Macroscopic Models of Superconductivity

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Abstract

After giving a description of the basic physical phenomena to be modelled, we begin by formulating a sharp-interface free-boundary model for the destruction of superconductivity by an applied magnetic field, under isothermal and anisothermal conditions, which takes the form of a vectorial Stefan model similar to the classical scalar Stefan model of solid/liquid phase transitions and identical in certain two-dimensional situations. This model is found sometimes to have instabilities similar to those of the classical Stefan model.

We then describe the Ginzburg-Landau theory of superconductivity, in which the sharp interface is 'smoothed out' by the introduction of an order parameter, representing the number density of superconducting electrons. By performing a formal asymptotic analysis of this model as various parameters in it tend to zero we find that the leading order solution does indeed satisfy the vectorial Stefan model. However, at the next order we find the emergence of terms analogous to those of 'surface tension' and 'kinetic undercooling' in the scalar Stefan model. Moreover, the 'surface energy' of a normal/superconducting interface is found to take both positive and negative values, defining Type I and Type II superconductors respectively.

We discuss the response of superconductors to external influences by considering the nucleation of superconductivity with decreasing magnetic field and with decreasing temperature respectively, and find there to be a pitchfork bifurcation to a superconducting state which is subcritical for Type I superconductors and supercritical for Type II superconductors. We also examine the effects of boundaries on the nucleation field, and describe in more detail the nature of the superconducting solution in Type II superconductors - the so-called 'mixed state'.

Finally, we present some open questions concerning both the modelling and analysis of superconductors.

Acknowledgements

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Contents

Chapter 1

Introduction

In 1911, H. Kamerlingh-Onnes, while investigating variation of the electrical resistivity of mercury with temperature, discovered that at a temperature of 4.2K the resistivity dropped sharply to zero [37]. The same phenomenon was later detected in other metals, and was termed *superconductivity*, with the materials being known as superconductors. The temperature at which superconductivity appears is characteristic of the material, and is known as the critical temperature, T_c. Although it cannot be shown that the resistivity is actually zero, closed currents have been made to circulate in a ring of superconducting material without observable decay for over two years, which leads to an estimate of the resistivity of some of these materials of no greater than $10^{-25}\,\Omega$m (compared to copper, which has a resistivity of about $10^{-8}\,\Omega$m).

In addition to the property of *perfect conductivity*, superconductors are also characterised by the property of *perfect diamagnetism*. This phenomenon was discovered by W. Meissner & R. Ochsenfeld in 1933, and is also known as the *Meissner effect* [48]. They observed that not only is a magnetic field excluded from a superconductor, i.e. if a magnetic field is applied to a superconducting material it does not penetrate into the material (as can be explained by perfect conductivity, due to the currents induced when the material is placed in the field), but also that a magnetic field is expelled from a superconductor, i.e. if an originally normal sample is placed in a magnetic field and then cooled through the critical temperature the magnetic field is expelled from the sample as it becomes superconducting (whereas a perfect conductor would trap the field, i.e when the field is removed the currents induced would hold the field within the sample). Fig. 1.1 shows the contrasting responses of perfect conductors and superconductors in the presence of an applied magnetic field.

In more detailed investigations it has been found that the field is zero only in the bulk of large samples, with the field decreasing from its given surface value to zero in a thin surface layer. The thickness of this layer is known as the penetration depth, and is usually of the order of 10^{-7}m.

The Meissner effect implies that any current in the superconducting material must flow on the surface of the material, since an internal current density would produce an internal magnetic field. (Indeed, it is by means of surface currents that a superconductor excludes a magnetic field.) Moreover, magnetic fields above a certain magnitude cannot be excluded from the material, so the Meissner effect also implies the existence of a critical magnetic field, H_c, above which the material ceases to be superconducting, even at temperatures below the critical temperature (see Fig. 1.2).

Furthermore all processes, e.g. passage through the critical temperature or critical magnetic field, are found to be reversible. Then simple thermodynamic arguments (see e.g. [54]) can be used to deduce that the transition from normally conducting (normal) to superconducting at zero magnetic field and current is not accompanied by a release of latent heat, i.e. the transition is what is known as *second order*. (In the presence of a magnetic field however, the transition is of first order, and is accompanied by a release of latent heat $\hat{l}(T) = -\mu T H_c dH_c/dT$ per unit volume, where, μ is the permeability, T is the absolute transition temperature, and we have assumed that the densities of the two phases are equal. Note that when there is no magnetic field the transition occurs at $T = T_c$, and that $H_c(T_c) = 0$.)

The simplest configuration in which to describe the above phenomena is that of a cylindrical wire with cross section Ω (Fig. 1.3), placed in an axial magnetic field $(0, 0, H_0)$. If H_0 is lower than the critical field H_c of the superconductor then the material will be in the superconducting state, with $\boldsymbol{H} = \boldsymbol{E} = \mathbf{0}$ in Ω. $\boldsymbol{H}$ is then discontinuous across $\partial\Omega$, which suggests a superconducting current sheet in $\partial\Omega$, perpendicular to the z-axis. If the field H_0 is now increased through H_c, the superconductor will gradually be converted to a normal conductor and the field will gradually penetrate it. While this conversion is occurring the material may consist

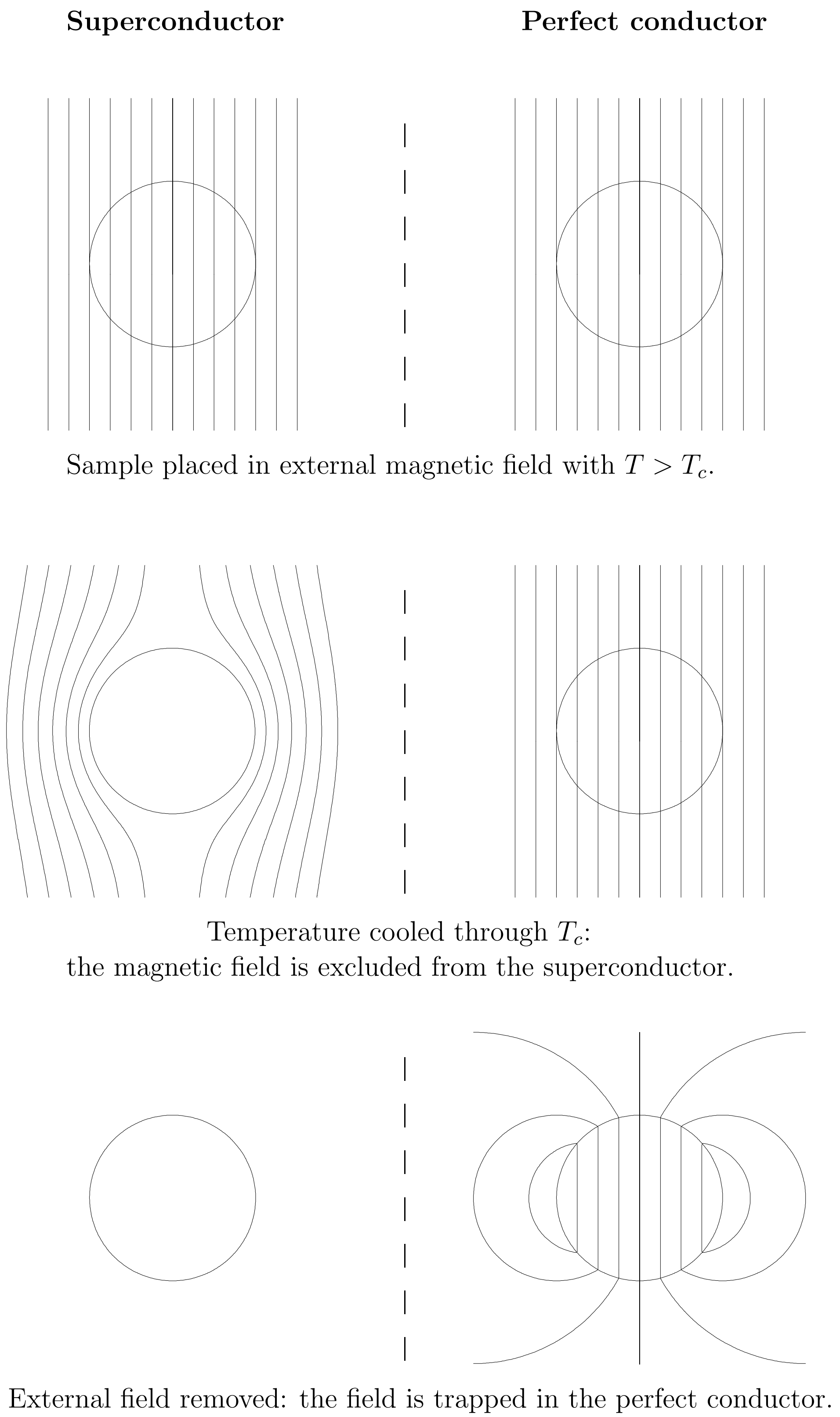

Figure 1.1: The contrasting responses of superconductors and perfect conductors in the presence of an applied magnetic field.

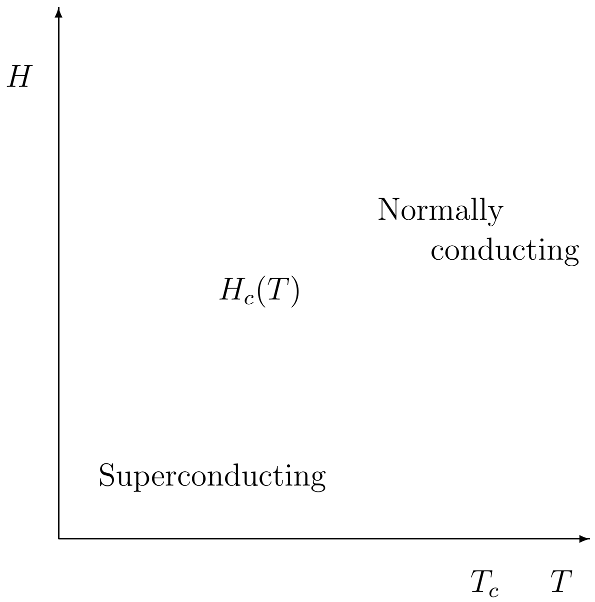

Figure 1.2: The response of a superconductor in the presence of an applied magnetic field.

Figure 1.3: Cylindrical superconducting wire in an applied axial magnetic field.

of a diminishing superconducting core surrounded by a normally conducting region in which a non-zero field exists (Fig. 1.4). Let us try and model this situation.

Figure 1.4: The destruction of superconductivity in a cylindrical wire by an applied axial magnetic field.

We assume that the latent heat and joule heating effects are negligible, and that the conversion occurs under isothermal conditions (in Section 2.4 we will relax this assumption and include thermal effects in the model). Here and throughout we assume that Maxwell's equations hold everywhere, with the displacement current being negligible. Thus the electric and magnetic fields $\boldsymbol{E}$, $\boldsymbol{H}$, the current density $\boldsymbol{j}$, and the charge density ϱ satisfy

$$\operatorname{div} \boldsymbol{E} = \frac{\varrho}{\varepsilon}, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{H} = 0, \tag{1.2}$$

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{j}, \tag{1.3}$$

$$\operatorname{curl} \boldsymbol{E} + \mu \frac{\partial \boldsymbol{H}}{\partial t} = \boldsymbol{0}, \tag{1.4}$$

where the permeability ε and the permittivity μ are assumed constant (but their values in the superconductor may differ from their values in an external non-superconducting material or vacuum). When the wire is in the normal state we assume Ohm's law

$$\boldsymbol{j} = \varsigma \boldsymbol{E}, \tag{1.5}$$

where ς is the constant electrical conductivity (which again may take different values in each material).

We nondimensionalise these equations by setting

$$\boldsymbol{H} = H_e \boldsymbol{H}', \quad \boldsymbol{E} = \frac{H_e}{\varsigma l_0} \boldsymbol{E}', \quad \varrho = \frac{\varepsilon_s H_e}{\varsigma l_0^2} \varrho',$$

$$\boldsymbol{j} = \frac{H_e}{l_0} \boldsymbol{j}', \quad \boldsymbol{x} = l_0 \boldsymbol{x}', \quad t = \mu_s \varsigma l_0^2 t',$$

where ε_s and μ_s are respectively the permittivity and permeability of the superconductor , l_0 is a typical length of the sample and H_e is a typical value of the external magnetic field. When we consider the Ginzburg-Landau equations in Chapter 3, H_e will be given specifically in terms of parameters in the equations and will turn out to be $\sqrt{2} H_c$ where H_c is the (dimensional) critical field of the sample. On dropping the primes, so that H_0 and H_c are henceforth dimensionless, this yields in Ω the dimensionless system

$$\text{div } \boldsymbol{E} = \varrho, \tag{1.6}$$

$$\text{div } \boldsymbol{H} = 0, \tag{1.7}$$

$$\text{curl } \boldsymbol{H} = \boldsymbol{j}, \tag{1.8}$$

$$\text{curl } \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} = 0, \tag{1.9}$$

with

$$\boldsymbol{j} = \boldsymbol{E}, \tag{1.10}$$

when the wire is in the normal state.

We now seek a solution $\boldsymbol{H} = (0, 0, H_3(x, y, t))$, $\boldsymbol{E} = (E_1(x, y, t), E_2(x, y, t), 0)$. We assume a configuration in which the part of the wire that is normal is separated from the remaining superconducting region by a smooth cylinder Γ.

In the superconducting region we have simply $H_3 = 0$. In the normal region (1.6)-(1.10) imply

$$\frac{\partial H_3}{\partial t} = \nabla^2 H_3, \tag{1.11}$$

with

$$H_3 = H_0, \text{ on } \partial\Omega.$$

As we approach Γ from the normal region we expect the magnitude of the magnetic field to tend to the critical magnetic field, i.e.

$$H_3 \to H_c. \tag{1.12}$$

Hence we expect a superconducting current sheet in Γ, perpendicular to the z-axis and of magnitude H_c. Finally, we derive a condition on the normal velocity v_n of Γ towards the normal region by writing (1.9) in the form

$$\frac{\partial H_3}{\partial t} + \text{div } (E_2, -E_1, 0) = 0,$$

and integrating over a small region in space and time containing part of Γ. After applying the divergence theorem we find

$$[H_3]_S^N \, v_n = [E_2 n_1 - E_1 n_2]_S^N \, ,$$

where, for definiteness, we take $\boldsymbol{n} = (n_1, n_2, 0)$ to be the outward normal vector to the superconducting region. Since $\boldsymbol{E} = \boldsymbol{H} = \mathbf{0}$ in the superconducting region, and $\boldsymbol{E} = \boldsymbol{j} = \text{curl } \boldsymbol{H}$ in the normal region, we therefore assert that

$$\frac{\partial H_3}{\partial n} = -H_c v_n, \tag{1.13}$$

as Γ is approached from the normal region.

The model (1.11)-(1.13) was written down in [38] in the case where Ω is circular. It is convenient in that it only involves $\boldsymbol{H}$ and not $\boldsymbol{E}$, and is in fact nothing more than a one-phase 'Stefan' model [16], which is itself the simplest macroscopic model that could be written down for an evolving phase boundary in the classical theory of melting or solidification. In its simplest dimensionless two-phase form, the Stefan model has

$$\frac{\partial T}{\partial t} = \nabla^2 T, \tag{1.14}$$

in both the solid and liquid phases, where T is the temperature. On the interface between the phases we have the temperature condition

$$T = T_m, \tag{1.15}$$

where T_m is the melting temperature, together with an energy balance for the velocity v_n of the phase boundary in the form

$$\left[\frac{\partial T}{\partial n}\right]_S^L = -L v_n, \tag{1.16}$$

where L is the latent heat. When T_m is constant, and this model is supplemented by suitable initial and boundary conditions, it is known to be well-posed just as long as neither superheating nor supercooling occurs, i.e. $T_{solid} < T_m$, $T_{liquid} > T_m$ [34]. However, when either of these conditions is violated the model appears to be ill-posed and thus needs to be regularised [17]. The most popular way of doing this is by writing

$$\gamma T_m = -\sigma\tilde{\kappa} - \alpha\sigma v_n, \tag{1.17}$$

where $\tilde{\kappa}$ is the mean curvature of the interface with a suitable sign, γ and α are positive constants (γ is the dimensionless entropy difference between the two phases), and σ is the 'surface energy' (so that $\sigma\tilde{\kappa}$ is the 'surface tension'). We can see heuristically the stabilising effects of (1.17) by noting that it implies that an order-one interface temperature is incompatible with a large interface curvature or normal velocity, and some well-posedness results are beginning to appear [21, 46].

A similar condition to (1.17) was proposed for the superconducting problem in [41], in which the interface condition (1.12) was modified to

$$H_3 = H_c - \frac{H_c}{2}\sigma\tilde{\kappa}, \tag{1.18}$$

as Γ is approached from the normal region, where σ is the (dimensionless) 'surface energy' of a normal/superconducting interface. The physical justification for the addition of such a term in the superconductivity model is not as clear as that for the solidification model, since the 'surface tension' of a normal/superconducting interface is difficult to demonstrate.

The layout of this thesis is strongly influenced by the analogy between models for solidification and models for superconductivity. A particularly useful link is provided by the so-called 'phase field' regularisation of (1.14)-(1.16) [12], whereby the phase boundary $T = T_m$ is smoothed by introducing an 'order parameter' $F \in [-1, 1]$ representing the mass fraction of material to have changed phase, say from liquid ($F = 1$) to solid ($F = -1$). Equation (1.14) is then modified to take account of the release of latent heat as F varies by writing

$$\frac{\partial T}{\partial t} + \frac{L}{2}\frac{\partial F}{\partial t} = \nabla^2 T. \tag{1.19}$$

In addition we need to append a 'Landau-Ginzburg' equation for F, obtained by relating the evolution of F to the variational derivative of a suitably chosen free energy functional, in the form

$$\alpha\xi^2\frac{\partial F}{\partial t} = \xi^2\nabla^2 F + \frac{1}{2a}(F - F^3) + 2T, \tag{1.20}$$

together with suitable initial and boundary conditions; α, ξ and a are all constants. In [10] it is conjectured that under rather general conditions there exists a unique smooth global solution to (1.19), (1.20) in arbitrary dimension. Numerical simulations have been performed [13] which also indicate their well-posedness. However, the most intriguing feature of (1.19), (1.20) from the viewpoint of superconductivity modelling is their ability to reduce formally to the classical and regularised Stefan models (1.14)-(1.17) as a, ξ and, in some cases, $\alpha \to 0$ [11]. When these parameters are small, the structure comprises liquid and solid regions separated by a thin transition layer in which F and T are smoothly varying 'travelling wave' solutions of certain approximations to (1.19), (1.20).

We note that in general the configuration of normal and superconducting domains in a sample will be much more complicated that that of the previous example, even in the steady state. Consider, for example, a circular superconducting cylinder of radius a in a transverse rather than axial magnetic field, as shown in Fig. 1.5.

In this case, for small values of the applied magnetic field there will be a complete Meissner effect as shown. Calculation of the magnetic field in this situation is a simple magnetostatics potential problem. We have $\boldsymbol{H} = \nabla\phi$, with

$$\begin{aligned} \nabla^2\phi &= 0, \quad r > a, \\ \frac{\partial\phi}{\partial r} &= 0, \quad \text{on } r = a, \\ \phi &\to H_0\, r\sin\theta, \quad \text{as } r \to \infty, \end{aligned}$$

with solution

$$\begin{aligned} \phi &= H_0\left(r + \frac{a^2}{r}\right)\sin\theta, \\ \boldsymbol{H} &= H_0\left(1 - \frac{a^2}{r^2}\right)\sin\theta\,\tilde{\boldsymbol{r}} + H_0\left(1 + \frac{a^2}{r^2}\right)\cos\theta\,\tilde{\boldsymbol{\theta}}, \end{aligned}$$

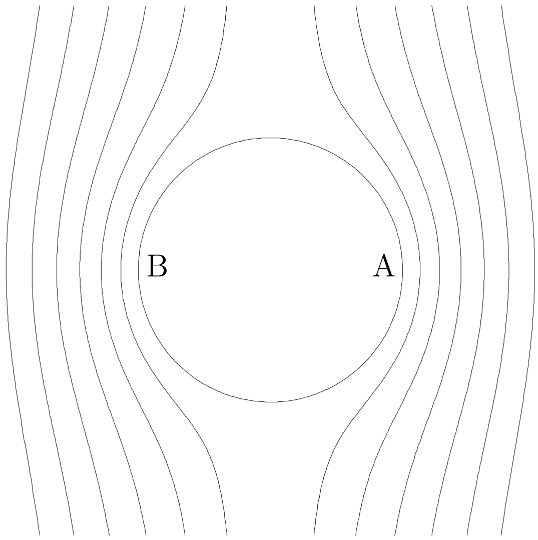

Figure 1.5: Cylindrical superconducting wire in an applied transverse magnetic field of small magnitude.

where $\tilde{\boldsymbol{r}}$ and $\tilde{\boldsymbol{\theta}}$ are unit vectors in the r and θ directions respectively. The maximum magnitude of the magnetic field occurs at the points $r = a$, $\theta = 0$, π, labelled A and B in the diagram, at which $\mid \boldsymbol{H} \mid = 2H_0$. Hence, when the applied field is increased to $H_c/2$, the field at the points A and B will actually be equal to H_c, and hence the material there will become normal again. However, if the whole wire were to become normal there would be no Meissner effect and the magnitude of the magnetic field would everywhere be equal to $H_c/2$ (assuming that μ takes the same value inside and outside the wire), which is less than H_c, and hence the material would become superconducting again. Thus for values of the applied magnetic field $H_c/2 < H_0 < H_c$ the material can be in neither a completely normal state nor a completely superconducting state, and must be in some intermediate state consisting of both normal and superconducting regions. The intermediate state in a single crystal may be a simple laminar structure, as seen for example in [56]. On the other hand, many intricate morphologies have also been observed [24, 62].

We shall begin our discussion of macroscopic superconductivity modelling with free boundary models analogous to (1.14)-(1.16) and then proceed to models in which the phase boundary is smoothed as in (1.19), (1.20). In Chapter 2 we shall write down the generalisation of (1.11)-(1.13) to a three-dimensional superconductor undergoing a phase change. This will take the form of a 'vectorial' Stefan problem. This vectorial Stefan problem will be shown sometimes to have instabilities which are similar to those which cause ill-posedness in the classical Stefan model (1.14)-(1.16) in superheated or supercooled situations. This means that the model is only capable of describing certain superconductor configurations. In particular, for intermediate states when both phases are present, we will only expect well-posedness when the normal region is expanding and the superconducting region is contracting. In these circumstances the model predicts the evolution of a smooth boundary Γ separating the two phases. However, the model also relies on the assumption that the wire forms normal and superconducting regions separated by thin transition layers. We shall see in Chapter 5 that this assumption is not valid for what are known as Type II superconductors, and this further constraint restricts the use of (1.11)-(1.13) to what are known as Type I superconductors.

Indeed, the simple account of the Meissner effect and the critical field given previously is also only accurate for Type I superconductors. Let us compare the magnetisation curves of Type I and Type II superconductors.

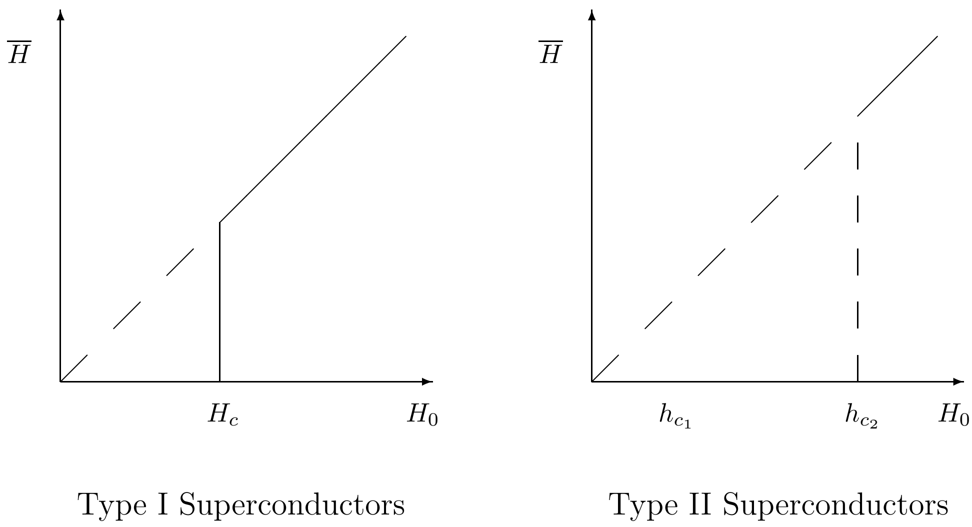

Figure 1.6: Average magnetic field $\overline{H}$ in bulk Type I and Type II superconductors as a function of the applied magnetic field H_0.

Fig. 1.6 shows the average magnetic field in bulk Type I and Type II superconductors as a function of the applied magnetic field. For Type I superconductors we see the phenomena mentioned previously, namely that the magnetic field is excluded from the sample for low values of the applied magnetic field, but above a certain critical field H_c the sample reverts to the normal state and the field completely penetrates it.

For Type II superconductors we see that for low values of the applied magnetic field ($H_0 < h_{c_1}$) the Meissner effect is complete and the field is excluded from the sample, and for high values of the applied magnetic field ($H_0 > h_{c_2}$) the sample reverts to the normal state and the magnetic field completely penetrates it. However, for intermediate values of the applied magnetic field ($h_{c_1} < H_0 < h_{c_2}$) the superconductor is neither completely superconducting nor completely normal, and the field partially penetrates it. In this regime the superconductor is in what is known as the *mixed state*. (Note that this state is very different from the intermediate state mentioned previously, which was dependent on the geometry of the sample, and which could be formed in Type I superconductors. The mixed state is an intrinsic property of Type II superconductors which occurs even in geometries where there is no distortion of the applied field.) Thus we see that for Type II superconductors there is not an abrupt transition from superconducting to normal as the applied field is increased through H_c, but rather a gradual transition as the applied field varies between two critical values h_{c_1} (the lower critical field) and h_{c_2} (the upper critical field). Fig. 1.2 should therefore be modified for a Type II superconductor to Fig. 1.7.

We can further highlight the difference between Type I and Type II superconductors by considering the experimental observation of the response of a superconductor as the external field is lowered and raised, shown in Fig. 1.8.

We see that the onset of superconductivity near H_{c_2} is reversible for a Type II superconductor. However, for a Type I superconductor there is a hysteresis loop. If the applied magnetic field is lowered under controlled conditions the normal state can be made to persist until a lower value of the applied magnetic field than the critical magnetic field, at which point there is an abrupt transition to the superconducting state. If the field is now raised the superconducting state will persisit until the applied field reaches the critical magnetic field.

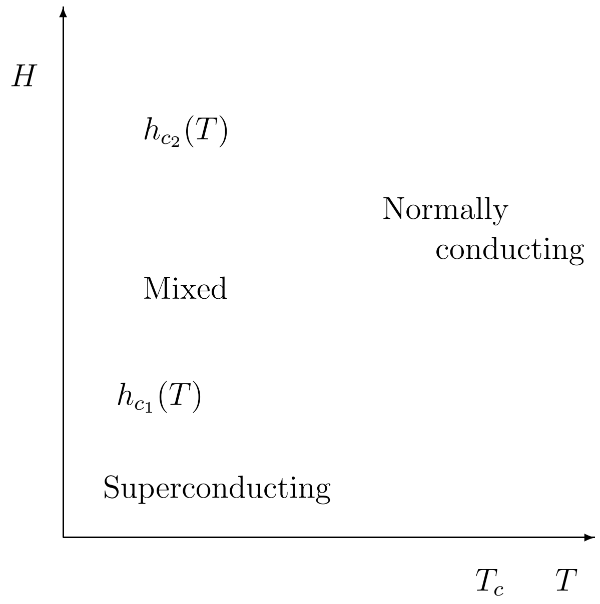

Figure 1.7: The response of a Type II superconductor in the presence of an applied magnetic field with no applied current.

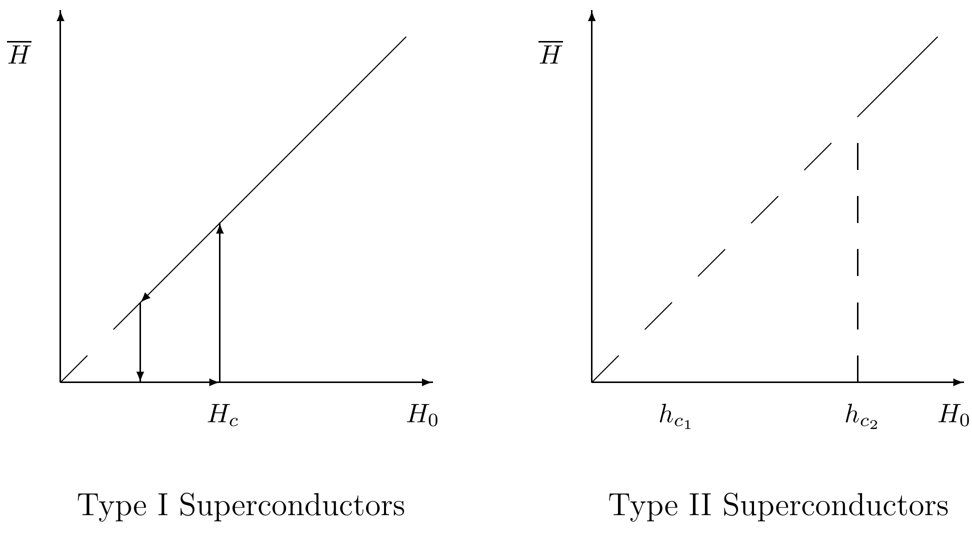

Figure 1.8: Response of Type I and Type II superconductors as the external magnetic field is raised and lowered.

In order to understand this hysteresis loop, the transition from normal to superconducting for Type I superconductors and the behaviour of Type II superconductors we must consider the phase boundary more carefully. A step in this direction was taken by London [45], who proposed that the superconducting region should not be modelled simply by writing $\boldsymbol{H} = \boldsymbol{0}$, but rather that there should be a distributed superconducting current $\boldsymbol{j}_s$ in that region, such that

$$\frac{\partial \boldsymbol{j}_s}{\partial t} \propto \boldsymbol{E}, \tag{1.21}$$

$$\text{curl}\, \boldsymbol{j}_s \propto \boldsymbol{H}. \tag{1.22}$$

Hence

$$\boldsymbol{j}_s \propto \boldsymbol{A}, \tag{1.23}$$

where $\boldsymbol{A}$ is a suitable chosen magnetic vector potential which, in the superconducting region, is only appreciable near Γ. This removes the discontinuity in the magnetic field at the interface, but the boundary between normal and superconducting regions in the material is still sharp.

In 1950 a model was written down by Ginzburg & Landau which smooths out the phase boundary altogether [28]. They considered the conducting electrons as a 'fluid' that could appear in two phases, namely superconducting and normal. They then applied the general Landau-Ginzburg theory of second order phase transitions, including terms to take into account how the electron 'fluid' motion is affected by a magnetic field. We introduce this model in Chapter 3. For Type I superconductors this theory will stand in relation to the vectorial Stefan model as does the phase field theory to the scalar Stefan model for solidification. However, it will also permit us to analyse Type II superconductors.

In Chapter 4 we shall study the asymptotic limit in which the Ginzburg-Landau theory reduces to the relatively simple vectorial Stefan model of Chapter 2.

In Chapter 5 we shall study the nucleation of superconductivity in decreasing fields at h_{c_2}, which will help to explain the magnetisation curves of Fig. 1.6 and the hysteresis in Fig. 1.8. In Chapter 6 we consider the effects of the presence of surfaces on the nucleation of superconductivity and in Chapter 7 we will examine the form of the mixed state in a bulk superconductor. In Chapter 8 we study the nucleation of superconductivity with decreasing temperature rather than decreasing magnetic field. Finally, in Chapter 9, we present the results of the thesis and some interesting open questions.

To close this introduction we remark that literature on the subject of superconductivity is both vast and varied, but the important papers from the point of view of this thesis are those of Ginzburg & Landau [28], Abrikosov [1], Keller [38], Millman & Keller [50], and on the phase field theory, Caginalp [12]. Recent reviews of the macroscopic theory of superconductivity have been given in [20] and [15].

Chapter 2

Free Boundary Models

In (1.11)-(1.13) we have already seen a simple configuration which permits a Stefan free boundary model to be formulated. We now generalise this model to three dimensions. Let the superconducting material occupy a region Ω, bounded by a surface $\partial\Omega$. We denote the region occupied by the normally conducting phase by Ω_N and the region occupied by the superconducting phase by Ω_S, with the free boundary, i.e. that portion of $\partial\Omega_N$ not contained in $\partial\Omega$, being denoted by Γ. Then $\Omega = \Omega_N \cup \Omega_S \cup \Gamma$. (See Fig. 2.1)

We take the region outside Ω to be a vacuum, in which we have the Maxwell equations

$$\begin{aligned} \text{curl } \boldsymbol{H} &= \boldsymbol{0}, & (2.1) \\ \text{div } \boldsymbol{H} &= 0. & (2.2) \end{aligned}$$

We assume the Meissner effect in the superconducting phase, so that

$$\boldsymbol{H} = \boldsymbol{0}, \qquad \text{in } \Omega_S, \tag{2.3}$$

and that Ohm's law applies in the normal phase, so that

$$\begin{aligned} \frac{\partial \boldsymbol{H}}{\partial t} &= -(\text{curl})^2 \boldsymbol{H} & (2.4) \\ &= \nabla^2 \boldsymbol{H}, & (2.5) \end{aligned}$$

in Ω_N, since

$$\text{div } \boldsymbol{H} = 0, \tag{2.6}$$

there. The generalisation of (1.12) is

$$|\boldsymbol{H}| \to H_c, \tag{2.7}$$

Figure 2.1: Destruction of superconductivity by an applied magnetic field.

as the phase boundary Γ is approached from the normal region.

We write (1.9) in the form

$$\frac{\partial \boldsymbol{H}}{\partial t} + \operatorname{div} \begin{pmatrix} 0 & -E_3 & E_2 \\ E_3 & 0 & -E_1 \\ -E_2 & E_1 & 0 \end{pmatrix} = \mathbf{0}.$$

We integrate this equation over a small region in space and time containing part of the boundary Γ and apply the divergence theorem to obtain

$$[\boldsymbol{E} \wedge \boldsymbol{n}]_S^N = -v_n \, [\boldsymbol{H}]_S^N \,,$$

where $\boldsymbol{n}$ is the unit normal to the interface Γ (taken to point towards the normal region) and v_n is the normal velocity of the interface, positive if the superconducting region is expanding. However, $\boldsymbol{E} = \operatorname{curl} \boldsymbol{H}$ in the normal region and $\boldsymbol{E} = \boldsymbol{H} = \mathbf{0}$ in the superconducting region. Hence we find that

$$\operatorname{curl} \boldsymbol{H} \wedge \boldsymbol{n} = -v_n \boldsymbol{H}, \qquad \text{on } \Gamma_N, \tag{2.8}$$

where Γ_N denotes the interface Γ approached from the normal region; this condition was written down in [3].

The conditions on the fixed boundary $\partial\Omega$ will be the usual conditions on $\boldsymbol{H}$ and $\boldsymbol{E}$ at an interface between two media, namely

$$
\begin{aligned}
[\boldsymbol{H}\cdot\boldsymbol{n}] &= \boldsymbol{0}, && (2.9)\\
[\boldsymbol{E}\wedge\boldsymbol{n}] &= [\text{curl } \boldsymbol{H}\wedge\boldsymbol{n}] = \boldsymbol{0}, && (2.10)\\
[\varepsilon\boldsymbol{E}\cdot\boldsymbol{n}] &= [\varepsilon\,\text{curl } \boldsymbol{H}\cdot\boldsymbol{n}] = \varrho_s, && (2.11)\\
[(1/\mu)\boldsymbol{H}\wedge\boldsymbol{n}] &= \boldsymbol{j}_s, && (2.12)
\end{aligned}
$$

where $[\,]$ denotes the jump in the enclosed quantity across $\partial\Omega$, $\boldsymbol{j}_s$ is the surface current density, and ϱ_s is the surface charge density. In the superconducting region μ and ε will be unity. In general ϱ_s will be zero, and $\boldsymbol{j}_s$ will only be non-zero on $\partial\Omega_S$. The boundary conditions (2.9)-(2.12) are not all independent. In particular, since the time derivative of equation (1.7) follows from equation (1.9), (2.9) gives extra information to (2.10) only in the case of static fields.

We must also give the initial condition

$$\boldsymbol{H} = \boldsymbol{H}_0(\boldsymbol{x}), \text{ when } t = 0, \qquad (2.13)$$

where

$$\text{div } \boldsymbol{H}_0 = 0, \qquad (2.14)$$

and the far field condition that

$$\boldsymbol{H} \to \boldsymbol{H}_\infty, \text{ as } |\boldsymbol{x}| \to \infty. \qquad (2.15)$$

Before we begin our analysis of (2.1)-(2.15) we note that all of (2.4)-(2.6) hold throughout Ω_N. It is a question of interest as to whether the condition (2.6) is a consequence of (2.1)-(2.4), and (2.7)-(2.15), or whether the extra condition

$$\text{div } \boldsymbol{H} = 0, \qquad \text{on } \Gamma_N,$$

needs to be appended to ensure that div $\boldsymbol{H} = 0$ everywhere. If this were the case then the free boundary would seem to be overdetermined. We now prove that, when we add the condition that

$$[\text{div } \boldsymbol{H}] = 0, \qquad (2.16)$$

i.e. that div $\boldsymbol{H}$ is continuous across the fixed boundary $\partial\Omega$, then (2.6) does indeed follow from (2.1)-(2.4), and (2.7)-(2.16), and also when (2.4) is replaced by (2.5) in the case when $v_n \geq 0$, which is the case of interest.

We let the fixed portion of the boundary of Ω_N be denoted by $\partial_N\Omega$ so that $\partial\Omega_N = \Gamma \cup \partial_N\Omega$. For ease of notation we denote div $\boldsymbol{H}$ by u.

We first note that

$$\mathrm{div}(u\boldsymbol{H}) = \boldsymbol{H}\cdot\nabla u + u^2.$$

Hence

$$\begin{aligned}\int_{\Omega_N}\left(\boldsymbol{H}\cdot\nabla u + u^2\right)dV &= \int_{\Omega_N}\mathrm{div}(u\boldsymbol{H})\,dV,\\ &= \int_{\partial\Omega_N} u\boldsymbol{H}\cdot\boldsymbol{n}\,dS,\\ &= 0,\end{aligned}$$

since $\boldsymbol{H}\cdot\boldsymbol{n} = 0$ on Γ, and $u = 0$ on $\partial_N\Omega$. Hence

$$\int_{\Omega_N} u^2\,dV = -\int_{\Omega_N}\boldsymbol{H}\cdot\nabla u\,dV.$$

Differentiating this equation with respect to t we have

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_N} u^2\,dV &= -\frac{d}{dt}\int_{\Omega_N}\boldsymbol{H}\cdot\nabla u\,dV,\\ &= -\int_{\Omega_N}\left[\frac{\partial\boldsymbol{H}}{\partial t}\cdot\nabla u + \boldsymbol{H}\cdot\frac{\partial}{\partial t}(\nabla u)\right]dV - \int_\Gamma(\boldsymbol{H}\cdot\nabla u)v_n\,dS.\end{aligned}$$

Taking the divergence of equation (2.4) yields

$$\frac{\partial u}{\partial t} = 0.$$

Hence, using (2.4) and (2.8) we have

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_N} u^2\,dV &= \int_{\Omega_N}\left((\mathrm{curl})^2\boldsymbol{H}\cdot\nabla u\right)dV + \int_\Gamma(\mathrm{curl}\ \boldsymbol{H}\wedge\boldsymbol{n})\cdot\nabla u\,dS,\\ &= \int_{\Omega_N}\mathrm{div}\left(\mathrm{curl}\ \boldsymbol{H}\wedge\nabla u\right)dV + \int_\Gamma(\mathrm{curl}\ \boldsymbol{H}\wedge\boldsymbol{n})\cdot\nabla u\,dS,\\ &= \int_{\partial\Omega_N}(\mathrm{curl}\ \boldsymbol{H}\wedge\nabla u)\cdot\boldsymbol{n}\,dS - \int_\Gamma(\mathrm{curl}\ \boldsymbol{H}\wedge\nabla u)\cdot\boldsymbol{n}\,dS,\\ &= \int_{\partial_N\Omega}(\mathrm{curl}\ \boldsymbol{H}\wedge\nabla u)\cdot\boldsymbol{n}\,dS,\\ &= 0,\end{aligned}$$

since curl $\boldsymbol{H} \wedge \boldsymbol{n} = \mathbf{0}$ on $\partial\Omega$. Thus

$$\frac{d}{dt}\int_{\Omega_N} u^2 \, dV = 0.$$

Hence

$$\int_{\Omega_N} u^2 \, dV = \int_{\Omega_N(t=0)} (\text{div } \boldsymbol{H}_0)^2 \, dV = 0,$$

by (2.13). Hence

$$\text{div } \boldsymbol{H} = 0, \text{ in } \Omega_N,$$

as required.

We now prove that div $\boldsymbol{H}$ is also zero when (2.4) is replaced by (2.5) and $v_n \geq 0$. As before we have

$$\int_{\Omega_N} u^2 \, dV = -\int_{\Omega_N} \boldsymbol{H} \cdot \nabla u \, dV.$$

Differentiating this equation with respect to t we have

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega_N} u^2 \, dV &= -\frac{d}{dt}\int_{\Omega_N} \boldsymbol{H} \cdot \nabla u \, dV, \\
&= -\int_{\Omega_N} \left[\frac{\partial \boldsymbol{H}}{\partial t} \cdot \nabla u + \boldsymbol{H} \cdot \frac{\partial}{\partial t}(\nabla u)\right] dV - \int_\Gamma (\boldsymbol{H} \cdot \nabla u) v_n \, dS, \\
&= -\int_{\Omega_N} \left[(\nabla^2 \boldsymbol{H} \cdot \nabla u) + \boldsymbol{H} \cdot \frac{\partial}{\partial t}(\nabla u)\right] dV \\
&\quad - \int_\Gamma (\text{curl } \boldsymbol{H} \wedge \boldsymbol{n}) \cdot \nabla u \, dS,
\end{aligned}$$

by (2.5) and (2.8). Now

$$\nabla^2 \boldsymbol{H} = \nabla u - (\text{curl})^2 \boldsymbol{H}.$$

Also

$$\begin{aligned}
\int_{\Omega_N} \text{div}\left(\frac{\partial u}{\partial t}\boldsymbol{H}\right) dV &= \int_{\Omega_N} u\frac{\partial u}{\partial t} + \boldsymbol{H} \cdot \nabla\left(\frac{\partial u}{\partial t}\right) dV, \\
&= \int_{\partial\Omega_N} \frac{\partial u}{\partial t}\boldsymbol{H} \cdot \boldsymbol{n} \, dS, \\
&= 0,
\end{aligned}$$

in the same way that $\int_{\partial\Omega_N} u\boldsymbol{H} \cdot \boldsymbol{n} \, dS = 0$. Hence

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega_N} u^2 \, dV &= \int_{\Omega_N} \left((\text{curl})^2 \boldsymbol{H} \cdot \nabla u - |\nabla u|^2 + u\frac{\partial u}{\partial t}\right) dV \\
&\quad - \int_\Gamma (\text{curl } \boldsymbol{H} \wedge \boldsymbol{n}) \cdot \nabla u \, dS, \\
&= \int_{\Omega_N} \frac{1}{2}\frac{\partial u^2}{\partial t} - |\nabla u|^2 \, dV,
\end{aligned}$$

since

$$\int_{\Omega_N} (\mathrm{curl})^2 \boldsymbol{H} \cdot \nabla u \, dV - \int_{\Gamma} (\mathrm{curl}\ \boldsymbol{H} \wedge \boldsymbol{n}) \cdot \nabla u \, dS = 0,$$

as before. We also have that

$$\frac{d}{dt} \int_{\Omega_N} u^2 \, dV \quad = \quad \int_{\Omega_N} \frac{\partial u^2}{\partial t} \, dV - \int_{\Gamma} u^2 v_n \, dS.$$

Hence

$$\frac{1}{2} \int_{\Omega_N} \frac{\partial u^2}{\partial t} \, dV = - \int_{\Omega_N} |\nabla u|^2 \, dV - \int_{\Gamma} u^2 v_n \, dS.$$

Hence, for $v_n \geq 0$,

$$\int_{\Omega_N} \frac{\partial u^2}{\partial t} \, dV \leq 0.$$

Hence

$$\frac{d}{dt} \int_{\Omega_N} u^2 \, dV \leq 0.$$

However, $\int_{\Omega_N} u^2 \, dV \geq 0$, and $\int_{\Omega_N} u^2 \, dV = 0$ initially. Hence

$$\int_{\Omega_N} u^2 \, dV = 0,$$

and therefore div $\boldsymbol{H} = 0$, as required.

Remark (Hairy Dog Theorem)

Before we begin our analysis of (2.1)-(2.15) we make one final remark concerning the boundary condition (2.7). In cases where we have a finite normal region entirely surrounded by a superconducting region, or vice versa (i.e. a free boundary Γ which does not meet the fixed boundary $\partial\Omega$), in either the time-dependent or steady situation, we have a smooth tangent vector field $\boldsymbol{H}$ of constant modulus on a smooth, closed surface Γ. A corollary to the Gauss-Bonnet theorem of differential geometry (sometimes known as the hairy dog theorem) implies that this can only be true if Γ is topologically equivalent to a torus, since a smooth tangent vector field on any other smooth, closed surface must have at least one stationary point.

This means that although there are many exact solutions (e.g. similarity solutions) in two dimensions, it is very difficult to find exact solutions in three dimensions, since there can be no spherically symmetric solutions.

2.1 Linear Stability Analysis

For a discussion of the linear stability analysis of the classical Stefan problem we refer to [42, 64]. We consider here the vector Stefan problem (2.1)-(2.16) in an infinite region, ignoring for the moment the conditions at infinity and the initial conditions. We have then

$$\begin{aligned} \nabla^2 \boldsymbol{H} &= \frac{\partial \boldsymbol{H}}{\partial t}, \text{ in the normal region,} & (2.17) \\ \boldsymbol{H} &= \boldsymbol{0}, \text{ in the superconducting region,} & (2.18) \\ |\boldsymbol{H}| &= H_c, \text{ on } \Gamma_N, & (2.19) \\ \text{curl } \boldsymbol{H} \wedge \boldsymbol{n} &= -v_n \boldsymbol{H}, \text{ on } \Gamma_N. & (2.20) \end{aligned}$$

A solution representing a plane wave travelling with constant velocity is given by

$$\boldsymbol{H} = \begin{cases} (0, H_c e^{-v(x-vt)}, 0) & x < vt \\ 0 & x > vt \end{cases} \tag{2.21}$$

where the boundary is given by $x = vt$ and the normal region is $x < vt$. We perform a linear stability analysis of this solution by considering perturbations to the boundary, which is given by $x = X$, of the form

$$X(y, z, t) = vt + \epsilon e^{\sigma t} \cos my \cos nz, \tag{2.22}$$

where $0 < \epsilon \ll 1$, and m and n are real and positive for definiteness. We expand the corresponding solution for $\boldsymbol{H}$ in powers of ϵ:

$$\boldsymbol{H} = (0, H_c e^{-v(x-vt)}, 0) + \epsilon \boldsymbol{H}^{(1)} + \cdots . \tag{2.23}$$

Substituting the expansion (2.23) into equation (2.17) and equating coefficients of ϵ yields

$$\nabla^2 \boldsymbol{H}^{(1)} = \frac{\partial \boldsymbol{H}^{(1)}}{\partial t}. \tag{2.24}$$

We will shortly change to co-ordinates moving with the free boundary, but we first calculate the boundary conditions for $\boldsymbol{H}^{(1)}$. We have

$$\begin{aligned} |\boldsymbol{H}(X, y, z)|^2 &= (\epsilon H_1^{(1)}(X, y, z) + \cdots)^2 + (H_c e^{-v(X-vt)} + \epsilon H_2^{(1)}(X, y, z) + \cdots)^2 \\ &\quad + (\epsilon H_1^{(1)}(X, y, z) + \cdots)^2 \end{aligned}$$

$$
\begin{aligned}
&= H_c^2 e^{-2\epsilon v e^{\sigma t}\cos my \cos nz} \\
&\quad + 2\epsilon H_c e^{-\epsilon v e^{\sigma t}\cos my\cos nz} H_2^{(1)}(vt + \epsilon e^{\sigma t}\cos my\cos nz, y, z) \\
&\quad + O(\epsilon^2) \\
&= H_c^2(1 - 2\epsilon v e^{\sigma t}\cos my\cos nz) + 2\epsilon H_c H_2^{(1)}(vt, y, z) + O(\epsilon^2) \\
&= H_c^2
\end{aligned}
$$

by (2.19). Equating coefficients of ϵ gives

$$H_2^{(1)}(vt, y, z) = vH_c e^{\sigma t}\cos my\cos nz. \tag{2.25}$$

The boundary is given by the equation $x - X = 0$. Hence the unit normal to the boundary is given by

$$\boldsymbol{n} = (-1, -\epsilon m e^{\sigma t}\sin my\cos nz, -\epsilon n e^{\sigma t}\cos my\sin nz) + O(\epsilon^2).$$

The velocity of the boundary is given by

$$\boldsymbol{v} = \left(\frac{\partial X}{\partial t}, 0, 0\right) = (v + \epsilon\sigma e^{\sigma t}\cos my\cos nz, 0, 0).$$

Hence

$$v_n = \boldsymbol{v}\cdot\boldsymbol{n} = -v - \epsilon\sigma e^{\sigma t}\cos my\cos nz + O(\epsilon^2). \tag{2.26}$$

Therefore

$$
\begin{aligned}
v_n \boldsymbol{H}(X, y, z) = \\
-(v + \epsilon\sigma e^{\sigma t}\cos my\cos nz)
\begin{pmatrix}
\epsilon H_1^{(1)}(X, y, z) \\
H_c e^{-\epsilon v e^{\sigma t}\cos my\cos nz} + \epsilon H_2^{(1)}(X, y, z) \\
\epsilon H_3^{(1)}(X, y, z)
\end{pmatrix}^T
\end{aligned}
$$

$$
= -\begin{pmatrix}
\epsilon v H_1^{(1)}(vt, y, z) \\
vH_c + \epsilon(\sigma - v^2)H_c e^{\sigma t}\cos my\cos nz + \epsilon H_2^{(1)}(vt, y, z) \\
\epsilon v H_3^{(1)}(vt, y, z)
\end{pmatrix}^T + O(\epsilon^2). \tag{2.27}
$$

We have

$$\text{curl } \boldsymbol{H} \wedge \boldsymbol{n} = \begin{pmatrix} \epsilon\left[\frac{\partial H_3^{(1)}}{\partial y} - \frac{\partial H_2^{(1)}}{\partial z}\right] \\ \epsilon\left[\frac{\partial H_1^{(1)}}{\partial z} - \frac{\partial H_3^{(1)}}{\partial x}\right] \\ -vH_c e^{-v(x-vt)} + \epsilon\left[\frac{\partial H_2^{(1)}}{\partial x} - \frac{\partial H_1^{(1)}}{\partial y}\right] \end{pmatrix}^T \wedge \begin{pmatrix} -1 \\ -\epsilon m e^{\sigma t}\sin my \cos nz \\ -\epsilon n e^{\sigma t}\cos my \sin nz \end{pmatrix}^T + O(\epsilon^2)$$

$$= -\begin{pmatrix} \epsilon v H_c e^{-v(x-vt)} m e^{\sigma t}\sin my \cos nz \\ -vH_c e^{-v(x-vt)} + \epsilon\left[\frac{\partial H_2^{(1)}}{\partial x} - \frac{\partial H_1^{(1)}}{\partial y}\right] \\ \left[\frac{\partial H_1^{(1)}}{\partial z} - \frac{\partial H_3^{(1)}}{\partial x}\right] \end{pmatrix}^T + O(\epsilon^2).$$

Hence

$$(\text{curl } \boldsymbol{H} \wedge \boldsymbol{n})(X,y,z) = -\begin{pmatrix} \epsilon v H_c e^{-\epsilon v e^{\sigma t}\cos my \cos nz} m e^{\sigma t}\sin my \cos nz \\ -vH_c e^{-\epsilon v e^{\sigma t}\cos my\cos nz} + \epsilon\left[\frac{\partial H_2^{(1)}}{\partial x}(X,y,z) - \frac{\partial H_1^{(1)}}{\partial y}(X,y,z)\right] \\ \left[\frac{\partial H_1^{(1)}}{\partial z}(X,y,z) - \frac{\partial H_3^{(1)}}{\partial x}(X,y,z)\right] \end{pmatrix}^T + O(\epsilon^2),$$

$$= -\begin{pmatrix} \epsilon v H_c m e^{\sigma t}\sin my \cos nz \\ -vH_c + \epsilon\left[v^2 H_c e^{\sigma t}\cos my \cos nz + \frac{\partial H_2^{(1)}}{\partial x}(vt,y,z) - \frac{\partial H_1^{(1)}}{\partial y}(vt,y,z)\right] \\ \left[\frac{\partial H_1^{(1)}}{\partial z}(vt,y,z) - \frac{\partial H_3^{(1)}}{\partial x}(vt,y,z)\right] \end{pmatrix}^T + O(\epsilon^2),$$

$$= \begin{pmatrix} \epsilon v H_1^{(1)}(vt,y,z) \\ vH_c + \epsilon(\sigma - v^2)H_c e^{\sigma t}\cos my \cos nz + \epsilon H_2^{(1)}(vt,y,z) \\ \epsilon v H_3^{(1)}(vt,y,z) \end{pmatrix}^T + O(\epsilon^2),$$

by (2.20) and (2.27). Equating powers of ϵ gives

$$H_1^{(1)}(vt,y,z) = -mH_c e^{\sigma t}\sin my \cos nz, \tag{2.28}$$

$$\frac{\partial H_2^{(1)}}{\partial x}(vt,y,z) - \frac{\partial H_1^{(1)}}{\partial y}(vt,y,z) = -vH_2^{(1)}(vt,y,z) - \sigma H_c e^{\sigma t}\cos my \cos nz, \tag{2.29}$$

$$\frac{\partial H_1^{(1)}}{\partial z}(vt,y,z) - \frac{\partial H_3^{(1)}}{\partial x}(vt,y,z) \quad = \quad vH_3^{(1)}(vt,y,z). \tag{2.30}$$

Using (2.28) and (2.25) we see

$$\frac{\partial H_2^{(1)}}{\partial x}(vt,y,z) \quad = \quad -(v^2+\sigma+m^2)H_c e^{\sigma t}\cos my \cos nz, \tag{2.31}$$

$$\frac{\partial H_3^{(1)}}{\partial x}(vt,y,z) \quad = \quad -vH_3^{(1)}(vt,y,z) + mnH_c e^{\sigma t}\sin my \sin nz. \tag{2.32}$$

Hence the problem we have to solve is

$$\nabla^2 \boldsymbol{H}^{(1)} = \frac{\partial \boldsymbol{H}^{(1)}}{\partial t}, \tag{2.33}$$

with the (fixed) boundary conditions

$$H_1^{(1)}(vt,y,z) \quad = \quad -mH_c e^{\sigma t}\sin my \cos nz, \tag{2.34}$$

$$H_2^{(1)}(vt,y,z) \quad = \quad vH_c e^{\sigma t}\cos my \cos nz, \tag{2.35}$$

$$\frac{\partial H_2^{(1)}}{\partial x}(vt,y,z) \quad = \quad -(v^2+\sigma+m^2)H_c e^{\sigma t}\cos my \cos nz, \tag{2.36}$$

$$\frac{\partial H_3^{(1)}}{\partial x}(vt,y,z) \quad = \quad -vH_3^{(1)}(vt,y,z) + mnH_c e^{\sigma t}\sin my \sin nz. \tag{2.37}$$

We look for a solution for $H_2^{(1)}$ of the form

$$H_2^{(1)} = F(x-vt)e^{\sigma t}\cos my \cos nz.$$

Letting $\eta = x - vt$ and $' \equiv d/d\eta$ we have

$$F'' + vF' - (n^2+m^2+\sigma)F \quad = \quad 0, \tag{2.38}$$

$$F(0) \quad = \quad vH_c, \tag{2.39}$$

$$F'(0) \quad = \quad -(v^2+\sigma+m^2)H_c. \tag{2.40}$$

Hence

$$F = Ae^{\lambda_1 \eta} + Be^{\lambda_2 \eta},$$

where λ_1, λ_2 are the roots of

$$\lambda^2 + v\lambda - (n^2+m^2+\sigma) = 0,$$

namely

$$\lambda_1 = \frac{-v+\sqrt{v^2+4(n^2+m^2+\sigma)}}{2},$$
$$\lambda_2 = \frac{-v-\sqrt{v^2+4(n^2+m^2+\sigma)}}{2}.$$

The boundary conditions (2.39), (2.40) imply

$$A+B = vH_c, \tag{2.41}$$
$$A\lambda_1+B\lambda_2 = -(v^2+\sigma+m^2)H_c. \tag{2.42}$$

We consider separately the cases $v>0$, $v<0$.

(1) $v>0$.

If we require that $H_2^{(1)}$ should be small compared to $H_2^{(0)} = H_c e^{-v(x-vt)}$ as $x-vt\to-\infty$ then we have either $\Re\{\lambda_2\}>-v$ or $B=0$. If $\Re\{\lambda_2\}>-v$ then

$$\Re\{v^2+4(n^2+m^2+\sigma)\} < \left(\Re\{\sqrt{v^2+4(n^2+m^2+\sigma)}\}\right)^2 < v^2,$$

i.e.

$$\Re\{\sigma\} < -(n^2+m^2).$$

Hence $\Re\{\sigma\}$ is negative. If we have $B=0$ then (2.41) implies $A=vH_c$, which in (2.42) implies

$$v\sqrt{v^2+4(n^2+m^2+\sigma)} = -v^2-2\sigma-2m^2. \tag{2.43}$$

Squaring gives

$$v^4+4v^2(n^2+m^2+\sigma) = v^4+4\sigma^2+4m^4+4v^2\sigma+4v^2m^2+8\sigma m^2,$$

i.e.

$$v^2n^2 = (\sigma+m^2)^2.$$

Hence

$$\sigma = -m^2\pm nv. \tag{2.44}$$

However, since we had to square our equation to obtain this answer we need to substitute (2.44) into (2.43) and check for consistency. We find

$$v\sqrt{(v\pm 2n)^2} = -v(v\pm 2n),$$

i.e. $v \pm 2n < 0$. Since $v > 0$ we must have

$$\sigma = -m^2 - nv, \qquad 2n > v. \tag{2.45}$$

Since v is positive we see that σ is negative and the solution is linearly stable.

(2) $v < 0$.

If we again require that $H_2^{(1)}$ should be small compared to $H_2^{(0)} = H_c e^{-v(x-vt)}$ as $x - vt \to -\infty$ then we must have $B = 0$. As before $\sigma = -m^2 \pm nv$, with $v \pm 2n < 0$. Thus

$$\sigma = \begin{cases} -m^2 \pm nv & 2n < -v, \\ -m^2 - nv & 2n > -v. \end{cases} \tag{2.46}$$

Hence, for $n/m^2 > -v$ there will be a solution with $\sigma > 0$ indicating an instability of the boundary.

We note the very different responses to perturbations in the y and z direction. We see that for a perturbation in the y-direction only ($n = 0$) the boundary is always stable. However, for a perturbation in the z direction only ($m = 0$) the boundary is stable if and only if $v > 0$, i.e. the normal region is advancing. Furthermore, when both perturbations are present the perturbation in the y-direction serves to increase the stability of the perturbation in the z-direction when $v > 0$, and decrease the instability of the perturbation in the z-direction when $v < 0$.

In (1) we found that no mode solutions decaying at infinity were possible when $2n > v$ (and yet such a perturbation may be applied to the system and it will evolve in time). A complete linear treatment for a given perturbation can be obtained by taking the Laplace transform in time, i.e. treating the problem as an initial value problem. If this approach is carried out we find that the free boundary is of the form

$$x = vt + \frac{\cos my \cos nz}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \hat{\sigma} e^{pt} \, dp,$$

where the integrand $\hat{\sigma}$ has poles at the roots of the dispersion relation (2.43). There is also a branch cut in the p plane from $-v^2/4 - n^2 - m^2$ to $-\infty$. When $2n > v$ the sole contribution to the integrand comes from this branch cut. Since $\Re\{p\} < 0$ there, this contribution decays with time.

Finally we solve for $H_1^{(1)}$, $H_3^{(1)}$. If we require that $H_1^{(1)}$, $H_3^{(1)}$ are bounded as $x - vt \to -\infty$ for $v > 0$ and are small compared with $H_c e^{-v(x-vt)}$ as $x - vt \to -\infty$

for $v < 0$ then, in order for div $\boldsymbol{H}$ to tend to zero as $x \to -\infty$ we require that $B = 0$. In this case

$$\begin{aligned} H_1^{(1)} &= Ce^{\lambda_1(x-vt)} \sin my \cos nz, \\ H_3^{(1)} &= De^{\lambda_1(x-vt)} \sin my \sin nz, \end{aligned}$$

where $\lambda_1 = n - v$. The boundary conditions (2.34), (2.37) imply

$$\begin{aligned} C &= -mH_c, \\ D(n-v) &= -vD + mnH_c, \end{aligned}$$

Hence $D = mH_c$ for $n \neq 0$. Note that for $n = 0$ we have $H_3^{(1)} = 0$.

We note that

$$\begin{aligned} \text{div } \boldsymbol{H}^{(1)} &= \frac{\partial H_1^{(1)}}{\partial x} + \frac{\partial H_2^{(1)}}{\partial y} + \frac{\partial H_3^{(1)}}{\partial z}, \\ &= -m(n-v)H_c e^{(n-v)(x-vt)} - vmH_c e^{(n-v)(x-vt)} + mnH_c e^{(n-v)(x-vt)}, \\ &= 0, \end{aligned}$$

as expected.

2.2 Steady State

The scalar Stefan problem has only a trivial steady state. Indeed, the steady state for the model (1.11)-(1.13) is simply

$$\begin{aligned} H = H_c, &\quad \text{in the normal region,} \\ H = 0, &\quad \text{in the superconducting region.} \end{aligned}$$

However, the full vectorial Stefan problem admits non-trivial steady states. The steady state problem on an infinite domain is

$$\text{curl } \boldsymbol{H} = \boldsymbol{0}, \text{ in the normal region,} \tag{2.47}$$

$$\text{div } \boldsymbol{H} = 0, \text{ in the normal region,} \tag{2.48}$$

$$\boldsymbol{H} = \boldsymbol{0}, \text{ in the superconducting region,} \tag{2.49}$$

$$\boldsymbol{H} \cdot \boldsymbol{n} = 0, \text{ on } \Gamma_N, \tag{2.50}$$

$$|\boldsymbol{H}| = H_c, \text{ on } \Gamma_N. \tag{2.51}$$

Equation (2.47) implies there exists a potential φ such that

$$\boldsymbol{H} = H_c \nabla \varphi. \tag{2.52}$$

Equations (2.48)-(2.51) now imply

$$\begin{aligned} \nabla^2 \varphi &= 0, \text{ in the normal region,} & (2.53)\\ \varphi &= 0, \text{ in the superconducting region,} & (2.54)\\ \frac{\partial \varphi}{\partial n} &= 0, \text{ on } \Gamma_N, & (2.55)\\ |\nabla \varphi| &= 1, \text{ on } \Gamma_N. & (2.56) \end{aligned}$$

The problem (2.53)-(2.56) can be identified with the classical problem of a jet or cavity in fluid dynamics by identifying φ with the fluid pressure [9], although for a bounded or semi-infinite superconducting body, the conditions at a fixed boundary will differ.

If we restrict ourselves to the two-dimensional case, with the magnetic field confined to the plane of interest, then we can solve (2.53)-(2.56) by the use of complex variables. In this case $\varphi = \varphi(x, y)$. We let $z = x + iy$ and

$$w(z) = \varphi(x, y) + i\eta(x, y),$$

where η is the harmonic conjugate of φ. Then w is a holomorphic function of z if and only if φ is harmonic.

Let the free boundary Γ be given by

$$z^* = g(z), \tag{2.57}$$

where g is analytic on Γ. g is known as a Schwarz function [18]. The free boundary conditions imply that

$$\frac{\partial w}{\partial s} = \frac{\partial \varphi}{\partial s} - \frac{\partial \varphi}{\partial n} = 1, \tag{2.58}$$

on Γ, where s is the arclength. We also have that

$$\frac{\partial w}{\partial s} = \frac{dw}{dz}\frac{\partial z}{\partial s},$$

on Γ. Equation (2.57) implies that

$$\frac{\partial z^*}{\partial s} = \frac{dg}{dz}\frac{\partial z}{\partial s},$$

and hence

$$\frac{\partial z^*}{\partial s}\frac{\partial z}{\partial s} = 1 = \frac{dg}{dz}\left(\frac{\partial z}{\partial s}\right)^2.$$

Thus (2.58) implies

$$\frac{dw}{dz} = \left(\frac{dg}{dz}\right)^{1/2}, \tag{2.59}$$

on Γ. By analytic continuation we have that equation (2.59) holds wherever both sides are analytic. Thus we can generate large numbers of possible free boundaries by solving the inverse problem, i.e. by specifying the boundary and solving equation (2.59) for the potential.

Equation (2.57) places restrictions on the function g, and it is a question of interest to see which functions are allowable. In particular we have that, on Γ

$$z = g(z)^*,$$

and hence

$$g(z)^* = g(g(z)^*)^* = z,$$

which is usually written as

$$g^* \circ g = z.$$

The simplest Schwarz function is just $g(z) = z$. This gives a plane boundary with $w(z) = \pm z$, $\varphi(x, y) = \pm x$, $\boldsymbol{H} = (\pm 1, 0, 0)$, and is the trivial case mentioned at the beginning of this section.

The only other rational g is a circle [18]

$$|z - a| = b,$$

which has

$$z^* = g(z) = a^* + \frac{b^2}{z - a}.$$

For a unit circle centred at the origin, $a = 0$, $b = 1$, and we have $w(z) = \pm i \log z$, $\varphi = \pm\theta$, $\boldsymbol{H} = \pm(1/r)\boldsymbol{e}_\theta$, where $z = re^{i\theta}$ and $\boldsymbol{e}_\theta$ is the unit vector in the θ direction. We note that in this case the normal state must occupy the region $r > 1$, since the solution with the normal state in the region $r < 1$ gives an unbounded field at the origin (we will return to this point in the conclusion when we consider a superconducting wire carrying an applied current). This is a general property of

Schwarz functions; it has been proved in [18] that if Γ is a simple closed curve then g has a singularity inside Γ. We note that even the solution with the normal region in $r > 1$ is likely to be physically unrealistic, since the magnetic field is everywhere less than the critical magnetic field.

2.3 Similarity Solutions

There are a number of similarity solutions to the vectorial Stefan problem (2.4), (2.7), (2.8). As noted in the introduction, in the two-dimensional case in which the field is perpendicular to the plane of interest the problem reduces to the classical Stefan problem in two dimensions, to which a number of similarity solutions are known to exist. We refer to [33] for a review.

We demonstrate here a new similarity solution for the two-dimensional case in which the field lies within the plane of interest and is azimuthal, depending on r only, i.e.

$$\boldsymbol{H} = H(r,t)\boldsymbol{e}_\theta.$$

If the boundary is given by $r = R(t)$, the equations for H, R are

$$\frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial H}{\partial r} - \frac{H}{r^2} = \frac{\partial H}{\partial t}, \qquad \begin{array}{c} r \geq R(t) \\ \text{or} \\ r \leq R(t) \end{array}, \tag{2.60}$$

$$H(R,t) = H_c, \tag{2.61}$$

$$\frac{\partial H}{\partial r}(R,t) = -H_c\left(\frac{dR}{dt} + \frac{1}{R}\right), \tag{2.62}$$

together with appropriate initial and fixed boundary conditions. There is a similarity variable $\eta = rt^{-1/2}$. Given suitable initial and fixed boundary conditions we have $H = H(\eta)$, $R(t) = ct^{1/2}$, where

$$\eta^2 H'' + \left(\eta + \frac{\eta^3}{2}\right)H' - H = 0, \qquad \begin{array}{c} \eta \geq c \\ \text{or} \\ \eta \leq c \end{array}, \tag{2.63}$$

$$H(c) = H_c, \tag{2.64}$$

$$H'(c) = -H_c\left(\frac{c}{2} + \frac{1}{c}\right), \tag{2.65}$$

where $' = d/d\eta$. One solution of (2.63) is the Toronto function $T(2,1,\eta/2)$. This can be written in terms of hypergeometric functions as $\eta\ {}_1F_1(1/2;2;-\eta^2/4)$, in terms of the Kummer function as $\eta M(1/2;2;-\eta^2/4)$, or in series form as

$$H(\eta) = \sum_{m=0}^{\infty} \frac{(-1)^m(2m)!}{16^m(m!)^2(m+1)!}\,\eta^{2m+1}.$$

An independent solution is given by

$$H(\eta) = e^{-\eta^2/4}\omega_{0,1}(-\eta^2/4),$$

where $\omega_{0,1}$ is the Cunningham function, which can be written in terms of hypergeometric functions as $H(\eta) = {}_2F_0(1/2,-1/2;;4/\eta^2)$. As $\eta \to \infty$,

$$\begin{aligned} T(2,1,\eta/2) &= \text{const.} + O(\eta^{-2}), \\ {}_2F_0(1/2,-1/2;;4/\eta^2) &= \text{const.} + O(\eta^{-1}). \end{aligned}$$

Thus if we are solving in $\eta \geq c$ both solutions are admissible and the general solution of (2.63) will be a linear combination of the two. Specifying the field at infinity will leave only one constant undetermined. The free boundary conditions will then determine the other constant and c. As $\eta \to 0$,

$$\begin{aligned} T(2,1,0) &= \text{const.}, \\ {}_2F_0(1/2,-1/2;;4/\eta^2) &\sim \text{const.}\eta^{-1}. \end{aligned}$$

Thus if we are solving in $\eta \leq c$ and require the field to be bounded at the origin the solution ${}_2F_0(1/2,-1/2;;4/\eta^2)$ is inadmissible. In this case the boundary conditions imply the following relation for c:

$$T'(2,1,c/2) = -(2c^{-1}+c)T(2,1,c/2).$$

A final case to consider is that of an annulus of normal material. In this case we have

$$\eta^2 H'' + \left(\eta + \frac{\eta^3}{2}\right) H' - H = 0, \quad c_1 \leq \eta \leq c_2, \tag{2.66}$$

$$H(c_1) = H_c, \tag{2.67}$$

$$H'(c_1) = -H_c\left(\frac{c_1}{2} + \frac{1}{c_1}\right), \tag{2.68}$$

$$H(c_2) = H_c, \tag{2.69}$$

$$H'(c_2) = -H_c\left(\frac{c_2}{2} + \frac{1}{c_2}\right), \tag{2.70}$$

The general solution is now

$$\alpha T(2, 1, \eta/2) + \beta\, {}_2F_0(1/2, -1/2; ; 4/\eta^2),$$

and we have four boundary conditions to determine the four constants α, β, c_1, c_2.

It is an open question as to whether a solution exists in each of the above cases, and if so how many solutions exist.

2.4 Thermal Effects

We can include thermal effects in the model (2.1)-(2.16) by allowing H_c in (2.7) to depend on the temperature T, and appending equations for T on either side of the free boundary, together with Stefan-type conditions on the free boundary itself. This has been done in one space dimension in [22]. Heat is generated via Ohmic heating in the normal region. In dimensional variables T satisfies

$$k\nabla^2 T = \rho c \frac{\partial T}{\partial t},$$

in the superconducting region, and

$$k\nabla^2 T = \rho c \frac{\partial T}{\partial t} - \frac{1}{\varsigma} \left| \boldsymbol{j}_N \right|^2,$$

in the normal region, with the interface conditions

$$\begin{aligned} [T]_S^N &= 0, \\ \left[k \frac{\partial T}{\partial n} \right]_S^N &= -\hat{l}(T) v_n, \end{aligned}$$

where k is the thermal conductivity, ρ is the density, c is the specific heat, and $\hat{l}(T)$ is the latent heat per unit volume. k, ρ, and c are assumed constant. We non-dimensionalise by setting

$$T = T_c + T_c T', \qquad (2.71)$$

$$\boldsymbol{H} = H_e \boldsymbol{H}', \qquad (2.72)$$

$$t = \mu_s \varsigma l_0^2 t', \qquad (2.73)$$

$$\boldsymbol{x} = l_0 \boldsymbol{x}', \qquad (2.74)$$

as before, where T_c is the critical temperature of the bulk superconductor in the absence of a magnetic field, and l_0 is a typical length of the sample. Ignoring the fixed boundary conditions and initial conditions, which remain as before but must be supplemented by conditions on T, the problem is then, on dropping the primes,

$$
\begin{aligned}
-(\mathrm{curl})^2 \boldsymbol{H} &= \frac{\partial \boldsymbol{H}}{\partial t}, \text{ in } \Omega_N, &(2.75)\\
\boldsymbol{H} &= \mathbf{0}, \text{ in } \Omega_S, &(2.76)\\
\nabla^2 T &= \beta \frac{\partial T}{\partial t} - \gamma\, |\mathrm{curl}\ \boldsymbol{H}\,|^2, \text{ in } \Omega_N, &(2.77)\\
\nabla^2 T &= \beta \frac{\partial T}{\partial t}, \text{ in } \Omega_S, &(2.78)\\
\mathrm{curl}\ \boldsymbol{H} \wedge \boldsymbol{n} &= -v_n \boldsymbol{H}, \text{ on } \Gamma_N, &(2.79)\\
|\,\boldsymbol{H}\,| &= H_c(T), \text{ on } \Gamma_N, &(2.80)\\
[T]_S^N &= 0, &(2.81)\\
\left[\frac{\partial T}{\partial n}\right]_S^N &= -\hat{L}(T) v_n, &(2.82)
\end{aligned}
$$

where

$$
\beta = \frac{\rho c}{\mu_s \varsigma k}, \qquad \gamma = \frac{H_e^2}{\varsigma k T_c}, \tag{2.83}
$$

measure the ratios of thermal to electromagnetic timescales and Ohmic heating to thermal conduction respectively, and

$$
\hat{L}(T) = \frac{\hat{l}(T)}{k T_c \mu_s \varsigma},
$$

is a dimensionless latent heat.

In one dimension, with $\boldsymbol{H} = (0, H(x,t), 0)$, $T = T(x,t)$, the free boundary given by $x = X(t)$, and the normal region given by $x < X(t)$, the model becomes

$$
\begin{aligned}
\frac{\partial^2 H}{\partial x^2} &= \frac{\partial H}{\partial t}, \quad x < X(t), &(2.84)\\
H &= 0, \quad x > X(t), &(2.85)\\
\frac{\partial^2 T}{\partial x^2} &= \beta \frac{\partial T}{\partial t} - \gamma \left(\frac{\partial H}{\partial x}\right)^2, \quad x < X(t), &(2.86)\\
\frac{\partial^2 T}{\partial x^2} &= \beta \frac{\partial T}{\partial t}, \quad x > X(t), &(2.87)\\
\frac{\partial H}{\partial x} &= -H_c(T) \frac{dX}{dt}, \quad x = X(t)^-, &(2.88)
\end{aligned}
$$

$$H = H_c(T), \quad x = X(t)^-, \tag{2.89}$$

$$[T]_{X^-}^{X^+} = 0, \tag{2.90}$$

$$\left[\frac{\partial T}{\partial n}\right]_{X^-}^{X^+} = \hat{L}(T)\frac{dX}{dt}, \tag{2.91}$$

In some respects this model resembles the one-phase alloy solidification problem with H playing the rôle of the impurity concentration [16, p.14], although we have the addition of a new nonlinear term due to the Joule heating. We also note that it bears a superficial resemblance to the 'thermistor' problem [68] with a step function conductivity, but a closer examination shows that the interface conditions for the two problems are quite different.

A similarity solution for the one-dimensional problem is given in [22]. In fact, all the similarity variables of the isothermal problem carry over to the anisothermal problem.

Chapter 3

Ginzburg-Landau Models

In 1950, seven years before the microscopic theory of Bardeen, Cooper and Schrieffer (BCS) [6] was published, Ginzburg and Landau proposed a macroscopic, phenomenological theory of superconductivity to describe the properties of superconductors for temperatures near the critical temperature [28]. The theory allows for a spatial variation in the number density of superconducting electrons by treating the electrons as a fluid that can exist in two phases - normal and superconducting - to which the general Landau-Ginzburg theory of second-order phase transitions is applied (with suitable modifications to take electromagnetic effects into account). It was later found that the Ginzburg-Landau theory could be derived as a formal limit of the BCS theory in the vicinity of the critical temperature [29].

In Section 3.3 we use the theory to calculate the surface energy of a normal/superconducting interface. For a certain range of parameter values (which describes what are now known as Type II superconductors) this energy turns out to be negative. This result was dismissed at the time (1950) as being physically unrealistic. Abrikosov later investigated the nature of possible solutions in this case [1], and about 10 years after that Type II superconductors were observed directly experimentally, and shown to exhibit the Abrikosov 'mixed state' [23]. It is one of the great achievements of the Ginzburg-Landau theory that it allowed for the possibility of Type II superconductors before their existence had been verified.

Before we introduce the theory of Ginzburg and Landau we need to define the magnetic vector potential, $\boldsymbol{A}$, and the electric scalar potential, Φ. Equation (1.2) implies the existence of $\boldsymbol{A}$ such that

$$\mu \boldsymbol{H} = \text{curl } \boldsymbol{A}. \tag{3.1}$$

Now equation (1.4) implies

$$\operatorname{curl}\left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) = \boldsymbol{0},$$

which implies that there exists Φ such that

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla\Phi. \tag{3.2}$$

$\boldsymbol{A}$ is unique up to the addition of a gradient. Once $\boldsymbol{A}$ is given Φ is unique up to the addition of a function of t.

3.1 Steady State

The Ginzburg-Landau theory considers only the steady state, in which $\boldsymbol{E} = \boldsymbol{0}$. As in the phase field theory, Ginzburg and Landau introduce a superconducting order parameter Ψ such that $|\Psi|^2$ represents the number density of superconducting charge carriers. However, the order parameter must in this case be *complex*, and can be thought of as an 'averaged macroscopic wavefunction' for the superconducting electrons. The connection made by Gor'kov between the microscopic theory and the macroscopic Ginzburg-Landau theory justified the need for Ψ to be complex-valued.

We proceed by expanding the Helmholtz free energy density $\mathcal{F}$ as a power series in $|\Psi|^2$, which is truncated after the second term since $|\Psi|^2$ is small near the critical temperature T_c. Thus, in the absence of a magnetic field we have

$$\mathcal{F}_{s0} = \mathcal{F}_{n0} + a(T)\,|\Psi|^2 + \frac{b(T)}{2}\,|\Psi|^4,$$

where $\mathcal{F}_{s0}$ ($\mathcal{F}_{n0}$) is the Helmholtz free energy density of the superconducting (normal) phase in the absence of a magnetic field. In stable equilibrium we require

$$\frac{\partial \mathcal{F}_{s0}}{\partial\,|\Psi|^2} = 0, \quad \frac{\partial^2 \mathcal{F}_{s0}}{\partial(|\Psi|^2)^2} > 0,$$

with $|\Psi|^2 = 0$ for $T \geq T_c$, $|\Psi|^2 > 0$ for $T < T_c$. It follows that $a(T_c) = 0$, $b(T_c) > 0$, $a(T) < 0$ for $T < T_c$. Since the theory supposes the temperature to be in the vicinity of T_c the coefficients a and b are usually expanded in powers of $T' = (T - T_c)/T_c$ and only the first non-zero terms retained. Then

$$a \sim \alpha T', \quad b \sim \beta, \qquad \alpha, \beta > 0.$$

Thus, in equilibrium, for $T < T_c$,

$$|\Psi|^2 = \Psi_0^2 = -\frac{a}{b} \sim -\frac{\alpha T'}{\beta}.$$

To calculate the Helmholtz free energy density in the presence of a magnetic field, $\mathcal{F}_{sH}$, we must add to the free energy density $\mathcal{F}_{s0}$ the magnetic field energy density $\frac{1}{2}\mu H_c^2$, and the energy associated with the possible appearance of a gradient in Ψ in the presence of the field. This last energy, at least for small values of $|\nabla\Psi|^2$, can, as a result of a series expansion with respect to $|\nabla\Psi|^2$ in which only the leading term is retained, be expressed in the form

$$\text{const.}\ |\nabla\Psi|^2\ . \tag{3.3}$$

The addition of a term of this form penalises variations of the order parameter Ψ and in the Landau-Ginzburg theory of second-order phase transitions such a term can be thought of as representing the 'surface energy'. On the other hand, the free energy should be gauge invariant (in the sense that if $\boldsymbol{A}$ is replaced by $\boldsymbol{A} + \nabla\omega$ then the phase of Ψ can be adjusted to make the resulting free energy density independent of ω), and we have yet to take into account the interaction between the magnetic field and the electric current associated with the presence of a gradient in Ψ. Ginzburg and Landau therefore postulated the addition of a term proportional to $i\boldsymbol{A}\Psi$ to $\nabla\Psi$, so that (3.3) becomes

$$\frac{1}{2m_s}\left|i\hbar\nabla\Psi + e_s\boldsymbol{A}\Psi\right|^2, \tag{3.4}$$

where e_s and m_s are the charge and mass respectively of the superconducting charge carriers, and $2\pi\hbar$ is Plank's constant. The microscopic pairing theory of superconductivity implies that $e_s = 2e$, where e is the electron mass. The value of m_s is somewhat more arbitrary, since any change in its definition may be compensated by a change in the magnitude of Ψ. However, it is customary to set $m_s = m$ or $m_s = 2m$. We adopt the latter convention, since in this case $|\Psi|^2$ represents the number density of superconducting charge carriers as stated above (i.e. half the number density of superconducting electrons). Following [60], we note that (3.4) may be written

$$\frac{1}{2m_s}\left(\hbar^2\,|\nabla f|^2 + |\hbar\nabla\chi - e_s\boldsymbol{A}|^2\,f^2\right),$$

where $\Psi = fe^{i\chi}$, with f and χ real, $f > 0$. The first term here is the previously mentioned surface energy term which penalises curvature in the level sets of f. The second term may be interpreted as a gauge-invariant 'kinetic energy density' associated with the superconducting currents. (We will see later that the superconducting current is given by $\boldsymbol{j}_s = (e_s f^2/m_s)(\hbar\nabla\chi - e_s\boldsymbol{A})$.)

The Helmholtz free energy density is now

$$\mathcal{F}_{sH} = \mathcal{F}_{no} + a(T)\,|\,\Psi\,|^2 + \frac{b(T)\,|\,\Psi\,|^4}{2} + \frac{1}{4m}\,|i\hbar\nabla\Psi + 2e\boldsymbol{A}\Psi|^2 + \frac{\mu\,|\,\boldsymbol{H}\,|^2}{2},$$

and is invariant under transformations of the type

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla\omega, \quad \Psi \to \Psi e^{2ie\hbar\omega}.$$

In the presence of a uniform applied magnetic field $\boldsymbol{H}_0$ the Gibbs free energy density $\mathcal{G}$ differs from the Helmholtz free energy density $\mathcal{F}$ due to the work done by the electromotive force induced by the applied field. This work (per unit volume) is given by $\mu\boldsymbol{H}\cdot\boldsymbol{H}_0$, so that the Gibbs free energy density is given by

$$\begin{aligned} \mathcal{G}_{sH} &= \mathcal{F}_{sH} - \mu\boldsymbol{H}\cdot\boldsymbol{H}_0, \\ &= \mathcal{F}_{no} + a(T)\,|\,\Psi\,|^2 + \frac{b(T)\,|\,\Psi\,|^4}{2} \\ &\qquad + \frac{1}{4m}\,|i\hbar\nabla\Psi + 2e\boldsymbol{A}\Psi|^2 + \frac{\mu\,|\,\boldsymbol{H}\,|^2}{2} - \mu\boldsymbol{H}\cdot\boldsymbol{H}_0. \end{aligned}$$

The basic thermodynamic postulate of the Ginzburg-Landau theory is that the total Gibbs free energy $\int \mathcal{G}_{sH}\,dV$, should be minimised.

In the normal state $\Psi = 0$, and $\int \mathcal{G}_{sH}\,dV$ is minimised by $\boldsymbol{H} = \boldsymbol{H}_0$, giving $\int \mathcal{G}_{sH}\,dV = \int \mathcal{F}_{no} - \frac{1}{2}\mu H_0^2\,dV$. In the perfect superconducting state in the absence of surface effects $\int \mathcal{G}_{sH}\,dV$ is minimised by $\boldsymbol{H} = \boldsymbol{0}$, $\Psi = -a/b$, giving $\int \mathcal{G}_{sH}\,dV = \int \mathcal{F}_{no} - \frac{a^2}{2b}\,dV$. We see that for low values of H_0 the superconducting state has a lower free energy, whereas for high values of H_0 the normal state has a lower free energy. Thus there is a critical magnetic field at which, in the absence of surface effects (i.e. for a bulk superconductor), the superconducting state becomes energetically more favourable. We see that this field is given by

$$H_c = \frac{|\,a(T)\,|}{\sqrt{\mu b(T)}}. \tag{3.5}$$

Varying $\int \mathcal{G}_{sH}\, dV$ with respect to Ψ^*, the conjugate of Ψ, and $\boldsymbol{A}$ we obtain the Ginzburg-Landau differential equations:

$$\frac{1}{4m}\left(i\hbar\nabla + 2e\boldsymbol{A}\right)^2 \Psi + a(T)\Psi + b(T)\Psi \mid \Psi \mid^2 = 0, \quad \text{in } \Omega, \tag{3.6}$$

$$\frac{1}{\mu}(\text{curl})^2\boldsymbol{A} = -\frac{ie\hbar}{2m}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) + \frac{2e^2}{m} \mid \Psi \mid^2 \boldsymbol{A} + \text{curl } \boldsymbol{H}_0, \quad \text{in } \Omega, \tag{3.7}$$

$$\frac{1}{\mu}(\text{curl})^2\boldsymbol{A} = \text{curl } \boldsymbol{H}_0, \quad \text{outside } \Omega, \tag{3.8}$$

with the natural boundary conditions

$$\boldsymbol{n}\cdot\left(i\hbar\nabla + 2e\boldsymbol{A}\right)\Psi = 0, \quad \text{on } \partial\Omega, \tag{3.9}$$

$$[(1/\mu)\text{curl } \boldsymbol{A}\wedge\boldsymbol{n}] = \boldsymbol{0}, \tag{3.10}$$

$$\text{curl } \boldsymbol{A} \to \boldsymbol{H}_0, \quad \text{as } r\to\infty, \tag{3.11}$$

where [ ] denotes the jump in the enclosed quantity across $\partial\Omega$, and r is the distance from the origin. Note that (3.8) simply states that the current in the external region is equal to the applied current, (3.10) is the usual boundary condition on the magnetic field at the interface of two magnetic media, and that (3.11) simply states that the field is equal to the applied field far from the origin. Since we are only considering situations in which there is no applied electric field, we have that curl $\boldsymbol{H}_0 = \varsigma\boldsymbol{E}_0 = \boldsymbol{0}$.

The conditions (3.9)-(3.11) are obtained if no supplementary requirements are imposed on Ψ and $\boldsymbol{A}$. In [27], using the microscopic theory, (3.9) has been shown to be modified to

$$\boldsymbol{n}\cdot\left(i\hbar\nabla + 2e\boldsymbol{A}\right)\Psi = -i\gamma\Psi, \quad \text{on } \partial\Omega, \tag{3.12}$$

at a boundary with another material; γ is very small for insulators and very large for magnetic materials, with normal metals lying in between. Such a term would arise from the variational approach if a term

$$\int_S \hbar\gamma \mid \Psi \mid^2 \, dS,$$

was added to the free energy. Such a term may describe what is known as the proximity effect, that is, the continuation of superconductivity a small distance into the material that lies adjacent to the superconductor.

In addition to (3.9)-(3.11) we also impose the condition

$$[\boldsymbol{n} \wedge \boldsymbol{A}] = \boldsymbol{0}. \tag{3.13}$$

This equation simply states that, although we are free to choose the gauge of $\boldsymbol{A}$ arbitrarily, we have chosen the same gauge both inside and outside Ω.

The Maxwell equation (1.3) and the relation (3.1) imply

$$\boldsymbol{j} = \frac{1}{\mu}(\mathrm{curl})^2 \boldsymbol{A},$$

where $\boldsymbol{j}$ is the current. Hence (3.7) is an expression for the superconducting current. We note that the natural boundary condition (3.9), or indeed the condition (3.12), implies

$$\boldsymbol{j} \cdot \boldsymbol{n} = 0,$$

since (3.7) may be written as

$$\boldsymbol{j} = \frac{e}{2m}\left(\Psi^* \left(i\hbar\nabla + 2e\boldsymbol{A}\right)\Psi + \Psi\left(-i\hbar\nabla + 2e\boldsymbol{A}\right)\Psi^*\right).$$

We will use two different non-dimensionalisations of equations (3.6)-(3.13), depending on whether we are considering isothermal conditions or not, since under isothermal conditions temperature may be completely scaled out of the problem, and the precise form of the coefficients a and b is irrelevant. It is in this nondimensionalised form that the equations are most widely used.

