

Some Contribution to
Analysis and Stochastic Analysis



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A thesis submitted for the degree of
Doctor of Philosophy
Hilary 2018

*To my parents Yaofang Liu and Xuelian Zhu, my supervisor
Prof. Zhongmin Qian, and my mentor Dr. Zhaodong Wang.*

Acknowledgements

I would like to express my gratefulness to my parents and my little sister, who have always been supportive and loving me. I want to thank my supervisor Prof. Zhongmin Qian for providing me with many insightful instructions and sharing with me his profound understanding of mathematics. I am very grateful to my mentor Dr. Zhaodong Wang, without whose support I will not have the opportunity to pursue my doctoral degree.

I also want to thank Prof. Terry Lyons and Prof. Ben Hambly for having many inspiring conversations with me and their helpful suggestions, and Prof. Masanori Hino, Dr. Michael Hinz and Prof. Alexander Teplyaev for discussions and communications.

I thank my colleagues Dr. Xi Geng, Jiawei Li, and Guangyu Xi, from whom I have had learnt much, for working with me in these years. I wish all the best to them.

Abstract

The dissertation consists of two parts. The first part (Chapter 1 to 4) is on some contributions to the development of a non-linear analysis on the quintessential fractal set Sierpinski gasket and its probabilistic interpretation. The second part (Chapter 5) is on the asymptotic tail decays for suprema of stochastic processes satisfying certain conditional increment controls.

Chapters 1, 2 and 3 are devoted to the establishment of a theory of backward problems for non-linear stochastic differential equations on the gasket, and to derive a probabilistic representation to some parabolic type partial differential equations on the gasket. In Chapter 2, using the theory of Markov processes, we derive the existence and uniqueness of solutions to backward stochastic differential equations driven by Brownian motion on the Sierpinski gasket, for which the major technical difficulty is the exponential integrability of quadratic processes of martingale additive functionals. A Feynman–Kac type representation is obtained as an application. In Chapter 3, we study the stochastic optimal control problems for which the system uncertainties come from Brownian motion on the gasket, and derive a stochastic maximum principle. It turns out that the necessary condition for optimal control problems on the gasket consists of two equations, in contrast to the classical result on \mathbb{R}^d , where the necessary condition is given by a single equation. The materials in Chapter 2 are based on a joint work with Zhongmin Qian (referenced in Chapter 2).

Chapter 4 is devoted to the analytic study of some parabolic PDEs on the gasket. Using a new type of Sobolev inequality which involves singular measures

developed in Section 4.2, we establish the existence and uniqueness of solutions to these PDEs, and derive the space-time regularity for solutions. As an interesting application of the results in Chapter 4 and the probabilistic representation developed in Chapter 2, we further study Burgers equations on the gasket, to which the space-time regularity for solutions is deduced. The materials in Chapter 4 are based on a joint work with Zhongmin Qian (referenced in Chapter 4).

In Chapter 5, we consider a class of continuous stochastic processes which satisfy the conditional increment control condition. Typical examples include continuous martingales, fractional Brownian motions, and diffusions governed by SDEs. For such processes, we establish a Doob type maximal inequality. Under additional assumptions on the tail decays of their marginal distributions, we derive an estimate for the tail decay of the suprema (Theorem 5.3.2), which states that the suprema decays in a manner similar to the margins of the processes. In Section 5.4, as an application of Theorem 5.3.2, we derive the existence of strong solutions to a class of SDEs. The materials in this chapter is based on the work [44] by the author (Section 5.2 and Section 5.3) and an ongoing joint project with Guangyu Xi (Section 5.4).

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Chapter 1

Introduction to Analysis on Fractals

1.1 Framework

Fractals are spaces which are very irregular compared to smooth manifolds and do not possess proper differential structures. Examples of fractals include Sierpinski gaskets, snow flakes, cantor sets, etc. These spaces appear as scaling limits of sequences of discrete structures, and satisfy certain self-similar properties. Many objects in nature (e.g. complex biology systems, polymeric materials, and etc.) have fractal features (see e.g. [50, 51, 24, 60, 69, 49, 11] for details). Analysis on fractals has attracted researchers' attentions over the last decades, not only for the reason that fractals are archetypal examples of spaces without suitable smooth structure, but also because fractals arise in interesting models in statistical physics. For example, lattice models such as the Ising model and their variants have been extensively studied in order to gain understanding of phase transitions of large particle systems at criticality, and a fractal model can be regarded as the scaling limit as the number of particles in the system tends to infinity.

Since B. Mandelbrot's introduction of the notion of fractals as a new class of mathematical objects modelling nature, a new mathematical area called fractal geometry developed quickly with the help of geometric measure theory and ergodic theory. Fractal geometry

concerns static aspects of fractals and measures on them, for example, Hausdorff dimensions and Hausdorff measures.

As evolutions take place on objects (e.g. \mathbb{R}^d) representing nature, dynamical phenomena can also occur on fractals. For example, one may ask how heat diffuse on media modelled by fractals, or how a medium with fractal structure vibrates when subject to forces. These problems are addressed by a theory called analysis on fractals, which concerns dynamical and analytic aspects of fractals.

On a domain in \mathbb{R}^d , heat transfer is described by the heat equation

$$\partial_t u = \Delta u,$$

where $u = u(t, x)$ with t, x being the time and space variables respectively, and $\Delta = \sum_{i=1}^d \partial_{x_i}^2$ is the Laplace operator. To answer the problem of heat transfer on fractals, one needs an appropriate concept to replace the Laplacian on Euclidean spaces. This is not possible from a classical viewpoint, since there exist no smooth differential structures on fractals in general. Therefore, a different approach must be employed to develop analysis on fractals. S. Goldstein [22], S. Kusuoka [40], and M. Barlow and E. Perkins [4] took the first step in the mathematical development of analysis on fractals. S. Goldstein, S. Kusuoka, and M. Barlow and E. Perkins independently constructed a diffusion process, called Brownian motion in literature, on the planar Sierpinski gasket as the scaling limit of a sequence of random walks on approximate lattices. Utilizing the theory of Markov processes and Dirichlet forms, the Laplacian on the Sierpinski gasket can be introduced as the non-positive self-adjoint operator associated with Brownian motion they constructed. An Aronson type heat kernel estimate was derived by M. Barlow and E. Perkins in [4]. J. Kigami [33] gave an alternative construction of the Dirichlet form on the gasket using sequences of finite differences. There have been many works on the study of diffusion processes and Laplacians on fractals (see e.g. [4, 21, 36, 3, 34], etc.), and linear analysis on

fractals has been relatively well established.

The objective of the remaining chapters is to develop a non-linear analysis on the archetypal fractal, the Sierpinski gasket. The materials are organized as follows.

Preliminaries for analysis on fractals are provided in the remaining part of Chapter 1. The presentation in Chapter 1 is specifically formulated for the Sierpinski gasket, and we make no attempt to present the theory of analysis on fractals at its full generality. A comprehensive discussion of the general theory can be found in the monograph [35].

In Chapter 2, we establish a theory of non-linear backward stochastic differential equations (with deterministic or stochastic durations) driven by Brownian motion on the Sierpinski gasket. We prove the existence and uniqueness of solutions to these backward stochastic differential equations (BSDEs), and as an application, a Feynman–Kac representation formula for weak solutions to semi-linear parabolic PDEs on the gasket is also established. Compared to the study of BSDEs on \mathbb{R}^d , the major technical difficulty in the study of our case is the exponential integrability of quadratic processes for martingale additive functionals. This chapter is based on the joint work [45] with Z. Qian.

In Chapter 3, we study a class of stochastic optimal control problems when the system uncertainties arise from the filtration determined by Brownian motion on the Sierpinski gasket. Utilizing the theory developed in Chapter 2, we derive a stochastic Pontryagin maximum principle for stochastic control problems on the gasket. The linear regulator problem is presented in Section 3.3 as an example.

Chapter 4 is devoted to the study of a class of non-linear parabolic partial differential equations (PDEs for short) on the Sierpinski gasket. Unlike PDEs in domains of \mathbb{R}^d , such parabolic type equations involving non-linear convection terms must take a different form, due to the fact that convection terms must be singular to the “linear part” determined by the heat semigroup. In order to study these parabolic type equations, a new kind of Sobolev inequality for the Dirichlet form on the gasket will be established in Section 4.2, and their multi-dimensional case in Section 4.3. These Sobolev inequalities, which are interesting

on their own, involve two L^p norms with respect to two mutually singular measures in contrast to the case of Euclidean spaces. By examining properties of singular convolutions of the associated heat semigroup, we derive the space-time regularity of solutions to these parabolic equations under a few technical conditions. The Burgers equation on the Sierpinski gasket is also studied, for which a maximum principle for solutions is derived using techniques from backward stochastic differential equations. Existence, uniqueness, and the space-time regularity of solutions to Burgers equations are obtained as well. This Chapter is based on the work [46] joint with Z. Qian.

1.2 Preliminaries

1.2.1 Sierpinski gaskets

Let $V_0 = \{p_1, p_2, p_3\}$ be the set of vertices of the unit equilateral triangle in \mathbb{R}^2 , i.e. $p_1 = (0, 0)$, $p_2 = (1, 0)$, $p_3 = (1/2, \sqrt{3}/2)$, and let $\mathbf{F}_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $i = 1, 2, 3$ be the contractions defined by

$$\mathbf{F}_i(x) = 2^{-1}(x + p_i), \quad x \in \mathbb{R}^2, \quad i = 1, 2, 3. \quad (1.2.1)$$

The sets V_m , $m \in \mathbb{N}_+$ are defined recursively by $V_m = \bigcup_{i=1,2,3} \mathbf{F}_i(V_{m-1})$, $m \geq 1$.

Definition 1.2.1. The *compact Sierpinski gasket* \mathbb{S} (Figure 1.2.1) is defined to be the closure $\mathbb{S} = \text{cl}(\bigcup_{m=0}^{\infty} V_m)$, and the *infinite Sierpinski gasket* $\hat{\mathbb{S}}$ (Figure 1.2.2) to be $\hat{\mathbb{S}} = \bigcup_{k=0}^{\infty} 2^k[\mathbb{S} \cup (-\mathbb{S})]$. For simplicity, we shall call the compact Sierpinski gasket \mathbb{S} the Sierpinski gasket or the gasket, that is, without the prefix ‘‘compact’’.

Clearly, $\hat{\mathbb{S}}$ can be written as a countable union $\hat{\mathbb{S}} = \bigcup_{i \in \mathbb{Z}} \tau_i(\mathbb{S})$ of sets with disjoint interiors, where $\tau_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $i \in \mathbb{Z}$ are translations on \mathbb{R}^2 , and for our purpose, the labelling of the translations τ_i is immaterial.

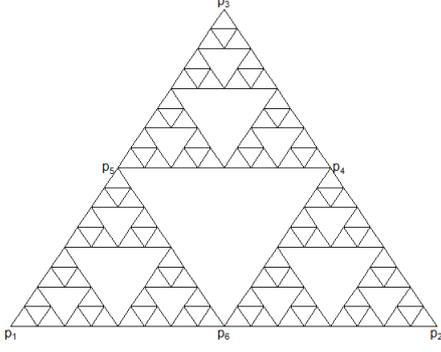


Figure 1.2.1: The Sierpinski gasket \mathbb{S}

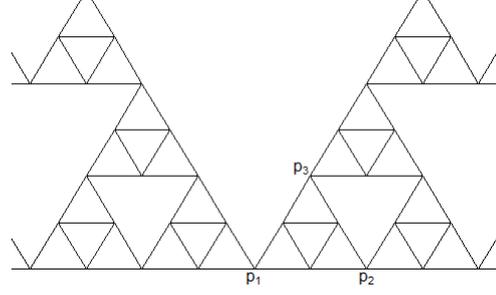


Figure 1.2.2: The infinite gasket $\hat{\mathbb{S}}$

Definition 1.2.2. The *shift space* W_* is defined to be the set of all infinite ordered sequence of symbols in the set $\{1, 2, 3\}$, i.e.

$$W_* = \{\omega = \omega_1\omega_2\omega_3 \dots : \omega_i \in \{1, 2, 3\}, i \in \mathbb{N}_+\}. \quad (1.2.2)$$

For any $\omega = \omega_1\omega_2\omega_3 \dots \in W_*$ and $m \in \mathbb{N}_+$, the truncation $[\omega]_m$ and the map $\mathbf{F}_{[\omega]_m} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are defined to be the finite sequence $[\omega]_m = \omega_1\omega_2 \dots \omega_m$ and the composition

$$\mathbf{F}_{[\omega]_m} = \mathbf{F}_{\omega_1 \dots \omega_{m-1} \omega_m} = \mathbf{F}_{\omega_m} \circ \mathbf{F}_{\omega_{m-1}} \circ \dots \circ \mathbf{F}_{\omega_1}, \quad (1.2.3)$$

respectively. As a convention, we define $\mathbf{F}_{[\omega]_0} = \text{Id}$ for all $\omega \in W_*$.

Remark 1.2.3. It should be noticed that the order of ω_i , $i = 1, \dots, m$ on the right hand side of (1.2.3) is the reverse of that of $[\omega]_m$.

For any $\omega \in W_*$, the triangle $\mathbf{F}_{[\omega]_m}(\mathbb{S})$ contracts to a single point as $m \rightarrow \infty$; in other words, the set $\bigcap_{m=0}^{\infty} \mathbf{F}_{[\omega]_m}(\mathbb{S})$ contains exactly one point. If we define $\pi(\omega)$ to be the unique element of $\bigcap_{m=0}^{\infty} \mathbf{F}_{[\omega]_m}(\mathbb{S})$, then the map $\pi : W_* \rightarrow \mathbb{S}$, $\omega \mapsto \pi(\omega)$ is surjective and gives a systematic labelling for points in \mathbb{S} .

Definition 1.2.4. The (*normalized*) *Hausdorff measure* μ on the Sierpinski gasket \mathbb{S} is the

unique Borel probability measure on \mathbb{S} such that

$$\mu(\mathbf{F}_{[\omega]_m}(\mathbb{S})) = 3^{-m} \text{ for all } \omega \in W_*, m \in \mathbb{N}_+. \quad (1.2.4)$$

The (*normalized*) Hausdorff measure $\hat{\mu}$ on the infinite gasket $\hat{\mathbb{S}}$ is defined to be the unique Borel measure on $\hat{\mathbb{S}}$ such that $(\hat{\mu} \circ \tau_i)|_{\mathbb{S}} = \mu$ for each $i \in \mathbb{Z}$.

1.2.2 Standard Dirichlet form and Brownian motion

On smooth manifolds, analysis can be established using the associated differential structures. However, there in general exist no differentiable structures on fractals, and for these spaces, analysis must be carried out by different approaches. As mentioned in Section 1.1, analysis on fractals can be established using the theory of Dirichlet forms, or equivalently, the theory of Markov processes (cf. [35]). Brownian motion on the Sierpinski gasket was constructed by S. Goldstein [22], S. Kusuoka [40], and M. Barlow and E. Perkins in [4]. Analytic construction of the Dirichlet form was given by J. Kigami [33]. In this subsection, we follow the approach proposed by J. Kigami [33] and introduce the standard Dirichlet form on the gasket \mathbb{S} by means of finite differences. The Dirichlet form we are going to construct is said to be “standard” as it is invariant under $(\pi/3)$ -rotations. Dirichlet forms on \mathbb{S} which are not rotationally invariant are also possible, and can be defined using the same procedure but with different renormalization factors (see, e.g. [35]).

Definition 1.2.5. For any functions u, v on $\cup_{m=0}^{\infty} V_m$, define recursively

$$\mathcal{E}^{(0)}(u, v) = 2^{-1} \sum_{x, y \in V_0} [u(x) - u(y)][v(x) - v(y)], \quad (1.2.5)$$

$$\mathcal{E}^{(m+1)}(u, v) = (5/3) \sum_{i=1,2,3} \mathcal{E}^{(m)}(u \circ \mathbf{F}_i, v \circ \mathbf{F}_i), \quad m = 0, 1, 2, \dots \quad (1.2.6)$$

Let us give some comments on Definition 1.2.5. Clearly, $\mathcal{E}^{(m)}(u, v)$ depends only on

the values of u, v on V_m , and can be regarded as a bilinear form defined for functions on V_m . The factor $1/2$ in (1.2.5) is merely out of convention, while the renormalization factor $5/3$ in (1.2.6) is chosen so that the sequence $\{\mathcal{E}^{(m)}\}_{m \in \mathbb{N}}$ is consistent in the sense of the following lemma (see [35, p. 71] for the proof).

Lemma 1.2.6. *Let $m \in \mathbb{N}$ and u be a function on V_m . The infimum*

$$\inf \{ \mathcal{E}^{(m+1)}(v, v) : v \text{ is a function on } V_{m+1} \text{ and } v|_{V_m} = u \}$$

is uniquely attained by a function h on V_{m+1} with $h|_{V_m} = u$. Moreover, $\mathcal{E}^{(m+1)}(h, h) = \mathcal{E}^{(m)}(u, u)$.

As a consequence of Lemma 1.2.6, we see that for any function u on $\cup_{m=0}^{\infty} V_m$, the sequence $\{\mathcal{E}^{(m)}(u, u)\}$ is non-decreasing in m . Therefore the limit $\lim_{m \rightarrow \infty} \mathcal{E}^{(m)}(u, u)$ exists (and may be infinite).

Definition 1.2.7. For any function u on $\cup_{m=0}^{\infty} V_m$, we define $\mathcal{E}(u, u)$ to be the limit

$$\mathcal{E}(u, u) = \lim_{m \rightarrow \infty} \mathcal{E}^{(m)}(u, u).$$

Let

$$\mathcal{F}(\mathbb{S}) = \left\{ u : u \text{ is a function on } \bigcup_{m=0}^{\infty} V_m \text{ with } \|u\|_{\mathcal{F}} < \infty \right\},$$

where

$$\|u\|_{\mathcal{F}}^2 \triangleq \mathcal{E}(u, u) + \|u\|_{L^2(\mu)}^2, \quad u \in \mathcal{F}(\mathbb{S}).$$

Occasionally we also use the notation $\mathcal{E}(u) = \mathcal{E}(u, u)$ for $u \in \mathcal{F}(\mathbb{S})$.

For any $u, v \in \mathcal{F}(\mathbb{S})$, the symmetric form $\mathcal{E}(u, v)$ is defined by the polarization

$$\mathcal{E}(u, v) = 4^{-1} [\mathcal{E}(u + v, u + v) - \mathcal{E}(u - v, u - v)].$$

By (1.2.6) and the definition of $\mathcal{E}(\cdot, \cdot)$, it is easily seen that \mathcal{E} satisfies the self-similar

property

$$\mathcal{E}(u, v) = (5/3) \sum_{i=1,2,3} \mathcal{E}(u \circ \mathbf{F}_i, v \circ \mathbf{F}_i). \quad (1.2.7)$$

The following theorem asserts that any function $u \in \mathcal{F}(\mathbb{S})$ uniquely extends to a continuous function on \mathbb{S} ; in other words, $\mathcal{F}(\mathbb{S}) \subseteq C(\mathbb{S})$. We refer the reader to [35, pp. 83–94] for the proof of Theorem 1.2.8 and Theorem 1.2.9.

Theorem 1.2.8. *Let u be a function on $\cup_{m=0}^{\infty} V_m$ and $\mathcal{E}(u, u) < \infty$. Then u is continuous on $\cup_{m=0}^{\infty} V_m$, and uniquely extends to a continuous function on \mathbb{S} . Therefore,*

$$\mathcal{F}(\mathbb{S}) = \{u \in C(\mathbb{S}) : \mathcal{E}(u, u) < \infty\}. \quad (1.2.8)$$

Moreover,

$$\text{osc}_{\mathbb{S}}(u) \leq C_* \mathcal{E}(u, u)^{1/2}, \quad \text{for all } u \in \mathcal{F}(\mathbb{S}), \quad (1.2.9)$$

where $\text{osc}_{\mathbb{S}}(u) = \max_{x, y \in \mathbb{S}} |u(x) - u(y)|$ is the oscillation of u on \mathbb{S} , and $C_* > 0$ is a universal constant.

Theorem 1.2.9. *The symmetric form $(\mathcal{E}, \mathcal{F}(\mathbb{S}))$ is a local Dirichlet form on $L^2(\mathbb{S}; \mu)$, and $\mathcal{F}(\mathbb{S}) \subseteq C(\mathbb{S})$.*

Definition 1.2.10. The Dirichlet form $(\mathcal{E}, \mathcal{F}(\mathbb{S}))$ on $L^2(\mathbb{S}; \mu)$ is called the *standard Dirichlet form* on \mathbb{S} . The non-negative self-adjoint operator \mathcal{L} associated with $(\mathcal{E}, \mathcal{F}(\mathbb{S}))$ is called the *standard Laplacian* on \mathbb{S} .

Remark 1.2.11. In Definition 1.2.10, the word “standard” refers to the spatial homogeneity of the Dirichlet form $(\mathcal{E}, \mathcal{F}(\mathbb{S}))$.

According to the theory of Dirichlet forms and Markov processes (cf. [20, Theorem A.2.2, p. 389]), we have the following definition.

Definition 1.2.12. The unique diffusion process $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \{X_t\}_{t \geq 0}, \{\mathbb{P}_x\}_{x \in \mathbb{S}})$ associated with the form $(\mathcal{E}, \mathcal{F}(\mathbb{S}))$ is called the *(reflected) Brownian motion* on \mathbb{S} , where $\{\mathcal{F}_t\}_{t \geq 0}$

is the filtration determined by $\{X_t\}_{t \geq 0}$, and \mathbb{P}_x is the law of $\{X_t\}_{t \geq 0}$ with initial point $X_0 = x \in \mathbb{S}$. The expectation with respect to \mathbb{P}_x will be denoted by \mathbb{E}_x .

Definition 1.2.13. For any Borel probability measure λ on \mathbb{S} , the probability measure \mathbb{P}_λ on Ω is defined to be

$$\mathbb{P}_\lambda(A) = \int_{\mathbb{S}} \mathbb{P}_x(A) \lambda(dx), \quad \text{for all } A \in \mathcal{B}(\mathbb{S}),$$

where $\mathcal{B}(\mathbb{S})$ is the Borel σ -field on \mathbb{S} . The corresponding expectation is denoted by \mathbb{E}_λ .

Remark 1.2.14. (i) Since $\mathcal{F}(\mathbb{S}) \subseteq C(\mathbb{S})$, the space $\mathcal{F}(\mathbb{S})$ is an algebra with its ring multiplication given by the pointwise multiplication of functions.

(ii) As a corollary of the inequality (1.2.9), we see that the empty set is the only subset of \mathbb{S} having zero capacity. Hence, the concept of “quasi-every” (“quasi-continuous” respectively) is identical to that of “every” (“continuous” respectively) for the Dirichlet form \mathcal{E} .

Combing the results of Lemma 1.2.6 and Theorem 1.2.8, we obtain the following.

Corollary 1.2.15. For any $m \in \mathbb{N}$ and any function u on V_m , there exists a unique function $h \in \mathcal{F}(\mathbb{S})$ such that $h|_{V_m} = u$ and

$$\mathcal{E}(h, h) = \mathcal{E}^{(m)}(u, u) = \inf \{ \mathcal{E}(v, v) : v \in \mathcal{F}(\mathbb{S}) \text{ and } v|_{V_m} = u \}.$$

Definition 1.2.16. The function h in Corollary 1.2.15 is called the m -harmonic function in \mathbb{S} with boundary value u . For the case when $m = 0$, the function h is simply called the harmonic function in \mathbb{S} . A function h is called *piecewise harmonic* if it is m -harmonic for some $m \in \mathbb{N}$.

Remark 1.2.17. (i) If h is an m -harmonic function, then $h \circ \mathbf{F}_{[\omega]_m}$ is a harmonic function for any $\omega \in W_*$.

(ii) For any $u \in \mathcal{F}(\mathbb{S})$, let u_m be the m -harmonic function with boundary value $u|_{V_m}$. Then $\mathcal{E}(u_m - u, u_m - u) = \mathcal{E}(u, u) - \mathcal{E}(u_m, u_m) \rightarrow 0$ as $m \rightarrow \infty$. Therefore, the family of piecewise harmonic functions is $\|\cdot\|_{\mathcal{F}}$ -dense in $\mathcal{F}(\mathbb{S})$.

For harmonic functions on the gasket, the following maximal principle is valid (see [35, Theorem 3.2.5] for the proof).

Lemma 1.2.18. *Suppose that h is a harmonic function in \mathbb{S} . Then*

$$\max_{\mathbb{S}} h = \max_{V_0} h.$$

Moreover, if $h(x) = \max_{\mathbb{S}} h$ for some $x \in \mathbb{S} \setminus V_0$, then h is a constant on \mathbb{S} .

The values of harmonic functions, and therefore m -harmonic function in view of Remark 1.2.17(i), can be calculated using the following lemma (cf. [35, p. 77]).

Lemma 1.2.19. *Let h be a harmonic function in \mathbb{S} . Then*

$$(h \circ \mathbf{F}_{[\omega]_m})|_{V_0} = \mathbf{A}_{[\omega]_m}(h|_{V_0}), \text{ for all } \omega = \omega_1\omega_2\omega_3 \dots \in W_*, m \in \mathbb{N}, \quad (1.2.10)$$

where

$$\mathbf{A}_{[\omega]_m} = \mathbf{A}_{\omega_m} \cdots \mathbf{A}_{\omega_2} \mathbf{A}_{\omega_1}, \quad (1.2.11)$$

and $\mathbf{A}_i : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $i = 1, 2, 3$ are the linear transformations of \mathbb{R}^3 with matrix representations

$$\mathbf{A}_1 = \frac{1}{5} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad \mathbf{A}_2 = \frac{1}{5} \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}, \quad \mathbf{A}_3 = \frac{1}{5} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}. \quad (1.2.12)$$

under the base consisting of indicator functions of the points $p_1 = (0, 0)$, $p_2 = (1, 0)$, $p_3 = (1/2, \sqrt{3}/2)$.

Remark 1.2.20. The matrices \mathbf{A}_i , $i = 1, 2, 3$ have the same eigenvalues $\{1/5, 3/5, 1\}$ and do not commute with each others.

The Dirichlet form on the infinite gasket $\hat{\mathbb{S}}$ can be defined in a similar way.

Definition 1.2.21. For any continuous function u on $\hat{\mathbb{S}}$, we define a symmetric form $(\hat{\mathcal{E}}, \mathcal{F}(\hat{\mathbb{S}}))$ by

$$\hat{\mathcal{E}}(u, u) = \sum_{i \in \mathbb{Z}} \mathcal{E}((u \circ \tau_i)|_{\mathbb{S}}, (u \circ \tau_i)|_{\mathbb{S}}), \quad (1.2.13)$$

$$\mathcal{F}(\hat{\mathbb{S}}) = \{u \in C(\hat{\mathbb{S}}) : \hat{\mathcal{E}}(u, u) < \infty\}. \quad (1.2.14)$$

Similar to the situation on \mathbb{S} , the symmetric form $(\hat{\mathcal{E}}, \mathcal{F}(\hat{\mathbb{S}}))$, called the *standard Dirichlet form* on $\hat{\mathbb{S}}$, is a local regular Dirichlet form on $L^2(\hat{\mathbb{S}}; \hat{\mu})$ satisfying the self-similar property

$$\hat{\mathcal{E}}(u, v) = (5/3) \hat{\mathcal{E}}(u \circ \mathbf{F}_1, v \circ \mathbf{F}_1), \quad u, v \in \mathcal{F}(\hat{\mathbb{S}}). \quad (1.2.15)$$

To simplify notation, we also denote $\hat{\mathcal{E}}(u) = \hat{\mathcal{E}}(u, u)$.

Definition 1.2.22. For any $x, y \in \hat{\mathbb{S}}$, define the *resistance metric* $R(x, y)$ by

$$R(x, y)^{-1} = \inf \{ \hat{\mathcal{E}}(u) : u \in \mathcal{F}(\hat{\mathbb{S}}), u(x) = 0, u(y) = 1 \}, \text{ if } x \neq y,$$

and $R(x, y) = 0$ if $x = y$.

For every $x, y \in \hat{\mathbb{S}}$, $R(x, y) < \infty$. If $x \neq y$, then there exists a unique $u \in \mathcal{F}(\hat{\mathbb{S}})$ such that $u(x) = 1$, $u(y) = 0$, $\hat{\mathcal{E}}(u) = R(x, y)^{-1}$ (see [35, Theorem 2.3.4]). Moreover, the resistance metric $R(\cdot, \cdot)$ is a metric on $\hat{\mathbb{S}}$ satisfying

$$C_*^{-1} |x - y|^{d_w - d_s} \leq R(x, y) \leq C_* |x - y|^{d_w - d_s}, \quad x, y \in \hat{\mathbb{S}} \quad (1.2.16)$$

for some universal constant $C_* \geq 1$, where $d_w = \log 5 / \log 2$ is the *walk dimension* of $\hat{\mathbb{S}}$, and

$$d_s = 2 \log 3 / \log 5$$

is the *spectral dimension* of $\hat{\mathbb{S}}$ (cf. [35, Lemma 3.3.5]). By the definition of $R(\cdot, \cdot)$,

$$|u(x) - u(y)| \leq R(x, y)^{1/2} \hat{\mathcal{E}}(u)^{1/2}, \quad u \in \mathcal{F}(\hat{\mathbb{S}}), \quad x, y \in \hat{\mathbb{S}}. \quad (1.2.17)$$

Since $u|_{\mathbb{S}} \in \mathcal{F}(\mathbb{S})$, $u \in \mathcal{F}(\hat{\mathbb{S}})$ and $\max_{\mathbb{S} \times \mathbb{S}} R < \infty$, (1.2.9) is in fact a corollary of (1.2.17).

1.2.3 Kusuoka measure, martingale representations, and gradients

As suggested by the formula (1.2.10) in Lemma 1.2.19, the matrices \mathbf{A}_i , $i = 1, 2, 3$ are closely related to the Dirichlet form \mathcal{E} . Indeed, as we shall see from (1.2.21) below, the relation between these matrices and the Dirichlet form \mathcal{E} can be expressed in terms of the Kusuoka measure defined below.

Definition 1.2.23. Let $\mathbf{P} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation

$$\mathbf{P}x = x - (x_1 + x_2 + x_3)/3, \quad \text{for all } x = (x_1, x_2, x_3) \in \mathbb{R}^3. \quad (1.2.18)$$

The *Kusuoka measure* ν on \mathbb{S} is defined to be the unique Borel probability measure on \mathbb{S} such that

$$\nu(\mathbf{F}_{[\omega]_m}(\mathbb{S}_0)) = 2^{-1} \cdot (5/3)^m \operatorname{tr}(\mathbf{A}_{[\omega]_m}^t \mathbf{P} \mathbf{A}_{[\omega]_m}), \quad \text{for all } \omega \in W_*, m \in \mathbb{N}. \quad (1.2.19)$$

The *Kusuoka measure* $\hat{\nu}$ on $\hat{\mathbb{S}}$ is defined to be the unique Borel measure on $\hat{\mathbb{S}}$ such that $(\hat{\nu} \circ \tau_i)|_{\mathbb{S}} = \nu$ for each $i \in \mathbb{Z}$.

Remark 1.2.24. (i) The factor 2^{-1} in (1.2.19) is a normalization factor so that $\nu(\mathbb{S}) = 1$, and the Kusuoka measure $\hat{\nu}$ on $\hat{\mathbb{S}}$ is the sum of the translations $\nu \circ \tau_i$ of the Kusuoka measure on \mathbb{S} .

(ii) The Kusuoka measure ν ($\hat{\nu}$ respectively) is singular to the Hausdorff measure μ ($\hat{\mu}$ respectively) (cf. [41]).

(iii) It is easily seen that $\mathbf{P}^t \mathbf{A}_i \mathbf{P} = \mathbf{P} \mathbf{A}_i = \mathbf{A}_i^t \mathbf{P} = \mathbf{P}^t \mathbf{A}_i^t \mathbf{P}$, $i = 1, 2, 3$. Therefore, the trace in (1.2.19) can be written as

$$\mathrm{tr}(\mathbf{A}_{[\omega]_m}^t \mathbf{P} \mathbf{A}_{[\omega]_m}) = \mathrm{tr}(\mathbf{Y}_{[\omega]_m}^t \mathbf{Y}_{[\omega]_m}),$$

where $\mathbf{Y}_{[\omega]_m} = \mathbf{Y}_{\omega_m} \cdots \mathbf{Y}_{\omega_2} \mathbf{Y}_{\omega_1}$ with $\mathbf{Y}_i = \mathbf{P}^t \mathbf{A}_i \mathbf{P}$, $i = 1, 2, 3$.

We are now in a position to state a representation theorem for martingale additive functionals on \mathbb{S} , which is a particular case of a representation theorem due to S. Kusuoka. The reader is referred to [41] for the proof, and to [26] for a generalization to a class of self-similar sets.

Theorem 1.2.25. *There exists a martingale additive functional $\{W_t\}_{t \geq 0}$ satisfying the following:*

(i) *The quadratic process $\langle W \rangle$ has ν as its Revuz measure (cf. [20, Theorem 5.1.3, pp. 228–230]), i.e. ν is the unique Borel measure on \mathbb{S} such that*

$$\int_{\mathbb{S}} f d\nu = \lim_{t \rightarrow 0} \frac{1}{t} \mathbb{E}_\mu \left(\int_0^t f(X_r) d\langle W \rangle_r \right), \text{ for all } f \in C(\mathbb{S});$$

(ii) *For any $u \in \mathcal{F}(\mathbb{S})$, there exists a unique $\zeta \in L^2(\mathbb{S}; \nu)$ such that*

$$M_t^{[u]} = \int_0^t \zeta(X_r) dW_r, \text{ for all } t \geq 0,$$

where $M^{[u]}$ is the martingale part of the additive functional $u(X_t) - u(X_0)$.

Remark 1.2.26. We should point out that the martingale additive functional $\{W_t\}_{t \geq 0}$ in Theorem 1.2.25 is not unique. In fact, for any Borel measurable function g on \mathbb{S} with $|g| = 1$, the martingale additive functional $\int_0^t g(X_r) dW_r$ also satisfies the properties (i) and (ii) in Theorem 1.2.25. Therefore, we have the freedom to choose the sign of the infinitesimal increment dW_t . To be definite, let us make a (canonical) choice of $\{W_t\}_{t \geq 0}$

as follows. Let h be the harmonic function in \mathbb{S} with boundary value

$$h|_{\nu_0}(x) = \begin{cases} 0, & \text{if } x = (0, 0), \\ 1, & \text{otherwise.} \end{cases}$$

For any martingale additive functional W_t satisfying the properties (i) and (ii) in Theorem 1.2.25, denote by $\nabla h \in L^2(\mathbb{S}; \nu)$ the integrand in the integral representation $M_t^{[h]} = \int_0^t \nabla h(X_r) dW_r$. The martingale additive functional $\{W_t\}_{t \geq 0}$ is chosen to be the unique one satisfying the properties in Theorem 1.2.25 and the additional condition that $\nabla h > 0$ ν -a.e. This can always be done by setting $W_t = \int_0^t \text{sgn}(\zeta_{W'}(X_r)) dW'_r$ for any W'_t satisfying the properties in Theorem 1.2.25.

Definition 1.2.27. The above described canonical choice of the martingale additive functional $\{W_t\}_{t \geq 0}$ is called the *Brownian martingale* on \mathbb{S} .

From now on, we shall always denote by $\{W_t\}_{t \geq 0}$ the Brownian martingale on \mathbb{S} . We are now in a position to introduce the definition of gradients of functions on \mathbb{S} with finite energy.

Definition 1.2.28. For any $u \in \mathcal{F}(\mathbb{S})$, the *gradient* ∇u of u is defined to be the unique element of $L^2(\mathbb{S}; \nu)$ such that

$$M_t^{[u]} = \int_0^t \nabla u(X_r) dW_r, \quad \text{for all } t \geq 0. \quad (1.2.20)$$

Suppose that $u \in \mathcal{F}(\mathbb{S})$. By the theory of Dirichlet forms, $\mathcal{E}(u, u) = \lim_{t \rightarrow 0} \frac{1}{t} \mathbb{E}_\mu[(M_t^{[u]})^2]$ (cf. [20, p. 247]). Using (1.2.20) and Theorem 1.2.25(i), we deduce that

$$\mathcal{E}(u, u) = \int_{\mathbb{S}} |\nabla u|^2 d\nu, \quad (1.2.21)$$

through which the Dirichlet form \mathcal{E} and the Kusuoka measure ν , and therefore the matrices \mathbf{A}_i , $i = 1, 2, 3$, are related.

Let $\Phi \in C^1(\mathbb{R}^m)$ and $\mathbf{u} = (u_1, \dots, u_m)$, $u_i \in \mathcal{F}(\mathbb{S})$, $i = 1, \dots, m$. Then $\Phi(\mathbf{u}) \in \mathcal{F}(\mathbb{S})$ and $\nabla \Phi(\mathbf{u}) = \sum_{i=1}^m \partial_i \Phi(\mathbf{u}) \cdot \nabla u_i$ ν -a.e. In particular, $\nabla(uv) = \nabla u \cdot v + u \cdot \nabla v$ ν -a.e. for all $u, v \in \mathcal{F}(\mathbb{S})$.

Definition 1.2.29. For each $u \in \mathcal{F}(\mathbb{S})$, the *energy measure* $\nu_{\langle u \rangle}$ of u is defined to be the unique Borel measure on \mathbb{S} such that

$$\int_{\mathbb{S}} \phi d\nu_{\langle u \rangle} = \mathcal{E}(\phi u, u) - 2\mathcal{E}(\phi, u^2), \quad \text{for all } \phi \in \mathcal{F}(\mathbb{S}). \quad (1.2.22)$$

For any $u, v \in \mathcal{F}(\mathbb{S})$, the *mutual energy measure* $\nu_{\langle u, v \rangle}$ of u and v is defined to be the polarization $\nu_{\langle u, v \rangle} = 4^{-1}(\nu_{\langle u+v \rangle} - \nu_{\langle u-v \rangle})$.

Remark 1.2.30. (i) According to [20, Lemma 5.3.3], the energy measure $\nu_{\langle u \rangle}$ of a function $u \in \mathcal{F}(\mathbb{S})$ coincides with the Revuz measure $\nu_{\langle M^{[u]} \rangle}$ of $\langle M^{[u]} \rangle$, i.e.

$$\int_{\mathbb{S}} \phi d\nu_{\langle u \rangle} = \lim_{t \rightarrow 0} \frac{1}{t} \mathbb{E}_{\mu} \left(\int_0^t \phi(X_r) d\langle M^{[u]} \rangle_r \right), \quad \text{for all } \phi \in \mathcal{F}(\mathbb{S}).$$

(ii) By (1.2.22) and the self-similar property (1.2.7) together with a standard approximation argument, we see that for any $\omega \in W_*$, $m \in \mathbb{N}$,

$$\nu_{\langle u \rangle}(\mathbf{F}_{[\omega]_m}(\mathbb{S})) = (5/3)^m \mathcal{E}(u \circ \mathbf{F}_{[\omega]_m}, u \circ \mathbf{F}_{[\omega]_m}). \quad (1.2.23)$$

The following is an immediate consequence of Theorem 1.2.25.

Corollary 1.2.31. *The measure ν is energy dominant, that is, $\nu_{\langle u \rangle} \ll \nu$ for all $u \in \mathcal{F}(\mathbb{S})$.*

Chapter 2

Backward Stochastic Differential

Equations on the Gasket

2.1 Introduction

The subject of this chapter is a class of backward stochastic differential equations (BSDEs), with deterministic or random durations, driven by the Brownian martingale $\{W_t\}_{t \geq 0}$ on the gasket \mathbb{S} . We shall give a Feynman–Kac representation (Theorem 2.4.5) for solutions to semi-linear parabolic equations on \mathbb{S} , of which the meaning will be clarified later. On one hand, the BSDEs considered in this thesis can be regarded as a specific case of those studied in [37, 38] associated with quite general (quasi-regular) Dirichlet forms. On the other hand, solutions to the BSDEs in our case can be formulated in a more specific way (more precisely, the martingale parts are given as stochastic integrals, and the exponential integrability assumption on quadratic processes as drifts can be verified), which is due to the concrete setting of the Dirichlet form on the gasket. The representation in Theorem 2.4.5 is an analogue of the result established in [57, 62, 2], which provides probabilistic solutions to parabolic equations in terms of BSDEs.

