Perimeter minimizing sets in RCD spaces

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Abstract

This thesis is about recent developments in the study of perimeter minimizing sets in RCD spaces, based on the works [94, 70] and an unpublished note.

After the introduction presented in *Chapter 1*, where we motivate the study of perimeter minimizing sets in RCD spaces and give an overview of our contributions, in *Chapter 2* we collect the background material relevant to our purposes.

In *Chapter 3* we present a monotonicity formula for perimeter minimizers in $\mathrm{RCD}(0, N)$ metric measure cones and derive a few applications, most notably a stratification result for the singular set of perimeter minimizing sets in non-collapsed $\mathrm{RCD}(K, N)$ spaces. We also show that if $(X, \mathsf{d}, \mathcal{H}^N)$ is a non-collapsed $\mathrm{RCD}(0, N)$ space with Euclidean volume growth containing an entire perimeter minimizer, then every blow-down of X contains a globally perimeter minimizing cone. This last result is new even in smooth Riemannian geometry. This chapter is based on a joint work with Mondino and Semola [94].

Chapter 4 is based on a joint work with Cucinotta [70]. We prove a regularity result for perimeter minimizing sets in spaces arising as pointed Gromov-Hausdorff limits of Riemannian manifolds with uniform two-sided bounds on the Ricci curvature. More specifically, we show that the Hausdorff dimension of the singular set of perimeter minimizers is at most $n - 5$, where n is the dimension of the ambient space. We also provide an example showing that our estimate is sharp.

Chapter 5 presents some results regarding Modica-Mortola-type approximations of the isoperimetric problem in non-collapsed RCD metric measure spaces. Specifically, we show how the perimeter functional can be approximated by suitable Cahn-Hilliard functionals [51, 141, 140]. We also explore the case where the approximating functionals are defined on a sequence of pointed Riemannian manifolds converging in the pointed

Gromov-Hausdorff sense to a limit space (X, d, p), on which the perimeter functional is defined.

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Chapter 1

Introduction

This thesis covers topics related to the structure of perimeter minimizers in RCD spaces. In the following chapters, we present results obtained in collaboration with Alessandro Cucinotta, Daniele Semola and Andrea Mondino in [70, 94] and in an unpublished work.

The aim of this chapter is to provide some motivations for the results presented in this thesis by offering a concise overview of the developments that underpin the results presented in the other chapters. It draws inspiration from [171], as well as [7, 59, 179, 109, 185].

1.1 An overview of Ricci curvature

Curvature is a fundamental concept in non-Euclidean geometry, describing how a space deviates from the Euclidean model. This deviation is captured by the full curvature tensor, a complex four-tensor whose complete understanding remains challenging.

A useful way of interpreting the curvature tensor is through the *sectional curvature*. Given a Riemannian manifold (M, g), the sectional curvature is a smooth real-valued function K on the Grassmannian of two-dimensional planes over the tangent space. A heuristic on the sectional curvature is the following: given two orthonormal tangent vectors $u, v \in T_pM$ at a point $p \in M$, the sectional curvature $K(u, v)$ (that is, applied to the plane spanned by u and v) provides the dominant term in the deviation from the Euclidean case of the distance between geodesics starting from p:

$$\mathsf{d}_g(\exp_p(tu), \exp_p(tv)) = \sqrt{2}t\left(1 - \frac{K(u,v)}{12}t^2 + O(t^3)\right)$$

as $t \to 0$. Therefore, positive sectional curvature corresponds to a contraction of distances. This can also be seen in Topogonov's theorem, which, briefly put, states that on a complete Riemannian manifold (M, g) with non-negative sectional curvature geodesic triangles are “fatter” than their Euclidean counterparts. More precisely, given three distinct points $p, q, r \in M$, let c_0, c_1, c_2 denote the minimal geodesics connecting p to q, p to r and q to r, respectively, all parametrized by arc length. Notice that $|c_0| \leq |c_1| + |c_2|$, where $|c_i|$ denotes the length of c_i for $i = 0, 1, 2$. Let $\alpha_i \in [0, \pi]$ for $i = 0, 1$ denote the angle at p and q formed by the geodesics, respectively. Then there exist $\tilde{p}, \tilde{q}, \tilde{r} \in \mathbb{R}^N$ with corresponding minimal geodesics $\tilde{c}_0, \tilde{c}_1, \tilde{c}_2$ such that $|c_i| = |\tilde{c}_i|$ for $i = 0, 1, 2$ and the corresponding angles $\tilde{\alpha}_i$ satisfy $\tilde{\alpha}_i \leq \alpha_i$, $i = 0, 1$. Moreover, $\mathsf{d}_{\text{Eucl}}(\tilde{r}, \tilde{c}_0(t)) \leq \mathsf{d}_g(r, c_0(t))$ for any $t \in [0, |c_0|]$. Conversely, if all geodesic triangles in (M, g) are “fatter” than the Euclidean ones, then (M, g) has non-negative sectional curvature. Therefore, Toponogov’s theorem fully characterizes non-negative sectional curvature just in metric terms.

Another important geometric object that quantifies the distortion from Euclidean spaces is the *Ricci curvature*. The Ricci curvature is a smooth tensor which is a symmetric bilinear form on the tangent space T_pM at each $p \in M$. It is an average of the sectional curvature in the following sense: given $u, e_2, ..., e_N \in T_pM$ such that $\{u, e_2, ..., e_N\}$ forms an orthonormal basis of T_pM, then

$$Ric(u, u) := \sum_{i=2}^{N} K(u, e_i),$$

where $N \in \mathbb{N}$ is the dimension of the manifold. If $u = 0$, set $Ric(u, u) = 0$.

Analogously to how the sectional curvature provides a way to quantify the distortion of distances from Euclidean geometry, the Ricci curvature determines the deviation of volumes. Namely, given a set of normal coordinates centered at $p \in M$, the volume element has the following expansion

$$d\mathrm{vol}_g = \left(1 - \frac{1}{6}\sum_{i,j=1}^{N} Ric_{ij} x_i x_j + O(|x|^3)\right),$$

where Ric_{ij} denotes the i, j-th coefficient of the Ricci curvature in local coordinates and x_i denotes the i-th coordinate of the point we are evaluating the volume form at in local coordinates. Consequently, if $Ric(u, u) > 0$, the area swept by a tightly focused family of geodesics emanating from p with direction close to u will have locally smaller volume than its Euclidean counterpart.

Another classical way of thinking of the Ricci curvature is as a negative Laplacian of the metric g. Indeed, given a suitable set of harmonic coordinates x_i about p, one has

$$Ric_{ij} = -\frac{1}{2}\Delta g_{ij} + \text{ lower order terms.}$$

This is often cited as a heuristic for the regularization properties of the Ricci flow

$$\frac{d}{dt} g_t = -2Ric_{g_t},$$

given the analogy with the heat equation.

We present two further formulas where the Ricci curvature codifies the distortion from the flat Euclidean behavior, which we borrow from [184, chapter 14].

Given a smooth function $\psi : M \to \mathbb{R}$, consider the family of deformations given by

$$T_t(x) := \exp(t\nabla\psi).$$

Let $\mathcal{J}(t) := \det(D_x T_t(x))$. We can interpret this quantity as the infinitesimal volume distortion of the map T_t. Up to regularity issues given by the presence of cut loci, one can infer that

$$\frac{d^2}{dt^2}\left((\mathcal{J}(t))^{1/n}\right) + \frac{Ric(\dot{\gamma},\dot{\gamma})}{n}(\mathcal{J}(t))^{1/n} \leq 0,$$

where $\dot{\gamma} := \frac{d}{dt}T_t(x)$. Moreover, the following inequality, known as the Bochner inequality, can be obtained

$$\Delta\frac{|\nabla\psi|^2}{2} - \nabla\psi\cdot\nabla\Delta\psi \geq \frac{(\Delta\psi)^2}{n} + Ric(\nabla\psi,\nabla\psi).$$

Let us point out that, contrary to results involving lower sectional curvature bounds, results regarding the lower Ricci curvature bounds are often coupled with upper bounds on the dimension of the manifold. We say that the Ricci curvature is bounded from below by $k \in \mathbb{R}$ if $Ric \geq kg$. The better known consequences of such bounds are Myers theorem [151], stating that a manifold with Ricci curvature positively bounded from below has finite diameter; the Bishop-Gromov inequality [110] on monotonicity of volume ratios; the splitting theorem due to Cheeger-Gromoll [64]; the heat kernel bounds obtained by Li-Yau [129]; several spectral gap results and the Lévy-Gromov isoperimetric inequality [110]. On the other hand, it was shown by Lohkamp [131] that any manifold of dimension greater or equal to three admits a complete metric having Ricci curvature bounded from above and below by negative constants. This suggests that having an upper bound on the Ricci curvature has no topological implication for a manifold.

In [110], Gromov was able to show that volume doubling (implied by the Bishop-Gromov inequality in the case of Riemannian manifolds with non-negative Ricci curvature) is a sufficient condition for precompactness in the topology induced by the pointed Gromov-Hausdorff convergence, which is an intrinsic notion of convergence for pointed metric spaces. This renowned observation inspired the study of limit points in the Gromov-Hausdorff topology of Riemannian manifolds with a uniform lower bound on Ricci curvature and an upper bound on dimension, known as Ricci limit spaces. The theory of Ricci limit spaces was developed by Cheeger-Colding in [61, 62, 63] and it soon attracted plenty of interest. Many authors contributed to the field, which is still an active field of research. We recommend the excellent survey [152] for an overview of the topic. The study of the properties of Ricci limit spaces has proved fruitful in understanding smooth Riemannian manifolds with a lower Ricci curvature bound. For instance, in [66, 117] the absence of certain types of singularities in Ricci limit spaces was one of the core steps needed to show uniform L^2 bounds on the Riemannian curvature for Riemannian manifolds having bounded Ricci curvature and volume bounded from below.

A natural question about Ricci limit spaces is whether lower bounds on the Ricci curvature and upper bounds on the dimension can be formulated in a synthetic sense (that is, without relying on smoothness) and whether Ricci limit spaces satisfy such a formulation in a meaningful way. This question gained further motivation from the success of Alexandrov space theory [2, 50, 172], where lower sectional curvature bounds are encoded through the triangle comparison inequalities of Toponogov's theorem.

1.2 A synthetic notion of Ricci curvature bounds

The key insight for formulating a synthetic notion of lower Ricci curvature bounds and upper dimension bounds emerged from optimal transport, which enables the localization of these conditions along individual directions. The origins of optimal transport trace back to Monge (1781) and Kantorovich (1942). At its core, optimal transport seeks to address the following problem. Given two Polish spaces (X, d) , (Y, ρ), a lower semicontinuous function $c : X \times Y \to [0, \infty]$ known as the cost function, and probability measures μ and ν on X and Y respectively, determine the existence and the properties of the distribution that achieves the infimum in

$$\inf_{\pi} \left\{ \int_{X\times Y} c(x, y)\, d\pi(x, y) \right\},$$

where the infimum is taken over all probability measures π on $X \times Y$ whose first and second marginals coincide with μ and ν, respectively.

A case of interest is when $X = Y$ and $c = \mathsf{d}^2$. A breakthrough was obtained by McCann [138], who realized that certain entropy functionals over $(\mathcal{P}_2(\mathbb{R}^N), W_2)$ are convex, where W_2 is the Wasserstein distance, which is naturally induced by optimal transport. Later, in [68, 163] it was shown that a Riemannian manifold (M, g) has a lower bound on the Ricci curvature if and only if the entropy functional

$$\mathcal{E}^\infty(\mu) := \begin{cases} \int_X \rho \ln(\rho)\, d\mathrm{vol}_g & \text{if } \mu = \rho\, d\mathrm{vol}_g, \\ \infty & \text{otherwise} \end{cases}$$

satisfies suitable convexity properties along W_2-geodesics in $\mathcal{P}_2(M)$. For instance, non-negative Ricci curvature corresponds to usual convexity. Consequently, in independent works, Sturm [179, 180] and Lott and Villani [132] recognized that the K-convexity of the relative entropy functional above could be used as a synthetic definition of Ricci curvature bounded from below by K, and introduced the $\mathrm{CD}(K, \infty)$ condition. The theory attracted plenty of interest and grew rapidly. For an overview, see [7] and the references therein. However, it was soon understood that Finslerian geometries where not excluded by the $\mathrm{CD}(K, N)$ condition. For instance, $(\mathbb{R}^N, \|\cdot\|, \mathcal{L}^N)$ satisfies the $\mathrm{CD}(0, N)$ condition regardless of the chosen norm $\|\cdot\|$. Moreover, contrary to Ricci limit spaces, no splitting theorem was available for $\mathrm{CD}(K, N)$ spaces. In [16] Ambrosio, Gigli and Savaré introduced the Riemannian Curvature Condition $\mathrm{RCD}(K, \infty)$ by requiring the Sobolev space $H^{1,2}$ to be a Hilbert space. Such a condition is known as infinitesimal Hilbertianity [99]. In [16] the authors required the reference measure to be finite and assumed a stronger version than the standard $\mathrm{CD}(K, \infty)$ condition. Soon afterwards, the paper [23] extended the results to the case of σ-finite measures, also relaxing the assumption to the standard $\mathrm{CD}(K, \infty)$ condition. The field of RCD spaces developed quickly and many properties of RCD spaces were soon established. Again, we recommend [7] for an overview of the theory.

In [183, Remark 4] it was pointed out by De Philippis, Mondino and Topping that, following the topological manifold regularity of three dimensional non-collapsed Ricci limit spaces obtained by M. Simon [174] and by M. Simon and Topping [175], there are examples of RCD spaces which are not the non-collapsed limit of smooth Riemannian manifolds with non-negative Ricci curvature. This suggests that the class of RCD spaces might be larger than that of Ricci limit spaces, implying that it does not provide a sharp synthetic characterization of the latter. However, recent developments have confirmed that the interest in RCD spaces is not limited to their relationship with the theory of Ricci limit spaces.

1.3 Finite perimeter sets in RCD spaces

Following the contributions [80, 142, 47], among others, a comprehensive understanding of the structure of $\mathrm{RCD}(K, N)$ spaces, up to sets of null reference measure, had been achieved. It seemed natural to pursue the analysis further and understand the structure of these spaces up to a codimension-1-negligible set. Such goal was initiated by Ambrosio, Brué and Semola in [11].

Let us start this section by giving a brief overview of the theory of minimal surfaces in the smooth setting. The Plateau problem consists in finding the surface of least area spanning a given contour. It consists of one of the most classical problems in the calculus of variations. It lies at the intersection of many branches of mathematics, and it has generated a vast amount of mathematical theory over the past one-hundred years. It is the prototype of various applications in mathematics and physics. Notably, Schoen and Yau [169, 170] relied on the theory that stemmed from it in their celebrated proof of the positive mass theorem. It has also been generalized in several ways. Its very formulation has proven challenging. For instance, how general are the surfaces that one should consider? Several different theories have been proposed. Any satisfactory variational theory should address two main problems: existence and regularity. These properties allow us to compute interesting geometric quantities and draw additional conclusions.

One of the first successful variational theories to tackle a version of the Plateau problem in codimension one is that of finite perimeter sets. It was originally developed by De Giorgi in [74] after Caccioppoli. Sets of finite perimeter are general enough to naturally provide an existence theory. For this reason, they are not necessarily smooth. Nonetheless, they satisfy an important structural property known as De Giorgi's structure theorem [74]. This theorem states that the boundary of a finite perimeter set, interpreted in a suitable measure-theoretic sense, is rectifiable (see also [136, chapter 15] for a modern introduction to the topic). An alternative approach was adopted by Federer and Fleming [93], who considered integral currents. Loosely speaking, k-dimensional currents are defined as the dual of k-forms. In codimension one, such theory is equivalent to that of finite perimeter sets as it can be shown that integral currents are countable integer combinations of boundaries of sets. The theory of integral currents provides a robust existence theory of minimizers in every codimension and to related variational problems. The regularity theory in codimension one was studied by several authors between the sixties and the nineties, most notably De Giorgi, Almgren, Fleming, Pitts, Simons, Federer, Bombieri, Giusti, Simon. Let us mention that, briefly put, a point x on the boundary of a finite perimeter set E is regular if there exists $r > 0$ such that $\partial E \cap B_r(x)$ is a smooth hypersurface. A point is said to be singular if it is not regular. A summary of the best theorems available in the smooth setting is as follows (see [77] for an introduction to the topic of regularity of minimizing currents). Here, N denotes the dimension of the ambient space.

- When $N \leq 7$ the singular set is empty. This result was obtained in subsequent works [74, 95, 75, 5, 176].
- When $N = 8$ the singular set consists of isolated points. See [92].
- When $N \geq 9$ the singular set has Hausdorff dimension less or equal to $N - 8$ ([92]) and is countably $N - 8$ rectifiable ([173]).
- For every $N \geq 8$ there are area minimizing integral currents in the Euclidean space $\mathbb{R}^N$ that have $N-8$ Hausdorff measure of their singular set strictly greater than zero ([40]).

Given a Riemannian manifold (M, g), a hypersurface $\Sigma \subset M$ and a smooth compactly supported vector field X, one may consider the first and second variation of the area Σ along X. This can be generalized to the setting of finite perimeter sets and currents. Σ is said to be stationary if its first variation is equal to zero along all compactly supported smooth vector fields. Moreover, Σ is said to be stable if it is stationary and its second variation is greater or equal than zero for all smooth compactly supported vector fields. An important tool used in the regularity theory of perimeter minimizing sets is the *monotonicity formula* obtained by testing the first variation formula with radial vector fields. It states that, given a stationary finite perimeter set $E \subset \mathbb{R}^N$ (analogous results hold in the setting of integral currents), the quantity

$$\frac{\mathrm{Per}(E, B_r(x))}{\omega_N r^{N-1}}$$

is increasing for all $x \in \mathbb{R}^N$ and $r > 0$. One of the consequences of the monotonicity formula is that the density

$$\Theta_{N-1}(E, x) := \lim_{r \longrightarrow 0} \frac{\mathrm{Per}(E, B_r(x))}{\omega_{N-1} r^{N-1}}$$

is defined at every point $x \in \mathbb{R}^N$. Another important consequence of the monotonicity formula is that the rescalings of perimeter minimizing sets at boundary points converge to perimeter minimizing cones, called tangent cones. In his fundamental work [74], De Giorgi realized that the existence of one flat tangent cone at a point is sufficient to conclude that such point is regular.

The study of the structure of the singular set of minimizing integral currents is one of the main subjects of regularity theory. A fundamental result in that direction, which also constitutes a first step of the results in higher codimension due to Almgren [6] is the stratification of the singular set of minimizing integral currents. Loosely speaking, given $k \in \mathbb{N}$, the k-singular stratum is the set of points whose tangent cones split at most $\mathbb{R}^k$. The Almgren stratification (cf. [188]; see also [92]) states that the k-stratum has at most Hausdorff dimension k. This result is widely used to analyze singularities in geometric analysis, for instance in the field of mean curvature flow and harmonic maps ([188]).

The regularity theory of minimal surfaces has been extended to different settings. When the codimension is greater than one, Almgren ([6]) showed that the singular set of minimizing integral currents of dimension $m \geq 2$ has Hausdorff dimension at most $m-2$. In [91] Federer showed that there exist minimizing integral currents of $m-2$ Hausdorff dimension greater than zero in $\mathbb{R}^{m+2}$ for every $m \geq 2$. In the setting of stationary varifolds, Allard [4] showed that the singular set is meager. For stable hypersurfaces a deep regularity theory has been developed in [167, 168, 189]. Many questions remain open in higher codimensions.

A related problem to that of minimal surfaces is that of approximating the area functionals by smoother functionals in an appropriate sense. To this end, Modica and Mortola [141, 140] showed that suitably rescaled phase-transition energies, known as Cahn-Hilliard energies, Γ-converge to the perimeter functional with a volume constraint. Without going into the details, let us mention that, under suitable compactness properties, if a sequence of functionals Γ-converges, then a subsequence of the minimizers of such functionals converges to a minimizer of the limit functional. The theory saw many developments. Notably, we report the result of Pacard and Ritoré [157], who showed that one may construct solutions to the Allen-Cahn equation (the Lagrange equation of the unconstrained Cahn-Hilliard energy) by assuming the existence of a stationary hypersurface and other suitable assumptions. On the other hand, Guaraco [112, 96], building on a deep regularity theory obtained in [115, 189, 182], was able to show that solutions to the Allen-Cahn equation converge to (unstable) minimal surfaces. This result provided an alternative proof of Yau's conjecture [190], which states that every three-dimensional manifold contains infinitely many (immersed) stationary surfaces. This conjecture was previously solved by [137, 116, 177] who relied on a min-max construction of stationary hypersurfaces due to Almgren and Pitts (see the introduction of [78]).

A natural first step in studying objects that minimize the area in the non-smooth setting is to study finite perimeter sets. In the non-collapsed $\mathrm{RCD}(K, N)$ setting, in [11] the authors studied sets of finite perimeter in RCD spaces. Let us point out that the problem of understanding the structure of RCD spaces up to a codimension-1-negligible set was also studied in [80, 118, 43], where the authors provided a satisfactory definition of the boundary of an RCD space, linked to the behavior of tangent cones, and investigated its structural properties.

In [11], the authors succeeded in obtaining stability properties of finite perimeter sets under a suitable notion of convergence along sequences of RCD spaces converging in the Gromov-Hausdorff sense. They also obtained that the tangent to a finite perimeter set is unique and equal to a half space up to a set of zero perimeter measure. The theory was taken further by Brué, Pasqualetto and Semola in [46, 44]. There, the authors showed that a De Giorgi-type structure theorem holds in the non-smooth setting of non-collapsed RCD spaces. Moreover, they obtained a Federer-type characterization of the reduced boundary of finite perimeter sets. An important role was played by vector calculus, introduced by Gigli in [102]. The authors of [46] succeeded in extending the theory up to sets having zero perimeter measure, allowing them to define unit normal vectors to sets of finite perimeter.

Minimal surfaces in the non-smooth setting were recently studied by Lytchak-Wenger [133, 135, 134] in the case of two-dimensional surfaces. In [121] Ketterer studied structural properties of $\mathrm{RCD}(0, N)$ spaces with subsets that have mean convex boundaries. In [144], Mondino and Semola extended the study of perimeter minimizing sets to RCD metric measure spaces of higher dimensions. One of the cornerstones of minimal surface theory in Euclidean spaces is the first variation formula. However, it is not clear in the non-smooth setting which one-parameter family of diffeomorphisms one should use. Moreover, it is not clear if the perturbed finite perimeter sets would even have locally finite perimeter. Therefore, an alternative approach was needed. A key insight used in [121, 144] to circumvent this problem is that, given a mean convex hypersurface in a Riemannian manifold with non-negative Ricci curvature, its interior equidistant hypersurfaces are mean convex too. See [109] for a proof. This fact inspired the authors of [144] to obtain Laplacian bounds on the distance from a perimeter minimizing set, which allowed them to show strong regularity properties of perimeter minimizing sets in non-collapsed RCD spaces. Namely,

- They showed an ε-regularity theorem, which states that if a point of the boundary of a perimeter minimizer is sufficiently close (in the Gromov-Hausdorff sense) to a rigid model, then there is a neighborhood of such point which is

C^α-homeomorphic to the ball $B_1(0) \subset \mathbb{R}^{N-1}$, where $N \in \mathbb{N}$ is the dimension of the ambient space. The Euclidean counterpart of this result is the ε-regularity theorem of De Giorgi [74], which states that a minimal boundary contained in a sufficiently small strip around a hyperplane is analytic.

- They proved sharp Hausdorff dimension estimates on the singular set of perimeter minimizers, showing also that in the set of regular points of the ambient space, the estimate achieved is the same as in the Euclidean case.
- They were able to prove quantitative regularity estimates using sharp perimeter bounds on the perimeter of sets equidistant from a minimal boundary.

The long term goals of the theory of perimeter minimizing sets in RCD metric measure spaces is to kickstart a theory of geometric analysis in non-smooth spaces with a lower bound on the Ricci curvature. Many questions remain open, such as whether it is possible to define critical points of the area functional and if the existence of metric currents is implied by the RCD condition. If these questions could be answered in the affirmative, then a host of problems in geometric analysis could be approached in the non-smooth setting, possibly leading to a better understanding of their Euclidean counterparts and of the structure of RCD spaces themselves.

A further motivation for the study of perimeter minimizing sets is their stability property (shown in [144]): if a sequence of non-collapsed RCD metric measure spaces each having a locally perimeter minimizing set converge in an appropriate sense (see Definition 2.4.8), then the limiting finite perimeter set is itself locally perimeter minimizing. This fact can be exploited to show results for perimeter minimizing sets in the class of smooth Riemannian manifolds satisfying a lower bound on the Ricci curvature and on the volume of unit balls, and an upper bound on the dimension. An arbitrarily bad behavior for perimeter minimizers in this class could be ruled out by arguing that otherwise, by compactness (see Corollary 2.4.15, which relies on Gromov's compactness theorem, and Proposition 2.4.33), there would exist a perimeter minimizing set in an RCD limiting space having a type of singularity that would contradict the regularity theory of perimeter minimizing sets in non-smooth spaces.

1.4 Contributions

In this section we summarize the new results reported in this thesis.

In Chapter 3 we present the results obtained in a joint work with Mondino and Semola [94]. We refine the analysis of the structure of perimeter minimizing sets in non-collapsed RCD metric measure spaces obtained in [144]. In such work, the authors showed sharp Hausdorff dimension estimates on the singular set of a perimeter minimizing set, among other regularity results for minimal boundaries in RCD metric measure spaces. The proof of such estimates relies on a Federer reduction argument [92] and a monotonicity formula for cones that are the product of at least one Euclidean factor and a cone with a Riemannian manifold with Ricci curvature bounded from below by $N-1$ as cross-section, where $N \in \mathbb{N}$ is the dimension of the cone (see [148, Theorem 9.3], [90, Theorem 5.4.3] and [147]). A more tailored monotonicity formula is needed to refine the analysis further and obtain a stratification of the singular set of minimal boundaries. It is a priori not known if such a formula exists in the setting of RCD metric measure spaces, as the Riemannian counterpart includes error terms which depend on the injectivity radius, which can be arbitrarily close to zero in the non-smooth setting. In [94] we establish a rigid monotonicity formula for (locally) perimeter minimizing sets in cones having an RCD metric measure space for cross-section. In the second part of the chapter, we present some applications of such result to the study of perimeter minimizers in RCD metric measure spaces. Firstly, we show how the monotonicity formula can be used to obtain a stratification of the singular set of a (locally) perimeter minimizing set in a non-collapsed RCD space. Namely, we show a sharp bound on the Hausdorff dimension of the singular strata (that is, points whose tangent split at most a fixed number of Euclidean factors). Lastly, using the monotonicity formula, we show that if $(X, \mathsf{d}, \mathcal{H}^N)$ is a non-collapsed $\mathrm{RCD}(0, N)$ space with Euclidean volume growth containing an entire perimeter minimizer, then every blow-down of X contains a globally perimeter minimizing cone. We highlight that this last result is new even in smooth Riemannian geometry. A natural question that could arise at this point is if a version of the Bernstein theorem holds for RCD. This was answered in the affirmative in some specific cases [71, 69]. A future direction could be to establish the rectifiability of the singular strata. This fact that is known for the singular set of Ricci limit spaces [65], but not for the singular set of RCD metric measure spaces themselves let alone for the strata of the singular set of a perimeter minimizing set.

The fourth chapter is based on a joint work with Cucinotta [70]. This work is inspired by and builds upon [66, Theorem 1.4], where the authors proved that the singular set of a non-collapsed limit space with a two-sided bound on the Ricci curvature has at most Hausdorff dimension $N-4$, where N is the dimension of the space. To put this in perspective, the Hausdorff dimension of the singular set of a non-collapsed $\mathrm{RCD}(K, N)$ without boundary can be at most $N-2$. In the deep work [65] the authors were able to improve such estimate by being able to exclude certain types of singularities that occur in the case where there is only a lower bound on the Ricci curvature. Let us recall that in [144] the sharp Hausdorff estimate on the singular set of a minimal boundary in an RCD metric measure space that was obtained is $N-3$. Additionally, the known examples that show that $N-3$ is sharp are based on the type of conical singularities that were excluded in [65] in the case of a two-sided bound on the Ricci curvature. Therefore, it seemed natural to wonder whether, by adding an upper bound on the Ricci curvature of the ambient space, one could improve the estimate on the Hausdorff dimension of the singular set of perimeter minimizing sets. In [70], we answer this question in the affirmative. We show that the Hausdorff dimension of the singular set of a perimeter minimizer in a non-collapsed limit space with a two-sided bound on the Ricci curvature is at most $N-5$. The proof is based on showing that certain types of singularities cannot arise in this setting. An application of the stratification result of [94] allows us to conclude. This result should be interpreted as a precise and quantified version of the intuitive guess that "perimeter minimizers wish to avoid most singularities of the ambient space". A future research direction is to examine whether the Hausdorff dimension of the singular set of minimal boundaries is always exactly one less than that of the ambient space. When is it true that if the ambient space has a singular set of Hausdorff dimension k, then the singular set of minimal boundaries in that space has a singular set of dimension at most $k-1$?

In the fifth chapter we study a Modica-Mortola-type approximations of the isoperimetric problem in the metric setting. See [1] for an introduction to the topic. The notion of Γ-convergence (sometimes referred to as epi-convergence) of functionals is first attributed to De Giorgi [76, 73]. Let $E_k : X \to \mathbb{R}$, $k \in \mathbb{N}$ be a sequence of functionals Γ-converging to $E_\infty : X \to \mathbb{R}$, where X is some function space. An important property of Γ-convergence is that, if $\{f_k\}_k \subset X$ converges to some function $f \in X$ and f_k minimimizes E_k for each $k \in \mathbb{N}$, then f minimizes E_∞. A classical instance of Γ-convergence is that of Modica and Mortola [141, 140], who showed that the Cahn-Hilliard phase transition energies [51] converge to the perimeter functional when suitably rescaled. In light of the previously mentioned property of Γ-convergence, this result yields itself to applications to the study of perimeter minimizers. The Cahn-Hilliard minimizers, which are smooth functions by elliptic PDE theory, constitute approximations of perimeter minimizing sets and can be even used as a procedure to construct them. For this reason, the field attracted plenty of interest quickly. However, the result of Modica and Mortola does not tell us much about the other critical points of the area functional, that is, minimal surfaces. More recently, related PDE approaches based on min-max theory [112, 96] (building on the regularity theory of [115, 189, 182]) have been used as an approach to tackle minimal surfaces problems, and have yielded a new proof of the existence of minimal hypersurfaces in closed Riemannian manifolds originally proved by Almgren, Pitts, and Schoen–Simon [159, 168] and an alternative proof of Yau's conjecture [190] to those based on the Almgren-Pitts theory [137, 116, 177]. This PDE approach to the construction of minimal surfaces has attracted the interest of the RCD community, since, as previously mentioned, in the RCD setting no satisfactory analogue of the first variation of area formula is available. The hope is that these novel PDE approaches can be adapted to the RCD setting and be used as a way to study minimal surfaces in such setting, and even lead to a better understanding of the structure of RCD metric measure spaces themselves. In this chapter, we aim at taking a step in this direction. We show some Γ-convergence results of the Cahn-Hilliard energy to the perimeter functional in the non-smooth setting. Firstly, we show that the Γ-liminf inequality holds in general in non-collapsed RCD metric measure spaces. We are only able to show the Γ-limsup inequality for volume-constrained perimeter minimizing sets by exploiting some regularity results regarding the Laplacian of the distance function obtained in the setting of non-collapsed RCD metric measure spaces in [34], which do not hold in general for finite perimeter sets. However, this result is sufficient to prove that any volume-constrained perimeter minimizer can be approximated in the perimeter sense by a sequence of functions minimizing the approximating Cahn-Hilliard functionals. Moreover, we prove that any sequence of minimizers of the approximating Cahn-Hilliard functionals have a subsequence which is converging in the BV sense to a volume-constrained perimeter minimizer with the same L^1 norm if the ambient space has a volume-constrained minimizer of the given volume. Lastly, exploiting the stability of the RCD condition under (pointed) Gromov-Hausdorff convergence, we "approximate the perimeter both in the functional and in the space sense". Namely, we deal with the case where the Cahn-Hilliard functionals are defined on a non-collapsing sequence of Riemannian manifolds having a lower bound on the Ricci curvature converging to some space (X, d). We are able to show that such functionals Γ-converge, up to a constant, to the perimeter functional on the limit space X.

Chapter 2

Preliminaries

The aim of this chapter is to provide context for the results presented in this thesis by offering a concise, though not exhaustive, overview of the underlying theory.

In *Section 1* we review some basic analysis tools in metric (measure) spaces. We also recall some definitions and basic results regarding the heat flow, infinitesimal Hilbertianity and optimal transport in the metric setting. In *Section 2* we report the concept of Gromov-Hausdorff convergence of sequences of metric (measure) spaces. This notion can be used to provide a metric on metric (measure) spaces. We also briefly present Ricci limit spaces and some of their properties which are relevant for the rest of the thesis. Briefly put, Ricci limits are the closure of the space of Riemannian manifolds with a lower bound on the Ricci curvature in the topology induced by the Gromov-Hausdorff convergence. In *Section 3* we report the Riemannian curvature dimension (RCD) condition. Loosely speaking, this consists of a synthetic notion of a lower bound on the Ricci curvature. We recall some of the properties enjoyed by metric measure spaces satisfying this property, review the construction of metric cones and briefly mention how higher order calculus and vector spaces (in a weak sense) can be defined on such spaces. Lastly, in *Section 4*, we review finite perimeter sets in RCD spaces and their structural properties. This theory mirrors the theory of finite perimeters sets in Euclidean spaces, albeit with some caveats. We also recall the known properties of perimeter minimizing sets in such setting.

2.1 Analysis on metric measure spaces

In this section we report some calculus rules in metric spaces needed for our purposes and provide some background on the theory. We then present the class of absolutely continuous curves in metric spaces. We also recall the properties of the heat flow and of optimal transport needed in this setting. Lastly, we report the infinitesimally

Hilbertian condition, which "rules out Finslerian geometries", and we present some of its consequences on metric calculus. For a more complete review of the theory, we recommend the books [107, 113].

2.1.1 Basic definitions and notation

In this work, a *metric measure space* (m.m.s. for short) is a triple $(X, \mathsf{d}, \mathfrak{m})$, where (X, d) is a complete and separable metric space and $\mathfrak{m}$ is a non-negative Borel measure on X of full support (i.e. $\operatorname{supp} \mathfrak{m} = X$), called the ambient or reference measure, which is finite on metric balls. We write $B_r(x)$ for the open ball centered at $x \in X$ of radius $r > 0$. We implicitly assume metric spaces to be proper unless explicitly stated.

We denote by $L^p(X; \mathfrak{m}) := \left\{u : X \to \mathbb{R} : \int |u|^p \, d\mathfrak{m} < \infty\right\}$ the space of p-integrable functions; sometimes, if it is clear from the context which space and measure we are considering, we will simply write L^p or $L^p(X)$.

Given a function $u : X \to \mathbb{R}$, we define its local Lipschitz constant at $x \in X$ as

$$\operatorname{lip}(u)(x) := \limsup_{y \to x} \frac{|u(x) - u(y)|}{\mathsf{d}(x, y)} \qquad \text{if } x \in X \text{ is an accumulation point,}$$

and $\operatorname{lip}(u)(x) = 0$ otherwise. We indicate by $\mathrm{LIP}(X)$ and $\mathrm{LIP}_{\mathrm{loc}}(X)$ the space of Lipschitz functions, and locally Lipschitz functions, respectively. We also denote by $\mathrm{C}_b(X)$ and $\mathrm{C}_{\mathrm{bs}}(X)$ the space of bounded continuous functions and the space of continuous functions with bounded support, respectively.

Definition 2.1.1 (Doubling Metric Measure Space)**.** *We say that a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is locally doubling if there exists a non-decreasing function $C : (0, \infty) \to (0, \infty)$ such that*

$$\mathfrak{m}(B_{2r}(x)) \leq C(R)\mathfrak{m}(B_r(x)), \qquad \textit{for all } 0 < r < R \textit{ and } x \in X.$$

Moreover, if C can be chosen to be constant, we will say that the space is doubling.

Let us point out that if a m.m.s. is doubling, then all spheres, i.e. sets of the form $\{y \in X : \mathsf{d}(y, x) = r\}$ are $\mathfrak{m}$-negligible for $\mathcal{L}^1$-a.e. $r \geq 0$, and $\mathfrak{m}$-a.e. $x \in X$. Moreover, the following characterization holds.

Proposition 2.1.2. *A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is doubling if and only if there exist constants $C', s > 0$ such that*

$$\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_R(y))} \geq C' \left(\frac{r}{R}\right)^s,$$

for any $R > r > 0$ and $y \in X$, $x \in B_R(y)$.

Given two subsets $A, B \subset X$ we say that their Hausdorff distance, denoted by $\mathsf{d}_{\mathrm{H}}(A, B)$, is

$$\mathsf{d}_{\mathrm{H}}(A, B) := \sup \left(\sup_{x \in A} \mathsf{d}_B(x), \sup_{x \in B} \mathsf{d}_A(x) \right),$$

where d_A and d_B are the distance function from the sets A and B, respectively.

We say that $(X, \mathsf{d}, \mathfrak{m})$ supports a 1-p Poincaré inequality if there exist constants $C_p > 0$, $\lambda \geq 1$ such that

$$\int_{B_r(x)} |u(y) - u_{B_r(x)}| \, d\mathfrak{m}(y) \leq C_p r \left(\int_{B_{\lambda r(x)}} \mathrm{lip}^p(u) \, d\mathfrak{m} \right)^{\frac{1}{p}}, \tag{2.1.1}$$

where $u_{B_r(x)} := \frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} u(x) \, d\mathfrak{m} = \fint_{B_r(x)} u(x) d\mathfrak{m}$.

Definition 2.1.3 (PI space)**.** *A PI space is a metric measure space* $(X, \mathsf{d}, \mathfrak{m})$ *which is doubling and supports a* 1-2 *Poincaré inequality.*

Given an integer $k \geq 0$, we denote $\omega_k := \frac{\pi^{k/2}}{\Gamma(1+k/2)}$, where

$$\Gamma(k) := \int_0^\infty t^{k-1} e^{-t} \, dt$$

is the Gamma function.

2.1.2 Hausdorff measures

This section is devoted to the notion of Hausdorff measures on metric spaces. These measures are one of the key objects of Geometric Measure Theory (see [90]).

Given a metric space (X, d), the diameter of a subset $A \subset X$ is

$$\mathrm{diam}(A) := \sup\{\mathsf{d}(x, y) : x, y \in A\}.$$

Definition 2.1.4 (Hausdorff measure)**.** *Let* (X, d) *be a metric space and* $A \subset X$*. We define the k-Hausdorff δ-pre-measure of A to be*

$$\mathcal{H}^k_\delta(A) := \inf \left\{ \frac{\omega_k}{2^k} \sum_{i \in I} \mathrm{diam}(A_i)^k : \, A \subset \bigcup_{i=1}^\infty A_i \text{ and } \mathrm{diam}(A_i) < \delta \right\}.$$

where the infimum is taken over all countable covers of A by sets $A_i \subset X$ *satisfying* $\mathrm{diam}(A_i) < \delta$ *for every* $i \in \mathbb{N}$*. The k-Hausdorff measure of the set A, denoted by* $\mathcal{H}^k(A)$*, is*

$$\mathcal{H}^k(A) = \sup_{\delta > 0} \mathcal{H}^k_\delta(A).$$

Remark 2.1.5. *Let us point out the following facts regarding Hausdorff measures.*

- *When k is a non-negative integer, the k-Hausdorff measure corresponds to the Lebesgue measure on $\mathbb{R}^k$.*
- *Both the Hausdorff measures and the Hausdorff pre-measures are outer measures, but the former is a measure on the Borel sets.*
- *For $k > k'$ it holds*
$$\mathcal{H}^k(A) > 0 \implies \mathcal{H}^{k'}(A) = \infty.$$
- *$\mathcal{H}^k_\delta$ is non-increasing as a function of δ.*

The k-∞-Hausdorff-pre-measure, which we sometimes denote by ∞-Hausdorff-pre-measure when $k \geq 0$ is clear from the context, is defined on subsets $A \subset X$ as
$$\mathcal{H}^k_\infty(A) = \lim_{\delta \to \infty} \mathcal{H}^k_\delta(A).$$

We here state, without proof, some properties of the k-∞-Hausdorff-pre-measure (see [90]).

Lemma 2.1.6 (Upper semi-continuity of $\mathcal{H}^k_\infty$)**.** *Let (X, d) be a metric space and let $\{A_i\}_i$ be a sequence of Borel subsets of X such that $\mathsf{d}_H(A_i, A) \to 0$ for some $A \subset X$ Borel and compact, where d_H stands for the Hausdorff distance. Then*
$$\mathcal{H}^k_\infty(A) \geq \limsup_{i \to \infty} \mathcal{H}^k_\infty(A_i).$$

Lemma 2.1.7 (Density of $\mathcal{H}^k_\infty$ [90, Theorem 2.10.17])**.** *Let (X, d) be a metric space and $E \subset X$ a Borel set. Then for every $k \geq 0$ there exists $C_k > 0$ such that for $\mathcal{H}^k$-almost every $x \in E$ we have*
$$\limsup_{r \to 0} \frac{\mathcal{H}^k_\infty(E \cap B_r(x))}{r^k} \geq C_k.$$

We report two more properties of k-∞-Hausdorff-pre-measure, whose proof is straightforward.

$$\mathcal{H}^k(X) = 0 \iff \mathcal{H}^{k+1}(X \times \mathbb{R}) = 0, \tag{2.1.2}$$
$$\mathcal{H}^k_\infty(X) = 0 \iff \mathcal{H}^k(X) = 0 \tag{2.1.3}$$

for any (X, d) metric space.

Lastly, we recall the notion of Hausdorff dimension, which is used to quantify the size of sets in some of the main results presented in this manuscript.

Definition 2.1.8. *Let $A \subset X$. Then the Hausdorff dimension of A is*

$$\dim_{\mathcal{H}}(A) = \inf \left\{k \geq 0 \,:\, \mathcal{H}^k(A) = 0\right\},$$

where we use the convention $\inf \emptyset = \infty$.

2.1.3 Absolutely continuous curves

In this section we report a few basic concepts regarding absolutely continuous curves in metric spaces and their properties. This section is based on [10, Chapter 9].

Definition 2.1.9 (Absolutely continuous curve)**.** *Let (X, d) be a metric space and $a, b \in \mathbb{R}$, $a < b$. A curve $\gamma : [a, b] \to X$ is said to be* absolutely continuous*, and we write $\gamma \in \mathrm{AC}([a, b]; X)$, if there exists $g \in L^1([a, b])$ such that*

$$\mathsf{d}(\gamma(s), \gamma(t)) \leq \int_s^t g(r)\, dr \quad \textit{for all } a \leq s \leq t \leq b.$$

Let us point out that, when $X = \mathbb{R}$, the definition above is equivalent to the classical notion of absolute continuity of functions.

Theorem 2.1.10 (Metric derivative)**.** *For any $\gamma \in \mathrm{AC}([a, b]; X)$ the limit*

$$\lim_{h \to 0} \frac{\mathsf{d}(\gamma(t, t + h))}{|h|} =: |\gamma'|(t)$$

exists for $\mathcal{L}^1$-a.e. $t \in (a, b)$. Moreover, up to $\mathcal{L}^1$-negligible sets, $|\gamma'|$ is the minimal g we can choose in Definition 2.1.9. We call $|\gamma'|$ the metric derivative of γ.

The *length* of a curve $\gamma \in \mathrm{AC}([a, b]; X)$ is $l(\gamma) := \int_a^b |\gamma'|(t)\, dt$. Moreover, we say that $\gamma \in AC([a, b]; X)$ has *constant speed* if $|\gamma'|$ is equivalent to a constant. Let us point out that, given $\gamma \in \mathrm{AC}([a, b]; X)$, there exists a reparametrization $\tilde{\gamma} \in \mathrm{AC}([0, 1]; X)$ with constant speed equal to $l(\gamma)$. That is, there exists an increasing onto map $\varphi : [0, 1] \to [a, b]$ such that $\tilde{\gamma} := \gamma \circ \varphi$ with $|\tilde{\gamma}'| = l(\gamma)$ $\mathcal{L}^1$-almost everywhere.

Definition 2.1.11 (The space $\mathrm{Geo}(X)$)**.** *A curve $\gamma \in \mathrm{AC}([a, b]; X)$ is a* geodesic *if $l(\gamma) = \mathsf{d}(\gamma(a), \gamma(b))$. The space of geodesics $\mathrm{Geo}(X)$ is the space of constant speed geodesics on $[0, 1]$.*

Throughout this thesis we denote by $e_t : C([0, 1]; X) \to X$ the *evaluation map* defined by

$$e_t(\gamma) := \gamma(t),$$

where $t \in [0, 1]$.

We say that a metric space (X, d) is *geodesic* if for all $x, y \in X$ there exists $\gamma \in \mathrm{Geo}(X)$ with $\gamma(0) = x$ and $\gamma(1) = y$.

2.1.4 Sobolev calculus

In this section we briefly introduce Sobolev functions on metric measure spaces. Let us point out that there are alternative ways to define Sobolev spaces on metric measure spaces. For their equivalence, see for instance [22]. We adopt the approach *à la Cheeger*, that is we view the norm of the gradient of a Sobolev function as the minimal relaxed slope of the Cheeger energy. We will be working on with the integrability exponent $p = 2$. For a more detailed treatment, see [12] and [107, 113].

Definition 2.1.12 (Cheeger energy)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. The Cheeger energy $Ch : L^2(X, \mathfrak{m}) \to [0, \infty]$ is the convex lower semi-continuous functional defined by*

$$Ch(f) := \frac{1}{2} \inf \left\{ \liminf_{i \to \infty} \int_X \operatorname{lip}^2(f_i) \, d\mathfrak{m} : \{f_i\}_i \subset L^2 \cap \operatorname{LIP}_b(X), \; f_i \to f \; \text{in } L^2 \right\}.$$

The Sobolev space $H^{1,2}(X)$ is the finiteness domain of the Cheeger energy. The vector space $H^{1,2}(X)$ is Banach when considered with the following norm

$$\|f\|_{H^{1,2}} = (\|f\|_{L^2}^2 + 2Ch(f))^{\frac{1}{2}}.$$

Moreover, if $(X, \mathsf{d}, \mathfrak{m})$ is doubling, $H^{1,2}$ is reflexive and standard analytic arguments show separability and density of Lipschitz functions. All these results, and more, can be found in [12].

We briefly also mention an alternative version that we will make use of in Section 2.3.9. We will use this notation to report the notion of tangent modules, which correspond to "vector fields" in the Euclidean setting. See [107, Chapter 2].

Definition 2.1.13 (Test Plan)**.** *A probability measure $\pi \in \mathcal{P}(\mathrm{AC}[0, 1], X)$ is said to be a test plan on X if*

- *there exists $C > 0$ such that $(e_t)_* \pi \leq C\mathfrak{m}$ for every $t \in [0, 1]$;*
- $\int_0^1 \int |\gamma(t)'|^2 d\pi(\gamma) \, dt < \infty$.

Definition 2.1.14 (Sobolev class)**.** *The* Sobolev class $S^2(X)$ *is defined as the space of all Borel functions $f : X \to \mathbb{R}$ for which there exists $G \in L^2(X)$, $G \geq 0$ $\mathfrak{m}$-a.e. such that*

$$\int |f(\gamma_1) - f(\gamma_0)| d\pi(\gamma) \leq G(\gamma(t))|\gamma'(t)| \, d\pi(\gamma) \, dt \quad \text{for every test plan } \pi \text{ on } X.$$

We say that $f \in H^{1,2}(X)$ if $f \in S^2(X) \cap L^2(X)$.

The analogy with the Euclidean case is straightforward: given $f \in C^1(\mathbb{R}^N)$ and $G \in C(\mathbb{R}^N)$ then $G \geq |\nabla f|$ if and only if

$$|f(\gamma(1)) - f(\gamma(0))| \leq \int_0^1 G(\gamma(t))|\gamma'(t)|\,dt \quad \text{for every } \gamma \in C^1([0,1], \mathbb{R}^N).$$

2.1.5 Infinitesimal Hilbertianity

In this section we report the definition of infinitesimal Hilbertianity of a metric measure space. We also recall some metric calculus rules that hold under such an assumption. This concept was introduced in [16], see also [99]. This condition rules out Finslerian geometries, which the CD condition (see Section 2.3.1 below) alone does not exclude.

Definition 2.1.15. *A metric measure space* $(X, \mathsf{d}, \mathfrak{m})$ *is said to be* infinitesimally Hilbertian *if its Cheeger energy is a quadratic form, or, equivalently, if* $H^{1,2}(X)$ *is a Hilbert space.*

For any function $f \in H^{1,2}(X)$ there exists a unique minimal upper gradient (see [107]) $|\nabla f| \in L^2(X)$ such that

$$Ch(f) = \frac{1}{2}\int_X |\nabla f|^2 d\mathfrak{m}.$$

Furthermore, if $(X, \mathsf{d}, \mathfrak{m})$ is also doubling and supports a 1-2 Poincaré inequality then $|\nabla f| = \text{lip}(f)$ $\mathfrak{m}$-almost everywhere for all $f \in H^{1,2}(X) \cap \text{LIP}_{\text{loc}}(X)$. This was first shown in [58]. See also [12]. See [99] for a proof of why, in the infinitesimally Hilbertian setting, this is allowed.

A consequence of infinitesimal Hilbertianity is the following.

Proposition 2.1.16. *If* $(X, \mathsf{d}, \mathfrak{m})$ *is infinitesimally Hilbertian the operator* $\Gamma : H^{1,2}(X) \times H^{1,2}(X) \to L^1(X)$ *defined by*

$$\Gamma(f,g) := \lim_{\varepsilon \to 0} \frac{|\nabla(f + \varepsilon g)|^2 - |\nabla f|^2}{2\varepsilon},$$

where the limit is understood in $L^1(X)$*, is well defined, symmetric and bilinear.*

For further information on Dirichlet forms, the reader may consult [41]. For the rest of this thesis, we will denote $\Gamma(f,g)$ by $\nabla f \cdot \nabla g$.

Let us present a few calculus rules for Sobolev functions. A more detailed presentation can be found in [107]. In the following f and g denote two functions in $H^{1,2}(X)$.

- *Locality*: $|\nabla f| = |\nabla g|$ $\mathfrak{m}$-almost everywhere on $\{f = g\}$.
- *Chain rule*: Let $\varphi \in \mathrm{LIP}(\mathbb{R})$ with $\varphi(0) = 0$. Then $|\nabla(\varphi \circ f)| = |\varphi' \circ f||\nabla f|$ $\mathfrak{m}$-almost everywhere.
- *Leibniz rule*: Let $h \in \mathrm{LIP}_b(X)$. Then $\nabla f \cdot \nabla(hg) = h\nabla f \cdot \nabla g + g\nabla f \cdot \nabla h$ $\mathfrak{m}$-almost everywhere.

Let us recall the definition of Laplacian in this setting.

Definition 2.1.17 (Laplacian). *We say that a function $f \in H^{1,2}(X)$ is in $D(\Delta)$ if there exists a function $g \in L^2(X)$ such that*

$$\int_X gh\, d\mathfrak{m} = -\int_X \nabla f \cdot \nabla h\, d\mathfrak{m} \quad \text{for all } h \in H^{1,2}(X).$$

We then define $\Delta f := g$.

Let us point out that $\Delta : D(\Delta) \to L^2(X)$ is a densely defined linear operator.

2.1.6 Heat flow

We recall the definition of heat flow in the setting of metric measure spaces. Because of its regularizing nature, the heat flow is often used to obtain what would correspond to "smooth fuctions" in the Euclidean setting.

Definition 2.1.18 (Heat flow). *The heat flow $P_t : L^2(X) \to L^2(X)$, $t > 0$ is the $L^2(X)$-gradient flow of the Cheeger energy.*

The Brezis-Komura theory provides the existence of gradient flows of lower semi-continuous convex functional on a Hilbert spaces. For more details, see [10].

If $(X, \mathsf{d}, \mathfrak{m})$ is assumed to be infinitesimally Hilbertian, then we can characterize the heat flow as the operator $P_t : L^2(X) \to L^2(X)$ such that for any $f \in L^2(X)$ $P_t f \in D(\Delta)$ for any $t \in (0, \infty)$, and for which the curve $[0, \infty) \ni t \to P_t f$ is continuous, absolutely continuous on $(0, \infty)$ and satisfies

$$\begin{cases} \frac{d}{dt} P_t f = \Delta P_t f & \text{for } \mathcal{L}^1\text{- a.e. } t > 0, \\ P_0 f = f \end{cases}$$

In the infinitesimally Hilbertian setting, it can also be shown (as first shown in the [16, proof of Theorem 6.1]) that the heat flow is a linear, continuous and self-adjoint contraction semigroup on $L^2(X)$ which extends to a linear and continuous contraction semigroup (which we still denote by P_t) on $L^p(X)$ for any $p \in [1, \infty)$. Let us also point out that P_t and Δ commute when they are well-defined.