Isothermal conditions

Under isothermal conditions, with $T < T_c$, we non-dimensionalise by setting

$$\Psi = \sqrt{\frac{|a|}{b}}\Psi', \qquad \boldsymbol{H} = |a|\sqrt{\frac{2}{b\mu_s}}\boldsymbol{H}', \qquad \boldsymbol{A} = |a|\, l_0\sqrt{\frac{2\mu_s}{b}}\boldsymbol{A}',$$

$$\boldsymbol{x} = l_0\boldsymbol{x}', \qquad \mu = \mu_s\mu',$$

where l_0 is a typical length of the sample and μ_s is the permeability of the superconducting material, to give, on dropping the primes

$$\left(\xi\nabla - \frac{i}{\lambda}\boldsymbol{A}\right)^2 \Psi = \Psi\,|\Psi|^2 - \Psi, \quad \text{in } \Omega, \tag{3.14}$$

$$-\lambda^2(\mathrm{curl})^2\boldsymbol{A} = \frac{\xi\lambda i}{2}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) + |\Psi|^2\,\boldsymbol{A}, \quad \text{in } \Omega, \tag{3.15}$$

$$(\mathrm{curl})^2 \boldsymbol{A} = \boldsymbol{0}, \qquad \text{outside } \Omega, \tag{3.16}$$

with boundary conditions

$$\boldsymbol{n} \cdot \left(\xi \nabla - \frac{i}{\lambda} \boldsymbol{A}\right) \Psi = -\frac{\Psi}{d}, \quad \text{on } \partial\Omega, \tag{3.17}$$

$$[(1/\mu)\mathrm{curl}\ \boldsymbol{A} \wedge \boldsymbol{n}] = \boldsymbol{0}, \tag{3.18}$$

$$[\boldsymbol{n} \wedge \boldsymbol{A}] = \boldsymbol{0}, \tag{3.19}$$

$$\mathrm{curl}\ \boldsymbol{A} \to \boldsymbol{H}_0, \qquad \text{as } r \to \infty, \tag{3.20}$$

where

$$\xi = \frac{\hbar}{2 l_0 \sqrt{m\,|a|}}, \qquad \lambda = \frac{1}{e l_0} \sqrt{\frac{mb}{2\,|a|\,\mu_s}}, \qquad d = \frac{2\sqrt{m\,|a|}}{\gamma}.$$

(note that $\mu = 1$ in the superconducting material Ω.) We note that in these dimensionless variables

$$H_c(T) = \frac{1}{\sqrt{2}}, \tag{3.21}$$

and that λ and $\xi \to \infty$ as $|T|^{-1/2}$ as $T \to 0$ (i.e. as the temperature tends to the critical temperature).

Here (3.14) has the form of a nonlinear Schrödinger equation. We use equations (3.14)-(3.20) in Chapters 5-7.

Anisothermal conditions

When the temperature is allowed to vary we non-dimensionalise (3.6)-(3.13) by setting

$$\Psi = \sqrt{\frac{\alpha}{\beta}} \Psi', \qquad \boldsymbol{H} = \alpha \sqrt{\frac{2}{\beta \mu_s}} \boldsymbol{H}', \qquad \boldsymbol{A} = \alpha l_0 \sqrt{\frac{2\mu_s}{\beta}} \boldsymbol{A}', \qquad \boldsymbol{x} = l_0 \boldsymbol{x}',$$

$$a(T) = \alpha a(T)', \qquad b(T) = \beta b(T)', \qquad \mu = \mu_s \mu'.$$

Dropping the primes we then have

$$\left(\xi \nabla - \frac{i}{\lambda} \boldsymbol{A}\right)^2 \Psi = a(T)\Psi + b(T)\Psi\,|\Psi|^2, \quad \text{in } \Omega, \tag{3.22}$$

$$-\lambda^2 (\mathrm{curl})^2 \boldsymbol{A} = \frac{\xi \lambda i}{2} \left(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*\right) + |\Psi|^2\, \boldsymbol{A}, \quad \text{in } \Omega, \tag{3.23}$$

$$(\mathrm{curl})^2 \boldsymbol{A} = \boldsymbol{0}, \qquad \text{outside } \Omega, \tag{3.24}$$

with boundary conditions

$$\boldsymbol{n} \cdot \left(\xi\nabla - \frac{i}{\lambda}\boldsymbol{A}\right)\Psi = -\frac{\Psi}{d}, \quad \text{on } \partial\Omega, \tag{3.25}$$

$$[(1/\mu)\text{curl } \boldsymbol{A} \wedge \boldsymbol{n}] = \boldsymbol{0}, \tag{3.26}$$

$$[\boldsymbol{n} \wedge \boldsymbol{A}] = \boldsymbol{0}, \tag{3.27}$$

$$\text{curl } \boldsymbol{A} \to \boldsymbol{H}_0, \quad \text{as } r \to \infty, \tag{3.28}$$

where

$$\xi = \frac{\hbar}{2l_0\sqrt{m\alpha}}, \qquad \lambda = \frac{1}{el_0}\sqrt{\frac{m\beta}{2\alpha\mu_s}}, \qquad d = \frac{2\sqrt{m\alpha}}{\gamma}.$$

We note that in these dimensionless variables

$$a(T) \sim T + \cdots, \qquad b(T) \sim 1 + \cdots,$$

and

$$H_c(T) = \frac{|\,a(T)\,|}{\sqrt{2b(T)}} \sim \frac{|\,T\,|}{\sqrt{2}}. \tag{3.29}$$

Note that λ and ξ are in this case temperature-independent.

In the steady state we will not consider situations in which the temperature varies spatially, but in Chapter 8 we will seek bifurcations from the normal state as the temperature is varied parametrically, using equations (3.22)-(3.28). For simplicity, in Chapter 8 we will linearise equation (3.22) in T to obtain

$$\left(\xi\nabla - \frac{i}{\lambda}\boldsymbol{A}\right)^2\Psi = T\Psi + \Psi\,|\,\Psi\,|^2, \quad \text{in } \Omega, \tag{3.30}$$

although we note that the analysis of Chapter 8 is also possible retaining a and b as unknown functions of T.

We see that λ and ξ are typical lengthscales for variations in $\boldsymbol{A}$ and Ψ respectively. λ and ξ are typically very small; in dimensional terms they are often of the order of 1 μm. Also the form of the solution of either (3.14)-(3.20) or (3.22)-(3.28) will depend only on the applied field $\boldsymbol{H}_0$, the geometry, and the ratio $\kappa = \lambda/\xi$, a material constant known as the Ginzburg-Landau parameter. Note that when we linearise in T, κ is the same in both our non-dimensionalisations.

Because of the gauge invariance of the equations we are able to choose the gauge of $\boldsymbol{A}$ to be convenient in our calculations. The usual choice is

$$\text{div } \boldsymbol{A} = 0. \tag{3.31}$$

However, even this condition does not determine $\boldsymbol{A}$ uniquely, since we may add the gradient of a harmonic function to $\boldsymbol{A}$.

As noted above, if we write $\Psi = fe^{i\chi}$, with f real, the equations are invariant under transformations of the type

$$\boldsymbol{A} \rightarrow \boldsymbol{A} + \nabla\omega, \quad \chi \rightarrow \chi + \frac{\omega}{\lambda\xi}$$

for any function ω. This invariance allows us to eliminate a variable from the equations by writing, in Ω,

$$\boldsymbol{Q} = \boldsymbol{A} - \xi\lambda\nabla\chi,$$

where $\boldsymbol{Q}$ is now independent of the gauge of $\boldsymbol{A}$. By defining $\boldsymbol{Q}$ to be a suitable gauge in the external region the equations then become the following:

Isothermal conditions

$$\xi^2\nabla^2 f = f^3 - f + \frac{f\,|\boldsymbol{Q}|^2}{\lambda^2}, \quad \text{in } \Omega, \tag{3.32}$$

$$-\lambda^2(\text{curl})^2\boldsymbol{Q} = f^2\boldsymbol{Q}, \quad \text{in } \Omega, \tag{3.33}$$

$$(\text{curl})^2\boldsymbol{Q} = \boldsymbol{0}, \quad \text{outside } \Omega, \tag{3.34}$$

with the boundary conditions

$$\boldsymbol{n}\cdot\nabla f = -\frac{f}{d}, \quad \text{on } \partial\Omega, \tag{3.35}$$

$$\boldsymbol{n}\cdot f\boldsymbol{Q} = 0, \quad \text{on } \partial\Omega, \tag{3.36}$$

$$[\boldsymbol{n}\wedge\boldsymbol{Q}] = \boldsymbol{0}, \tag{3.37}$$

$$[\boldsymbol{n}\wedge(1/\mu)\text{curl }\boldsymbol{Q}] = \boldsymbol{0}, \tag{3.38}$$

$$\text{curl }\boldsymbol{Q} \rightarrow \boldsymbol{H}_0, \text{ as } r\rightarrow\infty. \tag{3.39}$$

Anisothermal conditions

$$\xi^2\nabla^2 f = a(T)f + b(T)f^3 + \frac{f\,|\boldsymbol{Q}|^2}{\lambda^2}, \quad \text{in } \Omega, \tag{3.40}$$

$$-\lambda^2(\text{curl})^2\boldsymbol{Q} = f^2\boldsymbol{Q}, \quad \text{in } \Omega, \tag{3.41}$$

$$(\text{curl})^2\boldsymbol{Q} = \boldsymbol{0}, \quad \text{outside } \Omega, \tag{3.42}$$

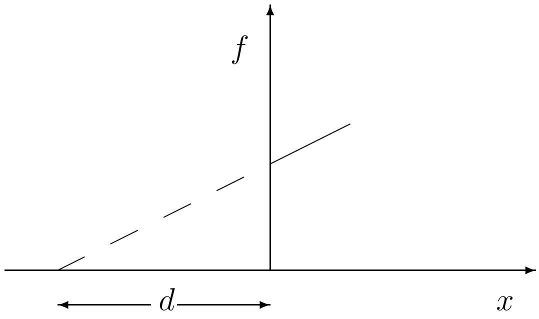

Figure 3.1: Schematic diagram of f near the boundary of the material ($x = 0$) showing the proximity effect.

with the boundary conditions

$$\begin{aligned} \boldsymbol{n} \cdot \nabla f &= -\frac{f}{d}, \quad \text{on } \partial\Omega, && (3.43) \\ \boldsymbol{n} \cdot f\boldsymbol{Q} &= 0, \quad \text{on } \partial\Omega, && (3.44) \\ [\boldsymbol{n} \wedge \boldsymbol{Q}] &= \boldsymbol{0}, && (3.45) \\ [\boldsymbol{n} \wedge (1/\mu)\text{curl } \boldsymbol{Q}] &= \boldsymbol{0}, && (3.46) \\ \text{curl } \boldsymbol{Q} &\to \boldsymbol{H}_0, \text{ as } r \to \infty. && (3.47) \end{aligned}$$

The equation $\text{div}(f^2\boldsymbol{Q}) = 0$ is also a consequence of (3.14)-(3.20) or (3.22)-(3.28), but is now a trivial deduction from (3.33) or (3.41).

We see now how equations (3.35) and (3.43) describe the proximity effect; d represents the distance (on the ξ lengthscale) that the superconducting electrons penetrate into the non-superconducting material (see Fig. 3.1).

3.1.1 The Importance of Phase: Fluxoid Quantization

Equation (3.33) gives an expression for the superconducting current,

$$\boldsymbol{j} = -\frac{|\Psi|^2 \boldsymbol{Q}}{\lambda^2}.$$

(Note that this equation is similar to the London equation (1.23), but differs in that $|\Psi|^2$, the number density of superconducting charge carriers, is allowed to vary spatially.) Let C be a closed curve lying in the material such that $|\Psi| \neq 0$

everywhere on C (i.e. the curve nowhere intersects a normal region), and let S be a surface bounded by C. Then

$$\begin{aligned} -\int_C \frac{\lambda^2}{|\Psi|^2}\boldsymbol{j}\cdot \boldsymbol{ds} &= \int_C \boldsymbol{Q}\cdot\boldsymbol{ds}, \\ &= \int_C \boldsymbol{A}\cdot\boldsymbol{ds} - \xi\lambda\int_C \nabla\chi\cdot\boldsymbol{ds}, \\ &= \int_S \boldsymbol{H}\cdot\boldsymbol{n}\,dS - 2\pi\xi\lambda N, \end{aligned}$$

where N is an integer, since the phase χ must change by an integer multiple of 2π around C if Ψ is single valued. Hence

$$\int_S \boldsymbol{H}\cdot\boldsymbol{n}\,dS + \int_C \frac{\lambda^2}{|\Psi|^2}\boldsymbol{j}\cdot\boldsymbol{ds} = 2\pi\xi\lambda N. \tag{3.48}$$

The left-hand side of this equation is known as the fluxoid through the surface S (and is often denoted by Φ'), with the first integral being the magnetic flux through S. We see that the fluxoid is quantized in multiples of $2\pi\xi\lambda$ (in these units), which is known as a quantum of fluxoid. In particular, the fluxoid through any normal cylinder or cylindrical hole is quantized. Also the fluxoid through any cross-section of an arbitrarily shaped hole or normal region is quantised (though if the hole is topologically equivalent to a sphere the fluxoid will be zero). Furthermore, if the curve C is taken to be far enough inside the completely superconducting material that $\boldsymbol{j} \to 0$ and the second integral is negligible, we have that the magnetic flux through S is quantized. Thus the magnetic flux through a hole is quantized if we include the flux that penetrates the superconductor (see Fig. 3.2). Note that this result implies that the superconductor cannot form arbitrarily small normal regions, since such a region must contain at least one quantum of flux, and the variation of the magnetic field is limited by the penetration depth λ.[1]

We will return to the idea of flux quantization in Chapter 7, when we consider 'vortex' solutions of the Ginzburg-Landau equations.

[1]Fluxoid quantization can also be used to explain the persistence of currents in a ring of superconducting material (even though such a state has a higher energy than that of no current). Our explanation follows that of [60]. The current in such a ring cannot decrease through fluctuations by arbitrarily small amounts, but only in finite jumps such that the fluxoid decreases by one or more integer multiples of $2\pi\xi\lambda$. If only a single or a few electrons were involved, this could easily be accomplished. However, we are requiring a quantum jump in the phase of Ψ, a *macroscopic* function. Such a change requires the *simultaneous* quantum jump of a very large ($> 10^{20}$) number of particles, and is of course extremely improbable, leading to an extremely long half life for circulating superconducting currents.

Figure 3.2: Quantization of flux in a hole in a superconductor. The curve C is taken to be sufficiently far inside the superconductor that $\boldsymbol{j} = 0$.

3.2 Evolution Model

It is not as easy to make the above model time-dependent as it is, say, with the phase field model, because of the coupling with Maxwell's equations. Fortunately an alternative approach is available, namely that of averaging the microscopic BCS theory [6]. The procedure requires that the temperature is close to T_c. It is described in [30] and results in the equations:

$$\hbar\frac{\partial\Psi}{\partial t} + 2ie\Psi\Phi + \frac{\tau_s}{3}\left[-\pi^2(T_c^2 - T^2) + \frac{|\Psi|^2}{2}\right]\Psi - D(\hbar\nabla - 2ie\boldsymbol{A})^2\Psi = 0, \quad \text{in } \Omega, \tag{3.49}$$

$$\boldsymbol{j} = \varsigma\boldsymbol{E} - 2\varsigma\tau_s\left[|\Psi|^2\,\boldsymbol{A} - \frac{i\hbar}{4e}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right)\right], \quad \text{in } \Omega, \tag{3.50}$$

with the boundary condition on Ψ remaining as before

$$\boldsymbol{n}\cdot(\hbar\nabla - 2ie\boldsymbol{A})\,\Psi \;=\; -\gamma\Psi, \quad \text{on } \partial\Omega; \tag{3.51}$$

here Φ is the electric scalar potential, τ_s and D are microscopic parameters (τ_s is the free flight time between collisions associated with the electron spin flip and D is a diffusion coefficient) and ς is the conductivity of the normal electrons. These equations define the coefficients a, b that appear in the steady Ginzburg-Landau equations in terms of the microscopic parameters.

Maxwell's equations imply that

$$(\text{curl})^2\boldsymbol{A} = -\varsigma_e\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right), \quad \text{outside } \Omega, \tag{3.52}$$

$$\nabla^2\Phi = 0, \quad \text{outside } \Omega, \tag{3.53}$$

since div $\boldsymbol{A} = 0$. Here ς_e is the conductivity of the external region and we have assumed in (3.53) that the charge density in the external region is zero in the case that the external conductivity is zero. Again, we have the usual boundary conditions on the magnetic vector potential that

$$[\boldsymbol{n} \wedge \boldsymbol{A}] = \boldsymbol{0}, \tag{3.54}$$

$$[\boldsymbol{n} \wedge (1/\mu)\text{curl } \boldsymbol{A}] = \boldsymbol{0}, \tag{3.55}$$

$$\text{curl } \boldsymbol{A} \to \boldsymbol{H}_0, \quad \text{as } r \to \infty. \tag{3.56}$$

We must now also add the usual boundary conditions on the electric scalar potential, namely

$$[\Phi] = 0, \tag{3.57}$$

$$[\varepsilon\partial\Phi/\partial n] = 0. \tag{3.58}$$

As before, we employ two different non-dimensionalisations.

Isothermal conditions

Under isothermal conditions we non-dimensionalise by setting

$$\boldsymbol{x} = l_0\boldsymbol{x}', \qquad \Psi = \pi\sqrt{2(T_c^2 - T^2)}\Psi',$$

$$\boldsymbol{H} = \frac{\tau_s\pi^2(T_c^2 - T^2)}{e}\sqrt{\frac{\varsigma}{3D\mu_s}}\boldsymbol{H}', \qquad \boldsymbol{A} = \frac{l_0\tau_s\pi^2(T_c^2 - T^2)}{e}\sqrt{\frac{\mu_s\varsigma}{3D}}\boldsymbol{A}',$$

$$t = \mu_s\varsigma l_0^2 t', \qquad \Phi = \frac{\tau_s\pi^2(T_c^2 - T^2)}{e\varsigma}\sqrt{\frac{\varsigma}{3D\mu_s}}\Phi',$$

$$\mu = \mu_s\mu', \qquad \varepsilon = \varepsilon_s\varepsilon',$$

where ε_s is the permittivity of the superconducting material. Dropping the primes this gives

$$\alpha\xi^2\frac{\partial\Psi}{\partial t} + \frac{\alpha\xi i}{\lambda}\Psi\Phi + \Psi\,|\,\Psi\,|^2 - \Psi - \left(\xi\nabla - \frac{i}{\lambda}\boldsymbol{A}\right)^2\Psi = 0, \quad \text{in } \Omega, \tag{3.59}$$

$$-\lambda^2(\text{curl})^2\boldsymbol{A} = \lambda^2\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right) + \frac{\xi\lambda i}{2}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) + |\Psi|^2\boldsymbol{A}, \quad \text{in } \Omega, \tag{3.60}$$

$$(\text{curl})^2\boldsymbol{A} = -\varsigma_e\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right), \quad \text{outside } \Omega, \tag{3.61}$$

$$\nabla^2\Phi = 0, \quad \text{outside } \Omega, \tag{3.62}$$

with boundary conditions

$$\boldsymbol{n}\cdot\left(i\xi\nabla + \frac{\boldsymbol{A}}{\lambda}\right)\Psi = -i\frac{\Psi}{d}, \quad \text{on } \partial\Omega, \tag{3.63}$$

$$[(1/\mu)\text{curl } \boldsymbol{A}\wedge\boldsymbol{n}] = \boldsymbol{0}, \tag{3.64}$$

$$[\boldsymbol{n}\wedge\boldsymbol{A}] = \boldsymbol{0}, \tag{3.65}$$

$$\text{curl } \boldsymbol{A} \to \boldsymbol{H}_0, \quad \text{as } r\to\infty, \tag{3.66}$$

$$[\Phi] = 0, \tag{3.67}$$

$$[\varepsilon\partial\Phi/\partial n] = 0, \tag{3.68}$$

where

$$\xi = \frac{\hbar}{l_0\pi}\sqrt{\frac{3D}{\tau_s(T_c^2 - T^2)}}, \qquad \lambda = \frac{1}{2\pi l_0\sqrt{\varsigma\tau_s\mu_s(T_c^2 - T^2)}},$$

$$\alpha = \frac{1}{\hbar\mu_s\varsigma D}, \qquad d = \frac{\pi}{\gamma}\sqrt{\frac{\tau_s(T_c^2 - T^2)}{3D}},$$

ς_e is now normalised with the normal conductivity of the superconducting material, and we have used the Maxwell equation (1.8) and the relations (3.1) and (3.2). (Note that $\varepsilon = 1$ in the superconducting material.) We now have expressions for λ and ξ in terms of the microscopic parameters. The new parameter α is taken to be of order one.

Anisothermal conditions

When the temperature is allowed to vary we non-dimensionalise by setting

$$\boldsymbol{x} = l_0\boldsymbol{x}', \qquad \Psi = 2\pi T_c\Psi',$$

$$\boldsymbol{H} = \frac{2\pi^2\tau_s T_c^2}{e}\sqrt{\frac{\varsigma}{3D\mu_s}}\boldsymbol{H}', \qquad \boldsymbol{A} = \frac{2l_0\pi^2\tau_s T_c^2}{e}\sqrt{\frac{\mu_s\varsigma}{3D}}\boldsymbol{A}',$$

$$t = \mu_s\varsigma l_0^2 t', \qquad \Phi = \frac{2\pi^2\tau_s T_c^2}{e\sqrt{3D\mu_s\varsigma}}\Phi',$$

$$\mu = \mu_s \mu', \qquad \varepsilon = \varepsilon_s \varepsilon'.$$

Dropping the primes gives

$$\alpha\xi^2 \frac{\partial \Psi}{\partial t} + \frac{\alpha\xi i}{\lambda}\Psi\Phi + b(T)\Psi \mid \Psi \mid^2 + a(T)\Psi - \left(\xi\nabla - \frac{i}{\lambda}\boldsymbol{A}\right)^2 \Psi = 0, \quad \text{in } \Omega, \tag{3.69}$$

$$-\lambda^2 (\text{curl})^2 \boldsymbol{A} = \lambda^2 \left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right) + \frac{\xi\lambda i}{2}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) + \mid \Psi \mid^2 \boldsymbol{A}, \quad \text{in } \Omega, \tag{3.70}$$

$$(\text{curl})^2 \boldsymbol{A} = -\varsigma_e \left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right), \quad \text{outside } \Omega, \tag{3.71}$$

$$\nabla^2 \Phi = 0, \quad \text{outside } \Omega, \tag{3.72}$$

with boundary conditions

$$\boldsymbol{n} \cdot \left(i\xi\nabla + \frac{\boldsymbol{A}}{\lambda}\right)\Psi = -i\frac{\Psi}{d}, \quad \text{on } \partial\Omega, \tag{3.73}$$

$$[(1/\mu)\text{curl } \boldsymbol{A} \wedge \boldsymbol{n}] = \boldsymbol{0}, \tag{3.74}$$

$$[\boldsymbol{n} \wedge \boldsymbol{A}] = \boldsymbol{0}, \tag{3.75}$$

$$\text{curl } \boldsymbol{A} \to \boldsymbol{H}_0, \quad \text{as } r \to \infty, \tag{3.76}$$

$$[\Phi] = 0, \tag{3.77}$$

$$[\varepsilon\partial\Phi/\partial n] = 0, \tag{3.78}$$

where

$$\alpha = \frac{1}{\hbar\mu_s\varsigma D}, \quad \xi = \frac{\hbar}{T_c l_0 \pi}\sqrt{\frac{3D}{2\tau_s}}, \quad \lambda = \frac{1}{2\pi l_0 T_c \sqrt{2\varsigma\tau_s\mu}}, \quad d = \frac{\pi T_c}{\gamma}\sqrt{\frac{2\tau_s}{3D}},$$

$$a(T) = T\left(1 + \frac{T}{2}\right), \quad b(T) = 1,$$

and we have again used the Maxwell equation (1.8) and the relations (3.1) and (3.2). As in the isothermal case we have expressions for λ and ξ in terms of the microscopic parameters.

In the time-dependent case we will consider situations in which the temperature varies in time and space as well as parametrically. In this case we must also have an equation to determine T in the form of a heat balance equation as in the phase field model. As noted in the introduction, thermodynamic arguments imply that there is a release of latent heat on the transition from normally conducting to superconducting in the presence of a magnetic field. Following the phase field model, we take the rate of release of latent heat to be proportional to the rate of change of the number density of superconducting electrons. We must also include a term in the heat balance equation to account for the Ohmic heating due to the normal current. Thus the equation we require is

$$k\nabla^2 T = \rho c \frac{\partial T}{\partial t} - l(T)\frac{\partial(|\Psi|^2)}{\partial t} - \frac{1}{\varsigma}|\boldsymbol{j}_N|^2, \tag{3.79}$$

which, on nondimensionalising, becomes

$$\begin{aligned}\nabla^2 T &= \beta\frac{\partial T}{\partial t} - L(T)\frac{\partial(|\Psi|^2)}{\partial t} - \gamma|\boldsymbol{j}_N|^2,\\ &= \beta\frac{\partial T}{\partial t} - L(T)\frac{\partial(|\Psi|^2)}{\partial t} - \gamma\left|\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right|^2,\end{aligned} \tag{3.80}$$

where β and γ are given by (2.83), and may be functions of all the variables, and

$$L(T) = \frac{4\pi^2 T_c l(T)}{\mu\varsigma k}.$$

We note that in deriving (3.49), (3.50), [30] assumed that Joule losses were small. It is not clear whether the equations would have the same form when we take Joule losses into account via (3.80), but we assume that this is the case. In particular we are assuming that the relaxation of the coefficients a and b occurs on a much shorter timescale than that of the diffusion of temperature or magnetic field, and can therefore be taken to be instantaneous.

We also note that equations (3.59)-(3.60) are the simplest time-dependent equations that could be written down. Equation (3.60) simply states that the total current is equal to the superconducting current plus the normal current. Furthermore, having added a time derivative to equation (3.14), we must also add a term proportional to $\Psi\Phi$ in order to preserve the gauge invariance of the equations. Just as ∇ was changed to $\nabla - (i/\xi\lambda)\boldsymbol{A}$, so must $\partial/\partial t$ be changed to $\partial/\partial t + (i/\xi\lambda)\Phi$.

Because the gauge invariance has been preserved, if we write $\Psi = fe^{i\chi}$, with f real, as before, we see that the equations are invariant under transformations of the type

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla\omega, \quad \Phi \to \Phi - \frac{\partial \omega}{\partial t}, \quad \chi \to \chi + \frac{\omega}{\xi\lambda},$$

where ω is an arbitrary function. Hence, if we write, in Ω,

$$\boldsymbol{Q} = \boldsymbol{A} - \xi\lambda\nabla\chi, \tag{3.81}$$

$$\Theta = \Phi + \xi\lambda\frac{\partial\chi}{\partial t}, \tag{3.82}$$

then Θ, $\boldsymbol{Q}$ are gauge invariant, and by defining Θ and $\boldsymbol{Q}$ to be suitable gauges in the external region we have the following alternative statements of the evolution problem:

Isothermal conditions

$$-\alpha\xi^2\frac{\partial f}{\partial t} + \xi^2\nabla^2 f = f^3 - f + \frac{f\,|\boldsymbol{Q}|^2}{\lambda^2}, \quad \text{in } \Omega, \tag{3.83}$$

$$\alpha f^2\Theta + \text{div}\left(f^2\boldsymbol{Q}\right) = 0, \quad \text{in } \Omega, \tag{3.84}$$

$$-\lambda^2(\text{curl})^2\boldsymbol{Q} = \lambda^2\left(\frac{\partial\boldsymbol{Q}}{\partial t} + \nabla\Theta\right) + f^2\boldsymbol{Q}, \quad \text{in } \Omega, \tag{3.85}$$

$$(\text{curl})^2\boldsymbol{Q} = -\varsigma_e\left(\frac{\partial\boldsymbol{Q}}{\partial t} + \nabla\Theta\right), \quad \text{outside } \Omega, \tag{3.86}$$

$$\frac{\partial(\text{div }\boldsymbol{Q})}{\partial t} + \nabla^2\Theta = 0, \quad \text{outside } \Omega, \tag{3.87}$$

with boundary conditions

$$\boldsymbol{n}\cdot\nabla f = -\frac{f}{d}, \quad \text{on } \partial\Omega, \tag{3.88}$$

$$\boldsymbol{n}\cdot f\boldsymbol{Q} = 0, \quad \text{on } \partial\Omega, \tag{3.89}$$

$$[\boldsymbol{n}\wedge\boldsymbol{Q}] = \boldsymbol{0}, \tag{3.90}$$

$$[\boldsymbol{n}\wedge(1/\mu)\text{curl }\boldsymbol{Q}] = \boldsymbol{0}, \tag{3.91}$$

$$\text{curl }\boldsymbol{Q} \to \boldsymbol{H}_0, \quad \text{as } r\to\infty, \tag{3.92}$$

$$[\Theta] = 0, \tag{3.93}$$

$$[\varepsilon\partial\Theta/\partial n] = 0. \tag{3.94}$$

Anisothermal conditions

$$-\alpha\xi^2\frac{\partial f}{\partial t} + \xi^2\nabla^2 f = a(T)f + b(T)f^3 + \frac{f\,|\boldsymbol{Q}|^2}{\lambda^2}, \quad \text{in } \Omega, \tag{3.95}$$

$$\alpha f^2\Theta + \text{div}\left(f^2\boldsymbol{Q}\right) = 0, \quad \text{in } \Omega, \tag{3.96}$$

$$-\lambda^2(\text{curl})^2\boldsymbol{Q} = \lambda^2\left(\frac{\partial\boldsymbol{Q}}{\partial t} + \nabla\Theta\right) + f^2\boldsymbol{Q}, \quad \text{in } \Omega, \tag{3.97}$$

$$(\text{curl})^2\boldsymbol{Q} = -\varsigma_e\left(\frac{\partial\boldsymbol{Q}}{\partial t} + \nabla\Theta\right), \quad \text{outside } \Omega, \tag{3.98}$$

$$\frac{\partial(\text{div }\boldsymbol{Q})}{\partial t} + \nabla^2\Theta = 0, \quad \text{outside } \Omega, \tag{3.99}$$

with boundary conditions

$$\boldsymbol{n}\cdot\nabla f = -\frac{f}{d}, \quad \text{on } \partial\Omega, \tag{3.100}$$

$$\boldsymbol{n}\cdot f\boldsymbol{Q} = 0, \quad \text{on } \partial\Omega, \tag{3.101}$$

$$[\boldsymbol{n}\wedge\boldsymbol{Q}] = \boldsymbol{0}, \tag{3.102}$$

$$[\boldsymbol{n}\wedge(1/\mu)\text{curl }\boldsymbol{Q}] = \boldsymbol{0}, \tag{3.103}$$

$$\text{curl }\boldsymbol{Q} \to \boldsymbol{H}_0, \quad \text{as } r\to\infty, \tag{3.104}$$

$$[\Theta] = 0, \tag{3.105}$$

$$[\varepsilon\partial\Theta/\partial n] = 0, \tag{3.106}$$

and heat balance equation

$$\nabla^2 T = \beta\frac{\partial T}{\partial t} - L(T)\frac{\partial(f^2)}{\partial t} - \gamma\left|\frac{\partial\boldsymbol{Q}}{\partial t} + \nabla\Theta\right|^2. \tag{3.107}$$

In the steady state, $\Theta = 0$, and these equations reduce to the steady state equations (3.32)-(3.39) and (3.40)-(3.47) respectively.

We will see that in certain situations (for example in Chapter 4) the formulation (3.83)-(3.94) is easier to work with, whereas in other situations (for example in Chapter 5) the formulation (3.59)-(3.68) is easier to work with. The former has the advantage of real variables, but the disadvantage that $\boldsymbol{Q}$ may be singular where f is zero. The latter has the freedom of the choice of the gauge of $\boldsymbol{A}$, but we must then work with complex variables.

We note that in the unsteady case the charge density need not vanish. It is given by

$$\text{div }\boldsymbol{E} = -\text{div}\left(\frac{\partial\boldsymbol{Q}}{\partial t} + \nabla\Theta\right) = \frac{1}{\lambda^2}\text{div}\left(f^2\boldsymbol{Q}\right) = -\frac{\alpha f^2\Theta}{\lambda^2}. \tag{3.108}$$

This is really a result of the fact that we are allowing superconducting currents to move (just as a moving charge density is seen as a current density, so can a moving current density, i.e. after a Lorentz transformation, be seen as a charge density). The current, which as mentioned earlier now has both superconducting and normal components, is given by

$$\boldsymbol{j} = -\frac{1}{\lambda^2} f^2 \boldsymbol{Q} + \boldsymbol{E}.$$

3.3 One-dimensional Problem

When we perform a formal asymptotic analysis of the Ginzburg-Landau equations in Chapter 4 we will assume that the solution comprises normal and superconducting domains separated by thin transition layers. We examine here a stationary, planar transition layer by considering the isothermal Ginzburg-Landau equations in one dimension. We will find in Chapter 4 that this is a local model for transition layers in general. In Section 3.3.1 we use the solution to calculate the surface energy of a planar normal/superconducting interface.

We take the field $\boldsymbol{H}$ to be directed along the z-axis and the magnetic vector potential $\boldsymbol{A}$ to be directed along the y-axis. We make the assumption that all functions are dependent on x only. Then $H_3 = dA_2/dx$, or simply $H = dA/dx$. Equation (3.15) now implies $\nabla\chi = 0$, in which case Ψ may be taken to be real. We then have

$$\xi^2 \Psi'' = \Psi^3 - \Psi + \frac{A^2\Psi}{\lambda^2}, \tag{3.109}$$

$$\lambda^2 A'' = \Psi^2 A, \tag{3.110}$$

with the boundary conditions

$$\Psi' = 0, \tag{3.111}$$

$$A' = H_0, \tag{3.112}$$

where $' \equiv d/dx$, and H_0 is the external magnetic field strength. We work on the length scale of the penetration depth by rescaling x and A with λ to obtain

$$\frac{1}{\kappa^2}\Psi'' = \Psi^3 - \Psi + A^2\Psi, \tag{3.113}$$

$$A'' = \Psi^2 A, \tag{3.114}$$

where $\kappa = \lambda/\xi$ is the Ginzburg-Landau parameter. We note that equations (3.113), (3.114) form a Hamiltonian system, with Hamiltonian given by

$$\mathcal{H} = \frac{\Psi^4}{2} - \Psi^2 + A^2\Psi^2 - \frac{(\Psi')^2}{\kappa^2} - (A')^2.$$

Hence

$$\frac{(\Psi')^2}{\kappa^2} + (A')^2 = \frac{\Psi^4}{2} - \Psi^2 + \Psi^2 A^2 + \text{const.} \tag{3.115}$$

In order that we have a local model for the transition region between normal and superconducting parts of a material we need to apply the boundary conditions

$$A \to 0, \quad \Psi \to 1, \qquad \text{as } x \to -\infty, \tag{3.116}$$

$$A' \to H_0, \quad \Psi \to 0, \qquad \text{as } x \to \infty, \tag{3.117}$$

where the field on the normal side of the region is equal to H_0. The equations admit a solution if and only if $H_0 = H_c$. To see this we note that the boundary conditions (3.116) imply that the constant in (3.115) is in this case equal to 1/2. Therefore

$$\frac{(\Psi')^2}{\kappa^2} + (A')^2 = \frac{\left(\Psi^2 - 1\right)^2}{2} + \Psi^2 A^2. \tag{3.118}$$

Hence, as $x \to \infty$, $A' \to 1/\sqrt{2}$, providing Ψ decays sufficiently quickly that $\Psi A \to 0$. Since $H_c = 1/\sqrt{2}$ in these units we see that in order for a normal/superconducting transition layer to exist the limiting value of the field in the normal region as the domain boundary is approached must be equal to H_c. A rigorous demonstration of this result is given in [14], where the existence and uniqueness of the solution when $H_0 = H_c$ is proved, and it is shown that the solution necessarily satisfies

$$0 < \Psi < 1, \quad A > 0, \quad \Psi' < 0, \quad A' > 0,$$

i.e. Ψ and A are monotonic.

We examine the asymptotic behaviour of Ψ and A as $x \to \pm\infty$. These results will be needed in Chapter 4. For the behaviour at $-\infty$ we set $\Psi = 1 + u$ and linearise about the solution $u = 0$, $A = 0$ to obtain

$$\begin{aligned} u'' &= 2\kappa^2 u, \\ A'' &= A. \end{aligned}$$

Hence

$$\begin{aligned}\Psi &\sim 1 + ae^{\kappa\sqrt{2}\,x},\\ A &\sim be^{x},\end{aligned}$$

as $x \to -\infty$. For the behavior as $x \to \infty$ we substitute $A \sim x/\sqrt{2}$ in (3.113) and retain only leading order terms to obtain

$$\Psi'' \sim \frac{\kappa^2 x^2 \Psi}{2}, \qquad \text{as } x \to \infty.$$

We seek a WKB approximation to Ψ as $x \to \infty$. We let $r = \epsilon x$ and let $\epsilon \to 0$ with r order one. We have

$$\epsilon^4 \Psi_{rr} = \frac{\kappa^2 r^2 \Psi}{2}.$$

Seeking an expansion

$$\Psi = \exp\left\{\frac{S_0}{\epsilon^2} + \frac{S_1}{\epsilon} + S_2 + \cdots\right\},$$

we find

$$\left(\frac{dS_0}{dr}\right)^2 = \frac{\kappa^2 r^2}{2}.$$

Hence

$$S_0 = -\frac{\kappa r^2}{2\sqrt{2}},$$

and

$$\Psi \sim d(r)\exp\left\{-\frac{\kappa r^2}{2\sqrt{2}\,\epsilon^2}\right\}, \text{ as } \epsilon \to 0.$$

Therefore

$$\Psi \sim d(x)\exp\left\{-\frac{\kappa x^2}{2\sqrt{2}}\right\}, \text{ as } x \to \infty. \tag{3.119}$$

In particular we see that the decay is sufficiently quick that $\Psi A \to 0$, as $x \to \infty$. We see now by (3.114) that $A'' = O(e^{-Kx^2})$ as $x \to \infty$, for some constant K. Hence we expect

$$A = x/\sqrt{2} + c + O(e^{-Kx^2}), \text{ as } x \to \infty, \tag{3.120}$$

for some constants c, K. Having found the form of A we may now find a more accurate expression for Ψ by substituting (3.119), (3.120) into (3.113) to give

$$\Psi'' = \kappa^2 \Psi\left(\frac{x^2}{2} + \sqrt{2}cx + c^2 - 1\right) + O(e^{-Kx^2}), \qquad \text{as } x \to \infty.$$

With $r = \epsilon x$ as before we have

$$\epsilon^4 \Psi_{rr} = \kappa^2 \Psi \left(\frac{r^2}{2} + \epsilon \sqrt{2} c r + \epsilon^2 (c^2 - 1) \right) + O(e^{-1/\epsilon^2}), \qquad \text{as } \epsilon \to 0.$$

Seeking an expansion

$$\Psi = \exp \left\{ \frac{S_0}{\epsilon^2} + \frac{S_1}{\epsilon} + S_2 + \cdots \right\},$$

we have

$$S_0 = -\frac{\kappa r^2}{2\sqrt{2}},$$

as before. Equating higher powers of ϵ we find

$$\begin{aligned} 2S_0' S_1' &= \kappa^2 c \sqrt{2}\, r, \\ S_0'' + (S_1')^2 + 2S_0' S_2' &= \kappa^2 (c^2 - 1). \end{aligned}$$

Hence

$$\begin{aligned} S_1 &= -\kappa c r, \\ S_2 &= \frac{1}{\sqrt{2}} \left(\kappa - \frac{1}{\sqrt{2}} \right) \log r. \end{aligned}$$

Therefore

$$\Psi \sim r^{\frac{1}{\sqrt{2}}\left(\kappa - \frac{1}{\sqrt{2}}\right)} \exp \left\{ -\frac{\kappa r^2}{2\sqrt{2}\epsilon^2} - \frac{\kappa c r}{\epsilon} \right\}, \qquad \text{as } \epsilon \to 0.$$

Therefore

$$\Psi \sim x^{\frac{1}{\sqrt{2}}\left(\kappa - \frac{1}{\sqrt{2}}\right)} \exp \left\{ -\frac{\kappa x^2}{2\sqrt{2}} - \kappa c x \right\}, \qquad \text{as } x \to \infty. \tag{3.121}$$

Finally, we note that when $\kappa = 1/\sqrt{2}$, $S_n = 0$ for all $n \geq 2$, i.e. the correction to (3.121) is a factor of order $1 + e^{-Kx^2}$.

3.3.1 Surface Energy of a Normal/Superconducting Interface

Let us now examine the surface energy associated with a plane boundary between normal and superconducting phases, which is defined in [28] to be the excess of the Gibbs free energy of such a transition region over the Gibbs free energy of the normal or superconducting phases at the critical field. The surface energy σ is therefore given by

$$\begin{aligned}\sigma &= \int_{-\infty}^{\infty} (\mathcal{G}_{sH} - \mathcal{G}_{nH})\, dx, \\ &= \int_{-\infty}^{\infty} \left(\mathcal{F}_{sH} - \mu H(x) H_c - \mathcal{F}_{n0} + \frac{\mu H_c^2}{2} \right) dx.\end{aligned}$$

Writing the free energy densities in terms of Ψ and A, the solution to (3.113)-(3.117), and non-dimensionalising σ with respect to $\mu H_c^2 L/2$ we find

$$\sigma = \lambda \int_{-\infty}^{\infty} \left((1 - \Psi^2)^2 + \frac{2(\Psi')^2}{\kappa^2} + 2\Psi^2 A^2 - 2A'(\sqrt{2} - A') \right) dx.$$

By (3.118) we have

$$\frac{(\Psi')^2}{\kappa^2} + (A')^2 = \frac{(\Psi^2 - 1)^2}{2} + \Psi^2 A^2,$$

which gives

$$\sigma = 4\lambda \int_{-\infty}^{\infty} \left(\frac{(\Psi')^2}{\kappa^2} - A' \left(\frac{1}{\sqrt{2}} - A' \right) \right) dx.$$

Now

$$\begin{aligned}\int_{-\infty}^{X} \left\{ \Psi^2 A^2 - A' \left(1/\sqrt{2} - A' \right) \right\} dx &= \int_{-\infty}^{X} \left\{ A'' A - A'/\sqrt{2} + (A')^2 \right\} dx, \\ &= A(X) \left\{ A'(X) - 1/\sqrt{2} \right\},\end{aligned}$$

by (3.114). Letting $X \to \infty$ we have

$$\int_{-\infty}^{\infty} \left\{ \Psi^2 A^2 - A' \left(1/\sqrt{2} - A' \right) \right\} dx = 0,$$

since $A \sim x/\sqrt{2}$+const.+$O(e^{-Kx^2})$, as $x \to \infty$. Hence we can write

$$\sigma = 4\lambda \int_{-\infty}^{\infty} \left(\frac{(\Psi')^2}{\kappa^2} - \Psi^2 A^2 \right) dx. \qquad (3.122)$$

Ginzburg & Landau use an approximate solution of (3.113)-(3.117) to approximate σ for small κ. They find

$$\sigma \approx \frac{4\sqrt{2}}{3}\xi \approx 1.89\xi, \text{ for } \sqrt{\kappa} \ll 1,$$

with the main contribution coming from near $x = -\infty$. The corresponding result for large κ is given in [60, p.116] as

$$\sigma \approx -\frac{8(\sqrt{2} - 1)}{3}\lambda \approx -1.104\lambda, \text{ for } \sqrt{\kappa} \gg 1.$$

We see that the surface energy is negative for large κ. Ginzburg and Landau claim on the basis of numerical integration that $\sigma = 0$ when $\kappa = 1/\sqrt{2}$. We now prove

Proposition 1 *For $\kappa < 1/\sqrt{2}$, $\kappa = 1/\sqrt{2}$, $\kappa > 1/\sqrt{2}$ we have respectively $\sigma > 0$, $\sigma = 0$, $\sigma < 0$.*

Proof

We consider the case $\kappa < 1/\sqrt{2}$. Define functions F and G by

$$F(x) = \Psi^2 - 1 + \sqrt{2}A', \tag{3.123}$$

$$G(x) = \kappa^{-1}\Psi' + \Psi A. \tag{3.124}$$

We aim to show that $F < 0$, $G < 0$. Differentiating (3.123), (3.124) we find

$$\begin{aligned} F' &= 2\Psi'\Psi + \sqrt{2}A'', \\ &= 2\Psi'\Psi + \sqrt{2}\Psi^2 A, \\ &= \sqrt{2}\Psi\left(\sqrt{2}\Psi' + \Psi A\right), \qquad (3.125) \\ G' &= \kappa^{-1}\Psi'' + \Psi' A + \Psi A', \\ &= \kappa\Psi\left(\Psi^2 - 1 + A^2\right) + \Psi' A + \Psi A', \\ &= \kappa\Psi\left(\Psi^2 - 1 + \kappa^{-1}A'\right) + \kappa A\left(\kappa^{-1}\Psi' + \Psi A\right), \qquad (3.126) \\ &= \kappa\Psi\left(\Psi^2 - 1 + \kappa^{-1}A'\right) + \kappa AG. \qquad (3.127) \end{aligned}$$

Now

$$\kappa < 1/\sqrt{2} \Rightarrow \kappa^{-1} > \sqrt{2} \Rightarrow \kappa^{-1}\Psi' < \sqrt{2}\,\Psi',$$

since $\Psi' < 0$. Hence

$$F' > \sqrt{2}\Psi G, \tag{3.128}$$

$$G' > \kappa\Psi F + \kappa AG, \tag{3.129}$$

since $\Psi > 0$. Also, (3.118) implies

$$2(\kappa^{-1}\Psi' + \Psi A)(\kappa^{-1}\Psi' - \Psi A) = (\Psi^2 - 1 + \sqrt{2}\,A')(\Psi^2 - 1 - \sqrt{2}\,A'),$$

i.e.

$$2G(\kappa^{-1}\Psi' - \Psi A) = F(\Psi^2 - 1 - \sqrt{2}\,A'). \tag{3.130}$$

We have that $\Psi' < 0$, $0 < \Psi < 1$, $A' > 0$. Hence

$$\begin{aligned} F > 0 &\Leftrightarrow G > 0, \\ F = 0 &\Leftrightarrow G = 0, \\ F < 0 &\Leftrightarrow G < 0. \end{aligned}$$

Suppose now, for a contradiction, that there is a point x_0 such that $G(x_0) \geq 0$. Then by (3.128), (3.129), and (3.130) we have $F(x_0) \geq 0$, $F'(x_0) > 0$, $G'(x_0) > 0$. Suppose now that there is a first point x_1 greater than x_0 such that $F'(x_1) = 0$. Then (3.128) implies $G(x_1) < 0$, whence (3.130) implies $F(x_1) < 0$. A contradiction is now easy to obtain. $F'(x_2) > 0$ implies there exists x_2 such that $F(x_2) > F(x_0) \geq 0$ and $x_0 < x_2 < x_1$. We now have $F(x_2) > F(x_0)$, $F(x_1) < 0 \leq F(x_0)$. Hence by the Intermediate Value Theorem there is a point x_3 such that $F(x_3) = F(x_0)$ and $x_2 < x_3 < x_1$. Now by Rolle's Theorem there is a point x_4 such that $F'(x_4) = 0$ and $x_0 < x_4 < x_3 < x_1$ which contradicts the minimality of x_1. Therefore there is no such point x_1 and we have $F'(x) > 0$ for all $x > x_0$. Hence $F(x) > F(x_0) \geq 0$ for all $x > x_0$. This contradicts the fact that $F(x) \to 0$ as $x \to \infty$, since $\Psi \to 0$ and $A' \to 1/\sqrt{2}$. Hence there does not exist x_0 such that $G(x_0) \geq 0$. Therefore

$$\begin{aligned} G(x) &< 0, \text{ for all } x, \\ F(x) &< 0, \text{ for all } x. \end{aligned}$$

Now

$$\begin{aligned} \sigma &= 4\lambda \int_{-\infty}^{\infty} \left\{ \kappa^{-2}(\Psi')^2 - \Psi^2 A^2 \right\} dx, \\ &= 4\lambda \int_{-\infty}^{\infty} \left\{ (\kappa^{-1}\Psi' + \Psi A)(\kappa^{-1}\Psi' - \Psi A) \right\} dx, \\ &= 4\lambda \int_{-\infty}^{\infty} \left\{ G(\kappa^{-1}\Psi' - \Psi A) \right\} dx. \end{aligned}$$

Since $G < 0$, $\Psi' < 0$, $\Psi > 0$, $A > 0$ we therefore have $\sigma > 0$ as required.

The case $\kappa > 1/\sqrt{2}$ is exactly similar. For $\kappa = 1/\sqrt{2}$ a similar proof shows that $F \equiv 0$, $G \equiv 0$, and so $\sigma = 0$.

Because of the above result we make the following mathematical definition.

Definition *Materials with* $\kappa < 1/\sqrt{2}$ *will be known as Type I superconductors. Materials with* $\kappa > 1/\sqrt{2}$ *will be known as Type II superconductors.*

To avoid ambiguity here κ refers to the isothermal parameter, although as mentioned previously when we linearise the equations in T the two parameters are equivalent.

At this point, having defined Type I and Type II superconductors mathemtically in terms of the Ginzburg-Landau parameter κ, it is convenient to introduce the following diagram of the response of a bulk superconductor (i.e. neglecting surface effects) in an applied magnetic field H_0 (Fig. 3.3).

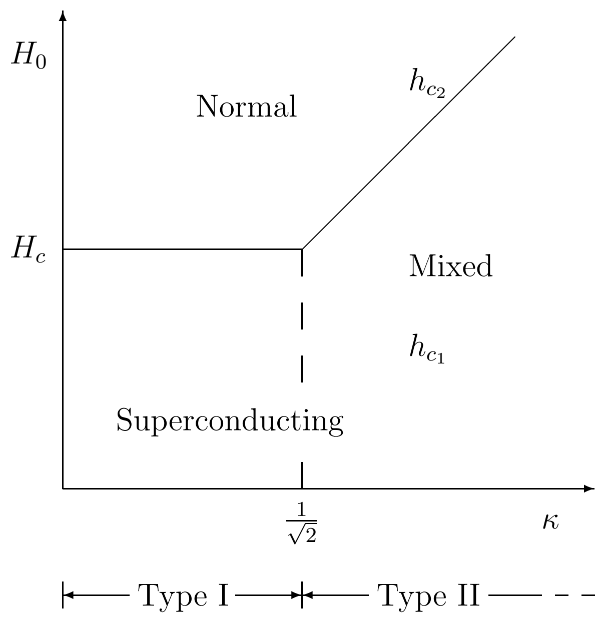

Figure 3.3: Response of a bulk superconductor as a function of the applied magnetic field H_0 and the Ginzburg-Landau parameter κ.

As the external magnetic field is raised for a Type I superconductor there is a transition from superconducting to normal at $H_0 = H_c$. For a Type II superconductor, however, there is first a transition to a mixed state when H_0 reaches a lower critical value h_{c_1}. The mixed state does not become fully normal until the field reaches the upper critical value h_{c_2}. In the next chapter, and in Chapter 7, we will attempt to justify this diagram, and to calculate the values of the critical fields h_{c_1} and h_{c_2}.

Finally we note that the result $F = G = 0$ above allows us to solve (3.113)-(3.117) explicitly in the case when $\kappa = 1/\sqrt{2}$. We then have

$$\sqrt{2}\,A' = 1 - \Psi^2, \tag{3.131}$$

$$\sqrt{2}\,\Psi' = -\Psi A. \tag{3.132}$$

Hence

$$A\frac{dA}{d\Psi} = \frac{\Psi^2 - 1}{\Psi},$$

and

$$A^2 = -\log\Psi^2 + \Psi^2 - C,$$

where C is constant. Therefore

$$2(\Psi')^2 = \Psi^4 - C\Psi^2 - \Psi^2\log\Psi^2,$$

and

$$x = \int_{\Psi(0)}^{\Psi} \frac{-\sqrt{2}\,d\Psi}{\Psi(\Psi^2 - C - \log\Psi^2)^{1/2}},$$

or

$$x = \int_{\Psi(0)^2}^{\Psi^2} \frac{-d(\Psi^2)}{\sqrt{2}\,\Psi^2(\Psi^2 - C - \log\Psi^2)^{1/2}}. \tag{3.133}$$

The boundary conditions as $x \to -\infty$ imply $C = 1$. Then

$$A = (\Psi^2 - 1 - \log\Psi^2)^{1/2},$$

and

$$H = A' = \frac{1 - \Psi^2}{\sqrt{2}}.$$

Ψ and H are shown in Fig. 3.4

We will return to the solutions corresponding to different values of C in Chapter 5. In Appendix B we will consider further the case $\kappa = 1/\sqrt{2}$, and show that a reduction of the Ginzburg-Landau equations in this case is also possible in two dimensions.

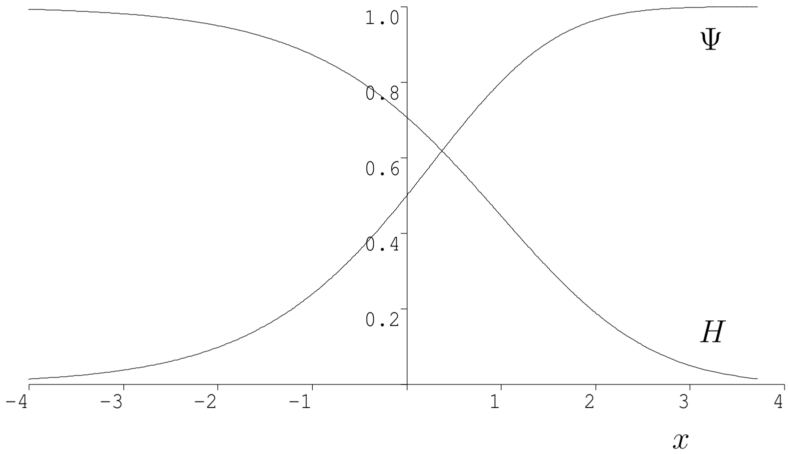

Figure 3.4: Variation of Ψ and H in a normal/superconducting transition region for $\kappa = 1/\sqrt{2}$.

Chapter 4

Asymptotic Solution of the Ginzburg-Landau model: Reduction to a Free-boundary Model

We consider in this chapter asymptotic solutions of the Ginzburg-Landau equations as λ and $\xi \to 0$. We assume that the material comprises normal and superconducting domains separated by thin transition layers. A local analysis of such a transition layer will reveal that, as claimed in the previous chapter, f and $|\boldsymbol{Q}|$ satisfy the stationary, planar transition layer equations (3.113)-(3.117) to leading order. The leading order outer solution will be found to satisfy the vectorial Stefan problem of Chapter 2.

Consideration of the first order terms in the outer solution will reveal the emergence of 'surface tension' and 'kinetic undercooling' terms, as in the modified Stefan model (1.17).

The matching conditions we use throughout the chapter between the inner and outer expansions are based on the principle [65]

$$(m \text{ term inner})(n \text{ term outer}) = (n \text{ term outer})(m \text{ term inner}),$$

and they are derived in Appendix A.

4.1 Asymptotic Solution of the Phase Field Model as a Paradigm for the Ginzburg-Landau Equations

We demonstrate here the asymptotic reduction of the phase field equations (1.19), (1.20) to the modified Stefan model (1.14)-(1.17) as a paradigm for the Ginzburg-Landau equations. The following analysis follows [11].

For simplicity we consider only the case of circular symmetry in two dimension, which has the advantage of allowing us to use familiar polar co-ordinates, while retaining all the essential ingredients of the general case. With $F = F(r,t)$, $T = T(r,t)$, equations (1.19), (1.20) become

$$\frac{\partial T}{\partial t} + \frac{L}{2}\frac{\partial F}{\partial t} = K\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right), \tag{4.1}$$

$$\alpha\xi^2\frac{\partial F}{\partial t} = \xi^2\left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r}\right) + \frac{1}{2a}(F - F^3) + 2T. \tag{4.2}$$

We let $c = \xi a^{-1/2}$, $\epsilon = \xi^2$, and consider the formal asymptotic limit ϵ, ξ, $a \to 0$, with α fixed. Writing the equations in terms of ϵ we have

$$\frac{\partial T}{\partial t} + \frac{L}{2}\frac{\partial F}{\partial t} = K\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right), \tag{4.3}$$

$$\alpha\epsilon^2\frac{\partial F}{\partial t} = \epsilon^2\left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r}\right) + \frac{c^2}{2}(F - F^3) + 2\epsilon T. \tag{4.4}$$

We define Γ to be the curve $F = 0$, and to be given by $r = R(t)$, i.e. $R(t)$ is such that $F(R(t),t) = 0$. At leading order Γ will be the 'interface' of the outer solution.

Outer regions

We denote the outer solution by F_o, T_o. Away from the interface we expand F_o, and T_o in powers of ϵ to obtain the outer expansions as

$$F_o(r,t,\epsilon) = F_o^{(0)}(r,t) + \epsilon F_o^{(1)}(r,t) + \cdots, \tag{4.5}$$

$$T_o(r,t,\epsilon) = T_o^{(0)}(r,t) + \epsilon T_o^{(1)}(r,t) + \cdots. \tag{4.6}$$

We also expand R in powers of ϵ:

$$R(t,\epsilon) = R^{(0)}(t) + \epsilon R^{(1)}(t) + \cdots. \tag{4.7}$$

Substituting the expansions (4.5)-(4.7) into equations (4.3), (4.4) and equations powers of ϵ yields at leading order

$$\frac{\partial T_o^{(0)}}{\partial t} + \frac{L}{2}\frac{\partial F_o^{(0)}}{\partial t} = K\left(\frac{\partial^2 T_o^{(0)}}{\partial r^2} + \frac{1}{r}\frac{\partial T_o^{(0)}}{\partial r}\right), \tag{4.8}$$

$$F_o^{(0)} - (F_o^{(0)})^3 = 0. \tag{4.9}$$

Hence $F_o^{(0)} = 0, \pm 1$. In the solid region we have $F_o^{(0)} = -1$, in the liquid region we have $F_o^{(0)} = 1$, and in both cases equation (4.8) reduces to the heat equation. Thus we have accomplished the first of our objectives.

Inner region

We denote the inner solution by F_i, T_i. We 'stretch out' the variable r near the interface by introducing local co-ordinates defined by

$$r - R(t) = \epsilon\rho, \qquad t = \tau,$$

so that

$$\frac{\partial}{\partial r} = \frac{1}{\epsilon}\frac{\partial}{\partial \rho}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \frac{1}{\epsilon}\frac{dR}{dt}\frac{\partial}{\partial \rho}.$$

In terms of the inner variables (ρ, τ) equations (4.3), (4.4) become

$$\frac{\partial T_i}{\partial \tau} - \frac{1}{\epsilon}\frac{dR}{dt}\frac{\partial T_i}{\partial \rho} + \frac{L}{2}\frac{\partial F_i}{\partial \tau} - \frac{L}{2\epsilon}\frac{dR}{dt}\frac{\partial F_i}{\partial \rho} = K\left(\frac{1}{\epsilon^2}\frac{\partial^2 T_i}{\partial \rho^2} + \frac{1}{\epsilon(R+\epsilon\rho)}\frac{\partial T_i}{\partial \rho}\right), \tag{4.10}$$

$$\alpha\epsilon^2\frac{\partial F_i}{\partial \tau} - \alpha\epsilon\frac{dR}{dt}\frac{\partial F_i}{\partial \rho} = \frac{\partial^2 F_i}{\partial \rho^2} + \frac{\epsilon}{(R+\epsilon\rho)}\frac{\partial F_i}{\partial \rho} + \frac{c^2}{2}(F_i - F_i^3) + 2\epsilon T_i. \tag{4.11}$$

We expand T_i and F_i in powers of ϵ as before to obtain the inner expansions as:

$$F_i(\rho, \tau, \epsilon) = F_i^{(0)}(\rho, \tau) + \epsilon F_i^{(1)}(\rho, \tau) + \cdots, \tag{4.12}$$

$$T_i(\rho, \tau, \epsilon) = T_i^{(0)}(\rho, \tau) + \epsilon T_i^{(1)}(\rho, \tau) + \cdots. \tag{4.13}$$

Substituting the expansions (4.12), (4.13) and (4.7) into equations (4.10), (4.11) and equating powers of ϵ yields at leading order

$$\frac{\partial^2 T_i^{(0)}}{\partial \rho^2} = 0, \tag{4.14}$$

$$\frac{\partial^2 F_i^{(0)}}{\partial \rho^2} + \frac{1}{2}\left(F_i^{(0)} - (F_i^{(0)})^3\right) = 0. \tag{4.15}$$

Hence

$$T_i^{(0)} = A(\tau)\rho + B(\tau).$$

The matching condition (A.15) derived in Appendix A implies

$$\lim_{\rho\to\pm\infty} T_i^{(0)}(\rho,\tau) = T_o^{(0)}(R_\pm^{(0)},t),$$

where $R_\pm$ denotes the interface approached from $r > R$ and $r < R$ respectively. This can only be satisfied if $A = 0$; otherwise $T_o^{(0)}$ would be unbounded at in interface. Hence $T_i^{(0)} = B(\tau)$, and

$$T_o^{(0)}(R_+^{(0)},t) = T_o^{(0)}(R_-^{(0)},t),$$

i.e., the outer temperature is continuous at the interface.

Using the matching condition (A.15) again we have

$$\lim_{\rho\to\pm\infty} F_i^{(0)}(\rho,\tau) = F_o^{(0)}(R_\pm^{(0)},t) = \pm 1,$$

where we have assumed that the liquid region lies in $r > R$. By definition of the interface $F_i(0,\tau) = 0$ and hence $F_i^{(0)}(0,\tau) = 0$. Therefore $F_i^{(0)}$ is given by

$$F_i^{(0)} = \tanh(c\rho/2). \tag{4.16}$$

Equating powers of ϵ in equations (4.10), (4.11) yields

$$K\frac{\partial^2 T_i^{(1)}}{\partial\rho^2} = \frac{L}{2}\frac{dR^{(0)}}{dt}\frac{dF_i^{(0)}}{d\rho}, \tag{4.17}$$

$$\begin{aligned} \mathcal{L}\,F_i^{(1)} &\equiv \frac{\partial^2 F_i^{(1)}}{\partial\rho^2} + \frac{1}{2}\left(F_i^{(1)} - 3(F_i^{(0)})^2 F_i^{(1)}\right) \\ &= -\alpha\frac{dR^{(0)}}{dt}\frac{dF_i^{(0)}}{d\rho} - \frac{1}{R^{(0)}}\frac{dF_i^{(0)}}{d\rho} - 2T_i^{(1)}. \end{aligned} \tag{4.18}$$

Integrating (4.17) over $(-\infty,\infty)$ we have

$$K\left[\frac{\partial T_i^{(1)}}{\partial\rho}\right]_{-\infty}^{\infty} = \frac{L}{2}\frac{dR}{dt}\left[F_i^{(0)}\right]_{-\infty}^{\infty} = -L\frac{dR}{dt}. \tag{4.19}$$

The matching condition (A.17) implies

$$\lim_{\rho\to\pm\infty}\frac{\partial T_i^{(1)}}{\partial\rho}(\rho,\tau) = \frac{\partial T_o^{(0)}}{\partial r}(R_\pm^{(0)},t).$$

Hence (4.19) implies

$$K\left[\frac{\partial T_o^{(0)}}{\partial \rho}\right]_-^+ = -L\frac{dR}{dt}. \tag{4.20}$$

We now evaluate the temperature at the interface using equation (4.18). We note that $dF_i^{(0)}/d\rho$ is a solution of $\mathcal{L}\, dF_i^{(0)}/d\rho = 0$ (with $dF_i^{(0)}/d\rho$ and $d^2F_i^{(0)}/d\rho^2$ vanishing as $\rho \to \pm\infty$). We therefore have a solution for $F_i^{(1)}$ if and only if an appropriate solvability condition is satisfied, namely that the right-hand side of (4.18) is orthogonal to $dF_i^{(0)}/d\rho$. We multiply by $dF_i^{(0)}/d\rho$ and integrate over $(-\infty, \infty)$ to obtain

$$\int_{-\infty}^{\infty} \frac{dF_i^{(0)}}{d\rho}\left[-2T_i^{(0)} - \alpha\frac{dR^{(0)}}{dt}\frac{dF_i^{(0)}}{d\rho} - \frac{1}{R^{(0)}}\frac{dF_i^{(0)}}{d\rho}\right] d\rho = 0.$$

Hence

$$2\left[T_i^{(0)}F_i^{(0)}\right]_{-\infty}^{\infty} = 4T_o^{(0)}(R^{(0)},t) = -\left(\alpha\frac{dR^{(0)}}{dt} + \frac{1}{R^{(0)}}\right)\sigma^{(0)}, \tag{4.21}$$

where

$$\sigma^{(0)} = \int_{-\infty}^{\infty}\left(\frac{dF_i^{(0)}}{d\rho}\right)^2 d\rho.$$

It is shown in [11] that $\sigma^{(0)}$ is the leading order approximation to the surface energy.

Thus we have retrieved the modified Stefan model as the leading order approximation to the Phase Field model, with this scaling of the parameters. Using other scalings we can retrieve the classical Stefan model, or a modified Stefan model with surface tension effects included but with no kinetic undercooling term [11]. (We can even retrieve the so-called Hele-Shaw problem.)

4.2 Asymptotic Solution of the Ginzburg-Landau Equations under Isothermal Conditions

We now proceed to try to relate the model (3.83)-(3.85) to to the free boundary models of Chapter 2 in a similar way. We have the following result.

Proposition 2 *In the formal asymptotic limit λ, $\xi \to 0$, with α and $\kappa = \lambda/\xi$ fixed one obtains the vectorial Stefan model (2.3)-(2.8) at leading order.*

As mentioned above, we make the assumption that the material comprises normal and superconducting regions separated by thin transition layers. A complete determination of the solution will involve initial and fixed boundary conditions. However, they will be left unspecified as our primary interest is rather the free boundary conditions.

The Ginzburg-Landau equations (3.83)-(3.85), together with the relation (3.1), are

$$-\frac{\alpha\lambda^2}{\kappa^2}\frac{\partial f}{\partial t} + \frac{\lambda^2}{\kappa^2}\nabla^2 f = f^3 - f + \frac{f\,|\boldsymbol{Q}|^2}{\lambda^2}, \tag{4.22}$$

$$\alpha f^2\Theta + \mathrm{div}(f^2\boldsymbol{Q}) = 0, \tag{4.23}$$

$$-\lambda^2 \mathrm{curl}\ \boldsymbol{H} = \lambda^2\frac{\partial \boldsymbol{Q}}{\partial t} + \lambda^2\nabla\Theta + f^2\boldsymbol{Q}, \tag{4.24}$$

$$\boldsymbol{H} = \mathrm{curl}\ \boldsymbol{Q}. \tag{4.25}$$

We define the $\Gamma(t)$ by

$$\Gamma(t) = \{\boldsymbol{r} \text{ such that } f(\boldsymbol{r},t) = \eta\}, \tag{4.26}$$

where η is to be specified later, but certainly $0 < \eta < 1$. At leading order Γ will be the 'interface' of the outer solution. The choice of η will not affect the interface conditions at leading order, and so any value of η will serve to prove the proposition. However, when we go on to consider the first order correction to the leading order solution the choice of η becomes relevant, and we wish to choose η to make the calculations as simple as possible. We note that there is no obvious choice for η as in the phase field, when symmetry suggests choosing $\eta = 0$. The natural choice for η is (by definition) the one that leads to the simplest first order problem. Such a situation also arises when considering shock waves (see e.g. [43]).

Outer Expansions

Away from the transition region we formally expand all functions in powers of λ to obtain the outer expansions, denoted by the subscript o, as

$$f_o(\boldsymbol{r},t,\lambda) = f_o^{(0)}(\boldsymbol{r},t) + \lambda f_o^{(1)}(\boldsymbol{r},t) + \cdots, \tag{4.27}$$

$$\Theta_o(\boldsymbol{r},t,\lambda) = \Theta_o^{(0)}(\boldsymbol{r},t) + \lambda\Theta_o^{(1)}(\boldsymbol{r},t) + \cdots, \tag{4.28}$$

$$\boldsymbol{Q}_o(\boldsymbol{r},t,\lambda) = \boldsymbol{Q}_o^{(0)}(\boldsymbol{r},t) + \lambda\boldsymbol{Q}_o^{(1)}(\boldsymbol{r},t) + \cdots, \tag{4.29}$$

$$\begin{aligned} \boldsymbol{H}_o(\boldsymbol{r},t,\lambda) &= \boldsymbol{H}_o^{(0)}(\boldsymbol{r},t) + \lambda \boldsymbol{H}_o^{(1)}(\boldsymbol{r},t) + \cdots, & (4.30) \\ \Gamma(t,\lambda) &= \Gamma^{(0)}(t) + \lambda \Gamma^{(1)}(t) + \cdots. & (4.31) \end{aligned}$$

We note that the expansions (4.27)-(4.30) may be discontinuous across $\Gamma^{(0)}(t)$, but will be smooth otherwise. Substituting (4.27)-(4.30) into (4.22)-(4.25) and equating powers of λ yields at leading order

$$\begin{aligned} f_o^{(0)} \,|\boldsymbol{Q}_o^{(0)}|^2 &= 0, & (4.32) \\ \alpha (f_o^{(0)})^2 \Theta_o^{(0)} + \operatorname{div}((f_o^{(0)})^2 \boldsymbol{Q}_o^{(0)}) &= 0, & (4.33) \\ (f_o^{(0)})^2 \boldsymbol{Q}_o^{(0)} &= \boldsymbol{0}, & (4.34) \\ \boldsymbol{H}_o^{(0)} &= \operatorname{curl} \boldsymbol{Q}_o^{(0)}. & (4.35) \end{aligned}$$

We see by (4.32) that either $f_o^{(0)} = 0$, or $\boldsymbol{Q}_o^{(0)} = \boldsymbol{0}$, corresponding to normal and superconducting regions respectively. We consider these cases separately.

Normal region

With $f_o^{(0)} \equiv 0$, $\boldsymbol{Q}_o^{(0)} \not\equiv \boldsymbol{0}$ we equate powers of λ at the next order in (4.22)-(4.24) to give

$$\begin{aligned} f_o^{(1)} \,|\boldsymbol{Q}_o^{(0)}|^2 &= 0, & (4.36) \\ \alpha (f_o^{(1)})^2 \Theta_o^{(0)} + \operatorname{div}((f_o^{(1)})^2 \boldsymbol{Q}_o^{(0)}) &= 0, & (4.37) \\ -\operatorname{curl} \boldsymbol{H}_o^{(0)} &= \frac{\partial \boldsymbol{Q}_o^{(0)}}{\partial t} + \nabla \Theta_o^{(0)} + (f_o^{(1)})^2 \boldsymbol{Q}_o^{(0)}. & (4.38) \end{aligned}$$

By (4.36) we have that $f_o^{(1)} \equiv 0$. Taking the curl of equation (4.38) and using equation (4.35) we have

$$-(\operatorname{curl})^2 \boldsymbol{H}_o^{(0)} = \frac{\partial \boldsymbol{H}_o^{(0)}}{\partial t}. \tag{4.39}$$

Noting that

$$\operatorname{div} \boldsymbol{H}_o^{(0)} = \operatorname{div} (\operatorname{curl} \boldsymbol{Q}_o^{(0)}) = 0,$$

we see that

$$\nabla^2 \boldsymbol{H}_o^{(0)} = \frac{\partial \boldsymbol{H}_o^{(0)}}{\partial t}. \tag{4.40}$$

At the next order in equations (4.22)-(4.25) we find

$$f_o^{(2)} \,|\boldsymbol{Q}_o^{(0)}|^2 = 0, \tag{4.41}$$

$$\alpha(f_o^{(2)})^2\Theta_o^{(0)} + \text{div}\left((f_o^{(2)})^2\boldsymbol{Q}_o^{(0)}\right) = 0, \tag{4.42}$$

$$-\text{curl}\ \boldsymbol{H}_o^{(1)} = \frac{\partial \boldsymbol{Q}_o^{(1)}}{\partial t} + \nabla\Theta_o^{(1)}, \tag{4.43}$$

$$\boldsymbol{H}_o^{(1)} = \text{curl}\ \boldsymbol{Q}_o^{(1)}. \tag{4.44}$$

As before we take the curl of equation (4.43) to give

$$\nabla^2\boldsymbol{H}_o^{(1)} = \frac{\partial \boldsymbol{H}_o^{(1)}}{\partial t}. \tag{4.45}$$

Thus we have

$$\nabla^2\boldsymbol{H}_o = \frac{\partial \boldsymbol{H}_o}{\partial t} + O(\lambda^2),\ \text{in the normal region.} \tag{4.46}$$

In fact, if we continue in this way, we find

$$\nabla^2\boldsymbol{H}_o = \frac{\partial \boldsymbol{H}_o}{\partial t} + O(\lambda^n),\ \text{in the normal region,} \tag{4.47}$$

for any n.

Superconducting region

With $f_o^{(0)} \not\equiv 0$, we have

$$\boldsymbol{Q}_o^{(0)} \equiv \boldsymbol{0},\ \ \boldsymbol{H}_o^{(0)} \equiv \boldsymbol{0},\ \ \Theta_o^{(0)} \equiv 0. \tag{4.48}$$

Equating powers of λ at the next order in each equation we have

$$0 = (f_o^{(0)})^3 - f_o^{(0)} + f_o^{(0)} \,|\boldsymbol{Q}_o^{(1)}|^2, \tag{4.49}$$

$$\alpha(f_o^{(0)})^2\Theta_o^{(1)} + \text{div}\left((f_o^{(0)})^2\boldsymbol{Q}_o^{(1)}\right) = 0, \tag{4.50}$$

$$\boldsymbol{0} = (f_o^{(0)})^2\boldsymbol{Q}_o^{(1)}, \tag{4.51}$$

$$\boldsymbol{H}_o^{(1)} = \text{curl}\ \boldsymbol{Q}_o^{(1)}. \tag{4.52}$$

Therefore

$$\boldsymbol{Q}_o^{(1)} \equiv \boldsymbol{0},\ \ \boldsymbol{H}_o^{(1)} \equiv \boldsymbol{0},\ \ \Theta_o^{(1)} \equiv 0,\ \ f_o^{(0)} \equiv 1. \tag{4.53}$$

Hence we have

$$\boldsymbol{H} = O(\lambda^2),\ \text{in the superconducting region.} \tag{4.54}$$

In fact, if we continue in this way, we find

$$\boldsymbol{H} = O(\lambda^n), \text{ in the superconducting region,} \tag{4.55}$$

for any n.

Inner Expansions

Let $\Gamma(t)$ be given by the surface

$$\boldsymbol{r} = (x, y, z) = \boldsymbol{R}\left(s_1(x, y, z), s_2(x, y, z), t\right),$$

i.e. $\boldsymbol{R}(s_1, s_2, t)$ is such that

$$f(\boldsymbol{R}(s_1, s_2, t), t) = \eta.$$

We parametrise the surface $\boldsymbol{R}$ such that s_1 and s_2 are the principal directions. We use the standard terminology for the first and second fundamental forms:

$$\begin{array}{lll} E = \boldsymbol{R}_1 \cdot \boldsymbol{R}_1 & F = \boldsymbol{R}_1 \cdot \boldsymbol{R}_2 & G = \boldsymbol{R}_2 \cdot \boldsymbol{R}_2 \\ L = \boldsymbol{R}_{11} \cdot \boldsymbol{n} & M = \boldsymbol{R}_{12} \cdot \boldsymbol{n} & N = \boldsymbol{R}_{22} \cdot \boldsymbol{n} \end{array}$$

where

$$\boldsymbol{n} = \frac{\boldsymbol{R}_1 \wedge \boldsymbol{R}_2}{|\boldsymbol{R}_1 \wedge \boldsymbol{R}_2|} = \frac{\boldsymbol{R}_1 \wedge \boldsymbol{R}_2}{(EG - F^2)^{1/2}},$$

is the unit normal, which we take to point away from the superconducting region, and

$$\boldsymbol{R}_1 \equiv \frac{\partial \boldsymbol{R}}{\partial s_1}, \boldsymbol{R}_2 \equiv \frac{\partial \boldsymbol{R}}{\partial s_2}, \text{etc.}$$

Since we chose s_1 and s_2 to be the principal directions, we have that $F = 0$, and $M = 0$. We define new variables ρ and τ by the equations

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{R}(s_1, s_2, t) + \lambda \rho \boldsymbol{n}, \\ t &= \tau. \end{aligned}$$

We then have a new local co-ordinate system (s_1, s_2, ρ, τ). We show that s_1, s_2, ρ are orthogonal co-ordinates. Using the above notation with the subscript 3 denoting $\partial/\partial\rho$ we have

$$\begin{aligned} \boldsymbol{r}_1 &= \boldsymbol{R}_1 + \lambda \rho \boldsymbol{n}_1, \\ \boldsymbol{r}_2 &= \boldsymbol{R}_2 + \lambda \rho \boldsymbol{n}_2, \\ \boldsymbol{r}_3 &= \lambda \boldsymbol{n}. \end{aligned}$$

Now $\boldsymbol{n}\cdot\boldsymbol{n} = 1$. Hence $\boldsymbol{n}_1\cdot\boldsymbol{n} = \boldsymbol{n}_2\cdot\boldsymbol{n} = 0$. Therefore $\boldsymbol{r}_1\cdot\boldsymbol{r}_3 = 0$, and $\boldsymbol{r}_2\cdot\boldsymbol{r}_3 = 0$. Also, since $\boldsymbol{R}_i \cdot \boldsymbol{n} = 0$, $i = 1, 2$, we have

$$\boldsymbol{R}_{ij} \cdot \boldsymbol{n} = -\boldsymbol{r}_i \cdot \boldsymbol{n}_j, \quad i, j = 1, 2.$$

Hence

$$\begin{aligned} \boldsymbol{r}_1 \cdot \boldsymbol{r}_2 &= \boldsymbol{R}_1 \cdot \boldsymbol{R}_2 + \lambda\rho \boldsymbol{R}_1 \cdot \boldsymbol{n}_2 + \lambda\rho \boldsymbol{R}_2 \cdot \boldsymbol{n}_1 + \lambda^2\rho^2 \boldsymbol{n}_1 \cdot \boldsymbol{n}_2, \\ &= F - 2\lambda\rho M + \lambda^2\rho^2 \boldsymbol{n}_1 \cdot \boldsymbol{n}_2, \\ &= \lambda^2\rho^2 \boldsymbol{n}_1 \cdot \boldsymbol{n}_2, \end{aligned}$$

since $F = M = 0$. However, since $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\boldsymbol{n}$ form an orthogonal triad we may write

$$\boldsymbol{n}_1 = a\boldsymbol{R}_1 + b\boldsymbol{R}_2 + c\boldsymbol{n},$$

where

$$\begin{aligned} a &= \frac{\boldsymbol{n}_1 \cdot \boldsymbol{R}_1}{\boldsymbol{R}_1 \cdot \boldsymbol{R}_1} = -\frac{\boldsymbol{n} \cdot \boldsymbol{R}_{11}}{E} = -\frac{L}{E}, \\ b &= \frac{\boldsymbol{n}_1 \cdot \boldsymbol{R}_2}{\boldsymbol{R}_2 \cdot \boldsymbol{R}_2} = -\frac{\boldsymbol{n} \cdot \boldsymbol{R}_{21}}{G} = -\frac{M}{G} = 0, \\ c &= \boldsymbol{n}_1 \cdot \boldsymbol{n} = 0. \end{aligned}$$

Hence $\boldsymbol{n}_1 = -LE^{-1}\boldsymbol{R}_1$. Similarly $\boldsymbol{n}_2 = -NG^{-1}\boldsymbol{R}_2$. Therefore

$$\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = \frac{LN}{EG}(\boldsymbol{R}_1 \cdot \boldsymbol{R}_2) = \frac{LNF}{EG} = 0.$$

Therefore $\boldsymbol{r}_1 \cdot \boldsymbol{r}_2 = 0$ also, and we have that s_1, s_2, ρ form orthogonal curvilinear co-ordinates. We calculate the scaling factors h_1, h_2, h_3 as given below:

$$\begin{aligned} h_1 &= (\boldsymbol{r}_1 \cdot \boldsymbol{r}_1)^{1/2}, \\ &= [(\boldsymbol{R}_1 + \lambda\rho\boldsymbol{n}_1) \cdot (\boldsymbol{R}_1 + \lambda\rho\boldsymbol{n}_1)]^{1/2}, \\ &= \left[(\boldsymbol{R}_1 \cdot \boldsymbol{R}_1) + 2\lambda\rho(\boldsymbol{R}_1 \cdot \boldsymbol{n}_1) + \lambda^2\rho^2(\boldsymbol{n}_1 \cdot \boldsymbol{n}_1)\right]^{1/2}, \\ &= \left[E - 2\lambda\rho L + \lambda^2\rho^2 L^2 E^{-1}\right]^{1/2}, \\ &= E^{1/2}\left[1 - \lambda\rho L E^{-1}\right]. \end{aligned}$$

Similarly

$$\begin{aligned} h_2 &= (\boldsymbol{r}_2 \cdot \boldsymbol{r}_2)^{1/2} = G^{1/2}\left[1 - \lambda\rho N G^{-1}\right], \\ h_3 &= (\boldsymbol{r}_3 \cdot \boldsymbol{r}_3)^{1/2} = \left(\lambda^2(\boldsymbol{n} \cdot \boldsymbol{n})\right)^{1/2} = \lambda. \end{aligned}$$

Let $\tilde{\kappa}_1$, $\tilde{\kappa}_2$ be the principal curvatures, in the s_1, s_2 directions respectively, positive if the centre of curvature lies in the superconducting region. Then, with $F = M = 0$, we have

$$\tilde{\kappa}_1 = -LE^{-1}, \quad \tilde{\kappa}_2 = -NG^{-1}.$$

Hence

$$h_1 = E^{1/2}(1 + \lambda\rho\tilde{\kappa}_1), \tag{4.56}$$

$$h_2 = G^{1/2}(1 + \lambda\rho\tilde{\kappa}_2), \tag{4.57}$$

$$h_3 = \lambda. \tag{4.58}$$

We can now use the general formulae for curl, div, $\boldsymbol{v} \cdot \nabla$ etc. in curvilinear coordinates given in Appendix C. In keeping with the above notation we set

$$\boldsymbol{Q}_i = Q_{i,1}\boldsymbol{e}_1 + Q_{i,2}\boldsymbol{e}_2 + Q_{i,3}\boldsymbol{n}, \tag{4.59}$$

$$\boldsymbol{H}_i = H_{i,1}\boldsymbol{e}_1 + H_{i,2}\boldsymbol{e}_2 + H_{i,3}\boldsymbol{n}, \tag{4.60}$$

where $\boldsymbol{e}_i = \boldsymbol{R}_i / |\boldsymbol{R}_i|$. Noting that $\partial/\partial t$ becomes $\partial/\partial\tau - \boldsymbol{v} \cdot \nabla$ in the new coordinates, where $\boldsymbol{v}$ is the velocity of the boundary, equations (4.22)-(4.25) become

$$\frac{\alpha\lambda v_n}{\kappa^2}\frac{\partial f_i}{\partial\rho} + \frac{1}{\kappa^2}\frac{\partial^2 f_i}{\partial\rho^2} + \frac{\lambda(\tilde{\kappa}_1 + \tilde{\kappa}_2)}{\kappa^2}\frac{\partial f_i}{\partial\rho} + O(\lambda^2) = f_i^3 - f_i + \frac{f_i}{\lambda^2}\left(Q_{i,1}^2 + Q_{i,2}^2 + Q_{i,3}^2\right), \tag{4.61}$$

$$\begin{aligned}\lambda\alpha f_i^2\Theta_i &+ \frac{1}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial\left[G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)f_i^2 Q_{i,1}\right]}{\partial s_1}\right)\\ &+ \frac{1}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial\left[E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)f_i^2 Q_{i,2}\right]}{\partial s_2}\right)\\ &+ \frac{1}{\lambda(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial\left[(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)f_i^2 Q_{i,3}\right]}{\partial\rho}\right) = 0,\end{aligned} \tag{4.62}$$

$$
\begin{aligned}
&\lambda\frac{\partial H_{i,2}}{\partial\rho}+\frac{\lambda^2\tilde{\kappa}_2 H_{i,2}}{1+\lambda\rho\tilde{\kappa}_2}-\frac{\lambda^2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial H_{i,3}}{\partial s_2}= \\
&\quad -v_n\lambda\frac{\partial Q_{i,1}}{\partial\rho}+\lambda^2\frac{\partial Q_{i,1}}{\partial\tau}-\frac{\lambda^2 v_1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial Q_{i,1}}{\partial s_1} \\
&\quad -\frac{\lambda^2 v_2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial Q_{i,1}}{\partial s_2}+\frac{\lambda^2 v_1\tilde{\kappa}_1 Q_{i,3}}{(1+\lambda\rho\tilde{\kappa}_1)} \\
&\quad -\frac{\lambda^2}{2(EG)^{1/2}}\left(\frac{v_2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial G}{\partial s_2}-\frac{v_1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial E}{\partial s_1}\right)Q_{i,2} \\
&\quad +\frac{\lambda^2}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial\Theta_i}{\partial s_1}+f_i^2 Q_{i,1},
\end{aligned}
\tag{4.63}
$$

$$
\begin{aligned}
&-\lambda\frac{\partial H_{i,1}}{\partial\rho}-\frac{\lambda^2\tilde{\kappa}_1 H_{i,1}}{1+\lambda\rho\tilde{\kappa}_1}+\frac{\lambda^2}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial H_{i,3}}{\partial s_1}= \\
&\quad -v_n\lambda\frac{\partial Q_{i,2}}{\partial\rho}+\lambda^2\frac{\partial Q_{i,2}}{\partial\tau}-\frac{\lambda^2 v_1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial Q_{i,2}}{\partial s_1} \\
&\quad -\frac{\lambda^2 v_2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial Q_{i,2}}{\partial s_2}+\frac{\lambda^2 v_2\tilde{\kappa}_2 Q_{i,3}}{(1+\lambda\rho\tilde{\kappa}_2)} \\
&\quad +\frac{\lambda^2}{2(EG)^{1/2}}\left(\frac{v_2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial G}{\partial s_2}-\frac{v_1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial E}{\partial s_1}\right)Q_{i,1} \\
&\quad +\frac{\lambda^2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial\Theta_i}{\partial s_2}+f_i^2 Q_{i,2},
\end{aligned}
\tag{4.64}
$$

$$
\begin{aligned}
&\frac{\lambda^2}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial\left[E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)H_{i,1}\right]}{\partial s_2}\right)- \\
&\quad \frac{\lambda^2}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial\left[G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)H_{i,2}\right]}{\partial s_1}\right)= \\
&\quad -v_n\lambda\frac{\partial Q_{i,3}}{\partial\rho}+\lambda^2\frac{\partial Q_{i,3}}{\partial\tau}-\frac{\lambda^2 v_1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial Q_{i,3}}{\partial s_1}-\frac{\lambda^2 v_2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial Q_{i,3}}{\partial s_2} \\
&\quad -\frac{\lambda^2 v_1\tilde{\kappa}_1 Q_{i,1}}{(1+\lambda\rho\tilde{\kappa}_1)}-\frac{\lambda^2 v_2\tilde{\kappa}_2 Q_{i,2}}{(1+\lambda\rho\tilde{\kappa}_2)}+\lambda\frac{\partial\Theta_i}{\partial\rho}+f_i^2 Q_{i,3},
\end{aligned}
\tag{4.65}
$$

$$
H_{i,1} = -\frac{1}{\lambda}\frac{\partial Q_{i,2}}{\partial\rho}-\frac{\tilde{\kappa}_2 Q_{i,2}}{1+\lambda\rho\tilde{\kappa}_2}+\frac{1}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial Q_{i,3}}{\partial s_2}, \tag{4.66}
$$

$$
H_{i,2} = \frac{1}{\lambda}\frac{\partial Q_{i,1}}{\partial\rho}+\frac{\tilde{\kappa}_1 Q_{i,1}}{1+\lambda\rho\tilde{\kappa}_1}-\frac{1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial Q_{i,3}}{\partial s_1}, \tag{4.67}
$$

$$H_{i,3} = \frac{1}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial\left[G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)Q_{i,2}\right]}{\partial s_1}\right) - \frac{1}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial\left[E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)Q_{i,1}\right]}{\partial s_2}\right), \quad (4.68)$$

where $\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_n\boldsymbol{n}$. We formally expand all functions in the inner variables in powers of λ to obtain the inner expansions as

$$f_i(s_1,s_2,\rho,\tau,\lambda) = f_i^{(0)}(s_1,s_2,\rho,\tau) + \lambda f_i^{(1)}(s_1,s_2,\rho,\tau) + \cdots, \quad (4.69)$$

$$\Theta_i(s_1,s_2,\rho,\tau,\lambda) = \Theta_i^{(0)}(s_1,s_2,\rho,\tau) + \lambda\Theta_i^{(0)}(s_1,s_2,\rho,\tau) + \cdots, \quad (4.70)$$

$$\boldsymbol{Q}_i(s_1,s_2,\rho,\tau,\lambda) = \boldsymbol{Q}_i^{(0)}(s_1,s_2,\rho,\tau) + \lambda\boldsymbol{Q}_i^{(1)}(s_1,s_2,\rho,\tau) + \cdots, \quad (4.71)$$

$$\boldsymbol{H}_i(s_1,s_2,\rho,\tau,\lambda) = \boldsymbol{H}_i^{(0)}(s_1,s_2,\rho,\tau) + \lambda\boldsymbol{H}_i^{(1)}(s_1,s_2,\rho,\tau) + \cdots. \quad (4.72)$$

We also expand

$$\boldsymbol{R}(s_1,s_2,\tau,\lambda) = \boldsymbol{R}^{(0)}(s_1,s_2,\tau) + \lambda\boldsymbol{R}^{(1)}(s_1,s_2,\tau) + \cdots,$$

which gives

$$E(s_1,s_2,\tau,\lambda) = E^{(0)}(s_1,s_2,\tau) + \lambda E^{(1)}(s_1,s_2,\tau) + \cdots,$$

etc. Substituting the expansions (4.69)-(4.72) into equations (4.61)-(4.68) and equating powers of λ we find at leading order

$$f_i^{(0)}\left((Q_{i,1}^{(0)})^2 + (Q_{i,2}^{(0)})^2 + (Q_{i,3}^{(0)})^2\right) = 0. \quad (4.73)$$

Matching the inner solution with the superconducting region gives $f_i^{(0)} \to 1$, as $\rho \to -\infty$. Hence $f_i^{(0)} \not\equiv 0$. Therefore

$$\boldsymbol{Q}_i^{(0)} \equiv \boldsymbol{0}. \quad (4.74)$$

Equating coefficients of λ^0 in equations (4.61)-(4.68) we have

$$\frac{1}{\kappa^2}\frac{\partial^2 f_i^{(0)}}{\partial\rho^2} = (f_i^{(0)})^3 - f_i^{(0)} + f_i^{(0)}\left\{(Q_{i,1}^{(1)})^2 + (Q_{i,2}^{(1)})^2 + (Q_{i,3}^{(1)})^2\right\}, \quad (4.75)$$

$$0 = \alpha(f_i^{(0)})^2\Theta_i^{(0)} + \frac{\partial\left((f_i^{(0)})^2 Q_{i,3}^{(1)}\right)}{\partial\rho}, \quad (4.76)$$

$$\frac{\partial H_{i,2}^{(0)}}{\partial\rho} = (f_i^{(0)})^2 Q_{i,1}^{(1)}, \quad (4.77)$$

$$
\begin{aligned}
-\frac{\partial H_{i,1}^{(0)}}{\partial \rho} &= (f_i^{(0)})^2 Q_{i,2}^{(1)}, && (4.78)\\
0 &= (f_i^{(0)})^2 Q_{i,3}^{(1)} + \frac{\partial \Theta_i^{(0)}}{\partial \rho}, && (4.79)\\
H_{i,1}^{(0)} &= -\frac{\partial Q_{i,2}^{(1)}}{\partial \rho}, && (4.80)\\
H_{i,2}^{(0)} &= \frac{\partial Q_{i,1}^{(1)}}{\partial \rho}, && (4.81)\\
H_{i,3}^{(0)} &= 0. && (4.82)
\end{aligned}
$$

The matching condition (A.15) of Appendix A implies

$$
F_o^{(0)}(\boldsymbol{R}_N^{(0)}, t) = \lim_{\rho \to \infty} F_i^{(0)}(s_1, s_2, \rho, \tau),
$$

where F is any of the functions under consideration. Coupled with the outer expansions this gives us the following boundary conditions on the inner variables:

$$
f_i^{(0)} \to 1,\ \boldsymbol{Q}_i^{(1)} \to 0,\ \boldsymbol{H}_i^{(0)} \to 0,\ \Theta_i^{(0)} \to 0, \qquad \text{as } \rho \to -\infty, \tag{4.83}
$$

$$
f_i^{(0)} \to 0, \qquad \text{as } \rho \to \infty, \tag{4.84}
$$

$$
f_i^{(0)}(s_1, s_2, 0, \tau) = \eta. \tag{4.85}
$$

We see that the choice of η simply fixes the translate of the leading order inner solution by specifying $f_i^{(0)}(s_1, s_2, 0, \tau)$. Our aim is to determine the values of $\boldsymbol{H}_i$, $\boldsymbol{Q}_i$ and Θ_i as $\rho \to \infty$. Using the matching condition (A.15) again we have

$$
H_{o,3}^{(0)}(\boldsymbol{R}_N^{(0)}, t) = \lim_{\rho \to \infty} H_{i,3}^{(0)}.
$$

Hence we see immediately by (4.82) and that we have

$$
H_{o,3}^{(0)}(\boldsymbol{R}_N^{(0)}, t) = 0. \tag{4.86}
$$

By (4.76) and (4.79) we have

$$
\frac{\partial^2 \Theta_i^{(0)}}{\partial \rho^2} = \alpha (f_i^{(0)})^2 \Theta_i^{(0)}. \tag{4.87}
$$

Equation (4.79) implies $\partial \Theta_i^{(0)} / \partial \rho \to 0$, as $\rho \to \infty$. We have $\Theta_i^{(0)} \to 0$, as $\rho \to -\infty$. Since equation (4.87) implies that $\Theta_i^{(0)}$ is convex we therefore have

$$
\Theta_i^{(0)} \equiv 0.
$$

Now by equation (4.79)

$$Q_{i,3}^{(1)} \equiv 0.$$

We multiply (4.78) by $-H_{i,1}^{(0)}$, (4.77) by $H_{i,2}^{(0)}$, (4.75) by $\partial f_i^{(0)}/\partial\rho$, add and integrate to give

$$\frac{1}{\kappa^2}\left(\frac{\partial f_i^{(0)}}{\partial\rho}\right) + \left(H_{i,1}^{(0)}\right)^2 + \left(H_{i,2}^{(0)}\right)^2 = \frac{\left((f_i^{(0)})^2-1\right)^2}{2} + (f_i^{(0)})^2\left\{(Q_{i,1}^{(1)})^2+(Q_{i,2}^{(1)})^2\right\},$$

where we have used the fact that $f_i^{(0)} \to 1$, $Q_{i,1}^{(1)} \to 0$, $Q_{i,2}^{(1)} \to 0$, as $\rho \to -\infty$. Letting ρ tend to infinity we have

$$\lim_{\rho\to\infty}\left\{(H_{i,1}^{(0)})^2+(H_{i,2}^{(0)})^2\right\}^{1/2} = \lim_{\rho\to\infty} |\boldsymbol{H}_i^{(0)}| = 1/\sqrt{2},$$

since $f_i^{(0)} \to 0$, $(f_i^{(0)})^2\left\{(Q_{i,1}^{(1)})^2+(Q_{i,2}^{(1)})^2\right\} \to 0$, as $\rho\to\infty$. Using matching condition (A.15) again we have

$$\left\{(H_{o,1}^{(0)})^2+(H_{o,2}^{(0)})^2\right\}^{1/2}(\boldsymbol{R}_N^{(0)},t) = |\boldsymbol{H}_o^{(0)}|\,(\boldsymbol{R}_N^{(0)},t) = 1/\sqrt{2}. \tag{4.88}$$

We note that the solution to the leading-order equations is not determined uniquely by the boundary conditions (4.83)-(4.85) since we have for $f_i^{(0)}$, $Q_{i,1}^{(1)}$, $Q_{i,2}^{(1)}$, three second-order equations with only five boundary conditions. However, we see that

$$\begin{aligned}\frac{\partial}{\partial\rho}\left[\frac{\partial Q_{i,1}^{(1)}}{\partial\rho}Q_{i,2}^{(1)} - \frac{\partial Q_{i,2}^{(1)}}{\partial\rho}Q_{i,1}^{(1)}\right] &= \frac{\partial^2 Q_{i,1}^{(1)}}{\partial\rho^2}Q_{i,2}^{(1)} - \frac{\partial^2 Q_{i,2}^{(1)}}{\partial\rho}Q_{i,1}^{(1)}, \\ &= (f_i^{(0)})^2 Q_{i,1}^{(1)}Q_{i,2}^{(1)} - (f_i^{(0)})^2 Q_{i,2}^{(1)}Q_{i,1}^{(1)}, \\ &= 0,\end{aligned}$$

by (4.77) and (4.78). Therefore

$$\frac{\partial Q_{i,1}^{(1)}}{\partial\rho}Q_{i,2}^{(1)} - \frac{\partial Q_{i,2}^{(1)}}{\partial\rho}Q_{i,1}^{(1)} = \text{const.} = 0,$$

by the boundary conditions as $\rho\to-\infty$. Hence

$$\frac{1}{Q_{i,1}^{(1)}}\frac{\partial Q_{i,1}^{(1)}}{\partial\rho} = \frac{1}{Q_{i,2}^{(1)}}\frac{\partial Q_{i,2}^{(1)}}{\partial\rho},$$

which, on integrating, gives

$$Q_{i,1}^{(1)} = C(s_1,s_2,\tau)Q_{i,2}^{(1)}, \tag{4.89}$$

where C is an unknown function of s_1, s_2, τ which will be determined by the outer solution in the normal region. Now if we let

$$Q_i^{(1)} = \mid \boldsymbol{Q}_i^{(1)} \mid = \sqrt{1+C^2}\, Q_{i,2}^{(1)},$$

we have

$$\frac{1}{\kappa^2}\frac{\partial^2 f_i^{(0)}}{\partial \rho^2} = (f_i^{(0)})^3 - f_i^{(0)} + f_i^{(0)}(Q_i^{(1)})^2, \tag{4.90}$$

$$\frac{\partial^2 Q_i^{(1)}}{\partial \rho^2} = (f_i^{(0)})^2 Q_i^{(0)}. \tag{4.91}$$

Therefore $f_i^{(0)}$ and $Q_i^{(1)}$ satisfy equations (3.113), (3.114) with boundary conditions (3.116), (3.117). Hence there is a unique solution for $f_i^{(0)}$ and $Q_i^{(1)}$. Moreover we have that $f_i^{(0)}$ decays exponentially as $\rho \to \infty$. Since no term can grow exponentially if it is to match with the outer region we conclude that any term involving $f_i^{(0)}$ as a numerator will tend to zero as $\rho \to \infty$. We also have

$$Q_i^{(1)} \sim \frac{\rho}{\sqrt{2}} + c + O(e^{-K\rho^2}), \text{ as } \rho \to \infty.$$

where c and K are constant. Here we make our choice of η, which we take to be such that $c = 0$. Thus we see that it is not f but $\mid \boldsymbol{Q} \mid$ which gives the natural choice for η. We have that $Q_i^{(1)} \sim 0$ as $\rho \to -\infty$, $Q_i^{(1)} \sim \rho/\sqrt{2} + c$ as $\rho \to \infty$. The condition that $c = 0$ simply states that we choose the origin of our inner coordinate (i.e. the centre of the transition layer) to be at the intersection of these two straight lines (see Fig. 4.1). The simplicity that this choice of η induces will become apparent when we consider the interface conditions at first order. Since $Q_{i,1}^{(1)}$, $Q_{i,2}^{(1)}$ are multiples of $Q_i^{(1)}$ we therefore have

$$Q_{i,1}^{(1)} \sim \rho H_{o,2}^{(0)}(\boldsymbol{R}_N^{(0)}, t) + O(e^{-K\rho^2}), \text{ as } \rho \to \infty, \tag{4.92}$$

$$Q_{i,2}^{(1)} \sim -\rho H_{o,1}^{(0)}(\boldsymbol{R}_N^{(0)}, t) + O(e^{-K\rho^2}), \text{ as } \rho \to \infty, \tag{4.93}$$

where we have made use of the matching condition (A.17), namely that

$$\frac{\partial F_o^{(0)}}{\partial n}(\boldsymbol{R}_N^{(0)}, t) = \lim_{\rho\to\infty} \frac{\partial F_i^{(1)}}{\partial \rho}(s_1, s_2, \rho, \tau).$$

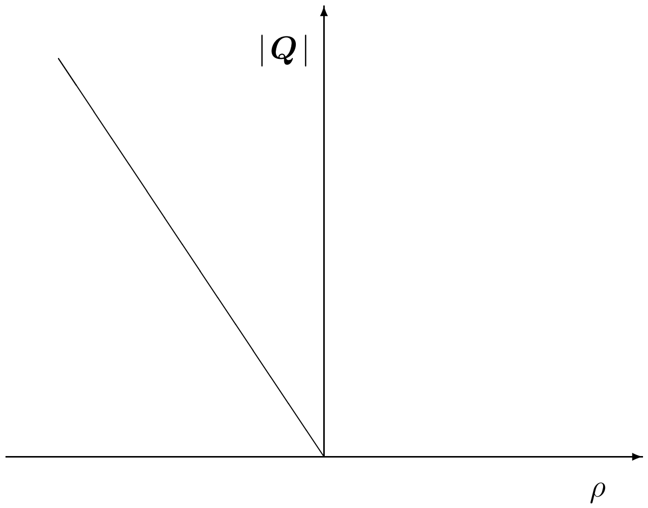

Figure 4.1: Variation of $|\boldsymbol{Q}|$ across a normal/superconducting transition region showing the choice of η.

