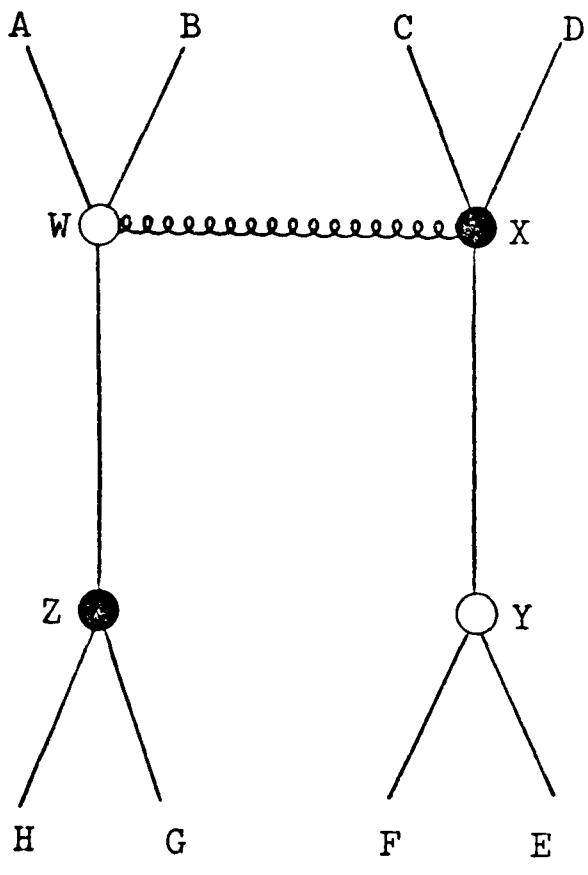
For the topological diagram in figure 1.4.5 the pair of spaces (X,A) is as follows:

$$X = \{(\begin{smallmatrix}W\\|\end{smallmatrix}, \begin{smallmatrix}|\\X\end{smallmatrix}, \begin{smallmatrix}Y\\|\end{smallmatrix}, \begin{smallmatrix}|\\Z\end{smallmatrix}) \in PT^* \times PT \times PT^* \times PT :$$

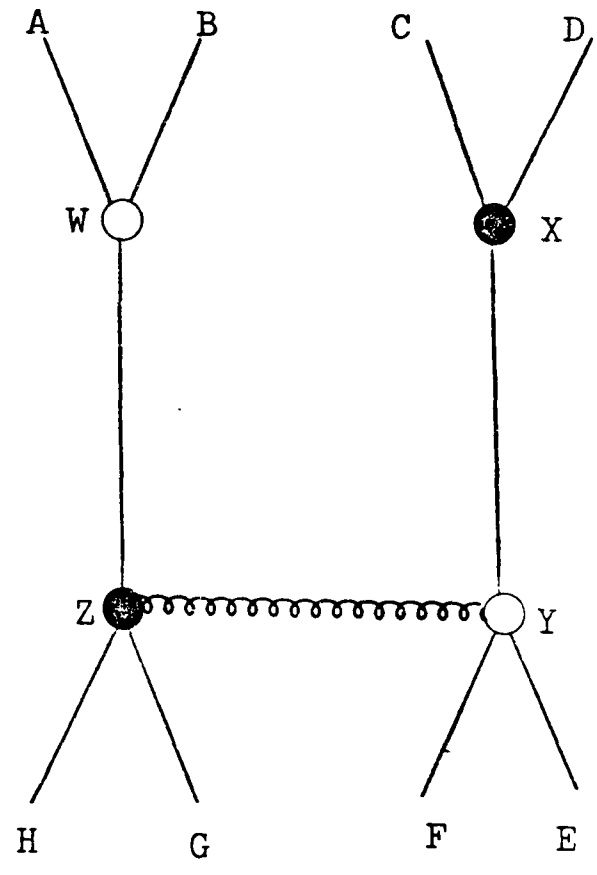
$$\begin{vmatrix} W & W & W & G & H \\ | & | & | & | & | \\ A & B & Z & Z & Z \end{vmatrix} \neq 0 \quad \text{and} \quad \begin{vmatrix} C & D & Y & Y & Y \\ | & | & | & | & | \\ X & X & X & E & F \end{vmatrix} \neq 0\} \qquad 1.4.6$$

$$A = \{(\begin{smallmatrix}W\\|\end{smallmatrix}, \begin{smallmatrix}|\\X\end{smallmatrix}, \begin{smallmatrix}Y\\|\end{smallmatrix}, \begin{smallmatrix}|\\Z\end{smallmatrix}) \in X : \begin{vmatrix} W \\ | \\ X \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} Y \\ | \\ Z \end{vmatrix} = 0\} \qquad 1.4.7.$$

X is the space in which the differential form for the twistor diagram in figure 1.4.4 is non singular, and A is the space in which it is zero. Therefore the contours must lie in X and their boundaries must lie in A. (Contours for the diagram in figure 1.4.2 must be closed because the corresponding space A is empty.) It will be convenient to have topological diagrams for spaces such as A in 1.4.7: it is the union of



and



where the <u>springy</u> lines restrict to the subspaces

$$\begin{matrix} W \\ | \\ X \end{matrix} = 0$$

and

$$\begin{matrix} Y \\ | \\ Z \end{matrix} = 0$$

respectively.

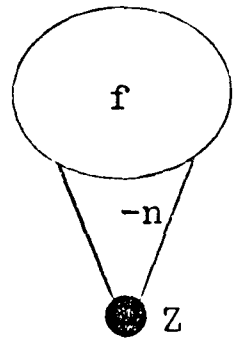
We have seen that given a twistor diagram we can define a pair of spaces (X,A), and that any contour for the integral represented by the diagram must lie in X and have boundary (if any) in A. Therefore to answer question (a) we calculate the relative singular homology group

$$H_{3n}(X,A;C)$$

where n is the number of variable vertices in the twistor diagram. These groups have been calculated for the scalar product diagram (see section 2.2) and for the special case of the box diagram (see chapter 3) in which all the twigs are integrated (as simple poles) first. In chapter 2 we use them to study the Möller scattering diagram. Their disadvantage, however, is that they only really tell us how many contours there are - they do not give us enough information about where the contours are. This is particularly true in the case of the box diagram, where the special case treated by Sparling only yields one of the three physical contours we require. These contours are constructed in chapter 3 using a detailed study of the translation procedure from the Feynman diagram to the twistor diagram. This approach is, in

general, more likely to answer question (b) than the homology theory.

We conclude this section with a little more notation. A twistor function $f(Z^{\alpha})$ homogeneous of degree n - 2 appears in a twistor diagram as



Chapter 2 : Homology Calculations

§2.1 Introduction

Unless otherwise stated all the diagrams in this chapter are topological diagrams. The aim of the chapter is to calculate and analyse the homology of the diagram

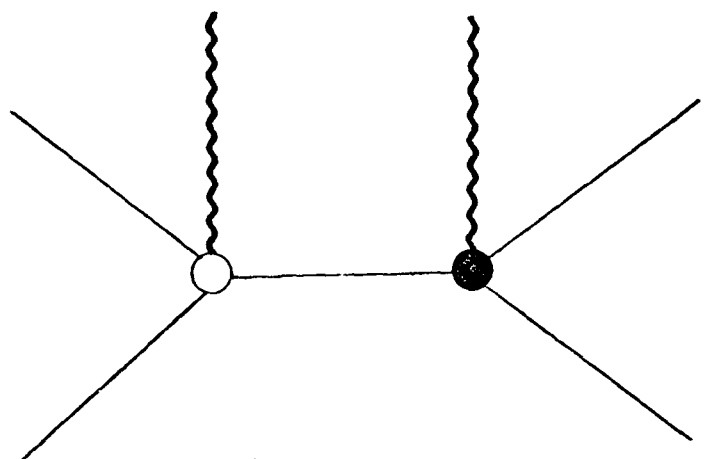
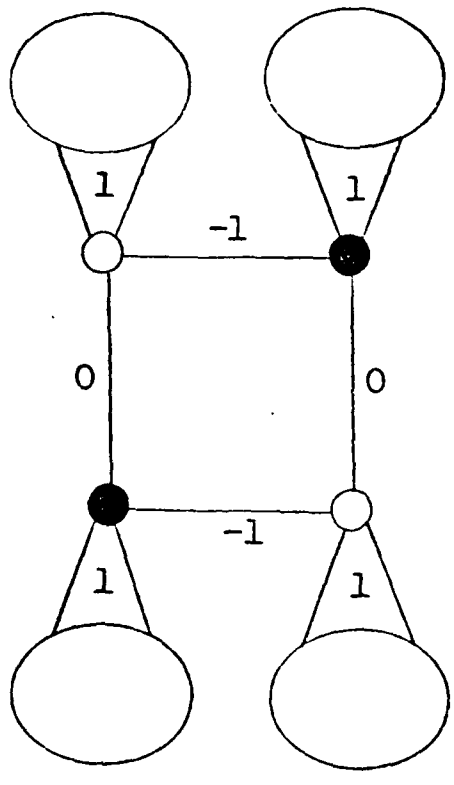
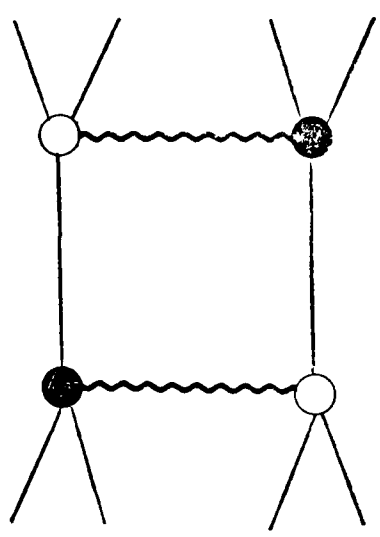


Figure 2.1.1

This diagram arises when we try to find contours for the twistor diagram



for massless Möller scattering. The problem of finding contours for this diagram is studied in (Hodges 1975) and mentioned in (Hodges and Huggett 1979). The corresponding topological diagram (for elementary states) is



Clearly any contours which are found for the diagram in figure 2.1.1 will be of use in constructing contours for the Möller scattering diagram.

Sections 2.2, 2.3, and 2.4 describe the three basic exact sequences we use. These sequences reduce the problem to that of the homology of fibre bundles, which is discussed in the next two sections. Then, in section 2.7, all the results are combined and the homology of the diagram is calculated.

We refer to (Vick 1973) and (Spanier 1966) for basic singular homology theory, and to (Sparling 1975) and (Ryman 1975) for other examples of the use of the theory in twistor diagrams.

§2.2 The Relative Homology Sequence

A non contractible n-dimensional closed contour γ in the topological space X is a representative cycle for an element $[\gamma]$ of the homology group $H_n(X;Z)$. From here on the Z will be omitted. If γ is not closed but has boundary in $A \subset X$ then $[\gamma]$ is an element of the relative homology group $H_n(X,A)$. These relative homology groups are the subject of this section. We calculate them in terms of the homology groups $H_*(X)$ and $H_*(A)$ using the following exact sequence

$$\ldots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i} H_n(X) \xrightarrow{p} H_n(X,A) \to \ldots . \qquad 2.2.1$$

called the relative homology sequence. ∂ is the boundary map, defined as follows. Given an element of $H_{n+1}(X,A)$ choose a representative γ and consider its boundary $\partial\gamma$. $[\partial\gamma]$ is an element of $H_n(A)$ and it is easy to check that the map

$$\partial: H_{n+1}(X,A) \to H_n(A)$$

$$[\gamma] \longmapsto [\partial\gamma]$$

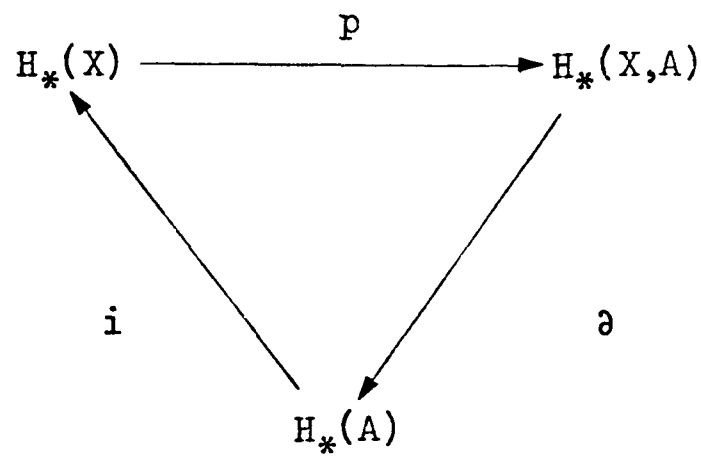
is well defined. i and p are induced by the obvious injections

$$i: C_n(A) \to C_n(X)$$

$$p: C_n(X) \to C_n(X,A)$$

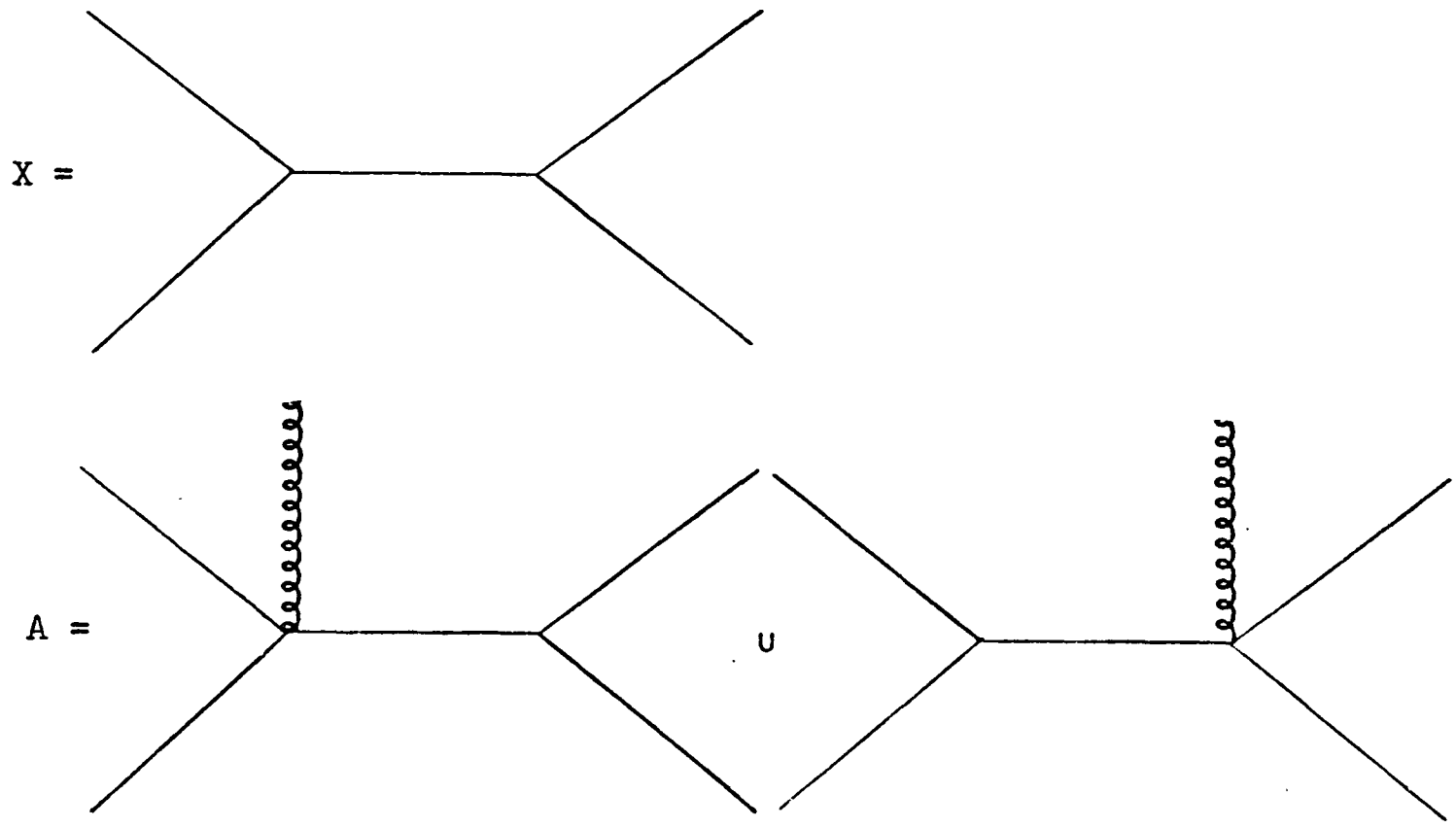
at chain level.

Sequence 2.2.1. is better drawn as the exact triangle:



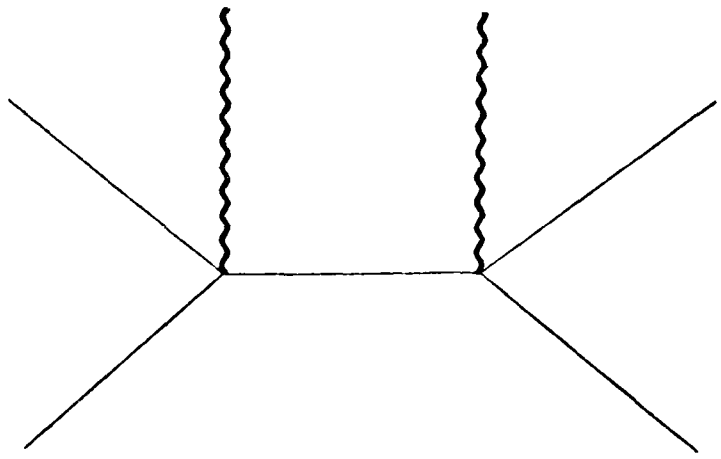
The maps i and p preserve dimension, while ∂ lowers dimension by one.

For example let

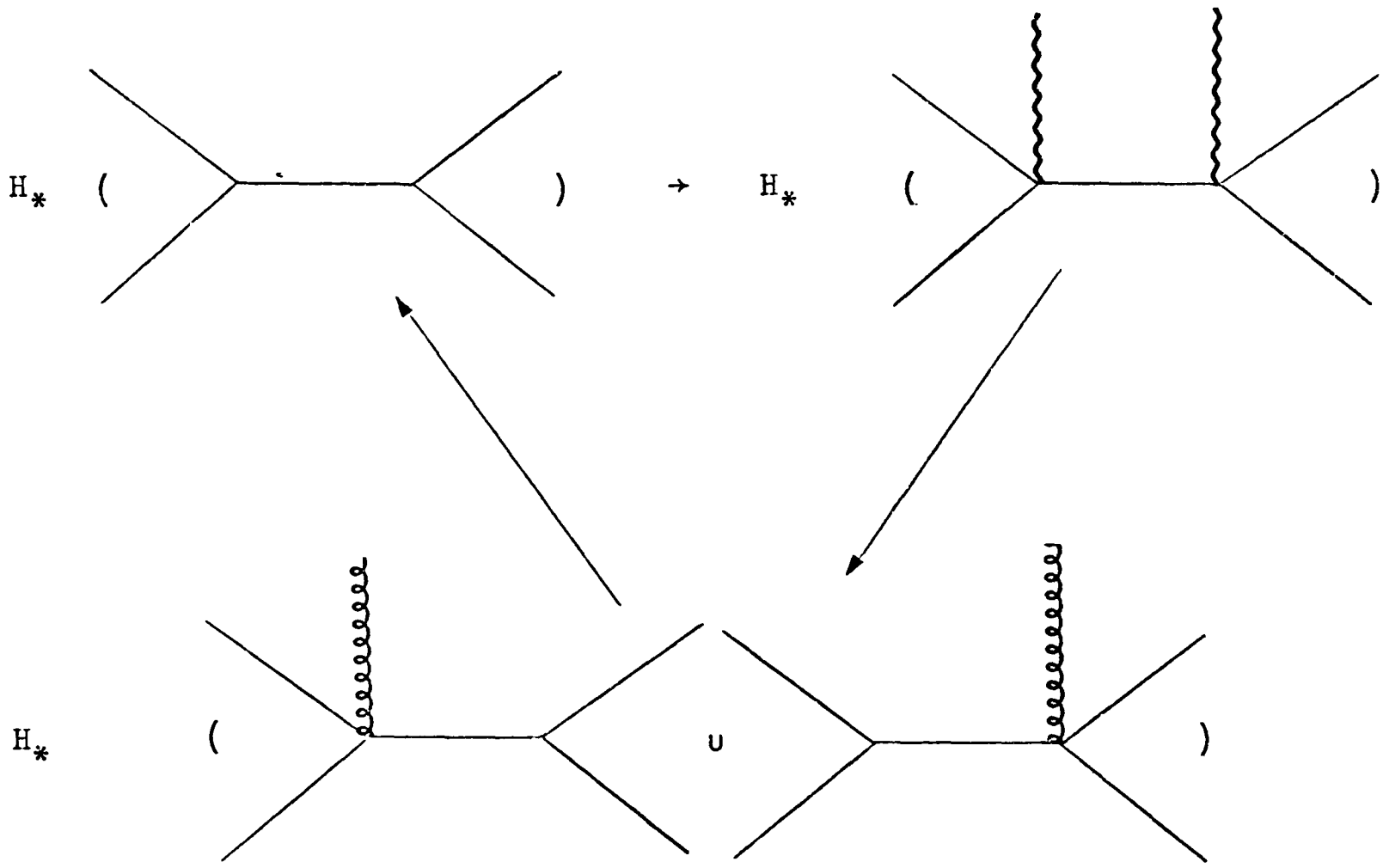


Then

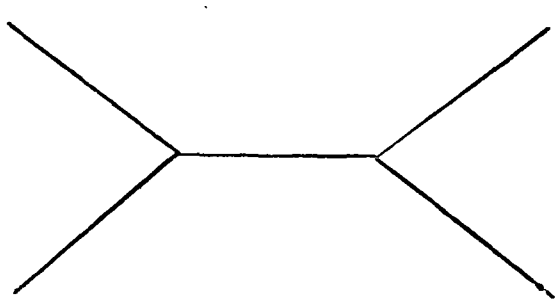
X,A =



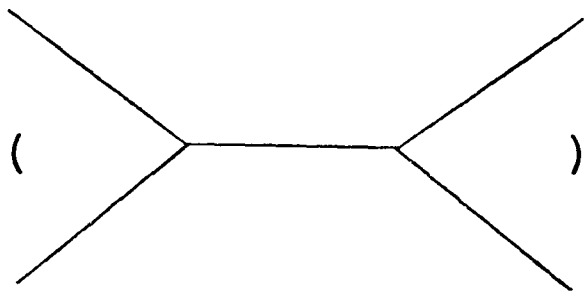
and we have the exact triangle



In (Sparling 1975) the homology of the scalar product space



is calculated. In particular, it is shown that

H_6 () = C ⊕ C and that

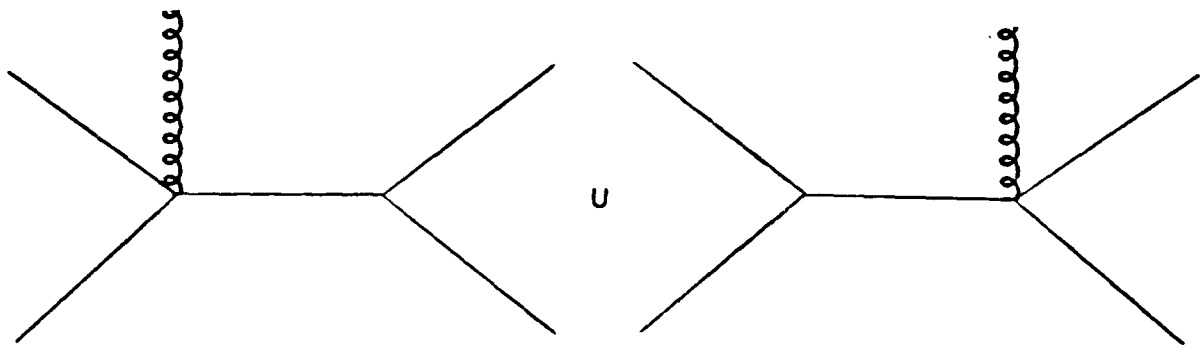
H_6 () = C .

Therefore one of the two contours for the scalar product (the loose contour) is also a contour for the space



The other one is called the tight contour, and it is the physical contour (see section 3.5).

The fact that we know the homology of the scalar product space means that it remains to calculate the homology of



§2.3 The Mayer Vietoris Sequence

We need the Mayer Vietoris sequence to work out the homology of the union of two spaces. Before describing it we define an excision. If $U \subset A \subset X$, the injection of $X - U$ into X induces the map

$$e : H_*(X - U, A - U) \to H_*(X,A)$$

If this map is an isomorphism it is called an excision.

Theorem

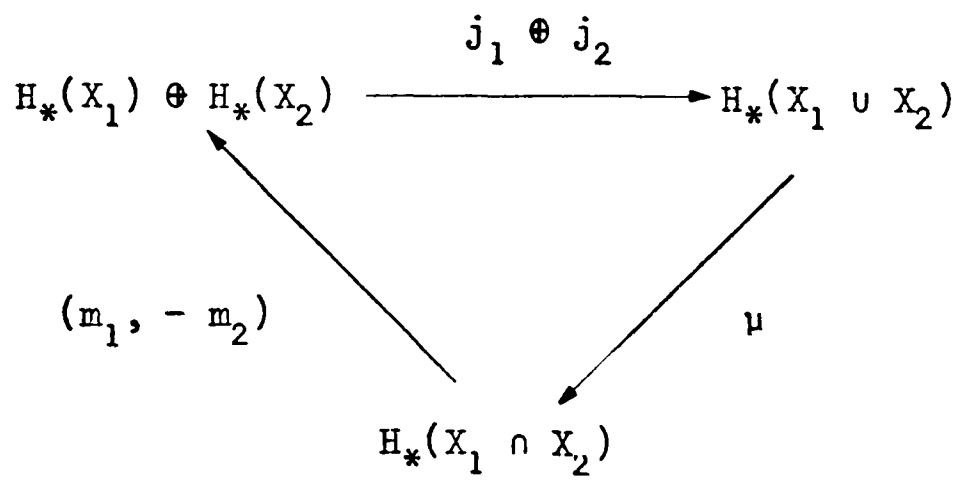
If $\bar{U} \subset \text{int } A$ then e is an excision. □

Now suppose X_1 and X_2 are topological spaces and consider the natural injections (which induce corresponding maps at the level of homology)

$$m_1 : X_1 \cap X_2 \to X_1 \qquad m_2 : X_1 \cap X_2 \to X_2$$

$$j_1 : X_1 \to X_1 \cup X_2 \qquad j_2 : X_2 \to X_1 \cup X_2.$$

The Mayer Vietoris sequence is the following triangle



However, the map μ can only be defined if the spaces X_1 and X_2 are such that the maps

$$e_1 : H_*(X_1, X_1 \cap X_2) \to H_*(X_1 \cup X_2, X_2)$$

$$e_2 : H_*(X_2, X_1 \cap X_2) \to H_*(X_1 \cup X_2, X_1)$$

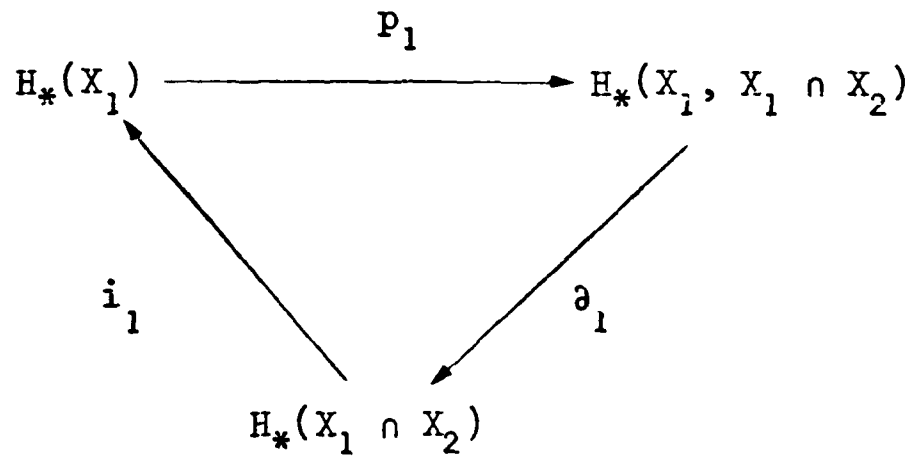
(induced by j_1 and j_2) are excisions. If they are we define μ by

$$\mu = \partial_1 e_1^{-1} p_1$$

where $$p_1 : H_*(X_1 \cup X_2) \to H_*(X_1 \cup X_2, X_2)$$

and $$\partial_1 : H_*(X_1, X_1 \cap X_2) \to H_*(X_1 \cap X_2)$$

come from the relative homology sequence



It can be shown (i) that

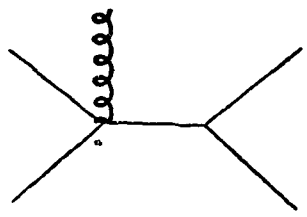
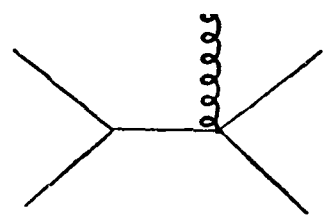
$$\mu = \partial_1 e_1^{-1} p_1 = - \partial_2 e_2^{-1} p_2,$$

so that the definition of μ is symmetrical in X_1 and X_2, and

(ii) that with this definition the Mayer - Vietoris triangle is exact.

The maps $(m_1, -m_2)$ and $j_1 \oplus j_2$ preserve dimension, while μ lowers dimension by one.

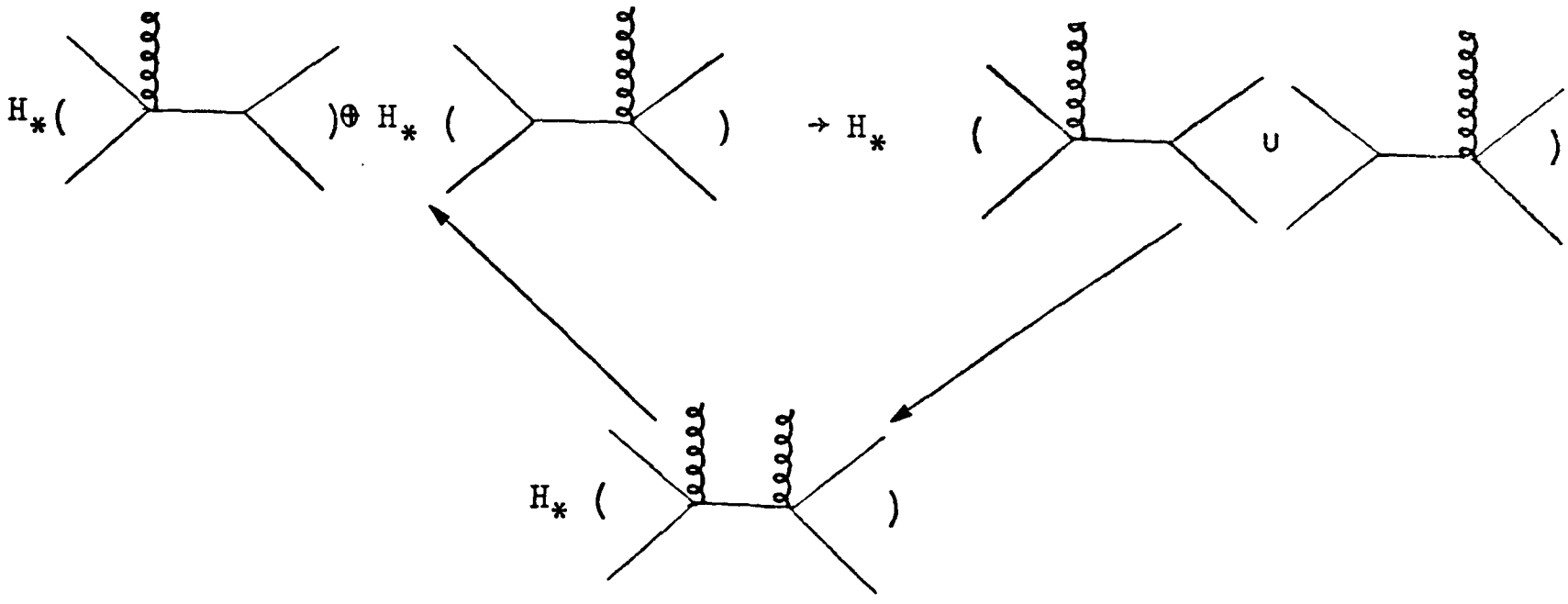
To return to our example, let

$X_1 =$  and $X_2 =$ 

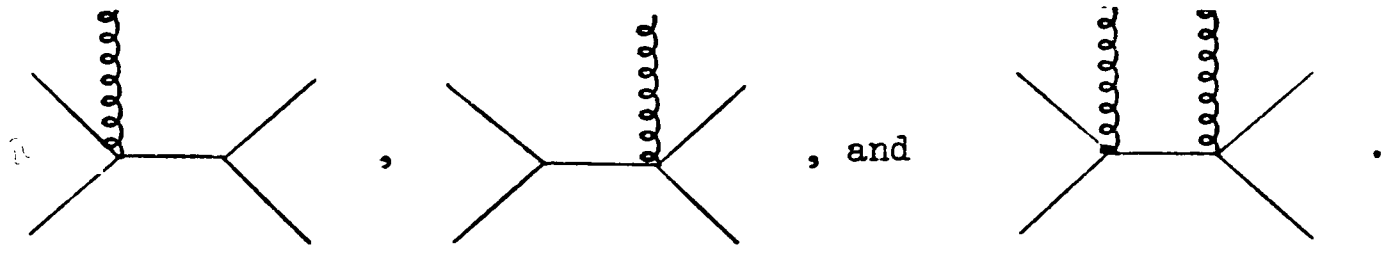
X_1 and X_2 are now closed submanifolds of a larger manifold, intersecting transversally. In particular, therefore, we have

$$\overline{X_2 - X_1} \subset \text{int } X_2 \qquad \text{and} \qquad \overline{X_1 - X_2} \subset \text{int } X_1$$

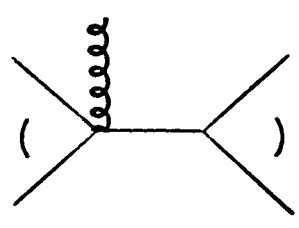
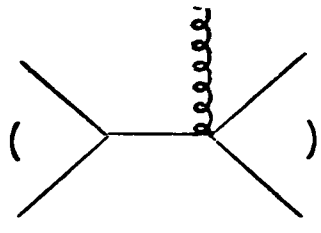
so that, from the excision theorem, e_1 and e_2 are excisions and we can use the Mayer - Vietoris sequence:



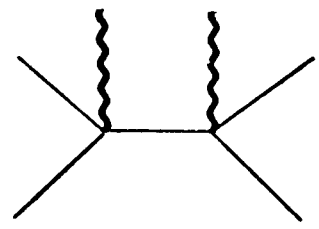
The next step is to calculate the homology of the spaces



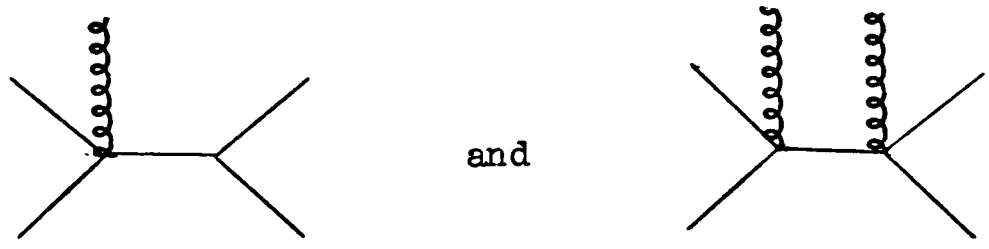
Notice that the first two are dual to each other, so that they have isomorphic homology groups:

H_*  $\cong$ H_*  .

(Another important point is that in (Sparling 1975) an alternative method of reducing the homology of



to the homology of



is described. This method makes use of the <u>adapted relative homology</u>

sequence, which can be thought of as a combination of the two sequences we have used so far.)

§2.4 The Leray Sequence

Let X be a complex manifold and let S be a submanifold of real codimension m. The Leray sequence is the exact triangle

$$\begin{array}{ccc} H_*(X - S) & \xrightarrow{\quad i \quad} & H_*(X) \\ & \delta \nwarrow \quad \swarrow \bar{\omega} & \\ & H_*(S) & \end{array}$$

The map i is induced by the injection of $C_*(X - S)$ into $C_*(X)$, and it preserves dimension. The map $\bar{\omega}$ is given by

$$\bar{\omega} : [\gamma] \longmapsto [\gamma \cap S] \quad \text{for} \quad [\gamma] \in H_n(X).$$

Notice that $[\gamma \cap S] \in H_{n-m}(S)$, so that $\bar{\omega}$ lowers dimension by m.
To construct the cobord map δ let N be a tubular neighbourhood of S isomorphic to the normal bundle of S in X. We use the following maps:

$\Phi : H_n(N, N - S) \xrightarrow{\cong} H_{n-m}(S)$ The Thom isomorphism.

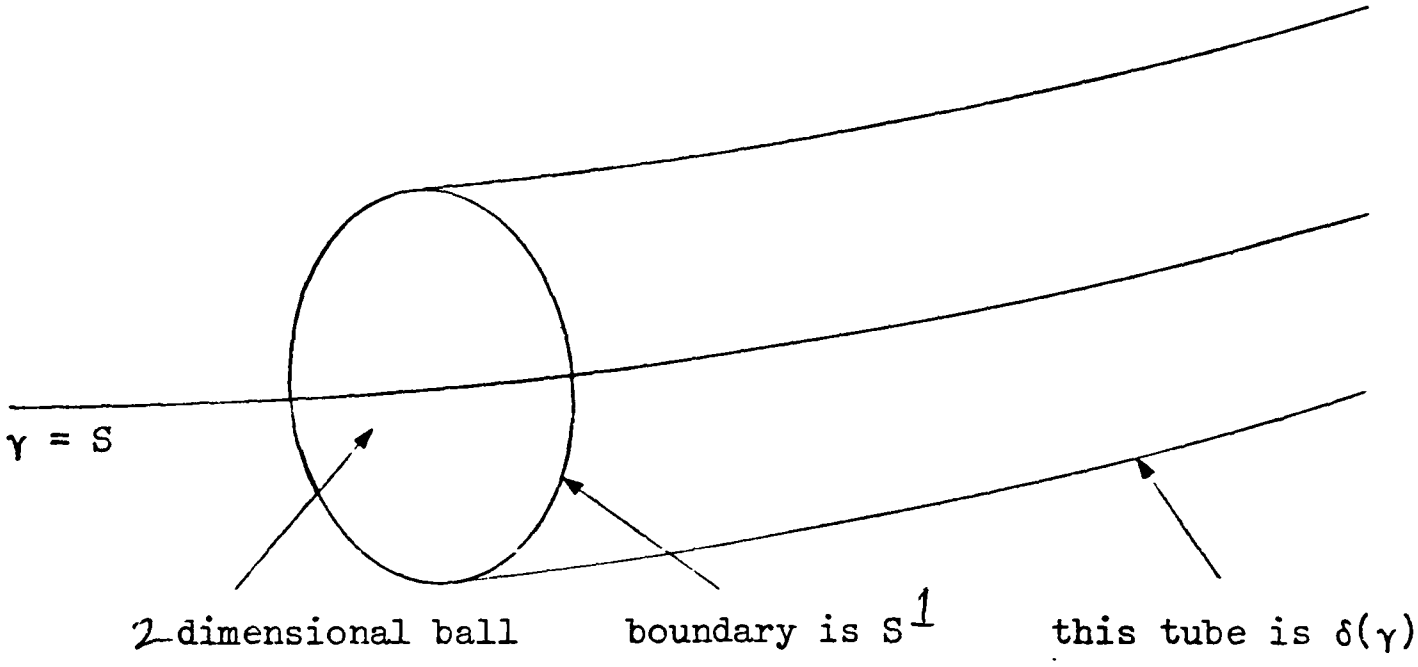
$e : H_n(N, N - S) \xrightarrow{\cong} H_n(X, X - S)$ Excision.

$\partial : H_n(X, X - S) \to H_{n-1}(X - S)$. The boundary map of the relative homology sequence.

