JOURNAL OF THE ASSOCIATION FOR MATHEMATICAL RESEARCH

AMR
ASSOCIATION FOR MATHEMATICAL RESEARCH

VOLUME 3 : ISSUE 2

2025

JOURNAL OF THE ASSOCIATION FOR MATHEMATICAL RESEARCH

PUBLISHED BY

Association for Mathematical Research, amathr.org,
Davis, CA; Jenkintown PA.

JOURNAL OF THE ASSOCIATION FOR MATHEMATICAL RESEARCH, 3(2):161–236, 2025
jamathr.org

On the GL(2n) eigenvariety: branching laws, Shalika families and p-adic L-functions

Daniel Barrera-Salazar * Mladen Dimitrov †
Andrew Graham ‡ Andrei Jorza § Chris Williams ¶

Received 9 Mar 2023; Revised 13 Sep 2024; Accepted 27 Feb 2025

Abstract:
In this paper, we prove that a $\mathrm{GL}(2n)$-eigenvariety is étale over the (pure) weight space at non-critical Shalika points, and construct multi-variable p-adic L-functions varying over the resulting Shalika components. Our constructions hold in tame level 1 and Iwahori level at p, and give p-adic variation of L-values (of regular algebraic cuspidal automorphic representations of $\mathrm{GL}(2n)$ admitting Shalika models) over the whole pure weight space. In the case of $\mathrm{GL}(4)$, these results have been used by Loeffler and Zerbes to prove cases of the Bloch–Kato conjecture for $\mathrm{GSp}(4)$.

Our main innovations are: (a) the introduction and systematic study of 'Shalika refinements' of local representations of $\mathrm{GL}(2n)$, and evaluation of their attached local twisted zeta integrals; and (b) the p-adic interpolation of representation-theoretic branching laws for $\mathrm{GL}(n)\times\mathrm{GL}(n)$ inside $\mathrm{GL}(2n)$. Using (b), we give a construction of multi-variable p-adic functionals on the overconvergent cohomology groups for $\mathrm{GL}(2n)$, interpolating the zeta integrals of (a). We exploit the resulting non-vanishing of these functionals to prove our main arithmetic applications.

Key words and phrases:
p-adic L-functions; Shalika Model; Eigenvarieties; $\mathrm{GL}(2n)$; Symplectic $\mathrm{GL}(N)$; Branching Laws; p-adic Families; p-adic Interpolation; Local Zeta Integral

*DBS was supported by ANID FONDECYT grant 11201025 and ANID PAI grant 77180007
†MD was supported by Agence Nationale de la Recherche grant ANR-18-CE40-0029
‡AG was supported in part by ERC-2018-COG-818856-HiCoShiVa
§AJ is grateful for support from CEMPI, Lille, the Center for Mathematics at Notre Dame, and the NSF
¶CW was supported by EPSRC Postdoctoral Fellowship EP/T001615/1

DOI: 10.56994/JAMR.003.002.002

1 Introduction

1.1 Motivation

The *Bloch–Kato conjectures* are amongst the most important open problems in modern algebraic number theory, and predict a deep link between arithmetic and analysis. Through decades of research, a fruitful approach to Bloch–Kato has been to find and prove p-adic reinterpretations; for every special case of the Bloch–Kato conjecture, there should be an analogous p-adic *Iwasawa Main Conjecture* relating p-adic arithmetic data to a p-adic L-function. These p-adic reinterpretations are usually more tractable than the original conjectures – for example, the Iwasawa Main Conjecture for elliptic curves has been proved in many cases (for example in [Kat04, SU14], but also in many other works). Moreover, understanding the p-adic picture can lead to proofs of special cases of Bloch–Kato.

Crucial to proofs of Bloch–Kato/Iwasawa Main Conjectures is a good understanding of p-adic L-functions, eigenvarieties, and p-adic L-functions over eigenvarieties. In this paper, we prove new results about these objects for $\mathrm{GL}_{2n}/\mathbf{Q}$. In particular, let π be a regular algebraic symplectic cuspidal automorphic representation (RASCAR) of $\mathrm{GL}_{2n}(\mathbf{A})$ that is everywhere spherical; here *symplectic* is the condition that π admits a Shalika model (i.e. is a functorial transfer from $\mathrm{GSpin}_{2n+1}(\mathbf{A})$). This ensures there is an integer w such that $\pi \cong \pi^\vee \otimes |\cdot|^{\mathsf{w}}$, i.e. π is essentially self-dual. Let $\tilde{\pi}$ be an Iwahoric p-refinement which is spin and of non-critical slope (see Definitions 6.2 and 10.6). As explained in §14, by [BDW, Thm. A] there exists a p-adic L-function $L_p(\tilde{\pi})$ attached to $\tilde{\pi}$, that is, a locally analytic distribution on $\mathbf{Z}_p^\times$ of controlled growth that interpolates its Deligne-critical L-values. In this paper:

(A) we prove that the Iwahoric GL_{2n}-eigenvariety is étale over the $(n+1)$-dimensional pure weight space at $\tilde{\pi}$, and that the unique connected component $\mathscr{C}$ through $\tilde{\pi}$ contains a very Zariski-dense set $\mathscr{C}^{\mathrm{Sha}}$ of classical Shalika points; and

(B) we construct an $(n+2)$-variable p-adic L-function $\mathcal{L}_p^{\mathscr{C}}$ interpolating $L_p(\tilde{\pi}_y)$ for $y \in \mathscr{C}^{\mathrm{Sha}}$.

We state these results in full in Theorems A and B respectively in §1.4 of the introduction.

We describe an application. In the special case of GL_4, part (B) fulfils the forward compatibility required by Loeffler and Zerbes in their recent tour-de-force work [LZb] proving new cases of the Bloch–Kato conjecture for GSp_4; in particular, the present paper is the 'forthcoming work' mentioned in §17.5 *op. cit.*, where this special case was first announced.

1.2 Set-up and previous work

Let π be as above, and let $\lambda = (\lambda_1, ..., \lambda_{2n})$ be its weight, a dominant algebraic character of the diagonal torus $T \subset G := \mathrm{GL}_{2n}$. Then $\mathsf{w} = \lambda_n + \lambda_{n+1}$ is the purity weight of λ (see §2.2). Let $L(\pi, s)$ be the standard L-function of π, normalised so that for $j \in \mathbf{Z}$, the value $L(\pi, j + \frac{1}{2})$ is Deligne-critical if and only if

$$j \in \mathrm{Crit}(\lambda) := \{j \in \mathbf{Z} : -\lambda_{n+1} \geqslant j \geqslant -\lambda_n\}. \tag{1.1}$$

Let $K = \mathrm{Iw}_G \prod_{\ell \neq p} G(\mathbf{Z}_\ell) \subset G(\widehat{\mathbf{Z}})$, where Iw_G is the Iwahori subgroup at p. Let S_K be the locally symmetric space for G of level K. As π is regular algebraic, it contributes to the compactly supported cohomology of S_K with coefficients in $\mathscr{V}_\lambda^\vee$ in degrees $n^2, n^2+1, ..., n^2+n-1$. Here V_λ is the algebraic representation of G of highest weight λ, and $\mathscr{V}_\lambda^\vee$ is the local system on S_K attached to its dual. Let $t = n^2+n-1$ (the top degree).

Our work builds on ideas of Grobner–Rahuram [GR14], of Dimitrov–Januszewski–Raghuram [DJR20], and particularly of Barrera–Dimitrov–Williams [BDW], all of which worked in the Q-parahoric setting, for Q the (n,n)-parabolic subgroup of G. As *op. cit.*, our methods are built upon the existence of *evaluation maps*, functionals on Betti cohomology groups. In particular:

- In [GR14], the authors constructed $\mathbf{C}$-valued evaluation maps
$$\mathrm{Ev}_j^{\lambda,\mathrm{GR}} : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(\mathbf{C})) \to \mathbf{C}$$
and used them to prove algebraicity for the Deligne-critical L-values $L(\pi, j+\frac{1}{2})$.
- In [DJR20], for χ finite order of conductor p^β, the authors used p-adic analogues
$$\mathrm{Ev}_{\chi,j}^{\lambda,\mathrm{DJR}} : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(\overline{\mathbf{Q}}_p)) \to \overline{\mathbf{Q}}_p$$
to construct p-adic L-functions for ordinary 'Shalika' Q-parahoric p-refinements $\tilde{\pi}^Q$ of π.
- In [BDW], the authors constructed (parahoric) overconvergent evaluations
$$\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega^Q) \to \mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega),$$
where Ω is a (2-dimensional) parahoric p-adic family of weights, $\mathscr{D}_\Omega^Q$ is a local system attached to a space of parahoric distributions, and $\mathcal{D}(\mathbf{Z}_p^\times, R)$ is the space of R-valued locally analytic distributions on $\mathbf{Z}_p^\times$; so $\mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$ is a space of 3-variable distributions. These interpolated the $\mathrm{Ev}_{\chi,j}^{\lambda,\mathrm{DJR}}$ for varying λ, χ and j. They were used to construct p-adic L-functions $L_p(\tilde{\pi}^Q)$ attached to finite slope Q-parahoric Shalika p-refinements $\tilde{\pi}^Q$, to construct 2-dimensional (parahoric) p-adic families through $\tilde{\pi}^Q$, and to vary $L_p(\tilde{\pi}^Q)$ over these families.

These papers work with GL_{2n} over totally real fields and do not require π everywhere spherical; for a detailed summary of these works, we refer the reader to [BDW, Intro.]. However, the Q-parahoric setting considered *op. cit.* cannot see variation in more than 2 weight variables, meaning our present results are necessary for the application to Bloch–Kato in [LZb].

1.3 New input

In our generalisation to Iwahoric families, substantial new ideas are required in two particular places: one automorphic (the computation of local zeta integrals at p), and one p-adic (the p-adic interpolation of classical representation-theoretic branching laws).

1.3.1 Shalika p-refinements and local zeta integrals

To study local zeta integrals attached to the (unramified) representation π_p, we introduce *Shalika p-refinements* of π_p. In this paper, we consider Iwahoric p-refinements, rather than the Q-parahoric p-refinements of [DJR20, BDW].

Let $\mathcal{H}_p$ be the Hecke algebra at p (see Definition 2.3). An (Iwahoric) *p-refinement* of π_p is a pair $\tilde{\pi}_p = (\pi_p, \alpha)$, where $\alpha : \mathcal{H}_p \longrightarrow \overline{\mathbf{Q}}$ is a system of Hecke eigenvalues appearing in $\pi_p^{\mathrm{Iw}_G}$. If π_p has regular Satake parameter, there are $(2n)!$ such p-refinements, indexed by elements of the Weyl group $\mathcal{W}_G = \mathrm{S}_{2n}$, all regular in the sense of Definition 2.4. Attached to any regular p-refinement $\tilde{\pi}_p$ is a certain family of twisted local zeta integrals at p. We call $\tilde{\pi}_p$ a *Shalika p-refinement* if one of these local zeta integrals is non-vanishing.

In [DJR20, BDW], it is implicitly predicted that a p-refinement should be Shalika if and only if it lies in a certain class of 'spin' p-refinements, i.e. those that interact well with the Shalika model. In the parahoric case, an ad-hoc definition of a spin refinement – there called a 'Q-regular Q-refinement' – is given in [DJR20, §3.3], inspired by [AG94]; and the relevant zeta integrals are shown to be non-vanishing.

In §6, we give a much more conceptual definition of spin p-refinements, generalising and justifying [DJR20, §3.3]. Since π_p admits a Shalika model, it is the functorial transfer of a representation Π_p of $\mathcal{G}(\mathbf{Q}_p)$, where $\mathcal{G} = \mathrm{GSpin}_{2n+1}$. Via a careful study of the root systems of G and $\mathcal{G}$, we construct a map $\jmath^\vee : \mathcal{H}_p \to \mathcal{H}_p^{\mathcal{G}}$ of Hecke algebras at p. We then say α is a *spin p-refinement* if there exists an eigensystem $\alpha^{\mathcal{G}}$ for $\mathcal{G}$ such that α factors as

$$\alpha : \mathcal{H}_p \xrightarrow{\jmath^\vee} \mathcal{H}_p^{\mathcal{G}} \xrightarrow{\alpha^{\mathcal{G}}} \overline{\mathbf{Q}},$$

and show the system $\alpha^{\mathcal{G}}$ then appears in $\Pi_p^{\mathrm{Iw}_{\mathcal{G}}}$, where $\mathrm{Iw}_{\mathcal{G}} \subset \mathcal{G}(\mathbf{Z}_p)$ denotes the Iwahori subgroup for $\mathcal{G}(\mathbf{Z}_p)$. When π_p (hence Π_p) is regular, there are $2^n n!$ such spin p-refinements.

We prove that the family of local zeta integrals attached to a spin p-refinement is non-vanishing, and hence that spin p-refinements are Shalika p-refinements. We study the converse (that Shalika refinements are spin refinements) in a sequel paper [BGW25]; there we prove it in many cases, and conjecture that it always holds.

We actually compute the local zeta integral in two different ways, with different benefits.

- In §5, we compute the local zeta integral at Iwahoric level, with the restriction that our method works only for ramified characters.
- The unramified case at Iwahoric level appears to be very difficult. We get around this in §9 by instead computing the unramified integral *at parahoric* level, via a totally different method.

The latter result strengthens the results of [DJR20, BDW] by proving that the p-adic L-functions they construct satisfy the expected interpolation property at the trivial character, a fundamental case these works omitted.

In later sections, we use interpolation at ramified characters to prove that the p-adic L-functions at Iwahori level and parahoric level agree; and thus we obtain interpolation at the trivial character for the p-adic L-functions of the present paper.

All of these local results are proved in §5–§9.

1.3.2 p-adic interpolation of branching laws

Over $\mathbf{Q}$, for any n the evaluation maps of [BDW] are valued in a space of distributions in only 3 variables. In the present paper we construct evaluation maps in the full expected $n+2$ variables.

Our key input is a p-adic interpolation, in $(n+2)$-variables, of classical representation-theoretic branching laws. More precisely: in [GR14, DJR20, BDW] the subgroup $H = \mathrm{GL}_n \times \mathrm{GL}_n$ (embedded diagonally in GL_{2n}) plays a distinguished role. For $j_1, j_2 \in \mathbf{Z}$, let $V^H_{(j_1,j_2)}$ denote the H-representation $\det_1^{j_1}\det_2^{j_2}$, the algebraic representation of highest weight $(j_1,...,j_1,j_2,...,j_2)$. Recall w is the purity weight of λ. Then we have the following reinterpretation of the Deligne-critical L-values (1.1):

(†) **Branching law:** $j \in \mathrm{Crit}(\lambda) \iff V^H_{(-j,\mathsf{w}+j)} \subset V_\lambda\big|_H$ *with multiplicity one.*

Example. Let $G = \mathrm{GL}_2, H = \mathrm{GL}_1 \times \mathrm{GL}_1$, and let π be a RACAR of weight $(k,0)$, corresponding to a classical newform f of weight $k+2$. Then (in our normalisations) π has Deligne-critical L-values $L(\pi, j+\frac{1}{2})$ with $-k \leqslant j \leqslant 0$. Here $\mathsf{w} = k$ and $V_\lambda = \mathrm{Sym}^k(\mathbf{C}^2)$, the space of homogeneous polynomials in two variables X,Y of degree k. We have $V_\lambda|_H = \oplus_{j=-k}^0[\mathbf{C}\cdot X^{-j}Y^{k+j}]$. The summand at j is the character $\left(\begin{smallmatrix} s & \\ & t\end{smallmatrix}\right) \mapsto s^{-j}t^{k+j}$ of H, and corresponds to the Deligne-critical L-value $L(\pi, j+\frac{1}{2})$.

Attached to H is a family of *automorphic cycles* $\{X_\beta\}_{\beta\geqslant 1}$, which are modified locally symmetric spaces for H whose dimension is crucially t. For $j \in \mathrm{Crit}(\lambda)$, the evaluation maps $\mathrm{Ev}^{\lambda,\mathrm{DJR}}_{\chi,j}$ of [DJR20] were constructed as a composition

$$\mathrm{H}^t_{\mathrm{c}}\big(S_K, \mathscr{V}_\lambda\big) \to \mathrm{H}^t_{\mathrm{c}}\Big(X_\beta, \mathscr{V}_\lambda|_H\Big) \to \mathrm{H}^t_{\mathrm{c}}\Big(X_\beta, \mathscr{V}^H_{(-j,\mathsf{w}+j)}\Big) \to \bigoplus_{(\mathbf{Z}/p^\beta)^\times} \overline{\mathbf{Q}}_p \to \overline{\mathbf{Q}}_p, \text{where:}$$

- the first map is a twisted pullback under the natural inclusion $X_\beta \to S_K$,
- the second is projection of the coefficients via the branching law (†),
- the third is integration of scalar-valued classes (of degree t) over the connected components of X_β (which have dimension t),
- and the last map is 'evaluation at χ', where χ has conductor p^β.

To interpolate these maps, in [BDW] the authors gave a p-adic interpolation of (†). There were two aspects to this: the interpolation for a fixed λ as j varies, proved in §5.2 *op. cit.*, used to prove existence of $L_p(\tilde{\pi})$; and the interpolation as λ varies in a (2-dimensional) Q-parabolic weight family $\mathscr{W}^Q$ in §6.2, used to construct parabolic families of RASCARs and 3-variable p-adic L-functions for these families.

The full pure weight space $\mathscr{W}^G_0$ for G has dimension $n+1$, whilst the families of [BDW] are only 2-dimensional. The trade-off made *op. cit.* was that variation in lower dimension allowed weaker assumptions on $\tilde{\pi}$, giving an 'optimal' notion of non-criticality for $\tilde{\pi}$. However, if one assumes stronger conditions on $\tilde{\pi}$, then one expects to be able to vary the p-adic L-function over the full $(n+1)$-dimensional pure weight space; and it is this, higher-dimensional, variation that is required for the application to the Bloch–Kato conjecture of [LZb].

To extend the results of [BDW] to get full variation, one needs to interpolate the branching law (†) as λ varies over $\mathscr{W}^G_0$. The approach of [BDW] has the parahoric, hence 2-dimensional, setting baked into it, so to interpolate in higher dimension requires new ideas.

In Proposition 11.13, we give a full interpolation of (†) over $\mathscr{W}^G_0$ (in the language of p-adic distributions). This result occupies the entirety of §11.

Our approach exploits properties of *spherical varieties*: in particular, we use crucially the existence of an 'open orbit element' $u \in G(\mathbf{Z}_p)$ such that $\overline{B}(\mathbf{Z}_p)u^{-1}H(\mathbf{Z}_p)$ is Zariski-open in $G(\mathbf{Z}_p)$ (for $\overline{B} \subset G$ the opposite Borel of lower triangular matrices). The approach should apply to much more general spherical pairs (G,H); this has now been studied, for example, in [Roc].

In §12, via §4, we use our p-adic branching laws to construct $(n+2)$-variabled evaluation maps

$$\mathrm{Ev}_\beta^\Omega : \mathrm{H}_c^t(S_K, \mathscr{D}_\Omega) \longrightarrow \mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega), \tag{1.2}$$

for an $(n+1)$-dimensional affinoid $\Omega \subset \mathscr{W}_0^G$. These maps interpolate (Iwahoric analogues of) $\mathrm{Ev}_{\chi,j}^{\lambda,\mathrm{DJR}}$ as λ varies in Ω, j varies over $\mathrm{Crit}(\lambda)$, and χ varies over finite order Hecke characters of conductor p^β.

1.4 Main constructions

We give two main applications of these results. Using the Shalika refinements of Definition 7.1, we give simple automorphic criteria for the non-vanishing of the evaluation maps (1.2). This non-vanishing puts tight restrictions on the structure of $\mathrm{H}_c^t(S_K, \mathscr{D}_\Omega)$, and thus – via [Han17] – on the structure of the G-eigenvariety $\mathscr{E}^G$. Exploiting ideas developed in [BDW], in Theorem 13.6 we prove the following. Recall that a RASCAR is a regular algebraic cuspidal automorphic representation that is symplectic (i.e. it admits a Shalika model).

Theorem A. *Let π be a RASCAR of $G(\mathbf{A})$ of weight $\lambda = (\lambda_1, \ldots, \lambda_{2n})$ that is everywhere spherical. Suppose that π_p is regular and that*

$$\lambda_n > \lambda_{n+1}.$$

Let $\tilde{\pi} = (\pi, \alpha)$ be spin p-refinement having non-critical slope (see Definitions 6.2 and 10.6).

Then the G-eigenvariety $\mathscr{E}^G$ is étale over $\mathscr{W}_0^G$ at $\tilde{\pi}$. There exists a neighbourhood of $\tilde{\pi}$ in $\mathscr{E}^G$ containing a very Zariski-dense set of classical points corresponding to RASCARs.

In other words, there exists an $(n+1)$-dimensional affinoid $\Omega \subset \mathscr{W}_0^G$ containing λ such that:

(a) $\tilde{\pi}$ varies in a unique p-adic family $\mathscr{C} \subset \mathscr{E}^G$ over Ω,

(b) $\mathscr{C}$ contains a very Zariski-dense set $\mathscr{C}^{\mathrm{Sha}}$ of classical points corresponding to RASCARs,

(c) and the weight map $w : \mathscr{C} \xrightarrow{\sim} \Omega$ is an isomorphism.

To prove this result, we observe that the weight condition $\lambda_n > \lambda_{n+1}$ implies existence of a non-vanishing Deligne-critical L-value. Since Ev_β^Ω interpolates these L-values, it is therefore non-vanishing. We use non-vanishing of Ev_β^Ω twice: once to produce existence of an $(n+1)$-dimensional family, and again to prove existence of a very Zariski-dense set of classical points attached to RASCARs.

Our second main result, under the same hypotheses, is the construction of an $(n+2)$-variable p-adic L-function over $\mathscr{C}$. We show that $\mathrm{Ev}_{\beta+1}^\Omega = \mathrm{Ev}_\beta^\Omega \circ U_p^\circ$, where U_p° is the full (normalised) Iwahori Hecke operator at p. We thus use (1.2) to attach a well-defined distribution $\mu^\Omega(\Phi) := (\alpha_p^\circ)^{-\beta}\mathrm{Ev}_\beta^\Omega(\Phi)$ to any finite-slope eigenclass $\Phi \in \mathrm{H}_c^t(S_K, \mathscr{D}_\Omega)$ with $U_p^\circ\Phi = \alpha_p^\circ\Phi$. Note this is independent of β. We show existence of a distinguished eigenclass $\Phi_\mathscr{C} \in \mathrm{H}_c^t(S_K, \mathscr{D}_\Omega)$ attached to the family $\mathscr{C}$, and then define $\mathcal{L}_p^\mathscr{C} := \mu^\Omega(\Phi_\mathscr{C}) \in \mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$. Under the Amice transform, we view $\mathcal{L}_p^\mathscr{C}$ as a rigid analytic function $\mathcal{L}_p^\mathscr{C}$ on $\mathscr{C} \times \mathscr{X}(\mathbf{Z}_p^\times)$, where $\mathscr{X}(\mathbf{Z}_p^\times)$ is the $\mathbf{Q}_p$-rigid space of characters on $\mathbf{Z}_p^\times$. In Theorem 13.8 we show:

Theorem B. *Let $\tilde{\pi}$ satisfy the hypotheses of Theorem A, and let $\mathscr{C} \subset \mathscr{E}^G$ be the (unique) corresponding p-adic family through $\tilde{\pi}$. There exists a rigid analytic function*

$$\mathcal{L}_p^{\mathscr{C}} : \mathscr{C} \times \mathscr{X}(\mathbf{Z}_p^\times) \longrightarrow \overline{\mathbf{Q}}_p$$

satisfying the following interpolation property: for all $y \in \mathscr{C}^{\mathrm{Sha}} \subset \mathscr{C}$, there exist $c_y^\pm \in \overline{\mathbf{Q}}_p^\times$ such that for all characters $\xi \in \mathscr{X}(\mathbf{Z}_p^\times)$ with $\xi(-1) = \pm 1$, we have

$$\mathcal{L}_p^{\mathscr{C}}(y, \xi) = c_y^\pm \cdot L_p(\tilde{\pi}_y, \xi). \tag{1.3}$$

Here $L_p(\tilde{\pi}_y, -)$ is the (1-variable) p-adic L-function from [BDW].

One could restate this interpolation property in terms of critical L-values as follows: for all $y \in \mathscr{C}^{\mathrm{Sha}}$ of weight $\lambda_y = w(y)$, all integers j with $-\lambda_{y,n+1} \geqslant j \geqslant -\lambda_{y,n}$, and all finite order characters χ of $\mathbf{Q}^\times \backslash \mathbf{A}^\times$ of p-power conductor (including trivial conductor), we have

$$\mathcal{L}_p^{\mathscr{C}}(y, \chi\chi_{\mathrm{cyc}}^j) = c_y^\pm \cdot \gamma_{(pm)} \cdot e_p(\tilde{\pi}_y, \chi, j) \cdot e_\infty(\pi_y, \chi, j) \cdot \frac{L^{(p)}\left(\pi_y \otimes \chi, j + \frac{1}{2}\right)}{\Omega_{\pi_y}^\pm}.$$

Here χ_{cyc} is the cyclotomic character, the sign is determined by $\chi(-1)(-1)^j = \pm 1$, $\gamma_{(pm)}$ is a volume constant, e_p and e_∞ are the Coates–Perrin-Riou factors at p and ∞, $L^{(p)}$ is the L-function with the local factor at p removed, and $\Omega_{\pi_y}^\pm$ are periods. All of this is explained in Theorem 12.13.

1.5 Application: Bloch–Kato for $\mathrm{GSp}(4)$

In [LZb], Loeffler and Zerbes prove new cases of the Bloch–Kato conjecture for Galois representations attached to Siegel modular forms of genus 2 (i.e. for automorphic representations of GSp_4). More precisely, if $\mathcal{F}_{\mathrm{new}}$ is a Siegel modular form of level 1 and sufficiently high weight, and $\mathcal{F}$ is an ordinary p-stabilisation, they prove the Bloch–Kato conjecture holds for the 4-dimensional spin Galois representation attached to $\mathcal{F}$ in analytic rank 0. This has also led to new understanding of the Bloch–Kato conjecture for symmetric cube modular forms [LZ23] and of Iwasawa theory for quadratic Hilbert modular forms [LZa].

In [LZb], the authors built on previous joint works with Skinner and Pilloni [LSZ22, LPSZ21] constructing Euler systems and p-adic L-functions for GSp_4. For applications to Bloch–Kato in analytic rank 0, one wants to show the Euler system of [LSZ22] is non-trivial. The main new input in [LZb] was an explicit reciprocity law relating the Euler system of [LSZ22] to a specific value of the p-adic L-function of [LPSZ21]. If this p-adic L-value does not vanish, then the Euler system is non-trivial and can be used to bound a Selmer group.

This non-vanishing is delicate, since the p-adic L-value seen by the explicit reciprocity law is *outside* the region of interpolation (so it does not directly relate to a Deligne-critical L-value). In [LZb, §17], Loeffler–Zerbes deform this into the region of interpolation – and thus prove the Bloch–Kato conjecture – conditional on the existence of a family of p-adic L-functions on GL_4, stated as Theorem 17.6.2 *op. cit*. This theorem, whose proof was deferred to 'forthcoming work' of the present authors, is a special case of Theorem B.

1.6 Remarks on assumptions

In the above, and the main body of the paper, we restrict to base field $\mathbf{Q}$ and π of tame level 1, a setting where all of our key new ideas are already present. These assumptions drastically simplify the notation and reduce technicality, allowing for a shorter, more conceptual article, whilst still including the results required by [LZb]. We indicate which of our various assumptions could be relaxed.

Firstly, all of these results can be modified in a conceptually straightforward, but notationally intricate, way to work for GL_{2n} over an arbitrary totally real field F. This was the setting treated in [BDW]; the reader could consult that paper for the extra details occurring in this case.

A second minor assumption is the weight condition that $\lambda_n > \lambda_{n+1}$. This rules out, for example, weight 2 modular forms in the case of GL_2. This condition is used in exactly one place: to guarantee existence of a non-central, and hence non-vanishing, Deligne-critical L-values. It could be replaced by instead assuming the existence of finite order Hecke characters $\chi^{\pm}$ of p-power conductor, with $\chi_\infty^{\pm}(-1)(-1)^{\lambda_n} = \pm 1$, such that $L\left(\pi \otimes \chi^{\pm}, \frac{1}{2} - \lambda_n\right) \neq 0$ (see [BDW, Def. 7.3]). One would expect this to always be true, and indeed it is known for weight 2 modular forms thanks to Rohrlich [Roh89].

Our most serious assumption is that π has tame level 1 (is everywhere spherical). We impose this for our global applications (Theorems A and B), as there is a subtlety in choosing local test vectors in Shalika models when π_ℓ is ramified. More precisely:

- One knows there exists a test vector in π_ℓ whose Friedberg–Jacquet local zeta integral computes exactly the local L-factor for π_ℓ. This test vector is smooth for *some* open compact subgroup $K_\ell \subset \mathrm{GL}_{2n}(\mathbf{Q}_\ell)$; but K_ℓ is inexplicit for ramified π_ℓ. To relate our evaluation maps to L-values (in order to show their non-vanishing) requires working at level $K = \mathrm{Iw}_G \prod_{\ell \neq p} K_\ell$.
- Thanks to work of [JPSS81], variation in families, however, works best at *new* level
$$K_1(\tilde{\pi}) = \mathrm{Iw}_G \prod_{\ell \neq p} K_{1,\ell}(m(\pi_\ell)),$$
where $K_{1,\ell}(m) \subset \mathrm{GL}_{2n}(\mathbf{Z}_\ell)$ is the open compact subgroup of all matrices whose bottom row is congruent to $(0,\dots,0,1) \pmod{\ell^m}$, and $m(\pi_\ell)$ is the minimal $m \in \mathbf{Z}_{\geqslant 0}$ with $\pi_\ell^{K_{1,\ell}(m)} \neq \{0\}$. Crucially, [JPSS81] showed that $\pi_\ell^{K_{1,\ell}(m(\pi_\ell))}$ is a line.

This leads to tensions between working at level K, where we see L-values, and level $K_1(\tilde{\pi})$, suitable for proving strong results about families. In tame level 1, we have $K = K_1(\tilde{\pi})$ and there is no problem. For simplicity – and since this issue is somewhat perpendicular to the main novelties of the present paper – we have presented all our proofs for tame level 1.

We briefly indicate how to drop this assumption in proving Theorem A, following the strategy employed in [BDW, §7], in the parahoric setting. There it was first shown that there exists a p-adic family of the correct dimension at level K, containing a very Zariski-dense set of classical points corresponding to RASCARs. Then, by studying conductors and the Local Langlands Correspondence, a delicate level-switching operation was carried out that transferred this family from level K to level $K_1(\tilde{\pi})$, where strong results can be deduced about its geometry. Since nothing about this argument used the parahoric setting, it applies equally well to our Iwahoric case, and the same methods prove the following generalisation of Theorem A (the base field still being $\mathbf{Q}$):

Theorem A′. *Let π be a RASCAR of $G(\mathbf{A})$ that is spherical and regular at p. Suppose that $\lambda_n > \lambda_{n+1}$, where π has weight $(\lambda_1,\dots,\lambda_{2n})$. Let $\tilde{\pi} = (\pi,\alpha)$ be a non-critical slope spin p-refinement of π. Then the level $K_1(\tilde{\pi})$ eigenvariety $\mathscr{E}^G(K_1(\tilde{\pi}))$ for G is étale over $\mathscr{W}_0^G$ at $\tilde{\pi}$. There exists a neighbourhood of $\tilde{\pi}$ in $\mathscr{E}^G(K_1(\tilde{\pi}))$ containing a very Zariski-dense set of classical points corresponding to RASCARs.*

However, to prove Theorem B requires much more control: we need not only that systems of Hecke eigenvalues vary p-adically, but also that we can vary all the local test vectors over the family. This is unconditionally possible in tame level 1. This could be relaxed to the assumption that π admits certain parahoric-fixed vectors at every finite place using the forthcoming work [DJ]. For a discussion on possible generalisations beyond this, see [BDW, §8].

1.7 Structure of the paper

This paper falls into three parts.

In Part I (§2–4), we fix notation and recall relevant automorphic results. In §4, we generalise the abstract construction of evaluation maps from [BDW, §4], showing that these evaluation maps compute classical L-values of RASCARs.

In Part II (§5–§9), we develop the theory of spin and Shalika refinements, and compute local zeta integrals (in two different ways). In these sections we prove all the results described in §1.3.1. Our Iwahoric results are summarised in §8.

In Part III (§10–§13), we build our p-adic machine on this automorphic foundation, reinterpreting the above in the context of overconvergent cohomology. The heart of is §11-12, where we give our main technical results on p-adic interpolation of branching laws. In §13, we obtain our main arithmetic applications, following strategies developed in [BDW].

Part I
Automorphic Results

2 Set-up and notation

2.1 Notation

We will use the following notation for almost all of this paper. In Sections 5, 6, 7 and 9, however, we will use purely local analogues of this notation, as fixed in the introduction to Part II.

Let $n \geqslant 1$ and let $G := \mathrm{GL}_{2n}$. We write B for the Borel subgroup of upper triangular matrices, $\overline{B}$ for the opposite Borel of lower triangular matrices and T for the maximal split torus of diagonal matrices. We have decompositions $B = TN$ and $\overline{B} = \overline{N}T$ where $N = N_{2n}$ and $\overline{N} = \overline{N}_{2n}$ are the unipotent radicals of B and $\overline{B}$. We also let $G_n = \mathrm{GL}_n$, with B_n, T_n, N_n etc. the analogous subgroups. Let $H := \mathrm{GL}_n \times \mathrm{GL}_n$, with an embedding $\iota : H \hookrightarrow G$, $\iota(h_1,h_2) = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$.

Let $\mathcal{W}_G = \mathrm{S}_{2n}$ (resp. $\mathcal{W}_n = \mathrm{S}_n$) be the Weyl group of G (resp. G_n), identified with the permutation subgroup of $G(\mathbf{Z})$ (resp. $G_n(\mathbf{Z})$). We write w_{2n} and w_n for the longest Weyl elements (i.e. the antidiagonal matrices with 1s on the antidiagonal).

Let $K_\infty = C_\infty Z_\infty$, where Z_∞ is the center and C_∞ is the maximal compact subgroup $O_{2n}(\mathbf{R}) \leqslant G(\mathbf{R})$. If A is a reductive real Lie group, then A° denotes the connected component of the identity.

Fix a rational prime p and an embedding $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. We fix a (non-canonical) extension of i_p to an isomorphism $i_p : \mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_p$. Let $\mathbf{Q}^{p\infty}$ be the maximal abelian extension of $\mathbf{Q}$ unramified outside $p\infty$, and let $\mathrm{Gal}_p := \mathrm{Gal}(\mathbf{Q}^{p\infty}/\mathbf{Q}) \cong \mathbf{Z}_p^\times$ be its Galois group.

If π is a regular algebraic cuspidal automorphic representation (RACAR) of $G(\mathbf{A})$, then we write $L(\pi, s)$ for its standard L-function.

Throughout, we work in 'tame level 1', that is, with the open compact level subgroup

$$K = \mathrm{Iw}_G \cdot \prod_{\ell \neq p} G(\mathbf{Z}_\ell) \subset G(\mathbf{A}_f), \tag{2.1}$$

so away from p we take maximal hyperspecial level and at p we take Iwahori level

$$\mathrm{Iw}_G := \{g \in G(\mathbf{Z}_p) : g \ (\mathrm{mod}\ p) \in B(\mathbf{F}_p)\} \subset G(\mathbf{Z}_p). \tag{2.2}$$

Let $\delta_B : T(\mathbf{A}) \to \mathbf{C}^\times$ be the standard modulus character

$$t = (t_1, ..., t_{2n}) \mapsto |t_1|^{2n-1}|t_2|^{2n-3} \cdots |t_{2n-1}|^{3-2n}|t_{2n}|^{1-2n}. \tag{2.3}$$

We repeatedly use that if π_ℓ is the local component at ℓ of a RACAR π, and $\pi_\ell^{G(\mathbf{Z}_\ell)} \neq 0$ (i.e. π_ℓ is spherical), then π_ℓ is a generic unramified principal series representation. By this, we mean there exists an unramified character $\theta : T(\mathbf{Q}_\ell) \to \mathbf{C}^\times$ such that

$$\pi_\ell = \mathrm{Ind}_B^G \theta \tag{2.4}$$

is the normalised parabolic induction[1] of θ to G (that is, the true induction of $\delta_B^{1/2}\theta$).

All our group actions will be on the left. If M is a R-module, with a left action of a group Γ, then we write $M^\vee = \mathrm{Hom}_R(M, R)$, with associated left dual action $(\gamma \cdot \mu)(m) = \mu(\gamma^{-1} \cdot m)$.

In later sections we work extensively with affinoid rigid spaces. For such a space X, we write $\mathcal{O}_X$ for the ring of rigid functions on X, so $X = \mathrm{Sp}(\mathcal{O}_X)$.

All Hecke characters in this paper are understood to be idelic, i.e. characters of $\mathbf{Q}^\times \backslash \mathbf{A}^\times$. Finite order Hecke characters correspond bijectively with Dirichlet characters.

2.2 Algebraic weights

Let $X^*(T)$ be the set of algebraic characters of T. Each element of $X^*(T)$ corresponds to an integral weight $\lambda = (\lambda_1, ..., \lambda_{2n}) \in \mathbf{Z}^{2n}$. If $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{2n}$, we say λ is *dominant*, and write $X_+^*(T) \subset X^*(T)$

[1] To see this, note by the Bernstein–Zelevinsky classification that any generic irreducible π_ℓ is isomorphic to an induction of a tensor product of unlinked segments (see [Zel80, Thm. 9.7(b)], stated in this form in [GH24, Thm. 8.4.4]). If π_ℓ is unramified, these segments are all unramified, and hence all are characters by [Mat13, Cor. 1.3]. But this says exactly that $\pi_\ell \cong \mathrm{Ind}_B^G \theta$ for some unramified character θ of $T(\mathbf{Q}_\ell)$.

for the subset of dominant weights. We say that λ is *pure* if there exists $\mathsf{w} \in \mathbf{Z}$, the *purity weight* of λ, such that $\lambda_i + \lambda_{2n-i+1} = \mathsf{w}$ for all $1 \leqslant i \leqslant n$; we write $\mathrm{pure}(\lambda) := \mathsf{w}$. We write $X_0^*(T) \subset X_+^*(T)$ for the subset of pure B-dominant integral weights, which are exactly those supporting cuspidal cohomology [Clo90, Lem. 4.9].

For $\lambda \in X_+^*(T)$, let V_λ be the algebraic irreducible representation of G of highest weight λ, and let $V_\lambda^\vee$ denote its linear dual, with its (left) dual action. We have an isomorphism $V_\lambda^\vee \cong V_{\lambda^\vee}$ where $\lambda^\vee = (-\lambda_{2n}, \dots, -\lambda_1)$. Given a pure dominant algebraic weight $\lambda \in X_0^*(T)$, let

$$\mathrm{Crit}(\lambda) := \{ j \in \mathbf{Z} : -\lambda_n \leqslant j \leqslant -\lambda_{n+1} \}. \tag{2.5}$$

If π is a RACAR for $G(\mathbf{A})$ of weight λ (which we take to mean cohomological with respect to $V_\lambda^\vee$), then $j \in \mathrm{Crit}(\lambda)$ if and only if the L-value $L(\pi, j + \frac{1}{2})$ is Deligne-critical (see [GR14, §6.1]).

2.3 Shalika models

Our main results come in the setting of RACARs that admit *Shalika models*. We recall relevant definitions and properties (see e.g. [GR14, §1,§3.1], [BDW, §2.6]).

2.3.1 Definition of Shalika models

Let $\mathcal{S} = \{ s = \left(\begin{smallmatrix} h & \\ & h \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1_n & X \\ & 1_n \end{smallmatrix}\right) : h \in \mathrm{GL}_n, X \in \mathrm{M}_n \}$ be the Shalika subgroup of GL_{2n}. Let ψ be the standard non-trivial additive character of $\mathbf{Q}\backslash\mathbf{A}$ from [DJR20, §4.1], and let η be a character of $\mathbf{Q}^\times\backslash\mathbf{A}^\times$. Let $\eta \otimes \psi$ be the character of $\mathcal{S}(\mathbf{A})$ given by $(\eta \otimes \psi)(s) = \eta(\det(h))\psi(\mathrm{tr}(X))$.

A cuspidal automorphic representation π of $G(\mathbf{A})$ is said to have an (η, ψ)-*Shalika model* if there exist $\varphi \in \pi$ and $g \in G(\mathbf{A})$ such that, letting Z_G denote the center of G, one has

$$\mathcal{S}_\psi^\eta(\varphi)(g) := \int_{Z_G(\mathbf{A})\mathcal{S}(\mathbf{Q})\backslash\mathcal{S}(\mathbf{A})} \varphi(sg)\ (\eta \otimes \psi)^{-1}(s) ds \neq 0. \tag{2.6}$$

If this non-vanishing holds, then:

- η^n is the central character of π,
- η has the form $\eta_0 |\cdot|^{\mathsf{w}}$, where η_0 has finite order and $\mathsf{w} = \mathrm{pure}(\lambda)$ is the purity weight of the weight λ of π,
- and $\mathcal{S}_\psi^\eta$ defines an intertwining $\pi \hookrightarrow \mathrm{Ind}_{\mathcal{S}(\mathbf{A})}^{G(\mathbf{A})}(\eta \otimes \psi)$.

If π has an (η, ψ)-Shalika model, then for each prime ℓ the local component π_ℓ has a local (η_ℓ, ψ_ℓ)-Shalika model [GR14, §3.2], that is, we have (non-canonical) intertwinings

$$\mathcal{S}_{\psi_\ell}^{\eta_\ell} : \pi_\ell \hookrightarrow \mathrm{Ind}_{\mathcal{S}(\mathbf{Q}_\ell)}^{\mathrm{GL}_{2n}(\mathbf{Q}_\ell)}(\eta_\ell \otimes \psi_\ell). \tag{2.7}$$

We fix a choice of intertwining $\mathcal{S}_{\psi_f}^{\eta_f}$ of π_f (or equivalently, via (2.6), an intertwining $\mathcal{S}_{\psi_\infty}^{\eta_\infty}$ of π_∞).

By [AG94, Prop. 1.3], if π_ℓ is spherical then it admits a (η_ℓ, ψ_ℓ)-Shalika model if and only if $\pi_\ell^\vee = \pi_\ell \otimes \eta_\ell^{-1}$, in which case η_ℓ is unramified.