Equating powers of λ at the next order in equations (4.61)-(4.68) we find

$$\frac{\alpha v_n^{(0)} + \tilde{\kappa}_1^{(0)} + \tilde{\kappa}_2^{(0)}}{\kappa^2} \frac{\partial f_i^{(0)}}{\partial \rho} + \frac{1}{\kappa^2} \frac{\partial^2 f_i^{(1)}}{\partial \rho^2} = 3(f_i^{(0)})^2 f_i^{(1)} - f_i^{(1)} + f_i^{(1)} \left((Q_{i,1}^{(1)})^2 + (Q_{i,2}^{(1)})^2 \right) + 2 f_i^{(0)} \left(Q_{i,1}^{(1)} Q_{i,1}^{(2)} + Q_{i,2}^{(1)} Q_{i,2}^{(2)} \right), \tag{4.94}$$

$$\alpha (f_i^{(0)})^2 \Theta_i^{(1)} + \frac{\partial \left[(f_i^{(0)})^2 Q_{i,3}^{(2)} \right]}{\partial \rho} + \frac{1}{(E^{(0)} G^{(0)})^{1/2}} \left(\frac{\partial \left[(G^{(0)})^{1/2} (f_i^{(0)})^2 Q_{i,1}^{(1)} \right]}{\partial s_1} \right) + \frac{1}{(E^{(0)} G^{(0)})^{1/2}} \left(\frac{\partial \left[(E^{(0)})^{1/2} (f_i^{(0)})^2 Q_{i,2}^{(1)} \right]}{\partial s_2} \right) = 0, \tag{4.95}$$

$$\frac{\partial H_{i,2}^{(1)}}{\partial \rho} + \tilde{\kappa}_2^{(0)} H_{i,2}^{(0)} = -v_n^{(0)} \frac{\partial Q_{i,1}^{(1)}}{\partial \rho} + 2 f_i^{(0)} f_i^{(1)} Q_{i,1}^{(1)} + (f_i^{(0)})^2 Q_{i,1}^{(2)}, \tag{4.96}$$

$$-\frac{\partial H_{i,1}^{(1)}}{\partial \rho} - \tilde{\kappa}_1^{(0)} H_{i,1}^{(0)} = -v_n^{(0)} \frac{\partial Q_{i,2}^{(1)}}{\partial \rho} + 2 f_i^{(0)} f_i^{(1)} Q_{i,2}^{(1)} + (f_i^{(0)})^2 Q_{i,2}^{(2)}, \tag{4.97}$$

$$-\frac{1}{(E^{(0)} G^{(0)})^{1/2}} \left\{ \frac{\partial \left[(G^{(0)})^{1/2} H_{i,2}^{(0)} \right]}{\partial s_1} - \frac{\partial \left[(E^{(0)})^{1/2} H_{i,1}^{(0)} \right]}{\partial s_2} \right\} = \frac{\partial \Theta_i^{(1)}}{\partial \rho} + (f_i^{(0)})^2 Q_{i,3}^{(2)}, \tag{4.98}$$

$$H_{i,1}^{(1)} = -\frac{\partial Q_{i,2}^{(2)}}{\partial \rho} - \tilde{\kappa}_2^{(0)} Q_{i,2}^{(1)}, \tag{4.99}$$

$$H_{i,2}^{(1)} = \frac{\partial Q_{i,1}^{(2)}}{\partial \rho} + \tilde{\kappa}_1^{(0)} Q_{i,1}^{(1)}, \tag{4.100}$$

$$H_{i,3}^{(1)} = \frac{1}{(E^{(0)}G^{(0)})^{1/2}} \left\{ \frac{\partial \left[(G^{(0)})^{1/2} Q_{i,2}^{(1)} \right]}{\partial s_1} - \frac{\partial \left[(E^{(0)})^{1/2} Q_{i,1}^{(1)} \right]}{\partial s_2} \right\}. \tag{4.101}$$

Letting $\rho \to \infty$ in equations (4.96) and (4.97) we have

$$\lim_{\rho\to\infty} \left[\frac{\partial H_{i,1}^{(1)}}{\partial \rho} + \tilde{\kappa}_1^{(0)} H_{i,1}^{(0)} \right] = -v_n^{(0)} \lim_{\rho\to\infty} H_{i,1}^{(0)},$$

$$\lim_{\rho\to\infty} \left[\frac{\partial H_{i,2}^{(1)}}{\partial \rho} + \tilde{\kappa}_2^{(0)} H_{i,2}^{(0)} \right] = -v_n^{(0)} \lim_{\rho\to\infty} H_{i,2}^{(0)},$$

since the terms involving $f_i^{(0)}$ tend to zero. Using the matching conditions (A.15) and (A.17) we have

$$\frac{\partial H_{o,1}^{(0)}}{\partial n} + \tilde{\kappa}_1^{(0)} H_{o,1}^{(0)} = -v_n^{(0)} H_{o,1}^{(0)}, \text{ on } \Gamma_N^{(0)}, \tag{4.102}$$

$$\frac{\partial H_{o,2}^{(0)}}{\partial n} + \tilde{\kappa}_2^{(0)} H_{o,2}^{(0)} = -v_n^{(0)} H_{o,2}^{(0)}, \text{ on } \Gamma_N^{(0)}. \tag{4.103}$$

At this stage we have derived all interface conditions to lowest order. If we were interested in a complete determination of the solution we could now solve the outer problem (4.40) with the boundary conditions (4.86), (4.88), (4.102), (4.103) to determine $\boldsymbol{H}_o^{(0)}$ and $\Gamma^{(0)}$. We show in Lemma 1 that the conditions (4.86), (4.102) and (4.103) are equivalent to the condition (2.8) when $v_n \neq 0$. The proposition is then proved.

We continue with the inner problem. The matching conditions (A.15), (A.16) imply

$$H_{i,3}^{(1)} \sim \rho \frac{\partial H_{o,3}^{(0)}}{\partial n}(\boldsymbol{R}_N^{(0)}, t) + \left\{ H_{o,3}^{(1)} + \boldsymbol{R}^{(1)} \cdot \nabla H_{o,3}^{(0)}(\boldsymbol{R}_N^{(0)}, t) \right\} + o(1), \text{ as } \rho \to \infty.$$

Equating the order one terms in equation (4.101) as $\rho \to \infty$ we have therefore

$$H_{o,3}^{(1)} + \boldsymbol{R}^{(1)} \cdot \nabla H_{o,3}^{(0)}(\boldsymbol{R}_N^{(0)}, t) = 0, \tag{4.104}$$

by (4.92), (4.93).

Multiplying (4.96) by $H_{i,2}^{(0)} = \partial Q_{i,1}^{(1)}/\partial\rho$, (4.97) by $-H_{i,1}^{(0)} = \partial Q_{i,2}^{(1)}/\partial\rho$, (4.94) by $\partial f_i^{(0)}/\partial\rho$, (4.75) by $\partial f_i^{(1)}/\partial\rho$, (4.77) by $H_{i,2}^{(1)} = \partial Q_{i,1}^{(2)}/\partial\rho + \tilde{\kappa}_1^{(0)} Q_{i,1}^{(1)}$, (4.78) by $-H_{i,1}^{(1)} = \partial Q_{i,2}^{(2)}/\partial\rho + \tilde{\kappa}_2^{(0)} Q_{i,2}^{(1)}$, adding and integrating gives

$$\frac{1}{\kappa^2}\frac{\partial f_i^{(0)}}{\partial\rho}\frac{\partial f_i^{(1)}}{\partial\rho} + H_{i,1}^{(0)}H_{i,1}^{(1)} + H_{i,2}^{(0)}H_{i,2}^{(1)} + \frac{(\alpha v_n^{(0)} + \tilde{\kappa}_1^{(0)} + \tilde{\kappa}_2^{(0)})}{\kappa^2}\int_{-\infty}^{\rho}\left(\frac{\partial f_i^{(0)}}{\partial\rho}\right)^2 d\rho$$
$$+ \tilde{\kappa}_1^{(0)}\int_{-\infty}^{\rho}\left(\frac{\partial Q_{i,2}^{(1)}}{\partial\rho}\right)^2 d\rho + \tilde{\kappa}_2^{(0)}\int_{-\infty}^{\rho}\left(\frac{\partial Q_{i,1}^{(1)}}{\partial\rho}\right)^2 d\rho =$$
$$(f_i^{(0)})^3 f_i^{(1)} - f_i^{(0)} f_i^{(1)} + (f_i^{(0)})^2\left(Q_{i,1}^{(1)}Q_{i,1}^{(2)} + Q_{i,2}^{(1)}Q_{i,2}^{(2)}\right)$$
$$+ f_i^{(0)} f_i^{(1)}\left\{(Q_{i,1}^{(1)})^2 + (Q_{i,2}^{(1)})^2\right\} + \tilde{\kappa}_1^{(0)}\int_{-\infty}^{\rho}(f_i^{(0)}Q_{i,1}^{(1)})^2\, d\rho$$
$$+ \tilde{\kappa}_2^{(0)}\int_{-\infty}^{\rho}(f_i^{(0)}Q_{i,2}^{(1)})^2\, d\rho - v_n^{(0)}\int_{-\infty}^{\rho}\left\{\left(\frac{\partial Q_{i,1}^{(1)}}{\partial\rho}\right)^2 + \left(\frac{\partial Q_{i,2}^{(1)}}{\partial\rho}\right)^2\right\} d\rho.$$

Letting $\rho \to \infty$ in this equation is equivalent to a solvability condition for the first order terms.

Now

$$\begin{aligned}\int_{-\infty}^{\rho}\left(\frac{\partial Q_{i,k}^{(1)}}{\partial\rho}\right)^2 d\rho &= Q_{i,k}^{(1)}\frac{\partial Q_{i,k}^{(1)}}{\partial\rho} - \int_{-\infty}^{\rho} Q_{i,k}^{(1)}\frac{\partial^2 Q_{i,k}^{(1)}}{\partial\rho^2}\, d\rho,\\ &= Q_{i,k}^{(1)}\frac{\partial Q_{i,k}^{(1)}}{\partial\rho} - \int_{-\infty}^{\rho}(f_i^{(0)}Q_{i,k}^{(1)})^2\, d\rho, \quad k = 1, 2.\end{aligned}$$

on integration by parts, and using (4.77), (4.78). Hence

$$\frac{1}{\kappa^2}\frac{\partial f_i^{(0)}}{\partial\rho}\frac{\partial f_i^{(1)}}{\partial\rho} + H_{i,1}^{(0)}H_{i,1}^{(1)} + H_{i,2}^{(0)}H_{i,2}^{(1)} + \frac{\alpha v_n^{(0)} + \tilde{\kappa}_1^{(0)} + \tilde{\kappa}_2^{(0)}}{\kappa^2}\int_{-\infty}^{\rho}\left(\frac{\partial f_i^{(0)}}{\partial\rho}\right)^2 d\rho$$
$$+ (v_n^{(0)} + \tilde{\kappa}_1^{(0)})Q_{i,2}^{(1)}\frac{\partial Q_{i,2}^{(1)}}{\partial\rho} + (v_n^{(0)} + \tilde{\kappa}_2^{(0)})Q_{i,1}^{(1)}\frac{\partial Q_{i,1}^{(1)}}{\partial\rho} =$$
$$(f_i^{(0)})^3 f_i^{(1)} - f_i^{(0)} f_i^{(1)} + (f_i^{(0)})^2\left(Q_{i,1}^{(1)}Q_{i,1}^{(2)} + Q_{i,2}^{(1)}Q_{i,2}^{(2)}\right)$$
$$+ f_i^{(0)} f_i^{(1)}\left\{(Q_{i,1}^{(1)})^2 + (Q_{i,2}^{(1)})^2\right\}$$
$$+ (v_n^{(0)} + \tilde{\kappa}_1^{(0)} + \tilde{\kappa}_2^{(0)})\int_{-\infty}^{\rho}(f_i^{(0)})^2\left\{(Q_{i,1}^{(1)})^2 + (Q_{i,2}^{(1)})^2\right\} d\rho. \quad (4.105)$$

We have

$$\begin{aligned}(v_n^{(0)} + \tilde{\kappa}_2^{(0)})Q_{i,1}^{(1)}\frac{\partial Q_{i,1}^{(1)}}{\partial \rho} &\sim (v_n^{(0)} + \tilde{\kappa}_2^{(0)})\rho(H_{o,2}^{(0)}\left(\boldsymbol{R}_N^{(0)}, t)\right)^2 + o(1), \text{ by (4.92)} \\ &= -\rho H_{o,2}^{(0)}(\boldsymbol{R}_N^{(0)}, t)\frac{\partial H_{o,2}^{(0)}}{\partial n}(\boldsymbol{R}_N^{(0)}, t) + o(1), \text{ by (4.102).}\end{aligned}$$

Similarly

$$(v_n^{(0)} + \tilde{\kappa}_1^{(0)})Q_{i,2}^{(1)}\frac{\partial Q_{i,2}^{(1)}}{\partial \rho} \sim -\rho H_{o,1}^{(0)}(\boldsymbol{R}_N^{(0)}, t)\frac{\partial H_{o,1}^{(0)}}{\partial n}(\boldsymbol{R}_N^{(0)}, t) + o(1).$$

Hence, letting $\rho \to \infty$ in (4.105) we have

$$\begin{aligned}&H_{o,1}^{(0)}(\boldsymbol{R}_N^{(0)}, t)\lim_{\rho\to\infty}\left\{H_{i,1}^{(1)} - \rho\frac{\partial H_{o,1}^{(0)}}{\partial n}(\boldsymbol{R}_N^{(0)}, t)\right\} + \\ &\qquad H_{o,2}^{(0)}(\boldsymbol{R}_N^{(0)}, t)\lim_{\rho\to\infty}\left\{H_{i,2}^{(1)} - \rho\frac{\partial H_{o,2}^{(0)}}{\partial n}(\boldsymbol{R}_N^{(0)}, t)\right\} = \\ &\qquad\qquad -v_n^{(0)}(\alpha\delta - \gamma) - (\tilde{\kappa}_1^{(0)} + \tilde{\kappa}_2^{(0)})(\delta - \gamma),\end{aligned}$$

where

$$\delta = \frac{1}{\kappa^2}\int_{-\infty}^{\infty}\left(\frac{\partial f_i^{(0)}}{\partial \rho}\right)^2 d\rho, \quad \gamma = \int_{-\infty}^{\infty}(f_i^{(0)})^2\left\{(Q_{i,1}^{(1)})^2 + (Q_{i,2}^{(1)})^2\right\} d\rho.$$

Using the matching condition (A.16) we have

$$\begin{aligned}&H_{o,1}^{(0)}(\boldsymbol{R}_N^{(0)}, t)\left\{H_{o,1}^{(1)}(\boldsymbol{R}_N^{(0)}, t) + (\boldsymbol{R}^{(1)}\cdot\nabla H_{o,1}^{(0)})(\boldsymbol{R}_N^{(0)}, t)\right\} + \\ &\qquad H_{o,2}^{(0)}(\boldsymbol{R}_N^{(0)}, t)\left\{H_{o,2}^{(1)}(\boldsymbol{R}_N^{(0)}, t) + (\boldsymbol{R}^{(1)}\cdot\nabla H_{o,2}^{(0)})(\boldsymbol{R}_N^{(0)}, t)\right\} = \\ &\qquad\qquad -v_n^{(0)}(\alpha\delta - \gamma) - (\tilde{\kappa}_1^{(0)} + \tilde{\kappa}_2^{(0)})(\delta - \gamma). \quad (4.106)\end{aligned}$$

Note that σ, the surface energy defined in Chapter 3, is given by $\sigma = 4\lambda(\delta - \gamma)$.

We let

$$P = \frac{1}{(E^{(0)}G^{(0)})^{1/2}}\left\{\frac{\partial\left[(G^{(0)})^{1/2}Q_{i,1}^{(1)}\right]}{\partial s_1} + \frac{\partial\left[(E^{(0)})^{1/2}Q_{i,2}^{(1)}\right]}{\partial s_2}\right\}.$$

Then (4.95) and (4.98) are

$$\alpha(f_i^{(0)})^2\Theta_i^{(1)} + \frac{\partial\left[(f_i^{(0)})^2Q_{i,3}^{(2)}\right]}{\partial \rho} + (f_i^{(0)})^2P = 0, \quad (4.107)$$

$$\frac{\partial \Theta_i^{(1)}}{\partial \rho} + (f_i^{(0)})^2Q_{i,3}^{(2)} + \frac{\partial P}{\partial \rho} = 0, \quad (4.108)$$

by (4.80) and (4.81), noting that $f_i^{(0)}$ is independent of s_1 and s_2. We note that

$$\frac{\partial^2 P}{\partial \rho^2} = (f_i^{(0)})^2 P,$$

by (4.77) and (4.78), and so

$$\frac{\partial^2 \Theta_i^{(1)}}{\partial \rho^2} = \alpha (f_i^{(0)})^2 \Theta_i^{(1)}, \tag{4.109}$$

as at leading order. Dividing (4.107) by $(f_i^{(0)})^2$ we have

$$\alpha \Theta_i^{(1)} + \frac{\partial \left[\log (f_i^{(0)})^2\right]}{\partial \rho} Q_{i,3}^{(2)} + \frac{\partial Q_{i,3}^{(2)}}{\partial \rho} + P = 0. \tag{4.110}$$

Now, by (3.121),

$$\begin{aligned} \Theta_i^{(1)} &= \text{const.}\rho + \text{const.} + O(e^{-K\rho^2}), \\ P &= \text{const.}\rho + O(e^{-K\rho^2}), \\ \frac{\partial \left[\log (f_i^{(0)})^2\right]}{\partial \rho} &\sim -\frac{\kappa \rho}{\sqrt{2}} + \frac{1}{\sqrt{2}\,\rho}\left(\kappa - \frac{1}{\sqrt{2}}\right), \end{aligned}$$

as $\rho \to \infty$, since $c = 0$. Therefore

$$Q_{i,3}^{(2)} \sim \text{const.}, \qquad \text{as } \rho \to \infty.$$

If we assume that $Q_{o,3}^{(4)}$ is bounded on $\Gamma^{(0)}$ then there can be no terms of order $1/\rho$ in $Q_{i,3}^{(2)}$ as $\rho \to \infty$. If we then equate the order one terms in (4.110) as $\rho \to \infty$ we find

$$\Theta_i^{(1)} = \text{const.}\rho + O(e^{-K\rho^2}), \qquad \text{as } \rho \to \infty. \tag{4.111}$$

Equating powers of λ at the next order in equations (4.63) and (4.64) we have

$$\begin{aligned} &\frac{\partial H_{i,2}^{(2)}}{\partial \rho} + \tilde{\kappa}_2^{(0)} H_{i,2}^{(1)} + \tilde{\kappa}_2^{(1)} H_{i,2}^{(0)} - \rho (\tilde{\kappa}_2^{(0)})^2 H_{i,2}^{(0)} - \frac{1}{(G^{(0)})^{1/2}} \frac{\partial H_{i,3}^{(1)}}{\partial s_2} = \\ &\quad - v_n^{(0)} \frac{\partial Q_{i,1}^{(2)}}{\partial \rho} - v_n^{(1)} \frac{\partial Q_{i,1}^{(1)}}{\partial \rho} + \frac{\partial Q_{i,1}^{(1)}}{\partial \tau} \\ &\quad - \frac{v_1^{(0)}}{(E^{(0)})^{1/2}} \frac{\partial Q_{i,1}^{(1)}}{\partial s_1} - \frac{v_2^{(0)}}{(G^{(0)})^{1/2}} \frac{\partial Q_{i,1}^{(1)}}{\partial s_2} \\ &\quad - \frac{1}{2(E^{(0)} G^{(0)})^{1/2}} \left(\frac{v_2^{(0)}}{(G^{(0)})^{1/2}} \frac{\partial G^{(0)}}{\partial s_2} - \frac{v_1^{(0)}}{(E^{(0)})^{1/2}} \frac{\partial E^{(0)}}{\partial s_1} \right) Q_{i,2}^{(1)} \\ &\quad + \frac{1}{(E^{(0)})^{1/2}} \frac{\partial \Theta_i^{(1)}}{\partial s_1} \\ &\quad + (f_i^{(0)})^2 Q_{i,1}^{(3)} + 2 f_i^{(0)} f_i^{(1)} Q_{i,1}^{(2)} \\ &\quad + \left(2 f_i^{(0)} f_i^{(2)} + (f_i^{(1)})^2\right) Q_{i,1}^{(1)}, \end{aligned} \tag{4.112}$$

$$
\begin{aligned}
-\frac{\partial H_{i,1}^{(2)}}{\partial \rho} - \tilde{\kappa}_1^{(0)} H_{i,1}^{(1)} - \tilde{\kappa}_1^{(1)} H_{i,1}^{(0)} + \rho(\tilde{\kappa}_1^{(0)})^2 H_{i,1}^{(0)} + \frac{1}{(E^{(0)})^{1/2}} \frac{\partial H_{i,3}^{(1)}}{\partial s_1} = \\
- v_n^{(0)} \frac{\partial Q_{i,2}^{(2)}}{\partial \rho} - v_n^{(1)} \frac{\partial Q_{i,2}^{(1)}}{\partial \rho} + \frac{\partial Q_{i,2}^{(1)}}{\partial \tau} \\
- \frac{v_1^{(0)}}{(E^{(0)})^{1/2}} \frac{\partial Q_{i,2}^{(1)}}{\partial s_1} - \frac{v_2^{(0)}}{(G^{(0)})^{1/2}} \frac{\partial Q_{i,2}^{(1)}}{\partial s_2} \\
+ \frac{1}{2(E^{(0)} G^{(0)})^{1/2}} \left(\frac{v_2^{(0)}}{(G^{(0)})^{1/2}} \frac{\partial G^{(0)}}{\partial s_2} - \frac{v_1^{(0)}}{(E^{(0)})^{1/2}} \frac{\partial E^{(0)}}{\partial s_1} \right) Q_{i,1}^{(1)} \\
+ \frac{1}{(G^{(0)})^{1/2}} \frac{\partial \Theta_i^{(1)}}{\partial s_2} \\
+ (f_i^{(0)})^2 Q_{i,2}^{(3)} + 2 f_i^{(0)} f_i^{(1)} Q_{i,2}^{(2)} \\
+ \left(2 f_i^{(0)} f_i^{(2)} + (f_i^{(1)})^2 \right) Q_{i,2}^{(1)}.
\end{aligned} \tag{4.113}
$$

We have from the matching conditions that

$$
\begin{aligned}
\frac{\partial H_{i,k}^{(2)}}{\partial \rho} &\sim \rho \frac{\partial^2 H_{o,k}^{(0)}}{\partial n^2}(\boldsymbol{R}_N^{(0)}, t) + \frac{\partial H_{o,k}^{(1)}}{\partial n}(\boldsymbol{R}_N^{(0)}, t) \\
&\qquad + (\boldsymbol{R}^{(1)} \cdot \nabla \left(\frac{\partial H_{o,k}^{(0)}}{\partial n} \right))(\boldsymbol{R}_N^{(0)}, t) + o(1), \\
H_{i,k}^{(1)} &\sim \rho \frac{\partial H_{o,k}^{(0)}}{\partial n}(\boldsymbol{R}_N^{(0)}, t) + H_{o,k}^{(1)}(\boldsymbol{R}_N^{(0)}, t) + (\boldsymbol{R}^{(1)} \cdot \nabla H_{o,k}^{(0)})(\boldsymbol{R}_N^{(0)}, t) + o(1), \\
Q_{i,1}^{(1)} &\sim \rho H_{o,2}^{(0)}(\boldsymbol{R}_N^{(0)}, t) + o(1), \\
Q_{i,2}^{(1)} &\sim -\rho H_{o,1}^{(0)}(\boldsymbol{R}_N^{(0)}, t) + o(1),
\end{aligned}
$$

as $\rho \to \infty$, for $k = 1, 2, 3$. We let $\rho \to \infty$ in equations (4.112), (4.113) and equate the order one terms, using equations (4.99), (4.100) and (4.116), to obtain

$$
\begin{aligned}
\frac{\partial H_{o,1}^{(1)}}{\partial n}(\boldsymbol{R}_N^{(0)}, t) + (\boldsymbol{R}^{(1)} \cdot \nabla \left(\frac{\partial H_{o,1}^{(0)}}{\partial n} \right))(\boldsymbol{R}_N^{(0)}, t) = \\
- (\tilde{\kappa}_1^{(0)} + v_n^{(0)}) \left\{ H_{o,1}^{(1)}(\boldsymbol{R}_N^{(0)}, t) + (\boldsymbol{R}^{(1)} \cdot \nabla H_{o,1}^{(0)})(\boldsymbol{R}_N^{(0)}, t) \right\} \\
- (\tilde{\kappa}_1^{(1)} + v_n^{(1)}) H_{o,1}^{(0)}(\boldsymbol{R}_N^{(0)}, t),
\end{aligned} \tag{4.114}
$$

$$
\begin{aligned}
\frac{\partial H_{o,2}^{(1)}}{\partial n}(\boldsymbol{R}_N^{(0)}, t) + (\boldsymbol{R}^{(1)} \cdot \nabla \left(\frac{\partial H_{o,2}^{(0)}}{\partial n} \right))(\boldsymbol{R}_N^{(0)}, t) = \\
- (\tilde{\kappa}_2^{(0)} + v_n^{(0)}) \left\{ H_{o,2}^{(1)}(\boldsymbol{R}_N^{(0)}, t) + (\boldsymbol{R}^{(1)} \cdot \nabla H_{o,2}^{(0)})(\boldsymbol{R}_N^{(0)}, t) \right\} \\
- (\tilde{\kappa}_2^{(1)} + v_n^{(1)}) H_{o,2}^{(0)}(\boldsymbol{R}_N^{(0)}, t).
\end{aligned} \tag{4.115}
$$

Notice how many of the terms in (4.112), (4.113) give no $O(1)$ contribution because of our choice of η.

We are now in a position to solve equation (4.45) with the (fixed) boundary conditions (4.104), (4.106), (4.114), (4.115) for $\boldsymbol{H}_o^{(1)}$ and $\boldsymbol{R}^{(1)}$.

Finally we calculate the values of $|\,\boldsymbol{H}_o\,|$, $H_{o,3}$, $\partial H_{o,1}/\partial n$ and $\partial H_{o,2}/\partial n$ on the boundary Γ_N. We have

$$\begin{aligned} H_{o,3}(\boldsymbol{R}_N,t) &= H_{o,3}^{(0)}(\boldsymbol{R}^{(0)}+\lambda\boldsymbol{R}^{(1)}+\cdots,t)+\lambda H_{o,3}^{(1)}(\boldsymbol{R}^{(0)}+\lambda\boldsymbol{R}^{(1)}+\cdots,t)+\cdots \\ &= H_{o,3}^{(0)}(\boldsymbol{R}_N^{(0)},t)+\lambda\left\{H_{o,3}^{(1)}+\boldsymbol{R}^{(1)}\cdot\nabla H_{o,3}^{(0)}(\boldsymbol{R}_N^{(0)},t)\right\}+\cdots \\ &= O(\lambda^2), \end{aligned} \tag{4.116}$$

by (4.86) and (4.104).

$$\begin{aligned} |\,\boldsymbol{H}_o(\boldsymbol{R}_N,t)\,|^2 &= (H_{o,1}(\boldsymbol{R}_N,t))^2+(H_{o,2}(\boldsymbol{R}_N,t))^2+O(\lambda^2) \\ &= \left(H_{o,1}^{(0)}(\boldsymbol{R}^{(0)}+\lambda\boldsymbol{R}^{(1)}+\cdots,t)\lambda H_{o,1}^{(1)}(\boldsymbol{R}^{(0)}+\lambda\boldsymbol{R}^{(1)}+\cdots,t)\right)^2 \\ &\quad+\left(H_{o,2}^{(0)}(\boldsymbol{R}^{(0)}+\lambda\boldsymbol{R}^{(1)}+\cdots,t)\lambda H_{o,2}^{(1)}(\boldsymbol{R}^{(0)}+\lambda\boldsymbol{R}^{(1)}+\cdots,t)\right)^2 \\ &= \left(H_{o,1}^{(0)}(\boldsymbol{R}_N^{(0)},t)\right)^2+\left(H_{o,2}^{(0)}(\boldsymbol{R}_N^{(0)},t)\right)^2 \\ &\quad+2\lambda H_{o,1}^{(0)}(\boldsymbol{R}_N^{(0)},t)\left[H_{o,1}^{(1)}(\boldsymbol{R}_N^{(0)},t)+(\boldsymbol{R}^{(1)}\cdot\nabla H_{o,1}^{(0)})(\boldsymbol{R}_N^{(0)},t)\right] \\ &\quad+2\lambda H_{o,1}^{(0)}(\boldsymbol{R}_N^{(0)},t)\left[H_{o,1}^{(1)}(\boldsymbol{R}_N^{(0)},t)+(\boldsymbol{R}^{(1)}\cdot\nabla H_{o,1}^{(0)})(\boldsymbol{R}_N^{(0)},t)\right]+\cdots \\ &= \frac{1}{2}-2\lambda\left\{v_n^{(0)}(\alpha\delta-\gamma)+(\tilde{\kappa}_1^{(0)}+\tilde{\kappa}_2^{(0)})(\delta-\gamma)\right\}+O(\lambda^2), \end{aligned}$$

by (4.88) and (4.106). Hence

$$\begin{aligned} |\,\boldsymbol{H}_o(\boldsymbol{R}_N,t)\,| &= \frac{1}{\sqrt{2}}-\sqrt{2}\lambda\left\{v_n^{(0)}(\alpha\delta-\gamma)+(\tilde{\kappa}_1^{(0)}+\tilde{\kappa}_2^{(0)})(\delta-\gamma)\right\}+O(\lambda^2) \\ &= H_c-2H_c\lambda\left\{v_n^{(0)}(\alpha\delta-\gamma)+(\tilde{\kappa}_1^{(0)}+\tilde{\kappa}_2^{(0)})(\delta-\gamma)\right\}+O(\lambda^2) \\ &= H_c-2H_c\lambda\, v_n^{(0)}(\alpha\delta-\gamma)-\frac{H_c}{2}(\tilde{\kappa}_1^{(0)}+\tilde{\kappa}_2^{(0)})\sigma+O(\lambda^2), \end{aligned} \tag{4.117}$$

where $H_c=1/\sqrt{2}$, and σ is the surface energy.

$$\begin{aligned} \frac{\partial H_{o,1}}{\partial n}(\boldsymbol{R}_N,t) &= \frac{\partial H_{o,1}^{(0)}}{\partial n}(\boldsymbol{R}^{(0)}+\lambda\boldsymbol{R}^{(1)}+\cdots,t) \\ &\qquad+\lambda\frac{\partial H_{o,1}^{(1)}}{\partial n}(\boldsymbol{R}^{(0)}+\lambda\boldsymbol{R}^{(1)}+\cdots,t)+\cdots \end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial H_{o,1}^{(0)}}{\partial n}(\boldsymbol{R}_N^{(0)}, t) \\
&\qquad + \lambda \left\{ \frac{\partial H_{o,1}^{(1)}}{\partial n}(\boldsymbol{R}_N^{(0)}, t) + (\boldsymbol{R}^{(1)} \cdot \nabla \left(\frac{\partial H_{o,1}^{(0)}}{\partial n} \right))(\boldsymbol{R}_N^{(0)}, t) \right\} + \cdots \\
&= (\tilde{\kappa}_1^{(0)} + v_n^{(0)}) H_{o,1}^{(0)}(\boldsymbol{R}_N^{(0)}, t) \\
&\qquad - (\tilde{\kappa}_1^{(0)} + v_n^{(0)}) \left\{ H_{o,1}^{(1)}(\boldsymbol{R}_N^{(0)}, t) + (\boldsymbol{R}^{(1)} \cdot \nabla H_{o,1}^{(0)})(\boldsymbol{R}_N^{(0)}, t) \right\} \\
&\qquad\quad - (\tilde{\kappa}_1^{(1)} + v_n^{(1)}) H_{o,1}^{(0)}(\boldsymbol{R}_N^{(0)}, t) + \cdots \\
&= (\tilde{\kappa}_1 + v_n) H_{o,1}(\boldsymbol{R}_N, t) + O(\lambda^2),
\end{aligned}
$$

by (4.102) and (4.114). Hence

$$
\frac{\partial H_{o,1}}{\partial n}(\boldsymbol{R}_N, t) = (\tilde{\kappa}_1 + v_n) H_{o,1}(\boldsymbol{R}_N, t) + O(\lambda^2). \tag{4.118}
$$

Similarly

$$
\frac{\partial H_{o,2}}{\partial n}(\boldsymbol{R}_N, t) = (\tilde{\kappa}_2 + v_n) H_{o,2}(\boldsymbol{R}_N, t) + O(\lambda^2). \tag{4.119}
$$

We see from (4.117) that, as in the phase field model, we have the emergence of 'surface tension' and 'kinetic undercooling' terms in the magnitude of the magnetic field at the interface. For Type I superconductors, where $\sigma > 0$, the 'surface tension' term will have a stabilising effect as in the phase field model. For Type II superconductors, where $\sigma < 0$, this term will be destabilising. The rôle of the 'kinetic undercooling' term depends also on the value of α. For $\alpha\delta - \gamma > 0$ it will be stabilising; for $\alpha\delta - \gamma < 0$ it will be destabilising.

We should not take the analogy with the phase field model too far, since in the phase field model a scaling of the parameters can be found in which these terms appear at leading order, whereas the 'surface tension' and 'kinetic undercooling' terms above appear only at first order, and so will not appreciably affect the solution unless the normal velocity or mean curvature of the boundary are of order $1/\lambda$. Thus, although the Ginzburg-Landau model seems to be a regularisation of the vectorial Stefan model, because the surface energy is very small we expect very intricate morphologies even from solutions to the Ginzburg-Landau equations. Numerical simulations [26, 44], and experimental observations [24, 62, 63], seem to agree with this.

There does not appear to be a scaling of the parameters in which the stabilising terms appear at leading order, since there is no variable well depth in the Ginzburg-Landau equations, so we cannot increase the size of the surface energy.

Finally we note that as well as there being a surface current density, as noted in Chapter 2, there may also be a surface charge density. We have that, from (3.108),

$$\varrho = -\frac{\alpha f^2 \Theta}{\lambda^2}.$$

Expanding ϱ in the inner region

$$\varrho_i = \frac{\varrho_i^{(0)}}{\lambda^2} + \frac{\varrho_i^{(1)}}{\lambda} + \cdots,$$

we find

$$\begin{aligned} \varrho_i^{(0)} &= -\alpha (f_i^{(0)})^2 \Theta_i^{(0)} = 0, \\ \varrho_i^{(1)} &= -\alpha (f_i^{(0)})^2 \Theta_i^{(1)}. \end{aligned}$$

Thus the surface charge density is given by

$$\varrho_s = -\int_{-\infty}^{\infty} \alpha (f_i^{(0)})^2 \Theta_i^{(1)}\, d\rho.$$

However, returning to (4.110) we see that if we assume that both $Q_{o,3}^{(4)}$ and $Q_{o,3}^{(5)}$ are bounded on $\Gamma^{(0)}$ then there can be no terms or order $1/\rho$ or $1/\rho^2$ in $Q_{i,3}^{(2)}$ as $\rho \to \infty$. For $\kappa \neq 1/\sqrt{2}$ this implies

$$Q_{i,3}^{(2)} = O(e^{-K\rho^2}), \qquad \text{as } \rho \to \infty.$$

(Note that by (3.121), when $\kappa = 1/\sqrt{2}$ it is possible that $Q_{i,3}^{(2)} = \text{const.} + O(e^{-K\rho^2})$, as $\rho \to \infty$.) For $\alpha \neq 1$ this implies $\Theta_i^{(1)} = O(e^{-K\rho^2})$, as $\rho \to \infty$. Since $\Theta_i^{(1)} \to 0$, as $\rho \to -\infty$, we have by the convexity of equation (4.109) that $\Theta_i^{(1)} \equiv 0$. Hence in this case $\varrho_s = 0$. Thus under the assumption that $Q_{o,3}^{(4)}$ and $Q_{o,3}^{(5)}$ are bounded on $\Gamma^{(0)}$, there can be a surface charge density at leading order only if $\kappa = 1/\sqrt{2}$ or $\alpha = 1$.

We complete this section by proving that the boundary conditions (4.86), (4.102) and (4.103) are indeed equivalent to the condition (2.8), when $v_n \neq 0$.

Lemma 1 (Origin of curvature terms) *The boundary conditions*

$$\begin{aligned} H_{o,3} &= 0, \qquad \text{on } \Gamma_N, & (4.120)\\ \frac{\partial H_{o,1}}{\partial n} + \tilde{\kappa}_1 H_{o,1} &= -v_n H_{o,1}, \qquad \text{on } \Gamma_N, & (4.121)\\ \frac{\partial H_{o,2}}{\partial n} + \tilde{\kappa}_2 H_{o,2} &= -v_n H_{o,2}, \qquad \text{on } \Gamma_N, & (4.122) \end{aligned}$$

can be written as the single boundary condition

$$(\text{curl } \boldsymbol{H}_o) \wedge \boldsymbol{n} = -v_n \boldsymbol{H}, \quad \text{on } \Gamma_N. \tag{4.123}$$

Proof. Near the boundary we transform co-ordinates to (s_1, s_2, n), where

$$\boldsymbol{r} = (x, y, z) = \boldsymbol{R}(s_1, s_2, \tau) + n\boldsymbol{n}(s_1, s_2, \tau).$$

As before (s_1, s_2, n) form a local orthogonal curvilinear co-ordinate system, with scaling factors

$$\begin{aligned} h_1 &= E^{1/2}[1 + n\tilde{\kappa}_1],\\ h_2 &= G^{1/2}[1 + n\tilde{\kappa}_2],\\ h_3 &= 1. \end{aligned}$$

In these co-ordinates curl $\boldsymbol{H}$ is given by

$$\begin{aligned} \text{curl } \boldsymbol{H} = &\\ &\frac{1}{G^{1/2}(1+n\tilde{\kappa}_2)}\left[\frac{\partial H_3}{\partial s_2} - \frac{\partial}{\partial n}\left(G^{1/2}(1+n\tilde{\kappa}_2)H_2\right)\right]\boldsymbol{e}_1\\ &-\frac{1}{E^{1/2}(1+n\tilde{\kappa}_1)}\left[\frac{\partial H_3}{\partial s_1} - \frac{\partial}{\partial n}\left(E^{1/2}(1+n\tilde{\kappa}_1)H_1\right)\right]\boldsymbol{e}_2\\ &-\frac{1}{(EG)^{1/2}(1+n\tilde{\kappa}_1)(1+n\tilde{\kappa}_2)}\left[\frac{\partial}{\partial s_1}\left(G^{1/2}(1+n\tilde{\kappa}_2)H_2\right)\right.\\ &\qquad\left. -\frac{\partial}{\partial s_2}\left(E^{1/2}(1+n\tilde{\kappa}_1)H_1\right)\right]\boldsymbol{n}. \end{aligned}$$

Hence

$$\begin{aligned} (\text{curl } \boldsymbol{H}) \wedge \boldsymbol{n} &= -\frac{1}{E^{1/2}(1+n\tilde{\kappa}_1)}\left[\frac{\partial H_3}{\partial s_1} - \frac{\partial}{\partial n}\left(E^{1/2}(1+n\tilde{\kappa}_1)H_1\right)\right]\boldsymbol{e}_1\\ &\quad -\frac{1}{G^{1/2}(1+n\tilde{\kappa}_2)}\left[\frac{\partial H_3}{\partial s_2} - \frac{\partial}{\partial n}\left(G^{1/2}(1+n\tilde{\kappa}_2)H_2\right)\right]\boldsymbol{e}_2. \end{aligned}$$

On the interface $n = 0$ we have $H_3 = 0$, hence $\partial H_3/\partial s_1 = 0$, $\partial H_3/\partial s_2 = 0$. Thus

$$(\text{curl } \boldsymbol{H}) \wedge \boldsymbol{n} = \left(\frac{\partial H_1}{\partial n} + \tilde{\kappa}_1 H_1\right) \boldsymbol{e}_1 + \left(\frac{\partial H_2}{\partial n} + \tilde{\kappa}_2 H_2\right) \boldsymbol{e}_2, \text{ on } \Gamma_N.$$

The boundary conditions (4.120)-(4.122) can then be seen to be equivalent to (4.123), for $v_n \neq 0$.

4.3 Asymptotic Solution of the Ginzburg-Landau Equations under Anisothermal Conditions

We now try to relate the Ginzburg-Landau model (3.95)-(3.97) to to the free boundary model (2.75)-(2.82) of Chapter 2, allowing the temperature to vary in time and space. As in the previous chapter a complete determination of the solution will involve initial and fixed boundary conditions. However, they will be left unspecified as our primary interest is rather the free boundary conditions.

The Ginzburg-Landau equations (3.95)-(3.97), together with the relation (3.1), and the heat balance equation (3.107) are

$$\begin{aligned}
-\frac{\alpha\lambda^2}{\kappa^2}\frac{\partial f}{\partial t} + \frac{\lambda^2}{\kappa^2}\nabla^2 f &= a(T)f + b(T)f^3 + \frac{f\,|\boldsymbol{Q}|^2}{\lambda^2}, && (4.124)\\
\alpha f^2\Theta + \text{div}(f^2\boldsymbol{Q}) &= 0, && (4.125)\\
-\lambda^2 \text{curl } \boldsymbol{H} &= \lambda^2\frac{\partial \boldsymbol{Q}}{\partial t} + \lambda^2\nabla\Theta + f^2\boldsymbol{Q}, && (4.126)\\
\boldsymbol{H} &= \text{curl } \boldsymbol{Q}. && (4.127)\\
\nabla^2 T &= \beta\frac{\partial T}{\partial t} - L(T)\frac{\partial (f^2)}{\partial t} - \gamma\left|\frac{\partial \boldsymbol{Q}}{\partial t} + \nabla\Theta\right|^2, && (4.128)
\end{aligned}$$

where β and γ may be functions of all the variables.

As before, we define $\Gamma(t)$ by

$$\Gamma(t) = \{\boldsymbol{r} \text{ such that } f(\boldsymbol{r}, t) = \eta\}, \tag{4.129}$$

where $0 < \eta < 1$.

Outer Expansions

Away from the transition region we formally expand all functions in powers of λ to obtain the outer expansions as

$$\begin{aligned} f_o(\boldsymbol{r},t,\lambda) &= f_o^{(0)}(\boldsymbol{r},t)+\lambda f_o^{(1)}(\boldsymbol{r},t)+\cdots, & (4.130)\\ \Theta_o(\boldsymbol{r},t,\lambda) &= \Theta_o^{(0)}(\boldsymbol{r},t)+\lambda \Theta_o^{(1)}(\boldsymbol{r},t)+\cdots, & (4.131)\\ \boldsymbol{Q}_o(\boldsymbol{r},t,\lambda) &= \boldsymbol{Q}_o^{(0)}(\boldsymbol{r},t)+\lambda \boldsymbol{Q}_o^{(1)}(\boldsymbol{r},t)+\cdots, & (4.132)\\ \boldsymbol{H}_o(\boldsymbol{r},t,\lambda) &= \boldsymbol{H}_o^{(0)}(\boldsymbol{r},t)+\lambda \boldsymbol{H}_o^{(1)}(\boldsymbol{r},t)+\cdots, & (4.133)\\ T_o(\boldsymbol{r},t,\lambda) &= T_o^{(0)}(\boldsymbol{r},t)+\lambda T_o^{(1)}(\boldsymbol{r},t)+\cdots, & (4.134)\\ \Gamma(t,\lambda) &= \Gamma^{(0)}(t)+\lambda \Gamma^{(1)}(t)+\cdots. & (4.135)\end{aligned}$$

We note that the expansions (4.130)-(4.133) may be discontinuous across $\Gamma^{(0)}(t)$, but will be smooth otherwise. Substituting (4.130)-(4.133) into (4.124)-(4.127) and equating powers of λ yields at leading order

$$\begin{aligned} f_o^{(0)}\,|\boldsymbol{Q}_o^{(0)}|^2 &= 0, & (4.136)\\ \mathrm{div}((f_o^{(0)})^2\boldsymbol{Q}_o^{(0)}) &= -\alpha(f_o^{(0)})^2\Theta_o^{(0)}, & (4.137)\\ (f_o^{(0)})^2\boldsymbol{Q}_o^{(0)} &= \mathbf{0}, & (4.138)\\ \boldsymbol{H}_o^{(0)} &= \mathrm{curl}\ \boldsymbol{Q}_o^{(0)}, & (4.139)\\ \nabla^2 T_o^{(0)} &= \beta\frac{\partial T_o^{(0)}}{\partial t} - L(T_o^{(0)})\frac{\partial (f_o^{(0)})^2}{\partial t} - \gamma\left|\frac{\partial \boldsymbol{Q}_o^{(0)}}{\partial t}+\nabla\Theta_o^{(0)}\right|^2. & (4.140)\end{aligned}$$

We see by (4.136) that either $f_o^{(0)} = 0$, or $\boldsymbol{Q}_o^{(0)} = \mathbf{0}$, corresponding to normal and superconducting regions respectively. We consider these cases separately.

Normal region

With $f_o^{(0)} \equiv 0$, $\boldsymbol{Q}_o^{(0)} \not\equiv \mathbf{0}$ we equate powers of λ at the next order in (4.124)-(4.126) to give

$$\begin{aligned} f_o^{(1)}\,|\boldsymbol{Q}_o^{(0)}|^2 &= 0, & (4.141)\\ \alpha(f_o^{(1)})^2\Theta_o^{(0)} + \mathrm{div}((f_o^{(1)})^2\boldsymbol{Q}_o^{(0)}) &= 0, & (4.142)\\ -\mathrm{curl}\ \boldsymbol{H}_o^{(0)} &= \frac{\partial \boldsymbol{Q}_o^{(0)}}{\partial t} + \nabla\Theta_o^{(0)} + (f_o^{(1)})^2\boldsymbol{Q}_o^{(0)}. & (4.143)\end{aligned}$$

By (4.141) we have that $f_o^{(1)} \equiv 0$. Taking the curl of equation (4.143) and using equation (4.139) we have

$$-(\text{curl})^2 \boldsymbol{H}_o^{(0)} = \frac{\partial \boldsymbol{H}_o^{(0)}}{\partial t},$$

or

$$\nabla^2 \boldsymbol{H}_o^{(0)} = \frac{\partial \boldsymbol{H}_o^{(0)}}{\partial t}. \tag{4.144}$$

Equation (4.140) becomes

$$\nabla^2 T_o^{(0)} = \beta \frac{\partial T_o^{(0)}}{\partial t} - \gamma \left| \text{curl } \boldsymbol{H}_o^{(0)} \right|^2 . \tag{4.145}$$

Superconducting region

With $f_o^{(0)} \not\equiv 0$, we have

$$\begin{aligned} \boldsymbol{Q}_o^{(0)} &\equiv \boldsymbol{0}, & (4.146)\\ \boldsymbol{H}_o^{(0)} &\equiv \boldsymbol{0}, & (4.147)\\ \Theta_o^{(0)} &\equiv 0. & (4.148) \end{aligned}$$

Equating powers of λ at the next order in each equation we have

$$\begin{aligned} 0 &= a(T_o^{(0)}) f_o^{(0)} b(T_o^{(0)})(f_o^{(0)})^3 + f_o^{(0)} \left| \boldsymbol{Q}_o^{(1)} \right|^2, & (4.149)\\ \text{div}\left((f_o^{(0)})^2 \boldsymbol{Q}_o^{(1)}\right) &= -\alpha (f_o^{(0)})^2 \Theta_o^{(1)}, & (4.150)\\ \boldsymbol{0} &= (f_o^{(0)})^2 \boldsymbol{Q}_o^{(1)}, & (4.151)\\ \boldsymbol{H}_o^{(1)} &= \text{curl } \boldsymbol{Q}_o^{(1)}. & (4.152) \end{aligned}$$

Therefore

$$\boldsymbol{Q}_o^{(1)} \equiv \boldsymbol{0}, \quad \boldsymbol{H}_o^{(1)} \equiv \boldsymbol{0}, \quad \Theta_o^{(1)} \equiv 0, \quad (f_o^{(0)})^2 = -\frac{a(T_o^{(0)})}{b(T_o^{(0)})}. \tag{4.153}$$

Equation (4.140) becomes

$$\begin{aligned} \nabla^2 T_o^{(0)} &= \beta \frac{\partial T_o^{(0)}}{\partial t} + L(T_o^{(0)}) \frac{\partial}{\partial t} \left(\frac{a(T_o^{(0)})}{b(T_o^{(0)})} \right), & (4.154)\\ &= \left[\beta + L(T_o^{(0)}) \frac{\partial}{\partial T_o^{(0)}} \left(\frac{a(T_o^{(0)})}{b(T_o^{(0)})} \right) \right] \frac{\partial T_o^{(0)}}{\partial t}. & (4.155) \end{aligned}$$

Inner Expansions

We note firstly that since equations (4.125)-(4.127) are identical to equations (4.23)-(4.25) the boundary condition

$$\text{curl}\, \boldsymbol{H}_o^{(0)} \wedge \boldsymbol{n}^{(0)} = -v_n^{(0)} \boldsymbol{H}_o^{(0)}, \qquad \text{on } \Gamma_N^{(0)}, \tag{4.156}$$

will hold exactly as in the previous section.

As before we define the inner variables by

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{R}(s_1, s_2) + \lambda \rho \boldsymbol{n}, \\ t &= \tau. \end{aligned}$$

where the interface $\Gamma(t)$ is given by the surface

$$\boldsymbol{r} = (x, y, z) = \boldsymbol{R}\left(s_1(x, y, z), s_2(x, y, z), t\right).$$

We have a local orthogonal curvilinear co-ordinate system (s_1, s_2, ρ) with scaling factors

$$h_1 = E^{1/2}(1 + \lambda\rho\kappa_1), \tag{4.157}$$

$$h_2 = G^{1/2}(1 + \lambda\rho\kappa_2), \tag{4.158}$$

$$h_3 = \lambda. \tag{4.159}$$

With

$$\boldsymbol{Q}_i = Q_{i,1}\boldsymbol{e}_1 + Q_{i,2}\boldsymbol{e}_2 + Q_{i,3}\boldsymbol{n}, \tag{4.160}$$

$$\boldsymbol{H}_i = H_{i,1}\boldsymbol{e}_1 + H_{i,2}\boldsymbol{e}_2 + H_{i,3}\boldsymbol{n}, \tag{4.161}$$

where $\boldsymbol{e}_i = \boldsymbol{R}_1 / \left|\boldsymbol{R}_1\right|$ as before, equations (4.124)-(4.128) become

$$\frac{\alpha\lambda v_n}{\kappa^2}\frac{\partial f_i}{\partial \rho} + \frac{1}{\kappa^2}\frac{\partial^2 f_i}{\partial \rho^2} + \frac{\lambda(\tilde{\kappa}_1 + \tilde{\kappa}_2)}{\kappa^2}\frac{\partial f_i}{\partial \rho} + O(\lambda^2) = a(T) f_i + b(T) f_i^3 + \frac{f_i}{\lambda^2}\left(Q_{i,1}^2 + Q_{i,2}^2 + Q_{i,3}^2\right), \tag{4.162}$$

$$\begin{aligned} &\lambda\alpha f_i^2 \Theta_i + \frac{1}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial\left[G^{1/2}(1+\lambda\rho\tilde{\kappa}_2) f_i^2 Q_{i,1}\right]}{\partial s_1}\right) \\ &+ \frac{1}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial\left[E^{1/2}(1+\lambda\rho\tilde{\kappa}_1) f_i^2 Q_{i,2}\right]}{\partial s_2}\right) \\ &+ \frac{1}{\lambda(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial\left[(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2) f_i^2 Q_{i,3}\right]}{\partial \rho}\right) = 0, \end{aligned} \tag{4.163}$$

$$
\begin{aligned}
\lambda\frac{\partial H_{i,2}}{\partial\rho} + \frac{\lambda^2\tilde{\kappa}_2 H_{i,2}}{1+\lambda\rho\tilde{\kappa}_2} - \frac{\lambda^2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial H_{i,3}}{\partial s_2} = \\
- v_n\lambda\frac{\partial Q_{i,1}}{\partial\rho} + \lambda^2\frac{\partial Q_{i,1}}{\partial\tau} - \frac{\lambda^2 v_1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial Q_{i,1}}{\partial s_1} \\
- \frac{\lambda^2 v_2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial Q_{i,1}}{\partial s_2} + \frac{\lambda^2 v_1\tilde{\kappa}_1 Q_{i,3}}{(1+\lambda\rho\tilde{\kappa}_1)} \\
- \frac{\lambda^2}{2(EG)^{1/2}}\left(\frac{v_2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial G}{\partial s_2} - \frac{v_1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial E}{\partial s_1}\right)Q_{i,2} \\
+ \frac{\lambda^2}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial\Theta_i}{\partial s_1} + f_i^2 Q_{i,1},
\end{aligned}
\tag{4.164}
$$

$$
\begin{aligned}
-\lambda\frac{\partial H_{i,1}}{\partial\rho} - \frac{\lambda^2\tilde{\kappa}_1 H_{i,1}}{1+\lambda\rho\tilde{\kappa}_1} + \frac{\lambda^2}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial H_{i,3}}{\partial s_1} = \\
- v_n\lambda\frac{\partial Q_{i,2}}{\partial\rho} + \lambda^2\frac{\partial Q_{i,2}}{\partial\tau} - \frac{\lambda^2 v_1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial Q_{i,2}}{\partial s_1} \\
- \frac{\lambda^2 v_2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial Q_{i,2}}{\partial s_2} + \frac{\lambda^2 v_2\tilde{\kappa}_2 Q_{i,3}}{(1+\lambda\rho\tilde{\kappa}_2)} \\
+ \frac{\lambda^2}{2(EG)^{1/2}}\left(\frac{v_2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial G}{\partial s_2} - \frac{v_1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial E}{\partial s_1}\right)Q_{i,1} \\
+ \frac{\lambda^2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial\Theta_i}{\partial s_2} + f_i^2 Q_{i,2},
\end{aligned}
\tag{4.165}
$$

$$
\begin{aligned}
\frac{\lambda^2}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial\left[E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)H_{i,1}\right]}{\partial s_2}\right) - \\
\frac{\lambda^2}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial\left[G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)H_{i,2}\right]}{\partial s_1}\right) = \\
- v_n\lambda\frac{\partial Q_{i,3}}{\partial\rho} + \lambda^2\frac{\partial Q_{i,3}}{\partial\tau} - \frac{\lambda^2 v_1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial Q_{i,3}}{\partial s_1} - \frac{\lambda^2 v_2}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial Q_{i,3}}{\partial s_2} \\
- \frac{\lambda^2 v_1\tilde{\kappa}_1 Q_{i,1}}{(1+\lambda\rho\tilde{\kappa}_1)} - \frac{\lambda^2 v_2\tilde{\kappa}_2 Q_{i,2}}{(1+\lambda\rho\tilde{\kappa}_2)} + \lambda\frac{\partial\Theta_i}{\partial\rho} + f_i^2 Q_{i,3},
\end{aligned}
\tag{4.166}
$$

$$
H_{i,1} = -\frac{1}{\lambda}\frac{\partial Q_{i,2}}{\partial\rho} - \frac{\tilde{\kappa}_2 Q_{i,2}}{1+\lambda\rho\tilde{\kappa}_2} + \frac{1}{G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)}\frac{\partial Q_{i,3}}{\partial s_2}, \tag{4.167}
$$

$$
H_{i,2} = \frac{1}{\lambda}\frac{\partial Q_{i,1}}{\partial\rho} + \frac{\tilde{\kappa}_1 Q_{i,1}}{1+\lambda\rho\tilde{\kappa}_1} - \frac{1}{E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)}\frac{\partial Q_{i,3}}{\partial s_1}, \tag{4.168}
$$

$$
\begin{aligned}
H_{i,3} &= \frac{1}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial\left[G^{1/2}(1+\lambda\rho\tilde{\kappa}_2)Q_{i,2}\right]}{\partial s_1}\right) \\
&\quad - \frac{1}{(EG)^{1/2}(1+\lambda\rho\tilde{\kappa}_1)(1+\lambda\rho\tilde{\kappa}_2)}\left(\frac{\partial\left[E^{1/2}(1+\lambda\rho\tilde{\kappa}_1)Q_{i,1}\right]}{\partial s_2}\right), \quad (4.169)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\lambda^2}\frac{\partial^2 T_i}{\partial\rho^2} + \frac{\tilde{\kappa}_1+\tilde{\kappa}_2}{\lambda}\frac{\partial T_i}{\partial\rho} = -\frac{v_n\beta(T_i)}{\lambda}\frac{\partial T_i}{\partial\rho} + \frac{v_n L(T_i)}{\lambda}\frac{\partial(f_i^2)}{\partial\rho} \\
&\quad - \left(-\frac{v_n}{\lambda}\frac{\partial Q_{i,1}}{\partial\rho} + \frac{\partial Q_{i,1}}{\partial\tau} - \frac{v_1}{E^{1/2}}\frac{\partial Q_{i,1}}{\partial s_1} - \frac{v_2}{G^{1/2}}\frac{\partial Q_{i,1}}{\partial s_2} + \frac{1}{E^{1/2}}\frac{\partial\Theta_i}{\partial s_1}\right)^2 \\
&\quad - \left(-\frac{v_n}{\lambda}\frac{\partial Q_{i,2}}{\partial\rho} + \frac{\partial Q_{i,2}}{\partial\tau} - \frac{v_1}{E^{1/2}}\frac{\partial Q_{i,2}}{\partial s_1} - \frac{v_2}{G^{1/2}}\frac{\partial Q_{i,2}}{\partial s_2} + \frac{1}{G^{1/2}}\frac{\partial\Theta_i}{\partial s_2}\right)^2 \\
&\quad - \left(-\frac{v_n}{\lambda}\frac{\partial Q_{i,3}}{\partial\rho} + \frac{\partial Q_{i,3}}{\partial\tau} - \frac{v_1}{E^{1/2}}\frac{\partial Q_{i,3}}{\partial s_1} - \frac{v_2}{G^{1/2}}\frac{\partial Q_{i,3}}{\partial s_2} + \frac{1}{\lambda}\frac{\partial\Theta_i}{\partial\rho}\right)^2 + O(1),
\end{aligned}
\quad (4.170)
$$

where $\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_n\boldsymbol{n}$. We formally expand all functions in the inner variables in powers of λ to obtain the inner expansions as

$$
\begin{aligned}
f_i(s_1,s_2,\rho,\tau,\lambda) &= f_i^{(0)}(s_1,s_2,\rho,\tau) + \lambda f_i^{(1)}(s_1,s_2,\rho,\tau) + \cdots, && (4.171) \\
\Theta_i(s_1,s_2,\rho,\tau,\lambda) &= \Theta_i^{(0)}(s_1,s_2,\rho,\tau) + \lambda\Theta_i^{(0)}(s_1,s_2,\rho,\tau) + \cdots, && (4.172) \\
\boldsymbol{Q}_i(s_1,s_2,\rho,\tau,\lambda) &= \boldsymbol{Q}_i^{(0)}(s_1,s_2,\rho,\tau) + \lambda\boldsymbol{Q}_i^{(1)}(s_1,s_2,\rho,\tau) + \cdots, && (4.173) \\
\boldsymbol{H}_i(s_1,s_2,\rho,\tau,\lambda) &= \boldsymbol{H}_i^{(0)}(s_1,s_2,\rho,\tau) + \lambda\boldsymbol{H}_i^{(1)}(s_1,s_2,\rho,\tau) + \cdots, && (4.174) \\
T_i(s_1,s_2,\rho,\tau,\lambda) &= T_i^{(0)}(s_1,s_2,\rho,\tau) + \lambda T_i^{(1)}(s_1,s_2,\rho,\tau) + \cdots. && (4.175)
\end{aligned}
$$

We also expand

$$
\boldsymbol{R}(s_1,s_2,\tau,\lambda) = \boldsymbol{R}^{(0)}(s_1,s_2,\tau) + \lambda\boldsymbol{R}^{(1)}(s_1,s_2,\tau) + \cdots,
$$

which gives

$$
E(s_1,s_2,\tau,\lambda) = E^{(0)}(s_1,s_2,\tau) + \lambda E^{(1)}(s_1,s_2,\tau) + \cdots,
$$

etc. Substituting the expansions (4.171)-(4.175) into equations (4.162)-(4.170) and equating powers of λ we find at leading order

$$
f_i^{(0)}\left((Q_{i,1}^{(0)})^2 + (Q_{i,2}^{(0)})^2 + (Q_{i,3}^{(0)})^2\right) = 0. \quad (4.176)
$$

Matching the inner solution with the superconducting region gives

$$(f_i^{(0)})^2 \to -a(T_o^{(0)}(\boldsymbol{R}_S^{(0)}, t))/b(T_o^{(0)}(\boldsymbol{R}_S^{(0)}, t)), \qquad \text{as } \rho \to -\infty.$$

Hence $f_i^{(0)} \not\equiv 0$. Therefore

$$\boldsymbol{Q}_i^{(0)} \equiv \boldsymbol{0}. \tag{4.177}$$

Equating coefficients of λ at the next order in equations (4.162)-(4.169) we have

$$\begin{aligned}
\frac{1}{\kappa^2}\frac{\partial^2 f_i^{(0)}}{\partial \rho^2} &= a(T_i^{(0)})f_i^{(0)} + b(T_i^{(0)})(f_i^{(0)})^3 \\
&\qquad + f_i^{(0)}\left\{(Q_{i,1}^{(1)})^2 + (Q_{i,2}^{(1)})^2 + (Q_{i,3}^{(1)})^2\right\}, && (4.178)\\
\frac{\partial\left((f_i^{(0)})^2 Q_{i,3}^{(1)}\right)}{\partial \rho} &= -\alpha (f_i^{(0)})^2 \Theta_i^{(0)}, && (4.179)\\
\frac{\partial H_{i,2}^{(0)}}{\partial \rho} &= (f_i^{(0)})^2 Q_{i,1}^{(1)}, && (4.180)\\
-\frac{\partial H_{i,1}^{(0)}}{\partial \rho} &= (f_i^{(0)})^2 Q_{i,2}^{(1)}, && (4.181)\\
0 &= (f_i^{(0)})^2 Q_{i,3}^{(1)} + \frac{\partial \Theta_i^{(0)}}{\partial \rho}, && (4.182)\\
H_{i,1}^{(0)} &= -\frac{\partial Q_{i,2}^{(1)}}{\partial \rho}, && (4.183)\\
H_{i,2}^{(0)} &= \frac{\partial Q_{i,1}^{(1)}}{\partial \rho}, && (4.184)\\
H_{i,3}^{(0)} &= 0, && (4.185)\\
\frac{\partial^2 T_i^{(0)}}{\partial \rho^2} &= 0. && (4.186)
\end{aligned}$$

The outer expansions imply the boundary conditions

$$f_i^{(0)} \to -\frac{a(T_o^{(0)}(\boldsymbol{R}_S^{(0)}, t))}{b(T_o^{(0)}(\boldsymbol{R}_S^{(0)}, t))},\ \boldsymbol{Q}_i^{(1)} \to 0,\ \boldsymbol{H}_i^{(0)} \to 0,\ \Theta_i^{(0)} \to 0,\ \text{as } \rho \to -\infty, \tag{4.187}$$

$$f_i^{(0)} \to 0,\ \text{as } \rho \to \infty, \tag{4.188}$$

$$f_i^{(0)}(s_1, s_2, 0, \tau) = \eta. \tag{4.189}$$

Our aim is to determine the values of $\boldsymbol{H}_i$, $\boldsymbol{Q}_i$ and Θ_i as $\rho \to \infty$. Exactly as in the previous chapter we find $\Theta_i^{(0)} \equiv 0$, $Q_{i,3}^{(1)} \equiv 0$.

Integrating (4.186) we have

$$T_i^{(0)} = A\rho + B.$$

Since $T_i^{(0)}$ must be bounded as $\rho \to \pm\infty$ if the temperature on the interface is bounded and we are to match with the outer region we must have $A = 0$. Then

$$T_i^{(0)} = B = T_o^{(0)}(\boldsymbol{R}_N^{(0)}, t) = T_o^{(0)}(\boldsymbol{R}_S^{(0)}, t). \tag{4.190}$$

We multiply (4.181) by $-H_{i,1}^{(0)}$, (4.180) by $H_{i,2}^{(0)}$, (4.178) by $\partial f_i^{(0)}/\partial\rho$, add and integrate to give

$$\frac{1}{\kappa^2}\left(\frac{\partial f_i^{(0)}}{\partial \rho}\right) + \left(H_{i,1}^{(0)}\right)^2 + \left(H_{i,2}^{(0)}\right)^2 =$$
$$\frac{b(B)}{2}\left((f_i^{(0)})^2 + \frac{a(B)}{b(B)}\right)^2 + (f_i^{(0)})^2\left\{(Q_{i,1}^{(1)})^2 + (Q_{i,2}^{(1)})^2\right\},$$

where we have used the fact that $(f_i^{(0)})^2 \to -a(B)/b(B)$, $Q_{i,1}^{(1)} \to 0$, $Q_{i,2}^{(1)} \to 0$, as $\rho \to -\infty$. Letting ρ tend to infinity we have

$$\lim_{\rho\to\infty}\left\{(H_{i,1}^{(0)})^2 + (H_{i,2}^{(0)})^2\right\}^{1/2} = \lim_{\rho\to\infty} |\boldsymbol{H}_i^{(0)}| = \frac{|a(B)|}{\sqrt{2b(B)}},$$

since $f_i^{(0)} \to 0$, $(f_i^{(0)})^2\left\{(Q_{i,1}^{(1)})^2 + (Q_{i,2}^{(1)})^2\right\} \to 0$, as $\rho \to \infty$. Using matching condition (A.1) we have

$$\begin{aligned}\left\{(H_{o,1}^{(0)})^2 + (H_{o,2}^{(0)})^2\right\}^{1/2}(\boldsymbol{R}_N^{(0)}, t) = |\boldsymbol{H}_o^{(0)}|\,(\boldsymbol{R}_N^{(0)}, t) &= \frac{|a(B)|}{\sqrt{2b(B)}} \\ &= \frac{|a(T_o^{(0)}(\boldsymbol{R}^{(0)}, t))|}{\sqrt{2b(T_o^{(0)}(\boldsymbol{R}^{(0)}, t))}} \\ &= H_c(T_o^{(0)}(\boldsymbol{R}^{(0)}, t)), \quad (4.191)\end{aligned}$$

by (3.5).

Equating powers of λ at the next order in equation (4.170) yields

$$\frac{\partial^2 T_i^{(1)}}{\partial \rho^2} = L(T_i^{(0)})v_n^{(0)}\frac{\partial (f_i^{(0)})^2}{\partial \rho}. \tag{4.192}$$

Integrating over $(-\infty, \infty)$ gives

$$\begin{aligned}\left[\frac{\partial T_i^{(1)}}{\partial \rho}\right]_{-\infty}^{\infty} &= L(B)v_n^{(0)}\left[(f_i^{(0)})^2\right]_{-\infty}^{\infty} \\ &= L(B)v_n^{(0)}\frac{a(B)}{b(B)}.\end{aligned}$$

Matching with the outer solution implies

$$\left[\frac{\partial T_o^{(0)}}{\partial n}\right]_S^N = v_n^{(0)} L(T_o^{(0)}(\boldsymbol{R}^{(0)},t))\frac{a(T_o^{(0)}(\boldsymbol{R}^{(0)},t))}{b(T_o^{(0)}(\boldsymbol{R}^{(0)},t))}. \tag{4.193}$$

We can now solve the outer problem (4.144), (4.145), (4.147), (4.155) with the interface conditions (4.156), (4.190), (4.191), (4.193).

Let us examine the problem for temperatures close to T_c which is the situation in which the Ginzburg-Landau equations are supported by the microscopic theory. In this case we can expand all functions of temperature in powers of T and keep only the leading terms. We have $a(T) \sim T,\ b(T) \sim 1$. We take $L(T) \sim L =$const., $\beta(T) \sim \beta =$const., $\gamma(T) \sim \gamma =$const. Then

$$\begin{aligned}
\nabla^2 \boldsymbol{H}_o^{(0)} &= \frac{\partial \boldsymbol{H}_o^{(0)}}{\partial t}, \quad \text{in the normal region}, && (4.194)\\
\nabla^2 T_o^{(0)} &= \beta\frac{\partial T_o^{(0)}}{\partial t} - \gamma\,|\text{curl } \boldsymbol{H}_o^{(0)}|^2, \quad \text{in the normal region}, && (4.195)\\
\boldsymbol{H}_o^{(0)} &= 0, \quad \text{in the superconducting region}, && (4.196)\\
\nabla^2 T_o^{(0)} &= (\beta + L)\frac{\partial T_o^{(0)}}{\partial t}, \quad \text{in the superconducting region}, && (4.197)\\
\text{curl } \boldsymbol{H}_o^{(0)} \wedge \boldsymbol{n}^{(0)} &= -v_n^{(0)}\boldsymbol{H}_o^{(0)}, \ \text{on } \Gamma_N^{(0)}, && (4.198)\\
|\,\boldsymbol{H}_o^{(0)}| &= -\frac{T}{\sqrt{2}}, \ \text{on } \Gamma_N^{(0)}, && (4.199)\\
[T]_S^N &= 0, && (4.200)\\
\left[\frac{\partial T}{\partial n}\right]_S^N &= v_n^{(0)} LT. && (4.201)
\end{aligned}$$

We stated in the introduction that the dimensional latent heat $\hat{l}$ is given by

$$\hat{l}(\tilde{T}) = -\mu \tilde{T} H_c \frac{dH_c}{d\tilde{T}},$$

where ρ is the density, μ is the permeability, and $\tilde{T}$ is the absolute temperature. Hence, on non-dimensionalizing

$$\hat{l} = kT_c\mu\varsigma\hat{L},$$

as in Section 2.4, and linearising in T we find

$$\hat{L}(T) = -LT,$$

where $L = \alpha^2/2\beta k T_c \mu_s \varsigma$. Thus we see that (4.201) is in agreement with (2.82), for temperatures close to T_c. We also note that the model (4.194)-(4.201) contains an extra term $L\partial T/\partial t$ in equation (4.197) which does not appear in the model (2.75)-(2.82). This term is a source term and is due to the fact that the number of superconducting electrons, and hence the latent heat, is proportional to T near T_c. Thus a change in temperature in the superconducting region will produce a release or absorbtion of latent heat. This effect was not taken into consideration in the model (2.75)-(2.82).

Chapter 5

Nucleation of Superconductivity in Decreasing Fields

Up to this point we have been assuming that the transformation of a superconductor occurs via the propagation of phase boundaries inwards from the surface of the sample, which separate regions of nearly normal material from regions of nearly superconducting material. We consider now a completely different scenario, by looking for solutions of the Ginzburg-Landau equations with $|\Psi| \ll 1$. In these solutions, which are steady state solutions, $|\Psi|$ grows gradually as the applied field h is decreased, in contrast to the abrupt rise in $|\Psi|$ to 1 as h crosses H_c that was described in the previous chapter.

Having then described the two possible methods of changing phase, a stability analysis of the normal state will give us an insight as to which of the two will occur in different parameter regimes. We will find that for bulk Type I superconductors the change of phase occurs by the method described earlier, whereas for bulk Type II superconductors the change of phase occurs by the method of the present chapter.

5.1 Nucleation of Superconductivity in Decreasing Fields

5.1.1 Superconductivity in a Body of Arbitrary Shape in an External Magnetic Field

This problem was considered in [50]. We work through their analysis as we will find it helpful when we go on to consider the stability of the solution.

Consider a superconducting body occupying a region Ω bounded by a surface $\partial\Omega$, placed in an originally uniform magnetic field of strength h. We work on the lengthscale of the penetration depth by rescaling length and $\boldsymbol{A}$ with λ. Under isothermal conditions the steady state Ginzburg-Landau equations, together with boundary and other conditions, (3.14)-(3.20), are then

$$
\begin{aligned}
((i/\kappa)\nabla + \boldsymbol{A})^2\Psi &= \Psi(1-|\Psi|^2), \quad \text{in } \Omega, && (5.1)\\
-(\mathrm{curl})^2\boldsymbol{A} &= (i/2\kappa)(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) + |\Psi|^2\boldsymbol{A}, \quad \text{in } \Omega, && (5.2)\\
(\mathrm{curl})^2\boldsymbol{A} &= \boldsymbol{0}, \quad \text{outside } \Omega, && (5.3)\\
\boldsymbol{n}\cdot((i/\kappa)\nabla + \boldsymbol{A})\Psi &= -(i/d)\Psi, \quad \text{on } \partial\Omega, && (5.4)\\
[\boldsymbol{n}\wedge\boldsymbol{A}] &= \boldsymbol{0}, && (5.5)\\
[\boldsymbol{n}\wedge(1/\mu)\mathrm{curl}\,\boldsymbol{A}] &= \boldsymbol{0}, && (5.6)\\
\mathrm{curl}\,\boldsymbol{A} &\to h\hat{\boldsymbol{z}}, \quad \text{as } r\to\infty. && (5.7)
\end{aligned}
$$

Here $\hat{\boldsymbol{z}}$ is a unit vector in the z-direction, and r is the distance from the origin. As in Chapter 3, $\boldsymbol{n}$ is the outward normal on $\partial\Omega$, and [] stands for the jump in the enclosed quantity across $\partial\Omega$. Equation (5.7) states that the field reduces to the applied field far from the body. In [50] the case $d = \infty$ is considered, that is, when the non-superconducting region is a vacuum, but the extra complication introduced by finite d is not great. As in Chapter 3 we choose the gauge of $\boldsymbol{A}$ by imposing the extra condition

$$\mathrm{div}\,\boldsymbol{A} = 0, \tag{5.8}$$

which proves convenient in later calculations. We note that even though (5.8) does not fix the solution uniquely (since, as mentioned in Chapter 3, $(e^{i\kappa\eta}\Psi, \boldsymbol{A} + \nabla\eta)$ will also be a solution, where η is any harmonic function) this does not affect the physical quantities since curl $\boldsymbol{A}$ and $|\Psi|$ are invariant under such a transformation.

The solution of (5.1)-(5.8) which corresponds to the normal state is

$$\Psi \equiv 0, \qquad \boldsymbol{A} = h\boldsymbol{A}_N. \tag{5.9}$$

Inserting (5.9) into (5.1)-(5.8) yields the following equations for $\boldsymbol{A}_N$:

$$
\begin{aligned}
(\mathrm{curl})^2\boldsymbol{A}_N &= \boldsymbol{0}, \quad \text{except on } \partial\Omega, && (5.10)\\
[\boldsymbol{n}\wedge\boldsymbol{A}_N] &= \boldsymbol{0}, && (5.11)
\end{aligned}
$$

$$
\begin{aligned}
[\boldsymbol{n} \wedge (1/\mu)\text{curl } \boldsymbol{A}_N] &= \boldsymbol{0}, & (5.12)\\
\text{curl } \boldsymbol{A}_N &\to \hat{\boldsymbol{z}}, \quad \text{as } r \to \infty, & (5.13)\\
\text{div } \boldsymbol{A}_N &= 0. & (5.14)
\end{aligned}
$$

Equations (5.10)-(5.14) correspond to the problem of determining the vector potential for a permeable body carrying no current placed in an originally uniform unit external magnetic field, and have well known methods of solution in magnetostatics.

[50] go on to seek a superconducting solution (i.e. one in which $\Psi \not\equiv 0$) in which $|\Psi| \ll 1$, which depends continuously on a parameter ϵ (which measures the magnitude of $|\Psi|^2$), and which reduces to (5.9) for $\epsilon = 0$. They introduce ϵ, ψ and $\boldsymbol{a}$ through the equations

$$
\begin{aligned}
\Psi &= \epsilon^{1/2}\psi, & (5.15)\\
\boldsymbol{A} &= h\boldsymbol{A}_N + \epsilon\boldsymbol{a}, \quad 0 < \epsilon \ll 1. & (5.16)
\end{aligned}
$$

The relative scalings of ψ and $\boldsymbol{a}$ here are motivated by requiring that $\Psi \mid \Psi \mid^2$ balances with $\boldsymbol{A} - h\boldsymbol{A}_N$ in equation (5.1) (the problem is similar to that of the buckling of an Euler strut [39]).