Given $[\gamma] \in H_{n-m}(S)$ Φ^{-1} maps each point x of γ to a small m dimensional ball in the fibre of N above x. The boundary ∂ of this m dimensional ball is a sphere S^{m-1} surrounding x. We define

$$\delta = \partial \, e \, \Phi^{-1} .$$

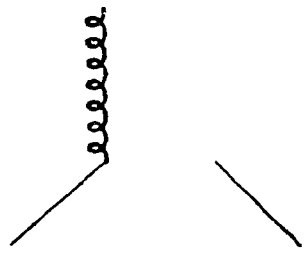
This map, which <u>raises</u> dimension by m - 1, is illustrated for m = 2, n = 1, and dim S = 1 below:

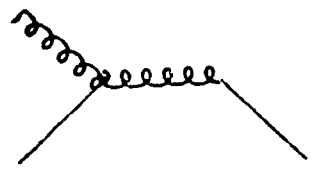


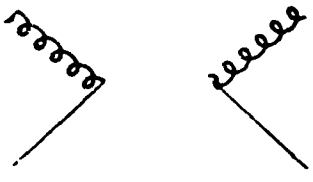
These maps are described in detail in (Leray 1959) and the sequence is proved to be exact. We use the Leray sequence to calculate the homology groups of X - S given the homology groups of X and S (these spaces being simpler than X - S). In fact, in our applications the original space X usually has several submanifolds $S_1, \ldots, S_p$ cut out of it, and we use the Leray sequence iteratively, once for each submanifold. This is best

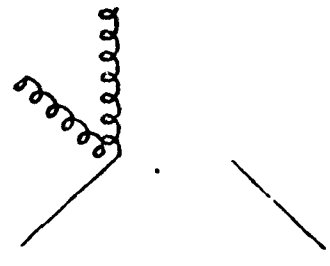
explained using the examples at the end of the last section.

Let

$X =$  ,

$S_1 =$  ,

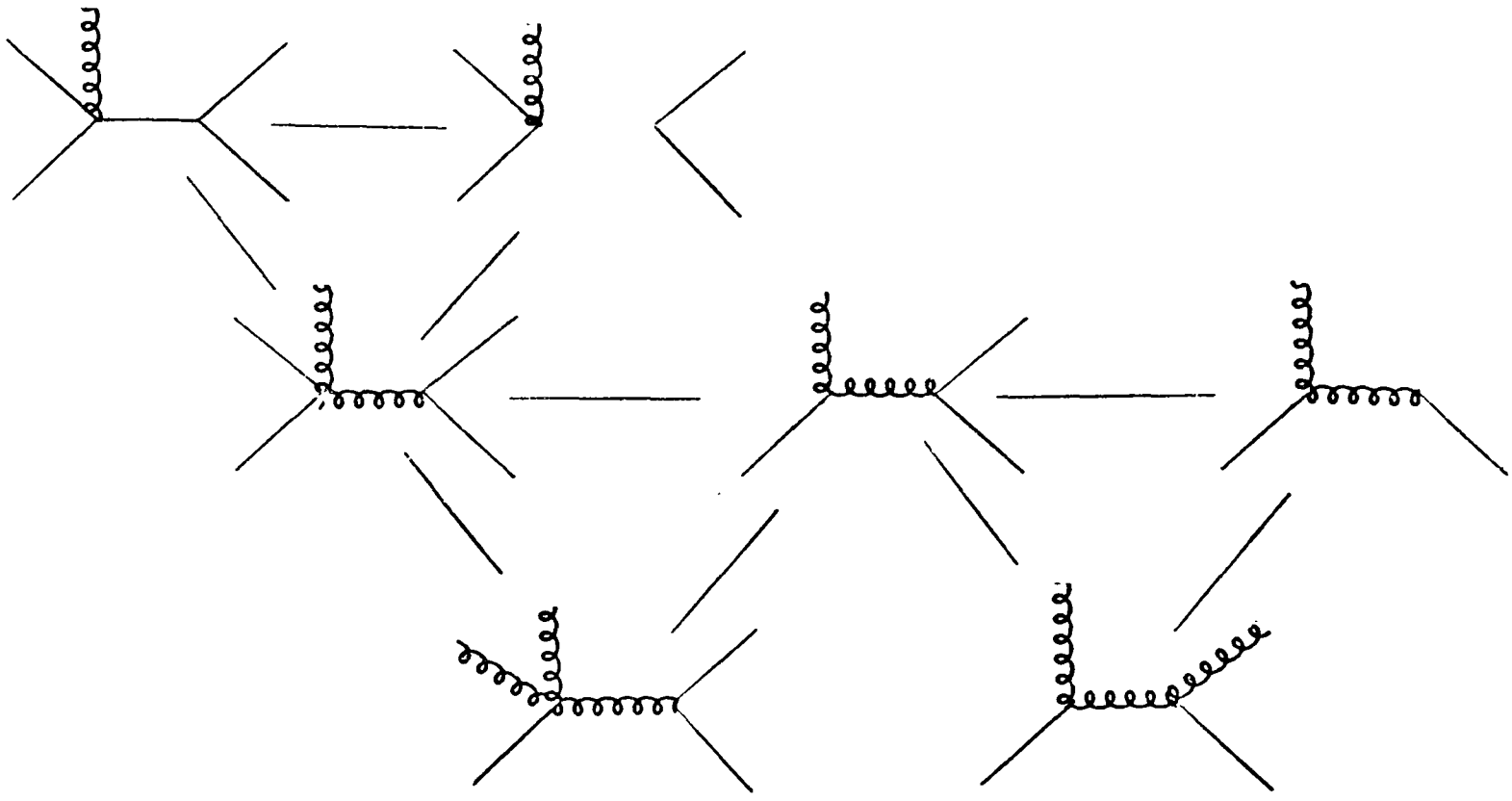
$S_2 =$  ,

and $S_3 =$  .

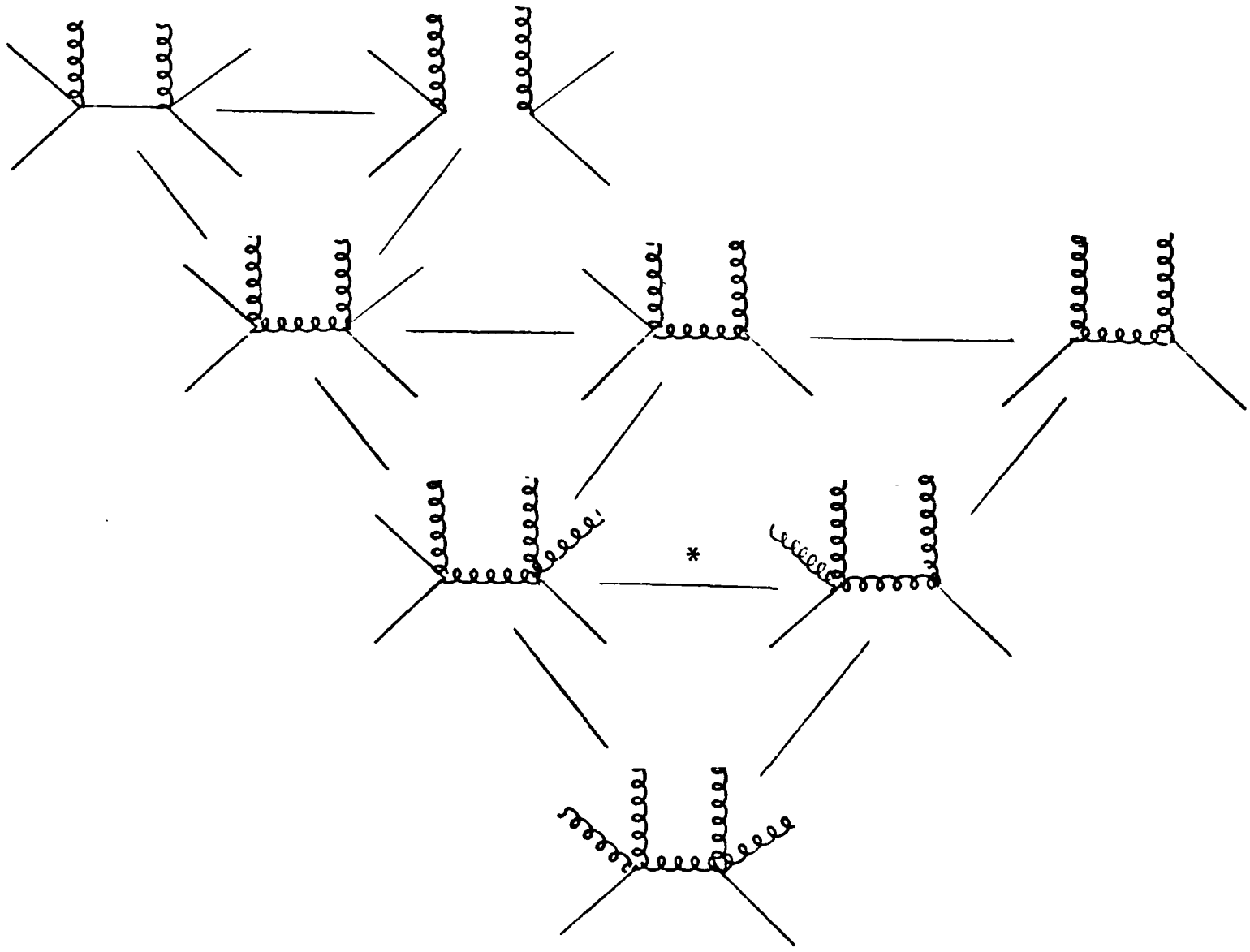
Then the array

$$
\begin{array}{ccccc}
H_*(X - S_1 \cup S_2 \cup S_3) & \longrightarrow & H_*(X - S_2 \cup S_3) & & \\
 & \nwarrow \quad \swarrow & & & \\
H_*(S_1 - S_2 \cup S_3) & \longrightarrow & H_*(S_1 - S_2) & \longrightarrow & H_*(S_1) \\
 & \nwarrow \quad \swarrow & & \nwarrow \quad \swarrow & \\
H_*(S_1 \cap S_3 - S_2) & & H_*(S_1 \cap S_2) & &
\end{array}
$$

becomes (dropping the H_* s):

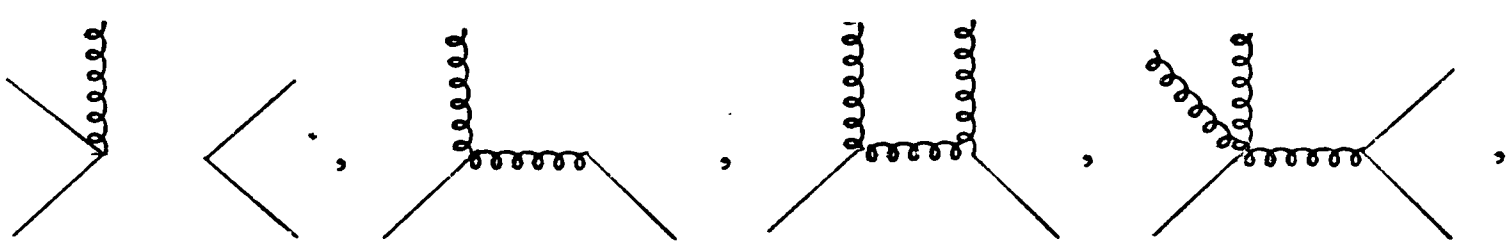


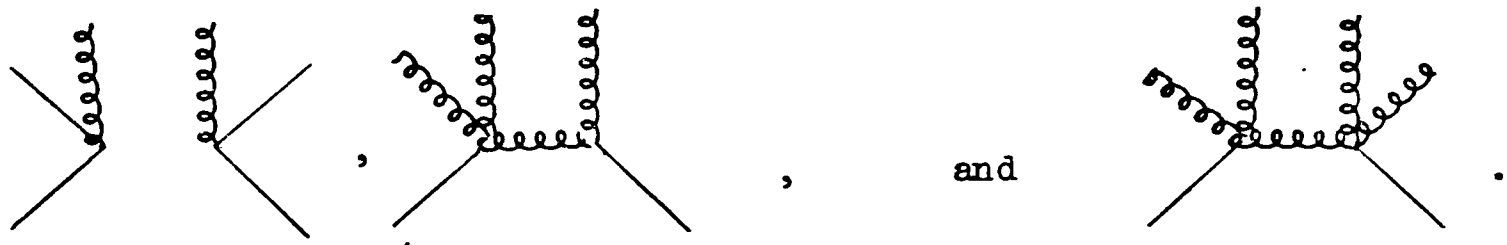
The other example is treated similarly:



Notice that the map marked * is actually a duality isomorphism followed by i. This type of array will often have these duality isomorphisms, which will henceforth be glossed over.

We now have to calculate the homology of the spaces





These spaces are either products or fibre bundles and we analyse their homology in the next section. Two points should be made while we are on the subject of Leray sequences, however.

Firstly, the homology of a space might not be uniquely determined by a particular array of Leray sequences. In that case we can either change the order in which the subspaces are treated or we can make use of the following array (where $m = m_1 + m_2$):

$$
\begin{array}{ccccccc}
\to & H_p(X - S_1 \cup S_2) & \to H_p(X - S_2) & \to & H_{p-m_1}(S_1 - S_2) & \to & H_{p-1}(X - S_1 \cup S_2) \to \\
 & \downarrow & \downarrow & & \downarrow & & \downarrow \\
\to & H_p(X - S_1) & \to \quad H_p(X) & \to & H_{p-m_1}(S_1) & \to & H_{p-1}(X - S_1) \to \\
 & \downarrow & \downarrow (-1)^{m_1 m_2} & & \downarrow & & \downarrow \\
\to & H_{p-m_2}(S_2 - S_1) & \to H_{p-m_2}(S_2) & \to & H_{p-m}(S_1 \cap S_2) & \to & H_{p-m_2-1}(S_2 - S_1) \to \\
 & \downarrow & \downarrow & & \downarrow (-1)^{m_1 m_2 + 1} & & \downarrow \\
\to & H_{p-1}(X - S_1 \cup S_2) & \to H_{p-1}(X - S_2) & \to & H_{p-m_1-1}(S_1 - S_2) & \to & H_{p-2}(X - S_1 \cup S_2) \to \\
 & \downarrow & \downarrow & & \downarrow & & \downarrow
\end{array}
$$

In (Ryman 1975) it is proved that given two submanifolds S_1 and S_2 (of codimension m_1 and m_2 respectively) all the squares in this array commute up to the given signs.

Secondly, we have so far only used the Leray sequence to make topological calculations. In (Leray 1959) - and in (Lascoux 1968) - it is proved that when $m = 2$ the dual of the cobord map

$$\delta: H_{n-2}(S) \to H_{n-1}(X - S)$$

is the residue map

$$\text{res: } H^{n-1}(X - S) \to H^{n-2}(S)$$

This residue map, which we describe in detail in a moment, is a generalisation of Cauchy's integral formula. As such it plays a central role in integrating twistor diagrams.

Suppose the m = 2 submanifold S is defined in some neighbourhood in X by the equation

$$s = 0$$

where s is a holomorphic function of the coordinates on that neighbourhood. A differential form ϕ which is smooth on X - S has a polar singularity of order p on S if p is the smallest number such that $s^p\phi$ is smooth throughout X.

Theorem

Let ϕ be a closed meromorphic form on X - S having a polar singularity of order 1 on S. Then in a neighbourhood of each point of S there exist smooth forms ψ and θ such that

$$\phi = \frac{ds}{s} \wedge \psi + \theta$$

and $\psi\big|_S$ is a closed holomorphic form on S depending only on ϕ. □

We define

$$\text{res}(\phi) = \psi\big|_S.$$

Theorem

Let ϕ be a smooth closed form on $X - S$. Its cohomology class $[\phi] \in H^{n-1}(X - S)$ contains forms having polar singularities of order 1 on S. The set of residue-forms of elements of $[\phi]$ forms a cohomology class $\mathrm{Res}[\phi] \in H^{n-2}(S)$. □

Theorem

$$\oint_{\delta\gamma} \phi = 2\pi i \oint_{\gamma} \mathrm{Res}[\phi]$$ □

where $\gamma \in H_{n-2}(S)$.

This is the generalisation of Cauchy's integral formula. Leray goes further and, in particular,

(i) proves these theorems for relative homology

(ii) shows that the residue map can be iterated (just as we iterated the cobord map).

We will return to the residue map in chapter 4.

§2.5 Fibre Bundles

In this section we define homotopies and fibre bundles. We also use the Künneth formula.

The continuous maps $h_o: X \to Y$ and $h_1: X \to Y$ are homotopic if there exists a continuous map

$$H : X \times I \to Y \qquad \text{(where } I = [0,1]\text{)}$$

such that $H(x,0) = h_0(x)$ and $H(x,1) = h_1(x) \quad \forall x \in X$.

The continuous maps $f: X \to Y$ and $g: Y \to X$ are homotopy inverses of each other if the composite maps fg and gf are homotopic to the appropriate identity maps.

Theorem

Homotopic maps $h_0: X \to Y$ and $h_1: X \to Y$ induce the same map

$$h_{0*} = h_{1*} : H_*(X) \to H_*(Y).$$

□

Corollary

If $f: X \to Y$ and $g: Y \to X$ are homotopy inverses then they induce the isomorphisms

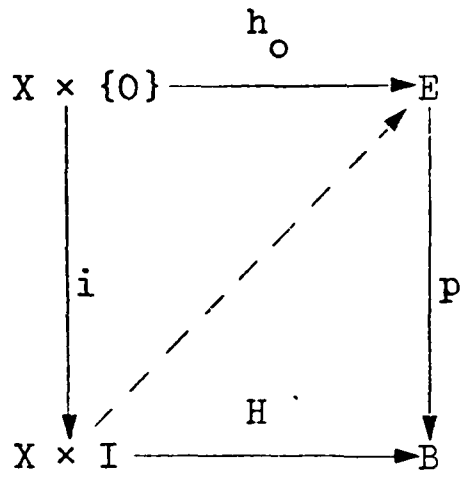
$$f_* : H_*(X) \tilde{=} H_*(Y)$$

and $$g_* : H_*(Y) \tilde{=} H_*(X) .$$

□

We use an important special case of this corollary, when Y is a subspace of X and g is the natural injection. Then f is called a deformation retraction. If Y is just one point in X then X is called contractible.

To define a fibration we consider the following commutative diagram



where p is a continuous map and H is a homotopy. If/there exists a continuous map

for any such diagram

$$L : X \times I \to E$$

(dashed in the diagram) making each triangle commute then p is called a <u>fibration</u>. A <u>fibre bundle</u> (E,B,F,p) consists of a total space E, a base space B, a fibre space F, and a bundle projection p: $E \to B$ which must be locally equivalent to the projection $B \times F \to B$. In other words there exists an open covering {U} of B and for each $U \in \{U\}$ a homeomorphism $\phi_u : U \times F \to p^{-1}(U)$ such that the composite map

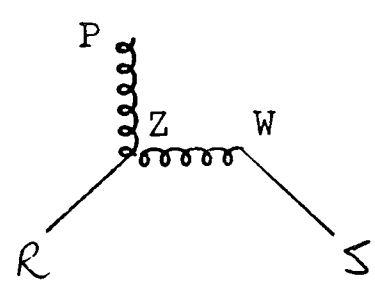
$$U \times F \xrightarrow{\phi_u} p^{-1}(U) \xrightarrow{p} U$$

is projection onto the first factor.

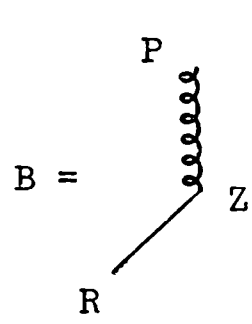
Theorem

If B is paracompact and Hausdorff then p is a fibration. □

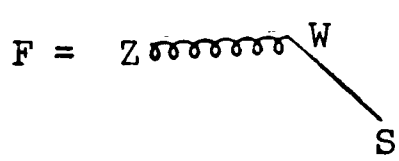
Now consider the space



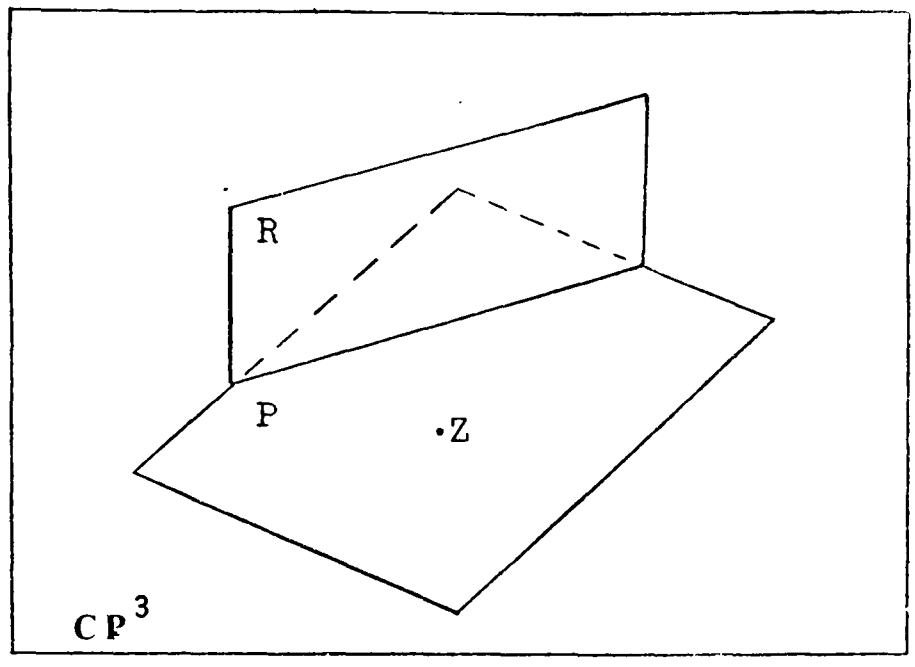
This is a fibre bundle with base



and fibre



(Z and S cannot coincide because Z lies on P whereas - if P, R and S are in general position - S does not.) B is the space of points Z in $\mathbf{CP}^3$ lying on the plane P and avoiding the plane R:



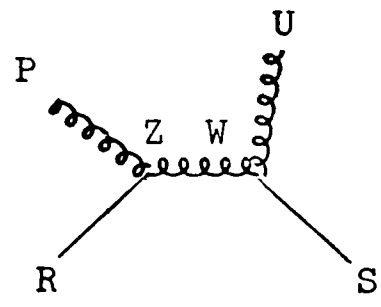
We can think of the line P ∩ R as being the line at infinity in P, so that the point Z is effectively restricted to C^2, which is contractible. Therefore

$$H_n(B) = 0 \quad n \neq 0, \qquad H_o(B) = C$$

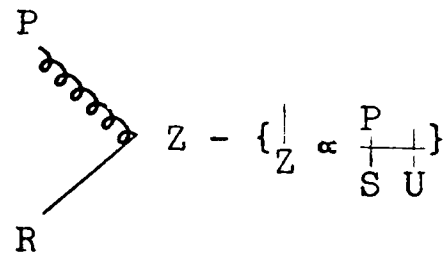
F is dual to B, so

$$H_n(F) = 0 \quad n \neq 0, \qquad H_o(F) = C$$

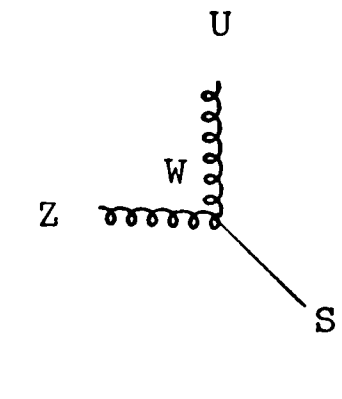
Similarly, the space



is a fibre bundle. Here the base is

B = 

and the fibre is

F = 

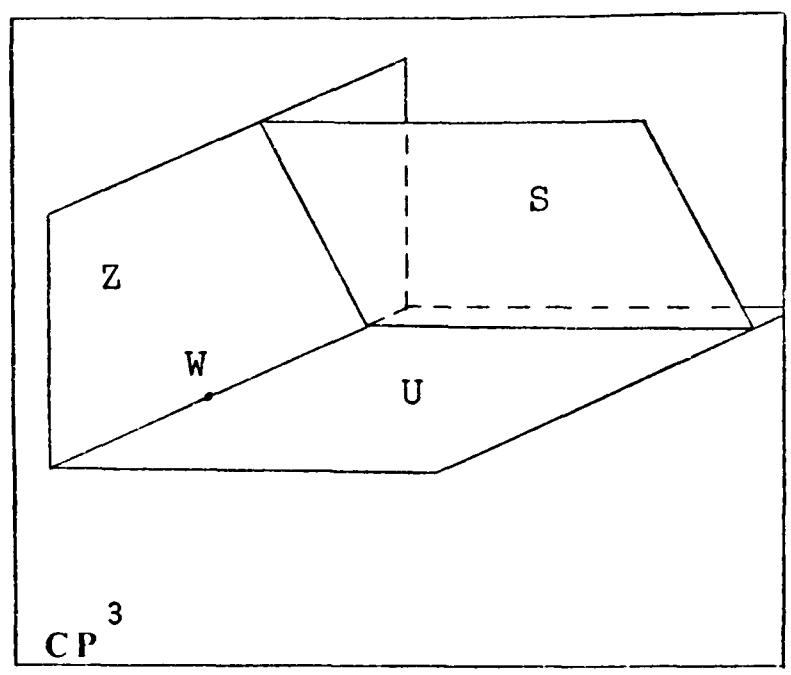
We take the point $\{Z \propto \begin{smallmatrix} P \\ SU \end{smallmatrix}\}$ out of B because if $Z \propto \begin{smallmatrix} P \\ SU \end{smallmatrix}$ the line ZU goes through the point S which makes it impossible for the plane W to contain Z and U but not S. B is the same as in the previous example except for this missing point. Therefore B is the space C^2 - {point}. Let Y be the unit sphere (S^3) around this point. There is a deformation retraction from B to Y, so that

$$H_*(B) = H_*(S^3)$$

which means

$$H_0(B) = H_3(B) = C \quad , \quad H_n(B) = 0 \quad n \neq 0 \text{ or } 3.$$

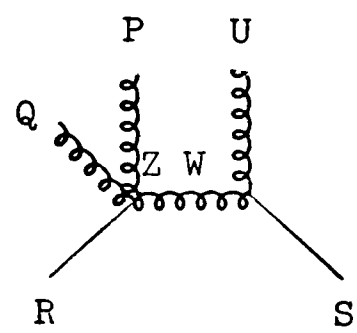
F is dual to the space of points W lying on two planes Z and U and avoiding the plane S:



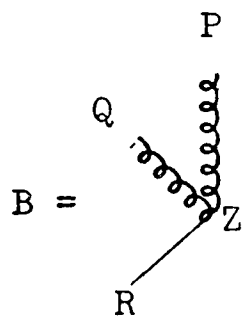
W is confined to the CP^1 $Z \cap U$ minus the point $Z \cap U \cap S$, and CP^1 - {point} is contractible. Hence

$$H_n(F) = 0 \quad n \neq 0, \qquad H_0(F) = C$$

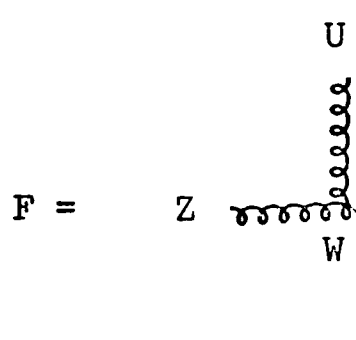
In the case of



the base is



and the fibre is

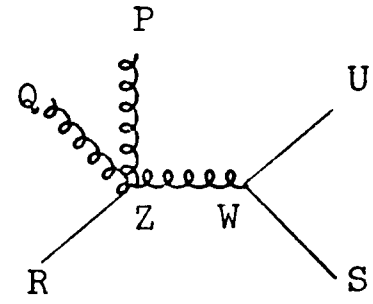


B is contractible and F is dual to B. (Z, U and S are independent because Z lies on the line $P \cap Q$ which does not intersect US as long as P, Q, U, and S are in general position.)

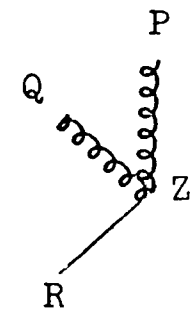
Therefore

$$H_n(B) = H_n(F) = 0 \quad n \neq 0, \qquad H_o(B) = H_o(F) = C$$

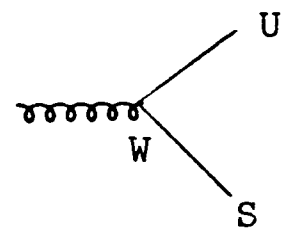
The last fibre space we consider here is



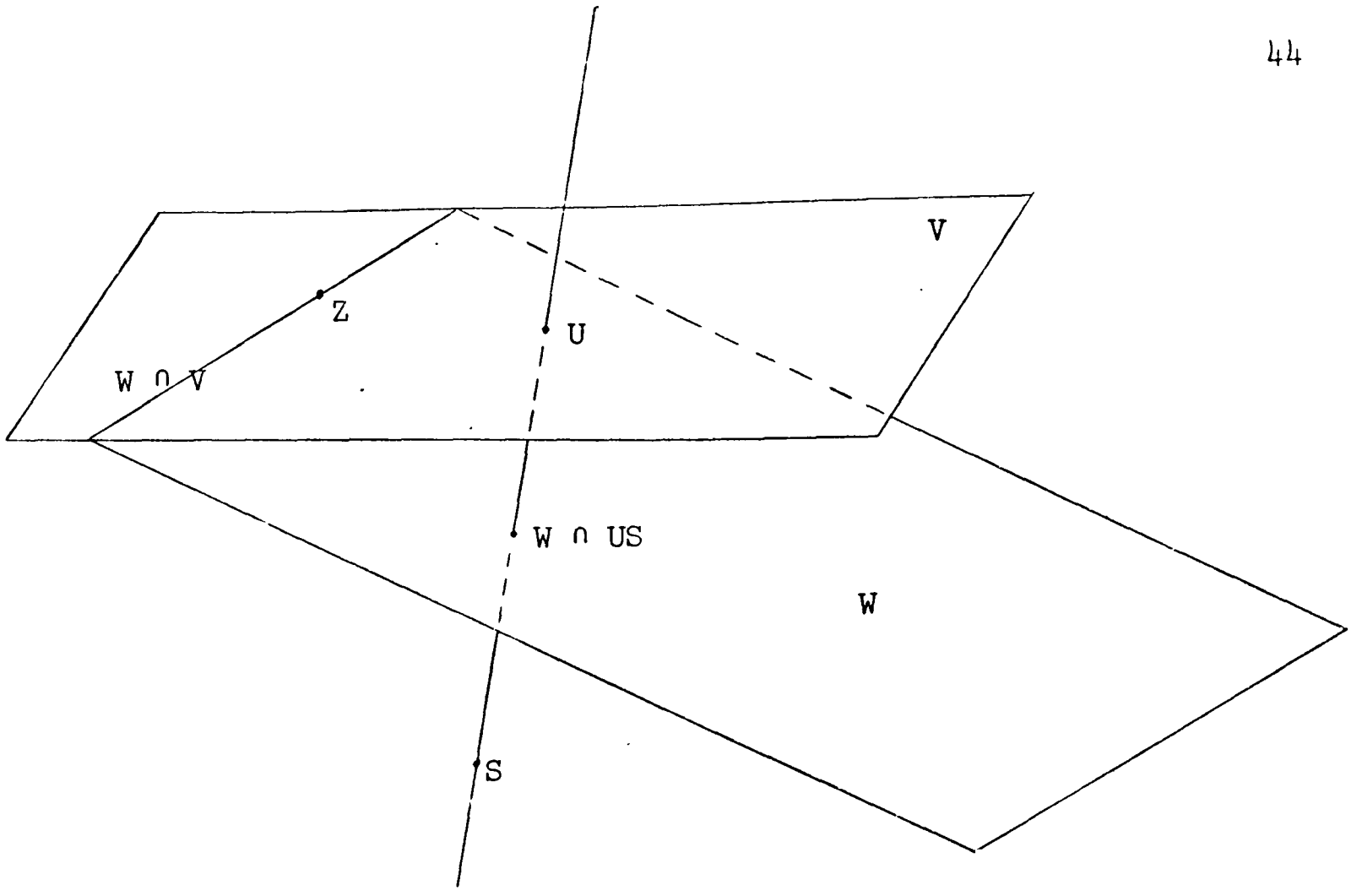
which has base

B = 

and fibre

F = 

Again the base is contractible and again Z, U, and S are independent because Z lies on $P \cap Q$. F is the space of planes W containing the point Z and avoiding the points U and S. Choose any plane V through U and Z avoiding S:

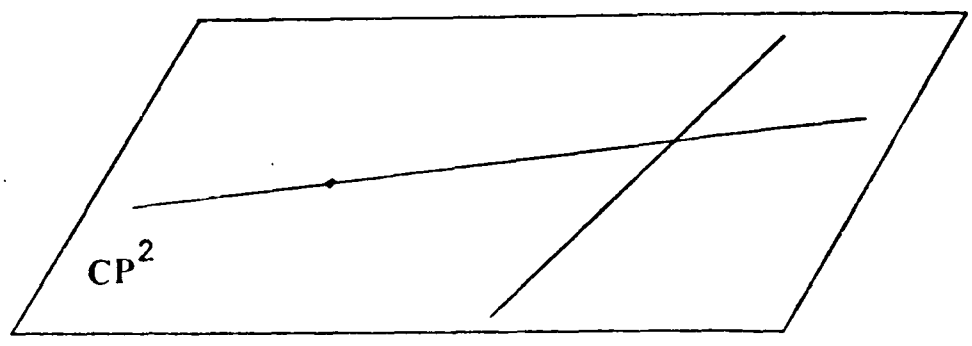


The position of W is now determined by two independent choices:

(i) the line $W \cap V$ through Z avoiding U

(ii) the point $W \cap US$ avoiding U and S.

Let X be the space of lines in (i) and let Y be the space of points in (ii). X is dual to the space of points in CP^2 lying on one line in the CP^2 and avoiding another:



Therefore X is dual to a CP^1 - {point}, which is contractible, so that $H_n(X) = 0 \quad n \neq 0$, $H_o(X) = C$. Y is the space CP^1 - {two points}.

There is a deformation retraction from Y to an S^1 separating these points. Therefore

$$H_o(Y) = H_1(Y) = C \ , \quad H_n(Y) = 0 \qquad n \geq 2.$$

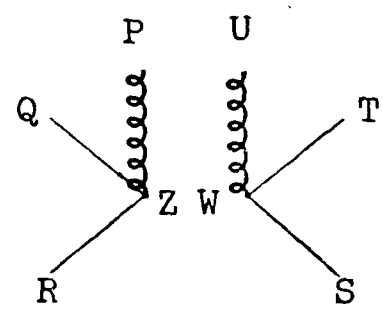
We have $F = X \times Y$, and from the Künneth formula for singular homology

$$H_p(X \times Y) \overset{\sim}{=} \bigoplus_{\substack{m,n \\ m+n=p}} H_m(X) \otimes H_n(Y).$$

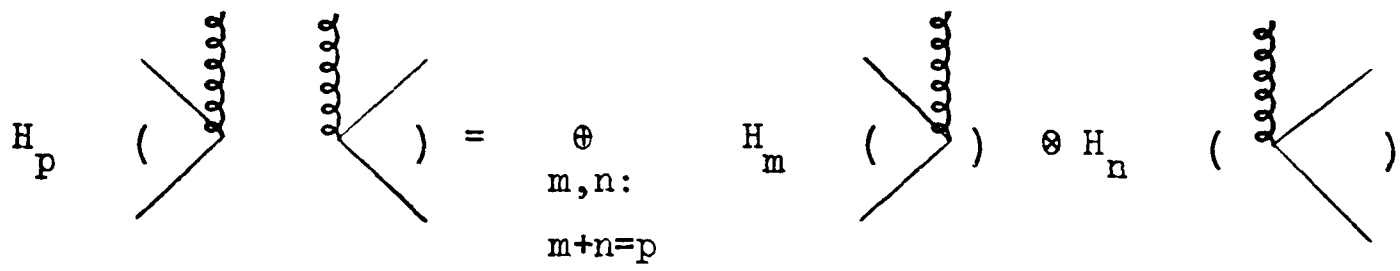
Therefore

$$H_o(F) = H_1(F) = C \ , \quad H_n(F) = 0 \qquad n \geq 2.$$

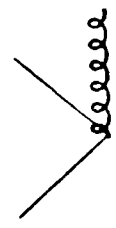
This result enables us to write down the homology of the space



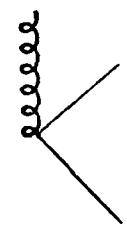
Again using the Künneth formula:



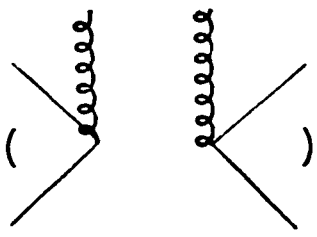
The space



is dual to



which is the same as the space F in the last example. Therefore

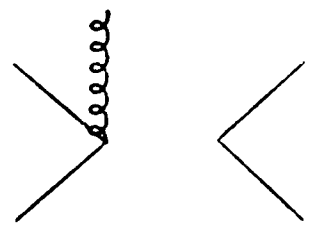
H_o  $= C$, $H_1 = C \oplus C$,

2.5.1.

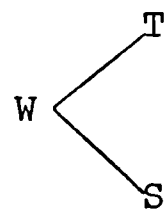
$H_2 = C$, and $H_n = 0 \quad n \geq 3$.

(We omit the space if there is no likelihood of confusion.)

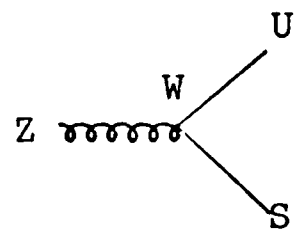
The next example



which is also a product, only differs in the factor



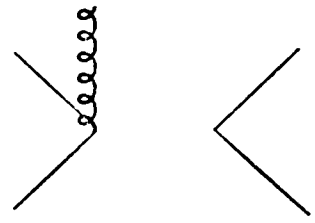
This space can be studied in exactly the same way as



except that no mention is made of the point Z. This means that the space X becomes the space of lines lying in a plane and avoiding a fixed point in that plane. Therefore X is still contractible, so that as before

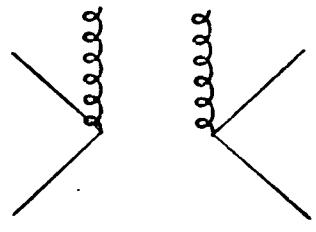
$$H_o\left(\langle\ \rangle\right) = H_1 = C\ , \qquad H_n = 0 \qquad n \geq 2$$

and therefore

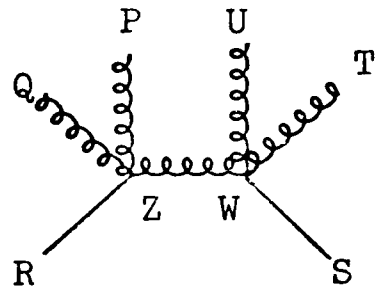


2.5.2

has the same homology as

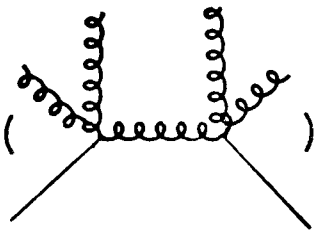


The last example is



2.5.3.

The point Z has to lie on the line $P \cap Q$. The plane W has to contain Z, U, and T, which are always independent because the lines $P \cap Q$ and UT do not intersect so long as P, Q, U, and T are in general position. So W is determined by Z. Z has to avoid R, and it also has to avoid forcing W through S, which means Z has to avoid the plane $\boxed{UTS}$. Therefore Z is confined to $\mathbf{CP}^1$ - {the two points $\boxed{PQR}$ and $\boxed{PQ}\,\boxed{UTS}$}, which has a homotopy retraction to S^1, so that

H_o  $= H_1 = \mathbf{C}, \qquad H_n = 0 \qquad n \geq 2.$

We have been able to write down the homology groups of the last three examples. For the other examples - the fibre bundles - we have only calculated the homology groups of the fibres and bases separately. It remains to relate the homology groups of the total space of a fibre bundle to those of its base and fibre.

§2.6 Spectral Sequences

After briefly describing the spectral sequence of a fibre bundle we shall use it to show that

(i) F contractible $\Rightarrow H_*(E) \overset{\sim}{=} H_*(B)$

(ii) B contractible $\Rightarrow H_*(E) \overset{\sim}{=} H_*(F)$

so that we can deal with the fibre bundles of the previous section. This is an example of using a piledriver (the spectral sequence) to crack an egg, but its introduction here is justified by its increasing use elsewhere in twistor theory.

We follow the description in (MacLane 1975) of a spectral sequence.

We first define a bigraded module E to be a family $\{E_{p,q}\}$ of modules, one for each pair of indices $p,q = 0, \pm1, \pm2, \ldots$. A differential $d: E \to E$ of bidegree $(-r, +r-1)$ is a family of homomorphisms $\{d: E_{p,q} \to E_{p-r,q+r-1}\}$, one for each pair p,q, with $d^2 = 0$.