We recall another characterisation of Shalika models:

Theorem 2.1 (Jacquet–Shalika, Asgari–Shahidi)**.** *Let π be a cuspidal automorphic representation of $G(\mathbf{A})$. Then π admits a (η,ψ)-Shalika model for some character η if and only if π is the Langlands functorial transfer of a globally generic cuspidal automorphic representation Π of $\mathrm{GSpin}_{2n+1}(\mathbf{A})$.*

Proof. By [JS90], having a global Shalika model is equivalent to a (partial) exterior square L-function having a pole at $s=1$. But in [AS06, AS14] this is shown to be equivalent to being such a functorial transfer. (For further details see [GR14, Prop. 3.1.4]). ☐

2.3.2 Friedberg–Jacquet integrals

Let π be a cuspidal automorphic representation of $G(\mathbf{A})$, and χ a finite order Hecke character for F. For $W \in \mathcal{S}_\psi^\eta(\pi)$ consider the *Friedberg–Jacquet* zeta integral

$$\zeta(s,W,\chi) := \int_{\mathrm{GL}_n(\mathbf{A})} W\left[\begin{pmatrix} h & \\ & I_n\end{pmatrix}\right] \chi(\det(h))\,|\det(h)|^{s-\frac{1}{2}}\,dh,$$

converging absolutely in a right-half plane and extending to a meromorphic function in $s\in\mathbf{C}$. When $W=\bigotimes_{\ell\leqslant\infty} W_\ell$ for $W_\ell \in \mathcal{S}_{\psi_\ell}^{\eta_\ell}(\pi_\ell)$, this decomposes into a product of local zeta integrals $\zeta_\ell(s,W_\ell,\chi_\ell)$. Suppose π is a RACAR admitting a (η,ψ)-Shalika model. If π_ℓ is spherical, then $\pi_\ell^{G(\mathbf{Z}_\ell)}$ is a line; let $W_\ell^\circ \in \mathcal{S}_{\psi_\ell}^{\eta_\ell}(\pi_\ell^{G(\mathbf{Z}_\ell)})$ be the spherical test vector normalised so that $W_\ell^\circ(1_{2n})=1$. Then by [FJ93, Props. 3.1, 3.2] W_ℓ° is a *Friedberg–Jacquet test vector*, i.e. for all unramified quasi-characters $\chi_\ell : \mathbf{Q}_\ell^\times \to \mathbf{C}^\times$ we have

$$\zeta_\ell\left(s+\tfrac{1}{2}, W_\ell^\circ,\chi_\ell\right) = L\left(\pi_\ell\otimes\chi_\ell, s+\tfrac{1}{2}\right). \tag{2.8}$$

We apply this to choose local test vectors at all finite $\ell \neq p$.

2.4 The Hecke algebra and p-refinements

Recall we took $K=\mathrm{Iw}_G\cdot\prod_{\ell\neq p} G(\mathbf{Z}_\ell)$.

2.4.1 Away from p

If $\nu\in X_*^+(T)$ is a dominant cocharacter, and $\ell\neq p$ a prime, define $T_{\nu,\ell} := [G(\mathbf{Z}_\ell)\cdot\nu(\ell)\cdot G(\mathbf{Z}_\ell)]$, a double coset operator.

Definition 2.2. The *spherical Hecke algebra* is the commutative algebra $\mathcal{H}' := \mathbf{Z}[T_{\nu,\ell} : \nu\in X_*^+(T), \ell\neq p]$ generated by all such operators.

If π is a RACAR with $\pi^K\neq 0$, then π_ℓ is spherical, and $\pi_\ell^{G(\mathbf{Z}_\ell)}$ is a line preserved by each $T_{\nu,\ell}$. Let E be a number field containing the Hecke field of π_f (which exists by [Clo90, Thm. 3.13]). Attached to π there is a homomorphism $\psi_\pi : \mathcal{H}'\otimes E\to E$ sending $T_{\nu,\ell}$ to its eigenvalue on $\pi_\ell^{G(\mathbf{Z}_\ell)}$. We define $\mathfrak{m}_\pi := \ker(\psi_\pi)$, a maximal ideal in $\mathcal{H}'\otimes E$. If M is a module on which $\mathcal{H}'\otimes E$ acts, we write M_π for its localisation at $\mathfrak{m}_\pi$.

2.4.2 The Hecke algebra at p

For $i = 1, ..., 2n-1$ define matrices $t_{p,r} \in T(\mathbf{Q}_p)$ by

$$t_{p,1} = \mathrm{diag}(p, 1, ..., 1),\ \ t_{p,2} = \mathrm{diag}(p, p, 1, ..., 1),\ \ ...,\ t_{p,2n-1} = \mathrm{diag}(p, ..., p, 1), \tag{2.9}$$

and let

$$t_p = t_{p,1} \cdots t_{p,2n-1} = \mathrm{diag}(p^{2n-1}, p^{2n-2}, ..., p, 1) \in T(\mathbf{Q}_p). \tag{2.10}$$

Then define operators $U_{p,r} = [\mathrm{Iw}_G \cdot t_{p,r} \cdot \mathrm{Iw}_G]$ and $U_p = [\mathrm{Iw}_G \cdot t_p \cdot \mathrm{Iw}_G] = U_{p,1} \cdots U_{p,2n-1}$.

Definition 2.3. The *Hecke algebra at p* is $\mathcal{H}_p := \mathbf{Z}[U_{p,r} : r = 1, ..., 2n-1]$, and the *full Hecke algebra* is $\mathcal{H} = \mathcal{H}' \otimes \mathcal{H}_p$.

2.4.3 p-refinements

Suppose π_p is spherical. In particular $\pi_p^{G(\mathbf{Z}_p)} \neq 0$, hence $\pi_p^{\mathrm{Iw}_G} \neq 0$.

Definition 2.4. A *p-refinement* of π_p is a pair $\tilde{\pi}_p = (\pi_p, \alpha)$, where $\alpha : \mathcal{H}_p \to \overline{\mathbf{Q}}$ is a system of $\mathcal{H}_p$-eigenvalues appearing in $\pi_p^{\mathrm{Iw}_G}$; i.e. if we set $\alpha_{p,r} = \alpha(U_{p,r})$, then there is an eigenvector $\varphi_p \in \pi_p^{\mathrm{Iw}_G}$ with $U_{p,r}\varphi_p = \alpha_{p,r}\varphi_p$ for each r. Such a $\tilde{\pi}_p$ is *regular* if the attached generalised eigenspace

$$\pi_p^{\mathrm{Iw}_G}[\![U_{p,r} - \alpha_{p,r} : r = 1, ..., 2n-1]\!] := \left\{ \varphi_p \in \pi_p^{\mathrm{Iw}_G} : \begin{array}{c} \exists\, m \geqslant 1 \text{ such that} \\ (U_{p,r} - \alpha_{p,r})^m \varphi_p = 0 \\ \text{for all } r = 1, \ldots, 2n-1 \end{array} \right\} \tag{2.11}$$

is 1-dimensional over $\mathbf{C}$. We will write $\varphi_p \in \tilde{\pi}_p$ as shorthand for $\varphi_p \in \pi_p^{\mathrm{Iw}_G}[\![U_{p,r} - \alpha_{p,r} : r = 1, ..., 2n-1]\!]$.

Let $\tilde{\pi}$ be a p-refinement. After possibly extending E, we extend ψ_π to a homomorphism

$$\psi_{\tilde{\pi}} : \mathcal{H} \otimes E \longrightarrow E, \quad U_{p,r} \mapsto \alpha_{p,r}. \tag{2.12}$$

We let $\mathfrak{m}_{\tilde{\pi}} := \ker(\psi_{\tilde{\pi}})$ be the corresponding maximal ideal of $\mathcal{H} \otimes E$. If M is a module upon which $\mathcal{H} \otimes E$ acts, we write $M_{\tilde{\pi}}$ for the localisation of M at $\mathfrak{m}_{\tilde{\pi}}$. Note that if L is a field containing E and M is a finite-dimensional L-vector space, then $M_{\tilde{\pi}}$ is the generalised eigenspace for $\mathcal{H} \otimes L$ corresponding to $\psi_{\tilde{\pi}}$.

Let $\pi_p = \mathrm{Ind}_B^G\, \theta$ be a generic unramified principal series representation (see (2.4)). We recall the standard classification of p-refinements for π_p. Recall $\mathcal{W}_G = \mathrm{S}_{2n}$ and δ_B from §2.1.

Proposition 2.5. [Che04, Lem. 4.8.4].

(i) The semisimplification of $\pi_p^{\mathrm{Iw}_G}$ as a $\mathcal{H}_p$-module is isomorphic to $\bigoplus_{\sigma \in \mathcal{W}_G} (\delta_B^{1/2}\theta^\sigma) \circ \mathrm{ev}_p$, where $\mathrm{ev}_p : \mathcal{H}_p \to T(\mathbf{Q}_p)$ is the map sending $U_{p,r} \mapsto t_{p,r}$. Thus if $\tilde{\pi}_p = (\pi_p, \alpha)$ is a p-refinement, then there exists $\sigma \in \mathcal{W}_G$ such that for each r,

$$\alpha_{p,r} = \alpha(U_{p,r}) = \left[\delta_B^{1/2}\theta^\sigma\right]^{w_{2n}}(t_{p,r}) = \prod_{j=1}^{r} p^{\frac{2n-2j+1}{2}} \theta_{\sigma(2n+1-j)}(p).$$

(ii) There are equivalences

$$\pi_p \text{ admits a regular } p\text{-refinement} \iff \text{its Satake parameter is regular}$$
$$\iff \text{every } p\text{-refinement of } \pi_p \text{ is regular}.$$

In this case, the choice of σ in (i) is unique, and via this correspondence there are exactly $(2n)!$ p-refinements of π_p, all of which are regular.

Remark 2.6. (i) This normalisation, where the character $\delta_B^{1/2}\theta^\sigma$ is conjugated by the Weyl group element w_{2n}, matches [DJR20] but might appear strange. Chenevier uses antidominant cocharacters, and switching to dominant characters is equivalent to conjugating by w_{2n}. This normalisation will be convenient in §7.

(ii) When the Satake parameter is regular, there is a bijection

$$\Delta_\theta : \{p\text{-refinements of } \pi_p\} \longrightarrow \mathcal{W}_G \tag{2.13}$$

induced by the above. This is *not* canonical, depending on the choice of character θ from which we induce. For $\tau \in \mathcal{W}_G$, replacing θ by θ^τ conjugates the image of Δ_θ by τ.

When π_p is the local component of a RACAR, the eigenvalues $\alpha_{p,r}$ are algebraic but not necessarily p-integral. To account for this, we make the following definition.

Definition 2.7. Let $\tilde{\pi}_p = (\pi_p, \alpha)$ be a p-refinement. Define *integral normalisations*

$$U_{p,r}^\circ := \lambda(t_{p,r})U_{p,r}, \qquad \alpha_{p,r}^\circ := \lambda(t_{p,r})\alpha_{p,r} = p^{\lambda_1+\cdots+\lambda_r}\alpha_{p,r}.$$

The $\alpha_{p,r}^\circ$ are p-integral (see Remark 10.4 or [BW21c, Rem. 3.23]).

3 Automorphic cohomology classes

For $K \subset G(\mathbf{A}_f)$ an open compact subgroup, the *locally symmetric space of level K* is

$$S_K = G(\mathbf{Q})\backslash G(\mathbf{A})/KK_\infty^\circ.$$

It is a $(2n-1)(n+1)$-dimensional real orbifold. We will now recall how to realise RACARs in the compactly supported Betti cohomology of S_K.

3.1 Local systems

We recall standard facts about local systems on S_K (e.g. [Urb11, §1], [BDW, §2.3]). If M is a left $G(\mathbf{Q})$-module such that $Z_G(\mathbf{Q}) \cap KK_\infty^\circ$ acts trivially, let $\mathcal{M}$ be the local system on S_K given by locally constant sections of $G(\mathbf{Q})\backslash[G(\mathbf{A}) \times M]/KK_\infty^\circ$, with action $\gamma(g,m)kz = (\gamma gkz, \gamma\cdot m)$. We denote such local systems with calligraphic letters.

If M is a left K-module, let $\mathscr{M}$ (with a script letter) be the local system on S_K given by locally constant sections of $G(\mathbf{Q})\backslash[G(\mathbf{A}) \times M]/KK_\infty^\circ$ with action $\gamma(g,m)kz = (\gamma gkz, k^{-1}\cdot m)$.

If M is a left $G(\mathbf{A})$-module, then it has actions of the subgroups $G(\mathbf{Q})$ and K, and there is an isomorphism $\mathcal{M} \xrightarrow{\sim} \mathscr{M}$ of associated local systems given by $(g,m) \mapsto (g, g_f^{-1}\cdot m)$. The key example of such M for this paper is $M = V_\lambda^\vee$, whence $\mathcal{V}_\lambda^\vee \xrightarrow{\sim} \mathscr{V}_\lambda^\vee$.

3.2 Hecke operators

Let $\gamma \in G(\mathbf{A}_f)$ and M be a left $G(\mathbf{Q})$-module (resp. K-module). We suppose γ acts on M. We have a natural projection map $p_{K,\gamma} : S_{\gamma K\gamma^{-1}\cap K} \to S_K$, and a double coset operator $[K\gamma K]$ on $\mathrm{H}^\bullet_c(S_K, \mathcal{M})$ (resp. $\mathrm{H}^\bullet_c(S_K, \mathscr{M})$) defined as the composition

$$[K\gamma K] := \mathrm{tr}(p_{K,\gamma}) \circ [\gamma] \circ p^*_{K,\gamma^{-1}}, \tag{3.1}$$

where tr is the trace and $[\gamma] : \mathrm{H}^\bullet_c(S_{K\cap\gamma^{-1}K\gamma}, \mathcal{M}) \to \mathrm{H}^\bullet_c(S_{\gamma K\gamma^{-1}\cap K}, \mathcal{M})$ is given on local systems by $(g,m) \mapsto (g\gamma^{-1}, \gamma \cdot m)$ (and similarly for $\mathscr{M}$).

3.2.1 Localisation at RACARs

Recall $K = \mathrm{Iw}_G \cdot \prod_{\ell \neq p} G(\mathbf{Z}_\ell)$ and $\mathcal{H}$ from §2.4. For appropriate M (e.g. $M = V_\lambda^\vee$), this acts on $\mathrm{H}^\bullet_c(S_K, \mathcal{M})$ and $\mathrm{H}^\bullet_c(S_K, \mathscr{M})$ via the process above. If π is a RACAR with $\pi^K \neq 0$, it therefore makes sense to localise $\mathrm{H}^\bullet_c(S_K, \mathcal{M})$ at $\mathfrak{m}_\pi$ as in §2.4.1. We denote the localisation by $\mathrm{H}^\bullet_c(S_K, -)_\pi$.

3.2.2 The action at infinity

We have $K_\infty/K_\infty^\circ = \{\pm 1\}$. This group has two characters $\varepsilon^\pm : K_\infty/K_\infty^\circ \to \{\pm 1\}$, where $\varepsilon^\pm$ sends -1 to ± 1. If M is a module on which K_∞/K_∞° acts and 2 acts invertibly – for example, the cohomology of S_K over a field of characteristic not 2 – then we have $M = M^+ \oplus M^-$, where $M^\pm$ are the eigenspaces where K_∞/K_∞° acts via $\varepsilon^\pm$. We obtain a (Hecke-equivariant) decomposition of the cohomology groups $\mathrm{H}^\bullet_c(S_K, -)$ into $\pm$-submodules (as the action of K_∞/K_∞° commutes with the $G(\mathbf{A}_f)$-action).

3.2.3 Integral normalisations

The module $V_\lambda^\vee$ comes equipped with the natural (algebraic) action of GL_{2n}, which we have been denoting with a $\cdot$. As we have already remarked, the resulting Hecke operators $U_{p,r} = U^\cdot_{p,r} = [K_p t_{p,r} K_p]$ on the cohomology of $\mathscr{V}_\lambda^\vee$ are not integrally normalised.

In §10.3, we will equip $V_\lambda^\vee(\overline{\mathbf{Q}}_p)$ with another natural action of $\mathrm{GL}_{2n}(\mathbf{Z}_p)$ and $t_{p,r}$, denoted $*$. Concretely, we will have $t_{p,r} * \mu = \lambda(t_{p,r})(t_{p,r} \cdot \mu)$. In light of Definition 2.7, if we let $U^*_{p,r}$ be the Hecke operator defined via (3.1) with the $*$-action instead of the $\cdot$-action, then $U^*_{p,r} = U^\circ_{p,r}$ is integrally normalised. This is all explained in detail in the remark at the end of [BDW, §3.3].

3.3 Cohomology classes attached to RACARs

Let $t = n^2 + n - 1$, which is the top degree of cohomology to which RACARs for $G(\mathbf{A})$ contribute. In particular, let π be a RACAR of $G(\mathbf{A})$; then we recall that there exists a Hecke-equivariant isomorphism

$$\pi_f^K \xrightarrow{\sim} \mathrm{H}^t_c(S_K, \mathscr{V}_\lambda^\vee(\overline{\mathbf{Q}}_p))^\pm_\pi, \tag{3.2}$$

for a unique $\lambda \in X_0^*(T)$. The isomorphism (3.2) is non-canonical, depending on our fixed choice of $i_p : \mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_p$ and a choice of basis $\Xi_\infty^\pm$ of the 1-dimensional $\mathbf{C}$-vector space $\mathrm{H}^t(\mathfrak{g}_\infty, K_\infty^\circ; \pi_\infty \otimes V_\lambda^\vee(\mathbf{C}))^\pm$, where $\mathfrak{g}_\infty := \mathrm{Lie}(G_\infty)$. This is all standard, explained e.g. in [BDW, §2.5].

Suppose π admits an (η,ψ)-Shalika model, and recall we chose an intertwining $\mathcal{S}_{\psi_f}^{\eta_f}:\pi_f\to\mathcal{S}_{\psi_f}^{\eta_f}(\pi_f)$. Combining with (3.2), we get a (non-canonical) Hecke-equivariant isomorphism

$$\Theta^{\pm}:\mathcal{S}_{\psi_f}^{\eta_f}(\pi_f^K)\xrightarrow{\sim}\mathrm{H}_{\mathrm{c}}^t(S_K,\mathscr{V}_\lambda^\vee(\overline{\mathbf{Q}}_p))_\pi^{\pm}. \tag{3.3}$$

Possibly enlarging the number field E, there is a natural E-rational subspace $\mathcal{S}_{\psi_f}^{\eta_f}(\pi_f^K,E)\subset\mathcal{S}_{\psi_f}^{\eta_f}(\pi_f^K)$. As in [GR14, Prop. 4.2.1] (cf. [BDW, §2.10]), there exist $\Omega_\pi^{\pm}\in\mathbf{C}^\times$ (canonical up to $E^\times$-multiple) and finite $L/\mathbf{Q}_p$ such that $\Theta^{\pm}/i_p(\Omega_\pi^{\pm})$ maps $\mathcal{S}_{\psi_f}^{\eta_f}(\pi_f^K,E)$ into $\mathrm{H}_{\mathrm{c}}^t(S_K,\mathscr{V}_\lambda^\vee(L))_\pi^{\pm}$. Moreover, for $\ell\neq p$ the spherical test vector W_ℓ° is E-rational.

4 Evaluation maps

Evaluation maps were crucial to the methods of [GR14, DJR20, BDW]. We give constructions of abstract evaluation maps, generalising [BDW] and [DJR20].

4.1 Automorphic cycles and abstract evaluation maps

In this section we generalise the abstract theory in [BDW, §4], where the evaluation maps were defined with respect to the parabolic Q with Levi H. These 'parahoric' evaluation maps can be interpolated over 2-dimensional parabolic subsets of weight space, but are not suitable for our goal of interpolation in $(n+1)$-weight variables. We now construct evaluation maps defined with respect to any standard parabolic $P\subset Q$. For the Iwahoric case, we are most interested in $P=B$. The proofs of [BDW, §4] go through almost identically with these modifications, so we are terse with details here.

Remark 4.1. Since the notation is heavy, we sketch the differences between our new definitions and those of [BDW]. Firstly, to better suit the more general theory, we replace the twisting operator $\xi=\left(\begin{smallmatrix}1_n & w_n\\ 0 & w_n\end{smallmatrix}\right)$ of [BDW, Def. 4.2] with u^{-1}, where

$$u=\begin{pmatrix}1_n & w_n\\ 0 & 1_n\end{pmatrix}\in G(\mathbf{Z}_p). \tag{4.1}$$

Unlike ξ, the element u^{-1} lies in Iw_G. We will show that the definitions/results of [BDW] are essentially unchanged with this switch.

In [DJR20, BDW], the evaluation maps for Q used the matrix $t_{p,n}=\mathrm{diag}(p,\dots,p,1,\dots,1)=:t_Q$ and operator $U_{p,n}^\circ=[K_pt_{p,n}K_p]=:U_Q^\circ$, a Q-controlling operator (in the sense of [BW21c, §2.5]). For a general parabolic P, we instead use a different matrix $t_P\in\mathrm{GL}_{2n}(\mathbf{Q}_p)$ (see Definition 4.2), giving a Hecke operator U_P° attached to P.

4.1.1 Automorphic cycles

Automorphic cycles are coverings of locally symmetric spaces for H that have real dimension equal to t, the top degree of cohomology to which RACARs for G contribute. This 'magical numerology' was exploited in [GR14, DJR20] to define classical evaluation maps and to give a cohomological interpretation of the Deligne-critical L-values of RASCARs.

Definition 4.2. Let $P \subset Q \subset \mathrm{GL}_{2n}$ be a standard parabolic with Levi $\mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_r}$. Define a block diagonal matrix

$$t_P = \mathrm{diag}(p^{r-1}\mathrm{I}_{m_1}, p^{r-2}\mathrm{I}_{m_2}, \cdots, p\mathrm{I}_{m_{r-1}}, \mathrm{I}_{m_r}),$$

where for an integer $m \geqslant 1$, we let I_m denote the $(m\times m)$ identity matrix. Note $t_P \in T_P^{++}$ as in [BW21c, §2.5]; and since $P \subset Q$, the first n diagonal entries are all a positive power of p.

For example, we have $t_Q = t_{p,n}$ (from (2.9)), and

$$t_B = \mathrm{diag}(p^{2n-1}, p^{2n-2}, ..., p, 1).$$

The most significant change in passing from the Q-evaluation maps (from [BDW]) to those here is the use of the matrix t_P rather than t_Q.

Define $J_P \subset \mathrm{GL}_{2n}(\mathbf{Z}_p)$ to be the parahoric subgroup for P.

Definition 4.3. Fix $m \in \mathbf{Z}_{\geqslant 0}$ prime to p, and let $K = K_pK^p \subset G(\mathbf{A}_f)$ be an open compact subgroup. We assume $N(\mathbf{Z}_p) \subset K_p \subset J_P$. For $\beta \in \mathbf{Z}_{\geqslant 0}$, define an open compact subgroup $L_\beta^P = L_p^{P,\beta}L^p \subset H(\mathbf{A}_f)$ by setting:

(i) $L_p^{P,\beta} := H(\mathbf{Z}_p) \cap K_p \cap (u^{-1}t_P^\beta)K_p(u^{-1}t_P^\beta)^{-1}$, and

(ii) $L^p := \{h \in H(\widehat{\mathbf{Z}}^{(p)}) : h \equiv 1 \pmod{m}\}$, the principal congruence subgroup of level m.

The *automorphic cycle of level* L_β^P is

$$X_\beta^P := H(\mathbf{Q})\backslash H(\mathbf{A})/L_\beta L_\infty^\circ,$$

where $L_\infty = H_\infty \cap K_\infty$ for $H_\infty = H(\mathbf{R})$. This is a real orbifold of dimension t [DJR20, (23)].

We will always take m to be the smallest positive integer such that $L^p \subset K^p \cap H(\mathbf{A}_f)$ and $H(\mathbf{Q}) \cap hL_\beta^P L_\infty^\circ h^{-1} = Z_G(\mathbf{Q}) \cap L_\beta^P L_\infty^\circ$ for all $h \in H(\mathbf{A})$ and for both $P = B, Q$ (compare [BDW, (4.1),(4.2)]). This means X_β^P is a real manifold [DJR20, (21)]. The impact of changing m is discussed in [BDW, §4.1].

Lemma 4.4. *We have*

$$\mathrm{vol}(L_p^{P,\beta}) = \delta_B(t_P^\beta)\cdot A_P,$$

where $A_P = \delta_B(t_P^{-1})\,\mathrm{vol}(L_p^{P,1})$ is a constant independent of β.

Proof. Let $N \subset G$ be the upper unipotent subgroup, and let $N^\beta := t_P^\beta N(\mathbf{Z}_p)t_P^{-\beta} \subset N(\mathbf{Z}_p)$. By [Loe21, Lem. 4.4.1], for $\beta \geqslant 1$ we have

$$[L_p^{P,\beta} : L_p^{P,\beta+1}] = [N^\beta : N^{\beta+1}] = \delta_B(t_P^{-1}),$$

the second equality being by definition of the modulus character δ_B. It follows that $\mathrm{vol}(L_p^{P,\beta}) = \delta_B(t_P^{\beta-1})\,\mathrm{vol}(L_p^{P,1})$, from which the result follows. □

Lemma 4.5. *If $(\ell_1, \ell_2) \in L_p^{P,\beta}$, then $\ell_2 \equiv w_n\ell_1w_n \pmod{p^\beta}$. Hence there is an isomorphism*

$$\det(L_p^{P,\beta}) \xrightarrow{\sim} (1+p^\beta\mathbf{Z}_p)\times\mathbf{Z}_p^\times, \quad (x,y)\mapsto(xy^{-1},y).$$

Proof. Similar to [DJR20, Lem. 2.1]. First, compute that for $\begin{pmatrix} \ell_1 & \\ & \ell_2 \end{pmatrix} \in L_\beta^{P,\beta}$, we need

$$t_P^{-\beta} u \begin{pmatrix} \ell_1 & \\ & \ell_2 \end{pmatrix} u^{-1} t_P^\beta = t_P^{-\beta} \begin{pmatrix} \ell_1 & w_n(\ell_2 - w_n\ell_1 w_n) \\ & \ell_2 \end{pmatrix} t_P^\beta \in K_p.$$

Since $P \subset Q$, each of the first n diagonal entries of t_P^β is congruent to $0 \pmod{p^\beta}$. In particular, after expanding we see $p^{-\beta}(\ell_2 - w_n\ell_1 w_n) \in \mathrm{GL}_n(\mathbf{Z}_p)$, so $\ell_2 \equiv w_n\ell_1 w_n \pmod{p^\beta}$, giving the first statement. We then have $\det(\ell_2) \equiv \det(\ell_1) \pmod{p^\beta}$, so to see the isomorphism, it suffices to prove surjectivity. But given $(a,b) \in (1+p^\beta\mathbf{Z}_p) \times \mathbf{Z}_p^\times$, we see $\ell_1 = \begin{pmatrix} 1_{n-1} & \\ & ab \end{pmatrix}, \ell_2 = \begin{pmatrix} b & \\ & 1_{n-1} \end{pmatrix}$ works (for any P). □

Corollary 4.6. *We have $L_p^{P,\beta} \subset L_p^{Q,\beta}$.*

Proof. By the proof of Lemma 4.5 (or [DJR20, Lem. 2.1]), we deduce $L_p^{Q,\beta} = \{(\ell_1,\ell_2) \in K_p : \ell_2 \equiv w_n\ell_1 w_n \pmod{p^\beta}\}$. But by Lemma 4.5, any element of $L_p^{P,\beta}$ satisfies this. □

By Lemma 4.5 and strong approximation for H, via the map

$$(h_1,h_2) \mapsto (\det(h_1)/\det(h_2), \det(h_2))$$

the cycle X_β^P decomposes into connected components indexed by

$$\pi_0(X_\beta^P) := \mathscr{C}\ell_{\mathbf{Q}}^+(p^\beta m) \times \mathscr{C}\ell_{\mathbf{Q}}^+(m) \cong (\mathbf{Z}/p^\beta m\mathbf{Z})^\times \times (\mathbf{Z}/m\mathbf{Z})^\times \tag{4.2}$$

(cf. [DJR20, (22)]). Here for an ideal $I \subset \mathbf{Z}$, we let $\mathscr{U}(I) := \left\{x \in \widehat{\mathbf{Z}}^\times : x \equiv 1 \pmod{I}\right\} \subset \widehat{\mathbf{Z}}^\times$ and let

$$\mathscr{C}\ell_{\mathbf{Q}}^+(I) = \mathbf{Q}^\times \backslash \mathbf{A}^\times / \mathscr{U}(I)\mathbf{R}_{>0} \cong (\mathbf{Z}/I)^\times \tag{4.3}$$

be the narrow ray class group of conductor I. For $\delta \in H(\mathbf{A}_f)$, we write $[\delta]$ for its associated class in $\pi_0(X_\beta^P)$ and denote the corresponding connected component

$$X_\beta^P[\delta] := H(\mathbf{Q})\backslash H(\mathbf{Q})\delta L_\beta^P H_\infty^\circ / L_\beta^P L_\infty^\circ.$$

As $L^p \subset K^p \cap H(\mathbf{A}_f)$, by definition of $L_p^{P,\beta}$ there is a proper map (see [Ash80, Lemma 2.7])

$$\iota_\beta^P : X_\beta^P \longrightarrow S_K, \qquad [h] \longmapsto [\iota(h)u^{-1}t_P^\beta]. \tag{4.4}$$

4.1.2 Abstract evaluation maps

Define $\Delta_P \subset \mathrm{GL}_{2n}(\mathbf{Q}_p)$ to be the semigroup generated by the parahoric subgroup $J_P \subset \mathrm{GL}_{2n}(\mathbf{Z}_p)$ and the matrices

$$t_{p,m_1}, \ \ t_{p,m_1+m_2}, \ \ \ldots, \ \ t_{p,m_1+\cdots+m_{r-1}},$$

where P has Levi $\mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_r}$ (and recalling $t_{p,r}$ from (2.9)). For example:

- $J_B = \mathrm{Iw}_G$, and Δ_B is generated by Iw_G and $t_{p,1}, t_{p,2}, \ldots, t_{p,2n-1}$.

– $J_Q = \{g \in \mathrm{GL}_{2n}(\mathbf{Z}_p) : g \pmod{p} \in Q(\mathbf{F}_p)\}$ and Δ_Q is generated by J_Q and $t_{p,n}$.

Let $K \subset G(\mathbf{A}_f)$ be an open compact subgroup such that $N_Q(\mathbf{Z}_p) \subset K_p \subset J_P$. Let M be a left Δ_P-module, with action denoted $*$. Then K acts on M via its projection to $K_p \subset \Delta_P$, giving a local system $\mathscr{M}$ on S_K via §3.1. The notation is suggestive: as in §3.2.3, using this $*$-action in (3.1) we get 'integrally normalised' Hecke operators $U_{p,r}^\circ$ on the cohomology $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{M})$.

The constructions here are almost identical to those of [BDW, §4.2] where they are motivated and explained in great detail; thus we give only the briefest description here.

For $\beta \in \mathbf{Z}_{\geqslant 0}$ and $\delta \in H(\mathbf{A}_f)$, define a congruence subgroup

$$\Gamma^P_{\beta,\delta} := H(\mathbf{Q}) \cap \delta L^P_\beta H^\circ_\infty \delta^{-1}. \tag{4.5}$$

This acts on M via

$$\gamma *_{\Gamma^P_{\beta,\delta}} m := (\delta^{-1}\gamma\delta)_f * m. \tag{4.6}$$

Let $M_{\Gamma^P_{\beta,\delta}} := M/\{m - \gamma *_{\Gamma^P_{\beta,\delta}} m : m \in M, \gamma \in \Gamma^P_{\beta,\delta}\}$ be the coinvariants of M by $\Gamma^P_{\beta,\delta}$.

Definition 4.7. The *evaluation map for M and P of level p^β at δ* is the composition

$$\begin{aligned}\mathrm{Ev}^M_{P,\beta,\delta} : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{M}) &\xrightarrow{\tau^{P,\circ}_\beta \circ (\iota^P_\beta)^*} \mathrm{H}^t_{\mathrm{c}}(X^P_\beta, \iota^*\mathscr{M}) \xrightarrow{c^*_\delta} \mathrm{H}^t_{\mathrm{c}}(\Gamma^P_{\beta,\delta}\backslash \mathcal{X}_H, c^*_\delta \iota^* \mathscr{M}) \\ &\xrightarrow{\mathrm{coinv}^P_{\beta,\delta}} \mathrm{H}^t_{\mathrm{c}}(\Gamma^P_{\beta,\delta}\backslash\mathcal{X}_H, \mathbf{Z}) \otimes M_{\Gamma^P_{\beta,\delta}} \xrightarrow[\sim]{-\cap\theta^P_\delta} M_{\Gamma^P_{\beta,\delta}},\end{aligned} \tag{4.7}$$

where:

- ι^P_β is the map from (4.4), and $\tau^{P,\circ}_\beta$ is the map $(\iota^P_\beta)^*\mathscr{M} \to \iota^*\mathscr{M}$ of local systems on X^P_β induced by $(h,m) \mapsto (h, u^{-1}t^\beta_P * m)$;
- $\Gamma^P_{\beta,\delta}$ acts on $\mathcal{X}_H := H^\circ_\infty / L^\circ_\infty$ by left translation, and there is an isomorphism

$$c_\delta : \Gamma^P_{\beta,\delta}\backslash\mathcal{X}_H \xrightarrow{\sim} X^P_\beta[\delta] \subset X_\beta, \quad [h_\infty]_\delta \mapsto [\delta h_\infty],$$

  where if $[h_\infty] \in \mathcal{X}_H$, we write $[h_\infty]_\delta$ for its image in $\Gamma^P_{\beta,\delta}\backslash\mathcal{X}_H$;
- $\mathrm{coinv}^P_{\beta,\delta}$ is the quotient map $M \twoheadrightarrow M_{\Gamma^P_{\beta,\delta}}$, which induces a map on cohomology with image in the cohomology of the trivial local system on $\Gamma^P_{\beta,\delta}\backslash\mathcal{X}_H$ attached to $M_{\Gamma^P_{\beta,\delta}}$;
- and $(-\cap\theta^P_\delta)$ is induced from cap product $(-\cap\theta^P_\delta) : \mathrm{H}^t_{\mathrm{c}}(\Gamma^P_{\beta,\delta}\backslash\mathcal{X}_H, \mathbf{Z}) \xrightarrow{\sim} \mathbf{Z}$, for θ^P_δ a fundamental class in the Borel–Moore homology $\mathrm{H}^{\mathrm{BM}}_t(\Gamma^P_{\beta,\delta}\backslash\mathcal{X}_H, \mathbf{Z}) \cong \mathbf{Z}$.

We choose the classes θ^P_δ compatibly in δ and P. Let θ^Q_δ be exactly as in [BDW, §4.2.3]. We have a natural map $\mathrm{pr}^B_Q : \Gamma^B_{\beta,\delta}\backslash\mathcal{X}_H \to \Gamma^Q_{\beta,\delta}\backslash\mathcal{X}_H$; we let $\theta^B_\delta = (\mathrm{pr}^B_Q)^*\theta^Q_\delta$.

Exactly as in [BDW, §4.3], we can track dependence of these maps as we allow M, β and δ to vary. Each of the following results is proved exactly as their given counterpart *op. cit.*:

Lemma 4.8. *(Variation in M) Let $\kappa : M \to N$ be a Δ_P-module map. There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{M}) \arrow[r, "\mathrm{Ev}^M_{P,\beta,\delta}"] \arrow[d, "\kappa_*"] & M_{\Gamma^P_{\beta,\delta}} \arrow[d, "\kappa"] \\
\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{N}) \arrow[r, "\mathrm{Ev}^N_{P,\beta,\delta}"] & N_{\Gamma^P_{\beta,\delta}}.
\end{tikzcd}$$

Proposition 4.9. *(Variation in δ) Let N be a left $H(\mathbf{A})$-module, with action denoted $*$, such that $H(\mathbf{Q})$ and H°_∞ act trivially. Let $\kappa : M \to N$ be a map of L^P_β-modules (with N an L^P_β-module by restriction). Then*

$$\mathrm{Ev}^{M,\kappa}_{P,\beta,[\delta]} := \delta * \left[\kappa \circ \mathrm{Ev}^M_{P,\beta,\delta}\right] : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{M}) \longrightarrow N$$

is well-defined and independent of the representative δ of $[\delta]$.

To vary β, we have a natural projection $\mathrm{pr}_\beta : X^P_{\beta+1} \longrightarrow X^P_\beta$, inducing a projection $\mathrm{pr}_\beta : \pi_0(X^P_{\beta+1}) \to \pi_0(X^P_\beta)$. The action of t_P on M yields an action of U°_P on $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{M})$, where $U^\circ_B = U^\circ_p$ and $U^\circ_Q = U^\circ_{p,n}$.

For compatibility in β, we need to assume additionally that $K_p = J_P$ is the parahoric for P.

Proposition 4.10. *(Variation in β) Let N and κ be as in Proposition 4.9. If $\beta > 0$, then as maps $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{M}) \to N$ we have*

$$\sum_{[\eta] \in \mathrm{pr}_\beta^{-1}([\delta])} \mathrm{Ev}^{M,\kappa}_{P,\beta+1,[\eta]} = \mathrm{Ev}^{M,\kappa}_{P,\beta,[\delta]} \circ U^\circ_P.$$

Proof. The proof follows almost exactly as in [BDW]. There is a unique point at which more detail is required. The left-hand square of diagram (4.14) *op. cit.* generalises to

$$\begin{tikzcd}
S_K & S_{K^0_P(p)} \arrow[l] \\
X^P_\beta \arrow[u, "\iota^P_\beta"] & X^P_{\beta+1} \arrow[l] \arrow[u, "\iota^{P,0}_\beta"']
\end{tikzcd},$$

where $K^0_P(p) = K \cap t_P K t_P^{-1}$, the map $\iota^{P,0}_\beta$ is induced by the map $[h] \mapsto [\iota(h)u^{-1}t^\beta_P]$, and where the horizontal maps are the natural projections. We need to show that this square is Cartesian in this generality. For this, since the vertical maps are embeddings, it is enough to show that the horizontal maps have the same degree; i.e., that $[K : K^0_P(p)] = [L^{P,\beta}_p : L^{P,\beta+1}_p]$ for any $\beta \geqslant 1$. Let N_P denote the unipotent radical of P and set $N_\beta = t^\beta_P N_P(\mathbf{Z}_p) t_P^{-\beta}$. By the Iwahori decomposition for K_p, we easily find that $[K : K^0_P(p)] = [N_P(\mathbf{Z}_p) : N_1] = [N_\beta : N_{\beta+1}]$ for any $\beta \geqslant 1$. On the other hand, the element u^{-1} is a representative of the (unique) Zariski dense H-orbit in the flag variety $G/\overline{P}$, where $\overline{P}$ denotes the opposite of P. Therefore the proof of [Loe21, Lem. 4.4.1] implies that $[L^{P,\beta}_p : L^{P,\beta+1}_p] = [N_\beta : N_{\beta+1}]$. Hence we have

$$[K : K^0_P(p)] = [L^{P,\beta}_p : L^{P,\beta+1}_p]$$

as required. □

4.2 Classical evaluation maps, Shalika models and L-values

The classical evaluation maps $\mathcal{E}_{\beta,\delta}^{j,\mathsf{w}}$ of [DJR20] were rephrased in the abstract language of Definition 4.7 in [BDW, §5]. We recap the construction, whilst again generalising it to parahoric level for a general parabolic P. When $P = Q$ this recovers [DJR20, BDW]; in this paper we are primarily interested in $P = B$ (which is new). We relate the values of these evaluation maps to critical L-values. Throughout, we assume $K_p = J_P$.

The definition of $\mathcal{E}_{\beta,\delta}^{j,\mathsf{w}}$ fundamentally used the following branching law from [GR14, Prop. 6.3.1] and [BDW, Lem. 5.2]. Let $\lambda \in X_0^*(T)$ be a pure algebraic weight, with purity weight w. For integers j_1, j_2, let $V_{(j_1,j_2)}^H$ denote the 1-dimensional $H(\mathbf{Z}_p)$-representation given by the character

$$H(\mathbf{Z}_p) \longrightarrow \mathbf{Z}_p^\times, \qquad (h_1,h_2) \longmapsto \det(h_1)^{j_1}\det(h_2)^{j_2}.$$

Lemma 4.11. *Let $j \in \mathbf{Z}$. Then $j \in \mathrm{Crit}(\lambda)$ if and only if* $\dim \mathrm{Hom}_{H(\mathbf{Z}_p)}\big(V_\lambda^\vee, V_{(j,-\mathsf{w}-j)}^H\big) = 1$.

For each $j \in \mathrm{Crit}(\lambda)$, fix some choice of non-trivial $H(\mathbf{Z}_p)$-map $\kappa_{\lambda,j} : V_\lambda^\vee(L) \to V_{(j,-\mathsf{w}-j)}^H \cong L$. We will make more precise choices in §11, but for now they can be arbitrary.

The *p-adic cyclotomic character* is

$$\chi_{\mathrm{cyc}} : \mathbf{Q}^\times\backslash\mathbf{A}^\times \longrightarrow \mathbf{Z}_p^\times, \quad y \mapsto \mathrm{sgn}(y_\infty)\cdot|y_f|\cdot y_p. \tag{4.8}$$

It is the p-adic character associated to the adelic norm [BW19, §2.2.2]. It is trivial on $\mathbf{R}_{>0}$.

It is simple to see that the $H(\mathbf{Z}_p)$-representation $V_{(j_1,j_2)}^H$ extends to $H(\mathbf{A})$ via the character

$$H(\mathbf{A}) \longrightarrow \mathbf{Z}_p^\times, \quad (h_1,h_2) \mapsto \chi_{\mathrm{cyc}}\big[\det(h_1)^{j_1}\det(h_2)^{j_2}\big].$$

Note the action of $L_\beta \subset H(\mathbf{A}_f)$ factors through projection to $H(\mathbf{Z}_p)$, so the map $\kappa_{\lambda,j} : V_\lambda^\vee(L) \to L$ chosen above is a map of L_β-modules. Moreover, $H(\mathbf{Q})$ and H_∞° act trivially on $V_{(j_1,j_2)}^H$, so we can use the formalism of Proposition 4.9.

Definition 4.12. Let $L/\mathbf{Q}_p$ be an extension. The *classical evaluation map for P of level p^β at δ* is

$$\mathcal{E}_{P,\beta,[\delta]}^{j,\mathsf{w}} := \mathrm{Ev}_{P,\beta,[\delta]}^{V_\lambda^\vee,\kappa_{\lambda,j}} : \mathrm{H}_c^t(S_K, V_\lambda^\vee(L)) \longrightarrow L.$$

Here $\mathrm{Ev}_{P,\beta,[\delta]}^{V_\lambda^\vee,\kappa_{\lambda,j}}$ was defined in Proposition 4.9, which shows $\mathcal{E}_{P,\beta,[\delta]}^{j,\mathsf{w}}$ is independent of the choice of δ representing the class $[\delta]$. We introduce the notation $\mathcal{E}_{P,\beta,[\delta]}^{j,\mathsf{w}}$ for consistency with [DJR20, BDW]; via [BDW, Lem. 5.3] this definition is consistent with that in [DJR20].