2.1.7 Optimal transport tools

We here recall some definitions and results from optimal transport theory. The books [184, 10] provide an excellent introduction to the topic for our purposes.

Given a metric space (X, d) and $p \in [1, \infty)$, we call the space of p-integrable probability measures the set of all Borel measures on X that have finite p-moments. We denote such space by $\mathcal{P}_p(X)$.

Definition 2.1.19 (Kantorovich-Wasserstein distance)**.** *Let (X, d) be a metric space, $p \in [1, \infty)$ and $\mu_0, \mu_1 \in \mathcal{P}_p(X)$. The p-Kantorovich-Wasserstein distance between μ_0 and μ_1 is*

$$W_p(\mu_0, \mu_1) = \left(\inf_\gamma \int \mathsf{d}^p(x, y)\, d\gamma(x, y) \right)^{\frac{1}{p}}, \tag{2.1.4}$$

where the infimum is taken over all $\gamma \in \mathcal{P}(X \times X)$ whose first and second marginals are μ_0 and μ_1, respectively. That is, $\pi^1_\# \gamma = \mu_0, \pi^2_\# \gamma = \mu_1$, where π^i, $i = 1, 2$ denote the projection to the first and second marginals, respectively.

It can be shown that the p-Wasserstein distance induces a metric on the space of Borel probability measures with p-finite moments. Moreover, if (X, d) is a Polish space, so is $(\mathcal{P}_p(X), W_p)$.

An elementary fact about Wasserstein geometry is that if (X, d) is a geodesic space, then so is $(\mathcal{P}_p(X), W_p)$.

Any geodesic $\{\mu_t\}_t$ of $(\mathcal{P}_2(X), W_2)$ can be lifted to a measure $\nu \in \mathcal{P}(\mathrm{Geo}(X))$ such that $(e_t)_\sharp \nu = \mu_t$ for all $t \in [0, 1]$. The measure ν is referred to as a dynamical optimal plan. Given two measures μ_0, $\mu_1 \in \mathcal{P}_2(X)$, we denote by $\mathrm{OptGeo}(\mu_0, \mu_1)$ the space of all $\nu \in \mathcal{P}(\mathrm{Geo})$ such that $(e_0, e_1)_\sharp \nu$ realizes the minimum in (2.1.4). If (X, d) is geodesic, then $\mathrm{OptGeo}(\mu_0, \mu_1)$ is non-empty for all μ_0, $\mu_1 \in \mathcal{P}_2(X)$.

For more results linking properties of metric spaces and their corresponding Wasserstein spaces, see [184].

2.2 Convergence of metric measure spaces

In this section, we recall the definition of Gromov-Hausdorff distance and associated convergence. Briefly put, the Gromov-Hausdorff distance between two metric spaces (X, d) and (Y, ρ) is the infimum over all possible embeddings i_X, i_Y in a common metric space (Z, d_Z) of the Hausdorff distance between $i_X(X)$ and $i_Y(Y)$.

We also discuss some of the basic properties of the Gromov-Hausdorff distance. The reader may consult [49, 184] for a thorough introduction to the topic. Let us point out that in [104] the authors showed that the $\mathrm{RCD}(K, \infty)$ condition is stable under pointed measured Gromov-Hausdorff convergence (see Section 2.3.2 below).

We then recall the notion of Ricci limit spaces, which are the Gromov-Hausdorff closure of the space of Riemannian manifolds with a lower bound on the Ricci curvature. These spaces have attracted plenty of interest partly due to Gromov's precompactness theorem, which implies that the space of Riemannian manifolds with a lower bound on the Ricci curvature is precompact in the Gromov-Hausdorff topology.

Lastly, we recall some definitions and results concerning the convergence of sequences of functions defined on a Gromov-Hausdorff converging sequence of metric spaces.

2.2.1 Basic definitions

Let (X, d) be a metric space and $x \in X$. We call the triple (X, d, x) a pointed metric space. Analogously, let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. and $x \in X$. We call the quadruple $(X, \mathsf{d}, \mathfrak{m}, x)$ a pointed metric measure space (or p.m.m.s. for short). For our purposes, we always suppose that the measures have full support. We say that the pointed m.m.s. $(X, \mathsf{d}, \mathfrak{m}, x)$ is isomorphic to (Y, ρ, μ, y) if there exists an isometry $T : X \to Y$ which is measure preserving, that is $T_{\#}\mathfrak{m} = \mu$, and such that $T(x) = y$. Our main goal in this section is to recall an intrinsic distance between pointed metric (measure) spaces considered up to isomorphisms. We say that a pointed metric measure space $(X, \mathsf{d}, \mathfrak{m}, x)$ is normalized if $\int_{B_1(x)}(1 - \mathsf{d}(\cdot, x))\, d\mathfrak{m} = 1$. Let us point out that there exists a unique constant $c > 0$ that turns a p.m.m.s. into a normalized one.

Definition 2.2.1 (Pointed (measured) Gromov-Hausdorff convergence)**.** *A sequence of pointed metric spaces $\{(X_k, \mathsf{d}_k, x_k)\}_k$ is said to converge in the pointed Gromov-Hausdorff topology to (X, d, x) if there exists a separable metric space (Z, d_Z) and isometric embeddings $i_k : X_k \to Z$ and $i : X \to Z$ such that, for any $\varepsilon > 0$ and $r > 0$ there exists $\bar{k} \in \mathbb{N}$ such that for for every $k \geq \bar{k}$ we have*

$$i\left(B_r^X(x)\right) \subset B_\varepsilon^Z\left(i_k\left(B_{r+\varepsilon}^{X_k}(x_k)\right)\right),$$

and

$$i_k\left(B_r^{X_k}(x_k)\right) \subset B_\varepsilon^Z\Big(i_k\Big(B_{r+\varepsilon}^{X}(x)\Big)\Big).$$

We denote the pointed Gromov-Hausdorff convergence by $X_k \xrightarrow{\mathrm{pGH}} X$ and we say that Z is the space realizing the convergence.

A sequence of p.m.m.s. $\{(X_k, \mathsf{d}_k, \mathfrak{m}_k, x_k)\}_k$ is said to converge in the pointed measured Gromov-Hausdorff topology to $(X, \mathsf{d}, \mathfrak{m}, x)$ if it converges in the pointed Gromov-Hausdorff sense and

$$\lim_{k\to\infty} \int_Z \varphi \, d((i_k)_\# \mathfrak{m}_k) = \int_Z \varphi \, d(i_\# \mathfrak{m}), \quad \text{for all } \varphi \in C_{bs}(Z),$$

where (Z, d_Z) and $i_k : X_k \to Z$, $i : X \to Z$ are the space and the immersions, respectively, realizing the pointed Gromov-Hausdorff convergence.

Let us recall the definition of tangent space in the setting of metric spaces. These are obtained through a blow-up procedure.

Definition 2.2.2 (Tangent space to a metric space)**.** *Let (X, d) be a metric space and let $x \in X$. We call the space of tangent spaces at x, denoted by $\mathrm{Tan}_x(X, \mathsf{d})$, to be the set of all (Y, ρ, y) such that there exists a sequence $1 > r_k > 0$, $r_k \to 0$ that satisfies*

$$(X, \mathsf{d}/r_k, x) \xrightarrow{pGH} (Y, \rho, y).$$

Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space and fix $x \in X$, $r > 0$. The rescaled normalized pointed metric measure space is $(X, \mathsf{d}/r, \mathfrak{m}_r^x, x)$ where

$$\mathfrak{m}_r^x := \left(\int_{B_r(x)} \left(1 - \frac{1}{r}\mathsf{d}(\cdot, x)\right) d\mathfrak{m} \right)^{-1} \mathfrak{m}.$$

Definition 2.2.3 (Tangent space to a metric measure space)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. and $x \in X$. The space $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ of tangent spaces at x is the set of p.m.m.s. (Y, ρ, μ, y) that satisfy*

$$\lim_{\text{pmGH}} (X, \mathsf{d}/r_k, \mathfrak{m}_{r_k}^x, x) = (Y, \rho, \mu, y) \tag{2.2.1}$$

for some sequence $(r_k)_k \subset (0, 1)$ with $r_k \to 0$.

The elements of the tangent spaces are sometimes referred to as *blow-ups*. When the limit in (2.2.1) is taken as $r \to \infty$ instead, the limit spaces are sometimes referred to as *blow-downs*. Blow-downs, if they exist, do not depend on the base point x.

The pointed (measured) Gromov-Hausdorff convergence is induced by a distance, called the (measured) Gromov-Hausdorff distance. We report the definition of (measured) Gromov-Hausdorff distance below.

Definition 2.2.4 (Pointed Measured Gromov-Hausdorff distance)**.** *Let* $(X, \mathsf{d}, \mathfrak{m}, x)$ *and* (Y, ρ, μ, y) *be two locally compact pointed metric measure spaces with measures of full support. Let* (Z, d_Z) *be a proper metric space and let* $\Psi_1 : X \to Z$ *and* $\Psi_2 : Y \to Z$ *be isometric embeddings. For any* $k \in \mathbb{N}$ *let*

$$\mathcal{D}_{k,\Psi_1,\Psi_2}((X, \mathsf{d}, \mathfrak{m}, x), (Y, \rho, \mu, y)) := \mathsf{d}_{\mathrm{H}} \left(\Psi_1 \left(X \cap \overline{B_k(x)} \right), \Psi_2 \left(Y \cap \overline{B_k(y)} \right) \right) \wedge 1 + \left| \log \left(\frac{\mathfrak{m}(B_k(x))}{\mu(B_k(y))} \right) \right| \wedge 1 + W_1^Z \left((\Psi_1)_\# \frac{\chi_{B_k(x)}}{\mathfrak{m}(B_k(x))} \mathfrak{m}, (\Psi_2)_\# \frac{\chi_{B_k(y)}}{\mu(B_k(y))} \mu \right),$$

where d_{H} *is the Hausdorff distance between compact subsets of* Z *and* W_1^Z *denotes the* 1*-Wasserstein distance in* $(Z, \mathsf{d}_Z \wedge 1)$*. We say that the distance between* $(X, \mathsf{d}, \mathfrak{m}, x)$ *and* (Y, ρ, μ, y) *is*

$$\mathsf{d}_{\mathrm{pmGH}}((X, \mathsf{d}, \mathfrak{m}, x), (y, \rho, \mu, y)) := \inf \left\{ \mathsf{d}_Z(\Psi_1(x), \Psi_2(y)) + \sum_{k=1}^{\infty} \frac{1}{2^k} \mathcal{D}_{k,\Psi_1,\Psi_2}((X, \mathsf{d}, \mathfrak{m}, x), (Y, \rho, \mu, y)) \right\},$$

where the infimum is taken over all proper metric space (Z, d_Z) *and isometric embeddings* Ψ_1*,* Ψ_2*.*

Let us briefly unpack definition 2.2.4. $\mathcal{D}_{k,\Psi_1,\Psi_2}$ above (and, therefore, the pointed measured Gromov-Hausdorff distance) takes into account both the metric structure, by considering the Hausdorff distance of embeddings of the spaces, and the measures through the 1-Wasserstein distance (which can be shown to metrize the topology of $\mathcal{P}_1$ induced by weak convergence of measures in duality with the bounded continuous functions defined on the space realizing the convergence). This is by no means the only intrinsic distance between metric measure spaces that can be found in the literature. For example, one may disregard the Hausdorff distance in definition 2.2.4, and obtain the pointed measured Gromov distance. If we assume the p.m.m.s. to be doubling, then the pointed measured Gromov distance is equivalent to the pointed measured Gromov-Hausdorff distance.

Lastly, let us report a compactness result that can be found in [49, Theorem 7.3.8], for instance. Let $\mathfrak{M}(X)$ the space of Borel subsets X endowed with the Hausdorff distance.

Theorem 2.2.5 (Blaschke [49, Theorem 7.3.8])**.** *If* X *is compact, so is* $\mathfrak{M}(X)$*.*

2.2.2 Ricci limit spaces

In this subsection we briefly review the theory of Ricci limit spaces. Loosely speaking, Ricci limit spaces are the Gromov-Hausdorff closure of the space of Riemannian manifolds with a lower bound on the Ricci curvature. The study of Ricci limit spaces, that is metric spaces arising as Gromov-Hausdorff limits of Riemannian manifolds with a uniform lower Ricci curvature bound was carried out in [61, 62, 63], among others. The theory had many contributors since then, and is still an active field of research. For Ricci limits with a two-sided bound on the Ricci curvature a non-exhaustive list of works where such spaces have been studied is [26, 36, 181, 24, 60], other than the works just mentioned. For an overview of topic, see [152].

Let $\mathcal{M}_C$ be the space of p.m.m.s. which are locally doubling at the reference point with a non-decreasing function $C : (0, \infty) \to (0, \infty)$. A remarkable result due to Gromov in [110] shows the following.

Proposition 2.2.6 (Gromov compactness)**.** *The space* $(\mathcal{M}_C, \mathsf{d}_{pmGH})$ *is compact.*

Let us point out that, in view of proposition 2.2.6, $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ is never empty if $(X, \mathsf{d}, \mathfrak{m})$ is locally doubling. However, $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ may contain more than one element.

An important class of spaces which is pre-compact with respect to the pGH distance is the space of Riemannian manifolds having a fixed dimension and a positive uniform lower bound on the Ricci curvature.

We report the definition of Ricci limit spaces used in this manuscript below. Given a Riemannian manifold (M, g), where g denotes the Riemannian metric on M, we write d to mean the distance on M induced by g and sometimes refer to a Riemannian manifold as (M, d).

Definition 2.2.7 (Non-collapsed limits of manifolds with two-sided bounds on the Ricci curvature)**.** *Let* $\{(M_k, \mathsf{d}_k, x_k)\}_k$ *be a sequence of pointed Riemannian manifolds of fixed dimension* $N \geq 2$ *such that*

$$\mathrm{Ric}_{M_k} \geq -(N-1). \tag{2.2.2}$$

Suppose there exists a metric space (X, d, x) *such that* $M_k \xrightarrow{pGH} X$*. Then we say that* (X, d, x) *is a (pointed) Ricci limit space. If condition* (2.2.2) *is strengthened to*

$$|\mathrm{Ric}_{M_k}| \leq N-1, \tag{2.2.3}$$

we say that X is a limit of manifolds with two-sided bounds on the Ricci curvature. Moreover, if

$$\mathrm{vol}(B_1(x_k)) \geq v \tag{2.2.4}$$

holds for some $v > 0$, we say that (X, d, x) is non-collapsed.

As previously mentioned, Ricci limit spaces were studied extensively in [61, 62, 63]. In particular, it was shown that the non-collapsing assumption (2.2.4) forces the Hausdorff dimension of the limit space and that of the approximating sequence to be the same.

In [61] it was shown that for non-collapsed Ricci limit spaces all tangent spaces are metric cones. Given a metric space (X, d), the *metric cone over X* is the set

$$C(X) := (X \times [0, +\infty))/\sim \quad \text{with } (x, 0) \sim (y, 0) \text{ for every } x, y \in X,$$

equipped with the cone metric. The point $(x, 0) \in C(X)$ is called the tip of $C(X)$. See [49] and Section 2.3.8 for more on the topic.

Definition 2.2.8. *Let (X, d) be a Ricci limit space of dimension $N \in \mathbb{N}$. The regular set of X, denoted by $\mathcal{R}$, is the set of points $x \in X$ such that*

$$\mathrm{Tan}_x(X, \mathsf{d}) = \{(\mathbb{R}^N, \mathsf{d}_{\mathrm{eucl}})\}.$$

We call the set of singular points $\mathcal{S} := X \setminus \mathcal{R}$.

In the remainder of this section, we report some structure theory results which are relevant to the present thesis. The following result follows from [61].

Theorem 2.2.9. *Let (X, d, x) be a non-collapsed limit of manifolds with a uniform lower bound on the Ricci curvature of dimension $N \geq 2$. Then*

$$\dim_{\mathcal{H}}(\mathcal{S}) \leq N - 2.$$

We now report the notion of singular stratum of a Ricci limit space.

Definition 2.2.10 (Singular strata of Ricci limit spaces)**.** *Let (X, d) be a non-collapsed Ricci limit space. For every $k \in \mathbb{N}$ the k-singular stratum $\mathcal{S}_k(X)$ is the set of points $x \in X$ such that no tangent space is isometric to $\mathbb{R}^{k+1} \times Y$ for some metric space (Y, ρ).*

It was proved in [61] that the Hausdorff dimension of the k-singular stratum of a non-collapsed Ricci limit space $\mathcal{S}_k(X)$ is always less or equal to k. The analogous result in the more general case of non-collapsed RCD space is reported in Theorem 2.3.25.

Theorem 2.2.11. *Let (X, d) be a non-collapsed Ricci limit space and $k \geq 2$. Then*

$$\dim_{\mathcal{H}}(\mathcal{S}_k(X)) \leq k.$$

In [65], the authors further improved the analysis of the singular strata of non-collapsed Ricci limit spaces. We report their result here.

Theorem 2.2.12. *Let (X, d) be a non-collapsed Ricci limit space. Then*

- *$\mathcal{S}_k(X)$ is k-rectifiable;*
- *for $\mathcal{H}^k$-a.e. $x \in \mathcal{S}_k(X)$ every tangent cone at x splits an $\mathbb{R}^k$ factor isometrically.*

The structure of singular strata of non-collapsed limits of manifolds with two-sided bounds on the Ricci curvature was studied in [66]. A key result used in the proof of Theorem 4.1.1 of Chapter 4 is Theorem 2.2.13 below, which was shown in [66, Theorem 5.12]. Briefly put, it states that by also assuming an upper bound on the Ricci curvature we may exclude certain types of singularities from arising and improve the Hausdorff dimension estimate of the singular set.

Theorem 2.2.13. *Let (X, d, x) be a non-collapsed limit of manifolds with two-sided bounds on the Ricci curvature of dimension $N \geq 2$. For every $\alpha \in (0, 1)$, X is an open $C^{1,\alpha}$ Riemannian manifold outside of $\mathcal{S}_{N-4}$. In particular,*

$$\dim_{\mathcal{H}}(\mathcal{S}) \leq N - 4.$$

The following result follows from [24, 60].

Theorem 2.2.14. *For every $N \in \mathbb{N}$, $N \geq 2$ and $\delta, \alpha \in (0, 1)$, there exists $\epsilon > 0$ satisfying the following. Let (M^N, g, p) be a pointed Riemannian manifold with $|\mathrm{Ric}_{M_k}| \leq \epsilon$, and such that $B_2(p)$ is ϵ-close in the GH-distance to the Euclidean ball $B_2(0^N)$. Then there exists harmonic coordinates in M around p such that*

$$\|g_{ij} - \delta_{ij}\|_{C^{1,\alpha}(B_1(0^N))} \leq \delta.$$

2.2.3 Convergence of functions

We report here some definitions regarding the convergence of functions defined on a sequence of pointed metric (measure) spaces. See [18, 17] and the book [113] for a more detailed introduction to the topic. Throughout this section, we consider a sequence of pointed metric (measure) spaces $\{(X_k, \mathsf{d}_k, x_k)\}_k$ ($\{(X_k, \mathsf{d}_k, \mathfrak{m}_k, x_k)\}_k$) converging in the pointed (measured) Gromov-Hausdorff sense to a pointed metric (measure) space (X, d, x) ($(X, \mathsf{d}, \mathfrak{m}, x)$). We also denote by (Z, d_Z) and $i_k : X_k \to Z, i : X \to Z$ the space and the embeddings realizing the convergence, respectively.

Definition 2.2.15 (Pointwise and uniform convergence)**.** *Let $f_k : X_k \to \mathbb{R}$ for $k \in \mathbb{N}$ and $f : X \to \mathbb{R}$. We say that $\{f_k\}_k$ converges to f pointwise if $f_k(x_k) \to f(x)$ for every sequence of points $x_k \in X_k$, $x \in X$ such that $i_k(x_k) \to i(x)$ in Z. Moreover, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f_k(x_k) - f(x)| \leq \varepsilon$ for every $k \geq \frac{1}{\delta}$ and $x_k \in X_k$, $x \in X$ with $\mathsf{d}_Z(i_k(x_k), i(x)) \leq \delta$, then we say that $f_k \to f$ uniformly.*

The following result is obtained similarly to the classical Ascoli-Arzelà theorem. Its proof can be found in [184, Proposition 27.20].

Proposition 2.2.16 (Ascoli-Arzelà)**.** *Fix $R, L > 0$. For any sequence of L-Lipschitz functions $f_k : B_R^{X_k}(x_k) \to \mathbb{R}$ such that $\sup_k |f_k(x_k)| < \infty$ there exists a subsequence that converges uniformly to some function $f : B_R^X(x) \to \mathbb{R}$ which is also L-Lipschitz.*

We now report the definition of L^p convergence along a converging sequence of pointed metric (measure) spaces. This notion of convergence was first studied in [104].

Definition 2.2.17 (L^p convergence)**.** *Let $p \in [1, \infty)$. We say that the sequence $\{f_k\}_k$, $f_k \in L^p(X_k)$ converges L^p-weakly to $f \in L^p(X)$ if $\limsup_{k\to\infty} \|f_k\|_{L^p(X_k)} < \infty$ and $f_k\mathfrak{m}_k \rightharpoonup f\mathfrak{m}$ as $k \to \infty$ weakly in duality with $C_{bs}(Z)$, where (Z, d_Z) is the space realizing the pointed measured Gromov-Hausdorff convergence of $\{X_k\}_k$ to X.*

Moreover, we say that the sequence $\{f_k\}_k$, converges L^p-strongly if

$$\limsup_{k\to\infty} \|f_k\|_{L^p(X_k)} = \|f\|_{L^p(X)}$$

also holds.

An analogous type of convergence can be defined for Sobolev functions.

Definition 2.2.18 ($H^{1,2}$ convergence)**.** *We say that $\{f_k\}_k$, $f_k \in H^{1,2}(X_k)$ converges to $f \in H^{1,2}(X)$ in the $H^{1,2}$-weak sense if $\{f_k\}_k$ converges in the L^2-strong sense to f and $\sup_k Ch_k(f_k) < \infty$.*

Furthermore, we say that $\{f_k\}_k$ converges in the $H^{1,2}$-strong sense if it converges in the $H^{1,2}$-weak sense and $Ch_k(f_k) \to Ch(f)$ as $k \to \infty$.

2.3 RCD metric measure spaces

In this section, we give a brief introduction to the topic of RCD spaces. The field has progressed rapidly since its inception, and it is now a deep theory. It is beyond the scope of this manuscript to provide an exhaustive survey of the topic. We refer to [7] and the references therein for an overview of the subject.

Briefly put, CD spaces are metric measure spaces with a synthetic notion of lower bounds on the Ricci curvature. The lower bounds are enforced using optimal transport tools by assuming that "given two measures absolutely continuous with respect to the ambient measure, the entropy along the geodesic connecting the two measures is convex". In the smooth setting, this constitutes a characterization of lower Ricci curvature bounds that does not require "taking derivates". The RCD condition is a refinement of the CD condition that rules out Finslerian geometries.

In the remainder of this section, we report some properties of RCD condition relevant to this thesis. We start by recalling its stability with respect to the pointed measured Gromov-Hausdorff distance and a splitting theorem. We then report some regularity theory that these spaces enjoy. We also recall some of the additional properties that the heat flow and the convergence of functions along sequences converging in the GH sense enjoy in the more regular setting (as compared to the bare metric setting) of RCD spaces. We then recall the construction and properties of metric cones over RCD spaces, a crucial component for the analysis of the singular sets presented in this thesis. Lastly, we revise the definition of tangent modules, which correspond to "vector fields" defined in an L^2 sense. Introduced by Gigli, they are an important tool for the analysis of perimeter minimizing sets in RCD spaces.

2.3.1 Curvature dimension conditions

In this section we briefly introduce the curvature dimension condition (commonly referred to as CD condition). The CD condition provides a synthetic notion of Ricci curvature lower bounds. The spaces satisfying these conditions were partly inspired by Alexandrov spaces, which correspond to metric spaces with a synthetic definition of lower sectional curvature bounds. The starting point of the CD theory is [138], in which displacement convexity was shown in $\mathbb{R}^N$. The work in [156] further developed the theory. Subsequently, the Riemannian manifolds case was covered in [68, 163]. This motivated Lott, Villani in [132] and independently Sturm in [179, 180] to introduce the CD condition 2.3.2.

For the rest of this thesis, we implicitly suppose that every metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies the following volume growth condition:

$$\mathfrak{m}(B_r(x)) \leq a \exp^{br^2} \qquad \text{for all } x \in X \text{ and } r > 0,$$

for some $x \in X$, $a, b \geq 0$ and for all $r > 0$.

The logarithmic relative entropy $\mathcal{E}^\infty : \mathcal{P}_2(X) \to (-\infty, \infty]$ is

$$\mathcal{E}^\infty(\mu) := \begin{cases} \int_X \rho \log(\rho) d\mathfrak{m} & \text{if } \mu = \rho\mathfrak{m}, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.3.1}$$

Definition 2.3.1 (CD(K,∞) condition)**.** *Let $K \in \mathbb{R}$. A m.m.s. satisfies the* CD(K,∞) *condition if $\mathcal{E}^\infty$ is geodesically K-convex on $(\mathcal{P}_2(X), W_2)$, that is, if for every μ_0, $\mu_1 \in \mathcal{P}_2(X)$ with $\mathcal{E}^\infty(\mu_0)$, $\mathcal{E}^\infty(\mu_1) < \infty$, there exists $(\mu_t)_{t\in[0,1]} \in \mathrm{Geo}(\mathcal{P}_2(X))$ connecting μ_0 to μ_1 such that*

$$\mathcal{E}^\infty(\mu_s) \leq (1-s)\mathcal{E}^\infty(\mu_0) + s\mathcal{E}^\infty(\mu_1) - K\frac{s(1-s)}{2}W_2^2(\mu_0, \mu_1). \tag{2.3.2}$$

An important regularity property shown in [161] is that CD(K,∞) m.m.s. satisfy the 1-1 Poincaré inequality (2.1.1).

The CD condition was extended to cover the case of an upper bound on the dimension. Given $N \in (1,\infty)$ we say that the Rényi relative entropy $\mathcal{E}^N : \mathcal{P}_2(X) \to [-\infty, 0]$ is

$$\mathcal{E}^N(\mu) := -\int_X \rho^{1-\frac{1}{N}} d\mathfrak{m} \tag{2.3.3}$$

where $\mu = \rho\mathfrak{m} + \mu^\perp$ is the singular decomposition of μ with respect to $\mathfrak{m}$.

In order to report the CD(K,N) condition for finite N, we need to recall the following functions. Given $K, N \in \mathbb{R}$, $\theta \in [0,+\infty)$ and $t \in [0,1]$, the functions $\tau_{K,N}^{(t)}(\theta)$ are defined as

$$\tau_{K,N}^{(t)}(\theta) := \begin{cases} +\infty & \text{if } K\theta^2 \geq (N-1)\pi^2, \\ t^{1/N}\left(\sin\left(\sqrt{\frac{K}{N-1}}t\theta\right)/\sin\left(\sqrt{\frac{K}{N-1}}\theta\right)\right)^{1-1/N} & \text{if } 0 < K\theta^2 < (N-1)\pi^2, \\ t & \text{if } K\theta^2 = 0 \\ & \text{ or } K\theta^2 < 0 \text{ and } N = 1, \\ t^{1/N}\left(\sinh\left(\sqrt{\frac{-K}{N-1}}t\theta\right)/\sinh\left(\sqrt{\frac{-K}{N-1}}\theta\right)\right)^{1-1/N} & \text{if } K\theta^2 < 0 \text{ and } N > 1. \end{cases}$$

Definition 2.3.2 (CD(K,N) condition)**.** *Let $K \in \mathbb{R}$ and $N \in (1,\infty)$. A m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ satisfies the* CD(K,N) *condition if for any $\mu_0 = \rho_0\mathfrak{m}$, $\mu_1 = \rho_1\mathfrak{m} \in \mathcal{P}_2(X)$ with compact support there exists $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ such that $(e_t)_\sharp\nu \ll \mathfrak{m}$ for all $t \in [0,1]$ and*

$$\mathcal{E}^{N'}(\mu_t) \leq \\ -\int_X \left[\tau_{K,N'}^{(1-t)}(\mathsf{d}(\gamma(0),\gamma(1)))\rho_0^{-\frac{1}{N'}}(\gamma(0)) + \tau_{K,N'}^{(t)}(\mathsf{d}(\gamma(0),\gamma(1)))\rho_1^{-\frac{1}{N'}}(\gamma(1))\right] d\nu(\gamma),$$

for any $N' \geq N$ and $t \in [0,1]$, where $\mu_t := (e_t)_\sharp\nu$.

An important geometric property $\mathrm{CD}(K,N)$ metric measure spaces satisfy is the Bishop-Gromov inequality (see [179]), namely

$$\frac{B_R(x)}{B_r(x)} \leq \frac{V_{K,N}(R)}{V_{K,N}(r)}, \tag{2.3.4}$$

for all $0 < r < R$ and $x \in X$, where $V_{K,N}(s)$ is the volume of a ball of radii $s > 0$ in the model space of dimension N and constant Ricci curvature K. As a consequence, $\mathrm{CD}(K,N)$ spaces are locally doubling and proper. Moreover, if $K \geq 0$, they are doubling with constant 2^N:

$$\mathfrak{m}(B_{2r}(x)) \leq 2^N \mathfrak{m}(B_r(x))$$

for all $x \in X$ and $r > 0$.

Lastly, let us point out the following scaling property of the CD condition.

Remark 2.3.3. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a $\mathrm{CD}(K,N)$ space and let $c_1, c_2 > 0$. Then the space $(X, c_1\mathsf{d}, c_2\mathfrak{m})$ satisfies the $\mathrm{CD}(K/c_1^2, N)$.*

2.3.2 RCD condition

In this section we briefly recall the Riemannian curvature dimension (RCD) condition. We also report some properties that follow from such condition which we use in the following chapters. The RCD condition was first introduced in the $N = \infty$ case in [16] and then proposed in the $N < \infty$ case in [99]. We refer the reader to the original papers [16, 99, 23, 89, 20, 54], or to the survey [7] for more details.

Definition 2.3.4 (RCD condition)**.** *A m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathrm{RCD}(K,N)$ (respectively, the $\mathrm{RCD}(K,\infty)$) condition if $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathrm{CD}(K,N)$ (respectively, the $\mathrm{CD}(K,\infty)$) and if it is infinitesimally Hilbertian.*

There are equivalent definitions to the RCD condition we have reported. Most notably, we have the following "Eulerian" equivalent condition. See [15], where the equivalence was shown for the case $N = \infty$ and [89, 20] for the $N < \infty$ case.

Theorem 2.3.5 (Equivalence of RCD and BE)**.** *A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathrm{RCD}(K,N)$ condition if and only if*

- *it is infinitesimally Hilbertian;*
- *any $f \in H^{1,2}(X)$ with $|\nabla f|(x) \leq 1$ for $\mathfrak{m}$-a.e. $x \in X$ admits a Lipschitz representative with Lipschitz constant 1;*

- *a weak Bochner inequality is satisfied: for any $f \in D(\Delta)$ with $\Delta f \in H^{1,2}(X)$ it holds*

$$\frac{1}{2}\int_X |\nabla f|^2 \Delta g \, d\mathfrak{m} \geq \int_X \left(\nabla f \cdot \nabla \Delta f + K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2 \right) g \, d\mathfrak{m}$$

for any $g \in D(\Delta) \cap L^\infty(X)^+$ with $\Delta g \in L^\infty(X)$.

Remark 2.3.6. *The Eulerian condition was originally shown to be equivalent to a variant of the* $\mathrm{RCD}(K, N)$ *condition known as* $\mathrm{RCD}^*(K, N)$*. In the search for better globalization properties, Bacher and Sturm [35] introduced the* $\mathrm{CD}^*(K, N)$ *condition. Even though* $\mathrm{CD}^*(K, N)$ *is weaker than* $\mathrm{CD}(K, N)$*, the two conditions were shown to be equivalent under the essentially non-branching assumption. Given a metric space (X, d), a geodesic $\gamma : [0, 1] \to X$ is said to be non-branching if the existence of $\tilde{\gamma} \in \mathrm{Geo}(X)$ and $t \in (0, 1)$ such that $\gamma|_{[0,t]} = \tilde{\gamma}_{[0,t]}$ implies $\gamma = \tilde{\gamma}$. A m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ is essentially non-branching if for any $\mu_1, \mu_2 \in \mathcal{P}_2(X)$, which are both absolutely continuous with respect to $\mathfrak{m}$, any optimal transport map $\mu \in \mathrm{OptGeo}(\mu_1, \mu_2)$ is concentrated on a set of non-branching geodesics. This assumption was also introduced in [162], and it is not stable under GH convergence. The essentially non-branching assumption was shown to be necessary for the equivalence to hold in [160]. In [54] the authors proved that in essentially non-branching spaces having finite reference measure, the* $\mathrm{CD}^*(K, N)$ *condition is equivalent to the* $\mathrm{CD}(K, N)$ *condition. This result was extended to the case of σ-finite measures in [130]. It follows from [23, 72, 162] that* $\mathrm{RCD}(K, N)$ *spaces are essentially non-branching. More recently, in [84] it was shown that* $\mathrm{RCD}^*(K, N)$ *are actually non-branching. Therefore,* $\mathrm{RCD}(K, N)$ *and* $\mathrm{RCD}^*(K, N)$ *are equivalent.*

2.3.3 Some properties of RCD spaces

Let us illustrate a few properties enjoyed by $\mathrm{RCD}(K, N)$ spaces. For the purposes of this thesis we are only concerned with finite dimensional RCD spaces. Therefore, we focus on the case $N < \infty$, unless explicitly stated. Let us remark once again that it goes beyond the scope of this thesis to be exhaustive. For a survey on the topic, see [7] and the references therein.

We begin by pointing out that the $\mathrm{RCD}(K, N)$ condition is stable under pmGH convergence. The following result was obtained in [104].

Theorem 2.3.7 (Stability of the RCD condition)**.** *Let $(X_k, \mathsf{d}_k, \mathfrak{m}_k, x_k)$ be* $\mathrm{RCD}(K_k, N_k)$ *spaces, $k \in \mathbb{N}$. Suppose that $K_k \to K$ and $N_k \to N$ for some $K \in \mathbb{R}$ and $N \geq 1$*

and that the sequence $\{(X_k, \mathsf{d}_k, \mathfrak{m}_k, x_k)\}_k$ *converges to some p.m.m.s.* $(X, \mathsf{d}, \mathfrak{m}, x)$ *as* $k \to \infty$*. Then* $(X, \mathsf{d}, \mathfrak{m})$ *is an* $\mathrm{RCD}(K, N)$ *space.*

Remark 2.3.8. *It follows from Gromov's compactness argument and the scaling properties of the* $\mathrm{RCD}(K, N)$ *that* $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ *is non-empty for any* $\mathrm{RCD}(K, N)$ *space* $(X, \mathsf{d}, \mathfrak{m})$ *and* $x \in X$*.*

However, tangent cones may not be unique. They do not even need to be homeomorphic.

Example 2.3.9 ([67])**.** *There exists a Ricci limit space of* (X^5, d) *such that there exists* $x \in X^5$ *with* $\mathrm{Tan}_x(X^5, \mathsf{d})$ *containing both* $\mathbb{R}^5$ *and the cone with* $(\mathrm{CP}^2 \# \overline{\mathrm{CP}^2})$ *as cross-section.*

A crucial geometrical result obtained in [100] is the splitting theorem in the RCD setting reported below. See [60] for an analogous result in the setting of Ricci limits.

Theorem 2.3.10 (Splitting)**.** *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be an* $\mathrm{RCD}(0, N)$ *metric measure space. Suppose that* X *contains a line, that is a curve* $\gamma : \mathbb{R} \to X$ *which is minimizing between any two of its points. Then there exists an* $\mathrm{RCD}(0, N-1)$ *metric measure space* (Y, ρ, μ) *such that* $(X, \mathsf{d}, \mathfrak{m})$ *is isomorphic to* $(\mathbb{R} \times Y, \mathsf{d}_{\mathrm{eucl}} \times \rho, \mathcal{L}^1 \times \mu)$*.*

For the purposes of this thesis, we use a slightly different version of the theorem above, which we report below.

Lemma 2.3.11 ([28, Lemma 1.21])**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(0, N)$ *space and let* $g \in D(\Delta)$ *be a function such that* $\Delta g = 0$ *and* $|\nabla g| = 1$*. Then* $(X, \mathsf{d}, \mathfrak{m})$ *is isomorphic to* $(Y \times \mathbb{R}, \mathsf{d}_Y \times \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_Y \times \mathcal{L}^1)$ *where* $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ *is an* $\mathrm{RCD}(0, N-1)$ *space.*

We sometimes say that a metric measure space ($\mathrm{RCD}(0, N)$ space) $(X, \mathsf{d}, \mathfrak{m})$ splits off a Euclidean factor when it is isomorphic to $(Y \times \mathbb{R}, \mathsf{d}_Y \times \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_Y \times \mathcal{L}^1)$ for some metric measure measure space ($\mathrm{RCD}(0, N-1)$ space) $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$.

2.3.4 Regularity theory of RCD metric measure spaces

In this section, we state a structure theory result which is the culmination of many different works [142, 47] and others. Let us first recall a few definitions.

Definition 2.3.12 (Regular points)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space with* $N \in \mathbb{N}$ *and let* $n \in \mathbb{N}$*. The set of* n*-regular points is*

$$\mathcal{R}_n := \{x \in X : \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}) = \{(\mathbb{R}^n, \mathsf{d}_{\mathrm{eucl}}, \mathcal{H}^n)\}\}$$

We call $\mathcal{R}_N$ the set of regular points, which we denote by $\mathcal{R}$. The set of singular points is $\mathcal{S} := X \setminus \mathcal{R}$.

Let us point out that the set of regular points $\mathcal{R}_n$ is Borel.

Definition 2.3.13 (Strong rectifiability). *Given $N \in \mathbb{N}$, a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ is said to be N-strongly rectifiable if for every $\varepsilon > 0$ there exists a countable collection $\{U_k\}_k$ of Borel subsets of X and of $(1+\varepsilon)$-bi-Lipschitz maps $\varphi_k : U_k \to \mathbb{R}^N$ such that*

$$\mathcal{H}^N(X \setminus (\cup_k U_k)) = 0.$$

Theorem 2.3.14 (Structure theory, [103, 81, 119, 142, 47, 106]). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *space with $K \in \mathbb{R}$, $N \geq 1$. Then there exists a unique $1 \leq n \leq N$, called the essential dimension of $(X, \mathsf{d}, \mathfrak{m})$, such that*

- *for $\mathfrak{m}$-a.e. $x \in X$, $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}) = \{(\mathbb{R}^n, \mathsf{d}_{\mathrm{eucl}}, \mathcal{H}^n)\}$;*
- *X is n-strongly rectifiable;*
- *there exists a non-negative density $\theta \in L^1_{\mathrm{loc}}(X, \mathcal{H}^n \llcorner \mathcal{R}_n)$ such that*

$$\mathfrak{m} = \theta \mathcal{H}^n \llcorner \mathcal{R}_n.$$

Let us remark that, for an $\mathrm{RCD}(K, N)$ metric measure space $(X, \mathsf{d}, \mathfrak{m})$ of essential dimension n, $(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, \mathcal{H}^k) \notin \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ for any $k > n$. This follows from the lower semicontinuity of the essential dimension [45, 124].

The essential dimension and the Hausdorff dimension of an $\mathrm{RCD}(K, N)$ space do not always coincide, as shown in [158].

Let us also point out the following dimensional gap result taken from [80, Theorem 1.4].

Theorem 2.3.15 (Dimensional gap). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *space with $N \in \mathbb{N}$. Then either $\dim_{\mathcal{H}}(X) = N$ or $\dim_{\mathcal{H}}(X) \leq N - 1$.*

RCD spaces also enjoy tensorization properties [16, 21, 132] and admit a universal cover [145]. Moreover, a fully developed second order calculus was developed by Gigli in [102]. See also [83].

2.3.5 Heat flow in RCD spaces

We now present some of the properties of the heat flow in $\mathrm{RCD}(K,\infty)$ spaces. The books [107, 10] provide excellent introductions to the topic. In the last part of this section, we recall some other important properties of RCD metric measure spaces and provide some references on the topic.

In [16, 23] it was shown that on $\mathrm{RCD}(K,\infty)$ metric measure spaces the dual heat semigroup $\bar{P}_t : (\mathcal{P}_2(X), W_2) \to (\mathcal{P}_2(X), W_2)$ of P_t defined by

$$\int_X f \, d\bar{P}_t\mu := \int_X P_t f \, d\mu \quad \text{for every } \mu \in \mathcal{P}_2(X) \text{ and } f \in \mathrm{LIP}_b(X),$$

is K-contractive, and maps probability measures into probability measures absolutely continuous with respect to the ambient measure for any $t > 0$. The authors in [16, 23] also showed some regularization properties of the heat flow on $\mathrm{RCD}(K,\infty)$ spaces. In particular, they were able to show the Bakry-Émery contraction estimate:

$$|\nabla P_t f|^2 \leq \exp^{-2Kt} P_t |\nabla f|^2 \quad \mathfrak{m}\text{-a.e.}$$

for any $t > 0$ and any function $f \in H^{1,2}(X)$. It was later proved in [166] that the Bakry-Émery contraction estimate extends to all exponents $p \in [1,\infty)$.

Another important regularization property of the heat flow is the L^∞-LIP regularization. Namely, for any $f \in L^\infty(X)$ it holds $P_t f \in \mathrm{LIP}(X)$ with

$$\sqrt{2\int_0^t \exp^{2Ks} ds}\, \mathrm{LIP}(P_t f) \leq \|f\|_{L^\infty(X)} \quad \text{for any } t > 0.$$

Lastly, we recall the Sobolev to Lipschitz property: any $f \in H^{1,2}(X)$ with $|\nabla f| \in L^\infty(X)$ admits a Lipschitz representative $\bar{f}$ such that $\mathrm{LIP}(\bar{f}) \leq \|\nabla f\|_{L^\infty(X)}$.

2.3.6 Stability properties of functions

Let $\{K_k\}_k \subset \mathbb{R}$ be a sequence converging to some $K \in \mathbb{R}$ as $k \to \infty$. In this section, we are concerned with the stability properties of functions along a fixed sequence of $\mathrm{RCD}(K_k,\infty)$ p.m.m.s. $\{(X_k, \mathsf{d}_k, \mathfrak{m}_k, x_k)\}_k$ converging to an $\mathrm{RCD}(K,N)$ p.m.m.s. $(X, \mathsf{d}, \mathfrak{m}, x)$ in the pmGH sense as $k \to \infty$. We assume that the convergence is realized by means of isometric embeddings of the spaces into a common separable metric space (Z, d_Z) and denote the corresponding embeddings by $i_k : X_k \to Z$. The main references for this section are [104, 18, 17].

We report [18, Proposition 3.3] here. See also [104, 114].

Proposition 2.3.16. *Let $p \in [1,\infty)$. The following hold:*

- *Let $\varphi \in \mathrm{LIP}(\mathbb{R})$ with $\varphi(0) = 0$. If f_k converge L^p-strongly to f, then $\varphi \circ f_k$ converge L^p-strongly to $\varphi \circ f$.*
- *If f_k, g_k converge L^p-strongly to f, g respectively, then $f_k + g_k$ converge L^p-strongly to $f + g$.*
- *If f_k, g_k converge L^2-strongly to f, g respectively, then $f_k g_k$ converge L^1-strongly to fg.*
- *If $\{f_k\}_k$ is uniformly bounded in L^∞ and is L^1-strongly convergent to f then $\lim_{k\to\infty} \|f_k\|_{L^p(X_k)} = \|f\|_{L^p(X)}$ for all $p \in [1,\infty)$.*

Proposition 2.3.17 ([18, Corollary 5.5])**.** *The following stability results hold:*

- *If $f_k \in H^{1,2}(X_k)$, $f_k \in D(\Delta_k)$ for all $k \in \mathbb{N}$ converge in L^2-strong to f and $\Delta_k f_k$ are uniformly bounded in L^2, then $f \in D(\Delta)$, $\Delta_k f_k$ converge L^2 weakly to Δf and f_k converge $H^{1,2}$ strongly to f.*
- *If f_k converge L^2 strongly to f, then $P_t f_k$ converge $H^{1,2}$ strongly to $P_t f$ for all $t > 0$.*

Proposition 2.3.18 ([18, Theorem 5.7])**.** *Let $f_k, g_k \in H^{1,2}(X_k)$ converge to $f, g \in H^{1,2}(X)$ in the $H^{1,2}$ strong sense, respectively. Then $\nabla f_k \cdot \nabla g_k$ converge to $\nabla f \cdot \nabla g$ in the L^1 strong sense.*

Theorem 2.3.19 ([104, Theorem 6.3], [18, Theorem 7.4])**.** *Suppose that $f_k \in H^{1,2}(X_k)$ are such that*

$$\sup_k \|f_k\|_{H^{1,2}(X_k)} < \infty.$$

Moreover, suppose that

$$\lim_{R\to\infty} \limsup_{k\to\infty} \int_{Z\setminus B_R(\bar z)} |f_k|^2 d((i_k)_\# \mathfrak{m}_k) = 0 \quad \textit{for some } \bar z \in Z.$$

Then there exists a subsequence, that we do not relabel, and a function $f \in H^{1,2}(X)$ such that f_k converge strongly in L^2 to f.

Lastly, let us report some useful results regarding local Sobolev convergence.

Lemma 2.3.20 ([17, Lemma 2.10])**.** *Let $f \in \mathrm{LIP}_c(B_R(x), \mathsf{d})$. Then there exists a sequence of functions $f_k \in \mathrm{LIP}_c(B_R(x_k), \mathsf{d}_k)$ satisfying*

$$\sup_k \||\nabla f_k|\|_{L^\infty(X_k)} < \infty$$

and converging to f strongly in the $H^{1,2}$ sense.

Theorem 2.3.21 ([17, Theorem 4.4]). *Suppose* $f_k \in D(\Delta_k, B_R(x_k))$ *with*

$$\sup_k(\|f_k\|_{H^{1,2}(B_R(x_k))} + \|\Delta f_k\|_{L^2(B_R(x_k))}) < \infty.$$

Let us also assume that there exists $f \in L^2(X)$ *such that* f_k *converge to* f *strongly in* L^2 *on* $B_R(x)$. *Then:*

- $f \in D(\Delta, B_R(y))$;
- $\Delta f_k \to \Delta f$ *on* $B_R(x)$ *weakly in* L^2;
- $|\nabla f_k|^2 \to |\nabla f|^2$ *on* $B_R(x)$ *strongly in* L^1.

2.3.7 Non-collapsed RCD spaces

We briefly mention the class of non-collapsed RCD metric measure spaces, which were introduced in [80]. Briefly put, non-collapsed $\mathrm{RCD}(K, N)$ m.m.s. have $\mathcal{H}^N$ as ambient measure and their essential dimension coincides with N.

Definition 2.3.22 (Non-collapsed RCD condition). *A m.m.s.* $(X, \mathsf{d}, \mathfrak{m})$ *satisfying the* $\mathrm{RCD}(K, N)$ *condition is said to be non-collapsed if* $\mathfrak{m} = \mathcal{H}^N$.

We start by reporting a volume rigidity result on non-collapsed spaces.

Theorem 2.3.23 (Volume rigidity, [80, Corollary 1.7]). *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be a non-collapsed* $\mathrm{RCD}(0, N)$ *space. Then for all* $x \in X$ *and* $r > 0$,

$$\mathcal{H}^N(B_r(x)) \leq \omega_N r^N. \tag{2.3.5}$$

Furthermore, if there exists $\bar{x} \in X$ *and* $\bar{r} > 0$ *such that the equality is achieved in* (2.3.5), *then* $B_{\bar{r}/2}(\bar{x})$ *is isometric to* $B^{\mathbb{R}^N}_{\bar{r}/2}(0)$. *Consequently,* $x \in \mathcal{R}$ *if and only if*

$$\lim_{r \to 0} \frac{\mathcal{H}^N(B_r(x))}{\omega_N r^N} = 1.$$

That is, $\mathrm{Tan}_x(X, \mathsf{d}, \mathcal{H}^N) = \{(\mathbb{R}^N, \mathsf{d}_{\mathrm{Eucl}}, \mathcal{L}^N)\}$ *if and only if the volume of the balls centered at* $x \in X$ *converges to the volume of balls in* $\mathbb{R}^N$.

Let us report the notion of strata of the singular set. In the Euclidean setting, this concept dates back to Federer [90]. Their study proved crucial in improving our understanding of the structure of the singular set.

Definition 2.3.24 (Singular Strata)**.** *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be an* $\mathrm{RCD}(K, N)$ *metric measure space and* $k = 1, ..., N$*. Then we define the* k*-singular stratum to be*

$$\mathcal{S}_k(X) := \{x \in X : \text{ for every tangent space } (Y, \rho, \mu, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}) \\ \mathsf{d}_{\mathrm{GH}}(\bar{B}_1^Y(y), \bar{B}_1^{\mathbb{R}^{k+1}\times Z}(0^{k+1}, z)) > 0 \text{ for any pointed space } (Z, \mathsf{d}_Z, z)\}$$

Let us point out that the following chain of inclusions holds:

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \ ... \subset \mathcal{S}_{N-1} \subset \mathcal{S}_{N-1} = \mathcal{S}.$$

In the non-collapsed setting, an estimate on the Hausdorff dimension of the singular strata holds.

Theorem 2.3.25 (Stratification, [80, Theorem 1.8])**.** *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be an* $\mathrm{RCD}(K, N)$ *metric measure space and* $k = 1, ..., N$*. Then*

$$\dim_{\mathcal{H}} \mathcal{S}_k \leq k.$$

We conclude this section by collecting some results regarding the boundary of non-collapsed $\mathrm{RCD}(K, N)$ metric measure spaces. See [80, 118, 43].

Definition 2.3.26 (Boundary)**.** *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be an* $\mathrm{RCD}(K, N)$*. The boundary of* X *is*

$$\partial X = \overline{\mathcal{S}_{N-1} \setminus \mathcal{S}_{N-2}}.$$

In [43] it was shown that this notion (proposed in [80]) is equivalent to that found in [118] in the following sense: the boundary in the former sense is empty if and only if the boundary in the latter sense is empty.

Theorem 2.3.27 (Boundary structure, [43, Theorem 1.4])**.** *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be an* $\mathrm{RCD}(-(N-1), N)$ *space with* $x \in X$ *such that* $\mathcal{H}^N(B_1(x)) > v > 0$*. Then, if* $(\mathcal{S}_{N-1} \setminus \mathcal{S}_{N-2}) \cap B_2(x) \neq \emptyset$*, the following hold*

- ∂X *is* $N-1$ *rectifiable and* $\mathcal{H}^{N-1}(B_r(x) \cap \partial X) \leq C(N, v) r^{N-1}$ *for any* $x \in \partial X \cap B_1(x)$ *and* $r \in (0, 1)$*;*
- $\mathcal{H}^N(B_r(\partial X)) \leq C r^{N-1}$ *for some* $C = C(n, v) > 0$ *and for any* $x \in X$ *and* $r \in (0, 1)$*;*
- *for any* $x \in \mathcal{S}_{N-1} \setminus \mathcal{S}_{N-2}$ *the tangent cone at* x *is unique and isomorphic to* $\mathbb{R}^N_+$*;*
- *for any* $0 < \alpha < 1$*, there exists a closed set* $C_\alpha \subset \mathcal{S}_{N-2}(X)$ *such that*

  1. $\dim_{\mathcal{H}}(X \setminus C_\alpha) \leq N - 2$*;*

2. *$X \setminus C_\alpha$ is a topological manifold with boundary and C^α-charts.*

We remark that Theorem 2.3.27 fails in codimension greater than 1. Specifically:

- The measure estimate for the full singular stratum and the volume estimates for the tubolar neighborhood of the full singular stratum fail, as there exists a two-dimensional Alexandrov spaces whose singular set $\mathcal{S}_0$ does not have locally finite $\mathcal{H}^0$-measure (see [65, Section 3.4]).
- In [67, Theorem 1.2] the authors showed that there exists a non-collapsed Ricci limit space with a point $x \in \mathcal{S}_{N-2} \setminus \mathcal{S}_{N-3}$ with multiple tangent cones.
- In [65, Example 3.2] the authors report an example, based on [128], of an N-dimensional Alexandrov spaces such that $\mathcal{S}_{N-2}$ is a Cantor set, which implies that no point has a neighborhood in which $\mathcal{S}_{N-2}$ is a topological manifold.

See the discussion in [43] after Theorem 1.4.

2.3.8 Metric cones

In this section we recall the construction of metric (measure) cones, that is cones having a metric measure space as cross-section endowed with an appropriate cone measure. For an introduction to the topic, see [49].