Insertion of (5.15), (5.16) into (5.1)-(5.8) yields

$$
\begin{aligned}
((i/\kappa)\nabla + h\boldsymbol{A}_N)^2\psi - \psi &= -\epsilon[|\psi|^2\psi + 2h\psi(\boldsymbol{A}_N\cdot\boldsymbol{a}) + 2(i/\kappa)(\boldsymbol{a}\cdot\nabla\psi)]\\
&\quad -\epsilon^2|\boldsymbol{a}|^2\psi, \quad \text{in } \Omega, & (5.17)\\
-(\text{curl})^2\boldsymbol{a} &= (1/2\kappa)(\psi^*\nabla\psi - \psi\nabla\psi^*)\\
&\quad + |\psi|^2(h\boldsymbol{A}_N + \epsilon\boldsymbol{a}), \quad \text{in } \Omega, & (5.18)\\
(\text{curl})^2\boldsymbol{a} &= \boldsymbol{0}, \quad \text{outside } \Omega, & (5.19)\\
\boldsymbol{n}\cdot((i/\kappa)\nabla + h\boldsymbol{A}_N)\psi + (i/d)\psi &= -\epsilon(\boldsymbol{n}\cdot\boldsymbol{a})\psi, \quad \text{on } \partial\Omega, & (5.20)\\
[\boldsymbol{n}\wedge\boldsymbol{a}] &= \boldsymbol{0}, & (5.21)\\
[\boldsymbol{n}\wedge(1/\mu)\text{curl } \boldsymbol{a}] &= \boldsymbol{0}, & (5.22)\\
\text{curl } \boldsymbol{a} &\to \boldsymbol{0}, \ \text{as } r\to\infty, & (5.23)\\
\text{div } \boldsymbol{a} &= 0. & (5.24)
\end{aligned}
$$

We expand h, $\boldsymbol{a}$ and ψ in powers of ϵ:

$$h = h^{(0)} + \epsilon h^{(1)} + \cdots, \tag{5.25}$$

$$\boldsymbol{a} = \boldsymbol{a}^{(0)} + \epsilon \boldsymbol{a}^{(1)} + \cdots, \tag{5.26}$$

$$\psi = \psi^{(0)} + \epsilon \psi^{(1)} + \cdots. \tag{5.27}$$

The problem is now to determine the coefficients in these expansions. We substitute the expansions (5.25)-(5.27) into equations (5.17)-(5.24) and equate powers of ϵ. At leading order we have

$$((i/\kappa)\nabla + h^{(0)}\boldsymbol{A}_N)^2\psi^{(0)} - \psi^{(0)} = 0, \quad \text{in } \Omega, \tag{5.28}$$

$$\begin{aligned} -(\text{curl})^2\boldsymbol{a}^{(0)} &= (i/2\kappa)(\psi^{(0)*}\nabla\psi^{(0)} - \psi^{(0)}\nabla\psi^{(0)*}) \\ &\quad + h^{(0)} \mid \psi^{(0)} \mid^2 \boldsymbol{A}_N, \quad \text{in } \Omega, \end{aligned} \tag{5.29}$$

$$(\text{curl})^2\boldsymbol{a}^{(0)} = \boldsymbol{0}, \quad \text{outside } \Omega, \tag{5.30}$$

$$\boldsymbol{n} \cdot ((i/\kappa)\nabla + h^{(0)}\boldsymbol{A}_N)\psi^{(0)} = -(i/d)\psi^{(0)}, \quad \text{on } \partial\Omega, \tag{5.31}$$

$$[\boldsymbol{n} \wedge \boldsymbol{a}^{(0)}] = \boldsymbol{0}, \tag{5.32}$$

$$[\boldsymbol{n} \wedge (1/\mu)\text{curl } \boldsymbol{a}^{(0)}] = \boldsymbol{0}, \tag{5.33}$$

$$\text{curl } \boldsymbol{a}^{(0)} \to \boldsymbol{0}, \text{ as } r \to \infty, \tag{5.34}$$

$$\text{div } \boldsymbol{a}^{(0)} = 0. \tag{5.35}$$

Equations (5.28) and (5.31) form an unconventional eigenvalue problem for $h^{(0)}$. The eigenvalues are independent of the gauge of $\boldsymbol{A}_N$. In the examples we consider they will also be real and discrete, as was postulated by [50]. The upper critical field h_{c_2} is defined to be the largest positive eigenvalue. Let the normalised eigenfunction corresponding to $h^{(0)}$ be θ, i.e. θ is such that

$$\int_\Omega \mid \theta \mid^2 \, dV = 1.$$

Then $\psi^{(0)} = \beta\theta$ where β is constant, and $\boldsymbol{a}^{(0)} = \mid \beta \mid^2 \hat{\boldsymbol{a}}^{(0)}$, where

$$-(\text{curl})^2\hat{\boldsymbol{a}}^{(0)} = (i/2\kappa)(\theta^*\nabla\theta - \theta\nabla\theta^*) + h^{(0)} \mid \theta \mid^2 \boldsymbol{A}_N, \quad \text{in } \Omega, \tag{5.36}$$

$$(\text{curl})^2\hat{\boldsymbol{a}}^{(0)} = \boldsymbol{0}, \quad \text{outside } \Omega, \tag{5.37}$$

$$[\boldsymbol{n} \wedge \hat{\boldsymbol{a}}^{(0)}] = \boldsymbol{0}, \tag{5.38}$$

$$[\boldsymbol{n} \wedge (1/\mu)\text{curl } \hat{\boldsymbol{a}}^{(0)}] = \boldsymbol{0}, \tag{5.39}$$

$$\text{curl}\ \hat{\boldsymbol{a}}^{(0)} \rightarrow \mathbf{0}, \text{ as } r \rightarrow \infty, \tag{5.40}$$

$$\text{div}\ \hat{\boldsymbol{a}}^{(0)} = 0, \tag{5.41}$$

which is the problem of determining the vector potential $\hat{\boldsymbol{a}}^{(0)}$ due to a permeable body carrying a specified real current distribution, since the right-hand side of (5.36) is known. Again, well known methods of solution are available.

We have now determined the upper critical field h_{c_2} and leading order approximations to ψ and $\boldsymbol{a}$, once we have determined the constant β.

Equating coefficients of ϵ in (5.17)-(5.24) yields

$$\begin{aligned}
((i/\kappa)\nabla + h^{(0)}\boldsymbol{A}_N)^2\psi^{(1)} - \psi^{(1)} &= -\,|\,\psi^{(0)}\,|^2\,\psi^{(0)} - 2h^{(1)}h^{(0)}\,|\,\boldsymbol{A}_N\,|^2\,\psi^{(0)} \\
&\quad - 2h^{(1)}(i/\kappa)(\boldsymbol{A}_N\cdot\nabla\psi^{(0)}) \\
&\quad - 2h^{(0)}\psi^{(0)}(\boldsymbol{A}_N\cdot\boldsymbol{a}^{(0)}) \\
&\quad - 2(i/\kappa)(\boldsymbol{a}^{(0)}\cdot\nabla\psi^{(0)}), \text{ in } \Omega, \qquad (5.42)\\
-\,(\text{curl})^2\boldsymbol{a}^{(1)} &= (i/2\kappa)(\psi^{(0)*}\nabla\psi^{(1)} + \psi^{(1)*}\nabla\psi^{(0)}) \\
&\quad - (i/2\kappa)(\psi^{(0)}\nabla\psi^{(1)*} + \psi^{(1)}\nabla\psi^{(0)*}) \\
&\quad + h^{(0)}\boldsymbol{A}_N(\psi^{(0)}\psi^{(1)*} + \psi^{(0)*}\psi^{(1)}) \\
&\quad + h^{(1)}\,|\,\psi^{(0)}\,|^2\,\boldsymbol{A}_N \\
&\quad + |\,\psi^{(0)}\,|^2\,\boldsymbol{a}^{(0)}, \text{ in } \Omega, \qquad (5.43)\\
(\text{curl})^2\boldsymbol{a}^{(1)} &= \mathbf{0}, \text{ outside } \Omega, \qquad (5.44)\\
\boldsymbol{n}\cdot((i/\kappa)\nabla + h^{(0)}\boldsymbol{A}_N)\psi^{(1)} + (i/d)\psi^{(1)} &= -\,\boldsymbol{n}\cdot(\boldsymbol{a}^{(0)} + h^{(1)}\boldsymbol{A}_N)\psi^{(0)}, \\
&\qquad\qquad \text{on } \partial\Omega, \qquad (5.45)\\
[\boldsymbol{n}\wedge\boldsymbol{a}^{(1)}] &= \mathbf{0}, \qquad (5.46)\\
[\boldsymbol{n}\wedge(1/\mu)\text{curl}\ \boldsymbol{a}^{(1)}] &= \mathbf{0}, \qquad (5.47)\\
\text{curl}\ \boldsymbol{a}^{(1)} &\rightarrow \mathbf{0}, \text{ as } r\rightarrow\infty, \qquad (5.48)\\
\text{div}\ \boldsymbol{a}^{(1)} &= 0. \qquad (5.49)
\end{aligned}$$

Assuming $\psi^{(1)}$ and $h^{(1)}$ were known these equations would again correspond to the problem of determining the vector potential due to a permeable body carrying a known current distribution. Thus $\boldsymbol{a}^{(1)}$ is fixed once $\psi^{(1)}$ and $h^{(1)}$ are given.

Now (5.42) and (5.45) are inhomogeneous versions of (5.28) and (5.31) and therefore have a solution if and only if an appropriate solvability condition is satisfied. This condition is derived by multiplying both sides of (5.42) by $\psi^{(0)*}$ and integrating over Ω. After considerable manipulation the result is

$$h^{(1)} = \frac{\int_\Omega |\,\psi^{(0)}\,|^4 \ dV - 2\int_\Omega \boldsymbol{a}^{(0)} \cdot (\mathrm{curl})^2 \boldsymbol{a}^{(0)}\, dV}{2\int_\Omega \boldsymbol{A}_N \cdot (\mathrm{curl})^2 \boldsymbol{a}^{(0)}\, dV}, \tag{5.50}$$

$$= |\,\beta\,|^2 \, \frac{\int_\Omega |\,\theta\,|^4 \ dV - 2\int_\Omega \hat{\boldsymbol{a}}^{(0)} \cdot (\mathrm{curl})^2 \hat{\boldsymbol{a}}^{(0)}\, dV}{2\int_\Omega \boldsymbol{A}_N \cdot (\mathrm{curl})^2 \hat{\boldsymbol{a}}^{(0)}\, dV}. \tag{5.51}$$

Thus $|\,\beta\,|$ is given in terms of $h^{(1)}$. Note that in order for a superconducting solution to exist this equation also determines the sign of $h^{(1)}$. When $|\,\beta\,|$ is given by (5.50), (5.42) and (5.45) have a solution $\psi^{(1)}$, which will still contain an undetermined constant. This constant is determined by a solvability condition for the second order equations.

We have now determined a solution in the form

$$h = h^{(0)} + \epsilon h^{(1)} + \cdots \tag{5.52}$$

$$\Psi = \epsilon^{1/2}[\psi^{(0)} + \epsilon \psi^{(1)} + \cdots] \tag{5.53}$$

$$\boldsymbol{A} = h\boldsymbol{A}_N + \epsilon[\boldsymbol{a}^{(0)} + \epsilon \boldsymbol{a}^{(1)} + \cdots] \tag{5.54}$$

Equation (5.54) leads to a magnetic field

$$\mathrm{curl}\ \boldsymbol{A} = \boldsymbol{H} = h\,\mathrm{curl}\ \boldsymbol{A}_N + \epsilon\,\mathrm{curl}\ \boldsymbol{a}^{(0)} + \cdots \tag{5.55}$$

If $h^{(1)} < 0$ we have a solution for all values of the external field slightly below a certain critical value $h^{(0)}$. If $\hat{\boldsymbol{z}} \cdot \mathrm{curl}\ \boldsymbol{a}^{(0)} < 0$ in Ω, as is the case in the one-dimensional example which follows, then the magnetic field within the body would be less than its value in the normal state. This would be the beginning of the Meissner effect.

Since θ is normalised we have that $\|\Psi\| = (\int_\Omega |\,\Psi\,|^2 \ dV)^{1/2} = \epsilon^{1/2}\,|\,\beta\,|$ and so $\|\Psi\|$ increases as $|\,h^{(0)} - h\,|^{1/2}$ for h close to $h^{(0)}$, as shown in Fig. 5.1.

5.1.2 One-dimensional Example

We demonstrate the techniques of the previous section with a one-dimensional example. We consider the case of an infinite superconducting body, when explicit solutions can be found.

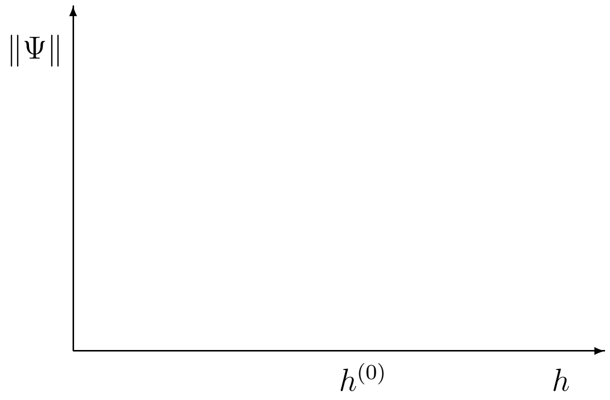

Figure 5.1: Pitchfork bifurcation from $\Psi \equiv 0$ at $h = h^{(0)}$.

As shown in Section 3.3, in one dimension we may choose Ψ to be real and the vector potential $\boldsymbol{A}$ to be directed along the y-axis, so that $\boldsymbol{A} = (0, A, 0)$. The Ginzburg-Landau equations (3.113), (3.114), together with the appropriate boundary conditions are

$$\kappa^{-2}\Psi'' = \Psi^3 - \Psi + \Psi A^2, \tag{5.56}$$

$$A'' = \Psi^2 A, \tag{5.57}$$

$$\Psi' \to 0, \quad \text{as } |x| \to \infty, \tag{5.58}$$

$$A' \to h, \quad \text{as } |x| \to \infty, \tag{5.59}$$

where $' \equiv d/dx$. The solution corresponding to the normal state is

$$\Psi \equiv 0, \qquad A = hx.$$

(We note that the equations and boundary conditions are translationally invariant. Thus, although we could add an arbitrary constant to A such a solution is simply a translation of this solution.) We introduce ϵ through the equations

$$\Psi = \epsilon^{1/2}\psi, \tag{5.60}$$

$$A = hx + \epsilon a, \tag{5.61}$$

as before. Substituting (5.60), (5.61) into (5.56)-(5.59) we find

$$\kappa^{-2}\psi'' = \epsilon\psi^3 - \psi + \psi(h^2x^2 + 2\epsilon hxa + \epsilon^2 a^2), \tag{5.62}$$

$$a'' = \psi^2(hx + \epsilon a), \tag{5.63}$$

$$\psi' \rightarrow 0, \quad \text{as } |x| \rightarrow \infty, \tag{5.64}$$
$$a' \rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \tag{5.65}$$

We expand h, ψ and a in powers of ϵ as before:

$$h = h^{(0)} + \epsilon h^{(1)} + \cdots, \tag{5.66}$$
$$\psi = \psi^{(0)} + \epsilon \psi^{(1)} + \cdots, \tag{5.67}$$
$$a = a^{(0)} + \epsilon a^{(1)} + \cdots. \tag{5.68}$$

Substituting expansions (5.66)-(5.68) into (5.62)-(5.65) and equating powers of ϵ, we find at leading order

$$\kappa^{-2}\psi^{(0)\prime\prime} = -\psi^{(0)} + (h^{(0)})^2 x^2 \psi^{(0)}, \tag{5.69}$$
$$a^{(0)\prime\prime} = h^{(0)} x (\psi^{(0)})^2, \tag{5.70}$$
$$\psi^{(0)\prime} \rightarrow 0, \quad \text{as } |x| \rightarrow \infty, \tag{5.71}$$
$$a^{(0)\prime} \rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \tag{5.72}$$

Equation (5.69) with the boundary condition (5.71) corresponds to Schrödinger's equation with an energy well, and has solutions

$$\exp\left(\frac{-\kappa^2 x^2}{2(2n+1)}\right) H_n\left(\frac{\kappa\sqrt{2}x}{\sqrt{2n+1}}\right),$$

when $h^{(0)} = \kappa/(2n+1)$, where H_n is the Hermite polynomial, given by

$$H_0(u) = 1, \quad H_n(u) = (-1)^n e^{u^2} \frac{d^n}{du^n}\left(e^{-u^2}\right).$$

The upper critical field is given by the largest of these eigenvalues, namely $h^{(0)} = \kappa$. This is where we begin to notice the difference between Type I and Type II superconductors. Noting that $H_c = 1/\sqrt{2}$ in these units we have

$$h_{c_2} \gtrless H_c \text{ as } \kappa \gtrless 1/\sqrt{2}.$$

For a Type II superconductor $\kappa > 1/\sqrt{2}$, $h_{c_2} > H_c$, and, as the external magnetic field is lowered, the bifurcation point will be reached before the thermodynamic critical field (see Fig. 3.3). For a Type I superconductor however, $\kappa < 1/\sqrt{2}$, $h_{c_2} < H_c$, and before the external field reaches the bifurcation point there is the possibility that the superconductor will undergo a phase change to the superconducting state by means of an inwardly propagating phase boundary as described in Chapter 4.

The eigensolution corresponding to $h^{(0)} = \kappa$ is

$$\psi^{(0)} = \beta \frac{\kappa^{1/2}}{\pi^{1/4}} e^{-\frac{\kappa^2 x^2}{2}}, \tag{5.73}$$

where as before we have normalised the eigenfunction so that

$$\int_{-\infty}^{\infty} \psi^2 \, dx = \mid \beta \mid^2 .$$

Substituting (5.73) into (5.70) yields

$$a^{(0)\prime\prime} = \mid \beta \mid^2 \frac{\kappa^2 x}{\pi^{1/2}} e^{-\kappa^2 x^2}.$$

Hence

$$a^{(0)} = - \mid \beta \mid^2 \frac{1}{2\pi^{1/2}} \int_{-\infty}^{x} e^{-\kappa^2 \xi^2} \, d\xi, \tag{5.74}$$

where again the arbitrary constant is irrelevant. Notice that $a^{(0)\prime} < 0$ so that the magnetic field in the sample is less than the applied field (the Meissner effect.) Equating coefficients of ϵ in equations (5.62)-(5.65) we find

$$\begin{aligned} \kappa^{-2}\psi^{(1)\prime\prime} + \psi^{(1)} - (h^{(0)})^2 x^2 \psi^{(1)} &= (\psi^{(0)})^3 + 2h^{(0)}h^{(1)}x^2\psi^{(0)} \\ &\quad + 2h^{(0)} x a^{(0)} \psi^{(0)}, & (5.75) \\ a^{(1)\prime\prime} &= (\psi^{(0)})^2 (h^{(1)}x + a^{(0)}) + 2h^{(0)} x \psi^{(0)} \psi^{(1)}, & (5.76) \\ \psi^{(1)\prime} &\to 0, \quad \text{as } \mid x \mid \to \infty, & (5.77) \\ a^{(1)\prime} &\to 0, \quad \text{as } \mid x \mid \to \infty. & (5.78) \end{aligned}$$

As in the previous section, $\psi^{(0)}$ is the solution of the homogeneous version of equation (5.75) with the boundary condition (5.77), namely equation (5.69) and (5.71). Thus there is a solution for $\psi^{(1)}$ if and only if the appropriate solvability condition is satisfied. As before this condition is obtained by multiplying both sides of (5.75) by $\psi^{(0)}$ and integrating over $(-\infty, \infty)$ to give

$$\int_{-\infty}^{\infty} \psi^{(0)} [(\psi^{(0)})^3 + 2h^{(0)}h^{(1)}x^2\psi^{(0)} + 2h^{(0)} x a^{(0)} \psi^{(0)}] \, dx = 0.$$

Note that the arbitrary constant in the expression for $a^{(0)}$ does not affect this condition since $\int_{-\infty}^{\infty} x(\psi^{(0)})^2\, dx = 0$. Inserting our expressions for $h^{(0)}$ and $\psi^{(0)}$ into this we find

$$
\begin{aligned}
-\frac{2h^{(1)}\kappa^2}{\pi^{1/2}} \int_{-\infty}^{\infty} x^2 e^{-\kappa^2 x^2}\, dx = & \\
& |\beta|^2 \int_{-\infty}^{\infty} \left[\frac{\kappa^2}{\pi} e^{-2\kappa^2 x^2} - \frac{\kappa^2}{\pi} x e^{-\kappa^2 x^2} \left(\int_{-\infty}^{x} e^{-\kappa^2 \xi^2}\, d\xi \right) \right] dx, \\
= & \quad |\beta|^2 \int_{-\infty}^{\infty} \left[\frac{\kappa^2}{\pi} e^{-2\kappa^2 x^2} - \frac{1}{2\pi} e^{-2\kappa^2 x^2} \right] dx, \\
= & \quad |\beta|^2 \frac{1}{\pi} \left[\kappa^2 - \frac{1}{2} \right] \int_{-\infty}^{\infty} e^{-2\kappa^2 x^2}\, dx,
\end{aligned}
$$

on integration by parts. Thus $h^{(1)}$ is given by

$$h^{(1)} = \frac{|\beta|^2}{\sqrt{2\pi}} \left[\frac{1}{2} - \kappa^2 \right]. \tag{5.79}$$

Here again we notice a difference between Type I and Type II superconductors. We have

$$h^{(1)} < 0 \text{ if and only if } \kappa > 1/\sqrt{2}.$$

Thus for Type II superconductors we have a solution for all values of the applied magnetic field slightly below the critical value. For Type I superconductors however we have a solution for all values of the applied magnetic field slightly above the critical value. As before we have $\|\Psi\| = \epsilon^{1/2} \mid \beta \mid$ and hence for $\kappa \neq 1/\sqrt{2}$, $\|\Psi\|$ increases as $\mid h - \kappa \mid^{1/2}$ for h close to κ. The response diagrams for κ less than, equal to and greater than $1/\sqrt{2}$ are shown in Fig. 5.2.

When $\kappa = 1/\sqrt{2}$ we have $h^{(1)} = 0$, and the question arises as to what the behaviour of Ψ is in the neighbourhood of $h = \kappa$ in this case. We would expect that $\|\Psi\|$ would increase as $|h-\kappa|^{1/2n}$ where n is the least integer such that $h^{(n)} \neq 0$. However, we find that $h^{(n)} = 0\ \forall n$. In fact we find that the response diagram is vertical, i.e. when $h = \kappa$ there exists a family of solutions of arbitrary amplitude (similar vertical bifurcations have been studied recently in [52], for example). We demonstrate this fact by recalling that in Chapter 3 we found that in one dimension, when $\kappa = 1/\sqrt{2}$, solutions of the Ginzburg-Landau equations are given by solutions of the following pair of first order equations:

$$\sqrt{2}A' = 1 - \Psi^2, \tag{5.80}$$

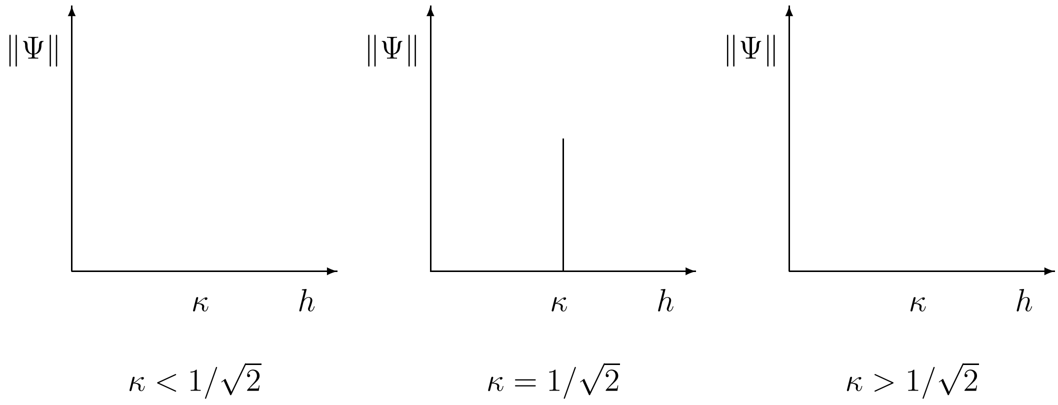

Figure 5.2: The response diagrams for κ less than, equal to and greater than $1/\sqrt{2}$.

$$\sqrt{2}\Psi' = -\Psi A. \tag{5.81}$$

When we impose the boundary conditions

$$A' \to 1/\sqrt{2}, \text{ as } x \to \pm\infty, \tag{5.82}$$

$$\Psi' \to 0, \text{ as } x \to \pm\infty, \tag{5.83}$$

these equations have solutions

$$x = \int_{\Psi(0)^2}^{\Psi^2} \frac{-d(\Psi^2)}{\sqrt{2}\,\Psi^2(\Psi^2 - C - \log\Psi^2)^{1/2}}, \tag{5.84}$$

where, in order for the solution to be bounded, C must be greater than 1 but is otherwise arbitrary. A number of these solutions, for different values of C, are shown in Fig. 5.3 (recall that when $C = 1$ we obtain the transition layer solution of Fig. 1.5). The solutions take the form of a superconducting blip that grows into a completely superconducting region separated from the remaining normal region by two phase boundaries. However, we should remember that these are all steady state solutions, and the diagrams in Fig. 5.3 do not represent the evolution of a small blip into two travelling waves. It is not clear whether the solutions along the superconducting branch in Figs. 5.2a,c take the same from as those in Fig. 5.3.

We consider further the question of the behaviour at $\kappa = 1/\sqrt{2}$ in Appendix B.

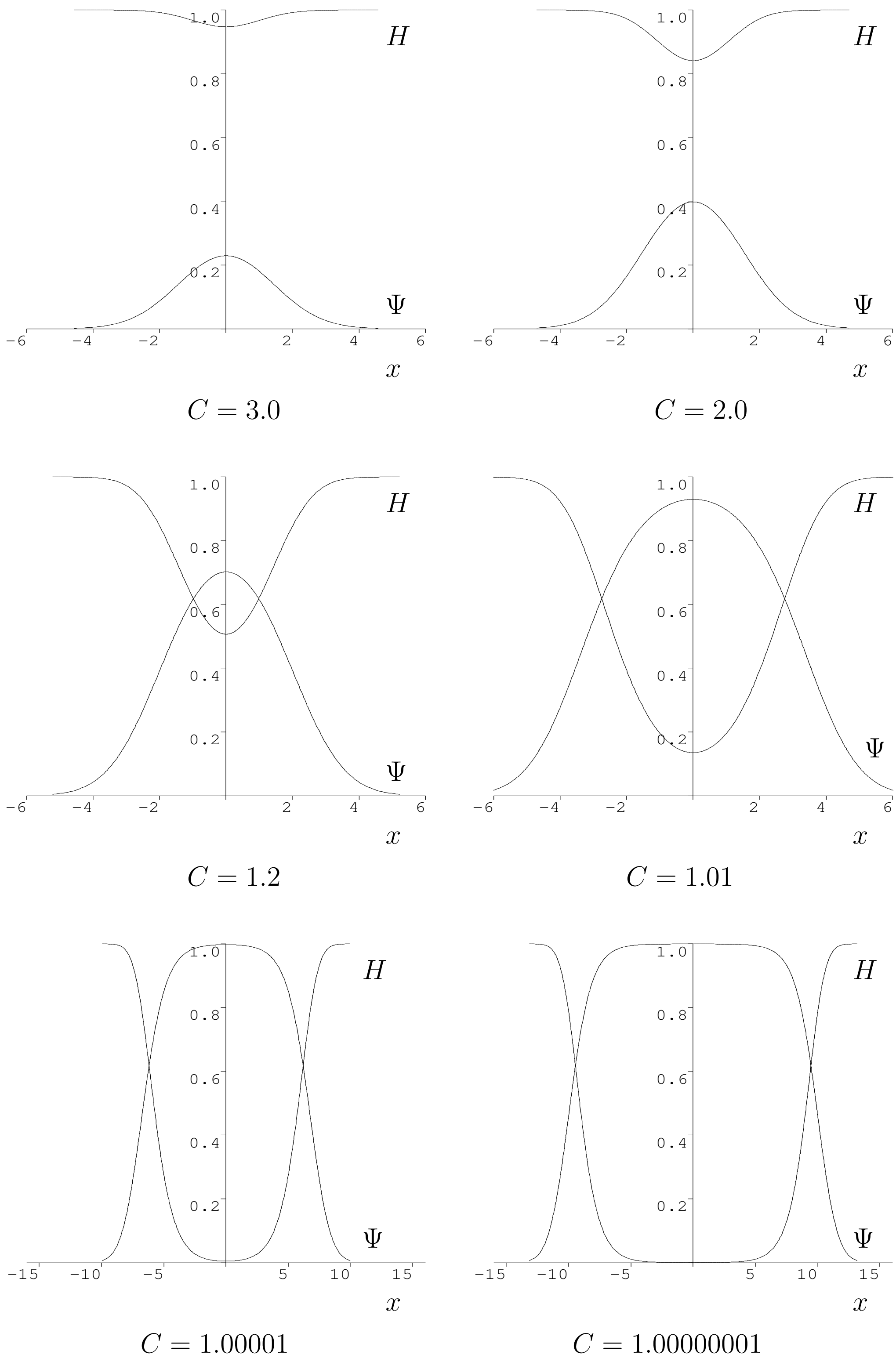

Figure 5.3: Solutions bifurcating from $\Psi \equiv 0$ at $h = 1/\sqrt{2}$ for $\kappa = 1/\sqrt{2}$.

5.2 Linear Stability of the Solution Branches

We now begin our analysis of the linear stability of the solution branches in Fig. 5.1.

We first demonstrate the technique with a one-dimensional example.

5.2.1 One-dimensional Example

As in the previous section we work on the length scale of the penetration depth by rescaling length and $\boldsymbol{A}$ with λ. We work also on the timescale of the relaxation of the order parameter by rescaling time with λ^2. In one dimension the time-dependent isothermal Ginzburg-Landau equations, (3.59)-(3.68), are then

$$-\frac{\alpha}{\kappa^2}\frac{\partial \Psi}{\partial t} + \frac{1}{\kappa^2}\frac{\partial^2 \Psi}{\partial x^2} = \Psi^3 - \Psi + A^2\Psi, \tag{5.85}$$

$$\frac{\partial^2 A}{\partial x^2} = \frac{\partial A}{\partial t} + \Psi^2 A, \tag{5.86}$$

$$\frac{\partial \Psi}{\partial x} \to 0, \qquad \text{as } |x| \to \infty, \tag{5.87}$$

$$\frac{\partial A}{\partial x} \to 0, \qquad \text{as } |x| \to \infty. \tag{5.88}$$

We examine firstly the linear stability of the solution corresponding to the normal state, $\Psi \equiv 0, A = hx$. We make a small perturbation about this solution by setting

$$\Psi = \delta e^{\sigma t}\psi(x), \tag{5.89}$$

$$A = hx + \delta e^{\sigma t} a(x), \quad 0 < \delta \ll 1. \tag{5.90}$$

Substituting (5.89), (5.90) into equations (5.85)-(5.88) and linearising in δ (to give the leading order behaviour of an asymptotic expansion in powers of δ) yields

$$-(\alpha\sigma/\kappa^2)\psi + (1/\kappa^2)\psi'' = -\psi + h^2x^2\psi, \tag{5.91}$$

$$a'' = \sigma a, \tag{5.92}$$

$$\psi' \to 0, \qquad \text{as } |x| \to \infty, \tag{5.93}$$

$$a' \to 0, \qquad \text{as } |x| \to \infty. \tag{5.94}$$

where $' \equiv d/dx$. The self-adjoint eigenvalue problem (5.91) with the boundary condition (5.93) has a discrete spectrum (see, for example, [61]), and has solutions

$$e^{-h\kappa x^2/2} H_n(\sqrt{2h\kappa}\, x),$$

where

$$\sigma = \sigma_n = \frac{\kappa h}{\alpha}\left(\frac{\kappa}{h} - 2n - 1\right).$$

When $h > \kappa = h_{c_2}$ all the eigenvalues σ are negative and the normal state solution is linearly stable. When $h < \kappa$ at least one eigenvalue is positive, and the normal state is linearly unstable.

Let us now examine the linear stability of the one-dimensional superconducting solution branches. We make a small perturbation of the form

$$\Psi = \Psi_0(x) + \delta e^{\sigma t}\Psi_1(x), \quad (5.95)$$

$$A = A_0(x) + \delta e^{\sigma t} A_1(x), \quad (5.96)$$

where (Ψ_0, A_0) is the steady state superconducting solution given by (5.66), (5.68). Substituting (5.95), (5.96) into the equations (5.85)-(5.88) yields:

$$-(\alpha\sigma/\kappa^2)\Psi_1 + (1/\kappa^2)\Psi_1'' = 3\Psi_0^2\Psi_1 - \Psi_1 + 2\Psi_0 A_0 A_1 + A_0^2\Psi_1, \quad (5.97)$$

$$A_1'' = \sigma A_1 + 2\Psi_0\Psi_1 A_0 + \Psi_0^2 A_1, \quad (5.98)$$

$$\Psi_1' \to 0, \quad \text{as } |x| \to \infty, \quad (5.99)$$

$$A_1' \to 0, \quad \text{as } |x| \to \infty. \quad (5.100)$$

We examine the stability near the bifurcation point by introducing ϵ as before:

$$\Psi_0 = \epsilon^{1/2}\psi_0, \quad (5.101)$$

$$A_0 = hx + \epsilon a_0, \quad (5.102)$$

$$\Psi_1 = \epsilon^{1/2}\psi_1, \quad (5.103)$$

$$A_1 = \epsilon a_1. \quad (5.104)$$

Note that since we are expanding in powers of ϵ after linearising in δ we are assuming that $\delta \ll \epsilon$. In particular we will equate coefficients of $\epsilon\delta$ in the equations while neglecting terms of order δ^2. Hence we require that at least $\delta = o(\epsilon)$. If we wish to be definite about the relative sizes of ϵ and δ we may take, for example, $\delta = \epsilon^2$.

Substituting (5.101)-(5.104) into (5.97)-(5.100) and linearising in δ yields:

$$\begin{aligned}-(\alpha\sigma/\kappa^2)\psi_1 + (1/\kappa^2)\psi_1'' = {} & 3\epsilon\psi_0^2\psi_1 - \psi_1 \\ & + 2\epsilon\psi_0 a_1(hx + \epsilon a_0) \\ & + \psi_1(h^2x^2 + 2\epsilon h x a_0 + a_0^2),\end{aligned} \quad (5.105)$$

$$a_1'' = \sigma a_1 + 2\psi_0\psi_1(hx + \epsilon a_0) + \epsilon\psi_0^2 a_1, \quad (5.106)$$

$$\psi_1' \to 0, \qquad \text{as } |x| \to \infty, \quad (5.107)$$

$$a_1' \to 0, \qquad \text{as } |x| \to \infty. \quad (5.108)$$

In Section 5.1.2 we wrote down h, ψ_0, and a_0 in terms of a power series in ϵ. We also expand ψ_1, a_1 and σ in powers of ϵ (so that the variations in σ balance the variations in h) to give:

$$h = h^{(0)} + \epsilon h^{(1)} + \cdots, \quad (5.109)$$

$$\psi_0 = \psi_0^{(0)} + \epsilon\psi_0^{(1)} + \cdots, \quad (5.110)$$

$$a_0 = a_0^{(0)} + \epsilon a_0^{(1)} + \cdots, \quad (5.111)$$

$$\psi_1 = \psi_1^{(0)} + \epsilon\psi_1^{(1)} + \cdots, \quad (5.112)$$

$$a_1 = a_1^{(0)} + \epsilon a_1^{(1)} + \cdots, \quad (5.113)$$

$$\sigma = \sigma^{(0)} + \epsilon\sigma^{(1)} + \cdots. \quad (5.114)$$

Substituting (5.109)-(5.114) into (5.105)-(5.108) and equating powers of ϵ we find at leading order

$$-(\alpha\sigma^{(0)}/\kappa^2)\psi_1^{(0)} + (1/\kappa^2)\psi_1^{(0)''} = -\psi_1^{(0)} + (h^{(0)})^2 x^2 \psi_1^{(0)}, \quad (5.115)$$

$$a_1^{(0)''} = \sigma^{(0)} a_1^{(0)} + 2h^{(0)} x \psi_0^{(0)} \psi_1^{(0)}, \quad (5.116)$$

$$\psi_1^{(0)'} \to 0, \qquad \text{as } |x| \to \infty, \quad (5.117)$$

$$a_1^{(0)'} \to 0, \qquad \text{as } |x| \to \infty. \quad (5.118)$$

Equation (5.115) with the boundary conditions (5.118) is exactly equation (5.91) with corresponding boundary conditions (5.94). Hence there are non-zero solutions for $\psi_1^{(0)}$ when

$$\sigma^{(0)} = \frac{\kappa h^{(0)}}{\alpha}\left(\frac{\kappa}{h^{(0)}} - 2n - 1\right).$$

For the solution branches bifurcating at eigenvalues $h^{(0)} < \kappa$ we see that there is at least one positive eigenvalue for σ, namely the case $n = 0$. Thus there is at least one unstable mode, and the superconducting solution branch will be linearly unstable. For the solution branch bifurcating at $h^{(0)} = \kappa$ all the eigenvalues $\sigma^{(0)}$ are negative except for the eigenvalue $\sigma^{(0)} = 0$. To determine the stability of this mode we need to proceed to higher orders in our expansions.

When $\sigma^{(0)} = 0$ we have $\psi_1^{(0)} \propto e^{-\kappa^2 x^2/2}$. Since all the equations are linear in $\psi_1, \boldsymbol{a}_1$ and ϕ_1 by construction, the constant of proportionality is irrelevant and we take it to be $\beta\kappa^{1/2}/\pi^{1/4}$, so that $\psi_1^{(0)} = \psi_0^{(0)}$ (in effect this defines δ.)

We recall the previously found expansions for h, ψ_0 and a_0:

$$h = \kappa + (\epsilon\beta^2/\sqrt{2\pi})(1/2 - \kappa^2) + \cdots, \tag{5.119}$$

$$\psi_0 = \beta\frac{\kappa^{1/2}}{\pi^{1/4}} e^{-\kappa^2 x^2/2} + \epsilon\psi_0^{(1)} + \cdots, \tag{5.120}$$

$$a_0 = -\frac{\beta^2}{2\sqrt{\pi}} \int_{-\infty}^{x} e^{-\kappa^2 \tilde{x}^2}\, d\tilde{x} + \epsilon a_0^{(1)} + \cdots. \tag{5.121}$$

Substituting the expressions for $\psi_1^{(0)}$, $\psi_0^{(0)}$, $\sigma^{(0)}$ and $h^{(0)}$ into equations (5.116), (5.118) gives

$$a_1^{(0)\prime\prime} = (2\kappa^2\beta^2/\sqrt{\pi}) x e^{-\kappa^2 x^2}, \tag{5.122}$$

$$a_1^{(0)\prime} \to 0, \quad \text{as } |x| \to \infty. \tag{5.123}$$

Hence

$$a_1^{(0)} = -\frac{\beta^2}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-\kappa^2 \tilde{x}^2}\, d\tilde{x} + \text{const.} \tag{5.124}$$

The arbitrary constant here is due to the translational invariance of the equations and we may take it to be zero without loss of generality (this simply fixes the translate.) Note that $a_1^{(0)} = 2a_0^{(0)}$. Equating powers of ϵ in equations (5.105), (5.108) yields

$$\begin{aligned}(1/\kappa^2)\psi_1^{(1)\prime\prime} + \psi_1^{(1)} - (h^{(0)})^2 x^2 \psi_1^{(1)} &= (\alpha\sigma^{(1)}/\kappa^2)\psi_1^{(0)} + 3(\psi_0^{(0)})^2\psi^{(1)} \\ &\quad + 2h^{(0)} x \psi_0^{(0)} a_1^{(0)} + 2h^{(0)} h^{(1)} x^2 \psi_1^{(0)} \\ &\quad + 2h^{(0)} x \psi_1^{(0)} a_0^{(0)}.\end{aligned} \tag{5.125}$$

$$\psi_1^{(1)\prime} \to 0 \qquad \text{as } |x| \to \infty. \tag{5.126}$$

Hence

$$\begin{aligned}(1/\kappa^2)\psi_1^{(1)\prime\prime} + \psi_1^{(1)} - \kappa^2 x^2 \psi_1^{(1)} &= \frac{\alpha\sigma^{(1)}\beta}{\pi^{1/4}\kappa^{3/2}} e^{-\kappa^2 x^2/2} + \frac{3\kappa^{3/2}\beta^3}{\pi^{3/4}} e^{-3\kappa^2 x^2/2} \\ &\quad - \frac{3\kappa^{3/2} x \beta^3}{\pi^{3/4}} e^{-\kappa^2 x^2/2} \int_{-\infty}^{x} e^{-\kappa^2 \tilde{x}^2}\, d\tilde{x} \\ &\quad + \frac{2(1 - 2\kappa^2)\kappa^{3/2} x^2 \beta^3}{2\sqrt{2}\pi^{3/4}} e^{-\kappa^2 x^2/2},\end{aligned} \tag{5.127}$$

$$\psi_1^{(1)\prime} \to 0 \quad \text{as } |x| \to \infty. \tag{5.128}$$

Now, $e^{-\kappa^2 x^2/2}$ satisfies the homogeneous version of equations (5.127), (5.128). Hence there is a solution for $\psi_1^{(1)}$ if and only if an appropriate solvability condition is satisfied. To derive this condition we multiply (5.127) by $e^{-\kappa^2 x^2/2}$ and integrate over $(-\infty, \infty)$ to give

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} \left[\frac{\alpha\sigma^{(1)}\pi^{1/2}}{\kappa^2} e^{-\kappa^2 x^2} + 3\kappa\beta^2 e^{-2\kappa^2 x^2} \right. \\ &\qquad \left. -3\kappa\beta^2 x e^{-\kappa^2 x^2} \int_{-\infty}^{x} e^{-\kappa^2 \tilde{x}^2}\, d\tilde{x} + \frac{(1-2\kappa^2)\kappa\beta^2 x^2}{\sqrt{2}} e^{-\kappa^2 x^2} \right] dx \\ &= \int_{-\infty}^{\infty} \left[\frac{\alpha\sigma^{(1)}\pi^{1/2}}{\kappa^2} e^{-\kappa^2 x^2} + 3\kappa\beta^2 e^{-2\kappa^2 x^2} - \frac{3\beta^2}{2\kappa} e^{-2\kappa^2 x^2} + \frac{(1-2\kappa^2)\beta^2}{2\sqrt{2}\kappa} e^{-\kappa^2 x^2} \right] dx, \end{aligned}$$

on integration by parts. Thus

$$\begin{aligned} 0 &= \frac{\alpha\sigma^{(1)}\pi^{1/2}}{\kappa^2} + \frac{3\kappa\beta^2}{\sqrt{2}} - \frac{3\beta^2}{2\sqrt{2}\kappa} + \frac{(1-2\kappa^2)\beta^2}{2\sqrt{2}\kappa} \\ &= \frac{\alpha\sigma^{(1)}\pi^{1/2}}{\kappa^2} - \frac{(1-2\kappa^2)\beta^2}{\sqrt{2}\kappa}. \end{aligned}$$

Thus

$$\sigma^{(1)} = \frac{\kappa(1-2\kappa^2)\beta^2}{\alpha\sqrt{2\pi}}, \tag{5.129}$$

or

$$\sigma^{(1)} = \frac{2\kappa h^{(1)}}{\alpha}. \tag{5.130}$$

Note that $\sigma^{(1)} < 0$ if and only if $\kappa > 1/\sqrt{2}$, i.e. if the superconductor is of Type II. Thus for Type I superconductors the bifurcation at $h = \kappa$ is subcritical, for Type II superconductors it is supercritical. Fig. 5.4 shows the stability of the solution branches in the response diagrams of Type I and Type II superconductors.

When $\kappa = 1/\sqrt{2}$ we find $\sigma^{(1)} = 0$. This was expected, since we know that in this case there exist solutions of arbitrary amplitude at $h = \kappa$. Infact, we expect $\sigma^{(n)} = 0 \quad \forall n$. The stability of vertical bifurcations is an interesting, and as yet unapproached, problem. We will return to this point in the next section when we consider the weakly nonlinear stability of the normal state solution.

5.2.2 Linear Stability of the Solution Branches for a Body of Arbitrary Shape

We now apply the above techniques to check the stability of the solution branches for a body of arbirtary shape in an applied magnetic field.

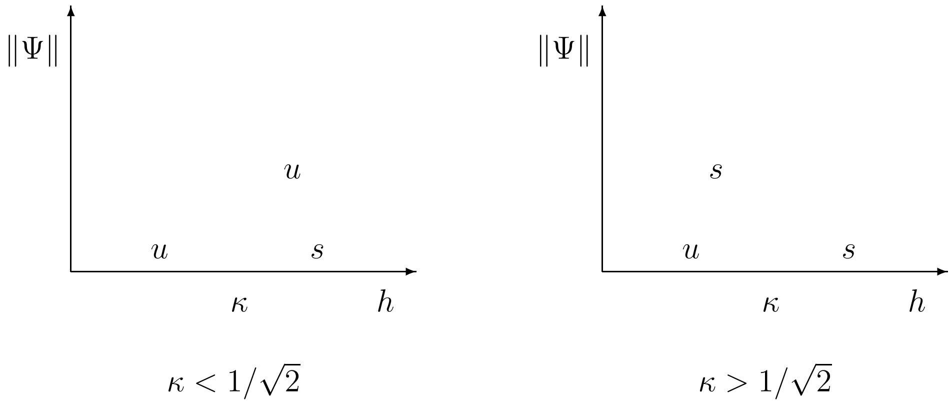

Figure 5.4: Stability of the solution branches in the response diagrams of Type I and Type II superconductors.

Linear Stability of the Normal State We examine first the stability of the normal state. As in the previous section we work on the length scale of the penetration depth by rescaling length and $\boldsymbol{A}$ with λ, and on the timescale of the relaxation of the order parameter by rescaling time with λ^2. The time-dependent Ginzburg-Landau equations, together with boundary and other conditions, (3.59)-(3.68), are then

$$\begin{aligned}
\frac{\alpha}{\kappa^2}\frac{\partial \Psi}{\partial t} + \frac{\alpha i}{\kappa}\Psi\Phi + \left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2 \Psi &= \Psi(1- |\Psi|^2), \quad \text{in } \Omega, && (5.131)\\
-(\text{curl})^2\boldsymbol{A} - \frac{\partial \boldsymbol{A}}{\partial t} - \nabla\Phi &= \frac{i}{2\kappa}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) \\
&\qquad + |\Psi|^2 \boldsymbol{A}, \quad \text{in } \Omega, && (5.132)\\
-(\text{curl})^2\boldsymbol{A} &= \varsigma_e\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right), \quad \text{outside } \Omega, && (5.133)\\
\nabla^2\Phi &= 0, \ \text{outside } \Omega, && (5.134)\\
\boldsymbol{n}\cdot((i/\kappa)\nabla + \boldsymbol{A})\Psi + (i/d)\Psi &= 0, \quad \text{on } \partial\Omega, && (5.135)\\
[\boldsymbol{n}\wedge\boldsymbol{A}] &= \boldsymbol{0}, && (5.136)\\
[\boldsymbol{n}\wedge(1/\mu)\text{curl } \boldsymbol{A}] &= \boldsymbol{0}, && (5.137)\\
[\Phi] &= 0, && (5.138)\\
\left[\varepsilon\frac{\partial\Phi}{\partial n}\right] &= 0, && (5.139)\\
\text{curl } \boldsymbol{A} &\to h\hat{\boldsymbol{z}}, \quad \text{as } r\to\infty, && (5.140)\\
\Phi &\to 0, \quad \text{as } r\to\infty, && (5.141)\\
\text{div } \boldsymbol{A} &= 0. && (5.142)
\end{aligned}$$

We make a small perturbation about the normal solution (5.9), by setting

$$\Psi = \delta e^{\sigma t}\Psi_1, \tag{5.143}$$

$$\boldsymbol{A} = h\boldsymbol{A}_N + \delta e^{\sigma t}\boldsymbol{A}_1, \tag{5.144}$$

$$\Phi = \delta e^{\sigma t}\Phi_1, \quad 0 < \delta \ll 1. \tag{5.145}$$

Substituting (5.143)-(5.145) into (5.131)-(5.142) and linearising in δ (to give the leading order behaviour of an asymptotic expansion in powers of δ) yields

$$\frac{\alpha}{\kappa^2}\sigma\Psi_1 + \left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)^2 \Psi_1 = \Psi_1, \quad \text{in } \Omega, \tag{5.146}$$

$$-(\text{curl})^2\boldsymbol{A}_1 = \sigma\boldsymbol{A}_1 + \nabla\Phi_1, \quad \text{in } \Omega, \tag{5.147}$$

$$-(\text{curl})^2\boldsymbol{A}_1 = \varsigma_e\left(\sigma\boldsymbol{A}_1 + \nabla\Phi_1\right), \quad \text{outside } \Omega, \tag{5.148}$$

$$\nabla^2\Phi_1 = 0, \text{ outside } \Omega, \tag{5.149}$$

$$\boldsymbol{n}\cdot((i/\kappa)\nabla + h\boldsymbol{A}_N)\Psi_1 + (i/d)\Psi_1 = 0, \quad \text{on } \partial\Omega, \tag{5.150}$$

$$[\boldsymbol{n}\wedge\boldsymbol{A}_1] = \boldsymbol{0}, \tag{5.151}$$

$$[\boldsymbol{n}\wedge(1/\mu)\text{curl } \boldsymbol{A}_1] = \boldsymbol{0}, \tag{5.152}$$

$$[\Phi_1] = 0, \tag{5.153}$$

$$\left[\frac{\partial\Phi_1}{\partial n}\right] = 0, \tag{5.154}$$

$$\text{curl } \boldsymbol{A}_1 \to \boldsymbol{0} \text{ as } r\to\infty, \tag{5.155}$$

$$\Phi_1 \to 0 \text{ as } r\to\infty, \tag{5.156}$$

$$\text{div } \boldsymbol{A}_1 = 0. \tag{5.157}$$

Writing

$$L = \left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)^2 - 1,$$

we have by equation (5.146)

$$\begin{aligned}
&\int_\Omega \Psi_1^* L\Psi_1 - \Psi_1 L^*\Psi_1^*\, dV = \\
&\quad \int_\Omega \left\{\left(-\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)\Psi_1^*\left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)\Psi_1 - \Psi_1^*\Psi_1\right\} dV \\
&\quad - \int_\Omega \left\{\left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)\Psi_1\left(-\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)\Psi_1^* - \Psi_1\Psi_1^*\right\} dV \\
&\quad + \frac{i}{\kappa}\int_{\partial\Omega}\left\{\Psi_1\left(-\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)\Psi_1^* + \Psi_1^*\left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)\Psi_1\right\}\cdot\boldsymbol{n}\, dS, \\
&\quad = 0,
\end{aligned}$$

on integration by parts. Hence, as in the previous section, L is self-adjoint on the space of smooth functions on Ω satisfying the boundary condition (5.150). It then follows that the eigenvalues of L are real and discrete, and that the eigenfunctions corresponding to distinct eigenvalues are orthogonal. Thus for each given h, (5.146) and (5.150) determine a discrete set of eigenvalues for σ. For $h > h_{c_2}$ all these eigenvalues will be negative and the normal state is linearly stable. For $h < h_{c_2}$ we expect at least one of these eigenvalues to be positive, and the normal state to be linearly unstable. To see this we multiply (5.146) by Ψ_1^* and integrate over Ω to give

$$\begin{aligned}
\frac{\alpha\sigma}{\kappa^2}\int_\Omega |\Psi_1|^2\, dV &= \int_\Omega |\Psi_1|^2\, dV - \int_\Omega \Psi_1^* \left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)^2 \Psi_1\, dV, \\
&= \int_\Omega |\Psi_1|^2\, dV - \int_\Omega \left|\left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)\Psi_1\right|^2 dV \\
&\qquad - \frac{i}{\kappa}\int_{\partial\Omega} \Psi_1^* \left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)\Psi_1\, dS, \\
&= \int_\Omega |\Psi_1|^2\, dV - \int_\Omega \left|\left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)\Psi_1\right|^2 dV \\
&\qquad - \frac{1}{\kappa d}\int_{\partial\Omega} |\Psi_1|^2\, dS.
\end{aligned}$$

(Note that this also shows that σ is real.) As $h \to \infty$ we see that the second term on the right-hand side dominates the first, and hence all the eigenvalues σ are negative for large h. We expect the eigenvalues σ to each depend continuously on h, and one of the eigenvalues σ will pass through zero when and only when h passes through an eigenvalue of (5.28), (5.31). The largest of these eigenvalues is h_{c_2}. Thus for $h > h_{c_2}$ all the eigenvalues σ are negative. As h passes through h_{c_2} the largest eigenvalue σ will pass through zero, and hence we expect that for $h < h_{c_2}$ there will be at least one positive eigenvalue σ.

Linear Stability of the Superconducting Branch We now consider a small perturbation of the previously found superconducting solution. We set

$$\Psi = \Psi_0 + \delta e^{\sigma t}\Psi_1, \tag{5.158}$$

$$\boldsymbol{A} = \boldsymbol{A}_0 + \delta e^{\sigma t}\boldsymbol{A}_1, \tag{5.159}$$

$$\Phi = \delta\Phi_1 e^{\sigma t}, \quad 0 < \delta \ll 1, \tag{5.160}$$

where $(\Psi_0, \boldsymbol{A}_0)$ is the steady superconducting solution given by (5.52)-(5.54). Substituting (5.158)-(5.160) into (5.131)-(5.142) and linearising in δ yields

$$\begin{aligned}
\frac{\alpha}{\kappa^2}\sigma\Psi_1 + \frac{\alpha}{\kappa}i\Psi_0\Phi_1 + \left(\frac{i}{\kappa}\nabla + \boldsymbol{A}_0\right)^2 \Psi_1 &= -\frac{2i}{\kappa}\boldsymbol{A}_1 \cdot \nabla\Psi_0 - 2\boldsymbol{A}_0 \cdot \boldsymbol{A}_1\Psi_0 \\
&\quad + \Psi_1 - 2\,|\,\Psi_0\,|^2\,\Psi_1 \\
&\quad - \Psi_0^2\Psi_1^*, \quad \text{in } \Omega, \qquad (5.161) \\
-(\text{curl})\boldsymbol{A}_1 - \sigma\boldsymbol{A}_1 - \nabla\Phi_1 &= (i/2\kappa)(\Psi_0^*\nabla\Psi_1 + \Psi_1^*\nabla\Psi_0) \\
&\quad - (i/2\kappa)(\Psi_0\nabla\Psi_1^* + \Psi_1\nabla\Psi_0^*) \\
&\quad + (\Psi_0\Psi_1^* + \Psi_0^*\Psi_1)\boldsymbol{A}_0 \\
&\quad + |\,\Psi_0\,|^2\,\boldsymbol{A}_1, \quad \text{in } \Omega, \qquad (5.162) \\
-(\text{curl})^2\boldsymbol{A_1} &= \varsigma_e\left(\sigma\boldsymbol{A}_1 + \nabla\Phi_1\right), \quad \text{outside } \Omega, (5.163) \\
\nabla^2\Phi_1 &= 0, \quad \text{outside } \Omega, \qquad (5.164) \\
\boldsymbol{n}\cdot((i/\kappa)\nabla + \boldsymbol{A}_0)\Psi_1 + \boldsymbol{n}\cdot\boldsymbol{A}_1\Psi_0 &= -(i/d)\Psi_1, \quad \text{on } \partial\Omega, \qquad (5.165) \\
[\boldsymbol{n}\wedge\boldsymbol{A}_1] &= \boldsymbol{0}, \qquad (5.166) \\
[\boldsymbol{n}\wedge(1/\mu)\text{curl }\boldsymbol{A}_1] &= \boldsymbol{0}, \qquad (5.167) \\
[\Phi_1] &= 0, \qquad (5.168) \\
\left[\varepsilon\frac{\partial\Phi_1}{\partial n}\right] &= 0, \qquad (5.169) \\
\text{curl }\boldsymbol{A_1} &\to \boldsymbol{0} \text{ as } r\to\infty, \qquad (5.170) \\
\Phi_1 &\to 0 \text{ as } r\to\infty, \qquad (5.171) \\
\text{div }\boldsymbol{A}_1 &= 0. \qquad (5.172)
\end{aligned}$$

We examine the stability close to the bifurcation point by introducing ϵ as before:

$$\Psi_0 = \epsilon^{1/2}\psi_0, \qquad (5.173)$$

$$\boldsymbol{A}_0 = h\boldsymbol{A}_N + \epsilon\boldsymbol{a}_0, \qquad (5.174)$$

$$\Psi_1 = \epsilon^{1/2}\psi_1, \qquad (5.175)$$

$$\boldsymbol{A}_1 = \epsilon\boldsymbol{a}_1, \qquad (5.176)$$

$$\Phi_1 = \epsilon\phi_1. \qquad (5.177)$$

As in the one-dimensional case we require $\delta \ll \epsilon$ (e.g. $\delta = \epsilon^2$). Substituting (5.173)-(5.177) into (5.161)-(5.172) yields

$$\begin{aligned}
(\alpha\sigma/\kappa^2)\psi_1 - \psi_1 + ((i/\kappa) + h\boldsymbol{A}_N)^2\psi_1 &= -(\epsilon\alpha i/\kappa)\psi_0\phi_1 - 2\epsilon\,|\psi_0|^2\,\psi_1 \\
&\quad - \epsilon\psi_1^*\psi_0^2 - (2\epsilon i/\kappa)\boldsymbol{a}_0\cdot\nabla\psi_1 \\
&\quad - 2\epsilon h\boldsymbol{a}_0\cdot\boldsymbol{A}_N\psi_1 - (2\epsilon i/\kappa)\boldsymbol{a}_1\cdot\nabla\psi_0 \\
&\quad - 2\epsilon h\boldsymbol{A}_N\cdot\boldsymbol{a}_1\psi_0 - \epsilon^2\,|\boldsymbol{a}_0|^2\,\psi_1 \\
&\quad - 2\epsilon^2\boldsymbol{A}_0\cdot\boldsymbol{a}_1\psi_0, \quad \text{in } \Omega, \qquad (5.178) \\
-(\mathrm{curl})^2\boldsymbol{a}_1 - \sigma\boldsymbol{a}_1 - \nabla\phi_1 &= (i/2\kappa)(\psi_0^*\nabla\psi_1 + \psi_1^*\nabla\psi_0) \\
&\quad - (i/2\kappa)(\psi_0\nabla\psi_1^* + \psi_1\nabla\psi_0^*) \\
&\quad + (\psi_0\psi_1^* + \psi_0^*\psi_1)h\boldsymbol{A}_N \\
&\quad \epsilon(\psi_0\psi_1^* + \psi_0^*\psi_1)\boldsymbol{a}_0 \\
&\quad + \epsilon\,|\psi_0|^2\,\boldsymbol{a}_1, \quad \text{in } \Omega, \qquad (5.179) \\
-(\mathrm{curl})^2\boldsymbol{a}_1 &= \varsigma_e(\sigma\boldsymbol{a}_1 + \nabla\phi_1), \quad \text{outside } \Omega, \quad (5.180) \\
\nabla^2\phi_1 &= 0, \quad \text{outside } \Omega, \qquad (5.181) \\
\boldsymbol{n}\cdot((i/\kappa)\nabla + h\boldsymbol{A}_N)\psi_1 + (i/d)\psi_1 &= -\epsilon\boldsymbol{n}\cdot\boldsymbol{a}_1\psi_0 \\
&\qquad - \epsilon\boldsymbol{n}\cdot\boldsymbol{a}_0\psi_1, \quad \text{on } \partial\Omega, \qquad (5.182) \\
[\boldsymbol{n}\wedge\boldsymbol{a}_1] &= \boldsymbol{0}, \qquad (5.183) \\
[\boldsymbol{n}\wedge(1/\mu)\mathrm{curl}\,\boldsymbol{a}_1] &= \boldsymbol{0}, \qquad (5.184) \\
[\phi_1] &= 0, \qquad (5.185) \\
\left[\varepsilon\frac{\partial\phi_1}{\partial n}\right] &= 0, \qquad (5.186) \\
\mathrm{curl}\,\boldsymbol{a}_1 &\to \boldsymbol{0}, \quad \text{as } r\to\infty, \qquad (5.187) \\
\phi_1 &\to 0, \quad \text{as } r\to\infty, \qquad (5.188) \\
\mathrm{div}\,\boldsymbol{a}_1 &= 0. \qquad (5.189)
\end{aligned}$$

In the previous section we obtained expansions in powers of ϵ for $\boldsymbol{a}_0$, ψ_0 and h near $h = h^{(0)}$. As in the one-dimensional case we expand also $\boldsymbol{a}_1$, ψ_1, ϕ_1 and σ in powers of ϵ to give

$$h = h^{(0)} + \epsilon h^{(1)} + \cdots, \qquad (5.190)$$

$$\psi_0 = \psi_0^{(0)} + \epsilon\psi_0^{(1)} + \cdots, \qquad (5.191)$$

$$a_0 = a_0^{(0)} + \epsilon a_0^{(1)} + \cdots, \tag{5.192}$$

$$\psi_1 = \psi_1^{(0)} + \epsilon \psi_1^{(1)} + \cdots, \tag{5.193}$$

$$a_1 = a_1^{(0)} + \epsilon a_1^{(1)} + \cdots, \tag{5.194}$$

$$\phi_1 = \phi_1^{(0)} + \epsilon \phi_1^{(1)} + \cdots, \tag{5.195}$$

$$\sigma = \sigma^{(0)} + \epsilon \sigma^{(1)} + \cdots. \tag{5.196}$$

Substituting the expansions (5.190)-(5.196) into equations (5.178)-(5.189) and equating powers of ϵ we find at leading order

$$(\alpha\sigma^{(0)}/\kappa^2)\psi_1^{(0)} - \psi_1^{(0)} = -((i/\kappa) + h^{(0)}\boldsymbol{A}_N)^2\psi_1^{(0)}, \quad \text{in } \Omega, \tag{5.197}$$

$$\begin{aligned} -(\text{curl})^2\boldsymbol{a}_1^{(0)} - \sigma^{(0)}\boldsymbol{a}_1 - \nabla\phi_1^{(0)} = {} & (i/2\kappa)(\psi_0^{(0)*}\nabla\psi_1^{(0)} + \psi_1^{(0)*}\nabla\psi_0^{(0)}) \\ & - (i/2\kappa)(\psi_0^{(0)}\nabla\psi_1^{(0)*} + \psi_1^{(0)}\nabla\psi_0^{(0)*}) \\ & + (\psi_0^{(0)}\psi_1^{(0)*} + \psi_0^{(0)*}\psi_1^{(0)})h^{(0)}\boldsymbol{A}_N, \ \text{in } \Omega, \end{aligned} \tag{5.198}$$

$$-(\text{curl})^2\boldsymbol{a}_1^{(0)} = \varsigma_e(\sigma^{(0)}\boldsymbol{a}_1^{(0)} + \nabla\phi_1^{(0)}), \quad \text{outside } \Omega, \tag{5.199}$$

$$\nabla^2\phi_1^{(0)} = 0, \quad \text{outside } \Omega, \tag{5.200}$$

$$\boldsymbol{n}\cdot((i/\kappa)\nabla + h^{(0)}\boldsymbol{A}_N)\psi_1^{(0)} = -(i/d)\psi_1^{(0)}, \ \text{on } \partial\Omega, \tag{5.201}$$

$$[\boldsymbol{n}\wedge\boldsymbol{a}_1^{(0)}] = \boldsymbol{0}, \tag{5.202}$$

$$[\boldsymbol{n}\wedge(1/\mu)\text{curl } \boldsymbol{a}_1^{(0)}] = \boldsymbol{0}, \tag{5.203}$$

$$[\phi_1^{(0)}] = 0, \tag{5.204}$$

$$\left[\varepsilon\frac{\partial\phi_1^{(0)}}{\partial n}\right] = 0, \tag{5.205}$$

$$\text{curl } \boldsymbol{a}_1^{(0)} \to \boldsymbol{0} \text{ as } r \to \infty, \tag{5.206}$$

$$\phi_1^{(0)} \to 0 \text{ as } r \to \infty, \tag{5.207}$$

$$\text{div } \boldsymbol{a}_1^{(0)} = 0. \tag{5.208}$$

Equations (5.197) and (5.201) are exactly equations (5.146) and (5.150). As before, if $h^{(0)} < h_{c_2}$ then there exists an unstable mode. Hence the solution branches bifurcating from eigenvalues $h^{(0)} < h_{c_2}$ are linearly unstable. It remains to determine the stability of the solution branch bifurcating from $h^{(0)} = h_{c_2}$. When $h^{(0)} = h_{c_2}$ all the eigenvalues for $\sigma^{(0)}$ are negative except for the eigenvalue $\sigma^{(0)} = 0$. We must proceed to higher orders in our expansions to determine the stability of this mode. We note that for $\sigma^{(0)} = 0$, $\psi_1^{(0)}$ satisfies the same equation and boundary conditions as $\psi_0^{(0)}$, and hence $\psi_1^{(0)} \propto \psi_0^{(0)}$. As in the one-dimensional case, the constant of proportionality is irrelevant and we take it to be unity (in effect this defines δ.) Substituting into equations (5.198)-(5.200) and (5.202)-(5.208) we find

$$\begin{aligned} -(\mathrm{curl})^2\boldsymbol{a}_1^{(0)} - \nabla\phi_1^{(0)} &= (i/\kappa)(\psi_0^{(0)*}\nabla\psi_0^{(0)} - \psi_0^{(0)}\nabla\psi_0^{(0)*}) + 2\,|\,\psi_0^{(0)}\,|^2\, h_{c_2}\boldsymbol{A}_N \\ &= -2(\mathrm{curl})^2\boldsymbol{a}_0^{(0)}, \text{ in } \Omega, \end{aligned} \tag{5.209}$$

by equation (5.29). Taking the divergence of this equation we find

$$\nabla^2\phi_1^{(0)} = 0, \text{ in } \Omega.$$

This, together with equation (5.200) and boundary conditions (5.204), (5.205) and (5.207), implies

$$\phi_1^{(0)} \equiv 0.$$

Now by comparing equations (5.209), (5.199), and (5.208) and boundary conditions (5.202), (5.203), and (5.206) with equations (5.29), (5.30), and (5.35) and boundary conditions (5.32)-(5.34) we see that

$$\boldsymbol{a}_1^{(0)} = 2\boldsymbol{a}_0^{(0)}. \tag{5.210}$$

Equating powers of ϵ in equations (5.178) and (5.182) we find

$$\begin{aligned} -\psi_1^{(1)} + ((i/\kappa) + h_{c_2}\boldsymbol{A}_N)^2\psi_1^{(1)} &= -(\alpha\sigma^{(1)}/\kappa^2)\psi_1^{(0)} - 2\,|\,\psi_0^{(0)}\,|^2\,\psi_1^{(0)} \\ &\quad - \psi_1^{(0)*}(\psi_0^{(0)})^2 - (2i/\kappa)(\boldsymbol{a}_0^{(0)}\cdot\nabla\psi_1^{(0)}) \\ &\quad - 2h_{c_2}(\boldsymbol{a}_0^{(0)}\cdot\boldsymbol{A}_N)\psi_1^{(0)} - (2i/\kappa)(\boldsymbol{a}_1^{(0)}\cdot\nabla\psi_0^{(0)}) \\ &\quad - 2h_{c_2}(\boldsymbol{a}_1^{(0)}\cdot\boldsymbol{A}_N)\psi_0^{(0)} - 2h_{c_2}h^{(1)}\,|\,\boldsymbol{A}_N\,|^2\,\psi_1^{(0)} \\ &\quad - (2i/\kappa)h^{(1)}(\boldsymbol{A}_N\cdot\nabla\psi_1^{(0)}), \text{ in } \Omega, \end{aligned} \tag{5.211}$$

$$\begin{aligned} \boldsymbol{n}\cdot((i/\kappa)\nabla + h_{c_2}\boldsymbol{A}_N)\psi_1^{(1)} + (i/d)\psi_1^{(1)} &= -\boldsymbol{n}\cdot\boldsymbol{a}_1^{(0)}\psi_0^{(0)} - \boldsymbol{n}\cdot\boldsymbol{a}_0^{(0)}\psi_1^{(0)} \\ &\quad - h^{(1)}\boldsymbol{n}\cdot\boldsymbol{A}_N\psi_1^{(0)}, \text{ on } \partial\Omega. \end{aligned} \tag{5.212}$$

Inserting the solutions for $\psi_1^{(0)}$ and $\boldsymbol{a}_1^{(0)}$ we have

$$\begin{aligned} -\psi_1^{(1)} + ((i/\kappa) + h_{c_2}\boldsymbol{A}_N)^2\psi_1^{(1)} &= -(\alpha\sigma^{(1)}/\kappa^2)\psi_0^{(0)} - 3\,|\,\psi_0^{(0)}\,|^2\,\psi_0^{(0)} \\ &\quad - (6i/\kappa)\boldsymbol{a}_0^{(0)}\cdot\nabla\psi_0^{(0)} - 6h_{c_2}\boldsymbol{a}_0^{(0)}\cdot\boldsymbol{A}_N\psi_0^{(0)} \\ &\quad - (2i/\kappa)h^{(1)}\boldsymbol{A}_N\cdot\nabla\psi_0^{(0)} \\ &\quad -2h_{c_2}h^{(1)}\,|\,\boldsymbol{A}_N\,|^2\,\psi_0^{(0)}, \text{ in } \Omega, \end{aligned} \tag{5.213}$$

$$\boldsymbol{n}\cdot((i/\kappa)\nabla + h_{c_2}\boldsymbol{A}_N)\psi_1^{(1)} + (i/d)\psi_1^{(1)} \;=\; -3\boldsymbol{n}\cdot\boldsymbol{a}_0^{(0)}\psi_0^{(0)} - h^{(1)}\boldsymbol{n}\cdot\boldsymbol{A}_N\psi_0^{(0)}, \quad \text{on } \partial\Omega. \tag{5.214}$$

Now, $\psi_0^{(0)}$ is a solution of the inhomogeneous version of equation (5.213) and boundary condition (5.214), namely (5.28) and (5.31). Hence there is a solution for $\psi_1^{(1)}$ if and only if an appropriate solvability condition is satisfied. To derive this condition we multiply (5.213) by $\psi_0^{(0)*}$ and integrate over Ω. We find that

$$\begin{aligned}
\text{LHS} \;=\;& \\
&\int_\Omega \psi_0^{(0)*}\left[-(1/\kappa^2)\nabla^2\psi_1^{(1)} + (2i/\kappa)h_{c_2}\boldsymbol{A}_N\cdot\nabla\psi_1^{(1)} + h_{c_2}^2\,|\boldsymbol{A}_N|^2\,\psi_1^{(1)} - \psi_1^{(1)}\right]dV,\\
=\;& \int_\Omega \psi_1^{(1)}\left[-(1/\kappa^2)\nabla^2\psi_0^{(0)*} - (2i/\kappa)h_{c_2}\boldsymbol{A}_N\cdot\nabla\psi_0^{(0)*} + h_{c_2}^2\,|\boldsymbol{A}_N|^2\,\psi_0^{(0)*} - \psi_0^{(0)*}\right]dV,\\
&+\int_{\partial\Omega}\left[-(1/\kappa^2)(\psi_0^{(0)*}\nabla\psi_1^{(1)} - \psi_1^{(1)}\nabla\psi_0^{(0)*}) + (2i/\kappa)h_{c_2}\psi_0^{(0)*}\psi_1^{(1)}\boldsymbol{A}_N\right]\cdot\boldsymbol{n}\,dS,
\end{aligned}$$

by Green's Theorem,

$$\begin{aligned}
=\;& (i/\kappa)\int_{\partial\Omega}\left[(i/\kappa)\nabla\psi_1^{(1)} + h_{c_2}\psi_1^{(1)}\boldsymbol{A}_N\right]\psi_0^{(0)*}\cdot\boldsymbol{n}\,dS\\
&+(i/\kappa)\int_{\partial\Omega}\left[-(i/\kappa)\nabla\psi_0^{(0)*} + h_{c_2}\psi_0^{(0)*}\boldsymbol{A}_N\right]\psi_1^{(1)}\cdot\boldsymbol{n}\,dS,
\end{aligned}$$

since the integral over Ω is zero by (5.28),

$$\begin{aligned}
=\;& -(i/\kappa)\int_{\partial\Omega}\psi_0^{(0)*}\left[(i/d)\psi_1^{(1)} + 3\boldsymbol{n}\cdot\boldsymbol{a}_0^{(0)}\psi_0^{(0)} + h^{(1)}\boldsymbol{n}\cdot\boldsymbol{A}_N\psi_0^{(0)}\right]dS\\
&+(i/\kappa)\int_{\partial\Omega}(i/d)\psi_0^{(0)*}\psi_1^{(1)}\,dS,
\end{aligned}$$

by (5.31) and (5.214),

$$=\; -(i/\kappa)\int_{\partial\Omega}|\,\psi_0^{(0)}\,|^2\,(3\boldsymbol{a}_0^{(0)} + h^{(1)}\boldsymbol{A}_N)\cdot\boldsymbol{n}\,dS.$$

$$\begin{aligned}
\text{RHS} \;=\;& \\
&-\int_\Omega\Big[3\,|\,\psi_0^{(0)}\,|^4 + (\alpha\sigma^{(1)}/\kappa^2)\,|\,\psi_0^{(0)}\,|^2\\
&\qquad + (6i/\kappa)\psi_0^{(0)*}\boldsymbol{a}_0^{(0)}\cdot\nabla\psi_0^{(0)} + (2i/\kappa)h^{(1)}\psi_0^{(0)*}\boldsymbol{A}_N\cdot\nabla\psi_0^{(0)}\\
&\qquad\qquad + 6h_{c_2}\,|\,\psi_0^{(0)}\,|^2\,\boldsymbol{A}_N\cdot\boldsymbol{a}_0^{(0)} + 2h_{c_2}h^{(1)}\,|\,\psi_0^{(0)}\,|^2|\,\boldsymbol{A}_N\,|^2\Big]\,dV,\\
=\;& -(\alpha\sigma^{(1)}\,|\,\beta\,|^2\,/\kappa^2)\\
&-\int_\Omega\Big[3\,|\,\psi_0^{(0)}\,|^4 + (2i/\kappa)\psi_0^{(0)*}\nabla\psi_0^{(0)}\cdot(3\boldsymbol{a}_0^{(0)} + h^{(1)}\boldsymbol{A}_N)\\
&\quad + 2(3\boldsymbol{a}_0^{(0)} + h^{(1)}\boldsymbol{A}_N)\cdot(-\text{curl}^2\boldsymbol{a}_0^{(0)} - (i/2\kappa)(\psi_0^{(0)*}\nabla\psi_0^{(0)} - \psi_0^{(0)}\nabla\psi_0^{(0)*}))\Big]\,dV,
\end{aligned}$$

by (5.29),

$$
\begin{aligned}
&= -(\alpha\sigma^{(1)} \mid \beta \mid^2 / \kappa^2) - \int_\Omega \Big[3 \mid \psi_0^{(0)} \mid^4 \\
&\quad + (3\boldsymbol{a}_0^{(0)} + h^{(1)}\boldsymbol{A}_N) \cdot ((i/\kappa)(\psi_0^{(0)*}\nabla\psi_0^{(0)} + \psi_0^{(0)}\nabla\psi_0^{(0)*}) - 2\mathrm{curl}^2\boldsymbol{a}_0^{(0)}) \Big] \, dV, \\
&= -(\alpha\sigma^{(1)} \mid \beta \mid^2 / \kappa^2) \\
&\quad - \int_\Omega \Big[3 \mid \psi_0^{(0)} \mid^4 + (3\boldsymbol{a}_0^{(0)} + h^{(1)}\boldsymbol{A}_N) \cdot ((i/\kappa)\nabla \mid \psi_0^{(0)} \mid^2 - 2\mathrm{curl}^2\boldsymbol{a}_0^{(0)}) \Big] \, dV, \\
&= -(\alpha\sigma^{(1)} \mid \beta \mid^2 / \kappa^2) + \int_\Omega \Big[-3 \mid \psi_0^{(0)} \mid^4 + 2(\mathrm{curl})^2\boldsymbol{a}_0^{(0)} \cdot (3\boldsymbol{a}_0^{(0)} + h^{(1)}\boldsymbol{A}_N) \Big] \, dV \\
&\quad - (i/\kappa) \int_{\partial\Omega} \mid \psi_0^{(0)} \mid^2 (3\boldsymbol{a}_0^{(0)} + h^{(1)}\boldsymbol{A}_N) \cdot \boldsymbol{n} \, dS,
\end{aligned}
$$

by the divergence theorem, since div $\boldsymbol{A}_N$ and div $\boldsymbol{a}_0^{(0)}$ are both zero. Equating the left-hand side to the right-hand side we have

$$
(\alpha\sigma^{(1)} \mid \beta \mid^2 / \kappa^2) = \int_\Omega \Big[-3 \mid \psi_0^{(0)} \mid^4 + 2(\mathrm{curl})^2\boldsymbol{a}_0^{(0)} \cdot (3\boldsymbol{a}_0^{(0)} + h^{(1)}\boldsymbol{A}_N) \Big] \, dV.
$$

Hence

$$
\begin{aligned}
(\alpha\sigma^{(1)}/\kappa^2) \mid \beta \mid^2 &= -2 \int_\Omega \Big[\mid \psi^{(0)} \mid^4 - 2\boldsymbol{a}^{(0)} \cdot (\mathrm{curl})^2\boldsymbol{a}^{(0)} \Big] \, dV \\
&= -4h^{(1)} \int_\Omega \boldsymbol{A}_N \cdot (\mathrm{curl})^2\boldsymbol{a}_0^{(0)} \, dV, \qquad (5.215)
\end{aligned}
$$

since we have

$$
h^{(1)} = \frac{\int_\Omega \mid \psi^{(0)} \mid^4 \, dV - \int_\Omega 2\boldsymbol{a}^{(0)} \cdot (\mathrm{curl})^2\boldsymbol{a}^{(0)} \, dV}{2 \int_\Omega \boldsymbol{A}_N \cdot (\mathrm{curl})^2\boldsymbol{a}^{(0)} \, dV}.
$$

Thus

$$
\begin{aligned}
\sigma^{(1)} &= -(4\kappa^2 h^{(1)}/\alpha \mid \beta \mid^2) \int_\Omega \boldsymbol{A}_N \cdot \mathrm{curl}^2\boldsymbol{a}_0^{(0)} \, dV, \qquad (5.216) \\
&= -(4\kappa^2 h^{(1)}/\alpha) \int_\Omega \boldsymbol{A}_N \cdot \mathrm{curl}^2\hat{\boldsymbol{a}}_0^{(0)} \, dV \qquad (5.217)
\end{aligned}
$$

We see that the sign of $\sigma^{(1)}$ depends on both the sign of $h^{(1)}$ and the sign of $\boldsymbol{A}_N \cdot \mathrm{curl}^2\hat{\boldsymbol{a}}_0^{(0)}$.

A quick calculation of (5.216) for the one-dimensional solution shows

$$
\begin{aligned}
-(4\kappa^2 h^{(1)}/\alpha\beta^2) \int_\Omega \boldsymbol{A}_N \cdot (\mathrm{curl})^2\boldsymbol{a}_0^{(0)} \, dV &= (4\kappa^2 h^{(1)}/\alpha\sqrt{\pi}) \int_{-\infty}^{\infty} \kappa^2 x^2 e^{-\kappa^2 x^2} \, dx \\
&= 2\kappa h^{(1)}/\alpha,
\end{aligned}
$$

in agreement with equation (5.130).

Let us now try and relate the above result about classical stability to the problem of free energy minimisation. If D represents the region outside Ω then the total dimensionless Gibbs free energy of the superconducting state is given by

$$\begin{aligned}\mathcal{G}_{sH} \;=\;& \int_\Omega \left|\frac{1}{\kappa}\nabla\Psi - i\boldsymbol{A}\Psi\right|^2 - |\Psi|^2 + \frac{|\Psi|^4}{2} + |\mathrm{curl}\ \boldsymbol{A}|^2 - 2h\,\mathrm{curl}\ \boldsymbol{A}\cdot\mathrm{curl}\ \boldsymbol{A}_N\,dV\\ &+ \frac{1}{\mu_e}\int_D |\mathrm{curl}\ \boldsymbol{A}|^2 - 2h\,\mathrm{curl}\ \boldsymbol{A}\cdot\mathrm{curl}\ \boldsymbol{A}_N\,dV + \int_{\partial\Omega}\frac{|\Psi|^2}{\kappa d}\,dS,\end{aligned}$$

where μ_e is the permeability of the external region normalised with that of the superconducting region, and we have added the final term to account for the modified boundary condition (5.4) (as noted in Chapter 3). In the normal state $\Psi\equiv 0$, $\boldsymbol{A} = h\boldsymbol{A}_N$ and the total dimensionless Gibbs free energy is then

$$\mathcal{G}_{nH} \;=\; -h^2\int_\Omega |\mathrm{curl}\ \boldsymbol{A}_N|^2\ dV - \frac{h^2}{\mu_e}\int_D |\mathrm{curl}\ \boldsymbol{A}_N|^2\ dV.$$

Hence

$$\begin{aligned}\mathcal{G}_{sH} - \mathcal{G}_{nH} \;=\;& \int_\Omega \left|\frac{1}{\kappa}\nabla\Psi - i\boldsymbol{A}\Psi\right|^2 - |\Psi|^2 + \frac{|\Psi|^4}{2} + |\mathrm{curl}(\boldsymbol{A} - h\boldsymbol{A}_N)|^2\ dV\\ &+ \frac{1}{\mu_e}\int_D |\mathrm{curl}(\boldsymbol{A} - h\boldsymbol{A}_N)|^2\ dV + \int_{\partial\Omega}\frac{|\Psi|^2}{\kappa d}\,dS\end{aligned}$$

$$\begin{aligned}=\;& \int_\Omega \Psi^*\left(\frac{1}{\kappa}\nabla - i\boldsymbol{A}\right)^2\Psi - |\Psi|^2 + \frac{|\Psi|^4}{2} + (\boldsymbol{A} - h\boldsymbol{A}_N)\cdot(\mathrm{curl})^2(\boldsymbol{A} - h\boldsymbol{A}_N)\,dV\\ &+ \frac{1}{\mu_e}\int_D (\boldsymbol{A} - h\boldsymbol{A}_N)\cdot(\mathrm{curl})^2(\boldsymbol{A} - h\boldsymbol{A}_N)\,dV\\ &- \frac{1}{\kappa}\int_{\partial\Omega}\Psi^*\left(\frac{1}{\kappa}\nabla - i\boldsymbol{A}\right)\Psi\cdot\boldsymbol{n}\,dS + \int_{\partial\Omega}\frac{|\Psi|^2}{\kappa d}\,dS\\ &+ \int_{\partial\Omega}(\boldsymbol{A} - h\boldsymbol{A}_N)\wedge\mathrm{curl}(\boldsymbol{A} - h\boldsymbol{A}_N)\cdot\boldsymbol{n}\,dS\\ &+ \frac{1}{\mu_e}\int_{\partial D}(\boldsymbol{A} - h\boldsymbol{A}_N)\wedge\mathrm{curl}(\boldsymbol{A} - h\boldsymbol{A}_N)\cdot\boldsymbol{n}\,dS,\end{aligned}$$

by the divergence theorem, since

$$\mathrm{div}\ (\boldsymbol{F}\wedge\boldsymbol{G}) = \boldsymbol{G}\cdot\mathrm{curl}\ \boldsymbol{F} - \boldsymbol{F}\cdot\mathrm{curl}\ \boldsymbol{G},$$

and $\mathrm{curl}\ (\boldsymbol{A} - h\boldsymbol{A}_N)\to 0$, as $r\to\infty$. Hence

$$\mathcal{G}_{sH} - \mathcal{G}_{nH} \;=\; \int_\Omega -\frac{|\Psi|^4}{2} + (\boldsymbol{A} - h\boldsymbol{A}_N)\cdot(\mathrm{curl})^2(\boldsymbol{A} - h\boldsymbol{A}_N)\,dV,$$

by (5.1), (5.3)-(5.6) and (5.10)-(5.12). Substituting the equations (5.15), (5.16) into this expression yields

$$\mathcal{G}_{sH} - \mathcal{G}_{nH} \quad = \quad \epsilon^2 \int_\Omega -\frac{|\,\psi\,|^4}{2} + \boldsymbol{a} \cdot (\text{curl})^2 \boldsymbol{a} \, dV.$$

Now inserting the expansions (5.26), (5.27) yields the leading order approximation to $\mathcal{G}_{sH} - \mathcal{G}_{nH}$ as

$$\epsilon^2 \int_\Omega -\frac{|\,\psi^{(0)}\,|^4}{2} + \boldsymbol{a}^{(0)} \cdot (\text{curl})^2 \boldsymbol{a}^{(0)} \, dV.$$

Thus we see that $\sigma^{(0)} < 0$ if and only if $\mathcal{G}_{sH} < \mathcal{G}_{nH}$, i.e. the superconducting solution is stable if and only if it has a lower Gibbs free energy than that of the normal state.

5.3 Results

We have found that as the magnitude of the external magnetic field is varied there is a series of bifurcations from the normal state solution to superconducting solution branches. The largest positive eigenvalue is known as the upper critical magnetic field, h_{c_2}. The normal state solution is linearly stable for fields of magnitude greater than h_{c_2}, and linearly unstable for fields of magnitude less than h_{c_2}. All the superconducting solution branches other than the one bifurcating at h_{c_2} are linearly unstable near the bifurcation points. The stability of this solution branch depends on the value of κ and the geometry of the sample.

When the sample dimensions are large in comparison to the penetration depth it may be considered infinite when we are working on the length scale of the penetration depth. In this case the upper critical field is equal to κ, the Ginzburg-Landau parameter. For Type I superconductors we find that $h_{c_2} < H_c$, and the bifurcation at $h = h_{c_2}$ is subcritical, that is, the superconducting solution exists for values of the field slightly greater than h_{c_2} and is linearly unstable near the bifurcation point. For Type II superconductors we find that $h_{c_2} > H_c$, and the bifurcation at $h = h_{c_2}$ is supercritical, that is, the superconducting solution exists for values of the field slightly less than h_{c_2} and is linearly stable near the bifurcation point. (We note that even for a large superconducting body things may be different near the surface. We will examine the effects of the presence of a surface on the nucleation of superconductivity in the following chapter.)

We have thus far only examined the linear stability of solutions. We have not yet determined how, or indeed whether, a small perturbation to the unstable normal solution will grow into the stable superconducting solution for a Type II superconductor. To answer such questions as this we need to consider the nonlinear stability of the normal state solution.

5.4 Weakly-nonlinear Stability of the Normal State Solution

As previously, before we consider the general case, we introduce the techniques by means of a simpler one-dimensional example.

5.4.1 One-dimensional Example

We examine the evolution of a small perturbation of the normal state when the superconducting body is infinite via the one-dimensional time-dependent Ginzburg-Landau equations. We have

$$-\frac{\alpha}{\kappa^2}\frac{\partial \Psi}{\partial t} + \frac{1}{\kappa^2}\frac{\partial^2 \Psi}{\partial x^2} = \Psi^3 - \Psi + A^2\Psi, \tag{5.218}$$

$$\frac{\partial^2 A}{\partial x^2} = \frac{\partial A}{\partial t} + \Psi^2 A, \tag{5.219}$$

$$\frac{\partial \Psi}{\partial x} \to 0, \text{ as } |x| \to \infty, \tag{5.220}$$

$$\frac{\partial A}{\partial x} \to h, \text{ as } |x| \to \infty. \tag{5.221}$$

We consider the solution near the bifurcation point $h = \kappa$. To this end we set

$$h = \kappa + \epsilon h^{(1)}, \tag{5.222}$$

where $\epsilon > 0$. As usual we introduce a and ψ via the relations

$$A = hx + \epsilon a, \tag{5.223}$$

$$\Psi = \epsilon^{1/2}\psi. \tag{5.224}$$

Substituting (5.222)-(5.224) into (5.218)-(5.221) yields

$$-\frac{\alpha}{\kappa^2}\frac{\partial \psi}{\partial t} + \frac{1}{\kappa^2}\frac{\partial^2 \psi}{\partial x^2} = \epsilon\psi^3 - \psi + \psi[(\kappa + \epsilon h^{(1)})^2 x^2 + 2\epsilon(\kappa + \epsilon h^{(1)})xa + \epsilon^2 a^2], \tag{5.225}$$

$$
\begin{aligned}
\frac{\partial^2 a}{\partial x^2} &= \frac{\partial a}{\partial t} + \psi^2[(\kappa + \epsilon h^{(1)})x + \epsilon a], && (5.226)\\
\frac{\partial \psi}{\partial x} &\to 0, \text{ as } |x| \to \infty, && (5.227)\\
\frac{\partial a}{\partial x} &\to 0, \text{ as } |x| \to \infty. && (5.228)
\end{aligned}
$$

When we examined the linear stability of the normal state and superconducting state solutions near the bifurcation point we found that one mode had growth/decay timescale of $O(\epsilon^{-1})$ while all other modes had a decay timescale of $O(1)$. Thus we expect when we examine the nonlinear behaviour of the solution that there will be two timescales: an $O(1)$ timescale and an $O(\epsilon^{-1})$ timescale. (Note that in the case of nonlinear hydrodynamic stability [19] the problem often requires a multiple scales analysis, since the linearised, short-time solution is often a wave, and the long-time solution a modulated wave. In the present situation the short-time solution tends to a single real exponential as time increases, and we may proceed via matched asymptotic expansions, treating each timescale separately.)

