

Capacity and Power Scaling Laws for Finite Antenna MIMO Amplify-and-Forward Relay Networks

David Simmons, Justin P. Coon, and Naqeeb Warsi

Abstract

In this paper, we present a novel framework that can be used to study the scaling properties of linear multiple-input multiple-output (MIMO) d antenna amplify-and-forward (AF) relay networks. In particular, we study these networks as random dynamical systems (RDS) and calculate their Lyapunov exponents. Our main results are twofold: 1) the total transmit power at the n th node is given by $\|X_n^{(\alpha)}\|^2 = \Theta_{\mathbb{P}}\left(e^{n\lambda_{Q,1}^{(\alpha)}}\right)$, 2) the capacity of the i th eigenchannel at the n th node is given by $c_{i,n}^{(\alpha)} = \Theta_{\mathbb{P}}\left(e^{n\lambda_{\gamma,i}^{(\alpha)}}\right)$; where $f(n) = \Theta_{\mathbb{P}}(g(n))$ implies that $f(n)$ is equal to $g(n)$ to first order in the exponent, $\lambda_{Q,1}^{(\alpha)}$ is a Lyapunov exponent associated with the total transmit power, $\{\lambda_{\gamma,i}^{(\alpha)} : i = 1, \dots, d\}$ is the set of Lyapunov exponents associated with the SNR of the d eigenchannels, and $\alpha \in \{f, v\}$ signifies the forwarding strategy (i.e., f for fixed-gain and v for variable-gain). Unlike previous work, our analysis can be applied to systems with a finite number of antennas at each node and arbitrary per-hop channel fading; although, in this manuscript we focus on Rayleigh fading. Before concluding our work, we concentrate on some applications of our results. In particular, we show how they can be used to determine the exponential rate at which the eigenchannel capacities diverge away from each other, how this relates to the forwarding strategy and number of antennas at each relay, and the extra cost (in terms of power) that must be incurred for each extra data stream that is multiplexed over the n -hop network.

Index Terms

Relay network, amplify-and-forward, AF, MIMO, capacity, affine, random dynamical system, RDS, Lyapunov exponent, scaling, finite antenna.

The authors are with the Department of Engineering Science, University of Oxford, Parks Road, Oxford, OX1 3PJ, UK.
Email: david.simmons@eng.ox.ac.uk, justin.coon@eng.ox.ac.uk, naqeeb.ahmedwarsi@eng.ox.ac.uk

I. INTRODUCTION

Consider a multiple-input multiple-output (MIMO) link with d_S source antennas and d_D destination antennas. It is well known that, under some basic assumptions (i.e., independent channel fading between each antenna pair), the capacity will *almost surely* scale linearly with $\min\{d_S, d_D\}$, [1].

Now, consider an n -hop MIMO link, aided by $n - 1$ amplify-and-forward (AF) relay nodes, where each relay node is equipped with d antennas. Furthermore, assume that signals received at the i th relay node propagate only as far as the $(i + 1)$ th node (Fig. 1). The deployment of relays is interesting because it can increase the diversity gain [2], and extend the coverage area of the network [3]. The end-to-end capacity, $c_n^{(\alpha)}$, of such links has been studied in many works, [4]–[8]. In [4], [5] it was shown that the limit $\lim_{n \rightarrow \infty} \left[\lim_{d_D \rightarrow \infty} \left[c_n^{(\alpha)} / d_D \right] / n \right]$ exists *almost surely* and is strictly positive, provided $d/d_D = \Omega(n)$. This work ([4], [5]) also considered the aforementioned limit for other forwarding strategies; namely, decode-and-forward, compress-and-forward, and quantize-and-forward. In [9], the asymptotic (in matrix dimension) eigenvalue distribution of the channel's covariance matrix for multihop MIMO channels with noiseless relays was established. Using free probability theory and, again, under the premise that negligible noise was received at the relays, it was shown in [6] that when linear precoding was applied at each relay, $c_n^{(\alpha)}$ would converge *almost surely* to a limit as d grows large. The singular vectors of the optimal precoding matrices for such a network when noise is negligible at the relays was also established in [6]. Ergodic capacity and average bit error rate results were established in [7] for multihop AF MIMO networks when arbitrary signaling occurs at the source node and, again, d grows without bound. Meanwhile, in [8], $c_n^{(\alpha)}$ was assessed for general n -hop AF networks in terms of the limiting (in d) eigenvalue distribution of products of random matrices when noise is *not* negligible at the relay nodes. Related work on the diversity-multiplexing tradeoff, [10], for various MIMO multihop relaying strategies can be found in [11]–[13].

To the best of the authors' knowledge, all attempts to study the statistical behavior of the end-to-end capacity for n -hop AF MIMO networks have leveraged a viewpoint in which the number of antennas at each node grows large. To achieve this, results from random matrix theory [14] have commonly been employed; e.g., [15]–[17], which describe the asymptotic/limiting spectral properties of large random matrices. In practice, it is desirable to establish capacity scaling

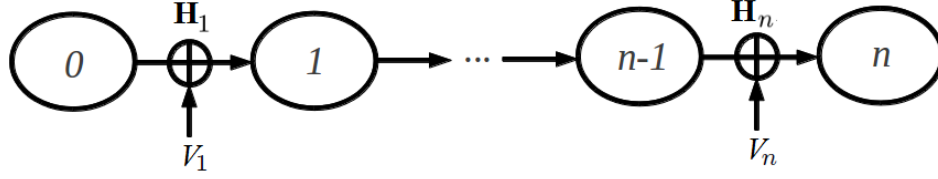


Fig. 1. An illustration of a linear relay network. The i th hop's channel is described by the channel matrix \mathbf{H}_i . The noise received at the j th node is described by the vector V_j . Nodes 0 and n are the source and destination, respectively.

laws when the number of antennas at each node is *finite*. In this paper, we establish such laws using the formalism of random dynamical systems (RDS), [18]. Such systems have also been used to study econometrics [19], biological systems [20], [21], chemical reactions [21], and the propagation of particles through fluidic media [22]. For relevant information on RDSs, the reader may refer to section II.

Going into more detail, the Lyapunov exponents [18] of RDSs are known to characterize the exponential growth/decay rates of the spectrum of finite dimensional random matrix products [18], [23], [24]. In this manuscript, we use these Lyapunov exponents to study the spectral properties of the n -hop AF MIMO network. The main conclusion of our paper is that the Lyapunov exponents of the network, which are obtained by studying the network as an RDS, can be used to evaluate the exponential growth/decay of the n th node transmit power and n th node end-to-end channel capacity when each of the nodes in the network has a finite number of antennas.

A. Key Results

One of the key insight that we provide in this paper is that n -hop AF MIMO systems can be studied from the viewpoint of RDS. To the best of the authors knowledge, this is the first time such an approach has been taken in the literature. This viewpoint then leads us to obtain the following results:

- In Lemma 2 and Theorem 2, we show that the d antenna MIMO AF network has associated with it an ordered set $\{\lambda_{\alpha\mathbf{H},1}, \dots, \lambda_{\alpha\mathbf{H},d}\}$ of Lyapunov exponents satisfying

$$\lambda_{\alpha\mathbf{H},1} > \dots > \lambda_{\alpha\mathbf{H},d},$$

where the $\alpha \in \{f, v\}$ term in the subscript denotes the forwarding strategy that is being implements (i.e., f for fixed-gain or v for variable-gain). From this ordered set, two other

sets of exponents are established. The first of these sets is constructed from elements of the form $\lambda_{\mathbf{Q},i}^{(\alpha)} = \max\{\lambda_{\alpha\mathbf{H},i}, 0\}$, and is associated with the instantaneous total transmit signal at the n th node. The second of these sets is constructed from elements of the form $\lambda_{\gamma,i}^{(\alpha)} = \min\{2\lambda_{\alpha\mathbf{H},i}, 0\}$, and is associated with the end-to-end SNR of the network's d eigenchannels.

- In Lemma 2, we show that the instantaneous transmit power, $\|X_n^{(\alpha)}\|^2$, at the n th node is given by

$$\|X_n^{(\alpha)}\|^2 = \Theta_{\mathbb{P}}\left(e^{2n\lambda_{\mathbf{Q},1}^{(\alpha)}}\right), \quad (1)$$

where, $f(n) = \Theta_{\mathbb{P}}(g(n))$ implies that $f(n)$ is equal to $g(n)$, to first order in the exponent [25, eq. (3.26)]. This is defined more rigorously in the notation subsection (subsection I-C).

- In Theorem 2, we show that the SNR and capacity of the i th eigenchannel at the n th node, $\mathcal{E}_{i,n}\left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)}\mathbf{R}_{\mathcal{N},n}^{(\alpha)-1}\right)$ and $c_{i,n}^{(\alpha)}$, are given, respectively, by

$$\mathcal{E}_{i,n}\left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)}\mathbf{R}_{\mathcal{N},n}^{(\alpha)-1}\right) = \Theta_{\mathbb{P}}\left(e^{n\lambda_{\gamma,i}^{(\alpha)}}\right) \text{ and } c_{i,n}^{(\alpha)} = \Theta_{\mathbb{P}}\left(e^{n\lambda_{\gamma,i}^{(\alpha)}}\right), \quad (2)$$

where $\mathbf{R}_{\mathcal{I},n}^{(\alpha)}$ and $\mathbf{R}_{\mathcal{N},n}^{(\alpha)}$ are the covariance matrices of the received information vector and noise vector at the n th node, respectively.

On top of our main information theoretic results, we also establish the following notable secondary information theoretic results:

- In Lemma 2, we show that to ensure the instantaneous transmit power *almost surely* displays no exponential growth, and that the end-to-end capacity of the dominant eigenchannel *almost surely* displays no exponential decay (i.e., from (1) and (2), $\lambda_{\alpha\mathbf{H},1} = 0$), the average transmit power *must* grow exponentially with n . Furthermore, this rate of growth can be reduced by:
 - 1) implementing variable-gain instead of fixed-gain,
 - 2) increasing the number of antennas at each node.
- In (49), we show that the exponential rate at which the capacities of the i th and j th ($i < j$) eigenchannels diverge away from each other is given by $n\left(\lambda_{\gamma,i}^{(\alpha)} - \lambda_{\gamma,j}^{(\alpha)}\right)$. When the i th and j th eigenchannel capacities are either both decaying or both not decaying, this divergence rate is shown to be independent of whether fixed-gain or variable-gain relaying is being performed. Furthermore, from Lemma 5 and Corollary 1, with $i = 1$, to ensure that this rate is asymptotically bounded away from infinity (so that multiplexing j streams

is asymptotically viable) we must either: 1) ensure that $\lambda_{\alpha\mathbf{H},j} \geq 0$, or 2) ensure that the number of antennas at each node grows like $d = \Omega(n)$. This result complements those presented in [4], [5].

- In Remark 3, we assign a transmit power cost to the n th node for each extra data stream that is multiplexed over the network. In particular, if i data streams are being multiplexed, then, to multiplex one extra stream, we must increase the n th relay's instantaneous transmit power by a factor of $\exp(n/(d-i))$.

On the way to proving the above mentioned results, we also obtain the following RDS results, which we believe are of independent interest.

- Let $\mathbf{A}_i \in \mathbb{C}^{d \times d}$ and $R_i \in \mathbb{C}^d$ for $i \in \mathbb{N}$ be random matrices and vectors, respectively, with $\mathbb{E} \log^+ \|\mathbf{A}_1\| < \infty$, $\mathbb{E} \log^+ \|R_1\| < \infty$, and $R_i \stackrel{a.s.}{\neq} 0$. Suppose that there exist $\alpha_j, \beta_j \in \mathbb{R}$ such that \mathbf{A}_1 is equal in distribution to $\alpha_j \mathbf{A}_j$ and R_1 is equal in distribution to $\beta_j R_j$. In Lemma 1, we show that the Lyapunov exponents of an affine RDS taking the form

$$X_n = \mathbf{A}_n X_{n-1} + R_n, \quad (3)$$

are strictly positive, and, consequently, are identical to those of

$$\begin{bmatrix} X_n \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_n & R_n \\ \mathbf{0}^T & 1 \end{bmatrix} \cdots \begin{bmatrix} \mathbf{A}_1 & R_1 \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ 1 \end{bmatrix}. \quad (4)$$

- In Theorem 1, we show that the Lyapunov exponents of (4) are given by the non-negative Lyapunov exponents of

$$\pi_n(\mathbf{A}) := \mathbf{A}_n \cdots \mathbf{A}_1. \quad (5)$$

Less formally, our RDS results provide us with a framework for determining *all* of the Lyapunov exponents of d dimensional affine RDSs, which will be crucial to our information theoretic analysis.