Recall $\pi_0(X_\beta^P) = (\mathbf{Z}/p^\beta m)^\times \times (\mathbf{Z}/m)^\times$ from (4.2). Write $\mathrm{pr}_1, \mathrm{pr}_2$ for the projections of $\pi_0(X_\beta^P)$ onto the first and second factors respectively, and let pr_β denote the natural composition

$$\mathrm{pr}_\beta : (\mathbf{Z}/p^\beta m)^\times \times (\mathbf{Z}/m)^\times \xrightarrow{\mathrm{pr}_1} (\mathbf{Z}/p^\beta m)^\times \twoheadrightarrow (\mathbf{Z}/p^\beta)^\times. \tag{4.9}$$

Definition 4.13. For η_0 be any character of $(\mathbf{Z}/m)^\times$, and $d \in (\mathbf{Z}/p^\beta)^\times$, define

$$\mathcal{E}^{j,\eta_0}_{P,\beta,d} := \sum_{[\delta]\in \mathrm{pr}_\beta^{-1}(d)} \eta_0^{-1}\big(\mathrm{pr}_2([\delta])\big)\ \mathcal{E}^{j,\mathsf{w}}_{P,\beta,[\delta]} \quad : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(L)) \to L.$$

(In our main application, we will take η_0 trivial; but the obstructions to taking more general η_0 are automorphic, not p-adic, so we develop the theory in full generality here).

Let χ be a finite order Hecke character of conductor $p^{\beta'}$, let $\beta = \max(1, \beta')$ and let $L(\chi)$ be the smallest extension of L containing $\mathrm{Im}(\chi)$. For $j \in \mathrm{Crit}(\lambda)$, define

$$\begin{aligned}\mathcal{E}^{j,\eta_0}_{P,\chi} = \sum_{[d]\in(\mathbf{Z}/p^\beta)^\times} \chi(d)\ \mathcal{E}^{j,\eta_0}_{P,\beta,d}\ :\ \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(L)) &\longrightarrow L(\chi), \\ \phi &\longmapsto \sum_{[\delta]\in(\mathbf{Z}/p^\beta m)^\times\times(\mathbf{Z}/m)^\times} \chi\big(\mathrm{pr}_\beta([\delta])\big)\cdot \eta_0^{-1}\big(\mathrm{pr}_2([\delta])\big)\cdot\Big(\delta * \Big[\kappa_{\lambda,j}\circ \mathrm{Ev}^{V_\lambda^\vee}_{P,\beta,\delta}(\phi)\Big]\Big).\end{aligned} \tag{4.10}$$

Remark 4.14. We see $\mathcal{E}^{j,\eta_0}_{P,\chi}$ is the composition

$$\begin{tikzcd} \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee) \arrow[r, "\oplus \mathrm{Ev}^{V_\lambda^\vee}_{P,\beta,\delta}"] \arrow[rr, bend left, "\oplus \mathcal{E}^{j,\mathsf{w}}_{P,\beta,[\delta]}"] \arrow[rrr, bend right, "\oplus \mathcal{E}^{j,\eta_0}_{P,\beta,d}"'] & \bigoplus_{[\delta]} (V_\lambda^\vee)_{\Gamma^P_{\beta,\delta}} \arrow[r, "v\mapsto\delta*\kappa_{\lambda,j}(v)"] & \bigoplus_{[\delta]} L \arrow[r, "\oplus \Xi_d^{\eta_0}"] & \bigoplus_d L \arrow[r, "\ell\mapsto\Sigma\chi(d)\ell_d"] & L(\chi), \end{tikzcd} \tag{4.11}$$

where the sums are over $[\delta] \in (\mathbf{Z}/p^\beta m)^\times \times (\mathbf{Z}/m)^\times$ or $d \in (\mathbf{Z}/p^\beta)^\times$, related by $d = \mathrm{pr}_\beta([\delta])$, and $\Xi_d^{\eta_0}$ is the η_0-averaging map

$$\Xi_d^{\eta_0} : (m_{[\delta]})_{[\delta]} \longmapsto \sum_{[\delta]\in\mathrm{pr}_\beta^{-1}(d)} \eta_0^{-1}(\mathrm{pr}_2([\delta]))\cdot m_{[\delta]}.$$

We give two applications of these maps. Let π be any RACAR of weight λ with attached maximal ideal $\mathfrak{m}_\pi \subset \mathcal{H}'$ as in §2.4.

Firstly, classical evaluation maps can detect existence of Shalika models:

Proposition 4.15. *Suppose there exists $\phi \in \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(\overline{\mathbf{Q}}_p))^\pm_\pi$ and some χ, j and η_0 such that*

$$\mathcal{E}^{j,\eta_0}_{P,\chi}(\phi) \neq 0. \tag{4.12}$$

Then π admits a global $(\eta_0|\cdot|^{\mathsf{w}}, \psi)$-Shalika model, where w is the purity weight of λ.

Proof. This is proved exactly as in [BDW, §5.3]. Whilst ξ is replaced by u^{-1}, and some non-zero volume factors change depending on P, the argument of proof is identical. $\square$

Secondly, we generalise [DJR20, §4], and show that up to a local zeta factor at p, these maps compute L-values. Let $K = J_P\prod_{\ell\neq p}\mathrm{GL}_{2n}(\mathbf{Z}_\ell)$, and let $W = W_p \otimes \bigotimes_{\ell\neq p} W_\ell^\circ \in \mathcal{S}^{\eta_f}_{\psi_f}(\pi_f^K, E)$, where we choose the normalised spherical vector at each $\ell \neq p$, and where W_p is an arbitrary E-rational vector in $\pi_p^{J_P}$. Recall the map $\Theta^\pm$ from (3.3), and the Friedberg–Jacquet integral $\zeta(s, W, \chi)$ from §2.3.2.

Theorem 4.16. *Let χ be a finite order Hecke character of conductor $p^{\beta'}$, let $\beta = \max(1,\beta')$, and let $j \in \mathrm{Crit}(\lambda)$. If $(-1)^j\chi_\infty\eta_\infty(-1) = \pm 1$, then*

$$\mathcal{E}_{P,\chi}^{j,\eta_0}\left(\frac{\Theta^{\pm}(W)}{\Omega_\pi^{\pm}}\right) = \delta_B(t_P^{-\beta})\Upsilon_P \cdot \lambda(t_P^{\beta}) \cdot \zeta_{\chi,j}(W_\infty) \times \frac{L^{(p)}(\pi\otimes\chi, j+1/2)}{\Omega_\pi^{\pm}} \cdot \zeta_p\Big(j+\tfrac{1}{2}, (u^{-1}t_P^{\beta})\cdot W_p, \chi_p\Big).$$

If $(-1)^j\chi_\infty\eta_\infty(-1) = \mp 1$, then $\mathcal{E}_{P,\chi}^{j,\eta_0}(\Theta^{\pm}(W)/\Omega_\pi^{\pm}) = 0$. Here all the signs are chosen consistently to be either the top or bottom sign.

Here we recall u from (4.1) and t_P from Definition 4.2, and:

- Υ_P is a non-zero rational volume constant independent of χ and j; we have

$$\Upsilon_P = \gamma \cdot A_P^{-1} \cdot p^{n^2} \cdot [\#\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})]^{-1},$$

  where γ is the constant from [DJR20, (77)] and A_P is the constant from Lemma 4.4. (Note that when $P = Q$ has Levi $\mathrm{GL}_n \times \mathrm{GL}_n$, we have $\Upsilon_Q = \gamma$).

- $\zeta_{\chi,j}(W_\infty)$ is an archimedean zeta integral that depends linearly on the choice of branching law $\kappa_{\lambda,j}$; this factor is non-zero by [Sun19, Thm. 5.5].

- $L^{(p)}(-)$ is the L-function with the local factor at p removed (that is, the product of all the local factors for all finite $\ell \neq p$).

Proof. A rephrasing of [DJR20, Prop. 4.6, Thm. 4.7] in this language is described in [BDW, §5.5]. As in [DJR20, Prop. 4.6], one first writes $\mathcal{E}_{\chi}^{j,\eta_0}(-)$ as an explicit integral over X_β^P, introducing the factor $\lambda(t_P^{\beta})$. One lifts this to an integral over $Z_G(\mathbf{A})H(\mathbf{Q})\backslash H(\mathbf{A})$, introducing the volume constant $\delta_B(t_P^{-\beta})\Upsilon_P$ (as we divide by $\mathrm{vol}(L_\beta^P)$, using Lemma 4.4). By [FJ93, Prop. 2.3] the integral equals a global Friedberg–Jacquet integral, which breaks into a product of local integrals. Away from p, the computation of these local zeta integrals is literally identical to that *op. cit.*; and at p, by definition the zeta integral is the one in the statement of the theorem (with t_P replacing t_Q). □

We will evaluate the local zeta integral at p for $P = B$ and specific choices of W_p in §5–9, and the integral at infinity for specific choices of branching law in Theorem 12.13.

Part II

Local Theory: Shalika p-refinements

For the remainder of the paper, unless otherwise specified we specialise to $P = B$ and consider Iwahori level.

Let us summarise what we have done so far. We took π to be a RASCAR that is everywhere spherical. A *p-refinement* of π was a choice of Hecke eigenspace $\tilde{\pi}_p$ in $\pi_p^{\mathrm{Iw}_G}$. To any choice of $W_p \in \mathcal{S}(\tilde{\pi}_p)$, before Theorem 4.16 we associated a (global) cohomology class in $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee)$. In that theorem, we computed its image under a scalar-valued functional, and showed that it took the form

$$\Big[\text{non-zero scalar}\Big] \times \Big[\text{critical } L\text{-value for } \pi\Big] \times \zeta_p\Big(j+\tfrac{1}{2}, (u^{-1}t_p^\beta)\cdot W_p, \chi_p\Big).$$

Over the next few sections, we compute the third term in this product – the local zeta integral at p – for 'nice' choices of W_p. This is a significant computation, spanning over several sections, so we briefly sketch the steps.

- In §5, when χ_p is ramified, we compute $\zeta_p(s,(u^{-1}t_p^\beta)\cdot W_p,\chi_p)$ as an explicit non-zero multiple of a specific value of W_p (depending on β).
- We are interested in finding p-refinements containing Hecke eigenvectors W_p for which this value is non-zero. We call such p-refinements *Shalika p-refinements*.
- In §6, we begin a systematic combinatorial study of p-refinements, and introduce 'spin p-refinements', a class of p-refinements for GL_{2n} that 'come from GSpin_{2n+1}'.
- In §7, we write down explicit eigenvectors attached to spin p-refinements. We precisely evaluate relevant values of these eigenvectors, and thus deduce that spin p-refinements are Shalika p-refinements. (In fact, we expect that the converse is true as well; we hope to return to this in a sequel to this paper).
- In §8, we summarise all of the above, and fold the local theory back into the global results of Part I.
- Our above computations worked with Iwahori-invariant W_p, but required χ_p to be ramified. Finally, in §9, we compute $\zeta_p(s,(u^{-1}t_p^\beta)\cdot W_p,\chi_p)$ for arbitrary χ_p, in particular allowing χ_p unramified. In this section, we use different methods and instead assume W_p is invariant for the parahoric subgroup of type (n,n).

Notation. Since it will be entirely focused on local theory, in Part II we henceforth drop subscripts p. In particular, we let π be a generic unramified principal series representation of $\mathrm{GL}_{2n}(\mathbf{Q}_p)$ admitting an (η,ψ)-Shalika model, for $\eta : \mathbf{Q}_p^\times \to \mathbf{C}^\times$ a smooth character and $\psi : \mathbf{Q}_p \to \mathbf{C}^\times$ the usual additive character (e.g. [DJR20, §4.1]). Note π is spherical. We continue to write Iw_G for the Iwahori subgroup of $\mathrm{GL}_{2n}(\mathbf{Z}_p)$. We write $\zeta(-)$ in place of $\zeta_p(-)$. We keep the notation of §2.4, with matrices $t_{p,r}$ and Hecke operators $U_{p,r}$ on π^{Iw_G}. A p-refinement $\tilde{\pi} = (\pi,\alpha)$ is a choice of Hecke eigensystem α occurring in π^{Iw_G}.

5 The local zeta integral at Iwahori level

We now give our first reduction of the local zeta integral

$$\zeta(s,(u^{-1}t_p^\beta)\cdot W,\chi) = \int_{\mathrm{GL}_n(\mathbf{Q}_p)} W\left[\begin{pmatrix} x & \\ & 1\end{pmatrix} u^{-1}t_p^\beta\right] \chi(\det(x))\,|\det(x)|^{s-\frac{1}{2}}\,dx. \tag{5.1}$$

The main aim of this section is Proposition 5.2, which computes this in terms of a specific value of W. First, we reduce the support of the zeta integral:

Lemma 5.1. *Suppose $W \in \mathcal{S}_\psi^\eta(\pi)$ is fixed under the action of Iw_G. Then the function*

$$\mathrm{GL}_n(\mathbf{Q}_p) \to \mathbf{C}, \qquad x \mapsto W\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)$$

is supported on $M_n(\mathbf{Z}_p) \cap \mathrm{GL}_n(\mathbf{Q}_p)$.

Proof. For $y \in M_n(\mathbf{Z}_p)$, right translation by $\left(\begin{smallmatrix} 1 & y \\ & 1 \end{smallmatrix}\right) \in \mathrm{Iw}_G$ gives

$$W\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right) = W\left[\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y \\ & 1 \end{smallmatrix}\right)\right] = \psi(\mathrm{tr}(xy))W\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right).$$

If $x \notin \mathrm{GL}_n(\mathbf{Q}_p)\backslash M_n(\mathbf{Z}_p)$, then we can choose y such that $\mathrm{tr}(xy) \notin \mathbf{Z}_p$, so $\psi(\mathrm{tr}(xy)) \neq 1$. $\square$

Let $d^\times c$ be the Haar measure on $\mathbf{Q}_p^\times$ giving $\mathbf{Z}_p^\times$ volume 1. Let $\chi : \mathbf{Q}_p^\times \to \mathbf{C}^\times$ be a finite order character of conductor p^β. In practice χ will be the local component at p of a Hecke character of p-power conductor, which forces $\chi(p) = 1$; we thus impose this condition throughout. For such a character, denote its Gauss sum by

$$\tau(\chi) = p^\beta(1-p^{-1})\int_{\mathbf{Z}_p^\times} \chi(c)\psi(p^{-\beta}c)d^\times c. \tag{5.2}$$

Recall $t_p = \mathrm{diag}(p^{2n-1}, p^{2n-2}, ..., p, 1)$. We write this in the form

$$t_p = \begin{pmatrix} p^n z & \\ & z \end{pmatrix}, \qquad z := \mathrm{diag}(p^{n-1}, p^{n-2}, ..., p, 1) \in T_n(\mathbf{Q}_p).$$

Let $\mathrm{Iw}_{G_n} \subset G_n(\mathbf{Z}_p)$ denote the upper-triangular Iwahori subgroup of $G_n = \mathrm{GL}_n$.

Proposition 5.2. *Let χ have conductor $p^\beta > 1$, and let $W \in \mathcal{S}_\psi^\eta\left(\pi^{\mathrm{Iw}_G}\right)$. Then*

$$\zeta\left(s, (u^{-1}t_p^\beta)\cdot W, \chi\right) = \Upsilon' \cdot \eta(\det z^\beta) \cdot p^{-\beta\frac{n^2+n}{2}} \cdot p^{\beta n(s-1/2)} \cdot \tau(\chi)^n \cdot \chi(\det(-w_n)) \cdot W\begin{pmatrix} w_n z^{2\beta} & \\ & 1 \end{pmatrix},$$

where $\Upsilon' = \mathrm{vol}(\mathrm{Iw}_{G_n}) \cdot (1-p^{-1})^{-n} \cdot p^{(n^2-n)/2}$ is a scalar independent of W, χ and β.

Proof. Recall $u = \left(\begin{smallmatrix} 1 & w_n \\ 0 & 1 \end{smallmatrix}\right)$ was defined in (4.1). Then observe that

$$\begin{pmatrix} x & \\ & 1 \end{pmatrix} u^{-1}t_p^\beta = \begin{pmatrix} z^\beta & \\ & z^\beta \end{pmatrix}\begin{pmatrix} 1 & -z^{-\beta}xw_nz^\beta \\ & 1 \end{pmatrix}\begin{pmatrix} p^{n\beta}z^{-\beta}xz^\beta & \\ & 1 \end{pmatrix}.$$

Substituting this into (5.1), and using the Shalika transformation property, we reduce to

$$\begin{aligned}\zeta(s, (u^{-1}t_p^\beta)\cdot W, \chi) &= \eta(\det z^\beta) \\ &\times \int_{\mathrm{GL}_n(\mathbf{Q}_p)} \psi\Big[-\mathrm{tr}(z^{-\beta}xw_nz^\beta)\Big] W\begin{pmatrix} p^{n\beta}z^{-\beta}xz^\beta & \\ & 1 \end{pmatrix} \chi(\det x)|\det x|^{s-1/2}dx.\end{aligned}$$

We make the change of variables $y = p^{n\beta} z^{-\beta} x z^{\beta}$. As dx is a left and right Haar measure, we have $dy = dx$. Recalling that $\chi(p) = 1$ and $|p| = 1/p$, we get

$$\begin{aligned}\zeta(s,(u^{-1}t_p^{\beta})\cdot W,\chi) = \eta(\det z^{\beta}) p^{n^2\beta(s-1/2)} \\ \times \int_{\mathrm{GL}_n(\mathbf{Q}_p)} \psi\Big[-\mathrm{tr}(p^{-n\beta} z^{\beta} y z^{-\beta} w_n)\Big] I(y) dy.\end{aligned} \tag{5.3}$$

Here:

- we define

$$I(y) := W\begin{pmatrix} y & \\ & 1\end{pmatrix}\chi(\det y)|\det y|^{s-1/2}, \tag{5.4}$$

- in the trace term, we have conjugated by z^{β},
- and we note that

$$|\det x| = |\det p^{n\beta} z^{-\beta} y z^{\beta}| = \det\begin{pmatrix} |p^{n\beta}| & & \\ & \ddots & \\ & & |p^{n\beta}|\end{pmatrix}\cdot|\det y| = p^{-n^2\beta}|\det y|.$$

We now cut down the support of this integral. Firstly, by Lemma 5.1 we can immediately reduce the support to $\mathrm{GL}_n(\mathbf{Q}_p)\cap M_n(\mathbf{Z}_p)$. To go further, we exploit Iwahori invariance of W.

Notation 5.3. (1) Let A denote the set of all diagonal $n\times n$-matrices of the form

$$\gamma = \mathrm{diag}(c_{11},\dots,c_{nn}), \qquad c_{ii}\in\mathbf{Z}_p^{\times}.$$

(2) Let B_β denote the additive group of all $n\times n$-matrices δ with

$$\delta_{i,j} = \begin{cases} c_{i,j} & \text{if } i<j \\ 0 & \text{if } i=j \\ p^{\beta}c_{i,j} & \text{if } i>j \end{cases}, \qquad c_{ij}\in\mathbf{Z}_p.$$

We will consider matrices of the form $\alpha = \gamma+\delta \in M_n(\mathbf{Z}_p)$ for $\gamma\in A$ and $\delta\in B_\beta$. Note that α is in the depth p^{β} Iwahori subgroup $\mathrm{Iw}_{G_n}(p^{\beta})\subset\mathrm{GL}_n(\mathbf{Z}_p)$ (the matrices that are upper-triangular modulo p^{β}). We set

$$\varepsilon = \begin{pmatrix}\alpha^{-1} & \\ & 1\end{pmatrix}\in\mathrm{Iw}_G\subset\mathrm{GL}_{2n}(\mathbf{Z}_p).$$

Now we translate the argument of the zeta integral by ε. By Iwahori invariance, we get

$$\begin{aligned}\zeta(s,(u^{-1}t_p^{\beta})\cdot W,\chi) &= \zeta(s,(u^{-1}t_p^{\beta}\varepsilon)\cdot W,\chi) \\ &= \eta(\det z^{\beta})p^{n^2\beta(s-1/2)} \\ &\quad\times\int_{\mathrm{GL}_n(\mathbf{Q}_p)\cap M_n(\mathbf{Z}_p)}\psi\Big[-\mathrm{tr}(p^{-n\beta}z^{\beta}yz^{-\beta}w_n)\Big]W\begin{pmatrix}y\alpha^{-1} & \\ & 1\end{pmatrix}\chi(\det y)|\det y|^{s-1/2}dy.\end{aligned} \tag{5.5}$$

Make the change of variables $x = y\alpha^{-1}$; then this becomes

$$= \eta(\det z^\beta) p^{n^2\beta(s-1/2)} \chi(\det\gamma) \int_{\mathrm{GL}_n(\mathbf{Q}_p)\cap M_n(\mathbf{Z}_p)} \psi\Big[-\mathrm{tr}(p^{-n\beta} z^\beta x\alpha z^{-\beta} w_n)\Big] I(x)dx. \tag{5.6}$$

Here we used that $\det(\alpha) = \det(\gamma) \pmod{p^\beta}$, that χ has conductor p^β, and that $|\det(\gamma)| = 1$.

Now we have

$$\psi\Big[\mathrm{tr}(-p^{-n\beta} z^\beta x\alpha z^{-\beta} w_n)\Big] = \psi\Big[\mathrm{tr}(-p^{-n\beta} z^\beta x\gamma z^{-\beta} w_n)\Big] \cdot \psi\Big[\mathrm{tr}(-p^{-n\beta} z^\beta x\delta z^{-\beta} w_n)\Big]. \tag{5.7}$$

We cut the support down first by averaging over $\delta \in B_\beta$, then over $\gamma \in A$.

Step 1: Average over B_β. For $x \in \mathrm{GL}_n(\mathbf{Q}_p) \cap M_n(\mathbf{Z}_p)$, define

$$\begin{aligned} F_x : B_\beta &\longrightarrow \mathbf{C}^\times \\ \delta &\longmapsto \psi\Big[\mathrm{tr}(-p^{-n\beta} z^\beta x\delta z^{-\beta} w_n)\Big]. \end{aligned}$$

This is a group homomorphism by additivity of trace and ψ.

Lemma 5.4. *(i) There exists a finite index subgroup $B'_\beta \subset B_\beta$ such that F_x is trivial on B'_β for all $x \in \mathrm{GL}_n(\mathbf{Q}_p) \cap M_n(\mathbf{Z}_p)$.*

(ii) For any fixed x, F_x is the trivial function if and only if

$$x_{n+1-i,j} \in \begin{cases} p^{2\beta(n-i)+\beta}\mathbf{Z}_p & \text{if } i > j \\ p^{2\beta(n-i)}\mathbf{Z}_p & \text{if } i < j \end{cases} \tag{5.8}$$

Proof. (i) For δ sufficiently divisible by p, we have $F_x(\delta) = 1$ for all $x \in M_n(\mathbf{Z}_p)$.

(ii) Writing out the trace explicitly, one sees

$$\mathrm{tr}\Big[-p^{-n\beta} z^\beta x\delta z^{-\beta} w_n\Big] = -\sum_{i=1}^{n} \left(p^{\beta(1-2i)} x_{i,k} \left(\sum_{k<n+1-i} c_{k,n+1-i} + \sum_{k>n+1-i} p^\beta c_{k,n+1-i} \right) \right).$$

and uses the change of variables $i \mapsto n+1-i$. If (5.8) fails for some (i,j), then F_x will be non-trivial on the matrix δ which is zero apart from a 1 at (i,j), so F_x is not the trivial function. Conversely, if (5.8) does hold, then the trace above is always integral and F_x is trivial. □

Let $M'_\beta \subset \mathrm{GL}_n(\mathbf{Q}_p) \cap M_n(\mathbf{Z}_p)$ be the subset of matrices x satisfying the conditions in (5.8).

Corollary 5.5. *For any $\gamma \in A$, we have*

$$\zeta(s, (u^{-1}t_p^\beta)\cdot W, \chi) = \eta(\det z^\beta) p^{n^2\beta(s-1/2)} \chi(\det\gamma) \int_{M'_\beta} \psi\Big[\mathrm{tr}(-p^{-n\beta} z^\beta x\gamma z^{-\beta} w_n)\Big] \cdot I(x)dx.$$

Proof. Using (5.5) (in the first equality) and (5.6) and (5.7) (in the second), we have

$$\begin{aligned}\zeta(s,(u^{-1}t_p^{\beta})\cdot W,\chi) &= \frac{1}{[B_\beta : B'_\beta]}\sum_{\delta\in B_\beta/B'_\beta}\zeta(s,(u^{-1}t_p^{\beta}(\gamma+\delta))\cdot W,\chi)\\ &= \eta(\det z^{\beta})p^{n^2\beta(s-1/2)}\chi(\det\gamma)\int_{\mathrm{GL}_n(\mathbf{Q}_p)\cap M_n(\mathbf{Z}_p)}\psi\Big[-\mathrm{tr}(p^{-n\beta}z^{\beta}x\gamma z^{-\beta}w_n)\Big]\\ &\qquad\times\left[\frac{1}{[B_\beta : B'_\beta]}\sum_{\delta\in B_\beta/B'_\beta}F_x(\delta)\right]I(x)dx.\end{aligned}$$

When F_x is non-trivial, the square-bracketed term (hence the integrand) vanishes by character orthogonality; and when $F_x\equiv 1$, it is identically 1. We conclude since by Lemma 5.4(ii), F_x is trivial if and only if $x\in M'_\beta$. □

Step 2: Average over A. Equip $A\cong(\mathbf{Z}_p^\times)^n$ with the measure $d^\times A=\prod_{i=1}^n d^\times c_{i,i}$.

Lemma 5.6. *We have*

$$\chi(\det\gamma)\psi\Big[\mathrm{tr}(-p^{-n\beta}z^{\beta}x\gamma z^{-\beta}w_n)\Big]=\prod_{i=1}^n\chi(c_{i,i})\psi\left(-p^{-(2\beta(n-i)+\beta)}x_{n+1-i,i}c_{i,i}\right).$$

Therefore

$$\begin{aligned}&\int_{\gamma\in A}\chi(\det\gamma)\psi\Big[\mathrm{tr}(-p^{-n\beta}z^{\beta}x\gamma z^{-\beta}w_n)\Big]d^\times A\\ &\qquad=\prod_{i=1}^n\chi(-1)\int_{\mathbf{Z}_p^\times}\chi(c_{i,i})\psi\left(p^{-(2\beta(n-i)+\beta)}x_{n+1-i,i}c_{i,i}\right)d^\times c_{i,i}\\ &\qquad=\begin{cases}\frac{\chi(-1)^n\tau(\chi)^n}{p^{n\beta}(1-p^{-1})^n}\prod_{i=1}^n\chi(x'_{n+1-i,i})^{-1} & \text{if } x_{n+1-i,i}\in p^{2\beta(n-i)}\mathbf{Z}_p^\times\ \forall i\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

where $x'_{n+1-i,i}=x_{n+1-i,i}/p^{2\beta(n-i)}$ *and* $\tau(\chi)$ *is the Gauss sum from* (5.2).

Proof. We have

$$\chi(\det\gamma)=\prod_{i=1}^n\chi(c_{i,i})$$

and

$$\begin{aligned}\psi\Big[\mathrm{tr}(-p^{-n\beta}z^{\beta}x\gamma z^{-\beta}w_n)\Big]&=\prod_{i=1}^n\psi\left(-p^{\beta(1-2i)}x_{i,n+1-i}c_{n+1-i,n+1-i}\right)\\ &=\prod_{i=1}^n\psi\left(-p^{-(2\beta(n-i)+\beta)}x_{n+1-i,i}c_{i,i}\right),\end{aligned}$$

giving the first part. The rest follows from a simple change of variables. □

Let $M_\beta \subset M'_\beta$ be the subset where $x_{n+1-i,i} \in p^{2\beta(n-i)}\mathbf{Z}_p^\times$ for all i. Note that

$$M_\beta = w_n z^{2\beta}\mathrm{Iw}_{G_n}(p^\beta).$$

Corollary 5.7. *We have*

$$\zeta(s,(u^{-1}t_p^\beta)\cdot W,\chi) = \eta(\det z^\beta)p^{n^2\beta(s-1/2)}\frac{\chi(-1)^n\tau(\chi)^n}{p^{n\beta}(1-p^{-1})^n}\int_{M_\beta}\prod_{i=1}^n \chi(x'_{n+1-i,i})^{-1}I(x)dx.$$

Proof. This is similar to Corollary 5.5. Integrate the expression of that corollary over A, and reduce the support using Lemma 5.6. □

Now using that fact that $M_\beta = w_n z^{2\beta}\mathrm{Iw}_{G_n}(p^\beta)$ we write $x = w_n z^{2\beta}x''$ and note that $\det x'' \equiv \prod_{i=1}^n x'_{n+1-i,i}$ modulo p^β. Making the change of variables, we see that

$$\begin{aligned}
&\int_{M_\beta}\prod_{i=1}^n \chi(x'_{n+1-i,i})^{-1}I(x)dx\\
&\quad= \int_{\mathrm{Iw}_{G_n}(p^\beta)} W\begin{pmatrix} w_n z^{2\beta}x'' & \\ & 1\end{pmatrix}\prod_{i=1}^n \chi(x'_{n+1-i,i})^{-1}\chi(\det w_n z^{2\beta}x'')|\det w_n z^{2\beta}|^{s-1/2}dx''\\
&\quad= \chi(\det w_n)\cdot p^{-\beta n(n-1)(s-1/2)}\int_{\mathrm{Iw}_{G_n}(p^\beta)} W\begin{pmatrix} w_n z^{2\beta} & \\ & 1\end{pmatrix}dx''\\
&\quad= \chi(\det w_n)\cdot p^{-\beta n(n-1)(s-1/2)}\cdot \mathrm{vol}(\mathrm{Iw}_{G_n}(p^\beta))\cdot W\begin{pmatrix} w_n z^{2\beta} & \\ & 1\end{pmatrix}.
\end{aligned} \tag{5.9}$$

In the penultimate equality, we have used that

$$\det x'' \equiv \prod_{i=1}^n x'_{n+1-i,i} \pmod{p^\beta},$$

$\chi(p) = 1$, $|\det w_n| = 1$, and $|\det z| = p^{-n(n-1)/2}$. Finally note that

$$\mathrm{vol}(\mathrm{Iw}_{G_n}(p^\beta)) = p^{-(\beta-1)\frac{n^2-n}{2}}\mathrm{vol}(\mathrm{Iw}_{G_n}). \tag{5.10}$$

Putting (5.10) and (5.9) into Corollary 5.7, using that $\chi(-1)^n\chi(\det w_n) = \chi(\det(-w_n))$, completes the proof of Proposition 5.2. □

6 Spin p-refinements

As highlighted in the introduction to Part II, we want to answer:

For which p-refinements $\tilde\pi$ is there $W \in \mathcal{S}_\psi^\eta(\tilde\pi)$ with $\zeta(s,(u^{-1}t_p^\beta)\cdot W,\chi) \neq 0$?

Given Proposition 5.2, this is equivalent to asking when there exists $W \in \tilde{\pi}$ and $\beta \geqslant 1$ such that $W\begin{pmatrix} w_n z^{2\beta} & \\ & 1 \end{pmatrix} \neq 0$. We expect this to be true only for a special class of p-refinements, those that 'interact well with the Shalika model'.

In this section, we begin to make this assertion rigorous. We define 'spin' p-refinements as those 'that come from GSpin_{2n+1}', made precise in Proposition 6.14. In later sections we will show that for any spin p-refinement $\tilde{\pi}$, there exists $W \in \mathcal{S}_\psi^\eta(\tilde{\pi})$ such that $W\begin{pmatrix} w_n z^{2\beta} & \\ & 1 \end{pmatrix} \neq 0$. Our key application of spin p-refinements will ultimately be summarised in Corollary 8.2.

Notation 6.1. Let $\mathcal{G} := \mathrm{GSpin}_{2n+1}$, taking the split form. As π is a generic unramified principal series and admits a Shalika model, by [AS06] it is the functorial transfer of an unramified principal series representation Π of $\mathcal{G}(\mathbf{Q}_p)$. We shall describe Π explicitly in Proposition 6.11.

We first give a concrete definition of spin p-refinement. We justify it in §6.4, and give several equivalent formulations in Propositions 6.9 and 6.14. Let $\tilde{\pi} = (\pi, \alpha)$ be a p-refinement of π (as in §2.4.3), and for $1 \leqslant r \leqslant 2n-1$ write $\alpha_{p,r} := \alpha(U_{p,r})$.

Definition 6.2. We say $\tilde{\pi}$ is a *spin p-refinement* if $\alpha_{p,n+s} = \eta(p)^s \alpha_{p,n-s}$ for all $0 \leqslant s \leqslant n-1$.

Recall here η is the Shalika character (see page 184). We will show $\tilde{\pi}$ is spin if and only if α factors through an eigensystem occuring in $\Pi^{\mathrm{Iw}_{\mathcal{G}}}$, for $\mathrm{Iw}_{\mathcal{G}}$ the Iwahori subgroup of $\mathcal{G}(\mathbf{Z}_p)$ (see §6.4).

6.1 Conventions for Shalika models

As π is spherical, it can be written as a (normalised) induction from the upper-triangular Borel, i.e. there exists an unramified character $\theta : T(F) \to \mathbf{C}^\times$ such that $\pi = \mathrm{Ind}_B^G \theta$. This θ is well-defined up to conjugation by $\mathcal{W}_G$. Moreover, we have:

Proposition 6.3. *[AG94, Prop. 1.3]. The unramified principal series* $\mathrm{Ind}_B^G \theta$ *admits an* (η, ψ)*-Shalika model if and only if there is a decomposition* $\{1, ..., 2n\} = X_1 \sqcup X_2$*, where* $\#X_j = n$*, and a bijection* $\nu : X_1 \to X_2$ *such that* $\theta_i \theta_{\nu(i)} = \eta$ *for all* $i \in X_1$.

Given choices of a decomposition $X_1 \sqcup X_2$ and a bijection ν, we can find an identification $\pi = \mathrm{Ind}_B^G \theta$ with $\theta_i \theta_{\nu(i)} = \eta$.

The natural choice of decomposition is $X_1 = \{1, ..., n\}$, $X_2 = \{n+1, ..., 2n\}$. For this there will be *two* natural choices of ν, with their own advantages and disadvantages.

- In [AG94], ν is chosen so that $\nu(i) = n+i$. When $\theta_i \theta_{n+i} = \eta$, Ash–Ginzburg use this to define an explicit intertwining $\mathrm{Ind}_B^G \theta \to \mathcal{S}_\psi^\eta(\pi)$, which we describe in §7.2.
- In [AS06], ν is chosen so that $\nu(i) = 2n+1-i$. This is considerably more natural when discussing spin refinements, as will become clear later in this section.

Since we will later use Ash–Ginzburg's explicit intertwining, we make the following choice:

In the rest of the paper, assume $\pi = \mathrm{Ind}_B^G \theta$*, where* $\theta_i \theta_{n+i} = \eta$ *for* $1 \leqslant i \leqslant n$.

Note that the Ash–Ginzburg and Asgari–Shahidi choices are interchanged by conjugation by $\tau := \begin{pmatrix} 1 & \\ & w_n \end{pmatrix}$. In particular, given the choice of θ fixed above, we have $\theta_i^\tau \theta_{2n+1-i}^\tau = \eta$. This helps when it is more convenient to use Asgari–Shahidi's choice (see e.g. Remark 6.12).

6.2 Root systems for GL_{2n} and GSpin_{2n+1}

Recall the space of algebraic characters and cocharacters of the torus $T \subset G = \mathrm{GL}_{2n}$ are given by

$$X = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2 \oplus \cdots \mathbf{Z}e_{2n}, \quad X^\vee = \mathbf{Z}e_1^* \oplus \mathbf{Z}e_2^* \oplus \cdots \mathbf{Z}e_{2n}^*.$$

Write $\langle -,- \rangle_G$ for the natural pairing on $X \times X^\vee$. The corresponding root system is A_{2n-1}, with roots $R = \{\pm(e_i - e_j) : 1 \leqslant i < j \leqslant 2n\}$. The Weyl group $\mathcal{W}_G = \mathrm{S}_{2n}$ acts by permuting the e_i, with longest Weyl element w_{2n} the permutation that sends $e_i \mapsto e_{2n+1-i}$ for all i.

Let $X_0 \subset X$ be the space of *pure characters* $X_0 = \{\lambda \in X : \exists \mathsf{w} \in \mathbf{Z} \text{ such that } \lambda_i + \lambda_{2n-i+1} = \mathsf{w}\}$, and let

$$\mathcal{W}_G^0 := \{\sigma \in \mathcal{W}_G : \forall \lambda \in X_0,\ \lambda^\sigma \in X_0\} \subset \mathcal{W}_G. \tag{6.1}$$

Now fix a standard upper Borel subgroup $\mathcal{B}$ and maximal split torus $\mathcal{T}$ in $\mathcal{G} = \mathrm{GSpin}_{2n+1}$. This has rank $n+1$ [Asg02, Thm. 2.7]. We use calligraphic letters to denote objects for GSpin, and otherwise maintain the same notational conventions as before.

Proposition 6.4. *The root system for $\mathcal{G}$ is $(\mathcal{X}, \mathcal{R}, \mathcal{X}^\vee, \mathcal{R}^\vee)$, where*

$$\mathcal{X} = \mathbf{Z}f_0 \oplus \mathbf{Z}f_1 \oplus \cdots \oplus \mathbf{Z}f_n, \quad \mathcal{X}^\vee = \mathbf{Z}f_0^* \oplus \mathbf{Z}f_1^* \oplus \cdots \oplus \mathbf{Z}f_n^*,$$

with roots $\mathcal{R} = \{\pm f_i \pm f_j : 1 \leqslant i < j \leqslant n\} \cup \{f_i : 1 \leqslant i \leqslant n\}$ and positive roots $\{f_i : 1 \leqslant i \leqslant n\} \cup \{f_i \pm f_j : 1 \leqslant i < j \leqslant n\}$. The Weyl group $\mathcal{W}_{\mathcal{G}}$ has size $2^n \cdot n!$, generated by permutations $\sigma \in S_n$ and sign changes sgn_i, which act on roots and coroots respectively as (for $j \neq i$)

$$\sigma f_0 = f_0,\ \ \sigma f_i = f_{\sigma(i)},\quad \mathrm{sgn}_i f_0 = f_0 + f_i,\ \ \mathrm{sgn}_i(f_i) = -f_i,\ \ \mathrm{sgn}_j(f_i) = f_i, \tag{6.2}$$

$$\sigma f_0^* = f_0^*,\ \ \sigma f_i^* = f_{\sigma(i)}^*,\quad \mathrm{sgn}_i f_0^* = f_0^*,\ \ \mathrm{sgn}_i(f_i^*) = f_0^* - f_i^*,\ \ \mathrm{sgn}_j(f_i^*) = f_i^*.$$

In particular, $\mathcal{W}_{\mathcal{G}}$ is the semidirect product $\{\pm 1\}^n \rtimes S_n$.

Proof. The first part is [Asg02, Prop. 2.4], and the second [HS16, Lem. 13.2.2]. □

Write $\langle -,- \rangle_{\mathcal{G}}$ for the natural pairing on $\mathcal{X} \times \mathcal{X}^\vee$.

6.3 The maps $\jmath$ and $\jmath^\vee$

There is a natural injective map $\jmath : \mathcal{X} \hookrightarrow X$ given by

$$f_i \longmapsto e_i - e_{2n-i+1} \text{ for } 1 \leqslant i \leqslant n, \qquad f_0 \longmapsto e_{n+1} + \cdots + e_{2n}.$$

We may identify $\mathcal{X}$ with cocharacters of GSp_{2n}, and X with cocharacters of GL_{2n}. The map $\jmath$ is then the natural map on cocharacters induced by the inclusion $T_{\mathrm{GSp}_{2n}} \subset T$ of tori.

Proposition 6.5. *We have $X_0 = \jmath(\mathcal{X})$.*

Proof. Any linear combination of the $\jmath(f_i)$ is a pure weight with purity weight 0, and any such weight arises in this form; and scaling the purity weight to w corresponds to adding $\jmath(\mathsf{w}f_0)$. □

Proposition 6.6. *There is a map $\mathcal{W}_{\mathcal{G}} \to \mathcal{W}_G$ of Weyl groups, which we also call $\jmath$, such that:*

(i) $\jmath$ induces an isomorphism $\mathcal{W}_{\mathcal{G}} \xrightarrow{\jmath} \mathcal{W}_G^0 \subset \mathcal{W}_G$;

(ii) for all $\sigma \in \mathcal{W}_{\mathcal{G}}$ and $\mu \in \mathcal{X}$, we have $\jmath(\mu^\sigma) = \jmath(\mu)^{\jmath(\sigma)}$.

Proof. We know $\mathcal{W}_{\mathcal{G}} \cong \{\pm 1\}^n \rtimes S_n$ is generated by sign changes and permutations in $\{f_1, ..., f_n\}$, and $\mathcal{W}_G = S_{2n}$ is permutations in $\{e_1, ..., e_{2n}\}$. If $\sigma \in S_n \subset \mathcal{W}_{\mathcal{G}}$ is a permutation, then define

$$\jmath(\sigma) = \left(\sigma \ {}_{w_n \sigma w_n}\right) : (e_1, ..., e_n, e_{n+1}, ..., e_{2n}) \mapsto \left(e_{\sigma(1)}, ..., e_{\sigma(n)}, e_{2n-\sigma(n)+1}, ..., e_{2n-\sigma(1)+1}\right),$$

and if $\sigma = \varepsilon_i$ is the sign change at $i \in \{1, ..., n\}$, define $\jmath(\varepsilon_i)$ to be the permutation switching e_i and e_{2n-i+1}. A simple check shows this induces a well-defined homomorphism.

Suppose $\lambda \in X$ is pure with all the λ_i distinct. Let $\sigma \in \mathcal{W}_G = S_n$; if λ^σ is pure, then σ must preserve the relative positions of each pair $\{\lambda_i, \lambda_{2n-i+1}\}$. The only way to do this is to permute $i \in \{1, ..., n\}$ or to switch $\lambda_i, \lambda_{2n-i+1}$. These are exactly the permutations in $\jmath(\mathcal{W}_{\mathcal{G}})$, giving (i).

Part (ii) is a simple explicit check. $\square$

Corollary 6.7. *There is a short exact sequence $1 \to \{\pm 1\}^n \to \mathcal{W}_G^0 \to S_n \to 1$, which is split by $\jmath : S_n \to \mathcal{W}_G^0$. The image of $\{\pm 1\}^n$ is generated by the transpositions $(i, 2n+1-i)$ for $1 \leqslant i \leqslant n$.*

Dually, define also a map $\jmath^\vee : X^\vee \to \mathcal{X}^\vee$ by sending $\nu \in X^\vee$ to

$$\jmath^\vee(\nu) := \sum_{i=0}^{n} \left\langle \jmath(f_i), \nu \right\rangle_G \cdot f_i^*.$$

Then for all $\mu \in X$, we have

$$\langle \mu, \jmath^\vee(\nu) \rangle_{\mathcal{G}} = \langle \jmath(\mu), \nu \rangle_G \tag{6.3}$$

by construction. (Again, this map arises from our explicit realisation of $T_{\mathrm{GSp}_{2n}} \subset T$). Also let $\jmath^\vee : \mathcal{W}_G^0 \to \mathcal{W}_{\mathcal{G}}$ denote the inverse to $\jmath : \mathcal{W}_{\mathcal{G}} \cong \mathcal{W}_G^0$.