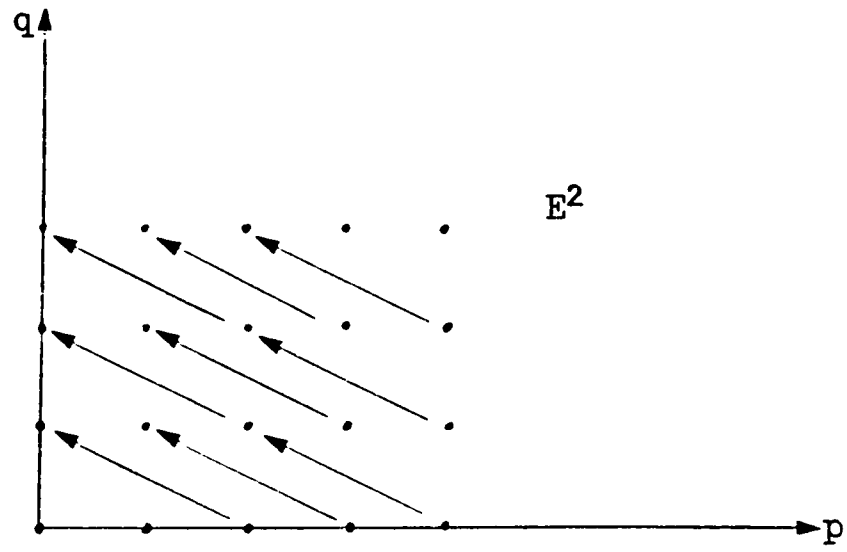
A <u>spectral sequence</u> $E = \{E^r, d^r\}$ is a sequence E^2, E^3, ... of bigraded modules, each with a differential

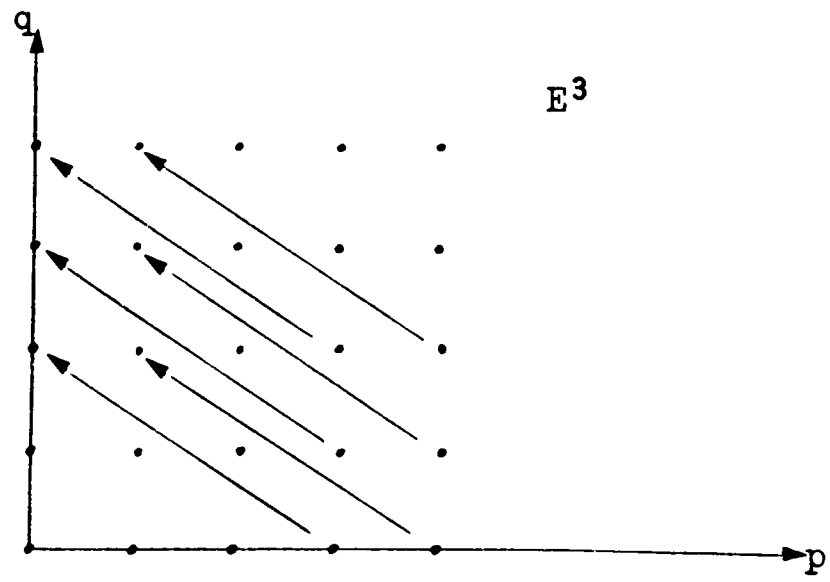
$$d^r: E^r_{p,q} \to E^r_{p-r,q+r-1}$$

of bidegree $(-r, r-1)$, satisfying

$$E^{r+1}_{p,q} = \ker [d: E^r_{p,q} \to E^r_{p-r,q+r-1}]/dE^r_{p+r,q-r+1}$$

A <u>first quadrant</u> spectral sequence is one for which $E^r_{p,q} = 0$ if $p < 0$ or $q < 0$. The E^2 and E^3 levels of such a spectral sequence are the lattices





where the lattice points are the modules $E^r_{p,q}$ and the arrows are the differentials.

At the E^r level of a first quadrant spectral sequence the differential is

$$E^r_{p+r,q-r+1} \xrightarrow{d^r} E^r_{p,q} \xrightarrow{d^r} E^r_{p-r,q+r-1}$$

and for $r > \max(p,q+1)$ this becomes

$$0 \to E^r_{p,q} \xrightarrow{d^r} 0$$

so that $E^{r+1}_{p,q} = E^r_{p,q}$. Therefore for fixed p,q $E^r_{p,q}$ is eventually constant in r, and is then denoted $E^\infty_{p,q}$.

Theorem 2.6.1 (Leray-Serre)

If (E,B,F,p) is an orientable fibre bundle there is for each n a nested family of subgroups of $H_n(E)$ (called a composition series)

$$0 < H_{o,n} < H_{1,n-1} < \ldots < H_{n-1,1} < H_{n,o} = H_n(E)$$

and a first quadrant spectral sequence such that

$$E^2_{p,q} = H_p(B,H_q(F)) \text{ and } E^\infty_{p,q} = H_{p,q}/H_{p-1,q+1}$$

□

We can use the universal coefficient theorem

$$H_p(B,H_q(F)) \tilde{=} H_p(B) \otimes H_q(F)$$

to calculate $E^2_{p,q}$. Theorem 2.6.1. is proved in (Spanier 1966) for the more general case of an orientable fibration, where we need to use locally constant sheaves (Ryman 1975).

In the special case when $B = S^k$ the spectral sequence becomes an ordinary long exact sequence called the Wang sequence. If, on the other hand, $F = S^k$, we obtain instead the Gysin sequence.

However, we study two even more drastic special cases. Firstly, suppose that the base B is contractible. Then the fibre bundle is orientable and the E^2 level becomes

q ↑			
⋮	⋮	⋮	
$H_2(F)$	0	0	...
$H_1(F)$	0	0	...
$H_o(F)$	0	0	... ⟶ p

so that

$$E^2_{p,q} = E^\infty_{p,q} = \begin{cases} 0 & p \neq 0 \\ H_q(F) & p = 0 . \end{cases}$$

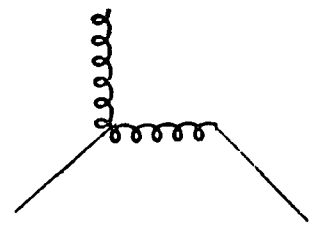
Therefore the composition series for $H_n(E)$ becomes

$$0 < H_n(F) < H_n(F) < \ldots < H_n(E)$$

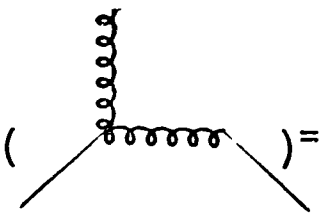
In other words $H_n(F) = H_n(E)$ for all n.

Secondly, suppose instead that the fibre F is contractible. Then a similar argument gives $H_n(B) = H_n(E)$ for all n.

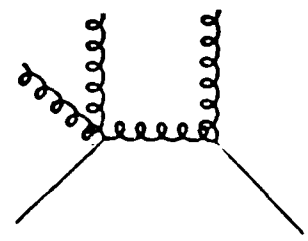
Now we turn to the examples of the previous section. Both the fibre and the base of the space



were contractible. Therefore

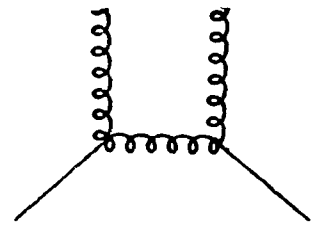
H_o () $= C$, $H_n = 0$ $n \neq 0$. 2.6.2.

The same thing happens with the space



2.6.3.

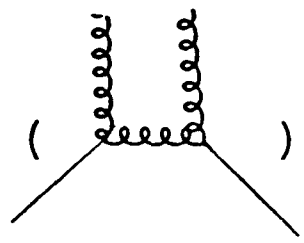
The fibre of the space



was also contractible, but the base had homology groups

$$H_o(B) = H_3(B) = C \quad , \qquad H_n(B) = 0 \qquad n \neq 0 \text{ or } 3.$$

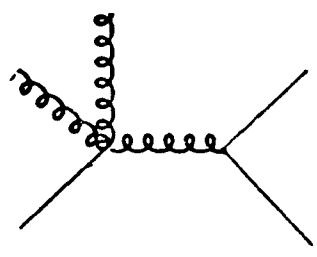
Therefore

H_o () $= H_3 = C$,

2.6.4.

$$H_n = 0 \qquad n \neq 0 \text{ or } 3.$$

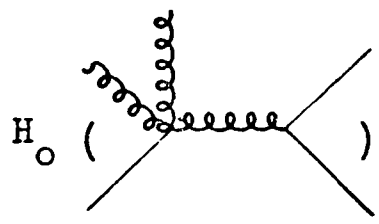
Finally, the space



had a contractible base, and its fibre F had homology

$$H_o(F) = H_1(F) = C \quad , \quad H_n(F) = 0 \qquad n \geq 2.$$

Therefore

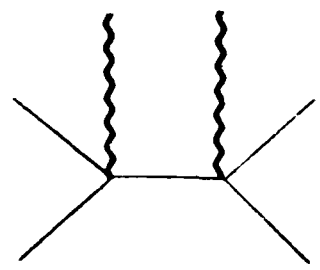
H_o () $= H_1 = C$,

2.6.5.

$$H_n = 0 \qquad n \geq 2.$$

§2.7 Calculation and Discussion

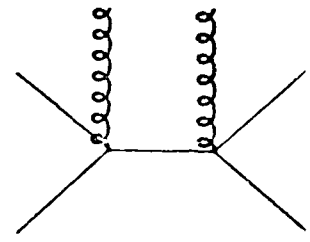
We have completed the process of breaking down the homology groups of the space



into those of simpler spaces and calculating the groups of all these simpler spaces. We next put all the results together.

Theorem 2.7.1

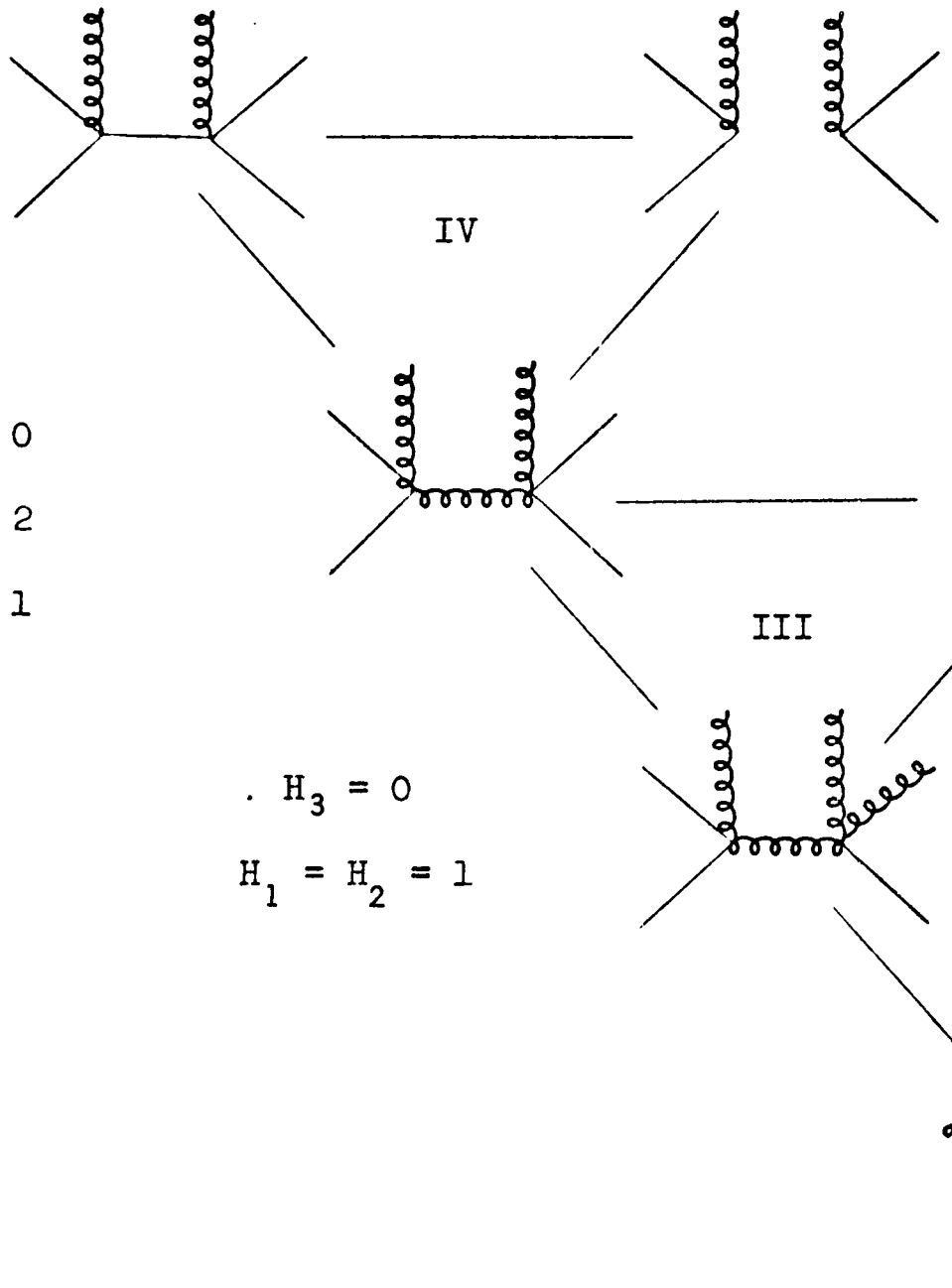
The space



has the following homology groups: $H_5 = 0$, $H_4 = 2$, and $H_3 = 1$. (We have omitted those not used later, and we have simplified the notation a little: $H_4 = 2$ should read $H_4 = C \oplus C$. In other words H_4 has 2 generators.)

Proof:

The following array of Leray sequences proves the result:



$H_5 = H_4 = H_3 = 0$

from 2.5.1

IV

$H_4 = 0$

$H_3 = 2$

$H_2 = 1$

$H_4 = H_2 = 0$

$H_3 = 1$

III

II

$H_4 = H_2 = 0$

$H_3 = 1$

from 2.6.3

. $H_3 = 0$

$H_1 = H_2 = 1$

$H_n = 0 \quad n \neq 0$

$H_o = 1$

from 2.6.3

I

$H_2 = 0$

$H_o = H_1 = 1$

from 2.5.3

Sequence I : we insert the known groups and obtain

$$0 \xrightarrow{\delta} H_3 \xrightarrow{i} 0 \xrightarrow{\cap} 1 \xrightarrow{\delta} H_2 \xrightarrow{i} 0 \xrightarrow{\cap} 1 \xrightarrow{\delta} H_1 \xrightarrow{i} 0$$

Hence the results quoted on the array.

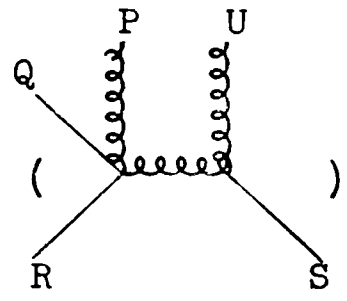
Sequence II :

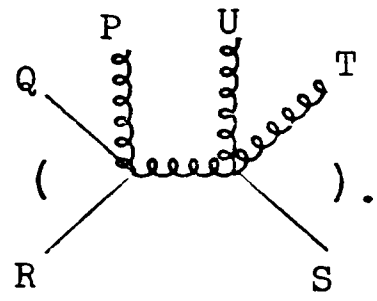
$$0 \xrightarrow{\delta^*} H_4 \xrightarrow{i} 0 \xrightarrow{\cap} 0 \xrightarrow{\delta} H_3 \xrightarrow{i} 1 \xrightarrow{\cap} 0 \xrightarrow{\delta} H_2 \xrightarrow{i} 0$$

Sequence III :

$$0 \xrightarrow{\delta} H_4 \xrightarrow{i} 0 \xrightarrow{\cap} 1 \xrightarrow{\delta} H_3 \xrightarrow{i} 1 \xrightarrow{\cap} 1 \xrightarrow{\delta} H_2 \xrightarrow{i} 0$$

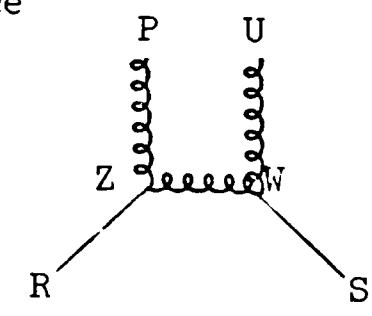
Clearly $H_4 = 0$, but we do not yet have enough information to decide on H_3 and H_2. Consider the intersection map

$\cap_{III} : H_3$ 

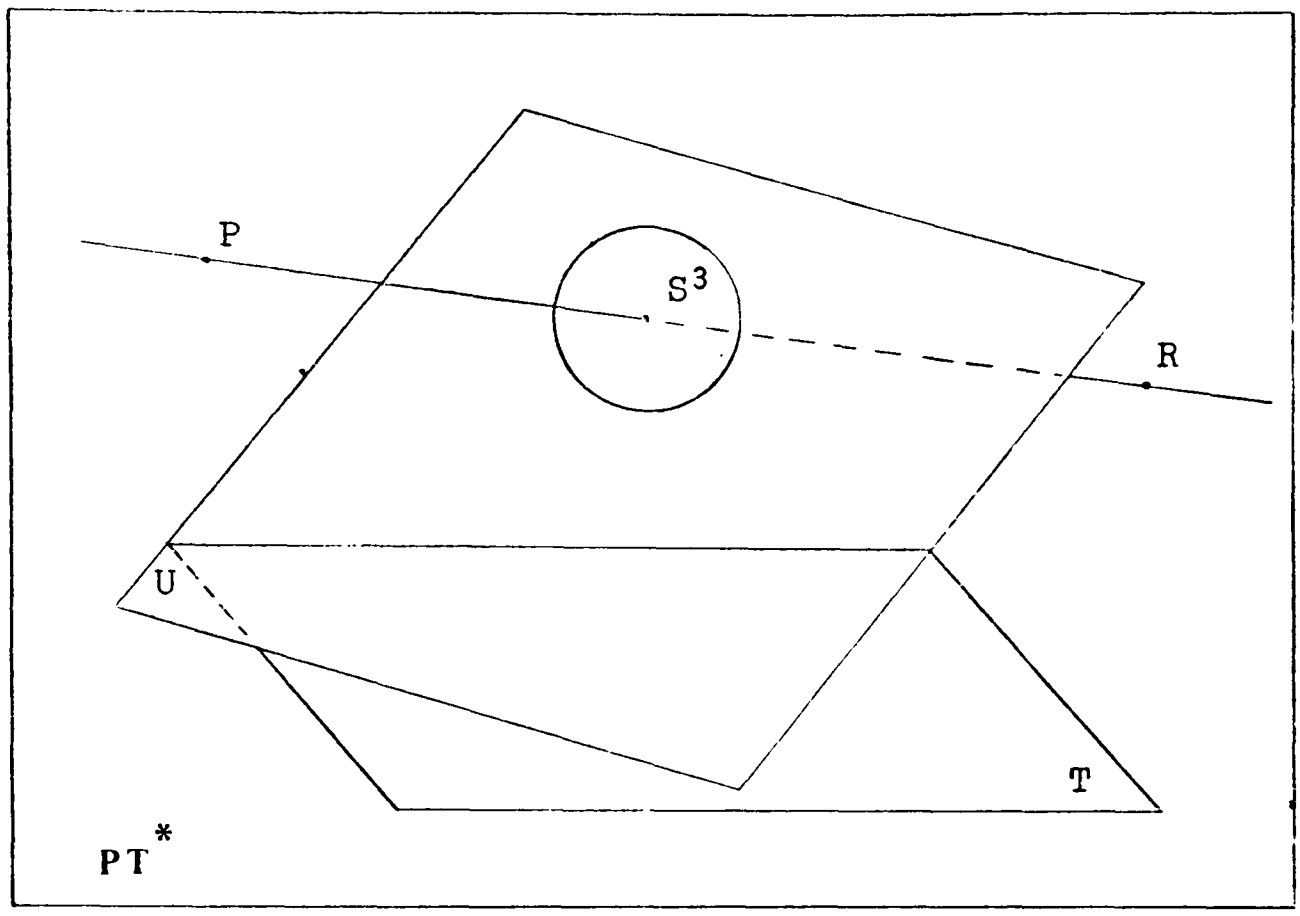
$\rightarrow$ H_1 

.

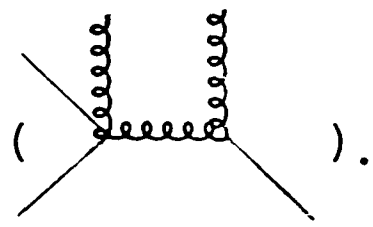
We have proved that the space



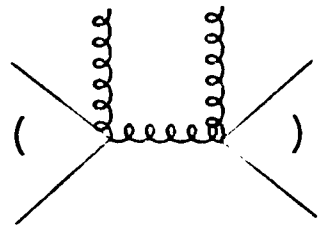
has a deformation retraction to an S^3. In dual twistor space we can think of this S^3 as lying in the plane U and surrounding the point of intersection of the line PR and the plane U.



Sequence II shows that this S^3 is also the generator for

H_3  .

The intersection map $\cap_{III}$ takes the intersection of this generator with the plane T. Even if the S^3 does intersect the line $U \cap T$, that intersection will be contractible. Therefore $\cap_{III}$ is the zero map, and so

H_3 () = 2, $H_2 = 1$.

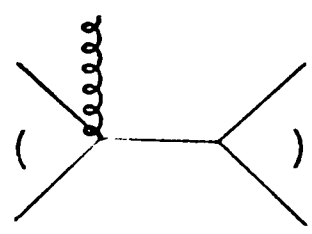
Sequence IV:

$$0 \to H_5 \to 0 \to 2 \to H_4 \to 0 \to 1 \to H_3 \to 0.$$

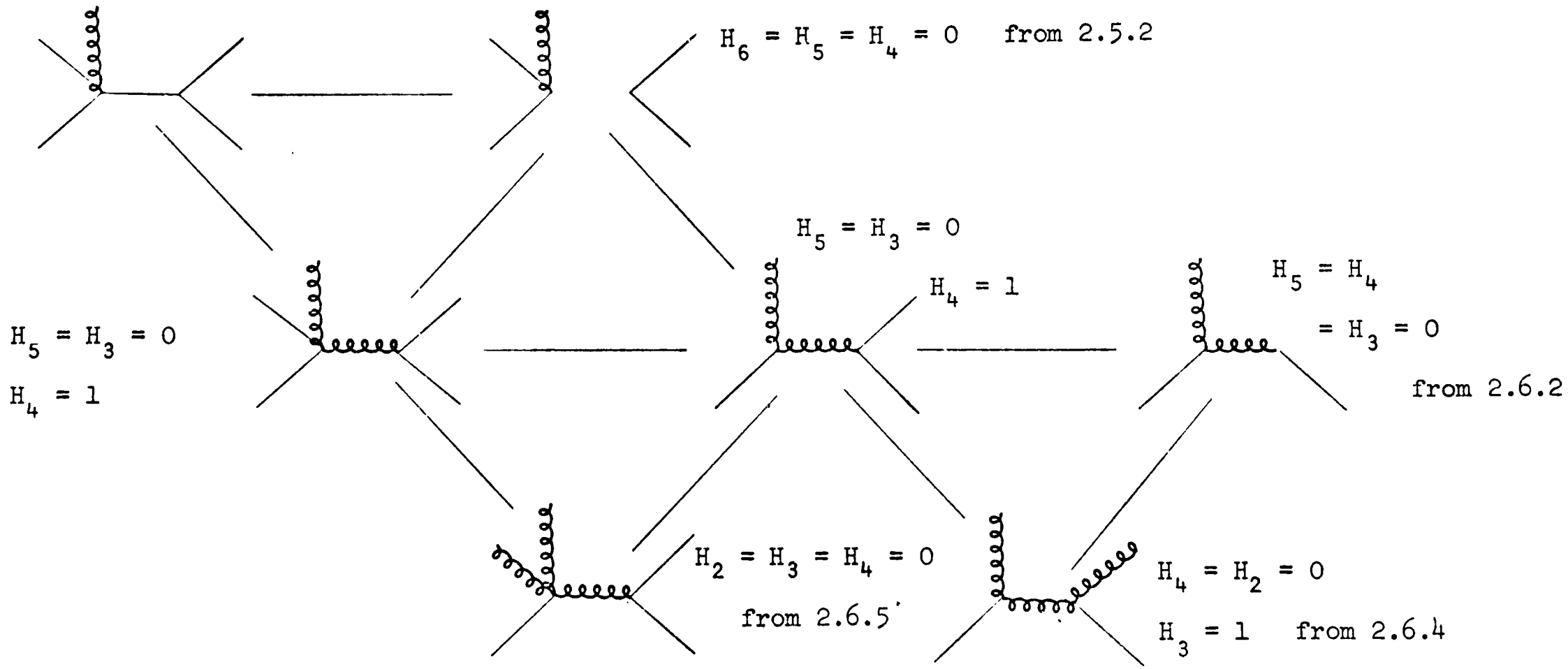
Hence the theorem. □

For the rest of this section we simply display the array of Leray sequences, without going into any more detail. There are no more cases like sequence III above which need a closer look.

Theorem 2.7.2

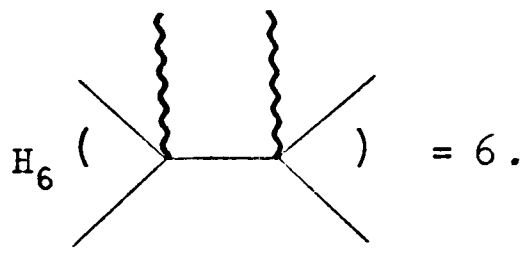
H_6 () = $H_4 = 0$, $H_5 = 1$.

Proof:

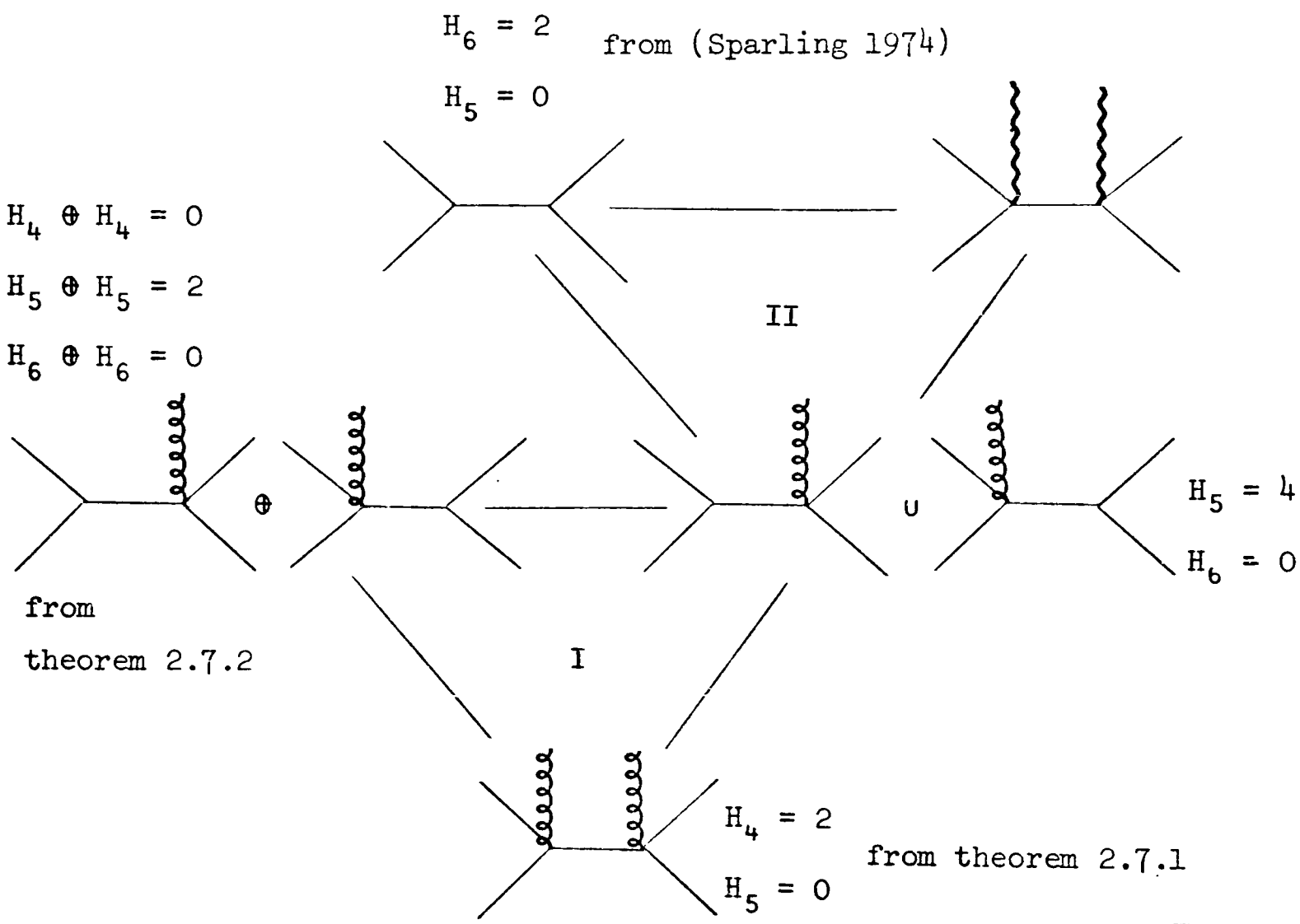

$H_6 = H_5 = H_4 = 0$ from 2.5.2
$H_5 = H_3 = 0$
$H_4 = 1$
$H_5 = H_3 = 0$
$H_4 = 1$
$H_5 = H_4$
$= H_3 = 0$
from 2.6.2
$H_2 = H_3 = H_4 = 0$
from 2.6.5
$H_4 = H_2 = 0$
$H_3 = 1$ from 2.6.4

□

Theorem 2.7.3

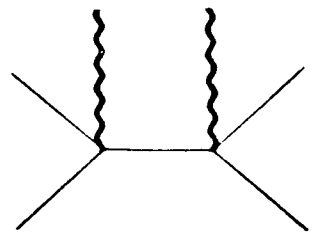


Proof:

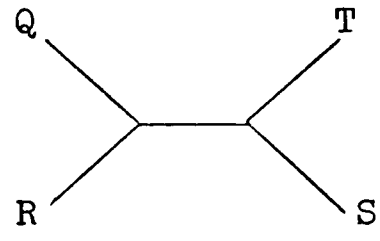


□

Sequence I is the Mayer-Vietoris sequence while sequence II is the relative homology sequence. Sequence II shows that two of the six independent contours for

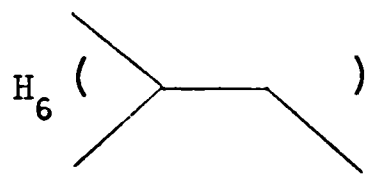


are the well-known scalar product contours for



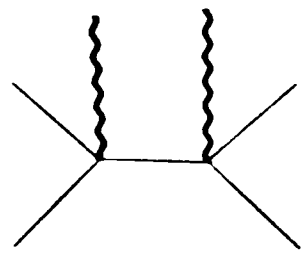
Only one of these contours (the loose contour) survives when <u>either</u> Q and R <u>or</u> T and S are moved into coincidence.

One way of proving this is to show that

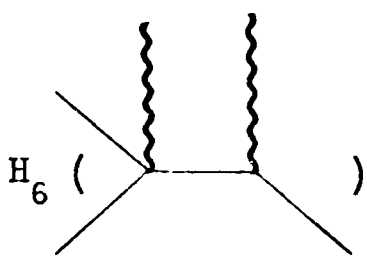


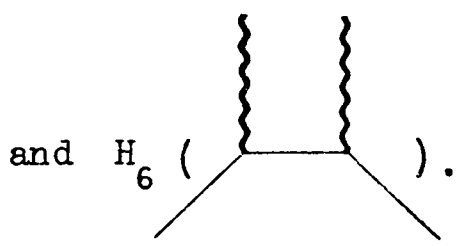
$= 1$.

Similarly, we can analyse the properties of the contours for

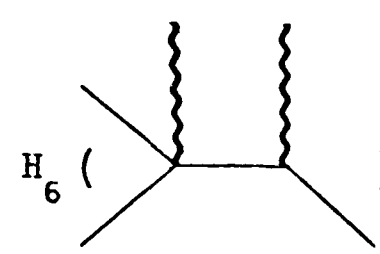


by calculating



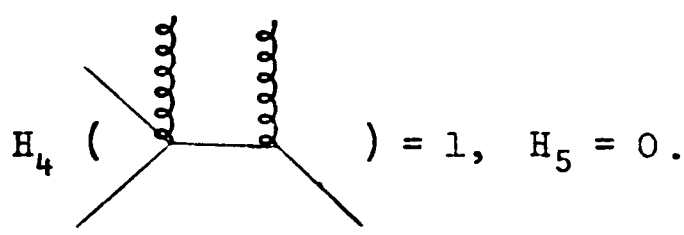
and 

Theorem 2.7.4

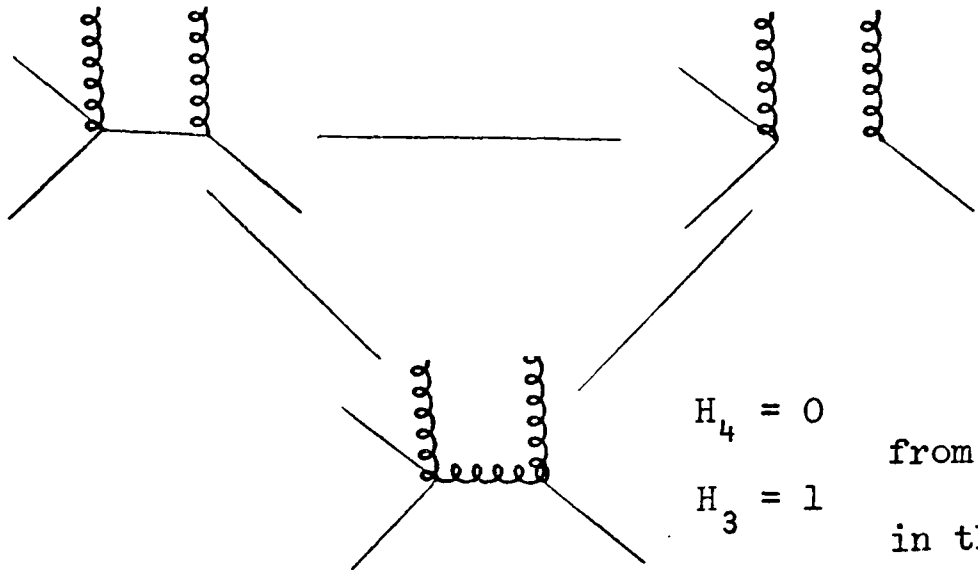


The proof makes use of the following three lemmas.

Lemma



Proof:



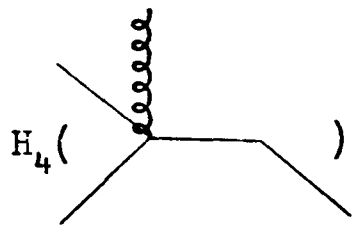
$H_4 = H_5 = 0$ from the Künneth formula

$H_4 = 0$

$H_3 = 1$

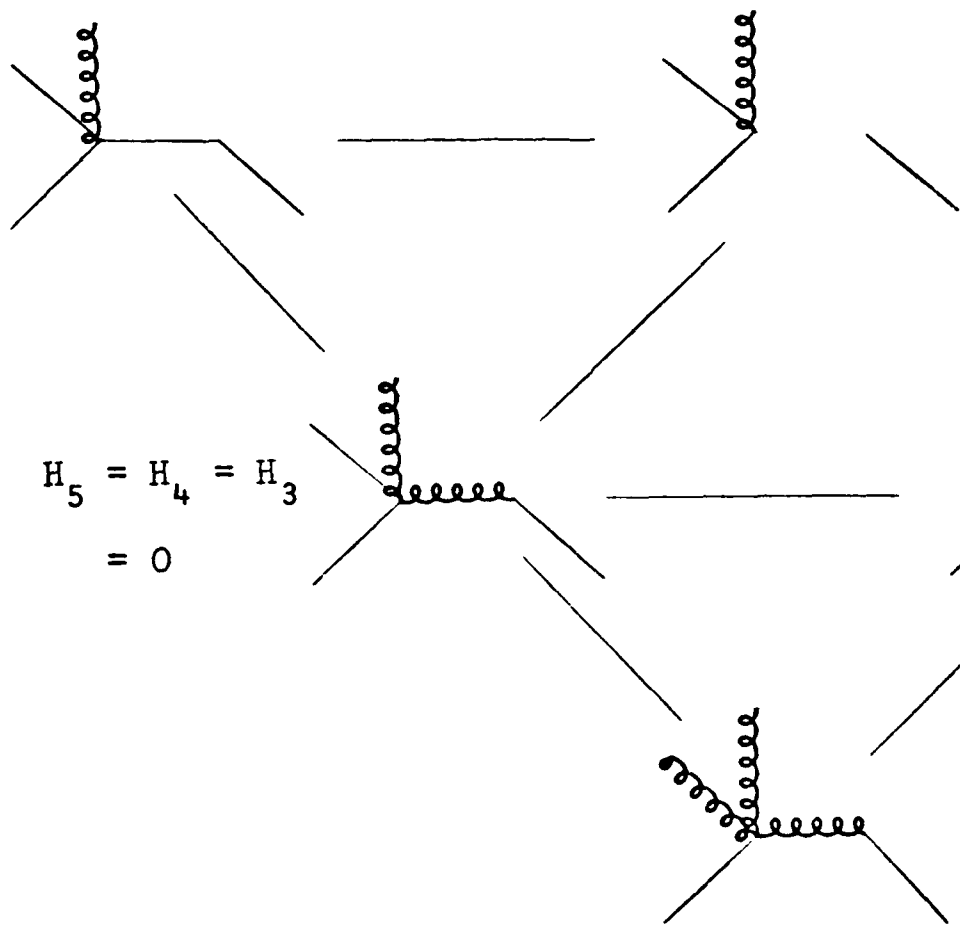
from the array in theorem 2.7.1 □

Lemma



$H_4($ $) = H_5 = H_6 = 0.$

Proof:



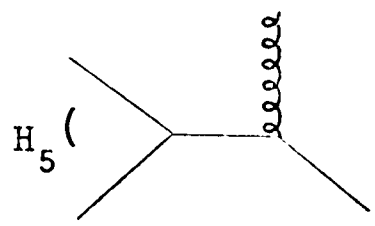
$H_4 = H_5 = H_6 = 0$ from the Künneth formula

$H_5 = H_4 = H_3 = 0$

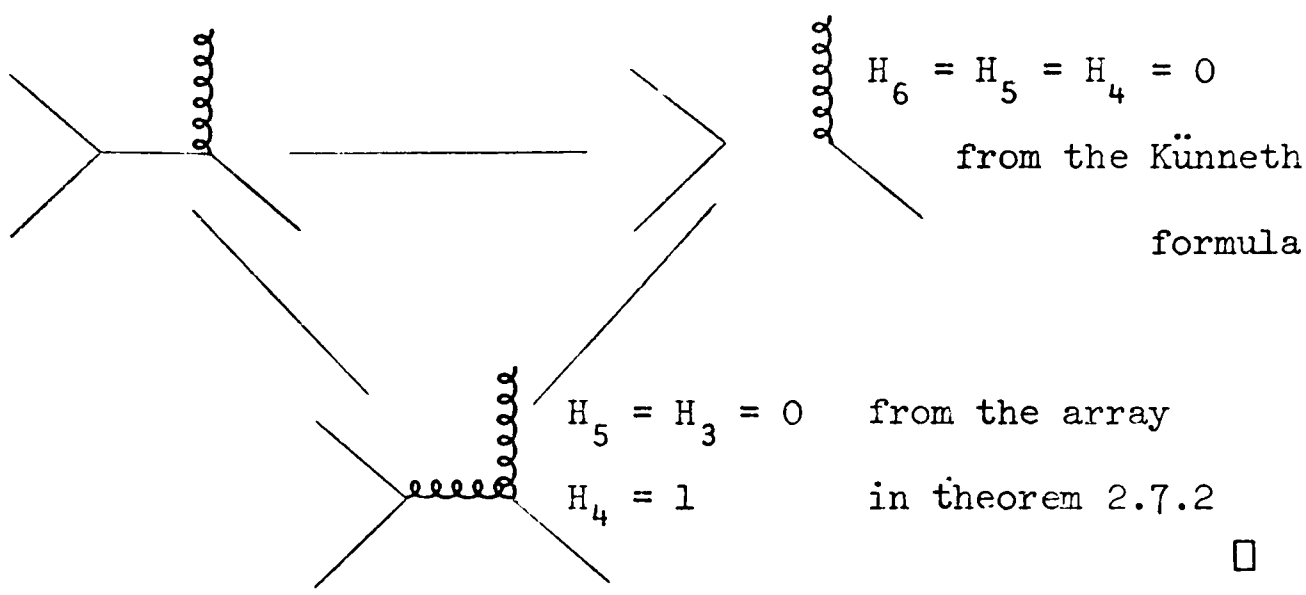
$H_5 = H_4 = H_3$ from 2.6.2

$H_4 = H_3 = H_2 = 0$ because both the fibre and the base are contractible □

Lemma



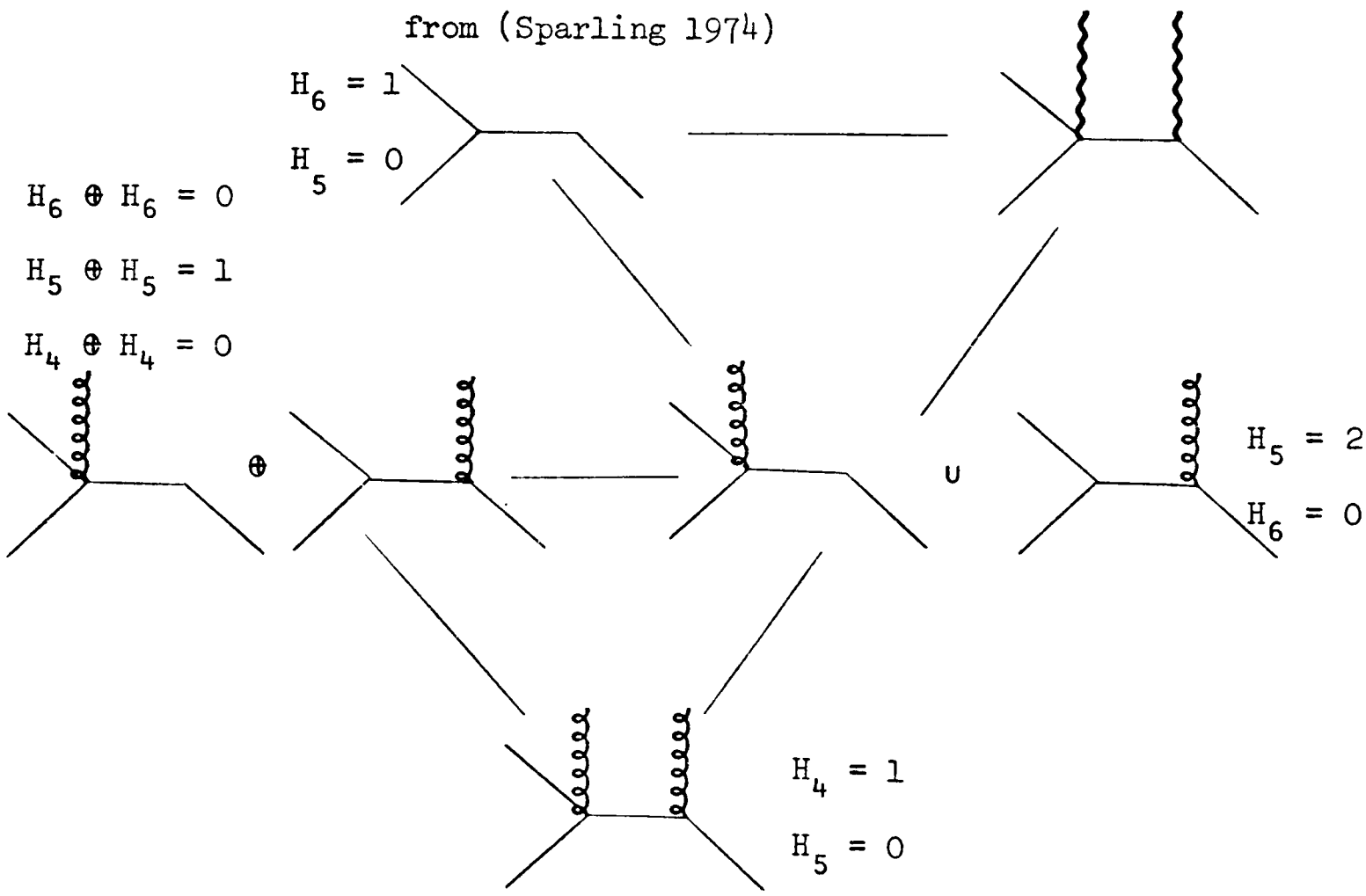
Proof:



□

Now we return to the theorem.