Linear BSDEs were first introduced by J. Bismut to establish a Pontryagin maximum

principle in stochastic control theory (see [5, 70] etc.), while the theory of non-linear BSDEs was developed in E. Pardoux and S. Peng [55]. The celebrated Feynman–Kac formula was also generalized to non-linear cases in [57] using BSDEs. There have been a large number of works on the theory of BSDEs. For example, BSDEs associated with non-symmetric second-order elliptic operators of divergence form on Euclidean spaces were studied in [62, 2] and applied to study semi-linear parabolic PDEs involving divergences of measurable vector fields, where an Itô–Fukushima decomposition for the diffusion process associated to the elliptic operator was derived in terms of forward-backward martingales, and a representation formula for solutions to parabolic PDEs was obtained. BSDEs and semi-linear parabolic equations on Hilbert spaces were investigated in [72] using methods from functional analysis and generalized Dirichlet forms. A martingale representation with countably many representing martingales for the infinite dimensional case was also proved in [72] in order to solve BSDEs on Hilbert spaces. In [37, 38], existence and uniqueness of solutions to a class of BSDEs associated with non-local regular (or quasi-regular) Dirichlet forms were established, together with a probabilistic representation of solutions to semi-linear elliptic equations perturbed by smooth measures. It was also shown in [37] that the probabilistic solutions yielded by BSDEs coincide with the notion of weak solutions (called solutions in the sense of duality in [37]) under a transience assumption on the Dirichlet form.

2.2 Several results on the Brownian martingale

In this section, we derive several results on the Kusuoka measure ν and the Brownian martingale $\{W_t\}_{t \geq 0}$ which will be needed in later sections. The main technical result in this section is the exponential integrability of the quadratic process $\langle W \rangle$ (Corollary 2.2.16), which is a sufficient condition for the Girsanov theorem and is crucial for the existence of solutions to BSDEs on the gasket.

2.2.1 Martingale representations for square integrable martingales

Similar to situations on Euclidean spaces, a martingale representation theorem for square integrable $\{\mathcal{F}_t\}$ -martingales is the cornerstone of BSDE theory on the Sierpinski gaskets, and a partial result in this direction is given by Theorem 1.2.25 for martingale additive functionals. The gap between martingale additive functionals and square integrable martingales can be filled by the following theorem, which states that, under certain conditions, the representation theorem for martingale additive functionals is equivalent to that for square integrable martingales (cf. [59] for the proof).

Theorem 2.2.1. *Let $(\Omega, \{\mathcal{G}_t\}_{t \geq 0}, \{Y_t\}_{t \geq 0}, \{\mathbb{Q}_x\}_{x \in E})$ be a continuous Hunt process in state space E , and λ be a Borel probability measure on E . Then any local martingale on $(\Omega, \{\mathcal{G}_t\}_{t \geq 0}, \mathbb{Q}_\lambda)$ is continuous. Suppose further that there exist a sub-algebra $\mathcal{A} \subseteq L^2(E; \lambda) \cap \mathcal{B}_b(E)$ and finitely many continuous martingales W^1, \dots, W^d such that the following are satisfied:*

(i) $\sigma(\mathcal{A}) = \mathcal{B}(E)$ and $R_\alpha(\mathcal{A}) \subseteq \mathcal{A}$ for each $\alpha > 0$, where $\sigma(\mathcal{A})$ is the σ -algebra generated by \mathcal{A} , and R_α is the α -resolvent of $\{Y_t\}$, i.e. R_α is the operator defined by

$$R_\alpha f(x) = \mathbb{E}_x \left(\int_0^\infty e^{-\alpha t} f(Y_t) dt \right);$$

(ii) For any $f \in \mathcal{A}$ and any $\alpha > 0$, there exist $\{\mathcal{G}_t\}$ -predictable processes f^1, \dots, f^d such that $M_t^{[R_\alpha f]} = \sum_{j=1}^d \int_0^t f^j dW_r^j$, $t \geq 0$ \mathbb{Q}_λ -a.s., where $M^{[R_\alpha f]}$ is the martingale part of $R_\alpha f(Y_t) - R_\alpha f(Y_0)$;

(iii) The covariance matrix $(\langle W^i, W^j \rangle_t)_{i,j}$ is strictly positive definite for all $t \geq 0$, \mathbb{Q}_λ -a.s.

Then, for any square integrable martingale M on $(\Omega, \{\mathcal{G}_t\}_{t \geq 0}, \mathbb{Q}_\lambda)$, there exist $\{\mathcal{G}_t\}$ -predictable processes f^1, \dots, f^d such that

$$M_t = M_0 + \sum_{j=1}^d \int_0^t f^j(r) dW_r^j, \quad t \geq 0 \quad \mathbb{P}_\lambda\text{-a.s.}$$

The processes f^1, \dots, f^d are unique in that if $\tilde{f}^1, \dots, \tilde{f}^d$ also satisfy the above representation formula, then

$$\sum_{i,j=1}^d \mathbb{E}_\lambda \left[\int_0^\infty (f^i(r) - \tilde{f}^i(r))(f^j(r) - \tilde{f}^j(r)) d\langle W^i, W^j \rangle_r \right] = 0.$$

For any Borel probability λ on \mathbb{S} , all the assumptions of Theorem 2.2.1 are easily seen to be satisfied by Brownian motion $\{X_t\}_{t \geq 0}$ with $\mathcal{F}(\mathbb{S})$ taken as the algebra \mathcal{A} . Therefore, we deduce the following representation theorem for square integrable martingales on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_\lambda)$.

Theorem 2.2.2. *For any square integrable martingale M on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_\lambda)$, there exists an $\{\mathcal{F}_t\}$ -predictable process f such that*

$$M_t = M_0 + \int_0^t f(r) dW_r, \quad t \geq 0 \quad \mathbb{P}_\lambda\text{-a.s.}$$

The process f is unique in that if \tilde{f} is another $\{\mathcal{F}_t\}$ -predictable process satisfying the above, then

$$\mathbb{E}_\lambda \left[\int_0^\infty [f(r) - \tilde{f}(r)]^2 d\langle W \rangle_r \right] = 0.$$

By Theorem 2.2.2, any $\{\mathcal{F}_t\}$ -adapted semi-martingales Y_t can be written in the form

$$Y_t = Y_0 + \int_0^t g(r) dr + \int_0^t f(r) d\langle W \rangle_r + M_t, \quad t \geq 0, \quad (2.2.1)$$

where M is a martingale on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_\mu)$. Since both $\int_0^t g(r) dr$ and $\int_0^t f(r) d\langle W \rangle_r$ have finite variations, one may ask about the uniqueness of the decomposition (2.2.1). We now show that the decomposition (2.2.1) is indeed unique, which follows immediately from the lemma below.

Lemma 2.2.3. *The Lebesgue-Stieltjes measure $d\langle W \rangle_t$ is singular with respect to the Lebesgue measure on $[0, \infty)$, \mathbb{P}_μ -a.s.*

Proof. Denote by \mathcal{P} the σ -field of predictable sets of $[0, \infty) \times \Omega$. Let Q be the unique measure on $([0, \infty) \times \Omega, \mathcal{P})$ satisfying $Q(\llbracket \sigma, \tau \rrbracket) = \mathbb{E}_\mu(\langle W \rangle_\tau - \langle W \rangle_\sigma)$, $\sigma, \tau \in \mathcal{T}_p$, where \mathcal{T}_p is the family of all $\{\mathcal{F}_t\}$ -predictable times, and $\llbracket \sigma, \tau \rrbracket = \{(t, \omega) \in [0, \infty) \times \Omega : \sigma(\omega) \leq t < \tau(\omega)\}$. By the Lebesgue decomposition, $Q = f \cdot (dt \times \mathbb{P}_\mu) + Q_s$, where $f \geq 0$ is a predictable process and is σ -integrable with respect to $dt \times \mathbb{P}_\mu$, and Q_s is a σ -finite positive measure singular to $dt \times \mathbb{P}_\mu$.

Let $B \in \mathcal{B}(\mathbb{S})$ be such that $\mu(B) = 1 = \nu(\mathbb{S} \setminus B) = 1$. By $\mathbb{E}_\mu(\int_0^T 1_{\mathbb{S} \setminus B}(X_r) dr) = T\mu(\mathbb{S} \setminus B) = 0$, we see that $1_B(X_t) = 1$, a.e. $t \geq 0$, \mathbb{P}_μ -a.s. Notice that, for any non-negative predictable process g ,

$$\int_{[0, T) \times \Omega} g(r) Q(dr, d\omega) = \mathbb{E}_\mu\left(\int_0^T g(r) d\langle W \rangle_r\right),$$

which can be easily shown by the definition of Q and a standard monotone class argument.

Therefore, by applying the above equality to $g(t) = 1_B(X_t)$,

$$\begin{aligned} \int_0^T \mathbb{E}_\mu(f(t)) dt &= \int_0^T \mathbb{E}_\mu(f(t) 1_B(X_r)) dt \leq \int_{[0, T) \times \Omega} 1_B(X_t(\omega)) Q(dt, d\omega) \\ &= \mathbb{E}_\mu\left(\int_0^T 1_B(X_t) d\langle W \rangle_t\right) = T\nu(B) = 0, \end{aligned}$$

which implies the conclusion of the lemma. \square

Corollary 2.2.4. *Let Y be a semi-martingale on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_\mu)$. Then the decomposition (1.2.12) is unique in that if (1.2.12) also holds with g, f, M replaced by $\tilde{g}, \tilde{f}, \tilde{M}$, then $\mathbb{E}_\mu(\int_0^\infty |g(r) - \tilde{g}(r)| dr) = \mathbb{E}_\mu(\int_0^\infty |f(r) - \tilde{f}(r)| d\langle W \rangle_r) = 0$.*

2.2.2 A time-dependent Itô–Fukushima decomposition

In this subsection, we give a time-dependent Itô–Fukushima decomposition (Lemma 2.2.8), which plays an important part in the derivation of the Feynman–Kac representation (Theorem 2.4.5). See [48, 18] for similar results in different settings. To formulate the decom-

position lemma, we first introduce the definition of the space $\mathcal{F}^{-1}(\mathbb{S})$, which is the analogue of H^{-1} spaces (i.e. the L^2 -dual of the Sobolev space $H^1 = W^{1,2}$) on \mathbb{R}^d .

Definition 2.2.5. For any $w \in L^2(\mathbb{S}; \mu)$, let

$$\|w\|_{\mathcal{F}^{-1}} = \sup \{ \langle u, w \rangle_\mu : u \in \mathcal{F}(\mathbb{S}), \|u\|_{\mathcal{F}} \leq 1 \}.$$

The space $\mathcal{F}^{-1}(\mathbb{S})$ is defined to be the $\|\cdot\|_{\mathcal{F}^{-1}}$ -completion of $L^2(\mathbb{S}; \mu)$.

Definition 2.2.6. Let $u \in L^2(0, T; \mathcal{F}(\mathbb{S}))$; that is, $u = u(t)$ is an $\mathcal{F}(\mathbb{S})$ -valued function on $[0, T]$ such that

$$\int_0^T [\mathcal{E}(u(t), u(t)) + \|u(t)\|_{L^2(\mu)}^2] dt < \infty.$$

We say that u has a *weak derivative* $\partial_t u$ in $L^2(0, T; \mathcal{F}^{-1}(\mathbb{S}))$, if $\partial_t u$ is an $\mathcal{F}^{-1}(\mathbb{S})$ -valued function on $[0, T]$ such that

$$\|u\|_{L^2(0, T; \mathcal{F}^{-1})}^2 = \int_0^T \|\partial_t u(t)\|_{\mathcal{F}^{-1}}^2 dt < \infty,$$

and

$$\int_0^T \langle u(t), \partial_t v(t) \rangle_\mu dt = - \int_0^T \langle \partial_t u(t), v(t) \rangle_\mu dt$$

for all $v \in C^1(0, T; \mathcal{F}(\mathbb{S}))$ with $v(0) = v(T) = 0$.

The following can be easily shown by a standard mollifier argument (cf. [12, Theorem 3, Section 5.9]).

Lemma 2.2.7. *Suppose $u \in L^2(0, T; \mathcal{F}(\mathbb{S}))$ has a weak derivative $\partial_t u \in L^2(0, T; \mathcal{F}^{-1}(\mathbb{S}))$. Then the function $t \mapsto \|u(t)\|_{L^2(\mu)}^2$, $t \in [0, T]$ is absolutely continuous and*

$$\frac{d}{dt} \|u(t)\|_{L^2(\mu)}^2 = 2 \langle \partial_t u(t), u(t) \rangle_\mu \quad \text{a.e. } t \in [0, T].$$

We can now state the decomposition lemma, which follows from Theorem 2.2.2 and

the result in [68, Theorem 4.5] applied to the (non-symmetric) generalized Dirichlet form

$$(u, v) \mapsto \Lambda(u, v) + \int_0^T \mathcal{E}(u(t), v(t)) dt,$$

where

$$\Lambda(u, v) = \int_0^T \langle u(t), \partial_t v(t) \rangle_\mu dt,$$

if $u \in L^2(0, T; \mathcal{F}(\mathbb{S}))$, $v \in L^2(0, T; \mathcal{F}(\mathbb{S})) \cap C^1(0, T; L^2(\mu))$, and

$$\Lambda(u, v) = - \int_0^T \langle \partial_t u(t), v(t) \rangle_\mu dt,$$

if $u \in L^2(0, T; \mathcal{F}(\mathbb{S})) \cap C^1(0, T; L^2(\mu))$, $v \in L^2(0, T; \mathcal{F}(\mathbb{S}))$. The reader is referred to [61, 68] and references therein for the theory of generalized Dirichlet forms.

Lemma 2.2.8. *Suppose $u \in C([0, T] \times \mathbb{S}) \cap L^2(0, T; \mathcal{F}(\mathbb{S}))$ and that u has a weak derivative $\partial_t u$ in $L^2(0, T; \mathcal{F}^{-1}(\mathbb{S}))$. Then*

$$u(t, X_t) = u(0, X_0) + \int_0^t \nabla u(r, X_r) dW_r + N_t, \quad t \in [0, T],$$

where N_t is a continuous processes with zero quadratic variation; that is,

$$\lim_{n \rightarrow \infty} \mathbb{E}_\mu \left[\sum_{i=1}^n (N_{t_i} - N_{t_{i-1}})^2 \right] = 0,$$

where $t_i = iT/n$, $i = 0, 1, \dots, n$.

2.2.3 Exponential integrability of the quadratic process

We now show the exponential integrability of $\langle W \rangle$. To this end, we shall need the following heat kernel estimate. See [35, Theorem 5.3.1] for the proof.

Lemma 2.2.9. *Brownian motion $\{X_t\}_{t \geq 0}$ admits a jointly continuous transition kernel*

$p(t, x, y)$ such that

$$C_{*,1} \max\{1, t^{-d_s/2}\} \leq \sup_{x,y \in \mathbb{S}} p(t, x, y) \leq C_{*,2} \max\{1, t^{-d_s/2}\},$$

for some universal constants $C_{*,1}, C_{*,2} > 0$, where $d_s = 2 \log 3 / \log 5$ is the spectral dimension of \mathbb{S} .

According to Lemma 2.2.9, the heat kernel $p(t, x, y)$ is jointly continuous, and therefore the definition below is legitimate.

Definition 2.2.10. For each Radon measure λ on \mathbb{S} and any $t > 0$, the function $P_t \lambda$ on \mathbb{S} is defined to be

$$P_t \lambda(x) \triangleq \int_{\mathbb{S}} p(t, x, y) \lambda(dy), \quad x \in \mathbb{S}. \quad (2.2.2)$$

Remark 2.2.11. If $\lambda = f\mu$ for some $f \in L^1(\mu)$, then $P_t \lambda = P_t f$. Therefore, Definition 2.2.10 coincides with the ordinary definition of the Markov operator P_t .

As a consequence of the joint continuity of the heat kernel $p(t, x, y)$, it is seen that $P_t \lambda \in C(\mathbb{S})$ and $|P_t \lambda| \leq |\lambda|(\mathbb{S}) < \infty$, where the last inequality follows from λ being a Radon measure.

Lemma 2.2.12. Suppose that $A = \{A_t\}_{t \geq 0}$ is a positive continuous additive functional such that $\nu_A(\mathbb{S}) < \infty$, where ν_A is the Revuz measure of A . For each $t > 0$ and each $f \in \mathcal{B}_b([0, \infty) \times \mathbb{S})$, the following is valid

$$\mathbb{E}_x \left(\int_0^t f(r, X_r) dA_r \right) = \int_0^t P_r(f(r, \cdot) \nu_A)(x) dr, \quad x \in \mathbb{S}. \quad (2.2.3)$$

Proof. Suppose first that $f(t) = f_0$, $t \geq 0$ for some $f_0 \in \mathcal{B}_b(\mathbb{S})$. By the definition of $P_r(f \nu_A)$,

$$\mathbb{E}_\mu \left[\left(\int_0^t f(r, X_r) dA_r \right) h(X_0) \right] = \int_0^t \langle P_r(f(r, \cdot) \nu_A), h \rangle_\mu dr = \left\langle \int_0^t P_r(f(r, \cdot) \nu_A) dr, h \right\rangle_\mu,$$

for all $h \in \mathcal{B}_b(\mathbb{S})$, and the above equality extends to general $f \in \mathcal{B}_b([0, \infty) \times \mathbb{S})$ by a monotone class argument. Now (2.2.3) follows immediately in view of the continuity of both sides of the equality. \square

Lemma 2.2.13. *Let $A^{(i)}$, $i = 1, \dots, n$ be positive continuous additive functionals with $\nu_i(\mathbb{S}) < \infty$, where ν_i is the Revuz measure of $A^{(i)}$, $i = 1, \dots, n$. Then, for each $t > 0$ and each $f_i \in \mathcal{B}_b([0, \infty) \times \mathbb{S})$, $i = 1, \dots, n$, and each $x \in \mathbb{S}$,*

$$\begin{aligned} & \mathbb{E}_x \left(f(X_t) \int_{0 < t_1 < \dots < t_n < t} f_1(t_1, X_{t_1}) \cdots f_n(t_n, X_{t_n}) dA_{t_1}^{(1)} \cdots dA_{t_n}^{(n)} \right) \\ &= \int_{0 < t_1 < \dots < t_n < t} P_{t_1}(\nu_1 f_1(t_1) P_{t_2-t_1}(\cdots \nu_n f_n(t_n) P_{t-t_n} f) \cdots)(x) dt_1 \cdots dt_{n-1} dt_n, \end{aligned} \quad (2.2.4)$$

where we have used the shorthand notation $f_i(t) = f_i(t, \cdot)$, and the measures ν_i are placed before functions due to notational consideration: the alternative writing $(f_1 \nu_1) P_t(\nu_2 f_2)$ places a measure ν_1 between functions f_1 and $P_t(\nu_2 f_2)$.

Proof. By Lemma 2.2.12,

$$\begin{aligned} \mathbb{E}_x \left(f(X_t) \int_0^t f_1(t_1, X_{t_1}) dA_{t_1}^{(1)} \right) &= \mathbb{E}_x \left(\int_0^t f_1(t_1, X_{t_1}) \mathbb{E}_x(f(X_t) | \mathcal{F}_{t_1}) dA_{t_1}^{(1)} \right) \\ &= \int_0^t P_{t_1}(\nu_1 f_1(t_1) P_{t-t_1} f)(x) dt_1, \end{aligned}$$

which proves (2.2.4) for $n = 1$. The conclusion for a general n follows easily from induction. \square

Proposition 2.2.14. *Let A be a positive continuous additive functional such that $\nu_A(\mathbb{S}) < \infty$, where ν_A is the Revuz measure of A .*

(a) For each $\beta > 0$,

$$\sup_{x \in \mathbb{S}} \mathbb{E}_x(e^{\beta A_t}) \leq E_{\gamma^s, 1}(C_* \nu_A(\mathbb{S}) \beta \max\{t, t^{\gamma^s}\}), \quad t \geq 0. \quad (2.2.5)$$

where $C_* > 0$ is a universal constant, $\gamma_s = 1 - d_s/2 = 1 - \log 3/\log 5$, and

$$\mathbb{E}_{a,b}(z) = \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(ap + b)}, \quad z \in \mathbb{C}, \quad a, b > 0$$

is the Mittag-Leffler function, with Γ being the Gamma function.

(b) For each $f \in L_+^1(\mu)$, $t \geq 0$, and $\beta > 0$,

$$\sup_{x \in \mathbb{S}} \mathbb{E}_x(f(X_t)e^{\beta A_t}) \leq \max\{1, t^{-d_s/2}\} \|f\|_{L^1(\mu)} \mathbb{E}_{\gamma_s, \gamma_s}(C_* \nu_A(\mathbb{S}) \beta \max\{t, t^{\gamma_s}\}).$$

Remark 2.2.15. The asymptotics of $\mathbb{E}_x(e^{\beta A_t})$ as $t \rightarrow 0$ can be seen from the following fact regarding the Mittag–Leffler function $\mathbb{E}_{a,b}$

$$\limsup_{r \rightarrow \infty} \frac{\log \log(\sup_{|z| \leq r} |\mathbb{E}_{a,b}(z)|)}{\log r} = \frac{1}{a}.$$

Proof. (a) Suppose first that $t \in (0, 1]$. For each $p \in \mathbb{N}_+$, by Lemma 2.2.13,

$$\begin{aligned} \mathbb{E}_x(A_t^p) &= p! \mathbb{E}_x \left(\int_{0 < t_1 < \dots < t_p < t} dA_{t_1} \cdots dA_{t_p} \right) \\ &= p! \int_{0 < t_1 < \dots < t_p < t} P_{t_1}(\nu_A P_{t_2-t_1}(\cdots \nu_A P_{t_p-t_{p-1}}(\nu_A) \cdots))(x) dt_1 \cdots dt_p. \end{aligned}$$

For each non-negative $f \in C(\mathbb{S})$, by Lemma 2.2.9,

$$\|P_r(f \nu_A)\|_{L^\infty} \leq \|f\|_{L^\infty} P_r \nu_A(x) \leq C_* \|f\|_{L^\infty} \nu_A(\mathbb{S}) r^{-d_s/2}, \quad r > 0, \quad (2.2.6)$$

so that

$$\|P_{t_1}(\nu_A P_{t_2-t_1}(\cdots \nu_A P_{t_p-t_{p-1}}(\nu_A) \cdots))\|_{L^\infty} \leq (C_* \nu_A(\mathbb{S}))^p [t_1(t_2-t_1) \cdots (t_p-t_{p-1})]^{-d_s/2},$$

which implies that

$$\begin{aligned}
& \frac{1}{p!} \mathbb{E}_x(A_t^p) \\
& \leq (C_* \nu_A(\mathbb{S}))^p \int_{0 < t_1 < \dots < t_p < t} [t_1(t_2 - t_1) \cdots (t_p - t_{p-1})]^{-d_s/2} dt_1 \cdots dt_p \quad (2.2.7) \\
& = (C_* \nu_A(\mathbb{S}) t^{\gamma_s})^p \int_{0 < \theta_1 < \dots < \theta_p < 1} [\theta_1(\theta_2 - \theta_1) \cdots (\theta_p - \theta_{p-1})]^{-d_s/2} d\theta_1 \cdots d\theta_p.
\end{aligned}$$

Let

$$\beta_0(\gamma_1, \dots, \gamma_p) = \int_{0 < \theta_1 < \dots < \theta_p < 1} \theta_1^{\gamma_1-1} (\theta_2 - \theta_1)^{\gamma_2-1} \cdots (\theta_p - \theta_{p-1})^{\gamma_p-1} d\theta_1 \cdots d\theta_p.$$

Then

$$\begin{aligned}
& \beta_0(\gamma_1, \dots, \gamma_p) \\
& = \int_{0 < \theta_2 < \dots < \theta_p < 1} \left(\int_0^{\theta_2} \theta_1^{\gamma_1-1} (\theta_2 - \theta_1)^{\gamma_2-1} d\theta_1 \right) (\theta_3 - \theta_2)^{\gamma_3-1} \cdots (\theta_p - \theta_{p-1})^{\gamma_p-1} d\theta_2 \cdots d\theta_p \\
& = \int_{0 < \theta_2 < \dots < \theta_p < 1} \left(\int_0^1 \theta^{\gamma_1-1} (1 - \theta)^{\gamma_2-1} d\theta \right) \theta_2^{\gamma_1+\gamma_2-1} (\theta_3 - \theta_2)^{\gamma_3-1} \cdots (\theta_p - \theta_{p-1})^{\gamma_p-1} d\theta_2 \cdots d\theta_p \\
& = B(\gamma_1, \gamma_2) \beta_0(\gamma_1 + \gamma_2, \gamma_3, \dots, \gamma_p),
\end{aligned}$$

where $B(\cdot, \cdot)$ is the Beta function. By induction, we see that

$$\begin{aligned}
\beta_0(\gamma_1, \dots, \gamma_p) & = B(\gamma_1, \gamma_2) B(\gamma_1 + \gamma_2, \gamma_3) \cdots B(\gamma_1 + \cdots + \gamma_{p-1}, \gamma_p) \int_0^1 \theta^{\gamma_1 + \cdots + \gamma_p - 1} d\theta \\
& = \frac{B(\gamma_1, \gamma_2) B(\gamma_1 + \gamma_2, \gamma_3) \cdots B(\gamma_1 + \cdots + \gamma_{p-1}, \gamma_p)}{\gamma_1 + \cdots + \gamma_p}.
\end{aligned}$$

Therefore, by (2.2.7),

$$\begin{aligned}
\frac{1}{p!} \mathbb{E}_x(A_t^p) & \leq (C_* \nu_A(\mathbb{S}) t^{\gamma_s})^p \beta(\gamma_s, \dots, \gamma_s) \\
& = \frac{(C_* \nu_A(\mathbb{S}) t^{\gamma_s})^p}{p \gamma_s} \prod_{i=1}^{p-1} B(i \gamma_s, \gamma_s) = \frac{(C_* \nu_A(\mathbb{S}) t^{\gamma_s})^p}{\Gamma(p \gamma_s + 1)},
\end{aligned}$$

for each $p \in \mathbb{N}$, $t \in (0, 1]$, which implies (2.2.5) for $t \in [0, 1]$.

Suppose $t \in (1, \infty)$. We first claim that

$$\sup_{x \in \mathbb{S}} \mathbb{E}_x(e^{\beta(A_{k+1}-A_k)}) \leq \mathbb{E}_{\gamma_s, 1}(C_* \nu_A(\mathbb{S})\beta), \quad \beta > 0, k \in \mathbb{N}. \quad (2.2.8)$$

In fact, for any $f \in L^1(\mu)$,

$$\begin{aligned} \left| \int_{\mathbb{S}} f(x) \mathbb{E}_x(e^{\beta(A_{k+1}-A_k)}) \mu(dx) \right| &= \left| \mathbb{E}_\mu(f(X_0)e^{\beta(A_{k+1}-A_k)}) \right| \\ &= \left| \mathbb{E}_\mu[f(X_0)\mathbb{E}_{X_k}(e^{\beta A_1})] \right| \leq \|f\|_{L^1(\mu)} \mathbb{E}_{\gamma_s, 1}(C_* \nu_A(\mathbb{S})\beta), \end{aligned}$$

which implies that $\mathbb{E}_x(e^{\beta(A_{k+1}-A_k)}) \leq \mathbb{E}_{\gamma_s, 1}(C_* \Gamma(\gamma_s) \nu_A(\mathbb{S})\beta)$, μ -a.e. $x \in \mathbb{S}$. In particular,

$$\sum_{p=0}^N \frac{1}{p!} \mathbb{E}_x((A_{k+1} - A_k)^p) \leq \mathbb{E}_{\gamma_s, 1}(C_* \nu_A(\mathbb{S})\beta), \quad N \in \mathbb{N}, \mu\text{-a.e. } x \in \mathbb{S}.$$

By Lemma 2.2.13, for each $N \in \mathbb{N}$, the function $x \mapsto \sum_{p=0}^N \frac{1}{p!} \mathbb{E}_x((A_{k+1} - A_k)^p)$ is continuous. Therefore, the above holds for each $N \in \mathbb{N}$ and each $x \in \mathbb{S}$. Letting $N \rightarrow \infty$ proves (2.2.8). Now let $n \in \mathbb{N}_+$ be such that $n < t \leq n+1 \leq 2t$, by Hölder's inequality and (2.2.8), we have

$$\mathbb{E}_x(e^{\beta A_t}) \leq \prod_{i=0}^n [\mathbb{E}_x(e^{(n+1)\beta(A_{i+1}-A_i)})]^{1/(n+1)} \leq \mathbb{E}_{\gamma_s, 1}(C_* \nu_A(\mathbb{S})\beta t).$$

which proves (a).

(b) Suppose $t \in (0, 1]$. By Lemma 2.2.13,

$$\mathbb{E}_x(f(X_t)A_t^p) = p! \int_{0 < t_1 < \dots < t_p < t} P_{t_1}(\nu_A P_{t_2-t_1}(\dots \nu_A P_{t_p-t_{p-1}}(\nu_A P_{t-t_p} f) \dots))(x) dt_1 \dots dt_p.$$

By successive applications of (2.2.6),

$$\begin{aligned} &\|P_{t_1}(\nu_A P_{t_2-t_1}(\dots \nu_A P_{t_p-t_{p-1}}(\nu_A P_{t-t_p} f) \dots))\|_{L^\infty} \\ &\leq \|f\|_{L^1(\mu)} (C_* \nu_A(\mathbb{S}))^p [t_1(t_2 - t_1) \dots (t - t_p)]^{-d_s/2}, \end{aligned}$$

which gives

$$\begin{aligned} \frac{1}{p!} \mathbb{E}_x(f(X_t)A_t^p) &\leq (C_* \nu_A(\mathbb{S}))^p \int_{0 < t_1 < \dots < t_p < t} [t_1(t_2 - t_1) \cdots (t - t_p)]^{-d_s/2} dt_1 \cdots dt_p \\ &= t^{-d_s/2} \|f\|_{L^1(\nu)} (C_* \nu_A(\mathbb{S}) t^{\gamma_s})^p \int_{0 < \theta_1 < \dots < \theta_p < 1} [\theta_1(\theta_2 - \theta_1) \cdots (1 - \theta_p)]^{-d_s/2} d\theta_1 \cdots d\theta_p. \end{aligned}$$

The rest of the proof proceeds in a way similar to that in (a). \square

As a consequence of Proposition 2.2.14 and the fact that $\nu_{\langle W \rangle} = \nu$ is a probability measure, we deduce the exponential integrability of $\langle W \rangle_t$, $t \geq 0$.

Corollary 2.2.16. *For each $f \in L^1_+(\mu)$ and $\beta, t > 0$,*

$$\sup_{x \in \mathbb{S}} \mathbb{E}_x(f(X_t) e^{\beta \langle W \rangle_t}) \leq \max\{1, t^{-d_s/2}\} \|f\|_{L^1(\mu)} \mathbb{E}_{\gamma_s, \gamma_s}(C_* \beta \max\{t, t^{\gamma_s}\}),$$

for some universal constant $C_* > 0$.

According to Corollary 2.2.16, the quadratic process $\langle W \rangle$ satisfies Novikov's condition, and therefore Girsanov's theorem is valid.

Corollary 2.2.17. (a) *For any $\beta \in \mathbb{R}$, the process*

$$Z_t = e^{\beta W_t - \frac{1}{2} \beta^2 \langle W \rangle_t}, \quad t \geq 0,$$

is a martingale on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_x)$ for each $x \in \mathbb{S}$.

(b) *Let $x \in \mathbb{S}$, $T > 0$. Let $\{M_t\}_{t \geq 0}$ be a continuous local (square integrable, respectively) \mathbb{P}_x -martingale. Then $\tilde{M}_t = M_t - \langle M, W \rangle_t$, $0 \leq t \leq T$ is a continuous local (square integrable, respectively) \mathbb{Q}_x -martingale, where $\mathbb{Q}_x = Z_T \mathbb{P}_x$.*

Next, we show that the quadratic process $\langle W \rangle$ is also exponentially integrable up to the hitting time

$$\sigma_{V_0} \triangleq \inf\{t > 0 : X_t \in V_0\}. \quad (2.2.9)$$

We start with the following lemma.

Lemma 2.2.18. *Let h_i be the harmonic functions with boundary values $h_i|_{V_0} = 1_{\{p_i\}}$, $i = 1, 2, 3$, where $p_1 = (0, 0)$, $p_2 = (1, 0)$, $p_3 = (1/2, \sqrt{3}/2)$. Then*

$$\nu = 3^{-1}(\nu_{\langle h_1 \rangle} + \nu_{\langle h_2 \rangle} + \nu_{\langle h_3 \rangle}), \quad (2.2.10)$$

and therefore,

$$\langle W \rangle = 3^{-1}(\langle M^{[h_1]} \rangle + \langle M^{[h_2]} \rangle + \langle M^{[h_3]} \rangle), \quad (2.2.11)$$

where $\nu_{\langle h_i \rangle}$ are the energy measures of h_i , and $M^{[h_i]}$ are the martingale parts of $h_i(X_t) - h_i(X_0)$, $i = 1, 2, 3$.

Proof. Let $\chi_i = 1_{\{p_i\}}$, $\mathbf{Y}_i = \mathbf{P}^t \mathbf{A}_i \mathbf{P}$, $i = 1, 2, 3$. For any $\omega \in W_*$, since $h_i \circ \mathbf{F}_{[\omega]_m}$ is the harmonic function with boundary value $\mathbf{A}_{[\omega]_m} \chi_i$, by Corollary 1.2.15,

$$\mathcal{E}(h_i \circ \mathbf{F}_{[\omega]_m}) = \mathcal{E}^{(0)}(h_i \circ \mathbf{F}_{[\omega]_m}) = \frac{3}{2} (\mathbf{A}_{[\omega]_m} \chi_i)^t \mathbf{P} (\mathbf{A}_{[\omega]_m} \chi_i) = \frac{3}{2} \chi_i^t \mathbf{Y}_{[\omega]_m}^t \mathbf{Y}_{[\omega]_m} \chi_i, \quad i = 1, 2, 3.$$

Therefore, by (1.2.23), we deduce that

$$\nu_{\langle h_i \rangle}(\mathbf{F}_{[\omega]_m}(\mathbb{S})) = \left(\frac{5}{3}\right)^m \mathcal{E}(h_i \circ \mathbf{F}_{[\omega]_m}) = \frac{3}{2} \left(\frac{5}{3}\right)^m \chi_i^t \mathbf{Y}_{[\omega]_m}^t \mathbf{Y}_{[\omega]_m} \chi_i, \quad i = 1, 2, 3.$$

In particular,

$$\sum_{i=1,2,3} \nu_{\langle h_i \rangle}(\mathbf{F}_{[\omega]_m}(\mathbb{S})) = \frac{3}{2} \left(\frac{5}{3}\right)^m \text{tr}(\mathbf{Y}_{[\omega]_m}^t \mathbf{Y}_{[\omega]_m}) = 3\nu(\mathbf{F}_{[\omega]_m}(\mathbb{S})).$$

which implies (2.2.10). Moreover, (2.2.11) follows readily from (2.2.10) in view of the one-to-one correspondence between Revuz measures and positive additive functionals. \square

Lemma 2.2.19. *Let $h \in \mathcal{F}(\mathbb{S})$ be a harmonic function in \mathbb{S} . Then, for each $x \in \mathbb{S}$, $M_t = h(X_{t \wedge \sigma_{V_0}})$, $t \geq 0$ is a BMO martingale on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_x)$ with $\|M\|_{\text{BMO}} \leq \max_{V_0} |h|$,*

where the BMO norm $\|M\|_{\text{BMO}}$ is defined to be the smallest constant $K \geq 0$ such that

$$\sqrt{\mathbb{E}_x(\langle M \rangle_\infty - \langle M \rangle_\tau | \mathcal{F}_\tau)} \leq K, \quad \mathbb{P}_x\text{-a.s.}$$

for all stopping times τ .

Proof. We first show that $\{M_t\}_{t \geq 0}$ is a martingale. Since h is a harmonic function, we have

$h(x) = \mathbb{E}_x(h(X_{\sigma_{V_0}}))$, $x \in \mathbb{S}$. Therefore

$$\begin{aligned} M_t &= \mathbb{E}_{X_t}(h(X_{\sigma_{V_0}}))1_{\{t < \sigma_{V_0}\}} + h(X_{\sigma_{V_0}})1_{\{t \geq \sigma_{V_0}\}} \\ &= \mathbb{E}_x(h(X_{t+\sigma_{V_0} \circ \theta_t})1_{\{t < \sigma_{V_0}\}} | \mathcal{F}_t) + h(X_{\sigma_{V_0}})1_{\{t \geq \sigma_{V_0}\}} \\ &= \mathbb{E}_x(h(X_{\sigma_{V_0}})1_{\{t < \sigma_{V_0}\}} | \mathcal{F}_t) + h(X_{\sigma_{V_0}})1_{\{t \geq \sigma_{V_0}\}}. \end{aligned}$$

Note that $h(X_{\sigma_{V_0}})1_{\{t \geq \sigma_{V_0}\}} \in \mathcal{F}_t$. We see that $M_t = \mathbb{E}_x(h(X_{\sigma_{V_0}}) | \mathcal{F}_t)$, $t \geq 0$. This implies that $\{M_t\}_{t \geq 0}$ is a martingale on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_x)$.

By the maximum principle for harmonic functions (Lemma 1.2.18), $|M_t| \leq \max_{V_0} |h|$, $t \geq 0$. Noticing that $M_t = M_{t \wedge \sigma_{V_0}}$, $t \geq 0$, we deduce that for any stopping time τ ,

$$\mathbb{E}_x(\langle M \rangle_\infty - \langle M \rangle_\tau | \mathcal{F}_\tau) = \mathbb{E}_x(M_{\sigma_{V_0}}^2 - M_{\tau \wedge \sigma_{V_0}}^2 | \mathcal{F}_\tau) \leq \max_{V_0} |h|^2,$$

which implies $\|M\|_{\text{BMO}} \leq \max_{V_0} |h|$. □

Proposition 2.2.20. *There exists a $\beta > 0$ such that*

$$\sup_{x \in \mathbb{S}} \mathbb{E}_x(e^{\beta \langle W \rangle_{\sigma_{V_0}}}) < \infty.$$

Proof. Let h_i , $i = 1, 2, 3$ be the harmonic functions in Lemma 2.2.18. By Lemma 2.2.19 and the John–Nirenberg inequality for BMO martingales (cf. [25, Theorem 10.42, p. 288]),

there exists a $\beta > 0$ such that

$$\sup_{x \in \mathbb{S}} \mathbb{E}_x \left(e^{\beta \langle M^{[h_i]} \rangle_{\sigma_{V_0}}} \right) < \infty, \quad i = 1, 2, 3,$$

which, together with (2.2.11), implies the conclusion of the proposition. \square

2.3 Existence and uniqueness of solutions

2.3.1 BSDEs with deterministic durations

In this subsection, we prove the existence and uniqueness of solutions to a class of BSDEs on \mathbb{S} driven by the Brownian martingale W . Results and proofs in this subsection are also valid if the constant terminal time T is replaced by a bounded $\{\mathcal{F}_t\}$ -stopping time τ .

Let λ be a Borel probability on \mathbb{S} , and $T \in (0, \infty)$ be the terminal time. Suppose that ξ is an \mathcal{F}_T -measurable random variable, and that $g : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ and $f : [0, T] \times \mathbb{R}^2 \times \Omega \rightarrow \mathbb{R}$ are measurable functions satisfying the following adaptedness condition:

(C) The process $(t, \omega) \mapsto (g(t, y, \omega), f(t, y, z, \omega))$ is $\{\mathcal{F}_t\}$ -adapted for all $y, z \in \mathbb{R}$.

We consider the backward stochastic differential equation (BSDE) on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_\lambda)$

$$\begin{cases} dY_t = -g(t, Y_t) dt - f(t, Y_t, Z_t) d\langle W \rangle_t + Z_t dW_t, & t \in [0, T), \\ Y_T = \xi, \end{cases} \quad (2.3.1)$$

where, to simplify notation, we suppress the dependence on $\omega \in \Omega$ for g and f , i.e.

$$g(t, Y_t) = g(t, Y_t(\omega), \omega), \quad f(t, Y_t, Z_t) = f(t, Y_t(\omega), Z_t(\omega), \omega).$$

The BSDE (2.3.1) should be interpreted as the following (backward) integral equation

$$\begin{aligned}
Y_t = & \xi + \int_t^T g(r, Y_r) dr + \int_t^T f(r, Y_r, Z_r) d\langle W \rangle_r \\
& - \int_t^T Z_r dW_r, \quad t \in [0, T] \quad \mathbb{P}_\lambda\text{-a.s.}
\end{aligned} \tag{2.3.2}$$

To give a precise definition of solutions, we first introduce the Banach spaces for solutions. As we shall consider BSDEs with random durations (cf. Section 2.3.2), it is convenient to define once and for all here these Banach spaces for the general case when the durations can be random.