B. Paper Layout

Section II introduces the mathematical preliminaries and new RDS results that will be utilized throughout this paper. Section III introduces the system model. In section IV we calculate the Lyapunov exponents, and show that they can be used to characterize the network's transmit power and end-to-end channel capacity. Section V establishes applications of the results that are

obtained in section IV. Section VI provides numerical illustrations of the theory that has been developed. Finally, section VII concludes the paper.

C. Notation and Definitions

We use $\stackrel{d}{=}$ to denote equality in distribution, $:=$ to denote equality by definition and $\log^+(x) := \max\{0, \log x\}$. $\mathbf{0}$ is used to denote the column vector of zeros, where the dimension of $\mathbf{0}$ will be implied from the surrounding text. Matrices are always represented using uppercase boldface notation, vectors are always represented using uppercase non-boldface notation, and scalars are always represented using lowercase notation. $\mathcal{E}_i\{\mathbf{A}\}$ is used to denote the i th ordered eigenvalue of the matrix \mathbf{A} , where $\mathcal{E}_i(\mathbf{A}) \geq \mathcal{E}_j(\mathbf{A})$ implies $i \leq j$. \mathbf{A}^\dagger is used to denote the conjugate transpose of the matrix \mathbf{A} . Matrix products are defined in the following way:

$$\prod_{i=j}^n \mathbf{A}_i := \mathbf{A}_n \cdots \mathbf{A}_j, \quad (6)$$

and when $j = 1$ we sometimes use the definition

$$\pi_n(\mathbf{A}) := \prod_{i=1}^n \mathbf{A}_i. \quad (7)$$

The standard 2-norm of a matrix \mathbf{A} is denoted by $\|\mathbf{A}\|$, and its Frobenius norm is denoted by $\|\mathbf{A}\|_F$. The Landau notation $f(x) = o(g(x))$ is used to imply $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$. Also, we use the following notation:

$$f(n) = O(g(n)) \Rightarrow \exists K_1, n' > 0 \text{ s.t. } K_1 |g(n)| > |f(n)|, n > n'$$

$$f(n) = \Omega(g(n)) \Rightarrow \exists K_2, n' > 0 \text{ s.t. } K_2 |g(n)| < |f(n)|, n > n'$$

$$f(n) = \Theta(g(n)) \text{ if } f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n));$$

and, similar to the notation proposed in [26], for a strictly positive random variable $f(n)$ depending on n , and some $h(n) = |o(n)|$,

$$f(n) = O_{\mathbb{P}}(g(n)) \Rightarrow \lim_{n \rightarrow \infty} \mathbb{P}[f(n) \leq g(n)e^{h(n)}] = 1.$$

$$f(n) = \Omega_{\mathbb{P}}(g(n)) \Rightarrow \lim_{n \rightarrow \infty} \mathbb{P}[f(n) \geq g(n)e^{-h(n)}] = 1$$

$$f(n) = \Theta_{\mathbb{P}}(g(n)) \text{ if } f(n) = O_{\mathbb{P}}(g(n)) \text{ and } f(n) = \Omega_{\mathbb{P}}(g(n)).$$

II. RANDOM DYNAMICAL SYSTEM

In this section, we introduce the RDS results that will be relied upon heavily throughout this paper. The first subsection is devoted to presenting preexisting RDS theory, while the second

subsection presents a new result which will be used to calculate the Lyapunov exponents of affine systems.

A. Preliminary RDS Results

The study of dynamical systems is concerned with tracking the trajectory of a position (particle/state/point) through a state space. In the discrete case, this position is calculated through the repeated action of a deterministic map. Informally, an RDS occurs when this map is non-deterministic and drawn from a sample space according to some fixed probability distribution. Such systems are often used to study econometrics [19], biological systems [20], [21], chemical reactions [21], and the propagation of particles through fluidic media [22]. The formal and rather intricate definition of an RDS can be found in [18].

In this manuscript, we consider an RDS to be the action of a product of $d \times d$ ($d \in \mathbb{N}$) complex random matrices on an appropriately dimensioned vector (the initial state $X_0 \in \mathbb{C}^d$). The state of the RDS at time n ($X_n \in \mathbb{C}^d$) can then be written as either

$$X_n = \mathbf{A}_n \cdots \mathbf{A}_1 X_0, \quad (8)$$

or

$$X_n = \mathbf{A}_1 \cdots \mathbf{A}_n X_0 \quad (9)$$

where, in general, we assume that $\mathbf{A}_1, \dots, \mathbf{A}_n$ are independent and identically distributed (i.i.d.) up to an arbitrary positive scaling factor. Mathematically, this means that $\exists g_i > 0$ such that $\mathbf{A}_1 \stackrel{d}{=} g_i \mathbf{A}_i$ for all i . Eqs. (8) and (9) are referred to as forward and backward RDSs, respectively, and take their names from the forward [18, Def'n. 1.1.1] and backward [18, Rem. 1.1.10] cocycle properties that their random mappings satisfy. It is interesting and important to note that, unlike (8), (9) is somewhat unnatural, in the sense that it is anticausal; however, all of the RDS properties that are to be described for (8) will apply to (9), [18].

Suppose we wish to study the asymptotic behavior of

$$\|\pi_n(\mathbf{A})\| \quad (10)$$

as $n \rightarrow \infty$. A customary approach is to exponentiate the logarithm of the norm; i.e., write (10) as

$$\|\pi_n(\mathbf{A})\| = e^{n \frac{1}{n} \log \|\pi_n(\mathbf{A})\|} \quad (11)$$

and investigate the behavior of the exponent, specifically, the term $\frac{1}{n} \log \|\pi_n(\mathbf{A})\|$ as n grows large. In this manner, the exponential growth/decay rate of the system can be observed. If $\{\mathbf{A}_j\}$ was a set of scalars (i.e., $d = 1$), the law of large numbers could be employed to evaluate the limiting behavior of $\frac{1}{n} \log \|\pi_n(\mathbf{A})\|$; however, this is not the case for general d .

The question of whether $\frac{1}{n} \log \|\pi_n(\mathbf{A})\|$ tends to a limit does not have a clear answer in most cases. Under the condition that $\mathbb{E}[\log^+ \|\mathbf{A}_1\|] < \infty$ and $\lim_{n \rightarrow \infty} 1/n \sum_{i=1}^n \log^+ |g_i| < \infty$, however, the theorem of Furstenberg and Kesten [18] guarantees that the limit does exist. We then obtain the *Lyapunov index*:

Definition 1: The Lyapunov index is given by

$$\iota(\mathbf{A}) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|\pi_n(\mathbf{A})\|. \quad (12)$$

The Lyapunov index can be used to describe the exponential growth rate of $\|\pi_n(\mathbf{A})\|$. By evaluating the Lyapunov index at a specific initial position within the state space, we then obtain the *Lyapunov exponent*:

Definition 2: The Lyapunov exponent is given by

$$\lambda(\mathbf{A}, X) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|\pi_n(\mathbf{A})X\|. \quad (13)$$

The Lyapunov exponent can be used to describe the exponential growth rate of the norm of a trajectory through its state space, where the initial state of the trajectory is given by X .

Remark 1: In the definitions of the Lyapunov index and exponent ((12) and (13), respectively), if the system is linear then the \limsup can be replaced with a limit, [18].

Fact 1: From [18, pp. 114 – 115, Theorem 3.3.3], assuming $\mathbb{E}[\log^+ \|\mathbf{A}_1\|] < \infty$ and $-\infty < \lim_{n \rightarrow \infty} 1/n \sum_{i=1}^n \log^+ |\alpha_i| < \infty$ (where $\mathbf{A}_1 \stackrel{d}{=} \alpha_i \mathbf{A}_i$), the Lyapunov exponent has the following properties [18]:

- 1) $\lambda(\mathbf{A}, X) \in \mathbb{R} \cup \{-\infty\} \forall X \in \mathbb{C}^d$, where $\lambda(\mathbf{A}, \mathbf{0}) := -\infty$;
- 2) The number, p , of distinct values, λ_i , that $\lambda(\mathbf{A}, X)$ can take on for $X \in \mathbb{C}^d \setminus \{\mathbf{0}\}$ is at most d , and we have $-\infty \leq \lambda_p < \dots < \lambda_1 < \infty$.
- 3) The sets

$$V_i := \{X : \lambda(\mathbf{A}, X) \leq \lambda_i\} \quad (14)$$

are linear subspaces, form a filtration

$$\{0\} =: V_{p+1} \subset V_p \subset \dots \subset V_1 = \mathbb{C}^d \quad (15)$$

(where all inclusions are proper), and

$$\lambda_i = \lambda(\mathbf{A}, X) \Leftrightarrow X \in V_i \setminus V_{i+1}, \quad i = 1, \dots, p. \quad (16)$$

The integer $m(i) := \dim V_i - \dim V_{i-1}$ is the multiplicity of λ_i .

4) The limiting behavior for the ordered singular values of the matrix product $\pi_n(\mathbf{A})$ satisfies

$$\frac{1}{2n} \log \mathcal{E}_i \{ \pi_n(\mathbf{A}) \pi_n(\mathbf{A})^\dagger \} \rightarrow \lambda_i. \quad (17)$$

Consequently, the random variable $\mathcal{E}_i \{ \pi_n(\mathbf{A}) \pi_n(\mathbf{A})^\dagger \}$ satisfies

$$\mathcal{E}_i \{ \pi_n(\mathbf{A}) \pi_n(\mathbf{A})^\dagger \} = \Theta_{\mathbb{P}}(e^{2n\lambda_i}). \quad (18)$$

In what follows, we will often drop the functional notation $\lambda(\mathbf{A}, X)$ and simply write $\lambda_{\mathbf{A},i}$ to refer to the i th ordered Lyapunov exponent of the system corresponding to $\pi_n(\mathbf{A})$. When it is clear, we may also omit the subscript \mathbf{A} as we did in Fact 1.

B. On the Lyapunov Exponents of Affine RDS

Throughout this paper, we will often be concerned with the Lyapunov exponents of affine systems of the form

$$X_n = \mathbf{A}_n X_{n-1} + R_n, \quad (19)$$

where $\mathbf{A}_n \in \mathbb{C}^{d \times d}$ is a random matrix that satisfies $\mathbf{A}_n \stackrel{d}{=} -\mathbf{A}_n$ and $R_n \in \mathbb{C}^d$ with $R_n \stackrel{a.s.}{\neq} 0$.

The following theorem will be used in the calculation of these exponents.

Theorem 1: For $i \in \mathbb{N}$, consider the product of random matrices $\pi_n(\mathbf{M})$, where

$$\mathbf{M}_i = \begin{bmatrix} \mathbf{A}_i & R_i \\ \mathbf{0}^T & a_i \end{bmatrix} \in \mathbb{C}^{(d+1) \times (d+1)} \quad (20)$$

$\mathbb{E} \log^+ \|\mathbf{A}_i\| < \infty$, $\mathbb{E} \log^+ \|R_i\| < \infty$, and $\mathbb{E} \log^+ |a_i| < \infty$. Under the assumption that $\pi_n(\mathbf{A})$ has d distinct Lyapunov exponents, $\forall i \exists X_0 \in \mathbb{C}^d$ such that

$$\lambda \left(\mathbf{M}, [X_0^T \ 1]^T \right) = \max \{ \lambda_{\mathbf{A},i}, \lambda_{a,1} \}. \quad (21)$$

Proof: See Appendix A. ■

1) *Applications and/or Implications of Theorem 1:* We will now show that Theorem 1 can be used to calculate the Lyapunov exponents of (19). To do this, the affine structure of (19) will be captured by converting it into a linear (non-affine) $(d+1) \times (d+1)$ system of the following form:

$$\begin{bmatrix} X_n \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_n & R_n \\ \mathbf{0}^T & 1 \end{bmatrix} \cdots \begin{bmatrix} \mathbf{A}_1 & R_1 \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ 1 \end{bmatrix}. \quad (22)$$

One may naively assume that the Lyapunov exponents of (19) are trivially identical to those of (22). However, for this to be true, we must have $\lim_{n \rightarrow \infty} \frac{1}{n} \log \|X_n\| \geq 0$, because, clearly,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left\| \begin{bmatrix} X_n & 1 \end{bmatrix}^T \right\| \geq 0. \quad (23)$$

We now provide the following lemma, which tells us that the Lyapunov exponents of (19) are indeed strictly non-negative, and that, consequently, the Lyapunov exponents of (19) and (22) are identical.