We start by recalling the definition of a warped metric measure space. For more on the subject, including tensorization properties of the RCD spaces, see [16, 21, 102], among others. We follow [102] closely in our presentation here.

Definition 2.3.28 (Warped length of curves)**.** *Let (X, d), (Y, ρ) be two metric spaces, and let $\omega_{\mathsf{d}} : X \to [0, \infty)$ be a continuous function. Let $\gamma = (\gamma^X, \gamma^Y) \in \mathrm{AC}([0,1]; X) \times \mathrm{AC}([0,1]; Y)$. The warped length of curves on the product space $X \times Y$ is*

$$l_{\omega_{\mathsf{d}}} := \sup_{0<t_0<\ldots<t_N<1} \sum_{i=1}^{N} \sqrt{\mathsf{d}(\gamma^X(t_{i-1}), \gamma^X(t_i))^2 + \omega_{\mathsf{d}}(\gamma^X(t_{i-1}))\rho(\gamma^Y(t_{i-1}), \gamma^Y(t_i))^2}$$
$$= \int_0^1 \sqrt{|\dot{\gamma}^X|^2(t) + \omega_{\mathsf{d}}(\gamma^X(t))|\dot{\gamma}^Y|^2(t)}dt.$$

Suppose now that X, Y are length spaces, that is metric spaces in which the distance between two points is given by the infimum of the lengths of curves that join them. The following holds

$$\tilde{\mathsf{d}}((x, y), (x_1, y_1)) := \inf l_{\omega_{\mathsf{d}}},$$

where $(x,y),(x_1,y_1) \in X \times Y$ and the infimum is taken over $\gamma \in \mathrm{AC}([0,1]; X \times Y)$ with $\gamma(0) = (x,y)$ and $\gamma(1) = (x_1,y_1)$. It can be shown that $\tilde{\mathsf{d}}$ defines a pseudo-distance on $X \times Y$. In order to get the definiteness property of distances we need to consider the quotient space $X \times_\omega Y := X \times Y/\sim$, where $(x,y) \sim (x_1,y_1)$ if and only if $\mathsf{d}_\omega((x,y),(x_1,y_1)) = 0$.

We now recall the definition of warped product of metric measure spaces.

Definition 2.3.29 (Warped metric measure spaces). *Let $(X,\mathsf{d},\mathfrak{m})$, (Y,ρ,μ) be two metric measure spaces, and let $\omega_\mathsf{d}, \omega_\mathfrak{m} : X \to [0,\infty)$ be continuous functions. The ω_d-$\omega_\mathfrak{m}$ warped product of the metric measure spaces $(X,\mathsf{d},\mathfrak{m})$ and (Y,ρ,μ) is the metric measure space*

$$(X \times_\omega Y, \mathsf{d}_\omega, \mathfrak{m}_\omega),$$

where $(X \times_\omega Y, \mathsf{d}_\omega)$ is defined as in definition 2.3.28 and $\mathfrak{m}_\omega := \pi_*(\omega_\mathfrak{m}\mathfrak{m} \times \mu)$, where $\pi : X \times Y \to X \times_\omega Y$ is the projection map.

An N-metric measure cone over a measure metric space $(X,\mathsf{d}_X,\mathfrak{m}_X)$ is defined as the warped product

$$C(X) := ([0,\infty) \times_C X, \mathsf{d}_C, \mathfrak{m}_C)$$

obtained with $\omega_\mathsf{d}(r) = r^2$ and $\omega_\mathfrak{m}(r) = r^{N-1}$. See [102] for some background.
We denote by $O \in C(X)$ the tip of the cone, given by $\{O\} := \pi(\{0\} \times X) \subset C(X)$. In what follows we will use a slight abuse of notation and denote $\pi(t,x)$ by (t,x). In particular, $O = (0,x)$ for any $x \in X$. Moreover, we shall adopt the more intuitive notation d_C, $\mathfrak{m}_C$ to denote the distance and the reference measure on $C(X)$, when there is no risk of confusion.

There is an explicit expression for the distance between two points on a cone:

$$\mathsf{d}_C^2((r_1,x_1),(r_2,x_2)) = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\mathsf{d}_X(x_1,x_2) \wedge \pi). \tag{2.3.6}$$

In particular, we have

$$\mathsf{d}_C(O,(r,x)) = r\,. \tag{2.3.7}$$

The following result, due to [79], is the rigidity case of the Bishop-Gromov inequality. See also [120].

Theorem 2.3.30 (Volume cone implies metric cone). *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(0,N)$ space. Suppose that there exist $x \in X$ and $0 < r < R$ satisfying*

$$\mathfrak{m}(B_r(x)) = \left(\frac{r^N}{R^N}\right) \mathfrak{m}(B_R(x)).$$

Then one of the three occurs:

1. *$S_{\frac{R}{2}}(x)$, that is the sphere centered at x of radius $\frac{R}{2}$, contains exactly one point. Then (X, d) is isometric to a 1-manifold with boundary and x is a boundary point.*

2. *$S_{\frac{R}{2}}(x)$ contains exactly two points. Then (X, d) is isometric to a 1-manifold and x is an interior point.*

3. *$S_{\frac{R}{2}}(x)$ contains at least three points. Then $N \geq 2$ and there exists an* $\mathrm{RCD}(N-2, N-1)$ *m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m})$ such that $B_R(x)$ is locally isometric to a ball centered at the tip of $C(Z)$ of radius R. Moreover, the restriction of such local isometry to $B_{\frac{R}{2}}(x)$ is an isometry.*

A consequence of the result is the following fundamental fact about tangent cones of RCD spaces.

Theorem 2.3.31 (Tangent cones are metric measure cones)**.** *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be a non-collapsed* $\mathrm{RCD}(K, N)$ *space and $x \in X$. Then any tangent cone at x is a metric measure cone.*

The following result was obtained in [120].

Proposition 2.3.32. *Let $(X, \mathsf{d}_X, \mathfrak{m}_X)$ be a metric measure space and let $N \geq 2$. Then $(C(X), \mathsf{d}_C, \mathfrak{m}_C)$ is an* $\mathrm{RCD}(0, N)$ *m.m.s. if and only if $(X, \mathsf{d}_X, \mathfrak{m}_X)$ is* $\mathrm{RCD}(N-2, N-1)$ *and, in the case $N = 2$,* $\mathrm{diam}(X) \leq \pi$.

Remark 2.3.33. *If $N > 2$, then the diameter bound* $\mathrm{diam}(X) \leq \pi$ *follows already from the* $\mathrm{RCD}(N-2, N-1)$ *condition, by the Bonnet-Meyers theorem for* CD *spaces.*

Next, we show a result relating radial derivatives of functions defined on cones and the distance from the tip of the cone. The gradient of such distance function, in some sense, corresponds to the position vector field in the Euclidean setting. Some of the identities in the proof, obtained using basic calculus in the metric setting, will be used later in this work. The following result uses the BL characterization found in [102], Section 3.2.

Proposition 2.3.34. *Let $(C(X), \mathsf{d}_C, \mathfrak{m}_C)$ be an* $\mathrm{RCD}(0, N)$ *cone over some* $\mathrm{RCD}(N-2, N-1)$ *space $(X, \mathsf{d}_X, \mathfrak{m}_X)$ and $f \in H^{1,2}(X)$. Let $f^{(r)}(x) := f(r, x)$ and $f^{(x)}(r) := f(r, x)$, for $(r, x) \in C(X)$. Then*

$$|\nabla f^{(x)}|(r) = \frac{1}{2r} |\nabla f(r, x) \cdot \nabla \mathsf{d}_C^2(O, \cdot)(r, x)| \text{ for } \mathfrak{m}\text{-a.e. } x \in X \text{ and } \mathcal{L}^1\text{-a.e. } r \in (0, \infty) . \tag{2.3.8}$$

Proof. By [102], Section 3.2, $f^{(r)} \in H^{1,2}(X)$ and $f^{(x)} \in H^{1,2}(([0,\infty), \mathsf{d}_{\text{eucl}}, r^{N-1}dr))$ for $\mathcal{L}^1$-a.e. $r \in (0,\infty)$ and $\mathfrak{m}$-a.e. $x \in X$, respectively. By using the BL characterization of $H^{1,2}(C(X))$ functions (cf. [102], Section 3.2) and the polarization identity, we have

$$\begin{aligned}
&\nabla f(r,x) \cdot \nabla \mathsf{d}_C^2(O,\cdot)(r,x) = \\
&\quad = \frac{1}{2}|\nabla(f + \mathsf{d}_C^2(O,\cdot))|^2(r,x) - \frac{1}{2}|\nabla(f - \mathsf{d}_C^2(O,\cdot))|^2(r,x) \\
&\quad = \frac{1}{2}|\partial_r(f^{(x)}(r) + \mathsf{d}_C^2(r,x))|^2 - \frac{1}{2}|\partial_r(f^{(x)}(r) - \mathsf{d}_C^2(r,x))|^2 \qquad (2.3.9)\\
&\quad = \frac{1}{2}|\partial_r f^{(x)} + 2r|^2 - \frac{1}{2}|\partial_r f^{(x)}(r) - 2r|^2 \qquad (2.3.10)\\
&\quad = 2r\,\partial_r f^{(x)}(r) = 2r\,\mathrm{sign}(\partial_r f^{(x)}(r))\,|\nabla f^{(x)}|(r)\,;
\end{aligned}$$

where we have denoted $\mathsf{d}_C(O,(r,x))^{(x)}$ by $\mathsf{d}_{C(x)}(r)$. Moreover, we have used the fact that $|\nabla(f^{(r)} + (\mathsf{d}_C^2(O,\cdot))^{(r)}|^2(x) = |\nabla f^{(r)}|(x)$ since $(\mathsf{d}_C(O,\cdot))^{(r)}$ is constant. We have also exploited the identification between different notions of derivatives (by, say, [107, Theorem 2.1.37]) $|\nabla g| = |\partial_r g|$ for smooth functions $g : [0,\infty) \to \mathbb{R}$ and the explicit formula for the radial sections of the distance function from the origin given by (2.3.7). Let us also point out that (2.3.9) shows that

$$\nabla f(r,x) \cdot \nabla(\mathsf{d}_C^2(O,(r,x))) = \nabla f^{(x)}(r) \cdot \nabla(\mathsf{d}_C^2(x))(r)\,. \qquad (2.3.11)$$

Lastly, we point out that the following equality

$$|\nabla \mathsf{d}_C(O,\cdot)|(r,x) = |\partial_r \mathsf{d}_C(r,x)| = \frac{1}{2r}|\nabla \mathsf{d}_C^2(O,\cdot)|(r,x)\,, \qquad (2.3.12)$$

implies

$$|\nabla f^{(x)}|(r) = |\nabla f(r,x) \cdot \nabla(\mathsf{d}_C(O,(r,x)))|\,. \qquad (2.3.13)$$

See [79, Corollary 3.6]. □

2.3.9 Tangent modules

In this section we briefly recall the notion of tangent modules. These correspond to "vector fields defined in an L^2 sense". We start by reporting the definition of L^2-normed L^∞-modules, which constitutes the tool used by Gigli to construct a differential structure in non-smooth spaces. They were introduced in [98], inspired by [164, 165, 186, 187]. Subsequently, we recall the definition of cotangent module of a metric space, which provide an abstraction of the concept of "1-form on a Riemannian manifold". We also report the definition of tangent module of a metric measure space, obtained by duality from the cotangent space, and report the definition of the divergence operator. The book [107] provides an excellent introduction to the topic. Lastly, we briefly mention the existence of Sobolev vector fields (see [107, Chapter 6]) and, in the next section, spaces of vector fields having quasi-continuous representatives (see [83]).

Throughout this section, $(X, \mathsf{d}, \mathfrak{m})$ is a metric measure space.

Definition 2.3.35 ($L^2(X)$-normed $L^\infty(X)$-module)**.** *An $L^2(X)$-normed $L^\infty(X)$-module is a quadruple $(\mathcal{M}, \|\cdot\|_{\mathcal{M}}, \cdot, |\cdot|)$ with the following properties*

- $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ *is a Banach space.*
- *The multiplication by $L^\infty(X)$-functions $\cdot : L^\infty(X) \times \mathcal{M} \to \mathcal{M}$ is a bilinear map such that*

$$
\begin{aligned}
&f \cdot (g \cdot v) = (fg) \cdot v && \text{for every } f, g \in L^\infty(X) \text{ and } v \in \mathcal{M},\\
&\mathrm{id} \cdot v = v && \text{for every } v \in \mathcal{M} \text{ where } \mathrm{id} = 1 \ \mathfrak{m}\text{-almost everywhere.}
\end{aligned}
$$

- *The pointwise norm $|\cdot| : \mathcal{M} \to L^2(X)$ satisfies*

$$
\begin{aligned}
&|v| \geq 0 \ \mathfrak{m}\text{-a.e.} && \text{for every } v \in \mathcal{M},\\
&|f \cdot v| = |f||v| \ \mathfrak{m}\text{-a.e.} && \text{for every } v \in \mathcal{M} \text{ and } f \in L^\infty(X),\\
&\|v\|_{\mathcal{M}} = \||v|\|_{L^2(X)} && \text{for every } v \in \mathcal{M}.
\end{aligned}
$$

We now report the definition of the cotangent module of a metric measure space, whose elements, loosely speaking, correspond to ‘square integrable 1-forms’ when the ambient space is a Riemannian manifold.

Theorem 2.3.36 (Cotangent module)**.** *There exists a unique couple $(L^2(T^*X), d)$, where $L^2(T^*X)$ is an $L^2(X)$-normed $L^\infty(X)$-module and $d : S^2(X) \to L^2(T^*X)$ is a linear operator that satisfies*

- $|df| = |\nabla f|$ *holds $\mathfrak{m}$-a.e. for every $f \in S^2(X)$.*
- $L^2(T^*X)$ *is generated by $\{df : f \in S^2(X)\}$.*

The uniqueness part of the statement is to be intended up to module isomorphisms $\Phi : L^2(T^*X) \to \tilde{\mathcal{M}}$ such that $\Phi \circ d = \tilde{d}$.

As previously mentioned, the tangent module is defined as the dual of the cotangent module. Therefore, we need to introduce what the dual of an $L^2(X)$-normed $L^\infty(X)$-module is. In order to do so, we first report the notion of essential supremum.

Lemma 2.3.37 ([107, Lemma 3.2.1]). *Let $f_k : X \to \mathbb{R} \cup \{\pm\infty\}$ be given Borel functions, with $k \in I$. Then there is a unique (up to equality $\mathfrak{m}$-a.e.) Borel function $g : X \to \mathbb{R} \cup \{\pm\infty\}$ such that*

- *$g \geq f_k$ holds $\mathfrak{m}$-a.e. for every $k \in I$;*
- *If $h \geq f_k$ holds $\mathfrak{m}$-a.e. for every $k \in I$, then $h \geq g$ $\mathfrak{m}$-almost everywhere.*

Moreover, there exists an at most countable subfamily $\{f_{k_n}\}_{n\in\mathbb{N}} \subset \{f_k\}_{k\in I}$ such that $g = \sup_n f_{k_n}$. Such function $g =: \operatorname{ess\,sup}_k f_k$ is called the essential supremum *of the family $\{f_k\}_{k\in I}$.*

See [107, Lemma 3.2.1] for a proof of this fact.

Definition 2.3.38. *The dual of an $L^2(X)$-normed $L^\infty(X)$-module $\mathcal{M}$ is*

$$\mathcal{M}^* := \Big\{ L : \mathcal{M} \to L^1(X) \Big| L \text{ is linear and continuous}, \\ L(fv) = fL(v) \text{ for all } v \in \mathcal{M}, f \in L^\infty(X) \Big\}$$

endowed with the operator norm $\|L\|_* := \sup_{\|v\|\leq 1} \|L(v)\|_{L^1(X)}$. The product between a function $f \in L^\infty(X)$ and an element $L \in \mathcal{M}^*$ is defined as

$$(fL)(v) := fL(v) \quad \text{for every } v \in \mathcal{M}.$$

The pointwise norm of $L \in \mathcal{M}^*$ is defined as

$$|L|_* := \operatorname{ess\,sup} L(v),$$

where the essential supremum is taken over all $v \in \mathcal{M}$ such that $|v| \leq 1$ $\mathfrak{m}$-almost everywhere.

See [107, Proposition 3.2.2] for a proof of the fact that $\mathcal{M}^*$ is an $L^2(X)$-normed $L^\infty(X)$-module.

Definition 2.3.39 (Tangent module). *The tangent module $L^2(TX)$ is the $L^2(X)$-normed $L^\infty(X)$-module dual of $L^2(T^*X)$. We call its elements* vector fields.

We are now ready to report the notion of divergence of a vector field on a metric measure space.

Definition 2.3.40 (Divergence). *The space $D(\operatorname{div})$ is defined as the set of all vector fields $v \in L^2(TX)$ for which there exists $h \in L^2(X)$ such that*

$$-\int_X fh \, d\mathfrak{m} = \int_X df(v) \, d\mathfrak{m} \quad \text{for every } f \in H^{1,2}(X).$$

The function $h \in L^2(X)$, which is unique by density of $H^{1,2}(X)$ in $L^2(X)$, is denoted by $\operatorname{div}(v)$.

We point out that a similar construction can be done by assuming $|v| \in L^\infty(X)$. We refer to the module thus obtained as $L^\infty(TX)$.

In the last part of the section we suppose that $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, \infty)$ metric measure space, whose definition we reported in Section 2.3.2. In [98, 97], Gigli introduced second order calculus in $\mathrm{RCD}(K, \infty)$ metric measure spaces. An important notion he introduced is that of vector fields that have a square integrable covariant derivative. To report their definition, we start by recalling the concept of test functions and test vector fields, which correspond to smooth functions and smooth vector fields in the Euclidean setting.

The space of *test functions* on an $\mathrm{RCD}(K, \infty)$ space is defined as

$$\mathrm{Test}(X) := \left\{ f \in D(\Delta) \cap L^\infty(X) : |\nabla f| \in L^\infty(X), \Delta f \in H^{1,2}(X) \right\}.$$

These functions are dense in $H^{1,2}(X)$. Moreover, $\nabla f \cdot \nabla g \in H^{1,2}(X)$ for every $f, g \in \mathrm{Test}(X)$. Moreover, the class of *test vector fields* $\mathrm{Test}V(X)$ is defined as

$$\mathrm{Test}V(X) := \left\{ \sum_{k=1}^{m} g_k \nabla f_k : f_k, g_k \in \mathrm{Test}(X), m \in \mathbb{N} \right\}.$$

On a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$, for any vector field v and any $f, g \in C^\infty(M)$ it holds

$$\langle \nabla_{\nabla f} v, \nabla g \rangle = \langle \nabla \langle v, \nabla g \rangle, \nabla f \rangle - Hg(v, \nabla f),$$

where Hg is the Hessian of g. This fact was exploited to introduce a notion of covariant derivative of test vector fields in $\mathrm{RCD}(K, \infty)$ metric measure spaces, which was in turn used to define $H^{1,2}_C(TX)$. Namely, the space $H^{1,2}_C(TX)$ is defined as the closure of $\mathrm{Test}V(X) \subset L^2(TX)$ with respect to the norm

$$\|v\|^2_{H^{1,2}_C(TX)} := \|v\|^2_{L^2(TX)} + \||\nabla v|\|^2_{L^2(X)}.$$

See [107, Section 6.3] for an introduction to the topic.

2.3.10 Quasi-continuous vector fields

In this section we report the concept of quasi-continuous vector fields. All the results here presented were obtained in [83, 42]. Loosely speaking, quasi-continuous vector fields consist of vector fields that are continuous outside a set of arbitrarily small capacity. As pointed out in [83], in the RCD setting it is not clear what it means for a vector field to be continuous (or what the value of the vector field at a point is).

We also remark that a result in [82] suggests that it might be pointless to look for 'many' continuous vector fields even in Alexandrov spaces.

This section is structered as follows: we start by reporting the concept of capacity. We then recall the definition of $L^0(\mathrm{Cap})$, $QC(X)$ and $L^0_{\mathrm{Cap}}(TX)$. These correspond to Borel functions, quasi-continuous functions and Borel vector fields, respectively. We then revise the definition of $QC(TX)$, which, much like in $\mathbb{R}^N$ quasi-continuous function are the $L^0(\mathrm{Cap})$-closure of smooth ones, is the $L^0_{\mathrm{Cap}}(TX)$-closure of $\mathrm{Test}V(X)$. Lastly, we report the definition of the space $QC^\infty(TX)$ of quasi-continuous vector fields that are Cap-essentially bounded.

Throughout this chapter $(X, \mathsf{d}, \mathfrak{m})$ denotes an $\mathrm{RCD}(K, \infty)$ space, unless otherwise specified.

Definition 2.3.41 (Capacity)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space, and $E \subset X$. The* capacity *of the set E is defined as the quantity $\mathrm{Cap}(E) \in [0, +\infty) \cup \{+\infty\}$ given by*

$$\mathrm{Cap}(E) := \inf \|f\|_{H^{1,2}(X)},$$

where the infimum is taken over all $f \in H^{1,2}(X)$ such that $f \geq 1$ $\mathfrak{m}$-a.e. on some open neighborhood of E, and it is understood to be $+\infty$ when such class is empty.

Let us point out that Cap is an outer measure.

Definition 2.3.42 (The space $L^0(\mathrm{Cap})$)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. $L^0(\mathrm{Cap})$ is defined as the space of all equivalence classes up to* Cap*-a.e. equality of Borel functions on X. Given an increasing sequence $\{A_k\}_k$ of open subsets of X with finite capacity such that for any bounded set $B \subset X$ there is $k \in \mathbb{N}$ with $B \subset A_k$, the distance between two functions $f, g \in L^0(\mathrm{Cap})$ is*

$$\mathsf{d}_{\mathrm{Cap}}(f, g) := \sum_{k\in\mathbb{N}} \frac{1}{2^k(\mathrm{Cap}(A_k) \vee 1)} \int_{A_k} |f - g| \wedge 1 \, d\mathrm{Cap}.$$

Let us point out that $\mathsf{d}_{\mathrm{Cap}}(f, g)$ does not depend on the particular representatives of f and g. Moreover, $(L^0(\mathrm{Cap}), \mathsf{d}_{\mathrm{Cap}})$ is a complete metric space and the topology induced by $\mathsf{d}_{\mathrm{Cap}}$ does not depend on the choice of $\{A_k\}_k$. Since $\mathfrak{m}$ is absolutely continuous with respect to Cap, there is a natural projection $\mathrm{Pr} : L^0(\mathrm{Cap}) \to L^0(\mathfrak{m})$ that assigns the equivalence class Cap-almost everywhere of a function to its equivalence class $\mathfrak{m}$-almost everywhere. Analogously to Definition 2.3.42, we denote with $L^0(\mathfrak{m})$ the space of Borel functions defined $\mathfrak{m}$-almost everywhere.

Definition 2.3.43 (Quasi-continuous functions). *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. A function $f : X \to \mathbb{R}$ is quasi-continuous if for every $\varepsilon > 0$ there exists a set $E \subset X$ with $\mathrm{Cap}(E) < \varepsilon$ such that the function $f_{|X \setminus E} : X \setminus E \to \mathbb{R}$ is continuous.*

If two functions agree Cap-a.e. and one of them is quasi-continuous, then so is the other. We also point out that, as shown in [83, Theorem 1.20], if continuous functions in $H^{1,2}(X)$ are dense in $H^{1,2}(X)$, there exists a unique continuous map $\mathrm{QCR} : H^{1,2}(X) \to QC(X)$ such that $\mathrm{Pr} \circ \mathrm{QCR}$ is the inclusion map $H^{1,2}(X) \subset L^0(\mathfrak{m})$.

Next, we report the definition of $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module and of $L^0_{\mathrm{Cap}}(TX)$. The latter object correspond to Borel vector fields defined Cap-almost everywhere.

Definition 2.3.44 ($L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module). *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. We say that a quadruple $(\mathcal{M}, \tau, \cdot, |\cdot|)$ is a $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module over $(X, \mathsf{d}, \mathfrak{m})$ if the following conditions are satisfied.*

- *$(\mathcal{M}, \tau)$ is a topological vector space.*
- *The bilinear map $\cdot : L^0(\mathrm{Cap}) \times \mathcal{M} \to \mathcal{M}$ satisfies $f \cdot (g \cdot v) = (fg) \cdot v$ and $1 \cdot v = v$ for every $f, g \in L^0(\mathrm{Cap})$ and $v \in \mathcal{M}$.*
- *The map $|\cdot| : \mathcal{M} \to L^0(\mathrm{Cap})$, called the* pointwise norm, *satisfies*

$$
\begin{aligned}
&|v| \geq 0 && \text{for every } v \in \mathcal{M}, \text{ with equality if and oly if } v = 0,\\
&|v + w| \leq |v| + |w| && \text{for every } v, w \in \mathcal{M},\\
&|f \cdot v| = |f||v| && \text{for every } v \in \mathcal{M} \text{ and } f \in L^0(\mathrm{Cap}),
\end{aligned}
$$

  where all equalities and inequalities are intended Cap-almost everywhere.
- *The distance $\mathsf{d}_{\mathcal{M}}$ on $\mathcal{M}$, given by*

$$
\mathsf{d}_{\mathcal{M}}(v, w) := \sum_{k \in \mathbb{N}} \frac{1}{2^k(\mathrm{Cap}(A_k) \vee 1)} \int_{A_k} |v - w| \wedge 1 \, d\mathrm{Cap} \quad \text{for all } v, w \in \mathcal{M},
$$

  is complete and induces the topology τ.

Theorem 2.3.45 (Tangent $L^0(\mathrm{Cap})$-module, [83, Theorem 2.6]). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space. Then there exists a unique couple $(L^0_{\mathrm{Cap}}(TX), \tilde{\nabla})$, where $L^0_{\mathrm{Cap}}(TX)$ is an $L^0(\mathrm{Cap})$-module over X and the operator $\tilde{\nabla} : \mathrm{Test}(X) \to L^0_{\mathrm{Cap}}(TX)$ is linear, such that the following properties hold:*

- *For any $f \in \mathrm{Test}(X)$ we have that the equality $|\tilde{\nabla} f| = \mathrm{QCR}(|\nabla f|)$ holds Cap-a.e. on X.*

- *The space of* $\sum_{k\in\mathbb{N}} \chi_{E_k} \tilde{\nabla} f_k$*, with* $\{f_k\}_k \subset \mathrm{Test}(X)$ *and* $\{E_k\}_k$ *Borel partition of* X*, is dense in* $L^0_{\mathrm{Cap}}(TX)$*.*

Uniqueness is intended up to module isomorphism that fixes $\tilde{\nabla}$*. The space* $L^0_{\mathrm{Cap}}(TX)$ *is called* tangent $L^0(\mathrm{Cap})$-module *associated to* $(X, \mathsf{d}, \mathfrak{m})$*.*

Throughout the rest of this thesis, we use the abuse of notation $\nabla f = \tilde{\nabla} f$ when dealing with functions defined only Cap-almost everywhere.

Definition 2.3.46 (Quasi-continuous vector fields)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *metric measure space and let* $\mathrm{Test}(\tilde{V})(X) \subset L^0_{\mathrm{Cap}}(TX)$ *be defined as*

$$\mathrm{Test}(\tilde{V})(X) := \left\{ \sum_{k=0}^{n} \mathrm{QCR}(g_k) \tilde{\nabla} f_k : n \in \mathbb{N}, \{f_k\}_{k=1}^n, \{g_k\}_{k=1}^n \subset \mathrm{Test}(X) \right\}.$$

Then the space $QC(TX)$ *of* quasi-continuous vector fields *on* X *is defined as the* $\mathsf{d}_{L^0_{\mathrm{Cap}}(TX)}$*-closure of* $\mathrm{Test}\tilde{V}(X)$ *in* $L^0_{\mathrm{Cap}}(TX)$*.*

Let us point out that, if $v \in QC(TX)$, then $|v| \in QC(X)$. See [83, Proposition 2.12]. Lastly, we denote with $QC^\infty(TX)$ the space of Cap-*essentially bounded quasi-continuous vector fields*, which is defined as

$$QC^\infty(TX) := \{v \in QC(TX) : |v| \in L^\infty(\mathrm{Cap})\}.$$

2.4 Finite perimeter sets in RCD metric measure spaces

In this section we recall the main results achieved in the study of finite perimeter sets in RCD metric measure spaces. Sets of finite perimeter have been a very important tool in the developments of Geometric Measure Theory in Euclidean and Riemannian contexts in the last seventy years. In [11, 46, 44], and the more recent [42, 27], most of the classical Euclidean theory of sets of finite perimeter has been generalized to $\mathrm{RCD}(K, N)$ metric measure spaces. Moreover, [144] began a study of locally perimeter minimizing sets in the same setting (see also [105]). Due to the compactness of the class of $\mathrm{RCD}(K, N)$ spaces with respect to the (pointed) measured Gromov-Hausdorff topology, these developments have been important to address some questions of Geometric Measure Theory on smooth Riemannian manifolds. For instance, see [33, 34].

We begin this section with an overview of some properties of BV functions in RCD m.m.s. and their distributional gradient, which we use in the proof of the monotonicity formula Theorem 3.2.1 for an approximation of the perimeter of a minimal boundary. We then report a notion of convergence for sets defined on a sequence of RCD m.m.s. converging in pmGH sense. We then recall the theory of finite perimeter sets in RCD metric measure spaces. We report some tools, such as the coarea formula, Gauss-Green, cut-and-paste. We also report some of their properties, such as De Giorgi's structure theorem in the RCD setting. Subsequently, we recall some results concerning perimeter minimizing sets in RCD m.m.s., particularly their stability along L^1 convergent sequences and some of the regularity results obtained in [144]. Lastly, we report some results concerning isoperimetric sets in the non-smooth setting which are needed for our study of the Modica-Mortola approximation in Chapter 5.

2.4.1 BV functions

Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space, $u \in L^1(X)$ and $\Omega \subset X$ an open set. The total variation norm of u evaluated on Ω is defined by

$$|\nabla u|(\Omega) := \inf \left\{ \liminf_{j\to\infty} \int_\Omega \mathrm{lip}(u_j)(y)\, d\mathfrak{m} \right\}, \tag{2.4.1}$$

where the infimum is taken over all sequences $\{u_j\}_j \subset \mathrm{LIP}(X) \cap L^1(X)$ such that $u_j \to u$ in $L^1(X)$.

A function $u \in L^1(X)$ is said to have bounded variation if its total variation $|\nabla u|(X)$ is finite. In this case one can prove that $|\nabla u|$ can be extended to a Borel measure on X. The space of functions of bounded variation is denoted by $\mathrm{BV}(X)$.

We say that a function $u \in \mathrm{BV}_{\mathrm{loc}}(X)$ if $u \in L^1_{\mathrm{loc}}(X)$ and is such that for all open bounded sets $\Omega \Subset X$ it holds $f_{|\Omega} \in \mathrm{BV}(X)$ for some function $f_{|\Omega}$ with $f = f_{|\Omega}$ in Ω.

We report here a few results on BV functions and their associated vectorial variational measures. Given a measure $\mu \ll \mathrm{Cap}$, then one may consider the natural projection map $\pi_\mu : L^0(\mathrm{Cap}) \to L^0(\mu)$. In [46, Section 1.3] it was shown that, given an $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module $\mathcal{M}$ and the equivalence relation $\sim_\mu$ on $\mathcal{M}$ defined by $v \sim_\mu v$ if $|v-w| = 0$ μ-almost everywhere, then the quotient module $\mathcal{M}^0_\mu$ is an $L^0(\mu)$-normed $L^0(\mu)$-module. Moreover, we call $\mathcal{M}^p_\mu := \{v \in M^0_\mu : |v| \in L^p(\mu)\}$ where $p \in [1, \infty]$. In the particular case in which $\mathcal{M} = L^0_{\mathrm{Cap}}(TX)$ and μ is a Borel measure finite on balls such that $\mu \ll \mathrm{Cap}$, we set $L^p_\mu(TX) := (L^0_{\mathrm{Cap}}(TX))^p_\mu$. When $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, \infty)$ space the total variational measure of a function $f \in \mathrm{BV}(X)$ is absolutely continuous with respect to Cap (see [42] for a proof of this fact). Therefore, one may consider the tangent modules $L^p_{|\nabla f|}(TX)$. This will be particularly useful in the context of finite perimeter sets in the section below: it allows to talk about 'vector fields defined up to perimeter negligible sets'.

Proposition 2.4.1. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space and* $f \in \mathrm{BV}(X)$*. Then there exists a unique vector field* $\nu_f \in L^\infty_{|\nabla f|}(TX)$ *with* $|\nu_f| = 1$ $|\nabla f|$*-a.e. such that*

$$\int_X f \operatorname{div} v \, d\mathfrak{m} = -\int_X v \cdot \nu_f \, d|\nabla f| \,, \quad \textit{for all } v \in QC^\infty(TX) \cap D(\operatorname{div}). \tag{2.4.2}$$

In what follows, for $f \in \mathrm{BV}(X)$ we will denote $\nabla f := \nu_f |\nabla f|$. If E is a set of locally finite perimeter, we denote $\nu_E := \nu_{\chi_E}$. This definition of unit normal is consistent with the one introduced above via the Gauss-Green formula, see [46]. For a function $f : X \to \mathbb{R}$, let

$$f^\wedge = \operatorname{ap} \liminf_{y \longrightarrow x} f(y) = \sup \left\{ t \in \overline{\mathbb{R}} : \lim_{r \longrightarrow 0} \frac{\mathfrak{m}(B_r(x) \cap \{f < t\})}{\mathfrak{m}(B_r(x))} = 0 \right\}$$

$$f^\vee = \operatorname{ap} \limsup_{y \longrightarrow x} f(y) = \inf \left\{ t \in \overline{\mathbb{R}} : \lim_{r \longrightarrow 0} \frac{\mathfrak{m}(B_r(x) \cap \{f > t\})}{\mathfrak{m}(B_r(x))} = 0 \right\} ,$$

and, lastly,

$$\bar f = \frac{f^\wedge + f^\vee}{2} \,, \tag{2.4.3}$$

with the convention $\infty - \infty = 0$.

Lemma 2.4.2 (Leibniz rule for BV)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space and* $f, g \in \mathrm{BV}(X) \cap L^\infty(X)$*. Then* $fg \in \mathrm{BV}(X)$ *and*

$$\nabla(fg) = \bar f \nabla g + \bar g \nabla f \,. \tag{2.4.4}$$

In particular, $|\nabla(fg)| \le |\bar f||\nabla g| + |\bar g||\nabla f|$*.*

Proposition 2.4.3 (BV extension)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space,* $E \subset X$ *a set of locally finite perimeter (see Section 2.4.3 below) and* $f \in \mathrm{BV}(X) \cap L^\infty(E)$*. Then*

$$\tilde f(x) := \begin{cases} \bar f(x) & \text{if } x \in E \,, \\ 0 & \text{elsewhere} \end{cases}$$

belongs to $\mathrm{BV}(X)$ *and* $\nabla \tilde f = \bar f \nabla \chi_E + \nabla f|_E$*.*

Proof. The result immediately follows by applying (2.4.4) with $g = \chi_E$. □

Lemma 2.4.4 (Cut and paste of BV Functions)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *m.m.s. and* $E \subset X$ *a set of locally finite perimeter. Let* $f \in \mathrm{BV}(E)$ *and* $g \in \mathrm{BV}(X \setminus E)$*. Let* $h : X \to \mathbb{R}$ *be defined as*

$$h(x) := \begin{cases} f(x) & \textit{if } x \in E \,; \\ g(x) & \textit{if } x \in X \setminus E \,. \end{cases}$$

Then $h \in \mathrm{BV}(X)$*. Moreover, called* $\bar f, \bar g$ *the representatives given by* (2.4.3)*, it holds*

$$\nabla h = \nabla f|_E + \nabla g|_{X \setminus E} + (\bar f - \bar g) \nabla \chi_E .$$

Proof. Let $\tilde{f}$ and $\tilde{g}$ be the extensions by zero given by Proposition 2.4.3. Then $h = \tilde{f} + \tilde{g}$. $\square$

Lastly, we recall a compactness criteria for BV functions along sequences of RCD spaces converging in pmGH, shown in [18, Proposition 7.5].

Definition 2.4.5 (Infinitesimal isoperimetric profile). *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space with $\mathfrak{m}(X) < \infty$. We say that $\mathcal{I} : (0, \infty) \to (0, 1/2]$ is an infinitesimal isoperimetric profile for $(X, d, \mathfrak{m})$ if for all $\varepsilon > 0$ the following implication holds*

$$\mathfrak{m}(A) \leq \mathcal{I}(\varepsilon) \Longrightarrow \mathfrak{m}(A) \leq \varepsilon \mathrm{Per}(A) \tag{2.4.5}$$

for any Borel set $A \subset X$.

Proposition 2.4.6 ([18, Proposition 7.5]). *Let $\{(X_k, \mathsf{d}_k, \mathfrak{m}_k, x_k)\}_k$ be a sequence of pointed $\mathrm{RCD}(K, \infty)$ metric measure spaces with $\mathfrak{m}_k(X_k) < \infty$ for all $k \in \mathbb{N}$ converging in the pmGH topology to $(X, \mathsf{d}, \mathfrak{m}, x)$ and having a common infinitesimal isoperimetric profile. Suppose that the sequence $\{f_k\}_k$, $f_k \in \mathrm{BV}(X)$ for all $k \in \mathbb{N}$ satisfy*

$$\sup_k \left\{ \int_{X_k} |f_k| d\mathfrak{m}_k + |Df_k|(X_k) \right\} < \infty. \tag{2.4.6}$$

Then there exists $f \in \mathrm{BV}(X)$ and a subsequence (k_l) such that f_{k_l} converges L^1-strongly to f. Moreover, it holds

$$|\nabla f|(X) \leq \liminf_{l \to \infty} |Df_{k_l}|_{k_l}(X_{k_l}). \tag{2.4.7}$$

The following criteria for having a common infinitesimal isoperimetric profile is taken from [18, Theorem 7.2].

Theorem 2.4.7 ([18, Theorem 7.2]). *The class of spaces $(X, \mathsf{d}, \mathfrak{m})$ with $\mathfrak{m}(X) < \infty$ having an isoperimetric profile includes:*

- $\mathrm{RCD}(K,\infty)$ *spaces with $K > 0$.*
- $\mathrm{RCD}(K,\infty)$ *spaces with finite diameter.*

2.4.2 L^1 convergence of sets

In this section, we report the definition of L^1 convergence of sets and recall how such convergence is induced by a metric. These definitions can be found in [11].

Let $(X, \mathsf{d}, \mathfrak{m}, x)$, (Y, ρ, μ, y) be pointed metric measure spaces and let $E \subset X$, $F \subset Y$. We say that the quintuples $(X, \mathsf{d}, \mathfrak{m}, x, E)$ and (Y, ρ, μ, y, F) are isomorphic if there exists a pointed metric measure isomorphism $i : X \to Y$ such that $\mu(i(E) \Delta F) = 0$.

We report here a notion of convergence of sets. We refer to [104, 18, 11] for more details.

Definition 2.4.8 (Convergence of sets)**.** *Let $\{(X_k, \mathsf{d}_k, \mathfrak{m}_k, x_k)\}_k$ be a sequence of pointed metric measure spaces converging in the pointed measured Gromov-Hausdorff sense to $(X, \mathsf{d}, \mathfrak{m}, x)$. Let $E_k \subset X_k$ and $E \subset X$ be Borel sets such that $\mathfrak{m}_k(E_k) < \infty$, $\mathfrak{m}(E) < \infty$ for each $k \in \mathbb{N}$. We say that the sequence $\{E_k\}_k$ converges in the L^1 strong sense to E if the measures $\chi_{E_k}\mathfrak{m}_k \rightharpoonup \chi_E \mathfrak{m}$ in duality with $\mathrm{C_{bs}}(Z)$, where (Z, d_Z) is the space realizing the convergence, and $\mathfrak{m}_k(E_k) \to \mathfrak{m}(E)$ as $k \to \infty$.*

We also say that a sequence of Borel sets $\{E_k\}_k$, $E_k \subset X_k$ for all $k \in \mathbb{N}$ converges strongly in L^1_{loc} to a Borel set $E \subset X$ if $E_k \cap B_r(x_k)$ converges in the strong L^1 sense to $E \cap B_r(x)$ for all $r > 0$.

This convergence can be metrized by a distance $\mathcal{D}$ defined on (isomorphic classes of) quintuples $(X, \mathsf{d}, \mathfrak{m}, x, E)$, see [11, Lemma A.4]. Let us briefly recall its definition.

Definition 2.4.9 (Distance between sets)**.** *Let $\Xi_1 := (X, \mathsf{d}, \mathfrak{m}, x, E)$ and $\Xi_2 := (Y, \rho, \mu, y, F)$ be two pointed $\mathrm{RCD}(K, N)$ spaces with $E \subset X$ and $F \subset Y$ sets of locally finite perimeter. Let (Z, d_Z) be a proper metric measure space and $\Psi_1 : (X, \mathsf{d}) \to (Z, \mathsf{d}_z)$, $\Psi_2 : (Y, \rho) \to (Z, \mathsf{d}_Z)$ be isometric embeddings. For any $k \in \mathbb{N}$ we define*

$$\begin{aligned}
\mathcal{D}_{k,\Psi_1,\Psi_2}(\Xi_1, \Xi_2) :=& \\
&\mathsf{d}_{\mathrm{H}}\left(\Psi_1\left(X \cap \overline{B_k(x)}\right), \Psi_2\left(Y \cap \overline{B_k(y)}\right)\right) \wedge 1 \\
&+ \left|\log\left(\frac{\mathfrak{m}(B_k(x))}{\mu(B_k(y))}\right)\right| \wedge 1 + W_1^Z\left((\Psi_1)_\# \frac{\chi_{B_k(x)}}{\mathfrak{m}(B_k(x))}\mathfrak{m}, (\Psi_2)_\# \frac{\chi_{B_k(y)}}{\mu(B_k(y))}\mu\right) \\
&+ \left|\log\left(\frac{\mathfrak{m}(B_k(x) \cap E)}{\mu(B_k(y) \cap F)}\right)\right| \wedge 1 \\
&+ W_1^Z\left((\Psi_1)_\# \frac{\chi_{B_k(x)}}{\mathfrak{m}(B_k(x) \cap E)}\mathfrak{m}, (\Psi_2)_\# \frac{\chi_{B_k(y)}}{\mu(B_k(y) \cap F)}\mu\right),
\end{aligned}$$

where d_{H} is the Hausdorff distance between compact subsets of Z and W_1^Z denotes the 1-Wasserstein distance in $(Z, \mathsf{d}_Z \wedge 1)$. We say that the distance between Ξ_1 and Ξ_2 is

$$\mathcal{D}(\Xi_1, \Xi_2) := \inf\left\{\mathsf{d}_Z(\Psi_1(x), \Psi_2(y)) + \sum_{k=1}^{\infty} \frac{1}{2^k}\mathcal{D}_{k,\Psi_1,\Psi_2}(\Xi_1, \Xi_2)\right\},$$

where the infimum is taken over all proper metric spaces (Z, d_Z) and isometric embeddings Ψ_1, Ψ_2.

We point out that Definition 2.4.9 is similar to Definition 2.2.4, but also takes into account a distance between sets.

It can be shown that $\mathcal{D}$ is a distance on the space of quintuples $(X, \mathsf{d}, \mathfrak{m}, x, E)$ considered up to isomorphisms. Furthermore, a sequence $\{(X_k, d_k, \mathfrak{m}_k, x, E_k)\}_k$ converges in the topology induced by $\mathcal{D}$ to $(X, \mathsf{d}, \mathfrak{m}, x, E)$ if and only if $\{(X_k, d_k, \mathfrak{m}_k, x_k)\}$ converges in the pointed measured Gromov-Hausdorff sense to $(X, \mathsf{d}, \mathfrak{m}, x)$ and $\{E_k\}_k$ converges to E in the L^1_{loc} sense. See [11, Lemma A.4] for a proof of these facts.

2.4.3 Finite perimeter sets in RCD spaces

Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. A set $E \subset X$ is of (locally) finite perimeter if $\chi_E \in \mathrm{BV}(X)$ ($\chi_E \in \mathrm{BV}_{\text{loc}}(X)$). We denote the perimeter measure of a locally finite perimeter set E by $\mathrm{Per}(E, \cdot) := |\nabla \chi_E|(\cdot)$.

The following lemma follows immediately from the definition of locally finite perimeter sets.

Lemma 2.4.10. *Consider two metric measure spaces $(X, \mathsf{d}, \mathcal{H}^N)$ and $(Y, \rho, \mathcal{H}^N)$ for some $N \in \mathbb{N}$ and suppose there exists a bijective isometry $i : X \to Y$. Then for every $A, B \subset X$ we have*

$$\mathrm{Per}(A, B) = \mathrm{Per}(i(A), i(B)).$$

An important tool for what follows is the coarea formula (cf. [139, Theorem 2.16]).

Theorem 2.4.11 (Coarea Formula)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space and $u \in \mathrm{BV}(X)$. Then $\{u > r\}$ has finite perimeter for $\mathcal{L}^1$-a.e. r and for any Borel function $f : X \to \mathbb{R}$ it holds*

$$\int_X f \, d|\nabla u| = \int_{\mathbb{R}} \left(\int_X f \, d\mathrm{Per}(\{u > r\}) \right) dr \,, \tag{2.4.8}$$

where the term $d|\nabla u|$ in the right hand side denotes that the integral is computed with respect to the total variation measure $|\nabla u|$.

Given a function $u \in \mathrm{BV}(X)$, we define the set $E_t := \{x \in X : u(x) \geq t, \, t \in \mathbb{R}\}$. As a consequence of the coarea formula, it holds

$$|\nabla u|(C) = \int_{\mathbb{R}} \mathrm{Per}(E_t; C) \, dt \,.$$

Related to the coarea formula (2.4.8) is the following local, relative isoperimetric inequality, which can be found in [126, Theorem 3.3].

Theorem 2.4.12. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ metric measure space. Then there exists $\lambda > 1$, $C > 0$ and $r_0 > 0$ such that*

$$\min\{\mathfrak{m}(E \cap B_r(x), \mathfrak{m}(B_r(x) \setminus E)\} \leq C r \mathrm{Per}(E, B_{\lambda r}(x)),$$

for all $x \in X$, $0 < r < r_0$.

We now look at a particular type of locally finite perimeter sets: cones inside cones. Let $(C(X), \mathsf{d}_C, \mathfrak{m}_C)$ be as in Proposition 2.3.32, $r > 0$, $x \in X$. An important family of sets for the purposes of this manuscript is the following

$$C(B_r^X(x)) := \{(t, y) \in C(X) : y \in B_r^X(x)\} . \tag{2.4.9}$$

Lemma 2.4.13. *The sets $C(B_r^X(x)) \subset C(X)$ are of locally finite perimeter for any $r > 0$.*

Proof. We prove the claim by exhibiting an explicit sequence of Lipschitz functions converging to $\chi_{C(B_r^X(x))}$. Let

$$f_k(t, y) = \begin{cases} 1 & \text{if } y \in B_r^X(x) ; \\ kr + 1 - k\mathsf{d}(y, x) & \text{if } y \in B_{r+\frac{1}{k}}^X(x) \setminus B_r^X(x) ; \\ 0 & \text{if } y \in X \setminus B_{r+\frac{1}{k}}^X(x) . \end{cases} \tag{2.4.10}$$

Clearly, f_k is Lipschitz with bounded support, $|f_k| \leq 1$, hence $f_k \in H^{1,2}_{\mathrm{loc}}(C(X))$. Moreover, $f_k \to \chi_{C(B_r^X(x))}$ in $L^1_{\mathrm{loc}}(C(X))$. Indeed, for $p = (s, z) \in C(X)$, $R > 0$

$$\begin{aligned} \int_{B_R(p)} |f_k - \chi_{C(B_r^X(x))}| \, d\mathfrak{m}_C &= \int_{B_R(p)} |f_k| \chi_{C\left(B_{r+\frac{1}{k}}^X(x) \setminus B_r^X(x)\right)} \, d\mathfrak{m}_C \\ &\leq \mathfrak{m}_C \left(B_R(p) \cap C \left(B_{r+\frac{1}{k}}^X(x) \setminus B_r^X(x) \right) \right) . \end{aligned}$$

Using the Bishop-Gromov inequality [180], we have

$$\mathfrak{m} \left(B_{r+\frac{1}{k}}^X(x) \setminus B_r^X(x) \right) = \mathfrak{m}(B_r^X(x)) \left(\frac{\mathfrak{m}\left(B_{r+\frac{1}{k}}^X(x)\right)}{\mathfrak{m}(B_r^X(x))} - 1 \right) \leq \mathfrak{m}(B_r^X(x)) \left(\frac{N-1}{rk} + O\left(\frac{1}{k^2}\right) \right) .$$

Therefore,

$$\mathfrak{m}_C \left(B_R(p) \cap C \left(B_{r+\frac{1}{k}}^X(x) \setminus B_r^X(x) \right) \right) = O\left(\frac{1}{k}\right) , \tag{2.4.11}$$

which implies $f_k \to \chi_{C(B_r^X(x))}$ in $L^1_{\mathrm{loc}}(C(X))$.

Let us now show that

$$\limsup_{k \to \infty} \int_{B_R(p)} \mathrm{lip} f_k \, d\mathfrak{m}_C < \infty , \tag{2.4.12}$$

for any $p = (s, z) \in C(X)$, $R > 0$, which directly implies that $C(B_r^X(x))$ is a set of locally finite perimeter.

It is elementary to check that

$$\operatorname{lip} f_k(t, x) = \begin{cases} k & \text{if } y \in B^X_{r+\frac{1}{k}}(x) \setminus B^X_r(x)\,; \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.4.13}$$

Therefore, using (2.4.11), we obtain

$$\int_{B_R(p)} \operatorname{lip} f_k \, d\mathfrak{m}_C = k\, \mathfrak{m}_C \left(B_R(p) \cap C \left(B^X_{r+\frac{1}{k}}(x) \setminus B^X_r(x) \right) \right) = O(1)\,, \quad \text{as } k \to \infty\,.$$

□

Definition 2.4.14 (Tangent to a set of locally finite perimeter). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *m.m.s. and $E \subset X$ be a set of locally finite perimeter.*
We say that $(Y, \rho, \mu, y, F) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$ if $(Y, \rho, \mu, y) \in \mathrm{Tan}_x(X)$ and $F \subset Y$ is a set of locally finite perimeter of positive measure such that χ_E converges in the L^1_{loc} sense of Definition 2.4.8 to F along the blow-up sequence associated to the tangent Y.

Let us report an important compactness result for sets of finite perimeter, which in particular guarantees the existence of tangent cones.

Corollary 2.4.15 ([11, Corollary 3.4]). *Let $\{(X_k, \mathsf{d}_k, \mathfrak{m}_k, x_k)\}$ be a sequence of* $\mathrm{RCD}(K, N)$ *p.m.m.s. converging in the pmGH topology to $(X, \mathsf{d}, \mathfrak{m}, x)$. Let $E_k \subset X_k$ be sets of locally finite perimeter such that*

$$\sup_k \mathrm{Per}(E_k, B_R(x_k)) < \infty \quad \textit{for every } R > 0.$$

Then there exists a subsequence, which we do not relabel, and a locally finite perimeter set $E \subset X$ such that $E_k \to E$ in the L^1_{loc} sense (see Definition 2.4.8).

We report a result on the uniqueness (up to a set of zero perimeter measure) of tangents to sets of locally finite perimeter, which can be found in [11], appendix A.

Proposition 2.4.16 (Uniqueness of Tangents). *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an* $\mathrm{RCD}(K, N)$ *metric measure space and $E \subset X$ a set of locally finite perimeter. Then for* Per_E*-almost every $x \in X$ it holds*

$$\mathrm{Tan}_x(X, \mathsf{d}, \mathcal{H}^N, E) = \{(\mathbb{R}^N, \mathsf{d}_{\mathrm{eucl}}, c\mathcal{L}^N, 0^N, \{x_N > 0\})\}.$$

The following result was also obtained in [11], appendix A.

Theorem 2.4.17 (Iterated tangents)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a an* $\mathrm{RCD}(K, N)$ *m.m.s. and let* $E \subset X$ *be a set of locally finite perimeter. For* Per_E*-almost every* $x \in X$ *we have that if* $(Y, \rho, \mu, y, F) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$ *then*

$$\mathrm{Tan}_{\bar{y}}(Y, \rho, \mu, F) \subset \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) \qquad \text{for every } \bar{y} \in \operatorname{supp} \mathrm{Per}_F.$$

We report here a De Giorgi-type structure theorem for finite perimeter sets obtained in [11] and [46]. To simplify our discussion, we restrict ourselves to the non-collapsed RCD setting. We start by recalling the notion of reduced boundary, which hinges on our notion of blow up in the RCD setting definition 2.4.8.