Definition 2.3.1. Let τ be an $\{\mathcal{F}_t\}$ -stopping time, λ be a Borel probability measure on \mathbb{S} , and $\beta = (\beta_0, \beta_1) \in \mathbb{R}_+^2$. We define $\mathcal{V}_\lambda^\beta[0, \tau]$ to be the Banach space of all pairs (y, z) of $\{\mathcal{F}_t\}$ -adapted processes such that

$$\begin{aligned}
\|(y, z)\|_{\mathcal{V}_\lambda^\beta[0, \tau]}^2 = & \mathbb{E}_\lambda \left[\sup_{0 \leq t \leq \tau} \left(y_t^2 e^{2\beta_0 t + 2\beta_1 \langle W \rangle_t} + \int_t^\tau y_r^2 e^{2\beta_0 r + 2\beta_1 \langle W \rangle_r} dr \right. \right. \\
& \left. \left. + \int_t^\tau (y_r^2 + z_r^2) e^{2\beta_0 r + 2\beta_1 \langle W \rangle_r} d\langle W \rangle_r \right) \right] < \infty.
\end{aligned}$$

We shall simply write $\mathcal{V}_x^\beta[0, \tau]$ when $\lambda = \delta_x$ is the Dirac measure concentrated at $x \in \mathbb{S}$.

Definition 2.3.2. Let $\beta \in \mathbb{R}_+^2$. We say that the pair (Y, Z) is a *solution in $\mathcal{V}_\lambda^\beta[0, T]$* to the BSDE (2.3.1) if $(Y, Z) \in \mathcal{V}_\lambda^\beta[0, T]$ and satisfies (2.3.2). The solution (Y, Z) is said to be *unique in $\mathcal{V}_\lambda^\beta[0, T]$* if $\|(Y - \bar{Y}, Z - \bar{Z})\|_{\mathcal{V}_\lambda^\beta[0, T]} = 0$ whenever (\bar{Y}, \bar{Z}) is also a solution to (2.3.1) in $\mathcal{V}_\lambda^\beta[0, T]$.

Notice that the uniqueness of solutions introduced in Definition 2.3.2 is a concept for processes in the space $\mathcal{V}_\lambda^\beta[0, T]$. In fact, uniqueness can be defined for solutions not necessarily in $\mathcal{V}_\lambda^\beta[0, T]$. More precisely, we have the following definition for uniqueness of solutions.

Definition 2.3.3. The BSDE (2.3.1) is said to *admit at most one solution* if

$$Y_t = \bar{Y}_t, \quad t \in [0, T], \quad \text{and} \quad \int_0^T (Z_r - \bar{Z}_r)^2 d\langle W \rangle_r = 0, \quad \mathbb{P}_\lambda\text{-a.s.}$$

whenever (Y, Z) and (\bar{Y}, \bar{Z}) are pairs of $\{\mathcal{F}_t\}$ -adapted processes satisfying (2.3.2).

For convenience, we list the technical assumptions below which will be referred to for several times.

Assumption 2.3.4. Let $\beta_1 > 0$ be a given constant.

(i)

$$\mathbb{E}_\lambda(\xi^2 e^{2\beta_1 \langle W \rangle_T}) < \infty; \quad (\text{A.1})$$

(ii) For all $t \in [0, T]$ and all $y, \bar{y}, z, \bar{z} \in \mathbb{R}$,

$$|g(t, y, \omega) - g(t, \bar{y}, \omega)| \leq \frac{K_0}{2} |y - \bar{y}| \quad \mathbb{P}_\lambda\text{-a.s.}, \quad (\text{A.2})$$

$$|f(t, y, z, \omega) - f(t, \bar{y}, \bar{z}, \omega)| \leq \frac{K_0}{2} |y - \bar{y}| + K_1 |z - \bar{z}| \quad \mathbb{P}_\lambda\text{-a.s.}, \quad (\text{A.3})$$

where $K_0, K_1 > 0$ are some constants;

(iii)

$$\begin{aligned} & \mathbb{E}_\lambda \left(\int_0^T g(r, 0)^2 e^{2\beta_1 \langle W \rangle_r} dr \right) \\ & + \mathbb{E}_\lambda \left(\int_0^T f(r, 0, 0)^2 e^{2\beta_1 \langle W \rangle_r} d\langle W \rangle_r \right) < \infty, \end{aligned} \quad (\text{A.4})$$

where we have suppressed the explicit dependence of g, f on $\omega \in \Omega$ for simplicity.

Let us start with the simple case of (2.3.1) when g, f do not depend on y or z ; that is,

$$\begin{cases} dY_t = -g(t)dt - f(t)d\langle W \rangle_t + Z_t dW_t, & t \in [0, T), \\ Y_T = \xi, \end{cases} \quad (2.3.3)$$

where $g(t) = g(t, \omega)$ and $f(t) = f(t, \omega)$ are $\{\mathcal{F}_t\}$ -adapted processes.

Lemma 2.3.5. Let $\beta = (\beta_0, \beta_1) \in [1, \infty)^2$, and let $\xi \in \mathcal{F}_T$ satisfying (A.1). Suppose that $g(t), f(t)$ are $\{\mathcal{F}_t\}$ -adapted processes such that

$$\mathbb{E}_\lambda \left(\int_0^T g(r)^2 e^{2\beta_1 \langle W \rangle_r} dr \right) < \infty, \quad \mathbb{E}_\lambda \left(\int_0^T f(r)^2 e^{2\beta_1 \langle W \rangle_r} d\langle M^1 \rangle_r \right) < \infty.$$

Then the BSDE (2.3.3) admits a unique solution (Y, Z) in $\mathcal{V}_\lambda^\beta[0, T]$. Moreover,

$$\begin{aligned} \|(Y, Z)\|_{\mathcal{V}_\lambda^\beta[0, T]}^2 \leq C_* & \left[\mathbb{E}_\lambda(\xi^2 e^{2\beta_0 T + 2\beta_1 \langle W \rangle_T}) + \mathbb{E}_\lambda \left(\int_0^T g(r)^2 e^{2\beta_0 r + 2\beta_1 \langle W \rangle_r} dr \right) \right. \\ & \left. + \mathbb{E}_\lambda \left(\int_0^T f(r)^2 e^{2\beta_0 r + 2\beta_1 \langle W \rangle_r} d\langle W \rangle_r \right) \right], \end{aligned} \quad (2.3.4)$$

for some universal constant $C_* > 0$.

Proof. Let

$$Y_t = \mathbb{E}_\lambda \left(\xi + \int_t^T g(r) dr + \int_t^T f(r) d\langle W \rangle_r \mid \mathcal{F}_t \right), \quad t \in [0, T].$$

Then $Y_T = \xi$, and

$$Y_t + \int_0^t g(r) dr + \int_0^t f(r) d\langle W \rangle_r = \mathbb{E}_\lambda \left(\xi + \int_0^T g(r) dr + \int_0^T f(r) d\langle W \rangle_r \mid \mathcal{F}_t \right)$$

is a martingale on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_\lambda)$. By Theorem 2.2.2, there exists a unique $\{\mathcal{F}_t\}$ -predictable process Z such that

$$Y_t - Y_0 + \int_0^t g(r) dr + \int_0^t f(r) d\langle W \rangle_r = \int_0^t Z_r dW_r, \quad t \in [0, T] \quad \mathbb{P}_\lambda\text{-a.s.}$$

which, together with $Y_T = \xi$, implies that

$$Y_t = \xi + \int_t^T g(r) dr + \int_t^T f(r) d\langle W \rangle_r - \int_t^T Z_r dW_r, \quad t \in [0, T] \quad \mathbb{P}_\lambda\text{-a.s.}$$

Therefore, (Y, Z) is a solution to the BSDE (2.3.3).

We now turn to the proof of (2.3.4), from which the uniqueness of the solution in $\mathcal{V}_\lambda^\beta[0, T]$ follows immediately. Let $e_t = \exp(2\beta_0 t + 2\beta_1 \langle W \rangle_t)$. By Itô's formula,

$$\begin{aligned} Y_t^2 e_t &= \xi^2 e_T + \int_t^T (-2\beta_0 Y_r^2 + 2Y_r g(r)) e_r dr \\ &\quad + \int_t^T (-2\beta_1 Y_r^2 + 2Y_r f(r) - Z_r^2) e_r d\langle W \rangle_r - 2 \int_t^T Y_r Z_r e_r dW_r, \end{aligned}$$

which, together with Young's inequality, implies that

$$\begin{aligned} &Y_t^2 e_t + 2\beta_0 \int_t^T Y_r^2 e_r dr + \int_t^T (2\beta_1 Y_r^2 + Z_r^2) e_r d\langle W \rangle_r \\ &\leq \xi^2 e_T + \int_t^T \left(\beta_0 Y_r^2 + \frac{1}{\beta_0} g(r)^2 \right) e_r dr + \int_t^T \left(\beta_1 Y_r^2 + \frac{1}{\beta_1} f(r)^2 \right) e_r d\langle W \rangle_r \quad (2.3.5) \\ &\quad - 2 \int_t^T Y_r Z_r e_r dW_r. \end{aligned}$$

Taking expectations on both sides of the above inequality and using a localization argument gives that

$$\begin{aligned} &\mathbb{E}_\lambda \left(\int_0^T Z_r^2 e_r d\langle W \rangle_r \right) \\ &\leq \mathbb{E}_\lambda \left(\xi^2 e_T + \frac{1}{\beta_0} \int_0^T g(r)^2 e_r dr + \frac{1}{\beta_1} \int_0^T f(r)^2 e_r d\langle W \rangle_r \right). \end{aligned} \quad (2.3.6)$$

By (2.3.5) again,

$$\begin{aligned} &Y_t^2 e_t + \beta_0 \int_t^T Y_r^2 e_r dr + \int_t^T (\beta_1 Y_r^2 + Z_r^2) e_r d\langle W \rangle_r \\ &\leq \xi^2 e_T + \frac{1}{\beta_0} \int_t^T g(r)^2 e_r dr + \frac{1}{\beta_1} \int_t^T f(r)^2 e_r d\langle W \rangle_r + 2 \left| \int_t^T Y_r Z_r e_r dW_r \right|. \end{aligned} \quad (2.3.7)$$

By Doob's maximal inequality and Young's inequality,

$$\begin{aligned} &\mathbb{E}_\lambda \left(\sup_{0 \leq t \leq T} \left| \int_t^T Y_r Z_r e_r dW_r \right| \right) \leq 2 \mathbb{E}_\lambda \left(\sup_{0 \leq t \leq T} \left| \int_0^t Y_r Z_r e_r dW_r \right| \right) \\ &\leq 2 \mathbb{E}_\lambda \left[\left(\int_0^T Y_r^2 Z_r^2 e_r^2 d\langle W \rangle_r \right)^{1/2} \right] \leq \frac{1}{4} \|(Y, Z)\|_{\mathcal{V}_\lambda^\beta[0, T]}^2 + 4 \mathbb{E}_\lambda \left(\int_0^T Z_r^2 e_r d\langle W \rangle_r \right), \end{aligned} \quad (2.3.8)$$

By the above, (2.3.7) and (2.3.6), we obtain that

$$\begin{aligned} \|(Y, Z)\|_{\mathcal{V}_\lambda^\beta[0, T]}^2 &\leq \frac{1}{2} \|(Y, Z)\|_{\mathcal{V}_\lambda^\beta[0, T]}^2 + 5 \mathbb{E}_\lambda \left(\xi^2 e_T + \frac{1}{\beta_0} \int_0^T g(r)^2 e_r dr \right. \\ &\quad \left. + \frac{1}{\beta_1} \int_0^T f(r)^2 e_r d\langle W \rangle_r \right). \end{aligned}$$

which, together with a localization argument if necessary, completes the proof. \square

The following a priori estimate allows us to construct convergent sequences of pairs of processes by iterating the BSDE (2.3.3).

Lemma 2.3.6. *Let $\beta = (\beta_0, \beta_1) \in [1, \infty)^2$. Suppose that ξ, g, f satisfy (A.1)–(A.4). For any $(y, z) \in \mathcal{V}_\lambda^\beta[0, T]$, according to Lemma 2.3.5, the BSDE*

$$\begin{cases} dY_t = -g(t, y_t)dt - f(t, y_t, z_t)d\langle W \rangle_t + Z_t dW_t, & t \in [0, T], \\ Y_T = \xi, \end{cases}$$

admits a unique solution (Y, Z) in $\mathcal{V}_\lambda^\beta[0, T]$. Let $F : \mathcal{V}_\lambda^\beta[0, T] \rightarrow \mathcal{V}_\lambda^\beta[0, T]$ be the solution map $(y, z) \mapsto (Y, Z)$. If $(\bar{y}, \bar{z}) \in \mathcal{V}_\lambda^\beta[0, T]$ and $(\bar{Y}, \bar{Z}) = F(\bar{y}, \bar{z})$, then

$$\|(\hat{Y}, \hat{Z})\|_{\mathcal{V}_\lambda^\beta[0, T]} \leq 3\sqrt{2}K_\beta \|(\hat{y}, \hat{z})\|_{\mathcal{V}_\lambda^\beta[0, T]},$$

where $\hat{\eta} = \eta - \bar{\eta}$ for $\eta = y, z, Y, Z$, and

$$K_\beta^2 = \frac{K_0^2}{\beta_0} + \frac{K_1^2}{\beta_1}. \quad (2.3.9)$$

Moreover, F is a $\|\cdot\|_{\mathcal{V}_\lambda^\beta[0, T]}$ -contraction when β_0, β_1 are sufficiently large ($\beta_i \geq 36K_i^2$, $i = 0, 1$ will suffice).

Proof. Let $\hat{g}_t = g(t, y_t) - g(t, \bar{y}_t)$, $\hat{f}_t = f(t, y_t, z_t) - f(t, \bar{y}_t, \bar{z}_t)$. Then

$$|\hat{g}_t| \leq \frac{K_0}{2} |\hat{y}_t|, \quad |\hat{f}_t| \leq \frac{K_0}{2} |\hat{y}_t| + K_1 |\hat{z}_t|,$$

and

$$d\hat{Y}_t = -\hat{g}_t dt - \hat{f}_t d\langle W \rangle_t + \hat{Z}_t dW_t, \quad \hat{Y}_T = 0.$$

Let $e_t = \exp(2\beta_0 t + 2\beta_1 \langle W \rangle_t)$. In a way similar to the derivation of (2.3.7), we deduce that

$$\begin{aligned} & \hat{Y}_t^2 e_t + \beta_0 \int_t^T \hat{Y}_r^2 e_r dr + \int_t^T (\beta_1 \hat{Y}_r^2 + \hat{Z}_r^2) e_r d\langle W_r \rangle_r \\ & \leq K_\beta^2 \int_t^T \hat{y}_r^2 e_r dr + K_\beta^2 \int_t^T \hat{z}_r^2 e_r d\langle W \rangle_r - 2 \int_t^T \hat{Y}_r \hat{Z}_r e_r dW_r, \end{aligned} \quad (2.3.10)$$

where $K_\beta > 0$ is given by (2.3.9). Proceeding as in the proof of Lemma 2.3.5, we obtain that

$$\|(\hat{Y}, \hat{Z})\|_{\mathcal{V}_\lambda^\beta[0,T]}^2 \leq \frac{1}{2} \|(\hat{Y}, \hat{Z})\|_{\mathcal{V}_\lambda^\beta[0,T]}^2 + 9K_\beta^2 \|(\hat{y}, \hat{z})\|_{\mathcal{V}_\lambda^\beta[0,T]}^2,$$

which completes the proof. \square

We are now in a position to formulate and prove our main result in this subsection.

Theorem 2.3.7. *Let $\beta = (\beta_0, \beta_1) \in [1, \infty)^2$. Suppose that (A.1)–(A.3) are satisfied. Then*

(a) *The BSDE (2.3.1) admits at most one solution.*

(b) *If, in addition, (A.1) and (A.4) hold for sufficiently large β_0, β_1 ($\beta_i > 36K_i^2$, $i = 0, 1$ will suffice), then (2.3.1) admits a unique solution $(Y, Z) \in \mathcal{V}_\lambda^\beta[0, T]$. Moreover,*

$$\begin{aligned} \|(Y, Z)\|_{\mathcal{V}_\lambda^\beta[0,T]}^2 & \leq C \mathbb{E}_\lambda \left(\xi^2 e^{2\beta_0 T + 2\beta_1 \langle W \rangle_T} + \int_0^T g(r, 0)^2 e^{2\beta_0 r + 2\beta_1 \langle W \rangle_r} dr \right. \\ & \quad \left. + \int_0^T f(r)^2 e^{2\beta_0 r + 2\beta_1 \langle W \rangle_r} d\langle W \rangle_r \right), \end{aligned} \quad (2.3.11)$$

where $C > 0$ is a constant depending only on K_0, K_1, β .

Remark 2.3.8. Results of Theorem 2.3.7 can be extended with no essential difficulties to the case when T is replaced by a bounded stopping time τ (see Section 2.3.2 below for the definition of solutions to BSDEs with random durations).

Proof. (a) Suppose that (Y, Z) and (\bar{Y}, \bar{Z}) are two pairs of $\{\mathcal{F}_t\}$ -adapted processes satisfying (2.3.2). Denote $\hat{\eta} = \eta - \bar{\eta}$ for $\eta = y, z, Y, Z$, and let $\hat{g}_t = g(t, Y_t) - g(t, \bar{Y}_t)$,

$\hat{f}_t = f(t, Y_t, Z_t) - f(t, \bar{Y}_t, \bar{Z}_t)$. Similar to the derivation of (2.3.10), we have

$$\begin{aligned} & \hat{Y}_t^2 e_t + \beta_0 \int_t^T \hat{Y}_r^2 e_r dr + \int_t^T (\beta_1 \hat{Y}_r^2 + \hat{Z}_r^2) e_r d\langle W_r \rangle_r \\ & \leq K_\beta^2 \int_t^T \hat{Y}_r^2 e_r dr + K_\beta^2 \int_t^T \hat{Z}_r^2 e_r d\langle W \rangle_r - 2 \int_t^T \hat{Y}_r \hat{Z}_r e_r dW_r, \end{aligned}$$

where $K_\beta > 0$ is given by (2.3.9). Setting $\beta_i = 4K_i^2$, $i = 0, 1$ in the above gives that

$$\mathbb{E}_\lambda \left(\hat{Y}_t^2 e_t + \frac{1}{2} \int_t^T \hat{Y}_r^2 e_r dr + \int_t^T \left(\frac{1}{2} \hat{Y}_r^2 + \hat{Z}_r^2 \right) e_r d\langle W \rangle_r \right) \leq 0, \quad t \in [0, T],$$

which completes the proof of (a).

(b) Suppose, in addition, that (A.4) is satisfied. Let $(Y^{(0)}, Z^{(0)}) = (0, 0)$. By virtue of Lemma 2.3.5, the pair $(Y^{(n)}, Z^{(n)}) \in \mathcal{V}_\lambda^\beta[0, T]$, $n \in \mathbb{N}_+$ can be defined inductively to be the unique solution in $\mathcal{V}_\lambda^\beta[0, T]$ of the BSDE

$$\begin{cases} dY_t^{(n)} = -g(t, Y_t^{(n-1)}) dt - f(t, Y_t^{(n-1)}, Z_t^{(n-1)}) d\langle W \rangle_t + Z_t^{(n)} dW_t, & t \in [0, T], \\ Y_T^{(n)} = \xi. \end{cases}$$

By Lemma 2.2.13,

$$\begin{aligned} & \| (Y^{(n+1)} - Y^{(n)}, Z^{(n+1)} - Z^{(n)}) \|_{\mathcal{V}_\lambda^\beta[0, T]} \\ & \leq K_\beta \| (Y^{(n)} - Y^{(n-1)}, Z^{(n)} - Z^{(n-1)}) \|_{\mathcal{V}_\lambda^\beta[0, T]}, \quad n \in \mathbb{N}_+, \end{aligned} \tag{2.3.12}$$

where $K_\beta > 0$ is given by (1.2.20). By Lemma 2.3.5,

$$\begin{aligned} \| (Y^{(1)}, Z^{(1)}) \|_{\mathcal{V}_\lambda^\beta[0, T]}^2 & \leq 10 \left[\mathbb{E}_\lambda \left(\xi^2 e^{2\beta_0 T + 2\beta_1 \langle W \rangle_T} \right) + \mathbb{E}_\lambda \left(\int_0^T g(r, 0)^2 e^{2\beta_0 r + 2\beta_1 \langle W \rangle_r} dr \right) \right. \\ & \quad \left. + \mathbb{E}_\lambda \left(\int_0^T f(r)^2 e^{2\beta_0 r + 2\beta_1 \langle W \rangle_r} d\langle W \rangle_r \right) \right]. \end{aligned} \tag{2.3.13}$$

Choose $\beta_0, \beta_1 > 0$ sufficiently large so that $K_\beta < 1$ (for example, $\beta_i > 36K_i^2$, $i = 0, 1$).

By (2.3.12) and (2.3.13), we conclude that $(Y^{(n)}, Z^{(n)})$, $n \in \mathbb{N}_+$ is a Cauchy sequence in

$\mathcal{V}_\lambda^\beta[0, T]$. Moreover, $\lim_{n \rightarrow \infty} \|(Y^{(n)} - Y, Z^{(n)} - Z)\|_{\mathcal{V}_\lambda^\beta[0, T]} = 0$ for some $(Y, Z) \in \mathcal{V}_\lambda^\beta[0, T]$ satisfying (2.3.11).

Clearly, (Y, Z) is a solution to (2.3.1), and the proof is completed. \square

2.3.2 BSDEs with random durations

The subject of this subsection is BSDEs on the gasket with random durations on the filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_\lambda)$, which take the form

$$\begin{cases} dY_t = -g(t, Y_t)dt - f(t, Y_t, Z_t)d\langle W \rangle_t + Z_t dW_t, & t \in [0, \tau), \\ Y_\tau = \xi, \end{cases} \quad (2.3.14)$$

where τ is an $\{\mathcal{F}_t\}$ -stopping time, and ξ is an \mathcal{F}_τ -measurable random variable. As in the previous subsection, the coefficients g, f are assumed to satisfy the càdlàg condition (C). We prove the existence and uniqueness of solutions to (2.3.14), and derive estimates for the solutions. As in [57] (see also [70, Section 7.3.2]), we shall use the method of continuity borrowed from the theory of partial differential equations.

We first introduce the definition of solutions to (2.3.14). Our definition is in analogy to that in [70, Definition 3.5, p. 362] given for BSDEs driven by Brownian motions on Euclidean spaces.

Definition 2.3.9. Let $\beta = (\beta_0, \beta_1) \in \mathbb{R}_+^2$. We say that (Y, Z) is a *solution to (2.3.14)* in $\mathcal{V}_\lambda^\beta[0, \tau]$ (cf. Definition 2.3.1) if $(Y, Z) \in \mathcal{V}_\lambda^\beta[0, \tau]$ and satisfies that

$$\begin{aligned} Y_{t \wedge \tau} = & Y_{T \wedge \tau} + \int_{t \wedge \tau}^{T \wedge \tau} g(r, Y_r) dr + \int_{t \wedge \tau}^{T \wedge \tau} f(r, Y_r, Z_r) d\langle W \rangle_r \\ & - \int_{t \wedge \tau}^{T \wedge \tau} Z_r dW_r, \quad \text{for all } 0 \leq t \leq T < \infty \text{ } \mathbb{P}_\lambda\text{-a.s.}, \end{aligned} \quad (2.3.15)$$

and that

$$\lim_{T \rightarrow \infty} \mathbb{E}_\lambda \left(|Y_{T \wedge \tau} - \xi|^2 e^{2\beta_0(T \wedge \tau) + 2\beta_1 \langle W \rangle_{T \wedge \tau}} \right) = 0. \quad (2.3.16)$$

The solution (Y, Z) is said to be *unique in* $\mathcal{V}_\lambda^\beta[0, \tau]$ if $\|(Y - \bar{Y}, Z - \bar{Z})\|_{\mathcal{V}_\lambda^\beta[0, \tau]} = 0$ for any solution (\bar{Y}, \bar{Z}) to (2.3.14) in $\mathcal{V}_\lambda^\beta[0, \tau]$.

Clearly, if the stopping time τ is bounded, then Definition 1.2.5 coincides with Definition 1.2.1 with T replaced by τ (cf. Corollary 2.2.16). Similar to BSDEs with deterministic durations, the concept of uniqueness of solutions in Definition 2.3.9 only concerns solutions in the space $\mathcal{V}_\lambda^\beta[0, \tau]$, and uniqueness of solutions may be discussed for more general solutions.

Definition 2.3.10. The BSDE (2.3.14) is said to *admit at most one solution*, if

$$Y_t = \bar{Y}_t, \quad t \in [0, \tau], \quad \text{and} \quad \int_0^\tau (Z_r - \bar{Z}_r)^2 d\langle W \rangle_r = 0, \quad \mathbb{P}_\lambda\text{-a.s.},$$

for any two pairs (Y, Z) and (\bar{Y}, \bar{Z}) of $\{\mathcal{F}_t\}$ -adapted processes satisfying (2.3.15) and (2.3.16).

For convenience, let us gather some technical assumptions to which we shall refer for several times.

Assumption 2.3.11. Let $(\beta_0, \beta_1) \in \mathbb{R}_+^2$ be given.

(i)

$$\mathbb{E}_\lambda(\xi^2 e^{2\beta_0\tau + 2\beta_1\langle W \rangle_\tau}) < \infty; \quad (\text{A}' .1)$$

(ii) For all $t \in [0, \tau(\omega))$ $y, \bar{y}, z, \bar{z} \in \mathbb{R}$, and \mathbb{P}_λ -a.e. $\omega \in \Omega$,

$$|g(t, y, \omega) - g(t, \bar{y}, \omega)| \leq \frac{K_0}{2} |y - \bar{y}| \quad \mathbb{P}_\lambda\text{-a.s.}, \quad (\text{A}' .2)$$

$$|f(t, y, z, \omega) - f(t, \bar{y}, \bar{z}, \omega)| \leq \frac{K_0}{2} |y - \bar{y}| + K_1 |z - \bar{z}| \quad \mathbb{P}_\lambda\text{-a.s.}, \quad (\text{A}' .3)$$

$$(y - \bar{y})(g(t, y, \omega) - g(t, \bar{y}, \omega)) \leq -\kappa_0 |y - \bar{y}|^2, \quad (\text{A}' .4)$$

$$(y - \bar{y})(f(t, y, z, \omega) - f(t, \bar{y}, z, \omega)) \leq -\kappa_1 |y - \bar{y}|^2; \quad (\text{A}' .5)$$

(iii)

$$\begin{aligned} & \mathbb{E}_\lambda \left(\int_0^\tau g(t, 0)^2 e^{2\beta_0 t + 2\beta_1 \langle W \rangle_t} dt \right) \\ & + \mathbb{E}_\lambda \left(\int_0^\tau f(t, 0, 0)^2 e^{2\beta_0 t + 2\beta_1 \langle W \rangle_t} d\langle W \rangle_t \right) < \infty, \end{aligned} \quad (\text{A'.6})$$

for some constants $\kappa_0, \kappa_1 \in \mathbb{R}$ satisfying

$$\beta_0 - \kappa_0 > 0, \quad \beta_1 - \kappa_1 + \frac{K_1^2}{2} > 0. \quad (\text{A'.7})$$

As in the previous subsection, we start with the simple case when g and f do not depend on y or z ; that is,

$$\begin{cases} dY_t = -g(t)dt - f(t)d\langle W \rangle_t + Z_t dW_t, & t \in [0, \tau), \\ Y_\tau = \xi, \end{cases} \quad (2.3.17)$$

where $g(t) = g(t, \omega)$ and $f(t) = f(t, \omega)$ are $\{\mathcal{F}_t\}$ -adapted processes.

Lemma 2.3.12. *Let $\beta = (\beta_0, \beta_1) \in [1, \infty)^2$, and let $\xi \in \mathcal{F}_\tau$ satisfy (A'.1). Suppose that $g(t), f(t)$ are $\{\mathcal{F}_t\}$ -adapted processes such that*

$$\mathbb{E}_\lambda \left(\int_0^\tau g(t)^2 e^{2\beta_0 t + 2\beta_1 \langle W \rangle_t} dt \right) + \mathbb{E}_\lambda \left(\int_0^\tau f(t)^2 e^{2\beta_0 t + 2\beta_1 \langle W \rangle_t} d\langle W \rangle_t \right) < \infty. \quad (2.3.18)$$

Then the BSDE (2.3.17) admits a unique solution (Y, Z) in $\mathcal{V}_\lambda^\beta[0, \tau]$. Moreover,

$$\begin{aligned} \|(Y, Z)\|_{\mathcal{V}_\lambda^\beta[0, \tau]}^2 & \leq C \left[\mathbb{E}_\lambda \left(\xi^2 e^{2\beta_0 \tau + 2\beta_1 \langle W \rangle_\tau} \right) + \mathbb{E}_\lambda \left(\int_0^\tau g(t)^2 e^{2\beta_0 t + 2\beta_1 \langle W \rangle_t} dt \right) \right. \\ & \quad \left. + \mathbb{E}_\lambda \left(\int_0^\tau f(t)^2 e^{2\beta_0 t + 2\beta_1 \langle W \rangle_t} d\langle W \rangle_t \right) \right], \end{aligned} \quad (2.3.19)$$

where $C > 0$ is a constant depending only on β .

Proof. Let

$$M_t = \mathbb{E}_\lambda \left(\xi + \int_0^\tau g(r)dr + \int_0^\tau f(r)d\langle W \rangle_r \mid \mathcal{F}_t \right), \quad t \geq 0.$$

Then, by (A'.1) and (2.3.18), $\{M_t\}_{t \geq 0}$ is an $\{\mathcal{F}_t\}$ -adapted square integrable martingale with $\mathbb{E}_\lambda(M_\tau^2) < \infty$. By Theorem 2.2.2, there exists a unique $\{\mathcal{F}_t\}$ -predictable process Z such that $M_t - M_0 = \int_0^t Z_r dW_r$, $t \geq 0$. Let

$$Y_t = M_{t \wedge \tau} - \int_0^{t \wedge \tau} g(r) dr - \int_0^{t \wedge \tau} f(r) dW_r, \quad t \geq 0.$$

Then (2.3.15) is satisfied. Let

$$e_t = \exp(2\beta_0 t + 2\beta_1 \langle W \rangle_t), \quad t \geq 0.$$

By the definitions of Y_t and M_t , we have

$$|Y_{T \wedge \tau} - \xi|^2 e_{T \wedge \tau} = \left| \mathbb{E}_\lambda \left(\xi + \int_{T \wedge \tau}^\tau g(r) dr + \int_{T \wedge \tau}^\tau f(r) d\langle W \rangle_r \mid \tilde{\mathcal{F}}_{T \wedge \tau}^\lambda \right) - \xi \right|^2 e_{T \wedge \tau},$$

which implies that

$$\begin{aligned} \mathbb{E}_\lambda(|Y_{T \wedge \tau} - \xi|^2 e_{T \wedge \tau}) &\leq 3\mathbb{E}_\lambda \left[\left| \mathbb{E}_\lambda(\xi \mid \tilde{\mathcal{F}}_{T \wedge \tau}^\lambda) - \xi \right|^2 e_{T \wedge \tau} \right] \\ &\quad + 3\mathbb{E}_\lambda \left(\int_{T \wedge \tau}^\tau g(r)^2 e_r dr \right) + 3\mathbb{E}_\lambda \left(\int_{T \wedge \tau}^\tau f(r)^2 e_r d\langle W \rangle_r \right). \end{aligned}$$

By the Lebesgue dominated convergence theorem, the last two expectations on the right hand side of the above converge to zero as $T \rightarrow \infty$. For the first expectation on the right hand side of the above, notice that $\lim_{T \rightarrow \infty} \mathbb{E}_\lambda(\xi \mid \tilde{\mathcal{F}}_{T \wedge \tau}^\lambda) = \xi$ \mathbb{P}_λ -a.s. by the martingale convergence theorem, which, together with (A'.1) and the dominated convergence theorem, implies that

$$\lim_{T \rightarrow \infty} \mathbb{E}_\lambda \left[\left| \mathbb{E}_\lambda(\xi \mid \tilde{\mathcal{F}}_{T \wedge \tau}^\lambda) - \xi \right|^2 e_{T \wedge \tau} \right] = 0.$$

This completes the proof of (2.3.16).

The proof of (2.3.19) is similar to that of its deterministic counterpart (2.3.4). As a corollary of the estimate (2.3.19), we see that (Y, Z) is the unique solution to (2.3.17) in

$\mathcal{V}_\lambda^\beta[0, \tau]$.

□

Next, we consider the following BSDE parametrized by $\alpha \in [0, 1]$:

$$\begin{cases} dY_t = -(g_0(t) + \alpha g(t, Y_t))dt \\ \quad - (f_0(t) + \alpha f(t, Y_t, Z_t))d\langle W \rangle_t + Z_t dW_t, \quad t \in [0, \tau), \\ Y_\tau = \xi, \end{cases} \quad (2.3.20)$$

where $g_0(t)$ and $f_0(t)$ are $\{\mathcal{F}_t\}$ -adapted processes, and f, g are functions satisfying the càdlàg condition (C) (cf. Section 2.3.1). The following a priori estimate will be crucial to our use of method of continuation for the proof of existence and uniqueness of solutions to BSDEs with random durations.

Lemma 2.3.13. *Let $\xi, \bar{\xi} \in \mathcal{F}_\tau$ satisfy (A'.1), and let g_0, f_0 and \bar{g}_0, \bar{f}_0 be $\{\mathcal{F}_t\}$ -adapted processes satisfying (2.3.18). Suppose that g, f satisfy (A'.2)–(A'.7) with $\kappa_0 = 0, \kappa_1 = K_1^2/2$. Let (Y, Z) be a solution in $\mathcal{V}_\lambda^\beta[0, \tau]$ to (2.3.20), and (\bar{Y}, \bar{Z}) be a solution in $\mathcal{V}_\lambda^\beta[0, \tau]$ to the BSDE given by replacing (ξ, g_0, f_0) by $(\bar{\xi}, \bar{g}_0, \bar{f}_0)$ in (2.3.20). Then*

$$\|(\hat{Y}, \hat{Z})\|_{\mathcal{V}_\lambda^\beta[0, \tau]}^2 \leq C \mathbb{E}_\lambda \left(\int_0^\tau \hat{g}_0(r)^2 e^{2\beta_0 r + 2\beta_1 \langle W \rangle_r} dr + \int_0^\tau \hat{f}_0(r)^2 e^{2\beta_0 r + 2\beta_1 \langle W \rangle_r} d\langle W \rangle_r \right), \quad (2.3.21)$$

for some constant $C > 0$ depending only on β , where $\hat{\eta} = \eta - \bar{\eta}$ for $\eta = g_0, f_0, Y, Z$.

Proof. Let $e_t = \exp(2\beta_0 t + 2\beta_1 \langle W \rangle_t)$, and $\hat{g}(t) = g(t, Y_t) - g(t, \bar{Y}_t)$, $\hat{f}(t) = f(t, Y_t, Z_t) - f(t, \bar{Y}_t, \bar{Z}_t)$. By Itô's formula, (A'.2)–(A'.5), and Young's inequality, we have

$$\begin{aligned} \hat{Y}_{t \wedge \tau}^2 e_{t \wedge \tau} &\leq \hat{Y}_{T \wedge \tau}^2 e_{T \wedge \tau} + \int_{t \wedge \tau}^{T \wedge \tau} (2\hat{Y}_r \hat{g}_0(r) - 2\beta_0 \hat{Y}_r^2) e_r dr \\ &\quad + \int_{t \wedge \tau}^{T \wedge \tau} [2\hat{Y}_r \hat{f}_0(r) + 2\alpha K_1 |\hat{Y}_r| |\hat{Z}_r| - (2\alpha \kappa_1 + 2\beta_1) \hat{Y}_r^2 - \hat{Z}_r^2] e_r d\langle W \rangle_r \\ &\quad - 2 \int_{t \wedge \tau}^{T \wedge \tau} \hat{Y}_r \hat{Z}_r e_r dW_r \end{aligned}$$

$$\begin{aligned}
&\leq \hat{Y}_{T \wedge \tau}^2 e_{T \wedge \tau} - \beta_0 \int_{t \wedge \tau}^{T \wedge \tau} \hat{Y}_r^2 e_r dr + \frac{1}{\beta_0} \int_{t \wedge \tau}^{T \wedge \tau} \hat{g}_0(r)^2 e_r dr \\
&\quad + \int_{t \wedge \tau}^{T \wedge \tau} \left[(a - 2\alpha\kappa_1 + b\alpha^2 K_1^2 - 2\beta_1) \hat{Y}_r^2 + \left(\frac{1}{b} - 1 \right) \hat{Z}_r^2 + \frac{1}{a} \hat{f}_0(r)^2 \right] e_r d\langle W \rangle_r \\
&\quad - 2 \int_{t \wedge \tau}^{T \wedge \tau} \hat{Y}_r \hat{Z}_r e_r dW_r,
\end{aligned}$$

where a and b are positive constants to be determined.

Since $\kappa_1 = K_1^2/2$, we may choose $b > 1$ sufficiently close to 1, and choose accordingly $a > 0$ sufficiently small such that $a - 2\kappa_1 + bK_1^2 - 2\beta_1 < 0$. Since the function $\alpha \mapsto a - 2\alpha\kappa_1 + b\alpha^2 K_1^2 - 2\beta_1$ is convex and is negative at $\alpha = 0$ and $\alpha = 1$, we see that $a - 2\alpha\kappa_1 + b\alpha^2 K_1^2 - 2\beta_1 < 0$ for each $\alpha \in [0, 1]$. With such a and b , the estimate (2.3.21) follows easily from an argument similar to the proof of (2.3.4). \square

Corollary 2.3.14. *Let g and f satisfy (A'.2)–(A'.7). Then there exists an $\epsilon_0 > 0$, depending only on K_0, K_1 and β , such that the following holds: If, for some $\alpha \in [0, 1]$, (2.3.20) admits a unique solution (Y, Z) in $\mathcal{V}_\lambda^\beta[0, \tau]$ such that*

$$\begin{aligned}
\|(Y, Z)\|_{\mathcal{V}_\lambda^\beta[0, \tau]}^2 &\leq C \mathbb{E}_\lambda \left(\xi^2 e^{2\beta_0 \tau + 2\beta_1 \langle W \rangle_\tau} + \int_0^\tau (g_0(t)^2 + g(t, 0)^2) e^{2\beta_0 t + 2\beta_1 \langle W \rangle_t} dt \right. \\
&\quad \left. + \int_0^\tau (f_0(t)^2 + f(t, 0, 0)^2) e^{2\beta_0 t + 2\beta_1 \langle W \rangle_t} d\langle W \rangle_t \right),
\end{aligned} \tag{2.3.22}$$

for any ξ satisfying (A'.1) and any g_0, f_0 satisfying (2.3.18), where $C > 0$ is a constant depending only on K_0, K_1 and β , then the same is valid when replacing α by $\alpha + \epsilon$ with $\epsilon \in [0, \epsilon_0]$ and $\alpha + \epsilon \leq 1$. Moreover, the estimate (2.3.22) holds for some (possibly different) constant $C > 0$ depending only on K_0, K_1 and β .