Lemma 1: For X_n given by (19), we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|X_n\| \geq 0.$$

Consequently, the Lyapunov exponents of the affine system, (19), and those of the linear-to-affine converted system, (22), are identical.

Proof: See Appendix B. ■

From Fact 2 (below), it can be seen that the Lyapunov exponents of (22) (and consequently (19)) must belong to $\{\max\{\lambda_{\mathbf{A},i}, 0\} : i = 1, \dots, d\}$.

Fact 2: [27, Theorem 5], consider the product of random matrices $\pi_n(\mathbf{M})$, where

$$\mathbf{M}_i = \begin{bmatrix} \mathbf{A}_i & R_i \\ \mathbf{0}^T & a_i \end{bmatrix} \in \mathbb{C}^{(d+1) \times (d+1)}, \quad (24)$$

$\mathbb{E} \log^+ \|\mathbf{A}_i\| < \infty$, $\mathbb{E} \log^+ \|R_i\| < \infty$, and $\mathbb{E} \log^+ |a_i| < \infty$. Then the Lyapunov exponents of $\pi_n(\mathbf{M})$ are given by $\{\lambda_{\mathbf{A},i} : i = 1, \dots, n\} \cup \lambda_{a,1}$. Furthermore, the Lyapunov exponents of $\pi_n(\mathbf{M})$ are independent of the statistics of R_i .

Notice that Fact 2 tells us nothing about how initial states of the form $\begin{bmatrix} X_0^T & 1 \end{bmatrix}^T$ (c.f. (22)), will affect the Lyapunov analysis. Consequently, we require a theorem that deals with such initial states. This provides the rational behind Theorem 1.

III. SYSTEM MODEL

Let us present the signaling model used in this paper. Consider an n -hop AF relay network, as depicted in Fig. 1. We assume that each node has $d \geq 1$ transmit and receive antennas. Independent frequency-flat Rayleigh fading [28] is assumed to take place between each node pair. Thus, the channel for the i th hop can be described by a $d \times d$ random matrix, \mathbf{H}_i , whose elements are zero-mean, circularly symmetric complex Gaussian (ZMCG) [28] with total variance μ_i ; i.e., for $h_{ab,i} \sim \mathcal{CN}(0, \mu_i)$, we have

$$\mathbf{H}_i = \begin{bmatrix} h_{11,i} & h_{12,i} & \cdots & h_{1d,i} \\ h_{21,i} & h_{22,i} & \cdots & h_{2d,i} \\ \vdots & & \ddots & \vdots \\ h_{d1,i} & h_{d2,i} & \cdots & h_{dd,i} \end{bmatrix}. \quad (25)$$

At each node (apart from the zeroth node) we assume noise is introduced into the system. We use $V_j \in \mathbb{C}^d$ to denote the vector of noise terms introduced at the j th relay. The elements of V_j correspond to the noise samples received at each antenna of node j and are independent ZMCG random variables with total variance n_0 .

An information vector

$$X_0 = [x_{0,1}, \dots, x_{0,d}]^T \quad (26)$$

is constructed at the source (node 0). Without loss of generality, we assume its elements have a mean of zero and average power given by $\mathbb{E}[|x_{0,i}|^2] = p_0/d$. The i th element of X_0 is then transmitted from the i th antenna of node 0.

We assume the j th relay node receives the transmission only from the $(j-1)$ th node in one time slot. This relay then applies a scalar gain, g_j , to the received signal on each of its antennas and transmits in the next time slot. Thus, the relays operate in a half-duplex manner. The gain for the j th relay is either a fixed-gain parameter, depending only upon the average statistics of the channel matrix of the previous hop, given by

$$f_j = \sqrt{\frac{p_j}{p_{j-1}d\mu_j + dn_0}}; \quad (27)$$

or a variable-gain parameter given by [29, eq. (7)]

$$v_j = \sqrt{\frac{p_j}{\frac{p_{j-1}}{d} \|\mathbf{H}_j\|_F^2 + dn_0}}. \quad (28)$$

The term p_j is selected by the relay, and represents the average transmit power at node j . Also, we assume that $\lim_{n \rightarrow \infty} (1/n) \log(p_n/p_0) < \infty$. This assumption implies that the average transmit power does not grow at a super-exponential rate. Similar assumptions have also been made in [5].

The information bearing content of the signal (herein referred to as the *information component*) at the n th node is given by

$$\mathcal{I}_n^{(\alpha)} = \prod_{j=1}^n \alpha_j \mathbf{H}_j X_0, \quad (29)$$

where $\alpha \in \{f, v\}$ dependent upon whether fixed-gain or variable-gain is being implemented. Similarly, the total transmitted signal at the n th node is given by

$$X_n^{(\alpha)} = \mathcal{I}_n^{(\alpha)} + \underbrace{\alpha_n \sum_{i=1}^n \prod_{j=i+1}^n \alpha_{j-1} \mathbf{H}_j V_i}_{\mathcal{N}_n^{(\alpha)}}, \quad (30)$$

where $\prod_{j=n+1}^n \alpha_{j-1} \mathbf{H}_j V_n := V_n$ and $\mathcal{N}_n^{(\alpha)}$ denotes the accumulated noise at node n . Owing to the discussion in section II-B, (30) can be re-expressed in matrix form as

$$\begin{bmatrix} X_n^{(\alpha)} \\ \sqrt{n_0} \end{bmatrix} = \prod_{j=1}^n \mathbf{Q}_j^{(\alpha)} \begin{bmatrix} X_0 \\ \sqrt{n_0} \end{bmatrix}, \quad (31)$$

where

$$\mathbf{Q}_j^{(\alpha)} := \begin{bmatrix} \alpha_j \mathbf{H}_j & \alpha_j Z_j \\ \mathbf{0}^T & 1 \end{bmatrix} \quad (32)$$

and $Z_i = V_i/\sqrt{n_0}$.

IV. CAPACITY AND POWER SCALING

In this section, we will use the network's Lyapunov exponents to establish the scaling behavior of the end-to-end capacity and n th node transmit power. The following theorem relates the Lyapunov exponents to the network's end-to-end capacity.

Theorem 2: Let $\lambda_{\alpha \mathbf{H}, i}$ be the Lyapunov exponent of $\mathcal{I}_n^{(\alpha)}$ (which is defined in (29)). Let $c_n^{(\alpha)} = \log \det \left(\mathbf{I}_d + \mathbf{R}_{\mathcal{I}, n}^{(\alpha)} \mathbf{R}_{\mathcal{N}, n}^{(\alpha)-1} \right) = \sum_{i=1}^d c_{n,i}^{(\alpha)}$ (nats/channel use) denote the capacity of the n -hop AF network (for more details, see [5]), where

$$\mathbf{R}_{\mathcal{I}, n}^{(\alpha)} = \left(p_0 \prod_{i=1}^n g_{\alpha, i}^2 \right) \mathbf{H}_n \cdots \mathbf{H}_1 \mathbf{H}_1^\dagger \cdots \mathbf{H}_n^\dagger, \quad (33)$$

$$\mathbf{R}_{\mathcal{N}, n}^{(\alpha)} = n_0 \left(\mathbf{I}_d + \sum_{l=2}^n \prod_{i=l}^n g_{\alpha, i}^2 \mathbf{H}_n \cdots \mathbf{H}_l \mathbf{H}_l^\dagger \cdots \mathbf{H}_n^\dagger \right) \quad (34)$$

are covariance matrices; and $c_{n,i}^{(\alpha)} = \log \left(1 + \mathcal{E}_i \left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)} \mathbf{R}_{\mathcal{N},n}^{(\alpha)-1} \right) \right)$ is the capacity of the i th eigenchannel at the n th node. Then the following statements hold.

- A. $\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{E}_i \left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)} \mathbf{R}_{\mathcal{N},n}^{(\alpha)-1} \right) \stackrel{a.s.}{=} \min\{0, 2\lambda_{\alpha\mathbf{H},i}\} =: \lambda_{\gamma,i}^{(\alpha)}$. Hence, the SNR of the i th eigenchannel is given by $\mathcal{E}_i \left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)} \mathbf{R}_{\mathcal{N},n}^{(\alpha)-1} \right) = \Theta_{\mathbb{P}} \left(e^{n\lambda_{\gamma,i}^{(\alpha)}} \right)$.
- B. $\lim_{n \rightarrow \infty} \frac{1}{n} \log c_{n,i}^{(\alpha)} \stackrel{a.s.}{=} \min\{0, 2\lambda_{\alpha\mathbf{H},i}\} =: \lambda_{\gamma,i}^{(\alpha)}$. Thus, the capacity of the i th eigenchannel is given by $c_{n,i}^{(\alpha)} = \Theta_{\mathbb{P}} \left(e^{n\lambda_{\gamma,i}^{(\alpha)}} \right)$.

Proof: See Appendix C. ■

The next lemma will evaluate the Lyapunov exponent $\lambda_{\alpha\mathbf{H},i}$, and establish how it relates to the Lyapunov exponents of $X_n^{(\alpha)}$ and the average transmit power at the n th node. This lemma will in turn allow us to establish a trade off between capacity decay and power growth across the network. It will also have implications on gain design.

Lemma 2: With $\mathcal{I}_n^{(\alpha)}$ given by (29), $X_n^{(\alpha)}$ given by (30), and the average transmit power at the n th node given by p_n , the following statements hold.

- A. The i th Lyapunov exponent of $\mathcal{I}_n^{(\alpha)}$ is given by

$$\lambda_{\alpha\mathbf{H},i} = \frac{1}{2} \left(L(\alpha^2 \mu) + \psi(d - i + 1) \right), \quad i = 1, \dots, d; \quad (35)$$

where $\psi(\cdot)$ is the digamma function [30, eq. (6.3.1)] and

$$L(\alpha^2 \mu) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \log(\alpha_i^2 \mu_i). \quad (36)$$

The i th Lyapunov exponent of $X_n^{(\alpha)}$ is given by

$$\lambda_{\mathbf{Q},i}^{(\alpha)} := \lambda(\mathbf{Q}, [X_0^T \sqrt{n_0}]) = \max\{0, \lambda_{\alpha\mathbf{H},i}\}. \quad (37)$$

Hence, the information power and total transmit power are given by

$$\|\mathcal{I}_n^{(\alpha)}\|^2 = \Theta_{\mathbb{P}} \left(e^{2n\lambda_{\alpha\mathbf{H},1}} \right), \quad (38)$$

$$\|X_n^{(\alpha)}\|^2 = \Theta_{\mathbb{P}} \left(e^{2n\lambda_{\mathbf{Q},1}^{(\alpha)}} \right). \quad (39)$$

- B. For fixed-gain, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{p_n}{p_0} \geq \max\{2\lambda_{f\mathbf{H},1} + 2\log d - \psi(d), 0\}; \quad (40)$$

for variable-gain, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{p_n}{p_0} \geq \max\{2\lambda_{v\mathbf{H},1} + \psi(d^2) - \psi(d) + \log d, 0\}; \quad (41)$$

where equality is maintained only when $n_0 = 0$.

Proof: See Appendix D. ■

A. A Brief Discussion of Theorem 2 and Lemma 2

From the first statement of Lemma 2, by ensuring $\lambda_{\alpha\mathbf{H},d} < \dots < \lambda_{\alpha\mathbf{H},1} = 0$, we can avoid exponential growth in the instantaneous transmit power. However, in this setup Theorem 2 tells us that all but the first eigenchannel will display an exponentially decaying capacity. Conversely, by ensuring $\lambda_{\alpha\mathbf{H},1} > \dots > \lambda_{\alpha\mathbf{H},q} \geq 0 > \lambda_{\alpha\mathbf{H},q+1} > \dots > \lambda_{\alpha\mathbf{H},d}$, we can stop the end-to-end capacity of the upper q eigenchannels from *almost surely* decaying exponentially. However, in this scenario, we must allow for exponential growth in the instantaneous power across the network. Thus, there is a clear tradeoff to be had between multiplexing multiple data streams across the network, and growth in the instantaneous transmit power at the n th node.