Proposition 6.8. *For all $\nu \in X^\vee$ and $\sigma \in \mathcal{W}_G^0$, we have $\jmath^\vee(\nu^\sigma) = \jmath^\vee(\nu)^{\jmath^\vee(\sigma)}$.*

Proof. If $\sigma \in \mathcal{W}_G^0$, then $\sigma = \jmath(\rho)$ for some $\rho \in \mathcal{W}_{\mathcal{G}}$, and $\jmath^\vee(\sigma) = \rho$. Then compute that

$$\begin{aligned} \jmath^\vee(\nu^\sigma) := \sum_{i=0}^{n} \langle \jmath(f_i), \nu^\sigma \rangle_G f_i^* = \sum_{i=0}^{n} \langle \jmath(f_i)^{\jmath(\rho^{-1})}, \nu \rangle_G f_i^* &= \sum_{i=0}^{n} \langle \jmath(f_i^{\rho^{-1}}), \nu \rangle_G f_i^* \\ &= \sum_{i=0}^{n} \langle \jmath(f_i), \nu \rangle_G (f_i^*)^\rho = \jmath^\vee(\nu)^{\jmath^\vee(\sigma)}, \end{aligned}$$

where the penultimate equality is a simple check. $\square$

6.4 Spin refinements via GSpin

Via the Satake isomorphism, we may describe Hecke operators in terms of cocharacters. Note $U_{p,r}$ is attached to the cocharacter

$$\nu_r := e_1^* + \cdots + e_r^* \in X^\vee, \tag{6.4}$$

since $t_{p,r}^G := t_{p,r} = \nu_r(p)$. Define $t_{p,r}^{\mathcal{G}} := \jmath^\vee(\nu_r)(p) \in \mathcal{T}(F)$; by definition, for $0 \leqslant s \leqslant n-1$

$$\jmath^\vee(\nu_{n-s}) = f_1^* + \cdots + f_{n-s}^*, \qquad \jmath^\vee(\nu_{n+s}) = \jmath^\vee(\nu_{n-s}) + sf_0^*. \tag{6.5}$$

Let $\mathrm{Iw}_{\mathcal{G}} \subset \mathcal{G}(\mathbf{Z}_p)$ be the Iwahori subgroup for $\mathcal{G}$, and for $1 \leqslant r \leqslant n$, let $\mathcal{U}_{p,r} := [\mathrm{Iw}_{\mathcal{G}}\, t_{p,r}^{\mathcal{G}}\, \mathrm{Iw}_{\mathcal{G}}]$. Also let $\mathcal{V}_p := [\mathrm{Iw}_{\mathcal{G}}\, f_0^*(p)\, \mathrm{Iw}_{\mathcal{G}}]$. From (6.5), we see $\mathcal{U}_{p,n+s} = \mathcal{V}_p^s \cdot \mathcal{U}_{p,n-s}$.

Let $\mathcal{H}_p^{\mathcal{G}} := \mathbf{Z}_p[\mathcal{V}_p, \mathcal{U}_{p,r} : 1 \leqslant r \leqslant n]$ be the Hecke algebra for $\mathcal{G}$. Then $\jmath^\vee$ induces a map

$$\mathcal{H}_p \longrightarrow \mathcal{H}_p^{\mathcal{G}}, \qquad U_{p,n-s} \longmapsto \mathcal{U}_{p,n-s}, \;\; U_{p,n+s} \longmapsto \mathcal{V}_p^s \cdot \mathcal{U}_{p,n-s},$$

for each $0 \leqslant s \leqslant n-1$. Comparing with Definition 6.2, we obtain:

Proposition 6.9. *A p-refinement (π, α) is a spin p-refinement if and only if α factors through*

$$\mathcal{H}_p \xrightarrow{\jmath^\vee} \mathcal{H}_p^{\mathcal{G}} \xrightarrow{\alpha^{\mathcal{G}}} \overline{\mathbf{Q}},$$

for some character $\alpha^{\mathcal{G}}$ with $\alpha^{\mathcal{G}}(\mathcal{V}_p) = \eta(p)$.

Remark 6.10. We could add the operator $U_{p,2n} = [\mathrm{Iw}_G\, \mathrm{diag}(p,...,p)\, \mathrm{Iw}_G]$ to $\mathcal{H}_p$; it acts by $\eta(p)^n$ on π as its central character is η^n. We would then have $\jmath^\vee(U_{p,2n}) = \mathcal{V}_p^n$. In particular, the requirement that $\alpha^{\mathcal{G}}(\mathcal{V}_p) = \eta(p)$ is natural.

Recall π on $G(\mathbf{Q}_p)$ is the transfer of Π on $\mathcal{G}(\mathbf{Q}_p)$, and:

$$\textit{From now on, we assume that the Satake parameter of } \pi \textit{ is regular.} \tag{6.6}$$

We now show in this case that if (π, α) is a spin refinement, then the system of eigenvalues $\alpha^{\mathcal{G}}$ occurs in $\Pi^{\mathrm{Iw}_{\mathcal{G}}}$.

Recall in §6.1 we fixed an unramified character θ of $T(\mathbf{Q}_p)$ such that $\pi = \mathrm{Ind}_B^G\, \theta$ and $\theta_i\theta_{n+i} = \eta$ (cf. [DJR20, (43)], where θ is denoted $|\cdot|^{(2n-1)/2}\lambda$). Recall $\tau = \begin{pmatrix} 1_n & \\ & w_n \end{pmatrix} \in \mathcal{W}_G$, and that θ^τ satisfies the Asgari–Shahidi condition $\theta_i^\tau \theta_{2n+1-i}^\tau = \eta$. From [AS06, p.177(i)] and [AS14, Prop. 5.1], we see:

Proposition 6.11. *There is an unramified character $\theta_{\mathcal{G}}$ of $\mathcal{T}(\mathbf{Q}_p)$ such that:*

(i) $\Pi = \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}\, \theta_{\mathcal{G}}$ *is a (normalised) parabolic induction, and*

(ii) $\jmath(\theta_{\mathcal{G}}) = \theta^\tau$.

Remark 6.12. The τ is necessary as we are using the convention of Ash–Ginzburg (see §6.1). From this, we see that θ^τ is a more natural convention for GSpin computations. If we chose θ as in Asgari–Shahidi, we could have removed τ from (ii) and henceforth in this section.

Note that the Asgari–Shahidi condition $\theta_i\theta_{2n+1-i} = \eta$ is preserved by $\mathcal{W}_G^0$, reflecting its natural place in the spin world. On the other hand the Ash–Ginzburg condition is preserved instead by $\tau^{-1}\mathcal{W}_G^0\tau$.

By (2.13) and (6.6), our fixed choice of θ (hence θ^τ) fixes a bijection

$$\Delta_{\theta^\tau} : \{p\text{-refinements of } \pi\} \longrightarrow \mathcal{W}_G.$$

Lemma 6.13. *A p-refinement $\tilde{\pi} = (\pi, \alpha)$ is spin if and only if $\Delta_{\theta^\tau}(\tilde{\pi}) \in \mathcal{W}_G^0$.*

Proof. Conjugating Proposition 6.3 by τ, we see for each i, we have $\theta_i^\tau \cdot \theta_{2n+1-i}^\tau = \eta$ as characters of $\mathbf{Q}_p^\times$. Let $\sigma = \Delta_{\theta^\tau}(\tilde{\pi})$. By definition of $\mathcal{W}_G^0$, we see if $\sigma \in \mathcal{W}_G^0$ then

$$\theta_{\sigma(i)}^\tau \cdot \theta_{\sigma(2n+1-i)}^\tau = \eta, \tag{6.7}$$

whilst as the Satake parameter is regular, if $\sigma \notin \mathcal{W}_G^0$, then (6.7) fails for some i. Thus (6.7) holds for all i if and only if $\sigma \in \mathcal{W}_G^0$.

From the explicit description of $\alpha_{p,r} = \alpha(U_{p,r})$ from Proposition 2.5, we see that $\alpha_{p,n+s} = \eta(p)^s \alpha_{p,n-s}$ if and only if (6.7) holds for all i. The result follows. □

A *p-refinement of* Π is a tuple $\tilde{\Pi} = (\Pi, \alpha^{\mathcal{G}})$, where $\alpha^{\mathcal{G}} : \mathcal{H}_p^{\mathcal{G}} \to \overline{\mathbf{Q}}$ is a system of Hecke eigenvalues appearing in $\Pi^{\mathrm{Iw}_{\mathcal{G}}}$. We say $\tilde{\Pi}$ is *regular* if this system of eigenvalues appears in $\Pi^{\mathcal{G}}$ without multiplicity, i.e. the generalised eigenspace is a line. As in Proposition 2.5, after fixing the unramified character $\theta_{\mathcal{G}}$, such p-refinements correspond to elements $\sigma \in \mathcal{W}_{\mathcal{G}}$.

The following is our main motivation for the definition of spin p-refinement.

Proposition 6.14. *Suppose the Satake parameter of π is regular, and let $\tilde{\pi} = (\pi, \alpha)$ be a p-refinement. Then $\tilde{\pi}$ is a spin p-refinement if and only if there exists a p-refinement $(\Pi, \alpha^{\mathcal{G}})$ of Π such that $\alpha = \alpha^{\mathcal{G}} \circ \jmath^\vee$ as characters $\mathcal{H}_p \to \overline{\mathbf{Q}}$.*

Proof. By Proposition 6.9, $\tilde{\pi}$ is spin if and only if α factors through some $\alpha^{\mathcal{G}}$; so suffices to show that in this case, the system $\alpha^{\mathcal{G}}$ occurs in $\Pi^{\mathrm{Iw}_{\mathcal{G}}}$. Let $\sigma = \Delta_{\theta^\tau}(\tilde{\pi})$. By Lemma 6.13, $\sigma \in \mathcal{W}_G^0$.

Denote half the sum of the positive roots for G and $\mathcal{G}$ by

$$\rho_G = \left(\tfrac{2n-1}{2}, \tfrac{2n-3}{2}, \cdots, \tfrac{-(2n-3)}{2}, \tfrac{-(2n-1)}{2}\right), \qquad \rho_{\mathcal{G}} = \tfrac{2n-1}{2} f_1 + \tfrac{2n-3}{2} f_2 + \cdots + \tfrac{1}{2} f_n. \tag{6.8}$$

Note $\jmath(\rho_{\mathcal{G}}) = \rho_G$. By rewriting the formulation of Proposition 2.5, the $U_{p,r}$-eigenvalue of $\tilde{\pi}$ can be written as

$$\alpha_{p,r} = q^{\langle \rho_G, \nu_{p,r} \rangle_G} p^{\langle \theta^\tau, \nu_{p,r}^\sigma \rangle_G},$$

where we identify $\theta^\tau(\nu_{p,r}^\sigma(p)) = p^{\langle \theta^\tau, \nu_{p,r}^\sigma \rangle_G}$ under the natural extension of $\langle -, - \rangle_G$.

Since $\sigma \in \mathcal{W}_G^0$, by Proposition 6.6 it is of the form $\jmath(w)$ for some $\omega \in \mathcal{W}_{\mathcal{G}}$. Let $\tilde{\Pi} = (\Pi, \tilde{\alpha})$ be the p-refinement corresponding to ω; then by considering the characteristic polynomial of $\mathcal{U}_{p,r}$ on $\Pi_p^{\mathrm{Iw}_{\mathcal{G}}}$ (see [OST23, Cor. 3.16]), we see that the $\mathcal{U}_{p,r}$-eigenvalue attached to $\tilde{\Pi}$ is

$$\begin{aligned} \tilde{\alpha}(\mathcal{U}_{p,r}) = q^{\langle \rho_{\mathcal{G}}, \jmath^\vee(\nu_{p,r}) \rangle_{\mathcal{G}}} p^{\langle \theta_{\mathcal{G}}, \jmath^\vee(\nu_{p,r}^\omega) \rangle_{\mathcal{G}}} &= q^{\langle \rho_G, \nu_{p,r} \rangle_G} p^{\langle \theta_{\mathcal{G}}, \jmath^\vee(\nu_{p,r})^{\jmath^\vee(\sigma)} \rangle_{\mathcal{G}}} \\ &= q^{\langle \rho_G, \nu_{p,r} \rangle_G} p^{\langle \theta^\tau, \nu_{p,r}^\sigma \rangle_G} = \alpha_{p,r}, \end{aligned}$$

where in the second equality we have used $\jmath(\rho_{\mathcal{G}}) = \rho_G$ with (6.3), and in the third we have used Proposition 6.8 with (6.3). In particular, $\tilde{\alpha}(\mathcal{U}_{p,r}) = \alpha^{\mathcal{G}}(\mathcal{U}_{p,r})$ for all r.

It remains to show $\tilde{\alpha}(\mathcal{V}_p) = \alpha^{\mathcal{G}}(\mathcal{V}_p)$. Note $\alpha^{\mathcal{G}}(\mathcal{V}_p) = \eta(p)$ by Proposition 6.9. Also, $f_0^*(p)$ is central in $\mathcal{G}(\mathbf{Q}_p)$ by [AS06, Prop. 2.3], and the central character of Π is η by p.178 *op. cit*. Hence $\mathcal{V}_p$ acts on Π by $\eta(p)$. It follows that $\tilde{\alpha}(\mathcal{V}_p) = \eta(p)$, and we conclude that $\tilde{\alpha} = \alpha^{\mathcal{G}}$, as required. □

Remark 6.15. We finally indicate how spin p-refinements relate to the notion of Q-regular Q-refinement in [DJR20, Def. 3.5]. This was defined to be an element $T \in \mathcal{W}_G/\mathcal{W}_H$, equivalent to a choice of n-element subset $S_T \subset \{1,...,2n\}$, satisfying two conditions. Their condition (i) is our definition of regularity, and their condition (ii) guarantees that T lies in the image of the composition $\mathcal{W}_G^0 \hookrightarrow \mathcal{W}_G \to \mathcal{W}_G/\mathcal{W}_H$. One sees that spin p-refinements are in bijection with $T \in \mathcal{W}_G/\mathcal{W}_H$ satisfying (ii) together with an ordering of S_T.

7 Shalika p-refinements

We now define another class of p-refinements. Let $\tilde{\pi} = (\pi, \alpha)$ be a p-refinement. Recall we write $f \in \tilde{\pi}$ as shorthand for $f \in \pi^{\mathrm{Iw}_G}[\![U_{p,r} - \alpha(U_{p,r}) : 1 \leqslant r \leqslant 2n-1]\!]$ (and similarly for $W \in \mathcal{S}_\psi^\eta(\tilde{\pi})$). Recall that $z = \mathrm{diag}(p^{n-1}, p^{n-2}, ..., p, 1)$.

Definition 7.1. We say $\tilde{\pi}$ is a *Shalika p-refinement* if there exist $W \in \mathcal{S}_\psi^\eta(\tilde{\pi})$ and $\beta \geqslant 1$ such that

$$W \begin{pmatrix} w_n z^{2\beta} & \\ & 1 \end{pmatrix} \neq 0. \tag{7.1}$$

Note that via Proposition 5.2, this implies that the local zeta integral $\zeta(s, (u^{-1}t_{\mathfrak{p},n}^{\beta}) \cdot W, \chi)$ is non-vanishing for any smooth character $\chi : \mathbf{Q}_p^\times \to \mathbf{C}^\times$ of conductor p^β. Accordingly this generalises the condition [BDW, §2.8, condition (C2)] required to construct a p-adic L-function for $\tilde{\pi}$ via the methods of [BDW].

The following connects this new definition to the previous section.

Expectation 7.2. *A refinement $\tilde{\pi}$ is a Shalika refinement if and only if it is a spin refinement.*

Our main results of this section (Proposition 7.14 and Corollary 7.15) show one direction of this when π is regular: that all spin p-refinements are Shalika p-refinements. Moreover in this case we precisely compute the value (7.1).

Remark 7.3. Our expectation is motivated by folklore conjectures on classical families in eigenvarieties. More precisely, in §13 we will show that through any non-critical slope Shalika refinement $\tilde{\pi}$, the GL_{2n}-eigenvariety is étale over the pure weight space, and the component through $\tilde{\pi}$ contains a Zariski-dense set of symplectic points. The folklore conjecture says this should not be possible unless the family is transfer from GSpin_{2n+1}, whence we expect $\tilde{\pi}$ to have been a spin refinement by the results of the previous section. We have elaborated on this in the sequel paper [BGW25]; in particular, the above expectation is generalised to arbitrary parabolics in Conjecture C *op. cit.*, and substantial results towards it are proved in Theorem D. As a special case, Expectation 7.2 holds for non-critical slope refinements of regular weight.

7.1 Explicit eigenvectors for principal series

If π is regular, then $\tilde{\pi} = (\pi, \alpha)$ is a line; so one can use any non-zero eigenvector $W \in \mathcal{S}_{\psi}^{\eta}(\tilde{\pi})$ to test the condition for $\tilde{\pi}$ be a Shalika refinement. For this, we must write down explicit α-eigenvectors in $\mathcal{S}_{\psi}^{\eta}(\tilde{\pi})$. First, we consider α-eigenvectors in principal series representations.

Recall that in §6.1 we fixed an identification $\pi = \mathrm{Ind}_B^G \theta$, for θ chosen as in Ash–Ginzburg (i.e. $\theta_i\theta_{n+i} = \eta$ for all i). Recall also that our choice of θ fixes a bijection $\Delta_\theta : \{p\text{-refinements of } \pi\} \xrightarrow{\sim} \mathcal{W}_G$ (see (2.13)). For $\sigma \in \mathcal{W}_G$, let $\tilde{\pi}_\sigma = (\pi, \alpha_\sigma) := \Delta_\theta^{-1}(\sigma)$; every p-refinement is of the form $\tilde{\pi}_\sigma$ for some σ.

For a general $\sigma \in \mathcal{W}_G$, it turns out to be non-trivial to write down an explicit α_σ-eigenvector in $\mathrm{Ind}_B^G \theta$. However, it is very easy to write down such an eigenvector in the (different, but isomorphic) principal series representation $\mathrm{Ind}_B^G \theta^\sigma$. In particular, let

$$f^\sigma \in \mathrm{Ind}_B^G \theta^\sigma$$

be the unique function that is:

- Iwahori-invariant,
- supported on the big Bruhat cell $B(\mathbf{Q}_p) \cdot w_{2n} \cdot \mathrm{Iw}_G$, and
- normalised so that $f^\sigma(w_{2n}) = 1$.

Proposition 7.4. *$f^\sigma \in \mathrm{Ind}_B^G \theta^\sigma$ is an α_σ-eigenvector; i.e. for $r = 1, \ldots, 2n-1$, we have*

$$U_{p,r} f^\sigma = \alpha_\sigma(U_{p,r}) f^\sigma.$$

Proof. For such r, let P_r be the maximal standard parabolic subgroup with Levi $\mathrm{GL}_r \times \mathrm{GL}_{2n-r}$, and let J_r be the associated parahoric subgroup. Parahoric decomposition gives $J_r = N_{P_r}(\mathbf{Z}_p) \cdot (P_r^-(\mathbf{Z}_p) \cap J_r)$, where $N_{P_r} \subset P_r$ is the unipotent radical. Intersecting with Iw_G shows $\mathrm{Iw}_G = N_{P_r}(\mathbf{Z}_p) \cdot (P_r^-(\mathbf{Z}_p) \cap \mathrm{Iw}_G)$. As $t_{p,r}^{-1}(P_r^-(\mathbf{Z}_p) \cap \mathrm{Iw}_G) t_{p,r} \subset (P_r^-(\mathbf{Z}_p) \cap \mathrm{Iw}_G) \subset \mathrm{Iw}_G$, we deduce

$$\mathrm{Iw}_G t_{p,r} \mathrm{Iw}_G = N_{P_r}(\mathbf{Z}_p)(P_r^-(\mathbf{Z}_p) \cap \mathrm{Iw}_G) \cdot t_{p,r} \cdot \mathrm{Iw}_G = N_{P_r}(\mathbf{Z}_p) \cdot t_{p,r} \cdot \mathrm{Iw}_G,$$

and in particular we can decompose the double coset into single cosets via

$$\mathrm{Iw}_G t_{p,r} \mathrm{Iw}_G = \bigsqcup_{m \in M_{r,2n-r}(\mathbf{Z}_p)/M_{r,2n-r}(p\mathbf{Z}_p)} \begin{pmatrix} 1_r & m \\ 0 & 1_{2n-r} \end{pmatrix} \cdot t_{p,r} \cdot \mathrm{Iw}_G. \tag{7.2}$$

We have Bruhat decomposition

$$\mathrm{GL}_n(\mathbf{Q}_p) = \bigsqcup_{\rho \in W_n} B_n(\mathbf{Q}_p) \cdot \rho \cdot \mathrm{Iw}_{G_n}, \tag{7.3}$$

so it suffices to compute $U_{p,r} f^\sigma(\rho)$ for $\rho \in \mathcal{W}_G$. By (7.2), we have

$$U_{p,r} f^\sigma(\rho) = \sum_{m \in M_{r,2n-r}(\mathbf{Z}_p)/M_{r,2n-r}(p\mathbf{Z}_p)} f^\sigma\left(\rho \begin{pmatrix} 1_r & m \\ 0 & 1_{2n-r} \end{pmatrix} t_{p,r}\right).$$

Claim 7.5. $\rho \left(\begin{smallmatrix} 1_r & m \\ 0 & 1_{2n-r}\end{smallmatrix}\right) t_{p,r} \in B(\mathbf{Q}_p)\cdot w_{2n}\cdot \mathrm{Iw}_G$ *if and only if* $\rho = w_{2n}$ *and* $m \in M_{r,2n-r}(p\mathbf{Z}_p)$.

Proof of claim: Let $t'_{p,r} = w_{2n}t_{p,r}w_{2n} \in T(\mathbf{Q}_p)$. Then

$$B(\mathbf{Q}_p)w_{2n}\mathrm{Iw}_G = B(\mathbf{Q}_p)t'_{p,r}w_{2n}\mathrm{Iw}_G = B(\mathbf{Q}_p)w_{2n}t_{p,r}\mathrm{Iw}_G = B(\mathbf{Q}_p)w_{2n}\mathrm{Iw}_G^{r,-}t_{p,r},$$

where $\mathrm{Iw}_G^{r,-} = t_{p,r}\mathrm{Iw}_G t_{p,r}^{-1}$. Thus

$$\rho \left(\begin{smallmatrix} 1_r & m \\ 0 & 1_{2n-r}\end{smallmatrix}\right) t_{p,r} \in B(\mathbf{Q}_p)w_{2n}\mathrm{Iw}_G \iff \rho \left(\begin{smallmatrix} 1_r & m \\ 0 & 1_{2n-r}\end{smallmatrix}\right) \in B(\mathbf{Q}_p)w_{2n}\mathrm{Iw}_G^{r,-}.$$

Conjugating (7.3) by $t_{p,r}$, we obtain $\mathrm{GL}_{2n}(\mathbf{Q}_p) = \bigsqcup_{\rho'\in\mathcal{W}_G} B(\mathbf{Q}_p)\rho'\mathrm{Iw}_G^{r,-}$. It follows immediately that $\rho \left(\begin{smallmatrix} 1_r & m \\ 0 & 1_{2n-r}\end{smallmatrix}\right)$ is in the cell $w_{2n}\mathrm{Iw}_G^{r,-}$, and the claim follows. $\square$

We return to Proposition 7.4. The claim implies $U_{p,r}f^\sigma(\rho) = 0$ unless $\rho = w_{2n}$, and

$$\begin{aligned} U_{p,r}f^\sigma(w_{2n}) &= f^\sigma\left(w_{2n}t_{p,r}\right) = f^\sigma\left(t'_{p,r}w_{2n}\right) \\ &= \delta_B^{1/2}\theta^\sigma(t'_{p,r})f^\sigma(w_{2n}) = (\delta_B^{1/2}\theta^\sigma)^{w_{2n}}(t_{p,r})f^\sigma(w_{2n}) = \alpha_\sigma(U_{p,r})f^\sigma(w_{2n}), \end{aligned}$$

where the last equality is the equation for $\alpha_\sigma(U_{p,r})$ in Proposition 2.5. $\square$

7.2 Local Shalika models à la Ash–Ginzburg

Let $\sigma \in \mathcal{W}_G$, corresponding to the p-refinement $\tilde{\pi}_\sigma = (\pi, \alpha_\sigma)$. We've written down explicit α_σ-eigenvectors $f^\sigma \in \mathrm{Ind}_B^G\,\theta^\sigma$. We know (abstractly) that as $\mathrm{GL}_{2n}(\mathbf{Q}_p)$-representations we have

$$\mathrm{Ind}_B^G\,\theta^\sigma \cong \mathrm{Ind}_B^G\,\theta = \pi \cong \mathcal{S}_\psi^\eta(\pi); \tag{7.4}$$

Since these isomorphisms are Hecke equivariant, by Proposition 7.4 the image of f^σ under any choice of isomorphisms (7.4) lies in $\mathcal{S}_\psi^\eta(\tilde{\pi}_\sigma)$. This gives an eigenvector with which we can test, via (7.1), if $\tilde{\pi}_\sigma$ is a Shalika refinement. To do this we must make (7.4) explicit.

We first describe the right-hand isomorphism, recalling [AG94]. When $\theta_i\theta_{n+i} = \eta$, Ash–Ginzburg defined and studied an explicit intertwining $\mathcal{S}_\psi^\eta$ of $\mathrm{Ind}_B^G\,\theta$ into its Shalika model. This intertwining is the reason we choose θ as in Ash–Ginzburg (see §6.1).

For $f \in \mathrm{Ind}_B^G\,\theta$ and $g \in \mathrm{GL}_{2n}(\mathbf{Q}_p)$, as in [AG94, (1.3)] we let

$$\mathcal{S}_\psi^\eta(f)(g) := \int_{\mathrm{GL}_n(\mathbf{Z}_p)}\int_{M_n(\mathbf{Q}_p)} f\left[\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & X \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} k & \\ & k\end{smallmatrix}\right)g\right]\psi^{-1}(\mathrm{tr}(X))\eta^{-1}(\det(k))dXdk. \tag{7.5}$$

Ash–Ginzburg show (before Lemma 1.4) that $\mathcal{S}_\psi^\eta$ converges absolutely for θ in a space Ω.

Proposition 7.6. *If* $\mathrm{Ind}_B^G\,\theta$ *is regular,* $\mathcal{S}_\psi^\eta(f)$ *can be analytically continued to a non-zero intertwining*

$$\mathcal{S}_\psi^\eta : \mathrm{Ind}_B^G\,\theta \longrightarrow \mathcal{S}_\psi^\eta(\mathrm{Ind}_B^G\,\theta) = \mathcal{S}_\psi^\eta(\pi).$$

Proof. This is proved across Lemmas 1.4–1.6 of [AG94]. For any $g \in \mathrm{GL}_{2n}(\mathbf{Q}_p)$, they consider f varying with θ in a flat family, and thus make sense of $\mathcal{S}_\psi^\eta(f)(g)$ as a function of θ. They then define a quantity $P(\theta)$ and show that $P(\theta)\mathcal{S}_\psi^\eta(f)(g)$ is polynomial in θ. Moreover $P(\theta)$ is non-zero when $\mathrm{Ind}_B^G\,\theta$ is regular. Thus $\mathcal{S}_\psi^\eta(f)$ can be analytically continued from $\theta \in \Omega$ to arbitrary θ, as claimed.

The map is easily seen to be $\mathrm{GL}_{2n}(\mathbf{Q}_p)$-equivariant. Finally $\mathcal{S}_\psi^\eta(f) \in \mathcal{S}_\psi^\eta(\mathrm{Ind}_B^G\,\theta)$ (justifying the notation) using [BFG92, p.72–73]. □

7.3 Spin p-refinements under intertwining maps

In Expectation 7.2, we stated that we expect Shalika and spin refinements are equivalent. Let, then, $\tilde{\pi}_\sigma$ be a spin p-refinement. Recall from §6.1 that we have identified $\pi = \mathrm{Ind}_B^G\,\theta$, but that we could have replaced θ by any of its conjugates under $\mathcal{W}_G$; we now fix a choice.

Lemma 7.7. *There exists an unramified character $\theta : T(\mathbf{Q}_p) \to \mathbf{C}^\times$ such that:*

- $\pi = \mathrm{Ind}_B^G\,\theta$,
- $\theta_i\theta_{n+i} = \eta$ *for* $1 \leqslant i \leqslant n$,
- *and* $\Delta_\theta(\tilde{\pi}) = \tau = \begin{pmatrix} 1 & \\ & w_n \end{pmatrix}$.

Proof. First fix any θ with $\theta_i\theta_{n+i} = \eta$. By Lemma 6.13, we know

$$\Delta_{\theta^\tau}(\tilde{\pi}_\sigma) = \Delta_\theta(\tilde{\pi}_\sigma)\cdot\tau = \sigma\tau \in \mathcal{W}_G^0,$$

so that $\sigma \in \mathcal{W}_G^0\tau$. After replacing θ with its conjugate by $\tau^{-1}(\sigma\tau)^{-1}\tau \in \tau^{-1}\mathcal{W}_G^0\tau$, we have $\Delta_\theta(\tilde{\pi}) = \tau$; and this preserves the the Ash–Ginzburg property $\theta_i\theta_{n+i} = \eta$ by Remark 6.12. □

Henceforth we fix θ as in Lemma 7.7. Motivated by (7.4) and the discussion around it, we now compute an explicit intertwining

$$M_{w_n} : \mathrm{Ind}_B^G\,\theta^\tau \xrightarrow{\sim} \mathrm{Ind}_B^G\,\theta, \tag{7.6}$$

after which we can combine with Proposition 7.6 to obtain an explicit eigenvector $\mathcal{S}_\psi^\eta \circ M_{w_n}(f^\tau)$ in the spin p-refinement $\mathcal{S}_\psi^\eta(\tilde{\pi}_\tau)$.

Since we will study this using an inductive argument, it is convenient to consider a slightly more general setting. Let $\rho \in \mathcal{W}_n$, and $\nu_\rho := \begin{pmatrix} 1 & \\ & \rho \end{pmatrix} \in \mathcal{W}_G$. There is an isomorphism

$$M_\rho : \mathrm{Ind}_B^G\,\theta^{\nu_\rho} \xrightarrow{\sim} \mathrm{Ind}_B^G\,\theta = \pi,$$

which is unique up-to-scalar by Schur's Lemma, and which we now make precise on the big cell eigenvector f^{ν_ρ}. Note $\tau = \nu_{w_n}$, hence the notation M_{w_n} in (7.6).

Definition 7.8. For each $w, \sigma \in \mathcal{W}_G$:

- Let $f_w^\sigma \in \mathrm{Ind}_B^G\,\theta^\sigma$ be the unique Iw_G-invariant function supported on $B(\mathbf{Q}_p)\cdot w\cdot \mathrm{Iw}_G$ such that $f_w^\sigma(w) = 1$.

- For $\delta \in \mathcal{W}_n$ (and $\nu_\delta = \left(\begin{smallmatrix}1 & \\ & \delta\end{smallmatrix}\right)$ as above), let

$$F_\delta^\sigma := f^\sigma_{\left(\begin{smallmatrix} & w_n \\ \delta w_n & \end{smallmatrix}\right)} = f^\sigma_{\nu_\delta w_{2n}} \in \mathrm{Ind}_B^G\,\theta^\sigma.$$

- If $\sigma = 1$, we drop the superscript and just write $F_\delta = F_\delta^1 \in \mathrm{Ind}_B^G\,\theta = \pi$.

Note $F^\sigma_{1_n} = f^\sigma_{w_{2n}} = f^\sigma$, the big cell eigenvector from Proposition 7.4.

Lemma 7.9. *After possibly rescaling $M_\rho : \mathrm{Ind}_B^G\,\theta^{\nu_\rho} \to \pi$, we have*

$$M_\rho(f^{\nu_\rho}) = F_\rho + \sum_{\ell(\delta)<\ell(\rho)} c_\delta F_\delta, \qquad \textit{with } c_\delta \in \mathbf{C}. \tag{7.7}$$

Here $\ell(\delta)$ denotes the Bruhat length of any $\delta \in \mathcal{W}_n$.

Proof. For any simple reflection $s \in \mathcal{W}_G$ and $\sigma \in \mathcal{W}_G$, we have an intertwining isomorphism

$$M_\sigma^s : \mathrm{Ind}_B^G\,\theta^\sigma \longrightarrow \mathrm{Ind}_B^G\,\theta^{s\sigma}.$$

By [Cas80, Thm. 3.4] (see [DJ] for more details) this can be normalised so that for any $\delta \in \mathcal{W}_n$, there is a constant $c_{\delta,s} \in \mathbf{C}$ (depending also on θ) such that

$$M_\sigma^s(F_\delta^\sigma) = \begin{cases} F_{s\delta}^{s\sigma} + c_{\delta,s}F_\delta^{s\sigma} & : \ell(s\delta) = \ell(\delta)+1, \\ p^{-1}F_{s\delta}^{s\sigma} + c_{\delta,s}F_\delta^{s\sigma} & : \ell(s\delta) = \ell(\delta)-1. \end{cases}$$

Writing $\rho = s_1\cdots s_r$, M_ρ is the composition

$$M_\rho = M^{s_r}_{s_{r-1}\cdots s_1\rho} \circ \cdots \circ M^{s_1}_\rho.$$

The lemma is then obtained by induction on $\ell(\rho)$ via basic properties of Bruhat length. □

We finally map into the Shalika model. For $\delta \in \mathcal{W}_n$, recall $\mathcal{S}_\psi^\eta$ from (7.5), and let

$$W_\delta := \mathcal{S}_\psi^\eta(F_\delta) \in \mathcal{S}_\psi^\eta(\pi).$$

This is well-defined, since $F_\delta \in \mathrm{Ind}_B^G\,\theta = \pi$ and we chose θ as in Ash–Ginzburg.

Proposition 7.10. *Let $\tilde{\pi} = (\pi,\alpha)$ be a regular spin p-refinement, and write $\pi = \mathrm{Ind}_B^G\,\theta$ as in Lemma 7.7. The α-eigenspace $\mathcal{S}_\psi^\eta(\tilde{\pi})$ is spanned by an eigenvector of the form*

$$W_0 = W_{w_n} + \sum_{w_n \neq \delta \in \mathcal{W}_n} c_\delta W_\delta \qquad \in \mathcal{S}_\psi^\eta(\tilde{\pi}). \tag{7.8}$$

Proof. By Proposition 7.4, we know $f^\tau \in \mathrm{Ind}_B^G\,\theta^\tau$ is an α-eigenvector. By Lemma 7.9, its image in Ind_B^G under the intertwining M_{w_n} has the form

$$M_{w_n}(f^\tau) = \left[F_{w_n} + \sum_{w_n \neq \delta \in \mathcal{W}_n} c_\delta F_\delta\right] \in \mathrm{Ind}_B^G\,\theta = \pi.$$

By definition the image of this under $\mathcal{S}_\psi^\eta$ is W_0 (from (7.8)), which hence gives a non-zero eigenvector in $\mathcal{S}_\psi^\eta(\tilde{\pi})$. By regularity this space is a line, so this eigenvector spans. □

7.4 Non-vanishing of the local zeta integral at Iwahori level

From Proposition 7.10 we've written down an explicit eigenvector $W_0 \in \mathcal{S}_\psi^\eta(\pi)$ in any spin p-refinement. We now deduce that all spin p-refinements are Shalika p-refinements, by showing $W_0\begin{pmatrix} w_n z^{2\beta} & \\ & 1\end{pmatrix} \neq 0$, recalling $z = \mathrm{diag}(p^{n-1}, p^{n-2}, \dots, 1)$. We also compute this value exactly, and hence – via Proposition 5.2 – complete the computation of the local zeta integral for W_0.

We first show that $W_\delta\begin{pmatrix} w_n z^{2\beta} & \\ & 1\end{pmatrix} = 0$ unless $\delta = w_n$ (which means, by Proposition 5.2, that the local zeta integral vanishes for W_δ unless $\delta = w_n$). To do so, we examine when the integrand of (7.5) lies in the support of $F_\delta \in \mathrm{Ind}_B^G(\theta)$.

Proposition 7.11. *Let $\delta \in \mathcal{W}_n$, $X \in M_n(\mathbf{Q}_p)$ and $k \in \mathrm{GL}_n(\mathbf{Z}_p)$, and let $\beta \geqslant 1$. Then*

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} k & \\ & k \end{pmatrix}\begin{pmatrix} w_n z^{2\beta} & \\ & 1 \end{pmatrix} \in B_n(\mathbf{Q}_p)\begin{pmatrix} & w_n \\ \delta w_n & \end{pmatrix}\mathrm{Iw}_G \tag{7.9}$$

if and only if:

- *$\delta = w_n$ is the longest Weyl element,*
- *$k \in B_n(\mathbf{Z}_p) w_n \mathrm{Iw}_{G_n}$ and*
- *$k^{-1}X \in w_n z^{2\beta} M_n(\mathbf{Z}_p)$.*

Proof. Suppose (7.9) holds, and write

$$\begin{aligned}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} k & \\ & k \end{pmatrix}\begin{pmatrix} w_n z^{2\beta} & \\ & 1 \end{pmatrix}& \\ = \begin{pmatrix} A & B \\ & D \end{pmatrix}\begin{pmatrix} & w_n \\ \delta w_n & \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}&,\end{aligned} \tag{7.10}$$

where

$$A, D \in B_n(\mathbf{Q}_p),\ \ B \in M_n(\mathbf{Q}_p),\ \ a, d \in \mathrm{Iw}_{G_n},\ \ c \in pM_n(\mathbf{Z}_p),\ \ b \in M_n(\mathbf{Z}_p).$$

Expanding this out, we get the equality of matrices:

$$\begin{pmatrix} & k \\ k w_n z^{2\beta} & Xk \end{pmatrix} = \begin{pmatrix} B\delta w_n a + A w_n c & B\delta w_n b + A w_n d \\ D\delta w_n a & D\delta w_n b \end{pmatrix}.$$

This implies the following:

(1) $B = -A w_n c a^{-1} w_n^{-1} \delta^{-1}$ (from the top left entry).

(2) $-A w_n (c a^{-1} b - d) = k$ ((1) and top right), whence $A \in B_n(\mathbf{Z}_p)$ and

$$k \in B_n(\mathbf{Z}_p) \cdot w_n \cdot \mathrm{Iw}_{G_n}.$$

(3) $D\delta w_n a = k w_n z^{2\beta}$ (bottom left) which implies

$$k^{-1}D = w_n z^{2\beta} a^{-1} w_n \delta^{-1} \in w_n z^{2\beta}\,\mathrm{GL}_n(\mathbf{Z}_p).$$

(4) $D\delta w_n b k^{-1} = X$ (bottom right), which by (3) implies

$$k^{-1}X = k^{-1}D \cdot \delta w_n b k^{-1} \in w_n z^{2\beta} M_n(\mathbf{Z}_p).$$

We treat the cases $\delta \neq w_n$ and $\delta = w_n$ separately.

Case 1: $\delta \neq w_n$**.** Suppose there exist X and k such that (7.9) holds. We will derive a contradiction.

For $r \in \{1,...,2n-1\}$, let P_r be the parabolic with Levi $\mathrm{GL}_r \times \mathrm{GL}_{n-r}$, with associated (opposite) parahoric subgroup $\overline{J}_r \subset \mathrm{GL}_{2n}(\mathbf{Z}_p)$.

Claim 7.12. *If $\delta \neq w_n$, there exists $r \in \{1,...,2n-1\}$ such that*

$$B_n(\mathbf{Q}_p)\cdot \delta w_n \cdot \overline{J}_r \cap B_n(\mathbf{Q}_p)\cdot \overline{J}_r = \varnothing.$$

Proof of claim: Let $r := \min(i : \delta w_n(i) \neq i)$, which exists since $\delta w_n \neq 1$. Let $\mathcal{W}_{r,n-r}$ be the Weyl group of $\mathrm{GL}_r \times \mathrm{GL}_{n-r}$; it is the subgroup of $\mathcal{W}_n$ that preserves both $\{1,...,r\}$ and $\{r+1,...,n\}$. In particular, $\delta w_n \notin \mathcal{W}_{r,n-r}$.

We have the opposite Bruhat decomposition

$$G(\mathbf{Q}_p) = \bigsqcup_{\sigma \in \mathcal{W}_n/\mathcal{W}_{r,n-r}} B(\mathbf{Q}_p)\sigma \overline{J}_r.$$

Since $\delta w_n \notin \mathcal{W}_{r,n-r}$, the cells $B(\mathbf{Q}_p)\delta w_n \overline{J}_r$ and $B(\mathbf{Q}_p)\overline{J}_r$ are disjoint, giving the claim.

For r as in the claim, let μ be such that $z^{2\beta} = t_{p,r}\cdot\mu$, recalling $t_{p,r} = \mathrm{diag}(p,\dots,p,1,\dots,1)$ with r lots of p. We see that

$$t_{p,r}\,\mathrm{Iw}_n\, t_{p,r}^{-1} \subset \overline{J}_r.$$

Note that the valuation of μ under any positive root is non-negative, so $\mu^{-1}\overline{N}_n(\mathbf{Z}_p)\mu \subset \overline{N}_n(\mathbf{Z}_p)$, where $\overline{N}_n$ is the lower triangular unipotent.

We now analyse (3) from the list above. Multiply both sides by $t_{p,r}^{-1}$ to get

$$D\delta w_n a t_{p,r}^{-1} = k w_n \mu.$$

We know from (2) that we can write $k = \alpha \cdot w_n \cdot \beta$, with $\alpha \in B_n(\mathbf{Z}_p)$ and $\beta \in \mathrm{Iw}_{G_n}$. We have

$$k w_n \mu = \alpha w_n \beta w_n \mu \in B_n(\mathbf{Z}_p)\cdot \overline{\mathrm{Iw}_{G_n}} \cdot \mu.$$

But by the Iwahori decomposition, we see that

$$B_n(\mathbf{Z}_p)\cdot \overline{\mathrm{Iw}_{G_n}}\cdot \mu \subset B_n(\mathbf{Q}_p)\cdot \overline{\mathrm{Iw}_{G_n}} \subset B_n(\mathbf{Q}_p)\cdot \overline{J}_r.$$

We also have

$$k w_n \mu = D\delta w_n a t_{p,r}^{-1} \in B_n(\mathbf{Q}_p)\cdot \delta w_n t_{p,r}^{-1}\cdot \overline{J}_r \subset B_n(\mathbf{Q}_p)\cdot \delta w_n \cdot \overline{J}_r$$

as the element δw_n normalises the torus.

We must therefore have

$$kw_n\mu \in \big(B_n(\mathbf{Q}_p)\cdot \delta w_n \cdot \overline{J}_r\big) \cap \big(B_n(\mathbf{Q}_p)\cdot \overline{J}_r\big) \neq \varnothing,$$

a contradiction. In particular, there do not exist X and k such that (7.9) holds if $\delta \neq w_n$.

Case 2: $\delta = w_n$. We have shown that if (7.9) holds, then

$$k \in B(\mathbf{Z}_p)w_n\mathrm{Iw}_{G_n} \qquad \text{and} \qquad k^{-1}X \in w_n z^{2\beta} M_n(\mathbf{Z}_p). \tag{7.11}$$

Conversely, suppose (7.11), and write $k = Aw_nd$, with $A \in B_n(\mathbf{Z}_p)$ and $d \in \mathrm{Iw}_{G_n}$. Via the Iwahori decomposition, we may assume $d \in N_n(\mathbf{Z}_p)$ is upper unipotent.