A. Short timescale : $t = O(1)$.

We denote the short-time solution by ψ_s, a_s. We again expand ψ_s and a_s in powers of ϵ:

$$
\begin{aligned}
\psi_s &= \psi_s^{(0)} + \epsilon \psi_s^{(1)} + \cdots, && (5.229)\\
a_s &= a_s^{(0)} + \epsilon a_s^{(1)} + \cdots. && (5.230)
\end{aligned}
$$

Substituting the expansions (5.229), (5.230) into equations (5.225)-(5.228) and equating powers of ϵ yields at leading order

$$
\begin{aligned}
-\frac{\alpha}{\kappa^2}\frac{\partial \psi_s^{(0)}}{\partial t} + \frac{1}{\kappa^2}\frac{\partial^2 \psi_s^{(0)}}{\partial x^2} &= -\psi_s^{(0)} + \kappa^2 x^2 \psi_s^{(0)}, && (5.231)\\
\frac{\partial^2 a_s^{(0)}}{\partial x^2} &= \frac{\partial a_s^{(0)}}{\partial t} + \kappa x (\psi_s^{(0)})^2, && (5.232)\\
\frac{\partial \psi_s^{(0)}}{\partial x} &\to 0, \text{ as } |x| \to \infty, && (5.233)\\
\frac{\partial a_s^{(0)}}{\partial x} &\to 0, \text{ as } |x| \to \infty. && (5.234)
\end{aligned}
$$

The solution to (5.231) with boundary condition (5.234) is

$$
\psi_s^{(0)}(x,t) = \sum_{n=0}^{\infty} \beta_n e^{\sigma_n t} \theta_n(x),
$$

where

$$\sigma_n = -2n\kappa^2/\alpha,$$

with corresponding eigenfunctions

$$\theta_n(x) = \frac{\kappa^{1/2} e^{-\kappa^2 x^2/2}}{(2^n n! \sqrt{\pi})^{1/2}} H_n(\sqrt{2}\,\kappa x),$$

and β_n are constants. The β_n must be chosen such that

$$\psi_s^{(0)}(x,0) = \sum_{n=0}^{\infty} \beta_n \theta_n(x),$$

Since the eigenfunctions are orthogonal we multiply by θ_m and integrate over $(-\infty, \infty)$ to obtain

$$\beta_m = \int_{-\infty}^{\infty} \psi_s^{(0)}(x,0)\theta_m(x)\,dx. \tag{5.235}$$

Hence

$$\psi_s^{(0)}(x,t) = \int_{-\infty}^{\infty} \psi_s^{(0)}(\xi,0) \left(\sum_{n=0}^{\infty} \theta_n(\xi)\theta_n(x) e^{\sigma_n t} \right) d\xi. \tag{5.236}$$

However, in one dimension we can write the Greens function in (5.236) in closed form. We find

$$\psi_s^{(0)}(x,t) = \int_{-\infty}^{\infty} \psi_s^{(0)}(\xi,0) G(\xi;x,t)\,d\xi, \tag{5.237}$$

where

$$G(\xi;x,t) = \frac{\kappa}{\sqrt{2\pi \sinh(2t)}} \exp\left\{ t - \frac{\kappa^2}{2}[(x^2+\xi^2)\coth(2t) - 2x\xi \mathrm{cosech}(2t)] \right\}. \tag{5.238}$$

When $\psi_s^{(0)}$ is given by (5.237) we can then solve (5.232), (5.234) for $a_s^{(0)}$. However this lowest order solution does not take into account the possible growth of the unstable mode, since the growth rate is $O(\epsilon)$. We expect that if we continue to higher orders in our expansion that we will find secular terms appearing, and that the expansion will cease to be valid when $t = O(\epsilon^{-1})$. Thus we need to consider the long-time behaviour of the solution.

B. Long timescale : $t = O(\epsilon^{-1})$

We define a new timescale by

$$\tau = \epsilon t.$$

Denoting the long-time solution by $\psi_l(x,\tau)$, $a_l(x,\tau)$ we have

$$-\frac{\alpha\epsilon}{\kappa^2}\frac{\partial\psi_l}{\partial\tau}+\frac{1}{\kappa^2}\frac{\partial^2\psi_l}{\partial x^2} = \epsilon\psi_l^3-\psi_l+\psi_l[(\kappa+\epsilon h^{(1)})^2x^2+2\epsilon\,(\kappa+\epsilon h^{(1)})\,xa_l+\epsilon^2a_l^2], \quad (5.239)$$

$$\frac{\partial^2 a_l}{\partial x^2} = \epsilon\frac{\partial a_l}{\partial\tau}+\psi_l^2[(\kappa+\epsilon h^{(1)})x+\epsilon a_l], \quad (5.240)$$

$$\frac{\partial\psi_l}{\partial x} \to 0, \text{ as } |x|\to\infty, \quad (5.241)$$

$$\frac{\partial a_l}{\partial x} \to h, \text{ as } |x|\to\infty. \quad (5.242)$$

We expand ψ_l and a_l in powers of ϵ as before:

$$\psi_l = \psi_l^{(0)}+\epsilon\psi_l^{(1)}+\cdots, \quad (5.243)$$

$$a_l = a_l^{(0)}+\epsilon a_l^{(1)}+\cdots. \quad (5.244)$$

Substituting the expansions (5.243), (5.244) into equations (5.239)-(5.242) and equating powers of ϵ yields at leading order

$$\frac{1}{\kappa^2}\frac{\partial^2\psi_l^{(0)}}{\partial x^2} = -\psi_l^{(0)}+\kappa^2x^2\psi_l^{(0)}, \quad (5.245)$$

$$\frac{\partial^2 a_l^{(0)}}{\partial x^2} = \kappa x(\psi_l^{(0)})^2, \quad (5.246)$$

$$\frac{\partial\psi_l^{(0)}}{\partial x} \to 0 \text{ , as } |x|\to\infty, \quad (5.247)$$

$$\frac{\partial a_l^{(0)}}{\partial x} \to h, \text{ as } |x|\to\infty. \quad (5.248)$$

Equations (5.245)-(5.248) are exactly equations (5.69)-(5.72) with $h^{(0)}=\kappa$, and have solution

$$\psi_l^{(0)} = \beta(\tau)\frac{\kappa^{1/2}}{\pi^{1/4}}e^{-\kappa^2x^2/2}, \quad (5.249)$$

$$a_l^{(0)} = -\beta(\tau)^2\frac{1}{2\pi^{1/2}}\int_{-\infty}^{x} e^{-\kappa^2\xi^2}\,d\xi, \quad (5.250)$$

where $\beta(\tau)$ is an unknown function of τ. The factor $\kappa^{1/2}/\pi^{1/4}$ has been inserted so that the eigenfunction is normalised to be consistent with the previous section.

To determine the function $\beta(\tau)$ we need to proceed to higher powers in our expansions in ϵ. Equating powers of ϵ in (5.239), (5.241) yields

$$\begin{aligned}
&\frac{1}{\kappa^2}\frac{\partial^2\psi_l^{(1)}}{\partial x^2}+\psi_l^{(1)}-\kappa^2x^2\psi_l^{(1)} \\
&= \frac{\alpha}{\kappa^2}\frac{\partial\psi_l^{(0)}}{\partial\tau}+(\psi_l^{(0)})^3+h^{(1)}2\kappa x^2\psi_l^{(0)}+2\kappa x\psi_l^{(0)}a_l^{(0)} \\
&= \frac{\alpha}{\kappa^{3/2}\pi^{1/4}}\frac{d\beta}{d\tau}e^{-\kappa^2x^2/2}+\frac{\beta^3\kappa^{3/2}}{\pi^{3/4}}e^{-3\kappa^2x^2/2} \\
&\quad +h^{(1)}\frac{2\kappa^{3/2}x^2\beta}{\pi^{1/4}}e^{-\kappa^2x^2/2}-\frac{x\beta^3\kappa^{3/2}}{\pi^{3/4}}e^{-\kappa^2x^2/2}\left(\int_{-\infty}^{x}e^{-\kappa^2\xi^2}\,d\xi\right), \qquad (5.251)
\end{aligned}$$

$$\frac{\partial\psi_l^{(1)}}{\partial x}\to 0,\ \text{as}\ |x|\to\infty. \qquad (5.252)$$

Since $e^{-\kappa^2x^2/2}$ is a solution of the homogeneous version of this equation, there is a solution for $\psi_l^{(1)}$ if and only if a solvability condition is satisfied. Multiplying (5.251) by $e^{-\kappa^2x^2/2}$ and integrating over $(-\infty,\infty)$ gives the condition:

$$\begin{aligned}
0 &= \int_{-\infty}^{\infty}\left[\frac{\alpha\pi^{1/2}}{\kappa}\frac{d\beta}{d\tau}e^{-\kappa^2x^2}+\beta^3\kappa^2e^{-2\kappa^2x^2}\right. \\
&\quad \left.+h^{(1)}2\kappa^2x^2\beta\pi^{1/2}e^{-\kappa^2x^2}-x\beta^3\kappa^2e^{-\kappa^2x^2}\left(\int_{-\infty}^{x}e^{-\kappa^2\xi^2}\,d\xi\right)\right]dx, \\
&= \int_{-\infty}^{\infty}\left[\frac{\alpha\pi^{1/2}}{\kappa}\frac{d\beta}{d\tau}e^{-\kappa^2x^2}+\beta^3\kappa^2e^{-\kappa^2x^2}+h^{(1)}\beta\pi^{1/2}e^{-\kappa^2x^2}-(\beta^3/2)e^{-2\kappa^2x^2}\right]dx,
\end{aligned}$$

on integration by parts. Hence

$$\frac{\alpha}{\kappa^2}\frac{d\beta}{d\tau}=\frac{\beta^3\kappa}{\sqrt{2\pi}}\left[\frac{1}{2\kappa^2}-1\right]-\frac{h^{(1)}}{\kappa}\beta. \qquad (5.253)$$

Equation (5.253) is often known as the Landau equation. The boundary condition for it comes from matching with the short-time solution. We have

$$\begin{aligned}
\beta(0)\frac{\kappa^{1/2}}{\pi^{1/4}}e^{-\kappa^2x^2/2} &= \lim_{t\to\infty}\psi_s^{(0)}, \\
&= \beta_0\frac{\kappa^{1/2}}{\pi^{1/4}}e^{-\kappa^2x^2/2}, \\
&= \frac{\kappa}{\sqrt{\pi}}e^{-\kappa^2x^2/2}\int_{-\infty}^{\infty}\psi_s^{(0)}(\xi,0)e^{-\kappa^2\xi^2/2}\,d\xi.
\end{aligned}$$

Hence

$$\beta(0)=\beta_0=\frac{\kappa^{1/2}}{\pi^{1/4}}\int_{-\infty}^{\infty}\psi_s^{(0)}(\xi,0)e^{-\kappa^2\xi^2/2}\,d\xi. \qquad (5.254)$$

To simplify the analysis, and since a similar equation arises in the general case, we set

$$p=\frac{\kappa}{\sqrt{2\pi}}\left[\frac{1}{2\kappa^2}-1\right],\ q=-\frac{h^{(1)}}{\kappa}.$$

Then

$$\begin{aligned} \frac{\alpha}{\kappa^2}\frac{d\beta}{d\tau} &= p\beta^3 + q\beta, \\ \beta(0) &= \beta_0. \end{aligned}$$

Solving for β we have

$$\frac{\kappa^2\, d\tau}{\alpha} = \frac{d\beta}{p\beta^3 + q\beta} = \left(\frac{1}{\beta} - \frac{\beta}{\beta^2 + q/p}\right)\frac{d\beta}{q}.$$

Integrating we find

$$\frac{q\kappa^2\tau}{\alpha} + \text{const.} = \log\left(\frac{|\beta|}{|\beta^2 + q/p|^{1/2}}\right).$$

Hence

$$\frac{\beta^2}{|\beta^2 + q/p|} = Ce^{(2q\kappa^2/\alpha)\tau}, \tag{5.255}$$

where

$$C = \frac{\beta_0^2}{|\beta_0^2 + q/p|}.$$

Hence

$$\beta^2 = \begin{cases} \frac{q}{p}\left(\frac{Ce^{(2\kappa^2 q/\alpha)\tau}}{1 - Ce^{(2\kappa^2 q/\alpha)\tau}}\right) & \text{if } q/p > 0, \\ \begin{cases} -\frac{q}{p}\left(\frac{Ce^{(2\kappa^2 q/\alpha)\tau}}{Ce^{(2\kappa^2 q/\alpha)\tau} - 1}\right) & \text{if } \beta_0^2 > -q/p \\ -\frac{q}{p}\left(\frac{Ce^{(2\kappa^2 q/\alpha)\tau}}{Ce^{(2\kappa^2 q/\alpha)\tau} + 1}\right) & \text{if } \beta_0^2 < -q/p \end{cases} & \text{if } q/p < 0. \end{cases} \tag{5.256}$$

Substituting in the values of p and q we find

$$\beta^2 = \begin{cases} \left|\frac{2\sqrt{2\pi}}{1-2\kappa^2}\right|\left(\frac{Ce^{-(2h^{(1)}\kappa/\alpha)\tau}}{1 - Ce^{-(2h^{(1)}\kappa/\alpha)\tau}}\right) & \text{if } \frac{2\sqrt{2\pi}\, h^{(1)}}{1-2\kappa^2} < 0 \\ \begin{cases} \left|\frac{2\sqrt{2\pi}}{1-2\kappa^2}\right|\left(\frac{Ce^{-(2h^{(1)}\kappa/\alpha)\tau}}{Ce^{-(2h^{(1)}\kappa/\alpha)\tau} - 1}\right) & \text{if } \beta_0^2 > \left|\frac{2\sqrt{2\pi}}{1-2\kappa^2}\right| \\ \left|\frac{2\sqrt{2\pi}}{1-2\kappa^2}\right|\left(\frac{Ce^{-(2h^{(1)}\kappa/\alpha)\tau}}{Ce^{-(2h^{(1)}\kappa/\alpha)\tau} + 1}\right) & \text{if } \beta_0^2 < \left|\frac{2\sqrt{2\pi}}{1-2\kappa^2}\right| \end{cases} & \text{if } \frac{2\sqrt{2\pi}\, h^{(1)}}{1-2\kappa^2} > 0 \end{cases},$$

where

$$C = \frac{\beta_0^2}{|\beta_0^2 - h^{(1)}\frac{2\sqrt{2\pi}}{1-2\kappa^2}|}.$$

There are four cases to consider.

A. Type I superconductors : $\kappa < 1/\sqrt{2}$

(1) $h < \kappa$ **:** $h^{(1)} < 0$**.** In this case

$$\beta^2 = \frac{2\sqrt{2\pi}}{1-2\kappa^2}\left(\frac{Ce^{-(2h^{(1)}\kappa/\alpha)\tau}}{1-Ce^{-(2h^{(1)}\kappa/\alpha)\tau}}\right).$$

The solution blows up in finite time $\tau = \tau_\infty = -(\alpha/2\kappa h^{(1)})\log(1/C)$, as shown in Fig. 5.5

(2) $h > \kappa$ **:** $h^{(1)} > 0$**.** In this case

$$\beta^2 = \begin{cases} \frac{2\sqrt{2\pi}}{1-2\kappa^2}\left(\frac{Ce^{-(2h^{(1)}\kappa/\alpha)\tau}}{Ce^{-(2h^{(1)}\kappa/\alpha)\tau}-1}\right) & \text{if } \beta_0^2 > \frac{2\sqrt{2\pi}}{1-2\kappa^2} \\ \frac{2\sqrt{2\pi}}{1-2\kappa^2}\left(\frac{Ce^{-(2h^{(1)}\kappa/\alpha)\tau}}{Ce^{-(2h^{(1)}\kappa/\alpha)\tau}+1}\right) & \text{if } \beta_0^2 < \frac{2\sqrt{2\pi}}{1-2\kappa^2}. \end{cases}$$

Note that $\beta^2 = \frac{2\sqrt{2\pi}}{1-2\kappa^2}$ is the (in this case unstable) steady state solution found previously. We see that if β_0 is small enough the solution will decay exponentially to zero. However, if β_0 is greater than a critical value the solution will blow up in finite time $\tau = \tau_\infty = (\alpha/2\kappa h^{(1)})\log C$. The dividing line between these two types of behaviour is the unstable steady state solution (see Fig. 5.6). Thus although the normal state solution is linearly stable in this parameter regime we see that it is unstable to sufficiently large initial perturbations.

B. Type II superconductors : $\kappa > 1/\sqrt{2}$.

(1) $h < \kappa$ **:** $h^{(1)} < 0$**.** In this case

$$\beta^2 = \begin{cases} \frac{2\sqrt{2\pi}}{2\kappa^2-1}\left(\frac{Ce^{-(2h^{(1)}\kappa/\alpha)\tau}}{Ce^{-(2h^{(1)}\kappa/\alpha)\tau}-1}\right) & \text{if } \beta_0^2 > \frac{2\sqrt{2\pi}}{2\kappa^2-1} \\ \frac{2\sqrt{2\pi}}{2\kappa^2-1}\left(\frac{Ce^{-(2h^{(1)}\kappa/\alpha)\tau}}{Ce^{-(2h^{(1)}\kappa/\alpha)\tau}+1}\right) & \text{if } \beta_0^2 < \frac{2\sqrt{2\pi}}{2\kappa^2-1}. \end{cases}$$

In either case we see that

$$\beta^2 \to \frac{2\sqrt{2\pi}}{2\kappa^2-1}, \text{ as } \tau \to \infty,$$

(see Fig. 5.7). Thus given any initial data the solution tends to the stable superconducting state solution as $\tau \to \infty$.

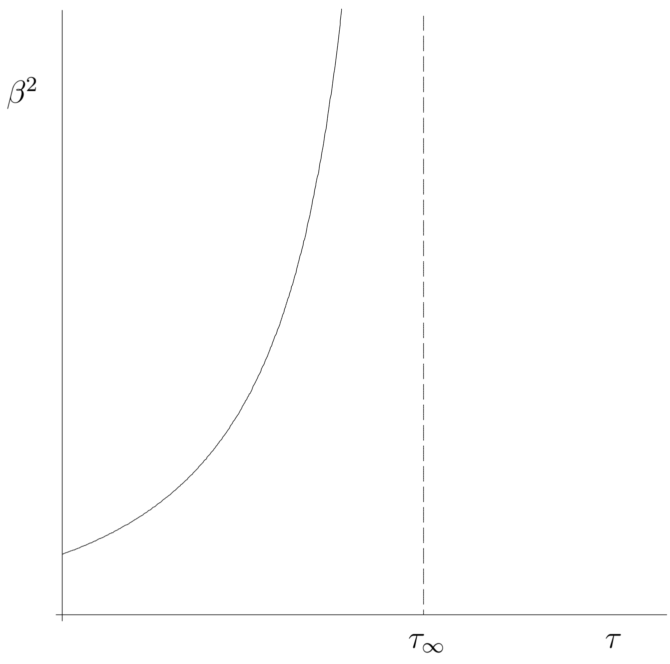

Figure 5.5: Response of a Type I superconductor with $h < \kappa$.

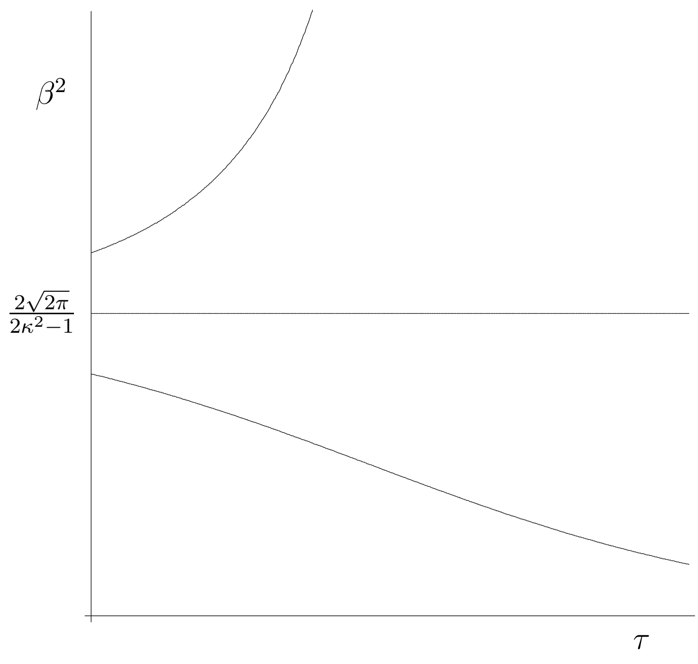

Figure 5.6: Response of a Type I superconductor with $h > \kappa$.

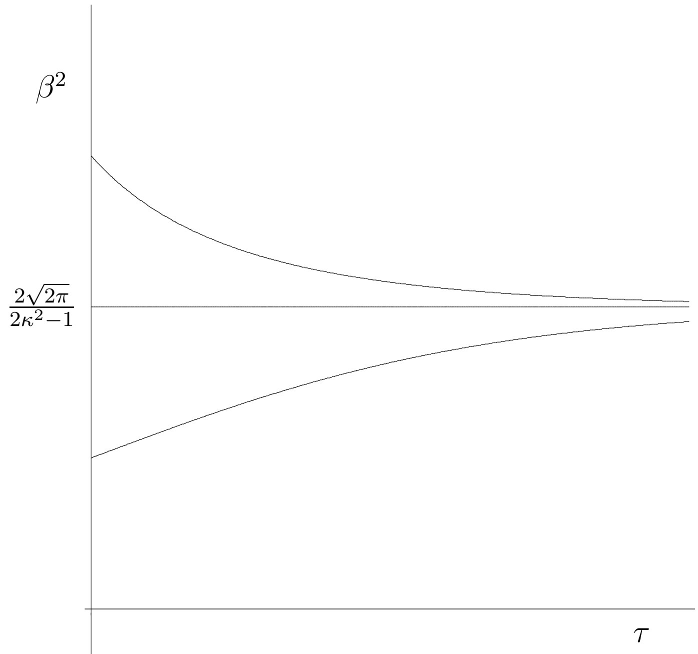

Figure 5.7: Response of a Type II superconductor with $h < \kappa$.

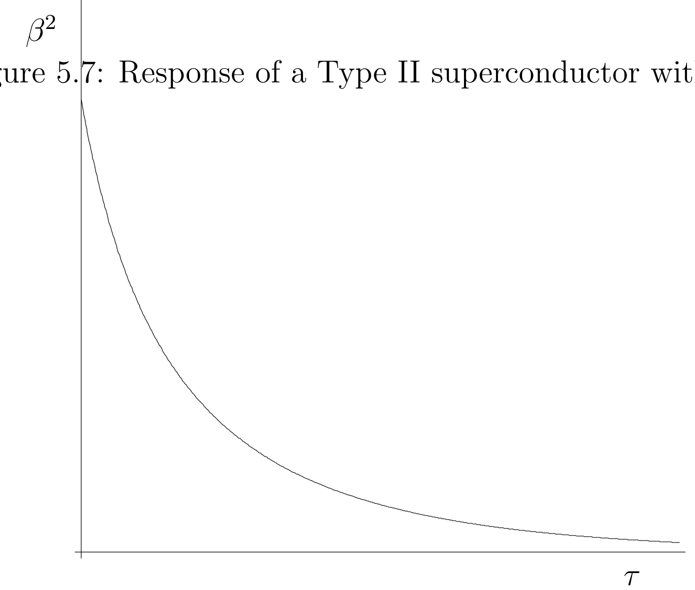

Figure 5.8: Response of a Type II superconductor with $h > \kappa$.

(2) $h > \kappa$ **:** $h^{(1)} > 0$**.** In this case

$$\beta^2 = \frac{2\sqrt{2\pi}}{2\kappa^2 - 1}\left(\frac{Ce^{-(2h^{(1)}\kappa/\alpha)\tau}}{1 - Ce^{-(2h^{(1)}\kappa/\alpha)\tau}}\right).$$

The solution decays exponentially to zero (see Fig. 5.8). Thus the normal state solution is both linearly and nonlinearly stable in this parameter regime.

The finite-time blow up of the solution under certain conditions is worth further comment. This does not mean that the solution of equations (5.218)-(5.221) is unbounded, rather that the expansion (5.243)-(5.244) in powers of ϵ ceases to be valid, since Ψ is no longer $\ll 1$. A complete determination of the solution would involve a new asymptotic expansion after choosing a new time variable

$$\epsilon t' = (\tau - \tau_\infty),$$

and treating Ψ as order one. The blow-up of the above solution would then provide a condition at $t' = -\infty$ for this solution. However, once Ψ becomes $O(1)$ we are faced with solving (5.218)-(5.221) in their entirety. We would conjecture that the solution would evolve into a superconducting region with $\Psi \approx 1$, separated from the surrounding normal region by two propagating phase boundaries, as described in Chapter 4.

Finally, we comment on the case $\kappa = 1/\sqrt{2}$. In this case $p = 0$, and so

$$\frac{d\beta}{d\tau} = q\beta.$$

For $h^{(1)} < 0$, $q > 0$ there is exponential growth, and for $h^{(1)} > 0$, $q < 0$ there is exponential decay.

5.4.2 Weakly-nonlinear Stability of the Normal State in a Body of Arbitrary Shape

We now use the above techniques to investigate the weakly-nonlinear stability of the normal state for a body of arbitrary shape in an external magnetic field.

We have the time-dependent Ginzburg-Landau equations (5.131)-(5.142):

$$\frac{\alpha}{\kappa^2}\frac{\partial \Psi}{\partial t} + \frac{\alpha i}{\kappa}\Psi\Phi + \left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2 \Psi = \Psi(1- |\Psi|^2), \text{ in } \Omega, \quad (5.257)$$

$$-(\mathrm{curl})^2 \boldsymbol{A} - \frac{\partial \boldsymbol{A}}{\partial t} - \nabla\Phi = \frac{i}{2\kappa}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) + |\Psi|^2 \boldsymbol{A}, \text{ in } \Omega, \quad (5.258)$$

$$-(\mathrm{curl})^2 \boldsymbol{A} = \varsigma_e \left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right), \text{ outside } \Omega, \quad (5.259)$$

$$\nabla^2\Phi = 0, \text{ outside } \Omega, \quad (5.260)$$

$$\boldsymbol{n}\cdot((i/\kappa)\nabla + \boldsymbol{A})\Psi + (i/d)\Psi = 0, \text{ on } \partial\Omega, \quad (5.261)$$

$$[\boldsymbol{n} \wedge \boldsymbol{A}] = \boldsymbol{0}, \quad (5.262)$$

$$[\boldsymbol{n} \wedge (1/\mu)\mathrm{curl}\ \boldsymbol{A}] = \boldsymbol{0}, \quad (5.263)$$

$$[\Phi] = 0, \quad (5.264)$$

$$\left[\varepsilon \frac{\partial \Phi}{\partial n}\right] = 0, \tag{5.265}$$

$$\text{curl } \boldsymbol{A} \rightarrow h\hat{\boldsymbol{z}}, \text{ as } r \rightarrow \infty, \tag{5.266}$$

$$\Phi \rightarrow 0, \text{ as } r \rightarrow \infty, \tag{5.267}$$

$$\text{div } \boldsymbol{A} = 0. \tag{5.268}$$

We seek a solution near the bifurcation point $h = h_{c_2}$. To this end we set

$$h = h_{c_2} + \epsilon h^{(1)}, \tag{5.269}$$

as in the one-dimensional case.

We introduce $\psi, \boldsymbol{a}$, and ϕ as before by setting

$$\Psi = \epsilon^{1/2}\psi, \tag{5.270}$$

$$\boldsymbol{A} = h\boldsymbol{A}_N + \epsilon \boldsymbol{a}, \tag{5.271}$$

$$\Phi = \epsilon\phi. \tag{5.272}$$

Substituting (5.269)-(5.272) into (5.257)-(5.268) yields

$$\begin{aligned}\frac{\alpha}{\kappa^2}\frac{\partial \psi}{\partial t} + \left(\frac{i}{\kappa}\nabla + (h_{c_2} + \epsilon h^{(1)})\boldsymbol{A}_N\right)^2 \psi - \psi = & -\epsilon\frac{\alpha i}{\kappa}\psi\phi + \epsilon\psi\,|\,\psi\,|^2 \\ & + 2\epsilon(h_{c_2} + \epsilon h^{(1)})\psi(\boldsymbol{A}_N \cdot \boldsymbol{a}) \\ & + \frac{2\epsilon i}{\kappa}(\boldsymbol{a} \cdot \nabla\psi) \\ & - \epsilon^2\,|\,\boldsymbol{a}\,|^2\,\psi, \quad \text{in } \Omega,\end{aligned} \tag{5.273}$$

$$\begin{aligned}-(\text{curl})^2\boldsymbol{a} - \frac{\partial \boldsymbol{a}}{\partial t} - \nabla\phi = & \frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) \\ & + |\,\psi\,|^2\,((h_{c_2} + \epsilon h^{(1)})\boldsymbol{A}_N + \epsilon\boldsymbol{a}), \\ & \text{in } \Omega,\end{aligned} \tag{5.274}$$

$$-(\text{curl})^2\boldsymbol{a} = \varsigma_e\left(\frac{\partial \boldsymbol{a}}{\partial t} + \nabla\phi\right), \text{ outside } \Omega, \tag{5.275}$$

$$\nabla^2\phi = 0, \text{ outside } \Omega, \tag{5.276}$$

$$\boldsymbol{n} \cdot ((i/\kappa)\nabla + (h_{c_2} + \epsilon h^{(1)})\boldsymbol{A}_N)\psi + (i/d)\psi = -\epsilon(\boldsymbol{n} \cdot \boldsymbol{a})\psi, \text{ on } \partial\Omega, \tag{5.277}$$

$$[\boldsymbol{n} \wedge \boldsymbol{a}] = \boldsymbol{0}, \tag{5.278}$$

$$[\boldsymbol{n} \wedge (1/\mu)\text{curl } \boldsymbol{a}] = \boldsymbol{0}, \tag{5.279}$$

$$[\phi] = 0, \tag{5.280}$$

$$\left[\varepsilon\frac{\partial \phi}{\partial n}\right] = 0, \tag{5.281}$$

$$\text{curl } \boldsymbol{a} \rightarrow \boldsymbol{0}, \text{ as } r \rightarrow \infty, \tag{5.282}$$

$$\phi \rightarrow 0, \text{ as } r \rightarrow \infty, \tag{5.283}$$

$$\text{div } \boldsymbol{a} = 0. \tag{5.284}$$

As in the one-dimensional example, when we examined the linear stability of the normal state and superconducting state solutions near the bifurcation point we found that one mode had growth/decay timescale of $O(\epsilon^{-1})$ while all other modes had a decay timescale of $O(1)$. Thus we expect when we examine the nonlinear behaviour of the solution that there will be two timescales: an $O(1)$ timescale and an $O(\epsilon^{-1})$ timescale.

A. Short timescale : $t = O(1)$.

We denote the short-time solution by $\psi_s(\boldsymbol{r}, t)$, $\boldsymbol{a}_s(\boldsymbol{r}, t)$, $\phi_s(\boldsymbol{r}, t)$, and expand all quantities in powers of ϵ as before:

$$\psi_s = \psi_s^{(0)} + \epsilon \psi_s^{(1)} + \cdots, \tag{5.285}$$

$$\boldsymbol{a}_s = \boldsymbol{a}_s^{(0)} + \epsilon \boldsymbol{a}_s^{(1)} + \cdots, \tag{5.286}$$

$$\phi_s = \phi_s^{(0)} + \epsilon \phi_s^{(1)} + \cdots. \tag{5.287}$$

Substituting the expansions (5.285)-(5.287) into equations (5.273)-(5.284) and equating powers of ϵ yields at leading order

$$\frac{\alpha}{\kappa^2}\frac{\partial \psi_s^{(0)}}{\partial t} + \left(\frac{i}{\kappa}\nabla + h_{c_2}\boldsymbol{A}_N\right)^2 \psi_s^{(0)} = \psi_s^{(0)}, \text{ in } \Omega, \tag{5.288}$$

$$-(\text{curl})^2 \boldsymbol{a}_s^{(0)} - \frac{\partial \boldsymbol{a}_s^{(0)}}{\partial t} - \nabla \phi_s^{(0)} = \frac{i}{2\kappa}(\psi_s^{(0)*}\nabla\psi_s^{(0)} - \psi_s^{(0)}\nabla\psi_s^{(0)*}) + h_{c_2} \, |\psi_s^{(0)}|^2 \, \boldsymbol{A}_N, \text{ in } \Omega, \tag{5.289}$$

$$-(\text{curl})^2 \boldsymbol{a}_s^{(0)} = \varsigma_e \left(\frac{\partial \boldsymbol{a}_s^{(0)}}{\partial t} + \nabla \phi_s^{(0)}\right), \text{ outside } \Omega, \tag{5.290}$$

$$\nabla^2 \phi_s^{(0)} = 0, \text{ outside } \Omega, \tag{5.291}$$

$$\boldsymbol{n} \cdot ((i/\kappa)\nabla + h_{c_2}\boldsymbol{A}_N)\psi_s^{(0)} = -(i/d)\psi_s^{(0)}, \text{ on } \partial\Omega, \tag{5.292}$$

$$[\boldsymbol{n} \wedge \boldsymbol{a}_s^{(0)}] = \boldsymbol{0}, \tag{5.293}$$

$$[\boldsymbol{n} \wedge (1/\mu)\text{curl } \boldsymbol{a}_s^{(0)}] = \boldsymbol{0}, \tag{5.294}$$

$$[\phi_s^{(0)}] = 0, \tag{5.295}$$

$$\left[\varepsilon\frac{\partial\phi_s^{(0)}}{\partial n}\right] = 0, \tag{5.296}$$

$$\text{curl } \boldsymbol{a}_s^{(0)} \rightarrow \boldsymbol{0}, \text{ as } r\rightarrow\infty, \tag{5.297}$$

$$\phi_s^{(0)} \rightarrow 0, \text{ as } r\rightarrow\infty, \tag{5.298}$$

$$\text{div } \boldsymbol{a}_s^{(0)} = 0. \tag{5.299}$$

Equation (5.288) with the boundary condition (5.292) has solution

$$\psi_s^{(0)}(x,t) = \sum_{n=-\infty}^{\infty}\beta_n e^{\sigma_n t}\theta_n(\boldsymbol{r}), \tag{5.300}$$

where σ_n are the eigenvalues of

$$\left(\frac{i}{\kappa}\nabla + h_{c_2}\boldsymbol{A}_N\right)^2\theta - \theta = -\frac{\alpha}{\kappa^2}\sigma\theta, \quad \text{in } \Omega, \tag{5.301}$$

$$\boldsymbol{n}\cdot((i/\kappa)\nabla + h_{c_2}\boldsymbol{A}_N)\theta = -(i/d)\theta, \quad \text{on } \partial\Omega, \tag{5.302}$$

with corresponding eigenfunctions θ_n, and β_n are constants. Note that equations (5.301), (5.302) are exactly equations (5.146), (5.150) with $h = h_{c_2}$, and hence the eigenvalues are real and the eigenfunctions corresponding to distinct eigenvalues are orthogonal. We know the largest eigenvalue is zero, so we specify $\sigma_0 = 0$. The β_n must be chosen such that

$$\sum_{n=-\infty}^{\infty}\beta_n\theta_n(\boldsymbol{r}) = \psi_s^{(0)}(\boldsymbol{r},0). \tag{5.303}$$

Multiplying (5.303) by $\theta_m^*(\boldsymbol{r})$ and integrating over Ω yields

$$\beta_m = \int_\Omega \psi_s^{(0)}(\boldsymbol{r},0)\theta_m^*(\boldsymbol{r})\,dV. \tag{5.304}$$

Thus

$$\psi_s^{(0)}(\boldsymbol{r},t) = \int_\Omega\left(\sum_{n=-\infty}^{\infty}\theta_n^*(\tilde{\boldsymbol{r}})e^{\sigma_n t}\theta_n(\boldsymbol{r})\right)\psi_s^{(0)}(\tilde{\boldsymbol{r}},0)\,d\tilde{V}. \tag{5.305}$$

We can then solve for $\boldsymbol{a}_s^{(0)}$ and $\phi_s^{(0)}$.

As in the one-dimensional case, this leading-order solution ignores the growth of the unstable mode since the growth happens on a timescale of $O(\epsilon^{-1})$. We expect that if we proceed to determine the first order terms that we will find secular terms appearing, and that the solution will cease to be valid when $t = O(\epsilon^{-1})$.

B. Long timescale : $t = O(\epsilon^{-1})$.

We now consider the long-time behaviour of the solution. We define

$$\tau = \epsilon t$$

and consider τ to be $O(1)$. We denote the long-time solution by $\psi_l(\boldsymbol{r},\tau)$, $\boldsymbol{a}_l(\boldsymbol{r},\tau)$, $\phi_l(\boldsymbol{r},\tau)$. Equations (5.273)-(5.284) become

$$\begin{aligned}
\epsilon\frac{\alpha}{\kappa^2}\frac{\partial\psi_l}{\partial\tau} + \left(\frac{i}{\kappa}\nabla + (h_{c_2}+\epsilon h^{(1)})\boldsymbol{A}_N\right)^2\psi_l - \psi_l &= -\epsilon\frac{\alpha i}{\kappa}\psi_l\phi_l + \epsilon\psi_l\,|\psi_l|^2 \\
&\quad + 2\epsilon(h_{c_2}+\epsilon h^{(1)})\psi_l(\boldsymbol{A}_N\cdot\boldsymbol{a}_l) \\
&\quad + \frac{2\epsilon i}{\kappa}(\boldsymbol{a}_l\cdot\nabla\psi_l) \\
&\quad - \epsilon^2\,|\boldsymbol{a}_l|^2\,\psi_l, \text{ in } \Omega, \qquad (5.306)\\
-(\mathrm{curl})^2\boldsymbol{a}_l - \epsilon\frac{\partial\boldsymbol{a}_l}{\partial\tau} - \nabla\phi_l &= \frac{i}{2\kappa}(\psi_l^*\nabla\psi_l - \psi_l\nabla\psi_l^*) \\
&\quad + |\psi_l|^2\,(h_{c_2}+\epsilon h^{(1)})\boldsymbol{A}_N \\
&\quad + \epsilon\,|\psi_l|^2\,\boldsymbol{a}_l, \text{ in } \Omega, \qquad (5.307)\\
-(\mathrm{curl})^2\boldsymbol{a}_l &= \varsigma_e\left(\epsilon\frac{\partial\boldsymbol{a}_l}{\partial\tau} + \nabla\phi_l\right), \\
&\qquad \text{outside } \Omega, \qquad (5.308)\\
\nabla^2\phi_l &= 0, \text{ outside } \Omega, \qquad (5.309)\\
\boldsymbol{n}\cdot((i/\kappa)\nabla + (h_{c_2}+\epsilon h^{(1)})\boldsymbol{A}_N)\psi_l + (i/d)\psi_l &= -\epsilon(\boldsymbol{n}\cdot\boldsymbol{a}_l)\psi_l, \text{ on } \partial\Omega, \qquad (5.310)\\
[\boldsymbol{n}\wedge\boldsymbol{a}_l] &= \boldsymbol{0}, \qquad (5.311)\\
[\boldsymbol{n}\wedge(1/\mu)\mathrm{curl}\,\boldsymbol{a}_l] &= \boldsymbol{0}, \qquad (5.312)\\
[\phi_l] &= 0, \qquad (5.313)\\
\left[\varepsilon\frac{\partial\phi_l}{\partial n}\right] &= 0, \qquad (5.314)\\
\mathrm{curl}\,\boldsymbol{a}_l &\to \boldsymbol{0}, \text{ as } r\to\infty, \qquad (5.315)\\
\phi_l &\to 0, \text{ as } r\to\infty, \qquad (5.316)\\
\mathrm{div}\,\boldsymbol{a}_l &= 0. \qquad (5.317)
\end{aligned}$$

We expand all quantities in powers of ϵ as before:

$$\psi_l = \psi_l^{(0)} + \epsilon\psi_l^{(1)} + \cdots, \qquad (5.318)$$

$$\boldsymbol{a}_l = \boldsymbol{a}_l^{(0)} + \epsilon\boldsymbol{a}_l^{(1)} + \cdots, \qquad (5.319)$$

$$\phi_l = \phi_l^{(0)} + \epsilon\phi_l^{(1)} + \cdots. \qquad (5.320)$$

Substituting the expansions (5.318)-(5.320) into equations (5.306)-(5.317) and equating powers of ϵ yields at leading order

$$
\begin{aligned}
\left(\frac{i}{\kappa}\nabla + h_{c_2}\boldsymbol{A}_N\right)^2 \psi_l^{(0)} - \psi_l^{(0)} &= 0, \text{ in } \Omega, && (5.321)\\
-(\mathrm{curl})^2\boldsymbol{a}_l^{(0)} - \nabla\phi_l^{(0)} &= \frac{i}{2\kappa}(\psi_l^{(0)*}\nabla\psi_l^{(0)} - \psi_l^{(0)}\nabla\psi_l^{(0)*}) \\
&\quad + h_{c_2}\,|\psi_l^{(0)}|^2\,\boldsymbol{A}_N, \text{ in } \Omega, && (5.322)\\
-(\mathrm{curl})^2\boldsymbol{a}_l^{(0)} &= \varsigma_e\nabla\phi_l^{(0)}, \text{ outside } \Omega, && (5.323)\\
\nabla^2\phi_l^{(0)} &= 0, \text{ outside } \Omega, && (5.324)\\
\boldsymbol{n}\cdot((i/\kappa)\nabla + h_{c_2}\boldsymbol{A}_N)\psi_l^{(0)} &= -(i/d)\psi_l^{(0)}, \text{ on } \partial\Omega, && (5.325)\\
[\boldsymbol{n}\wedge\boldsymbol{a}_l^{(0)}] &= \boldsymbol{0}, && (5.326)\\
[\boldsymbol{n}\wedge(1/\mu)\mathrm{curl}\ \boldsymbol{a}_l^{(0)}] &= \boldsymbol{0}, && (5.327)\\
[\phi_l^{(0)}] &= 0, && (5.328)\\
\left[\frac{\partial\phi_l^{(0)}}{\partial n}\right] &= 0, && (5.329)\\
\mathrm{curl}\ \boldsymbol{a}_l^{(0)} &\to \boldsymbol{0}, \text{ as } r\to\infty, && (5.330)\\
\phi_l^{(0)} &\to 0, \text{ as } r\to\infty, && (5.331)\\
\mathrm{div}\ \boldsymbol{a}_l^{(0)} &= 0. && (5.332)
\end{aligned}
$$

Equations (5.321) and (5.325) are exactly equations (5.28) and (5.31) with $h^{(0)} = h_{c_2}$, and as such have solution

$$
\psi_l^{(0)} = \beta(\tau)\theta_0, \tag{5.333}
$$

where $\beta(\tau)$ is an unknown function of τ and θ_0 is as before. Substituting this solution into (5.322) yields for $\boldsymbol{a}_l^{(0)}$ and $\phi_l^{(0)}$ the equations

$$
\begin{aligned}
-(\mathrm{curl})^2\boldsymbol{a}_l^{(0)} - \nabla\phi_l^{(0)} &= |\beta(\tau)|^2\left[(i/2\kappa)(\theta_0^*\nabla\theta_0 - \theta_0\nabla\theta_0^*) + h_{c_2}|\theta_0|^2\boldsymbol{A}_N\right], \\
&\quad \text{in } \Omega, && (5.334)\\
(\mathrm{curl})^2\boldsymbol{a}_l^{(0)} &= \varsigma_e\nabla\phi_l^{(0)}, \text{ outside } \Omega, && (5.335)\\
\nabla^2\phi_l^{(0)} &= 0, \text{ outside } \Omega, && (5.336)\\
[\boldsymbol{n}\wedge\boldsymbol{a}_l^{(0)}] &= \boldsymbol{0}, && (5.337)\\
[\boldsymbol{n}\wedge(1/\mu)\mathrm{curl}\ \boldsymbol{a}_l^{(0)}] &= \boldsymbol{0}, && (5.338)
\end{aligned}
$$

$$\begin{aligned}
[\phi_l^{(0)}] &= 0, &(5.339)\\
\left[\varepsilon\frac{\partial\phi_l^{(0)}}{\partial n}\right] &= 0, &(5.340)\\
\text{curl } \boldsymbol{a}_l^{(0)} &\to \mathbf{0}, \text{ as } r\to\infty, &(5.341)\\
\phi_l^{(0)} &\to 0, \text{ as } r\to\infty, &(5.342)\\
\text{div } \boldsymbol{a}_l^{(0)} &= 0. &(5.343)
\end{aligned}$$

By comparing (5.334) with (5.36) we see

$$-(\text{curl})^2\boldsymbol{a}_l^{(0)} - \nabla\phi_l^{(0)} = -\mid\beta(\tau)\mid^2 (\text{curl})^2\hat{\boldsymbol{a}}_0^{(0)}, \text{ in } \Omega, \tag{5.344}$$

where $\hat{\boldsymbol{a}}_0$ is the previously found steady-state superconducting solution, which is independent of τ. Taking the divergence of (5.344) we see

$$\nabla^2\phi_l^{(0)} = 0, \text{ in } \Omega,$$

which, with (5.336), (5.339), (5.340), and (5.342) implies

$$\phi_l^{(0)} \equiv 0. \tag{5.345}$$

We now see that the solution for $\boldsymbol{a}_l^{(0)}$ is

$$\boldsymbol{a}_l^{(0)} = \mid\beta(\tau)\mid^2 \hat{\boldsymbol{a}}_0^{(0)}. \tag{5.346}$$

To determine $\beta(\tau)$ we must proceed to higher orders in our expansions in ϵ. Equating powers of ϵ in (5.306), (5.310) yields

$$\begin{aligned}
\left(\frac{i}{\kappa}\nabla + h_{c_2}\boldsymbol{A}_N\right)^2\psi_l^{(1)} - \psi_l^{(1)} &= -\frac{\alpha}{\kappa^2}\frac{\partial\psi_l^{(0)}}{\partial\tau} - \mid\psi_l^{(0)}\mid^2\psi_l^{(0)}\\
&\quad - 2h_{c_2}h^{(1)}\mid\boldsymbol{A}_N\mid^2\psi_l^{(0)} - \frac{2ih^{(1)}}{\kappa}(\boldsymbol{A}_N\cdot\nabla\psi_l^{(0)})\\
&\quad + 2h_{c_2}(\boldsymbol{A}_N\cdot\boldsymbol{a}_l^{(0)})\psi_l^{(0)} + \frac{2i}{\kappa}(\boldsymbol{a}_l^{(0)}\cdot\nabla\psi_l^{(0)}),\\
&\quad \text{in } \Omega, &(5.347)\\
\boldsymbol{n}\cdot\left(\frac{i}{\kappa} + h_{c_2}\boldsymbol{A}_N\right)\psi_l^{(1)} - \frac{i}{d}\psi_l^{(1)} &= -\boldsymbol{n}\cdot(\boldsymbol{a}_l^{(0)} + h^{(1)}\boldsymbol{A}_N)\psi_l^{(0)}, \text{ on } \partial\Omega. &(5.348)
\end{aligned}$$

Substituting in our expressions for $\psi_l^{(0)}$ and $\boldsymbol{a}_l^{(0)}$ we find

$$\begin{aligned}\left(\frac{i}{\kappa}\nabla + h_{c_2}\boldsymbol{A}_N\right)^2\psi_l^{(1)} - \psi_l^{(1)} &= -\frac{\alpha}{\kappa^2}\frac{d\beta}{d\tau}\theta_0 - |\beta|^2\,\beta\,|\theta_0|^2\,\theta_0 \\ &\quad - 2\beta h_{c_2}h^{(1)}\,|\boldsymbol{A}_N|^2\,\theta_0 - \frac{2i\beta h^{(1)}}{\kappa}(\boldsymbol{A}_N\cdot\nabla\theta_0) \\ &\quad + 2\,|\beta|^2\,\beta h_{c_2}(\boldsymbol{A}_N\cdot\hat{\boldsymbol{a}}_0^{(0)})\theta_0 \\ &\quad + \frac{2i\,|\beta|^2\,\beta}{\kappa}(\hat{\boldsymbol{a}}_0^{(0)}\cdot\nabla\theta_0),\ \text{in }\Omega, \qquad (5.349)\\ \boldsymbol{n}\cdot\left(\frac{i}{\kappa} + h_{c_2}\boldsymbol{A}_N\right)\psi_l^{(1)} - \frac{i}{b}\psi_l^{(1)} &= -\beta\boldsymbol{n}\cdot(|\beta|^2\,\hat{\boldsymbol{a}}_0^{(0)} + h^{(1)}\boldsymbol{A}_N)\theta_0,\ \text{on }\partial\Omega. \qquad (5.350)\end{aligned}$$

As before, θ_0 is a solution of the homogeneous versions of equations (5.349), (5.350) and therefore there is a solution for $\psi_l^{(1)}$ if and only if an appropriate solvability condition is satisfied. This condition is derived by multiplying by θ_0^* and integrating over Ω. A calculation very similar to that preceding (5.216) yields

$$\begin{aligned}0 &= -\frac{\alpha}{\kappa^2}\frac{d\beta}{d\tau} - |\beta|^2\,\beta\int_\Omega |\theta_0|^4\ dV \\ &\quad + 2\,|\beta|^2\,\beta\int_\Omega \hat{\boldsymbol{a}}_0^{(0)}\cdot(\text{curl})^2\hat{\boldsymbol{a}}_0^{(0)}\,dV + h^{(1)}2\beta\int_\Omega \boldsymbol{A}_N\cdot(\text{curl})^2\hat{\boldsymbol{a}}_0^{(0)}\,dV.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\alpha}{\kappa^2}\frac{d\beta}{d\tau} &= |\beta|^2\,\beta\left[2\int_\Omega \hat{\boldsymbol{a}}_0^{(0)}\cdot(\text{curl})^2\hat{\boldsymbol{a}}_0^{(0)}\,dV - \int_\Omega |\theta_0|^4\ dV\right] \\ &\quad + 2h^{(1)}\beta\int_\Omega \boldsymbol{A}_N\cdot(\text{curl})^2\hat{\boldsymbol{a}}_0^{(0)}\,dV. \qquad (5.351)\end{aligned}$$

The boundary condition for this equation is given by matching with the short-time solution. We find

$$\beta(0)\theta_0 = \lim_{t\to\infty}\psi_s^{(0)} = \beta_0\theta_0,$$

since all the other eigenvalues σ_n in the expression (5.300) are negative. Hence

$$\beta(0) = \beta_0 = \int_\Omega \psi_s^{(0)}(\boldsymbol{r},0)\theta_0^*(\boldsymbol{r})\,dV. \qquad (5.352)$$

The coefficients in equation (5.351), although given by integrals of the steady state solution, are simply real numbers. As in the one-dimensional case, we simplify our expressions by writing

$$\begin{aligned}p &= 2\int_\Omega \hat{\boldsymbol{a}}_0^{(0)}\cdot(\text{curl})^2\hat{\boldsymbol{a}}_0^{(0)}\,dV - \int_\Omega |\theta_0|^4\ dV, \qquad (5.353)\\ q &= 2h^{(1)}\beta\int_\Omega \boldsymbol{A}_N\cdot(\text{curl})^2\hat{\boldsymbol{a}}_0^{(0)}\,dV. \qquad (5.354)\end{aligned}$$

Note that these are also the quantities that determine the sign of $h^{(1)}$ and the linear stability of the superconducting solution branch. We have

$$\frac{\alpha}{\kappa^2}\frac{d\beta}{d\tau} = p\left|\beta\right|^2\beta + q\beta.$$

Let

$$\beta = re^{i\vartheta},\ \beta_0 = r_0 e^{i\vartheta_0}.$$

Then

$$\frac{\alpha}{\kappa^2}\frac{dr}{d\tau}e^{i\vartheta} + \frac{\alpha i r}{\kappa^2}\frac{d\vartheta}{d\tau}e^{i\vartheta} = pr^3e^{i\vartheta} + qre^{i\vartheta}.$$

Hence

$$\begin{aligned} \frac{\alpha}{\kappa^2}\frac{dr}{d\tau} &= pr^3 + qr, \\ \frac{d\vartheta}{d\tau} &= 0, \\ r(0) = r_0 \quad &, \quad \vartheta(0) = \vartheta_0. \end{aligned}$$

Therefore

$$\vartheta \equiv \vartheta_0.$$

We see also that equation (5.355) is exactly equation (5.255) of the one dimensional case, and therefore has solution

$$r^2 = \begin{cases} \frac{q}{p}\left(\frac{Ce^{(2\kappa^2 q/\alpha)\tau}}{1-Ce^{(2\kappa^2 q/\alpha)\tau}}\right) & \text{if } q/p > 0, \\ \begin{cases} -\frac{q}{p}\left(\frac{Ce^{(2\kappa^2 q/\alpha)\tau}}{Ce^{(2\kappa^2 q/\alpha)\tau}-1}\right) & \text{if } r_0^2 > -q/p \\ -\frac{q}{p}\left(\frac{Ce^{(2\kappa^2 q/\alpha)\tau}}{Ce^{(2\kappa^2 q/\alpha)\tau}+1}\right) & \text{if } r_0^2 < -q/p \end{cases} & \text{if } q/p < 0. \end{cases} \tag{5.355}$$

where

$$C = \frac{r_0^2}{\left|r_0^2 + q/p\right|}.$$

The behaviour of these solutions is identical to that of the one-dimensional situation. In the first case $q/p > 0$, which will be the case when either $h > h_{c_2}$ and the superconducting solution exists for values of h slightly less than h_{c_2} (i.e. $h^{(1)}$ in (5.50) is negative), or $h < h_{c_2}$ and the superconducting solution exists for values of h slightly greater than h_{c_2} (i.e. $h^{(1)}$ in (5.50) is positive), we have

a. if $p < 0, q < 0$, the solution decays exponentially to zero.

b. if $p > 0, q > 0$, the solution blows up in finite time $\tau = (\alpha/2\kappa^2 q)\log(1/C)$.

In the second case, $q/p < 0$, which will be the case when either $h > h_{c_2}$ and the superconducting solution exists for values of h slightly greater than h_{c_2} (i.e. $h^{(1)}$ in (5.50) is positive), or $h < h_{c_2}$ and the superconducting solution exists for values of h slightly less than h_{c_2} (i.e. $h^{(1)}$ in (5.50) is negative), we have

a. $p > 0, q < 0, \begin{cases} \text{the solution decays exponentially to zero} & \text{if } r_0^2 < -q/p. \\ \begin{array}{l}\text{the solution blows up in finite time} \\ \tau = (\alpha/2\kappa^2 q)\log(1/C)\end{array} & \text{if } r_0^2 > -q/p. \end{cases}$

b. $p < 0, q > 0$, the solution tends to the steady state $r^2 = -q/p$ which is the previously found steady state superconducting solution.

Chapter 6

Surface Superconductivity

6.1 Nucleation at Surfaces

We have seen that as an external magnetic field is lowered a superconducting solution first appears in an infinite superconductor when $h = \kappa$. Any real superconducting body is of course finite, and it is of interest to consider the effects of the surface on the nucleation of superconductivity. If the superconducting body is large (compared to the penetration depth) we may rescale lengths with the penetration depth, measured from the surface, and consider surface boundary layers. The body then appears as a half space.

We consider here the problem of a superconducting half-space $x > 0$, in an external magnetic field which is parallel to the boundary. The problem was first considered in [55] for the case $d = \infty$, although they did not proceed further than finding the bifurcation point $h^{(0)}$. When the field is perpendicular to the boundary, or at any other angle, the problem is much more difficult, since there is no longer a one-dimensional solution. It is claimed in [55] that the nucleation field for a perpendicular magnetic field is exactly that of bulk nucleation, and that the parallel magnetic field is the one of greatest interest.

We take the field to be in the z-direction, so that we may still take the vector potential to be in the y-direction, $\boldsymbol{A} = (0, A, 0)$. We then look for a one-dimensional solution $A = A(x)$, $\Psi = F(x)$, where F is real. Here we are fixing the phase of Ψ by requiring that F is real, but allowing the gauge of A to be arbitrary. Because the equations are invariant under transformations of the form

$$\Psi \to e^{i\kappa c y}\Psi, \qquad A \to A + c,$$

this is equivalent to fixing the gauge of A by requiring that $A(0) = 0$, and seeking a solution $\Psi = e^{-i\kappa cy}F(x)$. In fact we prefer the latter viewpoint, since it is an extension of this idea that forms the basis of the following chapter. We note that since the superconducting body is unbounded the present problem is not covered by the preceding chapter.

We have the Ginzburg-Landau equations and boundary conditions:

$$\kappa^{-2}F'' = F^3 - F + (A+c)^2 F, \tag{6.1}$$

$$A'' = F^2(A+c), \tag{6.2}$$

$$F'(0) = (\kappa/d)F(0), \qquad F' \to 0, \text{ as } x \to \infty, \tag{6.3}$$

$$A'(0) = h, \qquad A' \to h, \text{ as } x \to \infty, \tag{6.4}$$

where $' \equiv d/dx$. The normal state solution is given by $F \equiv 0, A = hx$. As usual, we introduce ϵ through the quantities

$$F = \epsilon^{1/2} f, \tag{6.5}$$

$$A = hx + \epsilon a, \tag{6.6}$$

Substituting (6.5), (6.6) into (6.1)-(6.4) yields

$$\kappa^{-2}f'' = \epsilon f^3 - f + ((hx+c)^2 + 2\epsilon\,(hx+c)\,a + \epsilon^2 a^2)f, \tag{6.7}$$

$$a'' = (hx + c + \epsilon a)f^2, \tag{6.8}$$

$$f'(0) = (\kappa/d)f(0), \qquad f' \to 0, \text{ as } x \to \infty, \tag{6.9}$$

$$a'(0) = 0, \qquad a' \to 0, \text{ as } x \to \infty. \tag{6.10}$$

We expand f, a, h, and c in powers of ϵ

$$f = f^{(0)} + \epsilon f^{(1)} + \cdots, \tag{6.11}$$

$$a = a^{(0)} + \epsilon a^{(1)} + \cdots, \tag{6.12}$$

$$h = h^{(0)} + \epsilon h^{(1)} + \cdots, \tag{6.13}$$

$$c = c^{(0)} + \epsilon c^{(1)} + \cdots. \tag{6.14}$$

Substituting the expansions (6.11)-(6.14) into equations (6.7)-(6.10) yields at leading order

$$\kappa^{-2}f^{(0)\prime\prime} = -f^{(0)} + (h^{(0)}x + c^{(0)})^2 f^{(0)}, \tag{6.15}$$

$$a^{(0)\prime\prime} = (h^{(0)}x + c^{(0)})(f^{(0)})^2, \tag{6.16}$$

$$f^{(0)\prime}(0) = (\kappa/d)f^{(0)}(0), \qquad f^{(0)\prime} \to 0, \text{ as } x \to \infty, \tag{6.17}$$

$$a^{(0)\prime}(0) = 0, \qquad a^{(0)\prime} \to 0, \text{ as } x \to \infty. \tag{6.18}$$

We now have a double eigenvalue problem for $h^{(0)}$ and $c^{(0)}$. For each fixed $c^{(0)}$, equations (6.15) and (6.17) determine a set of eigenvalues for $h^{(0)}$. Equations (6.16) and (6.18) then determine $c^{(0)}$. Integrating (6.16) we find

$$a^{(0)\prime} = \int_0^x (h^{(0)}\xi + c^{(0)})(f^{(0)})^2\, d\xi. \tag{6.19}$$

Hence, by the boundary condition (6.18), $c^{(0)}$ must satisfy

$$\int_0^\infty (h^{(0)}\xi + c^{(0)})(f^{(0)})^2\, d\xi = 0. \tag{6.20}$$

We note that (6.15) and $f^{(0)\prime} \to 0$ as $x \to \infty \Rightarrow f^{(0)\prime\prime} \to 0$ as $x \to \infty$ and $x^2 f^{(0)} \to 0$ as $x \to \infty$. We multiply (6.15) by $f^{(0)\prime}$ and integrate over $[0, \infty)$ to give

$$\begin{aligned} 0 &= \int_0^\infty \kappa^{-2} f^{(0)\prime\prime} f^{(0)\prime}\, dx + \left(1 - (h^{(0)}x + c^{(0)})^2\right) f^{(0)} f^{(0)\prime}\, dx, \\ &= \left[\frac{(f^{(0)\prime})^2}{2\kappa^2}\right]_0^\infty + \left[\left(1 - (h^{(0)}x + c^{(0)})^2\right)\frac{(f^{(0)})^2}{2}\right]_0^\infty + \int_0^\infty (h^{(0)}x + c^{(0)})(f^{(0)})^2\, dx, \end{aligned}$$

on integration by parts. Hence

$$\begin{aligned} 0 &= \kappa^{-2}(f^{(0)\prime}(0))^2 + \left(1 - (c^{(0)})^2\right)(f^{(0)}(0))^2, \\ &= \left[d^{-2} + \left(1 - (c^{(0)})^2\right)\right](f^{(0)}(0))^2, \end{aligned}$$

by (6.20) and (6.17). Therefore, for $d \neq 0$,

$$(c^{(0)})^2 = 1 + d^{-2}. \tag{6.21}$$

The substitution $w = (2\kappa/h^{(0)})^{1/2}(h^{(0)}x + c^{(0)})$ converts the self-adjoint eigenvalue problem (6.15), (6.17) into

$$\frac{d^2 f^{(0)}}{dw^2} + \left(\frac{\mu^2}{2} - \frac{w^2}{4}\right) f^{(0)} = 0, \tag{6.22}$$

$$\frac{df^{(0)}}{dw}(\sqrt{2}\,\mu c^{(0)}) = \frac{\mu}{\sqrt{2}\,d} f^{(0)}(\sqrt{2}\,\mu c^{(0)}), \tag{6.23}$$

$$\frac{df^{(0)}}{dw} \to 0, \qquad \text{as } w \to \infty, \tag{6.24}$$

where $\mu^2 = \kappa/h^{(0)}$. Equation (6.22) is Weber's equation of index ν, where $\nu = \mu^2/2 - 1/2$. The solution which decays at infinity is

$$f^{(0)}(w) = \beta D_\nu(w), \tag{6.25}$$

where D_ν is the parabolic cylinder function. μ is given by the relation

$$D'_\nu(\sqrt{2}\,\mu c^{(0)}) = \frac{\mu}{\sqrt{2}\,d} D_\nu(\sqrt{2}\,\mu c^{(0)}). \tag{6.26}$$

We write $\sinh\gamma = 1/d$ so that $c^{(0)} = \pm\cosh\gamma$. Using the integral representation

$$D_\nu = e^{-w^2/4}\int_0^\infty t^{-\nu-1}e^{-t^2/2-wt}\,dt, \quad \nu < 0,$$

we obtain an implicit equation for μ:

$$\int_0^\infty e^{-(\tau\pm\mu\cosh\gamma)^2}\tau^{-\mu^2/2-1/2}(2\tau \pm \mu e^{\pm\gamma})\,d\tau = 0. \tag{6.27}$$

The alternative integral representation

$$D_\nu = e^{w^2/4}\int_0^\infty t^\nu e^{-t^2/2}\cos(wt - \nu\pi/2)\,dt, \quad \nu > -1,$$

yields for μ the equation

$$\int_0^\infty t^{\mu^2/2-1/2}e^{-t^2/2}\cos\left[\pm(\sqrt{2}\,\mu\cosh\gamma)t - (\mu^2-1)\pi/4\right]\times \\ \left[\mu^2 e^{\mp\gamma}\cosh\gamma + t^2 - \mu^2/2 - 1/2\right]\,d\tau = 0.$$

Figure 6.1 shows a plot of $D'_\nu(\sqrt{2}\,\mu c^{(0)}) - \frac{\mu}{\sqrt{2}\,\kappa d}D_\nu(\sqrt{2}\,\mu c^{(0)})$ against μ^2 for $\gamma = 0$, when the lower sign is taken and so $c^{(0)} = -1$. When the upper sign is taken there are no eigenvalues which are less than 1. Figure 6.2 is a plot of the lowest eigenvalue against γ. The eigenvalue $h^{(0)}$ corresponding to this lowest eigenvalue of μ is usually denoted by h_{c_3}. Not suprisingly the lowest value of μ^2 (and so the highest value of h_{c_3}) occurs when $\gamma = 0$, $d = \infty$, and the superconductor is adjacent to a vacuum. In this case the smallest eigenvalue is $\mu^2 \approx 0.59$ so that $h_{c_3} \approx 1.7\,\kappa$. Decreasing the value of d decreases the value of the nucleation field h_{c_3}. As $d \to 0$, the smallest eigenvalue $\to 1$. Thus we see that except when the superconductor is coated with a strong pairbreaker such as a normal metal, the value of the surface nucleation field is higher than the value of the bulk nucleation

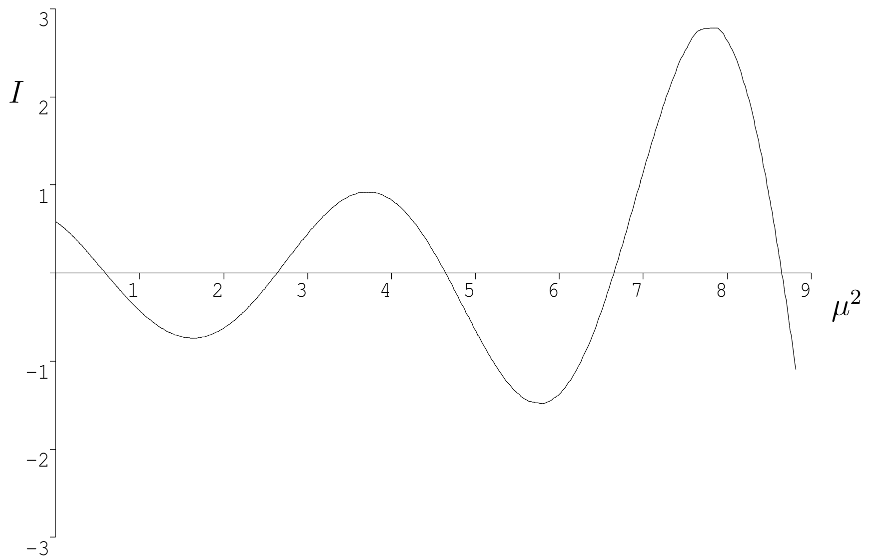

Figure 6.1: $I = D'_\nu(\sqrt{2}\,\mu c^{(0)}) - \frac{\mu}{\sqrt{2}\,\kappa d} D_\nu(\sqrt{2}\,\mu c^{(0)})$ against μ^2 for $\gamma = 0$.

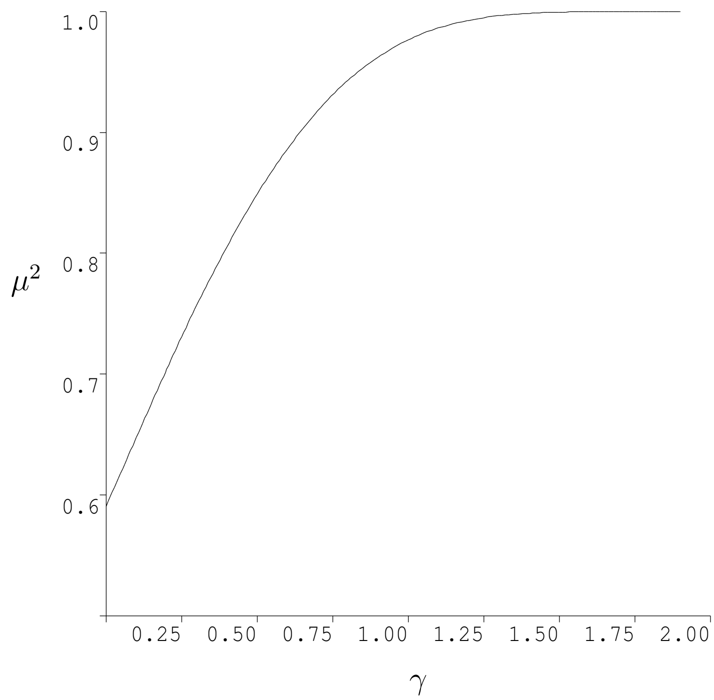

Figure 6.2: Lowest eigenvalue μ^2 as a function of γ.

field. This means that as the external field is lowered a superconducting sheath will first form on the surface of the sample.

Integrating (6.19) we now find

$$\begin{aligned} a^{(0)} &= \int_0^x \int_0^{\tilde{x}} (h^{(0)}\xi + c^{(0)})(f^{(0)}(\xi))^2 \, d\xi \, d\tilde{x}, \\ &= \int_0^x (x-\xi)(h^{(0)}\xi + c^{(0)})(f^{(0)}(\xi))^2 \, d\xi. \end{aligned} \quad (6.28)$$

We have now determined the leading order solution for f, a, h and c. Figures 6.3, 6.4 and 6.5 show the form of $f^{(0)}$ for different values of γ.

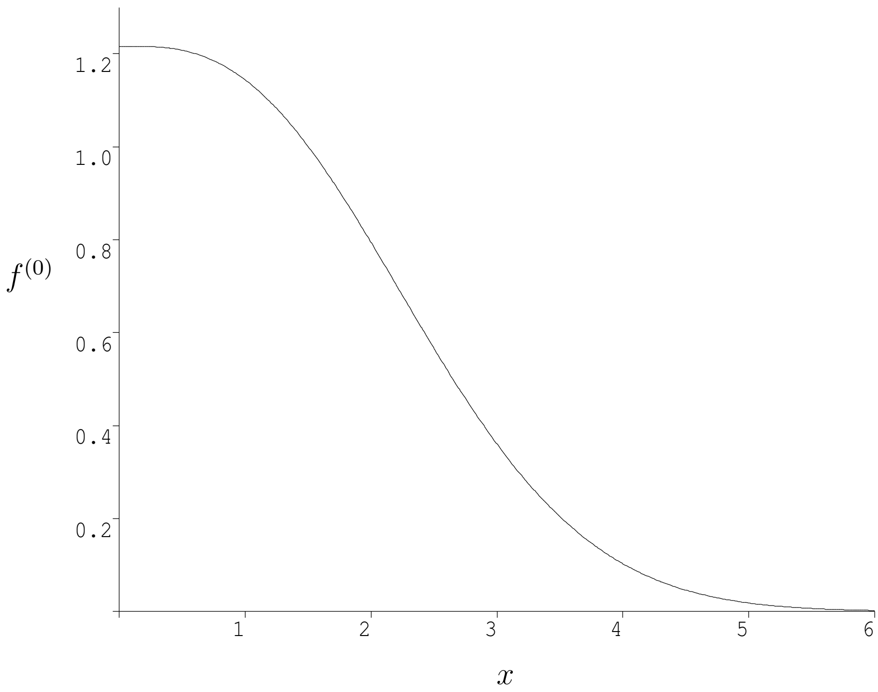

Figure 6.3: $f^{(0)}$ as a function of x for $\gamma = 0$. $f^{(0)}$ is concentrated in a region $x = O(1)$ from the boundary, hence the term 'surface superconductivity'.

We proceed with the first order terms. Equating coefficients of ϵ in (6.7)-(6.10) we find

$$\begin{aligned} \kappa^{-2} f^{(1)\prime\prime} + f^{(1)} - (h^{(0)}x + c^{(0)})^2 f^{(1)} &= (f^{(0)})^3 + 2(h^{(0)}x + c^{(0)})a^{(0)} f^{(0)} \\ &\quad + 2(h^{(0)}x + c^{(0)})(h^{(1)}x + c^{(1)}) f^{(0)}, \quad (6.29) \\ a^{(1)\prime\prime} &= (h^{(1)}x + c^{(1)} + a^{(0)})(f^{(0)})^2 \\ &\quad + 2(h^{(0)}x + c^{(0)}) f^{(1)} f^{(0)}, \quad (6.30) \end{aligned}$$

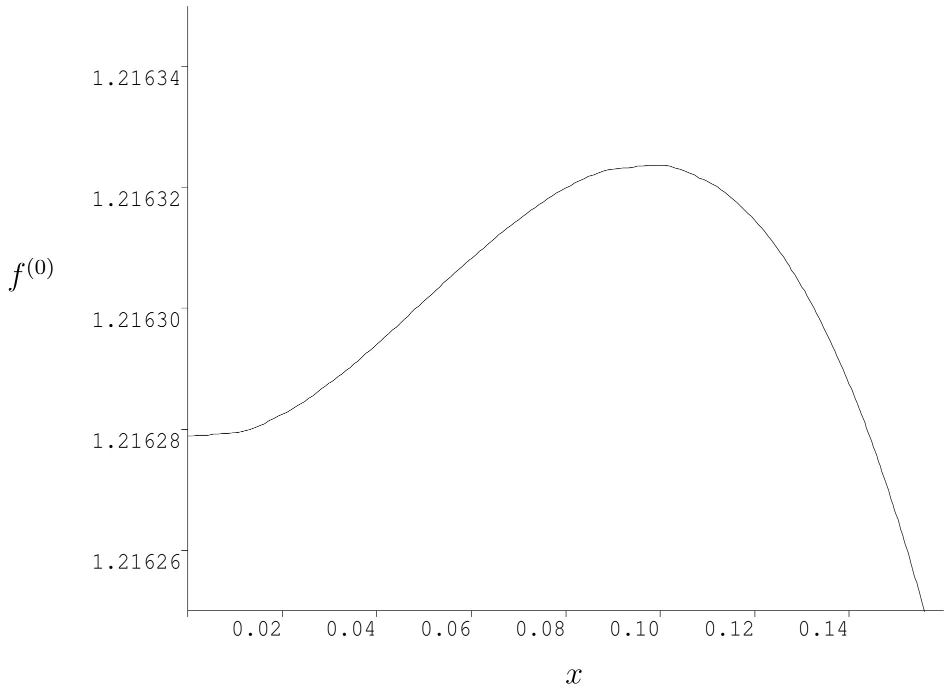

Figure 6.4: An enlargement of Fig. 6.3 near the boundary showing that $f^{(0)\prime}(0) = 0$ and that in fact $f^{(0)\prime\prime}(0) > 0$.

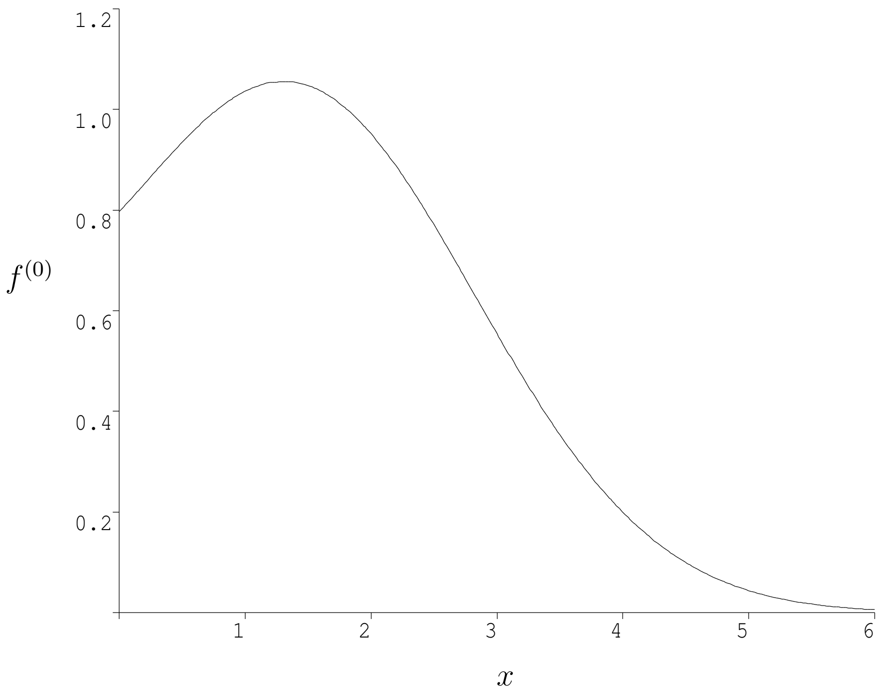

Figure 6.5: $f^{(0)}$ as a function of x for $\gamma = 0.5$.

$$f^{(1)\prime}(0) = (\kappa/d)f^{(1)}(0), \qquad f^{(1)\prime} \to 0, \text{ as } x \to \infty, \tag{6.31}$$

$$a^{(1)\prime}(0) = 0, \qquad a^{(1)\prime} \to 0, \text{ as } x \to \infty. \tag{6.32}$$

As before, $f^{(0)}$ is a solution of the inhomogeneous version of equations (6.29), (6.31) and hence there is a solution for $f^{(1)}$ if and only if an appropriate solvability condition is satisfied. We multiply by $f^{(0)}$ and integrate over $[0, \infty)$ to obtain

$$\begin{aligned} 0 &= \int_0^\infty \Big[(f^{(0)})^4 + 2(h^{(0)}x + c^{(0)})(h^{(1)}x + c^{(1)})(f^{(0)})^2 \\ &\qquad\qquad + 2(h^{(0)}x + c^{(0)})a^{(0)}(f^{(0)})^2\Big]\, dx \\ &= \int_0^\infty (f^{(0)})^4 + 2h^{(1)}x(h^{(0)}x + c^{(0)})(f^{(0)})^2 + 2(h^{(0)}x + c^{(0)})a^{(0)}(f^{(0)})^2\, dx \end{aligned}$$

by (6.20). Hence

$$h^{(1)} = -\frac{\int_0^\infty (f^{(0)})^4 + 2(h^{(0)}x + c^{(0)})a^{(0)}(f^{(0)})^2\, dx}{\int_0^\infty 2x(h^{(0)}x + c^{(0)})(f^{(0)})^2\, dx}. \tag{6.33}$$

As in the previous chapter, when we substitute in the solution (6.25) this equation gives β in terms of $h^{(1)}$, and also determines the sign of $h^{(1)}$. When $h^{(1)}$ is given by

(6.33), equations (6.29) and (6.31) will have a solution for $f^{(1)}$ (which, as with the leading order, will contain an unknown constant that is determined by a solvability condition for the second order terms). Note that $f^{(1)}$ is linear in $c^{(1)}$ since the right-hand side of (6.29) is.

Proceeding as in the case of the zero order terms we integrate (6.30) over $[0, \infty)$ to obtain

$$\int_0^\infty (h^{(1)}x + c^{(1)} + a^{(0)})(f^{(0)})^2 + 2(h^{(0)}x + c^{(0)})f^{(0)}f^{(1)}\,dx = 0. \tag{6.34}$$

Equations (6.30) and (6.32) will have a solution for $a^{(1)}$ if and only if $c^{(1)}$ satisfies this linear equation.

For a finite superconducting body, we can now modify Fig. 3.3 to include surface superconductivity (Fig. 6.6). We emphasize again, however, that surface superconductivity will only occur at h_{c_3} when the field is parallel to the surface of the sample.

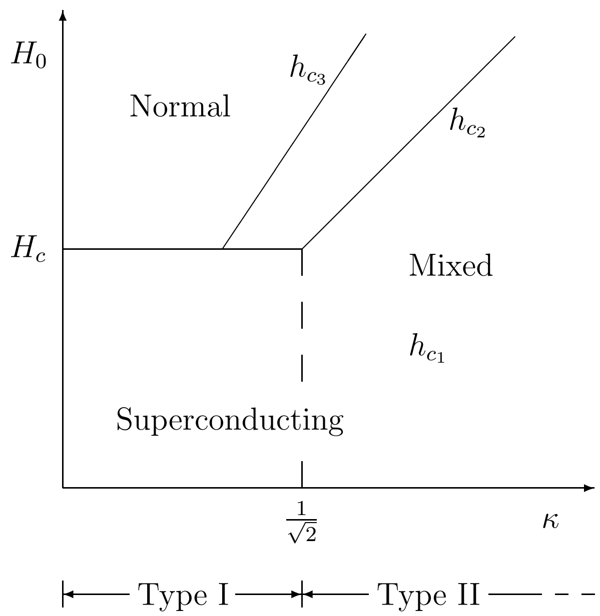

Figure 6.6: Response of a finite or semi-infinite superconductor as a function of the applied magnetic field H_0 and the Ginzburg-Landau parameter κ.

6.2 Linear Stability of the Solution Branches

We consider now the linear stability of the solution branches in the case when the external field is parallel to the surface of the sample. We have

$$-\frac{\alpha}{\kappa^2}\frac{\partial F}{\partial t}+\frac{1}{\kappa^2}\frac{\partial^2 F}{\partial x^2} = F^3-F+(A+c)^2F, \tag{6.35}$$

$$\frac{\partial^2 A}{\partial x^2} = \frac{\partial A}{\partial t}+F^2(A+c), \tag{6.36}$$

$$\frac{\partial F}{\partial x}(0)=\frac{\kappa F(0)}{d}, \qquad \frac{\partial F}{\partial x}\to 0, \text{ as } x\to\infty, \tag{6.37}$$

$$\frac{\partial A}{\partial x}(0)=0, \qquad \frac{\partial A}{\partial x}\to 0, \text{ as } x\to\infty. \tag{6.38}$$

We examine firstly the linear stability of the solution corresponding to the normal state, $F\equiv 0, A=hx$. We make a small perturbation about this solution by setting

$$F = \delta e^{\sigma t}f(x), \tag{6.39}$$

$$A = hx+\delta e^{\sigma t}a(x),\ 0<\delta\ll 1. \tag{6.40}$$

Substituting (6.39), (6.40) into equations (6.35)-(6.38) and linearising in δ (to give the leading-order behaviour of an asymptotic expansion in powers of δ) yields

$$-(\alpha\sigma/\kappa^2)f+(1/\kappa^2)f'' = -f+(hx+c)^2f, \tag{6.41}$$

$$a'' = \sigma a, \tag{6.42}$$

$$f'(0)=\frac{\kappa f(0)}{d}, \qquad f'\to 0, \text{ as } x\to\infty, \tag{6.43}$$

$$a'(0)=0, \qquad a'\to 0, \text{ as } x\to\infty. \tag{6.44}$$

where $'\equiv d/dx$. The operator of equation (6.41) with the boundary conditions (6.43) is again self-adjoint. For each fixed c, (6.41) and (6.43) determine a discrete set of real eigenvalues for σ. We see the solution to (6.42) satisfying (6.44) is $a=0$. Hence the leading-order behaviour of a is $O(\delta)$. Replacing (6.40) by

$$A=hx+\delta^2e^{\sigma t}a(x),$$

and equating powers of δ^2 in (6.36), (6.38) we find

$$a'' = \sigma a+(hx+c)^2f^2, \tag{6.45}$$

$$a'(0)=0, \qquad a'\to 0, \text{ as } x\to\infty. \tag{6.46}$$

These equations now determine c, and we again have a double eigenvalue problem, this time for σ and c.

When $h > h_{c_3}$ all the eigenvalues for σ can be shown to be negative and the normal-state solution is linearly stable. When $h < h_{c_3}$ at least one eigenvalue is positive, and the normal state is linearly unstable. To see this we multiply (6.41) by f and integrate over $[0, \infty)$ to give

$$\begin{aligned}\int_0^\infty (hx+c)^2 f^2\,dx + \frac{\alpha\sigma}{\kappa^2}\int_0^\infty f^2\,dx - \int_0^\infty f^2\,dx &= \frac{1}{\kappa^2}\int_0^\infty f'' f\,dx,\\ &= \left[\frac{f'f}{\kappa^2}\right]_0^\infty - \frac{1}{\kappa^2}\int_0^\infty (f')^2\,dx,\\ &= -\frac{f(0)^2}{\kappa d} - \frac{1}{\kappa^2}\int_0^\infty (f')^2\,dx.\end{aligned}$$

Hence

$$\frac{\alpha\sigma}{\kappa^2}\int_0^\infty f^2\,dx = \int_0^\infty f^2\,dx - \int_0^\infty (hx+c)^2 f^2\,dx - \frac{f(0)^2}{\kappa d} - \frac{1}{\kappa^2}\int_0^\infty (f')^2\,dx.$$

Letting $h \to \infty$ we see that the second term on the right-hand side of this equation dominates the first, and hence all the eigenvalues are negative for large h. We expect the eigenvalues σ, c to depend continuously on h, and one of the eigenvalues σ will pass through zero when and only when h and c pass through eigenvalues of (6.15)-(6.18). The largest of these eigenvalues for h is h_{c_3}. Thus for $h > h_{c_3}$ all the eigenvalues σ are negative. However, as h passes through h_{c_3} we expect the largest eigenvalue σ to pass through zero, and hence for $h < h_{c_3}$ there will be at least one positive eigenvalue, and the normal state will be linearly unstable.

Let us now examine the linear stability of the superconducting solution branches. We make a small perturbation of the form

$$F = F_0(x) + \delta e^{\sigma t} F_1(x), \tag{6.47}$$

$$A = A_0(x) + \delta e^{\sigma t} A_1(x), \tag{6.48}$$

where F_0, A_0 is the steady-state superconducting solution given by (6.5), (6.6). Substituting (6.47), (6.48) into the equations (6.35)-(6.38) and linearising in δ yields

$$-(\alpha\sigma/\kappa^2)F_1 + (1/\kappa^2)F_1'' = 3F_0^2 F_1 - F_1 + 2F_0(A_0+c)A_1 + (A_0+c)^2 F_1, \tag{6.49}$$

$$A_1'' = \sigma A_1 + 2F_0F_1(A_0+c) + F_0^2 A_1, \tag{6.50}$$

$$F_1'(0) = \frac{\kappa F_1(0)}{d}, \qquad F_1' \to 0, \text{ as } x \to \infty, \tag{6.51}$$

$$A_1'(0) = 0, \qquad A_1' \to 0, \text{ as } x \to \infty. \tag{6.52}$$

We examine the stability near the bifurcation point by introducing ϵ as before:

$$F_0 = \epsilon^{1/2} f_0, \tag{6.53}$$

$$A_0 = hx + \epsilon a_0, \tag{6.54}$$

$$F_1 = \epsilon^{1/2} f_1, \tag{6.55}$$

$$A_1 = \epsilon a_1. \tag{6.56}$$

Substituting (6.53)-(6.56) into (6.49)-(6.52) yields

$$\begin{aligned} -(\alpha\sigma/\kappa^2)f_1 + (1/\kappa^2)f_1'' &= 3\epsilon f_0^2 f_1 - f_1 \\ &\quad + 2\epsilon f_0 a_1(hx+\epsilon a_0) \\ &\quad + f_1\left\{(hx+c)^2 + 2\epsilon(hx+c)a_0 + a_0^2\right\}, \end{aligned} \tag{6.57}$$

$$a_1'' = \sigma a_1 + 2f_0 f_1(hx + c + \epsilon a_0) + \epsilon f_0^2 a_1, \tag{6.58}$$

$$f_1'(0) = \frac{\kappa f_1(0)}{d}, \qquad f_1' \to 0, \text{ as } x \to \infty, \tag{6.59}$$

$$a_1'(0) = 0, \qquad a_1' \to 0, \text{ as } x \to \infty. \tag{6.60}$$

In Section 6.1 we wrote down h, c, f_0, and a_0 in terms of a power series in ϵ. We again expand all quantities in powers of ϵ:

$$h = h^{(0)} + \epsilon h^{(1)} + \cdots, \tag{6.61}$$

$$c = c^{(0)} + \epsilon c^{(1)} + \cdots, \tag{6.62}$$

$$f_0 = f_0^{(0)} + \epsilon f_0^{(1)} + \cdots, \tag{6.63}$$

$$a_0 = a_0^{(0)} + \epsilon a_0^{(1)} + \cdots, \tag{6.64}$$

$$f_1 = f_1^{(0)} + \epsilon f_1^{(1)} + \cdots, \tag{6.65}$$

$$a_1 = a_1^{(0)} + \epsilon a_1^{(1)} + \cdots, \tag{6.66}$$

$$\sigma = \sigma^{(0)} + \epsilon \sigma^{(1)} + \cdots. \tag{6.67}$$

Substituting (6.61)-(6.67) into (6.57)-(6.60) and equating powers of ϵ we find at leading order

$$-(\alpha\sigma^{(0)}/\kappa^2)f_1^{(0)} + (1/\kappa^2)f_1^{(0)\prime\prime} = -f_1^{(0)} + (h^{(0)}x + c^{(0)})^2 f_1^{(0)}, \tag{6.68}$$

$$a_1^{(0)\prime\prime} = \sigma^{(0)} a_1^{(0)} + 2(h^{(0)}x + c^{(0)}) f_0^{(0)} f_1^{(0)}, \quad (6.69)$$

$$f_1^{(0)\prime}(0) = \frac{\kappa f_1^{(0)}(0)}{d}, \qquad f_1^{(0)\prime} \to 0, \text{ as } x \to \infty, \quad (6.70)$$

$$a_1^{(0)\prime}(0) = 0, \qquad a_1^{(0)\prime} \to 0, \text{ as } x \to \infty. \quad (6.71)$$

Equation (6.68) with the boundary conditions (6.70) is exactly equation (6.41) with corresponding boundary conditions (6.43). Hence, for the solution branches bifurcating at eigenvalues $h^{(0)} < h_{c_3}$ we see that there is at least one positive eigenvalue σ. Thus there is at least one unstable mode, and the superconducting solution branch will be linearly unstable. For the solution branch bifurcating at $h^{(0)} = h_{c_3}$ all the eigenvalues $\sigma^{(0)}$ are negative except for the eigenvalue $\sigma^{(0)} = 0$. To determine the stability of this mode we need to proceed to higher orders in our expansions. When $\sigma^{(0)} = 0$ we have $f_1^{(0)} \propto f_0^{(0)}$. As before, the constant of proportionality is unimportant since the equations are linear in f_1, a_1 by construction. We follow the previous chapter by setting $f_1^{(0)} = f_0^{(0)}$. Substituting this into (6.69) gives

$$a_1^{(0)\prime\prime} = 2(h^{(0)}x + c^{(0)})(f_0^{(0)})^2. \quad (6.72)$$

Hence

$$a_1^{(0)} = 2a_0^{(0)}. \quad (6.73)$$

Equating powers of ϵ in equations (6.57), (6.60) yields

$$\begin{aligned}(1/\kappa^2) f_1^{(1)\prime\prime} + f_1^{(1)} - (h^{(0)}x + c^{(0)})^2 f_1^{(1)} &= (\alpha\sigma^{(1)}/\kappa^2) f_1^{(0)} + 3(f_0^{(0)})^2 f^{(1)} \\ &\quad + 2(h^{(0)}x + c^{(0)}) f_0^{(0)} a_1^{(0)} \\ &\quad + 2(h^{(0)}x + c^{(0)})(h^{(1)}x + c^{(1)}) f_1^{(0)} \\ &\quad + 2(h^{(0)}x + c^{(0)} f_1^{(0)} a_0^{(0)}, \quad (6.74)\end{aligned}$$

$$f_1^{(1)\prime}(0) = \frac{\kappa f_1^{(1)}(0)}{d}, \qquad f_1^{(1)\prime} \to 0, \text{ as } x \to \infty. \quad (6.75)$$

Hence

$$\begin{aligned}(1/\kappa^2) f_1^{(1)\prime\prime} + f_1^{(1)} - (h_{c_3}x + c^{(0)}) f_1^{(1)} &= (\alpha\sigma^{(1)}/\kappa^2) f_1^{(0)} + 3(f_0^{(0)})^3 \\ &\quad + 2(h^{(0)}x + c^{(0)})(h^{(1)}x + c^{(1)}) f_0^{(0)} \\ &\quad + 6(h^{(0)}x + c^{(0)}) f_0^{(0)} a_0^{(0)}, \quad (6.76)\end{aligned}$$

$$f_1^{(1)\prime}(0) = \frac{\kappa f_1^{(1)}(0)}{d}, \qquad f_1^{(1)\prime} \to 0, \text{ as } x \to \infty. \quad (6.77)$$

Now, $f_0^{(0)}$ satisfies the homogeneous version of equations (6.76), (6.77). Hence there is a solution for $f_1^{(1)}$ if and only if an appropriate solvability condition is satisfied. To derive this condition we multiply (6.76) by $f_0^{(0)}$ and integrate over $[0,\infty)$ to give

$$\begin{aligned} 0 &= \int_0^\infty \left[\frac{\alpha\sigma^{(1)}}{\kappa^2}(f_0^{(0)})^2 + 3(f_0^{(0)})^4 \right. \\ & \qquad \left. + 6(h_{c_3}x + c^{(0)})(f_0^{(0)})^2 a_0^{(0)} + 2(h_{c_3}x + c^{(0)})h^{(1)}x(f_0^{(0)})^2 \right] dx, \end{aligned}$$

by (6.20). Hence

$$\begin{aligned} \int_0^\infty \frac{\alpha\sigma^{(1)}}{\kappa^2}(f_0^{(0)})^2\,dx &= 4h^{(1)}\int_0^\infty (h_{c_3}x + c^{(0)})x(f_0^{(0)})^2\,dx, \\ &= \frac{4h^{(1)}}{h_{c_3}}\int_0^\infty (h_{c_3}x + c^{(0)})^2(f_0^{(0)})^2\,dx, \end{aligned}$$

by (6.33) and (6.20). Hence $\sigma^{(1)} < 0$ if and only if $h^{(1)} < 0$.