Focusing on the second statement of Lemma 2, it can be seen that the terms $\log d - \psi(d)$ and $\psi(d^2) - \psi(d) + \log d$ in (40) and (41), respectively, are strictly non-negative. Thus, this statement tells us that, asymptotically, the average transmit power must grow at a greater exponential rate than the instantaneous power. Crucially, we find that exponential growth in p_n can be allowed for whilst avoiding (with high probability) exponential power growth at the relays. Said in a different way, as the network scales in size, the density function of the transmit power at the n th node becomes increasingly heavy tailed. Whilst most of the distribution's mass will be concentrated at the point governed by the Lyapunov exponent (cf. (39)), the distribution's heavy tail will push the average up exponentially. Combining this observation with Theorem 2, it can be seen that ensuring the first eigenchannel displays a non exponentially decaying capacity implies that the average transmit power will grow exponentially.

It can also be seen that, because $\log d - \psi(d) \geq \psi(d^2) - \psi(d) + \log d$, the lower bound on the exponential growth rate of the average transmit power for variable-gain is strictly less than that for fixed-gain, which suggests that the variable-gain network can sustain an approximately constant instantaneous power trend with a reduced growth in the average transmit power. Furthermore, as the number of antennas grows large, both bounds in Lemma 2 converge towards the Lyapunov exponents. Thus, ergodic behavior is induced as d grows large.

In summary, Theorem 2 and the first statement of Lemma 2 expose a fundamental trade off between capacity decay and instantaneous transmit power growth across the network. The

second statement of Lemma 2 has important implications on gain design for scaled networks. In particular, it implies that the average transmit power at each node should grow exponentially with the network if an approximately constant instantaneous power trend is to be maintained. These implications contrast with the system model proposed in [4], [5], where the capacity was assessed under strictly linear scaling of p_n . For the finite antenna system, we see that if linear scaling of p_n occurs, $\lim_{n \rightarrow \infty} \log(p_n/p_0)/n = 0$ and (from Lemma 2) $\lambda_{\alpha\mathbf{H},1} < 0$. As has been seen in the Theorem 2, $\lambda_{\alpha\mathbf{H},1} < 0$ will have serious implications on the network's end-to-end capacity.

V. APPLICATIONS OF THEOREM 2 AND LEMMA 2

In this section, we will study some applications of Theorem 2 and Lemma 2. In particular, we will study the rates at which the eigenchannel capacities diverge away from each other, and how this relates to:

- the forwarding strategy and number of antennas at each node,
- the growth in the instantaneous transmit power.

To discuss the above mentioned points, we will require the following preliminary definitions and lemmas.

A. Preliminary Definitions and Lemmas

Definition 3: The (i, j) th normalized channel capacity, $i \leq j$, is defined to be

$$\nu_{i,j,n}^{(\alpha)} := \frac{c_{i,n}^{(\alpha)}}{c_{j,n}^{(\alpha)}}. \quad (42)$$

Clearly, if $\nu_{1,j,n}^{(\alpha)} \approx 1$, the channel will be well suited for multiplexing j data streams, [28], [31], provided $c_{1,n}^{(\alpha)}$ is sufficiently large; otherwise, it will not.

Definition 4: For both fixed-gain and variable-gain, the (i, j) th *Lyapunov difference*, $i \leq j$, is defined to be

$$\phi_{i,j}^{(\alpha)} := \lambda_{\gamma,i}^{(\alpha)} - \lambda_{\gamma,j}^{(\alpha)}. \quad (43)$$

The following two lemmas are used to bound $\phi_{i,j}^{(\alpha)}$, and will be employed in the ensuing analysis.

Lemma 3: The (i, j) th Lyapunov difference is bounded as follows:

$$0 \leq \phi_{i,j} \leq 2(\lambda_{\alpha\mathbf{H},i} - \lambda_{\alpha\mathbf{H},j}) =: \bar{\phi}_{i,j}, \quad (44)$$

where lower equality is maintained if and only if $\lambda_{\alpha\mathbf{H},i} > \lambda_{\alpha\mathbf{H},j} \geq 0$, upper equality is maintained if and only if $\lambda_{\alpha\mathbf{H},j} < \lambda_{\alpha\mathbf{H},i} \leq 0$, and $\phi_{i,j}^{(\alpha)} = -2\lambda_{\alpha\mathbf{H},j}$ otherwise. Furthermore, the upper bound is independent of whether fixed-gain or variable-gain is being implemented.

Proof: See Appendix E. ■

Finally, we will also exploit the following lemma later in this section.

Lemma 4: For $i < j$, we have

$$\bar{\phi}_{i,j} = \sum_{k=d-j+1}^{d-i} \frac{1}{k} = (\mathcal{H}_{d-i} - \mathcal{H}_{d-j}), \quad (45)$$

where \mathcal{H}_i is the i th harmonic series defined to be

$$\mathcal{H}_i = \sum_{j=1}^i \frac{1}{j}. \quad (46)$$

Furthermore, by considering the first and last summands in (45), we can trivially construct the following bound:

$$\frac{j-i+1}{d-i} \leq \bar{\phi}_{i,j} \leq \frac{j-i+1}{d-j+1}. \quad (47)$$

Proof: This follows immediately from (35), Lemma 3, and applying the telescope property of the digamma function [30, eq. (6.3.5)]:

$$\psi(x+1) = \psi(x) + \frac{1}{x}. \quad (48)$$
■

B. Growth of $\nu_{i,j,n}^{(\alpha)}$ and $\|X_n\|^2$

We will now apply Theorem 2 and Lemma 2 to study $\nu_{i,j,n}^{(\alpha)}$ (Definition 3) and $\|X_n^{(\alpha)}\|^2$. Considering Theorem 2 first, from Definitions 3 and 4 we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \nu_{i,j,n}^{(\alpha)} = \phi_{i,j}^{(\alpha)} \iff \nu_{i,j,n}^{(\alpha)} = \Theta_{\mathbb{P}} \left(e^{n\phi_{i,j}^{(\alpha)}} \right). \quad (49)$$

We will now use (49) (in conjunction with Lemma 2) to study the following four problems:

- 1) The dependence of the growth in $\nu_{i,j,n}^{(\alpha)}$ on the forwarding strategy.
- 2) The dependence of the growth in $\nu_{i,j,n}^{(\alpha)}$ on the number of antennas at each node

- 3) The behavior of the network when either $\phi_{1,i}^{(\alpha)} = 0$ or $\lambda_{\alpha\mathbf{H},1} = 0$; i.e., when either $\nu_{i,j,n}^{(\alpha)}$ or $\|X_n^{(\alpha)}\|^2$ display no exponential growth, respectively.
- 4) The growth in $\nu_{i,i+1,n}^{(\alpha)}$ (i.e., rate at which adjacent eigenchannel capacities diverge away from each other), and the cost (in terms of instantaneous transmit power) associated with each extra multiplexed data stream.

1) *Growth of $\nu_{i,j,n}$ and the Forwarding Strategy* : Let us first establish how the forwarding strategy affects the growth of $\nu_{i,j,n}^{(\alpha)}$. As an immediate consequence of Lemma 3, it can be seen that when $\lambda_{\alpha\mathbf{H},j} < \lambda_{\alpha\mathbf{H},i} \leq 0$ the exponential growth of $\nu_{i,j,n}^{(\alpha)}$ will be independent of the forwarding strategy that has been implemented. The same holds true when $0 \leq \lambda_{\alpha\mathbf{H},j} < \lambda_{\alpha\mathbf{H},i}$, since we will have $\phi_{i,j,n}^{(\alpha)} = 0$. For $\lambda_{\alpha\mathbf{H},j} < 0 < \lambda_{\alpha\mathbf{H},i}$, we will have $\phi_{i,j,n}^{(\alpha)} = -2\lambda_{\alpha\mathbf{H},j}$. Consequently, in this scenario $\nu_{i,j,n}^{(\alpha)}$ is given by

$$\nu_{i,j,n}^{(f)} = \Theta_{\mathbb{P}} \left(e^{-2\lambda_{f\mathbf{H},j}} \right) = \Omega_{\mathbb{P}} \left(e^{-\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{pn}{p_0} + \log d - \psi(d-j+1)} \right), \quad (50)$$

for fixed-gain, and

$$\nu_{i,j,n}^{(v)} = \Theta_{\mathbb{P}} \left(e^{-2\lambda_{v\mathbf{H},j}} \right) = \Omega_{\mathbb{P}} \left(e^{-\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{pn}{p_0} - \log(d) + \psi(d^2) - \psi(d-j+1)} \right), \quad (51)$$

for variable-gain, where the second equalities of (50) and (51) follow from Lemma 7. Notice that, because $\log d^2 > \psi(d^2)$,

$$e^{-\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{pn}{p_0} + \log d - \psi(d-j+1)} \geq e^{-\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{pn}{p_0} - \log(d) + \psi(d^2) - \psi(d-j+1)}. \quad (52)$$

2) *Growth of $\nu_{i,j,n}$ and the Number of Antennas*: We will now establishing how the number of antennas at each node will affect the growth rate of $\nu_{i,j,n}^{(\alpha)}$. In particular, we will determine how $n\phi_{i,j}^{(\alpha)}$ (the term in the exponent of (49)) scales with n , and how the number of antennas relates to this. More specifically, in what follows (Lemma 5 and Corollary 1) we will determine conditions that give the following:

$$\lim_{n \rightarrow \infty} \left[n\phi_{i,j}^{(\alpha)} \right] = 0 \quad \Leftrightarrow \quad \phi_{i,j}^{(\alpha)} = o(1/n), \quad (53)$$

$$0 < \lim_{n \rightarrow \infty} \left[n\phi_{i,j}^{(\alpha)} \right] \leq K < \infty \quad \Leftrightarrow \quad \phi_{i,j}^{(\alpha)} = \Theta(1/n), \quad (54)$$

$$\lim_{n \rightarrow \infty} \left[n\phi_{i,j}^{(\alpha)} \right] = \infty \quad \Leftrightarrow \quad 1/\phi_{i,j}^{(\alpha)} = o(n). \quad (55)$$

Of course, if $\phi_{i,j}^{(\alpha)} = 0$ (i.e., $\lambda_{\alpha\mathbf{H},i} > \lambda_{\alpha\mathbf{H},j} \geq 0$) (53) is obtained trivially. We are therefore only interested in studying the behavior of $n\phi_{i,j}^{(\alpha)}$ when either $0 \geq \lambda_{\alpha\mathbf{H},i} > \lambda_{\alpha\mathbf{H},j}$ or $\lambda_{\alpha\mathbf{H},i} > 0 > \lambda_{\alpha\mathbf{H},j}$. We treat $0 \geq \lambda_{\alpha\mathbf{H},i} > \lambda_{\alpha\mathbf{H},j}$ in Lemma 5 and consider $\lambda_{\alpha\mathbf{H},i} > 0 > \lambda_{\alpha\mathbf{H},j}$ in its corollary.

Lemma 5: When $0 \geq \lambda_{\alpha\mathbf{H},i} > \lambda_{\alpha\mathbf{H},j}$, for both fixed-gain and variable-gain, to leading order about $d = \infty$ (i.e., $i < j$ fixed and $d \rightarrow \infty$) $\phi_{i,j}^{(\alpha)}$ is given by

$$\phi_{i,j}^{(\alpha)} = \bar{\phi}_{i,j} = \frac{(j-i)}{d} + O\left(\frac{1}{d}\right)^2. \quad (56)$$

Consequently,

- 1) the conditions that give (53) are $1/d = o(1/n)$,
- 2) the conditions that give (54) are $1/d = \Theta(1/n)$
- 3) the conditions that give (55) are $d = o(n)$.

Proof: Eq. (56) is obtained by performing a Taylor expansion of $\bar{\phi}_{i,j}$ about the point d and letting $d \rightarrow \infty$, with $i \leq j$ fixed. The following statements then follow immediately. ■

Corollary 1: It is only possible to maintain $\lambda_{\alpha\mathbf{H},i} > 0 > \lambda_{\alpha\mathbf{H},j}$ when $d = O(1)$. From Lemma 5, when this occurs $\lim_{n \rightarrow \infty} n\phi_{i,j}^{(\alpha)} = \infty$.

We will now discuss Lemma 5 and its corollary. These are seen to complement [5, Thrm. 4], in which it was shown that $\lim_{n \rightarrow \infty} [\lim_{d_D \rightarrow \infty} [c_n/d_D] / n]$ (where d_D is the number of destination antennas) will be strictly positive if and only if $d/d_D = \Theta(n^{1+\epsilon})$ for all $\epsilon \geq 0$ (note, the inequality for ϵ is not strict). In our work, if $d/j = \Theta(n^{1+\epsilon})$, for fixed j , $n\phi_{i,j}^{(\alpha)}$ will be bounded away from infinity $\forall i < j$ and consequently, from (49), $v_{i,j,n}^{(\alpha)}$ will *almost surely* display no exponential growth as n grows without bound¹. Clearly, avoiding exponential growth of $\nu_{1,j,n}^{(\alpha)}$ is required if we are to multiplex over the j upper eigenchannels. Crucially, these results provide us with an alternative perspective to [5] on how the number of antennas (more precisely, the scaling of the antennas) at each node affects the end-to-end capacity of the network.