Definition 2.4.18 (Reduced boundary)**.** *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be an* $\mathrm{RCD}(K, N)$ *space and* $E \subset X$ *a set of locally finite perimeter. For* $k = 1, ..., N$ *the reduced boundary* $\partial_k^* E$ *is defined as*

$$\partial_k^* E := \{x \in X : \mathrm{Tan}_x(X, \mathsf{d}, \mathcal{H}^N, E) = \{(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k \mathcal{L}^k, 0^k, \{x_k > 0\})\}\}.$$

We denote $\partial^* E := \partial_N^* E$.

Let us point out that the reduced boundary recalled in Definition 2.4.18 does not fully coincide with the reduced boundary in the classical Euclidean sense. Namely, it allows the possibility that different half spaces arise as blow-ups when rescaling along different sequences of radii converging to 0.

Theorem 2.4.19 (Rectifiability)**.** *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. and* $E \subset X$ *a set of locally finite perimeter. Then* $\partial^* E$ *is strongly* $(\mathrm{Per}_E, N-1)$*-rectifiable.*

Let us also report a stronger version of the above Theorem 2.4.19 specific to the non-collapsed setting.

Theorem 2.4.20 (De Giorgi structure theorem)**.** *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be an* $\mathrm{RCD}(K, N)$ *space and* $E \subset X$ *a set of locally finite perimeter. Then it holds*

1. Per_E *is concentrated on the reduced boundary* $\partial^* E$*;*
2. *the reduced boundary* $\partial^* E$ *is strongly* $(\mathrm{Per}_E, N-1)$*-rectifiable;*
3. $\mathrm{Per}_E = \mathcal{H}^{N-1}|_{\partial^* E}$.

An important tool for our analysis is the Gauss-Green formula for sets of finite perimeter in the RCD setting. We refer to [46] for the proof and for some background material. Given a set of finite perimeter E, for the *essentially bounded vector fields defined* Per_E*-a.e.* we use the following notation: $L^2_E(TX) := L^2_{|\nabla \chi_E|}(TX)$.

Theorem 2.4.21 (Gauss-Green formula, [46, Theorem 2.4]). *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. and let* $E \subset X$ *be a set of finite perimeter with* $\mathfrak{m}(E) < \infty$. *Then there exists a unique vector field* $\nu_E \in L^2_E(TX)$ *such that* $|\nu_E| = 1$ Per_E*-almost everywhere and*

$$\int_E \operatorname{div}(v) d\mathfrak{m} = -\int \operatorname{tr}_E(v) \cdot \nu_E \, d\mathrm{Per}(E),$$

for all $v \in H^{1,2}_C(TX) \cap D(\operatorname{div})$ *with* $|v| \in L^\infty(|\nabla \chi_E|)$.

Let us now report a version of the Gauss-Green formula for vector fields that are bounded and have measure-valued divergence, but that do not necessarily belong to $H^{1,2}_C(TX)$.

Definition 2.4.22 ([48, Definition 4.1]). *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *metric measure space. A vector field* $V \in L^\infty(TX)$ *is an* essentially bounded divergence measure vector field *if its distributional divergence is a finite Radon measure, that is if* $\operatorname{div}(V)$ *is a finite Radon measure such that*

$$\int_X g \, d\operatorname{div}(V) = -\int_X \nabla g \cdot V d\mathfrak{m}$$

for all $g \in \mathrm{LIP}_c(X)$. *We denote the class of essentially bounded divergence measure vector field by* $\mathcal{DM}^\infty(X)$.

A useful regularity result is the following.

Lemma 2.4.23 (Brue-Naber-Semola, Theorem 7.4). *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *metric measure space and* $V \in \mathcal{DM}^\infty(X)$. *Then* $\operatorname{div}(V)$ *is absolutely continuous with respect to* $\mathcal{H}^{N-1}$.

Since the divergence of a vector field $V \in \mathcal{DM}^\infty(X)$ may have a singular part with respect to the reference measure, it could assign positive mass to the boundary of a set of finite perimeter. Therefore, it is crucial for a Gauss-Green formula in this setting to specify whether we integrate the divergence of a vector field only over the interior of the set of finite perimeter or over its closure. Moreover, contrary to vector fields in $H^{1,2}_C(TX)$, $\mathcal{DM}^\infty(X)$-vector fields do not have a pointwise-a.e. defined representative over boundaries of sets of finite perimeter. However, it is possible to define the interior and exterior normal traces of a vector field $V \in \mathcal{DM}^\infty(X)$.

Let $E \subset X$ be a set of finite perimeter and let $V \in \mathcal{DM}^\infty(X)$. It was proved in [44, Section 5] and [48, Section 6.5] that there exist measures $\nabla\chi_E(\chi_E V)$ and $\nabla\chi_E(\chi_{E^C} V)$ such that

$$\nabla P_t \chi_E \cdot (\chi_E V) \rightharpoonup \nabla \chi_E(\chi_E V)$$

and

$$\nabla P_t \chi_E \cdot (\chi_{E^C} V) \rightharpoonup \nabla \chi_E(\chi_{E^C} V)$$

as $t \to 0$. Moreover, $\nabla \chi_E(\chi_E V)$ and $\nabla \chi_E(\chi_{E^C} V)$ are both absolutely continuous with respect to Per_E. Hence, we can consider their densities $(V \cdot \nu_E)_{\mathrm{int}}$ and $(V \cdot \nu_E)_{\mathrm{ext}}$ defined by

$$\nabla \chi_E(\chi_E V) = \frac{1}{2}(V \cdot \nu_E)_{\mathrm{int}} \mathrm{Per}_E$$

and

$$\nabla \chi_E(\chi_{E^C} V) = \frac{1}{2}(V \cdot \nu_E)_{\mathrm{ext}} \mathrm{Per}_E,$$

respectively.

The theorem below can be found in [48, Section 6.20] and [44, Theorem 5.2]. In the latter, sharp bounds were given.

Theorem 2.4.24 (Gauss-Green for $\mathcal{DM}^\infty$, [44, Theorem 5.2])**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *metric measure space. Moreover, let $E \subset X$ be a set of finite perimeter and let $V \in \mathcal{DM}^\infty(X)$. Then the following hold*

$$\int_{E^{(1)}} f \, d\mathrm{div}(V) + \int_E \nabla f \cdot V \, d\mathfrak{m} = -\int_{\partial^* E} f (V \cdot \nu_E)_{\mathrm{int}} d\mathrm{Per}_E,$$
$$\int_{E^{(1)} \cup \partial^* E} f \, d\mathrm{div}(V) + \int_E \nabla f \cdot V \, d\mathfrak{m} = -\int_{\partial^* E} f (V \cdot \nu_E)_{\mathrm{ext}} d\mathrm{Per}_E,$$

for all $f \in \mathrm{LIP}_c(X)$. Furthermore, we have

$$\|(V \cdot \nu_E)_{\mathrm{int}}\|_{L^\infty(|\nabla \chi_E|)} \le \|V\|_{L^\infty(E, \mathfrak{m})},$$
$$\|(V \cdot \nu_E)_{\mathrm{ext}}\|_{L^\infty(|\nabla \chi_E|)} \le \|V\|_{L^\infty(E^C, \mathfrak{m})}.$$

2.4.3.1 Operations on finite perimeter sets

We next recall a useful cut and paste result proved in [44, Theorem 4.11]. To this aim, let us first fix some notation. We denote by $\mathcal{H}^h$ the codimension one Hausdorff type measure induced by $\mathfrak{m}$ with gauge function $h(B_r(x)) := \mathfrak{m}(B_r(x))/r$, see [46] for further details. We write the total variation measure μ_E using the polar factorization $\mu_E = \nu_E \mathrm{Per}_E$. Given $t \in [0, 1]$, we also recall the following

$$E^{(t)} := \left\{ x \in X : \lim_{r \to 0} \frac{\mathfrak{m}(E \cap B_r(x))}{\mathfrak{m}(B_r(x))} = t \right\}.$$

We say that the set $E^{(1)}$ is the *measure theoretic interior* and $E^{(0)}$ is the *measure theoretic exterior* of a Borel subset $E \subset X$.

We begin the section by recalling the following Federer-type result.

Proposition 2.4.25 ([44, Proposition 4.2]). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *metric measure space and let $E \subset X$ be a set of finite perimeter with finite measure. Then we have*

- *For $\mathcal{H}^h$-a.e. $x \in X$*
$$\lim_{r \to 0} \frac{\mathfrak{m}(E \cap B_r(x))}{\mathfrak{m}(B_r(x))} \in \left\{0, \frac{1}{2}, 1\right\};$$
- *Up to a $\mathcal{H}^h$ negligible set it holds $\partial^* E = E^{(1/2)}$;*
- *For $\mathcal{H}^h$-a.e. $x \in X$ we have $\lim_{t \to 0} P_t \chi_E \in \left\{0, \frac{1}{2}, 1\right\}$;*
- *Up to a $\mathcal{H}^h$ negligible set it holds*
$$\partial^* E = \left\{x \in E : \lim_{t \to 0} P_t \chi_E(x) = \frac{1}{2}\right\}.$$

Let us now report [44, Theorem 4.11], which provides a formula for the perimeter measure of sets obtained through cut and paste operations on finite perimeter sets.

Theorem 2.4.26 (Cut and paste, [44, Theorem 4.11]). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *m.m.s. and E, $F \subset X$ be sets of finite perimeter. Then $E \cap F$, $E \cup F$ and $E \setminus F$ are sets of finite perimeter. Moreover, it holds*

$$\begin{aligned}
\mu_{E \cap F} &= \mu_E|_{F^{(1)}} + \mu_F|_{E^{(1)}} + \nu_E \mathcal{H}^h|_{\{\nu_E = \nu_F\}}\,; \\
\mu_{E \cup F} &= \mu_E|_{F^{(0)}} + \mu_F|_{E^{(0)}} + \nu_E \mathcal{H}^h|_{\{\nu_E = \nu_F\}}\,; \\
\mu_{E \setminus F} &= \mu_E|_{F^{(0)}} - \mu_F|_{E^{(1)}} + \nu_E \mathcal{H}^h|_{\{\nu_E = -\nu_F\}}\,.
\end{aligned}$$

Let us clarify that we interpret the first of the equations above (and the same applies for the other two) in the following sense: for any vector field $v \in H^{1,2}_C(TX) \cap D(\mathrm{div})$ such that $|v| \in L^\infty(X)$, it holds

$$\begin{aligned}
\int_{E \cap F} \mathrm{div}(v)\, d\mathfrak{m} = - \int_{F^{(1)}} \mathrm{tr}_E(v) \cdot \nu_E \, d\mathrm{Per}_E - \int_{E^{(1)}} \mathrm{tr}_F(v) \cdot \nu_F \, d\mathrm{Per}_F \\
- \int_{E^{1/2} \cap F^{1/2}} \mathrm{tr}_E(v) \cdot \nu_E \, d\mathrm{Per}_E.
\end{aligned}$$

We also report the following version of cut and paste operations for sets of finite perimeter for vector fields of essentially bounded divergence.

Proposition 2.4.27 ([44, Proposition 5.4]). *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *metric measure space. Let* $E, F \subset X$ *be sets of locally finite perimeter and let* $V \in \mathcal{DM}^\infty(X)$*. Then*

$$(V \cdot \nu_{E\cap F})_{\text{int}} = (V \cdot \nu_E)_{\text{int}}, \quad \mathrm{Per}_E - a.e. \text{ on } F^{(1)},$$
$$(V \cdot \nu_{E\cap F})_{\text{int}} = (V \cdot \nu_F)_{\text{int}}, \quad \mathrm{Per}_F - a.e. \text{ on } E^{(1)},$$
$$(V \cdot \nu_{E\cap F})_{\text{int}} = (V \cdot \nu_E)_{\text{int}}, \quad \mathrm{Per}_E - a.e. \text{ on } E^{(1/2)} \cap F^{(1/2)}.$$

Analogous results hold for the exterior normal traces and for the interior and exterior normal traces on $E \cup F$ *and on* $E \setminus F$*.*

Lastly, we report a technical result that states that outward-pointing unit normal to a sub-level set of a distance function is the gradient of the distance function at that point.

Proposition 2.4.28 ([44, Proposition 6.1]). *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *metric measure space. Let* $\Omega \subset X$ *be an open domain compactly contained in* $\Omega' \subset X$*, which is itself a domain. Let* $\varphi : \Omega' \to \mathbb{R}$ *be a* 1*-Lipschitz function such that*

- $|\nabla\varphi| = 1$ $\mathfrak{m}$*-a.e.;*
- φ *has measure valued Laplacian on* Ω' *with* $\mathfrak{m}$*-essentially bounded positive (or negative) part.*

Then for $\mathcal{L}^1$*-a.e.* t *such that* $\{\varphi = t\} \cap \Omega \neq \emptyset$ *it holds that* $\{\varphi < t\}$ *is a set of locally finite perimeter in* Ω*. Moreover, if holds*

$$\left(\nabla\varphi \cdot \nu_{\{\varphi<t\}}\right)_{\text{int}} = \left(\nabla\varphi \cdot \nu_{\{\varphi<t\}}\right)_{\text{ext}} = -1 \quad \mathrm{Per}_{\{\varphi<t\}}\text{-}a.e. \text{ on } \Omega.$$

2.4.4 Perimeter minimizing sets

In this subsection, we recall the notion of perimeter minimizing set in an $\mathrm{RCD}(K, N)$ space. We also report some regularity results relevant to our purposes.

Definition 2.4.29 (Local and Global Perimeter Minimizer). *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. A set of locally finite perimeter* $E \subset X$ *is a*

- Global perimeter minimizer *if it minimizes the perimeter for every compactly supported perturbation, i.e.*

  $$\mathrm{Per}(E, B_R(x)) \leq \mathrm{Per}(F; B_R(x))$$

  for all $x \in X$*,* $R > 0$ *and* $F \subset X$ *with* $F = E$ *outside* $B_R(x)$*;*

- Local perimeter minimizer *if for every* $x \in X$ *there exists* $r_x > 0$ *such that* E *minimizes the perimeter in* $B_{r_x}(x)$, *i.e. for all* $F \subset X$ *with* $F = E$ *outside* $B_{r_x}(x)$ *it holds*

$$\mathrm{Per}(E, B_{r_x}(x)) \le \mathrm{Per}(F; B_{r_x}(x)) .$$

We report here the definition of quasi-minimal set of finite perimeter.

Definition 2.4.30 (Quasi-minimal sets, [144, Definition 2.34])**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *metric measure space. Let* $E \subset X$ *be a finite perimeter set and let* $\Omega \subset X$ *be an open set. Given any* $\kappa \ge 1$ *we say that* E *is a* κ-quasi-minimal set *if for any* $U \Subset \Omega$ *and for all Borel sets* $F \subset X$ *such that* $E \Delta F \Subset U$ *it holds*

$$\mathrm{Per}(E, U) \le \kappa \mathrm{Per}(F, U) .$$

We say that E *is a* local κ-quasi-minimal *if for every* $x \in X$ *there exists* $r_x > 0$ *such that for all* $F \subset X$ *with* $F = E$ *outside* $B_{r_x}(x)$ *it holds*

$$\mathrm{Per}(E, B_{r_x}(x)) \le \mathrm{Per}(F; B_{r_x}(x)) .$$

Let us point out that

- perimeter minimizing sets are also quasi-minimal sets. The definitions coincide if $\kappa = 1$;
- solutions to the prescribed mean curvature problem are quasi-minimizers (under suitable assumptions);
- isoperimetric sets are quasi-minimizers, as shown in [31, Theorem 3.4].

For an overview of the theory of quasi-minimal sets in Euclidean spaces, see [136, Chapter 21].

The following result states that a quasi-minimal set of finite perimeter, up to a modification on a negligible set, has a measure theoretic boundary coinciding with the topological boundary. This result was first shown in the Euclidean setting in [74].

Theorem 2.4.31 ([122, Theorem 4.2])**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *metric measure space. Let* $E \subset \Omega$ *be a quasi-minimal set of finite perimeter. Then, up to modifying* E *on a set of* $\mathfrak{m}$*-negligible set, there exists* $\gamma > 0$ *such that*

$$\frac{\mathfrak{m}(E \cap B_r(x))}{\mathfrak{m}(B_r(x))} \ge \gamma, \quad \frac{\mathfrak{m}(E \setminus B_r(x))}{\mathfrak{m}(B_r(x))} \ge \gamma,$$

for any $r > 0$ *such that* $B_{2r}(x) \subset \Omega$.

A consequence of the above result and of the local relative isoperimetric inequality Theorem 2.4.12 is the following density estimate.

Lemma 2.4.32 (Density, [122, Lemma 5.1])**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. and let* $E \subset X$ *be a quasi-minimal set in* $\Omega \subset X$*. Then there exists* r_0*,* $C > 0$ *such that*

$$C^{-1}\frac{\mathfrak{m}(E \cap B_r(x))}{r} \leq \mathrm{Per}(E, B_r(x)) \leq C\frac{\mathfrak{m}(E \cap B_r(x))}{r},$$

for all $0 < r < r_0$ *and* $x \in \partial E \cap \Omega$ *whenever* $B_{2r}(x) \subset \Omega$*.*

Let us report a stability of minimality along L^1 convergence of sets. This result in particular shows that the class of locally perimeter minimizing sets in RCD spaces is non-empty, as it includes limits of locally perimeter minimizing sets in sequences of manifolds. Like in the Euclidean setting, it shows that the convergence of sets of finite perimeter can be improved if they are perimeter minimizers.

Proposition 2.4.33 (Stability of minimality, [11, Proposition 3.9])**.** *Let* $\{(X_k, \mathsf{d}_k, \mathfrak{m}_k, x_k)\}_k$ *be a sequence of* $\mathrm{RCD}(K, N)$ *metric measure spaces converging in the pmGH sense to* $(X, \mathsf{d}, \mathfrak{m}, x)$ *and let* (Z, d_Z) *be the space realizing the convergence. For any* $k \in \mathbb{N}$*, let* $\kappa_k \geq 1$ *and* $R_k > 0$ *be such that* $\kappa_k \to 1$ *and* $R_k \to \infty$*. Moreover, let* $E_k \subset X_k$ *be a* κ_k*-quasi-minimal set in* $B_{R_k}(x_k)$*. Assume that there exists* $E \subset X$ *such that* $E_k \to E$ *in the* L^1 *sense. Then*

- E *is an entire perimeter minimizer, that is*

$$\mathrm{Per}(E, B_r(x)) \leq \mathrm{Per}(F, B_r(x))$$

  for all $F \subset X$ *such that* $E\Delta F \Subset B_r(x) \Subset X$ *and for all* $r > 0$*;*
- $\mathrm{Per}_{E_k} \to \mathrm{Per}_E$ *in duality with* $C_c(Z)$ *as* $k \to \infty$*.*

In [144, Theorem 2.42] Proposition 2.4.33 was extended to a more general class of minimizers. The authors also proved the Hausdorff convergence of the topological boundaries (referred to as Kuratowski convergence) or, equivalently, of their measure-theoretic boundaries.

We also report an estimate on the Hausdorff dimension of the singular set of a local perimeter minimizer. See [11].

Definition 2.4.34 (Regular and singular sets of a locally finite perimeter set)**.** *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be an* $\mathrm{RCD}(K, N)$ *metric measure space and* $E \subset X$ *a locally finite perimeter set inside* $B_2(x)$*, for some* $x \in X$ *with* $\partial X \cap B_2(x) = \emptyset$*. Then the regular set* $\mathcal{R}^E$

and singular set $\mathcal{S}^E$ *of the locally perimeter minimizing set* E *are, respectively,*

$$\mathcal{R}^E := \{x \in \partial E : (\mathbb{R}^N, \mathsf{d}_{\text{eucl}}, \mathcal{L}^N, 0^N, \{x_N > 0\}) \in \text{Tan}_x(X, \mathsf{d}, \mathcal{H}^N, E)\};$$
$$\mathcal{S}^E = \partial E \setminus \mathcal{R}^E.$$

The following two regularity results were obtained in [144].

Theorem 2.4.35 (Topological regularity of the regular set, [144, Theorem 6.21])**.** *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be an* $\text{RCD}(K, N)$ *metric measure space and* $E \subset X$ *a locally perimeter minimizing set inside* $B_2(x)$*, for some* $x \in X$ *with* $\partial X \cap B_2(x) = \emptyset$*. Then for every* $\alpha \in (0, 1)$ *there exists a relatively open set* $O_\alpha \subset \partial E \cap B_1(x)$ *such that* $\mathcal{R}^E \subset O_\alpha$ *and* O_α *is* α*-bi-hölder homeomorphic to an open, smooth* $(N-1)$*-dimensional manifold.*

Theorem 2.4.36 (Hausdorff dimension of the singular set, [144, Theorem 6.30])**.** *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be an* $\text{RCD}(K, N)$ *metric measure space and* $E \subset X$ *a locally perimeter minimizing set inside* Ω*, where* $\Omega \subset X$ *is an open set. Then*

$$\dim_{\mathcal{H}}(\mathcal{S}^E \cap \mathcal{R} \cap \Omega) \leq N - 8.$$

Moreover, under the additional assumption $\partial X \cap \Omega = \emptyset$*, then it holds*

$$\dim_{\mathcal{H}}(\mathcal{S}^E \cap \Omega) \leq N - 3.$$

Let us mention that in [144, Theorem 6.8] an epsilon-regularity result for locally perimeter minimizing sets and a Minkowski-type estimate of the singular set was obtained [144, Theorem 6.39].

2.4.5 Isoperimetric regions

In this section we report some results regarding isoperimetric sets in the RCD setting, which we use in Chapter 5. The results recalled here are based on the works [143, 153, 56, 30, 31, 33, 34, 31].

Let us report the definition of a volume-constrained minimizer for compact variations, which we sometimes refer to volume-constrained perimeter minimizer.

Definition 2.4.37 (Volume-constrained minimizer for compact variations)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. A finite perimeter set* $E \subset X$ *is said to be a* volume-constrained minimizer for compact variations in X *if, for any finite perimeter set* $F \subset X$ *with* $\mathfrak{m}(F) = \mathfrak{m}(E)$ *and* $E \Delta F \Subset X$*, it holds*

$$\text{Per}(E) \leq \text{Per}(F).$$

Definition 2.4.38. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. The* isoperimetric profile *of* X *is*

$$I_X(v) := \inf \operatorname{Per}(A)$$

where the infimum is taken over all $A \subset X$ *Borel measurable sets with* $\mathfrak{m}(A) = v$ *and* $v \in (0, \mathfrak{m}(X))$*. A set* $E \subset X$ *such that* $I_X(\mathfrak{m}(E)) = \operatorname{Per}(E)$ *is said to be an isoperimetric region.*

Let us point out that isoperimetric regions are volume constrained minimizers for compact variations. When the ambient space $(X, \mathsf{d}, \mathfrak{m})$ is compact, the existence of isoperimetric regions of each volume follows by applying the direct methods in the calculus of variations. The non-compact case has been studied in [143, 153, 30, 31, 33, 34, 31], among others.

An important property satisfied by the isoperimetric profile in the case the ambient space is a smooth compact N-dimensional Riemannian manifold (M, g) having Ricci curvature bounded from below by $K \in \mathbb{R}$ is that the function $\psi_{K,N} := I_M$ satisfies

$$-\psi''_{K,N} = \frac{KN}{N-1} \psi_{K,N}^{\frac{2-N}{N}} \tag{2.4.14}$$

in a weak sense. This fact was shown in [150, 37, 38, 39, 154]. In [34], the authors showed that (2.4.14) holds when the ambient space is a non-collapsed $\operatorname{RCD}(K, N)$ metric measure space $(X, \mathsf{d}, \mathcal{H}^N)$ satisfying $\mathcal{H}^N(B_1(x)) \geq v_0$ for all $x \in X$ for some $v_0 > 0$. In particular, it holds for non-compact Riemannian manifolds with Ricci curvature bounded from below and volume of unit balls bounded away from zero.

Another celebrated result related to isoperimetric regions is the Lévy-Gromov isoperimetric inequality. We report here the version of it that can be found in [56], which holds for general $\operatorname{CD}^*(K, N)$ spaces.

Theorem 2.4.39 ([56, Theorem 1.1])**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\operatorname{RCD}(K, N)$ *metric measure space for some* $N \in \mathbb{N}$ *and* $K > 0$ *with* $\mathfrak{m}(X) = 1$*. Then for every Borel subset* $E \subset X$ *it holds*

$$\lim_{\varepsilon \to 0} \frac{\mathfrak{m}(E^\varepsilon) - \mathfrak{m}(E)}{\varepsilon} =: \mathfrak{m}^+(E) \geq \frac{\mathcal{H}^N(\partial B)}{\mathcal{H}^N(S)},$$

where B *is a spherical cap in the model sphere* S *(that is,* S *is the* N*-dimensional sphere of constant Ricci curvature equal to* K*) chosen so that* $\frac{\mathcal{H}^N(B)}{\mathcal{H}^N(S)} = \mathfrak{m}(E)$ *and* $E^\varepsilon := \{x \in X : \mathsf{d}(x, y) \leq \varepsilon \text{ for some } y \in E\}$ *is the* ε*-enlargement of* E*.*

Let us report a topological regularity result found in [31, Theorems 1.3, 1.4]. First, let us recall the definition of Ahlfors regularity.

Definition 2.4.40 (Ahlfors regular subset)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space, $S \subset X$ and $k \geq 0$. We say that S is k-Ahlfors regular if there exists constants $C \geq 1$ and $r_0 > 0$ such that*

$$C^{-1} r^k \leq \mathcal{H}^k(S \cap B_r(x)) \leq C r^k,$$

for every $x \in S$ and $r < r_0$ such that $B_r(x) \Subset X$.

Theorem 2.4.41. *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ space and let $E \subset X$ be a volume-constrained minimizer for compact variations. Then $E^{(1)}$ is open and $\partial^* E = \partial(E^{(1)})$ is locally uniformly $(N-1)$-Ahlfors regular in X. Suppose that $\mathcal{H}^N(B_1(x)) \geq v_0$ also holds for some $v_0 > 0$ and every $x \in X$ and that E is an isoperimetric region. Then $E^{(1)}$ is bounded and $\partial(E^{(1)})$ is $(N-1)$-Ahlfors regular in X.*

In the remainder of this work, whenever $(X, \mathsf{d}, \mathcal{H}^N)$ is an $\mathrm{RCD}(K, N)$ space and $E \subset X$ is a volume-constrained minimizer for compact variations, we assume that E coincides with its open representative $E^{(1)}$.

Let $N > 1$, $H, K \in \mathbb{R}$. Then the Jacobian function is (cf. [34, Equation 2.14])

$$J_{H,N,K}(r) := \left(\cos_{\frac{K}{N-1}}(r) + \frac{H}{N-1} \sin_{\frac{K}{N-1}}(r) \right)_+^{N-1}. \tag{2.4.15}$$

Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ metric measure space and let $E \subset X$ be a volume-constrained minimizer for compact variations in X. The distance function from the closed set $\bar{E}$ is denoted by $\mathsf{d}_{\bar{E}}$. Given $t \geq 0$, we denote by E_t the set $\{x \in X : \mathsf{d}_{\bar{E}}(x) \leq t\}$.

Let us recall [34, Proposition 3.11], which is one of the regularity results used in Chapter 5. It allows us to compare the perimeter of an isoperimetric set with the perimeter of its equidistant sets.

Proposition 2.4.42 ([34, Proposition 3.11])**.** *Let $(X, d, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ metric measure space and let $E \subset X$ be a volume constrained minimizer for compact variations. Then there exists $c \in \mathbb{R}$ such that for any $t \geq 0$ it holds*

$$\mathrm{Per}(E_t) \leq J_{c,K,N}(t) \mathrm{Per}(E). \tag{2.4.16}$$

Chapter 3

Monotonicity formula and stratification

In this chapter, we present the results obtained in a joint work with Mondino and Semola [94]. There we proved a monotonicity formula for perimeter minimizing sets in $\mathrm{RCD}(0, N)$ metric measure cones, together with the associated rigidity statement, namely that if the monotonicity function is constant, the perimeter minimizing set is itself a cone.
Among the applications, we established sharp Hausdorff dimension estimates for the singular strata of perimeter minimizing sets in non-collapsed RCD spaces. We also proved the existence of blow-down cones for global perimeter minimizers in non-collapsed $\mathrm{RCD}(0, N)$ spaces with Euclidean volume growth. The result is also new for Riemannian manifolds.

Section 1 is dedicated to an overview of the main results of the chapter and provides some context to the theory. In *Section 2* we show the monotonicity formula for perimeter minimizing sets passing through the tip of an $\mathrm{RCD}(0, N)$ cone. The *last section* is dedicated to some applications of the monotonicity formula.

3.1 Introduction

In this section, we briefly contextualize the setting of the chapter and discuss in more detail the main results reported and their relevance.

The work presented in this chapter refines the analysis of perimeter minimizing sets started in [144]. In the *second section* of this chapter we show a monotonicity formula for cones over $\mathrm{RCD}(N-1, N)$ metric measure spaces. The result states that, for an entire local perimeter minimizer E in a cone on an RCD m.m.s. passing through the tip of such cone, the quantity

$$r \to \frac{\mathrm{Per}(E, B_r)}{r^{N-1}} \tag{3.1.1}$$

is non-decreasing. We also obtain the analogue of the classical rigidity result that comes with this monotonicity formula: if the aforementioned function (3.1.1) were to be constant, then E would need to be a cone itself. This is an important lemma in the classical theory of analysis of singularities of local perimeter minimizers in $\mathbb{R}^n$. It allows to show that the blow-up of a perimeter minimizer in the Euclidean setting at a singular point is a singular cone. See [136, Theorem 28.6]. We use the monotonicity formula to bridge density estimates and a characterization of cones by means of properties of their normal vector field, a result which we show in the RCD metric measure spaces setting. Contrary to the Euclidean case, the proof makes use of both the rigidity and the monotonicity statements as the density results in the non-smooth setting are not as strong.

In the *last section* of this chapter we show an application of the monotonicity formula obtained in [94]. We refine the analysis of the singular set of perimeter minimizers of [144] by providing a dimensional estimate on the singular stratum of the local perimeter minimizer. For each integer $k \leq N$ the k-singular stratum is defined as the set of points of ∂E at which there does not exist a tangent having both the space and the perimeter minimizer split a factor of $\mathbb{R}^{k+1}$. The stratification of the singular set of non-collapsed RCD m.m.s. has been carried out in [80, Theorem 1.8] (reported as Theorem 2.3.25 in the preliminaries). We employ some of the techniques developed there. For RCD metric measure spaces, the k-singular stratum is defined to be the set of points of the space that do not have a tangent splitting $\mathbb{R}^{k+1}$. The dimensional estimate obtained in [80] is analogous to ours, showing that perimeter minimizers achieve, in this sense, the same regularity of the ambient space.

To perform analysis of singularities, analogously to Geometric Measure theory in $\mathbb{R}^N$, singular cones play an important role. In the non-smooth setting, we need to deal with the fact that blow-ups need to be performed both for the set of finite perimeter and the ambient space. It is a now classical fact that the tangent to an RCD m.m.s. is a cone over an RCD space (see Theorem 2.3.31), which follows straight from the rigidity in the Bishop-Gromov inequality and from the "volume cone implies metric cone" shown in [79], which is based on the previous work [120] and that we recalled in Theorem 2.3.30. Monotonicity formulas for perimeter minimizers hold in the Euclidean setting as well as in Riemannian manifolds, where one needs to take into account a remainder term depending on the norm of the curvature tensor. In the non-smooth setting, it is not clear whether a monotonicity formula for locally perimeter minimizing sets holds in general. For the purposes of dimensional estimates of the singular set and stratification, a monotonicity formula for cones over RCD spaces is sufficient. Indeed, it is possible to reduce ourselves to a local perimeter minimizer inside a cone by performing a first blow-up, where then our monotonicity formula holds.

3.1.1 Monotonicity Formula

The first main result of this chapter is a monotonicity formula for perimeter minimizers in cones over $\mathrm{RCD}(N-2, N-1)$ spaces, with the associated conical rigidity statement. We recall that, by [120], the metric measure cone over a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(0, N)$ metric measure space if and only if $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(N-2, N-1)$ metric measure space.

Our main result is the following:

Theorem 3.1.1 (Monotonicity Formula)**.** *Let $N \geq 2$ and let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(N-2, N-1)$ space (with $\mathrm{diam}(X) \leq \pi$, if $N = 2$). Let $C(X)$ be the metric measure cone over $(X, \mathsf{d}, \mathfrak{m})$ and let O denote its tip. Let $E \subset C(X)$ be a global perimeter minimizer. Then the function $\Phi : (0, \infty) \to \mathbb{R}$ defined by*

$$\Phi(r) := \frac{\mathrm{Per}(E, B_r(O))}{r^{N-1}},$$

is non-decreasing. Moreover, if there exist $0 < r_1 < r_2 < \infty$ such that $\Phi(r_1) = \Phi(r_2)$, then $E \cap \big(B_{r_2}(O) \setminus \overline{B_{r_1}(O)}\big)$ is a conical annulus, in the sense that there exists $A \subset X$ such that

$$E \cap \big(B_{r_2}(O) \setminus \overline{B_{r_1}(O)}\big) = C(A) \cap \big(B_{r_2}(O) \setminus \overline{B_{r_1}(O)}\big),$$

where $C(A) = \{(t, x) \in C(X) \colon x \in A\}$ is the cone over $A \subset X$. In particular, if Φ is constant on $(0, \infty)$, then E is a cone (in the sense that there exists $A \subset X$ such that $E = C(A)$).

The above monotonicity formula with rigidity generalizes the analogous, celebrated result in the Euclidean setting, see for instance [90, 149, 136]. On smooth Riemannian manifolds, it is well known that an almost monotonicity formula holds, with error terms depending on two-sided bounds on the Riemann curvature tensor and on lower bounds on the injectivity radius. For cones over smooth cross sections, the monotonicity formula is a folklore result, see for instance [25, 87]. Some special

cases of Theorem 3.1.1 have been discussed recently in [86, 144]. We also mention that in [57] an analogous monotonicity formula in RCD metric measure cones has been obtained for solutions of free boundary problems, generalizing a well known Euclidean result.
In the proof, we adapt one of the classical strategies in the Euclidean setting. The implementation is of course technically more demanding, in particular for the rigidity part, due to the low regularity of the present context.

The relevance of Theorem 3.1.1 for the applications, that we are going to discuss below, comes from the fact that tangent cones of non-collapsed $\mathrm{RCD}(K,N)$ metric measure spaces $(X,\mathsf{d},\mathcal{H}^N)$ and blow-downs of $\mathrm{RCD}(0,N)$ spaces $(X,\mathsf{d},\mathcal{H}^N)$ with Euclidean volume growth are metric measure cones, see [79, 80, 120] for the present setting and the earlier [60, 61, 49] for previous results in the case of Ricci limit spaces and Alexandrov spaces.

It is an open question whether an almost monotonicity formula holds for perimeter minimizers in general $\mathrm{RCD}(K,N)$ spaces, possibly under the non-collapsing assumption. In particular, we record the following:

Open question: let $(X,\mathsf{d},\mathcal{H}^N)$ be an $\mathrm{RCD}(K,N)$ space and let $E\subset X$ be a local perimeter minimizing set. Is it true that the limit

$$\lim_{r\to 0}\frac{\mathrm{Per}(E,B_r(x))}{r^{N-1}} \tag{3.1.2}$$

exists for all $x\in\partial E$?

3.1.2 Stratification of the singular set and other applications

It is well known that monotonicity formulas are an extremely powerful tool in the analysis of singularities of several problems in Geometric Analysis. We just mention here, for the sake of illustration and because of the connection with the developments of this chapter:

- the Hausdorff dimension estimates for the singular strata of area minimizing currents in codimension one, originally obtained in [92];
- the Hausdorff dimension estimates for the singular strata of $\mathrm{RCD}(K,N)$ spaces $(X,\mathsf{d},\mathcal{H}^N)$, obtained in [80] and earlier in [61] in the case of non-collapsed Ricci limit spaces.

The proofs of the aforementioned results are based on the so-called dimension reduction technique, which relies in turn on the validity of a monotonicity formula with associated conical rigidity statement.
In the present work, building on the top of Theorem 3.1.1 we establish analogous Hausdorff dimension estimates for the singular strata of perimeter minimizing sets in $\mathrm{RCD}(K,N)$ spaces $(X,\mathsf{d},\mathcal{H}^N)$. Below we introduce the relevant terminology and state our main results.

Definition 3.1.2 (Singular Strata)**.** *Let $(X,\mathsf{d},\mathcal{H}^N)$ be an $\mathrm{RCD}(K,N)$ space, $E\subset X$ a locally perimeter minimizing set and $0\le k\le N-3$ an integer. The k-singular stratum of E, $\mathcal{S}^E_k$, is defined as*

$$\begin{aligned}\mathcal{S}^E_k :=\{x\in\partial E\colon &\text{ no tangent space to } (X,\mathsf{d},\mathcal{H}^N,x,E) \text{ is of the form } (Y,\rho,\mathcal{H}^N,y,F),\\ &\text{with } (Y,\rho,y) \text{ isometric to } (Z\times\mathbb{R}^{k+1},\mathsf{d}_Z\times\mathsf{d}_{\mathrm{eucl}},(z,0)) \text{ for some pointed } (Z,\mathsf{d}_Z,z)\\ &\text{and } F=G\times\mathbb{R}^{k+1} \text{ with } G\subset Z \text{ global perimeter minimizer}\}.\end{aligned}$$

The above definition would make sense also in the cases when $k\ge N-2$. However, it seems more appropriate not to adopt the terminology *singular strata* in those instances, since they are used to define the boundary of a non-collapsed RCD space.

Definition 3.1.3 (Interior and Boundary Regularity Points)**.** *Let $(X,\mathsf{d},\mathcal{H}^N)$ be an $\mathrm{RCD}(K,N)$ space and let $E\subset X$ be a locally perimeter minimizing set. Given $x\in\partial E$, we say that x is an* interior regularity *point if*

$$\mathrm{Tan}_x(X,\mathsf{d},\mathcal{H}^N,E)=\{(\mathbb{R}^N,\mathsf{d}_{\mathrm{eucl}},\mathcal{H}^N,0,\mathbb{R}^N_+)\}\,. \tag{3.1.3}$$

The set of interior regularity points of E will be denoted by $\mathcal{R}^E$.
Given $x\in\partial E$, we say that x is a boundary regularity *point if*

$$\mathrm{Tan}_x(X,\mathsf{d},\mathcal{H}^N,E)=\{(\mathbb{R}^N_+,\mathsf{d}_{\mathrm{eucl}},\mathcal{H}^N,0,\{x_1\ge 0\})\}\,, \tag{3.1.4}$$

where x_1 is one of the coordinates of the $\mathbb{R}^{N-1}$ factor in $\mathbb{R}^N_+=\mathbb{R}^{N-1}\times\{x_N\ge 0\}$. The set of boundary regularity points of E will be denoted by $\mathcal{R}^E_{\partial X}$.

It was proved in [144] that the interior regular set $\mathcal{R}^E$ is topologically regular, in the sense that it is contained in a Hölder open manifold of dimension $N-1$. By a blow-up argument it is not hard to show that $\dim_{\mathcal{H}}\mathcal{R}^E_{\partial X}\le N-2$ (see Proposition 3.3.3). An application of the stratification of the singular set of perimeter minimizers and the definition of boundary of a non-collapsed RCD m.m.s. [43] is that the complement of $\mathcal{S}^E_{N-3}$ in ∂E consists of either interior or boundary regularity points.

Theorem 3.1.4. *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be an* $\mathrm{RCD}(K, N)$ *space and let* $E \subset X$ *be a locally perimeter minimizing set. Then*

$$\partial E \setminus \mathcal{S}^E_{N-3} = \mathcal{R}^E \cup \mathcal{R}^E_{\partial X}\,. \tag{3.1.5}$$

The main result of this section is the stratification of the singular set of locally perimeter minimizing sets Theorem 3.1.5 below. As mentioned in the introduction of this thesis, the stratification of the singular set of objects that minimize the area in a suitable sense in the Euclidean setting [92, 188] is a fundamental result for the regularity theory of such objects. It was used as a step in understanding the regularity of minimizing integral currents of codimension greater than one, and it is widely used to analyze singularities in geometric analysis (see, for instance [188]). Moreover, it furthers our understading of the structure of the singular set of minimal boundaries. The proof of Theorem 3.1.5 builds on the monotonicity formula Theorem 3.2.1: the monotonicity formulas for cones over Riemannian manifolds (see [148, Theorem 9.3], [90, Theorem 5.4.3] and [147]) used in [144] to show the sharp Hausdorff dimension bounds on the singular set of locally perimeter minimizing sets are not strong enough to obtain Theorem 3.1.5.

Theorem 3.1.5 (Stratification of the singular set)**.** *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be an* $\mathrm{RCD}(K, N)$ *space and* $E \subset X$ *a locally perimeter minimizing set. Then for any* $0 \le k \le N-3$ *it holds*

$$\dim_{\mathcal{H}} \mathcal{S}^E_k \le k\,. \tag{3.1.6}$$

We point out that the Hausdorff dimension estimate for the top dimensional singular stratum had already been established in [86] (for limits of sequences of codimension one area minimizing currents in smooth Riemannian manifolds with Ricci curvature and volume lower bounds) and independently in [144] (in the same setting of non-collapsed $\mathrm{RCD}(K, N)$ spaces having $\partial X = \emptyset$). Elementary examples illustrate that the Hausdorff dimension estimates above are sharp in the present setting.

With respect to the classical [92] or [61, 80], in the proof of Theorem 3.1.5 we need to handle the additional difficulty that a monotonicity formula does not hold directly in the ambient space.

Another application of the monotonicity formula with the associated rigidity is that if an $\mathrm{RCD}(0, N)$ space $(X, \mathsf{d}, \mathcal{H}^N)$ with Euclidean volume growth contains a global perimeter minimizer, then any asymptotic cone contains a perimeter minimizing cone.

Theorem 3.1.6. *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an* RCD$(0, N)$ *metric measure space with Euclidean volume growth, i.e. satisfying for some (and thus for every) $x \in X$:*

$$\lim_{r \to \infty} \frac{\mathcal{H}^N(B_r(x))}{r^N} > 0.$$

Let $E \subset X$ be a global perimeter minimizer. Then for any blow-down $(C(Z), \mathsf{d}_{C(Z)}, \mathcal{H}^N)$ of $(X, \mathsf{d}, \mathcal{H}^N)$ there exists a cone $C(W) \subset C(Z)$ which is a global perimeter minimizer.

The conclusion of Theorem 3.1.6 above seems to be new also in the more classical case of smooth Riemannian manifolds with non-negative sectional curvature, or non-negative Ricci curvature. We refer to [25] for earlier progress in the case of smooth manifolds with non-negative sectional curvature satisfying additional conditions on the rate of convergence to the tangent cone at infinity and on the regularity of the cross section, and to the more recent [87] for the case of smooth Riemannian manifolds with non-negative Ricci curvature and quadratic curvature decay.

Related to the open question that we raised above, to the best of our knowledge it is not currently known whether in the setting of Theorem 3.1.6 any blow-down of the perimeter minimizing set must actually be a cone.

3.2 Monotonicity formula

A classical and extremely powerful tool for studying sets which locally minimize the perimeter in Euclidean spaces is the monotonicity formula for the perimeter. The goal of this section is to generalize such monotonicity formula (with the associated rigidity statement) to perimeter minimizers in cones over non-collapsed RCD(K, N) spaces. In the next section, we draw some applications on the structure of the singular set of local perimeter minimizers.

Recall that given an RCD$(N-2, N-1)$ space $(X, \mathsf{d}_X, \mathfrak{m}_X)$ the metric-measure cone over X, denoted by $(C(X), \mathsf{d}_C, \mathfrak{m}_C)$, is an RCD$(0, N)$ space (if $N = 2$, we also assume that $\mathrm{diam}(X) \leq \pi$). We denote by $O = (0, x) \in C(X)$ the tip of the cone (see Section 2.3.8 for more details) and $B_r(O)$ the open metric ball centered at O of radius $r > 0$.

When we consider a local perimeter minimizer E, we shall always assume that $E = E^{(1)}$ is the open representative, given by the measure theoretic interior. See [122] for the relevant background.

Theorem 3.2.1 (Monotonicity formula)**.** *Let $N \geq 2$ and let $(X, \mathsf{d}, \mathfrak{m})$ be an* RCD$(N-2, N-1)$ *space (with* $\mathrm{diam}(X) \leq \pi$*, if $N = 2$). Let $C(X)$ be the metric measure* *cone over* $(X, d, \mathfrak{m})$*. Let* $E \subset C(X)$ *be a global perimeter minimizer in the sense of Definition 2.4.29. Then the function* $\Phi : (0, \infty) \to \mathbb{R}$ *defined by*

$$\Phi(r) := \frac{\mathrm{Per}(E, B_r(O))}{r^{N-1}}, \tag{3.2.1}$$

is non-decreasing. Moreover, if there exist $0 < r_1 < r_2 < \infty$ *such that* $\Phi(r_1) = \Phi(r_2)$*, then* $E \cap \left(B_{r_2}(O) \setminus \overline{B_{r_1}(O)}\right)$ *is a conical annulus, in the sense that there exists* $A \subset X$ *such that*

$$E \cap \left(B_{r_2}(O) \setminus \overline{B_{r_1}(O)}\right) = C(A) \cap \left(B_{r_2}(O) \setminus \overline{B_{r_1}(O)}\right),$$

where $C(A) = \{(t, x) \in C(X) : x \in A\}$ *is the cone over* $A \subset X$*. In particular, if* Φ *is constant on* $(0, \infty)$*, then* E *is a cone (in the sense that there exists* $A \subset X$ *such that* $E = C(A)$*; see Section 2.3.8).*

Remark 3.2.2. *In the case where* $E \subset C(X)$ *is a locally finite perimeter set, minimizing the perimeter for perturbations supported in* $B_{R+1}(O)$*, then the monotonicity formula holds on* $(0, R)$*, i.e. the function* Φ *defined in* (3.2.1) *is non-decreasing on* $(0, R)$*. Also the rigidity statement holds, for* $0 < r_1 < r_2 < R$*. The proofs are analogous.*

Proof of Theorem 3.2.1. Let us first give an outline of the argument. The first two steps are inspired by the approach used in the lecture notes [146], which provide a proof of the monotonicity formula for local perimeter minimizers in Euclidean spaces by-passing the first variation formula. Classical references for this approach are [90, 149].

The main idea is to approximate the characteristic function of E by regular functions f_k and approximate Φ by the corresponding Φ_{f_k}; show an almost-monotonicity formula for Φ_{f_k} and finally pass to the limit and get the monotonicity of Φ. This is achieved in steps 1-3. In step 4 we relate the derivative of Φ with a quantity characterizing cones as in Lemma 3.2.3.

Throughout the proof, we write B_r in place of $B_r(O)$ for the ease of notation.

Step 1: Approximation preliminaries.

In this step we show that, up to error terms, regular functions approximating χ_E preserve the minimality condition. The argument requires an initial approximation. Let $f \in \mathrm{LIP}(C(X)) \cap \mathrm{D_{loc}}(\Delta)(C(X))$ be non-negative. We introduce two functions a, $b : [0, \infty) \to [0, \infty)$ to quantify the errors in the approximation:

$$a(r) := \left| |Df|(B_r) - \mathrm{Per}(E, B_r) \right|, \quad b(r) := \int_{\partial B_r} |\mathrm{tr}^{\mathrm{ext}}_{\partial B_r} \chi_E - \mathrm{tr}_{\partial B_r} f| \, d\mathrm{Per}(B_r), \tag{3.2.2}$$

where $\mathrm{tr}^{\mathrm{ext}}_{\partial B_r}\chi_E$ is the trace of χ_E from the exterior of the ball B_r.
We remark that the interior and exterior normal traces can be defined by considering the precise representative of $\chi_E \cdot \chi_{B_r}$ and $\chi_E \cdot \chi_{X\setminus B_r}$ respectively. See [14] for the Euclidean theory.

Notice that $\mathrm{tr}^{\mathrm{ext}}_{\partial B_r} f = \mathrm{tr}_{\partial B_r} f = f|_{\partial B_r}$, since f is continuous. Fix $R > 0$. Let $0 < r < R$ and $g \in \mathrm{BV}_{\mathrm{loc}}(C(X))$ be any function such that

$$\mathrm{tr}^{\mathrm{int}}_{\partial B_r} g = \mathrm{tr}_{\partial B_r} f \quad \text{and} \quad g = \chi_E \text{ on } C(X) \setminus B_r \,,$$

where $\mathrm{tr}^{\mathrm{int}}_{\partial B_r} g$ is the trace of g from the interior of the ball B_r. The minimality of E implies

$$\mathrm{Per}(E, B_R) \leq \mathrm{Per}(\{q \in C(X) \,:\, g(q) > t\}; B_R),$$

for any $0 < t < 1$. Integrating in t and using the coarea formula (2.4.8) we obtain

$$\mathrm{Per}(E, B_R) \leq \int_0^1 \mathrm{Per}(\{q \in C(X) \,:\, g(q) > t\}; B_R)\, dt \leq |Dg|(B_R)\,.$$

Therefore, using Lemma 2.4.4 and the definition of g, we obtain

$$\begin{aligned}
\mathrm{Per}(E, B_r) &= \mathrm{Per}(E, B_R) - \mathrm{Per}(E, B_R \setminus B_r) \leq |Dg|(B_R) - \mathrm{Per}(E, B_R \setminus B_r) \\
&= |Dg|(B_r) + \int_{\partial B_r} |\mathrm{tr}^{\mathrm{ext}}_{\partial B_r}\chi_E - \mathrm{tr}_{\partial B_r} f|\, d\mathrm{Per}(B_r) = |Dg|(B_r) + b(r)\,.
\end{aligned}$$

Finally, for any such g it holds

$$|Df|(B_r) \leq \mathrm{Per}(E, B_r) + a(r) \leq |Dg|(B_r) + a(r) + b(r)\,. \tag{3.2.3}$$

Step 2. Main computation.
In this step we show the monotonicity, up to error terms, of an approximation of Φ, denoted below by Φ_f, obtained by replacing χ_E with the regular approximation f of step 1.

Fix f as in step 1 and $r > 0$. By [102],

$$|\nabla f|^2(t, x) = |\nabla f^{(x)}|^2(t) + t^{-2}|\nabla f^{(t)}|^2(x)\,, \text{for } \mathfrak{m}\text{-a.e. } x \in X \text{ and } \mathcal{L}^1\text{-a.e. } t > 0\,.$$

Let $h : C(X) \to \mathbb{R}$ be defined by $h(t, x) := f^{(r)}(x)$ for all $t > 0$. Notice that h is locally Lipschitz away from the origin and it is elementary to check that it has locally bounded variation.

By [102, 101], it holds

$$|Dh|(t,x) = \frac{r}{t}|\nabla f^{(r)}|(x)\,, \quad \text{for } \mathfrak{m}\text{-a.e. } x \text{ and } \mathcal{L}^1\text{-a.e. } t\,.$$

By integrating over B_r and using the coarea formula, we obtain

$$\begin{aligned}\int_{B_r} |Dh|(t,x)\, d\mathfrak{m}_C &= \int_0^r \int_{\partial B_t} |Dh|(t,x)\, d\mathrm{Per}(B_t)\, dt = \int_0^r t^{N-1} \int_X \frac{r}{t}|\nabla f^{(r)}|(x) d\mathfrak{m}\, dt \\ &= \int_0^r \frac{t^{N-2}}{r^{N-2}} \int_{\partial B_r} |\nabla f^{(r)}|(x)\, d\mathrm{Per}(B_r)\, \mathrm{d}t \\ &= \frac{r}{N-1} \int_{\partial B_r} |\nabla f^{(r)}|(x)\, d\mathrm{Per}(B_r)\,.\end{aligned} \tag{3.2.4}$$

Let us point out that the latter expression can be viewed as the integral on ∂B_r of the analog of the tangential derivative of f in the smooth case, while h is the radial extension of the values of f on ∂B_r to the whole of $C(X)$. Given $r > 0$, let us introduce the quantity

$$J(r) := \int_{B_r} |\nabla f|(t,x)\, d\mathfrak{m}_C = \int_0^r t^{N-1} \int_X |\nabla f|(t,x)\, d\mathfrak{m}\, dt\,,$$

which will approximate $r^{N-1}\Phi(r)$. Notice that J is a Lipschitz function, hence it is almost everywhere differentiable. Using the identity (3.2.4), we obtain that for a.e. r it holds

$$\begin{aligned}J'(r) &= \int_{\partial B_r} |\nabla f|(r,x)\, d\mathrm{Per}(B_r) \\ &= \frac{N-1}{r} \int_{B_r} |Dh|(t,x)\, d\mathfrak{m}_C + \int_{\partial B_r} \left(|\nabla f|(r,x) - |\nabla f^{(r)}|(x)\right)\, d\mathrm{Per}(B_r)\,.\end{aligned} \tag{3.2.5}$$

We notice that $\mathrm{tr}_{\partial B_r} h = \mathrm{tr}_{\partial B_r} f$. By defining $\tilde{h} : C(X) \to \mathbb{R}$ to be equal to h inside B_r and χ_E outside, we observe that (3.2.4) and (3.2.5) still hold if we replace h by $\tilde{h}$ (here it is key that B_r is the open ball). Therefore, in step 1 we can choose $g = \tilde{h}$ and (3.2.3) reads as

$$\int_{B_r} |D\tilde{h}|(t,x)\, d\mathfrak{m}_C + a(r) + b(r) \geq J(r)\,. \tag{3.2.6}$$

Substituting (3.2.6) into (3.2.5) and rearranging, yields

$$J'(r) - \frac{N-1}{r} J(r) \geq \int_{\partial B_r} \left(|\nabla f|(r,x) - |\nabla f^{(r)}|(x)\right)\, d\mathrm{Per}(B_r) - \frac{N-1}{r}(a(r) + b(r))\,. \tag{3.2.7}$$

With a slight abuse, in order to keep notation simple, below we will write $\nabla \mathsf{d}_C(O,(r,x))$ to denote $\nabla(\mathsf{d}_C(O,\cdot))(r,x)$. Using Proposition 2.3.34 together with the fact that $1-\sqrt{1-s} \geq \frac{s}{2}$, for $0 \leq s \leq 1$, the BL characterization of the norm [102, Def. 3.8] of $|\nabla f|$ and the identification between minimal weak upper gradients for different exponents on RCD spaces from [101], we have

$$\begin{aligned}\frac{|\nabla f|(r,x) - |\nabla f^{(r)}|(x)}{|\nabla f|(r,x)} &= 1 - \sqrt{1 - \frac{(\nabla f(r,x)\cdot \nabla \mathsf{d}_C(O,(r,x)))^2}{|\nabla f|(r,x)^2}} \\ &\geq \frac{(\nabla f(r,x)\cdot \nabla \mathsf{d}_C(O,(r,x)))^2}{2|\nabla f|(r,x)^2} \\ &= \frac{\left(\nabla f(r,x)\cdot \nabla(\frac{1}{2}\mathsf{d}_C^2(O,(r,x)))\right)^2}{2r^2\left(|\nabla f|(r,x)\right)^2},\end{aligned} \tag{3.2.8}$$

for $\mathfrak{m}$-a.e. $x \in X$ and a.e. $r \in (0,+\infty)$. Above, we understand that all the terms vanish on the set where $|\nabla f| = 0$.