Figure 6.7 shows a plot of the value of κ at which the solution becomes stable for different values of γ $(= \sinh^{-1}(1/d))$. We also show the value of κ at which the critical field h_{c_3} becomes greater than the critical field H_c, which is usually taken to be the criterion for the observation of surface superconductivity. As we can see the two values differ. Figure 6.8 shows the value of h_{c_3} at which the solution becomes stable with the critical field H_c as a reference. We see that in fact the solution is stable for values of h_{c_3} less than H_c.

We note here that it is also possible to perform a weakly-nonlinear stability analysis of the normal-state solution when the external field is parallel to the surface of the sample. The analysis mirrors that of the previous chapter, and the results are as expected.

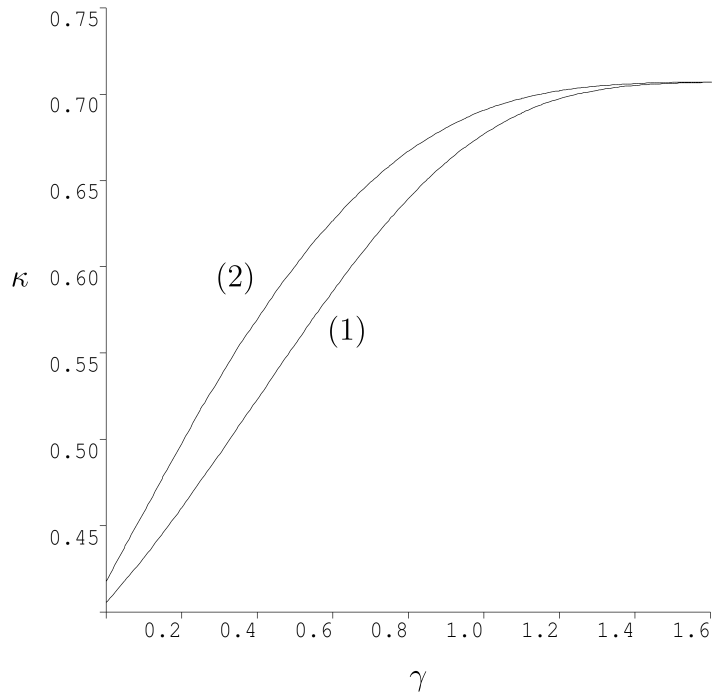

Figure 6.7: (1) The value of κ at which the superconducting solution becomes stable as a function of γ. (2) The value of κ at which the critical field h_{c_3} is equal to the thermodynamic critical field H_c.

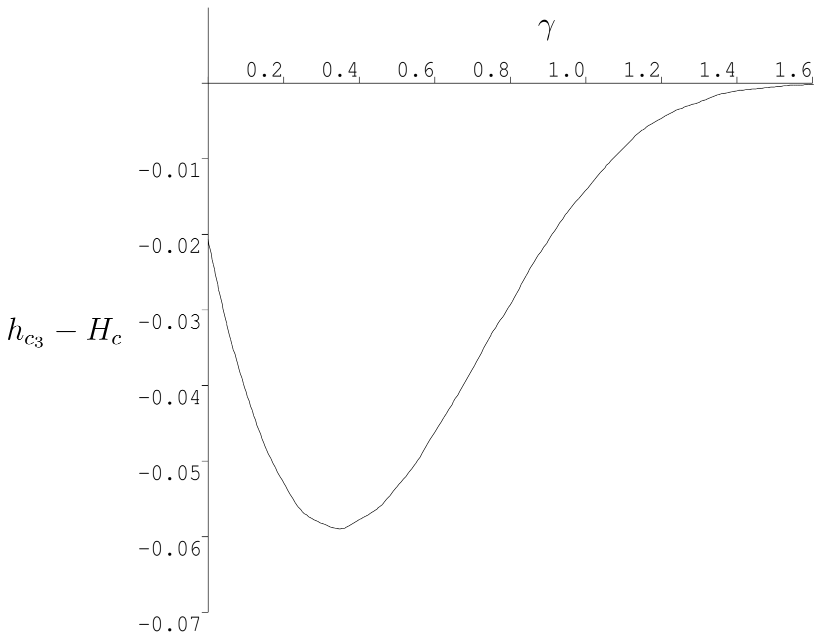

Figure 6.8: The value of h_{c_3} at which the superconducting solution becomes stable minus the thermodynamic critical field H_c, as a function of γ.

Chapter 7

The Mixed State

We have seen that as the external field is decreased there is a bifurcation from the normal solution to a superconducting solution. For a semi-infinite sample we have seen that superconductivity will nucleate first at the surface of the sample in the form of a surface superconducting layer. However, we have only considered the bifurcation for a bulk superconductor in one space dimension. Our aim now is to determine the nature of the superconducting solution in a bulk superconductor when we allow it to vary in two space dimensions.

Consider first a situation similar to that of the previous chapter, but where the superconductor is of a size comparable to the penetration depth (i.e. we have a superconducting film[1]), so that we are solving the equations in the region $-l \leq x \leq l$. As mentioned at the beginning of the previous chapter, we look for a solution of the form $\boldsymbol{A} = (0, A(x), 0)$, $\Psi = e^{-i\kappa c y} F(x)$. We then have

$$\kappa^{-2} F'' = F^3 - F + (A+c)^2 F, \tag{7.1}$$

$$A'' = F^2 (A+c), \tag{7.2}$$

$$F'(-l) = (\kappa/d) F(-l), \qquad F'(l) = -(\kappa/d) F(l), \tag{7.3}$$

$$A'(-l) = h, \qquad A'(l) = h, \tag{7.4}$$

where $' \equiv d/dx$. Writing

$$F = \epsilon^{1/2} f, \tag{7.5}$$

$$A = hx + \epsilon a, \tag{7.6}$$

[1] We will not discuss superconducting films in any detail in this thesis, but we note that there is a vast literature on the subject, and that they are especially important in relation to modern high-T_c superconducting devices.

Figure 7.1: Schematic diagram of the solution $\Psi = e^{-i\kappa c y}F(x) + e^{i\kappa c y}F(-x)$, showing the line of zeros of Ψ and the circulation of the current about each zero.

and expanding f, a, h and c in powers of ϵ as before yields the following equations for $f^{(0)}$:

$$\kappa^{-2} f^{(0)\prime\prime} = -f^{(0)} + (h^{(0)}x + c^{(0)})^2 f^{(0)},$$

$$f^{(0)\prime}(-l) = (\kappa/d) f^{(0)}(-l), \qquad f^{(0)\prime}(l) = -(\kappa/d) f^{(0)}(l).$$

We see that if $(f^{(0)}(x), c^{(0)})$ is a solution, then so is $(f^{(0)}(-x), -c^{(0)})$. Since the leading-order equations are linear, any linear combination of these solutions is also a solution of the leading-order equations (now in two dimensions, and subject to the usual solvability conditions when the first-order terms are considered). Consider the solution

$$\begin{aligned} \Psi(x, y) &= e^{-i\kappa c y}F(x) + e^{i\kappa c y}F(-x), \\ &= \{F(x) + F(-x)\}\cos(\kappa c y) + i\,\{F(x) - F(-x)\}\sin(\kappa c y). \end{aligned}$$

(see Fig. 7.1.) Note that $\Psi = 0$ at each of the points $x = 0$, $y = (2n+1)\pi/2\kappa c$, and around each of these points the phase of Ψ varies by 2π. Thus by superimposing two essentially one-dimensional solutions we have constructed a two-dimensional solution with the order parameter having a sequence of zeros about which the current is circulating (since $\boldsymbol{j} = -\,|\,\Psi\,|^2\,(\boldsymbol{A} - (1/\kappa)\nabla\chi) \sim (1/\kappa)\,|\,\Psi\,|^2\,\nabla\chi$ near each zero). The superposition of one-dimensional solutions of this type to give a two-dimensional solution is the basic idea behind the Mixed State of Abrikosov, which is the subject of this chapter.

7.1 Bifurcation to the Mixed State

We consider a bulk superconductor occupying all of space and examine the two-dimensional situation, with the applied field perpendicular to the plane of interest. This problem was first studied by Abrikosov [1]. We begin by reviewing his analysis in the framework of the systematic perturbation theory of the previous chapters. The steady state, isothermal Ginzburg-Landau equations, with length and $\boldsymbol{A}$ scaled with the penetration depth as usual, are

$$((i/\kappa)\nabla + \boldsymbol{A})^2\Psi = \Psi(1- \mid\Psi\mid^2), \tag{7.7}$$

$$-(\mathrm{curl})^2\boldsymbol{A} = \mid\Psi\mid^2 \boldsymbol{A} + (i/2\kappa)(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*). \tag{7.8}$$

We require that $\boldsymbol{H}$ and $((i/\kappa)\nabla + \boldsymbol{A})\Psi$ are periodic in x and y, with period L_x and L_y respectively. We choose the field $\boldsymbol{H}$ to be directed along the z-axis and choose the gauge $\boldsymbol{A}$ to be directed along the y-axis, so that $\boldsymbol{H} = (0, 0, H(x, y))$, $\boldsymbol{A} = (0, A(x, y), 0)$ and $H = \partial A/\partial x$. (Note that with this gauge div $\boldsymbol{A} \neq 0$. It is more convenient in the present situation to have a single scalar variable A than to have div $\boldsymbol{A} = 0$.)

The solution corresponding to the normal state is

$$\Psi \equiv 0, \; A = hx. \tag{7.9}$$

As before, we seek a solution in which $\mid\Psi\mid \ll 1$ which depends continuously on a parameter ϵ (measuring $\mid\Psi\mid^2$) and which reduces to (7.9) for $\epsilon = 0$. We introduce ϵ through the relations

$$\Psi = \epsilon^{1/2}\psi, \tag{7.10}$$

$$A = hx + \epsilon a. \tag{7.11}$$

Substituting (7.10), (7.11) into (7.7), (7.8) yields

$$-\frac{1}{\kappa^2}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + \frac{2i(hx + \epsilon a)}{\kappa}\frac{\partial\psi}{\partial y} + \frac{i\epsilon\psi}{\kappa}\frac{\partial a}{\partial y} + (hx+\epsilon a)^2\psi = \psi - \epsilon\psi \mid\psi\mid^2, \tag{7.12}$$

$$-\frac{\partial^2 a}{\partial x \partial y} = \frac{i}{2\kappa}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right), \tag{7.13}$$

$$\frac{\partial^2 a}{\partial x^2} = (hx + \epsilon a)\,|\,\psi\,|^2 + \frac{i}{2\kappa}\left(\psi^* \frac{\partial \psi}{\partial y} - \psi \frac{\partial \psi^*}{\partial y}\right), \tag{7.14}$$

with $\partial a/\partial x$, $\partial\psi/\partial x$ and $(i/\kappa)\partial\psi/\partial y + (hx + \epsilon a)\psi$ periodic. We expand all quantities in powers of ϵ as before

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \cdots, \tag{7.15}$$

$$a = a^{(0)} + \epsilon a^{(1)} + \cdots, \tag{7.16}$$

$$h = h^{(0)} + \epsilon h^{(1)} + \cdots. \tag{7.17}$$

Inserting the expansions (7.15)-(7.17) into equations (7.12)-(7.14) and equating powers of ϵ yields at leading order

$$-\frac{1}{\kappa^2}\left(\frac{\partial^2 \psi^{(0)}}{\partial x^2} + \frac{\partial^2 \psi^{(0)}}{\partial y^2}\right) + \frac{2ih^{(0)}x}{\kappa}\frac{\partial \psi^{(0)}}{\partial y} = \psi^{(0)} - (h^{(0)})^2 x^2 \psi^{(0)}, \tag{7.18}$$

$$-\frac{\partial^2 a^{(0)}}{\partial x \partial y} = \frac{i}{2\kappa}\left(\psi^{(0)*} \frac{\partial \psi^{(0)}}{\partial x} - \psi^{(0)} \frac{\partial \psi^{(0)*}}{\partial x}\right), \tag{7.19}$$

$$\frac{\partial^2 a^{(0)}}{\partial x^2} = h^{(0)}x\,|\,\psi^{(0)}\,|^2 + \frac{i}{2\kappa}\left(\psi^{(0)*} \frac{\partial \psi^{(0)}}{\partial y} - \psi^{(0)} \frac{\partial \psi^{(0)*}}{\partial y}\right), \tag{7.20}$$

with $\partial a^{(0)}/\partial x$, $\partial\psi^{(0)}/\partial x$ and $(i/\kappa)\partial\psi^{(0)}/\partial y + h^{(0)}x\psi^{(0)}$ periodic. Note that these boundary conditions imply that

$$\psi^{(0)}(x + L_x, y) = e^{i\kappa h^{(0)} L_x y}\psi^{(0)}(x, y), \tag{7.21}$$

$$\psi^{(0)}(x, y + L_y) = \psi^{(0)}(x, y). \tag{7.22}$$

We found previously that equation (7.18) had the one dimensional solutions

$$\psi^{(0)} = \exp\left(\frac{-\kappa^2 x^2}{2(2n+1)}\right)\mathrm{H}_n\left(\frac{\sqrt{2}\,\kappa x}{\sqrt{2n+1}}\right), \tag{7.23}$$

when $h^{(0)} = \kappa/(2n+1)$. In addition to (7.23), (7.18) is also satisfied by the function

$$\psi^{(0)} = \exp\left(iky - \frac{\kappa^2(x - (2n+1)k/\kappa^2)^2}{2(2n+1)}\right)\mathrm{H}_n\left(\frac{\sqrt{2}\,\kappa(x - (2n+1)k/\kappa^2)}{\sqrt{2n+1}}\right), \tag{7.24}$$

for any k. The largest eigenvalue is $h^{(0)} = \kappa$, with the set of corresponding eigenfunctions

$$\psi^{(0)} = \exp\left\{iky - \frac{\kappa^2}{2}\left(x - \frac{k}{\kappa^2}\right)^2\right\}. \tag{7.25}$$

Since we are looking for a solution which is periodic in y the general solution to (7.18) with $h^{(0)} = \kappa$ is

$$\psi^{(0)} = \sum_{-\infty}^{\infty} C_n e^{inky}\psi_n(x), \tag{7.26}$$

where

$$\psi_n(x) = \exp\left\{-\frac{\kappa^2}{2}\left(x - \frac{nk}{\kappa^2}\right)^2\right\}, \tag{7.27}$$

$L_y = 2\pi/k$, and C_n and k are as yet arbitrary. The condition (7.21) implies $L_x = kN/\kappa^2$, where N is an integer, and gives the following simple recursion relation for the C_n:

$$C_{n+N} = C_n, \text{ for all } n. \tag{7.28}$$

The resulting solution satisfies

$$\psi^{(0)}(x, y + 2\pi/k) = \psi^{(0)}(x, y), \tag{7.29}$$

$$\psi^{(0)}(x + kN/\kappa^2) = e^{ikNy}\psi^{(0)}(x, y). \tag{7.30}$$

Note that this implies that the phase of $\psi^{(0)}$ varies by $2\pi\kappa N$ around the boundary of the unit cell, which corresponds to the cell containing N zeros of $\psi^{(0)}$ (or vortices), and N quanta of fluxoid.

Substituting (7.26) into (7.19), (7.20) we find

$$-\frac{\partial a^{(0)}}{\partial x \partial y} = \frac{i}{2\kappa}\sum_{m,n=-\infty}^{\infty} C_n^* C_m k(m-n) e^{ik(m-n)y}\psi_m(x)\psi_n(x), \tag{7.31}$$

$$-\frac{\partial a^{(0)}}{\partial x^2} = \frac{1}{2\kappa}\sum_{m,n=-\infty}^{\infty} C_n^* C_m \left[2\kappa^2 x - k(m+n)\right] e^{ik(m-n)y}\psi_m(x)\psi_n(x). \tag{7.32}$$

Integrating (7.31), (7.32) we see

$$\begin{aligned}\frac{\partial a^{(0)}}{\partial x} &= -\frac{1}{2\kappa}\sum_{m,n=-\infty}^{\infty} C_n^* C_m e^{ik(m-n)y}\psi_m(x)\psi_n(x), \\ &= -\frac{1}{2\kappa}\,|\,\psi^{(0)}\,|^2\,.\end{aligned}$$

Hence

$$a^{(0)} = -\frac{1}{2\kappa}\int^x |\,\psi^{(0)}\,|^2\; dx. \tag{7.33}$$

Notice that the field in the sample is reduced (this is the Meissner effect).

Equating powers of ϵ in equation (7.12) yields

$$
\begin{aligned}
-\frac{1}{\kappa^2}\left(\frac{\partial^2\psi^{(1)}}{\partial x^2}+\frac{\partial^2\psi^{(1)}}{\partial y^2}\right)+2ix\frac{\partial\psi^{(1)}}{\partial y}+\kappa^2x^2\psi^{(1)}-\psi^{(1)}& \\
= \; & -\psi^{(0)}\mid\psi^{(0)}\mid^2-\frac{2ih^{(1)}x}{\kappa}\frac{\partial\psi^{(0)}}{\partial y}-\frac{2ia^{(0)}}{\kappa}\frac{\partial\psi^{(0)}}{\partial y} \\
& -\frac{i\psi^{(0)}}{\kappa}\frac{\partial a^{(0)}}{\partial y}-2\kappa x a^{(0)}\psi^{(0)}-2\kappa h^{(1)}x^2\psi^{(0)} \\
= \; & -\sum_{p,m,r}C_pC_m^*C_re^{ik(p-m+r)y}\psi_p\psi_m\psi_r \\
& +\frac{2h^{(1)}kx}{\kappa}\sum_p pC_pe^{ikpy}\psi_p \\
& -\frac{k}{\kappa^2}\sum_{p,m,r}pC_pC_m^*C_re^{ik(p-m+r)y}\psi_p\int^x\psi_m\psi_r\,dx \\
& -\frac{k}{2\kappa^2}\sum_{p,m,r}(r-m)C_pC_m^*C_re^{ik(p-m-r)y}\psi_p\int^x\psi_m\psi_r\,dx \\
& +x\sum_{p,m,r}C_pC_m^*C_re^{ik(p-m+r)y}\psi_p\int^x\psi_m\psi_r\,dx \\
& -2h^{(1)}\kappa x^2\sum_p C_pe^{ikpy}\psi_p \\
= \; & \sum_p C_p\frac{2h^{(1)}x}{\kappa}(-\kappa^2x+kp)e^{ikpy}\psi_p
\end{aligned}
$$

$$
+\sum_{p,m,r}C_pC_m^*C_r\left\{\left[x-\frac{k}{\kappa^2}\left(p+\frac{r-m}{2}\right)\right]e^{ik(p-m+r)y}\psi_p\int^x\psi_m\psi_r\,dx-\psi_p\psi_m\psi_r\right\}.
$$

Multiplying by e^{-inky} and integrating from $y=0$ to $y=2\pi/k$ we find

$$
\begin{aligned}
-\frac{1}{\kappa^2}\frac{\partial^2\psi_n^{(1)}}{\partial x^2}+\frac{k^2n^2\psi_n^{(1)}}{\kappa^2}-2xkn\psi_n^{(1)}+\kappa^2x^2\psi_n^{(1)}-\psi_n^{(1)}= & \\
\sum_n C_n\frac{2h^{(1)}x}{\kappa}(-\kappa^2x+kn)\psi_n & \\
+\sum_{n,m,r}C_{n-r+m}C_m^*C_r\left[x-\frac{k}{\kappa^2}\left(n+\frac{m-r}{2}\right)\right]\psi_{n-r+m}\int^x\psi_m\psi_r\,dx & \\
-\sum_{n,m,r}C_{n-r+m}C_m^*C_r\psi_{n-r+m}\psi_m\psi_r, & \qquad (7.34)
\end{aligned}
$$

where

$$
\psi^{(1)}=\sum_{n=-\infty}^{\infty}e^{ikny}\psi_n^{(1)}.
$$

Now ψ_n is a solution of the homogeneous version of this equation (with ψ_n and $d\psi_n/dx$ vanishing as $x \to \pm\infty$). Therefore there is a solution for $\psi_n^{(1)}$ if and only if the right-hand side is orthogonal to ψ_n for all n. Performing the necessary integration we find

$$\frac{1}{\sqrt{2}}\left(\frac{1}{2\kappa^2}-1\right)\sum C_{n-r+m}C_mC_r \exp\left\{-\frac{k^2}{2\kappa^2}\left[(r-n)^2+(r-m)^2\right]\right\} - \frac{h^{(1)}C_n}{\kappa} = 0. \tag{7.35}$$

This equation determines both $h^{(1)}$ and the allowed coefficients C_n. To determine $h^{(1)}$ we multiply by C_n^* and sum over n to give

$$\left(\frac{1}{2\kappa^2}-1\right)\overline{|\,\psi^{(0)}\,|^4} - \frac{h^{(1)}}{\kappa}\overline{|\,\psi^{(0)}\,|^2} = 0. \tag{7.36}$$

Hence

$$h^{(1)} = \kappa\left(\frac{1}{2\kappa^2}-1\right)\frac{\overline{|\,\psi^{(0)}\,|^4}}{\overline{|\,\psi^{(0)}\,|^2}}. \tag{7.37}$$

As in the one-dimensional case we see that $h^{(1)} < 0$ if and only if $\kappa > 1/\sqrt{2}$, i.e. for Type II superconductors. The average magnetic field in the specimen is given by

$$\begin{aligned}\overline{H} &= h^{(0)} + \epsilon h^{(1)} - \epsilon\overline{|\,\psi^{(0)}\,|^2}/2\kappa + \cdots,\\ &= h^{(0)} + \epsilon h^{(1)} - \epsilon h^{(1)}/(2\kappa^2-1)\beta + \cdots,\end{aligned}$$

where

$$\beta = \frac{\overline{|\,\psi^{(0)}\,|^4}}{(\overline{|\,\psi^{(0)}\,|^2})^2},$$

is independent of the amplitude of $\psi^{(0)}$. Thus $\overline{H}$ is linear in h near $h = \kappa$, with a gradient that tends to infinity as $\kappa \to 1/\sqrt{2}$, in agreement with the magnetization curves of Fig. 1.6.

Abrikosov calculates the free energy per unit volume and finds it to be proportional to

$$\frac{1}{2} + \overline{H}^2 - \frac{(\kappa - \overline{H})^2}{1 + (2\kappa^2 - 1)\beta}.$$

Thus, for $\kappa > 1/\sqrt{2}$ and fixed $\overline{H}$ the free energy is minimised by minimising β. This is the traditional way of deciding which of the mixed state solutions will be stable. In the following sections we will compare this to the classical stability of the solutions. We note that the solvability condition (7.35) can be written as

$$\frac{\partial \beta}{\partial C_n^*} = 0.$$

We now approach the problem of choosing the C_n.

The simplest case $N = 1$ was analysed by Abrikosov [1]. In this case $C_n = C$, for all n, and (7.35) simply gives C in terms of $h^{(1)}$. The resulting solution $\psi^{(0)}$ has one zero in the unit cell, at its centre $x = k/2\kappa^2, y = \pi/k$, and $|\psi^{(0)}|^2$ has the symmetry of a rectangular lattice. β is given by

$$\beta = \frac{k}{\sqrt{2\pi}\,\kappa} S_{1,0}^2,$$

where

$$S_{p,q} = \sum_{n=-\infty}^{\infty} e^{-\frac{k^2}{2\kappa^2}[pn+q]^2}.$$

Figure 7.2 shows β as a function of $R = k/\kappa\sqrt{\pi}$. The minimum value of β occurs

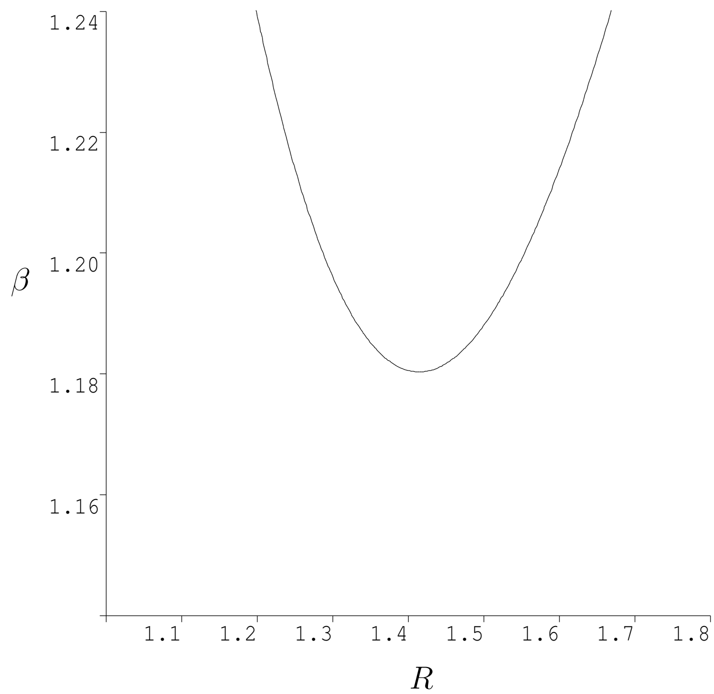

Figure 7.2: β as a function of $R = k/\kappa\sqrt{\pi}$, when $C_n = C \ \ \forall n$.

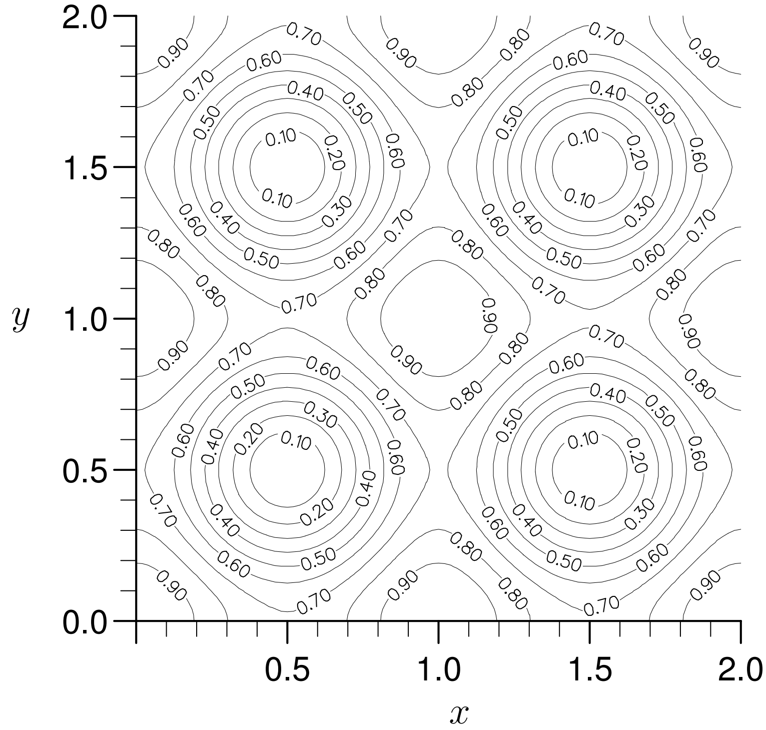

Figure 7.3: Mixed state solution $N = 1$, $C_n = C$, $\forall n$, when $k = \sqrt{2\pi}\,\kappa$. The diagram shows $|\Psi|^2$ as a function of x and y, which are labelled in multiples of the dimensions of the unit cell.

when $k = \kappa\sqrt{2\pi}$, giving

$$\beta = \sum_{n=-\infty}^{\infty} e^{-\pi n^2} \approx 1.18.$$

In this case the $|\,\psi^{(0)}\,|^2$ has the symmetry of a square lattice, as shown in Figs. 7.3 and 7.4.

The next simplest case $N = 2$ was considered in [40]. In this case (7.35) becomes

$$\frac{2\sqrt{2}\,\kappa h^{(1)} C_0}{(1-2\kappa^2)} = |\,C_0\,|^2\, C_0 S_{2,0}^2 + C_0^* C_1^2 S_{2,1}^2 + 2\,|\,C_1\,|^2\, C_0 S_{2,0} S_{2,1}, \tag{7.38}$$

$$\frac{2\sqrt{2}\,\kappa h^{(1)} C_1}{(1-2\kappa^2)} = |\,C_1\,|^2\, C_1 S_{2,0}^2 + C_1^* C_0^2 S_{2,1}^2 + 2\,|\,C_0\,|^2\, C_1 S_{2,0} S_{2,1}. \tag{7.39}$$

Let $C_1 = \eta C_0$. Then

$$\frac{2\sqrt{2}\,\kappa h^{(1)}}{|\,C_0\,|^2\,(1-2\kappa^2)} = S_{2,0}^2 + \eta^2 S_{2,1}^2 + 2\,|\,\eta\,|^2\, S_{2,0} S_{2,1}, \tag{7.40}$$

$$\frac{2\sqrt{2}\,\kappa\eta h^{(1)}}{|\,C_0\,|^2\,(1-2\kappa^2)} = \eta\,|\,\eta\,|^2\, S_{2,0}^2 + \eta^* S_{2,1}^2 + 2\eta S_{2,0} S_{2,1}. \tag{7.41}$$

Figure 7.4: Mixed state solution $N = 1$, $C_n = C$, $\forall n$, when $k = \sqrt{2\pi}\,\kappa$. The diagram shows $|\Psi|^2$ as a function of x and y, which are labelled in multiples of the dimensions of the unit cell.

We multiply (7.40) by η and subtract (7.41)

$$\eta(1-|\eta|^2)S_{2,0}^2 + (\eta^3 - \eta^*)S_{2,1} + 2\eta(|\eta|^2 - 1)S_{2,0}S_{2,1} = 0. \qquad (7.42)$$

We see that this equation has solutions

$$\eta = \pm 1,\ \pm i.$$

The case $\eta = 1$ corresponds to the case $N = 1$. The case $\eta = -1$ also corresponds to the $N = 1$ solution, but translated by π/k in y. Similarly the case $\eta = -i$ corresponds to a translation of the case $\eta = i$. Thus there is only one new case to consider, namely $\eta = i$, $C_1 = iC_0$, $C_{n+2} = C_n$ $\forall n$. In this case the unit cell has dimensions $L_x = 2k/\kappa^2$, $L_y = 2\pi/k$, and $\psi^{(0)}$ vanishes at the points $x = k/2\kappa^2$, $y = \pi/2k$, and $x = 3k/2\kappa^2, y = 3\pi/2k$. We see that $|\psi^{(0)}|^2$ has the symmetry of a rhombic lattice. When $k = \kappa\sqrt{\pi}$, $\psi^{(0)}$ has the symmetry of a square lattice, and [40] show that the solution is identical to the square lattice of the $N = 1$ case rotated by 45° and translated (Figs. 7.5 and 7.6).

β is given by

$$\begin{aligned}\beta &= \frac{2k}{\sqrt{2\pi}\,\kappa}\left\{\frac{(|C_0|^4 + |C_1|^4)S_{2,0}^2 + 4\,|C_0|^2|C_1|^2\,S_{2,0}S_{2,1} + 2\Re((C_0^*)^2C_1^2)S_{2,1}^2}{(|C_0|^2 + |C_1|^2)^2}\right\},\\ &= \frac{k}{\sqrt{2\pi}\,\kappa}\left\{S_{2,0}^2 + S_{2,1}(2S_{2,0} - S_{2,1})\right\}.\end{aligned}$$

Figure 7.7 shows β as a function of $R = k/\kappa\sqrt{\pi}$.

The minimum of β is obtained when $k = \kappa\sqrt{\pi\sqrt{3}}$, giving $\beta \approx 1.16$. In this case $|\psi^{(0)}|^2$ has the symmetry of a triangular lattice, as shown in Figs. 7.8 and 7.9.

Thus the square lattice of Abrikosov is continuously connected to a triangular lattice of lower energy by a pure shear deformation of the normal filament structure.

Let us now consider the cases $N = 3$ and $N = 4$. When $N = 3$, (7.35) becomes

$$\begin{aligned}\frac{2\sqrt{2}\,\kappa h^{(1)}C_0}{(1-2\kappa^2)} &= |C_0|^2\,C_0S_{3,0}^2 + C_0^*C_1C_2S_{3,1}^2 + C_0^*C_1C_2S_{3,2}^2\\ &\quad + C_0\,|C_1|^2\,S_{3,0}S_{3,2} + C_0\,|C_1|^2\,S_{3,1}S_{3,0} + C_1^*C_2^2S_{3,2}S_{3,1}\\ &\quad + C_0\,|C_2|^2\,S_{3,0}S_{3,1} + C_1^2C_2^*S_{3,1}S_{3,2} + C_0\,|C_2|^2\,S_{3,2}S_{3,0}, \qquad (7.43)\end{aligned}$$

$$\begin{aligned}\frac{2\sqrt{2}\,\kappa h^{(1)}C_1}{(1-2\kappa^2)} &= |C_1|^2\,C_1S_{3,0}^2 + C_1^*C_2C_0S_{3,1}^2 + C_1^*C_2C_0S_{3,2}^2\\ &\quad + C_1\,|C_2|^2\,S_{3,0}S_{3,2} + C_1\,|C_2|^2\,S_{3,1}S_{3,0} + C_2^*C_0^2S_{3,2}S_{3,1}\\ &\quad + C_1\,|C_0|^2\,S_{3,0}S_{3,1} + C_2^2C_0^*S_{3,1}S_{3,2} + C_1\,|C_0|^2\,S_{3,2}S_{3,0}, \qquad (7.44)\end{aligned}$$

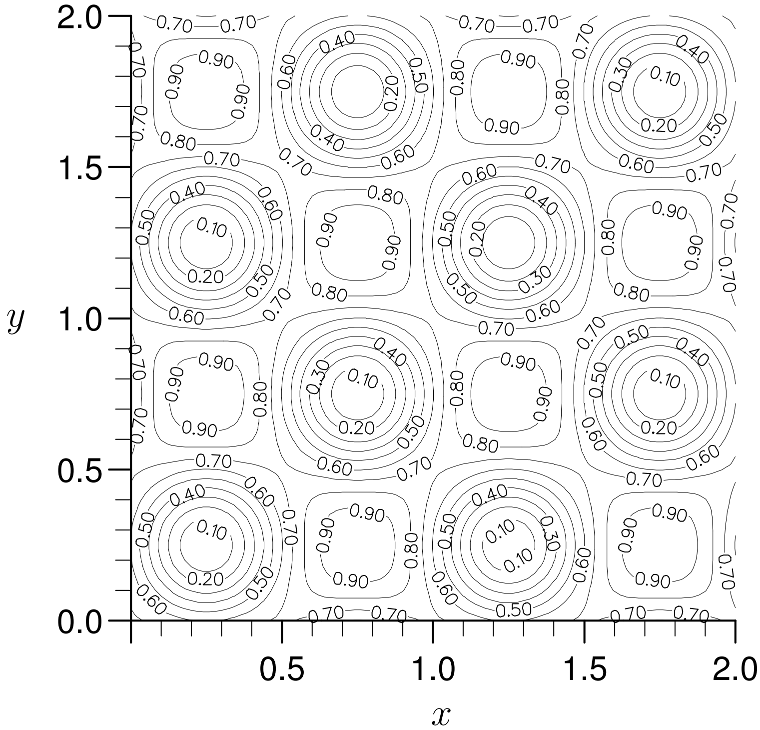

Figure 7.5: Mixed state solution $N = 2$, $C_1 = iC_0$, when $k = \sqrt{\pi}\,\kappa$. The diagram shows $|\Psi|^2$ as a function of x and y, which are labelled in multiples of the dimensions of the unit cell.

Figure 7.6: Mixed state solution $N = 2$, $C_1 = iC_0$, when $k = \sqrt{\pi}\,\kappa$. The diagram shows $|\Psi|^2$ as a function of x and y, which are labelled in multiples of the dimensions of the unit cell.

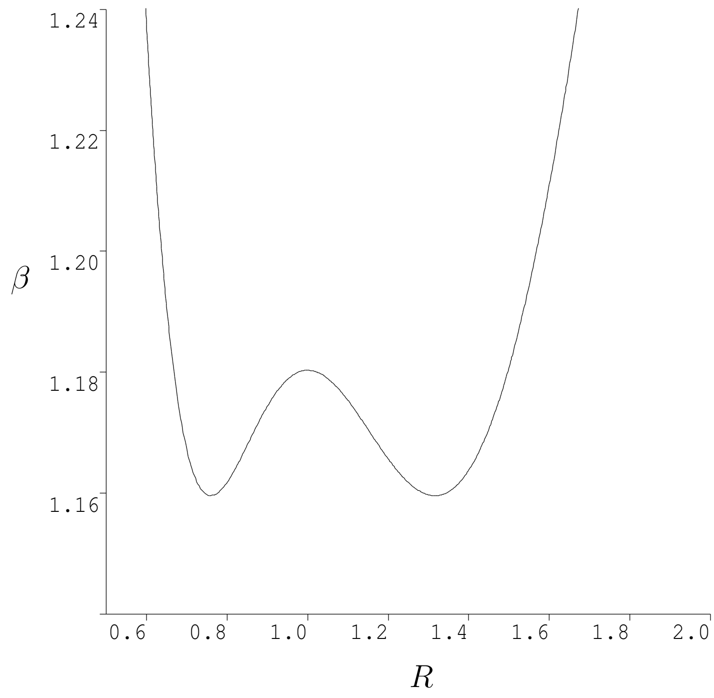

Figure 7.7: β as a function of $R = k/\kappa\sqrt{\pi}$, when $C_1 = iC_0$, $C_{n+2} = C_n \ \forall n$.

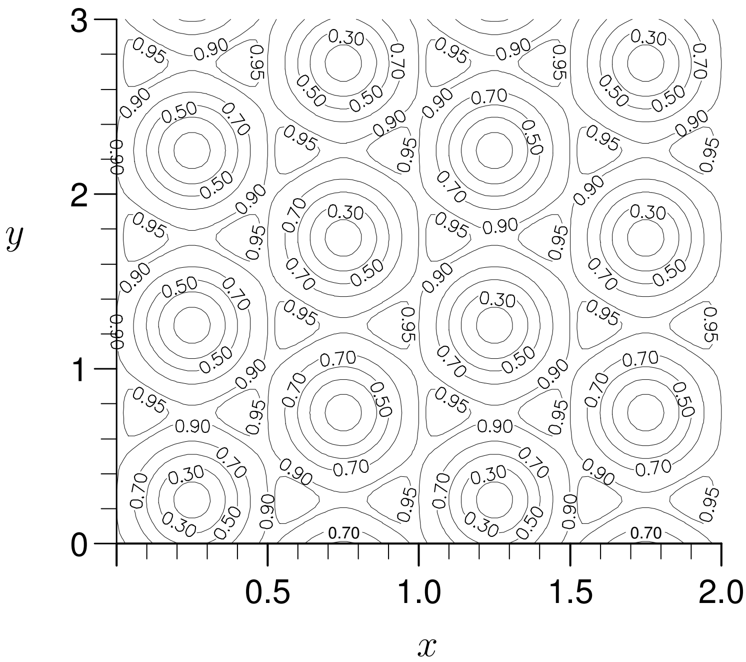

Figure 7.8: Mixed state solution $N = 2$, $C_1 = iC_0$, when $k = \sqrt{\pi\sqrt{3}}\,\kappa$. The diagram shows $|\Psi|^2$ as a function of x and y, which are labelled in multiples of the dimensions of the unit cell.

Figure 7.9: Mixed state solution $N = 2$, $C_1 = iC_0$, when $k = \sqrt{\pi\sqrt{3}}\,\kappa$. The diagram shows $|\Psi|^2$ as a function of x and y, which are labelled in multiples of the dimensions of the unit cell.

$$\begin{aligned}\frac{2\sqrt{2}\,\kappa h^{(1)}C_2}{(1-2\kappa^2)} &= |C_2|^2\,C_2S_{3,0}^2 + C_2^*C_0C_1S_{3,1}^2 + C_2^*C_0C_1S_{3,2}^2 \\ &\quad + C_2\,|C_0|^2\,S_{3,0}S_{3,2} + C_2\,|C_0|^2\,S_{3,1}S_{3,0} + C_0^*C_1^2S_{3,2}S_{3,1} \\ &\quad + C_2\,|C_1|^2\,S_{3,0}S_{3,1} + C_0^2C_1^*S_{3,1}S_{3,2} + C_2\,|C_1|^2\,S_{3,2}S_{3,0}, \quad (7.45)\end{aligned}$$

We let $C_1 = \xi C_0, C_2 = \eta C_0$. Then

$$\begin{aligned}\frac{2\sqrt{2}\,\kappa h^{(1)}}{|C_0|^2\,(1-2\kappa^2)} &= S_{3,0}^2 + \xi\eta S_{3,1}^2 + \xi\eta S_{3,2}^2 + (|\xi|^2 + |\eta|^2)S_{3,0}S_{3,2} \\ &\quad + (|\xi|^2 + |\eta|^2)S_{3,1}S_{3,0} + (\xi^*\eta^2 + \eta^*\xi^2)S_{3,2}S_{3,1}, \qquad (7.46)\end{aligned}$$

$$\begin{aligned}\frac{2\sqrt{2}\,\kappa\xi h^{(1)}}{|C_0|^2\,(1-2\kappa^2)} &= \xi\,|\xi|^2\,S_{3,0}^2 + \xi^*\eta S_{3,1}^2 + \xi^*\eta S_{3,2}^2 + \xi(1+|\eta|^2)S_{3,0}S_{3,2} \\ &\quad + \xi(1+|\eta|^2)S_{3,1}S_{3,0} + (\eta^2 + \eta^*)S_{3,2}S_{3,1}, \qquad (7.47)\end{aligned}$$

$$\begin{aligned}\frac{2\sqrt{2}\,\kappa\eta h^{(1)}}{|C_0|^2\,(1-2\kappa^2)} &= \eta\,|\eta|^2\,S_{3,0}^2 + \xi\eta^* S_{3,1}^2 + \xi\eta^* S_{3,2}^2 + \eta(1+|\xi|^2)S_{3,0}S_{3,2} \\ &\quad + \eta(1+|\xi|^2)S_{3,1}S_{3,0} + (\xi^2 + \xi^*)S_{3,2}S_{3,1}, \qquad (7.48)\end{aligned}$$

We multiply (7.46) by ξ and subtract (7.47), and multiply (7.46) by η and subtract (7.48) to give

$$\begin{aligned}\xi(1-|\xi|^2)S_{3,0}^2 + \left\{2\eta(\xi^2-\xi^*) + \eta^2(|\xi|^2-1) + \eta^*(\xi^3-1)\right\}S_{3,1}^2 & \\ + 2\xi(|\xi|^2-1)S_{3,0}S_{3,1} &= 0, \\ \eta(1-|\eta|^2)S_{3,0}^2 + \left\{2\xi(\eta^2-\eta^*) + \xi^2(|\eta|^2-1) + \xi^*(\eta^3-1)\right\}S_{3,1}^2 & \\ + 2\eta(|\eta|^2-1)S_{3,0}S_{3,1} &= 0,\end{aligned}$$

since $S_{3,1} = S_{3,2}$. We see that a solution is given by

$$\begin{aligned}|\xi| &= 1, \\ |\eta| &= 1, \\ 2\eta(\xi^2-\xi^*) + \eta^*(\xi^3-1) &= 0, \\ 2\xi(\eta^2-\eta^*) + \xi^*(\eta^3-1) &= 0,\end{aligned}$$

which in turn has solution

$$\xi, \eta = 1, \mu, \mu^2,$$

where μ is a complex root of unity. There are nine cases to consider, which we now tabulate.

		η		
		1	μ	μ^2
ξ	1	$N = 1$ case	New soln. 1	New soln. 2
	μ	1 translated in y	2 translated in y	$N = 1$ translated in y
	μ^2	2 translated in y	$N = 1$ translated in y	1 translated in y

Furthermore, solution 2 can be obtained from solution 1 by reflection in the x- axis and conjugation. Thus there is only one new solution to consider. When $C_1 = C_0$, $C_2 = \mu C_0$, the unit cell has dimensions $L_x = 3k/\kappa^2$, $L_y = 2\pi/k$ and $\psi^{(0)}$ has zeros at

$$x = k/2\kappa^2,\ y = \pi/k,$$
$$x = 3k/2\kappa^2,\ y = \pi/3k,$$
$$x = 5k/2\kappa^2,\ y = 5\pi/3k.$$

The solution again has a rhombic lattice structure. β is given by

$$\begin{aligned}\beta \;=\; & \frac{3k}{\sqrt{2\pi}\,\kappa}\frac{1}{(|C_0|^2+|C_1|^2+|C_2|^2)^2}\times \\ & \Big\{(|C_0|^4+|C_1|^4+|C_2|^4)S_{3,0}^2 \\ & +4(|C_0C_2|^2+|C_0C_1|^2+|C_1C_2|^2)S_{3,0}S_{3,1} \\ & +2(C_0^2C_1^*C_2^*+C_1^2C_2^*C_0^*+C_2^2C_0^*C_1^*)S_{3,1}^2 \\ & +2((C_2^*)^2C_0C_1+(C_0^*)^2C_1C_2+(C_1^*)^2C_2C_0)S_{3,1}^2\Big\}\,.\end{aligned}$$

When $C_1 = C_0$, $C_2 = \mu C_0$, β is given by

$$\beta = \frac{k}{\sqrt{2\pi}\,\kappa}\left\{3S_{3,0}^2 - 2(S_{3,0} - S_{3,1})^2\right\}.$$

Figure 7.10 shows β as a function of $R = k/\kappa\sqrt{\pi}$.

The minimum of β is obtained when $k \approx 1.355\sqrt{\pi}\,\kappa$, and is $\beta = 1.166$. Figures 7.11 and 7.12 show a plot of $|\,\Psi^{(0)}|^2$ in this case.

We see that the free energy of this solution lies between those of the square and triangular lattice solutions.

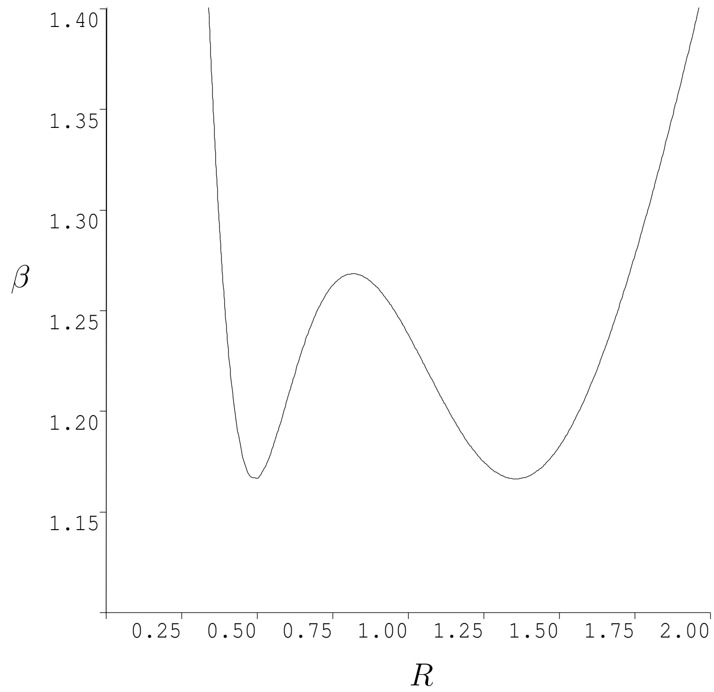

Figure 7.10: β as a function of $R = k/\kappa\sqrt{\pi}$, when $C_1 = C_0$, $C_2 = \mu C_0$, $C_{n+3} = C_n \;\; \forall n$.

Finally we examine the case $N = 4$. In this case equations (7.35) become

$$\begin{aligned}
\frac{2\sqrt{2}\,\kappa h^{(1)} C_0}{(1-2\kappa^2)} &= |C_0|^2\, C_0 S_{4,0}^2 + C_1 C_0^* C_3 S_{4,1}^2 + C_2^2 C_0^* S_{4,2}^2 \\
&\quad + C_3 C_0^* C_1 S_{4,3}^2 + C_0\,|C_1|^2\, S_{4,0} S_{4,3} + C_0\,|C_1|^2\, S_{4,0} S_{4,1} \\
&\quad + C_2 C_1^* C_3 S_{4,2} S_{4,1} + C_3 C_1^* C_2 S_{4,3} S_{4,2} + C_0\,|C_2|^2\, S_{4,0} S_{4,2} \\
&\quad + C_1^2 C_2^* S_{4,1} S_{4,3} + C_0\,|C_2|^2\, S_{4,0} S_{4,2} + C_3^2 C_2^* S_{4,3} S_{4,1} \\
&\quad + C_0\,|C_3|^2\, S_{4,0} S_{4,1} + C_1 C_3^* C_2 S_{4,1} S_{4,2} + C_2 C_3^* C_1 S_{4,2} S_{4,3} \\
&\quad + C_0\,|C_3|^2\, S_{4,3} S_{4,0},
\end{aligned}$$

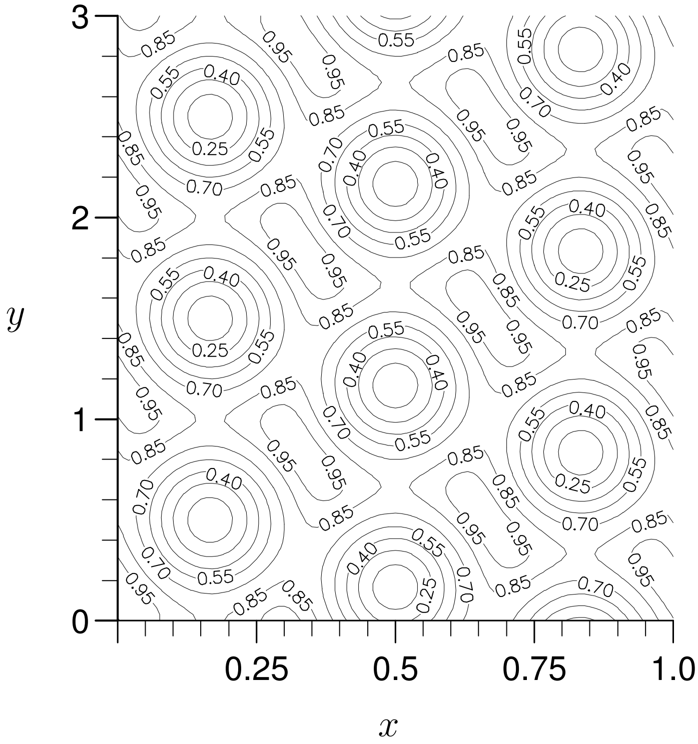

Figure 7.11: Mixed state solution $N = 3$, $C_1 = C_0$, $C_2 = \mu C_0$, when $k = 1.355\sqrt{\pi}\,\kappa$. The diagram shows $|\Psi|^2$ as a function of x and y, which are labelled in multiples of the dimensions of the unit cell.

$$\begin{aligned}
\frac{2\sqrt{2}\,\kappa h^{(1)} C_1}{(1-2\kappa^2)} \;=\; & |C_1|^2\, C_1 S_{4,0}^2 + C_2 C_1^* C_0 S_{4,1}^2 + C_3^2 C_1^* S_{4,2}^2 \\
& + C_0 C_1^* C_2 S_{4,3}^2 + C_1\, |C_2|^2\, S_{4,0} S_{4,3} + C_1\, |C_2|^2\, S_{4,0} S_{4,1} \\
& + C_3 C_2^* C_0 S_{4,2} S_{4,1} + C_0 C_2^* C_3 S_{4,3} S_{4,2} + C_1\, |C_3|^2\, S_{4,0} S_{4,2} \\
& + C_2^2 C_3^* S_{4,1} S_{4,3} + C_1\, |C_3|^2\, S_{4,0} S_{4,2} + C_0^2 C_3^* S_{4,3} S_{4,1} \\
& + C_1\, |C_0|^2\, S_{4,0} S_{4,1} + C_2 C_0^* C_3 S_{4,1} S_{4,2} + C_3 C_0^* C_2 S_{4,2} S_{4,3} \\
& + C_1\, |C_0|^2\, S_{4,3} S_{4,0},
\end{aligned}$$

Figure 7.12: Mixed state solution $N = 3$, $C_1 = C_0$, $C_2 = \mu C_0$, when $k = 1.355\sqrt{\pi}\,\kappa$. The diagram shows $|\Psi|^2$ as a function of x and y, which are labelled in multiples of the dimensions of the unit cell.

$$\begin{aligned}
\frac{2\sqrt{2}\,\kappa h^{(1)} C_2}{(1-2\kappa^2)} &= |C_2|^2\; C_2 S_{4,0}^2 + C_3 C_2^* C_1 S_{4,1}^2 + C_0^2 C_2^* S_{4,2}^2 \\
&\quad + C_1 C_2^* C_3 S_{4,3}^2 + C_2\; |C_3|^2\; S_{4,0} S_{4,3} + C_2\; |C_3|^2\; S_{4,0} S_{4,1} \\
&\quad + C_0 C_3^* C_1 S_{4,2} S_{4,1} + C_1 C_3^* C_0 S_{4,3} S_{4,2} + C_2\; |C_0|^2\; S_{4,0} S_{4,2} \\
&\quad + C_3^2 C_0^* S_{4,1} S_{4,3} + C_2\; |C_0|^2\; S_{4,0} S_{4,2} + C_1^2 C_0^* S_{4,3} S_{4,1} \\
&\quad + C_2\; |C_1|^2\; S_{4,0} S_{4,1} + C_3 C_1^* C_0 S_{4,1} S_{4,2} + C_0 C_1^* C_3 S_{4,2} S_{4,3} \\
&\quad + C_2\; |C_1|^2\; S_{4,3} S_{4,0}, \\
\frac{2\sqrt{2}\,\kappa h^{(1)} C_3}{(1-2\kappa^2)} &= |C_3|^2\; C_3 S_{4,0}^2 + C_0 C_3^* C_2 S_{4,1}^2 + C_1^2 C_3^* S_{4,2}^2 \\
&\quad + C_2 C_3^* C_0 S_{4,3}^2 + C_3\; |C_0|^2\; S_{4,0} S_{4,3} + C_3\; |C_0|^2\; S_{4,0} S_{4,1} \\
&\quad + C_1 C_0^* C_2 S_{4,2} S_{4,1} + C_2 C_0^* C_1 S_{4,3} S_{4,2} + C_3\; |C_1|^2\; S_{4,0} S_{4,2} \\
&\quad + C_0^2 C_1^* S_{4,1} S_{4,3} + C_3\; |C_1|^2\; S_{4,0} S_{4,2} + C_2^2 C_1^* S_{4,3} S_{4,1} \\
&\quad + C_3\; |C_2|^2\; S_{4,0} S_{4,1} + C_0 C_2^* C_1 S_{4,1} S_{4,2} + C_1 C_2^* C_0 S_{4,2} S_{4,3} \\
&\quad + C_3\; |C_2|^2\; S_{4,3} S_{4,0}.
\end{aligned}$$

We let $C_1 = \xi C_0$, $C_2 = \eta C_0$, $C_3 = \rho C_0$. Then

$$\begin{aligned}
\frac{2\sqrt{2}\,\kappa h^{(1)}}{|C_0|^2\;(1-2\kappa^2)} &= S_{4,0}^2 + \eta^2 S_{4,2}^2 \\
&\quad + (2\xi\rho + \xi^2\eta^* + \rho^2\eta^*) S_{4,1}^2 + 2(|\xi|^2 + |\rho|^2) S_{4,0} S_{4,1} \\
&\quad + 2\;|\eta|^2\; S_{4,0} S_{4,2} + 2\eta(\xi\rho^* + \xi^*\rho) S_{4,1} S_{4,2}, \\
\frac{2\sqrt{2}\,\kappa\xi h^{(1)}}{|C_0|^2\;(1-2\kappa^2)} &= \xi\;|\xi|^2\; S_{4,0}^2 + \xi^*\rho^2 S_{4,2}^2 \\
&\quad + (2\xi^*\eta + \eta^2\rho^* + \rho^*) S_{4,1}^2 + 2\xi(|\eta|^2 + 1) S_{4,0} S_{4,1} \\
&\quad + 2\xi\;|\rho|^2\; S_{4,0} S_{4,2} + 2\rho(\eta + \eta^*) S_{4,1} S_{4,2}, \\
\frac{2\sqrt{2}\,\kappa\eta h^{(1)}}{|C_0|^2\;(1-2\kappa^2)} &= \eta\;|\eta|^2\; S_{4,0}^2 + \eta^* S_{4,2}^2 \\
&\quad + (2\xi\eta^*\rho + \xi^2 + \rho^2) S_{4,1}^2 + 2\eta(|\xi|^2 + |\rho|^2) S_{4,0} S_{4,1} \\
&\quad + 2\eta S_{4,0} S_{4,2} + 2(\xi\rho^* + \xi^*\rho) S_{4,1} S_{4,2}, \\
\frac{2\sqrt{2}\,\kappa\rho h^{(1)}}{|C_0|^2\;(1-2\kappa^2)} &= \rho\;|\rho|^2\; S_{4,0}^2 + \xi^2\rho^* S_{4,2}^2 \\
&\quad + (2\eta\rho^* + \eta^2\xi^* + \xi^*) S_{4,1}^2 + 2\rho(|\eta|^2 + 1) S_{4,0} S_{4,1} \\
&\quad + 2\rho\;|\xi|^2\; S_{4,0} S_{4,2} + 2\xi(\eta + \eta^*) S_{4,1} S_{4,2}.
\end{aligned}$$

Solutions are given by

$$|\xi|^2 = |\eta|^2 = |\rho|^2 = 1,$$
$$\eta^2 = \frac{\xi^*\rho^2}{\xi} = \frac{\eta^*}{\eta} = \frac{\rho^*\xi^2}{\rho},$$
$$2\xi\rho = \xi^2\eta^* + \rho^2\eta^* = \frac{2\xi^*\eta}{\xi} + \frac{\eta^2\rho^*}{\xi} + \frac{\rho^*}{\xi} = \frac{2\xi\eta^*\rho}{\eta} + \frac{\xi^2}{\eta} + \frac{\rho^2}{\eta} = \frac{2\eta\rho^*}{\rho} + \frac{\eta^2\xi^*}{\rho} + \frac{\xi^*}{\rho},$$
$$2\xi\eta\rho^* + 2\xi^*\eta\rho = \frac{2\eta^*\rho}{\xi} + \frac{2\eta\rho}{\xi} = \frac{2\xi\rho^*}{\eta} + \frac{2\xi^*\rho}{\eta} = \frac{2\xi\eta^*}{\rho} + \frac{2\xi\eta}{\rho}.$$

These equations have solutions

$$\eta = \pm 1, \xi = \pm\rho = \pm 1, \pm i,$$

and

$$\eta = \pm 1, \xi = \mp\rho, |\xi| = 1,$$

Solutions in the first set are simply translations of the cases $N = 1$ and $N = 2$. β is given by

$$\begin{aligned}\beta \;=\; & \frac{4k}{\kappa}\frac{1}{(|C_0|^2 + |C_1|^2 + |C_2|^2 + |C_3|^2)^2} \times \\ & \Big\{(|C_0|^4 + |C_1|^4 + |C_2|^4 + |C_3|^4)S_{4,0}^2 \\ & + 4\Re\Big\{C_0C_1(C_1^{*2} + C_3^{*2}) + C_1C_3(C_0^{*2} + C_2^{*2})\Big\} \\ & + 4(|C_0|^2 + |C_2|^2)(|C_1|^2 + |C_3|^2)S_{4,0}S_{4,1} \\ & + 4(C_0C_2^* + C_0^*C_2)(C_1C_3^* + C_1^*C_3)S_{4,1}S_{4,2} \\ & + 2\Re(C_0^{*2}C_2^2 + C_1^{*2}C_3^2)S_{4,2}^2 \\ & + 4(|C_0C_2|^2 + |C_1C_3|^2)S_{4,0}S_{4,2}\Big\}.\end{aligned}$$

When $C_1 = \xi C_0$, $C_2 = \pm C_0$, $C_3 = \mp\xi C_0$, $|\xi| = 1$, then

$$\beta = \frac{k}{\sqrt{2\pi}\,\kappa}\Big\{(S_{4,0} + S_{4,2})^2 + 4S_{4,1}(S_{4,0} - S_{4,2})\Big\} \tag{7.49}$$
$$= \frac{k}{\sqrt{2\pi}\,\kappa}\Big\{S_{2,0}^2 + 4S_{4,1}(S_{4,0} - S_{4,2})\Big\}. \tag{7.50}$$

Figure 7.13 shows β as a function of $R = k/\kappa\sqrt{\pi}$. β is a minimum when $k \approx 1.380\sqrt{\pi}\kappa$, giving $\beta \approx 1.172$. Figures 7.14 and 7.15 show a plot of $|\Psi^{(0)}|^2$ in this case, with $\xi = 1$.

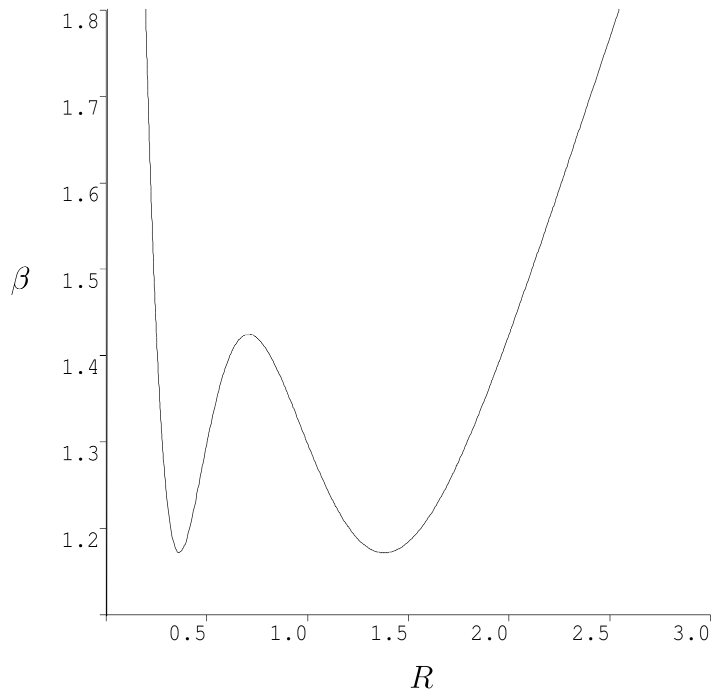

Figure 7.13: β as a function of $R = k/\kappa\sqrt{\pi}$, when $C_1 = C_0$, $C_2 = C_0$, $C_3 = -C_0$, $C_{n+4} = C_n \;\; \forall n$.

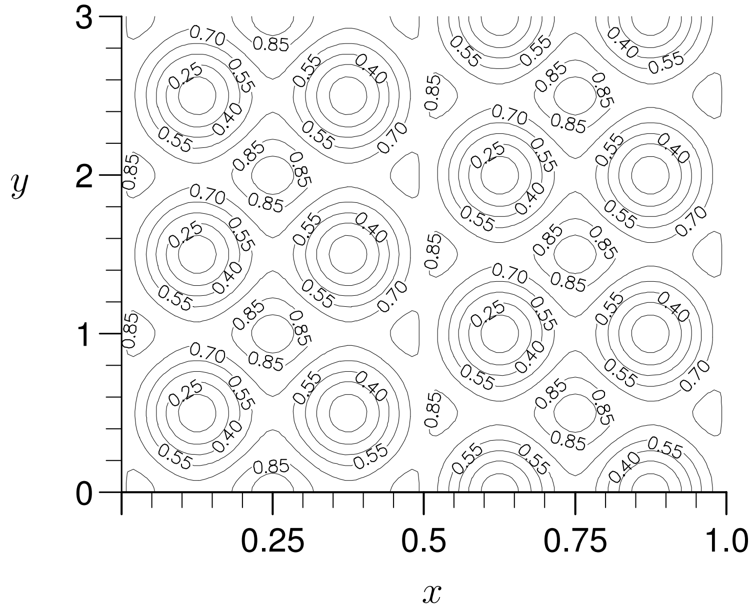

Figure 7.14: Mixed state solution $N = 4$, $C_1 = C_0$, $C_2 = C_0$, $C_3 = -C_0$, when $k = 1.380\sqrt{\pi}\,\kappa$. The diagram shows $|\Psi|^2$ as a function of x and y, which are labelled in multiples of the dimensions of the unit cell.

Figure 7.15: Mixed state solution $N = 4$, $C_1 = C_0$, $C_2 = C_0$, $C_3 = -C_0$, when $k = 1.380\sqrt{\pi}\,\kappa$. The diagram shows $|\Psi|^2$ as a function of x and y, which are labelled in multiples of the dimensions of the unit cell.

7.2 Linear Stability of the Mixed State

The analysis of the linear stability of the normal state with respect to two-dimensional disturbances is very similar to that of the infinite one-dimensional case, and the results are the same. We find that the normal state is linearly stable for $h > \kappa$ and linearly unstable for $h < \kappa$.

The more interesting question is that of the linear stability of the superconducting branches. Before we begin our analysis, we note that each solution contained an arbitrary parameter k, such that the period in the y direction was $2\pi/k$. In the previous section we varied k so as to minimize the free energy. However, the steady state solution existed for all k, and hence a classical linear stability analysis of the effects of perturbations of k would simply reveal neutral stability. Nonetheless, a linear stability analysis is useful to determine the stability of the various solutions to perturbations of the C_n, for fixed k and N. The problem can be thought of as an initial value problem, in which we perturb the C_n slightly from their equilibrium values and observe the evolution of the solution, requiring that the period in both the x and y-direction be fixed. We have the time-dependent Ginzburg-Landau equations:

$$\frac{\alpha}{\kappa^2}\frac{\partial \Psi}{\partial t} + \frac{\alpha i}{\kappa}\Psi\Phi + \left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2 \Psi = \Psi(1 - |\Psi|^2), \quad (7.51)$$

$$-(\mathrm{curl})^2\boldsymbol{A} - \frac{\partial \boldsymbol{A}}{\partial t} - \nabla\Phi = |\Psi|^2 \boldsymbol{A} + \frac{i}{2\kappa}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right), \quad (7.52)$$

with periodic boundary conditions

$$((i/\kappa)\nabla + \boldsymbol{A})\Psi(y + 2\pi/k) = ((i/\kappa)\nabla + \boldsymbol{A})\Psi(y), \quad (7.53)$$

$$\mathrm{curl}\ \boldsymbol{A}(y + 2\pi/k) = \mathrm{curl}\ \boldsymbol{A}(y), \quad (7.54)$$

$$\Phi(y + 2\pi/k) = \Phi(y), \quad (7.55)$$

$$((i/\kappa)\nabla + \boldsymbol{A})\Psi(x + Nk/\kappa^2) = ((i/\kappa)\nabla + \boldsymbol{A})\Psi(x), \quad (7.56)$$

$$\mathrm{curl}\ \boldsymbol{A}(x + Nk/\kappa^2) = \mathrm{curl}\ \boldsymbol{A}(x), \quad (7.57)$$

$$\Phi(x + Nk/\kappa^2) = \Phi(x). \quad (7.58)$$

We perturb about the previously found mixed state solution given by (7.15)-(7.17), (7.26) and (7.33), which we denote by Ψ_0, $\boldsymbol{A}_0$, by setting

$$\Psi = \Psi_0 + \delta e^{\sigma t}\Psi_1, \quad (7.59)$$

$$\boldsymbol{A} = \boldsymbol{A}_0 + \delta e^{\sigma t}\boldsymbol{A}_1, \quad (7.60)$$

$$\Phi = \delta e^{\sigma t}\Phi_1, \quad 0 < \delta \ll 1. \quad (7.61)$$

Inserting (7.59)-(7.61) into (7.51)-(7.58) and linearising in δ (to give the leading order behaviour of an asymptotic expansion in powers of δ) yields

$$\begin{aligned}&(\alpha i/\kappa)\Psi_0\Phi_1 + (\alpha\sigma/\kappa^2)\Psi_1 + ((i/\kappa)\nabla + \boldsymbol{A}_0)^2\Psi_1 \\ &\quad + (2i/\kappa)(\boldsymbol{A}_1\cdot\nabla\Psi_0) + (i/\kappa)\Psi_0(\nabla\cdot\boldsymbol{A}) + 2\Psi_0(\boldsymbol{A}_0\cdot\boldsymbol{A}_1) = \\ &\qquad\qquad \Psi_1 - 2\,|\,\Psi_0\,|^2\;\Psi_1 - \Psi_0^2\Psi_1^*, \quad (7.62)\end{aligned}$$

$$\begin{aligned}-(\mathrm{curl})^2\boldsymbol{A}_1 - \sigma\boldsymbol{A}_1 - \nabla\Phi_1 &= (i/2\kappa)(\Psi_0^*\nabla\Psi_1 + \Psi_1^*\nabla\Psi_0 - \Psi_0\nabla\Psi_1^* - \Psi_1\nabla\Psi_0^*) \\ &\quad + |\,\Psi_0\,|^2\;\boldsymbol{A}_1 + (\Psi_0\Psi_1^* + \Psi_0^*\Psi_1)\boldsymbol{A}_0, \quad (7.63)\end{aligned}$$

with $((i/\kappa)\nabla + \boldsymbol{A}_0)\Psi_1 + \boldsymbol{A}_1\Psi_0$, curl $\boldsymbol{A}_1$, and Φ_1 periodic.

We choose $\boldsymbol{A}$ to be directed along the y-axis as before and introduce ϵ as before by the equations

$$\Psi_0 = \epsilon^{1/2}\psi_0, \quad (7.64)$$

$$\boldsymbol{A}_0 = (0, hx + \epsilon a_0, 0), \quad (7.65)$$

$$\Psi_1 = \epsilon^{1/2}\psi_1, \quad (7.66)$$

$$\boldsymbol{A}_1 = (0, \epsilon a_1, 0), \quad (7.67)$$

$$\Phi_1 = \epsilon\phi_1. \quad (7.68)$$

Inserting (7.64)-(7.68) into (7.62)-(7.63) yields

$$\begin{aligned}&\frac{\alpha\sigma}{\kappa^2}\psi_1 + \frac{\epsilon\alpha i}{\kappa}\phi_1\psi_0 - \frac{1}{\kappa^2}\left(\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2}\right) + \frac{2ihx}{\kappa}\frac{\partial\psi_1}{\partial y} + h^2x^2\psi_1 \\ &\quad + \frac{2i\epsilon a_0}{\kappa}\frac{\partial\psi_1}{\partial y} + \epsilon^2 a_0^2\psi_1 + \frac{\epsilon i\psi_1}{\kappa}\frac{\partial a_0}{\partial y} + 2\epsilon hx a_0\psi_1 \\ &\quad + \frac{2i\epsilon a_1}{\kappa}\frac{\partial\psi_0}{\partial y} + \frac{i\epsilon\psi_0}{\kappa}\frac{\partial a_1}{\partial y} + 2\epsilon^2 a_0 a_1\psi_0 + 2\epsilon hx a_1\psi_0 = \\ &\qquad\qquad \psi_1 - 2\epsilon\,|\,\psi_0\,|^2\;\psi_1 - \epsilon\psi_0^2\psi_1^*, \quad (7.69)\end{aligned}$$

$$-\frac{\partial^2 a_1}{\partial x\partial y} - \frac{\partial\phi_1}{\partial x} = \frac{i}{2\kappa}\left(\psi_0^*\frac{\partial\psi_1}{\partial x} + \psi_1^*\frac{\partial\psi_0}{\partial x} - \psi_0\frac{\partial\psi_1^*}{\partial x} - \psi_1\frac{\partial\psi_0^*}{\partial x}\right), \quad (7.70)$$

$$\begin{aligned}\frac{\partial^2 a_1}{\partial x^2} - \sigma a_1 - \frac{\partial\phi_1}{\partial y} &= \frac{i}{2\kappa}\left(\psi_0^*\frac{\partial\psi_1}{\partial y} + \psi_1^*\frac{\partial\psi_0}{\partial y} - \psi_0\frac{\partial\psi_1^*}{\partial y} - \psi_1\frac{\partial\psi_0^*}{\partial y}\right) \\ &\quad + \epsilon\,|\,\psi_0\,|^2\;a_1 + (hx + \epsilon a_0)(\psi_0\psi_1^* + \psi_0^*\psi_1), \quad (7.71)\end{aligned}$$

with $\partial\psi_1/\partial x$, $(i/\kappa)\partial\psi_1/\partial y + (hx+\epsilon a_0)\psi_1 + \epsilon a_1\psi_0$, $\partial a_1/\partial x$, and ϕ_1 periodic. We expand all quantities in powers of ϵ as before:

$$\psi_0 = \psi_0^{(0)} + \epsilon\psi_0^{(1)} + \cdots, \tag{7.72}$$

$$a_0 = a_0^{(0)} + \epsilon a_0^{(1)} + \cdots, \tag{7.73}$$

$$\psi_1 = \psi_1^{(0)} + \epsilon\psi_1^{(1)} + \cdots, \tag{7.74}$$

$$a_1 = a_1^{(0)} + \epsilon a_1^{(1)} + \cdots, \tag{7.75}$$

$$h = h^{(0)} + \epsilon h^{(1)} + \cdots, \tag{7.76}$$

$$\sigma = \sigma^{(0)} + \epsilon\sigma^{(1)} + \cdots. \tag{7.77}$$

Inserting the expansions (7.72)-(7.77) into equations (7.69)-(7.71) and equating powers of ϵ yields at leading order

$$\frac{\alpha\sigma^{(0)}}{\kappa^2}\psi_1^{(0)} - \frac{1}{\kappa^2}\left(\frac{\partial^2\psi_1^{(0)}}{\partial x^2} + \frac{\partial^2\psi_1^{(0)}}{\partial y^2}\right) + \frac{2ih^{(0)}x}{\kappa}\frac{\partial\psi_1^{(0)}}{\partial y} + (h^{(0)})^2x^2\psi_1^{(0)} = \psi_1^{(0)}, \tag{7.78}$$

$$-\frac{\partial^2 a_1^{(0)}}{\partial x\partial y} - \frac{\partial\phi_1^{(0)}}{\partial x} = \frac{i}{2\kappa}\left(\psi_0^{(0)*}\frac{\partial\psi_1^{(0)}}{\partial x} + \psi_1^{(0)*}\frac{\partial\psi_0^{(0)}}{\partial x} - \psi_0^{(0)}\frac{\partial\psi_1^{(0)*}}{\partial x} - \psi_1^{(0)}\frac{\partial\psi_0^{(0)*}}{\partial x}\right), \tag{7.79}$$

$$\frac{\partial^2 a_1^{(0)}}{\partial x^2} - \sigma^{(0)}a_1^{(0)} - \frac{\partial\phi_1^{(0)}}{\partial y} = \frac{i}{2\kappa}\left(\psi_0^{(0)*}\frac{\partial\psi_1^{(0)}}{\partial y} + \psi_1^{(0)*}\frac{\partial\psi_0^{(0)}}{\partial y} - \psi_0^{(0)}\frac{\partial\psi_1^{(0)*}}{\partial y} - \psi_1^{(0)}\frac{\partial\psi_0^{(0)*}}{\partial y}\right) + h^{(0)}x(\psi_0^{(0)}\psi_1^{(0)*} + \psi_0^{(0)*}\psi_1^{(0)}), \tag{7.80}$$

with $\partial\psi_1^{(0)}/\partial x$, $(i/\kappa)\partial\psi_1^{(0)}/\partial y + h^{(0)}x\psi_1^{(0)}$, $\partial a_1^{(0)}/\partial x$, and $\phi_1^{(0)}$ periodic. As in one dimension, equation (7.78) has solutions when

$$\sigma^{(0)} = \frac{\kappa h^{(0)}}{\alpha}\left(\frac{\kappa}{h^{(0)}} - 2n - 1\right).$$

For $h^{(0)} < \kappa$ there is always an unstable mode. Hence the superconducting branches bifurcating from eigenvalues $h^{(0)} < \kappa$ are unstable. For $h^{(0)} = \kappa$ all modes are stable except the $n = 0$ mode which has $\sigma^{(0)} = 0$. To check the stability of this mode we must proceed to higher powers of ϵ in our expansions.

When $h^{(0)} = \kappa$ the solution of (7.78) with periodic boundary conditions is

$$\psi_1^{(0)} = \sum_{n=-\infty}^{\infty} B_n e^{inky} \psi_n(x), \tag{7.81}$$

where ψ_n is as before, and

$$B_{n+N} = B_n, \quad \forall n. \tag{7.82}$$

Substituting our expressions for $\psi_0^{(0)}$ and $\psi_1^{(0)}$ into equations (7.79), (7.80) yields

$$-\frac{\partial^2 a_1^{(0)}}{\partial x \partial y} = \frac{\partial \phi_1^{(0)}}{\partial x} + \frac{i}{2\kappa} \sum_{m,n=-\infty}^{\infty} (C_n^* B_m + C_m B_n^*) k(m-n) e^{i(m-n)ky} \psi_m \psi_n, \tag{7.83}$$

$$\frac{\partial^2 a_1^{(0)}}{\partial x^2} = \frac{\partial \phi_1^{(0)}}{\partial y} - \frac{1}{2\kappa} \sum_{m,n=-\infty}^{\infty} (C_n^* B_m + C_m B_n^*)[k(m+n) - 2\kappa^2 x] e^{i(m-n)ky} \psi_m \psi_n, \tag{7.84}$$

Taking the derivative with respect to x of (7.83), the derivative with respect to y of (7.84) and adding, we find

$$\frac{\partial^2 \phi_1^{(0)}}{\partial x^2} + \frac{\partial^2 \phi_1^{(0)}}{\partial y^2} = 0. \tag{7.85}$$

The solution on (7.85) with periodic boundary conditions is

$$\phi_1^{(0)} \equiv \text{const.}$$

Without loss of generality we may choose $\phi_1^{(0)} \equiv 0$. Now integrating equations (7.83), (7.84) gives

$$a_1^{(0)} = -\frac{1}{2\kappa} \sum_{m,n=-\infty}^{\infty} (C_n^* B_m + B_n^* C_m) e^{i(m-n)ky} \int^x \psi_m \psi_n \, dx. \tag{7.86}$$

Equating coefficients of ϵ in (7.69) yields

$$\begin{aligned} -\frac{1}{\kappa^2}\left(\frac{\partial^2 \psi_1^{(1)}}{\partial x^2} + \frac{\partial^2 \psi_1^{(1)}}{\partial y^2}\right) + 2ix\frac{\partial \psi_1^{(1)}}{\partial y} + \kappa^2 x^2 \psi_1^{(1)} - \psi_1^{(1)} = \\ -\frac{\alpha\sigma^{(1)}}{\kappa^2}\psi_1^{(0)} - 2\,|\,\psi_0^{(0)}|^2\,\psi_1^{(0)} - (\psi_0^{(0)})^2\psi_1^{(0)*} - \frac{2ia_0^{(0)}}{\kappa}\frac{\partial \psi_1^{(0)}}{\partial y} \\ -\frac{i\psi_1^{(0)}}{\kappa}\frac{\partial a_0^{(0)}}{\partial y} - 2\kappa x a_0^{(0)}\psi_1^{(0)} - \frac{2ia_1^{(0)}}{\kappa}\frac{\partial \psi_0^{(0)}}{\partial y} - \frac{i\psi_0^{(0)}}{\kappa}\frac{\partial a_1^{(0)}}{\partial y} \\ -2\kappa x a_1^{(0)}\psi_0^{(0)} - \frac{2ih^{(1)}x}{\kappa}\frac{\partial \psi_1^{(0)}}{\partial y} - 2\kappa h^{(1)} x^2 \psi_1^{(0)}, \end{aligned} \tag{7.87}$$

with $\psi_1^{(1)}$ periodic in y and $\mid \psi_1^{(1)} \mid$ periodic in x as before. Substituting in our expressions for $\psi_0^{(0)}$, $\psi_1^{(0)}$, $a_0^{(1)}$ and $a_1^{(0)}$ we find

$$\begin{aligned}
L\psi_1^{(1)} \quad = \quad & -\frac{\alpha\sigma^{(0)}}{\kappa^2}\sum_p B_p e^{ipky}\psi_p \\
& - 2\sum_{p,r,m} C_m C_p^* B_r e^{i(m-p+r)ky}\psi_m\psi_p\psi_r \\
& - \sum_{p,m,r} C_m C_r B_p^* e^{i(m-p+r)ky}\psi_m\psi_p\psi_r \\
& - \frac{k}{\kappa^2}\sum_{m,p,r} C_r(C_p^* B_m + C_m B_p^*) r e^{i(m-p+r)ky}\psi_r \int^x \psi_m\psi_p\, dx \\
& - \frac{k}{2\kappa^2}\sum_{m,p,r} C_r(C_p^* B_m + C_m B_p^*)(m-p) e^{i(m-p+r)ky}\psi_r \int^x \psi_m\psi_p\, dx \\
& - \frac{k}{\kappa^2}\sum_{m,p,r} B_r C_p^* C_m r e^{i(m-p+r)ky}\psi_r \int^x \psi_m\psi_p\, dx \\
& - \frac{k}{2\kappa^2}\sum_{m,p,r} B_r C_p^* C_m (m-p) e^{i(m-p+r)ky}\psi_r \int^x \psi_m\psi_p\, dx \\
& + x\sum_{m,p,r} B_r C_p^* C_m e^{i(m-p+r)ky}\psi_r \int^x \psi_m\psi_p\, dx \\
& + x\sum_{m,p,r} C_r(C_p^* B_m + B_p^* C_m) e^{i(m-p+r)ky}\psi_r \int^x \psi_m\psi_p\, dx \\
& + \frac{kxh^{(1)}}{\kappa}\sum_p B_p p e^{ipky}\psi_p \\
& - 2\kappa h^{(1)} x^2 \sum_p B_p e^{ipky}\psi_p,
\end{aligned}$$

where

$$L = -\frac{1}{\kappa^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 2ix\frac{\partial}{\partial y} + \kappa^2 x^2 - 1.$$

Hence

$$\begin{aligned}
L\psi_1^{(1)} = & -2h^{(1)}\kappa x \sum_p B_p \left(x - \frac{pk}{\kappa^2}\right) e^{ipky}\psi_p \\
& - \frac{\alpha\sigma^{(1)}}{\kappa^2}\sum_p B_p e^{ipky}\psi_p \\
& + \sum_{r,p,m} (2C_p^* C_m B_r + C_m C_r B_p^*)\left[x - \frac{k}{\kappa^2}\left(r - \frac{p-m}{2}\right)\right] e^{i(r-p+m)ky}\psi_r \int^x \psi_m\psi_p\, dx \\
& + \sum_{r,p,m} (2C_p^* C_m B_r + C_m C_r B_p^*) e^{i(r-p+m)ky}\psi_r\psi_m\psi_p.
\end{aligned}$$

As before, we multiply be e^{-inky} and integrate from $y = 0$ to $y = 2\pi/k$ to give

$$-\frac{1}{\kappa^2}\frac{\partial^2 \psi_{1,n}^{(1)}}{\partial x^2} + \frac{k^2 n^2 \psi_{1,n}^{(1)}}{\kappa^2} - 2xkn\psi_{1,n}^{(1)} + \kappa^2 x^2 \psi_{1,n}^{(1)} - \psi_{1,n}^{(1)} =$$

$$-2h^{(1)}\kappa x B_n \left(x - \frac{nk}{\kappa^2}\right)\psi_n - \frac{\alpha\sigma^{(1)}}{\kappa^2} B_n \psi_n$$

$$+ 2\sum_{p,m} C_p^* C_m B_{n-m+p}\left[x - \frac{k}{\kappa^2}\left(n + \frac{p-m}{2}\right)\right]\psi_{n-m+p}\int^x \psi_m \psi_p \, dx$$

$$+ \sum_{p,m} C_m C_{n-m+p} B_p^* \left[x - \frac{k}{\kappa^2}\left(n + \frac{p-m}{2}\right)\right]\psi_{n-m+p}\int^x \psi_m \psi_p \, dx$$

$$+ \sum_{p,m}(2C_p^* C_m B_{n-m+p} + C_m C_{n-m+p} B_p^*)\psi_{n-m+p}\psi_m \psi_p.$$

where

$$\psi_1^{(1)} = \sum_{n=-\infty}^{\infty} e^{inky}\psi_{1,n}^{(1)}.$$

Now ψ_n is a solution of the homogeneous version of this equation (with ψ_n and $d\psi_n/dx$ vanishing as $x \to \pm\infty$), and hence there is a solution for $\psi_{1,n}^{(1)}$ if and only if an appropriate solvability condition is satisfied. Multiplying by ψ_n and integrating we find this condition to be

$$\frac{\alpha\sigma^{(1)} B_n}{\kappa^2} = -\frac{h^{(1)} B_n}{\kappa}$$
$$+ \frac{1}{\sqrt{2}}\left(\frac{1}{2\kappa^2} - 1\right)\sum_{m,p} 2C_p C_m^* B_{n-p+m} \exp\left\{-\frac{k^2}{2\kappa^2}\left[(p-n)^2 + (p-m)^2\right]\right\}$$
$$+ \frac{1}{\sqrt{2}}\left(\frac{1}{2\kappa^2} - 1\right)\sum_{m,p} C_p C_{n-p+m} B_m^* \exp\left\{-\frac{k^2}{2\kappa^2}\left[(p-n)^2 + (p-m)^2\right]\right\}.$$

Equation (7.35) is

$$h^{(1)} C_n = \frac{\kappa}{\sqrt{2}}\left(\frac{1}{2\kappa^2} - 1\right)\sum_{p,m} C_p C_m^* C_{n-p+m} \exp\left\{-\frac{k^2}{2\kappa^2}\left[(p-n)^2 + (p-m)^2\right]\right\}.$$

Hence

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)} B_n}{(1-2\kappa^2)} &= \sum_{m,p} 2C_p C_m^* B_{n-p+m} \exp\left\{-\frac{k^2}{2\kappa^2}\left[(p-n)^2 + (p-m)^2\right]\right\} \\ &\quad + \sum_{m,p} C_p C_{n-p+m} B_m^* \exp\left\{-\frac{k^2}{2\kappa^2}\left[(p-n)^2 + (p-m)^2\right]\right\} \\ &\quad - \sum_{m,p} \frac{C_p C_m^* C_{n-p+m} B_n}{C_n} \exp\left\{-\frac{k^2}{2\kappa^2}\left[(p-n)^2 + (p-m)^2\right]\right\}.\end{aligned} \tag{7.88}$$

The problem now is to examine the growth rates of the various modes. Let us consider first the mode where $B_n = C_n$. Then

$$\frac{\sqrt{2}\,\alpha\sigma^{(1)}C_n}{(1-2\kappa^2)} = \sum_{m,p} C_p C_m^* C_{n-p+m} \exp\left\{-\frac{k^2}{2\kappa^2}\left[(p-n)^2+(p-m)^2\right]\right\}.$$

If we multiply by C_n^* and sum over n we find

$$\frac{\alpha\sigma^{(1)}\overline{|\,\psi_0^{(0)}\,|^2}}{\kappa^2} = 2\left(\frac{1}{2\kappa^2}-1\right)\overline{|\,\psi_0^{(0)}\,|^4}.$$

Hence

$$\sigma^{(1)} = \left(\frac{1-2\kappa^2}{\alpha}\right)\frac{\overline{|\,\psi_0^{(0)}\,|^4}}{\overline{|\,\psi_0^{(0)}\,|^2}} = \frac{2\kappa h^{(1)}}{\alpha}. \tag{7.89}$$

We see that

$$\sigma^{(1)} < 0 \text{ if and only if } \kappa > 1/\sqrt{2}.$$

Hence for a Type I superconductor, where $\kappa < 1/\sqrt{2}$, the mixed state is always unstable to this mode, whatever the value of N and the coefficients C_n. For a Type II superconductor the mode $B_n = C_n$ is always stable.