Proof. Suppose that (2.3.20) admits a unique solution in $\mathcal{V}_\lambda^\beta[0, \tau]$ satisfying (2.3.22) for some $\alpha \in [0, 1]$. Let $\epsilon > 0$ and $(Y_0, Z_0) = (0, 0)$. By (A'.6), $(Y^{(n)}, Z^{(n)})$, $n \in \mathbb{N}_+$ can be defined inductively using Lemma 2.3.12 as the unique solution in $\mathcal{V}_\lambda^\beta[0, \tau]$ to the BSDE

$$\begin{cases} dY_t^{(n)} = -[g_0(t) + \epsilon g(t, Y_t^{(n-1)}) + \alpha g(t, Y_t^{(n)})] dt \\ \quad - [f_0(t) + \epsilon f(t, Y_t^{(n-1)}, Z_t^{(n-1)}) + \alpha f(t, Y_t^{(n)}, Z_t^{(n)})] d\langle W \rangle_t \\ \quad + Z_t^{(n)} dW_t, \quad t \in [0, \tau), \\ Y_\tau^{(n)} = \xi. \end{cases}$$

According to (2.3.22) and Lemma 2.3.13, we have

$$\begin{aligned} \|(Y^{(1)}, Z^{(1)})\|_{\mathcal{V}_\lambda^\beta[0, \tau]}^2 &\leq C \mathbb{E}_\lambda \left(\xi^2 e^{2\beta_0\tau + 2\beta_1\langle W \rangle_\tau} + \int_0^\tau (g_0(t)^2 + g(t, 0)^2) e^{2\beta_0t + 2\beta_1\langle W \rangle_t} dt \right. \\ &\quad \left. + \int_0^\tau (f_0(t)^2 + f(t, 0, 0)^2) e^{2\beta_0t + 2\beta_1\langle W \rangle_t} d\langle W \rangle_t \right), \end{aligned} \quad (2.3.23)$$

and

$$\begin{aligned} &\|(Y^{(n+1)} - Y^{(n)}, Z^{(n+1)} - Z^{(n)})\|_{\mathcal{V}_\lambda^\beta[0, \tau]} \\ &\leq \epsilon C \|(Y^{(n)} - Y^{(n-1)}, Z^{(n)} - Z^{(n-1)})\|_{\mathcal{V}_\lambda^\beta[0, \tau]}, \quad n \in \mathbb{N}_+, \end{aligned}$$

where $C > 0$ is a constant depending only on K_0, K_1 and β ; in particular, C is independent of α or ϵ . Let $\epsilon_0 = (4C)^{-1/2}$. Then for each $\epsilon \in [0, \epsilon_0]$ with $\alpha + \epsilon \leq 1$,

$$\|(Y^{(n+1)} - Y^{(n)}, Z^{(n+1)} - Z^{(n)})\|_{\mathcal{V}_\lambda^\beta[0, \tau]} \leq 2^{-n} \|(Y^{(1)}, Z^{(1)})\|_{\mathcal{V}_\lambda^\beta[0, \tau]}, \quad n \in \mathbb{N}_+.$$

This implies that $\lim_{n \rightarrow \infty} \|(Y^{(n)} - Y, Z^{(n)} - Z)\|_{\mathcal{V}_\lambda^\beta[0, \tau]} = 0$ for some $(Y, Z) \in \mathcal{V}_\lambda^\beta[0, \tau]$.

Clearly, (Y, Z) is the unique solution in $\mathcal{V}_\lambda^\beta[0, \tau]$ to the BSDE given by replacing α by $\alpha + \epsilon$ in (2.3.20). Moreover, $\|(Y, Z)\|_{\mathcal{V}_\lambda^\beta[0, \tau]} \leq 2\|(Y^{(1)}, Z^{(1)})\|_{\mathcal{V}_\lambda^\beta[0, \tau]}$. This, together with (2.3.23), completes the proof. \square

We can now state and give the proof of the main result in this subsection.

Theorem 2.3.15. *Let $\beta = (\beta_0, \beta_1) \in [1, \infty)^2$. Suppose that (A'.1)–(A'.3) are satisfied.*

Then

(a) *The BSDE (2.3.14) admits at most one solution.*

(b) *If, in addition, the conditions (A'.4)–(A'.7) are satisfied, then the BSDE (2.3.14) admits*

a unique solution $(Y, Z) \in \mathcal{V}_\lambda^\beta[0, \tau]$, and

$$\begin{aligned} \|(Y, Z)\|_{\mathcal{V}_\lambda^\beta[0, \tau]}^2 &\leq C \mathbb{E}_\lambda \left(\xi^2 e^{2\beta_0\tau + 2\beta_1\langle W \rangle_\tau} + \int_0^\tau g(t, 0)^2 e^{2\beta_0t + 2\beta_1\langle W \rangle_t} dt \right. \\ &\quad \left. + \int_0^\tau f(t, 0, 0)^2 e^{2\beta_0t + 2\beta_1\langle W \rangle_t} d\langle W \rangle_t \right), \end{aligned} \quad (2.3.24)$$

for some constant $C > 0$ depending only on $\beta_0, \beta_1, \kappa_0, \kappa_1, K_0, K_1$.

Proof. (a) This can be proved similarly to Theorem 2.3.7(a).

(b) Suppose first that $\kappa_0 = 0, \kappa_1 = K_1^2/2$. By Lemma 2.3.12, when $\alpha = 0$, the BSDE

$$\begin{cases} dY_t = -\alpha g(t, X_t, Y_t) dt - \alpha f(t, X_t, Y_t, Z_t) d\langle W \rangle_t + Z_t dW_t, & t \in [0, \tau), \\ Y_\tau = \xi, \end{cases}$$

admits a unique solution $(Y, Z) \in \mathcal{V}_\lambda^\beta[0, \tau]$ satisfying

$$\|(Y, Z)\|_{\mathcal{V}_\lambda^\beta[0, \tau]}^2 \leq C \mathbb{E}_\lambda (\xi^2 e^{2\beta_0\tau + 2\beta_1\langle W \rangle_\tau}),$$

where and thereafter, $C > 0$ denotes a generic constant depending only on $\kappa_0, \kappa_1, K_0, K_1, \beta$ which may be different at various occasions.

By Corollary 2.3.14, there exists an $\epsilon_0 > 0$ depending only on K_0, K_1, β and satisfying the property stated therein. Successive applications of Corollary 2.3.14 shows that (2.3.14) admits a unique solution $(Y, Z) \in \mathcal{V}_\lambda^\beta[0, \tau]$ satisfying (2.3.24).

For the general case, let

$$\tilde{e}_t = \exp \left[-\kappa_0 t + (K_1^2/2 - \kappa_1) \langle W \rangle_t \right], \quad t \geq 0,$$

$$\tilde{\xi} = \xi \tilde{e}_\tau, \quad \tilde{g}(t, y) = g(t, y \tilde{e}_t) \tilde{e}_t^{-1}, \quad \tilde{f}(t, y, z) = f(t, y \tilde{e}_t, z \tilde{e}_t) \tilde{e}_t^{-1}.$$

Then \tilde{g}, \tilde{f} satisfy the assumptions of the above case. Let

$$\tilde{\beta}_0 = \beta_0 - \kappa_0 > 0, \quad \tilde{\beta}_1 = \beta_1 - \kappa_1 + \frac{K_1^2}{2} > 0.$$

Then the BSDE

$$\begin{cases} d\tilde{Y}_t = -\tilde{g}(t, \tilde{Y}_t)dt - \tilde{f}(t, \tilde{Y}_t, \tilde{Z}_t)d\langle W \rangle_t + \tilde{Z}_t dW_t, & t \in [0, \tau), \\ \tilde{Y}_\tau = \tilde{\xi}, \end{cases}$$

admits a unique solution $(\tilde{Y}, \tilde{Z}) \in \mathcal{V}_\lambda^{\tilde{\beta}}[0, \tau]$ satisfying that

$$\begin{aligned} \|(\tilde{Y}, \tilde{Z})\|_{\mathcal{V}_\lambda^{\tilde{\beta}}[0, \tau]}^2 &\leq C \mathbb{E}_\lambda \left(\tilde{\xi}^2 e^{2\tilde{\beta}_0\tau + 2\tilde{\beta}_1\langle W \rangle_\tau} + \int_0^\tau \tilde{g}(t, 0)^2 e^{2\tilde{\beta}_0t + 2\tilde{\beta}_1\langle W \rangle_t} dt \right. \\ &\quad \left. + \int_0^\tau \tilde{f}(t, 0, 0)^2 e^{2\tilde{\beta}_0t + 2\tilde{\beta}_1\langle W \rangle_t} d\langle W \rangle_t \right) \\ &= C \mathbb{E}_\lambda \left(\xi^2 e^{2\beta_0\tau + 2\beta_1\langle W \rangle_\tau} + \int_0^\tau g(t, 0)^2 e^{2\beta_0t + 2\beta_1\langle W \rangle_t} dt \right. \\ &\quad \left. + \int_0^\tau f(t, 0, 0)^2 e^{2\beta_0t + 2\beta_1\langle W \rangle_t} d\langle W \rangle_t \right). \end{aligned}$$

Let $Y_t = \tilde{Y}_t \tilde{e}_t^{-1}$, $Z_t = \tilde{Z}_t \tilde{e}_t^{-1}$, $t \geq 0$. It is easily seen that $(Y, Z) \in \mathcal{V}_\lambda^\beta[0, \tau]$ is a solution to (2.3.14), and (Y, Z) satisfies (2.3.24). Thus we have completed the proof. \square

2.3.3 An example: linear equations

Let λ be a Borel probability measure on \mathbb{S} , and τ be an $\{\mathcal{F}_t\}$ -stopping time such that $\tau \leq T$ \mathbb{P}_λ -a.s. for some constant $T > 0$. We present in this subsection worked-out solutions to linear BSDEs on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_\lambda)$ below

$$\begin{cases} dY_t = -aY_t dt - (bY_t + cZ_t)d\langle W \rangle_t + Z_t dW_t, & t \in [0, \tau), \\ Y_\tau = \xi, \end{cases} \quad (2.3.25)$$

where $a, b, c \in \mathbb{R}$ are constants, and $\xi \in L^p(\mathcal{F}_\tau, \mathbb{P}_\lambda)$ for some $p > 2$.

To solve (2.3.25), let

$$\Phi_t = \exp \left[at + \left(b - \frac{c^2}{2} \right) \langle W \rangle_t + cW_t \right], \quad t \geq 0.$$

Then, by Corollary 2.2.16 and Corollary 2.2.17(a), $\mathbb{E}_\lambda(\Phi_t^q) < \infty$ for any $t \geq 0$ and any

$q > 0$. By Itô's formula,

$$d\Phi_t = a\Phi_t dt + b\Phi_t d\langle W \rangle_t + c\Phi_t dW_t, \quad t \geq 0. \quad (2.3.26)$$

Furthermore, if (Y, Z) is the solution to (2.3.25) then, by (2.3.26),

$$d(\Phi_t Y_t) = Y_t d\Phi_t + \Phi_t dY_t + d\langle \Phi, Y \rangle_t = \Phi_t (cY_t + Z_t) dW_t.$$

Therefore,

$$\Phi_{t \wedge \tau} Y_{t \wedge \tau} = \Phi_\tau \xi - \int_{t \wedge \tau}^{\tau} \Phi_r (cY_r + Z_r) dW_r, \quad t \geq 0.$$

Taking conditional expectations on both sides of the above gives that

$$\Phi_t Y_t = \Phi_{t \wedge \tau} Y_{t \wedge \tau} = \mathbb{E}_\lambda(\Phi_\tau \xi | \mathcal{F}_{t \wedge \tau}^\lambda) = \mathbb{E}_\lambda(\Phi_\tau \xi | \mathcal{F}_t^\lambda), \quad t \geq 0.$$

Equivalently,

$$Y_t = \Phi_t^{-1} \mathbb{E}_\lambda(\Phi_\tau \xi | \mathcal{F}_t^\lambda), \quad t \geq 0. \quad (2.3.27)$$

Since $\xi \in L^p(\mathcal{F}_\tau, \mathbb{P}_\lambda)$ for some $p > 2$, $\Phi_\tau \xi \in L^2(\mathcal{F}_\tau, \mathbb{P}_\lambda)$. Therefore, by Theorem 2.2.2, there exists a unique $\{\mathcal{F}_t\}$ -predictable process $\zeta(t)$ such that

$$\mathbb{E}_\lambda(\Phi_\tau \xi | \mathcal{F}_t) = \mathbb{E}_\lambda(\Phi_\tau \xi) + \int_0^t \zeta(r) dW_r, \quad t \geq 0. \quad (2.3.28)$$

By a similar argument for (2.3.26), we have

$$d\Phi_t^{-1} = -a\Phi_t^{-1} dt - (b - c^2)\Phi_t^{-1} d\langle W \rangle_t - c\Phi_t^{-1} dW_t, \quad t \geq 0.$$

By (2.3.27), (2.3.28) and the above equation,

$$\begin{aligned} dY_t &= \Phi_t^{-1} \zeta(t) dW_t + \mathbb{E}_\lambda(\Phi_\tau \xi | \mathcal{F}_t^\lambda) d\Phi_t^{-1} - c\Phi_t^{-1} \zeta(t) d\langle W \rangle_t \\ &= -aY_t dt - [bY_t + c(\Phi_t^{-1} \zeta(t) - cY_t)] d\langle W \rangle_t + (\Phi_t^{-1} \zeta(t) - cY_t) dW_t. \end{aligned} \quad (2.3.29)$$

Let

$$Z_t = \Phi_t^{-1}\zeta(t) - cY_t = \Phi_t^{-1}[\zeta(t) - c\mathbb{E}_\lambda(\Phi_\tau\xi|\mathcal{F}_t)], \quad t \geq 0. \quad (2.3.30)$$

Then, by (2.3.29), (Y, Z) given by (2.3.27), (2.3.28), and (2.3.30) is the unique solution to (2.3.25).

2.4 A Feynman–Kac representation

In this section, we establish a Feynman–Kac representation for solutions to semi-linear parabolic equations on the gasket \mathbb{S} which will be formulated below. Recall that, under the framework of weak solutions, PDEs on \mathbb{R}^d can be regarded as equations of measures; for example, the (backward) parabolic equation $\partial_t u + \Delta u = -f(t, x, u, \nabla u)$ can be written as

$$(\partial_t u + \Delta u) dx = -f(t, x, u, \nabla u) dx, \quad (2.4.1)$$

where dx is the Lebesgue measure. As we have seen in Section 1.2.3 (cf. Definition 1.2.28), the situation on \mathbb{S} is different since gradients of functions on the gasket are only a.e. defined with respect to the Kusuoka measure μ , which is singular to the symmetric measure ν of the Dirichlet form $(\mathcal{E}, \mathcal{F}(\mathbb{S}))$. Therefore, the analogue of (2.4.1) on \mathbb{S} must be an equation involving singular measures, and take the form

$$(\partial_t u + \mathcal{L}u) d\mu = -g(t, x, u) d\mu - f(t, x, u, \nabla u) d\nu. \quad (2.4.2)$$

Before giving a precise interpretation of the above equation, we would like to point out that there are several formulations of non-linear PDEs on fractals which are different in essence to ours (see e.g. [67, 32, 29, 28, 30]). We shall return to this and discuss with more details in Chapter 4.

Definition 2.4.1. The space $\mathcal{F}(\mathbb{S} \setminus V_0)$ is defined to be $\mathcal{F}(\mathbb{S} \setminus V_0) = \{u \in \mathcal{F}(\mathbb{S}) : u|_{V_0} = 0\}$.

The restricted form $(\mathcal{E}, \mathcal{F}(\mathbb{S} \setminus V_0))$ is a Dirichlet form associated with the diffusion process $\{X_t^0\}_{t \geq 0}$, called the *killed Brownian motion*, given by

$$X_t^0 = \begin{cases} X_t, & \text{if } t < \sigma_{V_0}, \\ \Delta_\infty, & \text{if } t \geq \sigma_{V_0}, \end{cases}$$

where $\sigma_{V_0} = \inf\{t > 0 : X_t \in V_0\}$, and Δ_∞ is the cemetery of $\{X_t\}_{t \geq 0}$. The Markov semigroup associated with $\{X_t^0\}_{t \geq 0}$ is denoted by $\{P_t^0\}_{t \geq 0}$; that is,

$$P_t^0 f(x) = \mathbb{E}_x(f(X_t)1_{\{t < \sigma_{V_0}\}}), \quad x \in \mathbb{S} \setminus V_0.$$

As the following lemma states, the killed diffusion process $\{X_t^0\}_{t \geq 0}$ admits continuous transition kernels which satisfy heat kernel estimate similar to that for $\{X_t\}_{t \geq 0}$ in Lemma 2.2.9 (cf. [35, Theorem 5.3.1]).

Lemma 2.4.2. *The killed Brownian motion $\{X_t^0\}_{t \geq 0}$ admits (jointly) continuous transition kernels $p^0(t, x, y)$, $t > 0$ with respect to μ , and*

$$C_{*,3} \max\{1, t^{-d_s/2}\} \leq \sup_{x,y \in \mathbb{S} \setminus V_0} p^0(t, x, y) \leq C_{*,4} \max\{1, t^{-d_s/2}\},$$

for some universal constants $C_{*,3}, C_{*,4} > 0$.

Definition 2.4.3. Let $\varphi \in C^{1,0}([0, T] \times V_0)$ and $\psi \in L^2(\mu)$. A function u on $[0, T] \times \mathbb{S}$ is said to be a *weak solution* to the (Dirichlet) terminal–boundary value problem

$$\begin{cases} (\partial_t u + \mathcal{L}u) d\mu = -g(t, x, u) d\mu - f(t, x, u, \nabla u) d\nu, & \text{in } [0, T] \times \mathbb{S} \setminus V_0, \\ u(t, x) = \varphi(t, x) \text{ on } [0, T] \times V_0, \quad u(T) = \psi, \end{cases} \quad (2.4.3)$$

if the following are satisfied:

(WS.1) $u \in C([0, T] \times \mathbb{S}) \cap L^2(0, T; \mathcal{F}(\mathbb{S}))$, and u has weak derivative $\partial_t u$ in $L^2(0, T; \mathcal{F}^{-1}(\mathbb{S}))$;

(WS.2) For any $v \in \mathcal{F}(\mathbb{S} \setminus V_0)$,

$$\frac{d}{dt} \langle u(t), v \rangle_\mu - \mathcal{E}(u(t), v) = -\langle g(t, u(t)), v \rangle_\mu - \langle f(t, u(t), \nabla u(t)), v \rangle_\nu \quad \text{a.e. } t \in [0, T]; \quad (2.4.4)$$

where and throughout this section, we denote $u(t) = u(t, \cdot)$ for short;

(WS.3) $u(t) = \varphi(t)$ on V_0 for a.e. $t \in [0, T)$, and $\lim_{t \rightarrow T} u(t) = \psi$ in $L^2(\mu)$.

Remark 2.4.4. (i) We point out that the term $\langle f(t, u(t), \nabla u(t)), v \rangle_\nu$ in (2.4.4) is well-defined. In fact, ∇u is ν -a.e. defined and u is pointwise defined due to the fact that $\mathcal{F}(\mathbb{S}) \subseteq C(\mathbb{S})$.

(ii) The equation (2.4.4) is well-posed by virtue of Lemma 2.2.7.

(iii) In view of (WS.2) and the singularity of μ and ν , we see that if $f \neq 0$ then the PDE (2.4.3) does not admit a solution u such that $u \in C^{1,0}([0, T) \times \mathbb{S})$ and $u(t) \in \text{Dom}(\mathcal{L})$, $t \in [0, T)$. This suggests that the theory of PDEs on \mathbb{S} is quite different from that on \mathbb{R}^d .

To construct weak solutions to the PDE (2.4.3), a natural idea is to show that the solution mapping of a related linear equation is a contraction in some suitable Banach space, then iterate solutions to this linear equation. However, difficulties arise immediately due to the singularity of μ and ν . To address this difficulty, our idea is that, though calculus on fractals might be considerably different from that on \mathbb{R}^d , stochastic calculus however remains similar to its classical counterpart. Specifically, we have the following Feynman–Kac representation, which gives a BSDE approach for semi-linear parabolic PDEs on \mathbb{S} .

Theorem 2.4.5. *Let $\varphi \in C^{1,0}([0, T] \times V_0)$, $\psi \in L^2(\mu)$, and let $g \in C([0, T] \times \mathbb{S} \times \mathbb{R})$ and $f \in C([0, T] \times \mathbb{S} \times \mathbb{R}^2)$. If the PDE (2.4.3) admits a solution u , then, for each $s \in [0, T)$ and each $x \in \mathbb{S}$, the pair*

$$(Y_t^{(s)}, Z_t^{(s)}) = (u(t + s, X_t), \nabla u(t + s, X_t))$$

is the unique solution (in the sense of Theorem 2.3.7(a)) of the BSDE

$$\begin{cases} dY_t^{(s)} = -g(t+s, X_t, Y_t^{(s)})dt \\ \quad - f(t+s, X_t, Y_t^{(s)}, Z_t^{(s)})d\langle W \rangle_t + Z_t^{(s)}dW_t, \quad t \in [0, \sigma^{(s)}), \\ Y_{\sigma^{(s)}}^{(s)} = \Psi(\sigma^{(s)}, X_{\sigma^{(s)}}), \end{cases} \quad (2.4.5)$$

on $(\Omega, \{\mathcal{F}_t\}, \mathbb{P}_x)$ for each $x \in \mathbb{S}$, where $\sigma^{(s)} = (T-s) \wedge \sigma_{V_0}$, $s \in [0, T]$, and

$$\Psi(t, x) = \begin{cases} \varphi(t, x), & \text{if } (t, x) \in [0, T) \times V_0, \\ \psi(x), & \text{if } (t, x) \in \{T\} \times \mathbb{S} \setminus V_0. \end{cases}$$

Moreover, the solution to (2.4.3) is unique and has the representation

$$u(t, x) = \mathbb{E}_x(Y_0^{(t)}), \quad \text{for all } (t, x) \in [0, T) \times \mathbb{S}. \quad (2.4.6)$$

Proof. We prove the theorem by several steps.

Step 1. Let

$$\begin{aligned} g^{(s)}(r, x) &= g(r+s, x, u(r+s, x)), \\ f^{(s)}(r, x) &= f(r+s, x, u(r+s, x), \nabla u(r+s, x)). \end{aligned}$$

Then, for any $\eta \in \mathcal{F}(\mathbb{S} \setminus V_0)$,

$$\begin{aligned} & \frac{d}{dt} \mathbb{E}_\mu [u(t \wedge \sigma^{(s)} + s, X_{t \wedge \sigma^{(s)}}) \eta(X_0)] \\ &= \langle \partial_t u(t+s), P_t^0 \eta \rangle_\mu - \mathcal{E}(u(t+s), P_t^0 \eta) \quad \text{a.e. } t \in [0, T], \end{aligned} \quad (2.4.7)$$

and

$$\frac{d}{dt} \mathbb{E}_\mu \left[\left(\int_0^{t \wedge \sigma^{(s)}} g^{(s)}(r, X_r) dr \right) \eta(X_0) \right] = \langle g^{(s)}(t), P_t^0 \eta \rangle_\mu \quad \text{a.e. } t \in [0, T], \quad (2.4.8)$$

and

$$\frac{d}{dt}\mathbb{E}_\mu\left[\left(\int_0^{t\wedge\sigma^{(s)}} f^{(s)}(r, X_r)d\langle W\rangle_r\right)\eta(X_0)\right] = \langle f^{(s)}(t), P_t^0\eta\rangle_\nu \quad a.e. t \in [0, T], \quad (2.4.9)$$

where we denote $u(t) = u(t, \cdot)$ and similar notation is used on $f^{(s)}, g^{(s)}$.

Proof of Step 1. Without loss of generality, we may assume $s = 0$. Let $H(\varphi(t))$ be the harmonic function with boundary value $\varphi(t)$, and $u^0(t, x) = u(t, x) - H(\varphi(t))(x)$. Let $0 < \delta < T - t$. Since $u^0(t \wedge \sigma_{V_0}, X_{t \wedge \sigma_{V_0}}) = u^0(t, X_t)1_{\{t < \sigma_{V_0}\}}$ and $u^0(t) = 0$ on V_0 , by the μ -symmetry of $\{X_t^0\}$, we obtain that

$$\begin{aligned} & \mathbb{E}_\mu\left[(u^0((t + \delta) \wedge \sigma_{V_0}, X_{(t+\delta) \wedge \sigma_{V_0}}))\eta(X_0)\right] \\ &= \mathbb{E}_\mu\left[u^0(t + \delta, X_0^0)\eta(X_{t+\delta}^0)\right] = \langle u^0(t + \delta), P_{t+\delta}^0\eta\rangle_\mu. \end{aligned} \quad (2.4.10)$$

Similarly, $\mathbb{E}_\mu[u^0(t \wedge \sigma_{V_0}, X_{t \wedge \sigma_{V_0}})\eta(X_0)] = \langle u^0(t), P_t^0\eta\rangle_\mu$. Therefore,

$$\begin{aligned} & \mathbb{E}_\mu\left[(u((t + \delta) \wedge \sigma_{V_0}, X_{(t+\delta) \wedge \sigma_{V_0}}) - u(t \wedge \sigma_{V_0}, X_{t \wedge \sigma_{V_0}}))\eta(X_0)\right] \\ &= \langle u^0(t + \delta) - u^0(t), P_{t+\delta}^0\eta\rangle_\mu + \langle u^0(t), P_{t+\delta}^0\eta - P_t^0\eta\rangle_\mu \\ & \quad + \langle H[\varphi(t + \delta) - \varphi(t)], P_t^0\eta\rangle_\mu. \end{aligned} \quad (2.4.11)$$

Notice that $u^0(t) \in \mathcal{F}(\mathbb{S} \setminus V_0)$. We have

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \langle u^0(t), P_{t+\delta}^0\eta - P_t^0\eta\rangle_\mu = -\mathcal{E}(u^0(t), P_t^0\eta) = -\mathcal{E}(u(t), P_t^0\eta), \quad (2.4.12)$$

where we have used in the second equality the fact that $\mathcal{E}(H(\varphi(t)), v) = 0$ for any $v \in \mathcal{F}(\mathbb{S} \setminus V_0)$. By Lemma 2.2.7 and that $\lim_{\delta \rightarrow 0} \|P_{t+\delta}^0\eta - P_t^0\eta\|_{L^2(\mu)} = 0$, we deduce that

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \frac{1}{\delta} \langle u^0(t + \delta) - u^0(t), P_{t+\delta}^0\eta\rangle_\mu \\ &= \lim_{\delta \rightarrow 0} \frac{1}{\delta} \langle u(t + \delta) - u(t), P_{t+\delta}^0\eta\rangle_\mu - \langle H(\partial_t\varphi(t)), P_t^0\eta\rangle_\mu \\ &= \langle \partial_t u(t), P_t^0\eta\rangle_\mu - \langle H(\partial_t\varphi(t)), P_t^0\eta\rangle_\mu. \end{aligned} \quad (2.4.13)$$

The equality (2.4.7) now follows readily from (2.4.11), (2.4.12) and (2.4.13).

Similar to (2.4.10), we have

$$\mathbb{E}_\mu \left[\left(\int_{t \wedge \sigma_{V_0}}^{(t+\delta) \wedge \sigma_{V_0}} g^{(0)}(r, X_r) dr \right) \eta(X_0) \right] = \int_0^\delta \langle g^{(0)}(t+r), P_{t+r}^0 \eta \rangle_\mu dr.$$

Using the Lebesgue dominated convergence theorem and fact that $\lim_{r \rightarrow 0} P_{t+r}^0 \eta(x) = P_t^0 \eta(x)$, $x \in \mathbb{S} \setminus V_0$, we obtain (2.4.8).

We now prove (2.4.12). Similar to the above, using the μ -symmetry of $\{X_t^0\}$ again, we have

$$\mathbb{E}_\mu \left[\left(\int_{t \wedge \sigma_{V_0}}^{(t+\delta) \wedge \sigma_{V_0}} f^{(0)}(r, X_r) d\langle W \rangle_r \right) \eta(X_0) \right] = \mathbb{E}_\mu \left[\left(\int_0^\delta f^{(0)}(t_n+r, X_r^0) d\langle W \rangle_r \right) P_t^0 \eta(X_0^0) \right].$$

Now we apply Lemma 2.2.13 and conclude that

$$\frac{d}{dt} \mathbb{E}_\mu \left[\left(\int_0^t f^{(0)}(r, X_r^0) d\langle W \rangle_r \right) \eta(X_0) \right] = \lim_{\delta \rightarrow 0} \frac{1}{\delta} \int_0^\delta \langle f^{(0)}(r+t), P_t^0 \eta \rangle_\nu dr = \langle f^{(0)}(t), P_t^0 \eta \rangle_\nu.$$

This completes the proof of Step 1.

Step 2. *Let*

$$\begin{aligned} M_t^{(s)} &= u(t \wedge \sigma^{(s)} + s, X_{t \wedge \sigma^{(s)}}) - u(s, X_0) + \int_0^{t \wedge \sigma^{(s)}} g^{(s)}(r, X_r) dr \\ &\quad + \int_0^{t \wedge \sigma^{(s)}} f^{(s)}(r, X_r) d\langle W \rangle_r, \quad t \geq 0. \end{aligned}$$

Then $\{M_t^{(s)}\}$ is a \mathbb{P}_x -martingale for each $x \in \mathbb{S} \setminus V_0$.

Proof of Step 2. By Step 1, $\frac{d}{dt} \mathbb{E}_\mu [M_t^{(s)} \eta(X_{t_0}^0)] = 0$ for all $\eta \in \mathcal{F}(\mathbb{S} \setminus V_0)$ a.e. $t \geq t_0$, which, together with the continuity of $t \mapsto \mathbb{E}_x(M_t^{(s)})$, implies that

$$\mathbb{E}_x(M_t^{(s)}) = 0, \quad \text{for all } t \geq 0 \quad \mu\text{-a.e. } x \in \mathbb{S} \setminus V_0. \quad (2.4.14)$$

We claim that $x \mapsto \mathbb{E}_x(M_t^{(s)})$, $x \in \mathbb{S} \setminus V_0$ is continuous, and therefore, (2.4.14) holds for

all $t \geq 0$ and all $x \in \mathbb{S} \setminus V_0$. Notice that

$$M_t^{(s)} = u(t+s, X_t^0) - u(s, X_0^0) + \int_0^t g^{(s)}(r, X_r^0) dr + \int_0^t f^{(s)}(r, X_r^0) d\langle W \rangle_r, \quad 0 \leq t \leq T-s.$$

Hence, we deduce that

$$\mathbb{E}_x(M_t^{(s)}) = P_t^0(u^0(t+s))(x) - u^0(s, x) + \int_0^t [P_r^0(g^{(s)}(r))(x) + P_r^0(f^{(s)}(r)\nu)(x)] dr.$$

Therefore, it suffices to prove the continuity of

$$x \mapsto \int_0^t P_r^0(f^{(s)}(r)\nu)(x) dr, \quad x \in \mathbb{S} \setminus V_0.$$

By Lemma 2.4.2 and (A.3), we have

$$\begin{aligned} \|P_r^0(f^{(s)}(r)\nu)\|_{L^\infty} &\leq C \min\{1, r^{-d_s/2}\} \|f^{(s)}(r)\|_{L^1(\nu)} \\ &\leq C \min\{1, r^{-d_s/2}\} [\|f(r+s, 0, 0)\|_{L^1(\nu)} + \|u(r+s)\|_{L^1(\nu)} \\ &\quad + \|\nabla u(r+s)\|_{L^1(\nu)}] \\ &\leq C \min\{1, r^{-d_s/2}\} [1 + \max_{[0, T] \times \mathbb{S}} |u| + \mathcal{E}(u(r+s))^{1/2}] \end{aligned} \tag{2.4.15}$$

for all $r > 0$, where $C > 0$ is a constant depending only on K_0, K_1 and $\max_{[0, T] \times \mathbb{S}} |f(t, x, 0, 0)|$.

Notice that $\int_0^T \mathcal{E}(u(t)) dt < \infty$, and that, for each $r > 0$,

$$P_r^0(f^{(s)}(r)\nu)(x) = \int_{\mathbb{S}} f^{(s)}(r, y) p^0(r, x, y) \nu(dy)$$

is continuous in $x \in \mathbb{S} \setminus V_0$. Now the continuity of $x \mapsto \int_0^t P_r^0(f^{(s)}(r)\nu)(x) dr$ follows readily from (2.4.15) and the Lebesgue dominated convergence theorem. Therefore

$$\mathbb{E}_x(M_t^{(s)}) = 0, \quad \text{for all } t \geq 0, x \in \mathbb{S} \setminus V_0,$$

which, together with the Markov property of $\{X_t^0\}$, completes the proof of Step 2.

Step 3. For each $s \in [0, T)$ and each $x \in \mathbb{S} \setminus V_0$,

$$(Y_t^{(s)}, Z_t^{(s)}) = (u(t \wedge \sigma^{(s)} + s, X_{t \wedge \sigma^{(s)}}), \nabla u(t \wedge \sigma^{(s)} + s, X_{t \wedge \sigma^{(s)}}))$$

is the solution to the BSDE (2.4.5). Moreover, the representation (2.4.6) holds, and the solution to (2.4.3) is unique.

Proof of Step 3. By Lemma 2.2.8,

$$u(t + s, X_t) = u(s, X_0) + \int_0^t \nabla u(r + s, X_r) dW_r + N_t^{(s)}, \quad t \geq 0,$$

where $N^{(s)}$ is a continuous process with zero quadratic variation. Let

$$Q_t^{(s)} = N_{t \wedge \sigma^{(s)}}^{(s)} + \int_0^{t \wedge \sigma^{(s)}} g^{(s)}(r, X_r) dr + \int_0^{t \wedge \sigma^{(s)}} f^{(s)}(r, X_r) d\langle W \rangle_r, \quad t \geq 0.$$

Then

$$Q_t^{(s)} = M_t^{(s)} - \int_0^{t \wedge \sigma^{(s)}} \nabla u(r + s, X_r) dW_r, \quad t \geq 0, \quad (2.4.16)$$

and therefore $\{Q_t^{(s)}\}$ is a \mathbb{P}_x -martingale for all $x \in \mathbb{S} \setminus V_0$. Let $t_i = it/n$, $0 \leq i \leq n$. Then

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (Q_{t_i}^{(s)} - Q_{t_{i-1}}^{(s)})^2 = \langle Q^{(s)} \rangle_t, \quad \text{in } L^1(\mathbb{P}_x) \text{ for all } x \in \mathbb{S} \setminus V_0.$$

Since $\lim_{n \rightarrow \infty} \mathbb{E}_\mu \left[\sum_{i=1}^n (N_{t_i}^{(s)} - N_{t_{i-1}}^{(s)})^2 \right] = 0$, there exists a subsequence $\{n_k\}$ such that

$$\lim_{k \rightarrow \infty} \sum_{i=1}^{n_k} (N_{t_i}^{(s)} - N_{t_{i-1}}^{(s)})^2 = 0 \quad \mathbb{P}_\mu\text{-a.s.},$$

which implies that $\langle Q^{(s)} \rangle_t = 0$ \mathbb{P}_μ -a.s., as

$$t \mapsto \int_0^{t \wedge \sigma^{(s)}} g^{(s)}(r, X_r) dr$$

and

$$t \mapsto \int_0^{t \wedge \sigma^{(s)}} f^{(s)}(r, X_r) d\langle W \rangle_r$$

are of bounded variations. Therefore, $\mathbb{E}_x(\langle Q \rangle_t^{(s)}) = 0$ μ -a.e. $x \in \mathbb{S} \setminus V_0$. By an argument similar to the proof of Step 2, it can be shown that $x \mapsto \mathbb{E}_x(\langle Q \rangle_t^{(s)})$ is continuous. Thus, $\mathbb{E}_x(\langle Q \rangle_t^{(s)}) = 0$ for all $x \in \mathbb{S} \setminus V_0$. In particular, by (2.4.16),

$$\begin{aligned} u(t \wedge \sigma^{(s)} + s, X_{t \wedge \sigma^{(s)}}) &= u(s, X_0) - \int_0^{t \wedge \sigma^{(s)}} g^{(s)}(r, X_r) dr - \int_0^{t \wedge \sigma^{(s)}} f^{(s)}(r, X_r) d\langle W \rangle_r \\ &+ \int_0^{t \wedge \sigma^{(s)}} \nabla u(r + s, X_r) dW_r, \quad \mathbb{P}_x\text{-a.s. for all } x \in \mathbb{S} \setminus V_0. \end{aligned}$$

This implies that

$$(Y_t^{(s)}, Z_t^{(s)}) = (u(t \wedge \sigma^{(s)} + s, X_{t \wedge \sigma^{(s)}}), \nabla u(t \wedge \sigma^{(s)} + s, X_{t \wedge \sigma^{(s)}}))$$

is the unique solution to the BSDE (2.4.5) on $(\Omega, \{\mathcal{F}_t\}, \mathbb{P}_x)$ for $x \in \mathbb{S} \setminus V_0$, which is clearly also valid for $x \in V_0$. As a result, we obtain the representation (2.4.6).

The uniqueness of the solution to (2.4.3) follows immediately from the representation (2.3.1) and the uniqueness of solutions to (2.4.5). \square

Remark 2.4.6. It is well known that, solutions to BSDEs on \mathbb{R}^d correspond to viscosity solutions to the corresponding PDEs, which is a very weak formulation of solutions. Moreover, Theorem 2.4.5 shows that solutions to BSDEs on \mathbb{S} correspond to the solution to the PDE (2.4.3) whenever a solution exists. These justify to name the functions given by (2.4.6) the *viscosity solutions* to (2.4.3). The existence of such very weak solutions is guaranteed by Theorem 2.3.7. On the other hand, we also note that the existence of solutions to BSDEs does not imply the existence of weak solutions to the corresponding semi-linear parabolic equations. The existence of weak solutions to (2.4.3) will be investigated in Chapter 4.

Chapter 3

A Pontryagin Maximum Principle for Stochastic Controls on the Gasket

3.1 Introduction

Several interesting mathematical finance problems and production management problems are formulated as stochastic control problems, which are based upon the assumption that uncertainties in the models have their source from the Brownian filtration on \mathbb{R}^d . However, there exist objects which are better characterised by fractal models, e.g. transportation problem in complex biological systems, traffic lines in cities. For controlled dynamic systems on these models, the driving noise comes from diffusions on the corresponding fractals. In this chapter, as an application of the results in Chapter 2, we discuss stochastic optimal control problems on the gasket.

To present the motivation of stochastic optimal control problems, let us start with their deterministic counterparts. Consider the production plan of a manufacturing company. Let $x(t)$ be the amount of the company's inventory of its products, and $u(t)$ be the planned production rate of the company at time instant t . Suppose that the market demand for the product is a known deterministic function $z(t)$. Then $x(t)$ is governed by the differential

equation

$$dx(t) = (u(t) - z(t)) dt, \quad x(0) = x_0. \quad (3.1.1)$$

Let c be the production cost for per item, and a^+ , a^- the costs for production surplus and backlog respectively. The total cost $J(u)$ for the production plan $u(t)$ is given by

$$J(u) = a^+ x(T)^+ + a^- x(T)^- + cu(T),$$

where T is the terminal time. Minimizing the company's cost leads to the following (deterministic) optimal control problem

$$\begin{aligned} & \underset{u}{\text{minimize}} \quad J(u), \\ & \text{subject to: } x(t) \text{ satisfies (3.1.1).} \end{aligned}$$

In practice, the market demand $z(t)$ will not be a deterministic function and is subject to certain uncertainty/noise. Therefore, the controlling dynamic (3.1.1) is a stochastic differential equation driven by noise which generates the corresponding uncertainty, and the above deterministic control problem should be replaced by a stochastic control problem. In this chapter, we shall consider stochastic control problems where the uncertainty of the system comes from the filtration determined by Brownian motion on the Sierpinski gasket (see Section 3.2).

3.2 A stochastic maximum principle

In this section, we formulate a stochastic optimal control problem on the gasket \mathbb{S} , and derive a Pontryagin maximum principle (Theorem 3.2.13) for the problem. A crucial ingredient of our argument is an order comparison lemma (Lemma 3.2.5), which will be needed when performing stochastic Taylor expansions. It turns out that, in contrast to its counterpart on Euclidean spaces, the stochastic maximum principle on the Sierpinski gas-

ket consists of two equations of necessity rather than a single one (cf. [56] and [70, Section 3.2]), which is due to the singularity between measures.