For A and d as above, set $B = 0$ and $D = Az^{2\beta} \in B_n(\mathbf{Q}_p)$. Also set $c = 0$ and $a = z^{-2\beta}w_ndw_nz^{2\beta} \in \mathrm{Iw}_{G_n}$ (since $\beta \geqslant 1$). If we set $b = D^{-1}Xk \in M_n(\mathbf{Q}_p)$, then (7.10) holds. Clearly $\left(\begin{smallmatrix} A & B \\ 0 & D\end{smallmatrix}\right) \in B_n(\mathbf{Q}_p)$, so we are done if we can show $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{Iw}_G$. This will hold if $b \in M_n(\mathbf{Z}_p)$. Observe

$$\begin{aligned} b = D^{-1}Xk = z^{-2\beta}A^{-1}Xk = z^{-2\beta}w_ndk^{-1}Xk \\ \in z^{-2\beta}w_nd\cdot w_nz^{2\beta}M_n(\mathbf{Z}_p)k = aM_n(\mathbf{Z}_p)k \subset M_n(\mathbf{Z}_p), \end{aligned}$$

where in the second step we substitute D, the third we substitute $A^{-1} = w_ndk$, in the third we use (7.11), and in the fourth we substite $a = z^{-2\beta}w_ndw_nz^{2\beta}$. Thus $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{Iw}_G$, completing the proof. $\square$

From the proof, we also see the following:

Corollary 7.13. *Let Θ be any unramified character of $T(\mathbf{Q}_p)$, extended trivially to $B(\mathbf{Q}_p)$. If we have*

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} k & \\ & k \end{pmatrix}\begin{pmatrix} w_nz^{2\beta} & \\ & 1 \end{pmatrix} = \mathcal{B}\begin{pmatrix} & w_n \\ 1 & \end{pmatrix}\mathcal{I} \in B_n(\mathbf{Q}_p)\begin{pmatrix} & w_n \\ 1 & \end{pmatrix}\mathrm{Iw}_G, \tag{7.12}$$

then

$$\Theta(\mathcal{B}) = \Theta\begin{pmatrix} 1 & \\ & z^{2\beta} \end{pmatrix}.$$

Proof. The proposition gave necessary and sufficient conditions for (7.12). When they hold, we wrote down explicit values of $\mathcal{B} = \left(\begin{smallmatrix} A & \\ & D\end{smallmatrix}\right)$ and $\mathcal{I} = \left(\begin{smallmatrix} a & b \\ & d\end{smallmatrix}\right)$. By definition $A \in B_n(\mathbf{Z}_p)$ and $D = Az^{2\beta}$. The result follows easily. $\square$

We now put this all together. Recall that (without loss of generality, as at the start of §7.3) we have chosen θ so that identification $\Delta_\theta : \{p\text{-refinements}\} \xrightarrow{\sim} \mathcal{W}_G$ sends our fixed spin p-refinement $\tilde{\pi}$ to $\tau = \left(\begin{smallmatrix} & 1 \\ w_n & \end{smallmatrix}\right)$. By Proposition 2.5, the Hecke eigenvalue of $U_{p,r}$ on $\tilde{\pi}$ is

$$\alpha_{p,r} = \left[\delta_B^{1/2}\theta^\tau\right]^{w_{2n}}(t_{p,r}).$$

Note also that since $\tilde{\pi}$ is a spin p-refinement, by Definition 6.2 we have $\alpha_{p,n+s} = \eta(p)^s\alpha_{p,n-s}$ for $0 \leqslant s \leqslant n-1$. We also have the U_p-eigenvalue $\alpha_p = \alpha_{p,1}\cdots\alpha_{p,2n-1}$.

Proposition 7.14. *(i) If $\delta \neq w_n$, then $W_\delta\begin{pmatrix} w_n z^{2\beta} & \\ & 1\end{pmatrix} = 0$.*

(ii) We have

$$W_0\begin{pmatrix} w_n z^{2\beta} & \\ & 1\end{pmatrix} = W_{w_n}\begin{pmatrix} w_n z^{2\beta} & \\ & 1\end{pmatrix}$$
$$= \Upsilon'' \cdot p^{\beta n^2} \cdot \frac{\delta_B(t_p)^\beta}{\eta(\det z^\beta)} \cdot \left(\frac{\alpha_p}{\alpha_{p,n}}\right)^\beta \neq 0,$$

where $\Upsilon'' = \mathrm{vol}\left[B_n(\mathbf{Z}_p)\cdot w_n \cdot \mathrm{Iw}_{G_n}\right]$ is a constant independent of β.

Proof. In (7.5), we gave an integral representation of

$$W_\delta\begin{pmatrix} w_n z^{2\beta} & \\ & 1\end{pmatrix} = \mathcal{S}_\psi^\eta(F_\delta)\begin{pmatrix} w_n z^{2\beta} & \\ & 1\end{pmatrix}$$
$$= \int_{\mathrm{GL}_n(\mathbf{Z}_p)}\int_{M_n(\mathbf{Q}_p)} F_\delta\left[\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix} 1 & X\\ & 1\end{pmatrix}\begin{pmatrix} k & \\ & k\end{pmatrix}\begin{pmatrix} w_n z^{2\beta} & \\ & 1\end{pmatrix}\right]\psi^{-1}(\mathrm{tr}(X))\eta^{-1}(\det(k))dXdk.$$

(i) If $\delta \neq w_n$, then Proposition 7.11 shows the domain of F_δ in the integral is disjoint from the support of F_δ, hence the integrand (thus the integral) vanishes.

(ii) The first equality in (ii) follows from (7.8) and (i). Now, if $\delta = w_n$, then Proposition 7.11 means we can restrict the domain of the integral to $k \in B_n(\mathbf{Z}_p)w_n\mathrm{Iw}_{G_n}$ and $X \in kw_nz^{2\beta}M_n(\mathbf{Z}_p)$. Moreover:

- If $k \in B_n(\mathbf{Q}_p)w_n\mathrm{Iw}_{G_n}$ and $X \in kw_nz^{2\beta}M_n(\mathbf{Z}_p)$, then

$$F_{w_n}\left[\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix} 1 & X\\ & 1\end{pmatrix}\begin{pmatrix} k & \\ & k\end{pmatrix}\begin{pmatrix} w_n z^{2\beta} & \\ & 1\end{pmatrix}\right] = (\delta_B^{1/2}\theta)\left[\begin{pmatrix} 1 & \\ & z^{2\beta}\end{pmatrix}\right]F_{w_n}\begin{pmatrix} & w_n\\ 1 & \end{pmatrix}$$
$$= (\delta_B^{1/2}\theta)\left[\begin{pmatrix} 1 & \\ & z^{2\beta}\end{pmatrix}\right],$$

  using Corollary 7.13, Iwahori-invariance of F_{w_n}, and Definition 7.8.

- Since $X \in M_n(\mathbf{Z}_p)$, we have $\psi(\mathrm{tr}(X)) = 1$.
- $\eta(\det(k)) = 1$ as η is unramified.

In particular, the integral collapses to

$$W_{w_n}\begin{pmatrix} w_n z^{2\beta} & \\ & 1\end{pmatrix} = (\delta_B^{1/2}\theta)\left[\begin{pmatrix} 1 & \\ & z^{2\beta}\end{pmatrix}\right]\cdot\int_{B_n(\mathbf{Z}_p)w_n\mathrm{Iw}_{G_n}}\int_{kw_nz^{2\beta}M_n(\mathbf{Z}_p)} dXdk$$
$$= (\delta_B^{1/2}\theta)\left[\begin{pmatrix} 1 & \\ & z^{2\beta}\end{pmatrix}\right]\cdot \mathrm{vol}\left(B_n(\mathbf{Z}_p)\cdot w_n\cdot \mathrm{Iw}_{G_n}\right)\mathrm{vol}\left(z^{2\beta}M_n(\mathbf{Z}_p)\right), \tag{7.13}$$

where in the last step we made the change of variables $X \mapsto w_nk^{-1}Xkw_n$ before integrating. We easily see that

$$\mathrm{vol}\left(z^{2\beta}M_n(\mathbf{Z}_p)\right) = p^{\beta(n^2-n^3)} = p^{\beta n^2}\cdot\delta_B\begin{pmatrix} p^{n\beta} & \\ & 1\end{pmatrix}. \tag{7.14}$$

Now, note that $w_{2n}\tau\left(\begin{smallmatrix}1 & \\ & z\end{smallmatrix}\right)\tau^{-1}w_{2n}^{-1} = \left(\begin{smallmatrix}z & \\ & 1\end{smallmatrix}\right) = t_{p,1}\cdots t_{p,n-1}$ (from (2.9)). In particular,

$$\begin{aligned}
(\delta_B^{1/2}\theta)\left[\begin{pmatrix}1 & \\ & z^{2\beta}\end{pmatrix}\right] &= \delta_B^{1/2}\left[\begin{pmatrix}1 & \\ & z^{2\beta}\end{pmatrix}\right](\theta^\tau)^{w_{2n}}\left[t_{p,1}^{2\beta}\cdots t_{p,n-1}^{2\beta}\right]\\
&= \delta_B^{1/2}\left[\begin{pmatrix}1 & \\ & z^{2\beta}\end{pmatrix}\right](\delta_B^{-1/2})^{w_{2n}}\left[t_{p,1}^{2\beta}\cdots t_{p,n-1}^{2\beta}\right]\alpha_{p,1}^{2\beta}\cdots\alpha_{p,n-1}^{2\beta}\\
&= \delta_B^{1/2}\left[\begin{pmatrix}z^{2\beta} & \\ & z^{2\beta}\end{pmatrix}\right]\alpha_{p,1}^{\beta}\cdots\alpha_{p,n-1}^{\beta}\cdot[\eta(p)\alpha_{p,n+1}]^\beta\cdots[\eta(p)^{n-1}\alpha_{p,2n-1}]^\beta\\
&= \eta(\det z^\beta)^{-1}\delta_B\left[\begin{pmatrix}z^{\beta} & \\ & z^{\beta}\end{pmatrix}\right]\frac{\alpha_p^\beta}{\alpha_{p,n}^\beta}.
\end{aligned}\tag{7.15}$$

Here in the second equality we use that $\delta_B^{w_{2n}} = \delta_B^{-1}$, in the third we use the definition of spin refinements (Definition 6.2), and in the fourth that $\eta(p)\cdot\eta(p)^2\cdots\eta(p)^{n-1} = \eta(\det z)$ and $\alpha_{p,1}\cdots\alpha_{p,n-1}\cdot\alpha_{p,n+1}\cdots\alpha_{p,2n-1} = \alpha_p/\alpha_{p,n}$. Finally, we obtain (ii) after combining (7.13), (7.14) and (7.15), as

$$\delta_B(t_p)^\beta = \delta_B\begin{pmatrix}p^{n\beta} & \\ & 1\end{pmatrix}\delta_B\begin{pmatrix}z^\beta & \\ & z^\beta\end{pmatrix}. \qquad \square$$

Corollary 7.15. *Any regular spin p-refinement is a Shalika p-refinement.*

Proof. In Proposition 7.14, we started with an arbitrary regular spin p-refinement $\tilde{\pi}$, and exhibited in (ii) an eigenvector $W_0 \in \mathcal{S}_\psi^\eta(\tilde{\pi})$ with $W_0\begin{pmatrix}w_n z^{2\beta} & \\ & 1\end{pmatrix} \neq 0$. By definition, $\tilde{\pi}$ is thus a Shalika refinement. $\square$

8 Interlude: an automorphic summary

Everything so far has been classical/automorphic in nature. The rest of the paper is concerned with a p-adic interpolation of the previous sections. For the benefit of the reader, we now collect our running automorphic assumptions in one place, and summarise our main classical results.

Throughout the rest of the paper, we work with base field $\mathbf{Q}$, fix $K = \mathrm{Iw}_G\prod_{\ell\neq p}\mathrm{GL}_{2n}(\mathbf{Z}_\ell)$, and let π be a RACAR of $\mathrm{GL}_{2n}(\mathbf{A})$ of weight λ such that:

Conditions 8.1. (C1) π admits a global (η,ψ)-Shalika model, for a Hecke character η;

(C2) π_p is spherical and admits a regular spin p-refinement $\tilde{\pi}_p = (\pi_p, \alpha)$;

(C3) for each $\ell \neq p$, π_ℓ is spherical;

(C4) for each $r = 1, ..., 2n-1$, letting $\alpha_{p,r} = \alpha(U_{p,r})$ and $\alpha_{p,r}^\circ = \lambda(t_{p,r})\alpha_{p,r}$, we have

$$v_p(\alpha_{p,r}^\circ) = \left[v_p(\alpha_{p,r}) - \sum_{j=1}^{r}\lambda_{2n+1-j}\right] < \lambda_r - \lambda_{r+1} + 1.$$

We also write $\alpha_p = \alpha_{p,1}\cdots\alpha_{p,2n-1} = \alpha(U_p)$ and $\alpha_p^\circ = \alpha_{p,1}^\circ\cdots\alpha_{p,2n-1}^\circ$.

In this case we write $\tilde{\pi} = (\pi,\alpha)$ and call it a *p-refined RACAR satisfying (C1-4)*.

(C2) and (C3) imply that η is everywhere unramified, so $\eta = |\cdot|^{\mathsf{w}}$, where w is the purity weight of λ. (C4) ensures $\tilde{\pi}$ is a *non-critical slope* p-refinement, which we will explain in §10.4.

If $\tilde{\pi}$ satisfies (C1-4), choose

$$W_f = \otimes_\ell W_\ell \in \mathcal{S}_{\psi_f}^{\eta_f}(\pi_f, E) \tag{8.1}$$

as follows: for each $\ell \neq p$, let $W_\ell = W_\ell^\circ$ be the spherical test vector, and at p, let W_p be the vector $W_0 \in \mathcal{S}_{\psi_p}^{\eta_p}(\pi_p, E)$ from Proposition 7.10. Define

$$\phi_{\tilde{\pi}}^\pm := \Theta^\pm(W_f)/\Omega_\pi^\pm \in \mathrm{H}_c^t(S_K, \mathscr{V}_\lambda^\vee(L))_{\tilde{\pi}}^\pm, \tag{8.2}$$

where $\Theta^\pm/\Omega_\pi^\pm$ is defined in (3.3), and $(-)_{\tilde{\pi}}$ is the localisation at $\mathfrak{m}_{\tilde{\pi}}$ (see §2.4.3). Now recap:

- In Theorem 4.16, we showed that for χ of conductor $p^{\beta'}$ and $\beta = \max(\beta', 1)$,

$$\mathcal{E}_{B,\chi}^{j,\eta_0}\left(\phi_{\tilde{\pi}}^\pm\right) = \delta_B(t_B^{-\beta}) \cdot \Upsilon_B \cdot \lambda(t_B^\beta) \cdot \zeta_j(W_\infty^\pm) \cdot \frac{L^{(p)}(\pi \otimes \chi, j+1/2)}{\Omega_\pi^\pm} \cdot \zeta_p\Big(j + \tfrac{1}{2}, (u^{-1}t_B^\beta) \cdot W_p, \chi_p\Big).$$

- In Proposition 5.2, we showed that if χ_p is ramified,

$$\zeta_p\Big(j+\tfrac{1}{2}, (u^{-1}t_p^\beta\Big) \cdot W_p, \chi_p) = \Upsilon' \cdot \eta(\det z^\beta) \cdot p^{-\beta\frac{n^2+n}{2}} \cdot p^{\beta n j} \cdot \tau(\chi)^n \cdot \chi_p(\det(-w_n)) \cdot W_p\begin{pmatrix} w_n z^{2\beta} & \\ & 1 \end{pmatrix}.$$

- In Proposition 7.14, since $\tilde{\pi}$ is a spin p-refinement, we showed

$$W_p\begin{pmatrix} w_n z^{2\beta} & \\ & 1 \end{pmatrix} = \Upsilon'' \cdot p^{\beta n^2} \cdot \frac{\delta_B(t_p)^\beta}{\eta(\det z^\beta)} \cdot \left(\frac{\alpha_p}{\alpha_{p,n}}\right)^\beta.$$

  Note that we showed this by making specific 'good' identifications, namely fixing $\pi = \mathrm{Ind}_B^G\,\theta$ for an appropriate θ (chosen in Lemma 7.7) and then using the Shalika intertwining of (7.5). However, by Multiplicity One for Shalika models [Nie09, CS20], we immediately deduce this for all regular spin p-refinements $\tilde{\pi}$ independent of any choices.

Let $\gamma_{(pm)} := \Upsilon_B \cdot \Upsilon' \cdot \Upsilon''$ (recall that Υ_B depends on the constant γ appearing in [DJR20, (77)], which in turn depends on the fixed integer m in Definition 4.3). We deduce:

Corollary 8.2. *Let χ be a finite order Hecke character of conductor p^β, with $\beta \in \mathbf{Z}_{\geqslant 1}$, and let $j \in \mathrm{Crit}(\lambda)$. If $(-1)^j\chi_\infty\eta_\infty(-1) = \mp 1$, then $\mathcal{E}_\chi^{j,\eta_0}(\phi_{\tilde{\pi}}^\pm) = 0$. If $(-1)^j\chi_\infty\eta_\infty(-1) = \pm 1$, we have*

$$\mathcal{E}_{B,\chi}^{j,\eta_0}\left(\phi_{\tilde{\pi}}^\pm\right) = \gamma_{(pm)} \cdot \chi_p(\det(-w_n)) \cdot \left(\frac{\alpha_p^\circ}{\alpha_{p,n}}\right)^\beta \cdot Q'(\pi,\chi,j) \cdot \zeta_{\chi,j}(W_\infty) \cdot \frac{L^{(p)}(\pi \otimes \chi, j+1/2)}{\Omega_\pi^\pm},$$

where

$$Q'(\pi,\chi,j) := p^{\beta\left(nj + \frac{n^2-n}{2}\right)}\tau(\chi)^n.$$

Proof. We use $\alpha_p^\circ = \lambda(t_p)\alpha_p$. The rest is book-keeping. □

Finally, we record one more relevant result:

Proposition 8.3. *The L-vector space $\mathrm{H}_c^t(S_K, \mathscr{V}_\lambda^\vee(L))_{\tilde{\pi}}^\pm$ is 1-dimensional, generated by $\phi_{\tilde{\pi}}^\pm$.*

Proof. The generalised eigenspace in $\mathcal{S}_{\psi_f}^{\eta_f}(\pi_f^K)$ where $\mathcal{H}$ acts by $\psi_{\tilde{\pi}}$ is 1-dimensional; locally, at $\ell \neq p$ we have $\pi_\ell^{G(\mathbf{Z}_\ell)}$ is a line as π_ℓ is spherical, and at p, this is (C2). This line is generated by W_f by construction. The result now follows from Hecke-equivariance of $\Theta^\pm$. □

9 The local zeta integral at parahoric level

The local zeta integral we computed in Proposition 5.2 required the twisting character χ to be *ramified*. This is similar to previous papers [DJR20, BDW] on this topic, where p-adic L-functions were only shown to have the required interpolation property at ramified characters. However, this excludes the trivial character, which is typically the most interesting one. We finish Part II by computing the local zeta integral again in a different way which allows us to also handle *unramified* characters. Doing this at Iwahoric level appears to be very difficult. Instead, for this section only, we work at *Q-parahoric* level, for $Q \subset \mathrm{GL}_{2n}$ the parabolic with Levi $\mathrm{GL}_n \times \mathrm{GL}_n$ (the setting treated in [DJR20, BDW]). This allows us to directly strengthen the results of [DJR20, BDW] (see Proposition 14.3 below). In §12.4 we'll exploit the interpolation properties of p-adic L-functions to deduce the result at Iwahoric level, completing the present paper.

For compatibility with [DJR20, BDW], we work over a general local field rather than just over $\mathbf{Q}_p$. In particular, *for this section only* we adopt the following notation.

Notation 9.1. Let $F/\mathbf{Q}_p$ be a finite extension, with ring of integers $\mathcal{O}$ and maximal ideal $\mathfrak{p} = \varpi\mathcal{O}$, and let $q := \#\mathcal{O}/\mathfrak{p}$. Let δ be such that $\varpi^\delta\mathcal{O}$ be the different of $F/\mathbf{Q}_p$. We let $J \subset \mathrm{GL}_{2n}(\mathcal{O})$ denote the parahoric subgroup associated with Q, i.e., all elements which land in Q modulo $\mathfrak{p}$.

Let $\pi = \mathrm{Ind}_B^G\,\theta$ be a generic unramified principal series representation of $\mathrm{GL}_{2n}(F)$ admitting an (η,ψ)-Shalika model, for $\eta : F^\times \to \mathbf{C}^\times$ a smooth character and $\psi : F \to \mathbf{C}^\times$ the usual additive character (e.g. [DJR20, §4.1]). Note π is spherical. We will assume that (7.5) gives a non-trivial intertwining $\mathcal{S}_\psi^\eta : \mathrm{Ind}_B^G\,\theta \hookrightarrow \mathcal{S}_\psi^\eta(\pi)$ for an unramified character θ in the Ash–Ginzburg convention (cf. Proposition 7.6).

9.1 Normalisation of local data

Let $\tilde{\pi} = (\pi,\alpha)$ be a regular spin p-refinement of π (to Iwahori level), normalised without loss of generality as in the proof of Proposition 7.10. At parahoric level:

- Rather than Iw_G-invariant vectors we use J-invariant vectors.
- Attached to $\tilde{\pi}$ is a parahoric refinement $\tilde{\pi}^Q = (\pi, \alpha_{\mathfrak{p},n})$, for

$$\begin{aligned}\alpha_{\mathfrak{p},n} = \alpha(U_{\mathfrak{p},n}) &= \left[\delta_B^{1/2}\theta\right]^{w_{2n}}(t_{\mathfrak{p},n}) \\ &= \left[\delta_B^{1/2}\theta\right]\begin{pmatrix} 1_n & \\ & \varpi 1_n\end{pmatrix} = q^{n^2/2}\theta_{n+1}(\varpi)\cdots\theta_{2n}(\varpi)\end{aligned} \tag{9.1}$$

  (via Proposition 2.5), which we assume to be a Q-regular Q-refinement in the sense of [DJR20, Def. 3.5]. (In the notation *op. cit.*, it corresponds to the set $\tau = \{n+1,...,2n\}$).
- For a vector $W_\mathfrak{p} \in \pi_\mathfrak{p}^J$, the relevant twisted local zeta integral arising in Theorem 4.16 is $\zeta_\mathfrak{p}(j + \frac{1}{2}, (u^{-1}t_{\mathfrak{p},n}^\beta)\cdot W_\mathfrak{p}, \chi_\mathfrak{p})$.

Definition 9.2. Let $F_0 \in \mathrm{Ind}_B^G\,\theta$ be the unique J-invariant function supported on $B(F)\cdot w_{2n}\cdot J$ such that

$$F_0(w_{2n}) = \alpha_{\mathfrak{p},n}^\delta \cdot q^{-\delta n^2}.$$

Let $W_0 = \mathcal{S}_{\psi}^{\eta}(F_0) \in \mathcal{S}_{\psi}^{\eta}(\pi)$ be the Shalika vector associated with F_0 by (7.5).

This is the same choice of normalisation as [DJR20, §3.3] (hence in [BDW]). By Lemma 3.6 of that paper, we have $W_0(t_{\mathfrak{p},n}^{-\delta}) = 1$. In particular, $F_0 \in \tilde{\pi}^{Q}$ is non-zero, so generates this line.

9.2 The local zeta integral

Let $d^\times c$ be the Haar measure on $F^\times$ giving $\mathcal{O}^\times$ volume 1. Let $\chi : F^\times \to \mathbf{C}^\times$ be a finite order character of conductor $\mathfrak{p}^{\beta'}$, and denote its Gauss sum by

$$\tau(\chi) = \tau(\chi,\psi) = q^{\beta'}(1-q^{-1})\int_{\mathcal{O}^\times}\chi(c\varpi^{-\beta'-\delta})\psi(c\varpi^{-\beta'-\delta})d^\times c.$$

Our normalisation means that if χ arises from a Dirichlet character of p-power conductor, this recovers the usual Gauss sum

$$\tau(\chi) = \sum_{c\in(\mathbf{Z}/p^{\beta'})^\times}\chi(c)e^{2\pi i c/p^{\beta'}}.$$

In the rest of this section, we prove:

Proposition 9.3. *Let $\beta = \max(1,\beta')$. Then*

$$\zeta\Big(s,(u^{-1}t_{\mathfrak{p},n}^{\beta})\cdot W_0,\chi\Big) = q^{\beta n\left(s-\frac{n}{2}\right)+\delta n\left(s-\frac{n}{2}-1\right)}\cdot\chi(\det(-w_n))\cdot Q(\pi,\chi,s)$$

where

$$Q(\pi,\chi,s) = \begin{cases} q^{-\beta n}\frac{q^n}{(q-1)^n}\cdot\tau(\chi)^n & :\chi \text{ ramified},\\ \dfrac{\chi(\varpi)^{-n(\delta+1)}}{(1-q)^n}\displaystyle\prod_{i=n+1}^{2n}\frac{1-\theta_i\chi(\varpi)q^{1-s}}{1-\theta_i\chi(\varpi)q^{-s}} & :\chi \text{ unramified}.\end{cases}$$

Proof. We first rewrite the local zeta integrals. For simplicity, we write $G_n = \mathrm{GL}_n(F)$, $B_n = B_n(F)$, and $M_n = M_n(F)$. Since η is unramified, we have

$$\begin{aligned}\zeta(s,(u^{-1}t_{\mathfrak{p},n}^{\beta})\cdot W,\chi) &= \int_{G_n} W\left[\begin{pmatrix}h & \\ & 1\end{pmatrix}\begin{pmatrix}1 & -w_n\\ & 1\end{pmatrix}\begin{pmatrix}\varpi^\beta & \\ & 1\end{pmatrix}\right]\chi(\det h)|\det h|^{s-1/2}dh\\ &\overset{(7.5)}{=}\int_{G_n}\int_{K_n}\int_{M_n}F_0\left[\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix}1 & X\\ & 1\end{pmatrix}\begin{pmatrix}kh & \\ & k\end{pmatrix}\begin{pmatrix}1 & -w_n\\ & 1\end{pmatrix}\begin{pmatrix}\varpi^\beta & \\ & 1\end{pmatrix}\right]\\ &\qquad\times\chi(\det h)|\det h|^{s-1/2}\eta^{-1}\big(\det k\big)\psi\big(-\mathrm{tr}X\big)\,dXdkdh\\ &=\int_{G_n}\int_{K_n}\int_{M_n}F_0\left[\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix}kh\varpi^\beta & \\ & k\end{pmatrix}\begin{pmatrix}1 & \varpi^{-\beta}(h^{-1}k^{-1}Xk-w_n)\\ & 1\end{pmatrix}\right]\\ &\qquad\times\chi(\det h)|\det h|^{s-1/2}\psi\big(-\mathrm{tr}X\big)\,dXdkdh.\end{aligned}$$

Changing variables so that $kh\varpi^\beta$ becomes h and $\varpi^{-\beta}(h^{-1}k^{-1}Xk - w_n)$ becomes X will not change dh but changes dX to $|\det h|^n dX$. The integral becomes

$$\begin{aligned}
&= \int_{G_n}\int_{K_n}\int_{M_n} F_0\left[\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix} h & \\ & k\end{pmatrix}\begin{pmatrix} 1 & X\\ & 1\end{pmatrix}\right]\\
&\qquad \times \chi\Big(\det\varpi^{-\beta}k^{-1}h\Big)|\det\varpi^{-\beta}k^{-1}h|^{s-1/2}|\det h|^n\psi\Big(-\operatorname{tr}\left[hXk^{-1}+\varpi^{-\beta}hw_nk^{-1}\right]\Big)\,dXdkdh\\
&= q^{\beta n(s-1/2)}\int_{G_n}\int_{K_n}\int_{M_n} F_0\left[\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix} h & \\ & k\end{pmatrix}\begin{pmatrix} 1 & X\\ & 1\end{pmatrix}\right]\\
&\qquad\qquad \times \chi\Big(\det\varpi^{-\beta}k^{-1}h\Big)|\det h|^{s+n-1/2}\psi\Big(-\operatorname{tr}\left[hXk^{-1}+\varpi^{-\beta}hw_nk^{-1}\right]\Big)\,dXdkdh
\end{aligned}$$

Since $G_n = B_nK_n$ (with $B_n\cap K_n$ of volume 1) we may rewrite the above integral using $h = b\ell$ with $b\in B_n$ and $\ell\in K_n$ (so $|\det h| = |\det b|$):

$$\begin{aligned}
&= q^{\beta n(s-1/2)}\int_{B_n}\int_{K_n}\int_{K_n}\int_{M_n} F_0\left[\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix} b\ell & \\ & k\end{pmatrix}\begin{pmatrix} 1 & X\\ & 1\end{pmatrix}\right]\\
&\qquad\qquad \times \chi\Big(\det\varpi^{-\beta}k^{-1}b\ell\Big)|\det b|^{s+n-1/2}\psi\Big(-\operatorname{tr}\left[b\ell Xk^{-1}+\varpi^{-\beta}b\ell w_nk^{-1}\right]\Big)dXdkd\ell db\\
&= q^{\beta n(s-1/2)}\int_{B_n}\int_{K_n}\int_{K_n}\int_{M_n} \delta_B^{1/2}\theta\left[\begin{pmatrix} 1 & \\ & b\end{pmatrix}\right] F_0\left[\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix} \ell & \\ & k\end{pmatrix}\begin{pmatrix} 1 & X\\ & 1\end{pmatrix}\right]\\
&\qquad\qquad \times \chi\Big(\det\varpi^{-\beta}k^{-1}b\ell\Big)|\det b|^{s+n-1/2}\psi\Big(-\operatorname{tr}\left[b\ell Xk^{-1}+\varpi^{-\beta}b\ell w_nk^{-1}\right]\Big)dXdkd\ell db
\end{aligned}$$

Using J-invariance of F_0, we see that

$$F_0\left[\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix} \ell & \\ & k\end{pmatrix}\begin{pmatrix} 1 & X\\ & 1\end{pmatrix}\right] = F_0\left[\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix} 1 & \ell Xk^{-1}\\ & 1\end{pmatrix}\right].$$

The proof of [DJR20, Lemma 3.6] implies that this vanishes unless $X\in M_n(\mathcal{O})$, in which case it equals $F_0\left[\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\right] = F_0\left[w_{2n}\begin{pmatrix} w_n & \\ & w_n\end{pmatrix}\right] = F_0(w_{2n}) = \alpha_{\mathfrak{p},n}^\delta q^{-\delta n^2}$. The above integral becomes

$$\begin{aligned}
&= q^{\beta n(s-1/2)}\alpha_{\mathfrak{p},n}^\delta q^{-\delta n^2}\int_{B_n}\int_{K_n}\int_{K_n}\delta_B^{1/2}\theta\left[\begin{pmatrix} 1 & \\ & b\end{pmatrix}\right]\chi\Big(\det k^{-1}b\ell\varpi^{-\beta}\Big)\\
&\qquad\qquad \times |\det b|^{n+s-1/2}\psi\Big(-\operatorname{tr}\left[\varpi^{-\beta}b\ell w_nk^{-1}\right]\Big)\left(\int_{M_n(\mathcal{O})}\psi\Big(-\operatorname{tr} b\ell Xk^{-1}\Big)dX\right)dkd\ell db
\end{aligned}$$

Note that $\int_{M_n(\mathcal{O})}\psi(\operatorname{tr} AX)dX = 0$ unless $A\in\varpi^{-\delta}M_n(\mathcal{O})$, in which case the integral is $\operatorname{vol}(M_n(\mathcal{O})) = 1$. We conclude that the above inside integral vanishes unless $k^{-1}b\ell\in\varpi^{-\delta}M_n(\mathcal{O})$, i.e., if $b\in B_n\cap\varpi^{-\delta}M_n(\mathcal{O})$. The integral becomes

$$\begin{aligned}
&= q^{\beta n(s-1/2)}\alpha_{\mathfrak{p},n}^\delta q^{-\delta n^2}\int_{B_n\cap\varpi^{-\delta}M_n(\mathcal{O})}\int_{K_n}\int_{K_n}\delta_B^{1/2}\theta\left[\begin{pmatrix} 1 & \\ & b\end{pmatrix}\right]\\
&\qquad\qquad \times\chi\Big(\det k^{-1}b\ell\varpi^{-\beta}\Big)|\det b|^{n+s-1/2}\psi\Big(-\operatorname{tr}\left[\varpi^{-\beta}b\ell w_nk^{-1}\right]\Big)dkd\ell db.
\end{aligned}$$

Changing variables so that $\ell w_n k^{-1}$ becomes ℓ, and integrating out k, the integral becomes

$$= q^{\beta n(s-1/2)}\alpha_{\mathfrak{p},n}^{\delta} q^{-\delta n^2}\int_{B_n\cap\varpi^{-\delta}M_n(\mathcal{O})}\int_{K_n}\delta_B^{1/2}\theta\left[\begin{pmatrix}1 & \\ & b\end{pmatrix}\right] \times\chi\Big(\det b\ell\varpi^{-\beta}w_n\Big)|\det b|^{n+s-1/2}\psi\Big(-\operatorname{tr}\left[\varpi^{-\beta}b\ell\right]\Big)d\ell db.$$

Since $G_n = \bigsqcup_{\rho\in\mathcal{W}_n} B_n\rho\mathrm{Iw}_n$ we see $K_n = \bigsqcup_{\rho\in\mathcal{W}_n}(B_n\cap K_n)\rho\mathrm{Iw}_n$. In fact, since $B_n\cap K_n = N_n(\mathcal{O})T_n(\mathcal{O})$ and ρ normalises $T_n(\mathcal{O})$, we may replace $B_n\cap K_n$ with the unipotent radical $N_n(\mathcal{O})$ of $B_n(\mathcal{O})$. Write $\ell = n\rho i$ with $n\in N_n(\mathcal{O})$, $\rho\in\mathcal{W}_n$ and $i\in\mathrm{Iw}_n$. Using the Iwahori decomposition $I_n = N_n^-(\varpi\mathcal{O})B_n(\mathcal{O})$ we write $i=\overline{n}b_1$. The integral becomes

$$= q^{\beta n(s-1/2)}\alpha_{\mathfrak{p},n}^{\delta} q^{-\delta n^2}\sum_{\rho\in W_n}\int_{B_n\cap\varpi^{-\delta}M_n(\mathcal{O})}\int_{N_n(\mathcal{O})}\int_{N_n^-(\varpi\mathcal{O})}\int_{B(\mathcal{O})}\delta_B^{1/2}\theta\left[\begin{pmatrix}1 & \\ & b\end{pmatrix}\right] \times\chi\Big(\det b\rho b_1\varpi^{-\beta}w_n\Big)|\det b|^{n+s-1/2}\psi\Big(-\operatorname{tr}\left[\varpi^{-\beta}bn\rho\overline{n}b_1\right]\Big)db_1d\overline{n}dndb.$$

Changing variables so that b_1bn becomes b doesn't change the Haar measure, as $b_1\in B(\mathcal{O})$. Integrating out n and b_1, the integral becomes

$$= q^{\beta n(s-1/2)}\alpha_{\mathfrak{p},n}^{\delta} q^{-\delta n^2}\sum_{\rho\in W_n}\chi(\det\rho w_n\varpi^{-\beta})\int_{B_n\cap\varpi^{-\delta}M_n(\mathcal{O})}\int_{N_n^-(\varpi\mathcal{O})}\delta_B^{1/2}\theta\left[\begin{pmatrix}1 & \\ & b\end{pmatrix}\right] \times\chi\Big(\det b\Big)|\det b|^{n+s-1/2}\psi\Big(-\operatorname{tr}\left[\varpi^{-\beta}b\rho\overline{n}\right]\Big)d\overline{n}db.$$

Write

$$\overline{n} = \begin{pmatrix}1 & & 0\\ & \ddots & \\ x_{ij} & & 1\end{pmatrix}\qquad b = \varpi^{-\delta}\begin{pmatrix}t_1 & & u_{ij}\\ & \ddots & \\ 0 & & t_n\end{pmatrix},$$

where $t_i\in\mathcal{O}\backslash\{0\}$, $u_{ij}\in\mathcal{O}$ and $x_{ij}\in\varpi\mathcal{O}$. In this case, $db = \prod|t_i|^{i-n-1}\prod dt_i\prod du_{ij}$ and $d\overline{n} = \prod dx_{ij}$.

Fix $\rho\in W_n$. Writing $\rho\overline{n} = (m_{ij})$ we see that

$$\operatorname{tr} b\overline{n}\rho = \varpi^{-\delta}\left(\sum_{i=1}^{n} m_{ii}t_i + \sum_{i<j}u_{ij}m_{ji}\right).$$

If $\rho\neq 1$, there exist indices $i<j$ such that $m_{ji}=1$. Indeed, this is equivalent to there existing $j=\tau(i)>i$ where τ is the permutation of ρ. In this case, from the inner integral we may factor

$$\int_{\mathcal{O}}\psi(-\varpi^{-\delta-\beta}m_{ji}u_{ij})du_{ij} = 0,$$

as $\beta\geqslant 1$.

We conclude that all the terms of the sum vanish, except for the term $\rho = 1$. We see that the zeta integral becomes

$$= q^{\beta n(s-1/2)} \alpha_{\mathfrak{p},n}^{\delta} q^{-\delta n^2} \chi(\det w_n) \int \cdots \int \prod_{i=1}^{n} \theta_{n+i} |\cdot|^{s-1}(\varpi^{-\delta} t_i) \chi(\varpi^{-\beta-\delta} t_i) \psi\Big(-\varpi^{-\beta-\delta} t_i\Big)$$
$$\times \prod_{i<j} \psi\Big[-\varpi^{-\delta-\beta} x_{ji} u_{ji}\Big] \prod dx_{ij} dt_i du_{ij},$$

where $x_{ij} \in \varpi\mathcal{O}, u_{ij} \in \mathcal{O}, t_i \in \mathcal{O}\backslash\{0\}$.

The integral $\displaystyle\int_{\mathcal{O}} \psi\Big[\varpi^{-\delta-\beta} x_{ji} u_{ji}\Big] du_{ij}$ vanishes unless $x_{ji} \in \varpi^{\beta}\mathcal{O}$, in which case the integral is 1. Pulling out the resulting factor $\int \cdots \int \prod dx_{ij} = \mathrm{vol}(N^{-}(\varpi^{\beta}\mathcal{O})) = q^{-\beta\binom{n}{2}}$, the zeta integral is

$$= q^{\beta n(s-1/2)-\beta\binom{n}{2}} \alpha_{\mathfrak{p},n}^{\delta} q^{-\delta n^2} \chi(\det w_n)$$
$$\times \int \cdots \int \prod_{i=1}^{n} \theta_{n+i} |\cdot|^{s-1}(\varpi^{-\delta} t_i) \chi(\varpi^{-\beta-\delta} t_i) \psi\Big(-\varpi^{-\beta-\delta} t_i\Big) \prod dt_i.$$

Now we consider separately the cases where χ is ramified or unramified.

χ **ramified:** When χ is ramified with conductor β then

$$\int \theta_{n+i} |\cdot|^{s-1}(\varpi^{-\delta} t_i) \chi(\varpi^{-\beta-\delta} t_i) \psi\Big(-\varpi^{-\beta-\delta} t_i\Big) dt_i$$
$$= \chi(-1) \sum_{k=0}^{\infty} \theta_{n+i} |\cdot|^{s-1}(\varpi^{k-\delta}) \int_{\mathcal{O}^\times} \chi\Big(t_i \varpi^{k-\delta-\beta}\Big) \psi\Big(t_i \varpi^{k-\delta-\beta}\Big) dt_i$$
$$= \chi(-1) q^{\delta(s-1)} \cdot q^{-\beta} \frac{q}{q-1} \theta_{n+i}(\varpi)^{-\delta} \tau(\chi),$$

since the integral vanishes unless $k = 0$, whence it is the Gauss sum. Using $\chi(-1)^n \chi(\det w_n) = \chi(\det(-w_n))$, the zeta integral thus simplifies to

$$q^{\beta n(s-1/2)-\beta\binom{n}{2}} \cdot \alpha_{\mathfrak{p},n}^{\delta} q^{-\delta n^2} \cdot \chi(\det w_n) \prod_{i=n+1}^{2n} \left[\chi(-1) \theta_i(\varpi)^{-\delta} q^{\delta(s-1)} \cdot q^{-\beta} \frac{q}{q-1} \tau(\chi)\right]$$
$$= q^{\beta n(s-n/2)} \cdot \alpha_{\mathfrak{p},n}^{\delta} q^{-\delta n^2} \cdot \chi(\det(-w_n)) \cdot q^{\delta n(s-1)} \cdot \left[\prod_{i=n+1}^{2n} \theta_i(\varpi)^{-\delta}\right] \cdot Q(\pi, \chi, s).$$

Finally note that by (9.1), we have $\prod_{i=n+1}^{2n} \theta_i(\varpi)^{-\delta} = (q^{-n^2/2} \alpha_{\mathfrak{p},n})^{-\delta}$. Simplifying yields

$$= q^{\beta n(s-n/2)} q^{\delta n(s-1-n/2)} \cdot \chi(\det(-w_n)) \cdot Q(\pi, \chi, s), \tag{9.2}$$

as required.

χ **unramified:** If χ is unramified, then $\beta = 1$ and, dropping indices for simplicity,

$$\begin{aligned}
\int_{\mathcal{O}\setminus\{0\}} & \theta|\cdot|^{s-1}(t\varpi^{-\delta})\chi(t\varpi^{-\delta-1})\psi(-t\varpi^{-\delta-1})dt \\
&= \chi(-1)\sum_{k=0}^{\infty} q^{-k-(k-\delta)(s-1)}\theta(\varpi)^{k-\delta}\chi(\varpi)^{k-\delta-1}\int_{\mathcal{O}^\times}\psi(t\varpi^{k-\delta-1})d^\times t \\
&= \chi(-1)\frac{1}{1-q}q^{\delta(s-1)}\theta(\varpi)^{-\delta}\chi(\varpi)^{-\delta-1} + \sum_{k=1}^{\infty} q^{-k-(k-\delta)(s-1)}\theta(\varpi)^{k-\delta}\chi(\varpi)^{k-\delta-1} \\
&= \chi(-1)q^{-1-(1-\delta)(s-1)}\theta(\varpi)^{1-\delta}\chi(\varpi)^{-\delta}\left(\sum_{k=0}^{\infty}\left(\frac{\theta\chi(\varpi)}{q^s}\right)^k + \frac{q^s\theta\chi(\varpi)^{-1}}{1-q}\right) \\
&= \chi(-1)q^{-1-(1-\delta)(s-1)}\theta(\varpi)^{1-\delta}\chi(\varpi)^{-\delta}\left(\frac{1}{1-\frac{\theta\chi(\varpi)}{q^s}} + \frac{q^s\theta\chi(\varpi)^{-1}}{1-q}\right) \\
&= \chi(-1)\frac{1}{1-q}q^{\delta(s-1)}\theta(\varpi)^{-\delta}\chi(\varpi)^{-\delta-1}\frac{1-\frac{\theta\chi(\varpi)}{q^{s-1}}}{1-\frac{\theta\chi(\varpi)}{q^s}},
\end{aligned}$$

using that $\int_{\mathcal{O}^\times}\psi(t\varpi^{k-\delta-1})d^\times t$ is 0 if $k<0$, is 1 if $k>0$, and is $1/(1-q)$ if $k=0$.