Let us now examine the stability of the other modes. For the case $N = 1$ the only possible mode is that examined above. We consider the case $N = 2$, $C_{n+2} = C_n$, $B_{n+2} = B_n$ $\forall n$. Then (7.88) becomes

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)}B_0}{(1-2\kappa^2)} &= (|\,C_0\,|^2\,B_0 + C_0^2B_0^*)S_{2,0}^2\\ &\quad+\left(2C_0^*C_1B_1 + C_1^2B_0^* - \frac{C_0^*C_1^2B_0}{C_0}\right)S_{2,1}^2\\ &\quad+2(C_1^*C_0B_1 + C_0C_1B_1^*)S_{2,0}S_{2,1},\end{aligned} \tag{7.90}$$

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)}B_1}{(1-2\kappa^2)} &= (|\,C_1\,|^2\,B_1 + C_1^2B_1^*)S_{2,0}^2\\ &\quad+\left(2C_0C_1^*B_0 + C_0^2B_1^* - \frac{C_1^*C_0^2B_1}{C_1}\right)S_{2,1}^2\\ &\quad+2(C_0^*C_1B_0 + C_1C_0B_0^*)S_{2,0}S_{2,1}.\end{aligned} \tag{7.91}$$

We consider first the rectangular lattice of Abrikosov, so that $C_n = C$ $\forall n$. Then (7.90), (7.91) become

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)}B_0}{(1-2\kappa^2)} &= (|\,C\,|^2\,B_0 + C^2B_0^*)S_{2,0}^2\\ &\quad+(2\,|\,C\,|^2\,B_1 + C^2B_0^* - |\,C\,|^2\,B_0)S_{2,1}^2\\ &\quad+2(|\,C\,|^2\,B_1 + C^2B_1^*)S_{2,0}S_{2,1},\end{aligned} \tag{7.92}$$

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)}B_1}{(1-2\kappa^2)} &= (|C|^2\ B_1 + C^2 B_1^*)S_{2,0}^2 \\ &\quad + (2\,|C|^2\ B_0 + C^2 B_1^* - |C|^2\ B_1)S_{2,1}^2 \\ &\quad + 2(|C|^2\ B_0 + C^2 B_0^*)S_{2,0}S_{2,1}. \qquad (7.93)\end{aligned}$$

We let $B_1 = B_0 + D$. Then

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)}B_0}{(1-2\kappa^2)} &= (|C|^2\ B_0 + C^2 B_0^*)S_{2,0}^2 \\ &\quad + (2\,|C|^2\ B_0 + C^2 B_0^* - |C|^2\ B_0 + 2\,|C|^2\ D)S_{2,1}^2 \\ &\quad + 2(|C|^2\ B_0 + C^2 B_0^* + |C|^2\ D + C^2 D^*)S_{2,0}S_{2,1}, \qquad (7.94) \\ \frac{2\sqrt{2}\,\alpha\sigma^{(1)}(B_0+D)}{(1-2\kappa^2)} &= (|C|^2\ B_0 + C^2 B_0^* + |C|^2\ D + C^2 D^*)S_{2,0}^2 \\ &\quad + (|C|^2\ B_0 + C^2 B_0^* + C^2 D^* - |C|^2\ D)S_{2,1}^2 \\ &\quad + 2(|C|^2\ B_0 + C^2 B_0^*)S_{2,0}S_{2,1} \qquad (7.95)\end{aligned}$$

Subtracting (7.94) from (7.95) gives

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)}D}{(1-2\kappa^2)} &= (|C|^2\ D + C^2 D^*)S_{2,0}^2 \\ &\quad + (C^2 D^* - 3\,|C|^2\ D)S_{2,1}^2 \\ &\quad - 2(|C|^2\ D + C^2 D^*)S_{2,0}S_{2,1},\end{aligned}$$

or

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)}}{|C|^2\ (1-2\kappa^2)} &= (1+E)S_{2,0}^2 + (E-3)S_{2,1}^2 - 2(1+E)S_{2,0}S_{2,1}, \\ &= (1+E)(S_{2,0}-S_{2,1})^2 - 4S_{2,1}^2,\end{aligned}$$

where $E = CD^*/C^*D$, $|E| = 1$. We are interested in the case $\kappa > 1/\sqrt{2}$. Then, when $E = -1$, we see

$$\sigma^{(1)} = -\frac{\sqrt{2}\,S_{2,1}^2\ |C|^2\ (1-2\kappa^2)}{\alpha} > 0.$$

Thus the mode given by $E = -1$, $D/D^* = -C/C^*$ is unstable. Hence, when $N = 2$ the solution $C_n = C \quad \forall n$ is linearly unstable.

Let us consider now the solution $C_1 = iC_0$, $C_{n+2} = C_n\ \forall n$. Then (7.90), (7.91) become

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)}B_0}{(1-2\kappa^2)} &= (|C_0|^2\ B_0 + C_0^2 B_0^*)S_{2,0}^2 \\ &\quad + (2i\,|C_0|^2\ B_1 - C_0^2 B_0^* + |C_0|^2\ B_0)S_{2,1}^2 \\ &\quad + 2i(-\,|C_0|^2\ B_1 + C_0^2 B_1^*)S_{2,0}S_{2,1},\end{aligned}$$

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)}B_1}{(1-2\kappa^2)} &= (|\,C_0\,|^2\ B_1 - C_0^2B_1^*)S_{2,0}^2 \\ &\quad + (-2i\ |\,C_0\,|^2\ B_0 + C_0^2B_1^* - |\,C_0\,|^2\ B_1)S_{2,1}^2 \\ &\quad + 2i(|\,C_0\,|^2\ B_0 + C_0^2B_0^*)S_{2,0}S_{2,1}.\end{aligned}$$

Let $B_1 = iB_0 + iD$. Then

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)}B_0}{(1-2\kappa^2)} &= (|\,C_0\,|^2 B_0 + C_0^2B_0^*)S_{2,0}^2 \\ &\quad + (-\,|\,C_0\,|^2 B_0 - 2\,|\,C_0\,|^2 D - C_0^2B_0^*)S_{2,1}^2 \\ &\quad + 2(|\,C_0\,|^2 B_0 + C_0^2B_0^* + |\,C_0\,|^2 D + C_0^2D^*)S_{2,0}S_{2,1}, \quad (7.96) \\ \frac{2\sqrt{2}\,\alpha\sigma^{(1)}(B_0+D)}{(1-2\kappa^2)} &= (|\,C_0\,|^2 B_0 + C_0^2B_0^* + |\,C_0\,|^2 D + C_0^2D^*)S_{2,0}^2 \\ &\quad + (-\,|\,C_0\,|^2 B_0 - C_0^2B_0^* - C_0^2D^* + |\,C_0\,|^2 D)S_{2,1}^2 \\ &\quad + 2(|\,C_0\,|^2 B_0 + C_0^2B_0^*)S_{2,0}S_{2,1}. \quad (7.97)\end{aligned}$$

Subtracting (7.96) from (7.97) yields

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)}D}{(1-2\kappa^2)} &= (|\,C_0\,|^2\ D + C_0^2D^*)S_{2,0}^2 \\ &\quad + (3\,|\,C_0\,|^2\ D - C_0^2D^*)S_{2,1}^2 \\ &\quad - 2(|\,C_0\,|^2\ D + C_0^2D^*)S_{2,0}S_{2,1}.\end{aligned}$$

or

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)}}{|\,C_0\,|^2\ (1-2\kappa^2)} &= (1+E)S_{2,0}^2 + (3-E)S_{2,1}^2 - 2(1+E)S_{2,0}S_{2,1}, \\ &= (1+E)(S_{2,0}-S_{2,1})^2 + 2(1-E)S_{2,1}^2,\end{aligned}$$

where $E = C_0D^*/C_0^*D$. Since $|\,E\,| = 1$ we see that for $\kappa > 1/\sqrt{2}$, $\Re\{\sigma^{(1)}\} < 0$. Hence, when $N = 2$ the solution $C_1 = iC_0$, $C_{n+2} = C_n\ \forall n$, is linearly stable, for all k. Thus we are in agreement with the free energy argument of Abrikosov, namely that the solution $C_1 = iC_0$, $C_{n+2} = C_n\ \forall n$ is preferred to the solution $C_n = C\ \forall n$. However, we do not know whether this solution is stable for other values of N, e.g. $N = 4$.

We note that the one-dimensional solution found previously can be represented by (7.26), but with $C_0 = C$, $C_n = 0$, $n \neq 0$, so that in this case $N = \infty$. We now check the stability of this solution with respect to a two-dimensional perturbation. Equation (7.88) becomes

$$\frac{2\sqrt{2}\,\alpha\sigma^{(1)}B_n}{(1-2\kappa^2)} = \mid C\mid^2\, B_n(2e^{-\frac{k^2n^2}{2\kappa^2}}-1)+C^2B^*_{-n}e^{-\frac{k^2n^2}{\kappa^2}}. \tag{7.98}$$

When $n=0$ and $B_0\neq 0$ we have

$$\alpha\sigma^{(1)} = \frac{1}{2\sqrt{2}}(1-2\kappa^2)[1+E],$$

where $E = CB_0^*/C^*B_0$. Thus, since $\mid E\mid = 1$ we have $\sigma^{(1)} < 0$ when $\kappa > 1/\sqrt{2}$. Thus the one-dimensional solution is stable to any perturbation in which $B_0 \neq 0$. Consider the case $B_0 = 0$. Substituting $-n$ for n in (7.98) gives

$$\frac{2\sqrt{2}\,\alpha\sigma^{(1)}B_{-n}}{(1-2\kappa^2)} = \mid C\mid^2\, B_{-n}(2e^{-\frac{k^2n^2}{2\kappa^2}}-1)+C^2B^*_{n}e^{-\frac{k^2n^2}{\kappa^2}}. \tag{7.99}$$

Setting $B_{-n} = B_n + D$ and subtracting (7.98) from (7.99) yields

$$\frac{2\sqrt{2}\,\alpha\sigma^{(1)}}{\mid C\mid^2\,(1-2\kappa^2)} = 2e^{-\frac{k^2n^2}{2\kappa^2}}-1-\hat{E}e^{-\frac{k^2n^2}{\kappa^2}},$$

where $\hat{E} = CD^*/C^*D$, $\mid\hat{E}\mid = 1$. If we choose the B_n such that $\hat{E}=1$ we see

$$\sigma^{(1)} = -\frac{\mid C\mid^2\,(1-2\kappa^2)}{2\sqrt{2}\,\alpha}(1-e^{-\frac{k^2n^2}{2\kappa^2}})^2.$$

Thus $\sigma^{(1)} > 0$ when $\kappa > 1/\sqrt{2}$ for this mode. Hence the one-dimensional solution is linearly unstable with respect to a two-dimensional perturbation.

7.3 Weakly-nonlinear Stability of the Normal State with Periodic Boundary Conditions

Let us consider now how a small perturbation of the normal state will grow, taking into account the nonlinear terms. We have the time-dependent Ginzburg-Landau equations

$$\frac{\alpha}{\kappa^2}\frac{\partial\Psi}{\partial t}+\frac{\alpha i}{\kappa}\Psi\Phi+\left(\frac{i}{\kappa}\nabla+\boldsymbol{A}\right)^2\Psi = \Psi(1-\mid\Psi\mid^2), \tag{7.100}$$

$$-(\mathrm{curl})^2\boldsymbol{A}-\frac{\partial\boldsymbol{A}}{\partial t}-\nabla\Phi = \mid\Psi\mid^2\,\boldsymbol{A}+\frac{i}{2\kappa}\left(\Psi^*\nabla\Psi-\Psi\nabla\Psi^*\right), \tag{7.101}$$

with periodic boundary conditions

$$((i/\kappa)\nabla + \boldsymbol{A})\Psi(y + 2\pi/k) = ((i/\kappa)\nabla + \boldsymbol{A})\Psi(y), \tag{7.102}$$

$$\text{curl } \boldsymbol{A}(y + 2\pi/k) = \text{curl } \boldsymbol{A}(y), \tag{7.103}$$

$$\Phi(y + 2\pi/k) = \Phi(y), \tag{7.104}$$

$$((i/\kappa)\nabla + \boldsymbol{A})\Psi(x + Nk/\kappa^2) = ((i/\kappa)\nabla + \boldsymbol{A})\Psi(x), \tag{7.105}$$

$$\text{curl } \boldsymbol{A}(x + Nk/\kappa^2) = \text{curl } \boldsymbol{A}(x), \tag{7.106}$$

$$\Phi(x + Nk/\kappa^2) = \Phi(x). \tag{7.107}$$

as before. We seek a solution near the bifurcation point $h = \kappa$. To this end we set

$$h = \kappa + h^{(1)}\epsilon, \tag{7.108}$$

where $\epsilon > 0$. We choose $\boldsymbol{A}$ to be directed along the y-axis as before and introduce ψ, a and ϕ as before by the equations

$$\Psi = \epsilon^{1/2}\psi, \tag{7.109}$$

$$\boldsymbol{A} = (0, hx + \epsilon a, 0), \tag{7.110}$$

$$\Phi = \epsilon\phi. \tag{7.111}$$

Substituting (7.108)-(7.111) into (7.100)-(7.101) yields

$$\frac{\alpha}{\kappa^2}\frac{\partial\psi}{\partial t} + \frac{\epsilon\alpha i}{\kappa}\psi\phi - \frac{1}{\kappa^2}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + \frac{2i((\kappa + \epsilon h^{(1)})x + \epsilon a)}{\kappa}\frac{\partial\psi}{\partial y} + \frac{i\epsilon\psi}{\kappa}\frac{\partial a}{\partial y} + ((\kappa + \epsilon h^{(1)})x + \epsilon a)^2\psi = \psi - \epsilon\psi\,|\,\psi\,|^2, \tag{7.112}$$

$$-\frac{\partial^2 a}{\partial x \partial y} - \frac{\partial\phi}{\partial x} = \frac{i}{2\kappa}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right), \tag{7.113}$$

$$\frac{\partial^2 a}{\partial x^2} - \frac{\partial a}{\partial t} - \frac{\partial\phi}{\partial y} = ((\kappa + \epsilon h^{(1)})x + \epsilon a)\,|\,\psi\,|^2 + \frac{i}{2\kappa}\left(\psi^*\frac{\partial\psi}{\partial y} - \psi\frac{\partial\psi^*}{\partial y}\right), \tag{7.114}$$

with $\partial a/\partial x$, $\partial\psi/\partial x$ and $(i/\kappa)\partial\psi/\partial y + (hx + \epsilon a)\psi$, and ϕ periodic.

A. Short timescale : $t = O(1)$.

We denote the short-time solution by $\psi_s(\boldsymbol{r}, t)$, $a_s(\boldsymbol{r}, t)$, $\phi_s(\boldsymbol{r}, t)$. We expand all quantities in powers of ϵ as before:

$$\psi_s = \psi_s^{(0)} + \epsilon\psi_s^{(1)} + \cdots, \tag{7.115}$$

$$a_s = a_s^{(0)} + \epsilon a_s^{(1)} + \cdots, \tag{7.116}$$

$$\phi_s = \phi_s^{(0)} + \epsilon \phi_s^{(1)} + \cdots. \tag{7.117}$$

Inserting the expansions (7.115)-(7.117) into equations (7.112)-(7.114) and equating powers of ϵ yields at leading order

$$\frac{\alpha}{\kappa^2}\frac{\partial \psi_s^{(0)}}{\partial t} - \frac{1}{\kappa^2}\left(\frac{\partial^2 \psi_s^{(0)}}{\partial x^2} + \frac{\partial^2 \psi_s^{(0)}}{\partial y^2}\right) + \frac{2i\kappa x}{\kappa}\frac{\partial \psi_s^{(0)}}{\partial y} = \psi_s^{(0)} - \kappa^2 x^2 \psi_s^{(0)}, \tag{7.118}$$

$$-\frac{\partial^2 a_s^{(0)}}{\partial x \partial y} - \frac{\partial \phi_s^{(0)}}{\partial x} = \frac{i}{2\kappa}\left(\psi_s^{(0)*}\frac{\partial \psi_s^{(0)}}{\partial x} - \psi_s^{(0)}\frac{\partial \psi_s^{(0)*}}{\partial x}\right), \tag{7.119}$$

$$\frac{\partial^2 a_s^{(0)}}{\partial x^2} - \frac{\partial a_s^{(0)}}{\partial t} - \frac{\partial \phi_s^{(0)}}{\partial y} = \kappa x \mid \psi_s^{(0)} \mid^2 + \frac{i}{2\kappa}\left(\psi_s^{(0)*}\frac{\partial \psi_s^{(0)}}{\partial y} - \psi_s^{(0)}\frac{\partial \psi_s^{(0)*}}{\partial y}\right), \tag{7.120}$$

with $\partial a_s^{(0)}/\partial x$, $\partial \psi_s^{(0)}/\partial x$, $(i/\kappa)\partial \psi_s^{(0)}/\partial y + \kappa x \psi_s^{(0)}$, and $\phi_s^{(0)}$ periodic.

The solution of (7.118) with periodic boundary conditions is

$$\psi_s^{(0)} = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} C_{n,m} e^{\sigma_m t} e^{inky} \psi_{n,m}(x), \tag{7.121}$$

where

$$\sigma_m = -\frac{2m\kappa^2}{\alpha},$$

with corresponding set of eigenfunctions

$$\psi_{n,m} = \frac{1}{(2^m m!)^{1/2}} \exp\left\{-\frac{\kappa^2}{2}\left(x - \frac{kn}{\kappa^2}\right)^2\right\} \mathrm{H}_n\left(\sqrt{2}\,\kappa\left(x - \frac{kn}{\kappa^2}\right)\right),$$

where H_n is the Hermite polynomial, and the eigenfunctions have been scaled so that

$$\int_{-\infty}^{\infty} \psi_{n,m}\psi_{n,p}\,dx = \begin{cases} 0 & \text{if } m \neq p, \\ \sqrt{\pi}/\kappa & \text{if } m = p. \end{cases}$$

The C_n must be chosen so that

$$\psi_s^{(0)}(x,y,0) = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} C_{n,m} e^{inky} \psi_{n,m}(x).$$

Multiplying by $e^{-inky}\psi_{n,m}$ and integrating we see

$$C_{n,m} = \frac{k\kappa}{2\pi^{3/2}} \int_{-\infty}^{\infty} \int_0^{2\pi/k} e^{-inky} \psi_{n,m}(x) \psi_s^{(0)}(x,y,0)\,dy\,dx. \tag{7.122}$$

Note that $C_{n+N} = C_n$ for all n, as required. We may now solve for $a_s^{(0)}$ and $\phi_s^{(0)}$.

B. Long timescale : $t = O(\epsilon^{-1})$.

We now consider the long-time behaviour of the solution. We define

$$\tau = \epsilon t,$$

and consider τ to be $O(1)$. We denote the long-time solution by $\psi_l(\boldsymbol{r}, \tau)$, $a_l(\boldsymbol{r}, \tau)$, $\phi_l(\boldsymbol{r}, \tau)$. Equations (7.112)-(7.114) become

$$\begin{aligned}\frac{\epsilon\alpha}{\kappa^2}\frac{\partial\psi_l}{\partial\tau} + \frac{\epsilon\alpha i}{\kappa}\psi_l\phi_l - \frac{1}{\kappa^2}\left(\frac{\partial^2\psi_l}{\partial x^2} + \frac{\partial^2\psi_l}{\partial y^2}\right) + \frac{2i((\kappa + \epsilon h^{(1)})x + \epsilon a_l)}{\kappa}\frac{\partial\psi_l}{\partial y} & \\ + \frac{i\epsilon\psi_l}{\kappa}\frac{\partial a_l}{\partial y} + ((\kappa + \epsilon h^{(1)})x + \epsilon a_l)^2\psi_l &= \psi_l - \epsilon\psi_l\,|\,\psi_l\,|^2, \quad (7.123)\end{aligned}$$

$$-\frac{\partial^2 a_l}{\partial x\partial y} - \frac{\partial\phi_l}{\partial x} = \frac{i}{2\kappa}\left(\psi_l^*\frac{\partial\psi_l}{\partial x} - \psi_l\frac{\partial\psi_l^*}{\partial x}\right), \quad (7.124)$$

$$\frac{\partial^2 a_l}{\partial x^2} - \epsilon\frac{\partial a_l}{\partial\tau} - \frac{\partial\phi_l}{\partial y} = ((\kappa + \epsilon h^{(1)})x + \epsilon a_l)\,|\,\psi_l\,|^2 + \frac{i}{2\kappa}\left(\psi_l^*\frac{\partial\psi_l}{\partial y} - \psi_l\frac{\partial\psi_l^*}{\partial y}\right), \quad (7.125)$$

with $\partial a_l/\partial x$, $\partial\psi_l/\partial x$ and $(i/\kappa)\partial\psi_l/\partial y + (hx + \epsilon a_l)\psi_l$, and ϕ_l periodic.

We expand all quantities in powers of ϵ as before:

$$\psi_l = \psi_l^{(0)} + \epsilon\psi_l^{(1)} + \cdots, \quad (7.126)$$

$$a_l = a_l^{(0)} + \epsilon a_l^{(1)} + \cdots, \quad (7.127)$$

$$\phi_l = \phi_l^{(0)} + \epsilon\phi_l^{(1)} + \cdots. \quad (7.128)$$

Inserting the expansions (7.126)-(7.128) into equations (7.123)-(7.125) and equating powers of ϵ yields at leading order

$$-\frac{1}{\kappa^2}\left(\frac{\partial^2\psi_l^{(0)}}{\partial x^2} + \frac{\partial^2\psi_l^{(0)}}{\partial y^2}\right) + \frac{2i\kappa x}{\kappa}\frac{\partial\psi_l^{(0)}}{\partial y} = \psi_l^{(0)} - \kappa^2 x^2\psi_l^{(0)}, \quad (7.129)$$

$$-\frac{\partial^2 a_l^{(0)}}{\partial x\partial y} - \frac{\partial\phi_l^{(0)}}{\partial x} = \frac{i}{2\kappa}\left(\psi_l^{(0)*}\frac{\partial\psi_l^{(0)}}{\partial x} - \psi_l^{(0)}\frac{\partial\psi_l^{(0)*}}{\partial x}\right), \quad (7.130)$$

$$\frac{\partial^2 a_l^{(0)}}{\partial x^2} - \frac{\partial\phi_l^{(0)}}{\partial y} = \kappa x\,|\,\psi_l^{(0)}\,|^2 + \frac{i}{2\kappa}\left(\psi_l^{(0)*}\frac{\partial\psi_l^{(0)}}{\partial y} - \psi_l^{(0)}\frac{\partial\psi_l^{(0)*}}{\partial y}\right), \quad (7.131)$$

with $\partial a_l^{(0)}/\partial x$, $\partial\psi_l^{(0)}/\partial x$, $(i/\kappa)\partial\psi_l^{(0)}/\partial y + \kappa x\psi_l^{(0)}$, and $\phi_l^{(0)}$ periodic.

Equation (7.129) with periodic boundary conditions is exactly the steady-state problem and has solution

$$\psi_l^{(0)} = \sum_{-\infty}^{\infty} C_n(\tau) e^{inky} \psi_n(x), \tag{7.132}$$

where

$$\psi_n(x) = \exp\left\{ -\frac{\kappa^2}{2}\left(x - \frac{nk}{\kappa^2}\right)^2 \right\}, \tag{7.133}$$

as before, and $C_{n+N}(\tau) = C_n(\tau)$. Then, exactly as in the steady state, we find

$$\begin{aligned}
\phi_l^{(0)} &\equiv 0, \\
a_l^{(0)} &= -\frac{1}{2\kappa} \sum_{m,n=-\infty}^{\infty} C_n^*(\tau) C_m(\tau) e^{ik(m-n)y} \int^x \psi_m(x)\psi_n(x)\, dx, \\
&= -\frac{1}{2\kappa} \int^x |\psi_l^{(0)}|^2 \; dx.
\end{aligned}$$

Equating coefficients of ϵ in equation (7.123) yields

$$\begin{aligned}
-\frac{1}{\kappa^2}&\left(\frac{\partial^2 \psi_l^{(1)}}{\partial x^2} + \frac{\partial^2 \psi_l^{(1)}}{\partial y^2}\right) + 2ix\frac{\partial \psi_l^{(1)}}{\partial y} + \kappa^2 x^2 \psi_l^{(1)} - \psi_l^{(1)} \\
&= -\frac{\alpha}{\kappa^2}\frac{\partial \psi_l^{(0)}}{\partial \tau} - \psi_l^{(0)}\,|\psi_l^{(0)}|^2 - \frac{2ih^{(1)}x}{\kappa}\frac{\partial \psi_l^{(0)}}{\partial y} - \frac{2ia_l^{(0)}}{\kappa}\frac{\partial \psi_l^{(0)}}{\partial y} \\
&\quad - \frac{i\psi_l^{(0)}}{\kappa}\frac{\partial a_l^{(0)}}{\partial y} - 2\kappa x a_l^{(0)} \psi_l^{(0)} - h^{(1)} 2\kappa x^2 \psi_l^{(0)} \\
&= -\frac{\alpha}{\kappa^2}\sum_p \frac{dC_p}{d\tau} e^{ipky}\psi_p \\
&\quad - \sum_{p,m,r} C_p C_m^* C_r e^{ik(p-m+r)y} \psi_p \psi_m \psi_r \\
&\quad + \frac{2kh^{(1)}x}{\kappa}\sum_p pC_p e^{ikpy}\psi_p \\
&\quad - \frac{k}{\kappa^2}\sum_{p,m,r} pC_p C_m^* C_r e^{ik(p-m+r)y}\psi_p \int^x \psi_m \psi_r\, dx \\
&\quad - \frac{k}{2\kappa^2}\sum_{p,m,r} (r-m) C_p C_m^* C_r e^{ik(p-m-r)y}\psi_p \int^x \psi_m\psi_r\, dx \\
&\quad + x\sum_{p,m,r} C_p C_m^* C_r e^{ik(p-m+r)y}\psi_p \int^x \psi_m \psi_r\, dx \\
&\quad - h^{(1)} 2\kappa x^2 \sum_p C_p e^{ikpy}\psi_p
\end{aligned}$$

$$= \quad -\frac{\alpha}{\kappa^2}\sum_p \frac{dC_p}{d\tau}e^{ipky}\psi_p$$
$$+\frac{2h^{(1)}x}{\kappa}\sum_p C_p(-\kappa^2 x + kp)e^{ikpy}\psi_p$$
$$+\sum_{p,m,r} C_p C_m^* C_r \left\{\left[x - \frac{k}{\kappa^2}\left(p + \frac{r-m}{2}\right)\right] e^{ik(p-m+r)y}\psi_p \int^x \psi_m\psi_r\, dx - \psi_p\psi_m\psi_r\right\}.$$

Multiplying by e^{-ikny} and integrating from $y = 0$ to $y = 2\pi/k$ we find

$$-\frac{1}{\kappa^2}\left(\frac{\partial^2\psi_{l,n}^{(1)}}{\partial x^2} + \frac{\partial^2\psi_{l,n}^{(1)}}{\partial y^2}\right) + 2ix\frac{\partial\psi_{l,n}^{(1)}}{\partial y} + \kappa^2x^2\psi_{l,n}^{(1)} - \psi_{l,n}^{(1)} =$$
$$-\frac{\alpha}{\kappa^2}\frac{dC_n}{d\tau}\psi_n + C_n\frac{2h^{(1)}x}{\kappa}(-\kappa^2x + kn)\psi_n$$
$$+\sum_{m,r} C_{n-r+m}C_m^*C_r\left[x - \frac{k}{\kappa^2}\left(n + \frac{m-r}{2}\right)\right]\psi_{n-r+m}\int^x \psi_m\psi_r\, dx$$
$$-\sum_{m,r} C_{n-r+m}C_m^*C_r\psi_{n-r+m}\psi_m\psi_r, \tag{7.134}$$

where

$$\psi_l^{(1)} = \sum_{n=-\infty}^{\infty} e^{ikny}\psi_{l,n}^{(1)}.$$

Now ψ_n is a solution of the homogeneous version of this equation (with ψ_n and $d\psi_n/dx$ vanishing as $x \to \pm\infty$). Therefore there is a solution for $\psi_{l,n}^{(1)}$ if and only if the right-hand side is orthogonal to ψ_n for all n. Performing the necessary integration we find

$$\frac{\alpha}{\kappa^2}\frac{dC_n}{d\tau} = \frac{1}{\sqrt{2}}\left(\frac{1}{2\kappa^2} - 1\right)\sum_{r,m} C_{n-r+m}C_m^*C_r e^{\left\{-\frac{k^2}{2\kappa^2}\left[(r-n)^2+(r-m)^2\right]\right\}} - \frac{C_nh^{(1)}}{\kappa}. \tag{7.135}$$

The boundary conditions for these equations come from matching with the short-time solution. We have

$$\psi_l^{(0)}(x,y,0) = \sum_{n=-\infty}^{\infty} C_n(0)e^{inky}\psi_n(x) \quad = \quad \lim_{t\to\infty}\psi_s^{(0)}(x,y,t),$$
$$= \sum_{n=-\infty}^{\infty} C_{n,0}e^{inky}\psi_n(x).$$

Hence

$$C_n(0) = C_{n,0}, \qquad \forall n. \tag{7.136}$$

We multiply equation (7.135) by C_n^* and sum over n. The resulting equation is

$$\frac{\alpha}{2\kappa^2}\frac{d\overline{|\psi_l^{(0)}|^2}}{d\tau} = \left(\frac{1}{2\kappa^2} - 1\right)\overline{|\psi_l^{(0)}|^4} - \frac{h^{(1)}}{\kappa}\overline{|\psi_l^{(0)}|^2}, \tag{7.137}$$

or

$$\frac{\alpha}{2\kappa^2}\frac{d\overline{|\psi_l^{(0)}|^2}}{d\tau} = \beta\left(\frac{1}{2\kappa^2}-1\right)\left(\overline{|\psi_l^{(0)}|^2}\right)^2 - \frac{h^{(1)}}{\kappa}\overline{|\psi_l^{(0)}|^2}, \tag{7.138}$$

where $\beta = \overline{|\psi_l^{(0)}|^4}/\left(\overline{|\psi_l^{(0)}|^2}\right)^2$ as before. If β were constant this equation would have the same form as equation (5.253) of the one-dimensional case. Even for varying β we have that β is independent of the amplitude of $\psi_l^{(0)}$ and depends only on the relative sizes of the coefficients C_n. We see that $\beta \geq 1$ and so (7.138) will have the same qualitive features as (5.253).

We note that equation (7.135) can be written as

$$\frac{\alpha}{\kappa^2}\frac{dC_n}{d\tau} = \frac{Nk}{2\kappa\sqrt{\pi}}\left(\frac{1}{2\kappa^2}-1\right)\left(\overline{|\psi^{(0)}|^2}\right)^2\frac{\partial\beta}{\partial C_n^*}. \tag{7.139}$$

Hence

$$\begin{aligned}\frac{d\beta}{d\tau} &= \sum_{n=0}^{N-1}\frac{\partial\beta}{\partial C_n^*}\frac{dC_n^*}{d\tau} + \frac{\partial\beta}{\partial C_n}\frac{dC_n}{d\tau},\\ &= \frac{8\alpha\kappa\sqrt{\pi}}{\left(\overline{|\psi^{(0)}|^2}\right)^2 Nk(1-2\kappa^2)}\sum_{n=0}^{N-1}\left|\frac{dC_n}{d\tau}\right|^2.\end{aligned}$$

Hence

$$\frac{d\beta}{d\tau} \leq 0, \qquad \text{for } \kappa > 1/\sqrt{2}.$$

To proceed further we need to specify our choice of N. For $N = 1$ we have $C_n = C\ \forall n$, and

$$\frac{\alpha}{\kappa^2}\frac{dC}{d\tau} = \frac{1}{\sqrt{2}}\left(\frac{1}{2\kappa^2}-1\right)C\,|C|^2\,S_{1,0}^2 - \frac{Ch^{(1)}}{\kappa}. \tag{7.140}$$

This equation differs from (5.253) simply by the constant $S_{1,0}^2$, which can be removed by a suitable scaling of C, and so behaves in exactly the same way.

Let us now examine the case $N = 2$. In this case there are two possible steady-state solutions. Since the response is qualitively the same as in one dimension, the question of interest is which of the two steady-state solutions will be approached in the case when $\kappa > 1/\sqrt{2}$ and the applied magnetic field is slightly less than κ. We rescale C_0, C_1, τ for simplicity by setting

$$C_0' = \left[\frac{\kappa}{h^{(1)}\sqrt{2}}\left(1-\frac{1}{2\kappa^2}\right)\right]^{1/2}C_0,$$

$$
\begin{aligned}
C_1' &= \left[\frac{\kappa}{h^{(1)}\sqrt{2}}\left(1-\frac{1}{2\kappa^2}\right)\right]^{1/2} C_1, \\
\tau' &= \frac{\kappa h^{(1)}}{\alpha}\tau.
\end{aligned}
$$

Then, dropping the primes, (7.135) becomes

$$
\begin{aligned}
\frac{dC_0}{d\tau} &= -\left[|C_0|^2\, C_0 S_{2,0}^2 + C_0^* C_1^2 S_{2,1}^2 + 2\,|C_1|^2\, C_0 S_{2,0} S_{2,1}\right] + C_0, \quad (7.141)\\
\frac{dC_1}{d\tau} &= -\left[|C_1|^2\, C_1 S_{2,0}^2 + C_1^* C_0^2 S_{2,1}^2 + 2\,|C_0|^2\, C_1 S_{2,0} S_{2,1}\right] + C_1. \quad (7.142)
\end{aligned}
$$

The two steady states are given by

$$
\begin{aligned}
&\text{(a)} \quad C_1 = C_0, \quad |C_0| = \tfrac{1}{S_{1,0}^2}, \\
&\text{(b)} \quad C_1 = iC_0, \quad |C_0| = \tfrac{1}{(S_{1,0}^2 - 2S_{2,1}^2)^{1/2}}.
\end{aligned}
$$

Let $C_1 = \eta C_0$. Then

$$
\begin{aligned}
\frac{dC_0}{d\tau} &= -\,|C_0|^2\, C_0\left[S_{2,0}^2 + \eta^2 S_{2,1}^2 + 2\,|\eta|^2\, S_{2,0}S_{2,1}\right] + C_0, \quad (7.143)\\
\eta\frac{dC_0}{d\tau} + C_0\frac{d\eta}{d\tau} &= -\,|C_0|^2\, C_0\left[\eta\,|\eta|^2\, S_{2,0}^2 + \eta^* S_{2,1}^2 + 2\eta S_{2,0}S_{2,1}\right] + \eta C_0. \quad (7.144)
\end{aligned}
$$

We multiply (7.143) by η and subtract from (7.144)

$$
\frac{d\eta}{d\tau} = |C_0|^2\left[\eta(1-|\eta|^2)S_{2,0}^2 + (\eta^3 - \eta^*)S_{2,1}^2 + 2\eta(|\eta|^2 - 1)S_{2,0}S_{2,1}\right]. \quad (7.145)
$$

The steady-state solutions are $\eta = \pm 1,\ \pm i$. We let $\eta = re^{i\vartheta}$. Then

$$
\begin{aligned}
&\frac{dr}{d\tau}e^{i\vartheta} + ir\frac{d\vartheta}{d\tau}e^{i\vartheta} = \\
&\qquad |C_0|^2\left[re^{i\vartheta}(1-r^2)S_{2,0}^2 + r(r^2e^{3i\vartheta} - e^{-i\vartheta})S_{2,1}^2 + 2re^{i\vartheta}(r^2-1)S_{2,0}S_{2,1}\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{d\vartheta}{d\tau} &= |C_0|^2\, S_{2,1}^2(1+r^2)\sin 2\vartheta, \quad (7.146)\\
\frac{dr}{d\tau} &= |C_0|^2\, r(1-r^2)\left[S_{2,0}(S_{2,0} - 2S_{2,1}) - S_{2,1}^2\cos 2\vartheta\right]. \quad (7.147)
\end{aligned}
$$

Firstly we note that $S_{2,0} \gg S_{2,1}$ and so

$$
\begin{aligned}
&\frac{dr}{d\tau} < 0, \ \text{if } r > 1,\\
&\frac{dr}{d\tau} > 0, \ \text{if } r < 1.
\end{aligned}
$$

Hence we expect that

$$r \to 1, \text{ as } \tau \to \infty.$$

We also note that

$$\begin{aligned} \frac{d\vartheta}{d\tau} &> 0 \text{ for } 0 < \vartheta < \pi/2,\ \pi < \vartheta < 3\pi/2, \\ \frac{d\vartheta}{d\tau} &< 0 \text{ for } \pi/2 < \vartheta < \pi,\ 3\pi/2 < \vartheta < 2\pi/, \\ \frac{d\vartheta}{d\tau} &= 0 \text{ for } \vartheta = 0,\ \pi. \end{aligned}$$

Thus we expect

$$\begin{aligned} \vartheta &\to \pi/2, \text{ as } \tau \to \infty, \text{ if } 0 < \vartheta_0 < \pi, \\ \vartheta &\to 3\pi/2, \text{ as } \tau \to \infty, \text{ if } \pi < \vartheta_0 < 2\pi, \\ \vartheta &\equiv 0,\ \pi, \text{ if } \vartheta_0 = 0,\ \pi. \end{aligned}$$

Thus we see that only if the initial data has $C_1 = \pm C_0$ will this solution be approached. Any other initial data will result in the solutions $C_1 = \pm iC_0$ being approached as $\tau \to \infty$.

As a final note, if we assume that r has already reached the value 1 in (7.146), and that $|C_0|^2$ is constant, then we are solving (after a suitable rescaling)

$$\frac{d\vartheta}{d\tau} = \sin 2\vartheta,$$

which has solution

$$\vartheta = \tan^{-1}(Ae^{2\tau}),\ A = \tan\vartheta_0.$$

We see that

$$\begin{aligned} \vartheta &= 0, \pi, \text{ if } A = 0, \\ \vartheta &\to \pi/2, \text{ as } \tau \to \infty, \text{ if } A > 0, \\ \vartheta &\to 3\pi/2, \text{ as } \tau \to \infty, \text{ if } A < 0. \end{aligned}$$

7.4 Summary

Let us now summarize the above results.

We examined the bifurcation of the normal state solution to a periodic superconducting solution as the external field passes through the upper critical field h_{c_2}, which was found to be equal to κ, the Ginzburg-Landau parameter. Hence for Type I superconductors ($\kappa < 1/\sqrt{2}$) $h_{c_2} < H_c$, the thermodynamic critical field, while for Type II superconductors ($\kappa > 1/\sqrt{2}$) $h_{c_2} > H_c$. The superconducting solution was shown to exist for all values of the external field slightly less than κ for Type II superconductors, and all values of the external field slightly greater then κ for Type I superconductors.

For each of the values $N = 1, 2, 3, 4$, where N is the number of zeros of Ψ in the unit cell, we demonstrated the possible superconducting solutions, each of which depended on a parameter k, such that the period in the y-direction was $2\pi/k$, and the period in the x-direction was kN/κ^2. The traditional way of determining which solution is stable is to seek the solution with the lowest free energy, which was shown by Abrikosov to be equivalent, for Type II superconductors, to seeking the solution with the lowest value of $\beta = \overline{|\psi^{(0)}|^4}/(\overline{|\psi^{(0)}|^2})^2$. Of the solutions considered the lowest value of β was obtained by the solution $N = 2$, $C_1 = iC_0$, $C_{n+2} = C_n \ \forall n$, when $k = \sqrt{\pi\sqrt{3}}\,\kappa$. This solution corresponds to a triangular lattice of vortices (Fig. 7.8). Furthermore, for arbitrary k, the solution $N = 2$, $C_1 = iC_0$, $C_{n+2} = C_n \ \forall n$, has a lower free energy that the solution $N = 1$, $C_n = C \ \forall n$.

We then examined the classical linear stability of the solutions with fixed k and N, using the time-dependent Ginzburg-Landau equations. We found that the normal state is linearly stable for $h > \kappa$ and linearly unstable for $h < \kappa$. Moreover, the superconducting mixed states were all found to be unstable, for all k, for Type I superconductors. For Type II superconductors we examined the cases $N = 1$ and $N = 2$ only. For $N = 1$ there is only one possible superconducting solution. $C_n = C \ \forall n$, which was found to be linearly stable. For $N = 2$ there were two possible solutions. The solution $C_1 = iC_0$, $C_{n+2} = C_n \ \forall n$ was found to be linearly stable, while the solution $C_n = C \ \forall n$ was found to be linearly unstable. (Hence although the solution $C_n = C \ \forall n$ is found to be linearly stable when the period in the x direction is fixed at k/κ^2, i.e. when $N = 1$, it is found to be linearly unstable when the period in the x direction is fixed at $2k/\kappa^2$, i.e. when $N = 2$.) This is in agreement with the free energy arguments of Abrikosov.

An examination of the weakly-nonlinear stability of the normal state near $h = \kappa$, subject to periodic boundary conditions, revealed the same qualitative features as in one dimension. For Type I superconductors a small perturbation of the normal state blows up for $h < \kappa$. For $h > \kappa$ sufficiently small perturbations will decay to zero, while large perturbations will again blow up.

For Type II superconductors a perturbation of the normal state will decay to zero for $h > \kappa$. For $h < \kappa$ a perturbation will tend to one of the mixed state superconducting solutions. In the case $N = 2$, where there were two possible steady state solutions to approach, a perturbation was found to approach the solution $C_1 = iC_0$, $C_{n+2} = C_n \;\; \forall n$.

7.5 Transformation of the Mixed State to the Superconducting State: Structure of an Isolated Vortex

In Section 7.1 we found mixed state solutions corresponding to a regular array of vortices of superconducting current around nodal lines of Ψ (flux lines). Around each node the phase of Ψ varied by 2π, corresponding to each flux line containing one quantum of fluxoid.

It is natural to assume that for fields much lower than h_{c_2}, Ψ also has a lattice structure, but with a much larger period, and that the phase of Ψ also varies by 2π around each node. We suppose the flux lines to be sufficiently well-separated that the overlap is negligible and they can be treated in isolation (i.e. a separation $\gg \lambda$). We consider the problem of a single axially-symmetric filament by looking for a solution of the form

$$\Psi = f(r)e^{i\theta}, \tag{7.148}$$

$$\boldsymbol{A} = A(r)\hat{\boldsymbol{\theta}}, \tag{7.149}$$

where r, θ are polar co-ordinates and $\hat{\boldsymbol{\theta}}$ is the unit vector in the azimuthal direction. Substituting (7.148), (7.149) into equations (7.7), (7.8) yields

$$-\frac{1}{\kappa^2 r}\frac{d}{dr}\left(r\frac{df}{dr}\right) + Q^2 f = f - f^3, \tag{7.150}$$

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rQ)\right) = f^2 Q, \qquad (7.151)$$

$$f \to 1, \quad Q \to 0 \qquad \text{as } r \to \infty, \qquad (7.152)$$

$$Q \sim -\frac{1}{\kappa r}, \quad f \to 0 \qquad \text{as } r \to 0, \qquad (7.153)$$

where $Q = A - 1/\kappa r$. The magnetic field $H\hat{\boldsymbol{z}}$ is given by

$$H = -\frac{1}{r}\frac{d}{dr}(rQ),$$

and the superconducting current is given by

$$\boldsymbol{j}_s = -f^2 Q\hat{\boldsymbol{\theta}}.$$

The axial flux through the vortex is given by

$$\int_{R^2} H\, dS = \frac{2\pi}{\kappa},$$

i.e. the vortex contains one quantum of flux. Abrikosov determines the lower critical field h_{c_1}, at which the superconductor passes from the mixed state into the purely superconducting state, on the assumption that the transition is of second order (which seems to be the case at least for large κ), by equating the free energy of a superconductor with a single vortex to that of one with no vortices. He finds that, for $\kappa \gg 1$,

$$h_{c_1} \approx \frac{1}{2\kappa}(\log\kappa + 0.08).$$

This completes the description of the diagram Fig. 3.3.

There is an immediate generalisation of equations (7.150)-(7.153) to a vortex containing n quanta of flux. Then

$$\Psi = f(r)e^{in\theta}, \qquad Q = A - \frac{n}{\kappa r},$$

and the boundary condition (7.153) is modified to $Q \sim -n/\kappa r$ as $r \to 0$. In this case [7] have shown the existence of a C^2 solution on R^2, which is C^∞ on $R^2\backslash\{0\}$. They also prove that as $\kappa \to \infty$, $\kappa H \to G$, in a suitable function space, where G is the Green's function satisfing the *linear* equation

$$\nabla^2 G - G = -2\pi n\delta(r), \qquad (7.154)$$

$$G \to 0, \qquad \text{as } r \to \infty. \qquad (7.155)$$

Hence as $\kappa \to \infty$ the vortices play the rôle of singularities in the equation for the magnetic field. Since the equation is now linear we may add the contribution from several vortices to obtain

$$\nabla^2 H - H = -\frac{2\pi}{\kappa} \sum_j n_j \, \delta(\boldsymbol{x} - \boldsymbol{x}_j),$$

where the $\boldsymbol{x}_j$ is the position of the jth vortex, with vortex number n_j. This equation has solution

$$H = \frac{1}{\kappa} \sum_j n_j \, K_0(|\,\boldsymbol{x} - \boldsymbol{x}_j\,|),$$

where K_0 is the Hankel function of imaginary argument. In this limit the 'force' on a single quantum flux line can be calculated [60], and is given by

$$\boldsymbol{F} = -\boldsymbol{j} \wedge \boldsymbol{\Phi}', \tag{7.156}$$

where $\boldsymbol{j}$ is the total current density excluding the current due to the vortex in question, but including any applied current, and $\boldsymbol{\Phi}'$ is a vector in the direction of the flux line and one quantum of flux in magnitude (this is simply the Lorentz force). Thus, unless the total superconducting current density of the other vortices is zero the vortex will move. Such a situation can be achieved by regular arrays of vortices as discussed above. It is also in agreement with the fact that a triangular array of vortices is preferred to a square array, since under a repulsive force the vortices will try to maximise their nearest neighbour distance.

Moreover, even the triangular array will feel a force transverse to any applied current, so that the vortices will move unless 'pinned' in place by inhomogeneities in the medium. Flux motion is accompanied by a longitudinal electric field which leads to energy dissipation and an effective resistance of the wire. This situation has been modelled in [35] for fields near to the critical field h_{c_2}, using the time-dependent Ginzburg-Landau equations (3.59)-(3.60). In practice resistance will not return to the wire until the Lorentz force exceeds the pinning force on the vortices. Since in practical applications superconductors are required to carry high currents with very little resistance, this pinning force needs to be made as large as possible by introducing great numbers of imperfections into the material. The modelling of the movement of vortices through such 'dirty' materials is a difficult open question, for which good experimental data is as yet rather sparse.

The 'force' in equation (7.156) implies that in the limit $\kappa \to \infty$ two superconducting vortices will repel each other. This is in agreement with [36] who show numerically that the free energy of two fixed vortices increases with their separation for $\kappa < 1/\sqrt{2}$ and decreases with their separation for $\kappa > 1/\sqrt{2}$, implying that vortices will repel each other for $\kappa > 1/\sqrt{2}$, while for $\kappa < 1/\sqrt{2}$ they will attract each other. When $\kappa = 1/\sqrt{2}$ vortices neither attract nor repel each other, and in this case multivortex solutions to the equations have been shown to exist [58, 67]. However, numerical simulations [51] indicate that moving vortices which collide do interact non-trivially, even when $\kappa = 1/\sqrt{2}$ (the vortices seem to separate at rightangles to their original path of approach).

Finally, we note that when $\kappa = 1/\sqrt{2}$, solutions to equations (7.150), (7.151) are given by solutions of the following pair of first order equations:

$$\sqrt{2}\frac{df}{dr} = -fQ, \tag{7.157}$$

$$\frac{\sqrt{2}}{r}\frac{d}{dr}(rQ) = 1 - f^2. \tag{7.158}$$

As in the one-dimensional case such a reduction relies on the application of compatible boundary conditions. We see that in the present situation, (7.152), (7.153) are compatible with (7.157), (7.158). Using these reduced equations Abrikosov has shown that when $\kappa = 1/\sqrt{2}$, $h_{c_1} = 1/\sqrt{2} = h_{c_2} = H_c$ [2]. We consider further this reduction of the equations when $\kappa = 1/\sqrt{2}$ in Appendix B.

Chapter 8

Nucleation of superconductivity with decreasing temperature

We consider here the effects of placing a superconducting body in the normal state in an applied magnetic field, and then lowering its temperature. We consider only the isothermal case and treat the temperature as a parameter. In the time-dependent case this simplification requires that the effects of the latent heat and joule heating are negligible. Furthermore, we take the temperature to be close to the critical temperature, so that we may linearise the equations in T (this simplifies the analysis, although the same methods work in the more general case).

8.1 Superconductivity in a body of arbitrary shape in an external magnetic field

This problem was considered in [49] using bifurcation theory. Here we use the systematic perturbation theory of the previous chapters to examine the nucleation of superconductivity with decreasing temperature.

Consider a superconducting body occupying a region Ω bounded by a surface $\partial\Omega$, placed in an originally uniform magnetic field h. We work on the lengthscale of the penetration depth by rescaling length and $\boldsymbol{A}$ with λ. The steady-state Ginzburg-Landau equations, together with boundary and other conditions, (3.23)-(3.28), (3.30), are then

$$((i/\kappa)\nabla + \boldsymbol{A})^2\Psi = -\Psi(T + |\Psi|^2), \quad \text{in } \Omega, \tag{8.1}$$

$$-(\text{curl})^2\boldsymbol{A} = (i/2\kappa)(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) + |\Psi|^2 \boldsymbol{A}, \quad \text{in } \Omega, \tag{8.2}$$

$$(\text{curl})^2 \boldsymbol{A} = \boldsymbol{0}, \quad \text{outside } \Omega, \tag{8.3}$$

$$\boldsymbol{n} \cdot ((i/\kappa)\nabla + \boldsymbol{A})\Psi = -(i/d)\Psi, \quad \text{on } \partial\Omega, \tag{8.4}$$

$$[\boldsymbol{n} \wedge \boldsymbol{A}] = \boldsymbol{0}, \tag{8.5}$$

$$[\boldsymbol{n} \wedge (1/\mu)\text{curl}\boldsymbol{A}] = \boldsymbol{0}, \tag{8.6}$$

$$\text{curl}\boldsymbol{A} \to h\hat{\boldsymbol{z}}, \quad \text{as } r \to \infty. \tag{8.7}$$

Here, as before, $\hat{\boldsymbol{z}}$ is a unit vector in the z-direction, r is the distance from the origin, $\boldsymbol{n}$ is the outward normal on $\partial\Omega$, and $[\,]$ stands for the jump in the enclosed quantity across $\partial\Omega$. As in Chapter 5 we impose the gauge condition

$$\text{div } \boldsymbol{A} = 0, \tag{8.8}$$

which proves convenient in later calculations. The solution of (8.1)-(8.8) which corresponds to the normal state is

$$\Psi \equiv 0, \ \boldsymbol{A} = h\boldsymbol{A}_N, \tag{8.9}$$

where $\boldsymbol{A}_N$, as before, satisfies

$$(\text{curl})^2 \boldsymbol{A}_N = \boldsymbol{0}, \text{ except on } \partial\Omega, \tag{8.10}$$

$$[\boldsymbol{n} \wedge \boldsymbol{A}_N] = \boldsymbol{0}, \tag{8.11}$$

$$[\boldsymbol{n} \wedge (1/\mu)\text{curl } \boldsymbol{A}_N] = \boldsymbol{0}, \tag{8.12}$$

$$\text{curl } \boldsymbol{A}_N \to \hat{\boldsymbol{z}}, \text{ as } r \to \infty, \tag{8.13}$$

$$\text{div } \boldsymbol{A}_N = 0. \tag{8.14}$$

We now seek a superconducting solution (i.e. one in which $\Psi \not\equiv 0$) which depends continuously on a parameter ϵ, and which reduces to (8.9) for $\epsilon = 0$. As before we introduce ϵ, ψ and $\boldsymbol{a}$ through the equations

$$\Psi = \epsilon^{1/2}\psi, \tag{8.15}$$

$$\boldsymbol{A} = h\boldsymbol{A}_N + \epsilon\boldsymbol{a}, \ \epsilon > 0. \tag{8.16}$$

Insertion of (8.15), (8.16) into (8.1)-(8.8) yields

$$\begin{aligned} ((i/\kappa)\nabla + h\boldsymbol{A}_N)^2\Psi + T\psi = & -\epsilon[|\psi|^2 \psi + 2h\psi(\boldsymbol{A}_N \cdot \boldsymbol{a}) + 2(i/\kappa)(\boldsymbol{a} \cdot \nabla\psi)] \\ & - \epsilon^2 |\boldsymbol{a}|^2 \psi, \text{ in } \Omega, \end{aligned} \tag{8.17}$$

$$
\begin{aligned}
-(\text{curl})^2\boldsymbol{a} &= (1/2\kappa)(\psi^*\nabla\psi - \psi\nabla\psi^*) \\
&\quad + |\psi|^2\ (h\boldsymbol{A}_N + \epsilon\boldsymbol{a}),\ \text{in } \Omega, && (8.18) \\
(\text{curl})^2\boldsymbol{a} &= \boldsymbol{0},\ \text{outside } \Omega, && (8.19) \\
\boldsymbol{n}\cdot((i/\kappa)\nabla + h\boldsymbol{A}_N)\psi + (i/d)\psi &= -\epsilon(\boldsymbol{n}\cdot\boldsymbol{a})\psi,\ \text{on } \partial\Omega, && (8.20) \\
[\boldsymbol{n}\wedge\boldsymbol{a}] &= \boldsymbol{0}, && (8.21) \\
[\boldsymbol{n}\wedge(1/\mu)\text{curl } \boldsymbol{a}] &= \boldsymbol{0}, && (8.22) \\
\text{curl } \boldsymbol{a} &\to \boldsymbol{0},\ \text{as } r\to\infty, && (8.23) \\
\text{div } \boldsymbol{a} &= 0. && (8.24)
\end{aligned}
$$

We expand T, $\boldsymbol{a}$ and ψ in powers of ϵ

$$
\begin{aligned}
T &= T^{(0)} + \epsilon T^{(1)} + \cdots, && (8.25) \\
\boldsymbol{a} &= \boldsymbol{a}^{(0)} + \epsilon\boldsymbol{a}^{(1)} + \cdots, && (8.26) \\
\psi &= \psi^{(0)} + \epsilon\psi^{(1)} + \cdots. && (8.27)
\end{aligned}
$$

The problem is now to determine the coefficients in these expansions. We substitute the expansions (8.25)-(8.27) into equations (8.17)-(8.24) and equate powers of ϵ. At leading order we have

$$
\begin{aligned}
((i/\kappa)\nabla + h\boldsymbol{A}_N)^2\psi^{(0)} + T^{(0)}\psi^{(0)} &= 0,\ \text{in } \Omega, && (8.28) \\
-(\text{curl})^2\boldsymbol{a}^{(0)} &= (i/2\kappa)(\psi^{(0)*}\nabla\psi^{(0)} - \psi^{(0)}\nabla\psi^{(0)*}) \\
&\quad + h\,|\psi^{(0)}|^2\ \boldsymbol{A}_N,\ \text{in } \Omega, && (8.29) \\
(\text{curl})^2\boldsymbol{a}^{(0)} &= \boldsymbol{0},\ \text{outside } \Omega, && (8.30) \\
\boldsymbol{n}\cdot((i/\kappa)\nabla + h\boldsymbol{A}_N)\psi^{(0)} &= -(i/d)\psi^{(0)},\ \text{on } \partial\Omega, && (8.31) \\
[\boldsymbol{n}\wedge\boldsymbol{a}^{(0)}] &= \boldsymbol{0}, && (8.32) \\
[\boldsymbol{n}\wedge(1/\mu)\text{curl } \boldsymbol{a}^{(0)}] &= \boldsymbol{0}, && (8.33) \\
\text{curl } \boldsymbol{a}^{(0)} &\to \boldsymbol{0},\ \text{as } r\to\infty, && (8.34) \\
\text{div } \boldsymbol{a}^{(0)} &= 0. && (8.35)
\end{aligned}
$$

The self-adjoint eigenvalue problem (8.28) and (8.31) determines a discrete set of eigenvalues for $T^{(0)}$ which are independent of the gauge of $\boldsymbol{A}_N$. The critical temperature T_{c_2} is defined to be the largest of these eigenvalues. Let the normalised eigenfunction corresponding to $T^{(0)}$ be θ, i.e. θ is such that

$$\int_{\Omega} |\theta|^2 \; dV = 1.$$

Then $\psi^{(0)} = \beta\theta$ where β is constant, and $\boldsymbol{a}^{(0)} = |\beta|^2 \; \hat{\boldsymbol{a}}^{(0)}$, where

$$\begin{aligned}
-(\text{curl})^2\hat{\boldsymbol{a}}^{(0)} &= (i/2\kappa)(\theta^*\nabla\theta - \theta\nabla\theta^*) + h\,|\theta|^2\,\boldsymbol{A}_N, \text{ in } \Omega, && (8.36)\\
(\text{curl})^2\hat{\boldsymbol{a}}^{(0)} &= \boldsymbol{0}, \text{ outside } \Omega, && (8.37)\\
[\boldsymbol{n}\wedge\hat{\boldsymbol{a}}^{(0)}] &= \boldsymbol{0}, && (8.38)\\
[\boldsymbol{n}\wedge(1/\mu)\text{curl }\hat{\boldsymbol{a}}^{(0)}] &= \boldsymbol{0}, && (8.39)\\
\text{curl }\hat{\boldsymbol{a}}^{(0)} &\to \boldsymbol{0}, \text{ as } r\to\infty, && (8.40)\\
\text{div }\hat{\boldsymbol{a}}^{(0)} &= 0. && (8.41)
\end{aligned}$$

which is the problem of determining the vector potential $\hat{\boldsymbol{a}}^{(0)}$ due to a permeable body carrying a specified real current distribution, since the right-hand side of (8.36) is known. Again, well known methods of solution are available.

We have now determined the critical temperature T_{c_2} and leading order approximations to ψ and $\boldsymbol{a}$, once we have determined the constant β.

Equating coefficients of ϵ in (8.17)-(8.24) yields

$$\begin{aligned}
((i/\kappa)\nabla + h\boldsymbol{A}_N)^2\psi^{(1)} + T^{(0)}\psi^{(1)} &= -T^{(1)}\psi^{(0)} - |\psi^{(0)}|^2\,\psi^{(0)}\\
&\quad - 2h\psi^{(0)}(\boldsymbol{A}_N\cdot\boldsymbol{a}^{(0)})\\
&\quad - 2(i/\kappa)(\boldsymbol{a}^{(0)}\cdot\nabla\psi^{(0)}), \text{ in } \Omega, && (8.42)\\
-(\text{curl})^2\boldsymbol{a}^{(1)} &= (i/2\kappa)(\psi^{(0)*}\nabla\psi^{(1)} + \psi^{(1)*}\nabla\psi^{(0)})\\
&\quad - (i/2\kappa)(\psi^{(0)}\nabla\psi^{(1)*} + \psi^{(1)}\nabla\psi^{(0)*})\\
&\quad + h\boldsymbol{A}_N(\psi^{(0)}\psi^{(1)*} + \psi^{(0)*}\psi^{(1)})\\
&\quad + |\psi^{(0)}|^2\,\boldsymbol{a}^{(0)}, \text{ in } \Omega, && (8.43)\\
(\text{curl})^2\boldsymbol{a}^{(1)} &= \boldsymbol{0}, \text{ outside } \Omega, && (8.44)\\
\boldsymbol{n}\cdot((i/\kappa)\nabla + h\boldsymbol{A}_N)\psi^{(1)} + (i/d)\psi^{(1)} &= -(\boldsymbol{n}\cdot\boldsymbol{a}^{(0)})\psi^{(0)}, \text{ on } \partial\Omega, && (8.45)\\
[\boldsymbol{n}\wedge\boldsymbol{a}^{(1)}] &= \boldsymbol{0}, && (8.46)\\
[\boldsymbol{n}\wedge(1/\mu)\text{curl }\boldsymbol{a}^{(1)}] &= \boldsymbol{0}, && (8.47)\\
\text{curl }\boldsymbol{a}^{(1)} &\to \boldsymbol{0}, \text{ as } r\to\infty, && (8.48)\\
\text{div }\boldsymbol{a}^{(1)} &= 0. && (8.49)
\end{aligned}$$

Assuming $\psi^{(1)}$ and $T^{(1)}$ were known these equations would again correspond to the problem of determining the vector potential due to a permeable body carrying a known current distribution. Thus $\boldsymbol{a}^{(1)}$ is fixed once $\psi^{(1)}$ and $T^{(1)}$ are given.

Now (8.42) and (8.45) are inhomogeneous versions of (8.28) and (8.31) and therefore have a solution if and only if an appropriate solvability condition is satisfied. This condition is derived by multiplying both sides of (8.42) by $\psi^{(0)*}$ and integrating over Ω. We find

$$\begin{aligned}
\text{LHS} &= \int_\Omega \psi^{(0)*}\left[-(1/\kappa^2)\nabla^2\psi^{(1)} + (2i/\kappa)h\boldsymbol{A}_N\cdot\nabla\psi^{(1)}\right]dV \\
&\quad + \int_\Omega \psi^{(0)*}\left[h^2\mid\boldsymbol{A}_N\mid^2\psi^{(1)} + T^{(0)}\psi^{(1)}\right]dV, \\
&= \int_\Omega \psi^{(1)}\left[-(1/\kappa^2)\nabla^2\psi^{(0)*} - (2i/\kappa)h\boldsymbol{A}_N\cdot\nabla\psi^{(0)*}\right]dV \\
&\quad + \int_\Omega \psi^{(1)}\left[h^2\mid\boldsymbol{A}_N\mid^2\psi^{(0)*} + T^{(0)}\psi^{(0)*}\right]dV, \\
&\quad + \int_{\partial\Omega}\left[-(1/\kappa^2)(\psi^{(0)*}\nabla\psi^{(1)} - \psi^{(1)}\nabla\psi^{(0)*}) + (2i/\kappa)h\psi^{(0)*}\psi^{(1)}\boldsymbol{A}_N\right]\cdot\boldsymbol{n}\,dS,
\end{aligned}$$

by Greens Theorem,

$$\begin{aligned}
&= (i/\kappa)\int_{\partial\Omega}\left[(i/\kappa)\nabla\psi^{(1)} + h\psi^{(1)}\boldsymbol{A}_N\right]\psi^{(0)*}\cdot\boldsymbol{n}\,dS \\
&\quad +(i/\kappa)\int_{\partial\Omega}\left[-(i/\kappa)\nabla\psi^{(0)*} + h\psi^{(0)*}\boldsymbol{A}_N\right]\psi^{(1)}\cdot\boldsymbol{n}\,dS,
\end{aligned}$$

since the integral over Ω is zero by (8.28),

$$\begin{aligned}
&= (i/\kappa)\int_{\partial\Omega}\psi^{(0)*}\left[-(i/d)\psi^{(1)} - (\boldsymbol{n}\cdot\boldsymbol{a}_0^{(0)})\psi^{(0)}\right]dS \\
&\quad +(i/\kappa)\int_{\partial\Omega}(i/d)\psi^{(0)*}\psi^{(1)}\,dS,
\end{aligned}$$

by (8.31) and (8.45),

$$= -(i/\kappa)\int_{\partial\Omega}\mid\psi^{(0)}\mid^2(\boldsymbol{a}_0^{(0)}\cdot\boldsymbol{n})\,dS.$$

$$\begin{aligned}
\text{RHS} &= -\int_\Omega\mid\psi^{(0)}\mid^4 + T^{(1)}\mid\psi^{(0)}\mid^2 + (2i/\kappa)\psi^{(0)*}\boldsymbol{a}_0^{(0)}\cdot\nabla\psi^{(0)}\,dV \\
&\quad - 2\int_\Omega h\mid\psi^{(0)}\mid^2\boldsymbol{A}_N\cdot\boldsymbol{a}_0^{(0)}\,dV, \\
&= -\mid\beta\mid^2 T^{(1)} - \int_\Omega\mid\psi^{(0)}\mid^4 + (2i/\kappa)\psi^{(0)*}\nabla\psi^{(0)}\cdot\boldsymbol{a}_0^{(0)}\,dV \\
&\quad + 2\int_\Omega\boldsymbol{a}_0^{(0)}\cdot(\text{curl}^2\boldsymbol{a}_0^{(0)} + (i/2\kappa)(\psi^{(0)*}\nabla\psi^{(0)} - \psi^{(0)}\nabla\psi^{(0)*}))\,dV,
\end{aligned}$$

by (8.29),

$$
\begin{aligned}
&= -\mid\beta\mid^2 T^{(1)} \\
&\quad - \int_\Omega \mid\psi^{(0)}\mid^4 + \boldsymbol{a}_0^{(0)} \cdot ((i/\kappa)(\psi^{(0)*}\nabla\psi^{(0)} + \psi^{(0)}\nabla\psi^{(0)*}) - 2\text{curl}^2\boldsymbol{a}_0^{(0)})\, dV, \\
&= -\mid\beta\mid^2 T^{(1)} - \int_\Omega \mid\psi^{(0)}\mid^4 + \boldsymbol{a}_0^{(0)} \cdot ((i/\kappa)\nabla \mid\psi^{(0)}\mid^2 - 2\text{curl}^2\boldsymbol{a}_0^{(0)})\, dV, \\
&= -\mid\beta\mid^2 T^{(1)} + \int_\Omega - \mid\psi^{(0)}\mid^4 + 2(\text{curl})^2\boldsymbol{a}_0^{(0)} \cdot \boldsymbol{a}_0^{(0)}\, dV, \\
&\quad - (i/\kappa)\int_{\partial\Omega} \mid\psi^{(0)}\mid^2 (\boldsymbol{a}_0^{(0)} \cdot \boldsymbol{n})\, dS,
\end{aligned}
$$

by the divergence theorem, since div $\boldsymbol{A}_N$ and div $\boldsymbol{a}_0^{(0)}$ are both zero. Equating the left-hand side to the right-hand side we have

$$
\mid\beta\mid^2 T^{(1)} = \int_\Omega - \mid\psi^{(0)}\mid^4 + 2\boldsymbol{a}_0^{(0)} \cdot (\text{curl})^2\boldsymbol{a}_0^{(0)}\, dV,
$$

or

$$
T^{(1)} = \mid\beta\mid^2 \int_\Omega - \mid\theta\mid^4 + 2\hat{\boldsymbol{a}}_0^{(0)} \cdot (\text{curl})^2\hat{\boldsymbol{a}}_0^{(0)}\, dV. \tag{8.50}
$$

Thus $\mid\beta\mid^2$ is proportional to $T^{(1)}$. Note that in order for a superconducting solution to exist this equation also determines the sign of $T^{(1)}$. When $\mid\beta\mid^2$ is given by (8.50), (8.42) and (8.45) have a solution $\psi^{(1)}$, which will still contain an undertermined constant. This constant is determined by a solvability condition for the second-order terms.

We have now determined a solution in the form

$$
\begin{aligned}
T &= T^{(0)} + \epsilon T^{(1)} + \cdots & (8.51)\\
\Psi &= \epsilon^{1/2}[\psi^{(0)} + \epsilon\psi^{(1)} + \cdots] & (8.52)\\
\boldsymbol{A} &= h\boldsymbol{A}_N + \epsilon[\boldsymbol{a}^{(0)} + \epsilon\boldsymbol{a}^{(1)} + \cdots] & (8.53)
\end{aligned}
$$

As before, equation (8.53) leads to a magnetic field

$$
\text{curl } \boldsymbol{A} = \boldsymbol{H} = h\,\text{curl } \boldsymbol{A}_N + \epsilon\,\text{curl } \boldsymbol{a}^{(0)} + \cdots \tag{8.54}
$$

If $T^{(1)} < 0$ we have a solution for all values of the external field slightly below a certain critical value T_{c_2}. Notice that when $\int_\Omega \boldsymbol{A}_N \cdot (\text{curl})^2\boldsymbol{a}^{(0)}\, dV < 0$, which is the case in one dimension and which we expect to be true in all cases, then $T^{(1)} < 0$ if and only if $h^{(1)}$ given by (5.50) is < 0. Thus if the solution at given temperature exists for all fields less than a certain critical value, then the solution at for given field will exist for all temperatures less than a given value.

Since θ is normalised we have that $\|\Psi\| = \left(\int_\Omega \mid\Psi\mid^2\, dV\right)^{1/2} = \mid\beta\mid \epsilon^{1/2}$ and so $\|\Psi\|$ increases as $\mid T - T^{(0)}\mid^{1/2}$ for T close to $T^{(0)}$, as shown in Fig. 8.1.

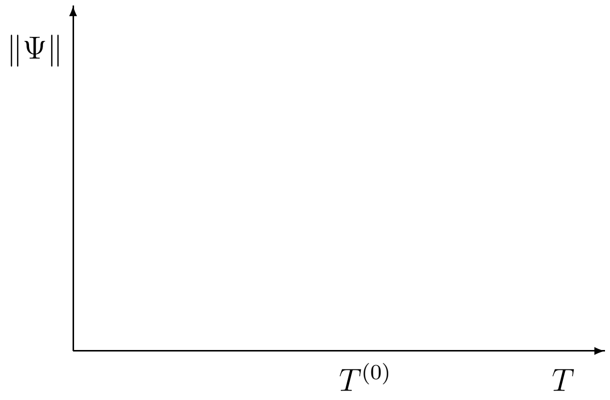

Figure 8.1: Pitchfork bifurcation from $\Psi \equiv 0$ at $T = T^{(0)}$.

8.2 Linear Stability of the Solution Branches

Let us now determine the linear stability of the solution branches in Fig. 8.1.

8.2.1 Linear Stability of the Normal State

We examine first the stability of the normal state. As in Chapter 5 we work on the lengthscale of the penetration depth by rescaling length and $\boldsymbol{A}$ with λ, and on the timescale of the relaxation of the order parameter by rescaling time with λ^2. The time-dependent Ginzburg-Landau equations, linearised in T, together with boundary and other conditions, (3.69)-(3.78) are then

$$\begin{aligned}
\frac{\alpha}{\kappa^2}\frac{\partial \Psi}{\partial t} + \frac{\alpha i}{\kappa}\Psi\Phi + \left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2 \Psi &= -\Psi(T + |\Psi|^2), \text{ in } \Omega, && (8.55)\\
-(\mathrm{curl})^2\boldsymbol{A} - \frac{\partial \boldsymbol{A}}{\partial t} - \nabla\Phi &= \frac{i}{2\kappa}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) \\
&\quad + |\Psi|^2 \boldsymbol{A}, \text{ in } \Omega, && (8.56)\\
-(\mathrm{curl})^2\boldsymbol{A} &= \varsigma_e\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right), \text{ outside } \Omega, && (8.57)\\
\nabla^2\Phi &= 0, \text{ outside } \Omega, && (8.58)\\
\boldsymbol{n}\cdot((i/\kappa)\nabla + \boldsymbol{A})\Psi + (i/d)\Psi &= 0, \text{ on } \partial\Omega, && (8.59)\\
[\boldsymbol{n}\wedge\boldsymbol{A}] &= \boldsymbol{0}, && (8.60)\\
[\boldsymbol{n}\wedge(1/\mu)\mathrm{curl}\ \boldsymbol{A}] &= \boldsymbol{0}, && (8.61)\\
[\Phi] &= 0, && (8.62)\\
\left[\varepsilon\frac{\partial\Phi}{\partial n}\right] &= 0, && (8.63)
\end{aligned}$$

$$\begin{aligned}
\text{curl } \boldsymbol{A} &\rightarrow h\hat{\boldsymbol{z}}, \text{ as } r \rightarrow \infty, && (8.64) \\
\Phi &\rightarrow 0, \text{ as } r \rightarrow \infty, && (8.65) \\
\text{div } \boldsymbol{A} &= 0. && (8.66)
\end{aligned}$$

We make a small perturbation about the normal solution (8.9), by setting

$$\begin{aligned}
\Psi &= \delta e^{\sigma t}\Psi_1, && (8.67) \\
\boldsymbol{A} &= h\boldsymbol{A}_N + \delta e^{\sigma t}\boldsymbol{A}_1, && (8.68) \\
\Phi &= \delta e^{\sigma t}\Phi_1, \quad 0 < \delta \ll 1. && (8.69)
\end{aligned}$$

Substituting (8.67)-(8.69) into (8.55)-(8.66) and linearising in δ yields

$$\begin{aligned}
\frac{\alpha}{\kappa^2}\sigma\Psi_1 + \left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)^2\Psi_1 &= -T\Psi_1, \text{ in } \Omega, && (8.70) \\
-(\text{curl})^2\boldsymbol{A}_1 &= \sigma\boldsymbol{A}_1 + \nabla\Phi_1, \text{ in } \Omega, && (8.71) \\
-(\text{curl})^2\boldsymbol{A}_1 &= \varsigma_e(\sigma\boldsymbol{A}_1 + \nabla\Phi_1), \text{ outside } \Omega, && (8.72) \\
\nabla^2\Phi_1 &= 0, \text{ outside } \Omega, && (8.73) \\
\boldsymbol{n}\cdot((i/\kappa)\nabla + h\boldsymbol{A}_N)\Psi_1 + (i/d)\Psi_1 &= 0, \text{ on } \partial\Omega, && (8.74) \\
[\boldsymbol{n}\wedge\boldsymbol{A}_1] &= \boldsymbol{0}, && (8.75) \\
[\boldsymbol{n}\wedge(1/\mu)\text{curl } \boldsymbol{A}_1] &= \boldsymbol{0}, && (8.76) \\
[\Phi_1] &= 0, && (8.77) \\
\left[\varepsilon\frac{\partial\Phi_1}{\partial n}\right] &= 0, && (8.78) \\
\text{curl } \boldsymbol{A}_1 &\rightarrow \boldsymbol{0} \text{ as } r \rightarrow \infty, && (8.79) \\
\Phi_1 &\rightarrow 0 \text{ as } r \rightarrow \infty, && (8.80) \\
\text{div } \boldsymbol{A}_1 &= 0. && (8.81)
\end{aligned}$$

For each given T (8.70) and (8.74) determine a discrete set of eigenvalues for σ. For $T > T_{c_2}$ all these eigenvalues will be negative. For $T < T_{c_2}$ at least one of these eigenvalues will be positive, indicating instability. Thus the normal state is linearly stable for $T > T_{c_2}$ and linearly unstable for $T < T_{c_2}$.

8.2.2 Stability of the superconducting branch

We now consider a small perturbation of the previously found superconducting solution. We set

$$\Psi = \Psi_0 + \delta e^{\sigma t}\Psi_1, \tag{8.82}$$

$$\boldsymbol{A} = \boldsymbol{A}_0 + \delta e^{\sigma t}\boldsymbol{A}_1, \tag{8.83}$$

$$\Phi = \delta\Phi_1 e^{\sigma t}, \quad 0 < \delta \ll 1, \tag{8.84}$$

where $(\Psi_0, \boldsymbol{A}_0)$ is the steady superconducting solution given by (8.51)-(8.53). Substituting (8.82)-(8.84) into (8.55)-(8.66) and linearising in δ yields

$$\begin{aligned}
\frac{\alpha}{\kappa^2}\sigma\Psi_1 + \frac{\alpha}{\kappa}i\Psi_0\Phi_1 + \left(\frac{i}{\kappa}\nabla + \boldsymbol{A}_0\right)^2 \Psi_1 + \frac{2i}{\kappa}\boldsymbol{A}_1\cdot\nabla\Psi_0 + 2\boldsymbol{A}_0\cdot\boldsymbol{A}_1\Psi_0 = \\
-T\Psi_1 - 2\,|\,\Psi_0\,|^2\;\Psi_1 - \Psi_0^2\Psi_1^*, \text{ in } \Omega,
\end{aligned} \tag{8.85}$$

$$\begin{aligned}
-(\text{curl})\boldsymbol{A}_1 - \sigma\boldsymbol{A}_1 - \nabla\Phi_1 &= (i/2\kappa)(\Psi_0^*\nabla\Psi_1 + \Psi_1^*\nabla\Psi_0) \\
&\quad - (i/2\kappa)(\Psi_0\nabla\Psi_1^* + \Psi_1\nabla\Psi_0^*) \\
&\quad + (\Psi_0\Psi_1^* + \Psi_0^*\Psi_1)\boldsymbol{A}_0 + |\,\Psi_0\,|^2\;\boldsymbol{A}_1, \text{ in } \Omega,
\end{aligned} \tag{8.86}$$

$$-(\text{curl})^2\boldsymbol{A_1} = \varsigma_e(\sigma\boldsymbol{A}_1 + \nabla\Phi_1), \text{ outside } \Omega, \tag{8.87}$$

$$\nabla^2\Phi_1 = 0, \text{ outside } \Omega, \tag{8.88}$$

$$\boldsymbol{n}\cdot((i/\kappa)\nabla + \boldsymbol{A}_0)\Psi_1 = -(i/d)\Psi_1 - \boldsymbol{n}\cdot\boldsymbol{A}_1\Psi_0, \text{ on } \partial\Omega, \tag{8.89}$$

$$[\boldsymbol{n}\wedge\boldsymbol{A}_1] = \boldsymbol{0}, \tag{8.90}$$

$$[\boldsymbol{n}\wedge(1/\mu)\text{curl }\boldsymbol{A}_1] = \boldsymbol{0}, \tag{8.91}$$

$$[\Phi_1] = 0, \tag{8.92}$$

$$\left[\varepsilon\frac{\partial\Phi_1}{\partial n}\right] = 0, \tag{8.93}$$

$$\text{curl }\boldsymbol{A_1} \to \boldsymbol{0} \text{ as } r\to\infty, \tag{8.94}$$

$$\Phi_1 \to 0 \text{ as } r\to\infty, \tag{8.95}$$

$$\text{div }\boldsymbol{A_1} = 0. \tag{8.96}$$

We examine the stability close to the bifurcation point by introducing ϵ as before

$$\Psi_0 = \epsilon^{1/2}\psi_0, \tag{8.97}$$

$$\boldsymbol{A}_0 = h\boldsymbol{A}_N + \epsilon\boldsymbol{a}_0, \tag{8.98}$$

$$\Psi_1 = \epsilon^{1/2}\psi_1, \tag{8.99}$$

$$\boldsymbol{A}_1 = \epsilon \boldsymbol{a}_1, \tag{8.100}$$

$$\Phi_1 = \epsilon\phi_1. \tag{8.101}$$

Substituting (8.97)-(8.101) into (8.85)-(8.96) yields

$$\begin{aligned}
(\alpha\sigma/\kappa^2)\psi_1 + T\psi_1 + ((i/\kappa) + h\boldsymbol{A}_N)^2\psi_1 &= -(\epsilon\alpha i/\kappa)\psi_0\phi_1 - 2\epsilon\,|\psi_0|^2\,\psi_1 \\
&\quad - \epsilon\psi_1^*\psi_0^2 - (2\epsilon i/\kappa)\boldsymbol{a}_0\cdot\nabla\psi_1 \\
&\quad - 2\epsilon h\boldsymbol{a}_0\cdot\boldsymbol{A}_N\psi_1 - (2\epsilon i/\kappa)\boldsymbol{a}_1\cdot\nabla\psi_0 \\
&\quad - 2\epsilon h\boldsymbol{A}_N\cdot\boldsymbol{a}_1\psi_0 - \epsilon^2\,|\boldsymbol{a}_0|^2\,\psi_1 \\
&\quad - 2\epsilon^2\boldsymbol{A}_0\cdot\boldsymbol{a}_1\psi_0, \text{ in } \Omega, \qquad (8.102)\\
-(\mathrm{curl})^2\boldsymbol{a}_1 - \sigma\boldsymbol{a}_1 - \nabla\phi_1 &= (i/2\kappa)(\psi_0^*\nabla\psi_1 + \psi_1^*\nabla\psi_0) \\
&\quad - (i/2\kappa)(\psi_0\nabla\psi_1^* + \psi_1\nabla\psi_0^*) \\
&\quad + (\psi_0\psi_1^* + \psi_0^*\psi_1)h\boldsymbol{A}_N \\
&\quad \epsilon(\psi_0\psi_1^* + \psi_0^*\psi_1)\boldsymbol{a}_0 \\
&\quad + \epsilon\,|\psi_0|^2\,\boldsymbol{a}_1, \text{ in } \Omega, \qquad (8.103)\\
-(\mathrm{curl})^2\boldsymbol{a}_1 &= \varsigma_e(\sigma\boldsymbol{a}_1 + \nabla\phi_1), \text{ outside } \Omega, \qquad (8.104)\\
\nabla^2\phi_1 &= 0, \text{ outside } \Omega, \qquad (8.105)\\
\boldsymbol{n}\cdot((i/\kappa)\nabla + h\boldsymbol{A}_N)\psi_1 + (i/d)\psi_1 &= -\epsilon\boldsymbol{n}\cdot\boldsymbol{a}_1\psi_0 \\
&\quad - \epsilon\boldsymbol{n}\cdot\boldsymbol{a}_0\psi_1, \text{ on } \partial\Omega, \qquad (8.106)\\
[\boldsymbol{n}\wedge\boldsymbol{a}_1] &= \boldsymbol{0}, \qquad (8.107)\\
[\boldsymbol{n}\wedge(1/\mu)\mathrm{curl}\ \boldsymbol{a}_1] &= \boldsymbol{0}, \qquad (8.108)\\
[\phi_1] &= 0, \qquad (8.109)\\
\left[\varepsilon\frac{\partial\phi_1}{\partial n}\right] &= 0, \qquad (8.110)\\
\mathrm{curl}\ \boldsymbol{a}_1 &\to \boldsymbol{0} \text{ as } r\to\infty, \qquad (8.111)\\
\phi_1 &\to 0 \text{ as } r\to\infty, \qquad (8.112)\\
\mathrm{div}\ \boldsymbol{a}_1 &= 0. \qquad (8.113)
\end{aligned}$$

In the previous section we obtained expansions in powers of ϵ for $\boldsymbol{a}_0$, ψ_0 and T near $T = T^{(0)}$. We expand also $\boldsymbol{a}_1$, ψ_1, ϕ_1 and σ in powers of ϵ to give

$$T = T^{(0)} + \epsilon T^{(1)} + \cdots, \tag{8.114}$$

$$\psi_0 = \psi_0^{(0)} + \epsilon\psi_0^{(1)} + \cdots, \tag{8.115}$$

$$a_0 = a_0^{(0)} + \epsilon a_0^{(1)} + \cdots, \tag{8.116}$$

$$\psi_1 = \psi_1^{(0)} + \epsilon\psi_1^{(1)} + \cdots, \tag{8.117}$$

$$a_1 = a_1^{(0)} + \epsilon a_1^{(1)} + \cdots, \tag{8.118}$$

$$\phi_1 = \phi_1^{(0)} + \epsilon\phi_1^{(1)} + \cdots, \tag{8.119}$$

$$\sigma = \sigma^{(0)} + \epsilon\sigma^{(1)} + \cdots. \tag{8.120}$$

Substituting the expansions (8.114)-(8.120) into equations (8.102)-(8.113) and equating powers of ϵ we find at leading order

$$(\alpha\sigma^{(0)}/\kappa^2)\psi_1^{(0)} + T^{(0)}\psi_1^{(0)} = -((i/\kappa) + h\boldsymbol{A}_N)^2\psi_1^{(0)}, \text{ in } \Omega, \tag{8.121}$$

$$\begin{aligned} -(\text{curl})^2\boldsymbol{a}_1^{(0)} - \sigma^{(0)}\boldsymbol{a}_1 - \nabla\phi_1^{(0)} &= (i/2\kappa)(\psi_0^{(0)*}\nabla\psi_1^{(0)} + \psi_1^{(0)*}\nabla\psi_0^{(0)}) \\ &\quad - (i/2\kappa)(\psi_0^{(0)}\nabla\psi_1^{(0)*} + \psi_1^{(0)}\nabla\psi_0^{(0)*}) \\ &\quad + (\psi_0^{(0)}\psi_1^{(0)*} + \psi_0^{(0)*}\psi_1^{(0)})h\boldsymbol{A}_N, \text{ in } \Omega, \end{aligned} \tag{8.122}$$

$$-(\text{curl})^2\boldsymbol{a}_1^{(0)} = \varsigma_e(\sigma^{(0)}\boldsymbol{a}_1 + \nabla\phi_1^{(0)}), \text{ outside } \Omega, \tag{8.123}$$

$$\nabla^2\phi_1^{(0)} = 0, \text{ outside } \Omega, \tag{8.124}$$

$$\boldsymbol{n}\cdot((i/\kappa)\nabla + h\boldsymbol{A}_N)\psi_1^{(0)} = -(i/d)\psi_1^{(0)}, \text{ on } \partial\Omega, \tag{8.125}$$

$$[\boldsymbol{n}\wedge\boldsymbol{a}_1^{(0)}] = \boldsymbol{0}, \tag{8.126}$$

$$[\boldsymbol{n}\wedge(1/\mu)\text{curl } \boldsymbol{a}_1^{(0)}] = \boldsymbol{0}, \tag{8.127}$$

$$[\phi_1^{(0)}] = 0, \tag{8.128}$$

$$\left[\varepsilon\frac{\partial\phi_1^{(0)}}{\partial n}\right] = 0, \tag{8.129}$$

$$\text{curl } \boldsymbol{a}_1^{(0)} \to \boldsymbol{0} \text{ as } r\to\infty, \tag{8.130}$$

$$\phi_1^{(0)} \to 0 \text{ as } r\to\infty, \tag{8.131}$$

$$\text{div } \boldsymbol{a}_1^{(0)} = 0. \tag{8.132}$$

Equations (8.121) and (8.125) are exactly equations (8.70) and (8.74). As before, if $T^{(0)} < T_{c_2}$ then there exists and unstable mode. Hence the solution branches bifurcating from eigenvalues $T^{(0)} < T_{c_2}$ are linearly unstable. It remains to determine the stability of the solution branch bifurcating from $T^{(0)} = T_{c_2}$. When $T = T_{c_2}$ all the eigenvalues for $\sigma^{(0)}$ are negative except for the eigenvalue $\sigma^{(0)} = 0$. We must proceed to higher order in our expansions to determine the stability of this mode. We note that for $\sigma^{(0)} = 0$, $\psi_1^{(0)}$ satisfies the same equation and boundary conditions as $\psi_0^{(0)}$, and hence $\psi_1^{(0)} \propto \psi_0^{(0)}$. Since all the equations are linear in $\psi_1, \boldsymbol{a}_1$ and ϕ_1 by construction, the constant of proportionality is irrelevant and we take it to be unity (in effect this defines δ). Substituting into equations (8.122)-(8.124) and (8.126)-(8.132) we find

$$\begin{aligned} -(\mathrm{curl})^2\boldsymbol{a}_1^{(0)} - \nabla\phi_1^{(0)} &= (i/\kappa)(\psi_0^{(0)*}\nabla\psi_0^{(0)} - \psi_0^{(0)}\nabla\psi_0^{(0)*}) + 2\,|\,\psi_0^{(0)}\,|^2\; h\boldsymbol{A}_N \\ &= -2(\mathrm{curl})^2\boldsymbol{a}_0^{(0)}, \text{ in } \Omega, \end{aligned} \tag{8.133}$$

by equation (8.29). Taking the divergence of this equation we find

$$\nabla^2\phi_1^{(0)} = 0, \text{ in } \Omega.$$

This, together with equation (8.124) and boundary conditions (8.128), (8.129) and (8.131), implies

$$\phi_1^{(0)} \equiv 0.$$

Now by comparing equations (8.133), (8.123), and (8.132) and boundary conditions (8.126), (8.127), and (8.130) with equations (8.29), (8.30), and (8.35) and boundary conditions (8.32)-(8.34) we see that

$$\boldsymbol{a}_1^{(0)} = 2\boldsymbol{a}_0^{(0)}, \tag{8.134}$$

is a solution.