Suppose that λ is a Borel probability measure on \mathbb{S} satisfying $\lambda \ll \mu$, and that (\mathbb{U}, ρ) is a given separable metric space, called the *decision space*. Let $h : \mathbb{R} \rightarrow \mathbb{R}$, $f_1 : [0, T] \times \mathbb{R} \times \mathbb{U} \rightarrow \mathbb{R}$, and $f_2 : [0, T] \times \mathbb{R} \times \mathbb{U} \rightarrow \mathbb{R}$ be Borel measurable functions. For any \mathbb{U} -valued $\{\mathcal{F}_t\}$ -adapted process $u(t)$, we introduce the *cost functional*

$$J(u) \triangleq \mathbb{E}_\lambda \left(h(x(T)) + \int_0^T f_1(t, x(t), u(t)) dt + \int_0^T f_2(t, x(t), u(t)) d\langle W \rangle_t \right),$$

where the *controlled process* $x(t)$ is given by the following SDE¹ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}_\lambda)$:

$$\begin{cases} dx(t) = b_1(t, x(t), u(t)) dt + b_2(t, x(t), u(t)) d\langle W \rangle_t \\ \quad + \sigma(t, x(t), u(t)) dW_t, \quad t \in (0, T], \quad \mathbb{P}_\lambda\text{-a.s.}, \\ x(0) = x_0, \end{cases} \quad (3.2.1)$$

where $\varphi : [0, T] \times \mathbb{R} \times \mathbb{U} \rightarrow \mathbb{R}$, $\varphi = b_1, b_2, \sigma$ are Borel measurable functions, and $x_0 \in \mathcal{F}_0$.

Remark 3.2.1. We should give some possible interpretation of terms included in the cost functional $J(u)$. Consider the following transportation problem. A delivery agent needs to deliver cargo to clients in a region. Once the agent arrives at some location in that region, he inform clients of his location for clients to collect their items. The agent may also collect more orders during the transportation. Then the terminal state $x(T)$ is the agent's location where he will inform his clients. The term $h(x(T))$ measures the cost of how well the agent balances the distances of all clients. The integrals in the cost functions relate to the cost of carry and the loss coming from not being able to collect more orders due to a sub-optimal choice of rout.

Definition 3.2.2. Denote by $\mathcal{A}[0, T]$ the family of all \mathbb{U} -valued progressively measurable

¹Existence and uniqueness of solutions to (3.2.1) with Lipschitz coefficients can be easily shown by an iteration argument similar to the proof of Theorem 2.3.7.

processes $u(t)$ such that

$$\mathbb{E}_\lambda \left(|h(x(T))| + \int_0^T |f_1(t, x(t), u(t))| dt + \int_0^T |f_2(t, x(t), u(t))| d\langle W \rangle_t \right) < \infty,$$

where $x(t)$ is the controlled process given by (3.2.1). Any process $u \in \mathcal{A}[0, T]$ will be called an *admissible control*, and the pair $(x(\cdot), u(\cdot))$ an *admissible pair*.

Assumption 3.2.3. (i) For $\varphi = b_1, b_2, \sigma, f_1, f_2, h$,

$$|\varphi(t, x, u) - \varphi(t, \hat{x}, \hat{u})| \leq M|x - \hat{x}| + \rho(u, \hat{u}),$$

$$|\varphi(t, 0, u)| \leq M, \quad t \in [0, T], \quad x, \hat{x} \in \mathbb{R}, \quad u, \hat{u} \in \mathbb{U};$$

(ii) For $\varphi = b_1, b_2, \sigma, f_1, f_2, h$,

$$\begin{aligned} & |\partial_x \varphi(t, x, u) - \partial_x \varphi(t, \hat{x}, \hat{u})| + |\partial_x^2 \varphi(t, x, u) - \partial_x^2 \varphi(t, \hat{x}, \hat{u})| \\ & \leq M|x - \hat{x}| + \rho(u, \hat{u}), \quad \text{for all } t \in [0, T], \quad x, \hat{x} \in \mathbb{R}, \quad u, \hat{u} \in \mathbb{U}, \end{aligned}$$

for some constant $M > 0$.

We consider the following optimization problem

$$\underset{u \in \mathcal{A}[0, T]}{\text{minimize}} \quad J(u), \tag{P}$$

subject to: $x(t)$ solves (3.2.1),

and establish a stochastic Pontryagin maximum principle under the assumptions in Assumption 3.2.3. Our argument is essentially a modification of those in [56, 70] while overcoming some difficulties concerning the the Brownian martingales on the Sierpinski gaskets. A crucial ingredient of our argument is an order comparison lemma (Lemma 3.2.5), which is needed when performing a stochastic Taylor expansion.

Definition 3.2.4. Let $\lambda \in \mathcal{P}(\mathbb{S})$, $k \geq 1$ and $E \in \mathcal{B}([0, \infty) \times \Omega)$ be a progressively measurable set. For each $I \in \mathcal{B}([0, \infty))$, we denote

$$m_{k,\lambda}(I; E) = \mathbb{E}_\lambda \left[\left(\int_I 1_E(t, \omega) d\langle W \rangle_t \right)^k \right].$$

Clearly, the map $I \mapsto |I| + m_{1,\lambda}(I; \Omega)$ is a Borel measure on $\mathcal{B}([0, \infty))$, where $|\cdot|$ is the one-dimensional Lebesgue measure. We denote by $\mathcal{B}_\lambda([0, \infty))$ the completion of $\mathcal{B}([0, \infty))$ with respect to the measure $|\cdot| + m_{1,\lambda}(\cdot; \Omega)$.

Lemma 3.2.5. Let $\lambda \in \mathcal{P}(\mathbb{S})$, and $E \in \mathcal{B}([0, \infty) \times \Omega)$ be a progressively measurable set. Let $\{I_\epsilon\}_{\epsilon > 0}$ be a family of $\mathcal{B}_\lambda([0, \infty))$ -measurable subsets of $[0, \infty)$ such that $\lim_{\epsilon \rightarrow 0} |I_\epsilon| = 0$. Then, for some universal constant $C_* > 0$,

$$m_{k+1,\lambda}(I_\epsilon; E) \leq C_*(k+1) |I_\epsilon|^{1-d_s/2} m_{k,\lambda}(I_\epsilon; E), \quad \text{for all } k \in \mathbb{N}_+. \quad (3.2.2)$$

In particular,

$$m_{l,\lambda}(I_\epsilon; E) = o(m_{k,\lambda}(I_\epsilon; E)), \quad \text{as } \epsilon \rightarrow 0, \quad (3.2.3)$$

for all $k \in \mathbb{N}_+$ and $l > k$.

Proof. Let $\phi_\epsilon(t) = \phi_\epsilon(t, \omega) = 1_{I_\epsilon}(t) 1_E(t, \omega)$. Then, for each $\epsilon > 0$, ϕ_ϵ is a bounded progressively measurable process. Clear, we have the following iterated integral representation

$$\begin{aligned} & \mathbb{E}_\lambda \left[\left(\int_0^\infty \phi_\epsilon(t) d\langle W \rangle_t \right)^k \right] \\ &= \mathbb{E}_\lambda \left[k! \int_{0 < t_1 < \dots < t_k < \infty} \phi_\epsilon(t_1) \cdots \phi_\epsilon(t_k) d\langle W \rangle_{t_1} \cdots d\langle W \rangle_{t_k} \right]. \end{aligned} \quad (3.2.4)$$

Since ϕ_ϵ is progressively measurable, we have $\phi_\epsilon(t) \in \mathcal{F}_t^\lambda$. Therefore, by (3.2.4) and the

tower property,

$$\begin{aligned}
& \mathbb{E}_\lambda \left[\left(\int_0^\infty \phi_\epsilon(t) d\langle W \rangle_t \right)^{k+1} \right] \\
&= \mathbb{E}_\lambda \left[(k+1)! \int_{0 < t_1 < \dots < t_{k+1} < \infty} \phi_\epsilon(t_1) \cdots \phi_\epsilon(t_{k+1}) d\langle W \rangle_{t_1} \cdots d\langle W \rangle_{t_{k+1}} \right] \\
&= \mathbb{E}_\lambda \left[(k+1)! \int_{0 < t_1 < \dots < t_k < \infty} \phi_\epsilon(t_1) \cdots \phi_\epsilon(t_k) \right. \\
&\quad \left. \times \mathbb{E}_\lambda \left(\int_{t_k}^\infty \phi_\epsilon(t_{k+1}) d\langle W \rangle_{t_{k+1}} \mid \mathcal{F}_{t_k}^\lambda \right) d\langle W \rangle_{t_1} \cdots d\langle W \rangle_{t_k} \right].
\end{aligned} \tag{3.2.5}$$

Recall that $\phi_\epsilon(t, \omega) \leq 1_{I_\epsilon}(t)$. By the Markov property and Lemma 2.2.12, we have

$$\begin{aligned}
\mathbb{E}_\lambda \left(\int_{t_k}^\infty \phi_\epsilon(t_{k+1}) d\langle W \rangle_{t_{k+1}} \mid \mathcal{F}_{t_k}^\lambda \right) &\leq \mathbb{E}_\lambda \left(\int_{t_k}^\infty 1_{I_\epsilon}(t_{k+1}) d\langle W \rangle_{t_{k+1}} \mid \mathcal{F}_{t_k}^\lambda \right) \\
&= \mathbb{E}_{X_{t_k}} \left(\int_{t_k}^\infty 1_{I_\epsilon}(t_{k+1}) d\langle W \rangle_{t_{k+1}-t_k} \right) \\
&= \int_{t_k}^\infty 1_{I_\epsilon}(t_{k+1}) (P_{t_{k+1}-t_k} \nu)(X_{t_k}) dt_{k+1},
\end{aligned} \tag{3.2.6}$$

where, $P_t \nu$ is as defined in Definition 1.2.13. By Lemma 2.2.9, $\|P_t \mu\|_{L^\infty} \leq C_* \max\{1, t^{-d_s/2}\}$, , $t > 0$. Therefore, for I_ϵ with $|I_\epsilon| \leq 1$, by (3.2.6),

$$\begin{aligned}
& \mathbb{E}_\lambda \left(\int_{t_k}^\infty \phi_\epsilon(t_{k+1}) d\langle W \rangle_{t_{k+1}} \mid \mathcal{F}_{t_k}^\lambda \right) \\
&\leq \int_{t_k}^\infty 1_{I_\epsilon}(t_{k+1}) (t_{k+1} - t_k)^{-d_s/2} dt_{k+1} \\
&\leq \int_{t_k}^{t_k + |I_\epsilon|} (t_{k+1} - t_k)^{-d_s/2} dt_{k+1} + |I_\epsilon|^{-d_s/2} \int_0^\infty 1_{I_\epsilon}(t_{k+1}) dt_{k+1} \\
&= C_* |I_\epsilon|^{1-d_s/2}.
\end{aligned}$$

Hence, by (3.2.5),

$$\begin{aligned}
& \mathbb{E}_\lambda \left[\left(\int_0^\infty \phi_\epsilon(t) d\langle W \rangle_t \right)^{k+1} \right] \\
&\leq C_* |I_\epsilon|^{1-d_s/2} \mathbb{E}_\lambda \left[(k+1)! \int_{0 < t_1 < \dots < t_k < \infty} \phi_\epsilon(t_1) \cdots \phi_\epsilon(t_k) d\langle W \rangle_{t_1} \cdots d\langle W \rangle_{t_k} \right].
\end{aligned}$$

By (3.2.4) again, we conclude that

$$\mathbb{E}_\lambda \left[\left(\int_0^\infty \phi_\epsilon(t) d\langle W \rangle_t \right)^{k+1} \right] \leq C_*(k+1) |I_\epsilon|^{1-d_s/2} \mathbb{E}_\lambda \left[\left(\int_0^\infty \phi_\epsilon(t) d\langle W \rangle_t \right)^k \right],$$

which is (3.2.2).

When l is an integer, the asymptotic (3.2.3) is a direct corollary of (3.2.2). For real-valued $l > k$, the conclusion follows easily from interpolation

$$m_{k+\theta, \lambda}(I_\epsilon; E) \leq m_{k, \lambda}(I_\epsilon; E)^{1-\theta} m_{k+1, \lambda}(I_\epsilon; E)^\theta, \quad \theta \in (0, 1).$$

□

We shall also need the following estimate for solutions to linear SDEs driven by the Brownian martingale W .

Lemma 3.2.6. *Let $\{Y_t\}$ be the solution to the SDE*

$$\begin{cases} dY_t = (a_1(t)Y_t + \alpha_1(t))dt + (a_2(t)Y_t + \alpha_2(t))d\langle W \rangle_t \\ \quad + (b(t)Y_t + \beta(t))dW_t, \quad t \in [0, T], \\ Y_0 = \xi. \end{cases}$$

Suppose that

$$|\varphi(t)| \leq M, \quad \text{for } \varphi = a_1, a_2, b,$$

where $M > 0$ is a constant. Then, for each $\lambda \in \mathcal{P}(\mathbb{S})$ and each $k > 1/2$,

$$\begin{aligned} \mathfrak{T}_{2k}(Y) \leq C \mathbb{E}_\lambda & \left[|\xi|^{2k} + \left(\int_0^T |\alpha_1(t)| dt \right)^{2k} \right. \\ & \left. + \left(\int_0^T |\alpha_2(t)| d\langle W \rangle_t \right)^{2k} + \left(\int_0^T |\beta(t)|^2 d\langle W \rangle_t \right)^k \right]. \end{aligned} \quad (3.2.7)$$

for some constant $C > 0$ depending only on k, M , where

$$\mathfrak{T}_{2k}(\varphi) = \mathbb{E}_\lambda \left(\sup_{0 \leq t \leq T} |\varphi(t)|^{2k} e_t^{-1} + \int_0^T |\varphi(t)|^{2k} e_t^{-1} d\langle W \rangle_t \right), \quad (3.2.8)$$

for any $k \geq 1$ and any progressively measurable process $\varphi(t)$, and

$$e_t = e^{k_1 t + k_2 \langle W \rangle_t}, \quad k_1 = 2kM, \quad k_2 = 8k^2(M+1)^2. \quad (3.2.9)$$

Therefore,

$$\begin{aligned} \mathfrak{T}_{2k}(Y) \leq C & \left\{ \mathbb{E}_\lambda(|\xi|^{2k}) + \|\alpha_1\|_{L^\infty(\mathfrak{M}_1)}^{2k} \mathbb{E}_\lambda \left[\left(\int_0^T 1\{\alpha_1(t) \neq 0\} dt \right)^{2k} \right] \right. \\ & + \|\alpha_2\|_{L^\infty(\mathfrak{M}_2)}^{2k} \mathbb{E}_\lambda \left[\left(\int_0^T 1\{\alpha_2(t) \neq 0\} d\langle W \rangle_t \right)^{2k} \right] \\ & \left. + \|\beta\|_{L^\infty(\mathfrak{M}_2)}^{2k} \mathbb{E}_\lambda \left[\left(\int_0^T 1\{\beta(t) \neq 0\} d\langle W \rangle_t \right)^k \right] \right\}, \end{aligned} \quad (3.2.10)$$

Proof. To simplify notation, denote $Y^m = |Y|^m \operatorname{sgn}(Y)$ for any $m > 0$. By Itô's formula,

$$\begin{aligned} |Y_t|^{2k} e_t^{-1} &= |\xi|^{2k} + \int_0^t [2kY_r^{2k-1}(a_1(r)Y_r + \alpha_1(r)) - k_1|Y_r|^{2k}] e_r^{-1} dr \\ &+ \int_0^t [2kY_r^{2k-1}(a_2(r)Y_r + \alpha_2(r)) + k(2k-1)Y_r^{2k-2}(b(r)Y_r \\ &+ \beta(r))^2 - k_2|Y_r|^{2k}] e_r^{-1} d\langle W \rangle_r \\ &+ \int_0^t 2kY_r^{2k-1}(b(r)Y_r + \beta(r)) e_r^{-1} dW_r \\ &\leq |\xi|^{2k} + \int_0^t 2k|Y_r|^{2k-1} |\alpha_1(r)| e_r^{-1} dr \\ &+ \int_0^t [(2kM + 4k^2M^2 - k_2)|Y_r|^{2k} + 2k|Y_r|^{2k-1} |\alpha_2(r)| \\ &+ 4k^2|Y_r|^{2k-2} |\beta(r)|^2] e_r^{-1} d\langle W \rangle_r \\ &+ \left| \int_0^t 2kY_r^{2k-1}(b(r)Y_r + \beta(r)) e_r^{-1} dW_r \right|. \end{aligned}$$

Denote $Z = \sup_{0 \leq t \leq T} |Y_t| e_t^{-1/(2k)}$. Then by the above inequality,

$$\begin{aligned} Z^{2k} &\leq |\xi|^{2k} + 2kZ^{2k-1} \left(\int_0^T |\alpha_1(t)| dt + \int_0^T |\alpha_2(t)| d\langle W \rangle_t \right) \\ &+ 4k^2 Z^{2k-2} \int_0^T |\beta(t)|^2 d\langle W \rangle_t + (2kM + 4k^2M^2 - k_2) \int_0^T |Y_r|^{2k} e_r^{-1} d\langle W \rangle_r \\ &+ \sup_{0 \leq t \leq T} \left| \int_0^t 2kY_r^{2k-1}(b(r)Y_r + \beta(r)) e_r^{-1} dW_r \right|. \end{aligned} \quad (3.2.11)$$

By the Burkholder–Davis–Gundy inequality and Young’s inequality,

$$\begin{aligned}
& \mathbb{E}_\lambda \left(\sup_{0 \leq t \leq T} \left| \int_0^t Y_r^{2k-1} (b(r)Y_r + \beta(r)) e_r^{-1} dW_r \right| \right) \\
& \leq C_* \mathbb{E}_\lambda \left[\left(\int_0^T (M^2 |Y_r|^{4k} + |Y_r|^{4k-2} |\beta(r)|^2) e_r^{-2} d\langle W \rangle_r \right)^{1/2} \right] \\
& \leq C_* \mathbb{E}_\lambda \left[M Z^k \left(\int_0^T |Y_r|^{2k} e_r^{-1} d\langle W \rangle_r \right)^{1/2} + Z^{2k-1} \left(\int_0^T |\beta(r)|^2 d\langle W \rangle_r \right)^{1/2} \right] \\
& \leq C_* \mathbb{E}_\lambda \left[(\epsilon_1 + \epsilon_2) Z^{2k} + \frac{M}{\epsilon_1} \int_0^T |Y_r|^{2k} e_r^{-1} d\langle W \rangle_r + \frac{1}{2k\epsilon_2^{2k-1}} \left(\int_0^T |\beta(r)|^2 d\langle W \rangle_r \right)^k \right].
\end{aligned}$$

where $C_* > 0$ is a universal constant. Choosing $\epsilon_1 = 1/4$ and $\epsilon_2 > 0$ sufficiently small gives

$$\begin{aligned}
& \mathbb{E}_\lambda \left(\sup_{0 \leq t \leq T} \left| \int_0^t Y_r^{2k-1} (b(r)Y_r + \beta(r)) e_r^{-1} dW_r \right| \right) \\
& \leq \frac{1}{2} \mathbb{E}_\lambda (Z^{2k}) + \mathbb{E}_\lambda \left[4M \int_0^T |Y_r|^{2k} e_r^{-1} d\langle W \rangle_r + C \left(\int_0^T |\beta(r)|^2 d\langle W \rangle_r \right)^k \right],
\end{aligned}$$

where $C > 0$ denotes a constant depending only on k, M . Since $k_2 > (4 + 2k)M + 4k^2 M^2$, (3.2.10) follows easily from the above and (3.2.11) and Young’s inequality. \square

We now turn to the derivation of the stochastic maximum principle. Suppose that $\bar{u} \in \mathcal{A}[0, T]$ is a minimizer of (1.2.7), and $\bar{x}(\cdot)$ is the corresponding controlled process. Let $\{I_\epsilon\}_{\epsilon > 0}$ be an arbitrary family of $\mathcal{B}_\lambda([0, \infty))$ -measurable subsets of $[0, T]$ such that $\lim_{\epsilon \rightarrow 0} |I_\epsilon| = 0$.

To formulate our result, we shall need the following definition.

Definition 3.2.7. We define the measure \mathfrak{M}_1 on $[0, \infty) \times \Omega$ to be

$$\mathfrak{M}_1 = dt \times \mathbb{P}_\lambda,$$

and the measure \mathfrak{M}_2 to be the unique measure on the optional σ -field² on $[0, \infty) \times \Omega$

²That is, the σ -field on $[0, \infty) \times \Omega$ generated by the family of all right continuous left limit processes.

satisfying

$$\mathfrak{M}_2(\llbracket \sigma_1, \sigma_2 \rrbracket) = \mathbb{E}_\lambda(\langle W \rangle_{\sigma_2} - \langle W \rangle_{\sigma_1}),$$

for any stochastic interval $\llbracket \sigma_1, \sigma_2 \rrbracket = \{(t, \omega) \in [0, \infty) \times \Omega : \sigma_1(\omega) \leq t < \sigma_2(\omega)\}$ with $\sigma_1, \sigma_2, \sigma_1 \leq \sigma_2$ being $\{\mathcal{F}_t\}$ -stopping times.

Remark 3.2.8. Since $\lambda \ll \mu$, by Lemma 2.2.3, the measures \mathfrak{M}_1 and \mathfrak{M}_2 are mutually singular.

Let $S_1, S_2 \subseteq [0, \infty) \times \Omega$ be disjoint optional sets such that \mathfrak{M}_1 is supported on S_1 and \mathfrak{M}_2 on S_2 . For arbitrary $u_1, u_2 \in \mathcal{A}[0, T]$, let

$$u^\epsilon(t, \omega) = \begin{cases} \bar{u}(t, \omega), & \text{if } (t, \omega) \in ([0, T] \setminus I_\epsilon) \times \Omega, \\ u_1(t, \omega), & \text{if } (t, \omega) \in (I_\epsilon \times \Omega) \cap S_1, \\ u_2(t, \omega), & \text{if } (t, \omega) \in (I_\epsilon \times \Omega) \cap S_2. \end{cases}$$

Let

$$E = \{(t, \omega) \in S_1 : \bar{u}(t, \omega) \neq u_1(t, \omega)\} \cup \{(t, \omega) \in S_2 : \bar{u}(t, \omega) \neq u_2(t, \omega)\}. \quad (3.2.12)$$

Then E is progressively measurable. Notice that if $\mathfrak{M}_2(E) = 0$, then $m_{k, \lambda}([0, \infty); E) = 0$ for all $k \in \mathbb{N}_+$.

In the remaining of this section, we denote by x^ϵ the controlled process corresponding to u^ϵ , and let $\xi^\epsilon = x^\epsilon - \bar{x}$.

Definition 3.2.9. The *first-order approximation* y^ϵ is defined to be the solution to the SDE

$$\begin{cases} dy^\epsilon(t) = \partial_x b_1(t) y^\epsilon(t) dt + \partial_x b_2(t) y^\epsilon(t) d\langle W \rangle_t \\ \quad + (\delta\sigma(t) + \partial_x \sigma(t) y^\epsilon(t)) dW_t, \\ y^\epsilon(0) = 0, \end{cases} \quad (3.2.13)$$

and the *second-order approximation* z^ϵ to be the solution to the SDE

$$\left\{ \begin{array}{l} dz^\epsilon(t) = \left[\partial_x b_1(t) z^\epsilon(t) + \delta b_1(t) + \frac{1}{2} \partial_x^2 b_1(t) y^\epsilon(t)^2 \right] dt \\ \quad + \left[\partial_x b_2(t) z^\epsilon(t) + \delta b_2(t) + \frac{1}{2} \partial_x^2 b_2(t) y^\epsilon(t)^2 \right] d\langle W \rangle_t \\ \quad + \left[\partial_x \sigma(t) z^\epsilon(t) + \delta(\partial_x \sigma)(t) y^\epsilon(t) + \frac{1}{2} \partial_x^2 \sigma(t) y^\epsilon(t)^2 \right] dW_t, \\ z^\epsilon(0) = 0, \end{array} \right. \quad (3.2.14)$$

where we have denoted

$$\varphi(t) = \varphi(t, \bar{x}(t), \bar{u}(t)), \quad \delta\varphi(t) = \varphi(t, \bar{x}(t), u^\epsilon(t)) - \varphi(t, \bar{x}(t), \bar{u}(t)),$$

for any function $\varphi : [0, \infty) \times \mathbb{R} \times \mathbb{U} \rightarrow \mathbb{R}$.

Lemma 3.2.10. *Let E be the progressively measurable set defined by (3.2.12). For each $k \in \mathbb{N}_+$, as $\epsilon \rightarrow 0$,*

$$\mathfrak{T}_{2k}(\xi^\epsilon) = \mathfrak{M}_1(E) O(|I_\epsilon|^k) + O(m_{k,\lambda}(I_\epsilon; E)), \quad (3.2.15)$$

$$\mathfrak{T}_{2k}(y^\epsilon) = \mathfrak{M}_1(E) O(|I_\epsilon|^k) + O(m_{k,\lambda}(I_\epsilon; E)), \quad (3.2.16)$$

$$\mathfrak{T}_{2k}(z^\epsilon) = \mathfrak{M}_1(E) O(|I_\epsilon|^{2k}) + O(m_{2k,\lambda}(I_\epsilon; E)), \quad (3.2.17)$$

$$\mathfrak{T}_{2k}(\xi^\epsilon(t) - y^\epsilon(t)) = \mathfrak{M}_1(E) O(|I_\epsilon|^{2k}) + O(m_{2k,\lambda}(I_\epsilon; E)), \quad (3.2.18)$$

$$\mathfrak{T}_{2k}(\xi^\epsilon(t) - y^\epsilon(t) - z^\epsilon(t)) = \mathfrak{M}_1(E) o(|I_\epsilon|^{2k}) + o(m_{2k,\lambda}(I_\epsilon; E)). \quad (3.2.19)$$

Proof. We only present the proof of (3.2.15) and (3.2.18), since the proof of (3.2.16) is similar to that of (3.2.15), while the proof of (3.2.17) and (3.2.19) are similar to that of (3.2.18). The difference between the proof of (3.2.15) and (3.2.18) is that the SDE for $\xi^\epsilon - y^\epsilon$ involves ξ^ϵ as bias terms $\alpha_1, \alpha_2, \beta$ in Lemma 3.2.6 (see (3.2.21)), which requires further estimate. This is also the case for z^ϵ and $\xi^\epsilon - y^\epsilon - z^\epsilon$, and hence their estimates are similar to that of $\xi^\epsilon - y^\epsilon$.

For any function $\varphi : [0, \infty) \times \mathbb{R} \times \mathbb{U} \rightarrow \mathbb{R}$, let

$$\tilde{\varphi}(t) = \int_0^1 \varphi[t, (1-\theta)\bar{x}(t) + \theta x^\epsilon(t), u^\epsilon(t)] d\theta, \quad t \geq 0.$$

By (3.2.1),

$$\begin{cases} d\xi^\epsilon(t) = [\widetilde{\partial_x b_1}(t)\xi^\epsilon(t) + \delta b_1(t)]dt + [\widetilde{\partial_x b_2}(t)\xi^\epsilon(t) + \delta b_2(t)]d\langle W \rangle_t \\ \quad + [\widetilde{\partial_x \sigma}(t)\xi^\epsilon(t) + \delta \sigma(t)]dW_t, \quad t \geq 0, \\ \xi^\epsilon(0) = 0. \end{cases}$$

Let $E_\epsilon = E \cap (I_\epsilon \times \Omega)$. Then $\text{supp}(\delta\varphi) \subseteq E_\epsilon$, $\varphi = b_1, b_2, \sigma$. Since b_1, b_2, σ are bounded, we see that

$$\mathbb{E}_\lambda \left[\left(\int_0^T |\delta b_1(t)| dt \right)^{4k} \right] = \mathfrak{M}_1(E) O(|I_\epsilon|^k).$$

Hence, by Lemma 3.2.6,

$$\begin{aligned} & \mathbb{E}_\lambda \left(\sup_{0 \leq t \leq T} |\xi^\epsilon(t)|^{4k} e_t^{-1} \right) \\ & \leq C \mathbb{E}_\lambda \left[\left(\int_0^T |\delta b_1(t)| dt \right)^{4k} + \left(\int_0^T |\delta b_2(t)| d\langle W \rangle_t \right)^{4k} + \left(\int_0^T |\delta \sigma(t)|^2 d\langle W \rangle_t \right)^{2k} \right] \\ & = \mathfrak{M}_1(E) O(|I_\epsilon|^k) + O(m_{k,\lambda}(I_\epsilon; E)), \end{aligned}$$

where $C > 0$ denotes a constant depending only on k, M , but might be different at various appearances. This completes the proof of (3.2.15). The proof of (3.2.16) is similar.

We now turn to the proof of (3.2.18). By the definition of $\tilde{\varphi}(t)$, we have

$$\tilde{\varphi}(t) - \varphi(t) = \delta\varphi(t) + O(|\xi^\epsilon|) = 1_{E_\epsilon}(t)O(1) + O(|\xi^\epsilon|). \quad (3.2.20)$$

Let $\eta^\epsilon = \xi^\epsilon - y^\epsilon$, and

$$\chi_1(t) = 1_{E_\epsilon}(t)O(|\xi^\epsilon(t)|) + O(|\xi^\epsilon(t)|^2).$$

Then, by (3.2.20) and the fact that $\delta\varphi = 1_{E_\epsilon}$ for $\varphi = b_1, b_2, \sigma$, we have

$$\begin{aligned}
d\eta^\epsilon &= [\partial_x b_1(t)\eta^\epsilon + 1_{E_\epsilon}(t)\mathcal{O}(1) + \chi_1(t)] dt \\
&+ [\partial_x b_2(t)\eta^\epsilon + 1_{E_\epsilon}(t)\mathcal{O}(1) + \chi_1(t)] d\langle W \rangle_t \\
&+ [\partial_x \sigma(t)\eta^\epsilon + \chi_1(t)] dW_t.
\end{aligned} \tag{3.2.21}$$

In order to apply Lemma 3.2.6, since the desired estimates involving $1_{E_\epsilon}(t)\mathcal{O}(1)$ follow directly from the definition, we need to estimate $\mathbb{E}_\lambda[(\int_0^T \chi_1(t) dt)^{2k}]$, $\mathbb{E}_\lambda[(\int_0^T \chi_1(t) d\langle W \rangle_t)^{2k}]$ and $\mathbb{E}_\lambda[(\int_0^T \chi_1(t)^2 d\langle W \rangle_t)^k]$.

We first estimate $\mathbb{E}_\lambda[(\int_0^T \chi_1(t) dt)^{2k}]$. For any $p > 1$, by the exponential integrability of $\langle W \rangle_T$ and (3.2.15),

$$\begin{aligned}
\mathbb{E}_\lambda\left[\left(\int_0^T 1_E(t)|\xi^\epsilon(t)| dt\right)^{2k}\right] &\leq \mathbb{E}_\lambda\left[\mathfrak{M}_1(E)|I_\epsilon|^{2k} e_T\left(\sup_{t\in[0,T]} |\xi^\epsilon(t)|^{2k} e_t^{-1}\right)\right] \\
&\leq C_p \mathfrak{M}_1(E)|I_\epsilon|^{2k} \mathbb{E}_\lambda\left(\sup_{t\in[0,T]} |\xi^\epsilon(t)|^{2pk} e_t^{-1}\right)^{1/p} \\
&\leq \mathfrak{M}_1(E)\mathcal{O}(|I_\epsilon|^{2k}),
\end{aligned} \tag{3.2.22}$$

where, to simplify notation, we allow the exponent parameters (i.e. k_1, k_2 in (3.2.9)) for e_t to be different from line to line, as long as this does not affect the asymptotic order. Moreover, for any $p > 1$,

$$\begin{aligned}
\mathbb{E}_\lambda\left[\left(\int_0^T |\xi^\epsilon(t)|^2 dt\right)^{2k}\right] &\leq \mathbb{E}_\lambda\left[e_T \sup_{t\in[0,T]} |\xi^\epsilon(t)|^{4k} e_t^{-1}\right] \\
&\leq C_p \mathbb{E}_\lambda\left(\sup_{t\in[0,T]} |\xi^\epsilon(t)|^{4pk} e_t^{-1}\right)^{1/p} \\
&\leq \mathcal{O}(m_{2pk,\lambda}(I_\epsilon; E)^{1/p}),
\end{aligned}$$

which implies that

$$\mathbb{E}_\lambda\left[\left(\int_0^T |\xi^\epsilon(t)|^2 dt\right)^{2k}\right] \leq \mathcal{O}(m_{2k,\lambda}(I_\epsilon; E)).$$

Therefore,

$$\mathbb{E}_\lambda \left[\left(\int_0^T \chi_1(t) dt \right)^{2k} \right] \leq \mathfrak{M}_1(E) O(|I_\epsilon|^{2k}) + O(m_{2k,\lambda}(I_\epsilon; E)). \quad (3.2.23)$$

Next, we estimate $\mathbb{E}_\lambda \left[\left(\int_0^T \chi_1(t) d\langle W \rangle_t \right)^{2k} \right]$. For any $p, q > 1$, by (3.2.15),

$$\begin{aligned} & \mathbb{E}_\lambda \left[\left(\int_0^T 1_{E_\epsilon}(t) |\xi^\epsilon(t)| d\langle W \rangle_t \right)^{2k} \right] \\ & \leq \mathbb{E}_\lambda \left[e_T^{2k} \left(\int_0^T 1_{E_\epsilon}(t) |\xi^\epsilon(t)| e_t^{-1} d\langle W \rangle_t \right)^{2k} \right] \\ & \leq C_p \mathbb{E}_\lambda \left[\left(\int_0^T 1_{E_\epsilon}(t) |\xi^\epsilon(t)| e_t^{-1} d\langle W \rangle_t \right)^{2pk} \right]^{1/p} \\ & \leq C_p \mathbb{E}_\lambda \left[\left(\int_0^T 1_{E_\epsilon}(t) d\langle W \rangle_t \right)^{2pk} \left(\sup_{t \in [0, T]} |\xi^\epsilon(t)| e_t^{-1} \right)^{2pk} \right]^{1/p} \\ & \leq C_p \mathbb{E}_\lambda \left[\left(\int_0^T 1_{E_\epsilon}(t) d\langle W \rangle_t \right)^{2pqk} \right]^{1/(pq)} \mathbb{E}_\lambda \left[\left(\sup_{t \in [0, T]} |\xi^\epsilon(t)| e_t^{-1} \right)^{2pq'k} \right]^{1/(pq')} \\ & \leq C_p m_{2pqk,\lambda}(I_\epsilon; E)^{1/(pq)} \mathfrak{T}_{4pq'k}(\xi^\epsilon)^{1/(pq')} \end{aligned}$$

which, in view of the fact that $\mathfrak{T}_{4pq'k}(\xi^\epsilon) = o(1)$, implies that

$$\mathbb{E}_\lambda \left[\left(\int_0^T 1_{E_\epsilon}(t) |\xi^\epsilon(t)| d\langle W \rangle_t \right)^{2k} \right] \leq o(m_{2k,\lambda}(I_\epsilon; E)). \quad (3.2.24)$$

Moreover, by the exponential integrability of $\langle W \rangle_T$ again,

$$\begin{aligned} & \mathbb{E}_\lambda \left[\left(\int_0^T |\xi^\epsilon(t)|^2 d\langle W \rangle_t \right)^{2k} \right] \\ & \leq \mathbb{E}_\lambda \left[e_T^{4k} \left(\int_0^T |\xi^\epsilon(t)| e^{-1} d\langle W \rangle_t \right)^{2k} \left(\sup_{t \in [0, T]} |\xi^\epsilon(t)| e_t^{-1} \right)^{2k} \right] \\ & \leq C_p \mathbb{E}_\lambda \left[\left(\int_0^T |\xi^\epsilon(t)| e^{-1} d\langle W \rangle_t \right)^{2pk} \left(\sup_{t \in [0, T]} |\xi^\epsilon(t)| e_t^{-1} \right)^{2pk} \right]^{1/p} \\ & \leq C_p \mathfrak{T}_{4pk}(\xi^\epsilon)^{1/p}, \end{aligned}$$

which, by (3.2.15), implies that

$$\mathbb{E}_\lambda \left[\left(\int_0^T |\xi^\epsilon(t)|^2 d\langle W \rangle_t \right)^{2k} \right] \leq \mathfrak{M}_1(E) O(|I_\epsilon|^{2k}) + O(m_{2k,\lambda}(I_\epsilon; E)).$$

Therefore,

$$\mathbb{E}_\lambda \left[\left(\int_0^T \chi_1(t) d\langle W \rangle_t \right)^{2k} \right] \leq \mathfrak{M}_1(E) O(|I_\epsilon|^{2k}) + O(m_{2k,\lambda}(I_\epsilon; E)). \quad (3.2.25)$$

We now estimate $\mathbb{E}_\lambda \left[\left(\int_0^T \chi_1(t)^2 d\langle W \rangle_t \right)^k \right]$. Similarly to the above, for any $p, q > 1$,

$$\begin{aligned} & \mathbb{E}_\lambda \left[\left(\int_0^T 1_{E_\epsilon}(t) |\xi^\epsilon(t)|^2 d\langle W \rangle_t \right)^k \right] \\ & \leq \mathbb{E}_\lambda \left[e_T^k \left(\int_0^T 1_{E_\epsilon}(t) |\xi^\epsilon(t)|^2 e_t^{-1} d\langle W \rangle_t \right)^k \right] \\ & \leq C_p \mathbb{E}_\lambda \left[\left(\int_0^T 1_{E_\epsilon}(t) |\xi^\epsilon(t)|^2 e_t^{-1} d\langle W \rangle_t \right)^{pk} \right]^{1/p} \\ & \leq C_p \mathbb{E}_\lambda \left[\left(\int_0^T 1_{E_\epsilon}(t) d\langle W \rangle_t \right)^{pk} \left(\sup_{t \in [0, T]} |\xi^\epsilon(t)|^2 e_t^{-1} \right)^{pk} \right]^{1/p} \\ & \leq C_p \mathbb{E}_\lambda \left[\left(\int_0^T 1_{E_\epsilon}(t) d\langle W \rangle_t \right)^{2pk} \right]^{1/(2p)} \mathbb{E}_\lambda \left[\left(\sup_{t \in [0, T]} |\xi^\epsilon(t)|^2 e_t^{-1} \right)^{2pk} \right]^{1/(2p)} \\ & \leq C_p [m_{2pk,\lambda}(I_\epsilon; E) \mathfrak{F}_{4pk}(\xi^\epsilon)]^{1/2p} \end{aligned} \quad (3.2.26)$$

which implies that

$$\mathbb{E}_\lambda \left[\left(\int_0^T 1_{E_\epsilon}(t) |\xi^\epsilon(t)|^2 d\langle W \rangle_t \right)^k \right] \leq \mathfrak{M}_1(E) O(|I_\epsilon|^{2k}) + O(m_{2k,\lambda}(I_\epsilon; E)).$$

Moreover, for any $p > 1$, by Young's inequality,

$$\begin{aligned} & \mathbb{E}_\lambda \left[\left(\int_0^T |\xi^\epsilon(t)|^4 d\langle W \rangle_t \right)^k \right] \\ & \leq \mathbb{E}_\lambda \left[e_T^{2k} \left(\int_0^T |\xi^\epsilon(t)|^4 e_t^{-2} d\langle W \rangle_t \right)^k \right] \\ & \leq C_p \mathbb{E}_\lambda \left[\left(\int_0^T |\xi^\epsilon(t)| e_t^{-1} d\langle W \rangle_t \right)^{pk} \left(\sup_{t \in [0, T]} |\xi^\epsilon(t)|^3 e_t^{-1} \right)^{pk} \right]^{1/p} \\ & \leq C_p \mathbb{E}_\lambda \left[\left(\int_0^T |\xi^\epsilon(t)| e_t^{-1} d\langle W \rangle_t \right)^{4pk} + \left(\sup_{t \in [0, T]} |\xi^\epsilon(t)|^3 e_t^{-1} \right)^{4pk/3} \right]^{1/p} \\ & \leq C_p \mathfrak{F}_{4pk}(\xi^\epsilon)^{1/p}, \end{aligned}$$

which, together with (3.2.15), implies that

$$\mathbb{E}_\lambda \left[\left(\int_0^T |\xi^\epsilon(t)|^4 d\langle W \rangle_t \right)^k \right] \leq \mathfrak{M}_1(E) O(|I_\epsilon|^{2k}) + O(m_{2k,\lambda}(I_\epsilon; E)).$$

Hence,

$$\mathbb{E}_\lambda \left[\left(\int_0^T \chi_1(t)^2 d\langle W \rangle_t \right)^k \right] \leq \mathfrak{M}_1(E) O(|I_\epsilon|^{2k}) + O(m_{2k,\lambda}(I_\epsilon; E)). \quad (3.2.27)$$

With the estimates (3.2.23), (3.2.25), and (3.2.27), we are now in a position to apply Lemma 3.2.6 and deduce (3.2.18). The proof of (3.2.17) is similar to that of (3.2.18), except that in the derivation, we need to use both (3.2.15), (3.2.16), and (3.2.18).