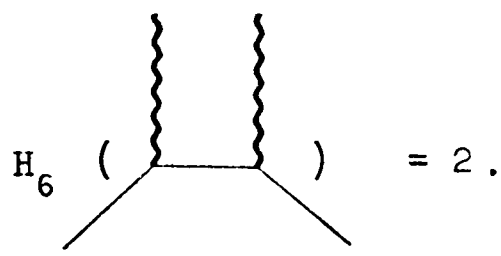
Proof:



□

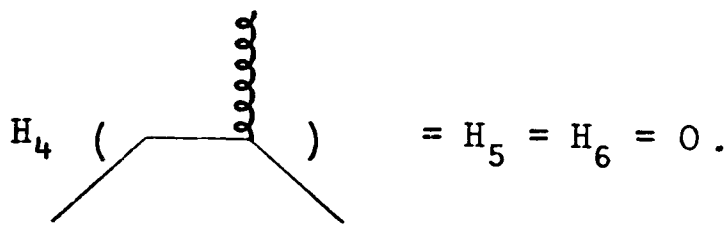
Finally:

Theorem 2.7.5

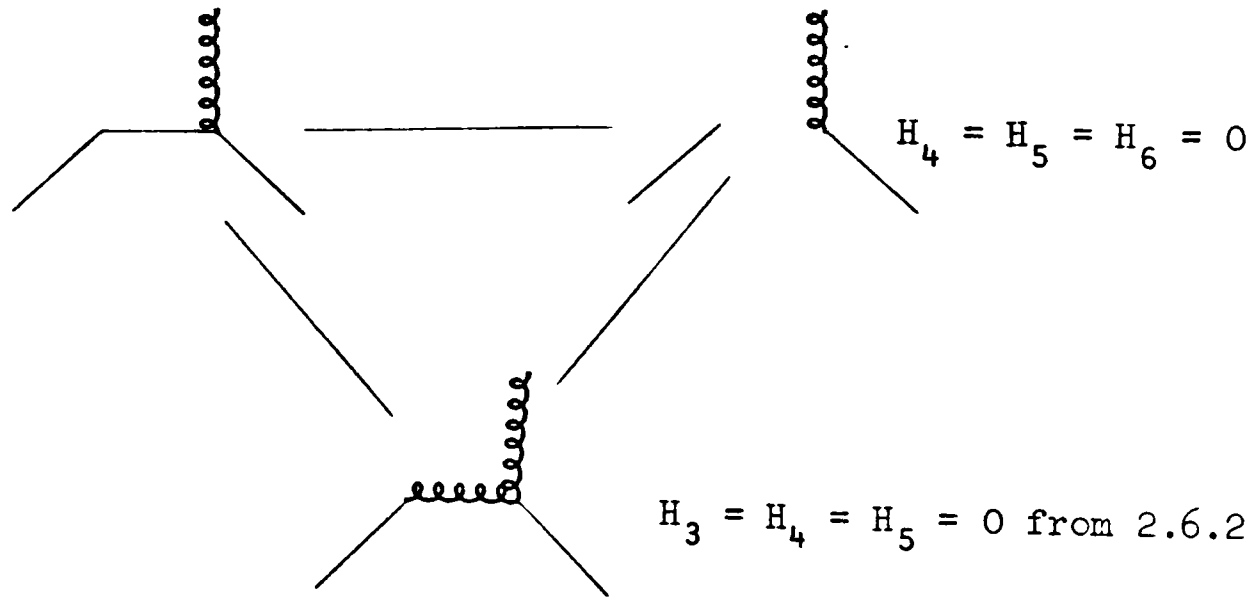


The proof uses the following two lemmas .

Lemma

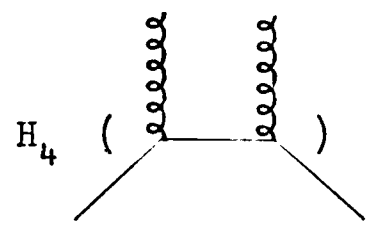


Proof:

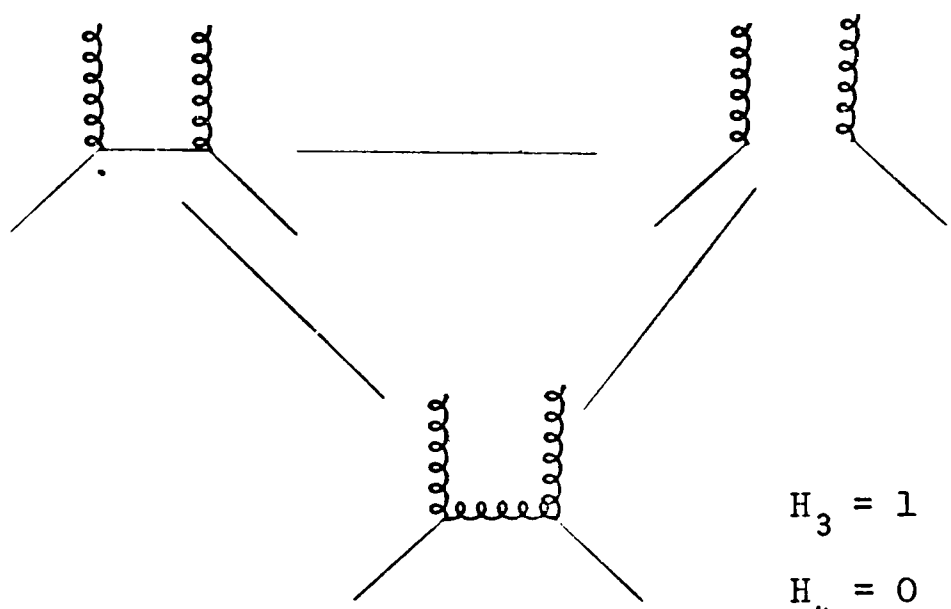


□

Lemma

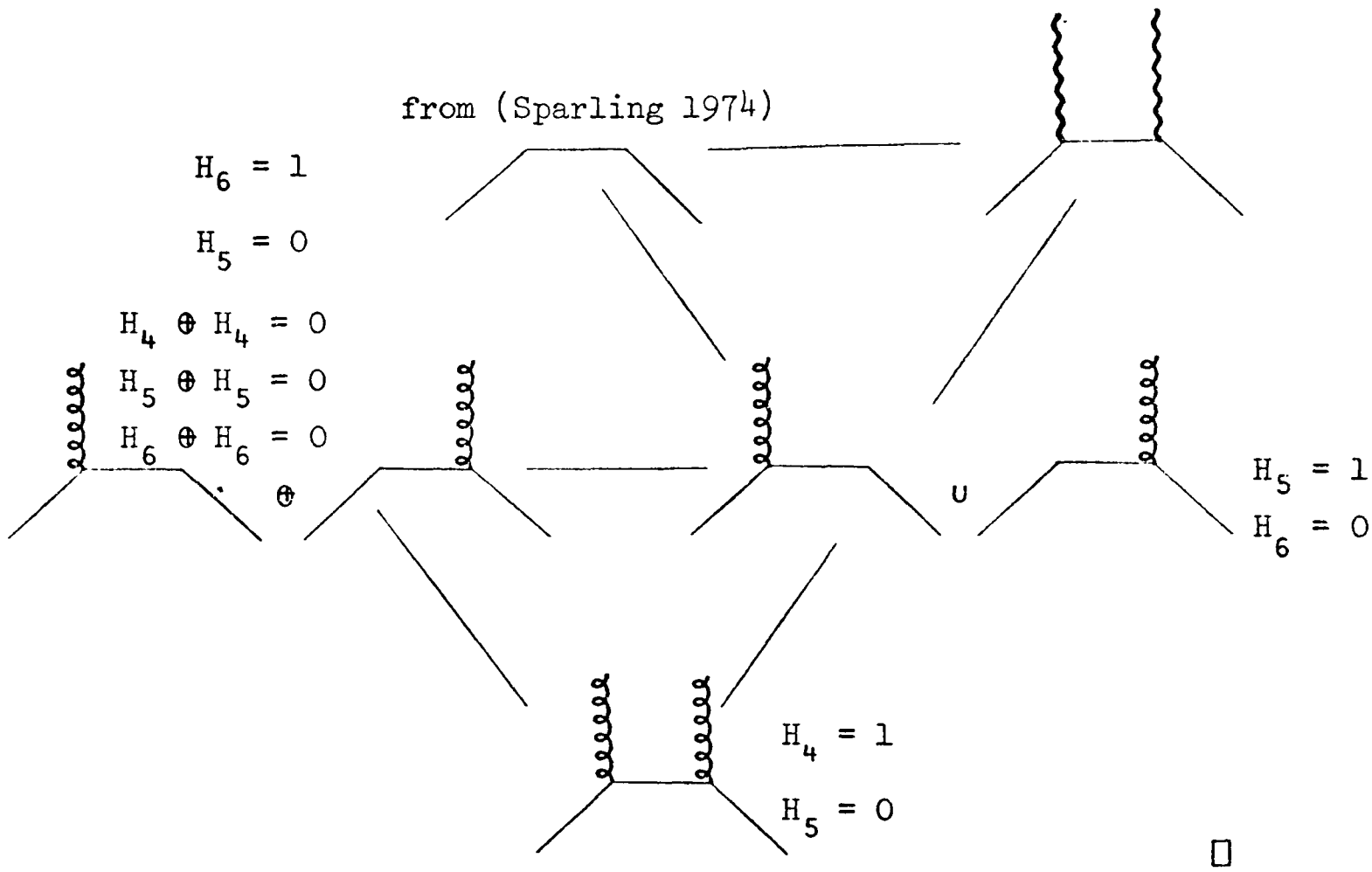


Proof:



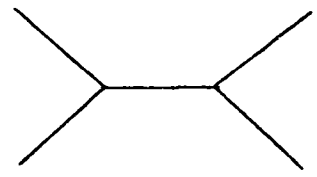
□

Proof of theorem:

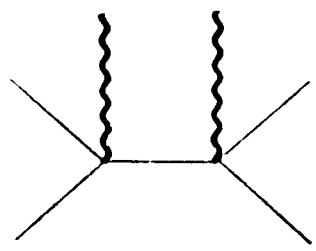
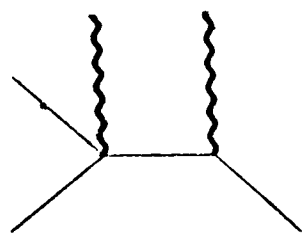
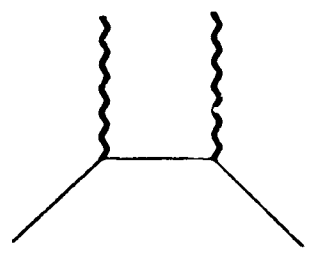


□

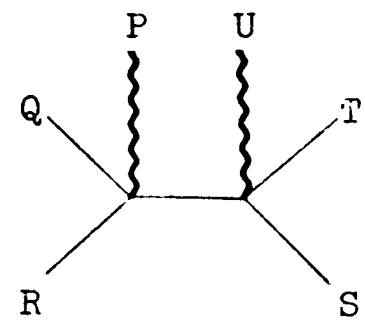
We can now construct the following table, where by new contours we mean contours <u>not</u> already known from the scalar product space



(In other words the new contours are those having non-trivial boundary.)

space	number of contours	new contours
	6	4
	3	2
	2	1

Therefore, of the 4 new contours for the space



one allows both $Q = R$ and $T = S$

one allows $Q = R$ only

one allows $T = S$ only

one allows neither $Q = R$ nor $T = S$.

It remains to actually construct these various contours and, as indicated in the introduction, to use them to construct contours for the massless Möller scattering diagram.

Chapter 3 : Three Contours for the Box Diagram

§3.1 Introduction

It has long been suspected that a full understanding of the box diagram

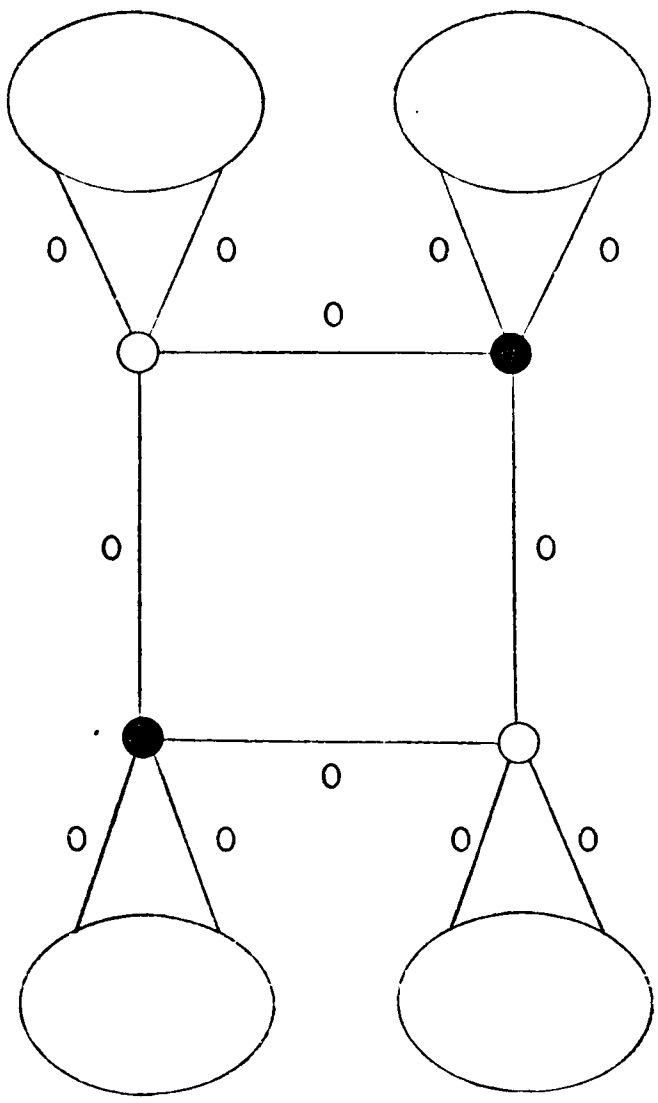


Figure 3.1.1

(which is the twistor version of the massless scalar ϕ^4 interaction) would involve the use of the technique of blowing up (see section 3.2). In particular, the line of research followed in (Huggett 1979a) was that in order to obtain three contours for the elementary state box diagram

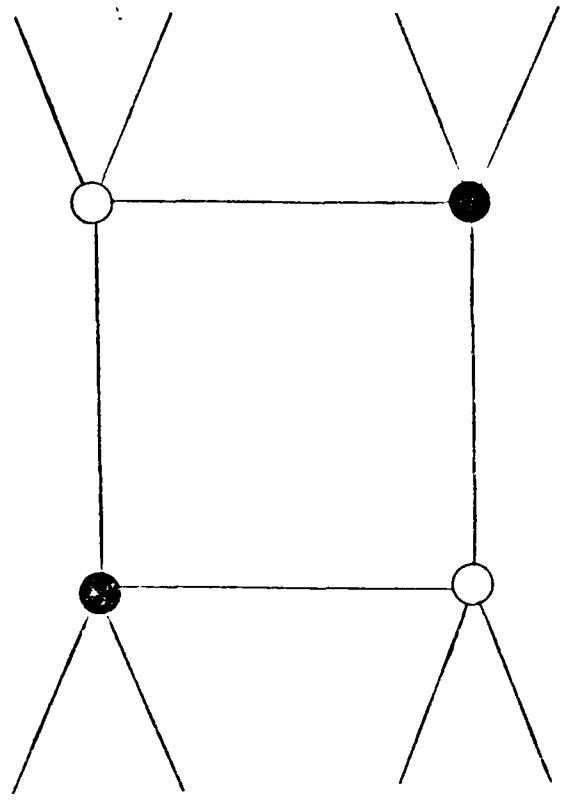


Figure 3.1.2

(one for each of the channels in the ϕ^4 interaction) we would have to blow up some part of the space represented by the topological diagram in figure 3.1.2. The need for these three contours follows from the desire to find some reflection in twistor theory of the crossing symmetry in scattering theory whereby the amplitudes for the various different channels in a process can be obtained by integrating the same function over different regions in momentum space (Hodges 1979).

In this chapter we study in detail how the box diagram is obtained from the spacetime ϕ^4 integral (see section 3.3), and we show (in section 3.4) that the three contours for the spacetime ϕ^4 integral (one for each channel) survive translation into twistors, so that the topological diagram in figure 3.1.2 has the three required contours as it stands. In fact we shall see that there is something like a blow up already involved in the translation procedure, but it neither destroys nor

creates any of the contours we are interested in.

In section 3.5 we describe the translation procedure for the scalar product between two spin $\frac{1}{2}$ massless fields. Comparison with the box diagram shows that both these cases involve a certain projection map which, it is argued, deserves more prominence in the study of twistor diagrams.

The work in section 3.4, which is largely due to Roger Penrose, has already appeared in (Huggett and Penrose 1980).

§3.2 Some Algebraic Geometry

We first of all describe the blowing up process (or quadratic transformation). This material is taken from (Griffiths and Harris 1978) and (Morrow and Kodaira 1971). Consider a polydisc Δ having coordinates $z = (z_1, \ldots, z_m)$ and suppose a hyperplane $V \subset \Delta$ is given by

$$\{z_{n+1} = \ldots = z_m = 0\} \quad .$$

Let $\ell = (\ell_{n+1} : \ell_{n+2} : \ldots : \ell_m)$ be homogeneous coordinates on P^{m-n-1} and define

$$\tilde{\Delta} \subset \Delta \times P^{m-n-1} \qquad \text{by}$$

$$\tilde{\Delta} = \{(z,\ell) : z_i \ell_j = z_j \ell_i \quad n+1 \leq i,j \leq m\} \quad .$$

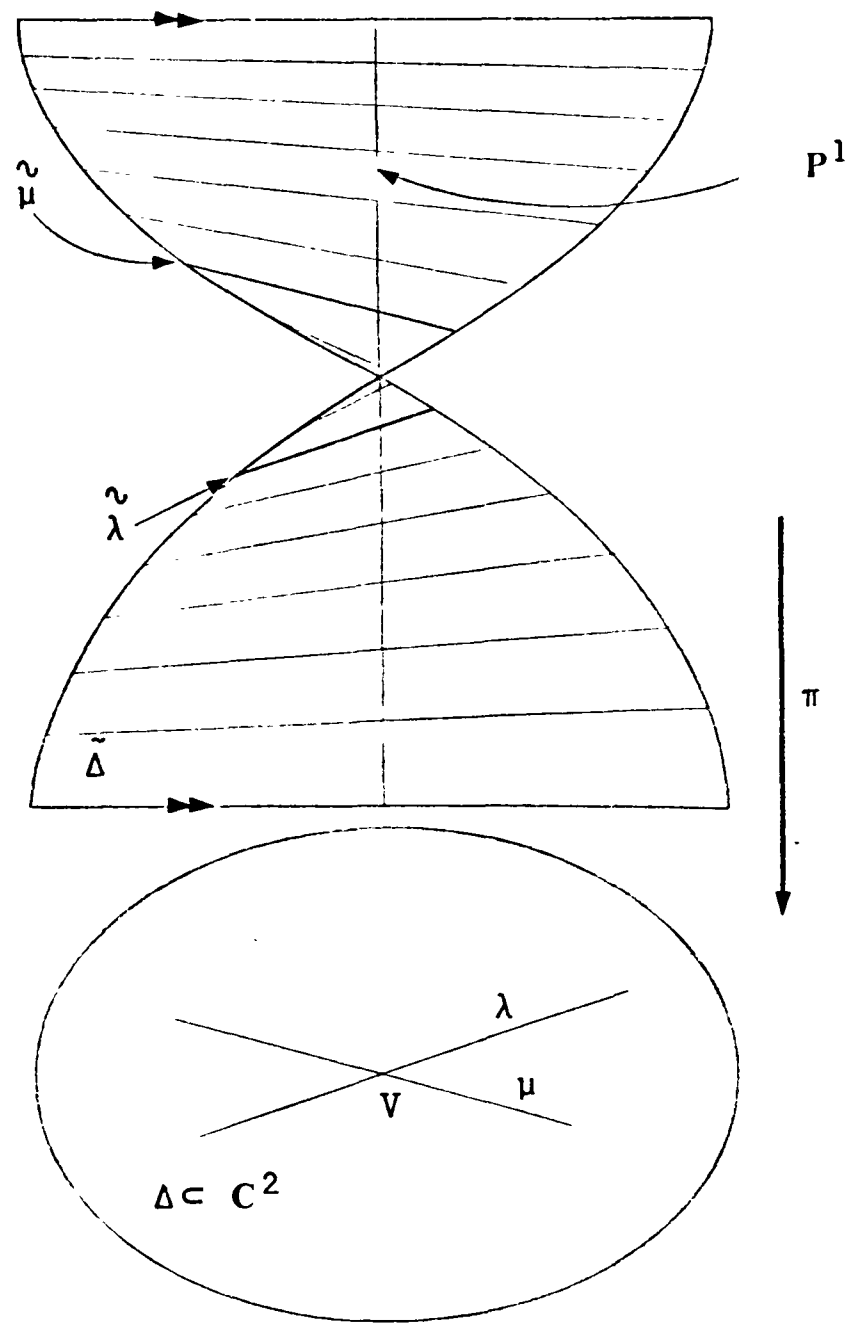
The projection $\pi: \tilde{\Delta} \to \Delta$ on the first factor is an isomorphism away

from V, while the inverse image of any point z in V is the whole projective space P^{m-n-1} . This is easiest to visualise in the example m = 2 and

$$V = \{z_1 = z_2 = 0\} .$$

Then $$\tilde{\Delta} = \{(z,\ell) : z_1 \ell_2 = z_2 \ell_1\} .$$

In $\tilde{\Delta}$ the point V has been replaced by the P^1 of lines



through V. To see this consider the two lines

$$z_2\lambda_1 = z_1\lambda_2$$

$$z_2\mu_1 = z_1\mu_2$$

through V in Δ. As the point z in Δ moves along the line λ towards V, its inverse image $\pi^{-1}(z)$ moves along the line $\tilde{\lambda}$ in $\tilde{\Delta}$. The ratio $z_1 : z_2$ stays fixed at $\lambda_1 : \lambda_2$, so that when z reaches V its inverse image $\pi^{-1}(z)$ can be chosen to be the point $(\lambda_1:\lambda_2) \in P^1$. If, instead, z were to approach V along the line μ its inverse image would move along the line $\tilde{\mu}$ and the natural choice for the inverse image of V would then be the point $(\mu_1:\mu_2) \in P^1$. So different points in the P^1 above V correspond to different lines through V.

To return to the general case, the manifold $\tilde{\Delta}$ together with the projection $\pi: \tilde{\Delta} \to \Delta$ is called the <u>blow up</u> of Δ along V. It can easily be shown that this blow up is independent of the coordinates chosen in Δ. This enables us to generalise the concept: let M be a complex manifold of complex dimension m and let N be a submanifold of complex dimension n. Then we can define the pair $(\tilde{M},\pi)$ to be the <u>blow up</u> of M along N. We can think of this process as replacing each point p of N by the space of directions in M at p normal to N. This gives us a way of <u>resolving</u> a singular variety: if W is a singular subvariety of M whereas its inverse image $\tilde{W}$ in $\tilde{M}$ is nonsingular then the pair $(\tilde{M},\pi)$ would be a <u>resolution</u> of W.

In some cases, however, not all the normal directions are necessary (Atiyah 1958), and then the resolution would not be a standard blow up. The resolution we will encounter in our discussion of the box diagram is an example of a resolution within a family (Brieskorn 1970).

We also need the "dimension theorem" (Mumford 1976).

Theorem

Let Z be an affine variety (of complex dimension n) and let X and Y be subvarieties of Z (of complex dimensions r and s respectively). Suppose Z is nonsingular along $X \cap Y$. Then

$$\dim W_i \geq r + s - n \qquad 1 \leq i \leq k$$

where W_i are the components of $X \cap Y$:

$$X \cap Y = W_1 \cup \ldots \cup W_k .$$

§3.3 The Translation Procedure

In this section we shall describe the translation procedure from the massless scalar ϕ^4 integral

$$\int_S \phi_1 \phi_2 \phi_3 \phi_4 \, d^4x \qquad 3.3.1$$

(where S is a four real dimensional contour in $\mathbb{CM}$)
to the twistor integral called the box diagram. This translation has never been studied in such detail before, and in particular it was never fully realised that it involves a resolution. It should be emphasised that this resolution is already there - we are not taking the box diagram and blowing it up, as was suggested in (Huggett 1979a).

We suppose the zero rest mass fields $\phi_1, \ldots, \phi_4$ are elementary states and choose the following twistor functions to generate them:

$$\phi_1(x) = \frac{1}{2\pi i}\oint_{S^1} \frac{W^A\, dW_A}{\rho_x\left(\begin{smallmatrix} W & W \\ | & | \\ A & B \end{smallmatrix}\right)} \qquad \phi_2(x) = \frac{1}{2\pi i}\oint_{S^1} \frac{X^{A'}\, dX_{A'}}{\rho_x\left(\begin{smallmatrix} C & D \\ | & | \\ X & X \end{smallmatrix}\right)}$$

$$\phi_3(x) = \frac{1}{2\pi i}\oint_{S^1} \frac{Y^B\, dY_B}{\rho_x\left(\begin{smallmatrix} Y & Y \\ | & | \\ E & F \end{smallmatrix}\right)} \qquad \phi_4(x) = \frac{1}{2\pi i}\oint_{S^1} \frac{Z^{B'}\, dZ_{B'}}{\rho_x\left(\begin{smallmatrix} G & H \\ | & | \\ Z & Z \end{smallmatrix}\right)}$$

where we have used the notation $W_\alpha = (W_A, W^{A'})$, $X^\alpha = (X^A, X_{A'})$, and so on, and where ρ_x means restrict the twistors to go through x^a (so that $\begin{smallmatrix} Y \\ | \\ E \end{smallmatrix}$, for example, should be written $Y_A(E^A - ix^{AA'} E_{A'})$). The integral 3.3.1 becomes

$$\int_\sigma \frac{W^A\, dW_A \wedge X^{A'}\, dX_{A'} \wedge Y^B\, dY_B \wedge Z^{B'}\, dZ_{B'} \wedge d^4x}{\rho_x\left(\begin{smallmatrix} W & W & C & D & Y & Y & G & H \\ | & | & | & | & | & | & | & | \\ A & B & X & X & E & F & Z & Z \end{smallmatrix}\right)} \qquad 3.3.2.$$

This integral is over the contour

$$\begin{array}{ccc} \sigma & \hookleftarrow & (S^1)^4 \\ \downarrow & & \\ S & & \end{array}$$

in the 8 complex dimensional space $\mathbf{B}$. $\mathbf{B}$ is a bundle over $\mathbf{CM}$ with fibre

$$\{(W_A, X_{A'}, Y_A, Z_{A'}) \in (\mathbf{CP}^1)^4\}$$

B has two important sub-bundles:

$$\mathbf{I} =: \{(W_A,\ X_{A'},\ Y_A,\ Z_{A'}; x^a) \in \mathbf{B} \ :\ W_A \propto Y_A \text{ and } X_{A'} \propto Z_{A'}\}$$

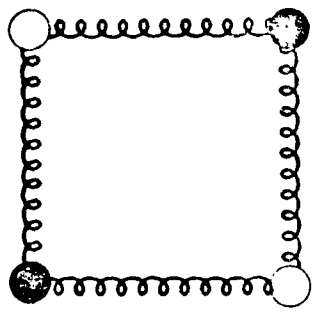
$$\mathbf{U} =: \{(W_A,\ X_{A'},\ Y_A,\ Z_{A'};\ x^a) \in \mathbf{B} \ :\ W_A \propto Y_A \text{ or } X_{A'} \propto Z_{A'}\}$$

which are the intersection and union respectively of the two sub-bundles

$$\mathbf{K} =: \{(W_A,\ X_{A'},\ Y_A,\ Z_{A'};\ x^a) \in \mathbf{B} \ : W_A \propto Y_A\}$$

$$\mathbf{L} =: \{(W_A,\ X_{A'},\ Y_A,\ Z_{A'};\ x^a) \in \mathbf{B} \ : X_{A'} \propto Z_{A'}\}$$

Now we come to the actual translation from a spacetime description to a twistor description. There is a projection π from **B** to the space



$$= \{ W ;\ X ;\ Y ;\ Z :\ \frac{W}{X} = \frac{W}{Z} = \frac{Y}{X} = \frac{Y}{Z} = 0\}$$

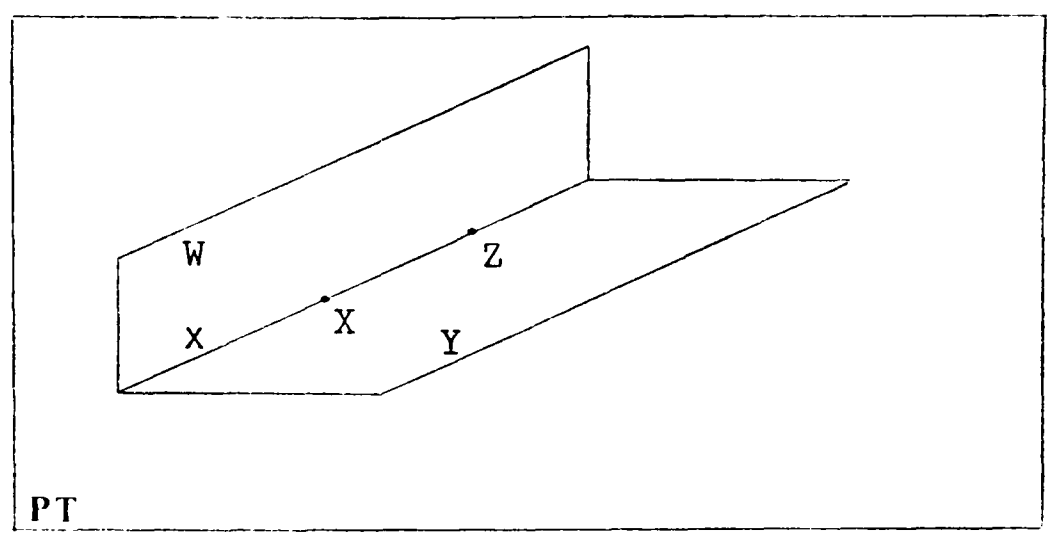
given by

$$\pi(W_A,\ X_{A'},\ Y_A,\ Z_{A'};\ x^a) = (W_A,\ -ix^{AA'} W_A\ ;\ ix^{AA'} X_{A'},\ X_A;\ Y_A,$$

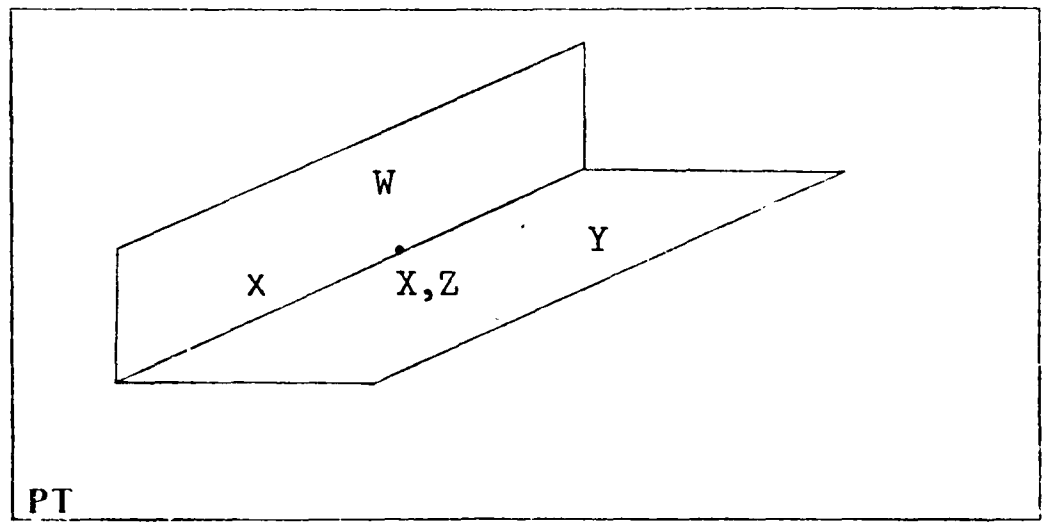
$$-ix^{AA'} Y_A;\ ix^{AA'} Z_{A'},\ Z_{A'}).$$

We can draw the images under π of the various sub-bundles of **B** as follows:

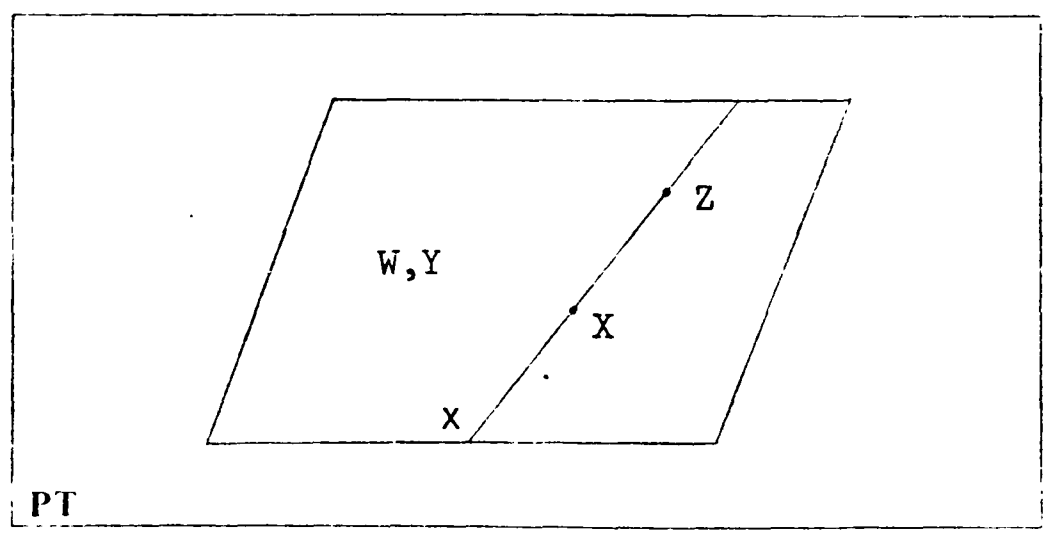
$\pi(\mathbf{B} - \mathbf{U})$ is



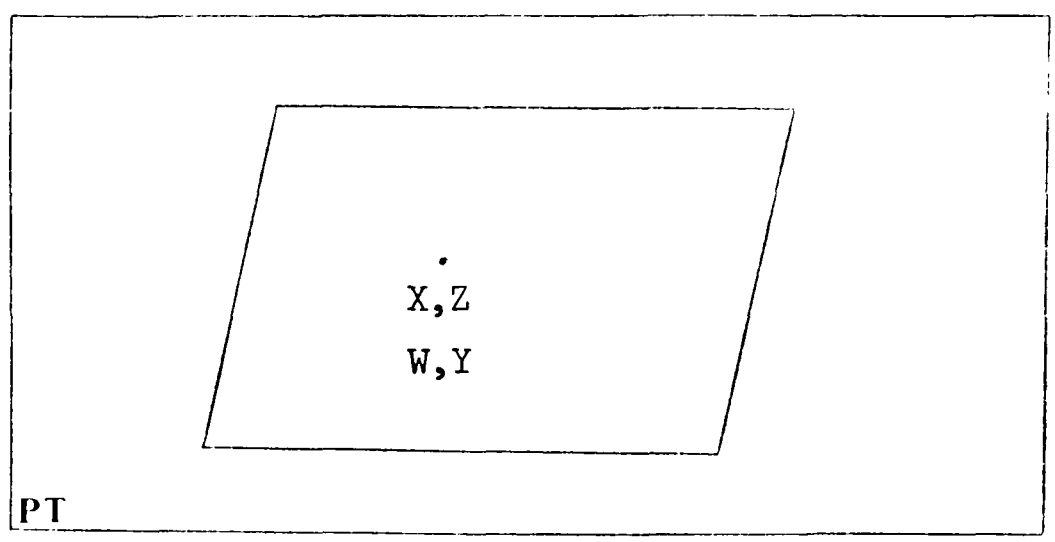
$\pi(\mathbf{L} - \mathbf{I})$ is



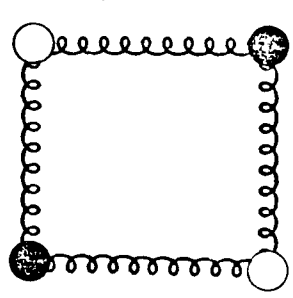
$\pi(\mathbf{K} - \mathbf{I})$ is



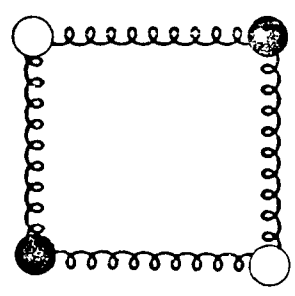
$\pi(\ \mathbf{I}\ )$ is



It is clear from these drawings that π is a biholomorphism on $\mathbf{B} - \mathbf{I}$ (because the images "remember" where the line x^a is). However, the inverse image of a point in $\pi(\ \mathbf{I}\ )$ is a $\mathbf{CP}^1$ of lines x^a lying in the plane $\begin{vmatrix} W & Y \\ \alpha & \end{vmatrix}$ and passing through the point $\begin{vmatrix} X & Z \\ \alpha & \end{vmatrix}$. Therefore the pair $(\mathbf{B}, \pi)$ is a resolution of the space



along $\pi(\ \mathbf{I}\ )$. It is not the standard blow up, in which the inverse image of a point in $\pi(\ \mathbf{I}\ )$ would be a $\mathbf{CP}^2$, because $\pi(\ \mathbf{I}\ )$ has five complex dimensions while



has eight. However, it is a resolution because

whereas the space above

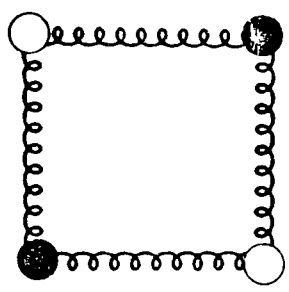
is <u>singular</u> along $\pi(\mathrm{I})$ (which we prove in a moment), I is a nonsingular subspace of B (which follows from its definition). To see that $\pi(\mathrm{I})$ is singular we notice that

$$\pi(\mathrm{I}) = \pi(\mathrm{K}) \cap \pi(\mathrm{L})$$

and then count dimensions:

$$\dim_{\mathbf{C}} \pi(\mathrm{I}) = 5$$

$$\dim_{\mathbf{C}} \pi(\mathrm{K}) = \dim_{\mathbf{C}} \pi(\mathrm{L}) = 7$$

$\dim_{\mathbf{C}}$  $= 8$

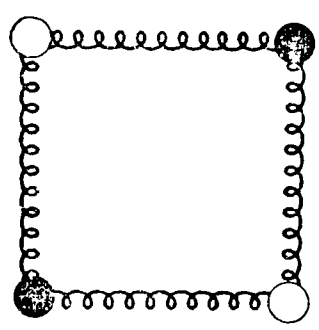
Now suppose $\pi(\mathrm{I})$ is a nonsingular. Then we get a contradiction from the dimension theorem (in the last section):

$$\dim_{\mathbf{C}} \pi(\mathrm{I}) \geq 7 + 7 - 8 \ .$$

The differential form in 3.3.2 is non-zero on I so when we blow I down this form will be singular on $\pi(\mathrm{I})$. Therefore if a contour in B is to survive the translation it must avoid I so that the restriction of the form to that contour remains nonsingular. It will be proved in the next section that all three contours in B

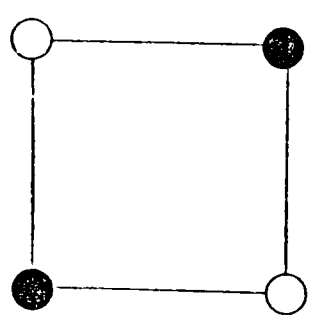
avoid I .