Let us now define the function

$$\Phi_f(r) := \frac{\int_{B_r} |\nabla f|(t,x)\, d\mathfrak{m}_C}{r^{N-1}} = \frac{J(r)}{r^{N-1}}$$

which will approximate the function Φ in the statement of the theorem. Notice that Φ_f is Lipschitz and differentiable almost everywhere by the coarea formula (2.4.8). Taking its derivative and using (3.2.7) and (3.2.8) we obtain that for a.e. r it holds

$$\begin{aligned}\Phi_f'(r) &= \frac{J'(r) - \frac{N-1}{r}J(r)}{r^{N-1}} \\ &\geq \int_{\partial B_r} \frac{\left(\nabla f(r,x)\cdot \nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(r,x))\right)\right)^2}{2r^{N+1}|\nabla f|(r,x)}\, d\mathrm{Per}(B_r) - \frac{N-1}{r^N}(a(r)+b(r)).\end{aligned} \tag{3.2.9}$$

Integrating (3.2.9) from $0 < r_1 < r_2 < \infty$, and using coarea formula, we get

$$\begin{aligned}\Phi_f(r_2) - \Phi_f(r_1) \geq &\int_{B_{r_2}\setminus \overline{B_{r_1}}} \frac{\left(\nabla f(r,x)\cdot \nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(r,x))\right)\right)^2}{2r^{N+1}|\nabla f|(r,x)}\, d\mathfrak{m}_C \\ &- \int_{r_1}^{r_2} \frac{N-1}{r^N}(a(r)+b(r))\, dr.\end{aligned} \tag{3.2.10}$$

Step 3. Approximation.

In this step we carry out an approximating argument, using step 2 and in particular (3.2.10). This allows us to conclude the monotonicity part of the theorem.

Let $\{f_k\}_{k\in\mathbb{N}} \subset \mathrm{LIP}(C(X)) \cap \mathrm{D}_{\mathrm{loc}}(\Delta)$ be a sequence of non-negative functions converging in $\mathrm{BV}_{\mathrm{loc}}(X)$ to χ_E. That is,

$$|f_k - \chi_E|_{L^1(B_r)} \overset{k\to\infty}{\longrightarrow} 0, \qquad |Df_k|(B_r) \overset{k\to\infty}{\longrightarrow} |D\chi_E|(B_r), \qquad \text{for all } r>0. \tag{3.2.11}$$

Such sequence can be easily constructed by approximation via the heat flow, see for instance [15] for analogous arguments.

Let us start by showing that the errors defined in (3.2.2) relative to f_k go to zero as k tends to ∞.

The term $a_k(r) := ||Df_k|(B_r) - \mathrm{Per}(E, B_r)| \overset{k\to\infty}{\longrightarrow} 0$ by $\mathrm{BV}_{\mathrm{loc}}$-convergence of f_k to χ_E, i.e. (3.2.11).

To deal with the error term $b_k(r) := \int_{\partial B_r} |\mathrm{tr}^{\mathrm{ext}}_{\partial B_r}\chi_E - \mathrm{tr}_{\partial B_r} f_k| d\mathrm{Per}(B_r)$, we can use the coarea formula (2.4.8) to show that

$$\int_{B_r} |f_k - \chi_E| \, d\mathfrak{m}_C = \int_0^r b_k(s) \, ds \, .$$

Together with the L^1-convergence of f_k to χ_E, this shows that $b_k(r) \to 0$ for $\mathcal{L}^1$-a.e. $r > 0$.

Lastly, let us show that $\Phi_f(r) \to \Phi(r)$ for $\mathcal{L}^1$-a.e. $r > 0$. By $\mathrm{BV}_{\mathrm{loc}}$ convergence of f_k to χ_E (3.2.11), we have

$$\lim_{k\to\infty} \Phi_{f_k}(r) = \frac{1}{r^{N-1}} \lim_{k\to\infty} \int_{B_r} |Df_k| d\mathfrak{m}_C = \frac{1}{r^{N-1}} \int_{B_r} d\mathrm{Per}(E) = \Phi(r) \, . \tag{3.2.12}$$

Consequently, letting $k \to \infty$ in the estimate (3.2.10) with f replaced by f_k, we obtain

$$\Phi(r_2) - \Phi(r_1) \geq 0 \, , \qquad \text{for } \mathcal{L}^1\text{-almost every } \; r_2 > r_1 > 0 \, , \tag{3.2.13}$$

thanks to the non-negativity of the term

$$\int_{B_{r_2}\setminus\overline{B_{r_1}}} \frac{\left(\nabla f_k(r,x) \cdot \nabla \left(\frac{1}{2}\mathsf{d}_C^2(O,(r,x))\right)\right)^2}{2r^{N+1}|\nabla f_k|(r,x)} \, d\mathfrak{m}_C \geq 0 \, . \tag{3.2.14}$$

To conclude that Φ is monotone, we need to extend (3.2.13) to every $r_2 > r_1 > 0$.

Let $\{r_k\}_{k\in\mathbb{N}}$ be any sequence such that $r_k \uparrow r$. Since B_r is open, $B_{r_k} \uparrow B_r$. Hence, by the inner regularity of measures

$$\mathrm{Per}(E, B_{r_k}) \to \mathrm{Per}(E, B_r) \, .$$

Let $r_2 > r_1 > 0$. Since the set of radii for which (3.2.13) holds is dense, we can find $\{r_{1,k}\}_{k\in\mathbb{N}}$ and $\{r_{2,l}\}_{l\in\mathbb{N}}$ for which (3.2.13) holds and such that $r_{1,k} \uparrow r_1$ and $r_{2,l} \uparrow r_2$. Then

$$0 \leq \lim_{l\to\infty} \Phi(r_{2,l}) - \lim_{k\to\infty} \Phi(r_{1,k}) = \Phi(r_2) - \Phi(r_1) \, .$$

Step 4. Rigidity.

In this step we focus on the rigidity part of the statement. We show that if there exist $r_2 > r_1 > 0$ such that $\Phi(r_1) = \Phi(r_2)$, then $E \cap \left(B_{r_2} \setminus \overline{B_{r_1}}\right)$ is a cone.

The first step is to prove the following claim:

$$\begin{aligned}\liminf_{k\to\infty} \int_{B_{r_2}\setminus\overline{B_{r_1}}} & \frac{\left(\nabla f_k(r,x)\cdot\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(r,x))\right)\right)^2}{2r^{N+1}|\nabla f_k|(r,x)}\, d\mathfrak{m}_C \\ & \geq \int_{B_{r_2}\setminus\overline{B_{r_1}}} \frac{\left(\nu_E(r,x)\cdot\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(r,x))\right)\right)^2}{2r^{N+1}}\, d\mathrm{Per}(E)\,,\end{aligned} \tag{3.2.15}$$

where ν_E is the unit normal to E, see Theorem 2.4.21. Subsequently, we will be able to conclude using the characterization of cones provided by Lemma 3.2.3.

The plan is to apply Lemma 3.2.6 to prove (3.2.15). Using the notation of Lemma 3.2.6, we define the measures

$$\mu_k := |\nabla f_k|\mathfrak{m}_C \llcorner_{(B_{r_2}\setminus\overline{B_{r_1}})}\,, \qquad \mu := \mathrm{Per}(E,\cdot)\llcorner_{(B_{r_2}\setminus\overline{B_{r_1}})}\,. \tag{3.2.16}$$

The $\mathrm{BV}_{\mathrm{loc}}$-convergence of f_k to χ_E (see (3.2.11)) ensures that $\mu_k \rightharpoonup \mu$ in duality with $\mathrm{C_b}(C(X))$. The functions

$$g_k := \frac{\nabla f_k\cdot\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)}{\sqrt{2}r^{\frac{N+1}{2}}|\nabla f_k|}\cdot\chi_{\{|\nabla f_k|>0\}} \in L^2(C(X);\mu_k)$$

satisfy (3.2.46). Indeed,

$$\nabla f_k(r,x)\cdot\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(r,x))\right) \leq \frac{1}{2}|\nabla f_k|(r,x)|\nabla\mathsf{d}_C^2(O,(r,x))| = r|\nabla f_k|(r,x)\,,$$

for $\mathfrak{m}_C$-almost every (r,x).

Therefore, using (2.3.12),

$$\begin{aligned}\|g_k\|^2_{L^2(C(X);\mu_k)} &= \int_{B_{r_2}\setminus\overline{B_{r_1}}} \frac{\left(\nabla f_k\cdot\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)\right)^2}{2r^{N+1}|\nabla f_k|}\cdot\chi_{\{|\nabla f_k|>0\}}d\mathfrak{m}_C \\ &\leq \int_{B_{r_2}\setminus\overline{B_{r_1}}} \frac{1}{2r^{N-1}}|\nabla f_k|\, d\mathfrak{m}_C < C < +\infty\,,\end{aligned}$$

for some $C > 0$ independent of $k \in \mathbb{N}$ thanks to the $\mathrm{BV}_{\mathrm{loc}}$-convergence (3.2.11). Consequently, Lemma 3.2.6 provides the existence of $g \in L^2(C(X);\mu)$ and a subsequence $k(l)$ such that

$$\frac{\nabla f_{k(l)}\cdot\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)}{\sqrt{2}r^{\frac{N+1}{2}}|\nabla f_{k(l)}|}\cdot\chi_{\{|\nabla f_{k(l)}|>0\}}\,\mu_{k(l)} \rightharpoonup g\,\mathrm{Per}(E,\cdot)\llcorner_{(B_{r_2}\setminus\overline{B_{r_1}})}\,, \tag{3.2.17}$$

in duality with $\mathrm{C_b}(C(X))$. Up to relabeling the approximating sequence f_k, we can suppose that the whole sequence satisfies (3.2.17). We next determine the limit function g.

Fix a test function $\varphi \in \mathrm{LIP} \cap \mathrm{D}(\Delta)(B_{r_2} \setminus \overline{B_{r_1}})$ with compact support contained in $B_{r_2} \setminus \overline{B_{r_1}}$.

We apply the Gauss-Green formula (Theorem 2.4.21) and use that φ has compact support in $B_{r_2} \setminus \overline{B_{r_1}}$ to obtain

$$\begin{aligned} &\int_{B_{r_2}\setminus\overline{B_{r_1}}} f_k \operatorname{div}\left(\frac{\varphi}{\sqrt{2}r^{\frac{N+1}{2}}}\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)\right)\, d\mathfrak{m}_C \\ &\qquad = -\int_{B_{r_2}\setminus\overline{B_{r_1}}} \frac{\varphi}{\sqrt{2}r^{\frac{N+1}{2}}}\left(\nabla f_k \cdot \nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)\right)\, d\mathfrak{m}_C \\ &\qquad = -\int_{B_{r_2}\setminus\overline{B_{r_1}}} \varphi \frac{\nabla f_k \cdot \nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)}{\sqrt{2}r^{\frac{N+1}{2}}|\nabla f_k|}\, d\mu_k\,. \end{aligned} \tag{3.2.18}$$

Using the L^1-convergence of f_k to χ_E and that

$$\left\|\frac{\operatorname{div}(\varphi\nabla(\frac{1}{2}\mathsf{d}_C^2(O,\cdot))}{\sqrt{2}r^{\frac{N+1}{2}}}\right\|_{L^\infty(B_{r_2}\setminus\overline{B_{r_1}})} < \infty\,,$$

we infer that

$$\begin{aligned} &\lim_{k\to\infty}\int_{B_{r_2}\setminus\overline{B_{r_1}}} f_k \operatorname{div}\left(\frac{\varphi}{\sqrt{2}r^{\frac{N+1}{2}}}\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)\right)\, d\mathfrak{m}_C \\ &\qquad = \int_E \operatorname{div}\left(\frac{\varphi}{\sqrt{2}r^{\frac{N+1}{2}}}\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)\right)\, d\mathfrak{m}_C \\ &\qquad = -\int_{\partial^* E} \frac{\varphi}{\sqrt{2}r^{\frac{N+1}{2}}}\, \nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)\cdot \nu_E\, d\mathrm{Per}(E)\,, \end{aligned} \tag{3.2.19}$$

where in the last equality we used the Gauss-Green formula (Theorem 2.4.21). Combining (3.2.18) and (3.2.19) we obtain

$$\lim_{k\to\infty}\int_{C(X)} \varphi\frac{\nabla f_k\cdot\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)}{\sqrt{2}r^{\frac{N+1}{2}}|\nabla f_k|}\, d\mu_k = \int_{\partial^* E}\frac{\varphi}{\sqrt{2}r^{\frac{N+1}{2}}}\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(\cdot))\right)\cdot \nu_E\, d\mathrm{Per}(E).$$

That is,

$$\frac{\nabla f_k\cdot\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot))\right)}{\sqrt{2}r^{\frac{N+1}{2}}|\nabla f_k|}\mu_k \rightharpoonup \frac{\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(\cdot))\right)\cdot\nu_E}{\sqrt{2}r^{\frac{N+1}{2}}}\mathrm{Per}(E) \tag{3.2.20}$$

in duality with $\mathrm{C}_c(B_{r_2}\setminus\overline{B_{r_1}})$, by approximation. By the uniqueness of the weak limit and from (3.2.17) we can conclude that

$$g = \frac{\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)\cdot\nu_E}{\sqrt{2}r^{\frac{N+1}{2}}} \qquad \mathrm{Per}(E)|_{B_{r_2}\setminus\overline{B_{r_1}}}\text{-almost everywhere}\,.$$

From (3.2.48) in Lemma 3.2.6, we have

$$\liminf_{k\to\infty} \|g_k\|^2_{L^2(C(X);\mu_k)} \geq \|g\|^2_{L^2(C(X);\mu)} .$$

That is, we have shown the claim (3.2.15).

We are now in position to improve the estimate (3.2.13) and use it to show the rigidity part of the theorem. By taking the inferior limit in (3.2.10), recalling (3.2.12) and that the error terms go to zero from step 3, we use (3.2.15) to infer

$$\Phi(r_2) - \Phi(r_1) \geq \int_{B_{r_2}\setminus\overline{B_{r_1}}} \frac{\left(\nu_E(r,x)\cdot\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(r,x))\right)\right)^2}{2r^{N+1}}\, d\mathrm{Per}(E) \geq 0, \tag{3.2.21}$$

for every $r_2 > r_1 > 0$. Since we are assuming $\Phi(r_1) = \Phi(r_2)$, if follows that

$$\nabla(\mathsf{d}_C(O,\cdot))\cdot\nu_E = 0 \quad \mathrm{Per}(E)\text{-a.e. on } B_{r_2}\setminus\overline{B_{r_1}} .$$

By applying Lemma 3.2.3, we can conclude that $E\cap\left(B_{r_2}\setminus\overline{B_{r_1}}\right)$ is a conical annulus.

□

Let us now prove a useful characterization of conical annuli contained in cones over RCD spaces. The characterization is based on the properties of the normal to the boundary of the subset. A subset is conical if and only if its normal is orthogonal to the gradient of the distance function from the tip of the ambient conical space. In case the ambient space is Euclidean, the result is classical (see for instance [136, Proposition 28.8]).

Lemma 3.2.3 (Characterization of conical annuli)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an* $\mathrm{RCD}(N-2,N-1)$ *space and let $C(X)$ be the cone over X. Let $E\subset C(X)$ be a locally finite perimeter set and let $0<r_1<r_2<\infty$. Then the measure theoretic interior $E^{(1)}\cap\left(B_{r_2}(O)\setminus\overline{B_{r_1}(O)}\right)$ is a conical annulus if and only if*

$$\nabla(\mathsf{d}_C(O,\cdot))\cdot\nu_E = 0 \quad \mathrm{Per}(E)\text{-a.e. on } B_{r_2}(O)\setminus\overline{B_{r_1}(O)}. \tag{3.2.22}$$

Proof. As in the proof of Theorem 3.2.1, to keep notation short we will write B_r to denote the open ball of radius $r>0$ and centered at the tip of the cone, i.e. $B_r = B_r(O)$. Also, we write $B_r^X(x)$ for the open ball in X, of center x and radius $r>0$. For simplicity of presentation we will show the equivalence only in the case $r_1=0$, $r_2=\infty$. The general case requires minor modifications. Moreover, in order to simplify the notation, we assume without loss of generality that $E=E^{(1)}$, as the condition (3.2.22) is independent of the chosen representative.

Step 1.

We start with some preliminary computations aimed at establishing the identity (3.2.28) below, which will be key in showing the characterization of conical annuli in $C(X)$.

Using the Gauss-Green and the coarea formulas, we will express the derivative of the function

$$u(s) := \mathfrak{m}_C(E \cap C(B_r^X(x)) \cap B_s)$$

(suitably rescaled) as the product between the unit normal of E and the gradient of the distance function from the tip of $C(X)$.

Let $x \in X$, $r, s > 0$. By Lemma 2.4.13 and Theorem 2.4.26 the set $F := E \cap C(B_r^X(x)) \cap B_s$ is a set of finite perimeter with

$$\begin{aligned} D\chi_F = & D\chi_E \llcorner_{C(B_r^X(x)) \cap B_s} + \nabla(\mathsf{d}_C(O,\cdot))\mathrm{Per}(B_s) \llcorner_{E \cap C(B_r^X(x))} \\ & + \nu_{C(B_r^X(x))} \mathrm{Per}(C(B_r^X(x))) \llcorner_{E \cap B_s}. \end{aligned}$$

Using the Gauss-Green formula (Theorem 2.4.21), the equality for the laplacian of the distance function from the tip on cones [79, Prop. 3.7], and cut and paste of sets of locally finite perimeter (Theorem 2.4.26), we obtain

$$\begin{aligned} N \cdot u(s) &= \int_F \Delta \left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) d\mathfrak{m}_C \\ &= \int_{C(B_r^X(x)) \cap B_s \cap \partial^* E} \nabla \left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) \cdot \nu_E \, d\mathrm{Per}(E) \\ &\quad + \int_{E \cap C(B_r^X(x)) \cap \partial B_s} \nabla \left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) \cdot \nu_{\partial B_s} \, d\mathrm{Per}(B_s) \\ &\quad + \int_{E \cap B_s \cap \partial C(B_r^X(x))} \nabla \left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) \cdot \nu_{C(B_r^X(x))} \, d\mathrm{Per}(C(B_r^X(x))). \end{aligned} \tag{3.2.23}$$

We now study separately the three integrals on the right hand side of (3.2.23), starting from the last one.

Fix a function $\varphi \in \mathrm{LIP}(C(X)) \cap \mathrm{D}(\Delta)$ with compact support. By applying the Gauss-Green Theorem 2.4.21 on the set of locally finite perimeter $C(B_r^X(x))$, we obtain

$$\begin{aligned} &\int_{\partial C(B_r^X(x))} \varphi \, \nu_{C(B_r^X(x))} \cdot \nabla \left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) d\mathrm{Per}(C(B_r^X(x))) \\ &= -\int_{C(B_r^X(x))} \nabla\varphi \cdot \nabla \left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) d\mathfrak{m}_C + \int_{C(B_r^X(x))} \varphi \Delta \left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) d\mathfrak{m}_C \\ &= \int_{B_r^X(x)} \int_0^\infty \partial_r \varphi^{(y)}(r) \cdot \partial_r \left(\frac{1}{2}\mathsf{d}^2_{(y)}(r)\right) r^{N-1} \, \mathrm{d}r \, d\mathfrak{m}(y) + \int_{C(B_r^X(x))} \varphi N d\mathfrak{m}_C \\ &= -\int_{B_r^X(x)} \int_0^\infty \varphi^{(y)}(r) N r^{N-1} \, dr \, d\mathfrak{m}(y) + \int_{C(B_r^X(x))} \varphi N d\mathfrak{m}_C = 0, \end{aligned}$$

where we have used (2.3.11), the definition of $\mathfrak{m}_C$ and integration by parts on $\mathbb{R}$. Since φ was arbitrary, we infer that

$$\nu_{C(B_r^X(x))} \cdot \nabla \left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) = 0 \quad \mathrm{Per}(C(B_r^X(x)))\text{-a.e.}$$

and thus

$$\int_{E\cap B_s\cap\partial C(B_r^X(x))} \nabla \left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) \cdot \nu_{C(B_r^X(x))}\, d\mathrm{Per}(C(B_r^X(x))) = 0\,. \tag{3.2.24}$$

Let us now deal with the second integral appearing in the right hand side of (3.2.23). By the chain rule, we have $\nabla\left(\frac{1}{2}\mathsf{d}^2(O,q)\right) = \mathsf{d}(O,q)\nabla\mathsf{d}(O,q)$. Therefore, we obtain

$$\begin{aligned}\int_{E\cap C(B_r^X(x))\cap\partial B_s} & \nabla \left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) \cdot \nu_{\partial B_s}\, d\mathrm{Per}(B_s) \\ &= \int_{E\cap C(B_r^X(x))\cap\partial B_s} \mathsf{d}(O,\cdot)\,\nabla\mathsf{d}(O,\cdot)\cdot \nu_{\partial B_s}\, d\mathrm{Per}(B_s) \\ &= s\mathrm{Per}(B_s; E\cap C(B_r^X(x)))\,,\end{aligned} \tag{3.2.25}$$

where we used [44, Prop. 6.1]. Inserting (3.2.24) and (3.2.25) into (3.2.23), yields

$$u(s) = \frac{1}{N}\int_{C(B_r^X(x))\cap B_s} \nabla \left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) \cdot \nu_E\, d\mathrm{Per}(E) + \frac{s}{N}\mathrm{Per}(B_s; E\cap C(B_r^X(x)))\,. \tag{3.2.26}$$

By the coarea formula (2.4.8), u is Lipschitz and differentiable almost everywhere and it holds

$$u(s) = \int_0^s \mathrm{Per}(B_t; E\cap C(B_r^X(x)))\, dt\,. \tag{3.2.27}$$

We now compute the derivative of $\frac{u(s)}{s^N}$. Combining (3.2.26) and (3.2.27), we obtain that for a.e. s it holds

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{u(s)}{s^N} = \frac{u'(s)}{s^N} - N\frac{u(s)}{s^{N+1}} = -\frac{\int_{C(B_r^X(x))\cap B_s} \nabla\left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right)\cdot\nu_E\, d\mathrm{Per}(E)}{s^{N+1}}. \tag{3.2.28}$$

Fix $0 < r < \min(s,1)$. In the following steps, we consider the sets

$$A((s,x),r) := C(B_r^X(x)) \cap B_{s(1+r)} \setminus B_{s(1-r)}\,. \tag{3.2.29}$$

By Lemma 3.2.4 below, the family of sets $\{A(q,r) : q \in C(X),\ r > 0\}$ generates the Borel σ-algebra of $C(X)$, since for any $q \in C(X)$, $r > 0$ there exist r_a, $r_b > 0$ such that

$$B(q,r_a) \subset A(q,r) \subset B(q,r_b)\,. \tag{3.2.30}$$

Step 2.

In this step we show that if E is a cone, then (3.2.22) holds with $r_1 = 0, r_2 = \infty$. We will first show that $\frac{u(s)}{s^N}$ is constant, and then conclude using (3.2.28).

Since E is a cone, there exists a set $F \subset X$ such that $E = \{(t,x) \in C(X) : x \in F,\ t \geq 0\}$. Note that $E \cap C(B_r^X(x))$ is a cone for any $x \in X$, $r > 0$. Thus, for any $s > 0$, it holds:

$$\mathfrak{m}_C(E \cap C(B_r^X(x)) \cap B_s) = \mathfrak{m}(F \cap B_r^X(x)) \int_0^s \rho^{N-1}\, d\rho = \frac{s^N}{N} \mathfrak{m}(F \cap B_r^X(x))\,,$$

yielding that $s \mapsto s^{-N} u(s)$ is constant.

From (3.2.28) it follows that for all $r > 0$, $p \in C(X)$ we have

$$\int_{A(p,r)} \nabla \left(\frac{1}{2} \mathsf{d}^2(O, \cdot) \right) \cdot \nu_E \, d\mathrm{Per}(E) = 0. \tag{3.2.31}$$

By the Lebesgue differentiation Theorem (see for instance [27, Rem. 2.19]), for Per(E)-a.e. $p \in C(X)$ it holds

$$\lim_{r \to 0} \fint_{B_r(p)} \left| \nabla \left(\frac{1}{2} \mathsf{d}^2(O, q) \right) \cdot \nu_E(q) - \nabla \left(\frac{1}{2} \mathsf{d}^2(O, p) \right) \cdot \nu_E(p) \right| d\mathrm{Per}(E)(q) = 0\,.$$

From (3.2.30) and (3.2.45) (see also [11, Proposition 4.8], which holds more generally for finite perimeter sets) we infer

$$\begin{aligned}
&\lim_{r \to 0} \fint_{A(p,r)} \left| \nabla \left(\frac{1}{2} \mathsf{d}^2(O, q) \right) \cdot \nu_E(q) - \nabla \left(\frac{1}{2} \mathsf{d}^2(O, p) \right) \cdot \nu_E(p) \right| d\mathrm{Per}(E)(q) \\
&\quad \leq C \lim_{r \to 0} \fint_{B_{r_b}(p)} \left| \nabla \left(\frac{1}{2} \mathsf{d}^2(O, q) \right) \cdot \nu_E(q) - \nabla \left(\frac{1}{2} \mathsf{d}^2(O, p) \right) \cdot \nu_E(p) \right| d\mathrm{Per}(E)(q) \\
&\quad = 0\,,
\end{aligned}$$

where r_a and r_b are as in Lemma 3.2.4. We can now conclude recalling (3.2.31):

$$\begin{aligned}
0 &= \lim_{r \to 0} \frac{1}{\mathrm{Per}(E, A(p,r))} \int_{A(p,r)} \nabla \left(\frac{1}{2} \mathsf{d}^2(O, \cdot) \right) \cdot \nu_E \, d\mathrm{Per}(E) \\
&= \nabla \left(\frac{1}{2} \mathsf{d}^2(O, p) \right) \cdot \nu_E(p)\,,
\end{aligned}$$

for Per(E)-a.e. p.

Step 3.

In this last step we show that (3.2.22) for $r_1 = 0, r_2 = \infty$ implies that E is a cone. We will show that given $(s, x) \in E$ and $\lambda > 0$, then $(\lambda s, x) \in E$.

Combining the assumption (3.2.22) with (3.2.28), we obtain that $s \mapsto u(s)/s^N = \mathfrak{m}_C(E \cap B_r^X(x) \cap B_s)/s^N$ is constant. Therefore, for $\lambda > 0$

$$\begin{aligned}\mathfrak{m}_C(E \cap A((s,x),r)) &= \mathfrak{m}_C\left(E \cap B_r^X(x) \cap B_{s(1+r)}\right) - \mathfrak{m}_C\left(E \cap B_r^X(x) \cap B_{s(1-r)}\right) \\ &= \frac{\mathfrak{m}_C\left(E \cap B_r^X(x) \cap B_{\lambda s(1+r)}\right)}{\lambda^N} - \frac{\mathfrak{m}_C\left(E \cap B_r^X(x) \cap B_{\lambda s(1-r)}\right)}{\lambda^N} \\ &= \frac{\mathfrak{m}_C(E \cap A((\lambda s,x),r))}{\lambda^N}.\end{aligned} \tag{3.2.32}$$

Moreover,

$$\begin{aligned}\mathfrak{m}_C(A((\lambda t,x),r)) &= \mathfrak{m}(B_r^X(x)) \int_{\lambda t(1-r)}^{\lambda t(1+r)} s^{N-1}\,\mathrm{d}s = \lambda^N \mathfrak{m}(B_r^X(x)) \int_{t(1-r)}^{t(1+r)} s^{N-1}\,ds \\ &= \lambda^N \mathfrak{m}_C(A((t,x),r)),\end{aligned} \tag{3.2.33}$$

The combination of (3.2.32) and (3.2.33) gives

$$\frac{\mathfrak{m}_C(E \cap A((s,x),r))}{\mathfrak{m}_C(A((s,x),r))} = \frac{\mathfrak{m}_C(E \cap A((\lambda s,x),r))}{\mathfrak{m}_C(A((\lambda s,x),r))}\,. \tag{3.2.34}$$

We next show that if $(s,x) \in E$ then $(\lambda s, x) \in E$. Thanks to (3.2.34), it is enough to show that $q \in E$ if and only if

$$\lim_{r \to 0} \frac{\mathfrak{m}_C(E \cap A(q,r))}{\mathfrak{m}_C(A(q,r))} = 1\,. \tag{3.2.35}$$

Assume by contradiction that (3.2.35) holds but $q \notin E$. Then,

$$\liminf_{r \to 0} \frac{\mathfrak{m}_C((X \setminus E) \cap B_r(q))}{\mathfrak{m}_C(B_r(q))} \geq \varepsilon > 0\,.$$

Using Lemma 3.2.4, we infer that

$$\begin{aligned}&\liminf_{r \to 0} \frac{\mathfrak{m}_C((X \setminus E) \cap A(q,r))}{\mathfrak{m}_C(A(q,r))} \\ &\geq \liminf_{r \to 0} \frac{\mathfrak{m}_C((X \setminus E) \cap B_{r_a(q,r)}(q))}{\mathfrak{m}_C(B_{r_a(q,r)}(q))} \cdot \frac{\mathfrak{m}_C(B_{r_a(q,r)}(q))}{\mathfrak{m}_C(B_{r_b(q,r)}(q))}\,.\end{aligned} \tag{3.2.36}$$

Since $C(X)$ is an $\mathrm{RCD}(0,N)$ space, the Bishop-Gromov monotonicity formula [180] gives

$$\mathfrak{m}_C(B_{r_a}(q)) \geq \left(\frac{r_a}{r_b}\right)^N \mathfrak{m}_C(B_{r_b}(q))\,.$$

Therefore, from (3.2.36) we may conclude

$$\begin{aligned}\liminf_{r \to 0} \frac{\mathfrak{m}_C((X \setminus E) \cap A(q,r))}{\mathfrak{m}_C(A(q,r))} &\geq \liminf_{r \to 0} \left(\frac{r_a(q,r)}{r_b(q,r)}\right)^N \cdot \liminf_{r \to 0} \frac{\mathfrak{m}_C((X \setminus E) \cap B_{r_a(q,r)}(q))}{\mathfrak{m}_C(B_{r_a(q,r)}(q)))} \\ &\geq C\varepsilon > 0\,,\end{aligned} \tag{3.2.37}$$

where $C := \liminf_{r\to 0} \left(\frac{r_a(q,r)}{r_b(q,r)}\right)^N > 0$ thanks to (3.2.39). Clearly, (3.2.37) contradicts (3.2.35). The proof that $q \in E$ implies (3.2.35) is analogous. □

The following technical lemmas was used in the proof of Lemma 3.2.3 above.

Lemma 3.2.4. *Let $N \geq 2$, let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(N-2, N-1)$ *space and let $(C(X), \mathsf{d}_C, \mathfrak{m}_C)$ be the cone over it. If $N = 2$, assume also that* $\mathrm{diam}(X) \leq \pi$. *For $x \in X$ and $0 < r < s$, consider the sets*

$$A((s,x),r) := C(B_r^X(x)) \cap B_{s(1+r)}(O) \setminus B_{s(1-r)}(O). \tag{3.2.38}$$

Then:

- *The family of sets $\{A(q,r) : q \in C(X),\, r > 0\}$ generates the Borel σ-algebra of $C(X)$;*
- *For any $q \in C(X)$, $r > 0$ there exist $r_a = r_a(q,r)$ and $r_b = r_b(q,r) > 0$ such that*
$$B_{r_a}(q) \subset A(q,r) \subset B_{r_b}(q)$$
and
$$\lim_{r\to 0} \frac{r_a}{r_b} = \frac{1}{4\sqrt{2}}. \tag{3.2.39}$$

Proof. The first claim follows from the second one; thus let us determine r_a and $r_b > 0$ that satisfy the second statement. To this aim, we compute the minimal and maximal distance of $q = (t,x)$ from the set $\partial A(q,r)$. Let us start from the minimal distance. We deal with the shell part first: given $(t(1+r), y) \in \partial B_{t(1+r)} \cap \partial A$ it holds, using (2.3.7)

$$\begin{aligned} \mathsf{d}_C^2((t,x),(t(1+r),y)) &= t^2 + t^2(1+r)^2 - 2t^2(1+r)\cos(\mathsf{d}(x,y)) \\ &\geq t^2 + t^2(1+r)^2 - 2t^2(1+r) = t^2r^2\,, \end{aligned} \tag{3.2.40}$$

where the equality is achieved at $y = x$. Let now $(s,y) \in \partial C(B_r^X(x)) \cap \partial A(q,r)$:

$$\mathsf{d}_C^2((t,x),(s,y)) = t^2 + s^2 - 2st\cos(r)\,. \tag{3.2.41}$$

This defines a differentiable function of $s \in [t(1-r), t(1+r)]$. Its derivative $\partial_s \mathsf{d}_C^2((t,x),(s,y)) = 2s - 2t\cos(r)$ is increasing and vanishes at $s = t\cos(r)$. Therefore, we have

$$\mathsf{d}_C^2(q, \partial A(q,r)) = t^2\sin^2(r)\,. \tag{3.2.42}$$

We may pick

$$r_a = r_a(q,r) := \frac{1}{2}\mathsf{d}_C(q, \partial A(q,r)) = \frac{1}{2}t\sin(r)\,. \tag{3.2.43}$$

Next, let us compute the maximal distance of q from $\partial A(q,r)$. Since the maximal distance is attained at the intersection of the shell with the side part of A (by monotonicity of both formulas (3.2.40) and (3.2.41) with respect to $\mathsf{d}(x,y)$ and s, respectively), we can simply compute the maximum by looking at the distance from the top shell. We again compute, for $\mathsf{d}(x,y)=r$,

$$\begin{aligned}\mathsf{d}_C^2((t,x),(t(1+r),y)) &= t^2+t^2(1+r)^2-2t^2(1+r)\cos(r)\\ &= t^2(2+2r+r^2-2(1+r)\cos(r))\,.\end{aligned}$$

Consequently, we may pick

$$r_b = r_b(q,r) := 2t\sqrt{(2+2r+r^2-2(1+r)\cos(r))}\,. \tag{3.2.44}$$

It is easy to check that $r_a, r_b > 0$ defined in (3.2.43), (3.2.44) satisfy (3.2.39). $\square$

Lemma 3.2.5. *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(N-2,N-1)$, $(C(X),\mathsf{d}_C,\mathfrak{m}_C)$ *be the metric measure cone having* X *as cross-section and* $E\subset C(X)$ *a locally finite perimeter cone. Then for any* $r>0$ *it holds*

$$\mathrm{Per}(E,B_{2r}) \le 2^{N+1}\mathrm{Per}(E,B_r). \tag{3.2.45}$$

Proof. Fix $\varepsilon>0$. There exists a sequence $\{f_k\}_k\subset \mathrm{LIP}(C(X))\cap L^1(C(X))$ such that

$$f_k \to \chi_E \text{ in } L^1(C(X)) \quad \text{and} \quad \liminf_{k\to\infty}\int_{B_r}\mathrm{lip}(f_k)\,d\mathfrak{m}_C \le \mathrm{Per}(E,B_r).$$

Let $\varphi : C(X)\to C(X)$ be the dilation defined by $\varphi(t,x)=(2t,x)$. We show that $f_k\circ\varphi\to\chi_E$ in $L^1(C(X))$. Indeed,

$$\begin{aligned}&\int_{C(X)}|\chi_E - f_k\circ\varphi|\,d\mathfrak{m}_C = \int_0^\infty t^{N-1}\int_X|\chi_E(t,x)-f_k(2t,x)|\,d\mathfrak{m}\,dt =\\ &\frac{1}{2^N}\int_0^\infty s^{N-1}\int_X|\chi_E(s/2,x)-f_k(s,x)|\,d\mathfrak{m}\,ds =\\ &\frac{1}{2^N}\int_{C(X)}|\chi_E-f_k|\,d\mathfrak{m}_C \overset{k\to\infty}{\longrightarrow} 0,\end{aligned}$$

where we have used $\chi_E(s/2,x)=\chi_E(s,x)$ for $\mathfrak{m}_C$-a.e. $(s,t)\in C(X)$, which holds since E is a cone.

Moreover,

$$\begin{aligned}&\int_{B_r}\mathrm{lip}(f_k)\,d\mathfrak{m}_C = \int_0^r t^{N-1}\int_X \mathrm{lip}(f_k)(t,x)\,d\mathfrak{m}\,dt =\\ &\frac{1}{2^N}\int_0^{2r} s^{N-1}\int_X\mathrm{lip}(f_k)(2s,x)\,d\mathfrak{m}\,ds =\\ &\frac{1}{2^{N+1}}\int_0^{2r}s^{N-1}\int_X\mathrm{lip}(f_k\circ\varphi)(s,x)\,d\mathfrak{m}\,ds = \frac{1}{2^{N+1}}\int_{B_{2r}}\mathrm{lip}(f_k\circ\varphi)(s,x)\,d\mathfrak{m}_C.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\operatorname{Per}(E, B_r) + \varepsilon &\geq \liminf_{k\to\infty} \int_{B_r} \operatorname{lip}(f_k)\, d\mathfrak{m}_C \\ &= \frac{1}{2^{N+1}} \liminf_{k\to\infty} \int_{B_{2r}} \operatorname{lip}(f_k \circ \varphi)(s,x)\, d\mathfrak{m}_C \geq \frac{1}{2^{N+1}} \operatorname{Per}(E, B_{2r}).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude (3.2.45). □

A useful technical tool, used to prove the rigidity statement of the monotonicity formula, is the following lemma (see [10, Lemma 10.1] for the proof).

Lemma 3.2.6 (Joint Lower Semicontinuity)**.** *Let (X, d) be a Polish space. Let $\mu, \mu_k \in \mathcal{M}_+(X)$ with $\mu_k \rightharpoonup \mu$ in duality with $\mathrm{C}_b(X)$. Let $g_k \subset L^2(X;\mu_k)$ be a sequence of functions such that*

$$\sup_{k\in\mathbb{N}} \|g_k\|_{L^2(X;\mu_k)} < \infty. \tag{3.2.46}$$

Then there exists a function $g \in L^2(X;\mu)$ and a subsequence $k(l)$ such that

$$g_{k(l)}\, \mu_{k(l)} \rightharpoonup g\mu \tag{3.2.47}$$

in duality with $\mathrm{C}_b(X)$ and

$$\liminf_{l\to\infty} \|g_{k(l)}\|_{L^2\left(X;\mu_{k(l)}\right)} \geq \|g\|_{L^2(X;\mu)}\,. \tag{3.2.48}$$

3.3 Stratification of the Singular Set and further applications

The first goal of this section is to prove sharp Hausdorff dimension estimates for the singular strata of locally perimeter minimizing sets in $\mathrm{RCD}(K,N)$ spaces $(X, \mathsf{d}, \mathcal{H}^N)$. The statement is completely analogous to the classical one for singular strata of minimizing currents in the Euclidean setting, see [92], and for the singular strata of non-collapsed Ricci limits [61] and non-collapsed RCD spaces [80]. The proof is based on the classical Federer's dimension reduction argument, and builds upon the monotonicity formula and associated rigidity for perimeter minimizing sets in $\mathrm{RCD}(0,N)$ metric measure cones, Theorem 3.2.1. Though, a difference between Theorem 3.3.5 and the aforementioned papers is that the monotonicity formula is available only at the level of blow-ups and not in the space X; this creates some challenges that are addressed in the proof.

The second main goal is to present an application of the monotonicity formula and the associated rigidity for cones, to the existence of perimeter minimizing cones in any blow-down of an $\mathrm{RCD}(0, N)$ space $(X, \mathsf{d}, \mathcal{H}^N)$ with Euclidean volume growth.

Below we introduce the relevant definition of singular strata and of interior and boundary regularity points for a locally perimeter minimizing set $E \subset X$, when $(X, \mathsf{d}, \mathcal{H}^N)$ is an $\mathrm{RCD}(K, N)$ metric measure space.

Definition 3.3.1 (Singular Strata). *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ space, $E \subset X$ a locally perimeter minimizing set in the sense of Definition 2.4.29 and $0 \le k \le N-3$ an integer. The k-singular stratum of E, $\mathcal{S}^E_k$, is defined as*

$$\begin{aligned} \mathcal{S}^E_k := \{x \in \partial E :\ & \text{no tangent space to } (X, \mathsf{d}, \mathcal{H}^N, x, E) \text{ is of the form } (Y, \rho, \mathcal{H}^N, y, F), \\ & \text{with } (Y, \rho, y) \text{ isometric to } (Z \times \mathbb{R}^{k+1}, \mathsf{d}_Z \times \mathsf{d}_{\mathrm{eucl}}, (z, 0)) \\ & \text{for some pointed metric space } (Z, \mathsf{d}_Z, z) \\ & \text{and } F = G \times \mathbb{R}^{k+1} \text{ with } G \subset Z \text{ global perimeter minimizer}\}. \end{aligned} \tag{3.3.1}$$

The above definition would make sense also in the cases when $k \ge N-2$. However, it seems more appropriate not to adopt the terminology *singular strata* in those instances.

Definition 3.3.2 (Interior and Boundary Regularity Points). *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ space and let $E \subset X$ be a locally perimeter minimizing set in the sense of Definition 2.4.29. Given $x \in \partial E$, we say that x is an* interior regularity *point if*

$$\mathrm{Tan}_x(X, \mathsf{d}, \mathcal{H}^N, E) = \{(\mathbb{R}^N, \mathsf{d}_{\mathrm{eucl}}, \mathcal{H}^N, 0, \mathbb{R}^N_+)\}\,. \tag{3.3.2}$$

The set of interior regularity points of E will be denoted by $\mathcal{R}^E$.
Given $x \in \partial E$, we say that x is a boundary regularity *point if*

$$\mathrm{Tan}_x(X, \mathsf{d}, \mathcal{H}^N, E) = \{(\mathbb{R}^N_+, \mathsf{d}_{\mathrm{eucl}}, \mathcal{H}^N, 0, \{x_1 \ge 0\})\}\,, \tag{3.3.3}$$

where x_1 is one of the coordinates of the $\mathbb{R}^{N-1}$ factor in $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \{x_N \ge 0\}$. The set of boundary regularity points of E will be denoted by $\mathcal{R}^E_{\partial X}$.

It was proved in [144] that the interior regular set $\mathcal{R}^E$ is topologically regular, in the sense that it is contained in a Hölder open manifold of dimension $N-1$. By a blow-up argument, in the next proposition, we show that $\dim_{\mathcal{H}} \mathcal{R}^E_{\partial X} \le N-2$.

Proposition 3.3.3. *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ space. Let $E \subset X$ be a locally perimeter minimizing set and let $\mathcal{R}^E_{\partial X}$ be the set of boundary regularity points of E, in the sense of Definition 3.3.2. Then*

$$\dim_{\mathcal{H}} \mathcal{R}^E_{\partial X} \le N-2\,. \tag{3.3.4}$$

Proof. We argue by contradiction. Assume there exists $k > N-2$, $k \in \mathbb{R}$ such that

$$\mathcal{H}^k\left(\mathcal{R}^E_{\partial X}\right) > 0\,. \tag{3.3.5}$$

Let $\varepsilon > 0$. We define the quantitative ε-singular set to be

$$\begin{aligned} &S^\varepsilon(E) := \\ &\quad \{x \in X:\ \mathcal{D}((B^X_r(x), \mathsf{d}, \mathcal{H}^N, x, E), (B^{\mathbb{R}^N}_r, \mathsf{d}_{\mathrm{eucl}}, 0, \mathbb{R}^N_+)) \geq \varepsilon r,\ \text{ for all } r \in (0,\varepsilon)\}\,. \end{aligned} \tag{3.3.6}$$

Recall that the definition of the distance $\mathcal{D}$ was reported in Definition 2.4.9. Notice that $S^{\varepsilon_1}(E) \supset S^{\varepsilon_2}(E)$ for $0 < \varepsilon_1 \leq \varepsilon_2$ and that

$$\partial E \setminus \mathcal{R}^E = \bigcup_{n\in\mathbb{N}} S^{\varepsilon_n}(E), \tag{3.3.7}$$

for any sequence $\varepsilon_n \to 0$. It is also clear that

$$\mathcal{R}^E_{\partial X} \subset \partial E \setminus \mathcal{R}^E. \tag{3.3.8}$$

The combination of (3.3.5), (3.3.7) and (3.3.8) implies that there exists $\bar\varepsilon > 0$ such that

$$\mathcal{H}^k\left(S^{\bar\varepsilon}(E) \cap \mathcal{R}^E_{\partial X}\right) > 0\,. \tag{3.3.9}$$

By [90, Theorem 2.10.17], there exists $x \in S^{\bar\varepsilon}(E) \cap \mathcal{R}^E_{\partial X}$ such that

$$\limsup_{r\to 0} \frac{\mathcal{H}^k_\infty\left(B_r(x) \cap S^{\bar\varepsilon}(E) \cap \mathcal{R}^E_{\partial X}\right)}{r^k} \geq C_k > 0\,, \tag{3.3.10}$$

where we denoted by $\mathcal{H}^k_\infty$ the k-dimensional ∞-pre-Hausdorff measure (recall Section 2.1.2).

By the very definition of $\mathcal{R}^E_{\partial X}$, for every sequence $r_i \to 0$, $E \subset (X, \mathsf{d}/r_i, \mathcal{H}^N/r_i^N, x)$ converges in the sense of Definition 2.4.8 to a quadrant $\{x_1 \geq 0\}$, where x_1 is one of the coordinates of the $\mathbb{R}^{N-1}$ factor in $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \{x_N \geq 0\}$.

Embedding the sequence of rescaled spaces X_i and their limit $\mathbb{R}^N_+$ into a proper realization of the pGH-convergence, by Blaschke's theorem (cf. [49, Theorem 7.3.8]) there exist a compact set $A \subset \mathbb{R}^N_+$ and a subsequence, which we do not relabel, such that $S^{\bar\varepsilon}(E) \cap \mathcal{R}^E_{\partial X} \cap B^i_1(x)$ converges to A in the Hausdorff sense.
Moreover, it is elementary to check that $A \subset S^{\bar\varepsilon}(\{x_1 \geq 0\})$ in $\mathbb{R}^N_+$. Therefore, we obtain

$$\begin{aligned} \mathcal{H}^k_\infty\left(S^{\bar\varepsilon}(\{x_1 \geq 0\})\right) &\geq \mathcal{H}^k_\infty(A) \geq \limsup_{i\to\infty} \mathcal{H}^k_\infty\left(S^{\bar\varepsilon}(E) \cap \mathcal{R}^E_{\partial X} \cap B^i_1(x)\right) \\ &= \limsup_{i\to\infty} \frac{\mathcal{H}^k_\infty\left(B_{r_i}(x) \cap S^{\bar\varepsilon}(E) \cap \mathcal{R}^E_{\partial X}\right)}{r_i^k} > 0\,, \end{aligned} \tag{3.3.11}$$

where we relied on the classical upper semicontinuity of the pre-Hausdorff measure with respect to Hausdorff convergence in the second inequality and on (3.3.10) in the last one. However, it is easy to check that $S^{\bar{\varepsilon}}(\{x_1 \geq 0\}) = \{x_1 = x_N = 0\}$ which has Hausdorff codimension 2, contradicting (3.3.11). □

The main result of this section is the stratification of the singular set of locally perimeter minimizing sets Theorem 3.3.5 below. As mentioned in the introduction of this thesis, the stratification of the singular set of objects that minimize the area in a suitable sense in the Euclidean setting [92, 188] is a fundamental result for the regularity theory of such objects. It was used as a step in understanding the regularity of minimizing integral currents of codimension greater than one, and it is widely used to analyze singularities in geometric analysis (see, for instance [188]). Moreover, it furthers our understading of the structure of the singular set of minimal boundaries. The proof of Theorem 3.3.5 builds on the monotonicity formula Theorem 3.2.1: the monotonicity formulas for cones over Riemannian manifolds (see [148, Theorem 9.3], [90, Theorem 5.4.3] and [147]) used in [144] to show the sharp Hausdorff dimension bounds on the singular set of locally perimeter minimizing sets are not strong enough to obtain Theorem 3.3.5.

An application of the stratification of the singular set of perimeter minimizers and the definition of boundary of a non-collapsed RCD m.m.s. [43] is that the complement of $\mathcal{S}^E_{N-3}$ in ∂E consists of either interior or boundary regularity points. We present such result here.

Theorem 3.3.4. *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ space and let $E \subset X$ be a locally perimeter minimizing set in the sense of Definition 2.4.29. Then*

$$\partial E \setminus \mathcal{S}^E_{N-3} = \mathcal{R}^E \cup \mathcal{R}^E_{\partial X} \,. \tag{3.3.12}$$

For ease of the reader, we restate here the stratification of the singular set of local perimeter minimizer 3.1.5.

Theorem 3.3.5 (Stratification of the singular set)**.** *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ space and $E \subset X$ a locally perimeter minimizing set. Then, for any $0 \leq k \leq N-3$ it holds*

$$\dim_{\mathcal{H}} \mathcal{S}^E_k \leq k \,. \tag{3.3.13}$$

Another application of the monotonicity formula with the associated rigidity is that if an $\mathrm{RCD}(0, N)$ space $(X, \mathsf{d}, \mathcal{H}^N)$ with Euclidean volume growth contains a global perimeter minimizer, then any asymptotic cone contains a perimeter minimizing cone.

Theorem 3.3.6. *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an* $\mathrm{RCD}(0,N)$ *metric measure space with Euclidean volume growth, i.e. satisfying for some (and thus for every) $x \in X$:*

$$\liminf_{r \to \infty} \frac{\mathcal{H}^N(B_r(x))}{r^N} > 0. \tag{3.3.14}$$

Let $E \subset X$ be a global perimeter minimizer in the sense of Definition 2.4.29. Then for any blow-down $(C(Z), \mathsf{d}_{C(Z)}, \mathcal{H}^N)$ of $(X, \mathsf{d}, \mathcal{H}^N)$ there exists a cone $C(W) \subset C(Z)$ which is a global perimeter minimizer.

Remark 3.3.7. *The conclusion of Theorem 3.3.6 above seems to be new also in the more classical case of smooth Riemannian manifolds with non-negative sectional curvature, or non-negative Ricci curvature. We refer to [25] for earlier progress in the case of smooth manifolds with non-negative sectional curvature satisfying additional conditions on the rate of convergence to the tangent cone at infinity and on the regularity of the cross section and to [87] for the case of smooth Riemannian manifolds with non-negative Ricci curvature and quadratic curvature decay.*

Proof of Theorem 3.3.4. Let us consider a point $x \in \partial E \setminus \mathcal{S}^E_{N-3}$. By the very definition of the singular stratum $\mathcal{S}^E_{N-3}$, there exists a tangent space to $(X, \mathsf{d}, \mathcal{H}^N, E)$ at x of the form $(\mathbb{R}^{N-2} \times Z, \mathsf{d}_{\mathrm{eucl}} \times \mathsf{d}_Z, \mathcal{H}^N, y, G \times \mathbb{R}^{N-2})$, where $(Z, \mathsf{d}_Z, \mathcal{H}^2)$ is an $\mathrm{RCD}(0,2)$ metric measure cone (because all tangent cones to an $\mathrm{RCD}(K,N)$ space $(X, \mathsf{d}, \mathcal{H}^N)$ are metric measure cones [80], recall Theorem 2.3.31) and $G \subset Z$ is a globally perimeter minimizing set (in the sense of Definition 2.4.29) thanks to [144, Theorem 2.42].