Substituting this in, we find the zeta integral is

$$\begin{aligned}
& q^{\beta n(s-1/2)-\beta\binom{n}{2}}\cdot\alpha_{\mathfrak{p},n}^{\delta}q^{-\delta n^2}\cdot\chi(\det w_n)\prod_{i=n+1}^{2n}\left[\chi(-1)\frac{q^{\delta(s-1)}}{1-q}\theta_i(\varpi)^{-\delta}\cdot\chi(\varpi)^{-\delta-1}\frac{1-\frac{\theta\chi(\varpi)}{q^{s-1}}}{1-\frac{\theta\chi(\varpi)}{q^s}}\right] \\
&= q^{\beta n(s-n/2)}\cdot\alpha_{\mathfrak{p},n}^{\delta}q^{-\delta n^2}\cdot\chi(\det(-w_n))\cdot q^{\delta n(s-1)}\cdot\left[\prod_{i=n+1}^{2n}\theta_i(\varpi)^{-\delta}\right]\cdot Q(\pi,\chi,s).
\end{aligned}$$

We conclude that this has the required shape via the same computations as in (9.2). ☐

Part III
p-adic Interpolation

10 Overconvergent cohomology

We recap the theory of overconvergent cohomology in p-adic families, as developed for example in [Urb11, Han17]. All of this material is explained in depth *op. cit.*, so we are terse with details.

10.1 Weight spaces

Recall $X^*(T)$, $X_0^*(T)$, $X^*(H)$ and $X_0^*(H)$ from §2.2. If $K \subset G(\mathbf{A}_f)$ is an open compact subgroup, let $\overline{Z}_K$ be the p-adic closure of $Z_G(\mathbf{Q})\cap K$.

Definition 10.1. (i) The *weight space* for G of level K is $\mathscr{W}_K^G := \mathrm{Spf}(\mathbf{Z}_p[[T(\mathbf{Z}_p)/\overline{Z}_K]])$. This is a rigid analytic space whose L-points, for $L \subset \mathbf{C}_p$ any extension of $\mathbf{Q}_p$, are given by

$$\mathscr{W}_K^G(L) = \mathrm{Hom}_{\mathrm{cont}}(T(\mathbf{Z}_p)/\overline{Z}_K, L^\times).$$

(ii) Any element of $\mathscr{W}_K^G(L) \cap X_+^*(T)$ is called an *algebraic weight*.

(iii) The *pure weight space* $\mathscr{W}_{K,0}^G$ is the Zariski-closure of $X_0^*(T) \cap \mathscr{W}_K^G$ in $\mathscr{W}_K^G$.

We view all of these weights as characters of $T(\mathbf{Z}_p)$ trivial on $\overline{Z}_K$. If λ is pure of weight w and $z \in \overline{Z}_K$, then $\lambda(z) \subset \{\pm 1\}$, so if λ is trivial on $\overline{Z}_K$, then this is also true of all weights in a neighbourhood of λ in the pure weight space. Since the level subgroup will always be fixed, we will henceforth always drop it from notation.

A weight $\lambda \in \mathscr{W}^G$ decomposes as $\lambda = (\lambda_1, ..., \lambda_{2n})$, where each λ_i is a character of $\mathbf{Z}_p^\times$. We see $\lambda \in \mathscr{W}_0^G$ if and only if there exists $\mathsf{w}_\lambda \in \mathrm{Hom}_{\mathrm{cts}}(\mathbf{Z}_p^\times, L^\times)$ such that $\lambda_i \cdot \lambda_{2n+1-i} = \mathsf{w}_\lambda$ for all $1 \leqslant i \leqslant n$. The space $\mathscr{W}_0^G$ has dimension $n+1$ (corresponding to changing $\lambda_1, ..., \lambda_n, \mathsf{w}_\lambda$).

If $\Omega \subset \mathscr{W}_0^G$ is an affinoid in the pure weight space, then Ω is equipped with a character $\chi_\Omega : T(\mathbf{Z}_p) \to \mathcal{O}_\Omega^\times$ such that for any $\lambda \in \Omega(L)$, the composition $T(\mathbf{Z}_p) \xrightarrow{\chi} \mathcal{O}_\Omega^\times \xrightarrow{\mathrm{sp}_\lambda} L^\times$ is the character attached to λ, where sp_λ is evaluation at λ. Moreover, write $\chi_\Omega = (\chi_{\Omega,1}, ..., \chi_{\Omega,2n})$, where each $\chi_{\Omega,i}$ is a character of $\mathbf{Z}_p^\times$; then since Ω is pure, there exists a character

$$\mathsf{w}_\Omega : \mathbf{Z}_p^\times \to \mathcal{O}_\Omega^\times \quad \text{such that } \mathsf{w}_\Omega(z) = \chi_{\Omega,i}(z) \cdot \chi_{\Omega,2n+1-i}(z)\ \forall z \in \mathbf{Z}_p^\times, 1 \leqslant i \leqslant n. \tag{10.1}$$

10.2 Algebraic and analytic induction

For $\lambda \in X_0^*(T)$, recall V_λ is the algebraic representation of G of highest weight λ. The L-points of V_λ can be described explicitly as the algebraic induction, whose points are algebraic functions $f : G(\mathbf{Q}_p) \to L$ such that

$$f(n^- t g) = \lambda(t) f(g) \quad \forall n^- \in \overline{N}(\mathbf{Q}_p), t \in T(\mathbf{Q}_p), g \in G(\mathbf{Q}_p). \tag{10.2}$$

The action of $\gamma \in G(\mathbf{Q}_p)$ on $f \in V_\lambda(L)$ is by $(\gamma \cdot f)(g) := f(g\gamma)$.

As $G(\mathbf{Z}_p)$ is Zariski-dense in $G(\mathbf{Q}_p)$, we can identify $V_\lambda(L)$ with the set of algebraic $f : G(\mathbf{Z}_p) \to L$ satisfying (10.2). We have an integral subspace $V_\lambda(\mathcal{O}_L)$ of f such that $f(G(\mathbf{Z}_p)) \subset \mathcal{O}_L$, and we let $V_\lambda^\vee(\mathcal{O}_L) = \mathrm{Hom}_{\mathcal{O}_L}(V_\lambda(\mathcal{O}_L), \mathcal{O}_L)$.

If I is a p-adic Lie group and R a $\mathbf{Q}_p$-algebra, let $\mathcal{A}(I,R)$ denote the space of locally analytic functions $I \to R$. Let $\mathrm{Iw}_G \subset G(\mathbf{Z}_p)$ be the Iwahori subgroup, and $\Omega \subset \mathscr{W}_0^G(L)$ an affinoid, with attached character χ_Ω; we allow $\Omega = \{\lambda\}$, whence $\chi_\Omega = \lambda$. Recall the analytic induction spaces:

Definition 10.2. Let $\mathcal{A}_\Omega$ be the space of $f \in \mathcal{A}(\mathrm{Iw}_G, \mathcal{O}_\Omega)$ such that

$$f(n^- t g) = \chi_\Omega(t) f(g) \text{ for all } n^- \in \overline{N}(p\mathbf{Z}_p),\ t \in T(\mathbf{Z}_p), \text{ and } g \in \mathrm{Iw}_G. \tag{10.3}$$

Via Iwahori decomposition and (10.3), restriction to $N(\mathbf{Z}_p)$ identifies $\mathcal{A}_\Omega(L)$ with $\mathcal{A}(N(\mathbf{Z}_p), \mathcal{O}_\Omega)$.

As any $f \in V_\lambda(L)$ is determined by its restriction to the Zariski-dense subgroup Iw_G, we see $V_\lambda(L)$ is the (finite Banach) subspace of $f \in \mathcal{A}_\lambda$ that are algebraic on Iw_G (e.g. [Urb11, §3.2.8]).

Definition 10.3. Define $\mathcal{D}_\Omega := \mathrm{Hom}_{\mathrm{cont}}(\mathcal{A}_\Omega, \mathcal{O}_\Omega)$, a compact Fréchet $\mathcal{O}_\Omega$-module.

If $\Sigma \subset \Omega$ is a closed affinoid, then $\mathcal{D}_\Omega \otimes_{\mathcal{O}_\Omega} \mathcal{O}_\Sigma \cong \mathcal{D}_\Sigma$.

10.3 Hecke actions and slope decompositions

Let $K \subset G(\mathbf{A}_f)$ be an open compact subgroup such that $K_p \subset \mathrm{Iw}_G$. Let $\Omega \subset \mathscr{W}_0^G$ be an affinoid. Recall $t_p = \mathrm{diag}(p^{2n-1}, p^{2n-2}, ..., p, 1)$, and let $\Delta_p \subset G(\mathbf{Q}_p)$ be the semigroup generated by Iw_G and t_p. There is a left-action of Δ_p on $\mathcal{D}_\Omega$ as follows:

– $k \in \mathrm{Iw}_G$ acts on $f \in \mathcal{A}_\Omega$ by $(k * f)(g) := f(gk)$, inducing a dual left action on $\mu \in \mathcal{D}_\Omega$ by $(k * \mu)(f(g)) := \mu(f(gk^{-1}))$.
– t_p acts on the left on $B(\mathbf{Z}_p)$ by $t_p * b := t_p b t_p^{-1}$. Since any $f \in \mathcal{A}_\Omega$ is uniquely determined by its restriction to $B(\mathbf{Z}_p)$, this induces a left action of t_p on $\mu \in \mathcal{D}_\Omega$ by $(t_p * \mu)(f(b)) = \mu(f(t_p b t_p^{-1}))$.

As $\bigcap_{i \geqslant 0} t_p^i N(\mathbf{Z}_p) t_p^{-i} = 1$, we have $t_p \in T^{++}$ in the notation of [Han17, §2]. Thus we get an $\mathcal{O}_\Omega$-linear controlling operator $U_p^\circ := [K_p t_p K_p]$ on the cohomology groups $\mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{D}_\Omega)$. Up to shrinking Ω, the $\mathcal{O}_\Omega$-module $\mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{D}_\Omega)$ admits a slope decomposition with respect to U_p° (see [Han17, Def. 2.3.1]). For $h \in \mathbf{Q}_{\geqslant 0}$ we let $\mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{D}_\Omega)^{\leqslant h}$ denote the subspace of elements of slope at most h, and note that it is an $\mathcal{O}_\Omega$-module of finite type.

Remark 10.4. The operator U_p° preserves the integral structure $\mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{D}_\lambda(\mathcal{O}_L)) \subset \mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{D}_\lambda(L))$. We also have a $*$-action of Δ_p on $V_\lambda^\vee(L)$, defined identically, giving an operator U_p° on $\mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(L))$ that preserves its natural integral subspace. If U_p denotes the automorphic Hecke operator from §2.4.3, one may check $U_p^\circ = \lambda(t_p) \cdot U_p$. This is all explained in the remark in [BDW, §3.5].

10.4 Non-critical slope refinements

Let $\lambda \in X_0^*(T)$ be a pure dominant integral weight, K as above, and $L/\mathbf{Q}_p$ a finite extension. The natural inclusion of $V_\lambda(L) \subset \mathcal{A}_\lambda(L)$ induces dually a surjection $r_\lambda : \mathcal{D}_\lambda(L) \longrightarrow V_\lambda^\vee(L)$, which is equivariant for the $*$-actions of Δ_p. This induces a map

$$r_\lambda : \mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{D}_\lambda(L)) \longrightarrow \mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(L)), \tag{10.4}$$

equivariant for the $*$-actions of Δ_p (hence U_p°) on both sides.

Definition 10.5. Let $\tilde{\pi} = (\pi, \alpha)$ be a p-refined RACAR of $G(\mathbf{A})$ of weight λ. We say $\tilde{\pi}$ is *non-critical* if r_λ restricts to an isomorphism

$$r_\lambda : \mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{D}_\lambda(L))_{\tilde{\pi}} \xrightarrow{\sim} \mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(L))_{\tilde{\pi}}$$

of generalised eigenspaces. We say $\tilde{\pi}$ is *strongly non-critical* if this is also true for $\mathrm{H}^\bullet$ (i.e., if $\tilde{\pi}$ is non-critical for $\mathrm{H}^\bullet$ *and* for $\mathrm{H}^\bullet_{\mathrm{c}}$ as in [BW21c, Rem. 4.6]).

Definition 10.6. Let $\tilde{\pi}$ be a p-refinement of π. For $1 \leqslant r \leqslant 2n-1$, let $\alpha_{p,r}^\circ = \lambda(t_{p,r})\alpha_{p,r}$, the corresponding $U_{p,r}^\circ$-eigenvalue. We say $\tilde{\pi}$ has *non-critical slope* if for each $1 \leqslant r \leqslant 2n-1$, we have

$$v_p\big(\alpha_{p,r}^\circ\big) < \lambda_r - \lambda_{r+1} + 1.$$

Theorem 10.7 (Classicality)**.** *If $\tilde{\pi}$ has non-critical slope, then it is strongly non-critical.*

Proof. This is [BW21c, Thm. 4.4, Rem. 4.6], explained in Examples 4.5 *op. cit.* □

11 p-adic interpolation of branching laws

A *branching law* describes how an irreducible representation of G decomposes upon restriction to H. Of particular interest to us is the branching law given by Lemma 4.11, rephrased below in Lemma 11.1, which provides a representation-theoretic interpretation of the Deligne-critical range. We now give a p-adic interpolation of this branching law.

11.1 Classical branching laws, revisited

Let $\lambda \in X_0^*(T)$ be a pure algebraic weight, with purity weight w. By dualising, we get the following equivalent formulation of Lemma 4.11:

Lemma 11.1. *For $j \in \mathbf{Z}$, we have $j \in \mathrm{Crit}(\lambda)$ if and only if $\dim \mathrm{Hom}_{H(\mathbf{Z}_p)}(V^H_{(-j,\mathsf{w}+j)}, V_\lambda) = 1$, that is if and only if $V_\lambda|_{H(\mathbf{Z}_p)}$ contains $V^H_{(-j,\mathsf{w}+j)}$ with multiplicity 1.*

We now give a conceptual description of the generators of $V^H_{(-j,\mathsf{w}+j)} \subset V_\lambda|_{H(\mathbf{Z}_p)}$. For $n \geqslant 2$, define weights

$$\alpha_1 = (1,0,...,0,-1),\ \ \alpha_2 = (1,1,0,...,0,-1,-1),\ \ ...,\ \ \alpha_{n-1} = (1,...,1,0,0,-1,...,-1),$$
$$\alpha_0 = (1,...,1,1,...,1), \quad \alpha_n = (1,...,1,0,...,0). \tag{11.1}$$

For $n = 1$, we define $\alpha_0 = (1,1)$ and $\alpha_1 = (1,0)$. For any n, if $\lambda = (\lambda_1,...,\lambda_{2n}) \in X_0^*(T)$ is a dominant pure algebraic weight, then we easily see that

$$\lambda = (\lambda_1 - \lambda_2)\alpha_1 + (\lambda_2 - \lambda_3)\alpha_2 + \cdots + (\lambda_n - \lambda_{n+1})\alpha_n + \lambda_{n+1}\alpha_0, \tag{11.2}$$

where each coefficient is a non-negative integer except perhaps λ_{n+1}, which can be negative.

Notation 11.2.
- (i) If $1 \leqslant i \leqslant n-1$, then $\mathrm{Crit}(\alpha_i) = \{0\}$. Since the purity weight of α_i is 0, by Lemma 11.1 there is a non-zero vector $v_{(i)} \in V_{\alpha_i}(\mathbf{Q}_p)$ upon which the action of $H(\mathbf{Z}_p)$ is trivial, and $v_{(i)}$ is unique up to $\mathbf{Q}_p^\times$-multiple.
- (ii) We have $\mathrm{Crit}(\alpha_n) = \{-1,0\}$, and the purity weight of α_n is 1. By Lemma 11.1, there exist non-zero vectors $v_{(n),1}, v_{(n),2} \in V_{\alpha_n}(\mathbf{Q}_p)$ such that the action of $(h_1,h_2) \in H(\mathbf{Z}_p)$ on $v_{(n),i}$ is by $\det(h_i)$. Again, these vectors are unique up to $\mathbf{Q}_p^\times$-multiple.
- (iii) The space V_{α_0} is a line, with basis $v_{(0)}(g) = \det(g)$.

All of the $v_{(i)}$ are explicit algebraic functions $G(\mathbf{Z}_p) \to \mathbf{Q}_p$, and we can multiply them in the ring $\mathbf{Q}_p[G]$. The following gives an explicit branching law from such a product.

Proposition 11.3. *Let $\lambda \in X_0^*(T)$ be a dominant pure algebraic weight and $j \in \mathrm{Crit}(\lambda)$. Then*

$$v_{\lambda,j} := (v_{(1)}^{\lambda_1-\lambda_2}) \cdot (v_{(2)}^{\lambda_2-\lambda_3}) \cdots (v_{(n-1)}^{\lambda_{n-1}-\lambda_n}) \cdot (v_{(n),1}^{-\lambda_{n+1}-j}) \cdot (v_{(n),2}^{\lambda_n+j}) \cdot (v_{(0)}^{\lambda_{n+1}})$$

is a generator of the line $V^H_{(-j,\mathsf{w}+j)}(\mathbf{Q}_p)$ inside the $H(\mathbf{Z}_p)$-representation $V_\lambda(\mathbf{Q}_p)|_{H(\mathbf{Z}_p)}$.

Proof. This product is algebraic, as the $v_{(i)}$ and $v_{(n),i}$ are algebraic by construction, and:

- for $1 \leqslant i \leqslant n-1$, we have $\lambda_i - \lambda_{i+1} \geqslant 0$, so $v_{(i)}^{\lambda_i - \lambda_{i+1}}$ is algebraic;
- since $\lambda_n \geqslant -j \geqslant \lambda_{n+1}$, we have $v_{(n),1}^{-\lambda_{n+1}-j}$ and $v_{(n),2}^{\lambda_n + j}$ are algebraic;
- and $v_{(0)}^{\lambda_{n+1}} = \det^{\lambda_{n+1}}$ is algebraic.

Thus their product is algebraic. If $t \in T(\mathbf{Z}_p)$, then by (11.2) we see

$$\lambda(t) = \alpha_1^{\lambda_1 - \lambda_2}(t) \cdots \alpha_{n-1}^{\lambda_{n-1}-\lambda_n}(t) \cdot \alpha_n(t)^{-\lambda_{n+1}-j} \cdot \alpha_n^{\lambda_n + j}(t) \cdot \alpha_0^{\lambda_{n+1}}(t),$$

so for $n^- \in \overline{N}(\mathbf{Z}_p)$ and $g \in G(\mathbf{Z}_p)$, we see $v_{\lambda,j}(n^- t g) = \lambda(t) v_{\lambda,j}(g)$ by multiplying together the analogous relations for the $v_{(i)}$. Finally, by Notation 11.2 we see $(h_1,h_2) \in H(\mathbf{Z}_p)$ acts on $v_{\lambda,j}$ by

$$\det(h_1)^{-\lambda_{n+1}-j} \cdot \det(h_2)^{\lambda_n + j} \cdot \det(h_1 h_2)^{\lambda_{n+1}} = \det(h_1)^{-j} \det(h_2)^{\mathsf{w}+j},$$

as required (using $\mathsf{w} = \lambda_n + \lambda_{n+1}$ in the final step). □

11.2 Support conditions for branching laws

Let w_n denote the antidiagonal $n \times n$ matrix with $(w_n)_{ij} = \delta_{i,n+1-j}$, and recall

$$u = \begin{pmatrix} 1_n & w_n \\ 0 & 1_n \end{pmatrix} \in G(\mathbf{Z}_p). \tag{11.3}$$

For $\beta \in \mathbf{Z}_{\geqslant 1}$, let

$$N^\beta(\mathbf{Z}_p) := N(p^\beta \mathbf{Z}_p) \cdot u = \left\{ n \in N(\mathbf{Z}_p) : n \equiv \begin{pmatrix} 1_n & w_n \\ 0 & 1_n \end{pmatrix} \pmod{p^\beta} \right\}.$$

Note this is *not* a subgroup of $N(\mathbf{Z}_p)$. We also emphasise that $N^\beta(\mathbf{Z}_p)$ is not the set of $\mathbf{Z}_p$-points of an algebraic group, and hope the notation does not cause confusion.

Let $\lambda \in X_0^*(T)$ be a pure dominant algebraic weight, let $j \in \mathrm{Crit}(\lambda)$, and let $v_{\lambda,j}$ be a generator of the line $V_{(-j,\mathsf{w}+j)}^H$ inside the $H(\mathbf{Z}_p)$-representation $V_\lambda(\mathbf{Q}_p)$, viewed as an explicit algebraic function $G(\mathbf{Z}_p) \to \mathbf{Q}_p$. The key examples we consider are $\lambda = \alpha_i$ from (11.1), with $v_{\lambda,j} = v_{(i)}$ from Notation 11.2. The aim of this subsection is to prove:

Proposition 11.4. *After possibly rescaling $v_{\lambda,j} \in V_{(-j,\mathsf{w}+j)}^H \subset V_\lambda(\mathbf{Q}_p)\big|_{H(\mathbf{Z}_p)}$, we have*

$$v_{\lambda,j}(N^\beta(\mathbf{Z}_p)) \subset 1 + p^\beta \mathbf{Z}_p \qquad \textit{for all } \beta \geqslant 1.$$

Recall $G_n = \mathrm{GL}_n$ and its subgroups B_n, $\overline{B}_n$, N_n, and $\overline{N}_n$ from §2.1. Let Iw_{G_n} be the Iwahori subgroup of $G_n(\mathbf{Z}_p)$, and let $J^\beta := \{g \in G_n(\mathbf{Z}_p) : g \pmod{p^\beta} \in T_n(\mathbf{Z}/p^\beta)\} \subset \mathrm{Iw}_{G_n}$. By the Iwahori factorisation, any element $1_n + p^\beta Y \in J^\beta$ has an Iwahori factorisation $1_n + p^\beta Y = RS$, where $R \in N_n(\mathbf{Z}_p)$ and $S \in \overline{B}_n(\mathbf{Z}_p)$.

Lemma 11.5. *If $X \in M_n(\mathbf{Z}_p)$, there exist $\overline{B} \in \overline{B}_{2n}(\mathbf{Z}_p), h \in H(\mathbf{Z}_p)$ with $\overline{B}, h \equiv 1_{2n} \pmod{p^\beta}$ and*

$$\begin{pmatrix} 1_n & w_n+p^\beta X \\ 0 & 1_n \end{pmatrix} = \overline{B} \cdot u \cdot h.$$

Proof. Fix $R \in B_n(\mathbf{Z}_p), S \in \overline{B}_n(\mathbf{Z}_p)$ such that $1+p^\beta w_n X = RS$. A simple check shows $R, S \equiv 1 \pmod{p^\beta}$. We see

$$\begin{pmatrix} w_nRw_n & \\ & S^{-1} \end{pmatrix} \in \overline{B}_{2n}(\mathbf{Z}_p), \qquad \begin{pmatrix} (w_nRw_n)^{-1} & \\ & S \end{pmatrix} \in H(\mathbf{Z}_p),$$

and

$$\begin{pmatrix} w_nRw_n & \\ & S^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1_n & w_n \\ 0 & 1_n \end{pmatrix} \cdot \begin{pmatrix} (w_nRw_n)^{-1} & \\ & S \end{pmatrix} = \begin{pmatrix} 1_n & w_n+p^n X \\ 0 & 1_n \end{pmatrix},$$

which has the claimed form. □

Now let $v_{\lambda,j}$ be as in Proposition 11.4. Since $v_{\lambda,j}$ transforms by $\lambda(\overline{B}) \in \mathbf{Z}_p^\times$ under left translation by $\overline{B} \in \overline{B}_{2n}(\mathbf{Z}_p)$, and transforms like $\det(h_1)^{-j}\det(h_2)^{\mathsf{w}+j} \in \mathbf{Z}_p^\times$ under right translation by $h=(h_1,h_2) \in H(\mathbf{Z}_p)$, we see

$$v_{\lambda,j}\big(\overline{B}_{2n}(\mathbf{Z}_p)\cdot u \cdot H(\mathbf{Z}_p)\big) \subset \mathbf{Z}_p^\times \cdot v_{\lambda,j}(u).$$

Suppose $v_{\lambda,j}(u)=0$; then $v_{\lambda,j}$ vanishes on the cell $\overline{B}_{2n}(\mathbf{Z}_p)\cdot u \cdot H(\mathbf{Z}_p)$. This cell is open and dense in $G(\mathbf{Z}_p)$ (e.g. [Loe21, §5.1.3]), forcing $v_{\lambda,j}=0$, which contradicts our assumptions. Thus $v_{\lambda,j}(u) \neq 0$, and we are free to rescale it by an element of $\mathbf{Q}_p^\times$ so that $v_{\lambda,j}(u)=1$. Further if $\overline{B}, h \equiv 1_{2n} \pmod{p^\beta}$, then $\lambda(\overline{B}), \det(h_1)^{-j}\det(h_2)^{\mathsf{w}+j} \equiv 1 \pmod{p^\beta}$. Combining all of this with Lemma 11.5, we deduce that for any $X \in M_n(\mathbf{Z}_p)$, we have

$$v_{\lambda,j}\left(\begin{pmatrix} 1_n & w_n+p^\beta X \\ 0 & 1_n \end{pmatrix}\right) \in 1+p^\beta \mathbf{Z}_p. \tag{11.4}$$

Proof. (Proposition 11.4). A general element of $N^\beta(\mathbf{Z}_p)$ looks like $n = \begin{pmatrix} A & w_n+p^\beta Y \\ 0 & B \end{pmatrix}$, where $A, B \in N_n(\mathbf{Z}_p)$ with $A \equiv B \equiv 1_n \pmod{p^\beta}$ and $Y \in M_n(\mathbf{Z}_p)$. Letting

$$X := \left(Y - w_n\left(\tfrac{B-1}{p^\beta}\right)\right)B^{-1} \in M_n(\mathbf{Z}_p),$$

we have

$$\begin{pmatrix} A & w_n+p^\beta Y \\ 0 & B \end{pmatrix} = \begin{pmatrix} 1_n & w_n+p^\beta X \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} A & \\ & B \end{pmatrix}.$$

Then $v_{\lambda,j}(n) = \det(A)^{-j}\det(B)^{\mathsf{w}+j}\cdot v_{\lambda,j}\left(\begin{pmatrix} 1_n & w_n+pY \\ 0 & 1_n \end{pmatrix}\right) \in 1+p^\beta\mathbf{Z}_p$ by (11.4). □

11.3 p-adic interpolation of branching laws

Now we p-adically interpolate. Assume the choices $v_{(0)}, v_{(1)}, \ldots, v_{(n-1)}, v_{(n),1}, v_{(n),2}$ in Notation 11.2 were all normalised so $v_{(i)}(u)=1$, whence $v_{(i)}(N^1(\mathbf{Z}_p)) \subset 1+p\mathbf{Z}_p \subset \mathbf{Z}_p^\times$, as in Proposition 11.4 (and its proof).

If R is a $\mathbf{Q}_p$-algebra and $\chi = (\chi_1, ..., \chi_{2n}) : T(\mathbf{Z}_p) \to R^\times$ is a character, then define a function

$$\begin{aligned} w_\chi : N^1(\mathbf{Z}_p) &\longrightarrow R^\times, \\ g &\longmapsto v_{(0)}(g)^{\chi_{n+1}} \cdot \left(\prod_{i=1}^{n-1} v_{(i)}(g)^{\chi_i - \chi_{i+1}}\right) \cdot v_{(n),1}(g)^{-\chi_{n+1}} \cdot v_{(n),2}(g)^{\chi_n}, \end{aligned} \tag{11.5}$$

where if $x \in \mathbf{Z}_p^\times$ we write x^{χ_i} as shorthand for $\chi_i(x)$. If $\beta \geqslant 1$, let

$$\mathrm{Iw}_G^\beta := \overline{N}(p\mathbf{Z}_p) \cdot T(\mathbf{Z}_p) \cdot N^\beta(\mathbf{Z}_p) \subset \mathrm{Iw}_G, \tag{11.6}$$

which again is not a subgroup. We may extend w_χ to a function $w_\chi : \mathrm{Iw}_G^1 \to R^\times$ via (10.3):

$$w_\chi(\overline{n} \cdot t \cdot n) = \chi(t) \cdot w_\chi(n), \quad \overline{n} \in \overline{N}(p\mathbf{Z}_p), t \in T(\mathbf{Z}_p), n \in N^1(\mathbf{Z}_p). \tag{11.7}$$

For $\beta \geqslant 1$, let

$$\mathrm{Iw}_H^\beta := H(\mathbf{Z}_p) \cap u^{-1}\mathrm{Iw}_G^\beta. \tag{11.8}$$

If $g \in \mathrm{Iw}_G^1$ and $h \in \mathrm{Iw}_H^1$, then a simple check shows $gh \in \mathrm{Iw}_G^1$, so we can consider $w_\chi(gh)$. The following will be important in the sequel: recalling $L_p^{B,\beta}$ from Definition 4.3, note that for appropriate choices of K_p (e.g. if $K_p = \mathrm{Iw}_G$) we have $L_p^{B,\beta} \subset \mathrm{Iw}_H^\beta$.

Lemma 11.6. *If $h = (h_1, h_2) \in \mathrm{Iw}_H^1$, then $w_\chi(gh) = \det(h_2)^{\chi_n + \chi_{n+1}} \cdot w_\chi(g)$.*

Proof. By definition, $w_\chi(gh)$ is a product of terms involving $v_{(i)}(gh)$, $v_{(n),i}(gh)$. By construction $v_{(i)}(gh) = v_{(i)}(g)$ for $1 \leqslant i \leqslant n-1$, and $v_{(0)}(gh) = \det(h_1)\det(h_2)v_{(0)}(g)$, and $v_{(n),i}(gh) = \det(h_i)v_{(n),i}(g)$ for $i = 1, 2$. We get factors of $\det(h_1)^{\chi_{n+1}}\det(h_2)^{\chi_{n+1}}$ from $v_{(0)}$, $\det(h_1)^{-\chi_{n+1}}$ from $v_{(n),1}$, and $\det(h_2)^{\chi_n}$ from $v_{(n),2}$, so

$$w_\chi(gh) = \det(h_1)^{\chi_{n+1}}\det(h_2)^{\chi_{n+1}} \cdot \det(h_1)^{-\chi_{n+1}} \cdot \det(h_2)^{\chi_n} \cdot w_\chi(g) = \det(h_2)^{\chi_n + \chi_{n+1}} \cdot w_\chi(g). \qquad \square$$

Remark 11.7. The definition of w_χ is heavily motivated by Proposition 11.3. Indeed we see that if $\lambda \in X_0^*(T)$ is a pure dominant algebraic weight and $j \in \mathrm{Crit}(\lambda)$, then for any $g \in N^1(\mathbf{Z}_p)$ we have

$$v_{\lambda,j}(g) = w_\lambda(g) \cdot \left(\frac{v_{(n),2}(g)}{v_{(n),1}(g)}\right)^j. \tag{11.9}$$

To p-adically vary branching laws, we take $R = \mathcal{O}_\Omega$, for $\Omega \subset \mathscr{W}_0^G$ an affinoid in the pure weight space, equipped with a character $\chi_\Omega : T(\mathbf{Z}_p) \to \mathcal{O}_\Omega^\times$ as in §10.1.

Definition 11.8. Let $f \in \mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$ be a locally analytic function. Define a function $v_\Omega(f) : N(\mathbf{Z}_p) \to \mathcal{O}_\Omega$ by

$$v_\Omega(f)(g) := \begin{cases} w_{\chi_\Omega}(g) \cdot f\left(\frac{v_{(n),2}(g)}{v_{(n),1}(g)}\right) & : g \in N^1(\mathbf{Z}_p), \\ 0 & : \text{otherwise}. \end{cases}$$

This is well-defined by the normalisations fixed at the start of §11.3, by Proposition 11.4, and the definition (11.5). Under the transformation law (10.3) and Iwahori decomposition, this extends to a unique element $[v_\Omega(f) : \mathrm{Iw}_G \to \mathcal{O}_\Omega] \in \mathcal{A}_\Omega$, with support on Iw_G^1 from (11.6).

We allow $\Omega = \{\lambda\}$ to be a point, in which case the above defines, for any $f \in \mathcal{A}(\mathbf{Z}_p^\times, L)$, a function $v_\lambda(f) : \mathrm{Iw}_G \to L$ in $\mathcal{A}_\lambda$.

Lemma 11.9. *If $\lambda \in X_0^*(T)$ and $g \in \mathrm{Iw}_G^1$, then for all $j \in \mathrm{Crit}(\lambda)$ we have*

$$\left[v_\lambda(z \mapsto z^j)\right](g) = v_{\lambda,j}(g).$$

Proof. For $g \in N^1(\mathbf{Z}_p)$, this follows by combining the definition of v_λ with (11.9). Both sides satisfy the same transformation law (11.7) to extend to Iw_G^1, so we have equality on the larger group too. □

Recall $\mathrm{w}_\Omega : \mathbf{Z}_p^\times \to \mathcal{O}_\Omega^\times$ from (10.1). Let $h = (h_1, h_2) \in H(\mathbf{Z}_p)$ act on $f \in \mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$ by

$$(h * f)(z) = \det(h_2)^{\mathrm{w}_\Omega} \cdot f\left(\tfrac{\det(h_2)}{\det(h_1)} \cdot z\right). \tag{11.10}$$

It follows that $\mathrm{Iw}_H^1 \subset H(\mathbf{Z}_p)$, as defined in (11.8), acts on $\mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$. It also acts on $\mathcal{A}_\Omega$ via its embedding into Iw_G.

Lemma 11.10. *The map $v_\Omega : \mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega) \to \mathcal{A}_\Omega$ is Iw_H^1-equivariant.*

Proof. Let $h = (h_1, h_2) \in \mathrm{Iw}_H^1$ and $g \in N^1(\mathbf{Z}_p)$. First note that by (10.1) we know $\mathrm{w}_\Omega = \chi_{\Omega,n} + \chi_{\Omega,n+1}$; so $\det(h_2)^{\mathrm{w}_\Omega} w_{\chi_\Omega}(g) = w_{\chi_\Omega}(gh)$ by Lemma 11.6.

Now let $f \in \mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$, and compute

$$\begin{aligned} v_\Omega(h * f)(g) &= w_{\chi_\Omega}(g) \cdot (h * f)\left(\frac{v_{(n),2}(g)}{v_{(n),1}(g)}\right) \\ &= \det(h_2)^{\mathrm{w}_\Omega} w_{\chi_\Omega}(g) \cdot f\left(\frac{\det(h_2) \cdot v_{(n),2}(g)}{\det(h_1) \cdot v_{(n),1}(g)}\right) = w_{\chi_\Omega}(gh) \cdot f\left(\frac{v_{(n),2}(gh)}{v_{(n),1}(gh)}\right). \end{aligned} \tag{11.11}$$

Since $gh \in \mathrm{Iw}_G^1$, we may write $gh = b'g'$, with $b' \in \overline{N}(p\mathbf{Z}_p) \cdot B(\mathbf{Z}_p)$ and $g' \in N^1(\mathbf{Z}_p)$. Then

$$\begin{aligned} [h * v_\Omega(f)](g) &= v_\Omega(f)(gh) = v_\Omega(f)(b'g') = \chi_\Omega(b') \cdot v_\Omega(f)(g') \\ &= \chi_\Omega(b') w_{\chi_\Omega}(g') f\left(\frac{v_{(n),2}(g')}{v_{(n),1}(g')}\right) = w_{\chi_\Omega}(b'g') f\left(\frac{v_{(n),2}((b')^{-1}gh)}{v_{(n),1}((b')^{-1}gh)}\right) = w_{\chi_\Omega}(gh) f\left(\frac{v_{(n),2}(gh)}{v_{(n),1}(gh)}\right), \end{aligned} \tag{11.12}$$

where in the last equality we use that $v_{(n),i}((b')^{-1}gh) = \alpha_n(b')^{-1} v_{(n),i}(gh)$ for both $i = 1, 2$. Combining (11.11) and (11.12) yields $v_\Omega(h * f)(g) = [h * v_\Omega(f)](g)$, as required. □

11.4 Branching laws for distributions

The overconvergent cohomology groups we consider have coefficients in $\mathcal{D}_\Omega$, not $\mathcal{A}_\Omega$, so we now dualise the above. Finally, we collate everything we have proved in the main result of this section (Proposition 11.13).

By Lemma 4.11, for $\lambda \in X_0^*(T)$ we have $j \in \mathrm{Crit}(\lambda)$ if and only if $\mathrm{Hom}_{H(\mathbf{Z}_p)}(V_\lambda^\vee, V^H_{(j,-\mathsf{w}-j)})$ is a line; and moreover, the choices made in §11.1 fix a generator

$$\kappa_{\lambda,j} : V_\lambda^\vee(\mathbf{Q}_p) \to V^H_{(j,-\mathsf{w}-j)}(\mathbf{Q}_p) \xrightarrow{\sim} \mathbf{Q}_p, \quad \mu \mapsto \mu(v_{\lambda,j}). \tag{11.13}$$

This is $H(\mathbf{Z}_p)$-equivariant, as if $h = (h_1,h_2) \in H(\mathbf{Z}_p)$, then

$$(h * \mu)(v_{\lambda,j}) = \mu(h^{-1} * v_{\lambda,j}) = \det(h_1)^j \det(h_2)^{-\mathsf{w}-j} \mu(v_{\lambda,j}). \tag{11.14}$$

We can base-extend this to any extension $L/\mathbf{Q}_p$ and consider $\kappa_{\lambda,j}$ as a map $V_\lambda^\vee(L) \to L$.

Similarly, we can dualise the map $v_\Omega : \mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega) \to \mathcal{A}_\Omega$ from Definition 11.8 to get

$$\kappa_\Omega : \mathcal{D}_\Omega \longrightarrow \mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega), \quad \mu \mapsto [f \mapsto \mu(v_\Omega(f))] \text{ for } f \in \mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega). \tag{11.15}$$

Definition 11.11. Let $\beta \geqslant 1$.

(1) We say that $f \in \mathcal{A}_\Omega$ has *support on* Iw_G^β if $f(g) = 0$ for all $g \notin \mathrm{Iw}_G^\beta$. By the definition of $\mathcal{A}_\Omega$, this is equivalent to asking $f(n) = 0$ for all $n \in N(\mathbf{Z}_p) \backslash N^\beta(\mathbf{Z}_p)$.

Similarly, we say f has *support away from* Iw_G^β if $f(g) = 0$ for all $g \in \mathrm{Iw}_G^\beta$, or equivalently if $f(n) = 0$ for all $n \in N^\beta(\mathbf{Z}_p)$.

(2) Let $\mathcal{A}_\Omega^\beta \subset \mathcal{A}_\Omega$ be the subspace of functions supported on Iw_G^β.

(3) If $f \in \mathcal{A}_\Omega$, define a function $f|_{\mathrm{Iw}_G^\beta} \in \mathcal{A}_\Omega^\beta \subset \mathcal{A}_\Omega$ by multiplying f by the indicator function of (the closed and open subset) Iw_G^β; explicitly,

$$f|_{\mathrm{Iw}_G^\beta}(g) := \begin{cases} f(g) & : g \in \mathrm{Iw}_G^\beta \\ 0 & : \text{otherwise.} \end{cases}$$

This is well-defined: it is locally analytic on $N(\mathbf{Z}_p)$ as $N^\beta(\mathbf{Z}_p)$ is closed and open in $N(\mathbf{Z}_p)$, it defines an element of $\mathcal{A}_\Omega$ by Iwahori decomposition and the definition of Iw_G^β, and by definition it has support on Iw_G^β.

(4) We say that $\mu \in \mathcal{D}_\Omega$ has *support on* Iw_G^β if $\mu(f) = \mu(f|_{\mathrm{Iw}_G^\beta})$ for all $f \in \mathcal{A}_\Omega$. Let $\mathcal{D}_\Omega^\beta \subset \mathcal{D}_\Omega$ be the subspace of distributions supported on Iw_G^β.

We similarly write $\mathcal{A}_\lambda^\beta$, $\mathcal{D}_\lambda^\beta$, etc. when $\Omega = \{\lambda\}$ is a point.

The following is a useful alternative criterion for support.

Lemma 11.12. *Let $\mu \in \mathcal{D}_\Omega$. We have*

$$\Big[\mu \in \mathcal{D}_\Omega^\beta\Big] \iff \Big[\mu(f) = 0 \textit{ for all } f \in \mathcal{A}_\Omega \textit{ with support away from } \mathrm{Iw}_G^\beta\Big].$$

Proof. Any $f \in \mathcal{A}_\Omega$ can be written as $f = f|_{\mathrm{Iw}_G^\beta} + f|_{\mathrm{Iw}_G \setminus \mathrm{Iw}_G^\beta}$, where $f|_{\mathrm{Iw}_G \setminus \mathrm{Iw}_G^\beta}$ has support away from Iw_G^β. Then $\mu(f) = \mu(f|_{\mathrm{Iw}_G^\beta}) + \mu(f|_{\mathrm{Iw}_G \setminus \mathrm{Iw}_G^\beta})$. The two conditions are now easily seen to be equivalent. □

There is a natural map $s_\lambda : V_\lambda(L) \to \mathcal{A}_\lambda^\beta(L)$ given by

$$s_\lambda(f)(g) = \begin{cases} f(g) & : g \in \mathrm{Iw}_G^\beta \\ 0 & : \text{ else.} \end{cases}$$

Abusing notation, for any β we continue to write $r_\lambda : \mathcal{D}_\lambda^\beta \to V_\lambda^\vee$ for its dual.

Proposition 11.13. *For each classical $\lambda \in \Omega$ and each $j \in \mathrm{Crit}(\lambda)$, the following diagram commutes:*

$$\begin{tikzcd} \mathcal{D}_\Omega^1 \arrow[r, "\mathrm{sp}_\lambda"] \arrow[d, "\kappa_\Omega"] & \mathcal{D}_\lambda^1(L) \arrow[r, "r_\lambda"] \arrow[d, "\kappa_\lambda"] & V_\lambda^\vee(L) \arrow[d, "\kappa_{\lambda,j}"] \\ \mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega) \arrow[r, "\mathrm{sp}_\lambda"] & \mathcal{D}(\mathbf{Z}_p^\times, L) \arrow[r, "\mu \mapsto \mu(z^j)"] & L. \end{tikzcd}$$

Proof. The first square commutes directly from the definitions and the fact that $\mathrm{sp}_\lambda \circ \chi_\Omega = \lambda$.