The last step in obtaining the box diagram is to go from the space



$- \pi(I)$

to

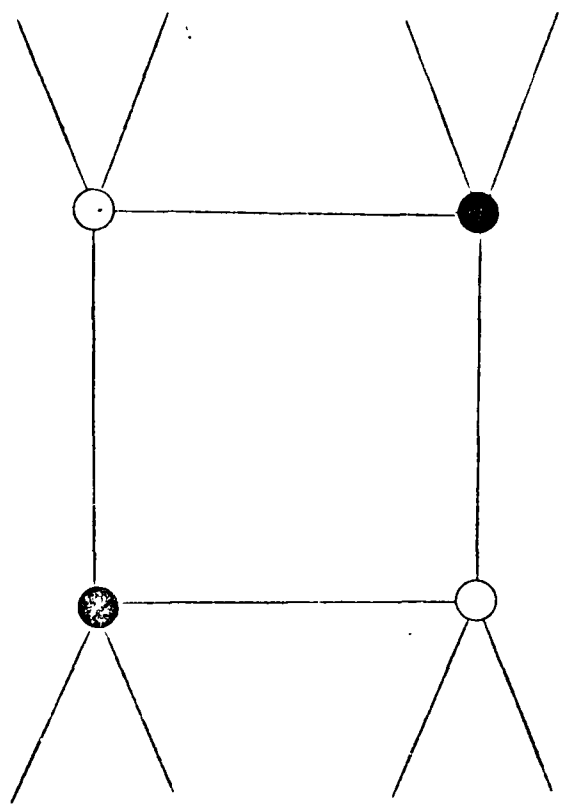


using Cauchy's theorem once for each of the four subspaces of the form

$$\{ \Big|_{X}^{W} = C \} \ .$$

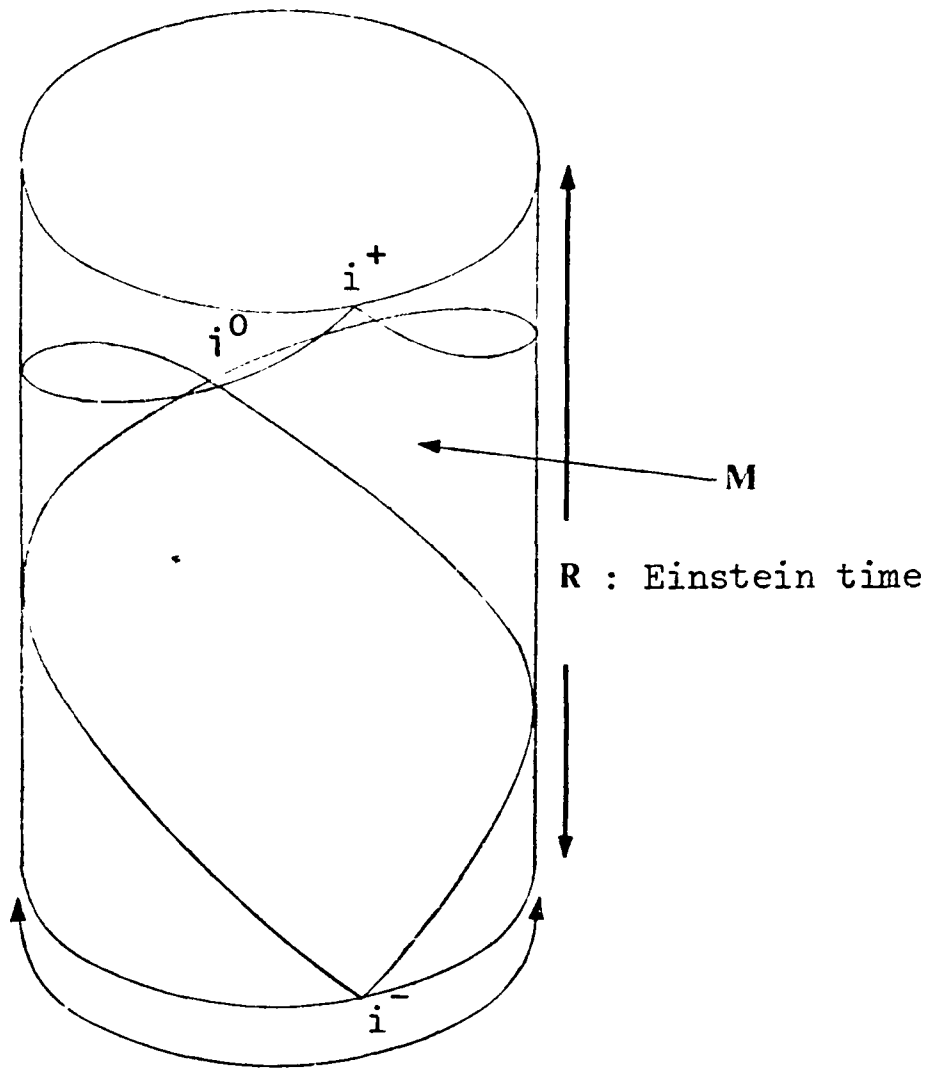
§3.4 Proof that the contours survive

This section is in two parts. In part (i) we demonstrate that the three contours (one for each channel) in B all avoid I , so that they also exist after the blow down map π. Then in part (ii) we show how to construct the corresponding contours in the box diagram



(i) We have essentially two cases to consider: case (a) is the contour for channel <13|24> (in which ϕ_1 and ϕ_3 have opposite frequency to ϕ_2 and ϕ_4) and case (b) is the contour for channel <12|34>. The contour for channel <14|23> can be treated as in case (b).

Case (a). Having moved the parameters A, B, ..., H until the lines AB and EF are in PT^+ (for positive frequency) and the lines C ∩ D and G ∩ H are in PT^- (for negative frequency) we must specify the location of the contour σ used in 3.3.2 and check that it does not intersect I. We choose S to be <u>real</u> compactified Minkowski space M. Recall that M can be thought of as an identified version of the Einstein cylinder (Penrose 1968):



S^3 : compactified spacelike hypersurface

When the identifications are made the Einstein time is compactified to S^1.

For each point $x^a \in \mathbf{M}$ there are four pairs of singularities (one in each of the four spin spaces above x^a in $\mathbf{B}$):

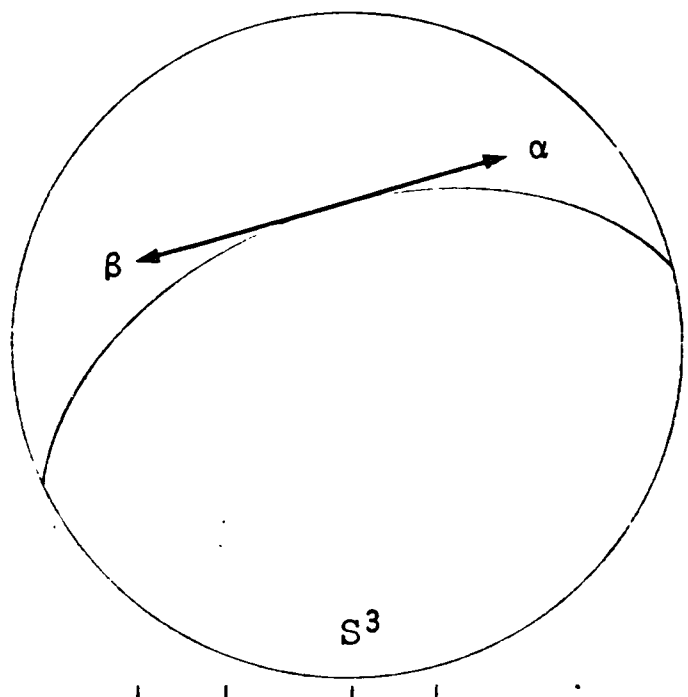
$$\rho_x\left(\begin{array}{cccccccc} W & W & C & D & Y & Y & G & H \\ | & | & | & | & | & | & | & | \\ A & B & X & X & E & F & Z & Z \end{array}\right)^{-1} .$$

We have to choose four S^1 contours (one separating each pair of

singularities) in such a way that the complete contour avoids I. In fact we make this choice for an even more specific location of the parameters, but we can then move the parameters away from that location while preserving the contour.

Recall that a non-null twistor $\underline{\underline{Z}}$ defines a Robinson congruence which itself defines an everywhere nonsingular and nowhere vanishing vector field on each of the S^3s. This vector field is constant in Einstein time (so that it does not matter which S^3 we study) and defines a rotation of the S^3 (right handed if $\underline{\underline{Z}} \in PT^+$ and left handed otherwise).

We choose $\underline{\underline{A}} = \underline{\underline{E}}$ and $\underline{\underline{B}} = \underline{\underline{F}}$ in PT^+ in such a way that the Robinson congruences α and β defined by A and B respectively point along opposite (right) rotations when projected into S^3:

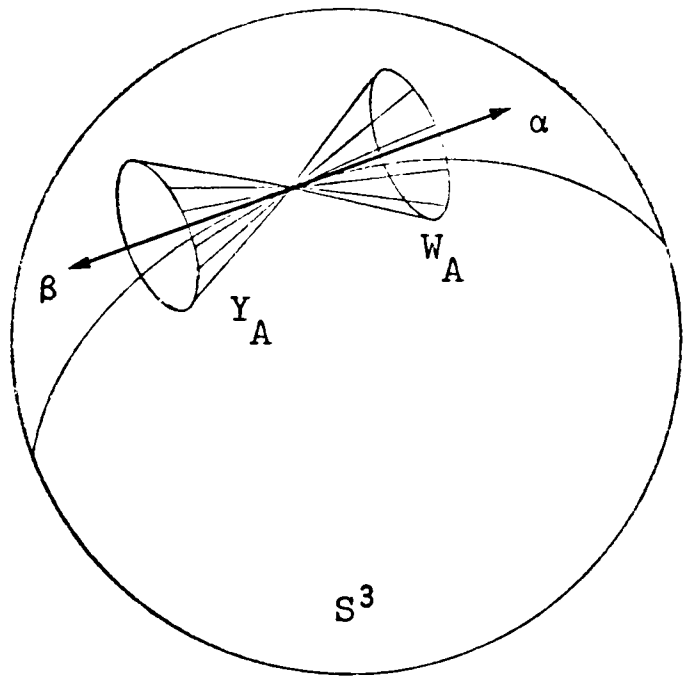


Similarly, we choose $\underline{\underline{C}} = \underline{\underline{G}}$ and $\underline{\underline{D}} = \underline{\underline{H}} \in PT^+$ so that the projections of the Robinson congruences γ and δ (defined by $\underline{\underline{C}}$ and $\underline{\underline{D}}$) are also opposite (right) rotations.

For each x^a we need $(W_A, X_{A'}, Y_A, Z_{A'})$ such that the flagpoles of W_A, $X_{A'}$, Y_A, $Z_{A'}$ separate the pairs of Robinson congruences α and β,

γ and δ, α and β, and γ and δ respectively. Also, to miss $\mathbf{I}$ we need $W_A \not\subset Y_A$ or $X_{A'} \not\subset Z_{A'}$ for each x^a, although in this case we can even arrange for $W_A \not\subset Y_A$ and $X_{A'} \not\subset Z_{A'}$ for each x^a, so that this contour avoids $\mathbf{U}$.

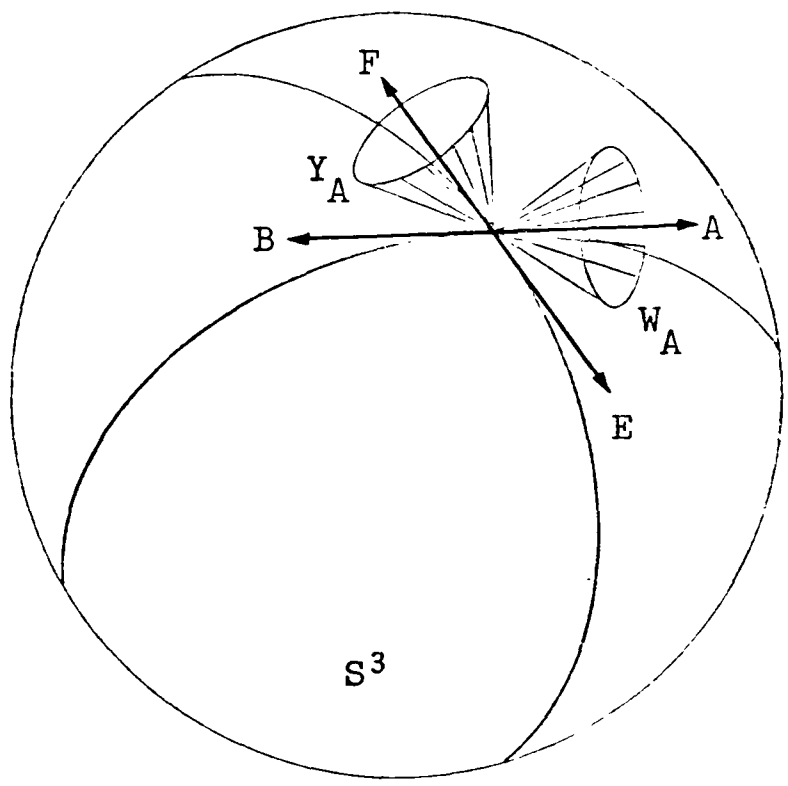
Let the W_A flagpole (when projected into S^3) execute a narrow cone about the α direction and let the Y_A flagpole do the same (independently) about the β direction:



We now have an $S^1 \times S^1$ contour at each point of the S^3. Treat $X_{A'}$ and $Z_{A'}$ similarly (and independently), and do all this for each value of the Einstein time. We obtain $S^3 \times S^1$s worth of $S^1 \times S^1 \times S^1 \times S^1$ contours avoiding $\mathbf{U}$, as required.

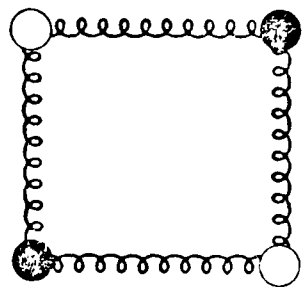
Case (b). For channel $\langle 12|34\rangle$ we move the parameters until the lines AB and $C \cap D$ are in $\mathbf{PT}^+$ and the lines EF and $G \cap H$ are in $\mathbf{PT}^-$. This means that A, B, $\bar{G}$, and $\bar{H}$ are in $\mathbf{PT}^+$ while E, F, $\bar{C}$, and $\bar{D}$ are in $\mathbf{PT}^-$. We still choose A and B so that α and β are opposite (right) rotations, but now we cannot choose $A = E$ because they

correspond to a right rotation and a left rotation respectively. So although we can make W_A and Y_A execute narrow cones about the directions on S^3 corresponding to $\hat{A}$ and $\hat{F}$ respectively (as before - $\hat{F}$ <u>was</u> equal to $\hat{B}$) W_A and Y_A will encounter one another when these directions get close. This will happen in the neighbourhood of a great circle on the S^3.

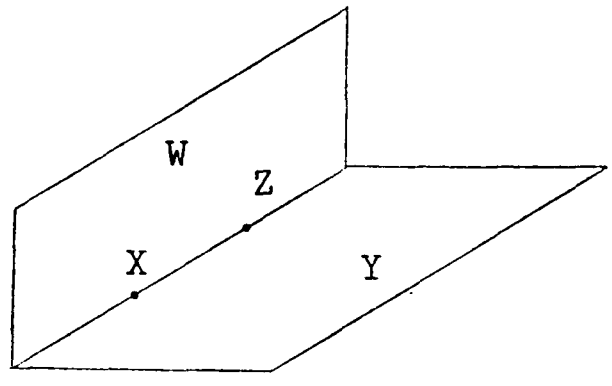


We can again treat X_A, and Z_A, similarly, so that they too will encounter one another near a great circle on the S^3. All we have to do to make the complete contour avoid I is to ensure that these two great circles do not intersect (so that at no stage do we have $W_A \propto Y_A$ <u>and</u> $X_A, \propto Z_A,$). This complete contour does, however, intersect U .

(ii) In case (a) of part (i) we constructed a contour (γ_a say) in B avoiding U . We now use π to map this contour into



Consider a typical point of $\pi(\gamma_a)$:



We can impose the conditions that

$$\int_W = 0 \qquad \text{and} \qquad \int_Y = 0$$

with the Cauchy integrals

$$\oint(\quad)\ \frac{d(\int_W)}{\int_W} \qquad \text{and} \qquad \oint(\quad)\ \frac{d(\int_Y)}{\int_Y}\ .$$

These use S^1 contours, one around the

$$\int_W$$

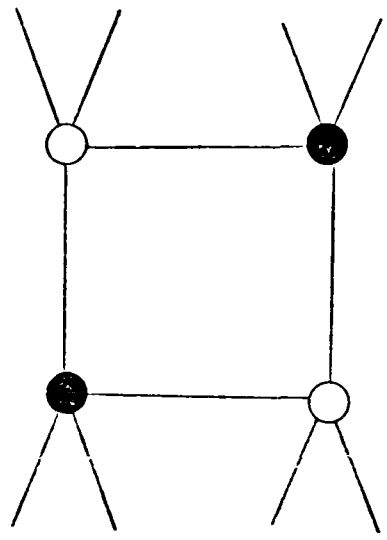
singularity and the other around the

$$\begin{matrix} Y \\ | \\ X \end{matrix}$$

singularity. We could hold the planes W and Y still while we make the point X execute a small torus $T^1 = S^1 \times S^1$ around them. Similarly, for the conditions

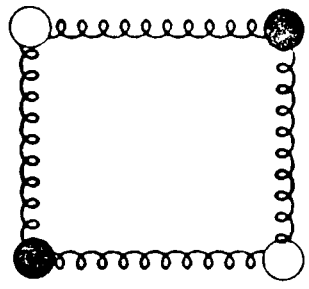
$$\begin{matrix} W \\ | \\ Z \end{matrix} = 0 \qquad \text{and} \qquad \begin{matrix} Y \\ | \\ Z \end{matrix} = 0$$

we construct another small torus. We now have an $S^1 \times S^1 \times S^1 \times S^1$ contour for each point of $\pi(\gamma_a)$. Therefore we have the contour for the box diagram

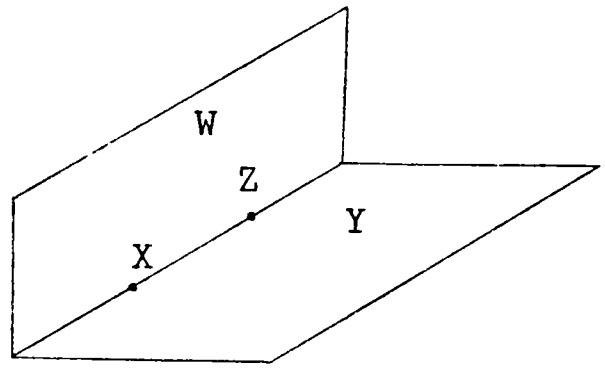


corresponding to channel <13|24>. This contour has been known for some time. In more abstract terms it is $\delta^4(\pi(\gamma_b))$, where δ is Leray's cobord map.

In case (b) of part (i) we constructed a contour (γ_b say) in B avoiding I but intersecting U . In fact γ_b intersects both K and L, which will be an important consideration. Again we study $\pi(\gamma_b)$ in



and consider a point of $\pi(\gamma_b)$ not in $\pi(\ U)$:



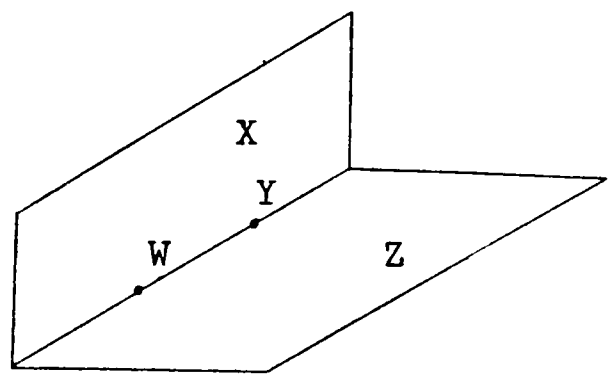
Exactly as for $\pi(\gamma_a)$ we can construct an $S^1 \times S^1 \times S^1 \times S^1$ contour for this point of $\pi(\gamma_b)$. This contour is the product of two tori, one above each of X and Z. However, in this case two complications appear when we move to another point of $\pi(\gamma_b)$:

(1) γ_b intersects L , so that X and Z may coincide

(2) γ_b intersects K , so that W and Y may coincide.

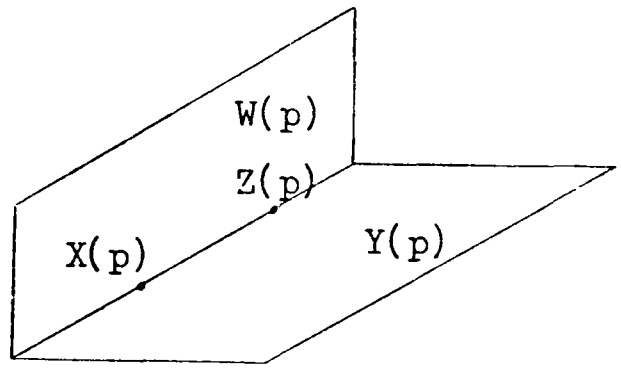
To overcome (1) we must choose the tori in such a way that they <u>coincide</u> when X = Z.

To deal with (2) we draw the previous picture in dual twistor space:



and think of the "points" W and Y as executing small tori around the "planes" X and Z in such a way that when W = Y these tori coincide.

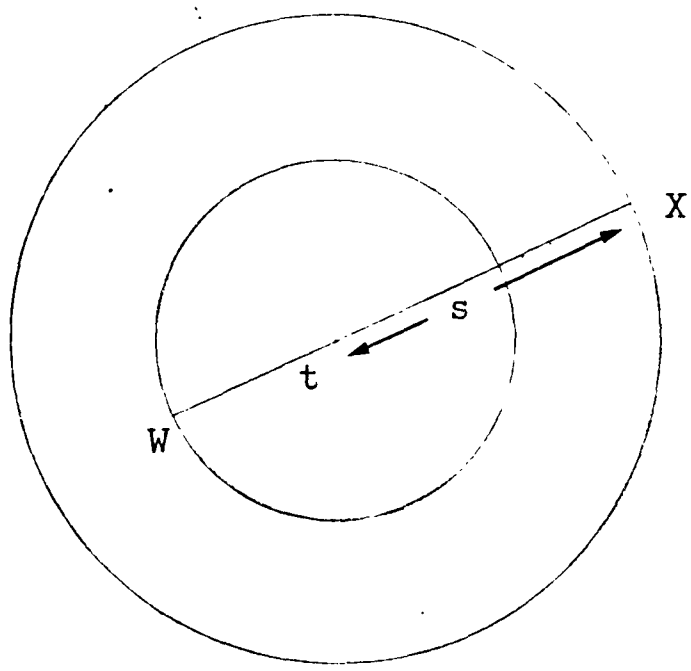
However, neither of these methods works for the <u>whole</u> of $\pi(\gamma_b)$ because γ_b intersects K <u>and</u> L (though not at the same time). So we have to combine them. For any point p in $\pi(\gamma_b)$



a line (CP^1) x^a is defined (either by XZ, or by W ∩ Y, or both). The space of planes through x^a is a CP^1. Let s be the Euclidean distance between W and Y on this CP^1, and let t be the distance between X and Z on the CP^1 x^a. Consider the condition

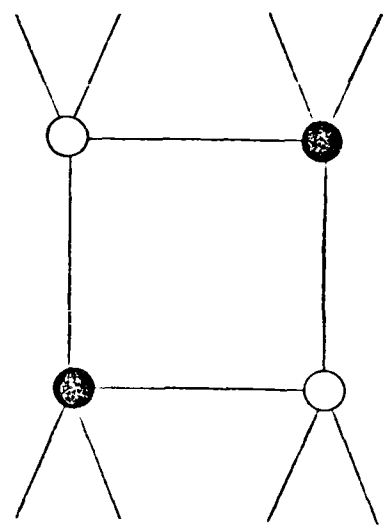
$$\begin{matrix} W \\ | \\ X \end{matrix} = 0 \ .$$

Instead of completely fixing either $\begin{matrix} W \\ | \end{matrix}$ or $\begin{matrix} | \\ X \end{matrix}$ at their initial positions $\begin{matrix} W(p) \\ | \end{matrix}$ and $\begin{matrix} | \\ X(p) \end{matrix}$ we make $\begin{matrix} W \\ | \end{matrix}$ execute an S^1 of radius t around $\begin{matrix} | \\ X(p) \end{matrix}$ and $\begin{matrix} | \\ X \end{matrix}$ execute an S^1 of radius s in such a way that $\begin{matrix} W \\ | \end{matrix}$ and $\begin{matrix} | \\ X \end{matrix}$ are always antipodal to each other:



We do exactly the same for the other three conditions.

Now suppose $\pi^{-1}(p)$ is in L . Then t = 0 and the points X and Z coincide. Also, the S^1s for W and Y have zero radius so that they are fixed and we can use the two (coincident) tori for the points X and Z. If, instead, $\pi^{-1}(p)$ is in K we have the dual to the foregoing, with t = 0 instead of s. Therefore this procedure works all over $\pi(\gamma_b)$, and so we have constructed the corresponding contour in



In fact there were really two contours constructed in case (b) of part (i), one for channel <12|34> and the other for channel <14|23>. The

procedure described above works for both of them so that we now have contours for all three ϕ^4 channels in the box diagram. Two points should be made here. Firstly, although sections 3.3 and 3.4 have discussed the box diagram with zeros on its internal lines (figure 3.1.1) the existence of the three contours is a topological result which holds given any positive integers on these lines. Secondly, as is shown in (Hodges 1975); contours for the three channels have the property that they sum to zero, so that only two of them are independent.

§3.5 Comparison with the Scalar Product

Consider the spacetime integral for the scalar product between two massless spin $\frac{1}{2}$ fields γ_A and $\chi_{A'}$:

$$\langle \bar{\chi} | \gamma \rangle = \kappa \int_{S^3} \gamma_A \, \chi_{A'} \, d^3x^{AA'} \qquad 3.5.1$$

where κ is some constant and the S^3 contour separates the singularities of the fields γ_A and $\chi_{A'}$ in complexified compactified Minkowski space. This is translated into twistors as follows. Suppose γ_A and $\chi_{A'}$ are generated by the twistor functions f and g (homogeneous of degree - 3):

$$\gamma_A(x^a) = \frac{1}{2\pi i} \oint_{S^1} f\left(\frac{W}{|}\right) W_A \, \Delta W$$

$$\chi_{A'}(x^a) = \frac{1}{2\pi i} \oint_{S^1} g\left(\frac{|}{Z}\right) Z_{A'} \, \Delta Z$$

Now (a) becomes

$$\langle \bar{\chi} | \gamma \rangle = \frac{\kappa}{(2\pi i)^2} \int_{\tau} f(\frac{W}{|}) \, g(\frac{|}{Z}) \, W_A \, Z_{A'} \, \Delta W \, \Delta Z \, d^3x^{AA'} \qquad 3.5.2.$$

This integral is over the contour

$$\begin{array}{ccc} \tau & \longleftarrow\!\!\!\!\!\!\rightarrow & (S^1)^2 \\ \downarrow & & \\ S^3 & & \end{array}$$

in the six complex dimensional space $\mathbf{D}$. $\mathbf{D}$ is a bundle over $\mathbf{CM}$ with fibre

$$\{(W_A, Z_{A'}) \in (\mathbf{CP}^1)^2\} \quad .$$

There is a projection

$$\pi : \mathbf{D} \rightarrow \quad =: \{(\frac{W}{|}, \frac{|}{Z}) : \frac{W}{|} = 0\}$$

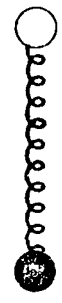
given by

$$\pi(W_A, Z_{A'}, x^a) = (W_A, -ix^{AA'} W_A; ix^{AA'} Z_{A'}, Z_{A'}).$$

The space

is the five complex dimensional space of complex null geodesics in CM.

The inverse image under π of <u>any</u> point in



is the complex null geodesic (CP^1) represented by that point:

$$\pi(W_A, Z_{A'}, x^a) = \pi(\alpha_A, \beta_{A'}, y^a)$$

$$\Leftrightarrow \quad \alpha_A = W_A, \quad \beta_{A'} = Z_{A'}, \quad \text{and} \quad x^{AA'} - y^{AA'} = W^A Z^{A'}$$

(so that they are null separated). In fact π is a fibre bundle projection, and not a blow down map as in the ϕ^4 case. Furthermore, (i) the contour τ is transverse to the fibres of π because the S^3 is a spacelike hypersurface, and (ii) the differential form

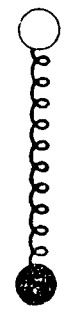
$$fg\ W_A Z_{A'}, \Delta W \wedge \Delta Z \wedge d^3x^{AA'}$$

is <u>constant</u> on the fibres of π and therefore defines a differential form on



These two facts allow us to perform the integral in 3.5.2 in the above space

instead of in **D** . Then (just as in the box diagram) we use a Cauchy integral to express the integral in

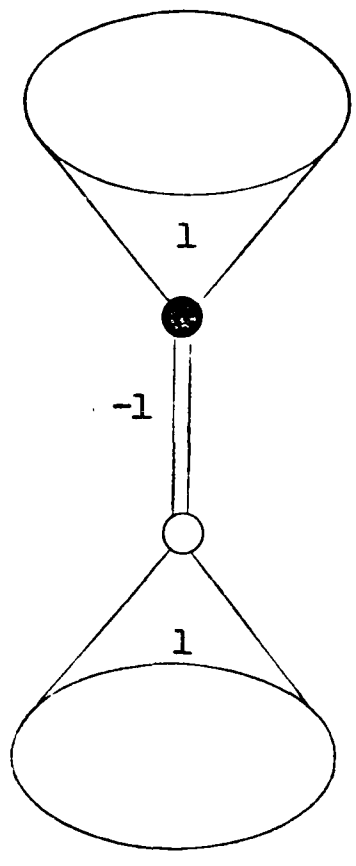


in terms of one in



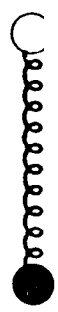
$$F(0) = \frac{1}{2\pi i} \oint \frac{F(z)}{z}\, dz \qquad \text{where} \qquad z = \frac{W}{Z} \quad .$$

If we follow the differential form through we find that the twistor scalar product for spin $\frac{1}{2}$ is

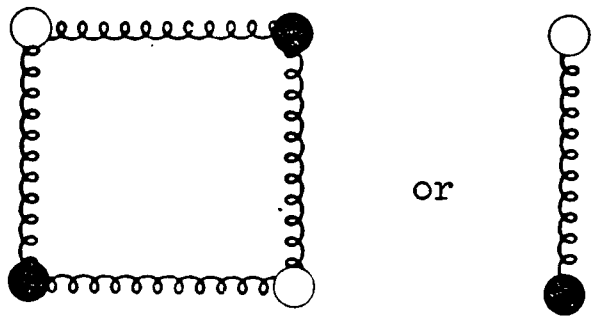


If we follow the contour through we obtain the tight contour (see section 2.2). Its topology is described at the beginning of section 5.3.

Nothing in this section is any different from the standard treatment (Penrose 1975) except that we have chosen to emphasise the projection π. In doing so two things emerge:

(i) In the standard treatment points $w^{AA'}$ on $\begin{smallmatrix} W \\ | \end{smallmatrix}$ and $z^{AA'}$ on $\begin{smallmatrix} | \\ Z \end{smallmatrix}$ are chosen in such a way that when $\begin{smallmatrix} W \\ | \\ Z \end{smallmatrix} = 0$ we have $w^{AA'} = z^{AA'} = x^{AA'}$ We can see now that this amounts to choosing a section of the projection π in order to be able to construct a contour in $\mathbf{D}$ given one in 

(ii) The translation into twistors of <u>any</u> spacetime integral of massless fields is going to involve a map π from some bundle over $\mathbf{CM}$ (whose fibre is a product of spin spaces) to a "springy" twistor diagram such as



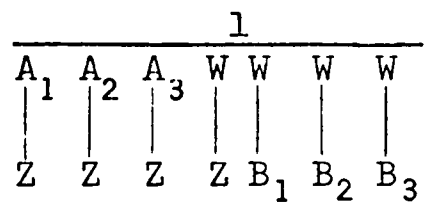
In a sense this map is the guts of the translation procedure so its properties are central to the geometry of twistor diagrams.

Chapter 4 : Sheaf Cohomology of Tree Diagrams

§4.1 Introduction

A tree diagram is a twistor diagram with no multiple lines and no loops. The absence of multiple lines implies that the contour to be used is a product of S^1s, while the absence of loops (and the fact that all vertices are of degree 4) means that the number of lines in a tree diagram with m vertices is 3m + 1. These two facts - together with the further restriction that all our diagrams have zeros on all their lines - make the cohomology of tree diagrams relatively simple, which is why we study them first.

Consider for example the diagram in figure 4.1.1, whose associated function is



This diagram was studied in (Sparling 1974) as a generalised scalar product and in (Hodges 1975) as part of the box diagram. It can easily be deduced from

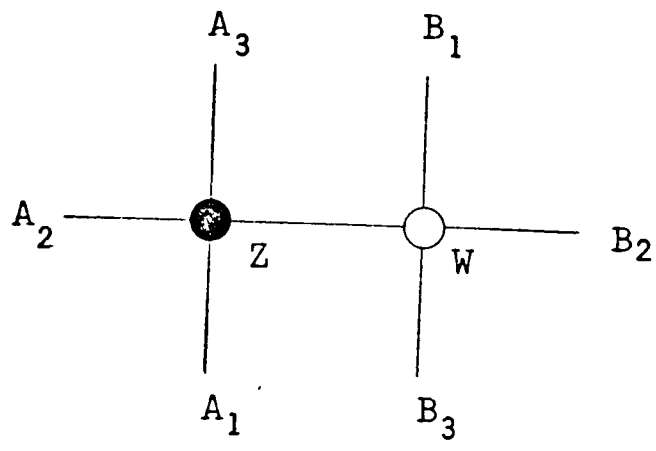


Figure 4.1.1

the results on page 11 - 13 of (Ryman 1975) that the associated space has six contours, only one of which separates all the external singularities. This contour - which is a product of S^1s - imposes an equivalence relation on homogeneous functions holomorphic in the associated space. We shall see that, just as in the case of zero rest mass fields, this equivalence relation corresponds to the freedom in choosing a representative for a cohomology class. Consequently, having chosen a contour in the associated space we can regard the diagram in figure 4.1.1 as defining a cohomology class.

After devoting two sections to sheaf cohomology theory we show how the one-vertex diagrams determine cohomology classes. Then, in section 4.5 we show how to construct a diagram from simpler pieces using the dot product. In section 4.6 the Dolbeault representatives are derived and discussed, and in section 4.7 the (Čech) cohomological interpretation of the residue map of chapter 2 is described and related to the dot product. Section 4.8 demonstrates how to evaluate cohomologically the general tree diagram and gives a "twistor transform" procedure for trees.

Most of the ideas in section 4.5 (and many elsewhere) are due to Matt Ginsberg and have already appeared in (Ginsberg and Huggett 1979). Mike Eastwood also deserves credit for some of the details in section 4.5. Some of the work in section 4.7 was described in (Huggett 1979b).

§4.2 Sheaves

Our brief account of sheaf cohomology - which is designed to do

little more than set up the notation we shall be using - is largely taken from (Griffiths and Harris 1978) and (Morrow and Kodaira 1971). Another good reference is (Wells 1980).

Let X be a topological space. S is a <u>sheaf</u> on X if for each open set $U \subset X$ there is a group $\Gamma(U,S)$, called the <u>sections</u> of S over U, such that for each pair $U \subset V$ of open sets there is a map

$$\rho_{V,U} : \Gamma(V,S) \to \Gamma(U,S),$$

called the <u>restriction</u> map, which satisfies

(i) for any triple $U \subset V \subset W$ of open sets

$$\rho_{W,U} = \rho_{V,U} \cdot \rho_{W,V}$$

(This implies that we can write ρ_U instead of $\rho_{V,U}$. Also, given an indexed set $\{U_i\}$ of open sets in X we shall write U_{ij} for $U_i \cap U_j$, ρ_i for ρ_{U_i}, ρ_{ij} for $\rho_{U_{ij}}$, and so on.)

(ii) for any pair of open sets $U, V \subset X$ and sections $\sigma \in \Gamma(U,S)$, $\tau \in \Gamma(V,S)$ such that

$$\rho_{U \cap V}(\sigma) = \rho_{U \cap V}(\tau)$$

there exists a section $\mu \in \Gamma(U \cup V, S)$ such that

$$\rho_U(\mu) = \sigma \qquad \text{and} \qquad \rho_V(\mu) = \tau$$

(iii) if $\nu \in \Gamma(U \cup V, S)$ and $\rho_U(\nu) = \rho_V(\nu) = 0$ then $\nu = 0$.

There are two types of examples we shall be using:

(a) Let X be a C^∞ manifold and define the sheaves A^p, Z^p, Z, R, and C on X by

$\Gamma(U, A^p)$ = C^∞ p-forms on U

$\Gamma(U, Z^p)$ = closed C^∞ p-forms on U

$\Gamma(U, Z)$, $\Gamma(U, R)$, or $\Gamma(U, C)$ = locally constant Z, R, or C-valued functions on U.