By Lemma 3.3.8 there are only two options. Either x is an interior point and a tangent space is $(\mathbb{R}^N, \mathsf{d}_{\mathrm{eucl}}, \mathcal{H}^N, 0, \mathbb{R}^N_+)$, or x is a boundary point and a tangent space is $(\mathbb{R}^N_+, \mathsf{d}_{\mathrm{eucl}}, \mathcal{H}^N, 0, \{x_1 \geq 0\})$.
In the first case, it was shown in [144] that the tangent space at x is unique and hence $x \in \mathcal{R}^E$. If the second possibility occurs, then by [43] we infer that the tangent cone to the ambient space $(X, \mathsf{d}, \mathcal{H}^N)$ is unique. The uniqueness of the tangent cone to the set of finite perimeter can be obtained with an argument completely analogous to the one used for interior points in [144], building on top of the classical boundary regularity theory (cf. for instance with [111]) instead of the classical interior regularity theory for perimeter minimizers in the Euclidean setting. Hence $x \in \mathcal{R}^E_{\partial X}$ is a boundary regularity point. □

Proof of Theorem 3.3.5. We argue by contradiction via Federer's dimension reduction argument. The proof is divided into four steps. In the first step we set up the contradiction argument and reduce to the case of entire perimeter minimizers inside

RCD$(0,N)$ metric measure cones. In the second step we make a further reduction to the case when the perimeter minimizer is a cone itself, building on top of Theorem 3.2.1. Via additional blow-up arguments we gain a splitting for the ambient space and for the perimeter minimizing set in step three, thus performing a dimension reduction. The argument is completed in step four. A key subtlety with respect to more classical situations is that the monotonicity formula holds only for perimeter minimizers centered at vertices of metric measure cones, resulting into the necessity of iterating the blow-ups.

Step 1.

We argue by contradiction. Suppose that the statement does not hold for some $0 \le k \le N-3$. Then there exists $k' > k$, $k' \in \mathbb{R}$ such that

$$\mathcal{H}^{k'}\left(\mathcal{S}^E_k\right) > 0\,. \tag{3.3.15}$$

Let $\varepsilon > 0$. We define the quantitative (k,ε)-singular stratum to be

$$\begin{aligned} S^E_{k,\varepsilon} := \{x \in X : \mathcal{D}((B^X_r(x), \mathsf{d}, \mathcal{H}^N, x, E), (B^{\mathbb{R}^{k+1}\times Z}_r, \mathsf{d}_{\mathrm{eucl}} \times \mathsf{d}_Z, \mathcal{H}^N, (0,z), F)) \ge \varepsilon r \\ \text{for all } r \in (0,\varepsilon),\ (Z, \mathsf{d}_Z, z) \text{ pointed metric spaces and} \\ F = \mathbb{R}^{k+1} \times G \text{ with } G \subset Z \text{ global perimeter minimizer}\}\,. \end{aligned} \tag{3.3.16}$$

Recall that the distance $\mathcal{D}$ was introduced in [11, Definition A.3], which we reported in Definition 2.4.9. Moreover, we notice that $S^E_{k,\varepsilon_1} \supset S^E_{k,\varepsilon_2}$ for $0 < \varepsilon_1 < \varepsilon_2$ and that $S^E_k = \bigcup_{n\in\mathbb{N}} S^E_{k,\varepsilon_n}$, for any sequence $\varepsilon_n \to 0$.

The contradiction argument assumption (3.3.15) implies that there exists $\bar{\varepsilon} > 0$ such that

$$\mathcal{H}^{k'}\left(S^E_{k,\bar{\varepsilon}}\right) > 0\,. \tag{3.3.17}$$

By [90, Theorem 2.10.17] (see Section 2.1.2), there exists $x \in S^E_{k,\bar{\varepsilon}}$ such that

$$\limsup_{r\to 0} \frac{\mathcal{H}^{k'}_\infty\left(B_r(x) \cap S^E_{k,\bar{\varepsilon}}\right)}{r^{k'}} \ge C_{k'} > 0\,, \tag{3.3.18}$$

where we denoted by $\mathcal{H}^{k'}_\infty$ the k'-dimensional ∞-pre-Hausdorff measure.

Then there exists a sequence $r_i \to 0$ such that $E \subset (X, \mathsf{d}/r_i, \mathcal{H}^N/r_i^N, x)$ converges in the sense of Definition 2.4.8 to a global perimeter minimizer (in the sense of Definition 2.4.29) $F \subset (C(Z), \mathsf{d}_C, \mathcal{H}^N)$. Here we used [11, Corollary 3.4] (Corollary 2.4.15) in combination with the density Lemma 2.4.32 for the compactness, [144, Theorem 2.42] (Theorem 2.4.33) for the perimeter minimality of F and [104] (Theorem

2.3.31) to infer that the ambient tangent space is a cone. Here $(Z, \mathsf{d}_Z, \mathcal{H}^{N-1})$ is an $\mathrm{RCD}(N-2, N-1)$ metric measure space.

Embedding the sequence of rescaled spaces X_i and their limit $C(Z)$ into a proper realization of the pmGH-convergence, by Blaschke's theorem (cf. [49, Theorem 7.3.8], reported in Theorem 2.2.5) there exist a compact set $A \subset C(Z)$ and a subsequence, which we do not relabel, such that $S^E_{k,\bar{\varepsilon}} \cap B^i_1(x)$ converges to A in the Hausdorff sense. Moreover, it is straightforward to check that $A \subset S^F_{k,\bar{\varepsilon}}$. Therefore, we obtain

$$\begin{aligned} \mathcal{H}^{k'}_\infty\left(S^F_{k,\bar{\varepsilon}}\right) \geq \mathcal{H}^{k'}_\infty(A) &\geq \limsup_{i\to\infty} \mathcal{H}^{k'}_\infty\left(S^E_{k,\bar{\varepsilon}} \cap B^i_1(x)\right) \\ &= \limsup_{i\to\infty} \frac{\mathcal{H}^{k'}_\infty\left(B_{r_i}(x) \cap S^E_{k,\bar{\varepsilon}}\right)}{r_i^{k'}} > 0\,, \end{aligned} \tag{3.3.19}$$

where we relied on the classical upper semicontinuity of the pre-Hausdorff measure with respect to Hausdorff convergence in the second inequality and on (3.3.18) in the last one.

Lastly, (3.3.19) implies that

$$\mathcal{H}^{k'}\left(B^{C(Z)}_1 \cap S^F_{k,\bar{\varepsilon}}\right) > 0\,. \tag{3.3.20}$$

Step 2.

In this step, by performing a second blow-up, we apply the monotonicity formula Theorem 3.2.1 to show that we can also suppose that the global perimeter minimizer is a cone (with respect to a vertex of the ambient cone). For the sake of clarity, we recall that the set of vertices of $C(Z)$ is the collection of all points $y \in C(Z)$ such that $C(Z)$ is a metric cone centered at y. Moreover, we remark that the set of vertices is isometric to $\mathbb{R}^k$ for some $0 \leq k \leq N$.

We claim that there is a point $O \in C(Z)$ such that O is a vertex of $C(Z)$ and the following density estimate holds:

$$\limsup_{r\to 0} \frac{\mathcal{H}^{k'}_\infty\left(B_r(O) \cap S^F_{k,\bar{\varepsilon}}\right)}{r^{k'}} \geq C_{k'} > 0\,. \tag{3.3.21}$$

If the claim does not hold, then by (3.3.20) there are points of density for $\mathcal{H}^{k'}_\infty$ restricted to $S^F_{k,\bar{\varepsilon}}$ and none of them belongs to the set of vertices of $C(Z)$. Hence we can repeat the argument in step 1, blowing up at a density point for $\mathcal{H}^{k'}_\infty$ restricted to $S^F_{k,\bar{\varepsilon}}$ which is not a vertex in the ambient cone. In this way, the dimension of the set of vertices of the ambient space, which is isometric to a Euclidean space, increases at least by one.

The procedure can be iterated until one of the following two possibilities occurs: the ambient is isometric to $\mathbb{R}^N$, with standard structure, in which case (3.3.20) contradicts the classical regularity theory, or there is a density point for $\mathcal{H}^{k'}_\infty$ restricted to $S^F_{k,\bar{\varepsilon}}$ which is also a vertex of $C(Z)$.

Let now O denote any such vertex of $C(Z)$. By the monotonicity formula Theorem 3.2.1 and the density estimates in Lemma 2.4.32, the map

$$r \mapsto \frac{\operatorname{Per}(F; B_r(O))}{r^{N-1}} \tag{3.3.22}$$

is monotone non-decreasing, bounded and bounded away from 0. Therefore, there exists the limit

$$0 < a := \lim_{r \longrightarrow 0} \frac{\operatorname{Per}(F; B_r(O))}{r^{N-1}} < \infty\,. \tag{3.3.23}$$

We perform a second blow-up at the tip $O \in C(Z)$ and obtain a global perimeter minimizer $G \subset C(Z)$. By (3.3.23) and the rigidity part of the monotonicity formula Theorem 3.2.1, G is a cone. Indeed, let $(C(Z), \mathsf{d}_C/r_i, \mathfrak{m}_C^{r_i})$ be an element of the blow-up sequence for some $\{r_i\}_i$, $r_i > 0$, $r_i \overset{i \to \infty}{\longrightarrow} 0$, and let Per_{r_i}, lip_{r_i} and $B^{r_i}_R(O)$ denote the perimeter measure, the Lipschitz constant functional and the ball of radius $R > 0$ centered at the tip of $(C(Z), \mathsf{d}_C/r_i, \mathfrak{m}_C^{r_i})$, respectively. Then

$$\begin{aligned}
&\operatorname{Per}_{r_i}(F, B^{r_i}_R(O)) = \inf \liminf_{k \longrightarrow \infty} \int_{B^{r_i}_R(O)} \operatorname{lip}_{r_i}(f_k)\, d\mathfrak{m}_C^{r_i} = \\
&\inf \liminf_{k \longrightarrow \infty} \frac{1}{{r_i}^N} \int_0^{Rr_i} \rho^{N-1} \int_X r_i \operatorname{lip}(f_k)\, d\mathfrak{m}\, d\rho = \\
&\frac{1}{{r_i}^{N-1}} \operatorname{Per}(F, B_{Rr_i}(O)),
\end{aligned} \tag{3.3.24}$$

where the infimum is takes over all sequences $\{f_k\}_k \subset \operatorname{LIP}(C(X)) \cap L^1_{\text{loc}}(C(X))$ converging in L^1_{loc} to χ_E and where we have used

$$\mathfrak{m}_C^{r_i} = \int_{B_{r_i}(O)} (1 - \mathsf{d}_C(O, y)/r_i)\, d\mathfrak{m}_C(y)\, \mathfrak{m}_C = \frac{{r_i}^N \mathfrak{m}(X)}{N(N+1)} \mathfrak{m}_C = {r_i}^N \mathfrak{m}_C. \tag{3.3.25}$$

The last equality in (3.3.25) follows from $\int_{B_1(O)} (1 - d(O, y))\, d\mathfrak{m}_C(y) = \frac{\mathfrak{m}(X)}{N(N+1)} = 1$, which holds since we are assuming that $(C(X), \mathsf{d}_C, \mathfrak{m}_C)$ is normalized.

By dividing both sides of (3.3.24) by R^{N-1}, we obtain

$$\frac{\operatorname{Per}_{r_i}(F, B^{r_i}_R(O))}{R^{N-1}} = \frac{\operatorname{Per}(F, B_{Rr_i}(O))}{R{r_i}^{N-1}}. \tag{3.3.26}$$

By [144, Theorem 2.42] it follows that the left-hand side of (3.3.26) converges to $\frac{\operatorname{Per}(G, B_R(O))}{R^{N-1}}$ as $i \to \infty$, whereas the right-hand side of (3.3.26) converges to $a > 0$

by (3.3.23). Therefore, we can apply the rigidity part of the monotonicity formula Theorem 3.2.1 to infer that G is a cone.

Moreover, by repeating the arguments in step 1, taking into account that O was chosen to be a density point for $\mathcal{H}^{k'}_\infty$ restricted to $S^F_{k,\bar{\varepsilon}}$, it holds

$$\mathcal{H}^{k'}\left(S^G_{k,\bar{\varepsilon}}\right) > 0\,. \tag{3.3.27}$$

It follows from (3.3.27) that there exists a point in $S^G_{k,\bar{\varepsilon}} \setminus \{O\}$.

Step 3.

The goal of this step is to show that the ambient cone and the perimeter minimizer both split a Euclidean factor by considering a blow-up of G at a density point for $\mathcal{H}^{k'}_\infty$ restricted to $S^G_{k,\bar{\varepsilon}}$ that is not a vertex. Roughly speaking, we will achieve this by showing that the unit normal of the blow-up is everywhere perpendicular to the gradient of a splitting function obtained with the help of Lemma 3.3.9 below, cf. [136, Lemma 28.13].

Our setup is that $G \subset C(Z)$ is a globally perimeter minimizing cone with vertex O, a vertex of the ambient cone. Moreover, $\mathcal{H}^{k'}(S^G_{k,\bar{\varepsilon}}) > 0$. In particular, by the very same arguments as in Step 2, there exist a point $O' \in C(Z)$, $O' \neq O$ and a sequence $r_i \to 0$ such that

$$\lim_{i\to\infty} \frac{\mathcal{H}^{k'}_\infty\left(B_{r_i}(O') \cap S^G_{k,\bar{\varepsilon}}\right)}{r_i^{k'}} \geq C_{k'} > 0\,. \tag{3.3.28}$$

Up to taking a subsequence that we do not relabel, we can assume that the sequence $(C(Z), \mathsf{d}_C/r_i, \mathcal{H}^N, O', G)$ converges to $(C(Z'), \mathsf{d}_{C'}, \mathcal{H}^N, O'', H)$, where $(C(Z'), \mathsf{d}_{C'}, \mathcal{H}^N)$ is an $\mathrm{RCD}(0, N)$ metric measure cone splitting an additional $\mathbb{R}$ factor with respect to $C(Z)$ and $H \subset C(Z')$ is a global perimeter minimizer. Moreover,

$$\mathcal{H}^{k'}\left(B_1(O'') \cap S^H_{k,\bar{\varepsilon}}\right) > 0\,. \tag{3.3.29}$$

Consider the sequence of functions $f_i : C(Z) \to \mathbb{R}$ defined as

$$f_i(z) := \frac{\mathsf{d}^2_C(O, z) - \mathsf{d}^2_C(O, O')}{r_i}\,, \tag{3.3.30}$$

that we view as functions on the rescaled metric measure space $(C(Z), \mathsf{d}_C/r_i, \mathcal{H}^N, O')$. By Lemma 3.3.9 below, the functions f_i converge to some splitting function $g : C(Z') \to \mathbb{R}$ in $H^{1,2}_{\mathrm{loc}}$, see [18] or Section 2.3.6 for the relevant background. Moreover, $\Delta_i f_i$ converge to 0 uniformly.

We claim that, for any function $\varphi \in \mathrm{LIP}(C(Z')) \cap H^{1,2}(C(Z'))$, it holds

$$\int_H \nabla\varphi \cdot \nabla g \, d\mathcal{H}^N = 0\,. \tag{3.3.31}$$

To see this, let $\varphi_i \in \mathrm{LIP}(X_i) \cap H^{1,2}(X_i)$ converging $H^{1,2}$-strongly to φ along the sequence $(C(Z), \mathsf{d}_C/r_i, \mathcal{H}^N, O')$, whose existence was shown in [18] (see Lemma 2.3.20). Then by using the Gauss-Green formula (Theorem 2.4.21) and the characterization of cones Lemma 3.2.3 we obtain

$$\begin{aligned} 0 &= \int_{\partial^* G} \varphi_i \nu_G^i \cdot \nabla_i f_i \, d\mathrm{Per}_i(G) \\ &= - \int_G \nabla_i \varphi_i \cdot \nabla_i f_i \, d\mathcal{H}^N - \int_G \varphi_i \cdot \Delta_i f_i \, \mathrm{d}\mathcal{H}^N \,, \end{aligned} \tag{3.3.32}$$

where the Hausdorff measure $\mathcal{H}^N$ is computed with respect to the rescaled distance d_C/r_i.
By (3.3.49) below and (3.3.32)

$$\int_G \nabla_i \varphi_i \cdot \nabla_i f_i \, d\mathcal{H}^N = - \int_G \varphi_i \cdot \Delta_i f_i \, \mathrm{d}\mathcal{H}^N \to 0 \,. \tag{3.3.33}$$

On the other hand, by [18, Theorem 5.7] (reported in Theorem 2.3.18), it follows that

$$\int_G \nabla_i \varphi_i \cdot \nabla_i f_i \, d\mathcal{H}^N \to \int_H \nabla \varphi \cdot \nabla g \, \mathrm{d}\mathcal{H}^N \,. \tag{3.3.34}$$

Combining (3.3.33) and (3.3.34) we obtain (3.3.31); see [46, 44] for analogous arguments.

Our next goal is to use (3.3.31) to show that the perimeter minimizer H splits a Euclidean factor in the direction of the ambient splitting induced by the splitting function g (see [28, Lemma 1.21], reported here as Lemma 2.3.11).

Let us set $Y := C(Z') = \mathbb{R} \times Y'$, and assume that $\mathbb{R}$ is the splitting induced by g.

Given any $\varphi \in W^{1,2}_{\mathrm{loc}}(Y)$ let us also denote $\varphi^{(t)}(y) := \varphi(t,y)$ and $\varphi^{(y)}(t) := \varphi(t,y)$. If $\varphi \in W^{1,2}_{\mathrm{loc}}(Y)$, then $\varphi^{(t)} \in H^{1,2}_{\mathrm{loc}}(Y')$ and $\varphi^{(y)} \in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$, for $\mathcal{L}^1$-a.e. t and $\mathcal{H}^{N-1}$-a.e. y respectively (see [102]). Up to the isomorphism given by the splitting induced by g, it holds

$$\nabla \varphi \cdot \nabla g(t,y) = \partial_t \varphi^{(y)}(t) \,, \quad \text{for } \mathcal{H}^N\text{-a.e. } (t,y) \in Y \,.$$

Let P_s denote the heat flow on Y. Then

$$\begin{aligned} \int_Y \mathrm{P}_s \chi_H(t,y) \nabla \varphi \cdot \nabla g(t,y) \, d\mathcal{H}^N &= \int_Y \mathrm{P}_s \chi_H(t,y) \partial_t \varphi^{(y)}(t) \, d\mathcal{H}^N \\ &= \int_H \mathrm{P}_s \partial_t \varphi^{(y)}(t) \, d\mathcal{H}^N = \int_H \partial_t (\mathrm{P}_s \varphi)^{(y)}(t) \, d\mathcal{H}^N, \end{aligned} \tag{3.3.35}$$

where in the second equality we have used the self-adjointess of the heat flow and in the last equality we have used Lemma 3.3.10.

By (3.3.31) and (3.3.35) it follows that

$$\int_Y \mathrm{P}_s\chi_H(t,y)\nabla\varphi\cdot\nabla g(t,y)\,d\mathcal{H}^N = \int_H \nabla(\mathrm{P}_s\varphi)\cdot\nabla g\,d\mathcal{H}^N = 0\,. \tag{3.3.36}$$

Since $\varphi \in \mathrm{LIP}(Y)\cap H^{1,2}(Y)$ is arbitrary, using the splitting $Y = \mathbb{R}\times Y'$, Fubini's theorem and integrating by parts the final expression of (3.3.35), we obtain from (3.3.36) that

$$\partial_t(\mathrm{P}_s\chi_H)^{(y)}(t) = 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } t\in\mathbb{R},\text{ for } \mathcal{H}^{N-1}\text{-a.e. } y\in Y'\,.$$

By the $L^1_{\mathrm{loc}}(Y)$ convergence of $\mathrm{P}_s\chi_H$ to χ_H for $s\to 0$ and the closedness of H, we conclude that $\chi_H^{(y)}$ is constant in t for every $y\in Y'$. That implies the existence of a set $H'\subset Y'$ such that

$$\chi_H(t,y) = \chi_{H'}(y)\,. \tag{3.3.37}$$

By Lemma 3.3.11, $H'\subset Y'$ is a set of locally finite perimeter.
Let us show that H' is a global perimeter minimizer, by following the classical Euclidean argument, cf. [136, Lemma 28.13]. Suppose not. Then there exist $\varepsilon>0$ and a set $H'_0\subset Y'$ such that $H'\Delta H'_0 \subset\subset B_r(y)$ for some $r>0$ and $y\in Y'$, such that

$$\mathrm{Per}(H'_0;B_r(y)) + \varepsilon \le \mathrm{Per}(H';B_r(y))\,. \tag{3.3.38}$$

Let $t>0$. We define the sets

$$I_t := \mathbb{R}\setminus(-t,t) \qquad H_0 := (H'_0\times(-t,t))\cup(H'\times I_t)\,. \tag{3.3.39}$$

At this stage, we can use the formulas for the cut and paste of sets of finite perimeter (Theorem 2.4.26), observe that $H\Delta H_0\subset B_r(y)\times(-t,t) := A$ and conclude by Lemma 3.3.11 that

$$\begin{aligned}\mathrm{Per}(H_0;A) - \mathrm{Per}(H;A) &= 2t(\mathrm{Per}(H'_0;B_r(y)) - \mathrm{Per}(H;B_r(y))) + 2\mathcal{H}^{N-1}(H'_0\Delta H')\\ &\le -2t\varepsilon + 2\mathcal{H}^{N-1}(B_r(y)) < 0\,,\end{aligned}$$

where we have chosen $t>0$ large enough so that $\mathcal{H}^{N-1}(B_r(y)) < t\varepsilon$.
Therefore, H' is a global perimeter minimizer, as we claimed.

If $k=0$, the above argument leads to a contradiction. Indeed we found a point in $\mathcal{S}^H_0$ such that some tangent space splits a line.

Step 4.
If $k>0$, then it is straightforward to see that $(t,y)\in S^H_{k,\bar\varepsilon}$ if and only if $y\in S^{H'}_{k-1,\bar\varepsilon}$

since the splitting $\mathbb{R}$ factor is invariant under blow-ups.
In particular, from the assumption that

$$\mathcal{H}^{k'}\left(B_1(O'')\cap S^H_{k,\bar\varepsilon}\right)>0\,, \tag{3.3.40}$$

we conclude that

$$\mathcal{H}^{k'-1}\left(B_1(O'')\cap S^{H'}_{k-1,\bar\varepsilon}\right)>0\,. \tag{3.3.41}$$

Therefore the steps from 1 to 3 prove that if there exist an $\mathrm{RCD}(K,N)$ metric measure space $(X,\mathsf{d},\mathcal{H}^N)$ and locally perimeter minimizing set $E\subset X$ such that for some $0\le k\le N-3$ it holds $\dim_{\mathcal{H}}(\mathcal{S}^E_k)>k$, then there exist an $\mathrm{RCD}(0,N-1)$ space $(X',\mathsf{d}',\mathcal{H}^{N-1})$ and a locally perimeter minimizing set $E'\subset X'$ such that $\dim_{\mathcal{H}}(\mathcal{S}^{E'}_{k-1})>k-1$. The dimension reduction can be iterated a finite number of times until we reduce to the case $k=0$, that we already discussed above. □

Proof of Theorem 3.3.6. First of all, up to modifying E on a set of measure zero if necessary, we can assume that E is open.

Step 1. Fix a point $x\in\partial E$. We claim that there exists $C>1$ such that

$$\frac{r^N}{C}\le\mathcal{H}^N(E\cap B_r(x))\le C\,r^N,\quad\text{for all } r>0, \tag{3.3.42}$$

$$\frac{r^{N-1}}{C}\le\mathrm{Per}(E,B_r(x))\le C\,r^{N-1},\quad\text{for all } r>0. \tag{3.3.43}$$

Recall that an $\mathrm{RCD}(0,N)$ space is globally doubling (thanks to the Bishop-Gromov inequality [180]) and satisfies a global Poincaré inequality [161] (see also Section 2.3.1). Since, by assumption, E mimimizes the perimeter on every metric ball then by [122, Theorem 4.2] (reported in Theorem 2.4.31) there exists a constant $\gamma_0>0$ (depending only on the doubling and Poincaré constants of $(X,\mathsf{d},\mathcal{H}^N)$) such that

$$\frac{\mathcal{H}^N(E\cap B_r(x))}{\mathcal{H}^N(B_r(x))}\ge\gamma_0\quad\text{and}\quad\frac{\mathcal{H}^N(B_r(x)\setminus E)}{\mathcal{H}^N(B_r(x))}\ge\gamma_0\quad\text{for all } r>0 \text{ and } x\in\partial E. \tag{3.3.44}$$

Recall that the ratio $\mathcal{H}^N(B_r(x))/r^N$ is monotone non-increasing by the Bishop-Gromov inequality, it is bounded from above by the value in $\mathbb{R}^N$ and it is bounded from below by a positive constant thanks to the assumption (3.3.14). Hence (3.3.42) follows from (3.3.44). The perimeter estimate (3.3.43) follows from (3.3.42) and [122, Lemma 5.1] (reported here as Lemma 2.4.32).

Step 2. The argument is similar to those involved in the proof of Theorem 3.3.5 above and therefore we only sketch it. Let $r_i\to\infty$ be any sequence such that $(X,\mathsf{d}/r_i,\mathcal{H}^N,x)$ converges to a tangent cone at infinity $(C(Z),\mathsf{d}_{C(Z)},\mathcal{H}^N,O)$ of

$(X, \mathsf{d}, \mathcal{H}^N)$. By the Ahlfors regularity estimates (3.3.42)-(3.3.43) and the compactness and stability [144, Theorem 2.42] (Theorem 2.4.33), the sequence $(X, \mathsf{d}/r_i, \mathcal{H}^N, E, x)$ converges to $(C(Z), \mathsf{d}_{C(Z)}, \mathcal{H}^N, O, F)$ for some non-empty perimeter minimizer $F \subset C(Z)$. At this stage, we are in position to apply Theorem 3.2.1 and obtain a perimeter minimizing cone in $C(Z)$, up to possibly taking an additional blow-down to apply the rigidity part of the monotonicity formula. □

In the remainder of the section, we present some technical results that have been used in the proof of Theorem 3.3.5.

Lemma 3.3.8. *Let $(Z, \mathsf{d}_Z, \mathcal{H}^2)$ be an* RCD(0, 2) *metric measure cone and let $G \subset Z$ be a globally perimeter minimizing set, in the sense of Definition 2.4.29. Then one of the following two possibilities occur:*

i) $(Z, \mathsf{d}_Z, \mathcal{H}^2)$ is isomorphic to $(\mathbb{R}^2, \mathsf{d}_{\text{eucl}}, \mathcal{H}^2)$ and G is a half-plane;

ii) $(Z, \mathsf{d}_Z, \mathcal{H}^2)$ is isomorphic to the half-plane $(\mathbb{R}^2_+, \mathsf{d}_{\text{eucl}}, \mathcal{H}^2)$ and G is a quadrant.

Proof. We distinguish two cases: if Z has no boundary, then we prove that it is isometric to $\mathbb{R}^2$ and i) must occur; if Z has non-empty boundary, then we prove that it is isometric to $\mathbb{R}^2_+$ and that ii) must occur.

Let us assume that $(Z, \mathsf{d}_Z, \mathcal{H}^2)$ has empty boundary. Then by [125] Z is isometric to a cone over $S^1(r)$ for some $0 < r \le 1$. Moreover, by Theorem 3.3.6 there exists a blow-down of G which is a global perimeter minimizing cone $C(A)$, with vertex in the origin and $A \subset S^1(r)$. We show that $A \subset S^1(r)$ is connected. Suppose not. Fix a point in $S^1(r)$ and consider the mapping $\varphi : S^1(r) \to 2\pi r$ that assigns to each point in $S^1(r)$ the clockwise arc-length distance from the fixed point. Let $a, b \in [0, 2\pi]$ be the infimum and supremum of $\varphi(A)$, respectively. Then one can check that, for sufficiently large $R > 0$ and small $\varepsilon > 0$, the set

$$B := \begin{cases} \varphi^{-1}([a, b)) & \text{inside } B_R(O), \\ A & \text{outside } B_R(O) \end{cases}$$

is a competitor for A in $B_{R+\varepsilon}(O)$ with $\text{Per}(A, B_{R+\varepsilon}) > \text{Per}(B, B_{R+\varepsilon})$.

Let $2\pi r\theta$ be the length of A, where $0 < \theta < 1$. Let $G' \subset Z$ be a set of finite perimeter such that $G' = G$ outside B_1 and $\partial G \cap B_1$ is composed by the geodesic connecting the two points in $\partial G \cap \partial B_1 = \{x_1, x_2\}$. Such geodesic is contained in B_1 as can be verified through the explicit form of the metric. Using (2.3.6)

$$\begin{aligned} \text{Per}(G'; B_1) &= \sqrt{2(1 - \cos(\mathsf{d}_{Z'}(x_1, x_2) \wedge \pi))} \\ &\le 2 = \text{Per}(G; B_1). \end{aligned} \tag{3.3.45}$$

Equality in (3.3.45) is achieved for $2\pi r\theta = \mathsf{d}_{S^1(r)}(x_1, x_2) \geq \pi$, that is for $1 \geq r\theta \geq \frac{1}{2}$. Let us notice, by symmetry of $S^1(r)$, that we may suppose that $\theta \leq \frac{1}{2}$. Indeed, for every fixed θ, we may find a comparison set with perimeter equal to the one constructed above corresponding to $1-\theta$. Hence equality is only achieved at $r = 1$, $\theta = \frac{1}{2}$, corresponding to the case where $Z = \mathbb{R}^2$ and $C(A)$ is a half space. Notice that once we have established that Z is isometric to $\mathbb{R}^2$, it is elementary that G must be a half-space.

In the case where Z has non-empty boundary, by [125] again, Z is isometric to a cone over a segment $[0, l]$ for some $0 < l \leq \pi$. The upper bound for the diameter is required in order for the cone to verify the $\mathrm{CD}(0, 2)$ condition. We claim that it must hold $l = \pi$.
As above, by Theorem 3.3.6, there exists a blow-down of G which is a global perimeter minimizing cone $C(A)$, with vertex in the origin and $A \subset [0, l]$ some set of finite perimeter. If A is not connected, then one can check as above that $C(A)$ is not globally perimeter minimizing. Notice also that the complement of a global perimeter minimizer is a global perimeter minimizer.
By minimality and symmetry we can suppose $G = C([0, l'])$, for some $0 < l' \leq \frac{l}{2}$. By considering a suitably constructed competitor in B_1, let us show that the only possibility is that Z is a half-space and $C(A)$ is a quadrant. Consider the set G' coinciding with G outside of B_1 and whose boundary inside B_1 is the geodesic minimizing the distance between $\partial B_1 \cap \partial G$ and ∂Z. Then $\mathrm{Per}(G'; B_1) \leq \mathrm{Per}(G; B_1)$, with equality achieved only if $l = \pi$ and $l' = \frac{\pi}{2}$. As above, once we have established that Z is isometric to $\mathbb{R}^2_+$, it is straightforward to check that G must be a quadrant. $\square$

It follows from the splitting Theorem 2.3.10 that any blow-up of a cone centered at a point different from the vertex splits a line. For our purposes it is important to observe that the blow-up of the squared distance function from the vertex is indeed a splitting function in the blow-up of the cone.

Lemma 3.3.9. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(N-2, N-1)$ space and let $(C(X), \mathsf{d}_{C(X)}, \mathfrak{m}_{C(X)})$ be the metric measure cone over X, with vertex $O \in C(X)$. Fix $O' \in C(X)$ with $O' \neq O$. Let $r_i \to 0$ and consider the sequence of rescaled spaces*

$$Y_i := (C(X), \mathsf{d}_{C(X)}/r_i, \mathfrak{m}_{C(X)}/\mathfrak{m}(B_{r_i}(O')), O')$$

converging in the pmGH topology to a tangent space Y of $C(X)$ at O'. Then the functions

$$f_i(\cdot) := \frac{\mathsf{d}^2_{C(X)}(O, \cdot) - \mathsf{d}^2_{C(X)}(O, O')}{r_i}, \tag{3.3.46}$$

viewed as functions $f_i : Y_i \to \mathbb{R}$*, have Laplacians uniformly converging to* 0 *and converge in* $H^{1,2}_{\mathrm{loc}}$ *to a splitting function* $g : Y \to \mathbb{R}$*, up to the extraction of a subsequence.*

Proof. Let us set

$$f(\cdot) := \mathsf{d}^2_{C(X)}(O, \cdot) - \mathsf{d}^2_{C(X)}(O, O') , \tag{3.3.47}$$

in order to ease the notation. On $C(X)$ it holds (see [79])

$$\Delta f = 2N , \quad |\nabla f(x)| = 2\mathsf{d}_{C(X)}(x, O) , \quad \text{for a.e. on } x \in C(X) . \tag{3.3.48}$$

By scaling, we obtain that

$$\Delta_i f_i = 2N r_i , \quad |\nabla f_i(x)| = 2\mathsf{d}_{C(X)}(x, O) , \quad \text{for a.e. } x \in Y_i , \tag{3.3.49}$$

where it is understood that the Laplacian and the minimal relaxed gradient are computed with respect to the metric measure structure
$(C(X), \mathsf{d}_{C(X)}/r_i, \mathfrak{m}_{C(X)}/\mathfrak{m}(B_{r_i}(O')), O')$. Notice that $x \mapsto 2\mathsf{d}_{C(X)}(x, O)$ is a $2r_i$-Lipschitz function on Y_i, by scaling.
Hence the functions $f_i : Y_i \to \mathbb{R}$ are locally uniformly Lipschitz, they satisfy $f_i(O') = 0$, and they have Laplacians locally uniformly converging to 0. Up to the extraction of a subsequence, thanks to a diagonal argument, we can assume that they converge locally uniformly and in $H^{1,2}_{\mathrm{loc}}$ to a function $g : Y \to \mathbb{R}$ in the domain of the local Laplacian, and that $\Delta_i f_i$ converge to Δg locally weakly in L^2, thanks to [18, 17] (see Theorem 2.3.21). We claim that g is a splitting function on Y, which amounts to say that $\Delta g = 0$ and $|\nabla g|$ is constant almost everywhere and not 0.

The fact that $\Delta g = 0$ follows from the weak convergence of the Laplacians and the identity $\Delta f_i = 2N r_i$ that we established above.
Analogously, employing the identity $|\nabla f_i(x)| = 2\mathsf{d}_{C(X)}(x, O)$ a.e. on Y_i, and the local $H^{1,2}$ convergence of f_i to g, we have that $|\nabla g| = 2\mathsf{d}(\cdot, O)$ a.e. on Y. □

The next result relates the heat flow on product spaces with one dimensional derivatives.

Lemma 3.3.10 (Heat flow and derivative in the splitting direction commute)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space and let* $X \times \mathbb{R}$ *be endowed with the standard product metric measure space structure. Let* $\varphi \in H^{1,2}(X \times \mathbb{R})$*. Then for every* $s > 0$ *it holds*

$$\mathrm{P}_s \partial_t \varphi(x, t) = \partial_t (\mathrm{P}_s \varphi)(x, t) , \tag{3.3.50}$$

for $\mathfrak{m}_X \otimes \mathcal{L}^1$*-a.e.* $(x, t) \in X \times \mathbb{R}$*.*

Proof. The statement follows from the tensorization of the Cheeger energy and of the heat flow for products of $\mathrm{RCD}(K,\infty)$ metric measure spaces, see for instance [15, 16], and from the classical commutation between derivative and heat semigroup on $\mathbb{R}$ endowed with the standard metric measure structure. $\square$

It is a well known fact of the Euclidean theory (see for instance [136]) that the perimeter enjoys natural tensorization properties, when taking an isometric product by an $\mathbb{R}$ factor. The next lemma establishes the RCD counterpart of this useful property.

Lemma 3.3.11 (Perimeter of Cylinders)**.** *Let $(X, \mathsf{d}_X, \mathfrak{m}_X)$ be an $\mathrm{RCD}(K,N)$ space and let $F \subset X$ be a Borel set. Under these assumptions, $E := F \times \mathbb{R} \subset X \times \mathbb{R}$ is a set of locally finite perimeter (where the product $X \times \mathbb{R}$ is endowed with the standard product metric measure structure), if and only if $F \subset X$ is a set of locally finite perimeter. Moreover, for any open set $A \subset X$ and for any $R > 0$ it holds*

$$R\,\mathrm{Per}(F; A) = \mathrm{Per}(E, A \times [0,R])\,. \tag{3.3.51}$$

Proof. By the very definition of perimeter it holds

$$\mathrm{Per}(E, A \times [0,R]) = \inf\left\{\liminf_{i\to\infty} \int_0^R \int_A \mathrm{lip}\,\varphi_i(t,x)\,d\mathfrak{m}_X\,dt\right\}\,,$$

where the infimum is taken over all sequences $\varphi_i \in \mathrm{LIP}_{\mathrm{loc}}(A \times [0,R])$ such that $\varphi_i \to \chi_E$ in $L^1_{\mathrm{loc}}(A \times [0,R])$. We are going to prove (3.3.51) and the first part of the statement will follow immediately.

Step 1. Let us start by showing the inequality

$$R\,\mathrm{Per}(F; A) \geq \mathrm{Per}(E, A \times [0,R])\,. \tag{3.3.52}$$

Let $\{\psi_i\}_i \subset \mathrm{LIP}_{\mathrm{loc}}(A)$ be a competitor for the perimeter of F in A, i.e. $\psi_i \to \chi_F$ in $L^1_{\mathrm{loc}}(A, \mathfrak{m}_X)$ and all the functions ψ_i are locally Lipschitz. Define $\phi(t,x) := \psi(x)$ for $0 \leq t \leq R$ and $x \in A$. Then by Fubini's Theorem $\{\phi_i\}_i$ is a competitor for the perimeter of E in $A \times [0,R]$. Therefore,

$$\begin{aligned}
\mathrm{Per}(F; A) &= \inf_{\{\psi_i\}_i}\left\{\liminf_{i\to\infty} \int_A \mathrm{lip}\,\psi_i(x)\,d\mathfrak{m}_X\right\} \\
&= \frac{1}{R}\inf_{\{\psi_i\}_i}\left\{\liminf_{i\to\infty} \int_0^R \int_A \mathrm{lip}\,\phi_i^{(t)}(x)\,d\mathfrak{m}_X\,dt\right\} \\
&\geq \frac{1}{R}\inf_{\{\varphi_i\}_i}\left\{\liminf_{i\to\infty} \int_0^R \int_A \mathrm{lip}\,\varphi_i(t,x)\,d\mathfrak{m}_X\,dt\right\} = \frac{1}{R}\mathrm{Per}(E; A \times [0,R])\,,
\end{aligned}$$

where the inequality follows from the fact that, on the right hand side, we are taking the infimum over a larger class.

Step 2. We prove the opposite inequality in (3.3.51).
Let us fix $\varepsilon > 0$. There exists a sequence $\{\varphi_i\}_i \subset \mathrm{LIP}_{\mathrm{loc}}(A \times [0, R])$ with $\varphi_i \to \chi_E$ in $L^1_{\mathrm{loc}}(A \times [0, R])$ such that

$$\liminf_{i \to \infty} \int_0^R \int_A \mathrm{lip}\, \varphi_i(t, x)\, dx\, dt \leq \mathrm{Per}(E; A \times [0, R]) + \varepsilon\,. \tag{3.3.53}$$

It is straightforward to check that $\mathrm{lip}\, \varphi_i^{(t)}(x) \leq \mathrm{lip}\, \varphi_i(t, x)$ for every $(t, x) \in \mathbb{R} \times X$. Moreover, the sequence $\{\varphi_i^{(t)}\}_i$ is a competitor for the variational definition of the perimeter of F in A for $\mathcal{L}^1$-almost every t, by the coarea formula. Therefore, by Fatou's lemma,

$$\begin{aligned} R\,\mathrm{Per}(F; A) &\leq \int_0^R \liminf_{i \to \infty} \int_A \mathrm{lip}\, \varphi_i^{(t)}(x)\, d\mathfrak{m}_X\, dt \\ &\leq \liminf_{i \to \infty} \int_0^R \int_A \mathrm{lip}\, \varphi_i^{(t)}(x)\, d\mathfrak{m}_X\, dt \\ &\leq \liminf_{i \to \infty} \int_0^R \int_A \mathrm{lip}\, \varphi_i(t, x)\, d\mathfrak{m}_X\, dt \leq \mathrm{Per}(E, A \times [0, R]) + \varepsilon\,. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude. □

Lastly, let us point out a few consequences of our proofs which will be used in the following chapter.

The first is a technical lemma. A proof of the equivalent statement in Euclidean spaces can be found in [136, Lemma 28.13], whereas a proof of Lemma 3.3.12 can be found by repeating the argument at the end of Step 3 in the proof of Theorem 3.3.4.

Lemma 3.3.12. *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be a non-collapsed $\mathrm{RCD}(K, N)$ space. Let $k \in \mathbb{N}$ and let $A \times \mathbb{R}^k \subset X \times \mathbb{R}^k$ be a perimeter minimizing set, then $A \subset X$ is also perimeter minimizing.*

We conclude the section by reporting a useful result regarding tangent spaces to perimeter minimizing sets in cones.

Lemma 3.3.13. *Let $(C(X), \mathsf{d})$ be a metric cone such that $(C(X), d, \mathcal{H}^N)$ is a non-collapsed $\mathrm{RCD}(0, N)$ metric measure space. If $E \subset C(X)$ is a perimeter minimizing set whose boundary contains the tip of the cone O, then there exists a set $A \subset X$ such that $C(A) \subset C(X)$ is a perimeter minimizing cone, whose boundary contains O, and such that*

$$(C(X), \mathsf{d}, O, C(A)) \in \mathrm{Tan}_p(C(X), \mathsf{d}).$$

The proof of this result follows from step 1 in the proof of Theorem 3.3.4: it is a consequence of the rigidity part of the monotonicity formula Theorem 3.1.1 and a density estimate for locally perimeter minimizing sets shown in [123, Lemma 5.1] (reported here in this manuscript as Lemma 2.4.32).

Chapter 4

On the dimension of the singular set of perimeter minimizers in spaces with a two-sided bound on the Ricci curvature

This work is based on a joint work with Cucinotta [70]. We show that the Hausdorff dimension of the singular set of perimeter minimizers in non-collapsed limits of manifolds with two-sided bounds on the Ricci curvature is at most $N-5$, where N is the dimension of the ambient space. The estimate is sharp.

We outline here the structure of the chapter. In *Section 1*, we present our contributions and give an overview of the problem and of related results. We also describe the main results our contributions are based on.

In *Section 2*, we review some classical results regarding the second variation formula of perimeter minimizers in Euclidean spaces. We decided to report them here for ease of the reader, as our results heavily rely on them.

Lastly, in *Section 3* we present the proof of our results.

4.1 Introduction

In this chapter, we consider non-collapsed limits of manifolds with two-sided bounds on the Ricci curvature. These are pointed metric spaces (X, d, x) arising as pointed Gromov-Hausdorff limits of sequences of pointed Riemannian manifolds (M_k^n, d_k, x_k), where d_k denotes the Riemannian distance, satisfying

$$|\mathrm{Ric}_{M_k^n}| \leq N-1, \quad \text{and} \quad \mathrm{vol}(B_1(x_k)) \geq v > 0 \quad \text{for every } k. \tag{4.1.1}$$

A non-exhaustive list of works where spaces satisfying the conditions above were initially studied is [26, 36, 181, 24, 60]. The study of metric spaces arising as Gromov-Hausdorff limits of Riemannian manifolds with a uniform lower Ricci curvature bound was carried out in [61, 62, 63], among others. See also Section 2.2.2.

We are concerned with perimeter minimizing sets in non-collapsed limits of manifolds with two-sided bounds on the Ricci curvature. The notion of perimeter in metric measure spaces was studied in [9, 8, 139, 19, 13], among others. In recent years, the theory of sets of finite perimeter was further studied in [11], [46], [44] (among others; see Section 2.4) in the setting of metric measure spaces with a synthetic notion of Ricci curvature lower bounds, known as RCD spaces. The Riemannian Curvature Dimension condition $\mathrm{RCD}(K, \infty)$ was introduced in [16] (see also [99, 23]) while its finite dimensional counterpart $\mathrm{RCD}(K, N)$ was formalized in [99]. For a thorough introduction to the topic we refer to the survey [7] and Section 2.3.2.

Some fundamental steps towards understanding perimeter minimizing sets in the context of RCD spaces and Ricci limit spaces were carried out in [144] and [86, 85] respectively. Other properties were then investigated in [94], [71], [69]. See Section 2.4.4 for a brief introduction to the topic. Let us point out that minimal hypersurfaces in Riemannian manifolds are locally boundaries of locally perimeter minimizing sets. Moreover, the study of sets of finite perimeter in the RCD setting, due to their convergence and stability properties, allows to deduce new results about classical area minimizing hypersurfaces and isoperimetric sets in Riemannian manifolds (see, for instance, [34], [32], [144], [71]).

One of the key advances in the study of perimeter minimizing sets in Euclidean spaces was understanding the Hausdorff dimension of their singular set. A fundamental result obtained by De Giorgi [74] and refined by Federer following the work of Simons shows that a perimeter minimizing set $E \subset \mathbb{R}^N$ is smooth outside a closed set of Hausdorff dimension at most $N - 8$. The regularity of perimeter minimizing sets in RCD spaces was studied in [144]. To report here the relevant result, we introduce some notation.

We recall that a point $x \in \partial E$ is regular if it is a regular point for the ambient space X, and additionally the tangent of E at x is a half space. This definition is slightly different from the classical one in the Euclidean setting: if one exploits the non-smooth definition in the smooth setting, the usual uniqueness requirement for the tangent space needs to be dropped, as different half spaces are all isometric.

The sharp estimate for the Hausdorff dimension of the singular set of local perimeter minimizers in non-collapsed $\mathrm{RCD}(K, N)$ spaces shown in [144] states the following: if $(X, \mathsf{d}, \mathcal{H}^N)$ is an $\mathrm{RCD}(K, N)$ space without boundary and $E \subset X$ is a locally perimeter minimizing set, then

$$\dim_{\mathcal{H}}(\mathcal{S}^E) \leq N - 3. \tag{4.1.2}$$

The Hausdorff dimension of the singular set of a non-collapsed $\mathrm{RCD}(K, N)$ space without boundary is at most $N - 2$, as shown in [142]. The analogous result for non-collapsed Ricci limit spaces was shown in [61]. In the case of non-collapsed limits of manifolds with two-sided bounds on the Ricci curvature, a stronger estimate on the Hausdorff dimension of the singular set holds. It was shown in [66] that if (X, d, x) is a non-collapsed limit of manifolds with two-sided bounds on the Ricci curvature of dimension N, then the Hausdorff dimension of its singular set is at most $N - 4$.

Comparing the estimates for $\dim_{\mathcal{H}}(\mathcal{S}^E)$ and $\dim_{\mathcal{H}}(\mathcal{S}(X))$ when $E \subset X$ is a locally perimeter minimizer in a non-collapsed $\mathrm{RCD}(K, N)$ space, a question that arises is whether, under the stronger assumption that X is a non-collapsed limit of manifolds with two-sided bounds on the Ricci curvature, it holds that $\dim_{\mathcal{H}}(\mathcal{S}^E) \leq N - 5$. This is the content of the main result of this note.

Theorem 4.1.1. *Let (X, d) be a non-collapsed limit of manifolds with two-sided bounds on the Ricci curvature of dimension N. If $E \subset X$ is a locally perimeter minimizing set, then $\mathcal{S}^E = \mathcal{S}^E_{N-5}$. In particular, it holds*

$$\dim_{\mathcal{H}}(\mathcal{S}^E) \leq N - 5. \tag{4.1.3}$$

Let us point out that Theorem 4.1.1 is sharp, as shown in Example 4.3.14.

Remark 4.1.2 (On the behavior of ∂E in $\mathcal{R}(X)$)**.** *Let $E \subset X$ be as in Theorem 4.1.1. By [144], the Hausdorff dimension of $\mathcal{S}^E \cap \mathcal{R}(X)$ is at most $N - 8$. By exploiting the double bound on the Ricci curvature, one obtains a finer description of $\partial E \cap \mathcal{R}(X)$. By [24, 60], the metric space X is a $C^{1,\alpha}$ manifold for every $\alpha \in (0, 1)$ outside of its singular set. By means of classical regularity results, one can then show that $\partial E \cap \mathcal{R}(X)$ is a $C^{1,\alpha}$ hypersurface of $\mathcal{R}(X)$ for every $\alpha \in (0, 1)$ outside of a closed set of Hausdorff dimension at most $N - 8$ (see Proposition 4.3.15).*

Before outlining the proof of Theorem 4.1.1, we mention that the key step in the proof is Theorem 4.1.3, which is a Bernstein-type theorem for cones over manifolds of constant curvature 1.

Theorem 4.1.3. *Let (M, g) be a manifold of constant sectional curvature equal to 1 and of dimension $N \leq 6$. Let $C(M)$ be the metric cone over M and let O be its tip. If $E \subset C(M)$ is a perimeter minimizing set such that $O \in \partial E$, then $M \cong S^N$, $C(M) \cong \mathbb{R}^{N+1}$ and $E \subset C(M) \cong \mathbb{R}^{N+1}$ is a half space.*

The proof of the previous theorem follows by adapting a classical result of Simons from [176]. Let us mention that in Theorem 4.1.3 the assumption on the dimension is sharp, as shown by Example 4.3.13.

We now outline the proof of Theorem 4.1.1. The estimate on the Hausdorff dimension (4.1.3) follows from the stratification result [94, Theorem 4.1] and $\mathcal{S}^E = \mathcal{S}^E_{N-5}$.

We divide the proof of $\mathcal{S}^E = \mathcal{S}^E_{N-5}$ in two steps. We first show that $\mathcal{S}^E = \mathcal{S}^E_{N-4}$. We rely on the following argument: suppose by contradiction that $x \in \mathcal{S}^E \setminus \mathcal{S}^E_{N-4}$. Then by [66, Theorem 5.12] the tangent space to X at x is isometric to $\mathbb{R}^N$. Moreover, by assumption, a tangent space to E at x is of the form $\mathbb{R}^{N-3} \times A$, where $A \subset \mathbb{R}^3$ is itself a perimeter minimizer. Therefore, by a classical Bernstein-type theorem (see, for instance, [136, Theorem 28.17]), it follows that A is a half space. This provides the desired contradiction and shows $\mathcal{S}^E = \mathcal{S}^E_{N-4}$.

In a second step of the proof we show $\mathcal{S}^E_{N-4} \setminus \mathcal{S}^E_{N-5} = \emptyset$. To this end, we suppose by contradiction that $x \in \mathcal{S}^E_{N-4} \setminus \mathcal{S}^E_{N-5}$. Then by [66] and [24] (see also [65, Theorem 1.16] or Theorem 4.3.3) a tangent space of X at x is $\mathbb{R}^{N-4} \times C(S^3/\Gamma)$, where $\Gamma \subset O(4)$ is a discrete group of isometries of the sphere acting freely.

Moreover, a tangent space to E at x is of the form $\mathbb{R}^{N-4} \times A$, where $A \subset C(S^3/\Gamma)$ is itself a perimeter minimizer.

Therefore, by Theorem 4.1.3, we are able to conclude $\Gamma = \{id_{S^3}\}$, $C(S^3/\Gamma) \cong \mathbb{R}^4$, and $A \cong \mathbb{R}^3 \times [0, +\infty)$, contradicting the initial assumption $x \in \mathcal{S}^E_{N-4} \setminus \mathcal{S}^E_{N-5}$.

4.2 Second variation formula in Euclidean spaces

In this section we collect some technical results on perimeter minimizing sets in Euclidean spaces that are used in the proof of Theorem 4.1.3. The topic is classical and we refer to [136] and [108] for an account of the theory.

Let us recall the notion of tangential derivatives to the boundary of a smooth open set $E \subset \mathbb{R}^N$. Let $\nu : \partial E \to S^{N-1}$ be the outward normal vector on ∂E and let $g \in C^\infty(\mathbb{R}^N)$. On a point $x \in \partial E$ the tangential derivative of g is defined as

$$\nabla_E g := \nabla g - \nu(\nabla g \cdot \nu).$$

Given any integer $1 \le i \le N$, the i-th component of the tangential derivative of g is defined as

$$\nabla_{E,i} g := \partial_i g - \nu_i (\nabla g \cdot \nu).$$

Similarly, the tangential Laplacian of g is defined as

$$\Delta_E g := \sum_{i=1}^{N} \nabla_{E,i} \nabla_{E,i} g.$$

Both $\nabla_E g$ and $\Delta_E g$ depend only on the restriction of g to ∂E; for this reason we consider tangential derivatives and tangential Laplacians of functions that are defined only on ∂E, assuming implicitly that we are extending the functions smoothly to $\mathbb{R}^N$ before applying such operators.