To see the second square commutes, let $\mu \in \mathcal{D}_\lambda^1(L)$; then

$$\begin{aligned} \kappa_\lambda(\mu)(z \mapsto z^j) = \int_{\mathbf{Z}_p^\times} z^j \cdot d\kappa_\lambda(\mu) = \int_{\mathrm{Iw}_G} v_\lambda(z^j) \cdot d\mu = \int_{\mathrm{Iw}_G^1} v_\lambda(z^j) \cdot d\mu = \int_{\mathrm{Iw}_G^1} v_{\lambda,j} \cdot d\mu \\ = \int_{\mathrm{Iw}_G} v_{\lambda,j} \cdot d\mu = \int_{\mathrm{Iw}_G} v_{\lambda,j} \cdot dr_\lambda(\mu) = \kappa_{\lambda,j} \circ r_\lambda(\mu), \end{aligned}$$

where for clarity we write $\int_X f \cdot d\mu$ for $\mu(f|_X)$, interpreted suitably for each term. Then the second equality is by definition of κ_λ, the third and fifth equalities follow as μ is supported on Iw_G^1, the fourth equality is Lemma 11.9, the sixth equality is by definition of r_λ as $v_{\lambda,j} \in V_\lambda$, and the seventh equality is the definition of $\kappa_{\lambda,j}$. □

Recall the $H(\mathbf{Z}_p)$-action on $\mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$ from (11.10). We equip $\mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$ with the left dual action. In the diagram of Proposition 11.13, recall the action of $H(\mathbf{Z}_p)$ on the bottom-right term is by $\det(h_1)^j \det(h_2)^{-\mathsf{w}-j}$. Finally we note:

Lemma 11.14. *Every map in the diagram of Proposition 11.13 is Iw_H^1-equivariant.*

Proof. For sp_λ and r_λ this follows straight from the definition; for κ_Ω and κ_λ this is Lemma 11.10; for $\kappa_{\lambda,j}$ this is (11.14); and for evaluation at z^j, this follows since for $\mu \in \mathcal{D}(\mathbf{Z}_p^\times, L)$ and $h \in H(\mathbf{Z}_p)$, we have $(h * \mu)(z^j) = \mu(h^{-1} * z^j) = \mu(\det(h_1)^j \det(h_2)^{-\mathsf{w}-j} \cdot z^j)$. □

12 Distribution-valued evaluation maps

We now define distribution-valued evaluation maps on the overconvergent cohomology by combining the p-adic branching laws of §11 with the abstract evaluation maps of §4.

12.1 Combining evaluations and branching laws

Recall $H(\mathbf{Z}_p)$ acts on $\mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$ by (11.10). We have an isomorphism

$$\mathbf{Z}_p^\times \cong \mathscr{C}\ell_{\mathbf{Q}}^+(p^\infty) := \mathbf{Q}^\times \backslash \mathbf{A}^\times / \prod_{\ell \neq p} \mathbf{Z}_\ell^\times \mathbf{R}_{>0}, \tag{12.1}$$

and we extend the $H(\mathbf{Z}_p)$-action on $f \in \mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$ to an action of $h = (h_1, h_2) \in H(\mathbf{A})$ by

$$(h * f)(z) = \chi_{\mathrm{cyc}}(\det(h_2))^{\mathrm{w}_\Omega} \cdot f\left(\tfrac{\det(h_2)}{\det(h_1)} \cdot z\right), \tag{12.2}$$

where translation of $z \in \mathbf{Z}_p^\times$ under $\det(h_2)\det(h_1)^{-1} \in \mathbf{A}^\times$ is defined by lifting z to $\mathbf{A}^\times$ under (12.1), translating, and projecting back to $\mathbf{Z}_p^\times$. Note H_∞° and $H(\mathbf{Q})$ again act trivially, and that any subgroup of $H(\widehat{\mathbf{Z}})$ acts through projection to $H(\mathbf{Z}_p)$.

Lemma 12.1. *If $\beta \geqslant 1$, then the map κ_Ω from (11.15) is L_β-equivariant.*

Proof. Recall that κ_Ω is the dual of $v_\Omega : \mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega) \to \mathcal{A}_\Omega$, so it suffices to prove v_Ω is L_β-equivariant. Recall Iw_H^1 from (11.8), and that v_Ω is Iw_H^1-equivariant by Lemma 11.10. A simple check shows that $L_p^\beta \subset \mathrm{Iw}_H^1$, so v_Ω is L_p^β-equivariant. But the L_β-action on both terms factors through projection to L_p^β, since $L_\beta \subset H(\mathbf{A})$ (for $\mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$) and by definition (for $\mathcal{A}_\Omega$). □

To define our distribution-valued evaluations, we take strong motivation from Definition 4.12, adapting it with the Borel B in place of the parabolic Q. Let $\Omega \subset \mathscr{W}_0^G$ be an affinoid in the pure weight space; we allow $\Omega = \{\lambda\}$ a single weight. Lemma 12.1 allows us to make the following definition:

Definition 12.2. Let $\beta \in \mathbf{Z}_{\geqslant 0}$ and $\delta \in H(\mathbf{A}_f)$, representing $[\delta] \in \pi_0(X_\beta)$. The *overconvergent evaluation map of level p^β at* $[\delta]$ is the map

$$\mathrm{Ev}_{B,\beta,[\delta]}^{\Omega} := \mathrm{Ev}_{B,\beta,[\delta]}^{\mathcal{D}_\Omega,\kappa_\Omega} = \delta * \left[\kappa_\Omega \circ \mathrm{Ev}_{B,\beta,\delta}^{\mathcal{D}_\Omega}\right] : \mathrm{H}_{\mathrm{c}}^t(S_K, \mathcal{D}_\Omega) \longrightarrow \mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega).$$

This is well-defined and independent of the choice of δ representing $[\delta]$ by Proposition 4.9. Here we also use that $H(\mathbf{Q})$ and H_∞° act trivially on $\mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$.

Proposition 12.3. *Suppose $\beta \geqslant 1$. Then for every $\lambda \in \Omega$, and $j \in \mathrm{Crit}(\lambda)$, we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{D}_\Omega) \arrow[r, "\mathrm{sp}_\lambda"] \arrow[d, "\mathrm{Ev}_{B,\beta,[\delta]}^{\Omega}"] & \mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{D}_\lambda(L)) \arrow[r, "r_\lambda"] \arrow[d, "\mathrm{Ev}_{B,\beta,[\delta]}^{\lambda}"] & \mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{V}_\lambda^\vee(L)) \arrow[d, "\mathcal{E}_{B,\beta,[\delta]}^{j,\mathrm{w}}"] \\
\mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega) \arrow[r, "\mathrm{sp}_\lambda"] & \mathcal{D}(\mathbf{Z}_p^\times, L) \arrow[r, "\mu \mapsto \mu(z^j)"] & L
\end{tikzcd}$$

Proof. First note that applying Lemma 4.8 first to $\mathcal{D}_\Omega \xrightarrow{\mathrm{sp}_\lambda} \mathcal{D}_\lambda(L)$ and then $\mathcal{D}_\lambda(L) \xrightarrow{r_\lambda} V_\lambda^\vee(L)$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{D}_\Omega) \arrow[r, "\mathrm{sp}_\lambda"] \arrow[d, "\mathrm{Ev}_{B,\beta,\delta}^{\mathcal{D}_\Omega}"] & \mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{D}_\lambda(L)) \arrow[r, "r_\lambda"] \arrow[d, "\mathrm{Ev}_{B,\beta,\delta}^{\mathcal{D}_\lambda}"] & \mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{V}_\lambda^\vee(L)) \arrow[d, "\mathrm{Ev}_{B,\beta,\delta}^{V_\lambda^\vee}"] \\
(\mathcal{D}_\Omega)_{\Gamma_{\beta,\delta}} \arrow[r, "\mathrm{sp}_\lambda"] & \mathcal{D}_\lambda(L)_{\Gamma_{\beta,\delta}} \arrow[r, "r_\lambda"] & V_\lambda^\vee(L)_{\Gamma_{\beta,\delta}}.
\end{tikzcd} \tag{12.3}$$

Recall $\mathcal{D}_\Omega^1 \subset \mathcal{D}_\Omega$ from Definition 11.11. Since $\beta \geqslant 1$, we have $\delta^{-1}\Gamma_{\beta,\delta}\delta \subset \mathrm{Iw}_H^1$ from (11.8), so the action of $\Gamma_{\beta,\delta}$ on $\mathcal{D}_\Omega$ preserves $\mathcal{D}_\Omega^1$, and we can consider $(\mathcal{D}_\Omega^1)_{\Gamma_{\beta,\delta}}$. Moreover, $H(\mathbf{Q})$ (hence $\Gamma_{\beta,\delta}$) acts trivially on $\mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$ and L, and κ_Ω and $\kappa_{\lambda,j}$ are Iw_H^1-equivariant; combining, each of these maps factors through the coinvariants, giving maps $(\mathcal{D}_\Omega^1)_{\Gamma_{\beta,\delta}} \to \mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$ and $V_\lambda^\vee(L)_{\Gamma_{\beta,\delta}} \to L$. Then by Proposition 11.13 (for the top squares) and Lemma 11.14 (for the bottom squares) we have a commutative diagram

$$\begin{tikzcd}
(\mathcal{D}_\Omega^1)_{\Gamma_{\beta,\delta}} \arrow[r, "\mathrm{sp}_\lambda"] \arrow[d, "\kappa_\Omega"] & \mathcal{D}_\lambda^1(L)_{\Gamma_{\beta,\delta}} \arrow[r, "r_\lambda"] \arrow[d, "\kappa_\lambda"] & V_\lambda^\vee(L)_{\Gamma_{\beta,\delta}} \arrow[d, "\kappa_{\lambda,j}"] \\
\mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega) \arrow[r, "\mathrm{sp}_\lambda"] \arrow[d, "\delta *-"] & \mathcal{D}(\mathbf{Z}_p^\times, L) \arrow[r, "\mu \mapsto \mu(z^j)"] \arrow[d, "\delta *-"] & L \arrow[d, "\delta *-"] \\
\mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega) \arrow[r, "\mathrm{sp}_\lambda"] & \mathcal{D}(\mathbf{Z}_p^\times, L) \arrow[r, "\mu \mapsto \mu(z^j)"] & L.
\end{tikzcd} \tag{12.4}$$

Since $\mathrm{Ev}_{B,\beta,[\delta]}^\Omega = \delta * [\kappa_\Omega \circ \mathrm{Ev}_{B,\beta,\delta}^{\mathcal{D}_\Omega}]$ and $\mathcal{E}_{B,\beta,[\delta]}^{j,\mathrm{w}} = \delta * [\kappa_{\lambda,j} \circ \mathrm{Ev}_{B,\beta,\delta}^{V_\lambda^\vee}]$, we can complete the proof by combining (12.3) with (12.4). This is possible by Lemma 12.4 below, noting if $\beta \geqslant 1$, then $\mathcal{D}_\Omega^\beta \subset \mathcal{D}_\Omega^1$. $\square$

Lemma 12.4. *Let $\beta \geqslant 1$ and $\Phi \in \mathrm{H}_\mathrm{c}^t(S_K, \mathscr{D}_\Omega)$. We have $\mathrm{Ev}_{B,\beta,\delta}^{\mathcal{D}_\Omega}(\Phi) \in (\mathcal{D}_\Omega^\beta)_{\Gamma_{\beta,\delta}}$.*

Proof. Note that $\mathrm{Ev}_{B,\beta,\delta}^{\mathcal{D}_\Omega}(\Phi) \in [(u^{-1}t_p^\beta) * \mathcal{D}_\Omega]_{\Gamma_{\beta,\delta}}$ by construction of evaluation maps. Thus it suffices to show that if $\mu \in \mathcal{D}_\Omega$, then $u^{-1}t_p^\beta * \mu \in \mathcal{D}_\Omega^\beta$. By Lemma 11.12, this is equivalent to showing that $(u^{-1}t_p^\beta * \mu)(f) = 0$ for all $f \in \mathcal{A}_\Omega$ with support away from Iw_G^β (that is, such that $f(n) = 0$ for all $n \in N^\beta(\mathbf{Z}_p)$).

Suppose $f \in \mathcal{A}_\Omega$ has support away from Iw_G^β. By definition (see §10.3), we have

$$(u^{-1}t_p^\beta * \mu)(f) = \mu(ut_p^\beta * f),$$

where $(ut_p^\beta * f)(n) = f(t_p^\beta n t_p^{-\beta} u)$ for any $n \in N(\mathbf{Z}_p)$. It is easily seen that $t_p^\beta n t_p^{-\beta} \equiv 1_{2n} \pmod{p^\beta}$ (e.g. from [BW21c, §2.5, Rem. 4.19]). By definition of $N^\beta(\mathbf{Z}_p)$ we deduce $t_p^\beta n t_p^{-\beta} u \in N^\beta(\mathbf{Z}_p)$. Since f vanishes on $N^\beta(\mathbf{Z}_p)$, we deduce that $ut_p^\beta * f$ vanishes on $N(\mathbf{Z}_p)$, so is the zero function in $\mathcal{A}_\Omega$. It follows that $(u^{-1}t_p^\beta * \mu)(f) = \mu(0) = 0$ for all f with support away from Iw_G^β, hence $u^{-1}t_p^\beta * \mu \in \mathcal{D}_\Omega^\beta$ by Lemma 11.12. This proves the lemma (and thus Proposition 12.3). $\square$

12.2 Further support conditions

Our ultimate goal is to interpolate the classical evaluation maps $\mathcal{E}_{B,\chi}^{j,\eta_0}$ from §4.2. In Remark 4.14, we described this map as a composition of four maps; and in Proposition 12.3, we have used the branching laws of §11 to interpolate the first two maps of this composition. We will combine over β and δ to interpolate the final two maps in Remark 4.14. First, we give a more precise description of the image of $\mathrm{Ev}_{\beta,[\delta]}^\Omega$.

Note that for $\beta \geqslant 1$, we have decompositions

$$\mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega) = \bigoplus_{d \in (\mathbf{Z}/p^\beta)^\times} \mathcal{A}(d + p^\beta \mathbf{Z}_p, \mathcal{O}_\Omega), \quad \mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega) = \bigoplus_{d \in (\mathbf{Z}/p^\beta)^\times} \mathcal{D}(d + p^\beta \mathbf{Z}_p, \mathcal{O}_\Omega). \tag{12.5}$$

As in Definition 11.11, a distribution $\mu \in \mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$ lies in the summand $\mathcal{D}(d + p^\beta \mathbf{Z}_p, \mathcal{O}_\Omega)$ if and only if $\mu(f) = \mu(f|_{d+p^\beta \mathbf{Z}_p})$ for all $f \in \mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$.

Lemma 12.5. *If $\mu \in \mathcal{D}_\Omega^\beta$, then $\kappa_\Omega(\mu) \in \mathcal{D}(1 + p^\beta \mathbf{Z}_p, \mathcal{O}_\Omega)$.*

Proof. Let $f \in \mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$. If $g \in N^\beta(\mathbf{Z}_p)$, then by Proposition 11.4, we know $v_{(n),i}(g) \in 1 + p^\beta \mathbf{Z}_p$. Thus by the definition of $v_\Omega(f)$, we see $v_\Omega(f)(g) = v_\Omega(f|_{1+p^\beta \mathbf{Z}_p})(g)$, that is,

$$v_\Omega(f)|_{N^\beta(\mathbf{Z}_p)} = v_\Omega(f|_{1+p^\beta \mathbf{Z}_p})|_{N^\beta(\mathbf{Z}_p)}.$$

By the transformation law (10.3), the function $v_\Omega(f)|_{\mathrm{Iw}_G^\beta} \in \mathcal{A}_\Omega$ depends only on $v_\Omega(f)|_{N^\beta(\mathbf{Z}_p)}$, so we deduce $v_\Omega(f)|_{\mathrm{Iw}_G^\beta} = v_\Omega(f|_{1+p\mathbf{Z}_p})|_{\mathrm{Iw}_G^\beta}$. Thus if μ has support on Iw_G^β, for any $f \in \mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$ we have

$$\begin{aligned} \kappa_\Omega(\mu)(f) = \mu[v_\Omega(f)] &= \mu[v_\Omega(f)|_{\mathrm{Iw}_G^\beta}] \\ &= \mu[v_\Omega(f|_{1+p^\beta \mathbf{Z}_p})|_{\mathrm{Iw}_G^\beta}] = \mu[v_\Omega(f|_{1+p^\beta \mathbf{Z}_p})] = \kappa_\Omega(\mu)(f|_{1+p^\beta \mathbf{Z}_p}), \end{aligned}$$

so $\kappa_\Omega(\mu)$ is supported on $1 + p^\beta \mathbf{Z}_p$, as required. □

Recall $\mathrm{pr}_\beta : \pi_0(X_\beta) \to (\mathbf{Z}/p^\beta)^\times$ from (4.9).

Corollary 12.6. *If $\Phi \in \mathrm{H}_c^t(S_K, \mathscr{D}_\Omega)$, we have $\mathrm{Ev}_{B,\beta,[\delta]}^\Omega(\Phi) \in \mathcal{D}(d + p^\beta \mathbf{Z}_p, \mathcal{O}_\Omega)$, where $d = \mathrm{pr}_\beta([\delta])$.*

Proof. Recall $\mathrm{Ev}_{B,\beta,[\delta]}^\Omega := \delta * [\kappa_\Omega \circ \mathrm{Ev}_{B,\beta,\delta}^{\mathcal{D}_\Omega}]$. By Lemma 12.4, we have $\mathrm{Im}(\mathrm{Ev}_{B,\beta,[\delta]}^{\mathcal{D}_\Omega}) \subset (\mathcal{D}_\Omega^\beta)_{\Gamma_{\beta,\delta}}$. Since κ_Ω factors through the coinvariants, Lemma 12.5 implies that $\kappa_\Omega[(\mathcal{D}_\Omega^\beta)_{\Gamma_{\beta,\delta}}] \subset \mathcal{D}(1 + p^\beta \mathbf{Z}_p, \mathcal{O}_\Omega)$.

Finally the action of δ on $\mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$ is by δ^{-1} on $\mathcal{A}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$, which includes translation on $z \in \mathbf{Z}_p^\times$ by $\det(\delta_1 \delta_2^{-1})$. As this is a representative of $d = \mathrm{pr}_\beta([\delta]) \in (\mathbf{Z}/p^\beta)^\times$, translation by $\det(\delta_1 \delta_2^{-1})$ sends $1 + p^\beta \mathbf{Z}_p$ to $d + p^\beta \mathbf{Z}_p$. This induces a map $\delta * - : \mathcal{D}(1 + p^\beta \mathbf{Z}_p, \mathcal{O}_\Omega) \to \mathcal{D}(d + p^\beta \mathbf{Z}_p, \mathcal{O}_\Omega)$. Combining all of the above gives the corollary. □

12.3 Interpolation of classical evaluations

Let η_0 be any character of $(\mathbf{Z}/m)^\times$. For $\beta \geqslant 1$ and $d \in (\mathbf{Z}/p^\beta \mathbf{Z})^\times$, motivated by Definition 4.13, we define a map

$$\begin{aligned} \mathrm{Ev}_{B,\beta,d}^{\Omega,\eta_0} : \mathrm{H}_c^t(S_K, \mathscr{D}_\Omega) &\longrightarrow \mathcal{D}(d + p^\beta \mathbf{Z}_p, \mathcal{O}_\Omega) \\ \Phi &\longmapsto \sum_{[\delta] \in \mathrm{pr}_\beta^{-1}(d)} \eta_0\big(\mathrm{pr}_2([\delta])\big)\, \mathrm{Ev}_{B,\beta,[\delta]}^\Omega(\Phi). \end{aligned}$$

Combining under (12.5), we finally obtain an evaluation map

$$\begin{aligned}\mathrm{Ev}_{B,\beta}^{\Omega,\eta_0} := \bigoplus_{d\in(\mathbf{Z}/p^\beta)^\times} \mathrm{Ev}_{B,\beta,d}^{\Omega,\eta_0} : \mathrm{H}_c^t(S_K,\mathscr{D}_\Omega) &\longrightarrow \mathcal{D}(\mathbf{Z}_p^\times,\mathcal{O}_\Omega) \\ \Phi &\longmapsto \sum_{[\delta]\in\pi_0(X_\beta)} \eta_0\big(\mathrm{pr}_2([\delta])\big)\ \mathrm{Ev}_{B,\beta,[\delta]}^{\Omega}(\Phi).\end{aligned} \tag{12.6}$$

Remark 12.7. We have an analogue of Remark 4.14; $\mathrm{Ev}_{B,\beta}^{\Omega,\eta_0}$ is the composition

$$\begin{tikzcd}[column sep=large] \mathrm{H}_c^t(S_K,\mathscr{D}_\Omega) \arrow[r, "\oplus \mathrm{Ev}_{B,\beta,\delta}^{\mathcal{D}_\Omega}"] \arrow[rr, bend left, "\oplus \mathrm{Ev}_{B,\beta,[\delta]}^{\Omega}"] \arrow[rrr, bend right, "\sum_d \mathrm{Ev}_{B,\beta,d}^{\Omega,\eta_0}"'] & \bigoplus_{[\delta]} (\mathcal{D}_\Omega^\beta)_{\Gamma_{\beta,\delta}} \arrow[r, "\delta * \kappa_\Omega"] & \bigoplus_{[\delta]} \mathcal{D}(\mathrm{pr}_\beta([\delta]) + p^\beta\mathbf{Z}_p,\mathcal{O}_\Omega) \arrow[r, "\sum_d \Xi_d^{\eta_0}"] & \mathcal{D}(\mathbf{Z}_p^\times,\mathcal{O}_\Omega), \end{tikzcd} \tag{12.7}$$

where again $\Xi_d^{\eta_0}$ sends a tuple $(m_{[\delta]})_{[\delta]}$ to $\sum_{[\delta]\in \mathrm{pr}_\beta^{-1}(d)} \eta_0(\mathrm{pr}_2([\delta]))\times m_{[\delta]}$.

Combining all of the results of this section, we finally deduce:

Proposition 12.8. *Suppose $\beta \geqslant 1$ and χ is a finite order Hecke character of conductor p^β. Then for every $\lambda\in\Omega$, and $j\in \mathrm{Crit}(\lambda)$, we have a commutative diagram*

$$\begin{tikzcd} \mathrm{H}_c^t(S_K,\mathscr{D}_\Omega) \arrow[r, "\mathrm{sp}_\lambda"] \arrow[d, "\mathrm{Ev}_{B,\beta}^{\Omega,\eta_0}"] & \mathrm{H}_c^t(S_K,\mathscr{D}_\lambda(L)) \arrow[r, "r_\lambda"] \arrow[d, "\mathrm{Ev}_{B,\beta}^{\lambda,\eta_0}"] & \mathrm{H}_c^t(S_K,\mathscr{V}_\lambda^\vee(L)) \arrow[d, "\mathcal{E}_{B,\chi}^{j,\eta_0}"] \\ \mathcal{D}(\mathbf{Z}_p^\times,\mathcal{O}_\Omega) \arrow[r, "\mathrm{sp}_\lambda"] & \mathcal{D}(\mathbf{Z}_p^\times,L) \arrow[r, "\mu\mapsto\mu\left[\chi(z)z^j\right]"] & L. \end{tikzcd}$$

Proof. By combining Proposition 12.3 with Corollary 12.6, and taking a direct sum over $[\delta]\in\pi_0(X_\beta)$, there is a commutative diagram

$$\begin{tikzcd} \mathrm{H}_c^t(S_K,\mathscr{D}_\Omega) \arrow[r, "\mathrm{sp}_\lambda"] \arrow[d, "\mathrm{Ev}_{B,\beta,[\delta]}^{\Omega}"] & \mathrm{H}_c^t(S_K,\mathscr{D}_\lambda(L)) \arrow[r, "r_\lambda"] \arrow[d, "\mathrm{Ev}_{B,\beta,[\delta]}^{\lambda}"] & \mathrm{H}_c^t(S_K,\mathscr{V}_\lambda^\vee(L)) \arrow[d, "\mathcal{E}_{B,\beta,[\delta]}^{j,\mathrm{w}}"] \\ \bigoplus_{[\delta]}\mathcal{D}(d+p^\beta\mathbf{Z}_p,\mathcal{O}_\Omega) \arrow[r, "\mathrm{sp}_\lambda"] & \bigoplus_{[\delta]}\mathcal{D}(d+p^\beta\mathbf{Z}_p,L) \arrow[r, "\mu\mapsto\mu(z^j)"] & \bigoplus_{[\delta]} L, \end{tikzcd} \tag{12.8}$$

where $d = \mathrm{pr}_\beta([\delta]) \in (\mathbf{Z}/p^\beta)^\times$. Also, there is a commutative diagram

$$\begin{tikzcd} \bigoplus_{[\delta]}\mathcal{D}(d+p^\beta\mathbf{Z}_p,\mathcal{O}_\Omega) \arrow[r, "\mathrm{sp}_\lambda"] \arrow[d, "\Xi_d^{\eta_0}"] & \bigoplus_{[\delta]}\mathcal{D}(d+p^\beta\mathbf{Z}_p,L) \arrow[r, "\mu\mapsto\mu(z^j)"] \arrow[d, "\Xi_d^{\eta_0}"] & \bigoplus_{[\delta]} L \arrow[d, "\Xi_d^{\eta_0}"] \\ \bigoplus_{d}\mathcal{D}(d+p^\beta\mathbf{Z}_p,\mathcal{O}_\Omega) \arrow[r, "\mathrm{sp}_\lambda"] \arrow[d, "\Sigma_d"] & \bigoplus_{d}\mathcal{D}(d+p^\beta\mathbf{Z}_p,L) \arrow[r, "\mu\mapsto\mu(z^j)"] \arrow[d, "\Sigma_d"] & \bigoplus_{d} L \arrow[d, "(\ell_d)\mapsto\sum_d\chi(d)\ell_d"] \\ \mathcal{D}(\mathbf{Z}_p^\times,\mathcal{O}_\Omega) \arrow[r, "\mathrm{sp}_\lambda"] & \mathcal{D}(\mathbf{Z}_p^\times,L) \arrow[r, "\mu\mapsto\mu\left[\chi(z)z^j\right]"] & L, \end{tikzcd} \tag{12.9}$$

where the direct sums are over $[\delta] \in \pi_0(X_\beta)$ and $d \in (\mathbf{Z}/p^\beta)^\times$. Indeed the top squares and bottom-left square all commute directly from the definitions; and the bottom right square commutes since for any $\mu \in \mathcal{D}(\mathbf{Z}_p^\times, L)$, we have

$$\int_{\mathbf{Z}_p^\times} \chi(z) z^j \cdot d\mu = \sum_{d \in (\mathbf{Z}/p^\beta)^\times} \chi(d) \int_{d+p^\beta \mathbf{Z}_p} z^j \cdot \mu.$$

Now, in line with Remarks 4.14 and 12.7, the proposition follows by combining (12.8) and (12.9). $\square$

12.4 p-adic L-functions attached to RASCARs

Proposition 12.9. *If* $\beta \geqslant 1$, *then* $\mathrm{Ev}_{B,\beta+1}^{\Omega,\eta_0} = \mathrm{Ev}_{B,\beta}^{\Omega,\eta_0} \circ U_p^\circ : \mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{D}_\Omega) \to \mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$.

Proof. This follows from Proposition 4.10 (cf. [BDW, Prop. 6.16]). $\square$

Definition 12.10. Let $\Phi \in \mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{D}_\Omega)$ be a U_p°-eigenclass with invertible eigenvalue $\alpha_p^\circ \in \mathcal{O}_\Omega^\times$, fix $\beta \geqslant 1$, and define

$$\mu^{\Omega,\eta_0}(\Phi) := (\alpha_p^\circ)^{-\beta} \cdot \mathrm{Ev}_{B,\beta}^{\Omega,\eta_0}(\Phi).$$

By Proposition 12.9, this is independent of the choice of β.

Now let $\tilde{\pi} = (\pi, \alpha)$ be a p-refined RACAR of weight λ satisfying Conditions 8.1; so it admits an $(|\cdot|^{\mathsf{w}}, \psi)$-Shalika model, with w the purity weight of λ. In particular, we have $\eta_0 = \mathbf{1}$ trivial. For K as in (2.1), let $\phi_{\tilde{\pi}}^\pm \in \mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{V}_\lambda^\vee(L))_{\tilde{\pi}}^\pm$ as (8.2). By (C4) and Theorem 10.7, $\tilde{\pi}$ is non-critical (Definition 10.5), hence $\phi_{\tilde{\pi}}^\pm$ lifts uniquely to an eigenclass $\Phi_{\tilde{\pi}}^\pm \in \mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{D}_\lambda)_{\tilde{\pi}}^\pm$ with U_p°-eigenvalue α_p°, recalling $\alpha_p^\circ = \lambda(t_p)\alpha_p$.

For $h \in \mathbf{Q}_{\geqslant 0}$, recall the notion of $\mu \in \mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$ having *growth of order* h [BDJ22, Def. 3.10].

Definition 12.11. Let $[\det(-w_n)] \in \mathcal{D}(\mathbf{Z}_p^\times, L)$ be the (bounded) Dirac measure defined by $\int_{\mathbf{Z}_p^\times} f \cdot d[\det(-w_n)] = f(\det(-w_n))$.

Let $\mathcal{L}_p^\pm(\tilde{\pi}) := \mu^{\lambda,\mathbf{1}}(\Phi_{\tilde{\pi}}^\pm) \in \mathcal{D}(\mathbf{Z}_p^\times, L)$. Let $\Phi_{\tilde{\pi}} = \Phi_{\tilde{\pi}}^+ + \Phi_{\tilde{\pi}}^-$, and define the *p-adic L-function* attached to $\tilde{\pi}$ to be

$$\mathcal{L}_p(\tilde{\pi}) = [\det(-w_n)] \cdot \mu^{\lambda,\mathbf{1}}(\Phi_{\tilde{\pi}}) = [\det(-w_n)] \cdot (\mathcal{L}_p^+(\tilde{\pi}) + \mathcal{L}_p^-(\tilde{\pi})).$$

(The presence of $[\det(-w_n)]$ will become clear). Let $\mathscr{X} := (\mathrm{Spf}\, \mathbf{Z}_p[\![\mathbf{Z}_p^\times]\!])^{\mathrm{rig}}$. The Amice transform allows us to consider $\mathcal{L}_p(\tilde{\pi}, -) : \mathscr{X} \to \overline{\mathbf{Q}}_p$ as an element of $\mathcal{O}(\mathscr{X})$.

Definition 12.12 (Modified Euler factors)**.** Let χ be a finite order character of conductor p^β with $\beta \geqslant 0$, and $j \in \mathrm{Crit}(\lambda)$. At p, let

$$e_p(\tilde{\pi}, \chi, j) := \left(\frac{p^{nj+\frac{n^2-n}{2}}}{\alpha_{p,n}}\right)^\beta \tau(\chi)^n \text{ for } \chi \neq \mathbf{1}, \qquad \text{and} \qquad e_p(\tilde{\pi}, \mathbf{1}, j) = \prod_{i=n+1}^{2n} \frac{1 - p^{-1}\alpha_{i,j}^{-1}}{1 - \alpha_{i,j}},$$

where $\alpha_{i,j} = \theta_i(p)/p^{j+1/2}$, recalling $\pi_p = \mathrm{Ind}_B^G \theta$. At ∞, let

$$e_\infty(\pi, \chi, j) := i^{-jn} \cdot L(\pi_\infty \otimes \chi_\infty, j + 1/2).$$

Theorem 12.13. *Let $\tilde{\pi}$ be a p-refined RACAR of weight λ_π satisfying Conditions 8.1. The distribution $\mathcal{L}_p(\tilde{\pi})$ has growth of order $h_p = v_p(\alpha_p^\circ)$. For every finite order character χ of $\mathbf{Q}^\times\backslash\mathbf{A}^\times$ of conductor p^β with $\beta \in \mathbf{Z}_{\geqslant 0}$, and all $j \in \mathrm{Crit}(\lambda)$, we have*

$$\mathcal{L}_p(\tilde{\pi}, \chi(z)z^j) := \int_{\mathbf{Z}_p^\times} \chi(z)z^j \cdot d\mathcal{L}_p(\tilde{\pi}) = \gamma_{(pm)} \cdot e_p(\tilde{\pi}, \chi, j) \cdot e_\infty(\pi, \chi, j) \cdot \frac{L^{(p)}\left(\pi \otimes \chi, j + \frac{1}{2}\right)}{\Omega_{\tilde{\pi}}^{\pm}},$$

where $\pm 1 = \chi_\infty(-1)(-1)^j$. Here $\gamma_{(pm)}$ is defined before Corollary 8.2, and all further notation is as in Definition 12.12 and Theorem 4.16.

Proof. The growth property follows as in [BDW, Prop. 6.20]. The Dirac measure is bounded so does not affect the growth.

For ramified χ, the interpolation is analogous to [BDW, Thm. 6.23]. In particular, we have

$$\begin{aligned}\int_{\mathbf{Z}_p^\times} \chi(z)z^j \cdot d\mu^{\lambda,\mathbf{1}}(\Phi_{\tilde{\pi}}) &= (\alpha_p^\circ)^{-\beta}\left[\mathrm{Ev}_{B,\beta}^{\lambda,\eta_0}(\Phi_{\tilde{\pi}})\right](\chi(z)z^j) \\ &= (\alpha_p^\circ)^{-\beta}\left[\mathcal{E}_{B,\chi}^{j,\eta_0} \circ r_\lambda\right](\Phi_{\tilde{\pi}}) \qquad \text{(by Prop. 12.8)} \\ &= (\alpha_p^\circ)^{-\beta}\,\mathcal{E}_{B,\chi}^{j,\eta_0}(\phi_{\tilde{\pi}}) \qquad \text{(by definition of } \Phi_{\tilde{\pi}}).\end{aligned}$$

We computed $\mathcal{E}_{B,\chi}^{j,\eta_0}(\phi_{\tilde{\pi}})$ in Corollary 8.2; the term $(\alpha_p^\circ)^{-\beta}$ here combines with the $(\alpha_p^\circ/\alpha_{p,n})^\beta$ there to give $\alpha_{p,n}^{-\beta}$. This is the difference between $Q'(\pi, \chi, j)$ there and $e_p(\tilde{\pi}, \chi, j)$ here, so we get the claimed factors at p.

We now have a term $\chi(\det(-w_n))\zeta_{\chi,j}(W_\infty)$. We recall that $\zeta_{\chi,j}(W_\infty)$ depends linearly on the choice of branching law $\kappa_{\lambda,j}$ and the choice $\Xi_\infty^\pm$ from (3.2). We may make the same choice of $\Xi_\infty^\pm$ as in [BDW], and then the analogous integral was computed in [BDW, §5.4] – using the period relations proved by Jiang–Sun–Tian and Geng [JST, Gen] – and shown to equal $e_\infty(\pi, \chi, j)$. To obtain the relation here, we need only compare the branching laws $\kappa_{\lambda,j}^{\mathrm{BDW}}$ of [BDW, §5.2] with the ones $\kappa_{\lambda,j}$ of §11 here. By definition, $\kappa_{\lambda,j}^{\mathrm{BDW}}$ (resp. $\kappa_{\lambda,j}$) is dual to an element $v_{\lambda,j}^{\mathrm{BDW}}$ (resp. $v_{\lambda,j}$) of V_λ. Since both lie in a line we know there exists C_j such that $v_{\lambda,j}^{\mathrm{BDW}} = C_j v_{\lambda,j}$. Both are algebraic functions on $G(\mathbf{Z}_p)$, and we evaluate them at $u = \begin{pmatrix} 1_n & w_n \\ 0 & 1_n \end{pmatrix}$.

By Lemmas 3.6 and 5.11 of [BDW], $v_{\lambda,j}^{\mathrm{BDW}}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (-1)^{jn}\det(w_n)^j = \det(-w_n)^j$. We have chosen (in Proposition 11.4) that $v_{\lambda,j}(u) = 1$ always. Thus $C_j = \det(-w_n)^j$, and we deduce

$$\zeta_{\chi,j}(W_\infty) = \det(-w_n)^j \cdot e_\infty(\pi, \chi, j).$$

Combining all of the above, for any ramified character we get exactly the claimed formula, but with an additional factor of $\chi(\det(-w_n))(\det(-w_n))^j$. This is removed when we multiply by the Dirac measure $[\det(-w_n)]$.

This leaves interpolation at the unramified character $\chi = \mathbf{1}$. We will complete the proof in this case in Proposition 14.3. □

Remark 12.14. The factors $e_p(\tilde{\pi}, \chi, j)$ and $e_\infty(\pi, \chi, j)$ are exactly as predicted by the Coates–Perrin-Riou conjecture from [Coa89]; for p, this is mostly explained in [AG94, §3].

13 Shalika families and p-adic L-functions

We fix a sufficiently large coefficient field $L/\mathbf{Q}_p$ and drop it from most of the notation. Let $\tilde{\pi}$ be a p-refined RACAR of weight λ_π satisfying (C1-4) of Conditions 8.1. Recall $K = \mathrm{Iw}_G \prod_{\ell \neq p} \mathrm{GL}_{2n}(\mathbf{Z}_\ell)$.

13.1 Existence and étaleness of Shalika families

This entire subsection is dedicated to the proof of Theorem 13.6 below, in which we will use our evaluation maps to construct Shalika families. The proof closely follows the proofs of [BDW, Thms. 7.6, 8.11].

Definition 13.1. Modify Definition 2.3 by defining normalised Hecke algebras $\mathcal{H}_p^\circ := \mathbf{Z}[U_{p,r}^\circ : r = 1,...,2n-1]$, and $\mathcal{H}^\circ := \mathcal{H}' \otimes \mathcal{H}_p^\circ$. At single weights the $\mathcal{H}$ and $\mathcal{H}^\circ$ eigenspaces in cohomology agree, but (unlike $\mathcal{H}$) the normalised algebra $\mathcal{H}^\circ$ acts on the cohomology in families.

Define $\mathbf{T}_{\Omega,\leqslant h}$ to be the image of $\mathcal{H}^\circ \otimes \mathcal{O}_\Omega$ in $\mathrm{End}_{\mathcal{O}_\Omega}\big(\mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{D}_\Omega)^{\leqslant h}\big)$. Define $\mathscr{E}_{\Omega,\leqslant h} := \mathrm{Sp}(\mathbf{T}_{\Omega,\leqslant h})$, a rigid analytic space.

Let $w : \mathscr{E}_{\Omega,\leqslant h} \to \Omega$ be the *weight map* induced by the structure map $\mathcal{O}_\Omega \to \mathbf{T}_{\Omega,\leqslant h}$. Also write $\mathbf{T}_{\Omega,h}^\pm$ and $\mathscr{E}_{\Omega,h}^\pm$ for the analogues using $\pm$-parts of the cohomology. By [JN19, Thm. 3.2.1], $\mathscr{E}_{\Omega,h}^\pm$ embeds as a closed subvariety of $\mathscr{E}_{\Omega,h}$, and $\mathscr{E}_{\Omega,h} = \mathscr{E}_{\Omega,h}^+ \sqcup \mathscr{E}_\Omega^-$.

By definition, $\mathscr{E}_{\Omega,\leqslant h}$ is a rigid space whose L-points y biject with non-trivial homomorphisms $\mathbf{T}_{\Omega,\leqslant h} \to L$, i.e. with systems of eigenvalues of $\psi_y^\circ : \mathcal{H}^\circ \to L$ appearing in $\mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{D}_\Omega)^{\leqslant h}$.

Definition 13.2. A point $y \in \mathscr{E}_{\Omega,h}$ is *classical* if there exists a cohomological automorphic representation π_y of $G(\mathbf{A})$ of weight $\lambda_y := w(y)$ such that ψ_y° appears in π_y^K, whence $\tilde{\pi}_y = (\pi_y, \alpha_y^\circ)$ is a p-refined RACAR (where $\alpha_y^\circ = \psi_y^\circ|_{\mathcal{H}_p^\circ}$). A classical point y is *cuspidal* if π_y is. A $(\mathbf{1},\psi)$-*Shalika point* is a classical cuspidal point y such that π_y admits an $(|\cdot|^{\mathsf{w}_y},\psi)$-Shalika model, for w_y the purity weight of λ_y.

A *classical family* in $\mathscr{E}_{\Omega,h}$ is an irreducible component $\mathscr{I}$ in $\mathscr{E}_{\Omega,h}$ containing a Zariski-dense set of classical points. A $(\mathbf{1},\psi)$-*Shalika family* is a classical family containing a Zariski-dense set of $(\mathbf{1},\psi)$-Shalika points.

Since $\tilde{\pi}$ satisfies Conditions 8.1, it is strongly non-critical by (C4) and Theorem 10.7. Let $\Lambda = \mathcal{O}_{\Omega,\mathfrak{m}_{\lambda_\pi}}$ be the algebraic localisation of $\mathcal{O}_\Omega$ at λ_π.

Lemma 13.3. *We have (Hecke-equivariant) isomorphisms*

$$\mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{D}_\Omega)_{\tilde{\pi}}^\pm \otimes \Lambda/\mathfrak{m}_{\lambda_\pi} \cong \mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{D}_{\lambda_\pi})_{\tilde{\pi}}^\pm \cong \mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{V}_{\lambda_\pi}^\vee)_{\tilde{\pi}}^\pm.$$

Proof. The first isomorphism is proved identically to [BDJ22, Lem. 2.9] (cf. [BDW, Prop. 7.8]). The second follows from non-criticality of $\tilde{\pi}$. □

Let $\mathscr{C}^\pm = \mathrm{Sp}(T^\pm)$ be the connected components of $\mathscr{E}_{\Omega,h}^\pm$ through $x_{\tilde{\pi}}^\pm$.

Corollary 13.4. *(i) There exist ideals $I_{\tilde{\pi}}^\pm \subset \Lambda$ such that $\mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{D}_\Omega)_{\tilde{\pi}}^\pm \cong \Lambda/I_{\tilde{\pi}}^\pm$.*

(ii) Possibly shrinking Ω, there exist ideals $I_{\mathscr{C}}^\pm \subset \mathcal{O}_\Omega$ such that $\mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{D}_\Omega)^{\leqslant h,\pm} \otimes_{\mathbf{T}_{\Omega,h}^\pm} T^\pm \cong \mathcal{O}_\Omega/I_{\mathscr{C}}^\pm$.

Proof. (i) By Proposition 8.3, the right-hand side in Lemma 13.3 is a line. Since $\mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\pm}_{\tilde\pi} \cong \mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\leqslant h,\pm}_{\tilde\pi}$ is finite over Λ, we may use Nakayama's lemma, whence $\mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\pm}_{\tilde\pi}$ is generated by one element over Λ; but every cyclic Λ-module has the form $\Lambda/I^{\pm}_{\tilde\pi}$ for some $I^{\pm}_{\tilde\pi}$.

(ii) This follows from rigid delocalisation of (i) (as in [BW21a, Cor. 4.7] and [BW21b, §2.7]). □

Proposition 13.5. *Suppose $\lambda_{\pi,n} > \lambda_{\pi,n+1}$. Then, up to shrinking Ω:*

(i) $T^{\pm}$ is free of rank one over $\mathcal{O}_\Omega$.

(ii) $\mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\leqslant h,\pm}\otimes_{\mathbf{T}^{\pm}_{\Omega,h}} T^{\pm}$ is free of rank one over $T^{\pm}$.

Proof. By the weight condition, as in [BDW, Lem. 7.4], there exist $\beta \geqslant 1$, $j \in \mathrm{Crit}(\lambda_\pi)$ and finite order Hecke characters $\chi^{\pm}$ of conductor p^β with $\chi^{\pm}_\infty(-1)(-1)^j = \pm 1$ such that $L^{(p)}(\pi\otimes\chi^{\pm}, j+\frac{1}{2}) \neq 0$.