(b) Let X be a complex manifold and suppose Y is a complex sub-manifold. Define the sheaves $\mathcal{O}$, $\mathcal{O}(p)$, $\mathcal{O}^*$, Ω^p, $A^{p,q}$, $Z^{p,q}$, and I_Y on X by

$\Gamma(U, \mathcal{O})$ = holomorphic functions on U

$\Gamma(U, \mathcal{O}(p))$ = holomorphic functions homogeneous of degree p on U (here X has to be part of a complex projective space)

$\Gamma(U, \mathcal{O}^*)$ = multiplicative group of non-zero holomorphic functions on U

$\Gamma(U, \Omega^p)$ = holomorphic p-forms on U

$\Gamma(U, A^{p,q})$ = C^∞ forms of type (p,q) on U

$\Gamma(U, Z^{p,q})$ = $\bar{\partial}$-closed C^∞ forms of type (p,q) on U

$\Gamma(U, I_Y)$ = holomorphic functions on U vanishing on $Y \cap U$

We next have to define exact sequences of sheaves. A map of sheaves $\beta: S \to T$ is given by a collection of homomorphisms

$$\{\beta_U : \Gamma(U,S) \to \Gamma(U,T)\}_{U \subset X}$$

such that for $U \subset V \subset X$, β_U and β_V commute with the restriction maps. The <u>kernel</u> of this map is the sheaf $\mathrm{Ker}(\beta)$ defined by

$$\Gamma(U,\mathrm{Ker}(\beta)) = \mathrm{Ker}(\beta_U: \Gamma(U,S) \to \Gamma(U,T))$$

We also need the sheaf $\mathrm{Coker}(\beta)$, whose definition is a little more involved. A section of $\mathrm{Coker}(\beta)$ over U is given by an open cover $\{U_i\}$ of U together with sections $\sigma_i \in \Gamma(U_i,T)$ such that for all i,j

$$\rho_{ij}\,\sigma_i - \rho_{ij}\,\sigma_j \in \beta_{U_{ij}}(\Gamma(U_{ij},S))$$

and we identify two such collections $\{U_i,\sigma_i\}$ and $\{V_j,\tau_j\}$ if for all $p \in U$ and U_i, V_j containing p there exists a set W with $p \in W \subset U_i \cap V_j$ such that

$$\rho_W\,\sigma_i - \rho_W\,\tau_j \in \beta_W(\Gamma(W,S)).$$

A short sequence of sheaf maps

$$0 \to R \xrightarrow{\alpha} S \xrightarrow{\beta} T \to 0$$

is <u>exact</u> if $R = \mathrm{Ker}\,\beta$ and $T = \mathrm{Coker}\,\alpha$.

A long sequence of sheaf maps

$$\cdots \to S_n \xrightarrow{\alpha_n} S_{n+1} \xrightarrow{\alpha_{n+1}} S_{n+2} \to \cdots$$

is exact if $\alpha_{n+1} \cdot \alpha_n = 0$ and the short sequence

$$0 \to \mathrm{Ker}(\alpha_n) \to S_n \to \mathrm{Ker}(\alpha_{n+1}) \to 0$$

is exact for each n. We use the following three examples of exact sequences on the complex manifold X:

The exponential sheaf sequence:

$$0 \to \mathbf{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0 \qquad 4.2.1$$

is exact, where i is the inclusion and $\exp(f) = e^{2\pi i f}$.

Suppose $Y \subset X$ is a complex submanifold of X. The sheaf $\mathcal{O}_Y$ on Y may be extended to a sheaf $\mathcal{O}_Y$ on X by

$$\Gamma(U, \mathcal{O}_Y) =: \Gamma(U \cap Y, \mathcal{O}_Y) \quad \text{for all} \quad U \subset X.$$

The sequence of sheaves on X

$$0 \to \mathcal{I}_Y \xrightarrow{i} \mathcal{O}_X \xrightarrow{r} \mathcal{O}_Y \to 0, \qquad 4.2.2$$

where i is inclusion and r restriction, is exact.

By the $\bar{\partial}$-Poincaré lemma the sequence

$$0 \to \Omega^p \xrightarrow{i} A^{p,0} \xrightarrow{\bar\partial} A^{p,1} \xrightarrow{\bar\partial} \dots \qquad 4.2.3$$

is exact.

§4.3 Čech Cohomology

Let S be a sheaf on the topological space X, and let $\underline{U} = \{U_i\}_I$ be a locally finite open cover. Define

$$C^0(\underline{U},S) = \prod_i \Gamma(U_i,S)$$

$$C^1(\underline{U},S) = \prod_{i\neq j} \Gamma(U_{ij},S)$$

$$\vdots$$

$$C^p(\underline{U},S) = \prod_{i_0\neq i_1\neq\dots\neq i_p} \Gamma(U_{i_0\dots i_p},S)$$

An element $f = \{f_{i_0\dots i_p} \in \Gamma(U_{i_0\dots i_p},S)\}_{i_0\neq i_1\neq\dots\neq i_p}$ of $C^p(\underline{U},S)$ is called a p-cochain. We define a coboundary operator

$$\delta : C^p(\underline{U},S) \to C^{p+1}(\underline{U},S)$$

by

$$(\delta f)_{i_0\dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \left. f_{i_0\dots\hat{i}_j\dots i_{p+1}} \right|_{U_{i_0\dots i_{p+1}}} \qquad 4.3.1$$

(where $\hat{i}_j$ means omit i_j).

f is called a <u>cocycle</u> if $\delta f = 0$ and a <u>coboundary</u> if $f = \delta g$ for some $g \in C^{p-1}(\underline{U};S)$. Define

$$Z^p(\underline{U},S) = \text{Ker } \delta \subset C^p(\underline{U},S)$$

and

$$\overset{v}{H}{}^p(\underline{U};S) = \frac{Z^p(\underline{U},S)}{\delta C^{p-1}(\underline{U},S)}$$

(for this we need $\delta^2 = 0$, which follows from 4.3.1).

Now suppose the covering $\underline{V} = \{V_j\}_J$ of X is a <u>refinement</u> of $\underline{U}$. Then for all $j \in J$ $\exists\ i \in I$ such that $V_j \subset U_i$ and we can choose a map

$$\phi : J \to I$$

such that $V_j \subset U_{\phi(j)}$ for all j.

Then we have a map

$$r_\phi : C^p(\underline{U},S) \to C^p(\underline{V},S)$$

given by

$$(r_\phi f)_{j_o \ldots j_p} = f_{\phi(j_o)\ldots\phi(j_p)}\Big|_{V_{j_o}\ldots j_p}$$

The maps r_ϕ and δ commute, so that we have a homomorphism

$$r : \overset{v}{H}{}^p(\underline{U};S) \to \overset{v}{H}{}^p(\underline{V};S)$$

(which is independent of the choice of $\underline{U}$).

We define the p^{th} Čech cohomology of S on X to be the direct limit of these refinements

$$\check{H}^p(X;S) = \varinjlim_{\underline{U}} \check{H}^p(\underline{U};S).$$

Clearly for any sheaf S

$$\check{H}^0(X;S) = \Gamma(X,S).$$

Also note that if we choose S = Z we have the isomorphism

$$\check{H}^*(X;Z) \tilde{=} H^*_{sing}(X;Z)$$

between Čech cohomology and singular cohomology. The definition of the Čech cohomology groups (which we will normally write $H^*(X;S)$) is very impractical. In applications we shall use the following theorem.

Theorem 4.3.2 (Leray)

$$H^q(U_{i_1\ldots i_p};S) = 0 \quad q > 0, \quad \forall_{i_1\ldots i_p} \Rightarrow H^*(\underline{U};S) = H^*(X;S).$$

□

A cover satisfying the conditions of this theorem is called acyclic.

We next describe how cohomology treats exact sequences of sheaves. Suppose

$$0 \to R \xrightarrow{\alpha} S \xrightarrow{\beta} T \to 0 \qquad 4.3.3$$

is a short exact sequence of sheaves on X. We can define maps

$$\alpha : C^p(\underline{U},R) \to C^p(\underline{U},S), \quad \beta : C^p(\underline{U},S) \to C^p(\underline{U},T)$$

(for any cover $\underline{U}$) which commute with δ. We therefore have maps

$$\alpha^* : H^p(X;R) \to H^p(X;S), \quad \beta^* : H^p(X;S) \to H^p(X;T) .$$

We can also define the connecting map

$$\delta^* : H^p(X;T) \to H^{p+1}(X;R)$$

using the exactness of 4.3.3 as follows. Given $f \in Z^p(\underline{U},T)$ we can find a refinement $\underline{V}$ of $\underline{U}$ and a cochain $g \in C^p(\underline{V},S)$ such that

$$\beta(g) = r_\phi f$$

(where r_ϕ is a refining map). Then

$$\beta(\delta g) = \delta\, \beta(g) = \delta\, r_\phi f = r_\phi\, \delta f = 0$$

so that we can find a refinement $\underline{W}$ of $\underline{V}$ and a cochain $h \in C^{p+1}(\underline{W},R)$ such that

$$\alpha h = \delta g$$

(missing out the refining map). Then

$$\alpha\delta h = \delta\alpha h = \delta^2 g = 0$$

and so $\delta h = 0$ and $h \in Z^{p+1}(\underline{W},R)$. We define

$$\delta^* [f] = [h]$$

(where [ ] denotes the cohomology class).

Theorem 4.3.4

The sequence

$$0 \to H^0(X;R) \xrightarrow{\alpha^*} H^0(X;S) \xrightarrow{\beta^*} H^0(X;T) \xrightarrow{\delta^*} H^1(X;R) \to \ldots$$

$$\ldots \to H^p(X;S) \xrightarrow{\beta^*} H^p(X;T) \xrightarrow{\delta^*} H^{p+1}(X;R) \xrightarrow{\alpha^*} H^{p+1}(X;S) \to \ldots$$

is exact. □

This theorem, which is fundamental in cohomology theory, in particular enables us to relate Čech cohomology to Dolbeault cohomology (which is defined in the proof of the following theorem).

Theorem 4.3.5

If X is a complex manifold

$$H^q(X;\Omega^p) \cong H^{p,q}(X;0).$$

Outline of proof:

We can break up the long exact sequence 4.2.3 into the short exact sequences:

$$0 \to \Omega^p \xrightarrow{i} A^{p,0} \xrightarrow{\bar\partial} Z^{p,1} \to 0$$

$$\vdots$$

$$0 \to Z^{p,q} \xrightarrow{i} A^{p,q} \xrightarrow{\bar\partial} Z^{p,q+1} \to 0$$

Then, since $H^r(X;A^{p,q}) = 0$ for $r > 0$ and all p,q ($A^{p,q}$ being a fine sheaf) the corresponding long exact sequences in cohomology yield

$$H^q(X;\Omega^p) \cong H^{q-1}(X;Z^{p,1})$$

$$\cong H^{q-2}(X;Z^{p,2})$$

$$\vdots$$

$$\cong H^1(X;Z^{p,q-1})$$

$$\cong H^0(X;Z^{p,q}) \Big/ \bar\partial H^0(X;A^{p,q-1})$$

and this last group is, by definition $H^{p,q}(X;0)$. We shall need an

extension of this result in section 4.6.

§4.4 One-Vertex Trees

In the rest of this chapter we generalise the concept of a twistor diagram so that the vertices represent $\mathbf{CP}^n$ (or its dual). Then $\mathcal{D}Z$ becomes the canonical n-form on $\mathbf{CP}^n$, and the one-vertex tree is that drawn in figure 4.4.1.

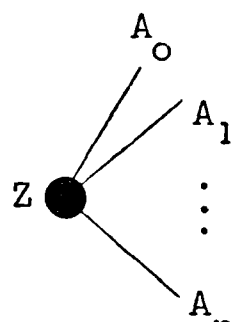


Figure 4.4.1

This diagram represents the contour integral

$$\oint_\gamma \frac{\mathcal{D}Z}{A_0|_Z\, A_1|_Z \cdots A_n|_Z} \qquad 4.4.2$$

where γ is a product of n S^1s. We can think of the associated function

$$f =: \frac{1}{A_0|_Z\, A_1|_Z \cdots A_n|_Z}$$

in 4.4.2 as an element of $\Gamma(A^c_{o\ldots n}, \mathcal{O}(-n-1))$ where

$$A^c_{o\ldots n} =: \{ \left|_Z \right. \in \mathbf{CP}^n : \left|^{A_o}_Z \right. \neq 0, \left|^{A_1}_Z \right. \neq 0, \ldots, \left|^{A_n}_Z \right. \neq 0 \} .$$

($A^c_{o\ldots n}$ is simply the associated space of the diagram in figure 4.4.1.) However, consider the integral

$$\oint_\gamma \frac{\mathcal{D}Z}{\left|^{A_1}_Z\right.^2 \left|^{A_2}_Z\right. \ldots \left|^{A_n}_Z\right.} .$$

It is zero because the $\left|^{A_o}_Z\right.$ pole is missing (so that part of γ can be shrunk to zero).

Therefore (irrespective of the representative chosen for γ)

$$\oint_\gamma \frac{\mathcal{D}Z}{\left|^{A_o}_Z\right. \left|^{A_1}_Z\right. \ldots \left|^{A_n}_Z\right.} = \oint_\gamma \left\{ \frac{1}{\left|^{A_o}_Z\right. \left|^{A_1}_Z\right. \ldots \left|^{A_n}_Z\right.} + \frac{1}{\left|^{A_1}_Z\right.^2 \left|^{A_2}_Z\right. \ldots \left|^{A_n}_Z\right.} \right\} \mathcal{D}Z .$$

So far as integration over γ is concerned, then,

$$\frac{1}{\left|^{A_o}_Z\right. \left|^{A_1}_Z\right. \ldots \left|^{A_n}_Z\right.} \quad \text{and} \quad \frac{\left|^{A_o}_Z\right. + \left|^{A_1}_Z\right.}{\left|^{A_o}_Z\right. \left|^{A_1}_Z\right.^2 \left|^{A_2}_Z\right. \ldots \left|^{A_n}_Z\right.}$$

are equivalent. Obviously the integral over γ of any function g

having one of the poles in f missing will be zero. Suppose g has the $A_{|p}$ pole missing. g still has to be holomorphic and homogeneous of degree - n - 1 so it is an element of

$$\Gamma(A^c_{o\ldots\ \hat{p}\ \ldots n},\ \mathcal{O}(-n-1)).$$

Therefore

$$\rho_{A^c_p} g \in \Gamma\,(A^c_{o\ldots n},\ \mathcal{O}(-n-1))$$

and the equivalence relation referred to above can be written

$$f \sim f + \rho_{A^c_p} g \in \Gamma(A^c_{o\ldots n},\ \mathcal{O}(-n-1)).$$

We can think of f as an n-cocycle, and we can define an n - 1 cochain from g by choosing

$$0 \in \Gamma(A^c_{o\ldots\ \hat{q}\ \ldots n},\ \mathcal{O}(-n-1)) \text{ for all } q \neq p.$$

Now we have

$$f \in C^n(\{A^c_i\},\ \mathcal{O}(-n-1))$$

$$g \in C^{n-1}(\{A^c_i\},\ \mathcal{O}(-n-1))$$

and the equivalence relation $\sim$ is exactly the coboundary freedom

$$f \sim f + \delta g.$$

Therefore the equivalence class containing f is an element of

$$\frac{C^n(\{A_i^c\},\ \mathcal{O}(-n-1))}{\delta C^{n-1}(\{A_i^c\},\ \mathcal{O}(-n-1))} = H^n(\{A_i^c\};\ \mathcal{O}(-n-1)).$$

We henceforth regard the diagram in figure 4.4.1 as defining this cohomology class.

In this case the direct limit map

$$H^n(\{A_i^c\};\ \mathcal{O}(\ -n-1)) \to H^n(\mathbf{P}^n;\ \mathcal{O}(-n-1))$$

is an isomorphism because the cover $\{A_i^c\}$ is acyclic. (In fact the sets A_i^c are Stein and $\mathcal{O}(-n-1)$ is a coherent sheaf.)

§4.5 The dot product

In the previous section we argued that the associated function of the diagram in figure 4.4.1 should be interpreted as an element of $H^n(\mathbf{P}^n;\ \mathcal{O}(-n-1))$. We used the Čech map

$$\check{C} : H^0(A^c_{0\ldots n};\ \mathcal{O}(-n-1)) \to H^n(\mathbf{P}^n;\ \mathcal{O}(-n-1)).$$

$$f \longmapsto [f]$$

Now suppose we break the diagram up as in figure 4.5.1.

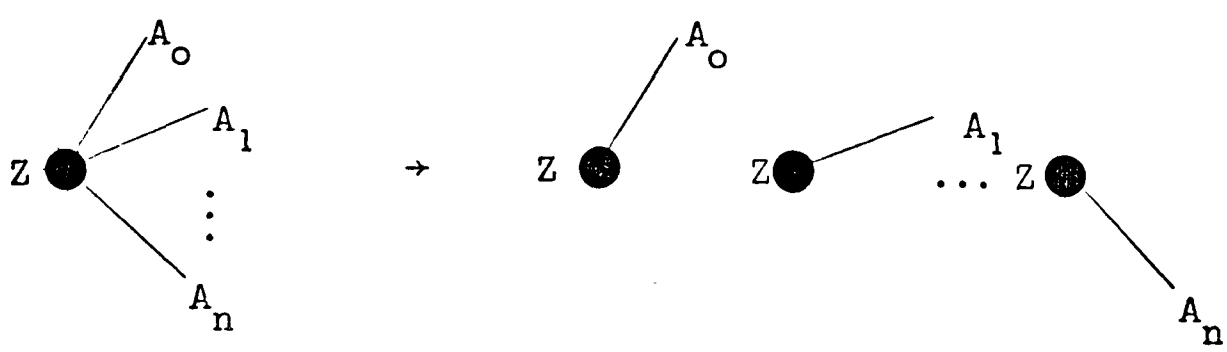


Figure 4.5.1

Each of the fragments in figure 4.5.1 is an element of $H^o(A_p^c;\ \mathcal{O}(-1))$ for some p. It will be useful when we consider more complicated diagrams to be able to construct those diagrams from fragments such as these. So in this case we construct an element of $H^n(\ P^n;\ \mathcal{O}(-\ n\ -\ 1))$ by

(i) restricting each $\left(\begin{array}{c} A_p \\ | \\ Z \end{array}\right)^{-1} \in H^o(A_p^c;\ \mathcal{O}(-\ 1))$ to $H^o(A^c_{o\ldots n};\ \mathcal{O}(-\ 1))$ and taking the cup product of all these elements to obtain

$$\frac{1}{\begin{array}{cccc} A_o & A_1 & \cdots & A_n \\ | & | & & | \\ Z & Z & & Z \end{array}} \in H^o(A^c_{o\ldots n};\ \mathcal{O}(-\ n\ -\ 1))$$

(ii) using the Čech map, as before.

However, the Čech map is the same as a succession of Mayer-Vietoris connecting maps. (The Mayer-Vietoris sequence in sheaf cohomology is

$$\ldots \to H^p(U;S) \oplus H^p(V;S) \to H^p(U \cap V;S) \xrightarrow{\delta^*} H^{p+1}(U \cup V;S) \to \ldots$$

and, as before, δ^* is called the connecting map.) In the case n = 2 for example we have the commutative triangle in figure 4.5.2.

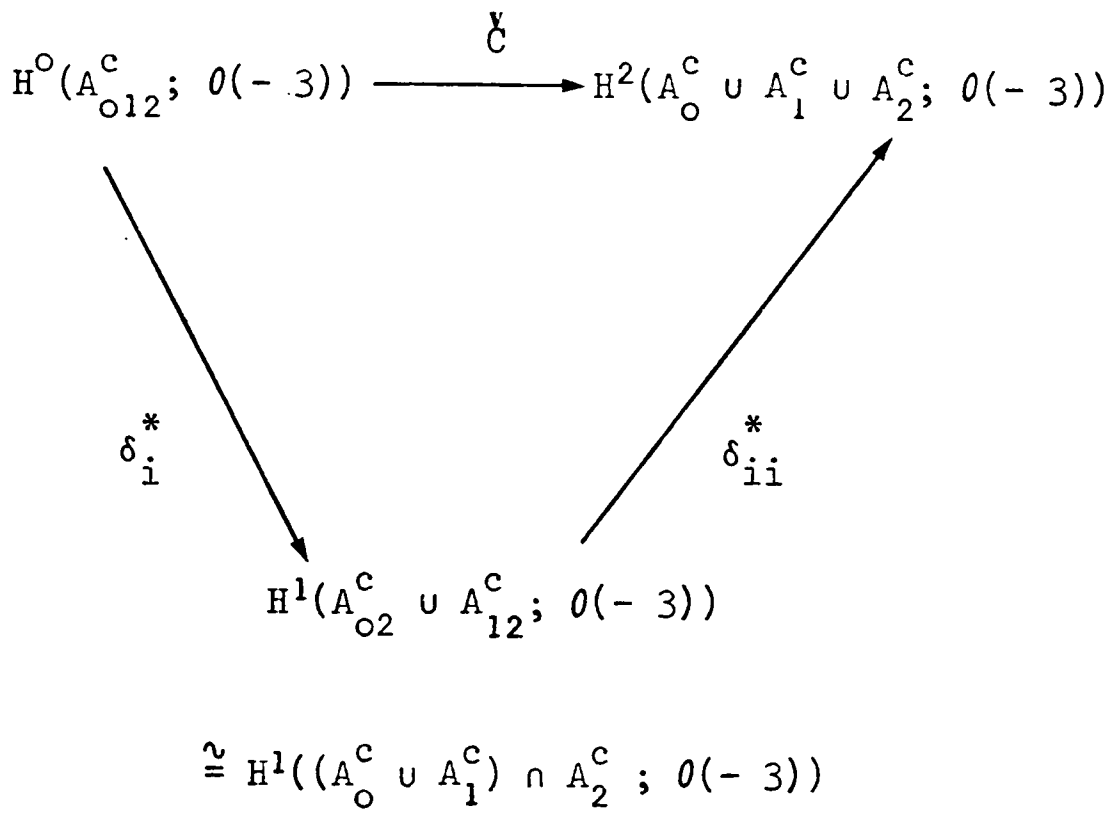
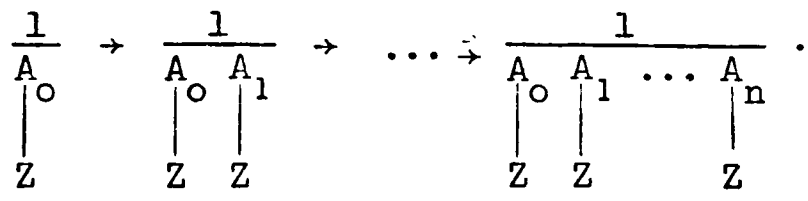
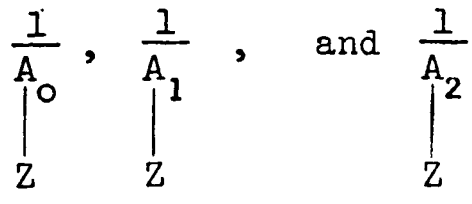


Figure 4.5.2

We can also split the cup product up - instead of taking the cup product of the n + 1 fragments in figure 4.5.1 all at once we can do it in n stages:



Each of these stages commutes with the relevant Mayer-Vietoris connecting map. This is best explained by considering the n = 2 example again. Instead of taking the cup product of <u>all three</u> functions



and then using the Čech map we could:

(i) take the cup product of

$$\frac{1}{A_o}\Big|_Z \quad \text{and} \quad \frac{1}{A_1}\Big|_Z$$

to obtain the element

$$\frac{1}{A_o \quad A_1}\Big|_Z \quad \Big|_Z$$

of

$$H^o(A^c_{o1};\ (-2))$$

(ii) use the Mayer-Vietoris δ^* to map this element to

$$\delta^*\left(\frac{1}{A_o \quad A_1}\Big|_Z \quad \Big|_Z\right) \in H^1(A^c_o \cup A^c_1;\ \mathcal{O}(-2))$$

(iii) take the cup product of

$$\delta^*\left(\frac{1}{A_o \quad A_1}\Big|_Z \quad \Big|_Z\right) \quad \text{and} \quad \frac{1}{A_2}\Big|_Z$$

to obtain an element of

$$H^1((A_o^c \cup A_1^c) \cap A_2^c;\ \mathcal{O}(-3))$$

(iv) use the Mayer-Vietoris δ^* (δ^*_{ii} in figure 4.5.2) to map this element into

$$H^2((A_o^c \cup A_1^c) \cup A_2^c;\ \mathcal{O}(-3)) \cong H^2(\mathbf{P}^2;\ \mathcal{O}(-3)).$$

We have used the commutativity of the diagram in figure 4.5.3 (where the maps δ^*_i and δ^*_{ii} are as in figure 4.5.2).

$$\begin{array}{ccccc}
H^0(A^c_{o1};\ (-2)) \times H^0(A^c_2;\ (-1)) & \xrightarrow{\cup} & H^0(A^c_{o12};\mathcal{O}(-3)) & \xrightarrow{C=\delta^*_{ii}\delta^*_i} & H^2(A^c_o \cup A^c_1 \cup A^c_2;\mathcal{O}(-3)) \\
\downarrow \delta^* \qquad\qquad \downarrow 1 & & & \nearrow \delta^*_{ii} & \\
H^1(A^c_o \cup A^c_1;\mathcal{O}(-2)) \times H^0(A^c_2;\mathcal{O}(-1)) & \xrightarrow{\cup} & H^1(A^c_{o2} \cup A^c_{12};\ \mathcal{O}(-3)) & &
\end{array}$$

Figure 4.5.3

In general, when we are constructing the diagram in figure 4.5.1 from its fragments, we use the commutativity of the diagram in figure 4.5.4 for $p = n - 1$, $p = n - 2$, ... , $p = 1$ successively.

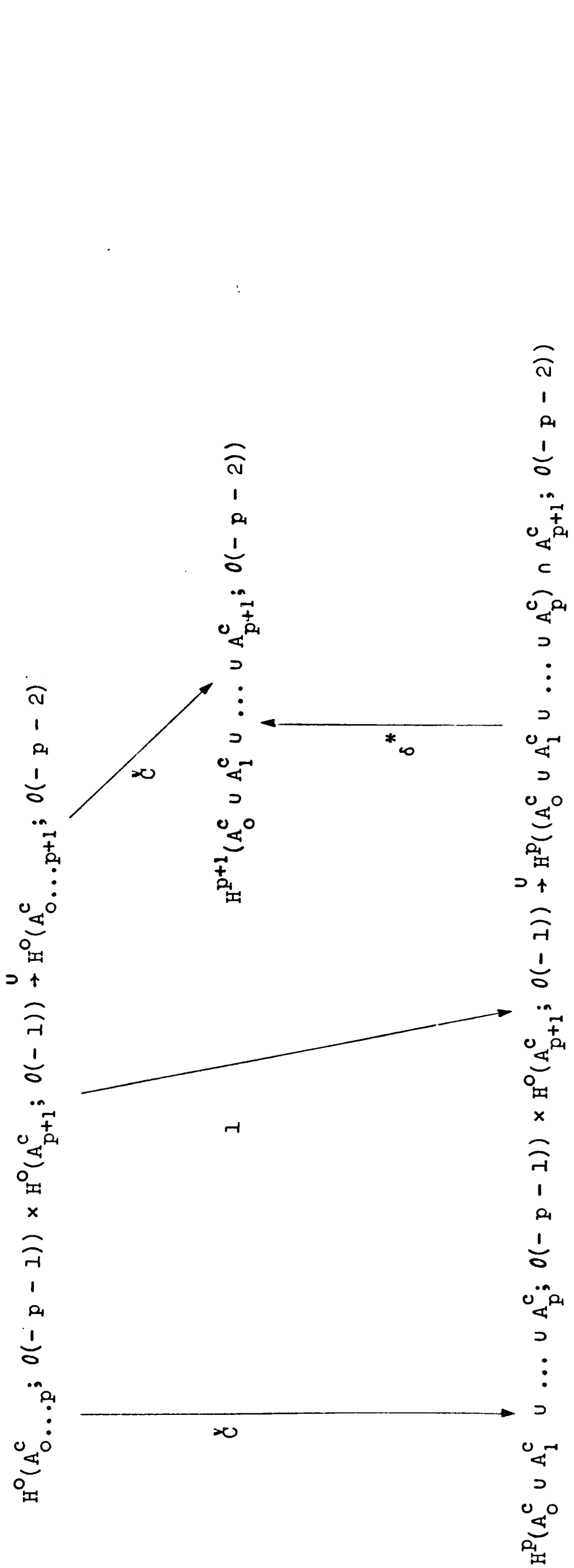

$H^o(A^c_{o\ldots p}; O(-p-1)) \times H^o(A^c_{p+1}; O(-1)) \xrightarrow{\cup} H^o(A^c_{o\ldots p+1}; O(-p-2))$
γ
$H^p(A^c_o \cup A^c_1 \cup \ldots \cup A^c_p; O(-p-1)) \times H^o(A^c_{p+1}; O(-1)) \xrightarrow{\cup} H^p((A^c_o \cup A^c_1 \cup \ldots \cup A^c_p) \cap A^c_{p+1}; O(-p-2))$
γ
1
δ^*
$H^{p+1}(A^c_o \cup A^c_1 \cup \ldots \cup A^c_{p+1}; O(-p-2))$

Figure 4.5.4

Now the construction can be done by "sticking on" the twigs one at a time (see figure 4.5.5):

(i) $\frac{1}{A_0 \mid o \mid Z} \rightarrow \frac{1}{A_0 \mid o \mid A_1 \mid Z \mid Z}$ cup product

(ii) $\frac{1}{A_0 \mid o \mid A_1 \mid Z \mid Z} \rightarrow \delta^*\left(\frac{1}{A_0 \mid o \mid A_1 \mid Z \mid Z}\right)$ Maye ietoris connecting map

(iii) $\delta^*\left(\frac{1}{A_0 \mid o \mid A_1 \mid Z \mid Z}\right) \rightarrow \frac{1}{A_2 \mid Z} \quad \delta^*\left(\frac{1}{A_0 \mid o \mid A_1 \mid Z \mid Z}\right)$ cup product

(iv) $\frac{1}{A_2 \mid Z} \; \delta^*\left(\frac{1}{A_0 \mid o \mid A_1 \mid Z \mid Z}\right) \rightarrow \delta^*\left\{\frac{1}{A_2 \mid Z} \quad \delta^*\left(\frac{1}{A_0 \mid o \mid A_1 \mid Z \mid Z}\right)\right\}$ Mayer-Vietoris connecting map

and so on.

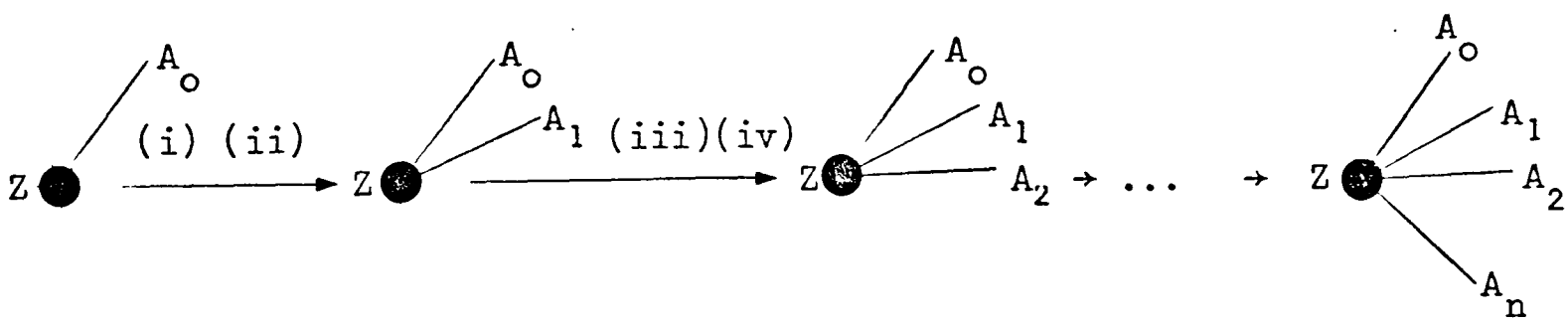


Figure 4.5.5

This procedure can usefully be regarded as an iteration of the composite map

$$H^p(A_o^c \cup A_1^c \cup \ldots \cup A_p^c; \mathcal{O}(-p-1))$$

$$\times H^o(A_{p+1}^c; \mathcal{O}(-1)) \xrightarrow{\delta^* \cup} H^{p+1}(A_o^c \cup A_1^c \cup \ldots \cup A_{p+1}^c; \mathcal{O}(-p-2))$$

called the dot product. In fact this is a special case of the dot product, which is defined in general between an element $\alpha \in H^p(A;S)$ and an element $\beta \in H^q(B;T)$ as follows. We take the cup product

$$\alpha \cup \beta \in H^{p+q}(A \cap B; \quad S \otimes T).$$

Then $$\alpha \cdot \beta =: \delta^*(\alpha \cup \beta) \in H^{p+q+1}(A \cup B; \quad S \otimes T)$$

is the dot product.

We can see from figure 4.5.5 that our construction scheme suggests that we interpret the fragment in figure 4.5.6 as an element of $H^p(A^c_{o\ldots p}; \mathcal{O}(-p-1))$.

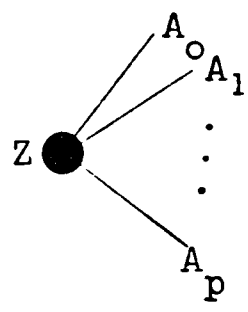


Figure 4.5.6

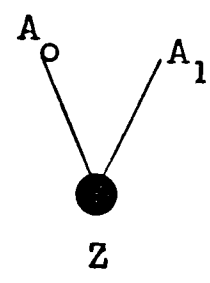
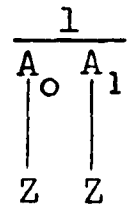


Figure 4.5.7

Now suppose n = 3. The fragment drawn in figure 4.5.7 is an element of $H^1(A_o^c \cup A_1^c; \mathcal{O}(-2))$ having the twistor elementary state



as a representative cocycle. In fact it was the description of these elementary states as representatives of elements of H^1 (see section 1.3) which led to this interpretation of diagram fragments in the first place.

We can also use the dot product to combine the fragments drawn in figure 4.5.8.

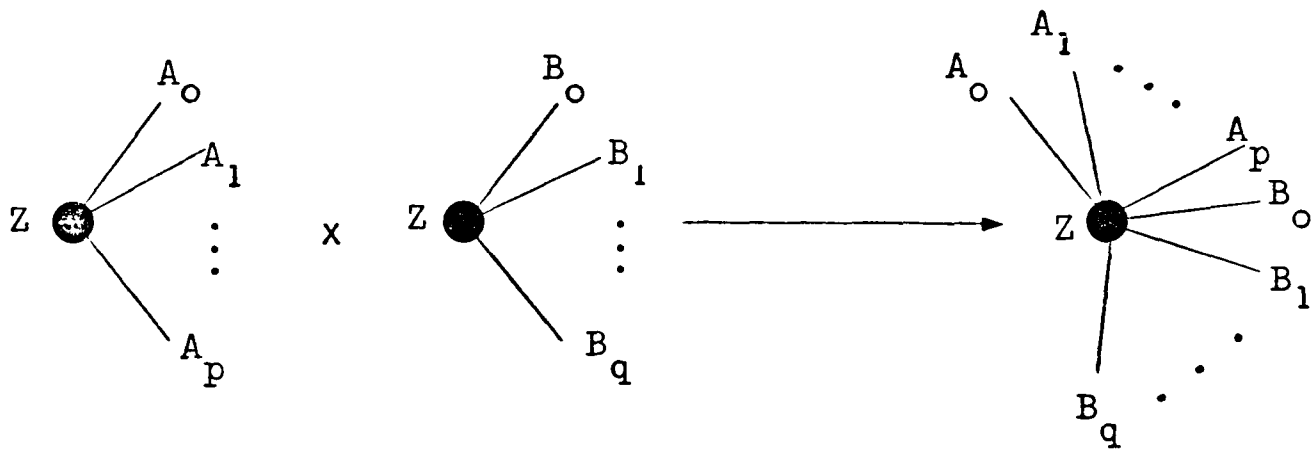


Figure 4.5.8

§4.6 The Dolbeault representation

Suppose we choose p = 0 in the long exact sequence of sheaves 4.2.3. We obtain the exact sequence

$$0 \to \mathcal{O} \xrightarrow{i} A^{0,0} \xrightarrow{\bar{\partial}} A^{0,1} \to \ldots \qquad 4.6.1.$$

This sequence is still exact (Morrow and Kodaira 1971) if we replace the functions in 4.6.1 by the corresponding sections of the vector bundle F on X:

$$0 \to \mathcal{O}(F) \xrightarrow{i} A^{0,0}(F) \xrightarrow{\bar{\partial}} A^{0,1}(F) \to \ldots \qquad 4.6.2.$$

Now let $X \subset \mathbf{CP}^n$ and $F = H^{-m}$ where H is the canonical bundle on $\mathbf{CP}^n$. Then a local section of F is a locally defined function homogeneous of degree m, and so $\mathcal{O}(F) = \mathcal{O}(m)$. Analogous results hold for the other sheaves, so that 4.6.2 becomes

$$0 \to \mathcal{O}(m) \xrightarrow{i} A^{0,0}(m) \xrightarrow{\bar{\partial}} A^{0,1}(m) \to \ldots \qquad 4.6.3.$$

We therefore have the following short exact sequences

$$\begin{aligned}
&0 \to \mathcal{O}(m) \xrightarrow{i} A^{0,0}(m) \xrightarrow{\bar{\partial}} Z^{0,1}(m) \to 0 \\
&0 \to Z^{0,1}(m) \xrightarrow{i} A^{0,1}(m) \xrightarrow{\bar{\partial}} Z^{0,2}(m) \to 0 \\
&\qquad\qquad \vdots \\
&0 \to Z^{0,p}(m) \xrightarrow{i} A^{0,p}(m) \xrightarrow{\bar{\partial}} Z^{0,p+1}(m) \to 0
\end{aligned} \qquad 4.6.4.$$

Theorem 4.6.5

$$H^q(X;\ \mathcal{O}(m)) \tilde{=} H^{o,q}(X;\ \mathcal{O}(m)).$$

Outline of proof:

Just as in Theorem 4.3.5 we have $H^r(X;\ A^{p,q}(m)) = 0$ for $r > 0$ and all p,q. Therefore the long exact sequences in cohomology corresponding to 4.6.4 give us the isomorphisms

$$H^q(X;\ \mathcal{O}(m)) \tilde{=} H^{q-1}(X;\ Z^{o,1}(m))$$

$$\tilde{=} H^{q-2}(X;\ Z^{o,2}(m))$$

$$\vdots$$

$$\tilde{=} H^1(X;\ Z^{o,q-1}(m))$$

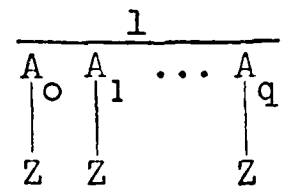
$$\tilde{=} H^o(X;\ Z^{o,q}(m))$$

$$\bar{\partial}\ H^o(X;A^{o,q-1}(m))$$

$$\tilde{=} H^{o,q}(X;\ \mathcal{O}(m)) \quad \text{by definition.} \qquad \square$$

Of course when m = 0 this theorem is identical to Theorem 4.3.5 with p = 0.

In this section we follow the representative cocycle



through all these isomorphisms from

$$H^q(A_o^c \cup \ldots \cup A_q^c;\ \mathcal{O}(-q-1)) \quad \text{to} \quad H^{o,q}(A_o^c \cup \ldots \cup A_q^c;\ \mathcal{O}(-q-1))$$

to obtain its Dolbeault representative. Each of the isomorphisms is the inverse of the connecting map for one of the short exact sequences in 4.6.4. Consider the first:

$$\delta^{*-1} : H^q(X;\ \mathcal{O}(-q-1)) \xrightarrow{\simeq} H^{q-1}(X; Z^{o,1}(-q-1)).$$

This comes from the short exact sequence

$$0 \to \mathcal{O}(-q-1) \xrightarrow{i} A^{o,o}(-q-1) \xrightarrow{\bar\partial} Z^{o,1}(-q-1) \to 0$$

via the array:

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^{q-1}(\underline{A}^c, \mathcal{O}(-q-1)) & \to & C^{q-1}(\underline{A}^c, A^{o,o}(-q-1)) & \xrightarrow{\bar\partial} & C^{q-1}(\underline{A}^c, Z^{o,1}(-q-1)) & \to & 0 \\
 & & \downarrow & & \downarrow \delta & & \downarrow & & \\
0 & \to & C^{q}(\underline{A}^c, \mathcal{O}(-q-1)) & \xrightarrow{i} & C^{q}(\underline{A}^c, A^{o,o}(-q-1)) & \to & C^{q}(\underline{A}^c, Z^{o,1}(-q-1)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

4.6.6.

The rows of this array are exact because $\underline{A}^{\mathcal{C}} = \{A_i^c\}_{i=0,\ldots,q}$ is a Leray cover (from the remarks at the end of section 4.4), and the squares commute. Let $\omega = \dfrac{1}{\begin{matrix} A_0 & \cdots & A_q \\ | & & | \\ Z & & Z \end{matrix}} \in Z^q(\underline{A}^c, \mathcal{O}(-q-1))$.