3) *Network behavior when $\phi_{1,i}^{(\alpha)} = 0$ or $\lambda_{\alpha\mathbf{H},1} = 0$:* Suppose we wish to ensure that the $(1,i)$ th normalized channel capacity displays no exponential growth; i.e., (from (49)) $\phi_{1,i}^{(\alpha)} = 0$. Furthermore, suppose this is achieved by ensuring that

$$\lambda_{\alpha\mathbf{H},1} > \lambda_{\alpha\mathbf{H},i} = 0. \quad (57)$$

Then (39) and Lemma 4 give us

$$\|X_n^{(\alpha)}\|^2 = \Theta_{\mathbb{P}}\left(e^{n(\mathcal{H}_{d-1}-\mathcal{H}_{d-i})}\right) \quad (58)$$

¹Note, for the work in [5] and our work, d_D and j can be thought of as the maximum number of data streams that can be multiplexed over the channel, respectively. This draws the connection between their work, where the scaling of the ratio d/d_D is assessed, and our work, where the scaling of d/j is assessed.

and

$$e^{\frac{ni}{d-1}} \leq e^{n(\mathcal{H}_{d-1}-\mathcal{H}_{d-i})} \leq e^{\frac{ni}{d-i+1}}. \quad (59)$$

Thus, ensuring $\phi_{1,i}^{(\alpha)} = 0$ implies that the transmit power must grow according to (58). This growth rate is strictly positive and bound according to (59). We can see that by increasing the number of antennas, d , for a fixed i , the rate at which the transmit power grows can be reduced. Conversely, by fixing d and increasing i (i.e., multiplexing more data streams), the rate at which the transmit power grows will increase.

Suppose instead we wish to ensure that the transmit power displays no exponential growth by setting $\lambda_{\alpha\mathbf{H},1} = 0$. From (49) and Lemmas 3 and 4, this gives

$$\nu_{1,i,n}^{(\alpha)} = \Theta_{\mathbb{P}} \left(e^{n(\mathcal{H}_{d-1}-\mathcal{H}_{d-i})} \right). \quad (60)$$

Thus, all of the growth properties that applied to $\|X_n^{(\alpha)}\|^2$ when $\lambda_{\alpha\mathbf{H},i} = 0$ apply to $\nu_{1,i,n}^{(\alpha)}$ when $\lambda_{\alpha\mathbf{H},1} = 0$.

Remark 2: Interestingly, from (58) and (60), it can be seen that there is a duality between the exponential growth rate of $\|X_n^{(\alpha)}\|^2$ and $\nu_{1,i,n}^{(\alpha)}$ when either $\lambda_{\alpha\mathbf{H},i} = 0$ or $\lambda_{\alpha\mathbf{H},1} = 0$, respectively. This duality property will be exploited below.

4) *Adjacent Eigenchannel Capacity Divergence and Individual Data Stream Cost:* For the final problem, let us consider the rate at which adjacent eigenchannel capacities diverge away from each other. Of course, we have already seen (Lemma 3 and (49)) that if $\lambda_{\alpha\mathbf{H},i} > \lambda_{\alpha\mathbf{H},i+1} \geq 0$ then $c_{i,n}$ and $c_{i+1,n}$ will not diverge away from each other. Thus, in what follows we consider the cases $0 \geq \lambda_{\alpha\mathbf{H},i} > \lambda_{\alpha\mathbf{H},i+1}$ and $\lambda_{\alpha\mathbf{H},i} > 0 > \lambda_{\alpha\mathbf{H},i+1}$.

When $0 \geq \lambda_{\alpha\mathbf{H},i} > \lambda_{\alpha\mathbf{H},i+1}$, by employing Lemma 4 we find that

$$\nu_{i,i+1,n}^{(\alpha)} = \Theta_{\mathbb{P}} \left(e^{\frac{n}{d-i}} \right). \quad (61)$$

Thus, the i th and $(i+1)$ th channel capacities diverge away from each other at an exponential rate $1/(d-i)$. When $\lambda_{\alpha\mathbf{H},i} \geq 0 > \lambda_{\alpha\mathbf{H},i+1}$ we find that

$$\nu_{i,i+1,n}^{(\alpha)} = \Theta_{\mathbb{P}} \left(e^{-2n\lambda_{\alpha\mathbf{H},i+1}} \right) = O_{\mathbb{P}} \left(e^{\frac{n}{d-i}} \right) \quad (62)$$

and the capacities diverge away from each other at an exponential rate $-2\lambda_{\alpha\mathbf{H},i+1}$, which is upper bounded by the exponential rate of (61).

Remark 3: By considering the discussion of duality in Remark 2, we can assign a cost (in terms of extra instantaneous power requirements) to each extra data stream that we attempt to multiplex. In particular, from (61) and because of the duality property, if we are multiplexing i data streams, then, to multiplex 1 more stream (whilst ensuring $\lambda_{\alpha\mathbf{H},i+1} = 0$), we must increase the n th relay's instantaneous transmit power by (approximately) a factor of $\exp(n/(d-i))$. Furthermore, we find that the cost of each extra eigenchannel increases with i .

VI. NUMERICAL ILLUSTRATION

In this section, we will illustrate the theory that has been presented in the previous sections. Figs. 2 and 3, respectively, illustrate the second statement of Theorem 2 for a 4×4 fixed-gain system and a 3×3 variable-gain system. It is important to note that plots illustrating the first statement of Theorem 2 are indistinguishable from Figs. 2 and 3. Interestingly, we see that convergence of $\frac{1}{n} \log c_{n,i}^{(\alpha)}$ to $\lambda_{\gamma,i}^{(\alpha)}$ occurs reasonably quickly, which attests to the utility of our methods.

Fig. 4 shows a plot of $\nu_{1,i,n}^{(v)}$ as a function of n for a 3×3 variable-gain system, while Fig. 5 shows a plot of $\bar{\phi}_{1,i}$ as a function of the number of antennas at each node. Again, for Fig. 4, convergence can be clearly seen. For Fig. 5, when d is large, the curves are seen to decay linearly on the log-log scale; i.e., they decay like $O(1/d)$ on a linear scale. This observation illustrates Lemma 5.

Finally, Fig. 6 shows an estimation of $\lambda_{\alpha\mathbf{H},i}^{(v)}$ and its upper bound (106) for a variable-gain network as a function of the transmit power at each node for a large network size, $n = 1000$. The choice of such a large n is only made to ensure that our results have converged significantly, where smaller values of n may exhibit less smooth plots. For this figure, we assume that the mean channel fading coefficient at the i th node is log-normally distributed. It is easy to see that the bound is very tight for large p_i/n_0 .

VII. CONCLUSION

In this paper, we have employed the formalism of RDSs to study the scaling properties of the transmit power and end-to-end channel capacity of finite antenna MIMO AF relay networks. By employing the RDS formalism, we have been able to associate Lyapunov exponents (which are classically used to characterize the stability of RDSs) with the MIMO AF relay network. Our study has revealed that the exponential growth and/or decay of the transmit power and

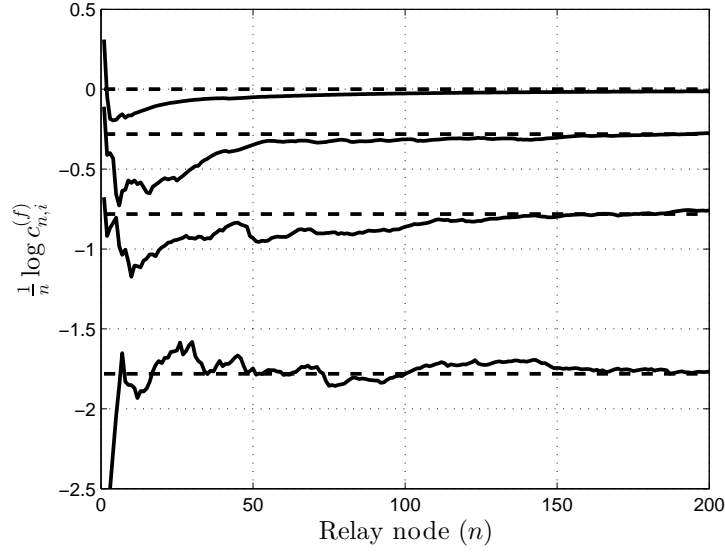


Fig. 2. Figure demonstrating the second statement of Theorem 2 for a 4×4 fixed-gain MIMO system. Dashed lines represent the Lyapunov exponents $\lambda_{\gamma,i}^{(f)}$, (2); solid lines represent instantaneous realizations of $\frac{1}{n} \log c_{n,i}^{(f)}$, where, starting from the top, $i = 1, \dots, 4$. For all $i = 1, \dots, n$, we set $p_i = n_0 = \mu_i = 1$.

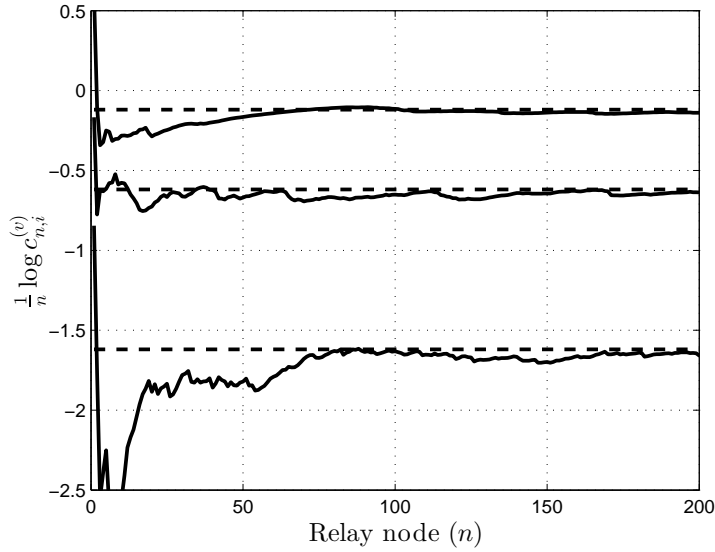


Fig. 3. Figure demonstrating the second statement of Theorem 2 for a 3×3 variable-gain MIMO system. Dashed lines represent the Lyapunov exponents $\lambda_{\gamma,i}^{(v)}$, (2); solid lines represent instantaneous realizations of $\frac{1}{n} \log c_{n,i}^{(v)}$, where, starting from the top, $i = 1, 2, 3$. For all $i = 1, \dots, n$, we set $p_i = n_0 = \mu_i = 1$.

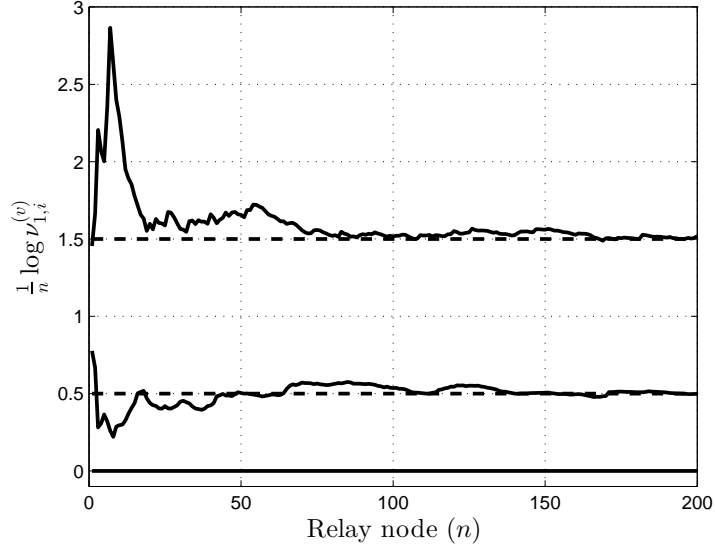


Fig. 4. Figure demonstrating (49). Dashed lines represent the Lyapunov difference $\lambda_{\gamma,1}^{(\alpha)} - \lambda_{\gamma,i}^{(\alpha)}$, (43); solid lines represent instantaneous realizations of $\frac{1}{n} \log \nu_{1,i,n}^{(v)}$, where, starting from the bottom, $i = 1, \dots, 4$. For all $i = 1, \dots, n$, we set $p_i = n_0 = \mu_i = 1$.