As the $\mathscr{C}^{\pm}$ are connected components, there exist idempotents $e^{\pm}$ such that $T^{\pm} = e^{\pm}\mathbf{T}^{\pm}_{\Omega,h}$, and $\mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\leqslant h,\pm}\otimes_{\mathbf{T}^{\pm}_{\Omega,h}} T^{\pm} = e^{\pm}\mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\leqslant h,\pm} \subset \mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\leqslant h,\pm}$. Restricting $\mathrm{Ev}^{\Omega,\mathbf{1}}_{B,\beta}$ from (12.6), we get

$$\mathrm{Ev}^{\Omega,\mathbf{1}}_{B,\beta} : \mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\leqslant h,\pm}\otimes_{\mathbf{T}^{\pm}_{\Omega,h}} T^{\pm} \to \mathcal{D}(\mathbf{Z}_p^\times,\mathcal{O}_\Omega).$$

Using Corollary 13.4, let $\Phi^{\pm}_{\mathscr{C}}$ be a generator of $\mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\leqslant h,\pm}\otimes_{\mathbf{T}^{\pm}_{\Omega,h}} T^{\pm}$ over $\mathcal{O}_\Omega$. Note that $\mathrm{Ann}_{\mathcal{O}_\Omega}(\Phi^{\pm}_{\mathscr{C}}) = I^{\pm}_{\mathscr{C}}$. Then via the interpolation of Theorem 12.13, we have

$$\int_{\mathbf{Z}_p^\times} \chi^{\pm}(z)z^j\cdot d\left(\mathrm{sp}_{\lambda_\pi}\circ \mathrm{Ev}^{\Omega,\mathbf{1}}_{B,\beta}(\Phi^{\pm}_{\mathscr{C}})\right) = (*)\times L^{(p)}(\pi\otimes\chi^{\pm}, j+\tfrac{1}{2}) \neq 0,$$

where $(*)$ is non-zero. As $\mathrm{sp}_{\lambda_\pi}\left(\mathrm{Ev}^{\Omega,\mathbf{1}}_{B,\beta}(\Phi^{\pm}_{\mathscr{C}})\right) \neq 0$, we deduce $\mathrm{Ev}^{\Omega,\mathbf{1}}_{B,\beta}(\Phi^{\pm}_{\mathscr{C}}) \neq 0$, that is, it is not the zero distribution. Since $\mathcal{D}(\mathbf{Z}_p^\times,\mathcal{O}_\Omega)$ is a torsion-free $\mathcal{O}_\Omega$-module, it follows that $\mathrm{Ann}_{\mathcal{O}_\Omega}(\Phi^{\pm}_{\mathscr{C}}) = 0$ (cf. [BDW, §7.3]). Thus $I^{\pm}_{\mathscr{C}} = 0$, and $\mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\leqslant h,\pm}\otimes_{\mathbf{T}^{\pm}_{\Omega,h}} T^{\pm} \cong \mathcal{O}_\Omega$.

We deduce $T^{\pm}$ is the image of $\mathcal{H}^\circ$ in $\mathrm{End}_{\mathcal{O}}(\Omega)$. Since this image is non-zero we deduce (i). Finally since the actions of $\mathcal{O}_\Omega$ and $T^{\pm}$ are compatible on $\mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\leqslant h,\pm}\otimes_{\mathbf{T}^{\pm}_{\Omega,h}} T^{\pm}$, and both $T^{\pm}$ and $\mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\leqslant h,\pm}\otimes_{\mathbf{T}^{\pm}_{\Omega,h}} T^{\pm}$ are free rank one $\mathcal{O}_\Omega$-modules, we deduce (ii). □

We finally arrive at the main result of this section.

Theorem 13.6. *Let $\tilde\pi$ be a p-refined RACAR of weight λ_π satisfying Conditions 8.1. Suppose $\lambda_{\pi,n} > \lambda_{\pi,n+1}$. Then for any $h \geqslant v_p(\alpha_p^\circ)$, we have:*

(i) There exists a point $x_{\tilde\pi} \in \mathcal{E}_{\Omega,h}$ attached to $\tilde\pi$, and $w : \mathcal{E}_{\Omega,h} \to \Omega$ is étale at $x_{\tilde\pi}$.

(ii) The connected component $\mathscr{C} = \mathrm{Sp}(T)$ in $\mathcal{E}_{\Omega,h}$ through $x_{\tilde\pi}$ contains a very Zariski-dense set $\mathscr{C}_{\mathrm{nc}}$ of classical points corresponding to p-refined RACARs $\tilde\pi_y$.

(iii) There exist Hecke eigenclasses $\Phi^{\pm}_{\mathscr{C}} \in \mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\pm}$ such that for every $y \in \mathscr{C}_{\mathrm{nc}}$, the specialisation $\mathrm{sp}_{\lambda_y}(\Phi^{\pm}_{\mathscr{C}})$ generates $\mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_{\lambda_y})^{\pm}_{\tilde\pi_y}$, where $\lambda_y := w(y)$.

(iv) Up to shrinking Ω, for each $y \in \mathscr{C}_{\mathrm{nc}}$ the p-refined RACAR $\tilde\pi_y$ satisfies Conditions 8.1.

Proof. (i) Lemma 13.3 and Proposition 8.3 show $\mathfrak{m}_{\tilde{\pi}}$ appears in $\mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\leqslant h,\pm}$, giving points $x^\pm_{\tilde{\pi}} \in \mathcal{E}^\pm_{\Omega,h}$. Moreover Proposition 13.5 shows $\mathscr{E}^\pm_{\Omega,h} \to \Omega$ is étale at $x^\pm_{\tilde{\pi}}$. Let $\mathscr{C}^\pm \subset \mathscr{E}^\pm_{\Omega,h}$ be the unique connected component through $x^\pm_{\tilde{\pi}}$. Using strong non-criticality of $\tilde{\pi}$, we deduce that $\mathscr{C}^\pm$ contain very[2] Zariski-dense sets $\mathscr{C}^\pm_{\mathrm{nc}}$ of cuspidal non-critical slope classical points by [BW21c, Prop. 5.15]. Now, as in [BDW, §8.3.4], we can exhibit a bijection between $\mathscr{C}^+_{\mathrm{nc}}$ and $\mathscr{C}^-_{\mathrm{nc}}$ and a canonical isomorphism $\mathscr{C}^+ \xrightarrow{\sim} \mathscr{C}^-$, whence $\mathscr{C} \cong \mathscr{C}^+ \cong \mathscr{C}^-$ is independent of sign. Part (i) follows immediately.

(ii) We let $\mathscr{C}_{\mathrm{nc}} = \mathscr{C}^+_{\mathrm{nc}} = \mathscr{C}^-_{\mathrm{nc}}$ be the set used in (i).

(iii) Let $\Phi^\pm_{\mathscr{C}}$ be $\mathcal{O}_\Omega$-module generators of

$$\mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\leqslant h,\pm} \otimes_{\mathbf{T}^\pm_{\Omega,h}} T^\pm \subset \mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\leqslant h,\pm} \subset \mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\pm},$$

which are Hecke eigenclasses by Proposition 13.5. For each $y \in \mathscr{C}_{\mathrm{nc}}$, let $\mathfrak{m}_y \subset T^\pm$ be the attached maximal ideal. Reduction modulo $\mathfrak{m}_{\lambda_y}$ induces a map

$$\mathrm{sp}_{\lambda_y} : \mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^{\leqslant h,\pm} \otimes_{\mathbf{T}^\pm_{\Omega,h}} T^\pm \twoheadrightarrow \mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_{\lambda_y})^\pm_{\tilde{\pi}_y},$$

which is surjective by combining étaleness of w at y with Lemma 13.3. By Proposition 13.5, we deduce $\mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_{\lambda_y})^\pm_{\tilde{\pi}_y}$ is a line, generated by $\mathrm{sp}_{\lambda_y}(\Phi^\pm_{\mathscr{C}})$.

(iv) Every $y \in \mathscr{C}_{\mathrm{nc}}$ has non-critical slope, hence satisfies (C4). Recall $\pi_y^K \cong \mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{V}^\vee_\lambda)^\pm_{\tilde{\pi}_y}$ by (3.2). As in (iii), the right-hand side is a line (using non-criticality), so $\pi_y^K \neq 0$. For each $\ell \neq p$, this ensures $\pi_\ell^{G(\mathbf{Z}_\ell)} \neq 0$, so $\pi_{y,\ell}$ is spherical, giving (C3).

Now, let $\beta, j, \chi^\pm$ be as in the proof of Proposition 13.5, and define a map

$$\begin{aligned}\mathrm{Ev}^\Omega_{B,\chi,j} : \mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{D}_\Omega)^\pm &\longrightarrow \mathcal{O}_\Omega\\ \Phi &\longmapsto \int_{\mathbf{Z}_p^\times} \chi(z) z^j \cdot d\mathrm{Ev}^{\Omega,\mathbf{1}}_{B,\beta}(\Phi).\end{aligned}$$

As in the proof of Proposition 13.5, we have $\mathrm{Ev}^\Omega_{B,\chi,j}(\Phi^\pm_{\mathscr{C}})(\lambda_\pi) = (*) \times L^{(p)}(\pi\otimes\chi^\pm, j+\frac{1}{2}) \neq 0$. Possibly shrinking Ω, we may thus suppose $\mathrm{Ev}^\Omega_{B,\chi,j}(\Phi^\pm_{\mathscr{C}})$ is everywhere non-vanishing on Ω. Let

$$\phi^\pm_{\tilde{\pi}_y} := r_{\lambda_y} \circ \mathrm{sp}_{\lambda_y}(\Phi^\pm_{\mathscr{C}}) \in \mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{V}^\vee_\lambda)^\pm_{\tilde{\pi}_y}.$$

Then by Proposition 12.8, we have

$$\mathcal{E}^{j,\mathbf{1}}_\chi(\phi^\pm_{\tilde{\pi}_y}) = \int_{\mathbf{Z}_p^\times} \chi(z) z^j d\left[\mathrm{Ev}^{\Omega,\mathbf{1}}_{B,\beta} \circ \mathrm{sp}_{\lambda_y}\right] = \mathrm{Ev}^\Omega_{B,\chi,j}(\Phi^\pm_{\mathscr{C}})(\lambda_y) \neq 0.$$

We deduce π_y satisfies (C1) by Proposition 4.15, i.e. π_y admits a $(|\cdot|^{\mathsf{w}_y},\psi)$-Shalika model.

It remains to show (C2). By [BGW25, Prop. 5.5], up to replacing $\mathscr{C}_{\mathrm{nc}}$ with a smaller (but still very Zariski-dense) subset, we may assume $\pi_{y,p}$ is spherical for all $y \in \mathscr{C}_{\mathrm{nc}}$. For such y, let $\alpha^\circ_{y,p,r} = \alpha^\circ_y(U^\circ_{p,r})$ be the $U_{p,r}$-eigenvalue of $\phi^\pm_{\tilde{\pi}_y}$. As $\mathrm{H}^t_{\mathrm{c}}(S_K,\mathscr{V}^\vee_\lambda)^\pm_{\tilde{\pi}_y}$ is a line, by (3.2) again we deduce that

[2] In [BW21c, Prop. 5.15], only Zariski-density of classical points is treated; but the same proof shows very Zariski-density, using that the classical weights are very Zariski-dense in Ω.

$\mathcal{S}_{\psi}^{|\cdot|^{w_y}}(\pi_{y,p}^{\mathrm{Iw}_G})[U_{p,r}^{\circ} - \alpha_{y,p,r}^{\circ} : r = 1,...,2n-1]$ is a line, so $\tilde{\pi}_y$ is a regular p-refinement. Let $W_{y,p}$ be a generator; then we can relate $W_{y,p}\left(\begin{smallmatrix} w_n & \\ & 1\end{smallmatrix}\right)$ to a non-zero multiple of $\mathrm{Ev}_{B,\chi,j}^{\Omega}(\phi_{\tilde{\pi}_y}^{\pm})$ exactly as in [BDW, §8.3.5]. In particular, $W_{y,p}\left(\begin{smallmatrix} w_n & \\ & 1\end{smallmatrix}\right) \neq 0$, so $\tilde{\pi}_y$ is Shalika. It then follows that $\tilde{\pi}_y$ is spin, by the non-critical slope special case of Expectation 7.2 proved in [BGW25]: particularly, combining [BGW25, Thm. 8.9, Prop. 8.7, Lem. 5.1]. □

13.2 p-adic L-functions in Shalika families

We finally give our main construction, of the variation of p-adic L-functions of RASCARs in pure weight families. Let $\tilde{\pi}$ of weight λ_π satisfy (C1-4) of Conditions 8.1, and suppose $\lambda_{\pi,n} > \lambda_{\pi,n+1}$. By Theorem 13.6(i), the eigenvariety for G is étale over weight space at $\tilde{\pi}$, and by Theorem 13.6(iv) its connected component $\mathscr{C}$ through $\tilde{\pi}$ contains a very Zariski-dense set $\mathscr{C}_{\mathrm{nc}}$ of classical points satisfying Conditions 8.1. Let $\Phi_{\mathscr{C}}^{\pm} \in \mathrm{H}_{\mathrm{c}}^{t}(S_K, \mathscr{D}_\Omega)^{\pm}$ be the classes of Theorem 13.6(iii). Possibly rescaling by $\mathcal{O}_\Omega^\times$, we may always assume $\mathrm{sp}_{\lambda_\pi}(\Phi_{\mathscr{C}}^{\pm}) = \Phi_{\tilde{\pi}}^{\pm}$.

Definition 13.7. Let $\mathcal{L}_p^{\mathscr{C},\pm} := [\det(-w_n)]\mu^{\Omega,\mathbf{1}}(\Phi_{\mathscr{C}}^{\pm})$, now considering the Dirac measure as being in $\mathcal{D}(\mathbf{Z}_p^\times, \mathcal{O}_\Omega)$. Also let $\Phi_{\mathscr{C}} = \Phi_{\mathscr{C}}^{+} + \Phi_{\mathscr{C}}^{-} \in \mathrm{H}_{\mathrm{c}}^{t}(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)$, a Hecke eigenclass, and define the *p-adic L-function over $\mathscr{C}$* to be

$$\mathcal{L}_p^{\mathscr{C}} := [\det(-w_n)]\mu^{\Omega,\mathbf{1}}(\Phi_{\mathscr{C}}) = \mathcal{L}_p^{\mathscr{C},+} + \mathcal{L}_p^{\mathscr{C},-} \in \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega).$$

Via the Amice transform (cf. Definition 12.11), we consider $\mathcal{L}_p^{\mathscr{C}}$ as a rigid function $\mathscr{C} \times \mathscr{X} \to \overline{\mathbf{Q}}_p$.

Theorem 13.8. *Let $y \in \mathscr{C}_{\mathrm{nc}}$ be a classical cuspidal point attached p-refined RACAR $\tilde{\pi}_y$ satisfying Conditions 8.1. There exist p-adic periods $c_y^{\pm} \in L^\times$ such that*

$$\mathcal{L}_p^{\mathscr{C},\pm}(y, -) = c_y^{\pm} \cdot \mathcal{L}_p^{\pm}(\tilde{\pi}_y, -)$$

as functions $\mathscr{X} \to \overline{\mathbf{Q}}_p$. In particular, $\mathcal{L}_p^{\mathscr{C}}$ satisfies the following interpolation: for any $j \in \mathrm{Crit}(w(y))$, and for any finite order Hecke character χ of conductor $p^\beta > 1$, we have

$$\mathcal{L}_p^{\mathscr{C}}(y,\ \chi(z)z^j) = c_y^{\pm} \cdot \gamma_{(pm)} \cdot e_p(\tilde{\pi}_y, \chi, j) \cdot e_\infty(\pi_y, \chi, j) \cdot L^{(p)}(\pi_y \times \chi, j + \tfrac{1}{2})/\Omega_{\pi_y}^{\pm}, \tag{13.1}$$

where $\chi_\infty(-1)(-1)^j = \pm 1$ and with notation as in Theorem 12.13. Finally we have $c_{x_{\tilde{\pi}}}^{\pm} = 1$.

The 'p-adic periods' $c_y^{\pm}$ p-adically align the natural algebraic structures in $\{\tilde{\pi}_y : y \in \mathscr{C}_{\mathrm{nc}}\}$.

Proof. Let y be as in the theorem. Using that y satisfies Conditions 8.1, let $W_{y,f} \in \mathcal{S}_{\psi_f}^{\eta_{y,f}}(\pi_{y,f}^K)$ be as defined as in (8.1), and pick complex periods $\Omega_{\pi_y}^{\pm}$ as in §8. Since $y \in \mathscr{C}$ is defined over L, as in §8, there exists a class

$$\phi_y^{\pm} := \Theta^{\pm}(W_{y,f})/i_p(\Omega_{\pi_y}^{\pm}) \in \mathrm{H}_{\mathrm{c}}^{t}(S_K, \mathscr{V}_{\lambda_y}^{\vee}(L))_{\tilde{\pi}_y}^{\pm}.$$

As y is non-critical, we can lift $\phi_y^{\pm}$ to a non-zero class $\Phi_y^{\pm} \in \mathrm{H}_{\mathrm{c}}^{t}(S_K, \mathscr{D}_{\lambda_y}(L))_{\tilde{\pi}_y}^{\pm}$. By Theorem 13.6(iii), this space is $L \cdot \mathrm{sp}_{\lambda_y}(\Phi_{\mathscr{C}}^{\pm})$, so there exists $c_y^{\pm} \in L^\times$ such that

$$\mathrm{sp}_{\lambda_y}(\Phi_{\mathscr{C}}^{\pm}) = c_y^{\pm} \cdot \Phi_y^{\pm}.$$

By definition, $\mathcal{L}_p^{\pm}(\tilde{\pi}_y) = [\det(-w_n)]\mathrm{Ev}^{B,\lambda_y,\mathbf{1}}(\Phi_y^{\pm})$. As evaluation maps commute with weight specialisation (Proposition 12.8), we deduce $\mathrm{sp}_{\lambda_y}(\mathcal{L}_p^{\mathscr{C},\pm}) = c_y^{\pm}\cdot\mathcal{L}_p^{\pm}(\tilde{\pi}_y)$, which when combined with Theorem 12.13 gives (13.1). Finally, our normalisation of $\Phi_{\mathscr{C}}^{\pm}$ ensures $c_{x_{\tilde{\pi}}}^{\pm} = 1$. □

14 Comparison to existing constructions

We finally show that the p-adic L-functions we've constructed at Iwahori level agree with previous constructions, and deduce their interpolation property at unramified characters.

Let $\tilde{\pi}$ be a non-critical slope regular (Iwahoric) spin p-refinement. Then $\tilde{\pi}^Q := (\pi,\alpha_{p,n})$ is a non-Q-critical slope Shalika Q-refinement as in [BDW, §2.7,§3.5], where being Shalika follows by combining (the proof of) [Roc23, Thm. 4] with [DJR20, Lem. 3.6]. Now [BDW, Thm. A] attaches a p-adic L-function $\mathcal{L}_p(\tilde{\pi}^Q)\in\mathcal{D}(\mathbf{Z}_p^\times, L)$ to $\tilde{\pi}^Q$.

Proposition 14.1. *There exists a constant $\Upsilon\in\mathbf{Q}^\times$, independent of π, such that*

$$\mathcal{L}_p(\tilde{\pi}) = \Upsilon\cdot\mathcal{L}_p(\tilde{\pi}^Q).$$

Proof. Since $\tilde{\pi}^Q$ is non-Q-critical slope, we have

$$v_p(\alpha_{p,n}^\circ) < \#\mathrm{Crit}(\lambda_\pi), \tag{14.1}$$

where λ_π is the weight of π. In this case, both distributions have sufficiently small growth that they are uniquely determined by their interpolation properties at ramified characters (see e.g. [Ngu22, Lem. 1.2.5]). Their respective interpolation properties agree exactly except at the volume terms at p (denoted γ in [DJR20, BDW] and $\gamma_{(pm)}$ in the present paper). In particular at every character, the interpolation formulas differ by a rational number Υ independent of π. The result follows. □

Remark 14.2. That the p-adic L-function depends only on the Q-refinement, not the full Iwahori refinement, should be expected; it is predicted by the Panchiskin condition from [Pan94].

Now let $\tilde{\pi}^Q$ be any non-Q-critical Q-refinement of π, and let $\mathcal{L}_p(\tilde{\pi}^Q)$ be the p-adic L-function attached by [BDW, Thm. A]. The following strengthens the results of [DJR20, BDW].

Proposition 14.3. *The p-adic L-function $\mathcal{L}_p(\tilde{\pi}^Q)$ satisfies the interpolation*

$$\mathcal{L}_p(\tilde{\pi}, z^j) = \Upsilon^{-1}\cdot\gamma_{(pm)}\cdot e_p(\tilde{\pi},\mathbf{1},j)\cdot e_\infty(\pi,\mathbf{1},j)\cdot\frac{L^{(p)}\big(\pi, j+\frac{1}{2}\big)}{\Omega_\pi^{\pm}},$$

for all $j\in\mathrm{Crit}(\lambda_\pi)$. Here $(-1)^j = \pm 1$ and all other notation is as in Theorem 12.13.

In particular, $\mathcal{L}_p(\tilde{\pi})$ satisfies the interpolation of Theorem 12.13 at $\chi = \mathbf{1}$.

Proof. Let $\phi_{\tilde{\pi}^Q}^{\pm}$ be the class defined in [BDW, §6.6]. Then, taking $\beta = 1$, we have

$$\begin{aligned}\mathcal{L}_p(\tilde{\pi}, z^j) &= (\alpha_{p,n}^\circ)^{-1}\cdot\mathcal{E}_{Q,1}^{j,\eta_0}(\phi_{\tilde{\pi}^Q}^{\pm})\\ &= \Upsilon_Q\cdot\frac{p^{n^2}}{\alpha_{p,n}}\cdot e_\infty(\pi,\mathbf{1},j)\cdot\frac{L^{(p)}(\pi,j+\frac{1}{2})}{\Omega_\pi^{\pm}}\cdot\zeta_p\Big(j+\tfrac{1}{2},(u^{-1}t_{p,n}^{\beta})\cdot W_p,\mathbf{1}\Big).\end{aligned}$$

Here the first equality is shown in [BDW, Thm. A]. The second is Theorem 4.16, noting that $\delta_B(t_Q^{-1}) = p^{n^2}$ and $\lambda(t_Q)/\alpha^\circ_{p,n} = 1/\alpha_{p,n}$. This local zeta integral was computed in Proposition 9.3. We find that this exactly agrees with the claimed formula (as by definition, Υ tracks the difference between the volume factor $\gamma_{(pm)}$ at Iwahori level and γ, from [DJR20], at parahoric level).

The final statement follows immediately from the first part and Proposition 14.1. □

Remark 14.4. Exactly the same proof shows more generally that the p-adic L-functions of [DJR20, BDW], for GL_{2n} over a general totally real field and at arbitrary tame level, satisfy the interpolation formula at unramified characters predicted by Coates–Perrin-Riou and Panchishkin.

Glossary of key notation/terminology

$(-)^\vee$ ... dual §2.1
$\langle -,-\rangle_G, \langle -,-\rangle_{\mathcal{G}}$ ... §6.2
$\mathcal{A}_\Omega$ ... Defn. 10.2
α ... system of eigenvalues §2.4.3
α_p° ... U_p°-eigenvalue
$\alpha^{\mathcal{G}}$ ... system of e'values for $\mathcal{G}$
$\alpha_0, \alpha_1, ..., \alpha_n$ ... basis of weights (11.1)
$\alpha_{p,r}$ ... $\alpha(U_{p,r})$
$B \subset G$ ... upper-triangular Borel
β ... integer $\geqslant 1$
$\overline{B} \subset G$ ... opposite Borel
$\mathscr{C}$ ... Shalika family, Thm. 13.6
$\mathrm{Crit}(\lambda)$ ... §2.2
$\mathrm{Cl}(I)$ ... (4.3)
$\mathcal{D}_\Omega$ ... Defn. 10.3
$\mathscr{D}_\Omega$ ... local system for $\mathcal{D}_\Omega$
$\mathcal{D}_\Omega^\beta$ ... Defn. 11.11
Δ_P ... semigroup, §4.1.2
Δ_θ ... function, (2.13)
δ_B ... §2.1
$\mathrm{Ev}_{M,-}^{P,-}$ ... evaluation maps §4.1.2
$\mathrm{Ev}_{B,-}^{\Omega}$ ... distribution-valued evaluation maps §12.1
$\mathcal{E}_{P,\chi}^{j,\eta_0}$ ... classical evaluation map §4.2
$\mathscr{E}_{\Omega,\leqslant h}, \mathscr{E}^{\pm}_{\Omega,\leqslant h}$ ... local pieces of eigenvariety, Defn. 13.1
e_i, e_i^* ... root data §6.2
$e_p(\tilde{\pi},\chi,j)$ ... Coates–Perrin-Riou factor, Thm. 12.13
η, η_0 ... Shalika characters §2.3
$F_\delta, F_\delta^\sigma$ ... Defn. 7.8
f_i, f_i^* ... root data §6.2
$f^\sigma \in \mathrm{Ind}_B^G \theta^\sigma$ ... §7.1
$f_w^\sigma \in \mathrm{Ind}_B^G \theta^\sigma$ ... Defn. 7.8
G ... GL_{2n}
$\mathcal{G}$ ... GSpin_{2n+1}
Gal_p ... §2.1
$\gamma_{(pm)}$ ... constant §8
H ... $\mathrm{GL}_n \times \mathrm{GL}_n$
$\mathrm{H}^\bullet_{\mathrm{c}}(S_K,-)_\pi$ ... §3.2.1
$\mathcal{H}'$ ... spherical Hecke algebra §2.4
$\mathcal{H}$ ... Hecke algebra §2.4
$\mathcal{H}^\circ, \mathcal{H}_p^\circ$ ... Defn. 13.1
$\mathcal{H}_p^{\mathcal{G}}$ ... Hecke algebra §6.4
$\iota: H \to G$ ... $(h_1,h_2) \mapsto \mathrm{diag}(h_1,h_2)$
ι_β^P ... (4.4)
$\mathrm{Iw}_G \subset \mathrm{GL}_{2n}(\mathbf{Q}_p)$ ... Iwahori subgroup
Iw_G^β ... (11.8)
$\mathrm{Iw}_{G_n} \subset \mathrm{GL}_n(\mathbf{Q}_p)$ ... Iwahori subgroup
i_p ... fixed isomorphism $\mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_p$
J_P ... parahoric §4.1.2
$\jmath, \jmath^\vee$ ... §6.3
$K \subset G(\mathbf{A}_f)$ ... level group
$\kappa_{\lambda,j}$ ... (11.13)
κ_Ω ... (11.15)
$L^{(p)}$ ... L-function without factor at p
$L_\beta^P, L_p^{P,\beta}, L^p$ ... Defn. 4.3
$\mathcal{L}_p(\tilde{\pi}), \mathcal{L}_p^{\pm}(\tilde{\pi})$ ... p-adic L-functions, Defn. 12.11
$\mathcal{L}_p^{\mathscr{C}}$ ... p-adic L-function in family, Defn. 13.7
$\lambda = (\lambda_1, ..., \lambda_{2n})$ ... weight
M_{w_n}, M_ρ ... intertwinings (7.6)
m ... auxiliary integer, Defn. 4.3
$\mathfrak{m}_\pi, \mathfrak{m}_{\tilde{\pi}}$ ... maximal ideals §2.4
$N^\beta(\mathbf{Z}_p)$ ... §11.2
non-critical slope ... Defn. 10.6
Ω ... affinoid in weight space
$\Omega_\pi^\pm \in \mathbf{C}^\times$ ... complex period
$P \subset G$ ... standard parabolic
pr_β ... (4.9)
π ... RASCAR of $G(\mathbf{A})$, Conditions 8.1
$\phi_{\tilde{\pi}}^{\pm}$ ... cohomology class (8.2)
$\varphi_p \in \tilde{\pi}_p$ ... §2.4.3
$\psi_{\tilde{\pi}}$ ... §2.4.3
$\sigma \in \mathcal{W}_G$ ... Weyl element
$\tilde{\pi}$ ... p-refinement §2.4.3

$Q \subset G$ ... parabolic with Levi H
$Q'(\pi,\chi,j)$ ... factor at p, Cor. 8.2
$\rho_G, \rho_{\mathcal{G}}$ ... half sum positive roots (6.8)
RASCAR ... p.162
r_λ ... (10.4)
$\mathcal{S}_\psi^\eta$ ... Shalika intertwining §2.3
S_K ... locally symm. space §3
Shalika refinement ... Defn. 7.1
spin refinement ... Defn. 6.2
$T \subset G$ ... diagonal torus
$\mathbf{T}_{\Omega,\leqslant h}, \mathbf{T}_{\Omega,\leqslant h}$ ... Hecke algebras, Defn. 13.1
$t_{p,r}$ ... (2.10)
t_P ... Defn. 4.2
τ ... $\begin{pmatrix} 1 & \\ & w_n \end{pmatrix} \in \mathrm{GL}_{2n}(\mathbf{Q}_p)$
$\tau(\chi)$ ... Gauss sum 5.2
$\Theta^\pm$ ... §3.3
θ ... $\pi_p = \mathrm{Ind}_B^G \theta$
$U_{p,r}$ ... Hecke operator §2.4.2
$U_{p,r}^\circ$ ... §2.4.3
u ... $\begin{pmatrix} 1 & w_n \\ 0 & 1 \end{pmatrix}$ (eqn. (4.1))
Υ_P ... constant, Thm. 4.16
Υ'' ... Prop. 7.14
V_λ ... algebraic representation §2.2
$V_{(j,-j-\mathsf{w})}^H$ ... §4.2
$v_{p,r}$ ... cocharacter §6.4
$v_\rho = \begin{pmatrix} 1 & \\ & \rho \end{pmatrix} \in \mathcal{W}_G$ ... $\rho \in \mathcal{W}_n$
$\mathcal{V}, \mathscr{V}$ ... local systems §3.1
$v_{(i)}, v_{(n),j}$ ... Notation 11.2
$v_{\lambda,j}$ ... Prop. 11.3
v_Ω ... Definition 11.8
W, W_ℓ, W_f ... Shalika vectors §2.3, (8.1)
$W_0, W_\delta \in \mathcal{S}_\psi^\eta(\pi_p)$ ... Prop. 7.10
W_ℓ° ... spherical vector §2.3
$\mathscr{W}^G, \mathscr{W}_0^G$ ... weight spaces, Defn. 10.1
$\mathcal{W}_G$ ... Weyl group of G (S_{2n})
$\mathcal{W}_{\mathcal{G}}$ ... Weyl group of $\mathcal{G}$
$\mathcal{W}_G^0$ ... §6.2
$\mathcal{W}_n$ ... Weyl group of GL_n (S_n)
w ... purity weight §2.2
w_Ω ... purity weight in family (10.1)
$w_{2n} \in \mathcal{W}_G$ ... longest Weyl element
$w_n \in \mathcal{W}_n$ ... longest Weyl element
w_χ ... character (11.5)
X_β^P ... Defn. 4.3
$X_\beta^P[\delta]$ ... §4.1.1
$\mathcal{X}, \mathcal{X}^\vee$ ... Prop. 6.4
$x_{\tilde{\pi}}$ ... point of eigenvariety, Thm. 13.6
Ξ ... averaging map (4.14)
χ (in §2-4, §8, §12-14) ... finite order Hecke character
χ (in §5-7) ... local character $\mathbf{Q}_p^\times \to C^\times$
χ (in §9) ... local character $F^\times \to \mathbf{C}^\times$
χ (in §10) ... character $T(\mathbf{Z}_p) \to R^\times$
χ_{cyc} ... cyclotomic character
y ... point of eigenvariety
z ... $\mathrm{diag}(p^{n-1}, p^{n-2}, \ldots, p, 1)$
$\zeta(s, W, \chi)$ ... Friedberg–Jacquet integral §2.3.2
$\zeta_j(W_\infty^\pm)$ ... zeta integral at infinity, Thm. 4.16

Acknowledgments

This paper owes a great debt to the observations of David Loeffler, and we thank him profusely. We thank Mahdi Asgari for patiently answering our questions about his work on GSpin_{2n+1}. We also thank the three anonymous referees for their valuable comments and corrections. Finally we thank the CIRM in Luminy and the CMND program on Eigenvarieties in Notre Dame, where parts of this work were carried out.

References

[AG94] Avner Ash and David Ginzburg. p-adic L-functions for $\mathrm{GL}(2n)$. *Invent. Math.*, 116(1-3):27–73, 1994. 164, 171, 190, 197, 198, 226

[AS06] Mahdi Asgari and Freydoon Shahidi. Generic transfer for general spin groups. *Duke Math. J.*, 132(1):137–190, 2006. 172, 190, 193, 195

[AS14] Mahdi Asgari and Freydoon Shahidi. Image of functoriality for general spin groups. *Manuscripta Math.*, 144(3-4):609–638, 2014. 172, 193

[Asg02] Mahdi Asgari. Local L-functions for split spinor groups. *Canad. J. Math.*, 54(4):673–693, 2002. 191

[Ash80] Avner Ash. Non-square-integrable cohomology of arithmetic groups. *Duke Math. J.*, 47(2):435–449, 1980. 178

[BDJ22] Daniel Barrera, Mladen Dimitrov, and Andrei Jorza. p-adic L-functions of Hilbert cusp forms and the trivial zero conjecture. *J. Eur. Math. Soc. (JEMS)*, 24(10):3439–3503, 2022. 225, 227

[BDW] Daniel Barrera Salazar, Mladen Dimitrov, and Chris Williams. On p-adic L-functions for GL_{2n} in finite slope Shalika families. Preprint: https://arxiv.org/abs/2103.10907. 162, 163, 164, 165, 166, 167, 168, 169, 171, 174, 175, 176, 177, 179, 180, 181, 182, 183, 195, 206, 207, 213, 225, 226, 227, 228, 229, 230, 231, 232

[BFG92] Daniel Bump, Solomon Friedberg, and David Ginzburg. Whittaker-orthogonal models, functoriality, and the Rankin-Selberg method. *Invent. Math.*, 109(1):55–96, 1992. 198

[BGW25] Daniel Barrera Salazar, Andrew Graham, and Chris Williams. On p-refined Friedberg–Jacquet integrals and the classical symplectic locus in the GL_{2n}-eigenvariety. *Res. Number Theory*, 11(51):1–57, 2025. 164, 195, 229, 230

[BW19] Daniel Barrera Salazar and Chris Williams. p-adic L-functions for GL_2. *Canad. J. Math.*, 71(5):1019–1059, 2019. 181

[BW21a] Daniel Barrera Salazar and Chris Williams. Families of Bianchi modular symbols: critical base-change p-adic L-functions and p-adic Artin formalism. *Selecta Math. (N.S.)*, 27(82), 2021. Appendix by Carl Wang-Erickson. 228

[BW21b] Daniel Barrera Salazar and Chris Williams. Overconvergent cohomology, p-adic L-functions and families for $\mathrm{GL}(2)$ over CM fields. *J. Théor. Nombres Bordeaux*, 33(3, part 1):659–701, 2021. 228

[BW21c] Daniel Barrera Salazar and Chris Williams. Parabolic eigenvarieties via overconvergent cohomology. *Math. Z.*, 299(1-2):961–995, 2021. 174, 176, 177, 213, 222, 229

[Cas80] W. Casselman. The unramified principal series of $\mathfrak{p}$-adic groups. I. The spherical function. *Compositio Math.*, 40(3):387–406, 1980. 199

[Che04] Gaëtan Chenevier. Familles p-adiques de formes automorphes pour GL_n. *J. Reine Angew. Math.*, 570:143–217, 2004. 173

[Clo90] Laurent Clozel. Motifs et formes automorphes: applications du principe de fonctorialité. In *Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988)*, volume 10 of *Perspect. Math.*, pages 77–159. Academic Press, Boston, MA, 1990. 171, 172

[Coa89] John Coates. On p-adic L-functions attached to motives over $\mathbf{Q}$. II. *Bol. Soc. Brasil. Mat. (N.S.)*, 20(1):101–112, 1989. 226

[CS20] Fulin Chen and Binyong Sun. Uniqueness of twisted linear periods and twisted Shalika periods. *Sci. China Math.*, 63(1):1–22, 2020. 205

[DJ] Mladen Dimitrov and Andrei Jorza. Parahoric level p-adic L-functions for automorphic representations of GL_{2n} with Shalika models. Preprint: http://arxiv.org/abs/2508.08176. 169, 199

[DJR20] Mladen Dimitrov, Fabian Januszewski, and A. Raghuram. L-functions of GL_{2n}: p-adic properties and nonvanishing of twists. *Compositio Math.*, 156(12):2437–2468, 2020. 163, 164, 165, 171, 174, 176, 177, 178, 181, 182, 183, 184, 193, 195, 205, 206, 207, 208, 231, 232

[FJ93] Solomon Friedberg and Hervé Jacquet. Linear periods. *J. Reine Angew. Math.*, 443:91–139, 1993. 172, 183

[Gen] Zhibin Geng. On the existence of twisted Shalika periods: the archimedean case. Preprint: https://arxiv.org/abs/2501.11917. 226

[GH24] Jayce R. Getz and Heekyoung Hahn. *An introduction to automorphic representations—with a view toward trace formulae*, volume 300 of *Graduate Texts in Mathematics*. Springer, Cham, [2024] ©2024. 170

[GR14] Harald Grobner and A. Raghuram. On the arithmetic of Shalika models and the critical values of L-functions for GL_{2n}. *Amer. J. Math.*, 136(3):675–728, 2014. With an appendix by Wee Teck Gan. 163, 165, 171, 172, 176, 181

[Han17] David Hansen. Universal eigenvarieties, trianguline Galois representations and p-adic Langlands functoriality. *J. Reine. Angew. Math.*, 730:1–64, 2017. 166, 211, 213

[HS16] Joseph Hundley and Eitan Sayag. Descent construction for GSpin groups. *Mem. Amer. Math. Soc.*, 243(1148):v+124, 2016. 191

[JN19] Christian Johansson and James Newton. Irreducible components of extended eigenvarieties and interpolating Langlands functoriality. *Math. Res. Lett.*, 26(1):159–201, 2019. 227

[JPSS81] Hervé Jacquet, Ilja Piatetski-Shapiro, and Joseph Shalika. Conducteur des représentations génériques du groupe linéaire. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(13):611–616, 1981. 168

[JS90] Hervé Jacquet and Joseph Shalika. Exterior square L-functions. In *Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988)*, volume 11 of *Perspect. Math.*, pages 143–226. Academic Press, Boston, MA, 1990. 172

[JST] Dihua Jiang, Binyong Sun, and Fangyang Tian. Period relations for standard L-functions of symplectic type. Preprint: `https://arxiv.org/abs/1909.03476`. 226

[Kat04] Kazuya Kato. p-adic Hodge theory and values of zeta functions of modular forms. *Astérisque*, 295:117–290, 2004. 162

[Loe21] David Loeffler. Spherical varieties and norm relations in Iwasawa theory. *J. Théor. Nombres Bordeaux*, 33(3, part 2):1021–1043, 2021. Iwasawa 2019 special issue. 177, 180, 216

[LPSZ21] David Loeffler, Vincent Pilloni, Christopher Skinner, and Sarah Livia Zerbes. Higher Hida theory and p-adic L-functions for GSp_4. *Duke Math. J.*, 170(18):4033–4121, 2021. 167

[LSZ22] David Loeffler, Christopher Skinner, and Sarah Livia Zerbes. Euler systems for $\mathrm{GSp}(4)$. *J. Eur. Math. Soc. (JEMS)*, 24(2):669–733, 2022. 167

[LZa] David Loeffler and Sarah Livia Zerbes. Iwasawa theory for quadratic Hilbert modular forms. Preprint: `https://arxiv.org/abs/2006.14491`. 167

[LZb] David Loeffler and Sarah Livia Zerbes. On the Bloch–Kato conjecture for $\mathrm{GSp}(4)$. Preprint: `https://arxiv.org/abs/2003.05960`. 162, 163, 165, 167, 168

[LZ23] David Loeffler and Sarah Livia Zerbes. On the Bloch-Kato conjecture for the symmetric cube. *J. Eur. Math. Soc. (JEMS)*, 25(8):3359–3363, 2023. 167

[Mat13] Nadir Matringe. Essential Whittaker functions for $GL(n)$. *Doc. Math.*, 18:1191–1214, 2013. 170

[Ngu22] Duc Nam Nguyen. *On functional equations of p-adic L-functions for* GL*(2)*. PhD thesis, Université de Lille, 2022. 231

[Nie09] Chufeng Nien. Uniqueness of Shalika models. *Canad. J. Math.*, 61(6):1325–1340, 2009. 205

[OST23] Masao Oi, Ryotaro Sakamoto, and Hiroyoshi Tamori. Iwahori-Hecke algebra and unramified local L-functions. *Math. Z.*, 303(3):Paper No. 59, 42, 2023. 194

[Pan94] Alexei A. Panchishkin. Motives over totally real fields and p-adic L-functions. *Ann. Inst. Fourier (Grenoble)*, 44(4):989–1023, 1994. 231

[Roc] Rob Rockwood. Spherical varieties and non-ordinary families of cohomology classes. *J. Number Theory*. To appear: `https://arxiv.org/abs/2204.07116`. 166

[Roc23] Rob Rockwood. Plus/minus p-adic L-functions for GL_{2n}. *Ann. Math. Qué.*, 47(1):177–193, 2023. 231

[Roh89] David E. Rohrlich. Nonvanishing of L-functions for $\mathrm{GL}(2)$. *Invent. Math.*, 97(2):381–403, 1989. 168

[SU14] Christopher Skinner and Eric Urban. The Iwasawa Main Conjectures for $\mathrm{GL}(2)$. *Invent. Math.*, 195 (1):1–277, 2014. 162

[Sun19] Binyong Sun. Cohomologically induced distinguished representations and cohomological test vectors. *Duke Math. J.*, 168(1):85–126, 2019. 183

[Urb11] Eric Urban. Eigenvarieties for reductive groups. *Ann. of Math. (2)*, 174(3):1685–1784, 2011. 174, 211, 212

[Zel80] A. V. Zelevinsky. Induced representations of reductive p-adic groups. II. On irreducible representations of $\mathrm{GL}(n)$. *Ann. Sci. École Norm. Sup. (4)*, 13(2):165–210, 1980. 170

AUTHORS

Daniel Barrera-Salazar

Universidad de Santiago de Chile
Alameda 3363, Estación Central
9170022 Santiago
Chile

daniel.barrera.s@usach.cl
https://sites.google.com/view/danielbarrerasalazar/math

Mladen Dimitrov

University of Lille
CNRS, UMR 8524 – Laboratoire Paul Painleve
59000 Lille
France

mladen.dimitrov@univ-lille.fr
https://pro.univ-lille.fr/mladen-dimitrov

Andrew Graham

University of Oxford
Andrew Wiles Building
Oxford
OX2 6GG
United Kingdom

andrew.graham@maths.ox.ac.uk
https://sites.google.com/view/andrew-graham

Andrei Jorza

University of Notre Dame
275 Hurley Hall
Notre Dame, IN 46556
United States

ajorza@nd.edu

Chris Williams

School of Mathematical Sciences, University of Nottingham
University Park Campus
Nottingham
NG7 ZRD
United Kingdom

chris.williams1@nottingham.ac.uk
https://chriswilliams1404.wixsite.com/website