Then

$$i(\omega) \in Z^q(\underline{A}^c, A^{0,0}(-q-1)) = \delta C^{q-1}(\underline{A}^c, A^{0,0}(-q-1))$$

(because $H^q(\underline{A}^c; A^{0,0}(-q-1)) = 0$). Therefore there is an element in $C^{q-1}(\underline{A}^c, A^{0,0}(-q-1))$ whose coboundary is $i(\omega)$. To construct this element we define the partition of unity for the cover $\underline{A}^c$:

$$p_i = \frac{\left\| \begin{matrix} A_i \\ | \\ Z \end{matrix} \right\|^2}{\left\| \begin{matrix} A_0 \\ | \\ Z \end{matrix} \right\|^2 + \left\| \begin{matrix} A_1 \\ | \\ Z \end{matrix} \right\|^2 + \ldots + \left\| \begin{matrix} A_q \\ | \\ Z \end{matrix} \right\|^2}$$

satisfying (i) p_i is C^∞ in X

(ii) $\sum_i p_i \equiv 1$

(iii) $\sup(p_i) \subset A_i^c$

Consider the cochain $\{p_i\omega\} \in C^{q-1}(\underline{A}^c, A^{0,0}(-q-1))$:

$$\delta\{p_i\omega\} = \delta\left\{\frac{\left(\left|\bar{A}_i\right|_Z\right)}{\left\|\left|A_0\right|_Z \left|A_1\right|_Z \cdots \left|\hat{A}_i\right|_Z \cdots \left|A_q\right|_Z\right\|\left|A\right|_Z\right\|^2}\right\}$$

$$= \omega$$

where $\left\|\left|A\right|_Z\right\|^2 =: \left|\left|A_0\right|_Z\right|^2 + \ldots + \left|\left|A_q\right|_Z\right|^2$.

Therefore the cocycle $\bar{\partial}\{p_i\omega\} \in C^{p-1}(\underline{A}^c, Z^{0,1}(-q-1))$ is an element of $\delta^{*-1}[\omega]$, as required.

The next δ^{*-1} map is identical except that instead of multiplying by p_i we multiply by p_j $(j \neq i)$ and skew symmetrise, obtaining the cocycle

$$\bar{\partial}\{p_j\,\bar{\partial}\{p_i\omega\} - p_i\,\bar{\partial}\{p_j\omega\}\} \in C^{p-2}(\underline{A}^c, Z^{0,2}(-q-1))$$

Continuing this process we finally obtain

$$\left\{\frac{n!\,(-1)^{i-1}\,\bar{\partial}p_0 \wedge \ldots \wedge \hat{\bar{\partial}}p_i \wedge \ldots \wedge \bar{\partial}p_q}{\left|A_0\right|_Z \cdots \left|A_q\right|_Z}\right\} \qquad 4.6.7.$$

To calculate 4.6.7 we need

$$\bar{\partial} p_i = \frac{(\left|\begin{smallmatrix}A_i\\Z\end{smallmatrix}\right|) d(\left|\begin{smallmatrix}\bar{A}_i\\Z\end{smallmatrix}\right|)}{\left\|\begin{smallmatrix}A\\Z\end{smallmatrix}\right\|^2} - \frac{\left|\begin{smallmatrix}A_i\\Z\end{smallmatrix}\right|^2 \cdot \sum_j \left|\begin{smallmatrix}A_j\\Z\end{smallmatrix}\right| d(\left|\begin{smallmatrix}\bar{A}_j\\Z\end{smallmatrix}\right|)}{\left\|\begin{smallmatrix}A\\Z\end{smallmatrix}\right\|^4}$$

Therefore, after a little calculation

$$\bigwedge_{j\neq i} \bar{\partial} p_j = \frac{\left|\begin{smallmatrix}A_o\\Z\end{smallmatrix}\right| \left|\begin{smallmatrix}A_1\\Z\end{smallmatrix}\right| \ldots \left|\begin{smallmatrix}A_q\\Z\end{smallmatrix}\right| (-1)^i \left(\sum_k (-1)^k \left(\left|\begin{smallmatrix}\bar{A}_k\\Z\end{smallmatrix}\right|\right) \bigwedge_{j\neq k} d(\left|\begin{smallmatrix}\bar{A}_j\\Z\end{smallmatrix}\right|)\right)}{\left\|\begin{smallmatrix}A\\Z\end{smallmatrix}\right\|^{2q+2}}$$

Therefore the 0-cocycle in 4.6.7 is equal to

$$c_n \sum_k (-1)^k \frac{(\left|\begin{smallmatrix}\bar{A}_k\\Z\end{smallmatrix}\right|) d(\left|\begin{smallmatrix}\bar{A}_o\\Z\end{smallmatrix}\right|) \wedge d(\left|\begin{smallmatrix}\bar{A}_1\\Z\end{smallmatrix}\right|) \wedge \ldots \wedge \widehat{d(\left|\begin{smallmatrix}\bar{A}_k\\Z\end{smallmatrix}\right|)} \wedge \ldots \wedge d(\left|\begin{smallmatrix}\bar{A}_q\\Z\end{smallmatrix}\right|)}{\left\|\begin{smallmatrix}A\\Z\end{smallmatrix}\right\|^{2q+2}}$$

$$= \frac{c_n \left[\begin{smallmatrix}\bar{A}_o & \bar{A}_1 \ldots & \bar{A}_q\end{smallmatrix}\right] \mathcal{D}\bar{Z}}{\left\|\begin{smallmatrix}A\\Z\end{smallmatrix}\right\|^{2q+2}} \qquad 4.6.8$$

where c_n is a constant depending only on n. This (o,q) form on X is the Dolbeault representative of the cocycle ω. It is closely

related to the Bochner-Martinelli kernel (Griffiths and Harris 1978) and will be discussed further in the next section.

§4.7 Residues

Figure 4.7.1 shows a diagram constructed from two fragments. The diagram is an element of

$$H^n(\mathbf{P}^n;\ \mathcal{O}(-n-1)) \cong H^{o,n}(\mathbf{P}^n;\ \mathcal{O}(-n-1))$$

which is isomorphic to $\mathbf{C}$ by Serre duality (Morrow and Kodaira 1971). We can therefore think of the dot product in figure 4.7.1 as the map which evaluates the pair of fragments.

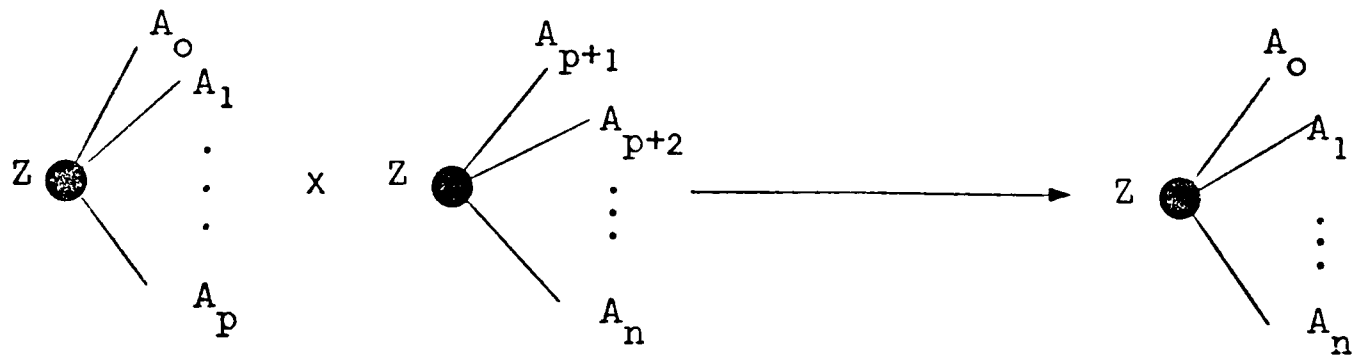


Figure 4.7.1

In this case the Serre duality is simple. An element of $H^{o,n}(\mathbf{P}^n;\ \mathcal{O}(-n-1))$ is a (o,n) form on $\mathbf{CP}^n$ homogeneous of degree $-n-1$ in Z. Take its cup product with the canonical (n,o) form $\mathcal{D}Z$ to obtain an (n,n) form on $\mathbf{CP}^n$ homogeneous of degree zero and integrate this form all over $\mathbf{CP}^n$ to get the complex number

corresponding to the original (o,n) form. For the diagram in figure 4.7.1 this (o,n) form is

$$\frac{\sum_{k=o}^{n} (-1)^k \left(\left|\bar{A}_k\right|_Z\right) \bigwedge_{\substack{\ell=o \\ \ell\neq k}}^{n} d\left(\left|\bar{A}_\ell\right|_Z\right)}{\left\|\left|A\right|_Z\right\|^2}$$

(up to a multiplicative constant) so we take the cup product with $\mathcal{D}Z$ and integrate:

$$\frac{1}{\left|A_o\ A_1\ \cdots\ A_n\right|} \int_{CP^n} \frac{\left(\sum_{k=o}^{n} (-1)^k \left(\left|A_k\right|_Z\right) \bigwedge_{\substack{\ell=o \\ \ell\neq k}}^{n} d\left(\left|A_\ell\right|_Z\right)\right) \left(\sum_{i=o}^{n} (-1)^i \left(\left|A_i\right|_Z\right) \bigwedge_{\substack{j=o \\ j\neq i}}^{n} \left(\left|A_j\right|_{dZ}\right)\right)}{\left\|\left|A\right|_Z\right\|^2}$$

$$= \frac{1}{\left|A_o\ A_1\ \cdots\ A_n\right|}$$

(up to a multiplicative constant). As expected, the result is the same as would be obtained from the contour integral

$$\frac{1}{(2\pi i)^n} \oint_{(S^1)^n} \frac{\mathcal{D}Z}{\left|A_o\right|_Z \left|A_1\right|_Z \cdots \left|A_n\right|_Z}$$

of the Čech representative. In fact there is a more general result

available. In (Griffiths and Harris 1978) the following theorem is proved.

Theorem 4.7.2

Let X be a neighbourhood of the origin in C^{n+1} and let $A = \bigcup_{i=o}^{n} A_i$, where the hyperplanes A_i are defined by

$$A_i = \{ \Big|_Z \in C^{n+1} : \Big|_Z^{A_i} = 0 \} \quad .$$

Define an n real dimensional contour in X - A by

$$\Gamma = \{ \Big|_Z \in C^{n+1} : \left| \Big|_Z^{A_i} \right| = \epsilon_i \}$$

and choose an element $\omega \in H^o(X - A;\ \mathcal{O}(-n-1))$.

Then

$$\frac{1}{(2\pi i)^{n+1}} \oint_\Gamma \omega\ d^{n+1}Z = \int_{S^{2n+1}} \omega_{\bar\partial} \wedge d^{n+1}Z \qquad 4.7.3$$

where (i) $\omega_{\bar\partial}$ is the Dolbeault representative of the Čech cohomology class $[\omega] \in H^n(X - \bigcap_{i=o}^{n} A_i;\ \mathcal{O}(-n-1))$ and

(ii) the sphere S^{2n+1} surrounds the origin. □

It is important to notice that while we shall choose

$$\omega = \frac{1}{\begin{matrix} A_0 & A_1 & \cdots & A_n \\ | & | & & | \\ Z & Z & & Z \end{matrix}}$$

the theorem holds for any $\omega \in H^0(X - A;\ \mathcal{O}(-n-1))$, so that ω could have multiple poles or essential singularities along the planes A_i. If we integrate out the phase factors on each side of 4.7.3 we obtain

$$\oint_{(S^1)^n} \omega\ \mathcal{D}Z = \int_{CP^n} \omega_{\bar{\partial}} \wedge \mathcal{D}Z \qquad 4.7.4$$

(up to a multiplicative constant), where we have used the Hopf fibration

$$\begin{array}{ccc} S^1 & \hookrightarrow & S^{2n+1} \\ & & \downarrow \\ & & CP^n \end{array}$$

on the right hand side. Equation 4.7.4 confirms that the dot product/Serre duality method of evaluating tree diagrams (as in figure 4.7.1) is equivalent to using Cauchy integrals. The differential form $\omega_{\bar{\partial}}$ is the same as that in 4.6.8, and will come up again in the next chapter.

In section 2.4 we described the Cauchy integral (generalised to more than one complex dimension) in the context of singular homology theory. We shall now show how the residue map described there is written in sheaf cohomology theory.

We use Dolbeault cohomology and consider the short exact sequence of sheaves on $X \subset \mathbf{CP}^n$:

$$0 \to \mathcal{O}(-n-1) \xrightarrow{s} \mathcal{O}(-n) \xrightarrow{r} \mathcal{O}_S(-n) \to 0 \qquad 4.7.5$$

where $s(f) = f\Big|_Z$ and r is the restriction to the hyperplane

$S = \{\Big|_Z \in X : f\Big|_Z = 0\}$. This sequence is a special case of sequence 4.2.2, and it yields the long exact sequence

$$\ldots \to H^{n-1}(X;\mathcal{O}_S(-n)) \xrightarrow{\delta^*} H^n(X;\mathcal{O}(-n-1)) \xrightarrow{s^*} H^n(X;\mathcal{O}(-n)) \to \ldots$$

4.7.6.

Suppose $[\overset{v}{\omega}] \in H^n(X;\mathcal{O}(-n-1))$ is a Čech cohomology element having a polar singularity of order 1 along the hyperplane S. This means that $s^*[\overset{v}{\omega}] = 0$ in 4.7.6.

Consider $[\overset{v}{\omega}\mathcal{D}Z] \in H^n(X;\Omega^n)$

and let $[\omega] \in H^{n,n}(X;\mathcal{O}(-n-1))$ be its Dolbeault representative.

$$\omega \in Z^{n,n}(X,\mathcal{O}(-n-1))$$

$$\therefore\ s\omega \in C^{n,n}(X,\mathcal{O}(-n))$$

However, $s^*[\overset{v}{\omega}] = 0$, and therefore

$$\omega = \frac{\bar{\partial}\mu}{s}$$

$$\therefore\ s\omega = \bar{\partial}\mu\ (= d\mu)$$

where $\mu \in C^{n,n-1}(X, \mathcal{O}(-n))$.

Now we have

$$r\mu \in Z^{n,n-1}(X, \mathcal{O}_s(-n))$$

because

$$\bar{\partial} r\mu = r\bar{\partial}\mu = rs\omega = 0.$$

Therefore

$$\delta^{*-1}[\omega] = \mathrm{res}[\omega] = [\mu] \text{ as required.} \qquad 4.7.7.$$

To finish this section we show that the map

$$\delta^* : H^{n-1}(S; \mathcal{O}(-n)) \to H^n(X; \mathcal{O}(-n-1)) \qquad 4.7.8$$

appearing in 4.7.6 is the same as taking the dot product with the cohomology class $\frac{1}{S}\Big|_Z$:

$$H^0(X - S; \mathcal{O}(-1)) \times H^{n-1}(S; \mathcal{O}(-n)) \to H^n(X; \mathcal{O}(-n-1))$$

We use Čech cohomology and choose the Stein (hence acyclic) cover $\underline{U} = \{U_i\}_{i=0,\ldots,n-1}$ for a neighbourhood V of S in X (V is necessary because S is closed in X). Then as long as X - S is Stein the cover $\underline{U}' = \{U_i, X - S\}$ is a Stein cover

for X and we have the isomorphisms

$$H^{n-1}(S;\ \mathcal{O}(-n)) \cong H^{n-1}(\underline{U};\ \mathcal{O}(-n))$$

$$H^{n}(X;\ \mathcal{O}(-n-1)) \cong H^{n}(\underline{U}';\ \mathcal{O}(-n-1)).$$

Given

$$\{\omega_{o\ldots n-1}\} \in Z^{n-1}(\underline{U},\ \mathcal{O}(-n))$$

choose an element

$$\{\lambda_{i_o\ldots i_{n-1}}\} \in C^{n-1}(\underline{U}',\ \mathcal{O}(-n))$$

such that $r\{\lambda_{i_o\ldots i_{n-1}}\} = \{\omega_{o\ldots n-1}\}$ by choosing

$$r\ \lambda_{o\ldots n-1} = \omega_{o\ldots n-1}$$

and

$$\lambda_{i_o\ldots i_{n-1}} = 0$$

if any one of the indices is n.

$$\delta\{\lambda_{i_o\ldots i_{n-1}}\} = \{\rho_{[i_n}\ \lambda_{i_o\ldots i_{n-1}]}\}$$

$$=\{\ \rho_{X-S}\ \lambda_{o\ldots n-1}\} \in C^{n}(\underline{U}',\ \mathcal{O}(-n))$$

$$r\delta\{\lambda_{i_o\ldots i_p}\} = \{r\rho_{X-S}\,\lambda_{o\ldots n-1}\} = 0$$

$$\therefore\quad \delta\{\lambda_{i_o\ldots i_{n-1}}\} = \{\int_Z^S \mu_{i_o\ldots i_n}\}$$

where $\{\mu_{i_o\ldots i_n}\} \in C^n(\underline{U}', \mathcal{O}(-n-1))$.

$$[\{\mu_{i_o\ldots i_n}\}] = \delta^*[\{\omega_{o\ldots n-1}\}]$$

and

$$\{\mu_{i_o\ldots i_n}\} = \int_Z^S \delta\{\lambda_{i_o\ldots i_{n-1}}\}$$

$$= \{\int_Z^S \rho_{X-S}\,\lambda_{o\ldots n-1}\}$$

which is a representative cocycle for the dot product

$$\left[\int_Z^S\right] \cdot [\{\omega_{o\ldots n-1}\}]$$

as required. In fact we can be a little more general - the same result clearly holds for the map

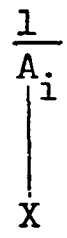
$$\delta^* : H^{p-1}(S; \mathcal{O}(-m)) \to H^p(X; \mathcal{O}(-m-1)).$$

This completes our study of residues in cohomology. It is worth noting that for some purposes the Dolbeault representation is more

convenient, while for others it is more practical to use Čech cohomology.

§4.8 The general Tree Diagram

Every vertex in a tree diagram is incident with n + 1 lines. Therefore a tree diagram on m vertices has mn + 1 lines altogether. Each of these lines is an element of an H^o cohomology group. For example, the twig



is an element of

$$H^o((\underbrace{W \times Y \times \ldots}_{m-1}) \times (X - A_i);\ (\mathcal{O}(o) \otimes \mathcal{O}(o) \otimes \ldots) \otimes \mathcal{O}(-1))$$

and the internal line



is an element of

$$H^{\circ}(\underbrace{X \times Y \times \ldots}_{m-2}) \times (W \times Z - \Delta);\ (\mathcal{O}(o) \otimes \mathcal{O}(o) \otimes \ldots) \otimes \mathcal{O}(-1) \otimes \mathcal{O}(-1))$$

where

$$\Delta = \{ \begin{vmatrix} W \\ Z \end{vmatrix} = 0 \}$$

(see figure 4.8.1, which is drawn for the case n = 3).

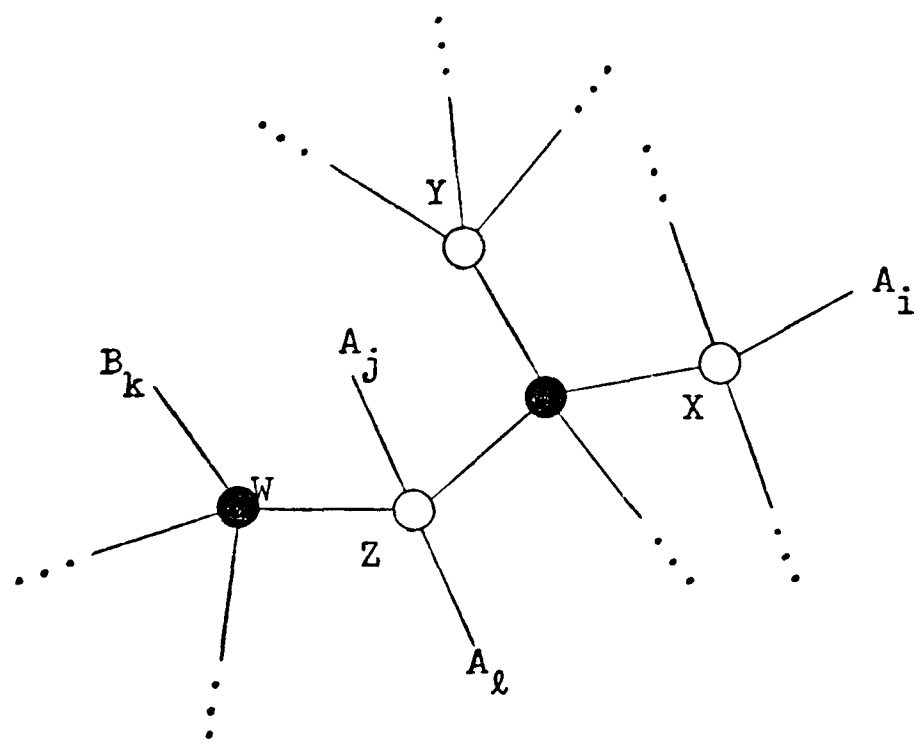


Figure 4.8.1

We therefore have mn + 1 elements of H°s, so that if we take their cup product followed by the Čech map we will obtain an element of

$$H^{mn}(\Pi;\ \mathcal{O}(-n-1) \otimes \mathcal{O}(-n-1) \otimes \ldots)$$

where Π is the union of all the spaces on which the H°s are defined.

In order for this to be a complex number we need to prove that

$$\Pi = \underbrace{W \times X \times Y \times Z \times \ldots}_{m}$$

The space $(W \times X \times Y \times Z \times \ldots) - \Pi$ is obtained by thinking of the tree diagram as a topological diagram and replacing all the lines by springs. Then the diagram in figure 4.8.1, for example, becomes that in figure 4.8.2.

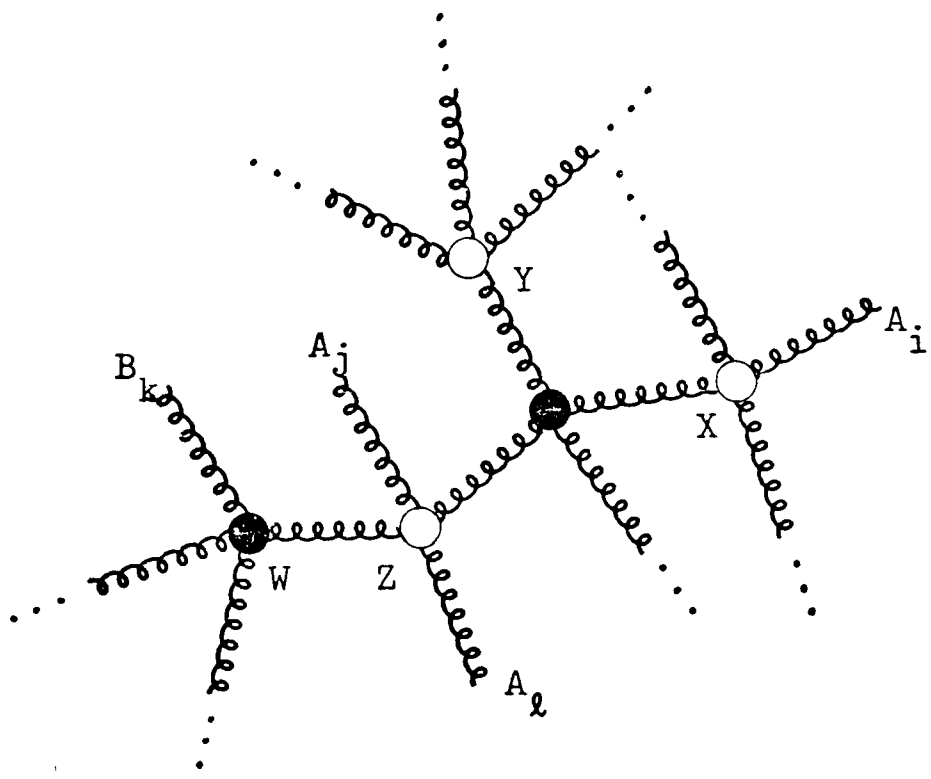


Figure 4.8.2

This space is empty if and only if the parameters $\{A_i, B_j\}$ are in general position. Therefore $\Pi = (W \times X \times Y \times Z \times \ldots)$ if and only if the parameters are in general position.

Now suppose the parameters <u>are</u> in general position. Then the diagram determines an element of

$$H^{mn}((W \times X \times Y \times Z \times \ldots);\ \mathcal{O}(-n-1) \otimes \mathcal{O}(-n-1) \otimes \ldots),$$

which is isomorphic to $\mathbf{C}$ by Serre duality. We can <u>calculate</u> the complex number represented by the diagram using a repeated application of the cohomological residue map. The diagram must have at least one vertex of the type in figure 4.8.3 (having at least n twigs).

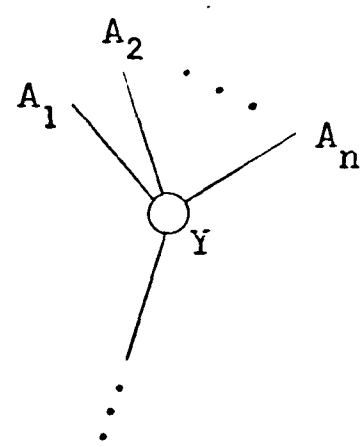
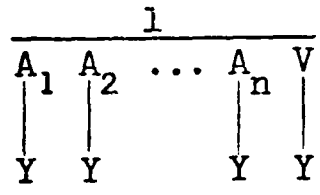
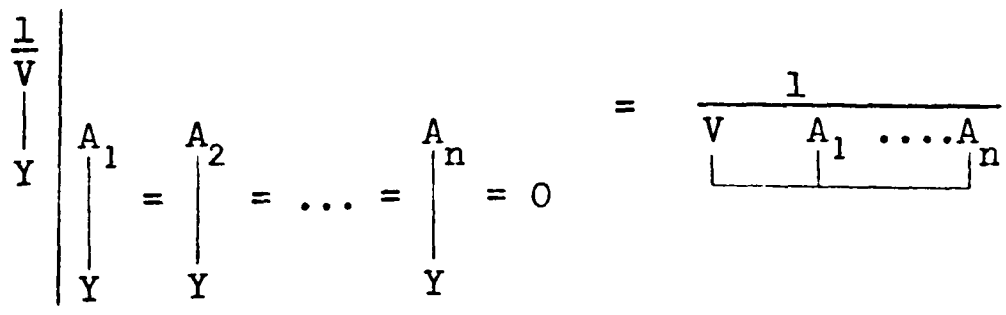


Figure 4.8.3

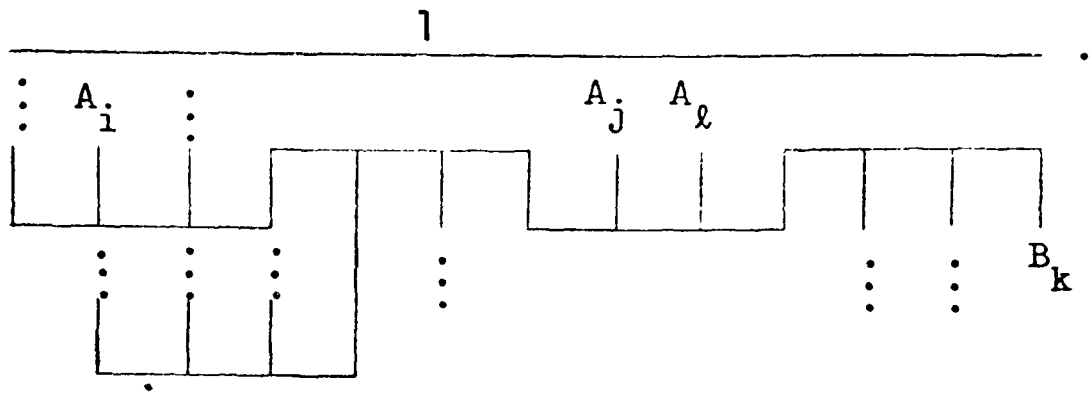
Then <u>part</u> of the representative cocycle for the element of H^{mn} will be



where $\overset{V}{|}$ may be a vertex or a parameter. Applying the cohomological residue map to each of the first n poles we obtain



so that we have integrated out the Y vertex completely, leaving a tree diagram with one less vertex. Continuing this process we obtain the usual answer, which for the diagram in figure 4.8.1 is



Now suppose n = 3 and $\overset{V}{|}$ is another vertex. Then figure 4.8.3 becomes figure 4.8.4.

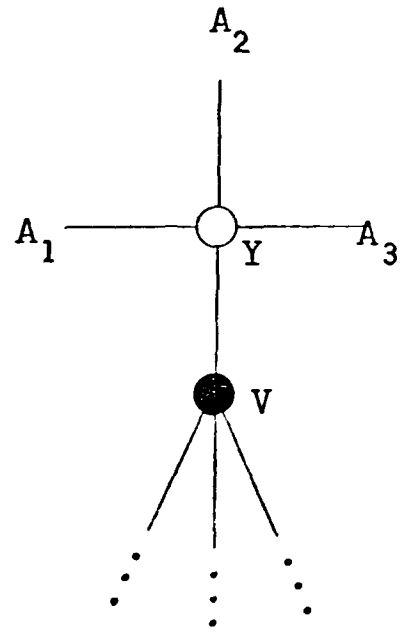


Figure 4.8.4

We shall rephrase our integration of the $\overset{|}{Y}$ vertex in such a way that it becomes a twistor transform (Penrose and MacCallum 1972) for tree diagrams. Think of the diagram in figure 4.8.4 as being constructed from the fragments in figure 4.8.5. The first two are elements of

$H^2(A_1^c \cup A_2^c \cup A_3^c;\ \mathcal{O}(-3))$ and $H^0((Y \times V - \Delta);\ \mathcal{O}(-1) \otimes \mathcal{O}(-1))$

respectively.

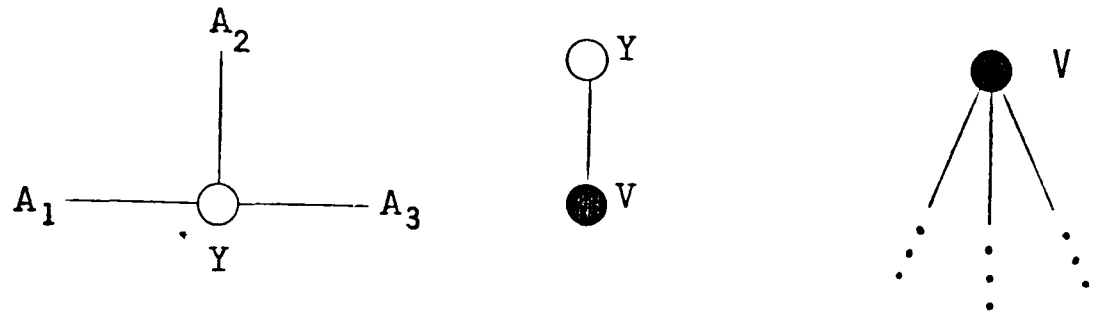


Figure 4.8.5

Extend the element of H^2 to an element of

$$H^2((A_1^c \cup A_2^c \cup A_3^c) \times V;\ \mathcal{O}(-3) \otimes \mathcal{O}(o))$$

by making it constant in V. Now take the dot product of this element and the element of H^0 to obtain an element of

$$H^3(\{(A_1^c \cup A_2^c \cup A_3^c) \times V\} \cup \{Y \times V - \Delta\};\ \mathcal{O}(-4) \otimes \mathcal{O}(-1)).$$

The space

$$Y \times S^c = Y \times \{V : \overbrace{V\ A_1\ A_2\ A_3} \neq 0\}$$

is a subset of

$$\{(A_1^c \cup A_2^c \cup A_3^c) \times V\} \cup \{Y \times V - \Delta\}.$$

Therefore we can restrict our element of H^3 so that it becomes an element of

$$H^3(Y \times S^c;\ \mathcal{O}(-4) \otimes \mathcal{O}(-1))$$

$$\cong H^3(Y;\ \mathcal{O}(-4)) \otimes H^0(S^c;\ \mathcal{O}(-1)) \qquad \text{by the Künneth formula}$$

$$\cong \mathbf{C} \otimes H^0(S^c;\ \mathcal{O}(-1)) \qquad \text{by Serre duality}$$

$$\cong H^0(S^c;\ \mathcal{O}(-1)) \qquad \text{because the tensor product is over } \mathbf{C}.$$

In terms of representative cocycles we have taken the dot product:

$$\left[\frac{1}{A_1\ A_2\ A_3}\right] \cdot \left[\frac{1}{V}\right] = \left[\frac{1}{A_1\ A_2\ A_3\ V}\right]$$

and then integrated:

$$\left[\frac{1}{A_1\ A_2\ A\ V}\right] \longrightarrow \left[\frac{1}{A_1\ A_2\ A_3\ V}\right]$$

We can think of this as a mapping (determined by $\frac{1}{V}$) from

$$H^2(A_1^c \cup A_2^c \cup A_3^c;\ \mathcal{O}(-3)) \quad \text{to} \quad H^0(S^c;\ \mathcal{O}(-1)) .$$

This is a twistor transform for tree diagrams. The cohomological description of the usual twistor transform is given in (Ginsberg 1980).

In general (n not necessarily 3) we have the mapping from

$$H^{n-1}(A_1^c \cup A_2^c \cup \ldots \cup A_n^c;\ \mathcal{O}(-n)) \quad \text{to} \quad H^0(S^c;\ \mathcal{O}(-1)).$$

Chapter 5 : Sheaf Cohomology of the Scalar Product

§5.1 Introduction

Not even the simplest physically important twistor diagram (the spin zero scalar product diagram in figure 5.1.1) is a tree diagram.

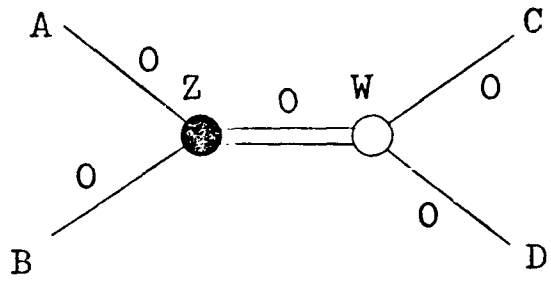


Figure 5.1.1

However, just as in the last chapter we can interpret each of the lines in this diagram as an element of some H^O group. In this way we obtain the elements

$$\begin{array}{c} 1 \\ \overline{A} \\ | \\ Z \end{array} \quad \in \quad H^O(A^C;\ \mathcal{O}(-1)),$$

$$\begin{array}{c} 1 \\ \overline{B} \\ | \\ Z \end{array} \quad \in \quad H^O(B^C;\ \mathcal{O}(-1)),$$

$$\begin{array}{c} 1 \\ \overline{W} \\ | \\ C \end{array} \quad \in \quad H^O(C^C;\ \mathcal{O}(-1)),$$

$$\frac{1}{\overset{W}{|}\atop D} \quad \epsilon \quad H^{o}(D^{c};\ \mathcal{O}(-1)), \quad \text{and the element}$$

$$\frac{1}{\overset{W}{|}\atop Z} \quad \epsilon \quad H^{o}(\ PT \times PT^{*} - \Delta,\ \ \mathcal{O}(-1) \otimes \mathcal{O}(-1)), \text{ where}$$

$$\Delta = \{(\underset{Z}{|},\ \overset{W}{|}) \ \epsilon \ \ PT \times PT^{*} : \underset{Z}{\overset{W}{|}} = 0\} \ ,$$

twice over - once for each of the two lines joining the Z and W vertices. The difficulty comes when we try to take the dot product of these last two lines. Their cup product is

$$\frac{1}{(\overset{W}{|}_{Z})^{2}} \quad \epsilon \quad H^{o}(\ PT \times PT^{*} - \Delta;\ \ \mathcal{O}(-2) \otimes \mathcal{O}(-2)) \ .$$

and so their dot product is

$$\delta^{*}\ \frac{1}{(\overset{W}{|}_{Z})^{2}}$$

which is zero because the Mayer-Vietoris connecting map

$$\delta^{*} : H^{p}(U \cap V;S) \to H^{p+1}(U \cup V;S)$$

is the zero map when $U = V$.

Consequently we cannot treat the scalar product diagram as we did the tree diagrams; something more is needed. This is not surprising when we recall that neither of the scalar product contours are products

of S's - they both have an S^2 contribution from the double line. Our description of tree diagrams only applied to contours which are products of S's.

However, it is very instructive to use the techniques of the last chapter as far as possible. For the moment we consider the double line as representing

$$\begin{array}{c} 1 \\ \hline W^2 \\ (|) \\ Z \end{array} \quad \in \quad H^0(\ PT \times PT^* - \Delta;\ \ \mathcal{O}(-2) \otimes \mathcal{O}(-2))$$

and we take the cup product of this and the four twigs

$$\begin{array}{c} 1/A \\ | \\ Z \end{array} \quad \in H^0(A^c \times PT^*;\ \mathcal{O}(-1) \otimes \mathcal{O}(0)),$$

$$\begin{array}{c} 1/B \\ | \\ Z \end{array} \quad \in H^0(B^c \times PT^*;\ \mathcal{O}(-1) \otimes \mathcal{O}(0)),$$

$$\begin{array}{c} 1/W \\ | \\ C \end{array} \quad \in H^0(\ PT \times C^c;\ \ \mathcal{O}(0) \otimes \mathcal{O}(-1)), \quad \text{and}$$

$$\begin{array}{c} 1/W \\ | \\ D \end{array} \quad \in H^0(\ PT \times D^c;\ \mathcal{O}(0) \otimes \mathcal{O}(-1))$$

to obtain the element

$$\frac{1}{\begin{matrix} A & B & W^{2} & W & W \\ | & | & (|) & | & | \\ Z & Z & Z & C & D \end{matrix}}$$

$$\in H^{0}(\{\ PT\times\ PT^{*} - \Delta\} \cap \{A^{c} \times\ PT^{*}\} \cap \{B^{c} \times PT^{*}\} \cap \{PT$$

$$\times C^{c}\} \cap \{\ PT\times D^{c}\};\ \mathcal{O}(-4) \otimes \mathcal{O}(-4)).$$

Now we use the Čech map

$$\check{C} : H^{0}(\{\ PT\times\ PT^{*} - \Delta\} \cap \{A^{c} \times\ PT^{*}\} \cap \{B^{c} \times\ PT^{*}\} \cap \{PT$$

$$\times C^{c}\} \cap \{\ PT\times D^{c}\};\ \mathcal{O}(-4) \otimes \mathcal{O}(-4))$$

$$\to H^{4}(\{\ PT\times\ PT^{*} - \Delta\} \cup \{A^{c} \times\ PT^{*}\} \cup \{B^{c} \times\ PT^{*}\} \cup\{\ PT\times C^{c}\} \cup\{\ PT$$

$$\times D^{c}\};\ \mathcal{O}(-4) \otimes \mathcal{O}(-4))$$

$$= H^{4}(\ PT\times\ PT^{*}$$

$$-\{(\begin{matrix}|\\ Z\end{matrix}, \begin{matrix}W\\ |\end{matrix}) \in\ PT \times\ PT^{*} : \begin{matrix}A\\ |\\ Z\end{matrix} = \begin{matrix}B\\ |\\ Z\end{matrix} = \begin{matrix}W\\ |\\ Z\end{matrix} = \begin{matrix}W\\ |\\ C\end{matrix} = \begin{matrix}W\\ |\\ D\end{matrix} = 0\};\ \mathcal{O}(-4) \otimes \mathcal{O}(-4))$$

which interprets

$$\frac{1}{\begin{matrix} A & B & W^{2} & W & W \\ | & | & (|) & | & | \\ Z & Z & Z & C & D \end{matrix}}$$

as a representative cocycle for an element of H^4.

Suppose $[\phi] \in H^1(U;\ \mathcal{O}(o) \otimes \mathcal{O}(o))$, where U is a neighbourhood of $\{(\begin{smallmatrix}|\\Z\end{smallmatrix}, \begin{smallmatrix}W\\|\end{smallmatrix}) \in \ PT\times \ PT^* : \begin{smallmatrix}A\\|\\Z\end{smallmatrix} = \begin{smallmatrix}B\\|\\Z\end{smallmatrix} = \begin{smallmatrix}W\\|\\Z\end{smallmatrix} = \begin{smallmatrix}W\\|\\C\end{smallmatrix} = \begin{smallmatrix}W\\|\\D\end{smallmatrix} = 0\}$. Then the dot product

$$[\phi] \quad \cdot \quad \left[\frac{1}{\begin{smallmatrix}A & B & (W)^2 & W & W\\ | & | & | & | & |\\ Z & Z & Z & C & D\end{smallmatrix}}\right]$$

is an element of $H^6(\ PT\times \ PT^*;\ \mathcal{O}(-4) \otimes \mathcal{O}(-4))$ which is isomorphic to **C** by Serre duality. Therefore if we can find an appropriate element $[\phi]$ we can integrate the scalar product diagram cohomologically by regarding the double line as representing the dot product between the elements

$$\frac{1}{(\begin{smallmatrix}W\\|\\Z\end{smallmatrix})^2} \in H^0$$

and $$[\phi] \in H^1 .$$

According to this scheme, then, which is due to Matt Ginsberg, a double line represents an element of some H^2 group.