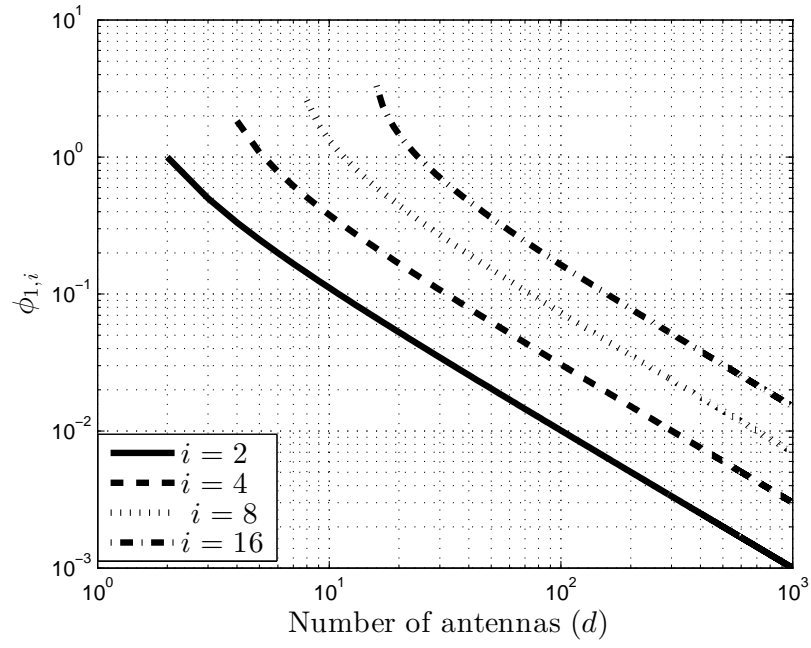


Fig. 5. Figure demonstrating Lemma 5 for different values of i as a function of the number of antennas.

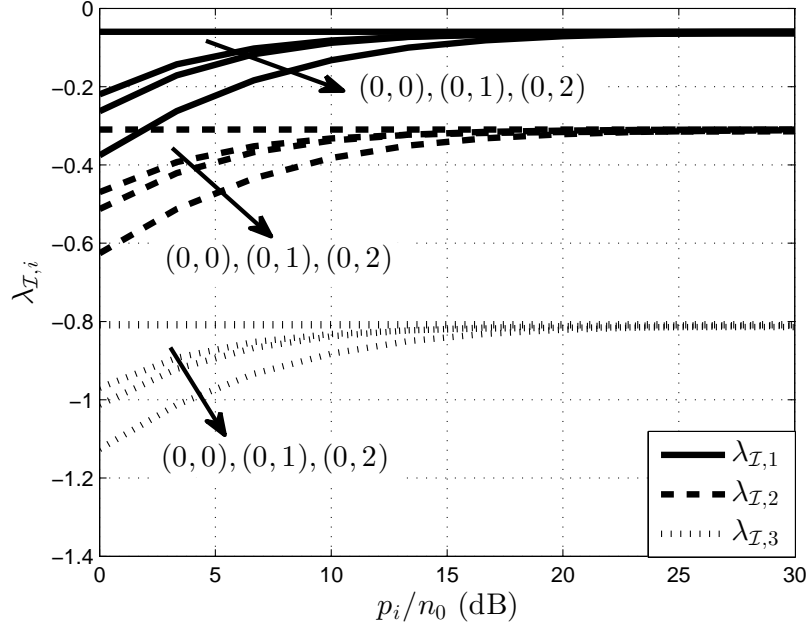


Fig. 6. Figure showing numerically estimated $\lambda_{v\mathbf{H},i}$ (curved lines) for large network ($n = 1000$) and its upper bound (straight lines), (106), for a non-homogeneous variable-gain 3×3 network. The average channel fading characteristics, μ_i , are assumed to be log-normally distributed with parameter pairs $(0, 1)$, $(0, 2)$ and $(0, 3)$; i.e., $\mu_i \sim \mathcal{LN}(a, b)$, $a = \mathbb{E} \log \mu_i$, $b = \mathbb{V} \log \mu_i$. The plot is taken as a function of the normalized transmit power at each node, with $p_i = p_{i-1} \forall i$.

end-to-end channel capacity are completely characterized by the network's Lyapunov exponents. Furthermore, our methods can be applied to systems with arbitrary channel fading statistics, provided $\mathbb{E} \log^+ \|\mathbf{H}_i\| < \infty$, where \mathbf{H}_i is the channel matrix for the i th hop; however, in this manuscript we focus explicitly on the Rayleigh fading scenario. We then establish growth laws for the eigenchannel capacity divergence, how this relates to the forwarding strategy and number of antennas at each node, and the cost (in terms of power) associated with multiplexing extra data streams.

APPENDIX A

PROOF OF THEOREM 1

Firstly, let

$$\prod_{i=1}^n \mathbf{M}_i [X_0^T \ 1]^T := [X_n^T \ 1]^T, \quad (63)$$

it is easy to see from Definition 2 that $\lambda(\mathbf{M}, [X_0^T \ 1]^T) \geq \lambda_{a,1}$. Thus, from Fact 2

$$\lambda(\mathbf{M}, [X_0^T \ 1]^T) \in \{\lambda_{\mathbf{A},i} \geq \lambda_{a,1}\} \cup \lambda_{a,1} =: \mathcal{L}. \quad (64)$$

The proof of the theorem now follows from Claim 1 (mentioned below).

Claim 1: With $\mathcal{Y} := \{[y_1 \ \cdots \ y_d \ 1]^T : y_i \in \mathbb{C}\}$, the mapping

$$\lambda(\mathbf{M}, \cdot) : \mathcal{Y} \rightarrow \mathcal{L} \quad (65)$$

is surjective.

Proof of Claim 1: If $\lambda_{\mathbf{A},1} < \lambda_{a,1}$ then $\mathcal{L} = \{\lambda_{a,1}\}$, $\lambda(\mathbf{M}, Y) = \lambda_{a,1} \ \forall Y \in \mathcal{Y}$ and the surjectivity of (65) is satisfied. Thus, w.l.o.g., we assume that $\exists k \leq d$ such that

$$\lambda_{\mathbf{A},1} > \cdots > \lambda_{\mathbf{A},k} \geq \lambda_{a,1} > \lambda_{\mathbf{A},k+1} > \cdots > \lambda_{\mathbf{A},d}. \quad (66)$$

In what follows, we consider the scenario in which $\lambda_{\mathbf{A},k} > \lambda_{a,1} > \lambda_{\mathbf{A},k+1}$. The proof can easily be extended to the case when $\lambda_{\mathbf{A},k} = \lambda_{a,1}$.

Consider the filtration,

$$\{0\} =: \mathcal{V}_{p+1} \subset \mathcal{V}_p \subset \cdots \subset \mathcal{V}_1 = \mathbb{C}^{d+1} \quad (67)$$

where $Y \in \mathcal{V}_i \setminus \mathcal{V}_{i+1} \Leftrightarrow \lambda(\mathbf{M}, Y) = \lambda_i$ (the existence of such a filtration is guaranteed by Fact 1.3). The proof of Claim 1 then follows immediately from Claim 2 (mentioned below).

Claim 2: Let \mathcal{V}_i be as in (67) and \mathcal{Y} be as in Claim 1. Then $(\mathcal{V}_i \setminus \mathcal{V}_{i+1}) \cap \mathcal{Y} \neq \emptyset$ for all $i = 1, \dots, k+1$, where $\lambda_{\mathbf{A},k} > \lambda_{a,1} > \lambda_{\mathbf{A},k+1}$.

Proof of Claim 2: Claim 2 follows immediately from Claim 3 (mentioned below).

Claim 3: Let \mathcal{V}_i be as in (67), \mathcal{Y} be as in Claim 1, and suppose that $\lambda_{\mathbf{A},k} > \lambda_{a,1} > \lambda_{\mathbf{A},k+1}$. Then:

- 1) for all $i \leq k$, $(\mathcal{V}_i \setminus \mathcal{V}_{i+1}) \cap \mathcal{Y} = \emptyset$ implies $(\mathcal{V}_l \setminus \mathcal{V}_{l+1}) \cap \mathcal{Y} = \emptyset$ for all $l < i$,
- 2) $(\mathcal{V}_1 \setminus \mathcal{V}_2) \cap \mathcal{Y} \neq \emptyset$.

Proof of Claim 3: We will begin by proving the first part of the claim. To do this, we first note the following: all the Lyapunov exponents have multiplicity 1 (i.e., they are distinct); consequently, from Fact 3, $\dim \mathcal{V}_j - \dim \mathcal{V}_{j+1} = 1 \ \forall j$ and

$$\dim \mathcal{V}_j = d + 2 - j. \quad (68)$$

Clearly,

$$(\mathcal{V}_i \setminus \mathcal{V}_{i+1}) \cap \mathcal{Y} = \emptyset \Leftrightarrow \mathcal{V}_i \cap \mathcal{Y} = \emptyset \text{ or } \mathcal{Y} \subseteq \mathcal{V}_{i+1} \subset \cdots \subset \mathcal{V}_1. \quad (69)$$

However, if $\mathcal{V}_i \cap \mathcal{Y} = \emptyset$ is satisfied, it can be seen that because \mathcal{V}_i is a vector space all vectors in \mathcal{V}_i must have their $(d+1)$ th element equal to zero. Thus,

$$\begin{aligned} \mathcal{V}_i \cap \mathcal{Y} &= \emptyset \\ \Rightarrow \mathcal{V}_i &= \{X = [y_1 \ \cdots \ y_d \ 0]^T : \lambda(\mathbf{M}, X) \leq \lambda_{\mathbf{A},i}\} \\ \Rightarrow \dim \mathcal{V}_i &= \dim \{X' = [y_1 \ \cdots \ y_d]^T : \lambda(\mathbf{A}, X') \leq \lambda_{\mathbf{A},i}\} \\ &= d + 1 - i. \end{aligned} \quad (70)$$

But from (68), $\dim \mathcal{V}_i = d + 2 - i$, so $\mathcal{V}_i \cap \mathcal{Y} = \emptyset$ gives us a contradiction so (69) becomes

$$(\mathcal{V}_i \setminus \mathcal{V}_{i+1}) \cap \mathcal{Y} = \emptyset \Leftrightarrow \mathcal{Y} \subseteq \mathcal{V}_{i+1} \subset \cdots \subset \mathcal{V}_1, \quad (71)$$

and

$$(\mathcal{V}_j \setminus \mathcal{V}_{j+1}) \cap \mathcal{Y} = \emptyset, \ \forall j \leq i. \quad (72)$$

This proves the first part of the Claim.

We will now prove the second part of the claim. From (71) we have $(\mathcal{V}_1 \setminus \mathcal{V}_2) \cap \mathcal{Y} = \emptyset \Leftrightarrow \mathcal{V}_2 \supseteq \mathcal{Y}$. But \mathcal{Y} contains an d dimensional subspace $\mathcal{A} := \{[y_1 \ \cdots \ y_d \ 0]^T : y_i \in \mathbb{C}\}$, and \mathcal{V}_2 is also d dimensional, so

$$\mathcal{V}_2 \supseteq \mathcal{Y} \supseteq \mathcal{A} \Rightarrow \mathcal{A} = \mathcal{V}_2 \Rightarrow \mathcal{A} = \mathcal{Y}. \quad (73)$$

But $\mathcal{A} \subset \mathcal{Y}$, so from (73) $\mathcal{V}_2 \supseteq \mathcal{Y}$ gives us a contradiction. Thus $\mathcal{V}_2 \not\supseteq \mathcal{Y}$, which (from (71)) gives

$$(\mathcal{V}_1 \setminus \mathcal{V}_2) \cap \mathcal{Y} \neq \emptyset. \quad (74)$$

This completes the proof.