The proof of (3.2.19) is also similar in essence to that of (3.2.18). By a second order Taylor expansion, it is not difficult to see that, for $\varphi = b_1, b_2, \sigma$,

$$\begin{aligned} & \varphi(t, x^\epsilon(t), u^\epsilon(t)) - \varphi(t) \\ &= \partial_x \varphi(t) \xi^\epsilon + \frac{1}{2} \partial_x^2 \varphi(t) (\xi^\epsilon)^2 + \delta \varphi(t) + \delta(\partial_x \varphi)(t) \xi^\epsilon + \delta(\partial_x^2 \varphi)(t) (\xi^\epsilon)^2 + O(|\xi^\epsilon|^3) \\ &= \partial_x \varphi(t) \xi^\epsilon + \frac{1}{2} \partial_x^2 \varphi(t) (\xi^\epsilon)^2 + \delta \varphi(t) + \delta(\partial_x \varphi)(t) \xi^\epsilon + 1_{E_\epsilon}(t) O(|\xi^\epsilon|^2) + O(|\xi^\epsilon|^3) \\ &= \partial_x \varphi(t) \xi^\epsilon + \frac{1}{2} \partial_x^2 \varphi(t) (y^\epsilon)^2 + \delta \varphi(t) + \delta(\partial_x \varphi)(t) y^\epsilon + 1_{E_\epsilon}(t) O(|\xi^\epsilon|^2) + O(|\xi^\epsilon|^3) \\ & \quad + 1_{E_\epsilon}(t) O(|\xi^\epsilon - y^\epsilon|) + O(|\xi^\epsilon - y^\epsilon| |y^\epsilon|) + O(|\xi^\epsilon - y^\epsilon|^2). \end{aligned}$$

Therefore,

$$\begin{aligned} & \varphi(t, x^\epsilon(t), u^\epsilon(t)) - \varphi(t) \\ &= \partial_x \varphi(t) \xi^\epsilon + \frac{1}{2} \partial_x^2 \varphi(t) (y^\epsilon)^2 + \delta \varphi(t) + 1_{E_\epsilon}(t) O(|\xi^\epsilon|) + O(|\xi^\epsilon|^3) \\ & \quad + 1_{E_\epsilon}(t) O(|\xi^\epsilon - y^\epsilon|) + O(|\xi^\epsilon - y^\epsilon| |y^\epsilon|) + O(|\xi^\epsilon - y^\epsilon|^2), \end{aligned} \quad (3.2.28)$$

for $\varphi = b_1, b_2$, and

$$\begin{aligned} & \sigma(t, x^\epsilon(t), u^\epsilon(t)) - \sigma(t) \\ &= \partial_x \sigma(t) \xi^\epsilon + \frac{1}{2} \partial_x^2 \sigma(t) (y^\epsilon)^2 + \delta \sigma(t) + \delta(\partial_x \sigma)(t) y^\epsilon + 1_{E_\epsilon}(t) O(|\xi^\epsilon|^2) + O(|\xi^\epsilon|^3) \\ & \quad + 1_{E_\epsilon}(t) O(|\xi^\epsilon - y^\epsilon|) + O(|\xi^\epsilon - y^\epsilon| |y^\epsilon|) + O(|\xi^\epsilon - y^\epsilon|^2). \end{aligned} \quad (3.2.29)$$

Let $\zeta^\epsilon = \xi^\epsilon - y^\epsilon - z^\epsilon$, and

$$\chi_2(t) = 1_{E_\epsilon}(t)O(|\xi^\epsilon|^2) + O(|\xi^\epsilon|^3) + 1_{E_\epsilon}(t)O(|\xi^\epsilon - y^\epsilon|) + O(|\xi^\epsilon - y^\epsilon||y^\epsilon|) + O(|\xi^\epsilon - y^\epsilon|^2).$$

Then, by substituting (3.2.28) and (3.2.29) into the SDE of ξ^ϵ , we have

$$\begin{aligned} d\zeta^\epsilon &= [\partial_x b_1(t)\zeta^\epsilon + 1_{E_\epsilon}(t)O(|\xi^\epsilon|) + \chi_2(t)]dt \\ &\quad + [\partial_x b_2(t)\zeta^\epsilon + 1_{E_\epsilon}(t)O(|\xi^\epsilon|) + \chi_2(t)]d\langle W \rangle_t \\ &\quad + [\partial_x b_2(t)\zeta^\epsilon + \chi_2(t)]dW_t. \end{aligned}$$

In view of (3.2.22) and (3.2.24), in order to apply Lemma 3.2.6, it suffices to estimate $\mathbb{E}_\lambda[(\int_0^T \chi_2(t) dt)^{2k}]$, $\mathbb{E}_\lambda[(\int_0^T \chi_2(t) d\langle W \rangle_t)^{2k}]$ and $\mathbb{E}_\lambda[(\int_0^T \chi_2(t)^2 d\langle W \rangle_t)^k]$, which can be done similarly to those of $\chi_1(t)$ in the above using the established estimates (3.2.15), (3.2.16), and (3.2.18). \square

Definition 3.2.11. Let (\bar{x}, \bar{u}) be a solution to the problem (P), and denote $\varphi(t) = \varphi(t, \bar{x}(t), \bar{u}(t))$ for $\varphi = b_1, b_2, f_1, f_2$. The *adjoint equations* are defined to be the BSDEs

$$\left\{ \begin{aligned} dp(t) &= -[\partial_x b_1(t)p(t) - \partial_x f_1(t)]dt \\ &\quad - [\partial_x b_2(t)p(t) + \partial_x \sigma(t)q(t) - \partial_x f_2(t)]d\langle W \rangle_t \\ &\quad + q(t)dW_t, \quad t \in [0, T], \mathbb{P}_\lambda\text{-a.s.}, \\ p(T) &= -\partial_x h(\bar{x}(T)), \end{aligned} \right. \quad (3.2.30)$$

and

$$\left\{ \begin{aligned} dP(t) &= -[2\partial_x b_1(t)P(t) + \partial_x^2 b_1(t)p(t) - \partial_x^2 f_1(t)]dt \\ &\quad - [(2\partial_x b_2(t) + \partial_x \sigma(t)^2)P(t) + \partial_x \sigma(t)Q(t) \\ &\quad + \partial_x^2 b_2(t)p(t) + \partial_x^2 \sigma(t)q(t) - \partial_x^2 f_2(t)]d\langle W \rangle_t \\ &\quad + Q(t)dW_t, \quad t \in [0, T], \mathbb{P}_\lambda\text{-a.s.}, \\ P(T) &= -\partial_x^2 h(\bar{x}(T)). \end{aligned} \right. \quad (3.2.31)$$

Remark 3.2.12. The adjoint equations (3.2.30) and (3.2.31) are introduced in order to exactly cancel the residuals y^ϵ, z^ϵ . This can be seen more clearly from the proof of Theorem

3.2.13.

Theorem 3.2.13. *Suppose that the assumptions in Assumption 3.2.3 are satisfied, and that (\bar{x}, \bar{u}) is a solution to (P). Let (p, q) and (P, Q) be solutions to the adjoint equations (3.2.30) and (3.2.31) respectively, and let $H_1(t, x, u)$, $H_2(t, x, u)$ be the Hamiltonians defined by*

$$H_1(t, x, u) = b_1(t, x, u)p(t) - f_1(t, x, u),$$

and

$$\begin{aligned} H_2(t, x, u) &= b_2(t, x, u)p(t) + \sigma(t, x, u)q(t) - f_2(t, x, u) \\ &\quad + \frac{1}{2}[\sigma(t, x, u) - \sigma(t, x, \bar{u}(t))]^2 P(t). \end{aligned}$$

Then

$$\begin{cases} H_1(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in \mathbb{U}} H_1(t, \bar{x}(t), u), & \mathfrak{M}_1\text{-a.e.}, \\ H_2(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in \mathbb{U}} H_2(t, \bar{x}(t), u), & \mathfrak{M}_2\text{-a.e.} \end{cases}$$

Proof. Let E be the progressively measurable set defined by (3.2.12). By definition of $J(\cdot)$,

$$\begin{aligned} &J(u^\epsilon) - J(\bar{u}) \\ &= \mathbb{E}_\lambda \left\{ \partial_x h(\bar{x}(T)) \xi^\epsilon(T) + \left(\int_0^1 \theta \partial_x^2 h(\bar{x}(T) + \theta \xi^\epsilon(T)) d\theta \right) \xi^\epsilon(T)^2 \right. \\ &\quad + \int_0^T \left[\delta f_1(t) + \partial_x f_1(t, \bar{x}(t), u^\epsilon(t)) \xi^\epsilon(t) \right. \\ &\quad \quad \left. + \left(\int_0^1 \theta \partial_x^2 f_1(t, \bar{x}(t) + \theta \xi^\epsilon(t), u^\epsilon(t)) d\theta \right) \xi^\epsilon(t)^2 \right] dt \\ &\quad + \int_0^T \left[\delta f_2(t) + \partial_x f_2(t, \bar{x}(t), u^\epsilon(t)) \xi^\epsilon(t) \right. \\ &\quad \quad \left. + \left(\int_0^1 \theta \partial_x^2 f_2(t, \bar{x}(t) + \theta \xi^\epsilon(t), u^\epsilon(t)) d\theta \right) \xi^\epsilon(t)^2 \right] d\langle W \rangle_t \left. \right\}. \end{aligned}$$

Notice that we have the following approximations

$$\xi^\epsilon = y^\epsilon + z^\epsilon + \mathfrak{M}_1(E) o(|I_\epsilon|) + o(m_{1,\lambda}(I_\epsilon; E)), \quad (3.2.32)$$

$$(\xi^\epsilon)^2 = (y^\epsilon)^2 + \mathfrak{M}_1(E) o(|I_\epsilon|) + o(m_{1,\lambda}(I_\epsilon; E)), \quad (3.2.33)$$

and

$$\begin{aligned} & \left(\int_0^1 \theta \varphi(t, \bar{x}(t) + \theta \xi^\epsilon(t), u^\epsilon(t)) d\theta \right) (\xi^\epsilon)^2 \\ &= \frac{1}{2} \varphi(t) (y^\epsilon)^2 + \mathfrak{M}_1(E) o(|I_\epsilon|) + o(m_{1,\lambda}(I_\epsilon; E)), \end{aligned} \quad (3.2.34)$$

for $\varphi = \partial_x^2 h, \partial_x^2 f_1, \partial_x^2 f_2$. The approximation (3.2.32) is straight forward. The approximation (3.2.33) follows from $|(\xi^\epsilon)^2 - (y^\epsilon)^2| \leq (|\xi|^\epsilon + |y^\epsilon|)|\xi^\epsilon - y^\epsilon|$ and (3.2.15), (3.2.16), (3.2.18) in Lemma 3.2.10. For (3.2.34), in view of $\text{supp}(\delta\varphi) \subseteq E_\epsilon = E \cap (I_\epsilon \times \Omega)$,

$$\begin{aligned} & \left(\int_0^1 \theta \varphi(t, \bar{x}(t) + \theta \xi^\epsilon(t), u^\epsilon(t)) d\theta \right) (\xi^\epsilon)^2 \\ &= \left(\int_0^1 \theta \varphi(t, \bar{x}(t), u^\epsilon(t)) d\theta \right) (\xi^\epsilon)^2 + O(|\xi^\epsilon|^3) \\ &= \left(\int_0^1 \theta \varphi(t) d\theta \right) (\xi^\epsilon)^2 + \frac{1}{2} \delta\varphi(\xi^\epsilon)^2 + O(|\xi^\epsilon|^3) \\ &= \frac{1}{2} \varphi(t) (\xi^\epsilon)^2 + O(1_{E_\epsilon} |\xi^\epsilon|^2) + O(|\xi^\epsilon|^3). \end{aligned}$$

By $\int_0^T 1_{E_\epsilon}(t) |\xi^\epsilon(t)|^2 dt \leq |I_\epsilon| e_T \sup_{t \in [0, T]} \xi^\epsilon(t) e_t^{-1}$ and Lemma 3.2.10, it is easily seen that

$$\mathbb{E}_\lambda \left(\int_0^T 1_{E_\epsilon}(t) |\xi^\epsilon(t)|^2 dt + \sup_{t \in [0, T]} |\xi^\epsilon(t)|^3 \right) = \mathfrak{M}_1(E) o(|I_\epsilon|) + o(m_{1,\lambda}(I_\epsilon; E)),$$

which yields the approximation (3.2.34). Therefore,

$$\begin{aligned} & J(u^\epsilon) - J(\bar{u}) \\ &= \mathbb{E}_\lambda \left\{ \partial_x h(\bar{x}(T))(y^\epsilon(T) + z^\epsilon(T)) + \frac{1}{2} \partial_x^2 h(\bar{x}(T)) y^\epsilon(T)^2 \right. \\ & \quad + \int_0^T \left[\delta f_1(t) + \partial_x f_1(t)(y^\epsilon(t) + z^\epsilon(t)) + \frac{1}{2} \partial_x^2 f_1(t) y^\epsilon(t)^2 \right] dt \\ & \quad \left. + \int_0^T \left[\delta f_2(t) + \partial_x f_2(t)(y^\epsilon(t) + z^\epsilon(t)) + \frac{1}{2} \partial_x^2 f_2(t) y^\epsilon(t)^2 \right] d\langle W \rangle_t \right\} \\ & \quad + \mathfrak{M}_1(E) o(|I_\epsilon|) + o(m_{1,\lambda}(I_\epsilon; E)), \end{aligned} \quad (3.2.35)$$

Next, we transform $\mathbb{E}_\lambda[\partial_x h(\bar{x}(T))(y^\epsilon(T) + z^\epsilon(T))]$ into a cumulative cost. By (3.2.30),

$$\begin{aligned}
& \mathbb{E}_\lambda[-\partial_x h(\bar{x}(T))(y^\epsilon(T) + z^\epsilon(T))] \\
&= \mathbb{E}_\lambda[p(T)(y^\epsilon(T) + z^\epsilon(T))] + \mathfrak{M}_1(E) o(|I_\epsilon|) + o(m_{1,\lambda}(I_\epsilon; E)) \\
&= \mathbb{E}_\lambda \left\{ \int_0^T \left(\delta b_1(t)p(t) + \partial_x f_1(t)(y^\epsilon(t) + z^\epsilon(t)) + \frac{1}{2} \partial_x^2 b_1(t)p(t)y^\epsilon(t)^2 \right) dt \right. \\
&\quad + \int_0^T \left(\delta b_2(t)p(t) + \delta \sigma(t)q(t) + \partial_x f_2(t)(y^\epsilon(t) + z^\epsilon(t)) \right. \\
&\quad \left. \left. + \frac{1}{2} [\partial_x^2 b_2(t)p(t)y^\epsilon(t)^2 + \partial_x^2 \sigma(t)q(t)] y^\epsilon(t)^2 + \delta(\partial_x \sigma)(t)q(t)y^\epsilon(t) \right) d\langle W \rangle_t \right\} \\
&\quad + \mathfrak{M}_1(E) o(|I_\epsilon|) + o(m_{1,\lambda}(I_\epsilon; E)). \tag{3.2.36}
\end{aligned}$$

Notice that the last integral term

$$\mathbb{E}_\lambda \left(\int_0^T \delta(\partial_x \sigma)(t)q(t)y^\epsilon(t) d\langle W \rangle_t \right)$$

in (3.2.36) is also of order $\mathfrak{M}_1(E) o(|I_\epsilon|) + o(m_{1,\lambda}(I_\epsilon; E))$. To see this, by Theorem 2.3.7, $\mathbb{E}_\lambda \left(\int_0^T q(t)^2 e_t d\langle W \rangle_t \right)$ is bounded. Therefore, for any $k \geq 2$, by $\text{supp}(\delta(\partial_x \sigma)) \subseteq E_\epsilon$, and the exponential integrability of $\langle W \rangle_T$, and Young's inequality,

$$\begin{aligned}
& \left| \mathbb{E}_\lambda \left(\int_0^T \delta(\partial_x \sigma)(t)q(t)y^\epsilon(t) d\langle W \rangle_t \right) \right| \\
&\leq C_k \mathbb{E}_\lambda \left(\int_0^T 1_{E_\epsilon}(t) y^\epsilon(t)^k e_t^{-1} d\langle W \rangle_t \right)^{1/k} \\
&\leq C_k \mathbb{E}_\lambda \left[\left(\int_0^T 1_{E_\epsilon}(t) d\langle W \rangle_t \right) \left(\sup_{t \in [0, T]} y^\epsilon(t)^k e_t^{-1} \right) \right]^{1/k} \\
&\leq C_k \mathbb{E}_\lambda \left[\left(\int_0^T 1_{E_\epsilon}(t) d\langle W \rangle_t \right)^2 \right]^{1/2} \mathbb{E}_\lambda \left[\left(\sup_{t \in [0, T]} y^\epsilon(t)^{2k} e_t^{-1} \right) \right]^{1/(2k)} \\
&\leq C_k m_{2,\lambda}(I_\epsilon; E)^{1/2} [\mathfrak{M}_1(E) o(|I_\epsilon|^{1/2}) + o(m_{k,\lambda}(I_\epsilon; E)^{1/(2k)})] \\
&\leq \mathfrak{M}_1(E) o(|I_\epsilon|) + o(m_{2,\lambda}(I_\epsilon; E)^{1/2}) + o(m_{k,\lambda}(I_\epsilon; E)^{1/k}) \\
&= \mathfrak{M}_1(E) o(|I_\epsilon|) + o(m_{1,\lambda}(I_\epsilon; E)).
\end{aligned}$$

Hence, the equality (3.2.36) can be further written as

$$\begin{aligned}
& \mathbb{E}_\lambda \left[-\partial_x h(\bar{x}(T))(y^\epsilon(T) + z^\epsilon(T)) \right] \\
&= \mathbb{E}_\lambda \left\{ \int_0^T \left[\delta b_1(t)p(t) + \partial_x f_1(t)(y^\epsilon(t) + z^\epsilon(t)) + \frac{1}{2} \partial_x^2 b_1(t)p(t)y^\epsilon(t)^2 \right] dt \right. \\
&\quad + \int_0^T \left[\delta b_2(t)p(t) + \delta \sigma(t)q(t) + \partial_x f_2(t)(y^\epsilon(t) + z^\epsilon(t)) \right. \\
&\quad \left. \left. + \frac{1}{2} \left[\partial_x^2 b_2(t)p(t)y^\epsilon(t)^2 + \partial_x^2 \sigma(t)q(t) \right] y^\epsilon(t)^2 \right] d\langle W \rangle_t \right\} \\
&\quad + \mathfrak{M}_1(E) o(|I_\epsilon|) + o(m_{1,\lambda}(I_\epsilon; E)).
\end{aligned} \tag{3.2.37}$$

Also, we transform $\mathbb{E}_\lambda[\partial_x^2 h(\bar{x}(T))y^\epsilon(T)^2]$ into a cumulative cost. By (3.2.31),

$$\begin{aligned}
& \mathbb{E}_\lambda \left[-\partial_x^2 h(\bar{x}(T))y^\epsilon(T)^2 \right] = \mathbb{E}_\lambda \left[P(T)y^\epsilon(T)^2 \right] \\
&= \mathbb{E}_\lambda \left\{ \int_0^T \left[\partial_x^2 f_1(t) - \partial_x^2 b_1(t)p(t) \right] y^\epsilon(t)^2 dt \right. \\
&\quad + \int_0^T \left(\left[\partial_x^2 f_2(t) - \partial_x^2 b_2(t)p(t) - \partial_x^2 \sigma(t)q(t) \right] y^\epsilon(t)^2 + \delta \sigma(t)^2 P(t) \right. \\
&\quad \left. \left. + \left[2\partial_x \sigma(t)P(t) + Q(t) \right] \delta \sigma(t)y^\epsilon(t) \right) d\langle W \rangle_t \right\}.
\end{aligned}$$

Similar to before, it can be shown that the term

$$\mathbb{E}_\lambda \left(\int_0^T \left[2\partial_x \sigma(t)P(t) + Q(t) \right] \delta \sigma(t)y^\epsilon(t) d\langle W \rangle_t \right)$$

is of order $\mathfrak{M}_1(E) o(|I_\epsilon|) + o(m_{1,\lambda}(I_\epsilon; E))$. Therefore,

$$\begin{aligned}
& \mathbb{E}_\lambda \left[-\partial_x^2 h(\bar{x}(T))y^\epsilon(T)^2 \right] \\
&= \mathbb{E}_\lambda \left\{ \int_0^T \left[\partial_x^2 f_1(t) - \partial_x^2 b_1(t)p(t) \right] y^\epsilon(t)^2 dt \right. \\
&\quad + \int_0^T \left[\left(\partial_x^2 f_2(t) - \partial_x^2 b_2(t)p(t) - \partial_x^2 \sigma(t)q(t) \right) y^\epsilon(t)^2 + \delta \sigma(t)^2 P(t) \right] d\langle W \rangle_t \left. \right\} \\
&\quad + \mathfrak{M}_1(E) o(|I_\epsilon|) + o(m_{1,\lambda}(I_\epsilon; E)).
\end{aligned} \tag{3.2.38}$$

Combining (3.2.35), (3.2.37), and (3.2.38), we arrive at

$$\begin{aligned}
& J(u^\epsilon) - J(\bar{u}) \\
&= \mathbb{E}_\lambda \left[\int_0^T \left(\delta f_1(t) - \delta b_1(t)p(t) \right) dt \right] \\
&+ \mathbb{E}_\lambda \left[\int_0^T \left(\delta f_2(t) - \delta b_2(t)p(t) - \delta \sigma(t)q(t) - \frac{1}{2} \delta \sigma(t)^2 P(t) \right) d\langle W \rangle_t \right] \\
&+ \mathfrak{M}_1(E) o(|I_\epsilon|) + o(m_{1,\lambda}(I_\epsilon; E)).
\end{aligned} \tag{3.2.39}$$

We now show that (3.2.39) and the optimality of \bar{u} implies that

$$\begin{cases} \delta f_1(t) - \delta b_1(t)p(t) \geq 0, & \mathfrak{M}_1\text{-a.e.}, \\ \delta f_2(t) - \delta b_2(t)p(t) - \delta \sigma(t)q(t) - \frac{1}{2} \delta \sigma(t)^2 P(t) \geq 0, & \mathfrak{M}_2\text{-a.e.} \end{cases} \tag{3.2.40}$$

Let \bar{u}_1 and \bar{u}_2 be progressively measurable processes such that

$$\begin{aligned}
H_1(t, \bar{x}(t), \bar{u}_1(t)) &= \max_{u \in \mathbb{U}} H_1(t, \bar{x}(t), u), \quad \mathfrak{M}_1\text{-a.e.}, \\
H_2(t, \bar{x}(t), \bar{u}_2(t)) &= \max_{u \in \mathbb{U}} H_2(t, \bar{x}(t), u), \quad \mathfrak{M}_2\text{-a.e.}
\end{aligned}$$

We first set $u_1 = \bar{u}_1$, $u_2 = \bar{u}$. Then $\mathfrak{M}_2(E) = 0$, and therefore $m_{2,\lambda}(I_\epsilon; E) = 0$. Moreover, (2.3.5) reduces to

$$J(u^\epsilon) - J(\bar{u}) = \mathbb{E}_\lambda \left[\int_0^T \left(\delta f_1(t) - \delta b_1(t)p(t) \right) dt \right] + \mathfrak{M}_1(E) o(|I_\epsilon|),$$

which clearly implies the first inequality in (3.2.40).

We now turn to the proof of the second inequality in (3.2.40). For any $a > 0$, let

$$E_a = \{(t, \omega) : H_2(t, \bar{x}(t), \bar{u}_2(t)) - H_2(t, \bar{x}(t), \bar{u}(t)) \geq a\}.$$

Set $u_1 = \bar{u}$, $u_2 = \bar{u}_2 1_{E_a} + \bar{u} 1_{E_a^c}$. Then $E = E_a$ and $\mathfrak{M}_1(E) = 0$. Therefore, (3.2.39)

reduces to

$$J(u^\epsilon) - J(\bar{u}) = \mathbb{E}_\lambda \left[\int_0^T \left(\delta f_2(t) - \delta b_2(t)p(t) - \delta \sigma(t)q(t) - \frac{1}{2} \delta \sigma(t)^2 P(t) \right) d\langle W \rangle_t \right] + o(m_{1,\lambda}(I_\epsilon; E_a)).$$

By the definition of E_a and u_2 , we have

$$\delta f_2(t) - \delta b_2(t)p(t) - \delta \sigma(t)q(t) - \frac{1}{2} \delta \sigma(t)^2 P(t) \leq -a, \quad \text{on } E_a.$$

Therefore,

$$0 \leq J(u^\epsilon) - J(\bar{u}) \leq -a m_{1,\lambda}(I_\epsilon; E_a) + o(m_{1,\lambda}(I_\epsilon; E_a)),$$

which clearly implies

$$\mathbb{E}_\lambda \left(\int_{I_\epsilon} 1_{E_a}(t, \omega) d\langle W \rangle_t \right) = 0, \quad \text{for all } I_\epsilon.$$

Therefore, $\mathfrak{M}_2(E_a) = \mathbb{E}_\lambda \left(\int_0^T 1_{E_a}(t, \omega) d\langle W \rangle_t \right) = 0$ in view of the arbitrariness of $\{I_\epsilon\}_{\epsilon>0}$.

This completes the proof. \square

3.3 An example: the linear regulator problems

In this section, we present a worked-out example of the control problem (P). Let λ be a Borel probability measure on \mathbb{S} and $\lambda \ll \mu$. Let $a > 0$ be a constant, and take $\mathbb{U} = \mathbb{R}$ as the decision space. We consider the following *linear regulator problem*, which has wide applications in mathematical finance and engineering (see [17, p. 23] and references therein).

$$\underset{u \in \mathcal{A}[0,1]}{\text{minimize}} \quad \mathbb{E}_\lambda \left(\frac{a}{2} \int_0^1 u(t)^2 dt + x(1)^2 \right), \quad (3.3.1)$$

subject to

$$\begin{cases} dx(t) = u(t)dt + u(t)d\langle W \rangle_t + u(t)dW_t, & t \in (0, 1], \\ x(0) = 1. \end{cases}$$

Suppose that (\bar{x}, \bar{u}) is an optimal pair for the problem (3.3.1). The adjoint equations are

$$\begin{cases} dp(t) = q(t)dW_t, & t \in [0, 1), \\ p(1) = -2\bar{x}(1), \end{cases} \quad (3.3.2)$$

and

$$\begin{cases} dP(t) = Q(t)dW_t, & t \in [0, 1), \\ P(1) = -2. \end{cases} \quad (3.3.3)$$

Clearly, $P(t) = -2$, $Q(t) = 0$ is the solution to (3.3.3). The Hamiltonians are

$$H_1(t, x, u) = up(t) - \frac{a}{2}u^2, \quad H_2(t, x, u) = u[p(t) + q(t)] - (u - \bar{u}(t))^2.$$

Let $\mathfrak{M}_1, \mathfrak{M}_2$ be the measures on $[0, \infty) \times \Omega$ given in Definition 3.2.7, and $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2$.

By Theorem 3.2.13,

$$\bar{u}(t) \frac{d\mathfrak{M}_1}{d\mathfrak{M}} = \frac{p(t)}{a} \frac{d\mathfrak{M}_1}{d\mathfrak{M}},$$

and

$$\bar{u}(t) \frac{d\mathfrak{M}_2}{d\mathfrak{M}} = \frac{p(t) + q(t) + 2\bar{u}(t)}{2} \frac{d\mathfrak{M}_2}{d\mathfrak{M}},$$

which implies that

$$q(t) = -p(t), \quad \mathfrak{M}_2\text{-a.e.}$$

It follows from the above and (3.3.2) that

$$p(t) = p(0) \exp\left(-W_t - \frac{1}{2}\langle W \rangle_t\right), \quad t \in [0, 1]. \quad (3.3.4)$$

Therefore, $(\bar{x}(\cdot), \bar{u}(\cdot), p(0))$ is given by the system

$$\begin{cases} d\bar{x}(t) = \frac{p(t)}{a}dt + \bar{u}(t)d\langle W \rangle_t + \bar{u}(t)dW_t, & t \in (0, 1], \\ \bar{x}(0) = 1, \quad \bar{x}(1) = -\frac{1}{2}p(1), \end{cases} \quad (3.3.5)$$

where $p(\cdot)$ is given by (3.3.4). Note that, compared to BSDEs, the system (3.3.5) takes the random variable $p(0)$ as a part of its solution so that the additional condition $\bar{x}(0) = 1$ is satisfied. Therefore, (3.3.5) is not a simple SDE or BSDE but a forward–backward type SDE. We now look for a solution to the form $\bar{x}(t) = \theta(t)p(t)$, where $\theta(t)$ is the solution to the BSDE of the form

$$\begin{cases} d\theta(t) = \xi_1(t)dt + \xi_2(t)d\langle W \rangle_t + \eta(t)dW_t, & t \in [0, 1), \\ \theta(1) = -\frac{1}{2}. \end{cases}$$

By Itô's formula,

$$d\bar{x}(t) = \xi_1(t)p(t)dt + [\xi_2(t) - \eta(t)]p(t)d\langle W \rangle_t + [\eta(t) - \theta(t)]p(t)dW_t, \quad t \in (0, 1].$$

Comparing the above with (3.3.5) gives that

$$\begin{aligned} \xi_1(t)p(t) \frac{d\mathfrak{M}_1}{d\mathfrak{M}} &= \frac{p(t)}{a} \frac{d\mathfrak{M}_1}{d\mathfrak{M}}, \\ [\xi_2(t) - \eta(t)]p(t) \frac{d\mathfrak{M}_2}{d\mathfrak{M}} &= \bar{u}(t) \frac{d\mathfrak{M}_2}{d\mathfrak{M}} = [\eta(t) - \theta(t)]p(t) \frac{d\mathfrak{M}_2}{d\mathfrak{M}}. \end{aligned}$$

Therefore,

$$\xi_1(t) = \frac{1}{a}, \quad \mathfrak{M}_1\text{-a.e.}$$

and

$$\xi_2(t) = 2\eta(t) - \theta(t), \quad \mathfrak{M}_2\text{-a.e.}$$

Furthermore, $\theta(t)$ is given by the BSDE

$$\begin{cases} d\theta(t) = \frac{1}{a}dt + [2\eta(t) - \theta(t)]d\langle W \rangle_t + \eta(t)dW_t, & t \in [0, 1], \\ \theta(1) = -\frac{1}{2}, \end{cases} \quad (3.3.6)$$

of which the solution (θ, η) exists and is unique by virtue of Theorem 2.3.7. For the moment, let us assume that $\theta(0) < 0$. Then by $\bar{x}(0) = \theta(0)p(0) = 1$, we have that $p(0) = 1/\theta(0)$ and

$$p(t) = \frac{1}{\theta(0)} \exp\left(-W_t - \frac{1}{2}\langle W \rangle_t\right). \quad (3.3.7)$$

The optimal pair (\bar{x}, \bar{u}) is given by

$$\bar{u}(t) = \frac{p(t)}{a} \frac{d\mathfrak{M}_1}{d\mathfrak{M}} + [\eta(t) - \theta(t)]p(t) \frac{d\mathfrak{M}_2}{d\mathfrak{M}}, \quad (3.3.8)$$

and

$$\begin{cases} d\bar{x}(t) = p(t) \left\{ \frac{1}{a}dt + [\eta(t) - \theta(t)]d\langle W \rangle_t + [\eta(t) - \theta(t)]dW_t \right\}, & t \in (0, 1], \\ \bar{x}(0) = 1, \end{cases} \quad (3.3.9)$$

where (θ, η) and p are given by (3.3.6) and (3.3.7).

It remains to show that $\theta(0) < 0$. Let $\Phi(t) = \exp(-2W_t - \langle W \rangle_t)$, $t \in [0, 1]$. By Itô's formula,

$$d\Phi(t) = \Phi(t)d\langle W \rangle_t - 2\Phi(t)dW_t.$$

Therefore,

$$d[\Phi(t)\theta(t)] = \frac{1}{a}\Phi(t)dt + \Phi(t)[\eta(t) - 2\theta(t)]dW_t,$$

which implies that $\Phi(t)\theta(t) - \frac{1}{a} \int_0^t \Phi(r)dr$ is a martingale. Therefore,

$$\begin{aligned} & \Phi(t)\theta(t) - \frac{1}{a} \int_0^t \Phi(r)dr \\ &= \mathbb{E}_\lambda \left(\Phi(1)\theta(1) - \frac{1}{a} \int_0^1 \Phi(r)dr \mid \mathcal{F}_t \right) \\ &= -\mathbb{E}_\lambda \left(\frac{1}{2}\Phi(1) + \frac{1}{a} \int_0^1 \Phi(r)dr \mid \mathcal{F}_t \right), \end{aligned}$$

which gives that

$$\theta(t) = -\Phi(t)^{-1} \mathbb{E}_\lambda \left(\frac{1}{2} \Phi(1) + \frac{1}{a} \int_t^1 \Phi(r) dr \mid \mathcal{F}_t \right).$$

This, together with the fact that $\Phi(t) > 0$, shows that $\theta(0) < 0$.

To summarise, we first solve the BSDE (3.3.6) to obtain $\theta(t)$. Then the optimal solution (\bar{x}, \bar{u}) is given by (3.3.8) and the BSDE (3.3.9) with $p(t)$ given by (3.3.7).

Chapter 4

Semi-linear Parabolic Equations on the Gasket

4.1 Introduction

Recently, there has been interest in the study of non-linear partial differential equations on fractals with non-linearities involving first-order derivatives (see e.g. [67, 6, 32, 29, 28, 30, 27, 46] and references therein). In this chapter, we establish the existence and uniqueness of solutions to the semi-linear parabolic PDEs on \mathbb{S} proposed in Section 2.4, and derive the space-time regularity of solutions. The non-linear PDEs studied in the aforementioned literature are different from those to be studied in this chapter (see Remark 4.4.1). A crucial ingredient of our argument is a new type of Sobolev inequality on the gasket (and the infinite gasket) involving different measures (which can be mutually singular). Moreover, we formulate and study Burgers equations on the gasket, which is an archetype of non-linear PDEs with non-Lipschitz coefficients, and also as a simplified model of flows in porous medium. The difficulty in our case is that there exists no suitable analogue of the Cole–Hopf transformation on the gasket. Instead we tackle the problem by using a Feynman–Kac representation and an iteration argument. We would like to draw the reader’s

attention to a related work [27] on Burgers equations, where the Cole–Hopf transformation remains valid. The main difference between our setting and that of [27] lies in the difference in definitions of Laplacians: The Laplacian used to formulate Burgers equations in [27] is called the Kirchhoff Laplacian, and is defined as $-\nabla\nabla^*$, while our Laplacian is given by $\mathcal{L} = -\nabla^*\nabla$, where $\nabla^* : \text{Dom}(\nabla^*) \cap L^2(\mathbb{S}; \nu) \rightarrow L^2(\mathbb{S}; \mu)$ is the adjoint of $\nabla : \mathcal{F}(\mathbb{S}) \rightarrow L^2(\mathbb{S}; \nu)$.

4.2 Sobolev inequality on the gasket

The objective of this section is to establish a Sobolev inequality involving different (probably mutually singular) measures on \mathbb{S} and $\hat{\mathbb{S}}$ (Theorem 4.2.6 and Theorem 4.2.11 respectively), which is crucial to our study of the PDE (2.4.2). A necessary and sufficient condition for the validity of this Sobolev inequality (Theorem 4.2.8 and Theorem 4.2.13) will be established as well.

To shed light on the motivation of these inequalities, consider the following simple parabolic PDE on \mathbb{S}

$$\partial_t u \, d\mu = \mathcal{L}u \, d\mu + \nabla u \, d\nu.$$

Let us assume that a weak solution u exists, and test the equation against the solution u to obtain

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2(\mu)}^2 = -\mathcal{E}(u(t), u(t)) + \langle u(t), \nabla u(t) \rangle_\nu,$$

From the above equality, and (1.2.21), and Young’s inequality, it follows that

$$\frac{d}{dt} \|u(t)\|_{L^2(\mu)}^2 \leq -\mathcal{E}(u(t), u(t)) + \|u(t)\|_{L^2(\nu)}^2.$$

For PDEs on Euclidean spaces, the measures μ and ν are equal to the Lebesgue measures, and therefore, the above differential inequality together with Grönwall’s inequality yields the energy estimates and the existence and uniqueness of solutions. However, on \mathbb{S} , the

measures μ and ν are mutually singular, and hence the L^2 -norms $\|\cdot\|_{L^2(\mu)}$ and $\|\cdot\|_{L^2(\nu)}$ are in general incomparable. Thus, Grönwall's inequality does not apply in this case. For PDEs involving gradients on \mathbb{S} , an appropriate comparison of $\|\cdot\|_{L^2(\mu)}$ and $\|\cdot\|_{L^2(\nu)}$ is necessary to obtaining energy estimates. In fact, for functions $u \in \mathcal{F}(\mathbb{S})$, the L^2 -norms $\|u\|_{L^2(\mu)}$ and $\|u\|_{L^2(\nu)}$ must be compared with the participation of (an arbitrarily small portion of) the Dirichlet energy $\mathcal{E}(u, u)$ (cf. Corollary 4.2.18). This type of comparison is possible due to the Sobolev inequality to be established in this section.

For convenience, the Dirichlet energy $\mathcal{E}(u, u)$ of $u \in \mathcal{F}(\mathbb{S})$ will be simply denoted by $\mathcal{E}(u)$, and C_* will always denote a generic universal constant which may be different on various occasions.

Definition 4.2.1. Let $S_{i,m} = 2^m \tau_i(\mathbb{S})$, $m, i \in \mathbb{Z}$. The energy of $u \in \mathcal{F}(\hat{\mathbb{S}})$ on $S_{i,m}$ is defined to be $\hat{\mathcal{E}}|_{S_{i,m}}(u) = (3/5)^m \mathcal{E}[(u \circ \tau_i \circ \mathbf{F}_1^{-m})|_{\mathbb{S}}]$.

Clearly, $\hat{\mathbb{S}}$ can be written as the union $\hat{\mathbb{S}} = \bigcup_{i \in \mathbb{Z}} S_{i,m}$ for each $m \in \mathbb{Z}$ with $\{S_{i,m}\}_i$ having disjoint interiors. Therefore, $\hat{\mathcal{E}}(u) = \sum_{i \in \mathbb{Z}} \hat{\mathcal{E}}|_{S_{i,m}}(u)$ for any $u \in \mathcal{F}(\hat{\mathbb{S}})$ in view of (1.2.13) and (1.2.15).

Definition 4.2.2. The constant $\delta_s > 0$ is defined by $1/\delta_s = 2/d_s - 1 = \log 5 / \log 3 - 1$.

The constant δ_s is introduced so that $5/3 = 3^{1/\delta_s}$. Therefore, for every i and m , by (1.2.9),

$$\begin{aligned} \text{osc}_{\mathbb{S}}(u \circ \tau_i \circ \mathbf{F}_1^{-m}) &\leq C_* \mathcal{E}[(u \circ \tau_i \circ \mathbf{F}_1^{-m})|_{\mathbb{S}}]^{1/2} \\ &= C_* (5/3)^{m/2} \hat{\mathcal{E}}|_{S_{i,m}}(u)^{1/2} = C_* \hat{\mu}(S_{i,m})^{1/(2\delta_s)} \hat{\mathcal{E}}|_{S_{i,m}}(u)^{1/2}, \end{aligned}$$

which implies that

$$\text{osc}_{S_{i,m}}(u) \leq C \hat{\mu}(S_{i,m})^{1/(2\delta_s)} \hat{\mathcal{E}}|_{S_{i,m}}(u)^{1/2}. \quad (4.2.1)$$

Definition 4.2.3. A subset $S \subseteq \hat{\mathbb{S}}$ is called a *dyadic triangle* if $S = S_{i,m}$ for some $m, i \in \mathbb{Z}$.