In the next section we identify this element and show that the contour integral of

$$\frac{\phi}{\begin{smallmatrix}A & B & (W)^2 & W & W\\ | & | & | & | & |\\ Z & Z & Z & C & D\end{smallmatrix}}$$

over a product of S's is equivalent to the integral of the differential form represented in figure 5.1.1 over the tight contour for the scalar product. Section 5.3 shows how to describe this tight contour evaluation cohomologically, and in the last section we discuss some possible future developments in this work.

§5.2 Integration of the log representative cocycle

We shall show (i) that we can evaluate the scalar product diagram by choosing

$$\phi = \log \left(\begin{array}{c} P \\ | \\ Z \end{array} \begin{array}{c} W \\ | \\ Q^* \end{array} \Big/ \begin{array}{c} Q \\ | \\ Z \end{array} \begin{array}{c} W \\ | \\ P^* \end{array} \right) \qquad 5.2.1$$

and (ii) that this evaluation is the same as integrating 5.1.1 over the tight contour. (We choose P and Q so that $\underbrace{A \quad B \quad P \quad Q}_{} \neq 0$. Then we define $\Big|_{P}^{*} = \underbrace{A \quad B \quad Q}$ and $\Big|_{Q}^{*} = \underbrace{A \quad B \quad P}$.) The representative cocycle in 5.2.1 was first introduced in (Ginsberg 1980) and it is also discussed in (Eastwood and Ginsberg 1980). To prove that

$$[\phi] \in H^1(\Omega^-; \mathcal{O}) \qquad 5.2.2$$

where

$$\Omega^- = \{ PT^- \times PT^{*-} \} \cap \{ \left(\begin{array}{c} | \\ Z \end{array}, \begin{array}{c} W \\ | \end{array} \right) \in PT \times PT^* : \begin{array}{c} W \\ | \\ Z \end{array} = 0 \}$$

we use the exact sequence of sheaves in 4.2.1 to generate the following exact sequence in cohomology:

$$H^1(\Omega^-; Z\ ) \to H^1(\Omega^-; \mathit{0}) \to H^1(\Omega^-; \mathit{0}^*) \to H^2(\Omega^-; Z\ ) \qquad 5.2.3.$$

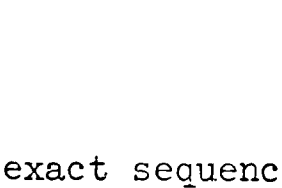
It is easy to show that

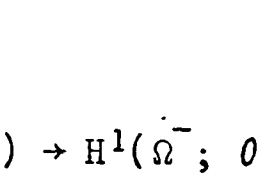
$$H^1(\Omega^-; Z\ ) = 0 \quad \text{and} \quad H^2(\Omega^-; Z\ ) = Z$$

so that 5.2.3 becomes

$$0 \to H^1(\Omega^-; \mathit{0}) \xrightarrow{\exp^*} H^1(\Omega^-; \mathit{0}^*) \xrightarrow{\delta^*} Z \qquad 5.2.4.$$

Now we choose the representative cocycles

$\psi_W =$ 

$\psi_Z =$ 

for elements in $H^1(\Omega^-; \mathit{0}^*)$.

$$\delta^*[\psi_W] = \delta^*[\psi_Z]$$

$$\therefore \quad \delta^*[\psi_Z/\psi_W] = 0$$

$$\therefore\ [\psi_Z/\psi_W] = \exp^*[\phi]$$

for some $[\phi] \in H^1(\Omega^-;\ 0)$, as required.

Next we have to check that the sets Ω^- and

$$\mathbf{PT}\times\ \mathbf{PT}^* - \{(\begin{smallmatrix}|\\Z\end{smallmatrix},\ \begin{smallmatrix}W\\|\end{smallmatrix}) \in\ \mathbf{PT}\times\ \mathbf{PT}^*\ :\ \begin{smallmatrix}A\\|\\Z\end{smallmatrix} = \begin{smallmatrix}B\\|\\Z\end{smallmatrix} = \begin{smallmatrix}W\\|\\Z\end{smallmatrix} = \begin{smallmatrix}W\\|\\C\end{smallmatrix} = \begin{smallmatrix}W\\|\\D\end{smallmatrix} = 0\}$$

cover $\mathbf{PT}\times\ \mathbf{PT}^*$.

This will only be true if the elementary states

$$\begin{smallmatrix}1\\\hline A\ B\\|\ \ |\\Z\ Z\end{smallmatrix} \qquad \text{and} \qquad \begin{smallmatrix}1\\\hline W\ W\\|\ \ |\\C\ D\end{smallmatrix}$$

have positive frequency. Therefore we must choose the lines $\begin{smallmatrix}A\ B\\|\ \ |\\+\!\!+\end{smallmatrix}$ and $\begin{smallmatrix}+\!\!+\\C\ D\end{smallmatrix}$ to be in $\mathbf{PT}^-$ and $\mathbf{PT}^{*-}$ respectively. Then Ω^- is a neighbourhood of

$$\{(\begin{smallmatrix}|\\Z\end{smallmatrix},\ \begin{smallmatrix}W\\|\end{smallmatrix}) \in\ \mathbf{PT}\times\ \mathbf{PT}^*\ :\ \begin{smallmatrix}A\\|\\Z\end{smallmatrix} = \begin{smallmatrix}B\\|\\Z\end{smallmatrix} = \begin{smallmatrix}W\\|\\Z\end{smallmatrix} = \begin{smallmatrix}W\\|\\C\end{smallmatrix} = \begin{smallmatrix}W\\|\\D\end{smallmatrix} = 0\}$$

and we have established part (i) of this section.

For part (ii) consider the representative cocycle

$$f = \frac{\log(\begin{smallmatrix}P & W\\|\ & |^*\\Z & Q\end{smallmatrix} \Big/ \begin{smallmatrix}Q & W\\|\ & |^*\\Z & P\end{smallmatrix})}{\begin{smallmatrix}A\ B\ \ W\\|\ \ |\ \ (|)\\Z\ Z\ \ Z\end{smallmatrix}^2 \begin{smallmatrix}W\ W\\|\ \ |\\C\ D\end{smallmatrix}}$$

for the element

$$[\phi]\ .\ \left[\frac{1}{A\ B\ W^{2}\ W\ W \;\;\; Z\ Z\ Z\ C\ D}\right] \in H^{6}(\ PT\times\ PT^{*};\ \mathcal{O}(-4)\otimes\mathcal{O}(-4)).$$

The corresponding integrand is f $\mathcal{D}Z \wedge \mathcal{D}W$, which we write in the form

$$\log\left(\frac{P\ W\ /\ Q\ W}{Z\ Q\ /\ Z\ P}\right)\ \frac{A\ B\ P\ Q}{Z\ dZ\ dZ\ dZ}\ \wedge\ \frac{W\ dW\ dW\ dW}{C\ D\ P\ Q}\ \Big/\ \frac{A\ B\ W^{2}\ W\ W}{Z\ Z\ Z\ C\ D}\ \frac{A\ B\ P\ Q}{C\ D\ P\ Q}$$

$$= \log\left(\frac{P\ W\ /\ Q\ W}{Z\ Q\ /\ Z\ P}\right)\ \frac{A\ B\ P\ Q}{Z\ dZ\ dZ\ dZ}\ \wedge\ \frac{W\ dW\ dW\ dW}{C\ D\ P\ Q}\ \Big/\ \frac{A\ B\ W^{2}\ W\ W}{Z\ Z\ Z\ C\ D}\ \left(\frac{A\ B\ P\ Q}{}\right)^{2}\ \frac{A\ B}{C\ D}$$

so that we can easily do the four simple pole integrals, obtaining

$$(2\pi i)^{4}\ \log\left(\frac{P\ W\ /\ Q\ W}{Z\ Q\ /\ Z\ P}\right)\ \frac{P\ Q}{Z\ dZ}\ \wedge\ \frac{W\ dW}{P\ Q}\ \Big/\ \left(\frac{A\ B\ P\ Q}{}\right)^{2}\ \frac{W^{2}}{Z}\ \frac{A\ B}{C\ D}\ \Bigg|_{A=B=W=W=0}^{Z\ Z\ C\ D} \qquad 5.2.3.$$

Before we integrate around the double pole at

$$\begin{matrix} W \\ | \\ Z \end{matrix} = 0$$

we write this integrand in different coordinates. The space

$$\{(\begin{matrix} | \\ Z \end{matrix}, \begin{matrix} W \\ | \end{matrix}) \in \mathbf{PT} \times \mathbf{PT}^* : \begin{matrix} A \\ | \\ Z \end{matrix} = \begin{matrix} B \\ | \\ Z \end{matrix} = \begin{matrix} W \\ | \\ C \end{matrix} = \begin{matrix} W \\ | \\ D \end{matrix} = 0\}$$

is $\mathbf{CP}^1 \times \mathbf{CP}_1$ with homogeneous coordinates

$$\cdot \; (\begin{matrix} P \\ | \\ Z \end{matrix} : \begin{matrix} Q \\ | \\ Z \end{matrix}, \begin{matrix} W \\ |_* \\ P \end{matrix} : \begin{matrix} W \\ |_* \\ Q \end{matrix}) \; .$$

We can use the basis $\begin{matrix} A \\ | \end{matrix}, \begin{matrix} B \\ | \end{matrix}, \begin{matrix} P \\ | \end{matrix}, \begin{matrix} Q \\ | \end{matrix}$ to expand the denominator of 5.2.3:

$$\begin{matrix} A & B & P & Q \\ \llcorner & | & | & \lrcorner \end{matrix} \quad \begin{matrix} W \\ | \\ Z \end{matrix} = \begin{matrix} W \\ |_* \\ A \end{matrix} \begin{matrix} A \\ | \\ Z \end{matrix} + \begin{matrix} W \\ |_* \\ B \end{matrix} \begin{matrix} B \\ | \\ Z \end{matrix} + \begin{matrix} W \\ |_* \\ P \end{matrix} \begin{matrix} P \\ | \\ Z \end{matrix} + \begin{matrix} W \\ |_* \\ Q \end{matrix} \begin{matrix} Q \\ | \\ Z \end{matrix} \quad .$$

But

$$\begin{matrix} A \\ | \\ Z \end{matrix} = \begin{matrix} B \\ | \\ Z \end{matrix} = 0.$$

Therefore

$$\begin{matrix} W \\ | \\ Z \end{matrix} = \frac{\begin{matrix} W \\ |_* \\ P \end{matrix} \begin{matrix} P \\ | \\ Z \end{matrix} + \begin{matrix} W \\ |_* \\ Q \end{matrix} \begin{matrix} Q \\ | \\ Z \end{matrix}}{\begin{matrix} A & B & P & Q \\ \llcorner & | & | & \lrcorner \end{matrix}}$$

and the integrand becomes

$$\frac{(2\pi i)^4 \log\left(\begin{vmatrix} P & W \\ Z & Q \end{vmatrix}_* \Big/ \begin{vmatrix} Q & W \\ Z & P \end{vmatrix}_*\right) \begin{vmatrix} P & Q \\ Z & dZ \end{vmatrix} \wedge \begin{vmatrix} W & dW \\ P & Q \end{vmatrix}_*}{\left(\begin{vmatrix} W & P \\ P & Z \end{vmatrix}_* + \begin{vmatrix} W & Q \\ Q & Z \end{vmatrix}_*\right)^2 \begin{vmatrix} A & B \\ C & D \end{vmatrix}} \qquad 5.2.4.$$

Choose spinor variables

$$(Z^0 : Z^1) = \left(\begin{vmatrix} P \\ Z \end{vmatrix} : \begin{vmatrix} Q \\ Z \end{vmatrix}\right), \quad (W_0 : W_1) = \left(\begin{vmatrix} W \\ P \end{vmatrix}_* : \begin{vmatrix} W \\ Q \end{vmatrix}_*\right).$$

Then 5.2.4 becomes

$$\frac{(2\pi i)^4 \log\left(\dfrac{Z^0 W_1}{Z^1 W_0}\right) \Delta Z \wedge \Delta W}{(Z^A W_A)^2 \begin{vmatrix} A & B \\ C & D \end{vmatrix}} \qquad 5.2.5.$$

To integrate around the double pole we fix Z^A and choose new coordinates for W_A:

$$\lambda_0 = Z^A W_A , \qquad \lambda_1 = V^A W_A$$

where V^A is any spinor distinct from Z^A. Then

$$\Delta\lambda = Z^A V_A \, \Delta W$$

and

$$W_o = \frac{V^1\lambda_o - Z^1\lambda_1}{Z^A V_A} \quad , \qquad W_1 = \frac{Z^o\lambda_1 - V^o\lambda_o}{Z^A V_A} \quad .$$

Now 5.2.5 becomes

$$\frac{(2\pi i)^4 \log\{\dfrac{Z_o(V^o\lambda_o - Z^o\lambda_1)}{Z_1(Z^1\lambda_1 - V^1\lambda_o)}\}\ \Delta Z \wedge \Delta\lambda}{(\lambda_o)^2\ Z^A V_A \begin{smallmatrix} A & B \\ C & D \end{smallmatrix}} \qquad 5.2.6.$$

Integrating around the double pole we obtain

$$\frac{(2\pi i)^5}{\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}} \quad \frac{\dfrac{d}{d\lambda_o} \log\{\dfrac{Z_o(V^o\lambda_o - Z^o\lambda_1)}{Z_1(Z^1\lambda_1 - V^1\lambda_o)}\}\Big|_{(\lambda_o:\lambda_1)=(0:1)}}{Z^A V_A} \Delta Z$$

$$= \frac{(2\pi i)^5}{\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}} \quad \frac{\{\dfrac{V^o}{V^o\lambda_o - Z^o\lambda_1} + \dfrac{V^1}{Z^1\lambda_1 - V^1\lambda_o}\}\Big|_{(\lambda_o:\lambda_1)=(0:1)}}{Z^A V_A} \Delta Z$$

$$= \frac{(2\pi i)^5}{\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}} \quad \frac{\{\dfrac{V^1}{Z^1} - \dfrac{V^o}{Z^o}\}\ \Delta Z}{Z^A V_A} \qquad 5.2.7$$

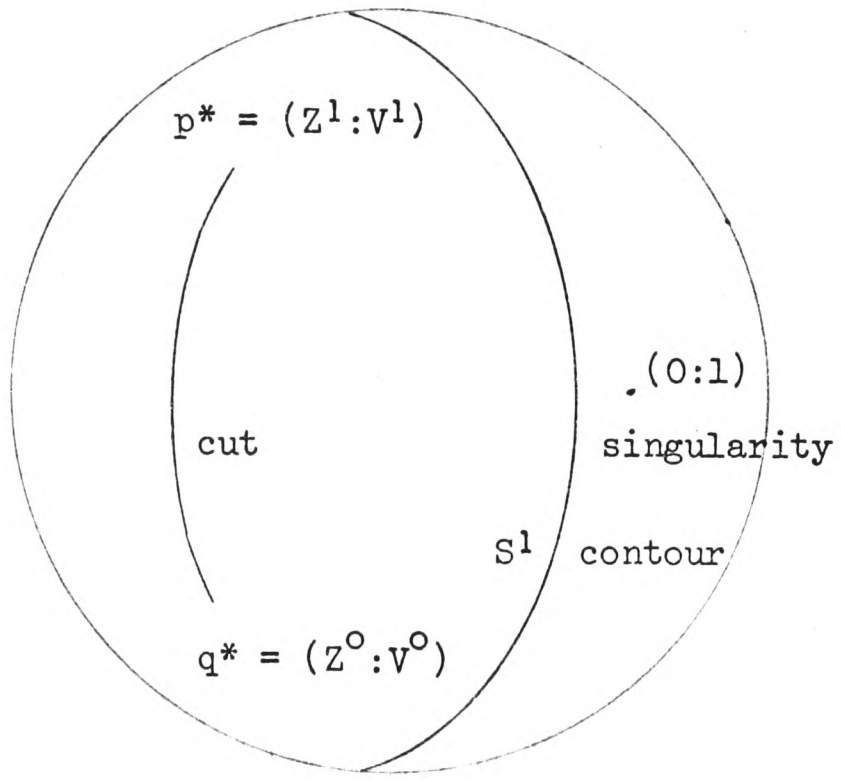
$$= \frac{(2\pi i)^5}{\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}} \quad \frac{\Delta Z}{Z^o Z^1}$$

and we could now integrate over an S^1 in Z^A to obtain the final result.

We have used an $S^1 \times S^1$ contour to integrate the variables W_A and Z^A. We now show that this contour is homologous to a cylindrical contour $[p^*, q^*] \times S^1$. For each fixed value of $(Z^o : Z^1)$ (not equal to $(0 : 1)$ or $(1 : 0)$) the function

$$\log\left\{\frac{Z_o(V^o\lambda_o - Z^o\lambda_1)}{Z_1(Z^1\lambda_1 - V^1\lambda_o)}\right\}$$

is defined everywhere on the $(\lambda_o : \lambda_1)$ CP^1 except along a cut from $(\lambda_o : \lambda_1) = (Z^o : V^o)$ to $(\lambda_o : \lambda_1) = (Z^1 : V^1)$. The S^1 contour used to integrate the λ_A variable avoided both this cut and the singularity at $\lambda_o = 0$. We can deform this S^1, as shown in figure 5.2.8, and then break it up into four pieces.


p* = (Z¹:V¹)
cut
S¹
q* = (Z⁰:V⁰)
(0:1)
singularity
contour

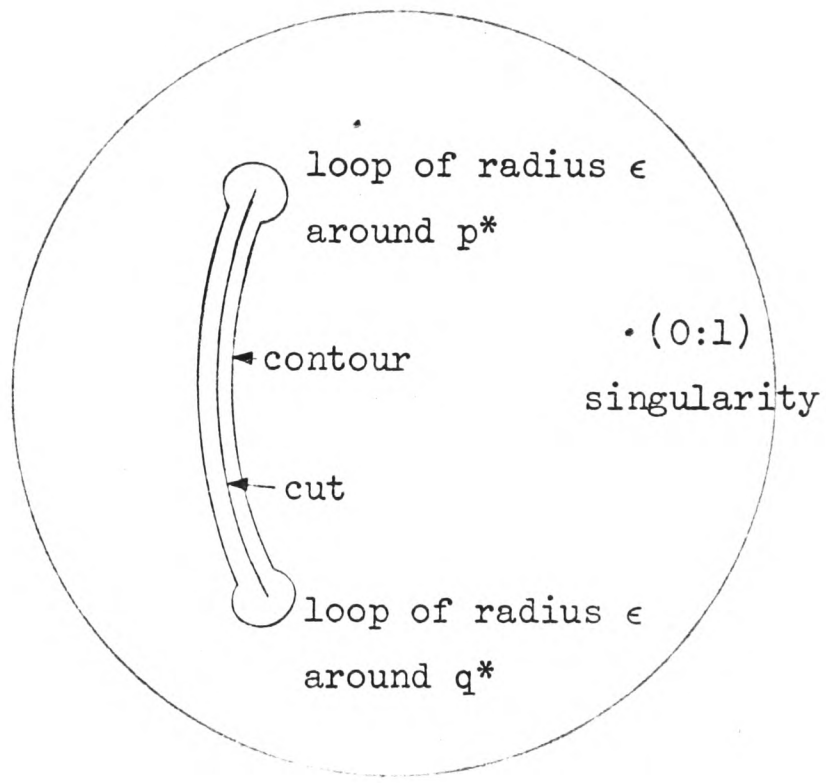

loop of radius ϵ
around p*
contour
cut
loop of radius ϵ
around q*
(0:1)
singularity

Figure 5.2.8

In the limit as the radius ϵ of the two small loops tends to zero the integrals over these loops contribute nothing, and we are left with

$$\int_{(Z^0:V^0)}^{(Z^1:V^1)} \frac{\log\{\dfrac{Z^0(V^0\lambda_0 - Z^0\lambda_1)}{Z^1(Z^1\lambda_1 - V^1\lambda_0)}\}\Delta\lambda}{(\lambda_0)^2} + \int_{(Z^1:V^1)}^{(Z^0:V^0)} \frac{[\log\{\dfrac{Z^0(V^0\lambda_0 - Z^0\lambda_1)}{Z^1(Z^1\lambda_1 - V^1\lambda_0)}\} + 2\pi i]\,\Delta\lambda}{(\lambda_0)^2}$$

$$= 2\pi i \int_{(Z^1:V^1)}^{(Z^0:V^0)} \frac{\Delta\lambda}{(\lambda_0)^2} = 2\pi i \frac{Z^A V_A}{Z^0 Z^1} \qquad \text{as in 5.2.7.}$$

We have converted an S^1 contour integral involving a logarithm to a boundary contour integral omitting the logarithm. The integration of the function

$$\frac{1}{A\ B\ (W)^2\ W\ W \;|\;Z\ Z\ Z\ C\ D}$$

over four S's (one for each of the simple poles) followed by the cylinder $[p^*,q^*] \times S^1$ is exactly the evaluation of the diagram in figure 5.2.9, which is the same as

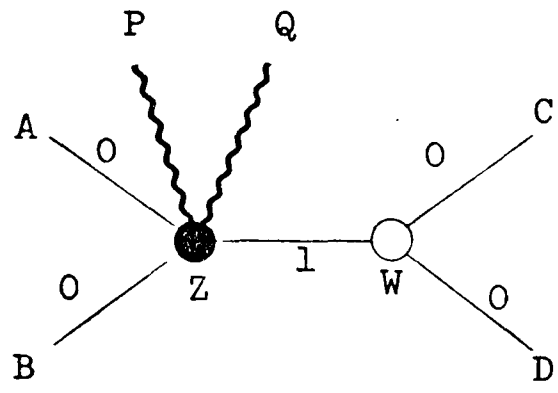


Figure 5.2.9

the integration of the spin zero scalar product diagram in figure 5.1.1 over the tight contour (Penrose 1979c). This completes part (ii).

§5.3 Cohomology of the tight contour

Now we return to the function

$$\frac{1}{\begin{matrix}A & B & W & W & W\\ | & | & (|) & | & |\\ Z & Z & Z & C & D\end{matrix}^{2}} \qquad 5.3.1$$

which, as we saw in section 5.1, we can regard as a representative cocycle for an element of

$$H^4\Big(\ PT\times\ PT^{*} - \{(\begin{matrix}|\\ Z\end{matrix}, \begin{matrix}W\\ |\end{matrix}) \in \ PT\times\ PT^{*} : \begin{matrix}A\\ |\\ Z\end{matrix} = \begin{matrix}B\\ |\\ Z\end{matrix} = \begin{matrix}W\\ |\\ Z\end{matrix} = \begin{matrix}W\\ |\\ C\end{matrix} = \begin{matrix}W\\ |\\ D\end{matrix} = 0\};\ \mathcal{O}(-4)\otimes\mathcal{O}(-4)\Big)$$

We can apply four residue maps to this element, one for each of the four simple poles in 5.3.1. In so doing we obtain the element

$$\mathrm{res}^4\left[\frac{1}{\begin{array}{cccccc} A & B & W & ^2 & W & W \\ | & | & (\,|\,) & & | & | \\ Z & Z & Z & & C & D \end{array}}\right] = \left[\frac{1}{\begin{array}{ccc} A & B & W \\ | & | & (\,|\,) \\ C & D & Z \end{array}^2} \;\middle|\; \begin{array}{cccc} A & B & W & W \\ | & | & | & | \\ Z & Z & C & D \end{array} = 0\right]$$

$$\in H^0(\{(\begin{smallmatrix} | \\ Z \end{smallmatrix}, \begin{smallmatrix} W \\ | \end{smallmatrix}) \in \mathbf{PT} \times \mathbf{PT}^* : \begin{smallmatrix} A \\ | \\ Z \end{smallmatrix} = \begin{smallmatrix} B \\ | \\ Z \end{smallmatrix} = \begin{smallmatrix} W \\ | \\ C \end{smallmatrix} = \begin{smallmatrix} W \\ | \\ D \end{smallmatrix} = 0, \; \begin{smallmatrix} W \\ | \\ Z \end{smallmatrix} \neq 0\}; \; \mathcal{O}(-2) \otimes \mathcal{O}(-2))$$

$$\tilde{=} H^0(\{(Z^A, W_A)^{\cdot} \in \mathbf{CP}^1 \times \mathbf{CP}_1 : Z^A W_A \neq 0\} ; \; \Omega^1 \otimes \Omega^1).$$

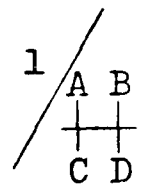
So far this is identical to the evaluation in section 5.2, except that we have omitted the logarithm term. The last part of the tight contour is an S^2, chosen by restricting the differential form

$$\frac{\Delta Z \wedge \Delta W}{\begin{array}{ccc} A & B & W \\ | & | & (\,|\,) \\ C & D & Z \end{array}^2} \in H^0(\{(\begin{smallmatrix} | \\ Z \end{smallmatrix}, \begin{smallmatrix} W \\ | \end{smallmatrix}) \in \mathbf{CP}^1 \times \mathbf{CP}_1 : \begin{smallmatrix} W \\ | \\ Z \end{smallmatrix} \neq 0\}; \; \Omega^1 \otimes \Omega^1)$$

to the space $W_o = \bar{Z}^o$, $W_1 = \bar{Z}^1$. We get the differential form

$$\frac{\Delta Z \wedge \Delta \bar{Z}}{\begin{array}{cc} A & B \\ | & | \\ C & D \end{array} (Z^o \bar{Z}^o + Z^1 \bar{Z}^1)^2} \in H^0(\mathbf{CP}^1; A^{1,1}) \qquad 5.3.2$$

which we integrate all over the $\mathbf{CP}^1$ to obtain



(up to a multiplicative constant):

$$H^0(\,CP^1;\, A^{1,1}) \to H^1(\,CP^1;\, \Omega^1) \cong C\ .$$

We now have two cohomological evaluations of the diagram in figure 5.1.1:

(i) the method just described, and

(ii) $\overset{\vee}{C}\left[\dfrac{\phi}{A\ B\ W^2\ W\ W \;\;(\,|\,)\;\; Z\ Z\ Z\ C\ D}\right] \in C$ (described in section 5.1).

Section 5.2 establishes explicitly that these two methods are equivalent, but we have yet to find an elegant <u>cohomological</u> proof of their equivalence.

The actual integration of

$$\overset{\vee}{C}\left[\dfrac{\phi}{A\ B\ W^2\ W\ W \;\;(\,|\,)\;\; Z\ Z\ Z\ C\ D}\right]$$

involved four residue maps. Suppose we use these maps <u>before</u> taking the dot product

$$[\phi]\ .\left[\dfrac{1}{A\ B\ W^2\ W\ W \;\;(\,|\,)\;\; Z\ Z\ Z\ C\ D}\right]\ .$$

Then methods (i) and (ii) can be compared as follows:

$$\left[\ \frac{1}{\begin{matrix} A & B & W^{2} & W & W \\ | & | & (\ |\ ) & | & | \\ Z & Z & Z & C & D \end{matrix}}\ \right] \quad \epsilon$$

$$H^4(\ PT\times\ PT^* - \{(\begin{matrix}|\\Z\end{matrix},\begin{matrix}W\\|\end{matrix}) \in\ PT\times\ PT^*: \begin{matrix}A\\|\\Z\end{matrix} = \begin{matrix}B\\|\\Z\end{matrix} = \begin{matrix}W\\|\\Z\end{matrix} = \begin{matrix}W\\|\\C\end{matrix} = \begin{matrix}W\\|\\D\end{matrix} = 0\};\mathcal{O}(-4)\otimes\mathcal{O}(-4))$$

$$\downarrow \text{res}^4$$

$$H^0(\{(\begin{matrix}|\\Z\end{matrix},\begin{matrix}W\\|\end{matrix}) \in\ PT\times\ PT^*: \begin{matrix}A\\|\\Z\end{matrix} = \begin{matrix}B\\|\\Z\end{matrix} = \begin{matrix}W\\|\\C\end{matrix} = \begin{matrix}W\\|\\D\end{matrix} = 0,\ \begin{matrix}W\\|\\Z\end{matrix} \neq 0\};\ \mathcal{O}(-2)\otimes\mathcal{O}(-2))$$

$$\left[\phi\ \middle|\ \begin{matrix}A\\|\\Z\end{matrix} = \begin{matrix}B\\|\\Z\end{matrix} = \begin{matrix}W\\|\\C\end{matrix} = \begin{matrix}W\\|\\D\end{matrix} = 0\right]$$

$$H^2(\{(\begin{matrix}|\\Z\end{matrix},\begin{matrix}W\\|\end{matrix}) \in\ PT\times\ PT^*: \begin{matrix}A\\|\\Z\end{matrix} = \begin{matrix}B\\|\\Z\end{matrix} = \begin{matrix}W\\|\\C\end{matrix} = \begin{matrix}W\\|\\D\end{matrix} = 0\};\ \mathcal{O}(-2)\otimes\mathcal{O}(-2)) \quad \cong$$

$$\cong H^2(\ CP^1\times\ CP_1\ ;\ \mathcal{O}(-2)\otimes\mathcal{O}(-2))$$

$$\cong C$$

$$H^0(\{(Z^A,W_A) \in\ CP^1\times\ CP_1 : Z^AW_A \neq 0\};\quad \Omega^1\otimes\Omega^1\ )$$

$$W = \bar{Z}$$

$$H^0(\ CP^1\ ;\ A^{1,1})$$

$$\cong C$$

This question is discussed further in the next section. Before leaving this section, however, we note that method (i) can also be used to evaluate the diagrams in figures 5.3.3 and 5.3.4. For the first of these we take the dot

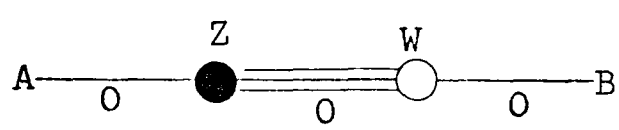


Figure 5.3.3

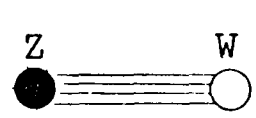


Figure 5.3.4

product of the elements

$$1/_{\substack{A\\|\\Z}} \in H^0(A^c \times PT^*;\ \mathcal{O}(-1) \otimes \mathcal{O}(o)),$$

$$1/_{\substack{W\\(|)\\Z}}{}^{3} \in H^0(\ PT\times\ PT^* - \Delta;\ \mathcal{O}(-3) \otimes \mathcal{O}(-3)),\ \text{and}$$

$$1/_{\substack{W\\|\\B}} \in H^0(\ PT\times B^c;\ \mathcal{O}(o) \otimes \mathcal{O}(-1))$$

to obtain an element

$$\frac{1}{\begin{matrix} A & W & 3 & W \\ | & (|) & & | \\ Z & Z & & B \end{matrix}} \quad \epsilon$$

$$H^2(\ PT\times\ PT^* - \{(\begin{matrix}|\\Z\end{matrix}, \begin{matrix}W\\|\end{matrix}) \in\ PT\times\ PT^* : \begin{matrix}A\\|\\Z\end{matrix} = \begin{matrix}W\\|\\Z\end{matrix} = \begin{matrix}W\\|\\B\end{matrix} = 0\};\ \mathcal{O}(-4) \otimes \mathcal{O}(-4)).$$

Next we use the residue map twice, once for each of the simple poles:

$$\text{res}^2 \left[\frac{1}{\begin{matrix} A & W & 3 & W \\ | & (|) & & | \\ Z & Z & & B \end{matrix}} \right] = \left[\frac{1}{\begin{matrix} A & W & 3 \\ | & (|) & \\ B & Z & \end{matrix}} \middle| \begin{matrix} A \\ | \\ Z \end{matrix} = \begin{matrix} W \\ | \\ B \end{matrix} = 0 \right]$$

$$\in H^0(\{(\begin{matrix}|\\Z\end{matrix}, \begin{matrix}W\\|\end{matrix}) \in\ PT\times\ PT^* : \begin{matrix}A\\|\\Z\end{matrix} = \begin{matrix}W\\|\\B\end{matrix} = 0,\ \begin{matrix}W\\|\\Z\end{matrix} \neq 0\};\ \mathcal{O}(-3) \otimes \mathcal{O}(-3))$$

$$\cong H^0(\{(\begin{matrix}|\\Z\end{matrix}, \begin{matrix}W\\|\end{matrix}) \in\ CP^2 \times CP_2 : \begin{matrix}W\\|\\Z\end{matrix} \neq 0\};\ \Omega^2 \otimes \Omega^2).$$

Finally we restrict to the CP^2 given by

$$W_o = \bar{Z}^o,\quad W_1 = \bar{Z}^1,\quad W_2 = \bar{Z}^2$$

and integrate the differential form all over this CP^2 :

$$\frac{\mathcal{D}Z \wedge \mathcal{D}W}{\begin{array}{cc} A & W \\ | & (|) \\ B & Z \end{array}^{3}} \to \frac{\mathcal{D}Z \wedge \mathcal{D}\bar{Z}}{\begin{array}{cc} A & \bar{Z} \\ | & (|) \\ B & Z \end{array}^{3}} \to \frac{1}{\begin{array}{c} A \\ | \\ B \end{array}} \qquad 5.3.5$$

(up to a multiplicative constant).

The evaluation of the diagram in figure 5.3.4 by this method is obvious. It is also worth noting that the differential forms

$$\frac{\Delta\bar{Z}}{(Z^0\bar{Z}^0 + Z^1\bar{Z}^1)^2} \qquad \text{and} \qquad \frac{\mathcal{D}\bar{Z}}{\begin{array}{c} \bar{Z} \\ (|) \\ Z \end{array}^{3}}$$

in 5.3.2 and 5.3.5 are the Bochner-Martinelli forms on $\mathbf{C}P^1$ and $\mathbf{C}P^2$ respectively (see section 4.6).

§5.4 Outlook

The comparison of methods (i) and (ii) in the previous section suggests that the calculation of a Dolbeault representative for the Čech cocycle

$$\phi = \log\left(\begin{array}{cc} P & W \\ | & | \\ Z & Q \end{array}^{*} \Big/ \begin{array}{cc} Q & W \\ | & | \\ Z & P \end{array}^{*}\right)$$

might lead to a cohomological proof of the equivalence of the two methods. In any event it would be interesting to know how to evaluate

the diagram in figure 5.3.3 using method (ii) - the fact that method (i) provides an evaluation procedure for this diagram implies the existence of some element

$$[\chi] \in H^3$$

such that

$$[\chi] \cdot \left[\frac{1}{A \quad W^3 \quad W \;\; (\,|\,) \;\; Z \quad Z \quad B} \right] \in H^6(\, PT \times PT^*;\, \mathcal{O}(-4) \otimes \mathcal{O}(-4)) \cong C$$

but it remains to construct χ.

Finally, we remark that methods (i) and (ii) correspond (except for the residue maps) to the two special cases (i) $p = 0$ and (ii) $p = q = n$, $V = M$ in the evaluation

$$H^p(M;\, d\Omega^{q-1}) \overset{V}{\to} C$$

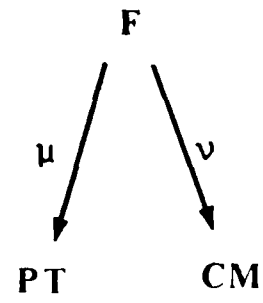
described in (Penrose 1980). Here M is an n complex dimensional complex manifold and V is a compact p + q real dimensional submanifold.

Conclusions

The calculations in chapter 2 provided a good example of the strengths and weaknesses of the use of homology theory to study the topology of twistor diagrams. Given a twistor diagram whose topological diagram is a tree we have a well defined procedure for finding out how many contours it has. If the topological diagram is not a tree we may not be able to break it up into fibre spaces. When that happens the homology of the complete diagram cannot be calculated using these techniques. This is true of the box diagram and the Möller scattering diagram and in both these cases certain parts of the topological diagrams have been studied using homology theory - see (Sparling 1975) for the box diagram and chapter 2 for Möller scattering. However, there is no guarantee that any of the contours whose existence is established in this way forms part of a physical contour for the corresponding twistor diagram. Indeed, even when we can calculate the homology of a complete topological diagram we still know very little about the various contours unless we actually construct them - which in some cases is quite difficult.

We conclude, therefore, that while homology theory is an important tool in the study of twistor diagrams we also need techniques for actually constructing contours. Some such techniques were described in chapter 3, where we showed - by constructing them - that there are in fact three contours for the box diagram, one for each channel. Previous work in twistor diagrams has assumed that different diagrams are needed for each channel in the ϕ^4 interaction. Consequently, the twistor

description of crossing symmetry was a little clumsy. The fact that all three channels are described by the box diagram therefore promises to simplify the theory considerably. Chapter 3 also illustrated the importance of the geometry of the translation procedure from a spacetime integral to a twistor diagram - at least in the early stages of the development of a twistor theory of interacting fields. It is in this translation that we saw the resolution process appearing naturally. In fact the translation procedures described in chapter 3 are similar to those used in (Eastwood, Penrose and Wells 1979) where, broadly speaking, the sheaf cohomology on $\mathbb{CM}$ is related to that on $\mathbb{PT}$ by means of the maps

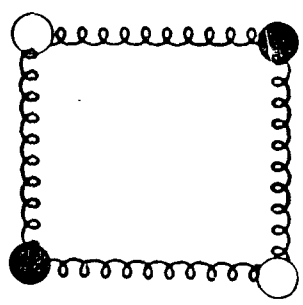


where F is the primed spin bundle on $\mathbb{CM}$ and

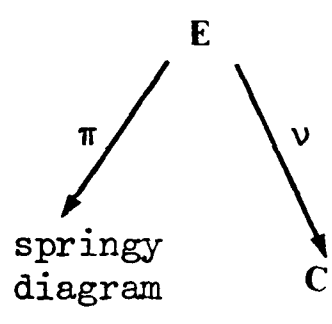
$$\nu(x^{AA'},\pi_{A'}) = x^{AA'}$$

$$\mu(x^{AA'},\pi_{A'}) = (ix^{AA'}\,\pi_{A'},\pi_{A'}).$$

In the case of the box diagram, for example, F is replaced by B, μ by π, and $\mathbb{PT}$ by

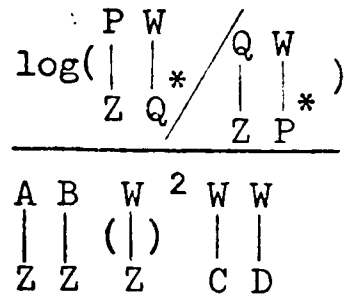


and in general F will become a bundle E of spin spaces, μ will be a blow-down or projection map π, and PT will be replaced by a topological diagram only containing springy lines. It would be instructive to study the Möller scattering diagram (among others) in this way, and it may also be interesting to study the sheaf cohomology of the maps

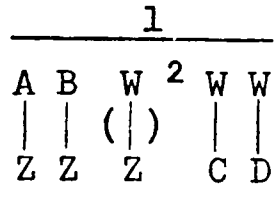


However, we have taken a slightly different approach to the cohomology of twistor diagrams. In chapter 4, where we discussed the cohomology of tree diagrams, the dot product and the residue map were introduced. These two important ideas - which were used in chapter 5 - allow us (i) to construct a diagram from its fragments and (ii) to integrate around simple poles. The dot product is put into a wider context in (Penrose 1980).

The last chapter analysed the cohomology of the spin zero scalar product. We integrated the function



and showed that this integral is the same as that of the function



over the tight scalar product contour. An alternative cohomological evaluation of this diagram (also corresponding to the tight contour) was exhibited, and it was left as an open problem to find a cohomological proof that these two methods of evaluation are equivalent. This alternative method of evaluation makes use of the restriction from

$$\left(\begin{smallmatrix}|\\Z\end{smallmatrix}, \begin{smallmatrix}W\\|\end{smallmatrix}\right) \in \mathbb{PT} \times \mathbb{PT}^* \text{ to } \left\{\left(\begin{smallmatrix}|\\Z\end{smallmatrix}, \begin{smallmatrix}W\\|\end{smallmatrix}\right) \in \mathbb{PT} \times \mathbb{PT}^* : W = \bar{Z}\right\} .$$

It is possible that the use of Dolbeault representatives for the various cohomology elements will eventually lead to a twistor proof of the positive definiteness of the scalar product. In any event the $W = \bar{Z}$ restriction deserves further investigation because it also appears in the "growing twistors" problem (Penrose 1979a) in which a relationship is sought between the two-twistor and one-twistor

descriptions of a given z.r.m. field.

Other outstanding problems in the sheaf cohomology of twistor diagrams include (i) the cohomological treatment of monodromy (Sparling, 1974) - where the sequence

$$0 \rightarrow Z \rightarrow \mathcal{O} \rightarrow \mathcal{O}^* \rightarrow 0$$

may be of use - and (ii) the cohomology of internal boundary lines - such as the lines labelled -1 in the massless Möller scattering diagram. In fact it is clear that the work in chapters 4 and 5 is really only the beginning of the subject.

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