APPENDIX B PROOF OF LEMMA 1

We have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{1}{n} \log \|X_n\| &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mathbf{A}_n X_{n-1} - R_n + 2R_n\| \\
&\leq \max \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mathbf{A}_n X_{n-1} - R_n\|, \lim_{n \rightarrow \infty} \frac{1}{n} \log \|2R_n\| \right\} \\
&\stackrel{a.s.}{=} \max \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \log \|X_n\|, 0 \right\}, \tag{75}
\end{aligned}$$

where the second line follows from Lemma 6 (below) and the last line follows from the symmetry of \mathbf{A}_n . If

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|X_n\| \geq 0,$$

our result is reached trivially; if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|X_n\| = \lambda < 0,$$

from Lemma 6 (below), the line above (75) holds with equality, which gives $\lambda = 0$. This contradicts our assumption that $\lambda < 0$. Therefore,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|X_n\| \geq 0.$$

This completes the proof.

Lemma 6: For $\alpha_n, \beta_n \in \mathbb{C}^d$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\alpha_n + \beta_n\| \leq \max \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\alpha_n\|, \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\beta_n\| \right\}, \tag{76}$$

where equality holds when

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\alpha_n\| \neq \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\beta_n\|. \tag{77}$$

Proof: For $\alpha_n, \beta_n \in \mathbb{C}^d$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\alpha_n + \beta_n\| \leq \max \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\alpha_n\|, \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\beta_n\| \right\} \tag{78}$$

since $\|\alpha_n + \beta_n\| \leq 2 \max\{\|\alpha_n\|, \|\beta_n\|\}$. To show that (78) holds with equality when

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\alpha_n\| \neq \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\beta_n\|, \tag{79}$$

w.l.o.g., we assume that $\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\alpha_n\| < \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\beta_n\|$. Eq. (78) then gives us

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\alpha_n + \beta_n\| \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\beta_n\| \quad (80)$$

$$\begin{aligned} &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\alpha_n + \beta_n - \alpha_n\| \\ &\leq \max \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\alpha_n + \beta_n\|, \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\alpha_n\| \right\} \end{aligned} \quad (81)$$

It follows that if $\lim_{n \rightarrow \infty} (1/n) \log \|\alpha_n + \beta_n\| < \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\alpha_n\|$ then (from (80) and (81)) $\lim_{n \rightarrow \infty} (1/n) \log \|\beta_n\| \leq \lim_{n \rightarrow \infty} (1/n) \log \|\alpha_n\|$ which contradicts our assumption. Consequently, $\lim_{n \rightarrow \infty} (1/n) \log \|\beta_n\|$ is sandwiched either side by $\lim_{n \rightarrow \infty} (1/n) \log \|\alpha_n + \beta_n\|$ and so must be equal to it. ■

APPENDIX C

PROOF OF THEOREM 2

Theorem 2 contains two statements. We prove these separately in the following two subsections.

A. First Statement

We prove the first statement in two parts. Each of these parts will involve manipulating the inverse of $(\mathbf{R}_{\mathcal{I},n}^{(\alpha)} \mathbf{R}_{\mathcal{N},n}^{(\alpha)-1})$, which is given by

$$\begin{aligned} \left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)} \mathbf{R}_{\mathcal{N},n}^{(\alpha)-1} \right)^{-1} &= \mathbf{R}_{\mathcal{N},n} \mathbf{R}_{\mathcal{I},n}^{(\alpha)-1} \\ &= \left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)-1} + \sum_{l=2}^n \mathbf{H}_n \cdots \mathbf{H}_l \mathbf{R}_{\mathcal{I},l-1}^{(\alpha)-1} \mathbf{H}_l^{-1} \cdots \mathbf{H}_n^{-1} \right), \end{aligned} \quad (82)$$

where, without loss of generality, we have assumed that $n_0 = 1$. The first part constructs an upper bound on the limit in question. The second part constructs a lower bound on the same limit, which is identical to the lower bound. This proves the first part of the theorem.

1) *Upper Bound* : Our aim is to show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{E}_i \left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)} \mathbf{R}_{\mathcal{N},n}^{(\alpha)-1} \right) \stackrel{a.s.}{\leq} \min\{0, 2\lambda_{\alpha \mathbf{H}, i}\}. \quad (83)$$

By noting that

$$\mathcal{E}_i \left(\mathbf{R}_{\mathcal{I},1}^{(\alpha)-1} \right) \stackrel{a.s.}{\neq} 0,$$

and does not depend on n , we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{E}_i \left(\mathbf{R}_{\mathcal{I},1}^{(\alpha)-1} \right) \stackrel{a.s.}{=} 0. \quad (84)$$

Also, from the definition of $\mathbf{R}_{\mathcal{I},n}^{(\alpha)}$ and property 4 of Fact 1, it is clear that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{E}_i \left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)-1} \right) \stackrel{a.s.}{=} -2\lambda_{\alpha\mathbf{H},i}. \quad (85)$$

By combining (84), (85), and

$$-\max\{0, -a\} = \min\{0, a\}, \quad a \in \mathbb{R}, \quad (86)$$

the upper bound of (83) follows immediately from the following claim (the proof of which is given at the end of Appendix 2).

Claim 4: The eigenvalues of $\left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)} \mathbf{R}_{\mathcal{N},n}^{(\alpha)-1} \right)^{-1}$ are bound in the following way:

$$\mathcal{E}_i \left(\left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)} \mathbf{R}_{\mathcal{N},n}^{(\alpha)-1} \right)^{-1} \right) \geq \max \left\{ \mathcal{E}_i \left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)-1} \right), \mathcal{E}_i \left(\mathbf{R}_{\mathcal{I},1}^{(\alpha)-1} \right) \right\}, \quad (87)$$

2) *Lower Bound:* We will now provide the second part of the proof (constructing the lower bound). To begin, let us introduce the following RDS², which will be exploited in a moment:

$$Y_n := \mathbf{M}_1^{(\alpha)} \cdots \mathbf{M}_n^{(\alpha)} \begin{bmatrix} \hat{Y}_0 \\ 1 \end{bmatrix}, \quad (88)$$

where

$$\mathbf{M}_i^{(\alpha)} := \begin{bmatrix} \frac{1}{\alpha_i} \mathbf{H}_i^{-1} & \hat{Y}_0 \\ \mathbf{0}^T & \pm 1 \end{bmatrix}. \quad (89)$$

To allow us to describe the mechanism by which the sign of ± 1 is chosen in the bottom right corner of (89), we must first establish the inner product of Y_n . The inner product of Y_n is given

²Note, (88) is a backward RDS as per (9).

by

$$\begin{aligned}
\|Y_n\|^2 = & \overbrace{\hat{Y}_0^\dagger \left(\overline{\mathbf{R}}_{\mathcal{I},n}^{(\alpha)-1} + \sum_{l=1}^{n-1} \overline{\mathbf{R}}_{\mathcal{I},l}^{(\alpha)-1} + \mathbf{I}_d \right)^\dagger \hat{Y}_0}^{\text{First inner product term}} \\
& \overbrace{\pm \hat{Y}_0^\dagger \left(\frac{1}{g_j} (\mathbf{H}_n^{-1})^\dagger \cdots (\mathbf{H}_1^{-1})^\dagger \mathbf{H}_1^{-1} \cdots \mathbf{H}_{n-1}^{-1} \prod_{j=1}^n \frac{1}{g_j} \prod_{j=1}^{n-1} \right.}^{\text{Second inner product term}} \\
& \pm \cdots \pm (\mathbf{H}_n^{-1})^\dagger \cdots (\mathbf{H}_1^{-1})^\dagger \mathbf{H}_1^{-1} \frac{1}{g_1} \prod_{j=1}^n \frac{1}{g_j} \\
& \left. \pm \cdots \pm (\mathbf{H}_1^{-1})^\dagger \frac{1}{g_1} \right) \hat{Y}_0, \tag{90}
\end{aligned}$$

where

$$\overline{\mathbf{R}}_{\mathcal{I},i}^{(\alpha)-1} := \mathbf{H}_i^{-1} \cdots \mathbf{H}_1^{-1} (\mathbf{H}_1^{-1})^\dagger \cdots (\mathbf{H}_i^{-1})^\dagger \prod_{j=1}^i \frac{1}{g_j^2}. \tag{91}$$

It is clear that the second inner product term is a real number that, at the moment, may be either positive or negative. However, there is nothing stopping us from ensuring that this is strictly positive by appropriately selecting the sign of ± 1 in (89); for, the RDS is permitted to remember the past, and predict the future [18]. This is the mechanism that we will use to select the sign. Furthermore, performing sign selection in this way will not affect the Lyapunov exponents of the system in question (Fact 2). We now have the following upper bound on the first inner product term of (90), which will be exploited later on:

$$\|Y_n\|^2 \geq \hat{Y}_0^\dagger \left(\overline{\mathbf{R}}_{\mathcal{I},n}^{(\alpha)-1} + \sum_{l=1}^{n-1} \overline{\mathbf{R}}_{\mathcal{I},l}^{(\alpha)-1} + \mathbf{I}_d \right)^\dagger \hat{Y}_0. \tag{92}$$

It can already be seen that (82) is remarkably similar to the first inner product term of (90). We will now show that this similarity is not superficial, and that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{E}_i \left(\left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)} \mathbf{R}_{\mathcal{N},n}^{(\alpha)-1} \right)^{-1} \right) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{E}_i \left(\overline{\mathbf{R}}_{\mathcal{I},n}^{(\alpha)-1} + \sum_{l=1}^{n-1} \overline{\mathbf{R}}_{\mathcal{I},l}^{(\alpha)-1} + \mathbf{I}_d \right). \tag{93}$$

To do this, note that

$$\mathcal{E}_i \left\{ \mathbf{H}_n \cdots \mathbf{H}_l \mathbf{R}_{\mathcal{I},l-1}^{(\alpha)-1} \mathbf{H}_l^{-1} \cdots \mathbf{H}_n^{-1} \right\} = \mathcal{E}_i \left\{ \mathbf{R}_{\mathcal{I},l-1}^{(\alpha)-1} \right\} \stackrel{d}{=} \mathcal{E}_i \left\{ \overline{\mathbf{R}}_{\mathcal{I},l-1}^{(\alpha)-1} \right\}. \tag{94}$$

Consequently,

$$\mathcal{E}_i \left(\left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)} \mathbf{R}_{\mathcal{N},n}^{(\alpha)-1} \right)^{-1} \right) \stackrel{d}{=} \mathcal{E}_i \left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)-1} + \sum_{l=1}^{n-1} \mathbf{R}_{\mathcal{I},l}^{(\alpha)-1} \right) \quad (95)$$

$$\stackrel{d}{=} \mathcal{E}_i \left(\overline{\mathbf{R}}_{\mathcal{I},n,1}^{(\alpha)-1} + \sum_{l=1}^{n-1} \overline{\mathbf{R}}_{\mathcal{I},l}^{(\alpha)-1} \right) \quad (96)$$

$$\leq \mathcal{E}_i \left(\overline{\mathbf{R}}_{\mathcal{I},n,1}^{(\alpha)-1} + \sum_{l=1}^{n-1} \overline{\mathbf{R}}_{\mathcal{I},l}^{(\alpha)-1} + \mathbf{I}_d \right), \quad (97)$$

where (95) follows from the first equality of (94), (96) follows from the second equality of (94), and (97) follows trivially from (96).

With (95) and (97), we have shown (93). The right hand side of (92) is known to be equal to the i th eigenvalue when \hat{Y}_0 is an i th unit eigenvector. Combining this fact with (97) gives us

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{E}_i \left(\left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)} \mathbf{R}_{\mathcal{N},n}^{(\alpha)-1} \right)^{-1} \right) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \|Y_n\|^2. \quad (98)$$

But the limit on the right hand side of (98), when Y_n is given by (88), is given by Theorem 1.

Thus, (98) and Theorem 1 give us

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{E}_i \left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)} \mathbf{R}_{\mathcal{N},n}^{(\alpha)-1} \right) \geq -\max \{-2\lambda_{\alpha \mathbf{H},i}, 0\}, \quad (99)$$

which can then be combined with (86) to yield the lower bound.