We are now in a position to formulate the main results of this section. Let $\hat{\sigma}$ be a Borel measure on $\hat{\mathbb{S}}$ satisfying the following condition: there exist constants $C_{\hat{\sigma}} \geq 1$ and $0 < \underline{\delta} \leq \bar{\delta} \leq \infty$, $\bar{\delta} \geq 1$ such that

$$\begin{cases} \hat{\sigma}(S) \leq C_{\hat{\sigma}} \hat{\mu}(S)^{1/\bar{\delta}}, & \text{if } 0 < \text{diam}(S) < 1, \\ \hat{\sigma}(S) \leq C_{\hat{\sigma}} \hat{\mu}(S)^{1/\underline{\delta}}, & \text{if } \text{diam}(S) \geq 1, \end{cases} \quad (\text{M.1})$$

for any dyadic triangle $S \subseteq \hat{\mathbb{S}}$, where $\text{diam}(A)$ denotes the diameter of $A \subseteq \hat{\mathbb{S}}$ with respect to the Euclidean metric.

Remark 4.2.4. (i) The restriction $\bar{\delta} \geq 1$ in the condition (M.1) is necessary in view of the countable additivity of measures.

(ii) We should point out that we only require (M.1) to be valid for dyadic triangles but not for general Borel sets. In fact, $\hat{\sigma}$ will be absolutely continuous with respect to $\hat{\mu}$ if (M.1) does hold for all Borel sets.

We would like to point out that the condition (M.1) is general enough to include many cases of interests, some important examples are listed below.

Example 4.2.5. (i) Dirac measures with $\underline{\delta} = \bar{\delta} = \infty$.

(ii) The Kusuoka measure $\hat{\mu}$ with $\underline{\delta} = 1$, $\bar{\delta} = \delta_s$ (cf. Corollary 4.2.9(b)).

(iii) Analogues on $\hat{\mathbb{S}}$ of $|x|^{-\theta} dx$ on \mathbb{R}^d with $0 \leq \theta < d$. For any cube $Q \subseteq \mathbb{R}^d$, we have $\int_Q |x|^{-\theta} dx \leq C |Q|^{1-\theta/d}$ for some constant $C > 0$ depending only on d . Therefore, the analogue on $\hat{\mathbb{S}}$ of $|x|^{-\theta} dx$ on \mathbb{R}^d would be a Borel measure $\hat{\sigma} \ll \hat{\mu}$ satisfying the condition (M.1) with $\underline{\delta}, \bar{\delta}$ given by $1/\underline{\delta} = 1/\bar{\delta} = 1 - \theta/d_s$. Here we have used d_s as the ‘‘Sobolev dimension’’ of $\hat{\mathbb{S}}$ (cf. Remark 4.2.7).

Theorem 4.2.6. *Let $1 \leq p \leq q \leq \infty$, $q \geq 2$. Suppose $\hat{\sigma}$ is a Borel measure on $\hat{\mathbb{S}}$ satisfying the condition (M.1). Then*

$$\|u\|_{L^q(\hat{\sigma})} \leq C \sum_{i=1,2} \hat{\mathcal{E}}(u)^{a_i/2} \|u\|_{L^p(\hat{\mu})}^{1-a_i}, \quad u \in \mathcal{F}(\hat{\mathbb{S}}), \quad (4.2.2)$$

where

$$a_1 = \left[\frac{1/p - 1/(q\bar{\delta})}{1/p + 1/(2\delta_s)} \right]^+, \quad a_2 = \left[\frac{1/p - 1/(q\delta)}{1/p + 1/(2\delta_s)} \right]^+, \quad (4.2.3)$$

and $C > 0$ is a constant depending only on the constant $C_{\hat{\sigma}}$ in (M.1). Moreover, if there exists a sequence $\{S_m\}_{m \in \mathbb{Z}}$ of dyadic triangles such that $\lim_{m \rightarrow -\infty} \text{diam}(S_m) = 0$ and $\lim_{m \rightarrow \infty} \text{diam}(S_m) = \infty$, and

$$1/\bar{\delta} = \lim_{m \rightarrow -\infty} \frac{\log \hat{\sigma}(S_m)}{\log \hat{\mu}(S_m)}, \quad 1/\delta = \lim_{m \rightarrow \infty} \frac{\log \hat{\sigma}(S_m)}{\log \hat{\mu}(S_m)}, \quad (M.2)$$

then the pair of exponents given by (4.2.3) is optimal in the following sense: If

$$\|u\|_{L^q(\hat{\sigma})} \leq C \sum_{i=1}^N \hat{\mathcal{E}}(u)^{b_i/2} \|u\|_{L^p(\hat{\mu})}^{1-b_i}, \quad u \in \mathcal{F}(\hat{\mathbb{S}}), \quad (4.2.4)$$

for some constants $b_i \in [0, 1]$, $1 \leq i \leq N$, $N \in \mathbb{N}_+$ and $C > 0$ independent of u , then

$$\min_i b_i \leq a_2 \leq a_1 \leq \max_i b_i.$$

Proof. Suppose first that $p \leq q < \infty$. Let $\hat{\mu}_m = \hat{\mu}(2^m \mathbb{S}) = 3^m$, $S_{i,m} = 2^m \tau_i(\mathbb{S})$ for any $m, i \in \mathbb{Z}$. Then $\hat{\mathbb{S}} = \bigcup_i S_{i,m}$. When $m \geq 0$, by (4.2.1), we have that

$$\begin{aligned} \int_{\hat{\mathbb{S}}} |u|^q d\hat{\sigma} &\leq 2^{q-1} \sum_i \left[\int_{S_{i,m}} \left| u - \frac{1}{\hat{\mu}_m} \int_{S_{i,m}} u d\hat{\mu} \right|^q d\hat{\sigma} + \left| \frac{1}{\hat{\mu}_m} \int_{S_{i,m}} u d\hat{\mu} \right|^q \hat{\sigma}(S_{i,m}) \right] \\ &\leq C^q \sum_i \left[\hat{\mu}_m^{q/(2\delta_s)} \hat{\mathcal{E}}|_{S_{i,m}}(u)^{q/2} \hat{\sigma}(S_{i,m}) + \frac{1}{\hat{\mu}_m^{q/p}} \left[\int_{S_{i,m}} |u|^p d\hat{\mu} \right]^{q/p} \hat{\sigma}(S_{i,m}) \right] \\ &\leq C^q \sum_i \left[\hat{\mu}_m^{q/(2\delta_s)+1/\delta} \hat{\mathcal{E}}|_{S_{i,m}}(u)^{q/2} + \hat{\mu}_m^{1/\delta-q/p} \left(\int_{S_{i,m}} |u|^p d\hat{\mu} \right)^{q/p} \right] \\ &\leq C^q \left\{ \hat{\mu}_m^{q/(2\delta_s)+1/\delta} \left[\sum_i \hat{\mathcal{E}}|_{S_{i,m}}(u) \right]^{q/2} + C^q \hat{\mu}_m^{1/\delta-q/p} \left[\sum_i \int_{S_{i,m}} |u|^p d\hat{\mu} \right]^{q/p} \right\} \\ &= C^q \left[\hat{\mu}_m^{q/(2\delta_s)+1/\delta} \hat{\mathcal{E}}(u)^{q/2} + \hat{\mu}_m^{1/\delta-q/p} \|u\|_{L^p(\hat{\mu})}^q \right], \end{aligned}$$

where and hereafter $C > 0$ denotes a generic constant depending only on the constant $C_{\hat{\sigma}}$

in (M.1). Therefore,

$$\|u\|_{L^q(\hat{\sigma})} \leq C \left[\hat{\mu}_m^{1/(2\delta_s)+1/(q\bar{\delta})} \hat{\mathcal{E}}(u)^{1/2} + \hat{\mu}_m^{1/(q\bar{\delta})-1/p} \|u\|_{L^p(\hat{\mu})} \right]. \quad (4.2.5)$$

Similarly, when $m \leq 0$, we have that

$$\|u\|_{L^q(\hat{\sigma})} \leq C \left[\hat{\mu}_m^{1/(2\delta_s)+1/(q\bar{\delta})} \hat{\mathcal{E}}(u)^{1/2} + \hat{\mu}_m^{1/(q\bar{\delta})-1/p} \|u\|_{L^p(\hat{\mu})} \right]. \quad (4.2.6)$$

Suppose $\hat{\mathcal{E}}(u)^{1/2} \geq \|u\|_{L^p(\hat{\mu})}$. Consider the following two cases:

Case 1: $p \leq q \leq p/\bar{\delta}$. Note that $p \leq p/\bar{\delta}$ forces $\bar{\delta} = 1$ and therefore $1/(q\bar{\delta}) = 1/p$, $a_1 = 0$.

Setting $m \rightarrow -\infty$ in (4.2.6) gives that

$$\|u\|_{L^q(\hat{\sigma})} \leq C \hat{\mathcal{E}}(u)^{a_1/2} \|u\|_{L^p(\hat{\mu})}^{1-a_1}.$$

Case 2: $q > p/\bar{\delta}$. Setting

$$m = \sup \{ m \leq 0 : \hat{\mu}_m^{1/(2\delta_s)+1/p} \leq \hat{\mathcal{E}}(u)^{-1/2} \|u\|_{L^p(\hat{\mu})} \leq 1 \} < \infty,$$

in (4.2.6), we obtain that

$$\|u\|_{L^q(\hat{\sigma})} \leq C \hat{\mathcal{E}}(u)^{a_1/2} \|u\|_{L^p(\hat{\mu})}^{1-a_1}, \text{ where } a_1 = \left[\frac{1/p - 1/(q\bar{\delta})}{1/p + 1/(2\delta_s)} \right]^+.$$

Suppose that $\hat{\mathcal{E}}(u)^{1/2} < \|u\|_{L^p(\hat{\mu})}$. We consider the two cases:

Case 1: $p \leq q \leq p/\bar{\delta}$. In this case, $a_2 = 0$. Setting $m = 0$ in (4.2.5) gives that

$$\|u\|_{L^q(\hat{\sigma})} \leq C \hat{\mathcal{E}}(u)^{a_2/2} \|u\|_{L^p(\hat{\mu})}^{1-a_2}.$$

Case 2: $q > p/\delta$. Setting

$$m = \inf \{m \geq 0 : \hat{\mu}_m^{1/(2\delta_s)+1/p} \geq \hat{\mathcal{E}}(u)^{-1/2} \|u\|_{L^p(\hat{\mu})} > 1\} < \infty,$$

in (4.2.5), we obtain that

$$\|u\|_{L^q(\hat{\sigma})} \leq C \hat{\mathcal{E}}(u)^{a_2/2} \|u\|_{L^p(\hat{\mu})}^{1-a_2}, \text{ where } a_2 = \left[\frac{1/p - 1/(q\delta)}{1/p + 1/(2\delta_s)} \right]^+.$$

This proves (4.2.2) for $q < \infty$. Setting $q \rightarrow \infty$ proves the case when $q = \infty$ as the constant C is independent of q .

Suppose in addition that the condition (M.2) is satisfied, we prove that (a_1, a_2) is the optimal pair of exponents. We first show that, for any dyadic triangle $S \subseteq \hat{\mathbb{S}}$, there exists an $h_S \in \mathcal{F}(\hat{\mathbb{S}})$ such that

$$C_*^{-1} \leq h_S \leq C_* \text{ on } S, \text{ supp}(h_S) \subseteq \tilde{S}, \text{ and } \hat{\mathcal{E}}(h_S) \leq C_* \hat{\mu}(S)^{-1/\delta_s}, \quad (4.2.7)$$

where $\tilde{S} = \{x \in \hat{\mathbb{S}} : \text{dist}(x, S) \leq \text{diam}(S)\}$.

To see this, suppose first that $S = 2^{-1}\mathbb{S}$ for some $m \in \mathbb{Z}$. Let h be the 1-harmonic function in \mathbb{S} with boundary value

$$h|_{\mathbb{V}_1}(x) = \begin{cases} 1, & \text{if } x = (0, 0), \\ 0, & \text{otherwise.} \end{cases}$$

Let $h(x) = h(-x)$ for $x \in -\mathbb{S}$, and $h(x) = 0$ for $x \in \hat{\mathbb{S}} \setminus [\mathbb{S} \cup (-\mathbb{S})]$. Then $h \in \mathcal{F}(\hat{\mathbb{S}})$ and satisfies (4.2.7). For a general dyadic triangle $S = 2^m \tau_i(\mathbb{S})$, $i, m \in \mathbb{Z}$, let $h_S = h \circ \tau_i^{-1} \circ \mathbf{F}_1^m$. Then $h_S \in \mathcal{F}(\hat{\mathbb{S}})$, and the property (4.2.7) follows from (1.2.13) and the self-similar property (1.2.15).

Suppose that (4.2.4) holds. Let $\{S_m\}_{m \in \mathbb{Z}}$ be the sequence of dyadic triangles in (M.2). For each $m \in \mathbb{Z}$, by the above, there exists an $h_m \in \mathcal{F}(\hat{\mathbb{S}})$ such that $h_m \sim 1$ on S_m ,

$\text{supp}(h_m) \subseteq \tilde{S}_m$ and $\hat{\mathcal{E}}(h_m) \lesssim \hat{\mu}(S_m)^{-1/\delta_s}$, where the notation $A \lesssim B$ means that $A \leq cB$ for some constant $c > 0$ independent of m , and $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$. In view of (M.2), it is easily seen that

$$\|h_m\|_{L^q(\hat{\sigma})} \simeq \hat{\mu}(S_m)^{1/(q\hat{\delta})}, \quad \|h_m\|_{L^p(\hat{\mu})} \simeq \hat{\mu}(S_m)^{1/p}, \quad m \rightarrow \infty,$$

and

$$\|h_m\|_{L^q(\hat{\sigma})} \simeq \hat{\mu}(S_m)^{1/(q\bar{\delta})}, \quad \|h_m\|_{L^p(\hat{\mu})} \simeq \hat{\mu}(S_m)^{1/p}, \quad m \rightarrow -\infty.$$

It follows from the above and (4.2.4) that

$$\hat{\mu}(S_m)^{1/(q\hat{\delta})} \lesssim \sum_{i=1}^N \hat{\mu}(S_m)^{-b_i/(2\delta_s) + (1-b_i)/p}, \quad m \rightarrow \infty,$$

and

$$\hat{\mu}(S_m)^{1/(q\bar{\delta})} \lesssim \sum_{i=1}^N \hat{\mu}(S_m)^{-b_i/(2\delta_s) + (1-b_i)/p}, \quad m \rightarrow -\infty,$$

These inequalities imply that $\min_i b_i \leq a_2 \leq a_1 \leq \max_i b_i$. \square

Remark 4.2.7. (i) Some comments are desired on the interpretation of the exponents appearing in the inequality (4.2.2). Recall that, on Euclidean space \mathbb{R}^d , the celebrated Gagliardo–Nirenberg inequality takes the form $\|D^j u\|_{L^q(\mathbb{R}^d)} \leq C \|D^m u\|_{L^r(\mathbb{R}^d)}^a \|u\|_{L^p(\mathbb{R}^d)}^{1-a}$, where $a \in [0, 1]$ is given by $\frac{1}{q} = \frac{j}{d} + \left(\frac{1}{r} - \frac{m}{d}\right)a + \frac{1-a}{p}$. The case corresponding to the setting of Dirichlet forms is the one when $j = 0$, $m = 1$ and $r = 2$, for which the exponent a is given by

$$a = \frac{1/p - 1/q}{1/p - 1/2 + 1/d}. \quad (4.2.8)$$

Some insights are gained by comparing (4.2.3) and (4.2.8):

(i.a) The exponents a_i , $i = 1, 2$ in (4.2.2) are determined by the harmonic structure on \hat{S} (or equivalently the Dirichlet form $\hat{\mathcal{E}}$), the configuration parameters $\underline{\delta}$ and $\bar{\delta}$ of the measure $\hat{\sigma}$, and the embedding parameters p and q .

(i.b) The ‘‘Sobolev dimension’’ d of $\hat{\mathbb{S}}$, if it exists, should depend only on the harmonic structure. This dependence is expressed in (4.2.3) as the denominator $1/p + 1/(2\delta_s)$. Comparing this to the denominator of (4.2.8), we see that the Sobolev dimension d should be given by $1/p - 1/2 + 1/d = 1/p + 1/(2\delta_s)$, i.e. $d = d_s$. This suggests the identification of the spectral dimension d_s as the effective Sobolev dimension of $\hat{\mathbb{S}}$.

(ii) The inequality (4.2.2) includes the analogue on $\hat{\mathbb{S}}$ of a specific case of the weighted Sobolev inequality on \mathbb{R}^d in [8]. The weighted Sobolev inequality established in [8] takes the form $\| |x|^\gamma u \|_{L^q(\mathbb{R}^d)} \leq C \| |x|^\alpha Du \|_{L^r(\mathbb{R}^d)}^a \| |x|^\beta u \|_{L^p(\mathbb{R}^d)}^{1-a}$, where $\alpha, \beta, \gamma < 0$ satisfy $1/r + \alpha/d > 0$, $1/p + \beta/d \geq 1/q + \gamma/d > 0$, and $\frac{1}{q} + \frac{\gamma}{d} = a\left(\frac{1}{r} + \frac{\alpha-1}{d}\right) + (1-a)\left(\frac{1}{p} + \frac{\beta}{d}\right)$. The case corresponding to setting of Dirichlet forms is the one when $\alpha = \beta = 0$, $r = 2$ and $1/p \geq 1/q + \gamma/d_s > 0$, for which the weighted inequality reads

$$\|u\|_{L^q(|x|^{\gamma q} dx)} \leq C \|Du\|_{L^2(dx)}^a \|u\|_{L^p(dx)}^{1-a}. \quad (4.2.9)$$

As remarked in Example 4.2.5(iii), the analogues on $\hat{\mathbb{S}}$ of $|x|^{\gamma q} dx$ on \mathbb{R}^d are Borel measures $\hat{\sigma}$ on $\hat{\mathbb{S}}$ satisfying the condition (M.1) with $\underline{\delta}, \bar{\delta}$ given by $1/\underline{\delta} = 1/\bar{\delta} = 1 + \gamma q/d_s$. Therefore, the analogue of (4.2.9) on $\hat{\mathbb{S}}$ should be $\|u\|_{L^q(\hat{\sigma})} \leq C \hat{\mathcal{E}}(u)^{a/2} \|u\|_{L^p(\hat{\mu})}^{1-a}$ with a given by $\frac{1}{q} + \frac{\gamma}{d_s} = a\left(\frac{1}{2} - \frac{1}{d_s}\right) + \frac{1-a}{p}$. This coincides with the result of (4.2.2) since the exponents for the measure $\hat{\sigma}$ are given by $a_1 = a_2 = \frac{1/p - 1/q - \gamma/d_s}{1/p + 1/d_s - 1/2} = a$.

According to Theorem 4.2.6, the condition (M.1) is sufficient for the derivation of the Sobolev inequality. The following theorem states that this condition is also necessary for the validity of the Sobolev inequality of the form (4.2.4) with $q < \infty$.

Theorem 4.2.8. *Let $\hat{\sigma}$ be a Borel measure on $\hat{\mathbb{S}}$. Suppose that there exist some constants $p, q \in (0, \infty)$, $b_i \in [0, 1]$, $1 \leq i \leq N$ and $C > 0$ such that (4.2.4) holds for all $u \in \mathcal{F}(\hat{\mathbb{S}})$. Then there exist constants $0 < \underline{\delta} \leq \bar{\delta} \leq \infty$ such that the condition (M.1) is satisfied.*

Proof. Suppose that (4.2.4) holds. For any dyadic triangle $S \subseteq \hat{\mathbb{S}}$, as shown in the proof of

Theorem 4.2.6, there exists a piecewise harmonic function $h_S \in \mathcal{F}(\hat{\mathbb{S}})$ such that

$$h_S \simeq 1 \text{ on } S, \text{ supp}(h_S) \subseteq \tilde{S}, \text{ and } \hat{\mathcal{E}}(h_S) \lesssim \hat{\mu}(S)^{-1/\delta_s},$$

where the notation \tilde{S} and the relations \lesssim and \simeq are the same as those in the proof of Theorem 4.2.6. Applying (4.2.4) to h_S gives that

$$\hat{\sigma}(S)^{1/q} \lesssim \sum_i \hat{\mu}(S)^{-b_i/(2\delta_s) + (1-b_i)/p}, \quad (4.2.10)$$

Since $q < \infty$, it follows from the above that

$$\sup \{ \hat{\sigma}(S) : S \text{ is a dyadic triangle with } \text{diam}(S) = 1 \} < \infty.$$

Therefore, the first part of (M.1) is satisfied with $\bar{\delta} = \infty$.

Furthermore, for any dyadic triangle S with $\text{diam}(S) \geq 1$, by (4.2.10), $\hat{\sigma}(S)^{1/q} \lesssim \hat{\mu}(S)^{1/p}$ as $\hat{\mu}(S) \geq 1$. Setting $\underline{\delta} = p/q$ completes the proof. \square

Applying Theorem 4.2.6 to the cases when $\hat{\sigma} = \hat{\mu}$ and when $\hat{\sigma} = \hat{\nu}$, we obtain the following.

Corollary 4.2.9. *Let $1 \leq p \leq q \leq \infty$, $q \geq 2$. Then*

(a) *The inequality (4.2.2) holds with $\hat{\sigma} = \hat{\mu}$ and $a_1 = a_2 = \frac{1/p-1/q}{1/p+1/(2\delta_s)} \in [0, 1)$. In particular,*

$$\max_{\hat{\mathbb{S}}} u \leq C \hat{\mathcal{E}}(u)^{a/2} \|u\|_{L^p(\hat{\mu})}^{1-a}, \quad u \in \mathcal{F}(\hat{\mathbb{S}}), \quad (4.2.11)$$

with $a = \frac{1/p}{1/p+1/(2\delta_s)}$. Conversely, the inequality (4.2.2) holds for all $u \in \mathcal{F}(\hat{\mathbb{S}})$ if and only if $a_1 = a_2 = \frac{1/p-1/q}{1/p+1/(2\delta_s)}$.

(b) *The inequality (4.2.2) holds with $\hat{\sigma} = \hat{\nu}$. The pair (a_1, a_2) given by (4.2.3) is optimal,*

where $\underline{\delta} = 1$ and $\bar{\delta} = \delta_s$.

Proof. The only thing needs a proof is that $\bar{\delta} = \delta_s$ in (b). Clearly,

$$1/\bar{\delta} = \inf_{\omega \in W_*} \liminf_{m \rightarrow \infty} \left[-\frac{1}{m \log 3} \log \nu(\mathbb{S}_{[\omega]_m}) \right].$$

We show that

$$\sup_{\omega \in W_*} \lim_{m \rightarrow \infty} [\text{tr}(\mathbf{A}_{[\omega]_m}^t \mathbf{P} \mathbf{A}_{[\omega]_m})]^{1/m} = (3/5)^2, \quad (4.2.12)$$

from which the conclusion follows immediately.

Let $\mathbf{Y}_i = \mathbf{P}^t \mathbf{A}_i \mathbf{P}$, $i = 1, 2, 3$. Then \mathbf{Y}_i , $i = 1, 2, 3$ have the same eigenvalues $\{0, 1/5, 3/5\}$. It is easily seen that $\mathbf{A}_{[\omega]_m}^t \mathbf{P} \mathbf{A}_{[\omega]_m} = \mathbf{Y}_{[\omega]_m}^t \mathbf{Y}_{[\omega]_m}$ for every $m \in \mathbb{N}_+$ and every $\omega \in W_m$, where $\mathbf{Y}_{[\omega]_m} = \mathbf{Y}_{\omega_m} \cdots \mathbf{Y}_{\omega_2} \mathbf{Y}_{\omega_1}$. Therefore, $\text{tr}(\mathbf{Y}_{[\omega]_m}^t \mathbf{Y}_{[\omega]_m}) \leq C_* (3/5)^{2m}$, which implies that

$$\sup_{\omega \in W_*} \lim_{m \rightarrow \infty} [\text{tr}(\mathbf{Y}_{[\omega]_m}^t \mathbf{Y}_{[\omega]_m})]^{1/m} \leq (3/5)^2.$$

For the reverse, let $\omega = 111 \dots \in W_*$. Then $\lim_{m \rightarrow \infty} [\text{tr}(\mathbf{Y}_{[\omega]_m}^t \mathbf{Y}_{[\omega]_m})]^{1/m} = (3/5)^2$.

This proves (4.2.12). \square

Remark 4.2.10. Setting $p = 1$, $q = 2$ in (4.2.11) gives the Nash inequality on $\hat{\mathbb{S}}$ (see [15, Theorem 4.1])

$$\|u\|_{L^2(\hat{\mu})}^{2+4/d_s} \leq C \hat{\mathcal{E}}(u) \|u\|_{L^1(\hat{\mu})}^{4/d_s}, \quad u \in \mathcal{F}(\hat{\mathbb{S}}).$$

Conclusions similar to that of Theorem 4.2.6 hold when the roles of $\hat{\sigma}$ and $\hat{\mu}$ are exchanged. More specifically, let $\hat{\sigma}$ be a Borel measure on $\hat{\mathbb{S}}$ satisfying the following condition: there exist constants $C_{\hat{\sigma}} \geq 1$ and $0 < \underline{\delta} \leq \bar{\delta} < \infty$ such that

$$\begin{cases} C_{\hat{\sigma}}^{-1} \hat{\mu}(S)^{1/\underline{\delta}} \leq \hat{\sigma}(S), & \text{if } 0 < \text{diam}(S) < 1, \\ C_{\hat{\sigma}}^{-1} \hat{\mu}(S)^{1/\bar{\delta}} \leq \hat{\sigma}(S), & \text{if } \text{diam}(S) \geq 1, \end{cases} \quad (\text{M}'1)$$

for any dyadic triangle $S \subseteq \hat{\mathbb{S}}$. For measures $\hat{\sigma}$ satisfying (M'.1), we have Theorem 4.2.11 and Theorem 4.2.13 below, of which the proofs will be omitted as they are similar to those of Theorem 4.2.6 and Theorem 4.2.8.

Theorem 4.2.11. *Let $1 \leq p \leq q \leq \infty$, $q \geq 2$. Suppose that $\hat{\sigma}$ is a Borel measure on $\hat{\mathbb{S}}$ satisfying the condition (M'.1). Then*

$$\|u\|_{L^q(\hat{\mu})} \leq C \sum_{i=1,2} \hat{\mathcal{E}}(u)^{a_i/2} \|u\|_{L^p(\hat{\sigma})}^{1-a_i}, \quad u \in \mathcal{F}(\hat{\mathbb{S}}), \quad (4.2.13)$$

where

$$a_1 = \left[\frac{1/(p\underline{\delta}) - 1/q}{1/(p\underline{\delta}) + 1/(2\underline{\delta}_s)} \right]^+, \quad a_2 = \left[\frac{1/(p\bar{\delta}) - 1/q}{1/(p\bar{\delta}) + 1/(2\bar{\delta}_s)} \right]^+, \quad (4.2.14)$$

and $C > 0$ is a constant depending only on the constant $C_{\hat{\sigma}}$ in (M'.1). Moreover, if there exists a sequence $\{S_m\}_{m \in \mathbb{Z}}$ of dyadic triangles such that $\lim_{m \rightarrow -\infty} \text{diam}(S_m) = 0$ and $\lim_{m \rightarrow \infty} \text{diam}(S_m) = \infty$, and

$$1/\underline{\delta} = \lim_{m \rightarrow -\infty} \frac{\log \hat{\sigma}(S_m)}{\log \hat{\mu}(S_m)}, \quad 1/\bar{\delta} = \lim_{m \rightarrow \infty} \frac{\log \hat{\sigma}(S_m)}{\log \hat{\mu}(S_m)}, \quad (\text{M'.2})$$

then the pair of exponents given by (4.2.14) is optimal in the following sense: if

$$\|u\|_{L^q(\hat{\mu})} \leq C \sum_{i=1}^N \hat{\mathcal{E}}(u)^{b_i/2} \|u\|_{L^p(\hat{\sigma})}^{1-b_i}, \quad u \in \mathcal{F}(\hat{\mathbb{S}}), \quad (4.2.15)$$

for some constants $b_i \in [0, 1]$, $1 \leq i \leq N$, $N \in \mathbb{N}_+$ and $C > 0$ independent of u , then

$$\min_i b_i \leq a_2 \leq a_1 \leq \max_i b_i.$$

Remark 4.2.12. Theorem 4.2.6 and Theorem 4.2.11 can be easily combined to yield the following

$$\|u\|_{L^q(\hat{\sigma}_2)} \leq C \sum_{i=1,2} \hat{\mathcal{E}}(u)^{a_i/2} \|u\|_{L^p(\hat{\sigma}_1)}^{1-a_i}, \quad u \in \mathcal{F}(\hat{\mathbb{S}}), \quad (4.2.16)$$

where $\hat{\sigma}_2$ satisfies (M.1) with $\bar{\delta} = \bar{\delta}_2$, $\underline{\delta} = \underline{\delta}_2$, $\hat{\sigma}_1$ satisfies (M'.1) with $\bar{\delta} = \bar{\delta}_1$, $\underline{\delta} = \underline{\delta}_1$, and

$$a_1 = \left[\frac{1/(p\underline{\delta}_1) - 1/(q\bar{\delta}_2)}{1/(p\underline{\delta}_1) + 1/(2\bar{\delta}_2)} \right]^+, \quad a_2 = \left[\frac{1/(p\bar{\delta}_1) - 1/(q\underline{\delta}_2)}{1/(p\bar{\delta}_1) + 1/(2\underline{\delta}_2)} \right]^+.$$

Theorem 4.2.13. *Let $\hat{\sigma}$ be a Borel measure on $\hat{\mathbb{S}}$. Suppose that there exist some constants $p, q \in (0, \infty)$, $b_i \in [0, 1]$, $1 \leq i \leq N$ and $C > 0$ such that (4.2.15) holds for all $u \in \mathcal{F}(\hat{\mathbb{S}})$. Then there exist constants $0 < \underline{\delta} \leq \bar{\delta} \leq \infty$ such that the condition (M'.1) is satisfied.*

Corollary 4.2.14. *The inequality (4.2.13) holds with $\hat{\sigma} = \hat{\nu}$. The pair (a_1, a_2) of exponents given by (4.2.14) is optimal, where the constants $\bar{\delta} = 1$ and $\underline{\delta}$ is given by $1/\underline{\delta} = 1/\delta_s + 2$.*

Remark 4.2.15. The value of $\underline{\delta}$ in Corollary 4.2.14 follows from the fact that

$$\inf_{\omega \in W_*} \lim_{m \rightarrow \infty} [\text{tr}(\mathbf{A}_{[\omega]_m}^t \mathbf{P} \mathbf{A}_{[\omega]_m})]^{1/m} = 3/25,$$

which will be postponed to Section 4.3 (cf. Corollary 4.3.26) in order to avoid digression from the main task of the current section.

We end this section with the corresponding Sobolev inequality on the compact gasket \mathbb{S} , whose proof shall be omitted. Let σ be a finite Borel measure on \mathbb{S} . For the compact gasket, only the first part of the condition (M.1) is relevant, i.e.

$$\sigma(\mathbb{S}_{[\omega]_m}) \leq C_\sigma \mu(\mathbb{S}_{[\omega]_m})^{1/\bar{\delta}}, \quad \text{for all } \omega \in W_* \text{ and all } m \in \mathbb{N}, \quad (4.2.17)$$

where $C_\sigma > 0$ and $\bar{\delta} \in [1, \infty]$ are constants depending only on the Borel measure σ . Similarly, we only need the first part of the condition (M'.1), i.e.

$$C_\sigma^{-1} \mu(\mathbb{S}_{[\omega]_m})^{1/\underline{\delta}} \leq \sigma(\mathbb{S}_{[\omega]_m}), \quad \text{for all } \omega \in W_* \text{ and all } m \in \mathbb{N}, \quad (4.2.18)$$

where $\underline{\delta} \in (0, \infty]$ is a constant depending only on the Borel measure σ .

Theorem 4.2.16. *Let $1 \leq p \leq q \leq \infty$, $q \geq 2$, and let σ be a finite Borel measure on \mathbb{S} .*

(a) *Suppose that σ satisfies (4.2.17). Then for any $u \in \mathcal{F}(\mathbb{S})$,*

$$\|u - c\|_{L^q(\sigma)} \leq C \mathcal{E}(u)^{a/2} \|u - c\|_{L^p(\mu)}^{1-a}. \quad (4.2.19)$$

where c is any constant satisfying $\min_{\mathbb{S}} u \leq c \leq \max_{\mathbb{S}} u$, and

$$a = \left[\frac{1/p - 1/(q\bar{\delta})}{1/p + 1/(2\delta_s)} \right]^+, \quad (4.2.20)$$

and $C > 0$ is a constant depending only on the constant C_σ in (4.2.17). Therefore, for any $u \in \mathcal{F}(\mathbb{S})$,

$$\|u\|_{L^q(\sigma)} \leq C \left[\mathcal{E}(u)^{a/2} \|u\|_{L^p(\mu)}^{1-a} + \|u\|_{L^p(\mu)} \right]. \quad (4.2.21)$$

Moreover, the exponent a given by (4.2.20) is optimal in the sense that if (4.2.19) holds for some $a \in [0, 1]$, then $a \geq \left[\frac{1/p - 1/(q\bar{\delta})}{1/p + 1/(2\delta_s)} \right]^+$.

(b) *Suppose that σ satisfies (4.2.18). Then the conclusions of (a) hold when σ and ν are exchanged and the exponent (4.2.20) is replaced by*

$$a = \left[\frac{1/(p\bar{\delta}) - 1/q}{1/(p\bar{\delta}) + 1/(2\delta_s)} \right]^+,$$

Remark 4.2.17. Setting $\sigma = \mu$, $\bar{\delta} = \delta = 1$ and $p = 1$, $q = 2$ in (4.2.21) gives the Nash inequality on \mathbb{S} (cf. [15, Theorem 4.4] or [35, Theorem 5.3.3])

$$\|u\|_{L^2(\mu)}^{2+4/d_s} \leq C [\mathcal{E}(u) + \|u\|_{L^1(\mu)}^2] \|u\|_{L^1(\mu)}^{4/d_s} \leq C [\mathcal{E}(u) + \|u\|_{L^2(\mu)}^2] \|u\|_{L^1(\mu)}^{4/d_s}, \quad u \in \mathcal{F}(\mathbb{S}),$$

from which an upper bound of the heat kernel follows automatically (cf. [10, Section 2.4] for more details).

Corollary 4.2.18. For any $u \in \mathcal{F}(\mathbb{S})$,

$$\|u\|_{L^2(\nu)} \leq C [\mathcal{E}(u)^{(d_s-1)/2} \|u\|_{L^2(\mu)}^{2-d_s} + \|u\|_{L^2(\mu)}]. \quad (4.2.22)$$

Therefore, for any $\epsilon > 0$,

$$\|u\|_{L^2(\nu)} \leq \epsilon \mathcal{E}(u)^{1/2} + C_\epsilon \|u\|_{L^2(\mu)},$$

where $C_\epsilon > 0$ is a constant depending only on ϵ .

4.3 Sobolev inequality on product gaskets

In this section, we consider products of Sierpinski gaskets and study Sobolev inequality involving singular measures on these spaces. The main difficulty in establishing these inequalities is that so far there is no appropriate analogue of the following Newton–Leibniz formula

$$f(x) - f(y) = \int_0^1 \langle \dot{\gamma}(s), \nabla f(\gamma(s)) \rangle ds, \quad x, y \in \mathbb{R}^n,$$

where $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ is the geodesic (parametrised by arc length) connecting x and y . To overcome this, our main idea is to exploit the self-similar property of Sierpinski spaces and derive the Sobolev inequality by an iteration argument.

Definition 4.3.1. For $n \in \mathbb{N}_+$, the n -fold (compact) Sierpinski gasket \mathbb{S}^n is defined to be the Cartesian product

$$\mathbb{S}^n = \underbrace{\mathbb{S} \times \dots \times \mathbb{S}}_{n \text{ folds}}$$

with the product topology, and the n -fold infinite Sierpinski gasket $\hat{\mathbb{S}}^n$ is defined similarly. Generic points in $\hat{\mathbb{S}}^n$ will be denoted as $\mathbf{x} = (x_1, \dots, x_n)$, $x_i \in \hat{\mathbb{S}}$, $1 \leq i \leq n$.

For any n -tuple $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}^n$, define the translation $\tau_{\mathbf{i}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$\tau_{\mathbf{i}} = \tau_{i_1} \times \cdots \times \tau_{i_n},$$

and define the contraction $\mathbf{F}_{\mathbf{i}} : \mathbb{S}^n \rightarrow \mathbb{S}^n$ by $\mathbf{F}_{\mathbf{i}} = \mathbf{F}_{(i_1, \dots, i_n)} = \mathbf{F}_{i_1} \times \cdots \times \mathbf{F}_{i_n}$; that is,

$$\mathbf{F}_{\mathbf{i}}(\mathbf{x}) = (\mathbf{F}_{i_1}(x_1), \dots, \mathbf{F}_{i_n}(x_n)), \quad \text{for all } \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{S}^n.$$

Remark 4.3.2. (i) Clearly, $\hat{\mathbb{S}}^n$ can be written as a countable union $\hat{\mathbb{S}}^n = \bigcup_{\mathbf{i} \in \mathbb{Z}^n} \tau_{\mathbf{i}}(\mathbb{S}^n)$ of sets with disjoint interiors.

(ii) Notice that the $\mathbf{F}_{i_1 \dots i_n}$ and $\mathbf{F}_{(i_1 \dots i_n)}$ are different notations: The former is a composition of contractions on \mathbb{S} , while the latter is the tensor product of these contractions.

Definition 4.3.3. For $n \in \mathbb{N}_+$, let

$$\mathbf{W}_* = \{\boldsymbol{\omega} = \omega_1 \omega_2 \omega_3 \dots : \omega_i \in \{1, 2, 3\}^n, i \in \mathbb{N}_+\}$$

be the family of infinite sequences $\boldsymbol{\omega} = \omega_1 \omega_2 \omega_3 \dots$ of n -tuples $\boldsymbol{\omega}_i = (\omega_{i1}, \dots, \omega_{in}) \in \{1, 2, 3\}^n$. For each $\boldsymbol{\omega} = \omega_1 \omega_2 \dots \in \mathbf{W}_*$ and $m \in \mathbb{N}_+$, we denote by

$$[\boldsymbol{\omega}]_m = \boldsymbol{\omega}_1 \dots \boldsymbol{\omega}_m = (\omega_{11}, \dots, \omega_{1n}) \dots (\omega_{m1}, \dots, \omega_{mn})$$

the truncation of $\boldsymbol{\omega}$ of length m , and define the map $\mathbf{F}_{[\boldsymbol{\omega}]_m} : \mathbb{S}^n \rightarrow \mathbb{S}^n$ by

$$\mathbf{F}_{[\boldsymbol{\omega}]_m} = \mathbf{F}_{\boldsymbol{\omega}_1 \dots \boldsymbol{\omega}_m} = \mathbf{F}_{\boldsymbol{\omega}_1} \circ \cdots \circ \mathbf{F}_{\boldsymbol{\omega}_m}.$$

Definition 4.3.4. The (n -fold) Hausdorff measure μ_n on \mathbb{S}^n is defined to be the product

$$\mu_n = \underbrace{\mu \times \cdots \times \mu}_{n \text{ folds}}.$$

The Hausdorff measure $\hat{\mu}_n$ on $\hat{\mathbb{S}}^n$ is defined similarly.

We start with two lemmas regarding elementary properties of gradients and harmonic functions in \mathbb{S} respectively.