We denote by $|\Pi_E| : \partial E \to \mathbb{R}$ the norm of the second fundamental form of ∂E in $\mathbb{R}^N$. This coincides with the square root of the sum of the squares of the principal curvatures of ∂E. One can check that $|\Pi_E|^2 = \sum_{i,j} (\nabla_{E,i} \nu_j)^2$ (see [108, Remark 10.6]). Whenever ∂E is not smooth, we assume that the aforementioned objects are defined in the largest smooth subset of ∂E. We mention that in [108] the objects ∇_E, $\nabla_{E,i}$, Δ_E, $|\Pi_E|$ are denoted respectively by $\delta, \delta_i, \mathcal{D}$, and c.

We say that a set $E \subset \mathbb{R}^N$ is a cone with tip O if it is invariant under dilations which fix O. This notion is consistent with the one of metric cone previously introduced. Without loss of generality, in the remainder of this work we suppose that O coincides with the origin $0 \in \mathbb{R}^N$.

Remark 4.2.1. *By inspecting the proof of [108, Lemma 10.9] one realizes that if $E \subset \mathbb{R}^N$ is a cone which is both smooth and has zero mean curvature in $\mathbb{R}^N \setminus \{0\}$, then $|\Pi_E|^2$ is homogeneous of degree -2.*

We now recall the second variation formula for sets with vanishing mean curvature, which can be found in [108, Identity (10.13)].

Proposition 4.2.2. *Let $E \subset \mathbb{R}^N$ be an open set such that ∂E is smooth and has zero mean curvature in an open set $A \subset\subset \mathbb{R}^N$. Let $\nu : A \to S^{N-1}$ be an extension of the outward unit normal of ∂E to A, and let $\zeta \in C_c^\infty(A)$. Define $F_t : A \to \mathbb{R}^N$ by $F_t(x) := x + t\zeta(x)\nu(x)$ and set $E_t := F_t(E)$. Then*

$$\left(\frac{d^2}{dt^2}\mathrm{Per}(A, E_t)\right)_{|t=0} = \int_{\partial E} (|\nabla_E \zeta|^2 - |\Pi_E|^2 \zeta^2)\, \mathrm{d}\mathcal{H}^{N-1}.$$

We conclude the section by recalling that tangential derivatives satisfy an integration by parts formula and that $\Delta_E |\Pi_E|^2$ is well behaved on minimal sets that are invariant under dilations. The next result can be found in [108, Lemma 10.8].

Lemma 4.2.3. *Let $E \subset \mathbb{R}^N$ be such that ∂E is a smooth hypersurface and let $\phi \in C_c^\infty(\mathbb{R}^N)$. Then*

$$\int_{\partial E} \nabla_{E,i}\phi \, \mathsf{d}\mathcal{H}^{N-1} = -\int_{\partial E} \phi\nu_i \, \mathsf{d}\mathcal{H}^{N-1}.$$

The next result can be found in [108, Lemma 10.9].

Lemma 4.2.4. *Let $E \subset \mathbb{R}^N$ be a cone which is both smooth and has vanishing mean curvature in $\mathbb{R}^N \setminus \{0\}$. Then in $\partial E \cap \mathbb{R}^N \setminus \{0\}$ we have*

$$\frac{1}{2}\Delta_E|\Pi_E|^2 \geq -|\Pi_E|^4 + |\nabla_E|\Pi_E||^2 + \frac{2|\Pi_E|^2}{|x|^2}.$$

4.3 Proof of the main results

The goal of this section is to prove Theorem 4.1.1 and Theorem 4.1.3, which we restate for the reader's convenience. The proofs will be given at the end of the section.

Theorem 4.3.1. *Let (X, d) be a non-collapsed limit of manifolds with two-sided bounds on the Ricci curvature of dimension N. If $E \subset X$ is a locally perimeter minimizing set, then $\mathcal{S}^E = \mathcal{S}^E_{N-5}$. In particular, it holds*

$$\dim_{\mathcal{H}}(\mathcal{S}^E) \leq N - 5. \tag{4.3.1}$$

Theorem 4.3.2. *Let (M, g) be a manifold of constant sectional curvature equal to 1 and of dimension $N \leq 6$. Let $C(M)$ be the metric cone over M and let O be its tip. If $E \subset C(M)$ is a perimeter minimizing set such that $O \in \partial E$, then $M \cong S^N$, $C(M) \cong \mathbb{R}^{N+1}$ and $E \subset C(M) \cong \mathbb{R}^{N+1}$ is a half space.*

We begin by reporting a result obtained in [117, Theorem 1.15] (see also [66, 65] and [24]), which is the starting point of the proof of Theorem 4.1.1. We also report the proof here for ease of the reader.

Theorem 4.3.3. *Let (X, d, x) be a non-collapsed limit of manifolds with two-sided bounds on the Ricci curvature. Then for any $x \in \mathcal{S}_{N-4} \setminus \mathcal{S}_{N-5}$ there exists a tangent space at x which is isometric to $\mathbb{R}^{N-4} \times C(S^3/\Gamma)$, where $\Gamma \subset O(4)$ is a discrete group acting freely.*

Proof. Let $x \in \mathcal{S}_{N-4} \setminus \mathcal{S}_{N-5}$. By the definition of singular strata Definition 2.2.10 there exists a tangent space to X at x isometric to $\mathbb{R}^{N-4} \times C(Y)$, for some metric space (Y, d_Y). We argue that Y is a $C^{1,\alpha}$ manifold. If not, by [26] $C(Y)$ would have a singular set of Hausdorff dimension at least 1 and, since $C(Y)$ is a Ricci limit space with a two-sided bound on the Ricci curvature, it would contradict Theorem 2.2.13. By [24] $C(Y)$ is smooth and Ricci flat outside of its tip. By relating the Ricci curvature of $C(Y)$ with that of Y (for instance, see [155, Corollary 43]) we can infer that $\mathrm{Ric}_Y = 2g_Y$. By a classification result of 3-dimensional Einstein manifolds (see [53], for instance) Y is isometric to S^3/Γ, where $\Gamma \subset O(4)$ is a discrete group acting freely. $\square$

Let us fix some notation. In this section $N \in \mathbb{N}$, $N \geq 2$ and Γ is a discrete group of isometries of S^{N-1} acting freely. Moreover, Γ induces an action on $\mathbb{R}^N$ given in polar coordinates by $g \cdot (\omega, r) := (g(\omega), r)$. We denote by $\pi : \mathbb{R}^N \to \mathbb{R}^N/\Gamma$ the projection on the quotient space. Since Γ acts freely on S^{N-1}, it follows that it also acts freely on $\mathbb{R}^N \setminus \{0\}$. Consequently, $\pi_{|\mathbb{R}^N \setminus \{0\}}$ is a covering of $(\mathbb{R}^N \setminus \{0\})/\Gamma$. Therefore, it is also a local isometry.

We say that an open set $U \subset (\mathbb{R}^N \setminus \{0\})/\Gamma$ is a *cover chart* if its preimage through π is a finite union of disjoint open sets $\{U_i\}_{i=1}^l$ (where l is the cardinality of Γ) such that $\pi_{|U_i} : U_i \to U$ is a bijective isometry for every i. Given a subset $E \subset \mathbb{R}^N$ and $g \in \Gamma$, we denote $g \cdot E := \{g \cdot e : e \in E\}$. Moreover, given a subset $E \subset \mathbb{R}^N$ $(\mathbb{R}^N/\Gamma)$ and $t > 0$, we define the rescaled set $E/t := \{x \in \mathbb{R}^N(\mathbb{R}^N/\Gamma) : tx \in E\}$.

Definition 4.3.4 (Γ-symmetric sets)**.** *We say that a set $E \subset \mathbb{R}^N$ is* Γ-symmetric *if for every $g \in \Gamma$ we have $g \cdot E = E$.*

The next lemma shows that Γ-symmetric sets arise as preimages via π of sets in $\mathbb{R}^N/\Gamma$.

Lemma 4.3.5. *If $E \subset \mathbb{R}^N$ is Γ-symmetric, then $\pi^{-1}(\pi(E)) = E$. Conversely, if $F \subset \mathbb{R}^N/\Gamma$, then $\pi^{-1}(F)$ is Γ-symmetric and $\pi(\pi^{-1}(F)) = F$.*

Proof. We start by showing that if $E \subset \mathbb{R}^N$ is Γ-symmetric, then $\pi^{-1}(\pi(E)) = E$.

Observe that $\pi^{-1}(\pi(E)) \supset E$ trivially, so that we only need to prove the other inclusion. If $x \in \pi^{-1}(\pi(E))$, then $\pi(x) = \pi(y)$ for some $y \in E$. Therefore, there exists $g \in \Gamma$ such that $g \cdot x = y$, giving that $x \in E$ as this set is Γ-symmetric.

Let us show that, if $F \subset \mathbb{R}^N/\Gamma$, then $\pi^{-1}(F)$ is Γ-symmetric and $\pi(\pi^{-1}(F)) = F$.

Consider $x \in \pi^{-1}(F)$. We show that for every $g \in \Gamma$ we have $g \cdot x \in \pi^{-1}(F)$. To this aim, note that $\pi(x) = \pi(g \cdot x)$ so that in particular $g \cdot x \in \pi^{-1}(x) \subset \pi^{-1}(F)$. Conversely, let $x \notin \pi^{-1}(F)$. We show that for every $g \in \Gamma$ it holds $g \cdot x \notin \pi^{-1}(F)$. Indeed, if $g \cdot x \in \pi^{-1}(F)$, then $x \in \pi^{-1}(F)$.

Finally, $\pi(\pi^{-1}(F)) = F$ since π is surjective. $\square$

Definition 4.3.6 (Γ-symmetric sets minimizing the perimeter against Γ-symmetric competitors)**.** *We say that a* Γ*-symmetric set* $E \subset \mathbb{R}^N$ *minimizes the perimeter against* Γ*-symmetric competitors if for every* Γ*-symmetric set* $A \subset \mathbb{R}^N$ *and* $r > 0$ *such that* $E\Delta A \subset\subset B_r(0)$ *we have*

$$\mathrm{Per}(E, B_r(0)) \leq \mathrm{Per}(A, B_r(0)).$$

The following key lemma allows us to compare the perimeter of subsets of $\mathbb{R}^N/\Gamma$ with the perimeter of their preimage through the projection map π.

Lemma 4.3.7. *If* $F \subset \mathbb{R}^N/\Gamma$ *and* $l \in \mathbb{N}$ *is the cardinality of* Γ*, then for every measurable set* $U \subset \mathbb{R}^N/\Gamma$*, it holds*

$$l\mathrm{Per}(F, U) = \mathrm{Per}(\pi^{-1}(F), \pi^{-1}(U)).$$

Proof. We claim that we can find a countable collection of disjoint measurable subsets $\{B_i\}_{i\in\mathbb{N}}$ of $\mathbb{R}^N/\Gamma$ which covers $(\mathbb{R}^N \setminus \{0\})/\Gamma$ and is such that each set B_i is contained in a cover chart of $(\mathbb{R}^N\setminus\{0\})/\Gamma$. To this aim, let $\{A_i\}_{i\in\mathbb{N}}$ be a covering of $(\mathbb{R}^N\setminus\{0\})/\Gamma$ with cover charts, which exists as every point has a neighborhood which is a cover chart. To obtain a disjoint cover we define $B_1 := A_1$ and $B_{i+1} := A_{i+1} \setminus \cup_{j=1}^{i} A_j$.

For every $i \in \mathbb{N}$ the preimage $\pi^{-1}(A_i)$ coincides with the disjoint union $\cup_{j=1}^{l} A_i^j$, and for every integer $1 \leq j \leq l$

$$\pi_{|A_i^j} : A_i^j \to A_i$$

is a bijective isometry. In particular, by Lemma 2.4.10

$$\mathrm{Per}(F, B_i \cap U) = \mathrm{Per}((\pi_{|A_i^j})^{-1}(F), (\pi_{|A_i^j})^{-1}(B_i \cap U)) \quad \text{for every } j = 1, ..., l.$$

Since

$$A_i^j \cap (\pi_{|A_i^j})^{-1}(F) = A_i^j \cap \pi^{-1}(F),$$

it holds

$$\mathrm{Per}(F, B_i \cap U) = \mathrm{Per}(\pi^{-1}(F), (\pi_{|A_i^j})^{-1}(B_i \cap U)) \quad \text{for every } j = 1, ..., l.$$

Summing over j we then get that for every $i \in \mathbb{N}$ it holds

$$l\mathrm{Per}(F, B_i \cap U) = \mathrm{Per}(\pi^{-1}(F), \pi^{-1}(U \cap B_i)).$$

The collection $\{\pi^{-1}(B_i)\}_{i\in\mathbb{N}}$ is also a disjoint cover of $\mathbb{R}^N \setminus \{0\}$ so that, taking into account that $\mathrm{Per}(F, \{0\}) = \mathrm{Per}(\pi^{-1}(F), \{0\}) = 0$, we obtain

$$\begin{aligned} l\mathrm{Per}(F, U) &= \sum_{i\in\mathbb{N}} l\mathrm{Per}(F, U \cap B_i) = \sum_{i\in\mathbb{N}} \mathrm{Per}(\pi^{-1}(F), \pi^{-1}(B_i \cap U)) \\ &= \mathrm{Per}(\pi^{-1}(F), \pi^{-1}(U)). \end{aligned}$$

□

The next lemma shows that there exists a correspondence between perimeter minimizers in $\mathbb{R}^N/\Gamma$ and Γ-symmetric sets minimizing the perimeter against Γ-symmetric competitors in $\mathbb{R}^N$.

Lemma 4.3.8. *Let $F \subset \mathbb{R}^N/\Gamma$ be a perimeter minimizing set, then $\pi^{-1}(F)$ is a Γ-symmetric set minimizing the perimeter against Γ-symmetric competitors in $\mathbb{R}^N$. Conversely, if $E \subset \mathbb{R}^N$ is a Γ-symmetric set minimizing the perimeter against Γ-symmetric competitors, then $\pi(E) \subset \mathbb{R}^N/\Gamma$ is a perimeter minimizing set.*

Proof. Let $F \subset \mathbb{R}^N/\Gamma$ be a perimeter minimizing set. Then $\pi^{-1}(F)$ is a Γ-symmetric set by Lemma 4.3.5. We now show that $\pi^{-1}(F)$ also minimizes the perimeter with respect to Γ-symmetric competitors. Let $r > 0$ and $E' \subset \mathbb{R}^N$ be a Γ-symmetric set such that $\pi^{-1}(F)\Delta E' \subset\subset B_r(0)$. Since $\pi(B_r(0)) = B_r(0) \subset \mathbb{R}^N/\Gamma$, using Lemma 4.3.7 we obtain

$$\mathrm{Per}(\pi^{-1}(F), B_r(0)) = l^{-1}\mathrm{Per}(F, B_r(0)) \leq l^{-1}\mathrm{Per}(\pi(E'), B_r(0)) = \mathrm{Per}(E', B_r(0)).$$

Since $r > 0$ is arbitrary, we conclude that $\pi^{-1}(F)$ is a Γ symmetric set minimizing the perimeter against Γ-symmetric competitors in $\mathbb{R}^N$.

In an analogous fashion, one can show that if $E \subset \mathbb{R}^N$ is a Γ-symmetric set minimizing the perimeter against Γ-symmetric competitors, then $\pi(E) \subset \mathbb{R}^N/\Gamma$ is a perimeter minimizing set. □

The next proposition shows that Γ-symmetric sets minimizing the perimeter against Γ-symmetric competitors in $\mathbb{R}^N$ are locally perimeter minimizing in $\mathbb{R}^N \setminus \{0\}$.

Proposition 4.3.9. *Let $E \subset \mathbb{R}^N$ be a Γ-symmetric set minimizing the perimeter against Γ-symmetric competitors in $\mathbb{R}^N$, then E is locally perimeter minimizing in $\mathbb{R}^N \setminus \{0\}$. In particular, in $\mathbb{R}^N \setminus \{0\}$ the set E admits an open and a closed representative sharing the same topological boundary. Moreover, if $N \leq 7$, then E has smooth boundary with vanishing mean curvature in $\mathbb{R}^N \setminus \{0\}$.*

Proof. By Lemma 4.3.8, the set $\pi(E) \subset \mathbb{R}^N/\Gamma$ is perimeter minimizing. Since the restricted projection map $\pi : \mathbb{R}^N \setminus \{0\} \to (\mathbb{R}^N/\Gamma) \setminus \{0\}$ is a local isometry, E is then locally perimeter minimizing. □

When we refer to a Γ-symmetric set minimizing the perimeter against Γ-symmetric competitors in $\mathbb{R}^N$, we implicitly mean its open representative. In the next lemma we deal with tangent spaces to sets of finite perimeter in $\mathbb{R}^N$ and $\mathbb{R}^N/\Gamma$. In both cases, when referring to elements of the tangent space at a point, we omit the distance.

Lemma 4.3.10. *Let $E \subset \mathbb{R}^N$ be a Γ-symmetric set which minimizes the perimeter against Γ-symmetric competitors and whose boundary contains 0. Then there exists a Γ-symmetric cone $E' \subset \mathbb{R}^N$ which minimizes the perimeter against Γ-symmetric competitors, whose boundary contains 0, and such that*

$$(\mathbb{R}^N, 0, E') \in \mathrm{Tan}_0(\mathbb{R}^N, E).$$

Proof. By Lemma 4.3.8, $\pi(E) \subset \mathbb{R}^N/\Gamma$ is a perimeter minimizing set. In particular, by Lemma 3.3.13, there exists a perimeter minimizing cone $\pi(E)^\infty \subset \mathbb{R}^N/\Gamma$, whose boundary contains 0, and such that

$$(\mathbb{R}^N/\Gamma, 0, \pi(E)^\infty) \in \mathrm{Tan}_0(\mathbb{R}^N/\Gamma, \pi(E)).$$

We claim that

$$(\mathbb{R}^N, 0, \pi^{-1}(\pi(E)^\infty)) \in \mathrm{Tan}_0(\mathbb{R}^N, E). \tag{4.3.2}$$

We fix $x \in (\mathbb{R}^N \setminus \{0\})/\Gamma$ and we consider a bounded cover chart $A \subset (\mathbb{R}^N \setminus \{0\})/\Gamma$ containing x. The preimage $\pi^{-1}(A)$ coincides with the disjoint union of open sets $\cup_{j=1}^l A_j$ such that the restricted maps $\pi_{|A_j} : A_j \to A$ are bijective isometries. Since $(\mathbb{R}^N/\Gamma, 0, \pi(E)^\infty) \in \mathrm{Tan}_0(\mathbb{R}^N/\Gamma, \pi(E))$, there exists a sequence $t_k \to 0$, independent of A, such that

$$\|\chi_{\pi(E)/t_k} - \chi_{\pi(E)^\infty}\|_{L^1(A)} \to 0 \quad \text{as } t_k \to 0.$$

Taking into account that $\pi^{-1}(\pi(E)/t_k) = E/t_k$, for every $j = 1, ..., l$ it holds

$$\|\chi_{E/t_k} - \chi_{\pi^{-1}(\pi(E)^\infty)}\|_{L^1(A_j)} \to 0 \quad \text{as } t_k \to 0. \tag{4.3.3}$$

Hence, we can construct a locally finite open cover $\{A_j\}_{j\in\mathbb{N}}$ of $\mathbb{R}^N \setminus \{0\}$, such that for every $j \in \mathbb{N}$ condition (4.3.3) is satisfied. Setting $B_1 := A_1$ and $B_j := A_j \setminus \cup_{i=1}^{j-1} B_i$ we obtain a refinement of the cover $\{A_j\}_{j\in\mathbb{N}}$ consisting of disjoint sets. Since $\{A_j\}_{j\in\mathbb{N}}$ is locally finite in $\mathbb{R}^N \setminus \{0\}$ also $\{B_j\}_{j\in\mathbb{N}}$ has this property. Hence, for every $r, \varepsilon > 0$ with

$r > \varepsilon$, there exists a finite subset $I_{r,\varepsilon} \subset \mathbb{N}$ such that $\{B_j\}_{j\in I_{r,\varepsilon}}$ covers $B_r(0) \setminus B_\varepsilon(0)$. Having fixed $r > \varepsilon > 0$ we then obtain

$$\begin{aligned} \|\chi_{E/t_k} - &\chi_{\pi^{-1}(\pi(E)^\infty)}\|_{L^1(B_r(0))} \\ &\le \|\chi_{E/t_k} - \chi_{\pi^{-1}(\pi(E)^\infty)}\|_{L^1(B_\varepsilon(0))} + \sum_{j\in I_{r,\varepsilon}} \|\chi_{E/t_k} - \chi_{\pi^{-1}(\pi(E)^\infty)}\|_{L^1(B_j)}. \end{aligned} \tag{4.3.4}$$

Since E/t_k and $\pi^{-1}(\pi(E)^\infty)$ are both perimeter minimizing, it follows from Theorem 2.4.31 that

$$\sup_{k\in\mathbb{N}} \|\chi_{E/t_k} - \chi_{\pi^{-1}(\pi(E)^\infty)}\|_{L^1(B_\varepsilon(0))} \le c(N)\mathcal{H}^N(B_\varepsilon(0)) \to 0 \quad \text{as } \varepsilon \to 0.$$

On the other hand, since $I_{r,\varepsilon} \subset \mathbb{N}$ is finite, it holds

$$\sum_{j\in I_{r,\varepsilon}} \|\chi_{E/t_k} - \chi_{\pi^{-1}(\pi(E)^\infty)}\|_{L^1(B_j)} \to 0 \quad \text{as } t_k \to 0.$$

Hence, passing to the limit in (4.3.4), one obtains

$$\|\chi_{E/t_k} - \chi_{\pi^{-1}(\pi(E)^\infty)}\|_{L^1(B_r(0))} \to 0 \quad \text{as } t_k \to 0,$$

proving claim (4.3.2).

Since $\pi(E)^\infty \subset \mathbb{R}^N/\Gamma$ is a cone, $\pi^{-1}(\pi(E)^\infty) \subset \mathbb{R}^N$ is also a cone. Since $0 \in \partial(\pi(E)^\infty) \subset \mathbb{R}^N/\Gamma$, then $0 \in \partial(\pi^{-1}(\pi(E)^\infty)) \subset \mathbb{R}^N$ as well. Finally, since $\pi(E)^\infty \subset \mathbb{R}^N/\Gamma$ is a perimeter minimizer, $\pi^{-1}(\pi(E)^\infty) \subset \mathbb{R}^N$ is a Γ-symmetric set minimizing the perimeter against Γ-symmetric competitors by Lemma 4.3.8. □

Definition 4.3.11 (Γ-symmetric functions)**.** *Given a Γ-symmetric set $E \subset \mathbb{R}^N$ we say that a function $f : E \to \mathbb{R}$ is Γ-symmetric if $f(x) = f(g \cdot x)$ for every $x \in E$ and $g \in \Gamma$.*

Lemma 4.3.12. *Let $E \subset \mathbb{R}^N$ be a Γ-symmetric cone which is smooth in $\mathbb{R}^N \setminus \{0\}$ and has zero mean curvature in $\mathbb{R}^N \setminus \{0\}$. There exists a smooth Γ-symmetric extension of $|\Pi_E|^2 : \partial E \setminus \{0\} \to \mathbb{R}$ to $\mathbb{R}^N \setminus \{0\}$.*

Proof. Since E is a cone which is smooth in $\mathbb{R}^N \setminus \{0\}$, the intersection $S := \partial E \cap S^{N-1}$ is a $N-2$ dimensional closed smooth manifold in S^{N-1}. We now consider the function $|\Pi_E|^2$ restricted to S. We show that it can be extended to a function $h : S^{N-1} \to \mathbb{R}$ with the property that for every $g \in \Gamma$ and every $x \in S^{N-1}$ it holds $h(g \cdot x) = h(x)$.

Let U be a tubular neighborhood of S in S^{N-1} and let $\pi_S : U \to S$ be the nearest point projection in S^{N-1}. We define $h_1 \in C^\infty(U)$ by $h_1(x) := |\Pi_E|^2(\pi_S(x))$. Let

$\eta \in C_c^\infty(U)$ be a function on U which depends only on the distance from S and is identically equal to 1 on S. We define $h \in C^\infty(S^{N-1})$ by

$$h(x) := \begin{cases} \eta(x)h_1(x) & x \in U \\ 0 & x \notin U. \end{cases}$$

Since $S \subset S^{N-1}$ is invariant under the action of Γ on S^{N-1}, so is U. Similarly, the restricted function $|\Pi_E|^2 \in C^\infty(S)$ has the property that for every $g \in \Gamma$ and every $x \in S$ it holds $|\Pi_E|^2(g \cdot x) = |\Pi_E|^2(x)$. Consequently, the extension $h_1 \in C^\infty(U)$ is also invariant under the action of Γ on U. The same is true for $h \in C^\infty(S^{N-1})$ given the choice of $\eta \in C_c^\infty(U)$.

Finally, we define the extension $f \in C^\infty(\mathbb{R}^N \setminus \{0\})$ in polar coordinates by $f(\omega, r) := r^{-2}h(\omega)$. Since $|\Pi_E|^2 : \partial E \setminus \{0\} \to \mathbb{R}$ is homogeneous of degree -2 by Remark 4.2.1, the function $f \in C^\infty(\mathbb{R}^N \setminus \{0\})$ is the desired extension. □

We here prove Theorem 4.1.3. Using the results we have shown so far we are able to reduce ourselves to studying Γ-symmetric sets in $\mathbb{R}^N$ which minimize the perimeter with respect to Γ-symmetric competitors. We are then able to follow the computations of Simons from [176] to conclude that such sets are half spaces. We mention that a more detailed exposition of the same computations can also be found in [108, Theorem 10.10].

Proof of Theorem 4.1.3. By a standard classification result regarding manifolds of constant sectional curvature (see, for instance, [53, Theorem 4.1]), M is isometric to S^N/Γ, where Γ is a discrete group of isometries of S^N acting freely. In conformity with the rest of this section, we consider the induced action of Γ on $\mathbb{R}^{N+1}$. We refer to the associated projection as $\pi : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}/\Gamma$.

Let us point out that $C(S^N/\Gamma)$ is isometric to $\mathbb{R}^{N+1}/\Gamma$. Therefore, to prove the statement of the theorem it is sufficient to show the following: if $F \subset \mathbb{R}^{N+1}/\Gamma$ is a perimeter minimizing set such that $0 \in \partial F$, then $\Gamma = \{id_{S^N}\}$, and $F \subset \mathbb{R}^{N+1}$ is a half space.

By Lemma 4.3.10 there exists a Γ-symmetric cone $G \subset \mathbb{R}^{N+1}$ which minimizes the perimeter against Γ-symmetric competitors, whose boundary contains 0, and such that

$$(\mathbb{R}^{N+1}, 0, G) \in \mathrm{Tan}_0(\mathbb{R}^{N+1}, \pi^{-1}(F)).$$

By Lemma 4.3.9 the cone G is smooth with vanishing mean curvature except at $\{0\}$. We follow the computations of [176] for perimeter minimizing cones in $\mathbb{R}^{N+1}$ to show that G is a half space.

If $g \in C_c^\infty(\mathbb{R}^{N+1} \setminus \{0\})$ is a Γ-symmetric function, then the set G_t (using the notation of Proposition 4.2.2) is Γ-symmetric. In particular, this holds if $g \in C_c^\infty(\mathbb{R}^{N+1} \setminus \{0\})$ is a radially symmetric function. Let us denote by $|\Pi_G|^2$ the Γ-symmetric extension of $|\Pi_G|^2$ to $\mathbb{R}^{N+1} \setminus \{0\}$ obtained in Lemma 4.3.12. Applying the second variation formula Proposition 4.2.2 to the Γ-symmetric product $g|\Pi_G|$, where $g \in C_c^\infty(\mathbb{R}^{N+1} \setminus \{0\})$ is a radially symmetric function, we have

$$\int_{\partial G} (|\nabla_G(g|\Pi_G|)|^2 - |\Pi_G|^2(g|\Pi_G|)^2)\, \mathsf{d}\mathcal{H}^N \geq 0.$$

Using Lemma 4.2.3 we obtain

$$\begin{aligned}\int_{\partial G} |\Pi_G|^4 g^2\, \mathsf{d}\mathcal{H}^N &\leq \int_{\partial G} \Big(|\Pi_G|^2|\nabla_G g|^2 + g^2|\nabla_G|\Pi_G||^2 + 2g|\Pi_G|\nabla_G|\Pi_G| \cdot \nabla_G g\Big)\, \mathsf{d}\mathcal{H}^N \\ &= \int_{\partial G} \Big(|\Pi_G|^2|\nabla_G g|^2 + g^2|\nabla_G|\Pi_G||^2 + \frac{1}{2}\nabla_G|\Pi_G|^2 \cdot \nabla_G g^2\Big)\, \mathsf{d}\mathcal{H}^N \\ &= \int_{\partial G} \Big(|\Pi_G|^2|\nabla_G g|^2 + g^2|\nabla_G|\Pi_G||^2 - \frac{1}{2}g^2\Delta_G|\Pi_G|^2\Big)\, \mathsf{d}\mathcal{H}^N.\end{aligned}$$

From Lemma 4.2.4 it follows that

$$\int_{\partial G} \left(|\nabla_G g|^2 - \frac{2g^2}{|x|^2}\right) |\Pi_G|^2\, \mathsf{d}\mathcal{H}^N \geq 0. \tag{4.3.5}$$

The same is true by approximation for every radial function $g \in C^\infty(\mathbb{R}^{N+1})$ such that

$$\int_{\partial G} \frac{g^2}{|x|^2}|\Pi_G|^2\, \mathsf{d}\mathcal{H}^N < +\infty. \tag{4.3.6}$$

Since $|\Pi_G|^2$ is homogeneous of degree -2 by Remark 4.2.1, the previous condition holds if $g \in C^\infty(\mathbb{R}^{N+1})$ satisfies

$$\int_{\partial G} \frac{g^2}{|x|^4}\, \mathsf{d}\mathcal{H}^N < +\infty. \tag{4.3.7}$$

A function $g \in C^\infty(\mathbb{R}^{N+1})$ of the form

$$g(x) := |x|^\alpha \max\{|x|, 1\}^\beta$$

satisfies (4.3.7) if

$$\begin{cases} \alpha > \frac{4-N}{2}, \\ \alpha + \beta < \frac{4-N}{2}. \end{cases} \tag{4.3.8}$$

Plugging such g in (4.3.5) we obtain

$$(\alpha^2 - 2)\int_{\partial G \cap B_1(0)} |x|^{2\alpha-2}|\Pi_G|^2\, \mathsf{d}\mathcal{H}^N + ((\alpha+\beta)^2 - 2)\int_{\partial G \setminus B_1(0)} |x|^{2(\alpha+\beta)-2}|\Pi_G|^2\, \mathsf{d}\mathcal{H}^N \geq 0. \tag{4.3.9}$$

Since $N \leq 6$ we can choose α and β compatible with (4.3.8) and such that $\alpha^2 - 2 \leq 0$ and $(\alpha+\beta)^2 - 2 \leq 0$. With such choice of α and β, from inequality (4.3.9) it follows that $|\Pi_G|^2$ is identically 0 on ∂G. Since the second fundamental form of ∂G in $\mathbb{R}^N$ vanishes, the outer unit normal vector to G is constant on ∂G. Hence, $G \subset \mathbb{R}^{N+1}$ is a half space.

Let us show that $\Gamma = \{id_{S^N}\}$. Since G is a half space, $\partial G \cap S^N = S^{N-1}$. Since G is Γ-symmetric, $\partial G \cap S^N$ is sent to itself by all elements of Γ. We claim that the poles with respect to $\partial G \cap S^N$ (that is, the two points on S^N at maximal distance from $\partial G \cap S^N$) are swapped by every element of Γ which is not the identity. Indeed, the poles cannot be fixed as the action of Γ is free. Furthermore, the distance between each pole and $\partial G \cap S^N$ must be preserved since all elements of Γ are isometries. In particular, all the elements of Γ which are not the identity swap the poles.

Since Γ acts on $\mathbb{R}^N$ isometrically, for every $g \in \Gamma$, $g \neq id_{S^N}$, there is a neighborhood $U \subset \mathbb{R}^N$ of one of the poles such that $U \subset G$ and $g \cdot U \subset \mathbb{R}^N \setminus G$. Consequently, any such element of Γ cannot map the half space G to itself. As G is Γ-symmetric, we conclude that Γ is trivial.

Finally, the initial set $F \subset \mathbb{R}^{N+1}/\Gamma \cong \mathbb{R}^{N+1}$ is a half space since there are no nontrivial perimeter minimizing sets in Euclidean spaces of dimension less than 8. □

Theorem 4.1.3 fails if $N \geq 7$ as shown by the next example.

Example 4.3.13. *Let $N = 7$ and let $\Gamma := \{id_{S^7}, -id_{S^7}\}$. Let $E \subset \mathbb{R}^8$ be the Simons cone, i.e.*

$$E = \{|x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2 > |x_5|^2 + |x_6|^2 + |x_7|^2 + |x_8|^2\} \subset \mathbb{R}^8.$$

Let us note that $E \subset \mathbb{R}^8$ is a Γ-symmetric set which minimizes the perimeter (and, in particular, it minimizes the perimeter against Γ-symmetric competitors). By Lemma 4.3.8, $\pi(E) \subset \mathbb{R}^8/\Gamma \cong C(S^7/\Gamma)$ is a perimeter minimizing set. Moreover, the boundary of $\pi(E)$ contains 0.

Let now $N = 7+k$ with $k \in \mathbb{N}$ and let $\Gamma := \{id_{S^{7+k}}, -id_{S^{k+7}}\}$. Let $E \times \mathbb{R}^k \subset \mathbb{R}^8 \times \mathbb{R}^k$ be the product of the Simons cone with the extra Euclidean factors. $E \times \mathbb{R}^k \subset \mathbb{R}^8 \times \mathbb{R}^k$ is a Γ-symmetric set which minimizes the perimeter (and, in particular, it minimizes the perimeter against Γ-symmetric competitors). By Lemma 4.3.8, $\pi(E) \subset \mathbb{R}^{8+k}/\Gamma \cong C(S^{7+k}/\Gamma)$ is a perimeter minimizing set. Moreover, the boundary of $\pi(E)$ contains 0.

Building on Theorem 4.1.3, we prove Theorem 4.1.1.

Proof of Theorem 4.1.1. The proof is divided in two steps: we start by showing $\mathcal{S}^E \setminus \mathcal{S}^E_{N-4} = \emptyset$, and then prove $\mathcal{S}^E_{N-4} \setminus \mathcal{S}^E_{N-5} = \emptyset$.

Let us show $\mathcal{S}^E \setminus \mathcal{S}^E_{N-4} = \emptyset$. Suppose by contradiction that $x \in \mathcal{S}^E \setminus \mathcal{S}^E_{N-4}$. Then there exists a pointed metric space (Y, d_y, y) and a perimeter minimizing set of the form $\mathbb{R}^{N-3} \times A \subset \mathbb{R}^{N-3} \times Y$ whose boundary contains $(0, y) \in \mathbb{R}^{N-3} \times Y$ such that

$$(\mathbb{R}^{N-3} \times Y, \mathbb{R}^{N-3} \times A, (0, y)) \in \mathrm{Tan}_x(X, E).$$

By Lemma 3.3.12, $A \subset Y$ is a perimeter minimizing set whose boundary contains y. Moreover, $\mathbb{R}^{N-3} \times Y \cong \mathbb{R}^N$ by Theorem 2.2.13 since

$$(\mathbb{R}^{N-3} \times Y) \in \mathrm{Tan}_x(X).$$

Therefore, $A \subset \mathbb{R}^3$ is a half space as it minimizes the perimeter and has non-empty boundary. We conclude that $x \notin \mathcal{S}^E \setminus \mathcal{S}^E_{N-4}$, a contradiction.

In the rest of the proof we show $\mathcal{S}^E_{N-4} \setminus \mathcal{S}^E_{N-5} = \emptyset$. Suppose by contradiction that there exists $x \in \mathcal{S}^E_{N-4} \setminus \mathcal{S}^E_{N-5}$.

By Theorem 4.3.3 the tangent space $\mathrm{Tan}_x(X)$ has an element isometric to $\mathbb{R}^{N-4} \times C(S^3/\Gamma)$, where $\Gamma \subset O(4)$ is a discrete group acting freely. Hence,

$$(\mathbb{R}^{N-4} \times C(S^3/\Gamma), \mathbb{R}^{N-4} \times F) \in \mathrm{Tan}_x(X, E),$$

where $O \in C(S^3/\Gamma)$ is the tip, and $\mathbb{R}^{N-4} \times F \subset \mathbb{R}^{N-4} \times C(S^3/\Gamma)$ is a perimeter minimizing set whose boundary contains $(0, O) \in \mathbb{R}^{N-4} \times C(S^3/\Gamma)$.

By Lemma 3.3.12, $F \subset C(S^3/\Gamma)$ is a perimeter minimizing set whose boundary contains O. By applying Theorem 4.1.3, we can then infer that $C(S^3/\Gamma)$ is isometric to $\mathbb{R}^4$ and that F is a half space, contradicting $x \in \mathcal{S}^E_{N-4} \setminus \mathcal{S}^E_{N-5}$. Therefore, $\mathcal{S}^E \setminus \mathcal{S}^E_{N-5} = \emptyset$ as claimed.

Since $\mathcal{S}^E = \mathcal{S}^E_{N-5}$, Theorem 3.3.5 implies that $\dim_{\mathcal{H}}(\mathcal{S}^E) \leq N - 5$. □

Theorem 4.1.1 is sharp: as shown in the following example, there exists a non-collapsed limit of manifolds with two-sided bounds on the Ricci curvature with a perimeter minimizing set E such that $\mathcal{S}^E_{N-5}$ is non-empty.

Example 4.3.14. *The cone $C(\mathbb{RP}^3)$, arising as the blow-down of the Eguchi-Hanson manifold (see [66, Example 2.15] for this construction; the Eguchi-Hanson metric was originally defined in [88, 52]), is a non-collapsed limit of manifolds with two-sided bounds on the Ricci curvature which is singular at the tip.*

By a standard calibration argument it is possible to show that the set

$$C(\mathbb{RP}^3) \times [0, +\infty) \subset C(\mathbb{RP}^3) \times \mathbb{R}$$

is perimeter minimizing. Moreover the tip of $C(\mathbb{RP}^3)$ belongs to $\mathcal{S}^E_{N-5}$.

As mentioned in Remark 4.1.2, the regularity of a set $E \subset X$ satisfying the conditions of Theorem 4.1.1 can be improved substantially if we consider the restriction of ∂E to the regular set $\mathcal{R}(X)$. The following proposition follows by repeating part of the proof of [32, Theorem A5], which in turn builds on a theorem of Allard [3, The Regularity Theorem p. 27] (which applies to minimizers of certain parametric integrands).

Proposition 4.3.15. *Let (X, d, x) be a non-collapsed limit of manifolds with two-sided bounds on the Ricci curvature of dimension N. Let $E \subset X$ be a locally perimeter minimizing set. The set $\partial E \cap \mathcal{R}(X)$ is a $C^{1,\alpha}$ hypersurface of $\mathcal{R}(X)$ for every $\alpha \in (0,1)$ outside of a closed set of Hausdorff dimension at most $N-8$.*

Proof. By [144, Theorem 6.23] (reported as Theorem 2.4.36), it holds

$$\dim_{\mathcal{H}}(\mathcal{S}^E \cap \mathcal{R}(X)) \le N - 8.$$

Let $\alpha \in (0,1)$ be fixed. By Theorem 2.2.13, $\mathcal{R}(X)$ is a $C^{1,\alpha}$ open manifold. To conclude, we fix $x_0 \in (\partial E \setminus \mathcal{S}^E) \cap \mathcal{R}(X)$ and we show that ∂E is a $C^{1,\alpha}$ hypersurface in a neighborhood of x_0.

Let $\epsilon > 0$ be fixed. Up to rescaling, by Theorem 2.2.14, we can identify a neighborhood of x_0 in X with the Euclidean ball $B_2(0^N) \subset \mathbb{R}^N$ equipped with a Riemannian metric g such that

$$\|g_{ij} - \delta_{ij}\|_{C^1(B_2(0^N))} < \epsilon. \tag{4.3.10}$$

With this identification, $E \subset B_2(0^N)$ and $0^N \in \partial E \setminus \mathcal{S}^E$. Up to restricting to a smaller ball, we can also assume that E is perimeter minimizing in $B_2(0^N)$.

Let $\mathcal{H}_e^{N-1}$ denote the Euclidean $N-1$ dimensional Hausdorff measure in $B_2(0^N)$. By the area formula, the perimeter measure of E in $(B_2(0^N), g)$, denoted $\mathrm{Per}(E, \cdot)$, coincides with

$$(\langle Cof_x(\nu_x), \nu_x \rangle)^{\frac{1}{2}} \mathcal{H}_e^{N-1} \llcorner \partial^* E, \tag{4.3.11}$$

where Cof_x is the cofactor matrix of (g_{ij}) at x, ν_x is the distributional Euclidean unit normal to E at x, $\partial^* E$ is the Euclidean reduced boundary of E, and $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product.

Since 0^N is a regular point for E, when one takes a blow up of $(B_2(0^N), g)$ in 0^N, E converges to a half space H. It follows from [144, Theorem 2.42] (reported as Theorem 2.4.33) that the perimeter of E converges weakly to the perimeter of H. Furthermore, ∂E converges in the Kuratowski sense to the boundary of H. Let $\pi : B_1(0^N) \to \mathbb{R}^N$ be the projection map on the normal line to H in $\mathbb{R}^N$.

By the aforementioned convergence properties of ∂E, for every $\epsilon_1 > 0$, there exists $r \in (0, 1)$ such that

$$1 - \epsilon_1 \leq \left| \frac{\text{Per}(E, B_r(0^N))}{\omega_{N-1} r^{N-1}} \right| \leq 1 + \epsilon_1, \quad |\pi(\partial E \cap B_{2r}(0^N))| \leq \epsilon_1 r. \tag{4.3.12}$$

It follows from (4.3.10), (4.3.12), the representation of the perimeter functional (4.3.11) and [3, The Regularity Theorem p. 27] that there exists a function $u \in C^{1,\alpha}(H \cap B_s(0^{N-1}))$ for a sufficiently small $s > 0$, such that $\partial E \cap B_s(0^N)$ coincides with the graph of u. $\square$

Chapter 5

Approximation of the perimeter functional in RCD spaces

In this chapter, our aim is to extend some of the classical results of Γ-convergence of Modica-Mortola to the setting of RCD metric measure spaces. The notion of Γ-convergence (sometimes referred to as epi-convergence) of functionals is first attributed to De Giorgi [76, 73]. Let $E_k : X \to \mathbb{R}$, $k \in \mathbb{N}$ be a sequence of functionals Γ-converging to $E_\infty : X \to \mathbb{R}$, where X is some function space. An important property of Γ-convergence is that, if $\{f_k\}_k \subset X$ converges to some function $f \in X$ and f_k minimimizes E_k for each $k \in \mathbb{N}$, then f minimizes E_∞. A classical instance of Γ-convergence is that of Modica and Mortola [141, 140], who showed that the Cahn-Hilliard phase transition energies [51] converge to the perimeter functional when suitably rescaled. In light of the previously mentioned property of Γ-convergence, this result yields itself to applications to the study of perimeter minimizers. The Cahn-Hilliard minimizers, which are smooth functions by elliptic PDE theory, constitute approximations of perimeter minimizing sets and can be even used as a procedure to construct them. For this reason, the field attracted plenty of interest quickly. However, Γ-convergence does not tell us much about the other critical points of the limiting functional, which in this case are minimal surfaces. More recently, related PDE approaches based on min-max theory [112, 96] (building on the regularity theory of [115, 189, 182]) have been used as an approach to tackle minimal surfaces problems, and have yielded a new proof of the existence of minimal hypersurfaces in closed Riemannian manifolds originally proved by Almgren, Pitts, and Schoen–Simon [159, 168] and an alternative proof of Yau's conjecture [190] to those based on the Almgren-Pitts theory [137, 116, 177]. This PDE approach to the construction of minimal surfaces has attracted the interest of the RCD community, since in such setting no satisfactory analogue of the first variation of area formula is available. The hope is that these novel PDE approaches can be adapted to the RCD setting and be used as a way to study minimal surfaces in such setting, and even lead to a better understanding of the structure of RCD metric measure spaces themselves. In this chapter, we aim at taking a step in this direction.

In Section 5.1 we recall the notion of Γ-convergence and introduce the standard result of Modica and Mortola in the smooth setting. We also define the Cahn-Hilliard energy in the non-smooth setting and report the results obtained in this chapter. In Section 5.2 we show, in the setting of non-collapsed compact $\mathrm{RCD}(K,N)$ metric measure spaces, that volume-constrained perimeter minimizing sets can be approximated by a sequence of functions converging to the characteristic function of such sets in L^1 and whose Cahn-Hilliard energies converge to the perimeter of the set up to a multiplicative constant. Moreover, we show that sequences of minimizers of the Cahn-Hilliard energies possess a subsequence converging to a volume-constrained perimeter minimizing set. In Section 5.3 we focus on Cahn-Hilliard functionals defined on a non-collapsing sequence of closed Riemannian manifolds with a uniform lower bound on the Ricci curvature converging in the pmGH sense to a Ricci limit space. We prove that the Cahn-Hilliard energies Γ-converge to the perimeter measure in the limit space.

5.1 Introduction

In this section we provide a brief overview of the results obtained in [141, 140]. We also present the main results obtained in sections 5.2 and 5.3.

The notion of Γ-convergence first appeared in [76, 73]. In [141, 140] the authors showed that suitably rescaled phase transition energies, commonly referred to as Cahn-Hilliard energies, Γ-converge to the perimeter functional. For an overview of the topic, see [1]. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Fix $0 < V < \mathrm{vol}(\Omega)$ and let $L^1_V(\Omega)$ be the set of all integrable functions $u : \Omega \to [0,1]$ such that $\int_\Omega u = V$. We denote by $\mathrm{BV}(\Omega; \{0,1\})$ the set of all functions with bounded variation taking values in $\{0,1\}$. We call Su the set of approximate discontinuity of u, that is the complement of the set of points $x \in \Omega$ such that there exists $z \in \mathbb{R}$ with

$$\lim_{r \to 0} \fint_{B_r(x)} |u(y) - z| dy = 0.$$

Lastly, let $W : \mathbb{R} \to [0,\infty)$ be a double-well potential with wells in 0 and 1. That is, we require W to be continuous, that $W(0) = W(1) = 0$, $W(t) > 0$ for every $t \in \mathbb{R} \setminus \{0,1\}$ and that $\lim_{t\to\infty} W(t) = \lim_{t\to-\infty} W(t) = \infty$. For instance, take $W(u) :=$

$\frac{1}{4}u^2(u-1)^2$. Let us recall the result of Γ-convergence of Modica and Mortola [141, 140], which states that the Cahn-Hilliard energies F_ε (5.1.2) Γ-converge to $c\mathrm{Per}(\cdot)$ where c is defined as

$$c := 2\int_0^1 \sqrt{W(s)}\, ds, \tag{5.1.1}$$

and depends only on the double-well potential W.

Theorem 5.1.1 (Modica-Mortola)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Fix $0 < V < \mathcal{L}^N(\Omega)$. For every $\varepsilon > 0$ let*

$$F_\varepsilon(u) := \begin{cases} \varepsilon \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_\Omega W(u) & \text{if } u \in H^{1,2}(\Omega) \cap L^1_V(\Omega), \\ +\infty & \text{otherwise;} \end{cases} \tag{5.1.2}$$

$$F(u) := \begin{cases} c\mathcal{H}^{N-1}(Su) & \text{if } u \in \mathrm{BV}(\Omega;\{0,1\}) \cap L^1_V(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{5.1.3}$$

where $c > 0$ was defined in (5.1.1).

The functionals F_ε Γ-converge to F in Ω as ε goes to 0. That is, the following conditions hold:

- Γ-liminf - *For every $u \in \mathrm{BV}(\Omega;\{0,1\}) \cap L^1_V(X)$ and every $\{u_\varepsilon\}_{\varepsilon>0}$, $u_\varepsilon \in L^1_V(X)$ for all $\varepsilon > 0$ and $u_\varepsilon \to u$ in $L^1(X)$ as $\varepsilon \to 0$ we have*

$$\liminf_{\varepsilon \longrightarrow 0} F_\varepsilon(u_\varepsilon) \geq F(u); \tag{5.1.4}$$

- Γ-limsup, recovery sequence - *For every $u \in \mathrm{BV}(\Omega;\{0,1\}) \cap L^1_V(X)$ there exists a family of functions $\{u_\varepsilon\}_\varepsilon$ such that $u_\varepsilon \in H^{1,2}(\Omega) \cap L^1_V(\Omega)$, $u_\varepsilon \to u$ in $L^1(\Omega)$ as $\varepsilon \to 0$ and*

$$\lim_{\varepsilon \longrightarrow 0} F_\varepsilon(u_\varepsilon) = F(u). \tag{5.1.5}$$

Moreover, the following compactness condition holds.

Proposition 5.1.2 (Compactness condition)**.** *Suppose we are given two sequences $\varepsilon_k > 0$, $\{u_{\varepsilon_k}\}_k \subset L^1(\Omega)$ with $\varepsilon_k \to 0$ as $k \to \infty$ such that*

$$\sup_{k\in\mathbb{N}} \|u_{\varepsilon_k}\|_{L^\infty(\Omega)} < \infty \quad \textit{and} \quad \sup_{k\in\mathbb{N}} F_{\varepsilon_k}(u_{\varepsilon_k}) < \infty.$$

is bounded. Then $\{u_{\varepsilon_k}\}_k$ is pre-compact in $L^1(\Omega)$.

An important property of Γ-convergence is that if u_ε is a global minimum of (5.1.2), then, as F_ε Γ-converges to F, the limit point provided by the compactness condition 5.1.2 is a minimizer of (5.1.3). See [140]. Therefore, the Modica-Mortola approximation theorem 5.1.1 is a powerful tool to study volume-constrained minimizers for compact variations.

Let us give a definition of the Cahn-Hilliard energy (cf. [51]) in the metric setting.

Definition 5.1.3. *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an* $\mathrm{RCD}(K, N)$ *metric measure space. Let $\varepsilon > 0$ and $0 < V < \mathcal{H}^N(X)$. The Cahn-Hilliard energy $F_\varepsilon : L^1(X) \to [0, +\infty) \cup \{+\infty\}$ is defined by*

$$F_\varepsilon(u) = \begin{cases} \int_X \left(\varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\right) d\mathfrak{m} & \text{if } u \in H^{1,2}(X) \cap L^1_V(X), \\ +\infty & \text{otherwise.} \end{cases} \tag{5.1.6}$$

We remark that the energy in (5.1.6) is composed of two terms:

- $\frac{1}{\varepsilon}\int_X W(u)\, \mathsf{d}\mathcal{H}^N$, whose minimization forces competitors to take values close to 0 and 1 (phase separation);
- $\varepsilon \int_X |\nabla u|^2 \,\mathsf{d}\mathcal{H}^N$, which penalizes inhomogeneous solutions.

This model was proposed by Cahn-Hilliard in [51] to describe phase transitions.

Let us give an overview of the main results presented in this chapter. In Section 5.2 we deal with the case where the ambient space is a compact non-collapsed $\mathrm{RCD}(K, N)$ metric measure space. To the best of our knowledge, the Modica-Mortola approximation has not previously been extended to metric ambient spaces. We focus on $\mathrm{RCD}(K, N)$ spaces due to recent advances in the theory of perimeter minimizing sets in this setting (see Section 2.4 for details). Our goal is for these results to serve as a first step toward adapting the min-max methods of Guaraco [112] and Gaspar-Guaraco [96] to establish the existence of unstable minimal surfaces. These methods provide an alternative to the approach developed by Almgren, Pitts [159], and Schoen-Simon [168]. Since they rely on variational formulas for the perimeter, it is unclear how to extend the latter framework to $\mathrm{RCD}(K, N)$ spaces. Indeed, it remains an open question whether the variation of a finite perimeter set along a vector field preserves finiteness of the perimeter for any $t > 0$.

The results reported in the remainder of this chapter were obtained independently by the author and have not been published previously. To the best of our knowledge, they are novel.

Let us recall the definition of volume-constrained minimizers for compact variations. We previously stated it as Definition 2.4.37, however here we also assume volume-constrained minimizers for compact variations to have finite volume. In Proposition 5.2.2 we show that the Γ-limsup inequality holds for such subsets.