B. Second Statement

From the first statement of Theorem 2, we have

$$\mathbb{P} \left[\log \left(e^{n\lambda_{\gamma,i}^{(\alpha)} - o(n)} + 1 \right) \leq c_{n,i}^{(\alpha)} \leq \log \left(e^{n\lambda_{\gamma,i}^{(\alpha)} + o(n)} + 1 \right) \right] \rightarrow 1,$$

which gives

$$\begin{aligned} \mathbb{P} \left[e^{n\lambda_{\gamma,i}^{(\alpha)} - o(n)} + O \left(e^{2n\lambda_{\gamma,i}^{(\alpha)}} \right) \leq c_n^{(\alpha)} \leq e^{n\lambda_{\gamma,i}^{(\alpha)} + o(n)} + O \left(e^{2n\lambda_{\gamma,i}^{(\alpha)}} \right) \right] &\rightarrow 1 \\ \Rightarrow \mathbb{P} \left[e^{n\lambda_{\gamma,i}^{(\alpha)} - o(n)} O(1) \leq c_n^{(\alpha)} \leq e^{n\lambda_{\gamma,i}^{(\alpha)} + o(n)} O(1) \right] &\rightarrow 1 \\ \Rightarrow \mathbb{P} \left[e^{n\lambda_{\gamma,i}^{(\alpha)} - o(n)} \leq c_n^{(\alpha)} \leq e^{n\lambda_{\gamma,i}^{(\alpha)} + o(n)} \right] &\rightarrow 1 \end{aligned}$$

where the first line follows from the Taylor expansion of $\log(1+x)$ about $x=0$ and the second line follows by factoring $e^{n\lambda_{\gamma,i}^{(\alpha)} \pm o(n)}$ from the left and right sides of the second line, respectively, and noting that $\lambda_{\gamma,i}^{(\alpha)} \leq 0$.

Proof of Claim 4: An immediate consequence of the dual Lidskii inequality [32] is that

$$\mathcal{E}_i(\mathbf{A} + \mathbf{B}) \geq \mathcal{E}_i(\mathbf{A}) + \mathcal{E}_d(\mathbf{B}), \quad (100)$$

which applies to $d \times d$ Hermitian matrices \mathbf{A} and \mathbf{B} . Combining (100) with the fact that the summands in (82) are positive definite (i.e., they have positive eigenvalues), gives us

$$\begin{aligned} \mathcal{E}_i \left(\left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)} \mathbf{R}_{\mathcal{N},n}^{(\alpha)-1} \right)^{-1} \right) &\geq \mathcal{E}_i \left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)-1} \right) \\ \text{and } \mathcal{E}_i \left(\left(\mathbf{R}_{\mathcal{I},n}^{(\alpha)} \mathbf{R}_{\mathcal{N},n}^{(\alpha)-1} \right)^{-1} \right) &\geq \mathcal{E}_i \left(\mathbf{H}_n \cdots \mathbf{H}_2 \mathbf{R}_{\mathcal{I},1}^{(\alpha)-1} \mathbf{H}_2^{-1} \cdots \mathbf{H}_n^{-1} \right). \end{aligned} \quad (101)$$

Claim 4 follows immediately from (101) after noting that

$$\mathcal{E}_i \left(\mathbf{H}_n \cdots \mathbf{H}_2 \mathbf{R}_{\mathcal{I},1}^{(\alpha)-1} \mathbf{H}_2^{-1} \cdots \mathbf{H}_n^{-1} \right) = \mathcal{E}_i \left(\mathbf{R}_{\mathcal{I},1}^{(\alpha)-1} \right).$$

APPENDIX D

PROOF OF LEMMA 2

Lemma 2 contains two statements. We prove these separately in the following two subsections.

A. First Statement

The Lyapunov exponents of the matrix product that describes the progression of \mathcal{I}_n , (29), follow immediately from [33, Proposition 1]. The Lyapunov exponents of $X_n^{(\alpha)}$, (29), then follow immediately from Theorem 1. Combining these with (18), we obtain (38) and (39).

B. Second Statement

For the second statement, we begin by showing that the limit is greater than or equal to zero for both fixed-gain and variable-gain:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{p_n}{p_0} \geq \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{g_n^2 n_0}{p_0} \stackrel{a.s.}{=} 0, \quad (102)$$

where the *almost sure* equality becomes an equality for fixed-gain. For fixed-gain, the stated result then follows immediately from (102) and (105). For variable-gain, the stated result then follows immediately from (102) and (106).

APPENDIX E

PROOF OF LEMMA 3

The lower bound follows trivially from (2) and (43). By noting that $\lambda_{\alpha\mathbf{H},1}^{(\alpha)} > \lambda_{\alpha\mathbf{H},i}^{(\alpha)} \geq 0 \Leftrightarrow \lambda_{\gamma,1}^{(\alpha)} = \lambda_{\gamma,i}^{(\alpha)} = 0$, we obtain equality of the bound. For the upper bound, we need to prove that

$$a - b \geq \min\{0, a\} - \min\{0, b\} \quad (103)$$

for $a \geq b$. To do this, we need to check the following three cases:

- 1) $a \geq 0, b \geq 0, a \geq b$;
- 2) $a \geq 0, b \leq 0$;
- 3) $a \leq 0, b \leq 0, a \geq b$;

which can be done trivially. Equality of the upper bound occurs when $b \leq a \leq 0$. Finally, to obtain the if and only if statements, we need to show that $a > 0$ and $b < 0$ implies that

$$a - b > \min\{0, a\} - \min\{0, b\} > 0, \quad (104)$$

which can be done trivially. The independence of fixed-gain or variable-gain implementation is trivial.

APPENDIX F

Lemma 7: For the fixed-gain network,

$$\lambda_{f\mathbf{H},j} \leq \frac{1}{2} \left(\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{p_n}{p_0} - \log d + \psi(d - j + 1) \right), \quad (105)$$

For the variable-gain network,

$$\lambda_{v\mathbf{H},i} \leq \frac{1}{2} \left(\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{p_n}{p_0} + \log(d) - \psi(d^2) + \psi(d - i + 1) \right). \quad (106)$$

Proof: The first equation, (105), is obtained from (35) by noting that

$$L(f^2\mu) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{p_n}{p_0} - \log d.$$

For the second equation, (106), we have

$$\begin{aligned} L(v^2\mu) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \log \left(\frac{p_i}{\frac{p_{i-1}}{d} \|\mathbf{H}_i\|_F^2 + dn_0} \mu_i \right) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \log \left(\frac{dp_i}{p_{i-1} \|\mathbf{H}_i\|_F^2} \mu_i \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{p_n}{p_0} + \log(d) - \psi(d^2). \end{aligned} \quad (107)$$

where the final line follows from $\mathbb{E} \log \|\mathbf{H}_i\|_F^2 / \mu_i = \psi(d^2)$. From (35), the stated result follows immediately. ■

REFERENCES

- [1] I. E. Telatar *et al.*, “Capacity of Multi-Antenna Gaussian Channels,” *European transactions on telecommunications*, vol. 10, no. 6, pp. 585–595, 1999.
- [2] J. Laneman, D. Tse, and G. Wornell, “Cooperative Diversity in Wireless Networks: Efficient Protocols and Outage Behavior,” *Information Theory, IEEE Transactions on*, vol. 50, pp. 3062 – 3080, dec. 2004.
- [3] A. Chandra, C. Bose, and M. Bose, “Wireless Relays for Next Generation Broadband Networks,” *Potentials, IEEE*, vol. 30, pp. 39 –43, march-april 2011.
- [4] J. Wagner and A. Wittneben, “On Capacity Scaling of (Long) MIMO Amplify-and-Forward Multihop Networks,” in *Signals, Systems and Computers, 2008 42nd Asilomar Conference on*, pp. 346–350, Oct 2008.
- [5] J. Wagner and A. Wittneben, “On Capacity Scaling of Multi-Antenna Multi-Hop Networks: The Significance of the Relaying Strategy in the “Long Network Limit”,” *Information Theory, IEEE Transactions on*, vol. 58, pp. 2107–2133, April 2012.
- [6] N. Fawaz, K. Zarifi, M. Debbah, and D. Gesbert, “Asymptotic Capacity and Optimal Precoding in MIMO Multi-Hop Relay Networks,” *Information Theory, IEEE Transactions on*, vol. 57, no. 4, pp. 2050–2069, 2011.
- [7] M. A. Girnyk, M. Vehkaperä, and L. K. Rasmussen, “Asymptotic Performance Analysis of a K-Hop Amplify-and-Forward Relay MIMO Channel,” *CoRR*, vol. abs/1410.5716, 2014.
- [8] S.-p. Yeh and O. Lévêque, “Asymptotic Capacity of Multi-Level Amplify-and-Forward Relay Networks,” in *Proceedings of the 2007 IEEE International Symposium on Information Theory*, no. LTHI-CONF-2006-011, 2007.
- [9] R. Muller, “On the Asymptotic Eigenvalue Distribution of Concatenated Vector-Valued Fading Channels,” *Information Theory, IEEE Transactions on*, vol. 48, pp. 2086–2091, Jul 2002.
- [10] L. Zheng and D. N. Tse, “Diversity and Multiplexing: a Fundamental Tradeoff in Multiple-Antenna Channels,” *Information Theory, IEEE Transactions on*, vol. 49, no. 5, pp. 1073–1096, 2003.
- [11] S. Borade, L. Zheng, and R. Gallager, “Amplify-and-Forward in Wireless Relay Networks: Rate, Diversity, and Network Size,” *Information Theory, IEEE Transactions on*, vol. 53, pp. 3302–3318, Oct 2007.
- [12] D. Gunduz, M. Khojastepour, A. Goldsmith, and H. Poor, “Multi-hop MIMO Relay Networks: Diversity-Multiplexing Trade-Off Analysis,” *Wireless Communications, IEEE Transactions on*, vol. 9, pp. 1738–1747, May 2010.
- [13] S. Yang and J.-C. Belfiore, “Diversity of MIMO Multihop Relay Channels,” *arXiv preprint arXiv:0708.0386*, 2007.
- [14] A. M. Tulino and S. Verdú, *Random Matrix Theory and Wireless Communications*, vol. 1. Now Publishers Inc, 2004.
- [15] Y. Q. Yin, “Limiting Spectral Distribution for a Class of Random Matrices,” *Journal of multivariate analysis*, vol. 20, no. 1, pp. 50–68, 1986.
- [16] J. W. Silverstein, “Strong Convergence of the Empirical Distribution of Eigenvalues of Large Dimensional Random Matrices,” *Journal of Multivariate Analysis*, vol. 55, no. 2, pp. 331–339, 1995.
- [17] V. A. Marčenko and L. A. Pastur, “Distribution of Eigenvalues for some Sets of Random Matrices,” *Sbornik: Mathematics*, vol. 1, no. 4, pp. 457–483, 1967.
- [18] L. Arnold, *Random Dynamical Systems*. Monographs in Mathematics, Springer, 1998.
- [19] R. Bhattacharya and M. Majumdar, *Random Dynamical Systems: Theory and Applications*. Cambridge University Press, 2007.

- [20] H. Weinberger, “Long-Time Behavior of a Class of Biological Models,” *SIAM journal on Mathematical Analysis*, vol. 13, no. 3, pp. 353–396, 1982.
- [21] S. H. Strogatz, “Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering (Studies in Nonlinearity),” 2001.
- [22] L. Yu, E. Ott, and Q. Chen, “Transition to Chaos for Random Dynamical Systems,” *Physical review letters*, vol. 65, no. 24, p. 2935, 1990.
- [23] M. S. Raghunathan, “A Proof of Oseledec’s Multiplicative Ergodic Theorem,” *Israel Journal of Mathematics*, vol. 32, no. 4, pp. 356–362, 1979.
- [24] V. I. Oseledec, “A Multiplicative Ergodic Theorem. Characteristic Lyapunov Exponents of Dynamical Systems,” *Trudy Moskovskogo Matematicheskogo Obshchestva*, vol. 19, pp. 179–210, 1968.
- [25] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. John Wiley & Sons, 2012.
- [26] S. Janson, “Probability Asymptotics: Notes on Notation,” *arXiv preprint arXiv:1108.3924*, 2011.
- [27] E. S. Key, “Lyapunov Exponents for Matrices with Invariant Subspaces,” *The Annals of Probability*, pp. 1721–1728, 1988.
- [28] A. Goldsmith, *Wireless Communications*. Cambridge university press, 2005.
- [29] R. H. Louie, Y. Li, H. Suraweera, and B. Vucetic, “Performance Analysis of Beamforming in Two Hop Amplify and Forward Relay Networks with Antenna Correlation,” *Wireless Communications, IEEE Transactions on*, vol. 8, pp. 3132–3141, June 2009.
- [30] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*. No. 55, Courier Corporation, 1964.
- [31] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*. Cambridge university press, 2005.
- [32] T. Tao, *Topics in Random Matrix Theory*, vol. 132. American Mathematical Soc., 2012.
- [33] P. Forrester, “Lyapunov Exponents for Products of Complex Gaussian Random Matrices,” *Journal of Statistical Physics*, vol. 151, no. 5, pp. 796–808, 2013.