Equating powers of ϵ in equations (8.102) and (8.106) we find

$$\begin{aligned} T^{(0)}\psi_1^{(1)} + ((i/\kappa) + h\boldsymbol{A}_N)^2\psi_1^{(1)} &= -T^{(1)}\psi_1^{(0)} - (\alpha\sigma^{(1)}/\kappa^2)\psi_1^{(0)} - 2\,|\,\psi_0^{(0)}\,|^2\;\psi_1^{(0)} \\ &\quad - \psi_1^{(0)*}(\psi_0^{(0)})^2 - (2i/\kappa)(\boldsymbol{a}_0^{(0)}\cdot\nabla\psi_1^{(0)}) \\ &\quad - 2h(\boldsymbol{a}_0^{(0)}\cdot\boldsymbol{A}_N)\psi_1^{(0)} - (2i/\kappa)(\boldsymbol{a}_1^{(0)}\cdot\nabla\psi_0^{(0)}) \\ &\quad - 2h(\boldsymbol{a}_1^{(0)}\cdot\boldsymbol{A}_N)\psi_0^{(0)}, \text{ in } \Omega, \end{aligned} \tag{8.135}$$

$$\boldsymbol{n}\cdot((i/\kappa)\nabla + h\boldsymbol{A}_N)\psi_1^{(1)} + (i/d)\psi_1^{(1)} = -\boldsymbol{n}\cdot\boldsymbol{a}_1^{(0)}\psi_0^{(0)} - \boldsymbol{n}\cdot\boldsymbol{a}_0^{(0)}\psi_1^{(0)}, \text{ on } \partial\Omega. \tag{8.136}$$

Inserting the solutions for $\psi_1^{(0)}$ and $\boldsymbol{a}_1^{(0)}$ we have

$$\begin{aligned} T^{(0)}\psi_1^{(1)} + ((i/\kappa) + h\boldsymbol{A}_N)^2\psi_1^{(1)} &= -T^{(1)}\psi_0^{(0)} - (\alpha\sigma^{(1)}/\kappa^2)\psi_0^{(0)} \\ &\quad - 3\,|\,\psi_0^{(0)}\,|^2\,\psi_0^{(0)} - (6i/\kappa)\boldsymbol{a}_0^{(0)}\cdot\nabla\psi_0^{(0)} \\ &\quad - 6h\boldsymbol{a}_0^{(0)}\cdot\boldsymbol{A}_N\psi_0^{(0)}, \text{ in } \Omega, \qquad (8.137) \\ \boldsymbol{n}\cdot((i/\kappa)\nabla + h\boldsymbol{A}_N)\psi_1^{(1)} + (i/d)\psi_1^{(1)} &= -3\boldsymbol{n}\cdot\boldsymbol{a}_0^{(0)}\psi_0^{(0)}, \text{ on } \partial\Omega. \qquad (8.138) \end{aligned}$$

Now, $\psi_0^{(0)}$ is a solution of the inhomogeneous version of equation (8.137) and boundary condition (8.138), namely (8.28) and (8.31). Hence there is a solution for $\psi_1^{(1)}$ if and only if an appropriate solvability condition is satisfied. To derive this condition we multiply (8.137) by $\psi_0^{(0)*}$ and integrate over Ω. We find that

$$\begin{aligned} \text{LHS} &= \\ &\int_\Omega \psi_0^{(0)*}\Big[-(1/\kappa^2)\nabla^2\psi_1^{(1)} + (2i/\kappa)h\boldsymbol{A}_N\cdot\nabla\psi_1^{(1)} + h^2\,|\,\boldsymbol{A}_N\,|^2\,\psi_1^{(1)} - T^{(0)}\psi_1^{(1)}\Big]dV, \\ &= \int_\Omega \psi_1^{(1)}\Big[-(1/\kappa^2)\nabla^2\psi_0^{(0)*} - (2i/\kappa)h\boldsymbol{A}_N\cdot\nabla\psi_0^{(0)*} \\ &\qquad\qquad + h^2\,|\,\boldsymbol{A}_N\,|^2\,\psi_0^{(0)*} - T^{(0)}\psi_0^{(0)*}\Big]\,dV, \\ &\quad + \int_{\partial\Omega}\Big[-(1/\kappa^2)(\psi_0^{(0)*}\nabla\psi_1^{(1)} - \psi_1^{(1)}\nabla\psi_0^{(0)*}) + (2i/\kappa)h\psi_0^{(0)*}\psi_1^{(1)}\boldsymbol{A}_N\Big]\cdot\boldsymbol{n}\,dS, \end{aligned}$$

by Greens Theorem,

$$\begin{aligned} &= (i/\kappa)\int_{\partial\Omega}\Big[(i/\kappa)\nabla\psi_1^{(1)} + h\psi_1^{(1)}\boldsymbol{A}_N\Big]\psi_0^{(0)*}\cdot\boldsymbol{n}\,dS \\ &\quad + (i/\kappa)\int_{\partial\Omega}\Big[-(i/\kappa)\nabla\psi_0^{(0)*} + h\psi_0^{(0)*}\boldsymbol{A}_N\Big]\psi_1^{(1)}\cdot\boldsymbol{n}\,dS, \end{aligned}$$

since the integral over Ω is zero by (8.28),

$$\begin{aligned} &= (i/\kappa)\int_{\partial\Omega}\psi_0^{(0)*}\Big[-(i/d)\psi_1^{(1)} - 3\boldsymbol{n}\cdot\boldsymbol{a}_0^{(0)}\psi_0^{(0)}\Big]\,dS \\ &\quad + (i/\kappa)\int_{\partial\Omega}(i/d)\psi_0^{(0)*}\psi_1^{(1)}\,dS, \end{aligned}$$

by (8.31) and (8.138),

$$= -(i/\kappa)\int_{\partial\Omega}|\,\psi_0^{(0)}\,|^2\,(3\boldsymbol{a}_0^{(0)}\cdot\boldsymbol{n})\,dS.$$

$$\begin{aligned} \text{RHS} &= \\ &-\int_\Omega\Big[3\,|\,\psi_0^{(0)}\,|^4 + T^{(1)}\,|\,\psi_0^{(0)}\,|^2\,(\alpha\sigma^{(1)}/\kappa^2)\,|\,\psi_0^{(0)}\,|^2 \\ &\qquad + (6i/\kappa)\psi_0^{(0)*}\boldsymbol{a}_0^{(0)}\cdot\nabla\psi_0^{(0)} + 6h\,|\,\psi_0^{(0)}\,|^2\,\boldsymbol{A}_N\cdot\boldsymbol{a}_0^{(0)}\Big]\,dV, \end{aligned}$$

$$
\begin{aligned}
= \quad & -(\alpha\,|\,\beta\,|^2\,\sigma^{(1)}/\kappa^2) -|\,\beta\,|^2\,T^{(1)} \\
& -\int_\Omega \Big[3\,|\,\psi_0^{(0)}\,|^4 + (2i/\kappa)\psi_0^{(0)*}(\nabla\psi_0^{(0)}\cdot 3\boldsymbol{a}_0^{(0)}) \\
& \qquad + 6\boldsymbol{a}_0^{(0)}\cdot(-\mathrm{curl}^2\boldsymbol{a}_0^{(0)} - (i/2\kappa)(\psi_0^{(0)*}\nabla\psi_0^{(0)} - \psi_0^{(0)}\nabla\psi_0^{(0)*}))\Big]\,dV,
\end{aligned}
$$

by (8.29),

$$
\begin{aligned}
= \quad & -(\alpha\,|\,\beta\,|^2\,\sigma^{(1)}/\kappa^2) -|\,\beta\,|^2\,T^{(1)} \\
& -\int_\Omega \Big[3\,|\,\psi_0^{(0)}\,|^4\; 3\boldsymbol{a}_0^{(0)}\cdot((i/\kappa)(\psi_0^{(0)*}\nabla\psi_0^{(0)} + \psi_0^{(0)}\nabla\psi_0^{(0)*}) - 2\mathrm{curl}^2\boldsymbol{a}_0^{(0)})\Big]\,dV, \\
= \quad & -(\alpha\,|\,\beta\,|^2\,\sigma^{(1)}/\kappa^2) -|\,\beta\,|^2\,T^{(1)} \\
& -\int_\Omega \Big[3\,|\,\psi_0^{(0)}\,|^4 + 3\boldsymbol{a}_0^{(0)}\cdot((i/\kappa)\nabla\,|\,\psi_0^{(0)}\,|^2 - 2\mathrm{curl}^2\boldsymbol{a}_0^{(0)})\Big]\,dV, \\
= \quad & -(\alpha\,|\,\beta\,|^2\,\sigma^{(1)}/\kappa^2) -|\,\beta\,|^2\,T^{(1)} + \int_\Omega \Big[-3\,|\,\psi_0^{(0)}\,|^4 + 6(\mathrm{curl})^2\boldsymbol{a}_0^{(0)}\cdot\boldsymbol{a}_0^{(0)}\Big]\,dV, \\
& -(i/\kappa)\int_{\partial\Omega} 3\,|\,\psi_0^{(0)}\,|^2\,(\boldsymbol{a}_0^{(0)})\cdot\boldsymbol{n}\,dS,
\end{aligned}
$$

by the divergence theorem, since div $\boldsymbol{A}_N$ and div $\boldsymbol{a}_0^{(0)}$ are both zero. Equating the left-hand side to the right-hand side we have

$$
(\alpha\,|\,\beta\,|^2\,\sigma^{(1)}/\kappa^2) +|\,\beta\,|^2\,T^{(1)} = \int_\Omega \Big[-3\,|\,\psi_0^{(0)}\,|^4 + 6(\mathrm{curl})^2\boldsymbol{a}_0^{(0)}\cdot\boldsymbol{a}_0^{(0)}\Big]\,dV.
$$

Hence

$$
(\alpha\sigma^{(1)}/\kappa^2) = 2T^{(1)} \tag{8.139}
$$

since we have

$$
|\,\beta\,|^2\,T^{(1)} = \int_\Omega -\,|\,\psi^{(0)}\,|^4 + 2\boldsymbol{a}^{(0)}\cdot(\mathrm{curl})^2\boldsymbol{a}^{(0)}\,dV
$$

Thus

$$
\sigma^{(1)} = (2\kappa^2 T^{(1)}/\alpha). \tag{8.140}
$$

We see that $\sigma^{(1)} < 0$ if and only if $T^{(1)} < 0$.

8.3 Weakly nonlinear stability of the normal state solution

We have the time-dependent Ginzburg-Landau equations:

$$
\frac{\alpha}{\kappa^2}\frac{\partial\Psi}{\partial t} + \frac{\alpha i}{\kappa}\Psi\Phi + \left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2\Psi \quad = \quad -\Psi(T +|\,\Psi\,|^2),\ \text{in }\Omega, \tag{8.141}
$$

$$
\begin{aligned}
-(\text{curl})^2\boldsymbol{A} - \frac{\partial \boldsymbol{A}}{\partial t} - \nabla\Phi &= \frac{i}{2\kappa}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) \\
&\quad + |\,\Psi\,|^2\ \boldsymbol{A},\ \text{in } \Omega, && (8.142) \\
-(\text{curl})^2\boldsymbol{A} &= \varsigma_e\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right),\ \text{outside } \Omega, && (8.143) \\
\nabla^2\Phi &= 0,\ \text{outside } \Omega, && (8.144) \\
\boldsymbol{n}\cdot((i/\kappa)\nabla + \boldsymbol{A})\Psi + (i/d)\Psi &= 0,\ \text{on } \partial\Omega, && (8.145) \\
[\boldsymbol{n}\wedge\boldsymbol{A}] &= \boldsymbol{0}, && (8.146) \\
[\boldsymbol{n}\wedge(1/\mu)\text{curl } \boldsymbol{A}] &= \boldsymbol{0}, && (8.147) \\
[\Phi] &= 0, && (8.148) \\
\left[\varepsilon\frac{\partial\Phi}{\partial n}\right] &= 0, && (8.149) \\
\text{curl } \boldsymbol{A} &\to h\hat{\boldsymbol{z}},\ \text{as } r\to\infty, && (8.150) \\
\Phi &\to 0,\ \text{as } r\to\infty, && (8.151) \\
\text{div } \boldsymbol{A} &= 0. && (8.152)
\end{aligned}
$$

We seek a solution near the bifurcation point $T = T_{c_2}$. To this end we set

$$T = T_{c_2} + \epsilon T^{(1)}, \tag{8.153}$$

as before.

We introduce ψ, $\boldsymbol{a}$, and ϕ as before by setting

$$
\begin{aligned}
\Psi &= \epsilon^{1/2}\psi, && (8.154) \\
\boldsymbol{A} &= h\boldsymbol{A}_N + \epsilon\boldsymbol{a}, && (8.155) \\
\Phi &= \epsilon\phi. && (8.156)
\end{aligned}
$$

Substituting (8.153)-(8.156) into (8.141)-(8.152) yields

$$
\begin{aligned}
\frac{\alpha}{\kappa^2}\frac{\partial\psi}{\partial t} + \left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)^2\psi + (T_{c_2} + \epsilon T^{(1)})\psi &= -\epsilon\frac{\alpha i}{\kappa}\psi\phi + \epsilon\psi\,|\,\psi\,|^2 \\
&\quad + 2\epsilon h\psi(\boldsymbol{A}_N\cdot\boldsymbol{a}) \\
&\quad + \frac{2\epsilon i}{\kappa}(\boldsymbol{a}\cdot\nabla\psi) \\
&\quad - \epsilon^2\,|\,\boldsymbol{a}\,|^2\ \psi,\ \text{in } \Omega, && (8.157) \\
-(\text{curl})^2\boldsymbol{a} - \frac{\partial\boldsymbol{a}}{\partial t} - \nabla\phi &= \frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) \\
&\quad + |\,\psi\,|^2\ (h\boldsymbol{A}_N + \epsilon\boldsymbol{a}), \\
&\qquad \text{in } \Omega, && (8.158)
\end{aligned}
$$

$$
\begin{aligned}
-(\mathrm{curl})^2\boldsymbol{a} &= \varsigma_e\left(\frac{\partial \boldsymbol{a}}{\partial t}+\nabla\phi\right),\ \text{outside }\Omega, && (8.159)\\
\nabla^2\phi &= 0,\ \text{outside }\Omega, && (8.160)\\
\boldsymbol{n}\cdot((i/\kappa)\nabla+h\boldsymbol{A}_N)\psi+(i/d)\psi &= -\epsilon(\boldsymbol{n}\cdot\boldsymbol{a})\psi,\ \text{on }\partial\Omega, && (8.161)\\
[\boldsymbol{n}\wedge\boldsymbol{a}] &= \boldsymbol{0}, && (8.162)\\
[\boldsymbol{n}\wedge(1/\mu)\mathrm{curl}\ \boldsymbol{a}] &= \boldsymbol{0}, && (8.163)\\
[\phi] &= 0, && (8.164)\\
\left[\varepsilon\frac{\partial\phi}{\partial n}\right] &= 0, && (8.165)\\
\mathrm{curl}\ \boldsymbol{a} &\to \boldsymbol{0},\ \text{as } r\to\infty, && (8.166)\\
\phi &\to 0,\ \text{as } r\to\infty, && (8.167)\\
\mathrm{div}\ \boldsymbol{a} &= 0. && (8.168)
\end{aligned}
$$

When we examined the linear stability of the normal-state solution near the bifurcation point we found that one mode had growth/decay rate of $O(\epsilon)$ while all other modes had a decay rate of $O(1)$. Thus we expect when we examine the nonlinear behaviour of the solution that there will be two timescales: and $O(1)$ timescale and an $O(\epsilon)$ timescale.

A. Short timescale : $t = O(1)$.

We denote the short-time solution by $\psi_s(\boldsymbol{r},t)$, $\boldsymbol{a}_s(\boldsymbol{r},t)$, $\phi_s(\boldsymbol{r},t)$, and expand all quantities in powers of ϵ as before:

$$
\begin{aligned}
\psi_s &= \psi_s^{(0)}+\epsilon\psi_s^{(1)}+\cdots, && (8.169)\\
\boldsymbol{a}_s &= \boldsymbol{a}_s^{(0)}+\epsilon\boldsymbol{a}_s^{(1)}+\cdots, && (8.170)\\
\phi_s &= \phi_s^{(0)}+\epsilon\phi_s^{(1)}+\cdots. && (8.171)
\end{aligned}
$$

Substituting the expansions (8.169)-(8.171) into equations (8.157)-(8.168) and equating powers of ϵ yields at leading order

$$
\begin{aligned}
\frac{\alpha}{\kappa^2}\frac{\partial\psi_s^{(0)}}{\partial t}+\left(\frac{i}{\kappa}\nabla+h\boldsymbol{A}_N\right)^2\psi_s^{(0)} &= -T_{c_2}\psi_s^{(0)},\ \text{in }\Omega, && (8.172)\\
-(\mathrm{curl})^2\boldsymbol{a}_s^{(0)}-\frac{\partial\boldsymbol{a}_s^{(0)}}{\partial t}-\nabla\phi_s^{(0)} &= \frac{i}{2\kappa}(\psi_s^{(0)*}\nabla\psi_s^{(0)}-\psi_s^{(0)}\nabla\psi_s^{(0)*})\\
&\quad +h\,|\,\psi_s^{(0)}|^2\,\boldsymbol{A}_N,\ \text{in }\Omega, && (8.173)
\end{aligned}
$$

$$-(\text{curl})^2 \boldsymbol{a}_s^{(0)} = \varsigma_e \left(\frac{\partial \boldsymbol{a}_s^{(0)}}{\partial t} + \nabla \phi_s^{(0)} \right), \text{ outside } \Omega, \tag{8.174}$$

$$\nabla^2 \phi_s^{(0)} = 0, \text{ outside } \Omega, \tag{8.175}$$

$$\boldsymbol{n} \cdot ((i/\kappa)\nabla + h\boldsymbol{A}_N)\psi_s^{(0)} = -(i/d)\psi_s^{(0)}, \text{ on } \partial\Omega, \tag{8.176}$$

$$[\boldsymbol{n} \wedge \boldsymbol{a}_s^{(0)}] = \boldsymbol{0}, \tag{8.177}$$

$$[\boldsymbol{n} \wedge (1/\mu)\text{curl } \boldsymbol{a}_s^{(0)}] = \boldsymbol{0}, \tag{8.178}$$

$$[\phi_s^{(0)}] = 0, \tag{8.179}$$

$$\left[\varepsilon \frac{\partial \phi_s^{(0)}}{\partial n} \right] = 0, \tag{8.180}$$

$$\text{curl } \boldsymbol{a}_s^{(0)} \to \boldsymbol{0}, \text{ as } r \to \infty, \tag{8.181}$$

$$\phi_s^{(0)} \to 0, \text{ as } r \to \infty, \tag{8.182}$$

$$\text{div } \boldsymbol{a}_s^{(0)} = 0. \tag{8.183}$$

Equation (8.172) with the boundary condition (8.176) has solution

$$\psi_s^{(0)}(x,t) = \sum_{n=-\infty}^{\infty} \beta_n e^{\sigma_n t} \theta_n(\boldsymbol{r}), \tag{8.184}$$

where σ_n are the eigenvalues of

$$\frac{\alpha\sigma}{\kappa^2}\theta + \left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N \right)^2 \theta = -T_{c_2}\theta, \text{ in } \Omega, \tag{8.185}$$

$$\boldsymbol{n} \cdot ((i/\kappa)\nabla + h\boldsymbol{A}_N)\theta = -(i/d)\theta, \text{ on } \partial\Omega, \tag{8.186}$$

with corresponding eigenfunctions θ_n, and β_n are constants. Note that equations (8.185), (8.186) are exactly equations (8.70), (8.74) with $T^{(0)} = T_{c_2}$. We know the largest eigenvalue is zero, so we specify $\sigma_0 = 0$. The β_n must be chosen such that

$$\sum_{n=-\infty}^{\infty} \beta_n \theta_n(\boldsymbol{r}) = \psi_s^{(0)}(\boldsymbol{r}, 0). \tag{8.187}$$

As in Chapter 5 the eigenfunctions corresponding to distinct eigenvalues are orthogonal. Then multiplying (8.187) by $\theta_m^*(\boldsymbol{r})$ and integrating over Ω yields

$$\beta_m = \int_\Omega \psi_s^{(0)}(\boldsymbol{r}, 0)\theta_m^*(\boldsymbol{r})\, dV. \tag{8.188}$$

Thus

$$\psi_s^{(0)}(\boldsymbol{r}, t) = \int_\Omega \left(\sum_{n=-\infty}^{\infty} \theta_n^*(\tilde{\boldsymbol{r}}) e^{\sigma_n t} \theta_n(\boldsymbol{r}) \right) \psi_s^{(0)}(\tilde{\boldsymbol{r}}, 0)\, d\tilde{V}. \tag{8.189}$$

We can then solve for $\boldsymbol{a}_s^{(0)}$ and $\phi_s^{(0)}$.

This leading-order solution ignores the growth of the unstable mode since the growth happens on a timescale of $O(\epsilon^{-1})$. We expect that if we proceed to determine the first-order terms that we will find secular terms appearing, and that the solution will cease to be valid when $t = O(\epsilon^{-1})$.

B. Long timescale : $t = O(\epsilon^{-1})$.

We now consider the long-time behaviour of the solution. We define

$$\tau = \epsilon t$$

and consider τ to be $O(1)$. We denote the long-time solution by $\psi_l(\boldsymbol{r}, \tau)$, $\boldsymbol{a}_l(\boldsymbol{r}, \tau)$, $\phi_l(\boldsymbol{r}, \tau)$. Equations (8.157)-(8.168) become

$$\begin{aligned}\epsilon\frac{\alpha}{\kappa^2}\frac{\partial\psi_l}{\partial\tau} + \left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)^2\psi_l + (T_{c_2} + \epsilon T^{(1)})\psi_l = \\ -\epsilon\left[\frac{\alpha i}{\kappa}\psi_l\phi_l + \psi_l\,|\psi_l|^2 + 2h\psi_l(\boldsymbol{A}_N\cdot\boldsymbol{a}_l) + \frac{2i}{\kappa}(\boldsymbol{a}_l\cdot\nabla\psi_l)\right] \\ -\epsilon^2\,|\boldsymbol{a}_l|^2\,\psi_l, \qquad \text{in } \Omega, \end{aligned} \tag{8.190}$$

$$-(\text{curl})^2\boldsymbol{a}_l - \epsilon\frac{\partial\boldsymbol{a}_l}{\partial\tau} - \nabla\phi_l = \frac{i}{2\kappa}(\psi_l^*\nabla\psi_l - \psi_l\nabla\psi_l^*) + |\psi_l|^2\,(h\boldsymbol{A}_N + \epsilon\boldsymbol{a}_l), \quad \text{in } \Omega, \tag{8.191}$$

$$-(\text{curl})^2\boldsymbol{a}_l = \varsigma_e\left(\epsilon\frac{\partial\boldsymbol{a}_l}{\partial\tau} + \nabla\phi_l\right), \text{ outside } \Omega, \tag{8.192}$$

$$\nabla^2\phi_l = 0, \text{ outside } \Omega, \tag{8.193}$$

$$\boldsymbol{n}\cdot((i/\kappa)\nabla + h\boldsymbol{A}_N)\psi_l + (i/d)\psi_l = -\epsilon(\boldsymbol{n}\cdot\boldsymbol{a}_l)\psi_l, \text{ on } \partial\Omega, \tag{8.194}$$

$$[\boldsymbol{n}\wedge\boldsymbol{a}_l] = \boldsymbol{0}, \tag{8.195}$$

$$[\boldsymbol{n}\wedge(1/\mu)\text{curl } \boldsymbol{a}_l] = \boldsymbol{0}, \tag{8.196}$$

$$[\phi_l] = 0, \tag{8.197}$$

$$\left[\varepsilon\frac{\partial\phi_l}{\partial n}\right] = 0, \tag{8.198}$$

$$\text{curl } \boldsymbol{a}_l \to \boldsymbol{0}, \text{ as } r\to\infty, \tag{8.199}$$

$$\phi_l \to 0, \text{ as } r\to\infty, \tag{8.200}$$

$$\text{div } \boldsymbol{a}_l = 0. \tag{8.201}$$

We expand all quantities in powers of ϵ as before:

$$\psi_l = \psi_l^{(0)} + \epsilon\psi_l^{(1)} + \cdots, \tag{8.202}$$

$$\boldsymbol{a}_l = \boldsymbol{a}_l^{(0)} + \epsilon\boldsymbol{a}_l^{(1)} + \cdots, \tag{8.203}$$

$$\phi_l = \phi_l^{(0)} + \epsilon\phi_l^{(1)} + \cdots. \tag{8.204}$$

Substituting the expansions (8.202)-(8.204) into equations (8.190)-(8.201) and equating powers of ϵ yields at leading order

$$\left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)^2 \psi_l^{(0)} + T_{c_2}\psi_l^{(0)} = 0, \text{ in } \Omega, \tag{8.205}$$

$$-(\mathrm{curl})^2\boldsymbol{a}_l^{(0)} - \nabla\phi_l^{(0)} = \frac{i}{2\kappa}(\psi_l^{(0)*}\nabla\psi_l^{(0)} - \psi_l^{(0)}\nabla\psi_l^{(0)*}) + h\,|\,\psi_l^{(0)}\,|^2\,\boldsymbol{A}_N, \text{ in } \Omega, \tag{8.206}$$

$$-(\mathrm{curl})^2\boldsymbol{a}_l^{(0)} = \varsigma_e\nabla\phi_l^{(0)}, \text{ outside } \Omega, \tag{8.207}$$

$$\nabla^2\phi_l^{(0)} = 0, \text{ outside } \Omega, \tag{8.208}$$

$$\boldsymbol{n}\cdot((i/\kappa)\nabla + h\boldsymbol{A}_N)\psi_l^{(0)} = -(i/d)\psi_l^{(0)}, \text{ on } \partial\Omega, \tag{8.209}$$

$$[\boldsymbol{n}\wedge\boldsymbol{a}_l^{(0)}] = \boldsymbol{0}, \tag{8.210}$$

$$[\boldsymbol{n}\wedge(1/\mu)\mathrm{curl}\;\boldsymbol{a}_l^{(0)}] = \boldsymbol{0}, \tag{8.211}$$

$$[\phi_l^{(0)}] = 0, \tag{8.212}$$

$$\left[\varepsilon\frac{\partial\phi_l^{(0)}}{\partial n}\right] = 0, \tag{8.213}$$

$$\mathrm{curl}\;\boldsymbol{a}_l^{(0)} \to \boldsymbol{0}, \text{ as } r\to\infty, \tag{8.214}$$

$$\phi_l^{(0)} \to 0, \text{ as } r\to\infty, \tag{8.215}$$

$$\mathrm{div}\;\boldsymbol{a}_l^{(0)} = 0. \tag{8.216}$$

Equations (8.205) and (8.209) are exactly equations (8.28) and (8.31) with $T^{(0)} = T_{c_2}$, and as such have solution

$$\psi_l^{(0)} = \beta(\tau)\theta_0, \tag{8.217}$$

where $\beta(\tau)$ is an unknown function of τ and θ_0 is as before. Substituting this solution into (8.206) yields for $\boldsymbol{a}_l^{(0)}$ and $\phi_l^{(0)}$ the equations

$$-(\mathrm{curl})^2\boldsymbol{a}_l^{(0)} - \nabla\phi_l^{(0)} = |\,\beta(\tau)\,|^2\,[(i/2\kappa)(\theta_0^*\nabla\theta_0 - \theta_0\nabla\theta_0^*) + h\,|\,\theta_0\,|^2\,\boldsymbol{A}_N], \text{ in } \Omega, \tag{8.218}$$

$$
\begin{aligned}
-(\text{curl})^2 \boldsymbol{a}_l^{(0)} &= \varsigma_e \nabla \phi_l^{(0)}, \text{ outside } \Omega, && (8.219)\\
\nabla^2 \phi_l^{(0)} &= 0, \text{ outside } \Omega, && (8.220)\\
[\boldsymbol{n} \wedge \boldsymbol{a}_l^{(0)}] &= \boldsymbol{0}, && (8.221)\\
[\boldsymbol{n} \wedge (1/\mu)\text{curl } \boldsymbol{a}_l^{(0)}] &= \boldsymbol{0}, && (8.222)\\
[\phi_l^{(0)}] &= 0, && (8.223)\\
\left[\varepsilon \frac{\partial \phi_l^{(0)}}{\partial n}\right] &= 0, && (8.224)\\
\text{curl } \boldsymbol{a}_l^{(0)} &\to \boldsymbol{0}, \text{ as } r \to \infty, && (8.225)\\
\phi_l^{(0)} &\to 0, \text{ as } r \to \infty, && (8.226)\\
\text{div } \boldsymbol{a}_l^{(0)} &= 0. && (8.227)
\end{aligned}
$$

By comparing (8.218) with (8.29) we see

$$
-(\text{curl})^2 \boldsymbol{a}_l^{(0)} - \nabla \phi_l^{(0)} = - \mid \beta(\tau) \mid^2 (\text{curl})^2 \hat{\boldsymbol{a}}_0^{(0)}, \text{ in } \Omega, \tag{8.228}
$$

where $\hat{\boldsymbol{a}}_0$ is the previously found steady-state superconducting solution, which is independent of τ. Taking the divergence of (8.228) we see

$$
\nabla^2 \phi_l^{(0)} = 0, \text{ in } \Omega,
$$

which, with (8.220), (8.223), (8.224), and (8.226) implies

$$
\phi_l^{(0)} \equiv 0. \tag{8.229}
$$

We now see that the solution for $\boldsymbol{a}_l^{(0)}$ is

$$
\boldsymbol{a}_l^{(0)} = \mid \beta(\tau) \mid^2 \boldsymbol{a}_0^{(0)}. \tag{8.230}
$$

To determine $\beta(\tau)$ we must proceed to higher orders in our expansions in ϵ. Equating powers of ϵ in (8.190), (8.194) yields

$$
\begin{aligned}
\left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)^2 \psi_l^{(1)} + T_{c_2}\psi_l^{(1)} &= -\frac{\alpha}{\kappa^2}\frac{\partial \psi_l^{(0)}}{\partial \tau} - T^{(1)}\psi_l^{(0)} - \mid \psi_l^{(0)} \mid^2 \psi_l^{(0)}\\
&\quad + 2h(\boldsymbol{A}_N \cdot \boldsymbol{a}_l^{(0)})\psi_l^{(0)} + \frac{2i}{\kappa}(\boldsymbol{a}_l^{(0)} \cdot \nabla \psi_l^{(0)}),\\
&\quad \text{in } \Omega, && (8.231)\\
\boldsymbol{n} \cdot \left(\frac{i}{\kappa} + h\boldsymbol{A}_N\right)\psi_l^{(1)} + \frac{i}{d}\psi_l^{(1)} &= -(\boldsymbol{n} \cdot \boldsymbol{a}_l^{(0)})\psi_l^{(0)}, \text{ on } \partial\Omega. && (8.232)
\end{aligned}
$$

Substituting in our expressions for $\psi_l^{(0)}$ and $\boldsymbol{a}_l^{(0)}$ we find

$$\begin{aligned}\left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)^2 \psi_l^{(1)} + T_{c_2}\psi_l^{(1)} &= -\frac{\alpha}{\kappa^2}\frac{d\beta}{d\tau}\theta_0 - T^{(1)}\beta\theta_0 - |\beta|^2\,\beta\,|\theta_0|^2\,\theta_0 \\ &\quad + 2\,|\beta|^2\,\beta h(\boldsymbol{A}_N\cdot\hat{\boldsymbol{a}}_0^{(0)})\theta_0 \\ &\quad + \frac{2i\,|\beta|^2\,\beta}{\kappa}(\hat{\boldsymbol{a}}_0^{(0)}\cdot\nabla\theta_0), \text{ in } \Omega, \qquad (8.233)\\ \boldsymbol{n}\cdot\left(\frac{i}{\kappa} + h\boldsymbol{A}_N\right)\psi_l^{(1)} + \frac{i}{d}\psi_l^{(1)} &= -\beta(\boldsymbol{n}\cdot|\beta|^2\,\hat{\boldsymbol{a}}_0^{(0)})\theta_0, \text{ on } \partial\Omega. \qquad (8.234)\end{aligned}$$

As before, θ_0 is a solution of the homogeneous versions of equations (8.233), (8.234) and therefore there is a solution for $\psi_l^{(1)}$ if and only if an appropriate solvability condition is satisfied. This condition is derived by multiplying by θ_0^* and integrating over Ω. A calculation very similar to that preceding (5.216) yields

$$\frac{\alpha}{\kappa^2}\frac{d\beta}{d\tau} = |\beta|^2\,\beta\left[2\int_\Omega \hat{\boldsymbol{a}}_0^{(0)}\cdot(\text{curl})^2\hat{\boldsymbol{a}}_0^{(0)}\,dV - \int_\Omega |\theta_0|^4\,dV\right] - T^{(1)}\beta. \qquad (8.235)$$

The boundary condition for this equation is given by matching with the short-time solution. We find

$$\beta(0)\theta_0 = \lim_{t\to\infty}\psi_s^{(0)} = \beta_0\theta_0,$$

since all the other eigenvalues σ_n in the expression (8.184) are negative. Hence

$$\beta(0) = \beta_0 = \int_\Omega \psi_s^{(0)}(\boldsymbol{r},0)\theta_0^*(\boldsymbol{r})\,dV. \qquad (8.236)$$

We see that equation (8.235) is very similar to equation (5.351). If we write

$$\begin{aligned} p &= 2\int_\Omega \hat{\boldsymbol{a}}_0^{(0)}\cdot(\text{curl})^2\hat{\boldsymbol{a}}_0^{(0)}\,dV - \int_\Omega |\theta_0|^4\,dV, \qquad (8.237)\\ q &= -T^{(1)}, \qquad (8.238)\end{aligned}$$

then the analysis following (5.351) holds and gives the solution to (8.235) as

$$r^2 = \begin{cases} \frac{q}{p}\left(\frac{Ce^{(2\kappa^2 q/\alpha)\tau}}{1-Ce^{(2\kappa^2 q/\alpha)\tau}}\right). & \text{if } q/p > 0, \\ \begin{cases} -\frac{q}{p}\left(\frac{Ce^{(2\kappa^2 q/\alpha)\tau}}{Ce^{(2\kappa^2 q/\alpha)\tau}-1}\right) & \text{if } r_0^2 > -q/p, \\ -\frac{q}{p}\left(\frac{Ce^{(2\kappa^2 q/\alpha)\tau}}{Ce^{(2\kappa^2 q/\alpha)\tau}+1}\right) & \text{if } r_0^2 < -q/p, \end{cases} & \text{if } q/p < 0. \end{cases} \qquad (8.239)$$

Let us examine the behaviour of these solutions. In the first case $q/p > 0$,which will be the case when either $T > T_{c_2}$ and the superconducting solution exists for values of T slightly less than T_{c_2} (i.e. $T^{(1)}$ in (8.50) is negative), or $T < T_{c_2}$ and the superconducting solution exists for values of T slightly greater than T_{c_2} (i.e. $T^{(1)}$ in (8.50) is positive), we have

a. if $p < 0, q < 0$, the solution decays exponentially to zero.

b. if $p > 0, q > 0$, the solution blows up in finite time $\tau = (\alpha/2\kappa^2 q)\log(1/C)$.

In the second case, $q/p < 0$, which will be the case when either $T > T_{c_2}$ and the superconducting solution exists for values of T slightly greater than T_{c_2} (i.e. $T^{(1)}$ in (8.50) is positive), or $T < T_{c_2}$ and the superconducting solution exists for values of T slightly less than T_{c_2} (i.e. $T^{(1)}$ in (8.50) is negative), we have

a. $p > 0, q < 0,$ $\begin{cases} \text{the solution decays exponentially to zero} & \text{if } r_0^2 < -q/p. \\ \text{the solution blows up in finite time } \tau = (\alpha/2\kappa^2 q)\log(1/C) & \text{if } r_0^2 > -q/p. \end{cases}$

b. $p < 0, q > 0$, the solution tends to the steady state $r^2 = -q/p$ which is the previously found steady-state superconducting solution.

Chapter 9

Conclusion

9.1 Results

We opened this thesis by formulating the simplest possible sharp-interface model for the change of phase of a superconducting material under isothermal and anisothermal conditions, which took the form of a vectorial Stefan model. The assumption of a sharp interface (or rather of normal and superconducting regions of extent much greater than the interface width) limits the applicability of this model to Type I superconductors. Examination of the model under isothermal conditions revealed instabilities similar to those of the classical Stefan model, which lead us to conjecture that the model is only well-posed when the normal region is expanding.

In Chapter 3 we introduced the Ginzburg-Landau model of superconductivity, which smooths out the boundary between normal and superconducting parts of a material by introducting a complex order parameter as a macroscopic wavefunction for the superconducting electrons, whose magnitude represents the number density of superconducting charge carriers. In Chapter 4 we showed that the vectorial Stefan model can be retrieved as a formal asymptotic limit of the Ginzburg-Landau model as the width of the interface tends to zero. Thus the Ginzburg-Landau model can be considered as a regularisation of the vectorial Stefan model. An examination of the magnitude of the magnetic field on the interface at first order in the aforementioned asymptotic limit revealed the emergence of 'surface tension' and 'kinetic undercooling' terms. For Type I superconductors these terms are expected to have a stabilising effect on the normal/superconducting interface. However, because of the very small size of the surface energy (linearly proportional to the interface width) these terms will not appreciably affect the interface until its curvature is of the order of its thickness. Thus we expect the Ginzburg-Landau equations to give intricate morphologies (in the 'switch-on' case), even for Type I superconductors. Experimental evidence [24, 62, 63] and numerical simulations [26, 44] support this conjecture.

In Section 3.3.1 we saw that for Type II superconductors the surface energy is in fact negative, and hence the 'surface tension' and 'kinetic undercooling' terms will have a destabilising effect on the interface. This negative surface energy leads to the formation of superconducting and normal domains that are of size comparable to the interface thickness, i.e. the material tends to have as many normal/superconducting transitions as possible. Such a state is known as a mixed state.

In Chapters 5 and 7 we examined the nucleation of superconductivity in decreasing magnetic fields and found that there is a bifurcation to a partially superconducting state when the applied magnetic field h is equal to the upper critical field h_{c_2}. We found that for Type II superconductors $h_{c_2} > H_c$ and the partially superconducting solution exists for values of the external magnetic field slightly less than h_{c_2} and is stable, i.e. the bifurcation is supercritical. For Type I superconductors $h_{c_2} < H_c$ and the partially superconducting solution exists for values of the external magnetic field slightly greater than h_{c_2} and is unstable, i.e. the bifurcation is subcritical. In Chapter 7 we demonstrated a variety of mixed state solutions for a bulk superconductor, and found the stable solution to be that of a triangular lattice of normal filaments in a superconducting matrix, both in terms of the minimum free energy and in terms of classical linear and nonlinear stability. The average magnetic field in the specimen $\overline{H}$ was found to depend linearly on the applied magnetic field h near h_{c_2}, in agreement with the magnetisation curve in Fig. 1.6*b*, and the gradient $d\overline{H}/dh$ was found to tend to infinity as $\kappa \to 1/\sqrt{2}$ as expected. The nucleation field h_{c_2} also forms the limit of the 'supercooling' of a bulk Type I superconductor, and explains the hysteresis shown in Fig. 1.6*a*.

In Chapter 6 we examined the effects of the presence of a surface on the nucleation of superconductivity in decreasing magnetic fields. We found that for fields parallel to the surface of a sample the nucleation field, h_{c_3}, is higher than that for bulk nucleation, h_{c_2}, whereas for fields perpendicular to the surface it is the same. Thus, in decreasing magnetic fields, superconductivity will first nucleate on the surface of a sample, where the field is parallel to it, in the form of a superconducting sheath (which may not cover the whole surface). This also implies that for finite samples it is h_{c_3}, and not h_{c_2}, which limits the supercooling of a Type I superconductor, and hence for a slowly quenched superconductor we expect superconducting regions to grow inwards from the surface of the sample. For a rapid quench below h_{c_2} the situation would be quite different, since then seeds of superconducting material may form in the bulk of the sample. Such a situation corresponds to what is known as spinodal decomposition in the Cahn-Hilliard theory of solid-solid phase transitions [25]. Numerical simulations seem to agree with these predictions [26, 44].

Thus we have seen the very different method of phase change for Type I and Type II superconductors. We have found that for Type II superconductors there is a continuous rise in the magnitude of the order parameter as h is decreased through h_{c_2} (or h_{c_3}). For Type I superconductors on the other hand there is an rapid increase in the order parameter as h is lowered through h_{c_2} (or h_{c_3}) and the change of phase takes place by means of propagating phase boundaries. If h is then raised again the superconducting state will persist until $h = H_c$, when again there will be an rapid decrease in the order parameter, as shown by the hysteresis loop in Fig. 9.1.

Finally we demonstrated the nucleation of superconductivity with decreasing temperature in the presence of an applied magnetic field, which corresponds simply to crossing the line $h_{c_2}(T)$ in Fig. 1.7 in a vertical rather than horizontal direction. The results are very similar, and both surface superconducting and mixed state solutions can be reached by decreasing the temperature rather than the magnetic field.

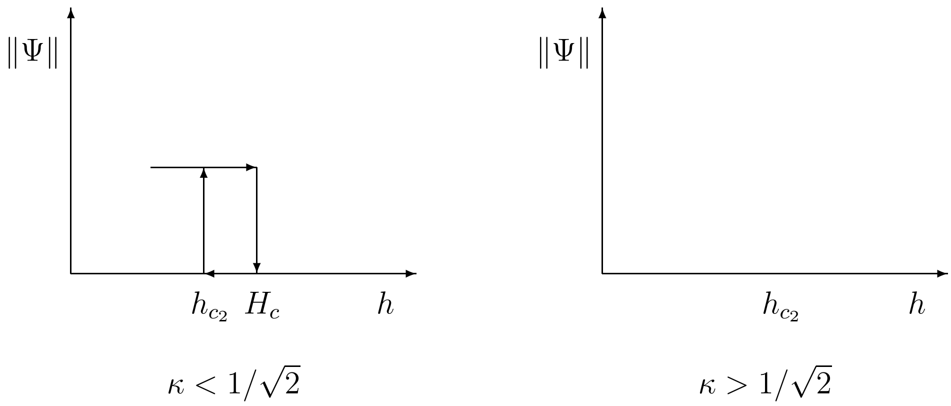

Figure 9.1: Onset of superconductivity for Type I and Type II superconductors. For Type I superconductors there is a hysteresis loop. For Type II superconductors there is no hysteresis.

9.2 Open Questions

9.2.1 Current-induced intermediate state in Type I superconductors

Very little mathematics has been done from the Ginzburg-Landau point of view on the application of an electric current to a superconductor, mainly because the interesting situations are nearly always unsteady. We give an example here of an interesting problem on which no general consensus has been reached. (We note that mathematical analyses have been performed on the carrying of a current by a superconducting multifilamentary composite, but that is a quite different problem [66].)

Consider a superconducting wire of Type I material carrying an applied current I, as in Fig. 9.2. While the wire is in the superconducting state the current density will be confined to a thin region around the surface of the wire of the order of the penetration depth. As the applied current I is increased the magnetic field at the surface of the wire due to the applied current will increase. When this field reaches the critical magnetic field the surface of the wire will begin to turn normal.

If the wire were to form a normal sheath around a superconducting core, as in Fig. 9.3, then the current would run down the edge of the core, having a smaller circumference than the wire, and thus the current density would be increased. Hence

Figure 9.2: Superconducting cylinder carrying an applied current.

Figure 9.3: Normal sheath surrounding a superconducting core carrying an applied current.

the magnetic field on the surface of the core would also be greater than the critical magnetic field and the core would continue to shrink. However, if the wire were to turn completely normal then the current would no longer be confined to the surface of the wire but would distribute itself evenly over the cross-section. This would lead to a reduced current density and therefore a magnetic field everywhere less than the critical magnetic field! Thus the wire can be neither wholly superconducting nor wholly normal, and must be again in some intermediate state consisting of normal and superconducting regions. (Note that the steady state solution for a circle given in Section 2.2 implies that the wire cannot form a superconducting sheath around a normal core.) It is generally agreed that the wire forms a normal sheath around a core in an intermediate state, and resistance returns to the wire, but at a lower value than that of the completely normal state. As the current is increased further the intermediate state shrinks until at some higher value of the current the wire becomes completely normal again.

The intermediate state is taken to be such that the magnetic field strength in the normal region is equal to the critical field and the resistance is equal to the fraction of normal material present. Various forms have been suggested. London [45] suggested a steady intermediate state of alternating normal and superconducting domains, which has approximately the desired properties. This state is shown schematically in Fig. 9.4a.

[5] have performed more detailed numerical studies of static models similar to London's. Gorter [31] suggested that the intermediate state may be unsteady (which experimental evidence seems to support) and consist of annular normal cylinders that form at the surface of the core and shrink inwards, as shown in Fig. 9.4b. The similarity solution given by (2.66)-(2.70) may represent such a cylinder. The cylinders may be unstable in the axial direction and break up into tori, as shown in Fig. 9.4c.

[4] has shown that in fact there is a family of possibilities, all possessing the same averaged properties, of which the above models of London and Gorter are the extremes.

(a) (b) (c)

Figure 9.4: Current induced intermediate state in a cylindrical superconducting wire. (a) The static structure proposed by London. (b) The moving structure proposed by Gorter. (c) Variation of the structure proposed by Gorter in which the shrinking normal regions are tori.

9.2.2 Melting of the Mixed State

Some of the most interesting open questions concern the behaviour of the mixed state of Abrikosov away from the nucleation field h_{c_2} (especially since it is in this form that superconductors are used in most practical applications). We first note that the theoretical discussion given previously is obviously an oversimplification. In a real material the vortex lines are not all straight and parallel, but are free to vary in the z-direction and even become entangled; the vortex lines in a real Type II superconductor in the mixed state will resemble cooked rather than uncooked spaghetti. Furthermore, as the applied magnetic field is reduced the vortices separate and exert less of an influence on each other. It is conjectured that at some lower value of the applied magnetic field the solid-like flux lattice of Abrikosov may 'melt' into a more liquid-like structure in which the vortex lines wander around. (There are even conjectures for glass-like vortex states in the presence of material defects such as pinning sites.) Thus the response diagram of a Type II superconductor in an applied magnetic field is modified from Fig. 1.7 to Fig. 9.5.

The modelling of this 'melting' of the vortex lattice and of the new 'vortex

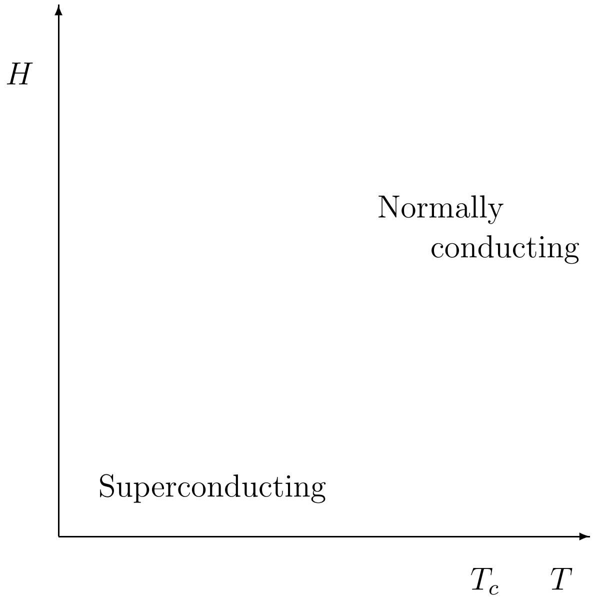

Figure 9.5: The conjectured response of a Type II superconductor in the presence of an applied magnetic field.

liquid' present challenging open problems, and some preliminary work has been started [47]. The homogenization of the vortex lines may be similar to that of dislocations, as given say in [32].

9.2.3 Application of the Ginzburg-Landau Equations to High-temperature Superconductors

Another debate focusses on the question of the applicability of the Ginzburg-Landau model, or some variant of it, to high-temperature superconducting materials, for which there is at present no established microscopic theory.

The Ginzburg-Landau model presented in Chapter 3 is valid only for low temperature superconductors. In principle there is no difficulty in extending the model to high-temperature superconductors, which are generally inhomogeneous, anisotropic and highly disordered. For example, as noted in [20], the constants a and b whose values depend on temperature are replaced by spatially varying scalar valued functions, and the constant m_s is replaced by a matrix, with possibly spatially varying entries (with $1/m_s$ replaced by m_s^{-1}). This would result in ξ and

λ being matrices, also with possibly spatially varying entries. However, in practice the functional form and values of a, b and m_s are not known.

Anisotropy is in general easier to model than inhomogeneity. Many of the high-temperature superconducting ceramics are made up of stacks of planes of atoms in which the superconducting electrons are confined. For these superconductors a simple anisotropic model is given by ξ and λ being diagonal matrices with constant entries. In this case, for example

$$\xi^2\nabla^2 f$$

is modified to

$$\xi_1^2\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)+\xi_2^2\frac{\partial^2 f}{\partial z^2}.$$

However, because of the problems associated with inhomogeneity, and in the absence of the support of a microscopic theory, no consensus has been reached on whether or not high-temperature superconductors can be modelled by anisotropic and inhomogeneous versions of the Ginzburg-Landau equations.

9.2.4 Further Open Questions

In addition to those problems mentioned above, a number of interesting questions have arisen in the course of the thesis, and we mention a few here.

The question of the existence of the similarity solution in Section 2.3 is similar to that of the corresponding Stefan problem. In the Stefan model there exists a solution to the transcendental equation for the interface velocity only if the degree of supercooling or superheating is not too large. It will be of interest to see whether there is a corresponding result for the problem of Section 2.3.

The anisothermal vectorial Stefan model (2.75)-(2.82) has yet to be studied in any detail, and many questions arise. One such is the question of whether the model can exhibit 'constitutional supercooling', as in the corresponding model of the alloy problem. Also in this model we have the appearance of a heating term $L\partial T/\partial t$. It will be of interest to examine the effect that this release of latent heat throughout the superconducting region has on the solution.

Finally, there is the question of the vertical bifurcation which appeared in Chapter 5 when $\kappa = 1/\sqrt{2}$. Normally nonlinearity guarantees the selection of at least a finite number of solutions. The examples of vertical bifurcations cited in the literature have not had an obvious physical interpretation, and it is to be hoped that some terms as yet unaccounted for will come to the rescue here.

Appendix A

Matching conditions

Here we derive the matching conditions used in Chapter 4. Let f be the function under consideration. We use the matching principle

$$(m \text{ term inner})(n \text{ term outer}) = (n \text{ term outer})(m \text{ term inner})$$

For a justification of this principle, which needs to be modified, for example, when applied to terms involving logarithms, we refer to [65]. In order to use this principle we need first to define outer variables (s_1, s_2, n) by

$$\boldsymbol{r} = (x, y, z) = \boldsymbol{R}(s_1, s_2, t) + n\boldsymbol{n}(s_1, s_2, t).$$

The outer expansion in terms of these variables is

$$\hat{f}_o = \hat{f}_o^{(0)}(s_1, s_2, n, t) + \lambda \hat{f}_o^{(1)}(s_1, s_2, n, t) + \lambda^2 \hat{f}_o^{(2)}(s_1, s_2, n, t) + \cdots .$$

The inner variables are defined by

$$\boldsymbol{r} = \boldsymbol{R}(s_1, s_2, t, \lambda) + \lambda\rho\boldsymbol{n}(s_1, s_2, t, \lambda), \quad \text{i.e. by } \lambda\rho = n.$$

We write the outer expansion in terms of the inner variables and expand in powers of λ:

$$\begin{aligned} \hat{f}_o &= \hat{f}_o^{(0)}(s_1, s_2, \lambda\rho, t) + \lambda \hat{f}_o^{(1)}(s_1, s_2, \lambda\rho, t) + \lambda^2 \hat{f}_o^{(2)}(s_1, s_2, \lambda\rho, t) + \cdots, \\ &= \hat{f}_o^{(0)}(s_1, s_2, 0, t) + \lambda\rho \frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1, s_2, 0, t) + \frac{\lambda^2\rho^2}{2} \frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1, s_2, 0, t) \\ &\quad + \lambda \hat{f}_o^{(1)}(s_1, s_2, 0, t) + \lambda^2 \rho \frac{\partial \hat{f}_o^{(1)}}{\partial n}(s_1, s_2, 0, t) \\ &\quad + \lambda^2 \hat{f}_o^{(2)}(s_1, s_2, 0, t) + \cdots . \end{aligned}$$

Hence

$$\begin{aligned}
(1ti)(1to) &= \hat{f}_o^{(0)}(s_1,s_2,0,t),\\
(2ti)(1to) &= \hat{f}_o^{(0)}(s_1,s_2,0,t)+\lambda\rho\frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1,s_2,0,t),\\
(3ti)(1to) &= \hat{f}_o^{(0)}(s_1,s_2,0,t)+\lambda\rho\frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1,s_2,0,t)\\
&\quad+\frac{\lambda^2\rho^2}{2}\frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1,s_2,0,t),
\end{aligned}$$

$$\begin{aligned}
(1ti)(2to) &= \hat{f}_o^{(0)}(s_1,s_2,0,t),\\
(2ti)(2to) &= \hat{f}_o^{(0)}(s_1,s_2,0,t)+\lambda\rho\frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1,s_2,0,t)+\lambda\hat{f}_o^{(1)}(s_1,s_2,0,t),\\
(3ti)(2to) &= \hat{f}_o^{(0)}(s_1,s_2,0,t)+\lambda\rho\frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1,s_2,0,t)\\
&\quad+\lambda\hat{f}_o^{(1)}(s_1,s_2,0,t)+\frac{\lambda^2\rho^2}{2}\frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1,s_2,0,t)\\
&\quad+\lambda^2\rho\frac{\partial \hat{f}_o^{(1)}}{\partial n}(s_1,s_2,0,t),
\end{aligned}$$

$$\begin{aligned}
(1ti)(3to) &= \hat{f}_o^{(0)}(s_1,s_2,0,t),\\
(2ti)(3to) &= \hat{f}_o^{(0)}(s_1,s_2,0,t)+\lambda\rho\frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1,s_2,0,t)+\lambda\hat{f}_o^{(1)}(s_1,s_2,0,t),\\
(3ti)(3to) &= \hat{f}_o^{(0)}(s_1,s_2,0,t)+\lambda\rho\frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1,s_2,0,t)\\
&\quad+\lambda\hat{f}_o^{(1)}(s_1,s_2,0,t)+\frac{\lambda^2\rho^2}{2}\frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1,s_2,0,t)\\
&\quad+\lambda^2\rho\frac{\partial \hat{f}_o^{(1)}}{\partial n}(s_1,s_2,0,t)+\lambda^2\hat{f}_o^{(2)}(s_1,s_2,0,t).
\end{aligned}$$

The inner expansion is

$$f_i = f_i^{(0)}(s_1,s_2,\rho,t)+\lambda f_i^{(1)}(s_1,s_2,\rho,t)+\lambda^2 f_i^{(2)}(s_1,s_2,\rho,t)+\cdots.$$

We write the inner expansion in terms of the outer variable $n=\lambda\rho$:

$$f_i = f_i^{(0)}(s_1,s_2,n\lambda^{-1},t)+\lambda f_i^{(1)}(s_1,s_2,n\lambda^{-1},t)+\lambda^2 f_i^{(2)}(s_1,s_2,n\lambda^{-1},t)+\cdots.$$

We expand each of these functions in powers of λ to obtain

$$
\begin{aligned}
f_i^{(0)}(s_1, s_2, n\lambda^{-1}, t) &= F_0^{(0)}(s_1, s_2, n, t) \\
&\quad + \lambda F_0^{(1)}(s_1, s_2, n, t) \\
&\qquad + \lambda^2 F_0^{(2)}(s_1, s_2, n, t) + \cdots, \\
\lambda f_i^{(1)}(s_1, s_2, n\lambda^{-1}, t) &= F_1^{(0)}(s_1, s_2, n, t) \\
&\quad + \lambda F_1^{(1)}(s_1, s_2, n, t) \\
&\qquad + \lambda^2 F_1^{(2)}(s_1, s_2, n, t) + \cdots, \\
\lambda^2 f_i^{(2)}(s_1, s_2, n\lambda^{-1}, t) &= F_2^{(0)}(s_1, s_2, n, t) \\
&\quad + \lambda F_2^{(1)}(s_1, s_2, n, t) \\
&\qquad + \lambda^2 F_2^{(2)}(s_1, s_2, n, t) + \cdots.
\end{aligned}
$$

We calculate the values of $F_m^{(n)}$ later. We now apply the matching principle.

$$
\begin{aligned}
& (1to)(1ti) = (1ti)(1to) \\
\Rightarrow\quad & F_0^{(0)} = \hat{f}_o^{(0)}(s_1, s_2, 0, t). \qquad \text{(A.1)}
\end{aligned}
$$

$$
\begin{aligned}
& (2to)(1ti) = (1ti)(2to) \\
\Rightarrow\quad & F_0^{(0)} + \lambda F_0^{(1)} = \hat{f}_o^{(0)}(s_1, s_2, 0, t) \\
\Rightarrow\quad & F_0^{(1)} = 0. \qquad \text{(A.2)}
\end{aligned}
$$

$$
\begin{aligned}
& (1to)(2ti) = (2ti)(1to) \\
\Rightarrow\quad & F_0^{(0)} + F_1^{(0)} = \hat{f}_o^{(0)}(s_1, s_2, 0, t) + \lambda\rho \frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1, s_2, 0, t) \\
\Rightarrow\quad & F_1^{(0)} = \lambda\rho \frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1, s_2, 0, t). \qquad \text{(A.3)}
\end{aligned}
$$

$$
\begin{aligned}
& (2to)(2ti) = (2ti)(2to) \\
\Rightarrow\quad & F_0^{(0)} + F_1^{(0)} + \lambda F_1^{(1)} = \\
& \hat{f}_o^{(0)}(s_1, s_2, 0, t) + \lambda\rho \frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1, s_2, 0, t) + \lambda \hat{f}_o^{(1)}(s_1, s_2, 0, t) \\
\Rightarrow\quad & F_1^{(1)} = \hat{f}_o^{(1)}(s_1, s_2, 0, t). \qquad \text{(A.4)}
\end{aligned}
$$

$$
\begin{aligned}
& (3to)(1ti) = (1ti)(3to) \\
\Rightarrow\quad & F_0^{(0)} + \lambda^2 F_0^{(2)} = \hat{f}_o^{(0)}(s_1, s_2, 0, t)
\end{aligned}
$$

$$\Rightarrow \quad F_0^{(2)} = 0. \tag{A.5}$$

$$(3to)(2ti) = (2ti)(3to)$$

$$\Rightarrow \quad F_0^{(0)} + F_1^{(0)} + \lambda F_1^{(1)} + \lambda^2 F_1^{(2)} =$$

$$\hat{f}_o^{(0)}(s_1, s_2, 0, t) + \lambda\rho \frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1, s_2, 0, t) + \lambda \hat{f}_o^{(1)}(s_1, s_2, 0, t)$$

$$\Rightarrow \quad F_1^{(2)} = 0. \tag{A.6}$$

$$(1to)(3ti) = (3ti)(1to)$$

$$\Rightarrow \quad F_0^{(0)} + F_1^{(0)} + F_2^{(0)} =$$

$$\hat{f}_o^{(0)}(s_1, s_2, 0, t) + \lambda\rho \frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1, s_2, 0, t) + \frac{\lambda^2\rho^2}{2}\frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1, s_2, 0, t)$$

$$\Rightarrow \quad F_2^{(0)} = \frac{\lambda^2\rho^2}{2}\frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1, s_2, 0, t). \tag{A.7}$$

$$(2to)(3ti) = (3ti)(2to)$$

$$\Rightarrow \quad F_0^{(0)} + F_1^{(0)} + F_2^{(0)} + \lambda F_1^{(1)} + \lambda^2 F_2^{(1)} =$$

$$\hat{f}_o^{(0)}(s_1, s_2, 0, t) + \lambda\rho \frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1, s_2, 0, t) + \lambda \hat{f}_o^{(1)}(s_1, s_2, 0, t)$$

$$+ \frac{\lambda^2\rho^2}{2}\frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1, s_2, 0, t) + \lambda^2\rho \frac{\partial \hat{f}_o^{(1)}}{\partial n}(s_1, s_2, 0, t)$$

$$\Rightarrow \quad F_2^{(1)} = \lambda\rho \frac{\partial \hat{f}_o^{(1)}}{\partial n}(s_1, s_2, 0, t). \tag{A.8}$$

$$(3to)(3ti) = (3ti)(3to)$$

$$\Rightarrow \quad F_0^{(0)} + F_1^{(0)} + F_2^{(0)} + \lambda F_1^{(1)} + \lambda F_2^{(1)} + \lambda^2 F_2^{(2)} =$$

$$\hat{f}_o^{(0)}(s_1, s_2, 0, t) + \lambda\rho \frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1, s_2, 0, t) + \lambda \hat{f}_o^{(1)}(s_1, s_2, 0, t)$$

$$+ \frac{\lambda^2\rho^2}{2}\frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1, s_2, 0, t) + \lambda^2\rho \frac{\partial \hat{f}_o^{(1)}}{\partial n}(s_1, s_2, 0, t) + \lambda^2 \hat{f}_o^{(2)}(s_1, s_2, 0, t)$$

$$\Rightarrow \quad F_2^{(2)} = \hat{f}_o^{(2)}(s_1, s_2, 0, t). \tag{A.9}$$

We now need to calculate the $F_m^{(n)}$ in terms of f_i. We have $n = \lambda\rho$. Thus taking the limit as $\lambda \to 0$ with n fixed is equivalent to taking the limit as $\rho \to \infty$. We have

$$\begin{aligned} F_0^{(0)} &= \lim_{\lambda \to 0} f_i^{(0)}(s_1, s_2, n\lambda^{-1}, t), \\ &= \lim_{\rho \to \infty} f_i^{(0)}(s_1, s_2, \rho, t). \end{aligned}$$

Hence (A.1) implies

$$\hat{f}_o^{(0)}(s_1, s_2, 0, t) = \lim_{\rho \to \infty} f_i^{(0)}(s_1, s_2, \rho, t). \tag{A.10}$$

Also

$$\begin{aligned} F_1^{(0)} &= \lim_{\lambda \to 0} \lambda f_i^{(1)}(s_1, s_2, n\lambda^{-1}, t), \\ &= \lim_{\rho \to \infty} \left(\frac{n}{\rho}\right) f_i^{(1)}(s_1, s_2, \rho, t), \\ &= n \lim_{\rho \to \infty} \frac{\partial f_i^{(1)}}{\partial \rho}(s_1, s_2, \rho, t). \end{aligned}$$

Hence (A.3) implies

$$\lambda \rho \frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1, s_2, 0, t) \quad = \quad n \lim_{\rho \to \infty} \frac{\partial f_i^{(1)}}{\partial \rho}(s_1, s_2, \rho, t),$$

i.e.

$$\frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1, s_2, 0, t) = \lim_{\rho \to \infty} \frac{\partial f_i^{(1)}}{\partial \rho}(s_1, s_2, \rho, t). \tag{A.11}$$

Also

$$\begin{aligned} F_2^{(0)} &= \lim_{\lambda \to 0} \lambda^2 f_i^{(2)}(s_1, s_2, n\lambda^{-1}, t), \\ &= \lim_{\rho \to \infty} \left(\frac{n}{\rho}\right)^2 f_i^{(2)}(s_1, s_2, \rho, t), \\ &= \frac{n^2}{2} \lim_{\rho \to \infty} \frac{\partial^2 f_i^{(2)}}{\partial \rho^2}(s_1, s_2, \rho, t). \end{aligned}$$

Hence (A.7) implies

$$\lambda^2 \rho^2 \frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1, s_2, 0, t) \quad = \quad n^2 \lim_{\rho \to \infty} \frac{\partial^2 f_i^{(2)}}{\partial \rho^2}(s_1, s_2, \rho, t),$$

i.e.

$$\frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1, s_2, 0, t) = \lim_{\rho \to \infty} \frac{\partial^2 f_i^{(2)}}{\partial \rho^2}(s_1, s_2, \rho, t). \tag{A.12}$$

We have

$$\begin{aligned} F_1^{(1)} &= \lim_{\lambda \to 0} \left\{ f_i^{(1)}(s_1, s_2, n\lambda^{-1}, t) - \lambda^{-1} F_1^{(0)} \right\}, \\ &= \lim_{\rho \to \infty} \left\{ f_i^{(1)}(s_1, s_2, \rho, t) - \frac{\rho}{n} F_1^{(0)} \right\}, \\ &= \lim_{\rho \to \infty} \left\{ f_i^{(1)}(s_1, s_2, \rho, t) - \rho \frac{\partial f_i^{(0)}}{\partial n}(s_1, s_2, 0, t) \right\}. \end{aligned}$$

Hence (A.4) implies

$$f_i^{(1)}(s_1,s_2,0,t) = \lim_{\rho\to\infty}\left\{ f_i^{(1)}(s_1,s_2,\rho,t) - \rho\frac{\partial f_i^{(0)}}{\partial n}(s_1,s_2,0,t)\right\}. \tag{A.13}$$

Also

$$\begin{aligned}
F_2^{(1)} &= \lim_{\lambda\to 0}\left\{\lambda f_i^{(2)}(s_1,s_2,n\lambda^{-1},t) - \lambda^{-1}F_2^{(0)}\right\}, \\
&= \lim_{\rho\to\infty}\left\{\frac{n}{\rho}f_i^{(2)}(s_1,s_2,\rho,t) - \frac{\rho}{n}F_2^{(0)}\right\}, \\
&= \lim_{\rho\to\infty}\left\{\frac{n}{\rho}f_i^{(2)}(s_1,s_2,\rho,t) - \frac{\rho n}{2}\frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1,s_2,0,t)\right\}, \\
&= \lim_{\rho\to\infty}\left\{\frac{n}{\rho}\left[f_i^{(2)}(s_1,s_2,\rho,t) - \frac{\rho^2}{2}\frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1,s_2,0,t)\right]\right\}, \\
&= n\lim_{\rho\to\infty}\left\{\frac{\partial f_i^{(2)}}{\partial \rho}(s_1,s_2,\rho,t) - \rho\frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1,s_2,0,t)\right\}.
\end{aligned}$$

Hence (A.8) implies

$$\lambda\rho\frac{\partial \hat{f}_o^{(1)}}{\partial n}(s_1,s_2,0,t) = n\lim_{\rho\to\infty}\left\{\frac{\partial f_i^{(2)}}{\partial \rho}(s_1,s_2,\rho,t) - \rho\frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1,s_2,0,t)\right\},$$

i.e.

$$\frac{\partial \hat{f}_o^{(1)}}{\partial n}(s_1,s_2,0,t) = \lim_{\rho\to\infty}\left\{\frac{\partial f_i^{(2)}}{\partial \rho}(s_1,s_2,\rho,t) - \rho\frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1,s_2,0,t)\right\}. \tag{A.14}$$

Thus far we have derived the matching conditions

$$\begin{aligned}
\hat{f}_o^{(0)}(s_1,s_2,0,t) &= \lim_{\rho\to\infty} f_i^{(0)}(s_1,s_2,\rho,t), && \text{(A.10)}\\
\frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1,s_2,0,t) &= \lim_{\rho\to\infty}\frac{\partial f_i^{(1)}}{\partial \rho}(s_1,s_2,\rho,t), && \text{(A.11)}\\
\frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1,s_2,0,t) &= \lim_{\rho\to\infty}\frac{\partial^2 f_i^{(2)}}{\partial \rho^2}(s_1,s_2,\rho,t), && \text{(A.12)}\\
\hat{f}_o^{(1)}(s_1,s_2,0,t) &= \lim_{\rho\to\infty}\left\{f_i^{(1)}(s_1,s_2,\rho,t) - \rho\frac{\partial f_i^{(0)}}{\partial n}(s_1,s_2,0,t)\right\}, && \text{(A.13)}\\
\frac{\partial \hat{f}_o^{(1)}}{\partial n}(s_1,s_2,0,t) &= \lim_{\rho\to\infty}\left\{\frac{\partial f_i^{(2)}}{\partial \rho}(s_1,s_2,\rho,t) - \rho\frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1,s_2,0,t)\right\}. && \text{(A.14)}
\end{aligned}$$

However, in Chapter 4 the outer variables we use are not (s_1, s_2, n) but (x, y, z). We need to rewrite (A.10)-(A.14) in terms of (x, y, z). We have that

$$
\begin{aligned}
\hat{f}_o(s_1, s_2, 0, t) &= f_o(\boldsymbol{R}, t), \\
&= f_o^{(0)}(\boldsymbol{R}^{(0)} + \lambda \boldsymbol{R}^{(1)} + \cdots, t) \\
&\qquad + \lambda f_o^{(1)}(\boldsymbol{R}^{(0)} + \lambda \boldsymbol{R}^{(1)} + \cdots, t) + \cdots, \\
&= f_o^{(0)}(\boldsymbol{R}^{(0)}, t) + \lambda \left\{ \boldsymbol{R}^{(1)} \cdot \nabla f_o^{(0)}(\boldsymbol{R}^{(0)}, t) + f_o^{(1)}(\boldsymbol{R}^{(0)}, t) \right\} + \cdots, \\
\frac{\partial \hat{f}_o}{\partial n}(s_1, s_2, 0, t) &= \frac{\partial f_o}{\partial n}(\boldsymbol{R}^{(0)}, t), \\
&= \frac{\partial f_o^{(0)}}{\partial n}(\boldsymbol{R}^{(0)} + \lambda \boldsymbol{R}^{(1)} + \cdots, t) \\
&\qquad + \lambda \frac{\partial f_o^{(1)}}{\partial n}(\boldsymbol{R}^{(0)} + \lambda \boldsymbol{R}^{(1)} + \cdots, t) + \cdots, \\
&= \frac{\partial f_o^{(0)}}{\partial n}(\boldsymbol{R}^{(0)}, t) \\
&\qquad + \lambda \left\{ \boldsymbol{R}^{(1)} \cdot \nabla \frac{\partial f_o^{(0)}}{\partial n}(\boldsymbol{R}^{(0)}, t) + \frac{\partial f_o^{(1)}}{\partial n}(\boldsymbol{R}^{(0)}, t) \right\} + \cdots .
\end{aligned}
$$

Hence

$$
\begin{aligned}
f_o^{(0)}(\boldsymbol{R}^{(0)}, t) &= \hat{f}_o^{(0)}(s_1, s_2, 0, t), \\
&= \lim_{\rho \to \infty} f_i^{(0)}(s_1, s_2, \rho, t), \qquad \text{(A.15)}
\end{aligned}
$$

$$
\begin{aligned}
&\boldsymbol{R}^{(1)} \cdot \nabla f_o^{(0)}(\boldsymbol{R}^{(0)}, t) + f_o^{(1)}(\boldsymbol{R}^{(0)}, t) \\
&\qquad = \hat{f}_o^{(1)}(s_1, s_2, 0, t), \\
&\qquad = \lim_{\rho \to \infty} \left\{ f_i^{(1)}(s_1, s_2, \rho, t) - \rho \frac{\partial f_i^{(0)}}{\partial n}(s_1, s_2, 0, t) \right\}, \qquad \text{(A.16)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f_o^{(0)}}{\partial n}(\boldsymbol{R}^{(0)}, t) &= \frac{\partial \hat{f}_o^{(0)}}{\partial n}(s_1, s_2, 0, t), \\
&= \lim_{\rho \to \infty} \frac{\partial f_i^{(1)}}{\partial \rho}(s_1, s_2, \rho, t), \qquad \text{(A.17)}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{R}^{(1)} \cdot \nabla \frac{\partial f_o^{(0)}}{\partial n}(\boldsymbol{R}^{(0)}, t) + \frac{\partial f_o^{(1)}}{\partial n}(\boldsymbol{R}^{(0)}, t) & \\
&= \frac{\partial \hat{f}_o^{(1)}}{\partial n}(s_1, s_2, 0, t), \\
&= \lim_{\rho \to \infty} \left\{ \frac{\partial f_i^{(2)}}{\partial \rho}(s_1, s_2, \rho, t) - \rho \frac{\partial^2 \hat{f}_o^{(0)}}{\partial n^2}(s_1, s_2, 0, t) \right\}. \quad \text{(A.18)}
\end{aligned}
$$

We note that if

$$
\begin{aligned}
f_i^{(0)} &\sim a^{(0)} + o(1), \\
f_i^{(1)} &\sim \rho b^{(1)} + a^{(1)} + o(1), \\
f_i^{(2)} &\sim \rho^2 c^{(2)} + \rho b^{(2)} + a^{(2)} + o(1),
\end{aligned}
$$

as $\rho \to \infty$, then

$$
\begin{aligned}
f_o(\boldsymbol{R}, t) &= a^{(0)} + \lambda a^{(1)} + \cdots, \\
\frac{\partial f_o}{\partial n}(\boldsymbol{R}, t) &= b^{(1)} + \lambda b^{(2)} + \cdots, \\
\frac{\partial^2 f_o}{\partial n^2}(\boldsymbol{R}, t) &= 2c^{(2)} + \cdots,
\end{aligned}
$$

etc.

Appendix B

Behaviour at $\kappa = 1/\sqrt{2}$

We have seen how the behaviour of the Ginzburg-Landau equations depends on κ, and how the value $\kappa = 1/\sqrt{2}$ is of particular interest. This value of κ separates so called Type I superconductors ($\kappa < 1/\sqrt{2}$) from Type II superconductors ($\kappa > 1/\sqrt{2}$). In Section 3.3.1 we found that the surface energy of a normal/superconducting interface was positive for Type I superconductors, negative for Type II superconductors, and zero at $\kappa = 1/\sqrt{2}$. In Section 5.2.1 we found that in one dimension the superconducting solution branch bifurcating from $h = h_{c_2}$ is stable for Type II superconductors, unstable for Type I superconductors, and that at $\kappa = 1/\sqrt{2}$ there is a singular bifurcation to a superconducting state.

In both of these situations we made use of the fact that solutions of the Ginzburg-Landau equations are given by solutions of the following pair of first-order ordinary differential equations:

$$\sqrt{2}\, f' = -fQ, \tag{B.1}$$

$$\sqrt{2}\, Q' = 1 - f^2, \tag{B.2}$$

in one dimension and with the application of compatible boundary conditions, to write the solution as a quadrature. To see why this reduction occurs we note that in one dimension the free energy density

$$\frac{(f')^2}{\kappa^2} + (Q')^2 + \frac{(f^2-1)^2}{2} + f^2Q^2,$$

may be written as

$$\left(\frac{f'}{\kappa} + fQ\right)^2 + \left(Q' + \frac{f^2-1}{2\kappa}\right)^2 + \frac{(f^2-1)^2}{2}\left(1 - \frac{1}{2\kappa^2}\right) + \frac{[Q(1-f^2)]'}{\kappa}.$$

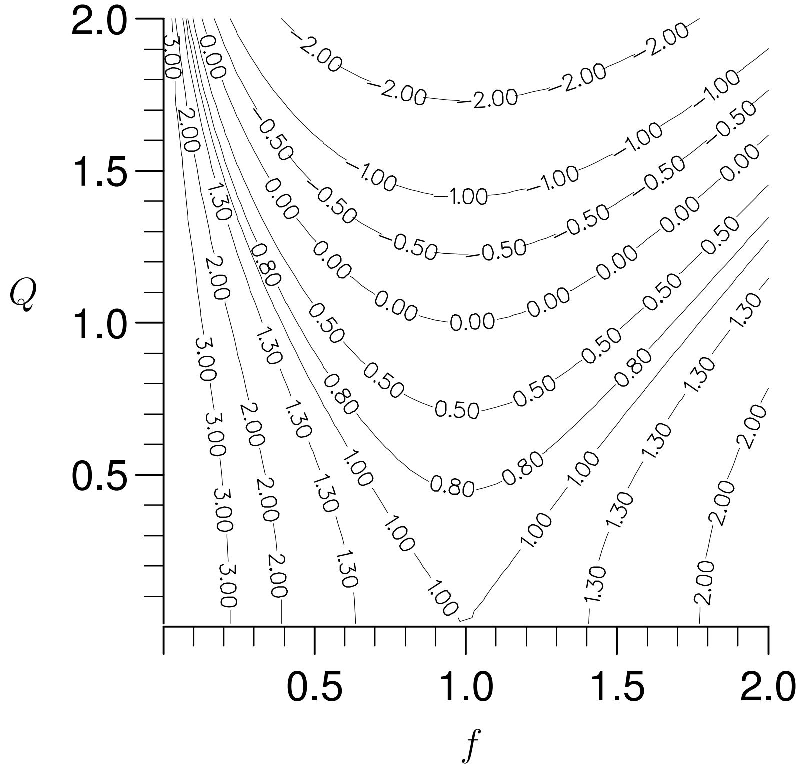

Figure B.1: Phase plane for equations (B.1), (B.2).

Hence, when $\kappa = 1/\sqrt{2}$, the equations exhibit 'self-duality symmetry' [8], in that the free energy density can be written as a sum of squares of first-order operators together with an exact differential. Thus solutions of the second-order equations are given by solutions of the first-order equations obtained by setting the first two terms equal to zero, namely equations (B.1), (B.2). The phase plane for equations (B.1), (B.2) is shown in Fig. B.1 for $f, Q > 0$ (the complete phase plane is obtained by reflection in the f and Q axes). The labels on the contours refer to the value of C in the solution given by (3.133). The normal/superconducting transition region solution of Section 3.3 is the separatrix joining the point $(1, 0)$ to $(0, \infty)$. The solutions with $C < 1$ passing through $(0, \infty)$ represent the solutions bifurcating from the normal state demonstrated in Section 5.1.2.

Similar solutions may be found in two dimensions with the magnetic field perpendicular to the plane of interest. In this case, with $\boldsymbol{Q} = (Q_1(x, y), Q_2(x, y), 0)$,

$\boldsymbol{H} = (0, 0, \partial Q_2/\partial x - \partial Q_1/\partial y)$, $f = f(x, y)$, the free energy density

$$\frac{1}{\kappa^2}\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] + \left(\frac{\partial Q_2}{\partial x} - \frac{\partial Q_1}{\partial y}\right)^2 + f^2\left(Q_1^2 + Q_2^2\right) + \frac{(f^2-1)^2}{2},$$

may be written as

$$\left(\frac{1}{\kappa}\frac{\partial f}{\partial x} + fQ_2\right)^2 + \left(\frac{1}{\kappa}\frac{\partial f}{\partial y} - fQ_1\right)^2 + \left(\frac{\partial Q_2}{\partial x} - \frac{\partial Q_1}{\partial y} + \frac{f^2-1}{2\kappa}\right)^2$$
$$+ \frac{(f^2-1)^2}{2}\left(1 - \frac{1}{2\kappa^2}\right) - \frac{1}{\kappa}\left\{\frac{\partial[(1-f^2)Q_2]}{\partial x} - \frac{\partial[(1-f^2)Q_1]}{\partial y}\right\}.$$

Again, when $\kappa = 1/\sqrt{2}$, this is a sum of squares of first-order operators plus a divergence, and solutions are given by setting the first three terms equal to zero:

$$\sqrt{2}\,\frac{\partial f}{\partial x} + fQ_2 = 0, \tag{B.3}$$
$$\sqrt{2}\,\frac{\partial f}{\partial y} - fQ_1 = 0, \tag{B.4}$$
$$\frac{\partial Q_2}{\partial x} - \frac{\partial Q_1}{\partial y} + \frac{f^2-1}{\sqrt{2}} = 0. \tag{B.5}$$

Hence $w = \log f^2$ satisfies the inhomogeneous Liouville equation

$$\nabla^2 w + 1 - e^w = 4\pi\sum_{i=1}^{n}\delta(\boldsymbol{x} - \boldsymbol{x}_i), \tag{B.6}$$

in the sense of distributions, where the $\boldsymbol{x}_i$ are the points at which f vanishes, and δ is the Dirac δ-function. This again requires the application of compatible boundary conditions.

We note that for a solution of (B.3)-(B.5) in an infinite region, when we impose the boundary conditions that $f \to 1$, $\boldsymbol{Q} \to 0$, as $r \to \infty$, the free energy is given by

$$\begin{aligned}\frac{1}{\kappa}\int \text{curl}\;[(1-f^2)\boldsymbol{Q}]\cdot\hat{\boldsymbol{z}}\,dS &= \frac{1}{\kappa}\int \text{curl}\;\boldsymbol{Q}\cdot\hat{\boldsymbol{z}}\,dS,\\ &= \frac{1}{\kappa}\int H\,dS,\\ &= \frac{1}{\kappa^2}\,[\nabla\chi] = \frac{2\pi N}{\kappa^2},\end{aligned}$$

since

$$\int \text{curl}\;(f^2\boldsymbol{Q})\cdot\boldsymbol{z}\,dS = \int (\text{curl})^2\boldsymbol{H}\cdot\boldsymbol{z}\,dS = 0,$$

by Stokes' theorem, and $\boldsymbol{Q} = \boldsymbol{A} - \nabla\chi/\kappa$ where $\boldsymbol{A}$ is nonsingular (note that Stokes' theorem does not apply to $\int \text{curl}\, \boldsymbol{Q} \cdot \hat{\boldsymbol{z}}\, dS$ since $\boldsymbol{Q}$ is singular where f is zero.) Thus the free energy is quantised, and is proportional to the number of superconducting vortices in the solution, or equivalently the number of flux quanta. [58] has shown that each finite energy solution of the Ginzburg-Landau equations in two dimensions is a solution of the reduced equations (B.3)-(B.5), and using these equations [57] has shown that the solutions containing N quanta of flux may be parametrised by the points of the plane where f vanishes together with their vortex numbers. Thus such a solution may be thought of as a superposition of N vortices. Since when $\kappa \neq 1/\sqrt{2}$ vortices attract or repel each other [36], we do not expect such multivortex solutions to exist for other values of κ other than when the number of vortices is infinite.

We note that the reduction relies on the fact that

$$|\,\text{curl}\ (Q_1, Q_2, 0)\,|^2 = \left(\text{div}\ (Q_2, -Q_1, 0)\right)^2,$$

in two dimensions, since with $\boldsymbol{P} = (Q_2, -Q_1, 0)$ we then have for the free energy density

$$2\,|\,\nabla f\,|^2 + f^2 P^2 + \frac{(f^2-1)^2}{2} + (\text{div}\ \boldsymbol{P})^2 = \\ (\sqrt{2}\nabla f + f\boldsymbol{P})^2 + \left(\text{div}\ \boldsymbol{P} + \frac{f^2-1}{\sqrt{2}}\right)^2 - \sqrt{2}\text{div}\ \left[(f^2-1)\boldsymbol{P}\right].$$

Since this is only possible in two dimensions there is no immediate generalisation of the reduction to three dimensions.

Appendix C

Operators in Curvilinear Coordinates

We list here for ease of reference expressions for the laplacian, curl, divergence, and gradient of functions in orthogonal curvilinear coordinates (x_1, x_2, x_3) with scaling factors h_1, h_2, h_3. Let $\boldsymbol{e}_i$ be the unit vector in the x_i direction and let $\boldsymbol{F} = F_1\boldsymbol{e}_1 + F_2\boldsymbol{e}_2 + F_3\boldsymbol{e}_3$. Then

$$\nabla F = \frac{1}{h_1}\frac{\partial F}{\partial x_1}\boldsymbol{e}_1 + \frac{1}{h_2}\frac{\partial F}{\partial x_2}\boldsymbol{e}_2 + \frac{1}{h_3}\frac{\partial F}{\partial x_3}\boldsymbol{e}_3, \tag{C.1}$$

$$\text{div } \boldsymbol{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3F_1)}{\partial x_1} + \frac{\partial(h_1h_3F_2)}{\partial x_2} + \frac{\partial(h_1h_2F_3)}{\partial x_3}\right], \tag{C.2}$$

$$\text{curl } \boldsymbol{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\boldsymbol{e}_1 & h_2\boldsymbol{e}_2 & h_3\boldsymbol{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}, \tag{C.3}$$

$$\nabla^2 F = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial F_1}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1h_3}{h_2}\frac{\partial F_2}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial F_3}{\partial x_3}\right)\right]. \tag{C.4}$$

We also calculate an expression for $\boldsymbol{v} \cdot \nabla$ in our local coordinate system. We have

$$\boldsymbol{e}_i \cdot [(\boldsymbol{v} \cdot \nabla)\boldsymbol{F}] = (\boldsymbol{v} \cdot \boldsymbol{F})(\boldsymbol{F} \cdot \boldsymbol{e}_i) - \boldsymbol{F} \cdot [(\boldsymbol{v} \cdot \nabla)\boldsymbol{e}_i].$$

Now

$$\boldsymbol{e}_1 = E^{-1/2}\boldsymbol{R}_1, \qquad \boldsymbol{e}_2 = G^{-1/2}\boldsymbol{R}_2, \qquad \boldsymbol{e}_3 = \boldsymbol{n}.$$

Hence

$$\frac{\partial \boldsymbol{e}_i}{\partial \rho} = 0, \quad i = 1, 2, 3.$$

We have

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}.$$

Hence

$$\frac{\partial e_i}{\partial s_k} \cdot \boldsymbol{e}_i = 0, \quad \frac{\partial e_i}{\partial s_k} \cdot \boldsymbol{e}_j = -\frac{\partial e_j}{\partial s_k} \cdot \boldsymbol{e}_i.$$

We also have

$$\begin{aligned}
\frac{\partial \boldsymbol{e}_1}{\partial s_1} &= -\frac{1}{2E^{3/2}}\frac{\partial E}{\partial s_1} + E^{-1/2}\boldsymbol{R}_{11}, \\
\frac{\partial \boldsymbol{e}_1}{\partial s_2} &= -\frac{1}{2E^{3/2}}\frac{\partial E}{\partial s_2} + E^{-1/2}\boldsymbol{R}_{12}.
\end{aligned}$$

Hence

$$\begin{aligned}
\boldsymbol{n} \cdot \frac{\partial \boldsymbol{e}_1}{\partial s_1} &= \frac{L}{E^{1/2}} = -E^{1/2}\tilde{\kappa}_1, \\
\boldsymbol{n} \cdot \frac{\partial \boldsymbol{e}_1}{\partial s_2} &= \frac{M}{E^{1/2}} = 0, \\
\boldsymbol{e}_2 \cdot \frac{\partial \boldsymbol{e}_1}{\partial s_2} = -\boldsymbol{e}_1 \cdot \frac{\partial \boldsymbol{e}_2}{\partial s_2} &= \frac{1}{(EG)^{1/2}}\boldsymbol{R}_{12} \cdot \boldsymbol{R}_2 = \frac{1}{2(EG)^{1/2}}\frac{\partial G}{\partial s_1}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\boldsymbol{n} \cdot \frac{\partial \boldsymbol{e}_2}{\partial s_2} &= \frac{N}{G^{1/2}} = -G^{1/2}\tilde{\kappa}_2, \\
\boldsymbol{n} \cdot \frac{\partial \boldsymbol{e}_2}{\partial s_1} &= \frac{M}{G^{1/2}} = 0, \\
\boldsymbol{e}_2 \cdot \frac{\partial \boldsymbol{e}_1}{\partial s_1} = -\boldsymbol{e}_1 \cdot \frac{\partial \boldsymbol{e}_2}{\partial s_1} &= -\frac{1}{(EG)^{1/2}}\boldsymbol{R}_{21} \cdot \boldsymbol{R}_1 = -\frac{1}{2(EG)^{1/2}}\frac{\partial E}{\partial s_2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial \boldsymbol{e}_1}{\partial s_1} &= -\frac{1}{2(EG)^{1/2}}\frac{\partial E}{\partial s_2}\boldsymbol{e}_2 - E^{1/2}\tilde{\kappa}_1\boldsymbol{n}, \\
\frac{\partial \boldsymbol{e}_1}{\partial s_2} &= \frac{1}{2(EG)^{1/2}}\frac{\partial G}{\partial s_1}\boldsymbol{e}_2, \\
\frac{\partial \boldsymbol{e}_2}{\partial s_1} &= \frac{1}{2(EG)^{1/2}}\frac{\partial E}{\partial s_2}\boldsymbol{e}_1, \\
\frac{\partial \boldsymbol{e}_2}{\partial s_2} &= -\frac{1}{2(EG)^{1/2}}\frac{\partial G}{\partial s_2}\boldsymbol{e}_1 - G^{1/2}\tilde{\kappa}_2\boldsymbol{n}.
\end{aligned}$$

We found in Chapter 4 that

$$\frac{\partial \boldsymbol{n}}{\partial s_1} = -\frac{L}{E^{1/2}}\boldsymbol{e}_1 = E^{1/2}\tilde{\kappa}_1\boldsymbol{e}_1, \qquad \frac{\partial \boldsymbol{n}}{\partial s_2} = -\frac{N}{G^{1/2}}\boldsymbol{e}_2 = G^{1/2}\tilde{\kappa}_2\boldsymbol{e}_2.$$

Hence

$$\begin{aligned}
\boldsymbol{e}_1 \cdot [(\boldsymbol{v} \cdot \nabla)\boldsymbol{F}] &= \left(\frac{v_1}{h_1}\frac{\partial}{\partial s_1} + \frac{v_2}{h_2}\frac{\partial}{\partial s_2} + \frac{v_n}{h_3}\frac{\partial}{\partial n}\right) F_1 \\
&\quad - \frac{F_2}{2(EG)^{1/2}}\left(-\frac{v_1}{h_1}\frac{\partial E}{\partial s_2} + \frac{v_2}{h_2}\frac{\partial G}{\partial s_1}\right) + \frac{F_3 v_1 E^{1/2}\tilde{\kappa}_1}{h_1}, \quad \text{(C.5)}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{e}_2 \cdot [(\boldsymbol{v} \cdot \nabla)\boldsymbol{F}] &= \left(\frac{v_1}{h_1}\frac{\partial}{\partial s_1} + \frac{v_2}{h_2}\frac{\partial}{\partial s_2} + \frac{v_n}{h_3}\frac{\partial}{\partial n}\right) F_2 \\
&\quad - \frac{F_1}{2(EG)^{1/2}}\left(\frac{v_1}{h_1}\frac{\partial E}{\partial s_2} - \frac{v_2}{h_2}\frac{\partial G}{\partial s_1}\right) + \frac{F_3 v_1 G^{1/2}\tilde{\kappa}_2}{h_1}, \quad \text{(C.6)}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{n} \cdot [(\boldsymbol{v} \cdot \nabla)\boldsymbol{F}] &= \left(\frac{v_1}{h_1}\frac{\partial}{\partial s_1} + \frac{v_2}{h_2}\frac{\partial}{\partial s_2} + \frac{v_n}{h_3}\frac{\partial}{\partial n}\right) F_3 \\
&\quad - F_1 \frac{v_1}{h_1} E^{1/2}\tilde{\kappa}_1 - F_2 \frac{v_2}{h_2} G^{1/2}\tilde{\kappa}_2. \quad \text{(C.7)}
\end{aligned}$$

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