Definition 5.1.4 (Volume-constrained minimizers for compact variations)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. A set* $E \subset X$ *is said to be* volume-constrained minimizer for compact variations *if* $\mathfrak{m}(E) \in (0, \infty)$ *and if for every* $F \subset X$ *such that* $E \Delta F \Subset K \subset X$*, where* K *is compact, and* $\mathfrak{m}(E \cap K) = \mathfrak{m}(F \cap K)$ *it holds* $\mathrm{Per}(E) \le \mathrm{Per}(F)$*.*

Let us also report the definition of isoperimetric sets.

Definition 5.1.5 (Isoperimetric sets)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. A set* $E \subset X$ *with* $\mathfrak{m}(E) < \infty$ *is said to be an* isoperimetric set *if for every* $F \subset X$ *with* $\mathfrak{m}(E) = \mathfrak{m}(F)$ *it holds* $\mathrm{Per}(E) \le \mathrm{Per}(F)$*.*

Let us point out that isoperimetric sets are also volume-constrained minimizers for compact variations. Moreover, the two definitions are equivalent if the ambient space is compact.

Proposition 5.1.6 (Compactness)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a compact* $\mathrm{RCD}(K, N)$ *metric measure space. Let* $\{u_\varepsilon\}_\varepsilon \subset H^{1,2}(X)$ *be a family of functions satisfying*

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^\infty(X)} < \infty \quad \text{and} \quad \sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon) < \infty. \tag{5.1.7}$$

Then there exists a finite perimeter set $A \subset X$ *and a subsequence* $\varepsilon_k \to 0$ *as* $k \to \infty$ *such that* u_{ε_k} *converges to* χ_A *in* $L^1(X)$ *as* $k \to \infty$*.*

Theorem 5.1.7 (Approximation of isoperimetric sets)**.** *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be an* $\mathrm{RCD}(K, N)$ *metric measure space and let* $E \subset X$ *be a volume-constrained minimizer for compact variations. Then there exists a sequence* $\{u_\varepsilon\}_\varepsilon \in H^{1,2}(X) \cap \mathrm{LIP}(X)$*,* $\varepsilon > 0$*, such that*

$$0 \le u_\varepsilon \le 1 \quad \text{and} \quad \int_X u_\varepsilon \, d\mathcal{H}^N = \mathcal{H}^N(E) \quad \text{for all } \varepsilon > 0. \tag{5.1.8}$$

Moreover,

$$\lim_{\varepsilon \longrightarrow 0} F_\varepsilon(u_\varepsilon) = c\mathrm{Per}(E), \tag{5.1.9}$$

where $c > 0$ *was defined in* (5.1.1)*.*

Theorem 5.1.8. *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be a compact* $\mathrm{RCD}(K, N)$ *metric measure space. Let* $0 < V < \mathcal{H}^N(X)$ *and* $u_\varepsilon \in H^{1,2} \cap \mathrm{LIP}(X)$ *be minimizers of problem* (5.1.6) *with* $\int_X u_\varepsilon \, d\mathcal{H}^N = V$ *for all* $\varepsilon > 0$*. There exists an isoperimetric set* $E \subset X$ *with* $\mathcal{H}^N(E) = V$ *and a subsequence* $\{\varepsilon_k\}_k$ *such that* $u_{\varepsilon_k} \to \chi_E$ *in* $L^1(X)$ *as* $k \to \infty$ *and*

$$\lim_{k \longrightarrow \infty} F_k(u_{\varepsilon_k}) = c\mathrm{Per}(E),$$

where $c > 0$ *was defined in* (5.1.1)*.*

Let us comment on these results. We are able to show a general Γ-liminf result for locally PI spaces in Proposition 5.2.1 (see [113] for an overview of calculus on PI spaces). We also show that the Γ-limsup inequality holds for volume-constrained perimeter minimizing subsets of non-collapsing $\mathrm{RCD}(K, N)$ metric measure spaces (Proposition 5.2.2). Sets of finite perimeter in $\mathrm{RCD}(K, N)$ m.m.s. do not possess the amount of regularity needed for our proof to work. The proof of the Γ-limsup inequality in the smooth setting found in [140] is based on a result which states that every finite perimeter set can be approximated both in the volume and in the perimeter sense by smooth sets (for instance, see [136, Theorem 13.8]). Smooth sets, in turn, can be approximated by their tubular neighborhoods as the enlargement goes to zero. See [140, Lemma 4], where this fact is presented.

The next result we present in Section 5.2 is Theorem 5.1.7, which states that any volume-constrained minimizer for compact variations in a non-collapsed $\mathrm{RCD}(K, N)$ m.m.s. can be approximated by a sequence of $H^{1,2} \cap L^1_V(X)$ functions, where V is the volume of the set being approximated, in the following sense: the functions converge in $L^1(X)$ to the characteristic function of the set and their Cahn-Hilliard energies converge to the perimeter of the set up to a multiplicative constant.
Lastly, we show Theorem 5.1.8, which states that in a compact non-collapsed $\mathrm{RCD}(K, N)$ m.m.s. sequences of minimizers of the Cahn-Hilliard energies of Definition 5.1.3 possess a subsequence converging in $L^1(X)$ to an isoperimetric set. Moreover, the Cahn-Hilliard energies of the approximating subsequence converge to the perimeter of the isoperimetric set up to a multiplicative constant depending only on the double-well potential W. In Theorem 5.1.8 we assume the ambient space to be compact because it allows us to apply Proposition 5.1.6 and since it guarantees the existence of an isoperimetric region of any admissible prescribed volume. The latter requirement is needed since the Γ-limsup inequality Proposition 5.2.2 holds only for volume-constrained minimizers for compact variations.
A future direction we would like to explore is to exploit a generalized existence result found in [153, 30], where the authors showed that isoperimetric regions in non-compact non-collapsed $\mathrm{RCD}(K, N)$ spaces exist in a suitable generalized sense (see also [143] for an earlier existence result for the case of non-compact manifolds with non-negative Ricci curvature which are locally asymptotically space forms, [33] for the existence of isoperimetric sets of large volumes in non-compact manifolds with non-negative Ricci curvature and Euclidean volume growth, [29] for a non-existence result in the case of a non-compact manifold of positive sectional curvature. See also [30] for an extension of the results obtained in [153] to the class of non-compact

RCD(K,N) spaces). Applying these result could allow us to treat the setting of non-compact ambient spaces.

In Section 5.3 we study a Modica-Mortola approximation in which the Cahn-Hilliard energies are defined on a sequence of non-collapsing closed N-dimensional Riemannian manifolds with a uniform lower bound on the Ricci curvature converging in the pmGH sense to some Ricci limit space $(X, \mathsf{d}, \mathcal{H}^N)$, for some $N \geq 2$.

For the remainder of this section, we suppose that $\{(X_k, \mathsf{d}_k, x_k)\}_k$ is a sequence of pointed closed N-dimensional Riemannian manifolds with

$$\mathrm{Ric}_k \geq K$$

for some $K \in \mathbb{R}$ and

$$\mathrm{vol}(B_1(x_k)) \geq v$$

for some $v > 0$ and $K \in \mathbb{R}$ endowed with their respective volume measures, converging in the pmGH to $(X, \mathsf{d}, \mathcal{H}^N, x)$. Moreover, given a sequence $\varepsilon_k \to 0$ and $0 < V < \mathcal{H}^N(X)$, we define the functionals $F_k : L^1_V(X_k) \to \mathbb{R}$ such that

$$F_k(u) = \begin{cases} \int_{X_k} \left(\varepsilon_k |\nabla u|^2 + \frac{1}{\varepsilon_k} W(u) \right) d\mathfrak{m} & \text{if } u \in H^{1,2}(X_k) \cap L^1_V(X), \\ +\infty & \text{otherwise.} \end{cases} \tag{5.1.10}$$

In Section 5.3 we show the following.

Theorem 5.1.9 (Γ-convergence). *Let $\{(X_k, \mathsf{d}_k, x_k)\}_k$ and $(X, \mathsf{d}, \mathcal{H}^N, x)$ be as above. Let $c > 0$ be as in (5.1.1). Then,*

1. *(Γ-liminf). Let $u \in L^1(X)$, $u_k \in L^1(X_k)$, $\|u_k\|_{L^1(X_k)} = V > 0$ for all k and u_k converge strongly in L^1 to u. Moreover, suppose that $\liminf_{k \to \infty} F_k(u_k) < \infty$. Then there exists a finite perimeter set $A \subset X$ such that $u = \chi_A$ $\mathcal{H}^N$-almost everywhere and*
$$c\mathrm{Per}(A) \leq \liminf_{k \to \infty} F_k(u_k).$$
2. *(Γ-limsup). Let $E \subset X$ be a finite perimeter set with $\mathcal{H}^N(E) < \infty$. Then there exists a sequence $u_k \in H^{1,2} \cap \mathrm{Lip}(X_k, [0,1])$ such that $\|u_k\|_{L^1(X_k)} = \mathcal{H}^N(E)$ for all $k \in \mathbb{N}$, u_k converge strongly in L^1 to u and*
$$\lim_{k \to \infty} F_k(u_k) = c\mathrm{Per}(u).$$

We also show that volume-constrained minimizers for compact variations in the limit space X can be approximated by a sequence of functions $u_k \in H^{1,2} \cap L^1_V(X_k)$ in the strong L^1 sense and such that the Cahn-Hilliard energies converge to the perimeter of the set being approximated up to a multiplicative constant. Moreover, we show that, when X is compact, a sequence of minimizers of the Cahn-Hilliard energies for a fixed prescribed volume having bounded energies converge to an isoperimetric subset of X strongly in L^1 and have their Cahn-Hilliard energies converge to the perimeter of the region up to a multiplicative constant depending only on the double-well potential W. Contrary to the fixed ambient case Theorem 5.1.8, here we recover the existence of isoperimetric sets through the proof since the Γ-limsup inequality holds for any finite perimeter set.

5.2 Fixed ambient space

The aim of this section is to prove Theorems 5.1.7 and 5.1.8. We start by showing the Γ-liminf inequality Proposition 5.2.1, which holds true for general locally PI spaces. We then show a Γ-limsup inequality Proposition 5.2.2, which holds only for volume-constrained minimizers of the perimeter. After establishing a compactness result in Proposition 5.1.6, we conclude the section by proving Theorems 5.1.7 and 5.1.8.

The proof of the Γ-liminf inequality below is an adaptation of the original argument found in [140] to the metric setting. See [113, 107] for an overview of calculus in metric measure spaces.

Proposition 5.2.1 (Γ-liminf)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a locally* PI *space and let $\{u_\varepsilon\}_\varepsilon \subset L^1(X)$ be a family of functions such that $u_\varepsilon \to u$ in $L^1(X)$ as $\varepsilon \to 0$. Suppose that*

$$\liminf_{\varepsilon \longrightarrow 0} F_\varepsilon(u_\varepsilon) < \infty. \tag{5.2.1}$$

Then there exists a finite perimeter set $E \subset X$ such that $u = \chi_E$ and

$$c\mathrm{Per}(E) \le \liminf_{\varepsilon \longrightarrow 0} F_\varepsilon(u_\varepsilon), \tag{5.2.2}$$

where $c > 0$ was defined in (5.1.1).

Proof. Let $\{u_\varepsilon\}_\varepsilon \subset L^1(X)$ be a family of functions such that

$$u_\varepsilon \to u \text{ in } L^1(X) \text{ as } \varepsilon \to 0 \tag{5.2.3}$$

for some function $u \in L^1(X)$ and such that

$$\liminf_{\varepsilon \longrightarrow 0} F_\varepsilon(u_\varepsilon) < \infty. \tag{5.2.4}$$

It follows from (5.2.3) that there exists a subsequence $\varepsilon_k \to 0$ as $k \to \infty$ such that $u_{\varepsilon_k}(x) \to u(x)$ for $\mathfrak{m}$-almost every $x \in X$. Therefore, by Fatou's lemma

$$\begin{aligned} \int_X W(u)\, d\mathfrak{m} &\leq \liminf_{k\longrightarrow\infty} \int_X W(u_{\varepsilon_k})\, d\mathfrak{m} \\ &\leq \liminf_{k\longrightarrow\infty} \varepsilon_k F_{\varepsilon_k}(u_{\varepsilon_k}). \end{aligned} \tag{5.2.5}$$

Moreover, from (5.2.4) it holds that

$$\liminf_{k\longrightarrow\infty} \varepsilon_k F_{\varepsilon_k}(u_{\varepsilon_k}) = 0. \tag{5.2.6}$$

Since W is non-negative, we get

$$W(u) = 0 \quad \mathfrak{m}\text{-almost everywhere,}$$

This implies that $u(x) \in \{0, 1\}$ for $\mathfrak{m}$-almost every $x \in X$.

Up to restricting ε to an appropriate subsequence $\{\varepsilon_k\}_k$ converging to 0, it follows from (5.2.4) that $F_{\varepsilon_k}(u_{\varepsilon_k}) < \infty$. In particular, we have that $u_{\varepsilon_k} \in H^{1,2}(X)$ for all $k \in \mathbb{N}$. We may also assume that $0 \leq u_{\varepsilon_k}(x) \leq 1$ for all $x \in X$. Indeed, we may substitute u_{ε_k} with the truncated sequence $\tilde{u}_{\varepsilon_k} := \min\{1, \max\{u_{\varepsilon_k}, 0\}\}$, which satisfies $F_{\varepsilon_k}(\tilde{u}_{\varepsilon_k}) \leq F_{\varepsilon_k}(u_{\varepsilon_k})$ and $\tilde{u}_{\varepsilon_k} \to \tilde{u} = u$ in $L^1(X)$ as $k \to \infty$ (since $|\tilde{u}_\varepsilon - \tilde{u}_\delta| \leq |u_\varepsilon - u_\delta|$ for all $\delta, \varepsilon > 0$). Consider the functions

$$\Phi(s) := \int_0^s \sqrt{W(t)}\, dt, \qquad w_\varepsilon = \Phi \circ u_\varepsilon.$$

Since both the function $\Phi \in C^1(\mathbb{R})$ and $\{u_\varepsilon\}_\varepsilon$ is uniformly bounded, it follows that

$$w_\varepsilon \to w := \Phi \circ u \quad \text{ in } L^1(X). \tag{5.2.7}$$

Moreover, the lower semicontinuity of the BV seminorm (see the definition of total variational measure (2.4.1)) yields

$$|\nabla w|(X) \leq \liminf_{k\longrightarrow\infty} \int_X |\nabla w_{\varepsilon_k}| d\mathfrak{m}. \tag{5.2.8}$$

By the coarea formula (2.4.8) it holds

$$\begin{aligned} |\nabla w|(X) &= \int_{\mathbb{R}} \int_X d\mathrm{Per}(\Phi(u) > t) dt \\ &= (\Phi(1) - \Phi(0))\mathrm{Per}(Su) = \frac{c}{2}\mathrm{Per}(E). \end{aligned} \tag{5.2.9}$$

On the other hand, since $u_{\varepsilon_k} \in H^{1,2}(X)$ and is pointwise uniformly bounded $\mathfrak{m}$-a.e. for all $k \in \mathbb{N}$, by the chain rule (cf. [107, Theorem 2.1.28] or Section 2.1.5) we have $\nabla w_{\varepsilon_k} = \Phi'(u_{\varepsilon_k})\nabla u_{\varepsilon_k}$ $\mathfrak{m}$-almost everywhere. Therefore,

$$\begin{aligned}\int_X |\nabla w_{\varepsilon_k}| d\mathfrak{m} &= \int_X \sqrt{W(u_{\varepsilon_k})}|\nabla u_{\varepsilon_k}| d\mathfrak{m} \\ &\leq \frac{1}{2}\left[\frac{1}{\varepsilon_k}\int_X W(u_{\varepsilon_k}) d\mathfrak{m} + \varepsilon_k \int_X |\nabla u_{\varepsilon_k}|^2\right] d\mathfrak{m} = \frac{1}{2} F_{\varepsilon_k}(u_{\varepsilon_k}),\end{aligned} \tag{5.2.10}$$

where in the second inequality we have used $ab \leq \frac{1}{2}(a^2 + b^2)$ for all $a, b \in \mathbb{R}$.
We conclude the proof of (5.2.2) by plugging in (5.2.9) and (5.2.10) in (5.2.8). □

Next, we show the Γ-limsup inequality Proposition 5.2.2. We reported the classical result in (5.1.5). The recent advances in calculus in metric measure spaces (see [107, 113] for an introduction to the topic) allow us to adapt part of the classical proof found in [140, Proposition 2] to our setting. However, the proof of the result in the Euclidean setting relies on the fact that any finite perimeter set can be approximated in the perimeter by smooth sets. Smooth sets, in turn, can then be approximated by their tubular neighborhoods as the enlargement goes to zero. In the non-smooth setting of RCD spaces, it is not clear what the counterpart of such a smooth approximation should be. We show that the Γ-limsup inequality (5.1.5) holds for volume-constrained minimizers for compact variations in non-collapsed RCD spaces, since they possess sufficient regularity properties (see Proposition 2.4.42 obtained in [34], which builds on previous results found in [55, 144]).

Proposition 5.2.2 (Γ-limsup inequality)**.** *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ metric measure space. Let $E \subset X$ be a volume-constrained minimizer for compact variations with $\mathcal{H}^N(E) < \infty$. There exists a sequence $\{u_\varepsilon\}_\varepsilon \subset H^{1,2}(X) \cap \mathrm{LIP}(X)$ with*

$$0 \leq u_\varepsilon \leq 1 \quad \textit{and} \quad \int_X u_\varepsilon \, d\mathcal{H}^N = \mathcal{H}^N(E) \quad \textit{for all } \varepsilon > 0, \tag{5.2.11}$$

such that

$$u_\varepsilon \to \chi_E \textit{ in } L^1(X) \textit{ as } \varepsilon \to 0, \tag{5.2.12}$$

and

$$c\mathrm{Per}(E) \geq \limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon), \tag{5.2.13}$$

where $c > 0$ was defined in (5.1.1).

Proof. This argument follows closely the proof of [140, Proposition 2].

Throughout the proof, we use the shorthand notation $u(x) := \chi_E(x)$. Let $\mathsf{d}_E : X \to \mathbb{R}$ denote the signed distance function from E. Since we are assuming E to be an open set (see discussion after Theorem 2.4.41), we have $u = \rho \circ \mathsf{d}_E$, where $\rho : \mathbb{R} \to \mathbb{R}$ is the Heaviside function

$$\rho(t) = 0 \text{ for } t < 0, \qquad \rho(t) = 1 \text{ for } t > 0.$$

Let us point out that $u = \rho \circ \mathsf{d}_E$ does not hold in general for finite perimeter sets.

We construct a family of functions $\{u_\varepsilon\}_\varepsilon$ such that $u_\varepsilon \to u$ as $\varepsilon \to 0$ by approximating ρ in a way that is compatible with the energies F_ε. Namely, to obtain the sharp upper bound in (5.2.13), we need the approximation ρ_ε of ρ to satisfy

$$(\varepsilon {\rho_\varepsilon}')^2 = \varepsilon + W(\rho_\varepsilon(t)) \quad \text{for all } \varepsilon > 0, \tag{5.2.14}$$

where the additional term ε on the right hand-side of the equation has been added to avoid constant solutions.

To construct an explicit solution to (5.2.14), we define $\psi_\varepsilon : [0,1] \mapsto \mathbb{R}$ to be

$$\psi_\varepsilon(t) := \int_0^t \frac{\varepsilon}{\sqrt{\varepsilon + W(s)}} ds. \tag{5.2.15}$$

Let us point out that ψ_ε in (5.2.15) is increasing as the integrand is positive. Let $\eta_\varepsilon := \psi_\varepsilon(1) \le \varepsilon^{1/2}$. We can invert ψ_ε to obtain a function $\phi_\varepsilon : [0, \eta_\varepsilon] \mapsto [0,1]$, which is of class C^1, since W is continuous. Moreover, it is straightforward to check that ϕ_ε satisfies

$$(\varepsilon {\phi_\varepsilon}')^2(t) = \varepsilon + W(\phi_\varepsilon(t)) \qquad \text{for } t \in [0, \eta_\varepsilon]. \tag{5.2.16}$$

We continuously extend ϕ_ε to $\mathbb{R}$ by imposing it to be constant outside $[0, \eta_\varepsilon]$. Let us remark that the function $\Phi_\varepsilon \in C(\mathbb{R})$ defined by

$$\Phi_\varepsilon(t) := \int_X \phi_\varepsilon(\mathsf{d}_E(x) + t) d\mathcal{H}^N(x)$$

is such that

$$\Phi_\varepsilon(0) \le \int_X \rho(\mathsf{d}_E(x)) d\mathcal{H}^N(x) \le \Phi_\varepsilon(\eta_\varepsilon),$$

because $\phi_\varepsilon(t) \le \rho(t) \le \phi_\varepsilon(t + \eta_\varepsilon)$ for all $t \in \mathbb{R}$. Therefore, for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\int_X \phi_\varepsilon(\mathsf{d}_E(x) + \delta_\varepsilon)\, d\mathcal{H}^N(x) = \Phi_\varepsilon(\delta_\varepsilon) = \int_X \rho(\mathsf{d}_E(x))\, d\mathcal{H}^N(x) = \mathcal{H}^N(E).$$

Let $\rho_\varepsilon(t) := \phi_\varepsilon(t+\delta_\varepsilon)$ for $t \in \mathbb{R}$ and $u_\varepsilon := \rho_\varepsilon \circ \mathsf{d}_E$. From $0 \le \rho_\varepsilon \le 1$ it follows that $0 \le u_\varepsilon \le 1$. Moreover, since $|\nabla \mathsf{d}_E|(x) = 1$ $\mathcal{H}^N$-almost everywhere,

$$\int_X |u_\varepsilon - u| d\mathcal{H}^N = \int_X |\rho_\varepsilon(\mathsf{d}_E(x)) - \rho(\mathsf{d}_E(x))||\nabla \mathsf{d}_E|(x)\, d\mathcal{H}^N(x). \tag{5.2.17}$$

Applying the coarea formula (2.4.8) to the right hand-side of (5.2.17) and recalling the definition of ρ_ε above, we obtain

$$\begin{aligned}\int_X |u_\varepsilon - u| d\mathcal{H}^N &= \int_{-\delta_\varepsilon}^{\eta_\varepsilon-\delta_\varepsilon} \int_X |\rho_\varepsilon(t) - \rho(t)| d\mathrm{Per}(\{\mathsf{d}_E > t\})\, dt \\ &\le \eta_\varepsilon \|\mathrm{Per}(E_t)\|_{L^\infty(-\delta_\varepsilon, \eta_\varepsilon-\delta_\varepsilon)},\end{aligned} \tag{5.2.18}$$

where $E_t := \{x \in X \,:\, \mathsf{d}_E(x) \le t\}$.
Since E is a volume-constrained minimizer for compact variations, it follows from [34, Proposition 3.11] (reported here as Proposition 2.4.42) that

$$\|\mathrm{Per}(E_t; X)\|_{L^\infty(-\delta_\varepsilon,\eta_\varepsilon-\delta_\varepsilon)} \le \sup_{t\in(-\delta_\varepsilon,\eta_\varepsilon-\delta_\varepsilon)} J_{c,K,N}(t)\mathrm{Per}(E). \tag{5.2.19}$$

By combining (5.2.18) with (5.2.19), we have

$$u_\varepsilon \to u \text{ in } L^1(X) \text{ as } \varepsilon \to 0.$$

To simplify the notation, we write

$$\gamma_\varepsilon := \|\mathrm{Per}(E_t)\|_{L^\infty(-\delta_\varepsilon,\eta_\varepsilon-\delta_\varepsilon)}.$$

It follows from (5.2.19) that

$$\limsup_{\varepsilon\to 0} \gamma_\varepsilon \le \mathrm{Per}(E). \tag{5.2.20}$$

Let us show that the sequence $\{u_\varepsilon\}_\varepsilon$, sometimes referred to as recovery sequence, satisfies the Γ-limsup inequality (5.1.5). By using the coarea formula (2.4.8) and the definition of $\rho_\varepsilon(t) = \phi_\varepsilon(t+\delta_\varepsilon)$, we estimate

$$\begin{aligned}F_\varepsilon(u_\varepsilon) &= \int_{\mathbb{R}} \int_X \left[\varepsilon({\rho_\varepsilon}'(t))^2 + \frac{1}{\varepsilon} W(\rho_\varepsilon(t))\right] d\mathrm{Per}(E_t) dt \\ &\le \gamma_\varepsilon \int_{-\delta_\varepsilon}^{\eta_\varepsilon-\delta_\varepsilon} \left[\varepsilon(\phi_\varepsilon'(t+\delta_\varepsilon))^2 + \frac{1}{\varepsilon} W(\phi_\varepsilon(t+\delta_\varepsilon))\right] dt \\ &\le \gamma_\varepsilon \int_0^{\eta_\varepsilon} \left[\varepsilon(\phi_\varepsilon'(t))^2 + \frac{1}{\varepsilon}(W(\phi_\varepsilon(t)) + \varepsilon)\right] dt.\end{aligned}$$

By combining with (5.2.16) we arrive at

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &\le \gamma_\varepsilon \int_0^{\eta_\varepsilon} 2\left(W(\phi_\varepsilon(t))+\varepsilon\right)^{\frac{1}{2}} \phi_\varepsilon{}'(t)\,dt \\ &= 2\gamma_\varepsilon \int_0^1 \sqrt{W(s)+\varepsilon}\,ds. \end{aligned}$$

By taking the limsup in the inequality and using (5.2.20), we conclude

$$\limsup_{\varepsilon\longrightarrow 0} F_\varepsilon(u_\varepsilon) \le 2\int_0^1 \sqrt{W(t)}\,dt\,\mathrm{Per}(E).$$

□

Remark 5.2.3. *Let $E \subset X$ be a volume-constrained minimizer for compact variations with $\mathcal{H}^N(E) < \infty$. If we relaxed this assumption on $E \subset X$ and assumed it were a finite perimeter set, the following steps of the proof of Theorem 5.2.2 would fail:*

- *The characteristic function is equal $\mathcal{H}^N$-a.e. to the distance from the set composed with the Heaviside function.*
- *The perimeter of E can be approximated by the perimeter of its enlargements.*

Next, we show the compactness result Proposition 5.1.6. It is one of the key steps needed to prove Theorem 5.1.8 and it is also of indepedent interest. This result states that if a family of functions has uniformly bounded Cahn-Hilliard energies, then it has a subsequence converging in L^1. Our approach will closely follow [140, Proposition 3]. Similar results have been shown under relaxed assumptions on the growth conditions of the double-well potential W, see for instance [127] and the references therein.

We report here the statement of Proposition 5.1.6 for the reader's convenience.

Proposition 5.2.4 (Compactness)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a compact $\mathrm{RCD}(K,N)$ metric measure space. Let $\{u_\varepsilon\}_\varepsilon \subset H^{1,2}(X)$ be a family of functions satisfying*

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^\infty(X)} < \infty \quad \textit{and} \quad \sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon) < \infty.$$

Then there exists a finite perimeter set $A \subset X$ and a subsequence $\varepsilon_k \to 0$ as $k \to \infty$ such that u_{ε_k} converges to χ_A in $L^1(X)$ as $k \to \infty$.

Proof. As in the proof of Proposition 5.2.1, let $w_\varepsilon := \Phi(u_\varepsilon)$, where

$$\Phi(t) := \int_0^t \sqrt{W(s)}ds.$$

Since $\{u_\varepsilon\}_\varepsilon$ is pointwise uniformly bounded and $\mathfrak{m}(X) < \infty$, it follows that $\{w_\varepsilon\}_\varepsilon$ is bounded in $L^1(X)$. Moreover, the bound (5.2.10) holds, which we report here for the reader's convenience.

$$\int_X |\nabla w_\varepsilon| d\mathfrak{m} \leq F_\varepsilon(u_\varepsilon) < \infty.$$

Applying [18, Proposition 7.5] (reported here as Proposition 2.4.6), there exists a function $w \in \mathrm{BV}(X)$ and a sequence $\varepsilon_k \to 0$ as $k \to \infty$ such that $w_{\varepsilon_k} \to w$ in $L^1(X)$.

Let $\Psi : [0, \infty) \to \mathbb{R}$ be the inverse of Φ. It is straightforward to check that Ψ is uniformly continuous on bounded sets. Let $u := \Psi(w)$. Since $u \in \mathrm{BV}(X)$ as $\{w_\varepsilon\}_\varepsilon$ is pointwise bounded, the sequence $u_{\varepsilon_k} = \Psi(w_{\varepsilon_k})$ converges to u in $L^1(X)$.
By Fatou's lemma it follows that

$$\int_X W(u) d\mathfrak{m} \leq \liminf_{k \to \infty} \int_X W(u_{\varepsilon_k}) d\mathfrak{m} \leq \liminf_{k \to \infty} \varepsilon_k F_k(u_{\varepsilon_k}) = 0,$$

where we have used the assumption (5.1.7) in the last equality.

Therefore, $u(x) \in \{0, 1\}$ for $\mathfrak{m}$-almost every $x \in X$. Since $u \in \mathrm{BV}(X, \{0, 1\})$, we conclude that there exists a finite perimeter set $A \subset X$ such that $u = \chi_A$ $\mathfrak{m}$-almost everywhere. $\square$

Let us point out that, in Proposition 5.1.6, instead of assuming pointwise uniform boundedness of $\{u_\varepsilon\}_\varepsilon$, we could have supposed a polynomial growth of the potential W at infinity. A proof of this fact follows from an adaptation of the original proof of [140, Proposition 3]. Let us comment on our compactness assumption: to preserve the volume constraint we need $L^1(X)$ convergence and not simply $L^1_{\mathrm{loc}}(X)$, and we achieve this by applying [18, Proposition 7.5] (reported here as Proposition 2.4.6). The compactness assuption on $(X, \mathsf{d}, \mathfrak{m})$ in Proposition 5.1.6 can be relaxed to those of [18, Proposition 7.5].

We now prove Theorem 5.1.7. We report the statement here for the reader's convenience.

Theorem 5.2.5 (Approximation of isoperimetric sets)**.** *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ metric measure space and let $E \subset X$ be a volume-constrained minimizer for compact variations with $\mathcal{H}^N(E) < \infty$. Then there exists a family $\{u_\varepsilon\}_\varepsilon \subset H^{1,2}(X) \cap \mathrm{LIP}(X)$, $\varepsilon > 0$, such that*

$$0 \leq u_\varepsilon \leq 1 \quad \text{and} \quad \int_X u_\varepsilon \, d\mathcal{H}^N = \mathcal{H}^N(E) \quad \text{for all } \varepsilon > 0. \tag{5.2.21}$$

Moreover,

$$\lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) = c\mathrm{Per}(E), \tag{5.2.22}$$

where $c > 0$ was defined in (5.1.1).

Proof. By Proposition 5.2.2 there exists a sequence $u_\varepsilon \in H^{1,2} \cap \mathrm{LIP}(X)$, $0 \le u_\varepsilon \le 1$, such that $u_\varepsilon \to \chi_E$ in $L^1(X)$, satisfying (5.2.21) and

$$\limsup_{\varepsilon > 0} F_\varepsilon(u_\varepsilon) \le c\mathrm{Per}(E). \tag{5.2.23}$$

By Proposition 5.2.1, there also holds

$$\liminf_{\varepsilon > 0} F_\varepsilon(u_\varepsilon) \ge c\mathrm{Per}(E). \tag{5.2.24}$$

By chaining (5.2.23) and (5.2.24), we obtain

$$\limsup_{\varepsilon > 0} F_\varepsilon(u_\varepsilon) \le c\mathrm{Per}(E) \le \liminf_{\varepsilon > 0} F_\varepsilon(u_\varepsilon),$$

allowing us to conclude (5.2.22). □

The following result shows that sequences of minimizers of (5.1.6) converge to volume-constrained minimizers for compact variations. Using the direct method in the calculus of variations it is possible to show that in a compact PI space isoperimetric sets exist for any volume. The analogous existence statement holds true also for minimizers of the Cahn-Hilliard energies F_ε for any $\varepsilon > 0$.

Theorem 5.2.6. *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be a compact $\mathrm{RCD}(K, N)$ metric measure space. Let $0 < V < \mathcal{H}^N(X)$ and $u_\varepsilon \in H^{1,2} \cap \mathrm{LIP}(X)$ be minimizers of problem (5.1.6) with $\int_X u_\varepsilon \, d\mathcal{H}^N = V$ for all $\varepsilon > 0$. Moreover, assume that $\sup_{\varepsilon > 0} F_\varepsilon(u_\varepsilon) < \infty$. Then there exists an isoperimetric set $E \subset X$ with $\mathcal{H}^N(E) = V$ and a subsequence $\{\varepsilon_k\}_k$ such that $u_{\varepsilon_k} \to \chi_E$ in $L^1(X)$ as $k \to \infty$ and*

$$\lim_{k \to \infty} F_k(u_{\varepsilon_k}) = c\mathrm{Per}(E),$$

where $c > 0$ was defined in (5.1.1).

Proof. By Proposition 5.1.6, there exist a sequence $\{\varepsilon_k\}_k$ with $\varepsilon_k \to 0$ as $k \to \infty$, and a finite perimeter set $E \subset X$ with $u_{\varepsilon_k} \to \chi_E$ in $L^1(X)$. Moreover, by Proposition 5.2.1, it holds

$$c\mathrm{Per}(E) \le \liminf_{k \to \infty} F_k(u_{\varepsilon_k}). \tag{5.2.25}$$

Suppose by contradiction that E is not isoperimetric. Then by the very definition of isoperimetric set there exists a subset $F \subset X$ of finite perimeter such that $\mathcal{H}^N(F) = V$ and such that

$$\mathrm{Per}(F) < \mathrm{Per}(E). \tag{5.2.26}$$

By Proposition 5.2.2, there exists a sequence $v_{\varepsilon_k} \in H^{1,2} \cap \mathrm{LIP}(X)$ with

$$c\mathrm{Per}(F) \geq \limsup_{k \to \infty} F_k(v_{\varepsilon_k}). \tag{5.2.27}$$

Combining (5.2.25), (5.2.26) and (5.2.27) with the minimality of u_{ε_k} we obtain

$$\begin{aligned} c\mathrm{Per}(E) > c\mathrm{Per}(F) &\geq \limsup_{k \to \infty} F_k(v_{\varepsilon_k}) \\ &\geq \liminf_{k \to \infty} F_k(v_{\varepsilon_k}) \geq \liminf_{k \to \infty} F_k(u_{\varepsilon_k}) \\ &\geq c\mathrm{Per}(E), \end{aligned}$$

which provides a contradiction. □

Let us comment again on the compactness assumption of Theorem 5.1.8. Such assumption is needed both to apply Proposition 5.1.6 and to guarantee existence of isoperimetric regions for every volume. Theorem 5.1.8 holds under more general assumptions when these two results can be applied. A promising direction for addressing the non-compact case is to leverage the generalized existence of isoperimetric regions established in [30] for non-compact $\mathrm{RCD}(K, N)$ spaces, building on the framework developed in [153].

5.3 Variable ambient space

In this section we consider a sequence of p.m.m.s. $\{(X_k, \mathsf{d}_k, \mathfrak{m}_k, x_k)\}_k$ converging in the pmGH topology to $(X, \mathsf{d}, \mathfrak{m}, x)$. Let $\varepsilon_k > 0$, $\varepsilon_k \to 0$. We define the Cahn-Hilliard energies $F_k : L^1(X_k) \to \mathbb{R}$ as follows

$$F_k(u) := \begin{cases} \int_{X_k} \left(\varepsilon_k |\nabla u|^2 + \frac{1}{\varepsilon_k} W(u) \right) d\mathcal{H}^N & \text{if } u \in H^{1,2}(X_k), \\ +\infty & \text{otherwise.} \end{cases}$$

Our aim in this section is to prove Theorem 5.1.9, which states that under some assumptions on the pointed metric measure spaces $(X_k, \mathsf{d}_k, \mathfrak{m}_k, x_k)$ (namely, that they are non-collapsing closed Riemannian manifolds with uniform lower bounds on the Ricci curvature) the energies F_k Γ-converge to $c\mathrm{Per}(\cdot)$, where $c > 0$ was defined in (5.1.1).

This section is structured analogously to Section 5.2 and similarly to [140]. We start by showing the Γ-liminf inequality, which holds under more general assumptions on the sequence $\{(X_k, \mathsf{d}_k, \mathfrak{m}_k, x_k)\}_k$ than Theorem 5.1.9. We then present a Γ-limsup result, which holds when $(X_k, \mathsf{d}_k, \mathfrak{m}_k, x_k)$ is a pointed closed Riemannian manifold with a lower bound on the Ricci curvature and the volume of a unit ball centered at x_k endowed with the standard volume measure for each $k \in \mathbb{N}$. After showing a compactness result in the case where X is compact, we show Corollary 5.3.6, which states that a sequence of minimizers of the Cahn-Hilliard energies in the approximating spaces X_k for $k \in \mathbb{N}$ have a subsequence converging to an isoperimetric subset of X both in the L^1 and in the energy sense.

In this section we make the additional assumption $W \in \mathrm{LIP}(\mathbb{R})$.

Proposition 5.3.1 (Γ-liminf)**.** *Let $\{(X_k, \mathsf{d}_k, \mathfrak{m}_k, x_k)\}_k$ be a sequence of* $\mathrm{RCD}(K, N)$ *spaces converging in the pmGH sense to $(X, \mathsf{d}, \mathfrak{m}, x)$. Let $\{u_k\}_k$ be a sequence of functions $u_k \in L^1(X_k)$ for all $k \in \mathbb{N}$ that converges L^1-strongly to $u \in L^1(X)$. Let $\{\varepsilon_k\}_k$, $\varepsilon_k > 0$ for all $k \in \mathbb{N}$ and $\varepsilon_k \to 0$ as $k \to \infty$. Assume that*

$$\liminf_{k \to \infty} F_k(u_k) < \infty.$$

Then there exists a finite perimeter set $A \subset X$ such that $u = \chi_A$. Moreover,

$$c\mathrm{Per}(A) \le \liminf_{k \to \infty} F_{\varepsilon_k}(u_k), \tag{5.3.1}$$

where $c > 0$ was defined in (5.1.1).

Proof. By considering a subsequence of $\{u_k\}_k$, which we do not relabel, we can suppose that

$$\sup_{k \in \mathbb{N}} F_{\varepsilon_k}(u_k) < \infty. \tag{5.3.2}$$

Since $W \in \mathrm{LIP}(\mathbb{R})$, it follows from [18, Proposition 3.3 a)] (reported here in Section 2.3.6) that $W(u_k)$ converges L^1-strongly to $W(u)$. In particular, we have

$$0 \le \int_X W(u)\, d\mathfrak{m} = \lim_{k \to \infty} \int_{X_k} W(u_k) d\mathfrak{m}_k \le \varepsilon_k F_k(u_k) \to 0 \quad \text{as } k \to 0.$$

Consequently, $u(x) \in \{0, 1\}$ for $\mathfrak{m}$-a.e. $x \in X$ or, equivalently, there exists a set $A \subset X$ such that $u = \chi_A$ $\mathfrak{m}$-almost everywhere.

Analogously to the proof of Proposition 5.2.1, we also assume that $0 \le u_k \le 1$ for all $k \in \mathbb{N}$. Indeed, the function $\Psi(t) := \max\{\min\{t, 1\}, 0\}$ for $t \in \mathbb{R}$ is 1-Lipschitz. Hence, by [18, Proposition 3.3 a)] (reported here in Section 2.3.6) $\Psi(u_k)$ converges

L^1-strongly to $\Psi(u) = u$. Moreover, $F_k(\Psi(u_k)) \leq F_k(u_k)$ for all $k \in \mathbb{N}$. Hence, if (5.3.1) holds for $\Psi(u_k)$, it also holds for u_k.
Let now

$$\Phi(t) := \int_0^t \sqrt{W(s)} ds.$$

Applying [18, Proposition 3.3 a)] (for instance, by modifying Φ to exploit the assumption $0 \leq u_k(x) \leq 1$ for all $x \in X_k$ and $k \in \mathbb{N}$) we obtain that $\Phi(u_k)$ converges L^1-strongly to $\Phi(u)$. From the chain rule (see Section 2.1.5) it holds

$$\begin{aligned} \int_{X_k} |\nabla \Phi(u_k)| d\mathfrak{m}_k &= \int_{X_k} \sqrt{W(u_k)} |\nabla u_k| \, d\mathfrak{m}_k \\ &\leq \int_{X_k} \left(\varepsilon_k |\nabla u_k| + \frac{1}{\varepsilon_k} W(u_k) \right) \, d\mathfrak{m}_k = F_k(u_k) < \infty. \end{aligned} \tag{5.3.3}$$

It follows from [18, Theorem 6.4] that

$$|D\Phi(u)|(X) \leq \liminf_{k \to \infty} \int_{X_k} |\nabla \Phi(u_k)| d\mathfrak{m}_k. \tag{5.3.4}$$

By the coarea formula (2.4.8), it also holds

$$|D\Phi(u)|(X) = \int_0^\infty \left| D\chi_{\{\Phi(\chi_A) > t\}} \right| dt = \int_{\Phi(0)}^{\Phi(1)} \left| D\chi_{\{\Phi(\chi_A) > t\}} \right| dt = \Phi(1) \mathrm{Per}(A). \tag{5.3.5}$$

Combining (5.3.3), (5.3.4) and (5.3.5) we are able to conclude

$$\liminf_{k \to \infty} F_k(u_k) \geq |D\Phi(u)|(X) d\mathfrak{m} = c \mathrm{Per}(A). \tag{5.3.6}$$

□

Let us now show that a Γ-limsup inequality also holds for the functionals F_k under the assumption that the approximating p.m.m.s. (X_k, d_k, x_k) are Riemannian manifolds with a uniform lower bound on the Ricci curvature and the volume of the unit ball centered at x_k for each $k \in \mathbb{N}$.

Proposition 5.3.2 (Γ-limsup). *Let $\{(X_k, \mathsf{d}_k, x_k)\}_k$ be a sequence of pointed closed Riemannian manifolds with*

$$\mathrm{Ric}_k \geq K$$

for some $K \in \mathbb{R}$ and

$$\mathrm{vol}(B_1(x_k)) > v$$

for some $v > 0$ of dimension N endowed with their respective N-dimensional Hausdorff measures. Suppose that there exists a pointed non-collapsed Ricci limit space

$(X, \mathsf{d}, \mathcal{H}^N, x)$ *such that* X_k *converges in the pmGH topology to* X. *Moreover, let* $E \subset X$ *be a finite perimeter set with* $\mathcal{H}^N(E) \in (0, \infty)$. *Then there exists a sequence* $\{u_k\}_k$ *with* $u_k \in H^{1,2} \cap \mathrm{Lip}(X_k, [0,1])$ *and* $\int_{X_k} u_k \, d\mathcal{H}^N = \mathcal{H}^N(E)$ *for all* $k \in \mathbb{N}$ *that converges* L^1*-strongly to* χ_E. *Moreover,*

$$c\mathrm{Per}(E) = \lim_{k \to \infty} F_k(u_k), \tag{5.3.7}$$

where $c > 0$ *was defined in* (5.1.1).

Proof. By [11, Theorem 3.8] there exists a sequence of finite perimeter sets $E_k \subset X_k$ converging to E in the L^1 sense (see Definition 2.4.8) and with $\mathrm{Per}(E_k) \to \mathrm{Per}(E)$ as $k \to \infty$. By the classical theory of Γ-convergence [141, 140, 178], for each $k \in \mathbb{N}$ there exists $u_k \in H^{1,2} \cap \mathrm{Lip}(X_k, [0,1])$ with the following property: there exist $M > 0$ and $w : [0, \infty] \to [0, M]$ such that $\lim_{\varepsilon \to 0} w(\varepsilon) = 0$ and

$$\int_{X_k} |\chi_{E_k} - u_k| d\mathfrak{m}_k \leq w(\varepsilon_k). \tag{5.3.8}$$

Moreover, $\int_{X_k} u_k \, d\mathcal{H}^N = \mathcal{H}^N(E)$.

It follows from (5.1.5) and a diagonal argument that there exists a subsequence, which we do not relabel, for which

$$|F_k(u_k) - c\mathrm{Per}(E_k)| \leq w(\varepsilon_k). \tag{5.3.9}$$

Therefore, we have

$$\begin{aligned} &|F_k(u_k) - c\mathrm{Per}(E)| \\ &\qquad \leq |F_k(u_k) - c\mathrm{Per}(E_k)| + c|\mathrm{Per}(E_k) - \mathrm{Per}(E)| \\ &\qquad \leq w(\varepsilon_k) + c|\mathrm{Per}(E_k) - \mathrm{Per}(E)|. \end{aligned} \tag{5.3.10}$$

Letting $k \to \infty$ allows us to conclude (5.3.7). □

Combining Theorem 5.3.1 and 5.3.2 we obtain Γ-convergence of F_ε to $c\mathrm{Per}$. We explicitly restate it here for ease of the reader.

Corollary 5.3.3 (Γ-convergence). *Let* $(X, \mathsf{d}, \mathcal{H}^N, x)$ *and* $\{(X_k, \mathsf{d}_k, x_k)\}_k$ *be as in Proposition* 5.3.2. *Let* $c > 0$ *be as in* (5.1.1). *Then,*

1. *(Γ-liminf). Let* $u \in L^1(X)$, $u_k \in L^1(X_k)$, $\|u_k\| = V > 0$ *for all* k *and* u_k *converge strongly in* L^1 *to* u. *Moreover, suppose that* $\liminf_{k \to \infty} F_k(u_k) < \infty$. *Then there exists a finite perimeter set* $A \subset X$ *such that* $u = \chi_A$ $\mathcal{H}^N$*-almost everywhere and*

$$c\mathrm{Per}(A) \leq \liminf_{k \to \infty} F_k(u_k).$$

2. *(Γ-limsup). Let $E \subset X$ be a finite perimeter set with $\mathcal{H}^N(E) < \infty$. Then there exists a sequence $u_k \in H^{1,2} \cap \mathrm{Lip}(X_k, [0,1])$ such that $\|u_k\|_{L^1(X_k)} = \mathcal{H}^N(E)$ for all $k \in \mathbb{N}$, u_k converge strongly in L^1 to u and*

$$\lim_{k \to \infty} F_k(u_k) = c\mathrm{Per}(u).$$

We now report a compactness criteria for sequences of functions having bounded Cahn-Hilliard energies F_k. The result plays a key role in showing that sequences of minimizers of F_k converge to some volume-constrained perimeter minimizer, which is the statement of Corollary 5.3.6. The result is also of independent interest.

Proposition 5.3.4 (Compactness). *Let $\{(X_k, \mathsf{d}_k)\}_k$ be a non-collapsing sequence of closed Riemannian manifolds with*

$$\mathrm{Ric}_k \geq K$$

for some $K \in \mathbb{R}$ of dimension N endowed with their respective N-dimensional Hausdorff measures. Suppose that there exists a non-collapsed compact Ricci limit space $(X, \mathsf{d}, \mathcal{H}^N)$ such that X_k converges in the mGH topology to X. Let $u_k \in L^1(X_k)$ be such that

$$\sup_{k \in \mathbb{N}} \|u_k\|_{L^\infty(X_k)} < \infty \quad \text{and} \quad \sup_k F_k(u_k) < \infty.$$

Then there exists a subsequence u_k, which we do not relabel, and a finite perimeter set $E \subset X$ such that $\{u_k\}_k$ converges to χ_E L^1-strongly.

The proof of Proposition 5.3.4 follows verbatim from the proof of Proposition 5.1.6. The compactness assumption is needed to apply [18, Proposition 7.5] (reported here as Proposition 2.4.6). Proposition 5.3.4 holds under the more general assumption of [18, Proposition 7.5]. A consequence of the Γ-convergence Corollary 5.3.3 is the following result, which states that any volume-constrained minimizers for compact variations can be approximated by functions both in the L^1 and in the Cahn-Hilliard energies senses.

Corollary 5.3.5. *Let $(X, \mathsf{d}, \mathcal{H}^N, x)$ and $\{(X_k, \mathsf{d}_k, x_k)\}_k$ be as in Proposition 5.3.2. Let $0 < V < \min\{\mathcal{H}^N(X), \inf_k \mathcal{H}^N(X_k)\}$ and let $E \subset X$ be a volume-constrained minimizer for compact variations with $\mathcal{H}^N(E) = V$. Then there exist $u_k \in H^{1,2}(X_k, [0,1])$, $\int_{X_k} u_k \, d\mathcal{H}^N = V$ for all $k \in \mathbb{N}$ such that $u_k \to \chi_E$ in the L^1 sense and $\lim_{k \to \infty} F_k(u_k) = c\mathrm{Per}(E)$, where $c > 0$ was defined in (5.1.1).*

Proof. From the Γ-limsup inequality Proposition 5.3.2 there exist $u_k \in H^{1,2}(X_k,[0,1])$, $\int_{X_k} u_k \, d\mathcal{H}^N = V$ for all $k \in \mathbb{N}$ such that $u_k \to \chi_E$ in the L^1 sense and

$$\limsup_{k \to \infty} F_k(u_k) \le c\mathrm{Per}(E).$$

The Γ-liminf inequality 5.3.1 allows us to show

$$\lim_{k \to \infty} F_k(u_k) = c\mathrm{Per}(E),$$

concluding the proof. □

Another application of the Γ-convergence Corollary 5.3.3 is that sequences of minimizers for F_k converge to perimeter minimizers.

Corollary 5.3.6. *Let $(X, \mathsf{d}, \mathcal{H}^N)$ and $\{(X_k, \mathsf{d}_k)\}_k$ be as in Proposition 5.3.4. Let $0 < V < \min\{\mathcal{H}^N(X), \inf_k \mathcal{H}^N(X_k)\}$ and let $u_k \in L^1(X_k)$ be a solution to the problem*

$$\inf \left\{ F_k(u) : u \in L^1(X_k), \int u \, d\mathcal{H}^N = V \ \textit{and} \ u \ge 0 \right\}$$

for each $k \in \mathbb{N}$. Moreover, suppose

$$\sup_k F_k(u_k) < \infty.$$

Then there exists a subsequence $\{u_k\}_k$, which we do not relabel, and an isoperimetric set $E \subset X$ with $\mathcal{H}^N(E) = V$ such that $\{u_k\}_k$ converges L^1-strongly to χ_E and

$$\lim F_k(u_k) \to c\mathrm{Per}(E),$$

where $c > 0$ was defined in (5.1.1).

Proof. By Proposition 5.3.4 there exists a subsequence of $\{u_k\}_k$, which we do not relabel, and $u \in L^1(X)$ with $0 \le u \le 1$ $\mathcal{H}^N$-almost everywhere, $\int_X u \, d\mathcal{H}^N = V$ and such that $\{u_k\}_k$ converges L^1-strongly to u. From the Γ-liminf inequality Proposition 5.3.1 it follows that $u = \chi_E$ $\mathcal{H}^N$-almost everywhere for some $E \subset X$. Moreover,

$$c\mathrm{Per}(E) \le \liminf_{k \to \infty} F_k(u_k). \tag{5.3.11}$$

From the Γ-limsup inequality Proposition 5.3.2 there exists a sequence $\{w_k\}_k \subset H^{1,2}(X)$ such that $w_k \to \chi_E$ L^1-strongly, $\int_{X_k} w_k \, d\mathcal{H}^N = V$ for all $k \in \mathbb{N}$ and

$$c\mathrm{Per}(E) \ge \limsup_{k \to \infty} F_k(w_k). \tag{5.3.12}$$

By chaining the inequalities (5.3.11) and (5.3.12) and using the minimality assumption of u_k for all $k \in \mathbb{N}$, we obtain

$$c\mathrm{Per}(E) \leq \liminf_{k \to \infty} F_k(u_k) \leq \limsup_{k \to \infty} F_k(w_k) \leq c\mathrm{Per}(E).$$

We conclude $c\mathrm{Per}(E) = \lim_{k\to\infty} F_k(u_k)$. Suppose by contradiction that E is not isoperimetric. Then there exists $F \subset X$ with $\mathcal{H}^N(F) = \mathcal{H}^N(E)$ and $\mathrm{Per}(E) > \mathrm{Per}(F)$. By Corollary 5.3.3 there exists $v_k \in H^{1,2}(X_k, [0,1])$, $\int_{X_k} v_k d\mathcal{H}^N = \mathcal{H}^N(F)$ for all $k \in \mathbb{N}$ such that $v_k \to \chi_F$ strongly in L^1 and $\lim_{k\to\infty} F_k(v_k) = c\mathrm{Per}(F)$. By the minimality of u_k it follows that

$$c\mathrm{Per}(E) = \lim_{k \to \infty} F_k(u_k) \leq \lim_{k \to \infty} F_k(v_k) = c\mathrm{Per}(F) < c\mathrm{Per}(E).$$

This provides a contradiction, showing that E is indeed isoperimetric. □

In contrast with Theorem 5.1.8, in Corollary 5.3.6 we do not need to assume that a volume-constrained perimeter minimizer for compact variations exists. This is a consequence of the full Γ-convergence-type result Corollary 5.3.3, which is not available in the fixed ambient space setting of Section 5.2. Indeed, the extra ingredient in Corollary 5.3.3 is the existence of an approximation of the RCD space by smooth Riemannian manifolds with Ricci bounded below.

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