Lemma 4.3.5. *Let $u \in \mathcal{F}(\mathbb{S})$. For any $\omega \in W_*$ and any $m \in \mathbb{N}_+$,*

(a)

$$\nabla(u \circ \mathbf{F}_{[\omega]_m}) = \left(\frac{3}{5}\right)^{m/2} (\nabla u \circ \mathbf{F}_{[\omega]_m}) \cdot \left[\frac{d(\nu \circ \mathbf{F}_{[\omega]_m})}{d\nu}\right]^{1/2}. \quad (4.3.1)$$

(b) For $\omega \in W_*$ and $m \in \mathbb{N}$,

$$\left(\frac{1}{15}\right)^m \leq \frac{d(\nu \circ \mathbf{F}_{[\omega]_m})}{d\nu} \leq \left(\frac{3}{5}\right)^m, \quad \nu\text{-a.e.} \quad (4.3.2)$$

Consequently, for any $r \geq 2$,

$$\begin{aligned} & \left(\frac{3^{r'/r}}{5}\right)^{m/r'} |\nabla u \circ \mathbf{F}_{[\omega]_m}| \cdot \left[\frac{d(\nu \circ \mathbf{F}_{[\omega]_m})}{d\nu}\right]^{1/r} \\ & \leq |\nabla(u \circ \mathbf{F}_{[\omega]_m})| \leq \left(\frac{3}{5}\right)^{m/r'} |\nabla u \circ \mathbf{F}_{[\omega]_m}| \cdot \left[\frac{d(\nu \circ \mathbf{F}_{[\omega]_m})}{d\nu}\right]^{1/r}. \end{aligned} \quad (4.3.3)$$

Proof. (a) For any $\omega, \omega' \in W_*$ and any $m, l \in \mathbb{N}_+$,

$$\begin{aligned} \int_{\mathbf{F}_{[\omega']_l}(\mathbb{S})} |\nabla(u \circ \mathbf{F}_{[\omega]_m})|^2 d\nu &= \left(\frac{5}{3}\right)^l \mathcal{E}(u \circ \mathbf{F}_{[\omega]_m} \circ \mathbf{F}_{[\omega']_l}) \\ &= \left(\frac{3}{5}\right)^m \int_{\mathbf{F}_{[\omega]_m} \circ \mathbf{F}_{[\omega']_l}(\mathbb{S})} |\nabla u|^2 d\nu \\ &= \left(\frac{3}{5}\right)^m \int_{\mathbf{F}_{[\omega']_l}(\mathbb{S})} |\nabla u \circ \mathbf{F}_{[\omega]_m}|^2 d(\nu \circ \mathbf{F}_{[\omega]_m}). \end{aligned}$$

Since the above equality holds for all sets of the form $\mathbf{F}_{[\omega']_l}(\mathbb{S})$, we deduce that

$$|\nabla(u \circ \mathbf{F}_{[\omega]_m})|^2 = \left(\frac{3}{5}\right)^m |\nabla u \circ \mathbf{F}_{[\omega]_m}|^2 \cdot \frac{d(\nu \circ \mathbf{F}_{[\omega]_m})}{d\nu}. \quad (4.3.4)$$

In particular,

$$|\nabla(h_1 \circ \mathbf{F}_i)| = \sqrt{\frac{3}{5}} |\nabla h_1 \circ \mathbf{F}_i| \cdot \frac{d(\nu \circ \mathbf{F}_i)}{d\nu}, \quad i = 1, 2, 3, \quad (4.3.5)$$

where h_1 is the harmonic function with boundary value $h_1|_{V_{0,0}} = 1_{\{p_1\}}$. Notice that, by (1.2.10), $h_1 \circ \mathbf{F}_1 = \frac{2}{5} + \frac{3}{5} h_1$ and $h_1 \circ \mathbf{F}_2 = h_1 \circ \mathbf{F}_3 = \frac{2}{5} h_1$. Therefore,

$$\nabla(h_1 \circ \mathbf{F}_1) = \frac{3}{5} \nabla h_1, \quad \nabla(h_1 \circ \mathbf{F}_2) = \nabla(h_1 \circ \mathbf{F}_3) = \frac{2}{5} \nabla h_1.$$

which implies that $\nabla(h_1 \circ \mathbf{F}_i) < 0$ ν -a.e., $i = 1, 2, 3$, since $\nabla h_1 < 0$ ν -a.e. (cf. Remark 1.2.26¹). It follows from (4.3.5) that

$$\nabla(h_1 \circ \mathbf{F}_i) = \sqrt{\frac{3}{5}} \nabla h_1 \circ \mathbf{F}_i \cdot \frac{d(\nu \circ \mathbf{F}_i)}{d\nu}, \quad i = 1, 2, 3.$$

By the above and induction, it is easily seen that (4.3.1) holds for h_1 . Moreover, (4.3.1) for general $u \in \mathcal{F}(\mathbb{S})$ follows from (4.3.1) for h_1 and polarisation of (4.3.4).

(b) We only need to prove (4.3.2). The inequality (4.3.3) follows from (4.3.2) immediately.

For any $\omega' \in W_*$ and any $l \in \mathbb{N}$, since \mathbf{Y}_i has eigenvalues $\{0, 1/5, 3/5\}$, we have

$$\begin{aligned} \int_{\mathbf{F}_{[\omega']_l}(\mathbb{S})} d(\nu \circ \mathbf{F}_{[\omega]_m}) &= \nu(\mathbf{F}_{[\omega]_m} \circ \mathbf{F}_{[\omega']_l}(\mathbb{S})) = \left(\frac{5}{3}\right)^{m+l} \operatorname{tr}(\mathbf{Y}_{[\omega']_l}^t \mathbf{Y}_{[\omega]_m}^t \mathbf{Y}_{[\omega]_m} \mathbf{Y}_{[\omega']_l}) \\ &\leq \left(\frac{5}{3}\right)^{m+l} \left(\frac{3}{5}\right)^{2m} \operatorname{tr}(\mathbf{Y}_{[\omega']_l}^t \mathbf{Y}_{[\omega']_l}) = \left(\frac{3}{5}\right)^m \nu(\mathbf{F}_{[\omega']_l}(\mathbb{S})). \end{aligned}$$

and similarly,

$$\int_{\mathbf{F}_{[\omega']_l}(\mathbb{S})} d(\nu \circ \mathbf{F}_{[\omega]_m}) \geq \left(\frac{5}{3}\right)^{m+l} \left(\frac{1}{5}\right)^{2m} \operatorname{trace}(\mathbf{Y}_{[\omega']_l}^t \mathbf{Y}_{[\omega']_l}) = \left(\frac{1}{15}\right)^m \nu(\mathbf{F}_{[\omega']_l}(\mathbb{S})).$$

Now (4.3.2) follows readily from the above and the Lebesgue differentiation theorem. \square

Lemma 4.3.6. *Let h_i , $i = 1, 2, 3$ be the harmonic functions in \mathbb{S} with boundary values*

¹Notice that the harmonic function h in Remark 1.2.26 is equal to $1 - h_1$ here.

$h_i|_{V_0} = 1_{\{p_i\}}$, $i = 1, 2, 3$. Then

(a) $h_1 + h_2 + h_3 = 1$, $|\nabla h_1|^2 + |\nabla h_2|^2 + |\nabla h_3|^2 = 3$ ν -a.e., and

$$|\nabla h_i| \leq \sqrt{2} \quad \nu\text{-a.e.}, \quad i = 1, 2, 3. \quad (4.3.6)$$

(b) $|\nabla h_i|$ has no strictly positive lower bound in any dyadic triangle $S = \mathbf{F}_{[\omega]_m}(\mathbb{S}) \subseteq \mathbb{S}$; that is,

$$\operatorname{ess\,inf}_S |\nabla h_i| = 0, \quad i = 1, 2, 3,$$

where the essential infimum is taken with respect to the Kusuoka measure ν .

Proof. (a) The two equalities in the statement are corollaries of the uniqueness of harmonic functions and the fact $\nu = \frac{1}{3}(\nu_{\langle h_1 \rangle} + \nu_{\langle h_2 \rangle} + \nu_{\langle h_3 \rangle})$. Since $h_1 + h_2 + h_3 = 1$, we have $\sum_{i=1,2,3} \nabla h_i = 0$, which together with $\sum_{i=1,2,3} |\nabla h_i|^2 = 3$ gives

$$3 = \sum_{i=1,2,3} |\nabla h_i|^2 = 2(|\nabla h_1|^2 + \nabla h_1 \nabla h_2 + |\nabla h_2|^2) \geq \frac{3}{2} |\nabla h_1|^2.$$

Therefore, $|\nabla h_i| \leq \sqrt{2}$ ν -a.e., $i = 1, 2, 3$.

(b) It suffices to prove $\operatorname{ess\,inf}_S |\nabla h_1| = 0$. We first show that $\operatorname{ess\,inf}_{\mathbb{S}} |\nabla h_1| = 0$. Let $\omega = 2333\dots \in W_*$. Then

$$\mathbf{Y}_{[\omega]_{m+1}} = \mathbf{Y}_3^m \mathbf{Y}_2 = \begin{bmatrix} \left(\frac{1}{5}\right)^{m+1} & -\frac{1}{30}\left(\frac{3}{5}\right)^m - \frac{3}{10}\left(\frac{1}{5}\right)^m & -\frac{1}{30}\left(\frac{3}{5}\right)^m + \frac{1}{10}\left(\frac{1}{5}\right)^m \\ -\left(\frac{1}{5}\right)^{m+1} & -\frac{1}{30}\left(\frac{3}{5}\right)^m + \frac{3}{10}\left(\frac{1}{5}\right)^m & -\frac{1}{30}\left(\frac{3}{5}\right)^m - \frac{1}{10}\left(\frac{1}{5}\right)^m \\ 0 & \frac{1}{15}\left(\frac{3}{5}\right)^m & \frac{1}{15}\left(\frac{3}{5}\right)^m \end{bmatrix}.$$

Therefore,

$$\frac{1}{\nu(\mathbf{F}_{[\omega]_{m+1}}(\mathbb{S}))} \int_{\mathbf{F}_{[\omega]_{m+1}}} |\nabla h_1|^2 d\nu = \frac{3}{2} \frac{\mathbf{e}_1^t \mathbf{Y}_{[\omega]_{m+1}}^t \mathbf{Y}_{[\omega]_{m+1}} \mathbf{e}_1}{\operatorname{tr}(\mathbf{Y}_{[\omega]_{m+1}}^t \mathbf{Y}_{[\omega]_{m+1}})} \leq 9^{-m+1}.$$

The above implies that

$$\nu\{|\nabla h_1| \leq 3^{-m+1}\} > 0.$$

Therefore, $\text{ess inf}_{\mathbb{S}} |\nabla h_1| = 0$.

Next, we show that $\text{ess inf}_{\mathbf{F}_i(\mathbb{S})} |\nabla h_1| = 0$, $i = 1, 2, 3$. As seen in the proof of Lemma 4.3.5(a), we have

$$\nabla(h_1 \circ \mathbf{F}_1) = \frac{3}{5}\nabla h_1, \quad \nabla(h_1 \circ \mathbf{F}_2) = \nabla(h_1 \circ \mathbf{F}_2) = \frac{2}{5}\nabla h_1.$$

This implies $\text{ess inf}_{\mathbb{S}} |\nabla(h_1 \circ \mathbf{F}_i)| = 0$. Since $\nu \circ \mathbf{F}_i$ and ν are equivalent measures by virtue of (4.3.2), it follows from (4.3.1) that

$$\text{ess inf}_{\mathbf{F}_i(\mathbb{S})} |\nabla h_1| = \text{ess inf}_{\mathbb{S}} |\nabla h_1 \circ \mathbf{F}_i| = 0.$$

By induction, we see that $\text{ess inf}_S |\nabla h_1| = 0$ holds for all dyadic triangles. This completes the proof of (b). \square

We now give the definition of Sobolev spaces $W^{1,r}$ on \mathbb{S}^n and $\hat{\mathbb{S}}^n$. For a generic point $\mathbf{x} = (x_1, \dots, x_i, \dots, x_n) \in \hat{\mathbb{S}}^n$, we denote

$$\mathbf{x}'_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

To simplify notations, we abuse notation to write $\mathbf{x} = (x_i, \mathbf{x}'_i)$ and

$$(\hat{\mu}_{i-1} \times \hat{\nu} \times \hat{\mu}_{n-i-1})(d\mathbf{x}) = (\hat{\nu} \times \hat{\mu}_{n-1})(dx_i, d\mathbf{x}'_i).$$

Definition 4.3.7. Let $r \geq 2$, and let $\mathcal{C}(\mathbb{S}^n)$ be the space of functions $u \in C(\mathbb{S}^n)$ such that $u(\cdot, \mathbf{x}'_i) \in \mathcal{F}(\mathbb{S}^n)$ for all $1 \leq i \leq n$ and $\mathbf{x}'_i \in \mathbb{S}^{n-1}$. For any $u \in \mathcal{C}(\mathbb{S}^n)$, define

$$\llbracket u \rrbracket_{W^{1,r}(\mathbb{S}^n)} = \left(\sum_{i=1}^n \int_{\mathbb{S}^{n-1}} |\nabla_i u(x_i, \mathbf{x}'_i)|^r (\nu \times \mu_{n-1})(dx_i, d\mathbf{x}'_i) \right)^{1/r},$$

with ∇_i being ∇ applied to the i -th component x_i . Let

$$\|u\|_{W^{1,r}(\mathbb{S}^n)} = \left(\|u\|_{L^r(\mathbb{S}^n; \mu_n)}^r + \llbracket u \rrbracket_{W^{1,r}(\mathbb{S}^n)}^r \right)^{1/r}.$$

The Sobolev space $W^{1,r}(\mathbb{S}^n)$ is defined to be the completion of

$$\{u \in C(\mathbb{S}^n) : \|u\|_{W^{1,r}(\mathbb{S}^n)} < \infty\}$$

with respect to the norm $\|\cdot\|_{W^{1,r}(\mathbb{S}^n)}$. Let

$$\begin{aligned} \llbracket u \rrbracket_{W^{1,r}(\hat{\mathbb{S}}^n)} &= \left[\sum_{i \in \mathbb{Z}^n} \llbracket (u \circ \tau_i)|_{\mathbb{S}^n} \rrbracket_{W^{1,r}(\mathbb{S}^n)}^r \right]^{1/r}, \\ \|u\|_{W^{1,r}(\hat{\mathbb{S}}^n)} &= \left(\|u\|_{L^r(\hat{\mathbb{S}}^n; \hat{\mu}_n)}^r + \llbracket u \rrbracket_{W^{1,r}(\hat{\mathbb{S}}^n)}^r \right)^{1/r}. \end{aligned}$$

Then Sobolev space $W^{1,r}(\hat{\mathbb{S}}^n)$ is defined to be the completion of

$$\{u \in C(\mathbb{S}^n) : \|u\|_{W^{1,r}(\hat{\mathbb{S}}^n)} < \infty\}$$

with respect to the norm $\|\cdot\|_{W^{1,r}(\hat{\mathbb{S}}^n)}$.

The proposition below states that the space $W^{1,r}(\mathbb{S}^n)$ is sufficiently large.

Proposition 4.3.8. *Let $n \geq 1$, $r \geq 2$. Then $W^{1,r}(\mathbb{S})$ contains all piecewise harmonic functions in \mathbb{S} . Moreover, $C(\mathbb{S}^n) \cap W^{1,r}(\mathbb{S}^n)$ is dense in $C(\mathbb{S}^n)$ with respect to the supremum norm.*

Proof. We first show that $W^{1,r}(\mathbb{S})$ contains all harmonic functions in \mathbb{S} . Let h_i , $i = 1, 2, 3$ be the harmonic functions with boundary values

$$h_i|_{V_0} = 1_{\{p_i\}}, \quad i = 1, 2, 3.$$

By Lemma 4.3.6, $\nabla h_i \in L^\infty(\mathbb{S}; \nu)$ and therefore $h_i \in W^{1,r}(\mathbb{S})$, $i = 1, 2, 3$. Furthermore,

$W^{1,r}(\mathbb{S})$ contains all harmonic functions in \mathbb{S} as any harmonic function is a linear combination of h_i , $i = 1, 2, 3$. By (4.3.3), we see that $W^{1,r}(\mathbb{S})$ contains all piecewise harmonic functions in \mathbb{S} .

Since the linear space generated by functions of the form

$$u(x) = u_1(x_1) \cdots u_n(x_n), \quad u_1, \dots, u_n \in C(\mathbb{S})$$

is dense in $C(\mathbb{S}^n)$, so is the linear space generated by functions of the above form with each u_i being piecewise harmonic. Clearly, $u_1(x_1) \cdots u_n(x_n) \in W^{1,r}(\mathbb{S}^n)$ when u_i , $i = 1, \dots, n$ are piecewise harmonic. This completes the proof. \square

The following Poincaré inequality on \mathbb{S}^n is the cornerstone of our arguments.

Lemma 4.3.9 (Poincaré inequality). *There exists a universal constant $C_* > 0$ (independent of n as well) such that*

$$\int_{\mathbb{S}^n} |u - [u]_{\mathbb{S}^n}|^2 d\mu_n \leq C_* \llbracket u \rrbracket_{W^{1,2}(\mathbb{S}^n)}^2, \quad u \in W^{1,2}(\mathbb{S}^n), \quad (4.3.7)$$

where $[u]_{\mathbb{S}^n} = \int_{\mathbb{S}^n} u d\mu_n$.

Proof. Let u_k be the function on \mathbb{S}^k defined by

$$u_k(x_1, \dots, x_k) = \int_{\mathbb{S}^{n-k}} u(x_1, \dots, x_n) \mu_{n-k}(dx_{k+1}, \dots, dx_n), \quad k = 0, 1, \dots, n.$$

In particular, $u_0 = [u]_{\mathbb{S}^n}$ and $u_n = u$. Clearly,

$$u_{k-1}(x_1, \dots, x_{k-1}) = \int_{\mathbb{S}} u_k(x_1, \dots, x_k) \mu(dx_k), \quad k = 1, \dots, n.$$

Moreover,

$$\int_{\mathbb{S}^n} |u - [u]_{\mathbb{S}^n}|^2 d\mu_n = \int_{\mathbb{S}^n} |u_n - u_0|^2 d\mu_n = \sum_{k=1}^n \int_{\mathbb{S}^k} |u_k - u_{k-1}|^2 d\mu_k \quad (4.3.8)$$

Now the Poincaré inequality (1.2.9) on \mathbb{S} gives that

$$\begin{aligned} \int_{\mathbb{S}} |u_k(x_1, \dots, x_k) - u_{k-1}(x_1, \dots, x_{k-1})|^2 \mu(dx_k) &\leq C_* \int_{\mathbb{S}} |\nabla_k u_k(x_1, \dots, x_k)|^2 \nu(dx_k) \\ &= C_* \int_{\mathbb{S}^{n-k+1}} |\nabla_k u(x_k, \mathbf{x}'_k)|^2 (\nu \times \mu_{n-k+1})(dx_k, \dots, dx_n), \quad k = 1, \dots, n. \end{aligned}$$

Therefore,

$$\int_{\mathbb{S}^k} |u_k - u_{k-1}|^2 d\mu_k \leq C_* \int_{\mathbb{S}^n} |\nabla_k u(x_k, \mathbf{x}'_k)|^2 (\nu \times \mu_{n-1})(dx_k, d\mathbf{x}'_k), \quad k = 1, \dots, n.$$

This, together with (4.3.8), completes the proof. \square

We can now prove the first inequality for Sobolev functions, which is the key technical ingredient for the derivation of the Sobolev inequality on product Sierpinski spaces.

Lemma 4.3.10. *Let $u \in C(\mathbb{S}^n) \cap W^{1,r}(\mathbb{S}^n)$, $n \geq 1$. If $r > 1 + (n-1)\delta_s$ and $r \geq 2$, then*

$$\text{osc}_{\mathbb{S}^n}(u) \leq C_n \|u\|_{W^{1,r}(\mathbb{S}^n)} \quad (4.3.9)$$

for some constant $C_n > 0$ depending only on n .

Remark 4.3.11. The assumption $r \geq 2$ is only to guarantee that the gradient ∇u has a proper definition. This is also the reason for having $r \geq 2$ in most of the results in this chapter.

Proof. For any $\omega = \omega_1 \omega_2 \omega_3 \cdots \in \mathbf{W}_*$ with $\omega_j = (\omega_{1j}, \dots, \omega_{nj}) \in \{1, 2, 3\}^n$, $j \in \mathbb{N}_+$, denote

$$\omega_i = \omega_{i1} \omega_{i2} \omega_{i3} \cdots \in \mathbf{W}_*, \quad 1 \leq i \leq n, \quad (4.3.10)$$

and

$$\omega'_i = (\omega_{11}, \dots, \omega_{1,i-1}, \omega_{1,i+1}, \dots, \omega_{1n})(\omega_{21}, \dots, \omega_{2,i-1}, \omega_{2,i+1}, \dots, \omega_{2n}) \cdots \quad (4.3.11)$$

By the Poincaré inequality (4.3.7) and (4.3.3),

$$\begin{aligned}
& \int_{\mathbb{S}^n} |u \circ \mathbf{F}_{[\omega]_m} - [u \circ \mathbf{F}_{[\omega]_m}]_{\mathbb{S}^n}|^2 d\mu_n \\
& \leq C_* \sum_{i=1}^n \int_{\mathbb{S}^n} |\nabla_i (u \circ \mathbf{F}_{[\omega]_m})(x_i, \mathbf{x}'_i)|^2 (\nu \times \mu_{n-1})(dx_i, d\mathbf{x}'_i) \\
& = C_* \left(\frac{3}{5}\right)^m \sum_{i=1}^n \int_{\mathbb{S}^n} |\nabla_i u \circ \mathbf{F}_{[\omega]_m}(x_i, \mathbf{x}'_i)|^2 [(\nu \circ \mathbf{F}_{[\omega]_m}) \times \mu_{n-1}](dx_i, d\mathbf{x}'_i) \\
& = C_* \left(\frac{3^n}{5}\right)^m \sum_{i=1}^n \int_{\mathbf{F}_{[\omega]_m}(\mathbb{S}^n)} |\nabla_i u(x_i, \mathbf{x}'_i)|^2 (\nu \times \mu_{n-1})(dx_i, d\mathbf{x}'_i) \\
& \leq C_* \left(\frac{3^n}{5}\right)^m \left[\sum_{i=1}^n \nu(\mathbf{F}_{[\omega]_m}(\mathbb{S})) \mu_{n-1}(\mathbf{F}_{[\omega]_m}(\mathbb{S}^{n-1})) \right]^{1-2/r} \llbracket u \rrbracket_{W^{1,r}(\mathbb{S}^n)}^2.
\end{aligned}$$

Since $\nu(\mathbf{F}_{[\omega]_m}(\mathbb{S})) \leq (3/5)^m$, we deduce that

$$\begin{aligned}
& \frac{1}{\mu_n(\mathbf{F}_{[\omega]_m}(\mathbb{S}^n))} \int_{\mathbf{F}_{[\omega]_m}(\mathbb{S}^n)} |u - [u]_{\mathbf{F}_{[\omega]_m}(\mathbb{S}^n)}| d\mu_n \\
& \leq \left[\int_{\mathbb{S}^n} |u \circ \mathbf{F}_{[\omega]_m} - [u \circ \mathbf{F}_{[\omega]_m}]_{\mathbb{S}^n}|^2 d\mu_n \right]^{1/2} \\
& \leq C_n 3^{-m[1/\delta_s - (n-1+1/\delta_s)/r]} \llbracket u \rrbracket_{W^{1,r}(\mathbb{S}^n)}.
\end{aligned}$$

For any $\mathbf{x} \in \mathbb{S}^n$, choose $\omega \in \mathbf{W}_*$ such that $\mathbf{F}_{[\omega]_m}(\mathbb{S}^n) \rightarrow \mathbf{x}$ as $m \rightarrow \infty$. Let $[u]_m = [u]_{\mathbf{F}_{[\omega]_m}(\mathbb{S}^n)}$ for $m \in \mathbb{N}$. Then, by the above inequality, we have

$$\begin{aligned}
|[u]_m - [u]_{m+1}| & \leq \frac{1}{\mu_n(\mathbf{F}_{[\omega]_{m+1}}(\mathbb{S}^n))} \int_{\mathbf{F}_{[\omega]_{m+1}}(\mathbb{S}^n)} |u - [u]_m| d\mu_n \\
& \leq \frac{3^n}{\mu_n(\mathbf{F}_{[\omega]_m}(\mathbb{S}^n))} \int_{\mathbf{F}_{[\omega]_m}(\mathbb{S}^n)} |u - [u]_m| d\mu_n \\
& \leq C_n 3^{-m[1/(r'\delta_s) - (n-1)/r]} \llbracket u \rrbracket_{W^{1,r}(\mathbb{S}^n)}.
\end{aligned}$$

Since $1/(r'\delta_s) - (n-1)/r > 0$, we may conclude that

$$\begin{aligned}
|[u]_k - [u]_0| & \leq \sum_{m=0}^{k-1} |[u]_m - [u]_{m+1}| \\
& \leq C_n \sum_{m=0}^{\infty} 3^{-m[1/(r'\delta_s) - (n-1)/r]} \llbracket u \rrbracket_{W^{1,r}(\mathbb{S}^n)} \\
& = C_n \llbracket u \rrbracket_{W^{1,r}(\mathbb{S}^n)}.
\end{aligned}$$

Using the Lebesgue differentiation theorem and setting $k \rightarrow \infty$ gives

$$\left| u(x) - \int_{\mathbb{S}^n} u d\mu_n \right| \leq C_n \llbracket u \rrbracket_{W^{1,r}(\mathbb{S}^n)},$$

which implies (4.3.9). \square

Remark 4.3.12. When $n = 1$, $p = 2$, the inequality (4.3.9) reduces to the inequality (1.2.9) on \mathbb{S} .

Proposition 4.3.13. *Suppose that $u \in W^{1,r}(\mathbb{S}^n)$, $r > 1 + (n-1)\delta_s$. Then $u \in C(\mathbb{S}^n)$, and for any $\omega \in \mathbf{W}_*$,*

$$\text{osc}_{\mathbb{S}^n}(u \circ \mathbf{F}_{[\omega]_m}) \leq C_n 3^{-m\alpha_r} \llbracket u \rrbracket_{W^{1,r}(\mathbf{F}_{[\omega]_m}(\mathbb{S}^n))}, \quad (4.3.12)$$

where

$$\alpha_r = 1/(r'\delta_s) - (n-1)/r. \quad (4.3.13)$$

Proof. For any $\omega \in \mathbf{W}_*$ and any $m \in \mathbb{N}$, by (4.3.9),

$$\text{osc}_{\mathbb{S}^n}(u \circ \mathbf{F}_{[\omega]_m}) \leq C_n \llbracket u \circ \mathbf{F}_{[\omega]_m} \rrbracket_{W^{1,r}(\mathbb{S}^n)}.$$

By (4.3.3) and using the notation (4.3.10), we deduce that

$$\begin{aligned} & \llbracket u \circ \mathbf{F}_{[\omega]_m} \rrbracket_{W^{1,r}(\mathbb{S}^n)}^r \\ &= \sum_{i=1}^n \int_{\mathbb{S}^n} |\nabla_i(u \circ \mathbf{F}_{[\omega]_m})(x_i, \mathbf{x}'_i)|^r (\nu \times \mu_{n-1})(dx_i, d\mathbf{x}'_i) \\ &\leq 3^{-m(r-1)/\delta_s} \sum_{i=1}^n \int_{\mathbb{S}^n} |\nabla_i u \circ \mathbf{F}_{[\omega]_m}(x_i, \mathbf{x}'_i)|^r [(\nu \circ \mathbf{F}_{[\omega]_m}) \times \mu_{n-1}](dx_i, d\mathbf{x}'_i) \\ &\leq 3^{-m[(r-1)/\delta_s - n + 1]} \sum_{i=1}^n \int_{\mathbf{F}_{[\omega]_m}(\mathbb{S}^n)} |\nabla_i u(x_i, \mathbf{x}'_i)|^r (\nu \times \mu_{n-1})(dx_i, d\mathbf{x}'_i) \\ &= 3^{-m[(r-1)/\delta_s - n + 1]} \llbracket u \rrbracket_{W^{1,r}(\mathbf{F}_{[\omega]_m}(\mathbb{S}^n))}^r, \end{aligned}$$

which completes the proof. □

To proceed further, we shall need the following result for the Kusuoka measure.

Lemma 4.3.14. *There exists an $\omega \in W_*$ such that*

$$\lim_{k \rightarrow \infty} \frac{\nu \circ \mathbf{F}_1^m(\mathbf{F}_{[\omega]_k}(\mathbb{S}))}{\nu(\mathbf{F}_{[\omega]_k}(\mathbb{S}))} = \inf_{\omega \in W_*, k \in \mathbb{N}_+} \frac{\nu \circ \mathbf{F}_1^m(\mathbf{F}_{[\omega]_k}(\mathbb{S}))}{\nu(\mathbf{F}_{[\omega]_k}(\mathbb{S}))} = \left(\frac{1}{15}\right)^m. \quad (4.3.14)$$

Proof. By definition,

$$\frac{\nu \circ \mathbf{F}_1^m(\mathbf{F}_{[\omega]_k}(\mathbb{S}))}{\nu(\mathbf{F}_{[\omega]_k}(\mathbb{S}))} = \frac{\text{tr}((\mathbf{Y}_1^m)^t \mathbf{Y}_{[\omega]_k}^t \mathbf{Y}_{[\omega]_k} \mathbf{Y}_1^m)}{\text{tr}(\mathbf{Y}_{[\omega]_k}^t \mathbf{Y}_{[\omega]_k})}, \quad \omega \in W_*.$$

Since the eigenvalues of \mathbf{Y}_1 are $\{0, 1/5, 3/5\}$, we deduce that

$$\text{tr}((\mathbf{Y}_1^m)^t \mathbf{Y}_{[\omega]_k}^t \mathbf{Y}_{[\omega]_k} \mathbf{Y}_1^m) \geq 5^{-2m} \text{tr}(\mathbf{Y}_{[\omega]_k}^t \mathbf{Y}_{[\omega]_k}),$$

which implies that

$$\inf_{\omega \in W_*, k \in \mathbb{N}_+} \frac{\nu \circ \mathbf{F}_1^m(\mathbf{F}_{[\omega]_k}(\mathbb{S}))}{\nu(\mathbf{F}_{[\omega]_k}(\mathbb{S}))} \geq \left(\frac{1}{15}\right)^m.$$

To show the reverse inequality, let

$$\mathbf{Q} = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Then

$$\mathbf{Q}^t \mathbf{Y}_i \mathbf{Q} = \begin{bmatrix} \mathbf{M}_i & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 1, 2, 3,$$

where

$$\mathbf{M}_1 = \begin{bmatrix} \frac{3}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} \frac{3}{10} & -\frac{\sqrt{3}}{10} \\ -\frac{\sqrt{3}}{10} & \frac{1}{2} \end{bmatrix}, \quad \mathbf{M}_3 = \begin{bmatrix} \frac{3}{10} & \frac{\sqrt{3}}{10} \\ \frac{\sqrt{3}}{10} & \frac{1}{2} \end{bmatrix}. \quad (4.3.15)$$

Denote

$$\mathbf{M}_{[\omega]_k} = \begin{bmatrix} M(k)_{11} & M(k)_{12} \\ M(k)_{21} & M(k)_{22} \end{bmatrix}, \quad k \in \mathbb{N}_+.$$

Then

$$\begin{aligned} \frac{\operatorname{tr}((\mathbf{Y}_1^m)^t \mathbf{Y}_{[\omega]_k}^t \mathbf{Y}_{[\omega]_k} \mathbf{Y}_1^m)}{\operatorname{tr}(\mathbf{Y}_{[\omega]_k}^t \mathbf{Y}_{[\omega]_k})} &= \frac{\operatorname{tr}((\mathbf{M}_1^m)^t \mathbf{M}_{[\omega]_k}^t \mathbf{M}_{[\omega]_k} \mathbf{M}_1^m)}{\operatorname{tr}(\mathbf{M}_{[\omega]_k}^t \mathbf{M}_{[\omega]_k})} \\ &= \left(\frac{3}{5}\right)^{2m} \frac{\sum_i M(k)_{i1}^2}{\sum_{i,j} M(k)_{ij}^2} + \left(\frac{1}{5}\right)^{2m} \frac{\sum_i M(k)_{i2}^2}{\sum_{i,j} M(k)_{ij}^2}. \end{aligned}$$

Let $\omega = 2333\dots \in W_*$. By simple computation,

$$\mathbf{M}_{[\omega]_k} = \mathbf{M}_3^{k-1} \mathbf{M}_2 = \begin{bmatrix} \frac{3}{10} \cdot \left(\frac{1}{5}\right)^{k-1} & \frac{\sqrt{3}}{10} \cdot \left(\frac{3}{5}\right)^{k-1} - \frac{2\sqrt{3}}{10} \cdot \left(\frac{1}{5}\right)^{k-1} \\ -\frac{\sqrt{3}}{10} \cdot \left(\frac{1}{5}\right)^{k-1} & \frac{3}{10} \cdot \left(\frac{3}{5}\right)^{k-1} + \frac{2}{10} \cdot \left(\frac{1}{5}\right)^{k-1} \end{bmatrix}.$$

Clearly,

$$\lim_{k \rightarrow \infty} \frac{\sum_i M(k)_{i1}^2}{\sum_{i,j} M(k)_{ij}^2} = 0, \quad \lim_{k \rightarrow \infty} \frac{\sum_i M(k)_{i2}^2}{\sum_{i,j} M(k)_{ij}^2} = 1,$$

and therefore

$$\lim_{k \rightarrow \infty} \frac{\operatorname{tr}((\mathbf{M}_1^m)^t \mathbf{M}_{[\omega]_k}^t \mathbf{M}_{[\omega]_k} \mathbf{M}_1^m)}{\operatorname{tr}(\mathbf{M}_{[\omega]_k}^t \mathbf{M}_{[\omega]_k})} = \left(\frac{1}{5}\right)^{2m},$$

This implies

$$\lim_{k \rightarrow \infty} \frac{\nu \circ \mathbf{F}_1^m(\mathbf{F}_{[\omega]_k}(\mathbb{S}))}{\nu(\mathbf{F}_{[\omega]_k}(\mathbb{S}))} = \left(\frac{1}{15}\right)^m,$$

which completes the proof. \square

As an immediate corollary of Lemma 4.3.14, we obtain the following.

Corollary 4.3.15. *Let $m \in \mathbb{N}$. Then*

$$\left\| \frac{d(\hat{\nu} \circ \mathbf{F}_1^{-m})}{d\hat{\nu}} \right\|_{L^\infty(\hat{\mathbb{S}}; \hat{\nu})} = \left\| \frac{d\hat{\nu}}{d(\hat{\nu} \circ \mathbf{F}_1^m)} \right\|_{L^\infty(\hat{\mathbb{S}}; \hat{\nu})} = 15^m. \quad (4.3.16)$$

The following proposition shows that a function $u \in W^{1,r}(\hat{\mathbb{S}}^n)$ with $r > 1 + (n - 1)\delta_s$, $r \geq 2$ has at most polynomial growth.

Proposition 4.3.16. *Suppose $u \in W^{1,r}(\hat{\mathbb{S}}^n)$, $r > 1 + (n - 1)\delta_s$, $r \geq 2$. Let $S_m = \mathbf{F}_{(1,\dots,1)}^{-m}(\mathbb{S}^n) = 2^m \mathbb{S}^n$, $m \in \mathbb{N}$. Then*

$$\text{osc}_{S_m}(u) \leq 3^{m\beta_r} \|u\|_{W^{1,r}(S_m)}, \quad (4.3.17)$$

where

$$\beta_r = (1/\delta_s + 1)/r' - n/r. \quad (4.3.18)$$

Proof. By (4.3.1),

$$\begin{aligned} \nabla_i u &= \nabla_i (u \circ \mathbf{F}_{(1,\dots,1)}^{-m} \circ \mathbf{F}_{(1,\dots,1)}^m) \\ &= \left(\frac{3}{5}\right)^{m/2} \nabla_i (u \circ \mathbf{F}_{(1,\dots,1)}^{-m}) \circ \mathbf{F}_{(1,\dots,1)}^m \cdot \left[\frac{d(\nu \circ \mathbf{F}_1^m)}{d\nu} \right]^{1/2}. \end{aligned}$$

Therefore,

$$\nabla_i (u \circ \mathbf{F}_{(1,\dots,1)}^{-m}) = 3^{m/(2\delta_s)} \nabla_i u \circ \mathbf{F}_{(1,\dots,1)}^{-m} \cdot \left[\frac{d(\nu \circ \mathbf{F}_1^{-m})}{d\nu} \right]^{1/2}.$$

It follows from the above and Corollary 4.3.15 that

$$|\nabla_i (u \circ \mathbf{F}_{(1,\dots,1)}^{-m})| \leq 3^{m[(1/\delta_s + 2)/r' - 1]} |\nabla_i u \circ \mathbf{F}_{(1,\dots,1)}^{-m}| \cdot \left[\frac{d(\nu \circ \mathbf{F}_1^{-m})}{d\nu} \right]^{1/r}.$$

Therefore,

$$\begin{aligned}
& \int_{\mathbb{S}^n} |\nabla_i (u \circ \mathbf{F}_{(1,\dots,1)}^{-m})(x_i, \mathbf{x}'_i)|^r (\nu \times \mu_{n-1})(dx_i, d\mathbf{x}'_i) \\
& \leq 3^{m[(1/\delta_s+2)(r-1)-r]} \int_{\mathbb{S}^n} |\nabla_i u \circ \mathbf{F}_{(1,\dots,1)}^{-m}(x_i, \mathbf{x}'_i)|^r [(\nu \circ \mathbf{F}_1^{-m}) \times \mu_{n-1}](dx_i, d\mathbf{x}'_i) \\
& = 3^{m[(1/\delta_s+1)(r-1)-n]} \int_{S_m} |\nabla_i u(x_i, \mathbf{x}'_i)|^r (\nu \times \mu_{n-1})(dx_i, d\mathbf{x}'_i),
\end{aligned}$$

which yields that

$$\llbracket u \circ \mathbf{F}_{(1,\dots,1)}^{-m} \rrbracket_{W^{1,r}(\mathbb{S}^n)} \leq 3^{m\beta_r} \llbracket u \rrbracket_{W^{1,r}(S_m)}.$$

Now (4.3.17) follows immediately from (4.3.9) and the above inequality. \square

To formulate our first main result, let us first introduce the setting that we shall work on. Let $\hat{\sigma}$ be a positive Radon measure on $\hat{\mathbb{S}}^n$ satisfying the following condition: there exist constants $0 < \underline{\delta} \leq \bar{\delta} \leq \infty$ with $\bar{\delta} \geq 1$ and $C_{\hat{\sigma}} > 0$ such that

$$\begin{cases} \hat{\sigma}(S) \leq C_{\hat{\sigma}} \hat{\mu}_n(S)^{1/\bar{\delta}}, & \text{if } 0 < \text{diam}(S) \leq 1, \\ \hat{\sigma}(S) \leq C_{\hat{\sigma}} \hat{\mu}_n(S)^{1/\underline{\delta}}, & \text{if } \text{diam}(S) > 1 \end{cases} \quad (\text{N})$$

for all dyadic triangles $S \subseteq \hat{\mathbb{S}}^n$.

Remark 4.3.17. Note that the restriction $\bar{\delta} \geq 1$ in (N) is necessary in view of the countable additivity of $\hat{\sigma}$ and $\hat{\mu}_n$ and that $\hat{\sigma}$ is finite on compact subsets.

We list some examples of the Radon measure $\hat{\sigma}$.

Example 4.3.18. (i) The Hausdorff measure μ_n , for which $\underline{\delta} = \bar{\delta} = 1$.

(ii) The product Kusuoka measure $\hat{\nu}_n = \hat{\nu} \times \dots \times \hat{\nu}$, for which the sharp constants $\underline{\delta}$ and $\bar{\delta}$ will be given later (see Corollary 2.2.4(a)).

(iii) Dirac measures, for which $\underline{\delta} = \bar{\delta} = \infty$.

(iv) Examples in (ii) and (iii) can be generalized to linear combinations of measures of the form $\hat{\sigma} = \hat{\sigma}_1 \times \hat{\sigma}_2$, where $\hat{\sigma}_1, \hat{\sigma}_2$ are Radon measures on $\hat{\mathbb{S}}^k$ and $\hat{\mathbb{S}}^{n-k}$ satisfying conditions

(N) on the corresponding spaces. Another particular case of such measures is

$$\hat{\sigma} = \sum_{i=1}^n \sum_{j=1,2,3} \hat{\mu}_{i-1} \times \delta_{p_j} \times \hat{\mu}_{n-i},$$

where δ_{p_j} is the Dirac measure concentrated at $p_j \in V_0$. Applying Theorem 4.3.19 below to the above measure $\hat{\sigma}$ gives the trace theorem for functions in $W^{1,r}(\hat{\mathbb{S}}^n)$.

We are now in a position to formulate the Sobolev inequality on the infinite product space $\hat{\mathbb{S}}^n$.

Theorem 4.3.19. *Suppose $\hat{\sigma}$ is a positive Radon measure on $\hat{\mathbb{S}}^n$ satisfying the condition (N). Let $r > 1 + (n-1)\delta_s$, $r \geq 2$, $p \geq 1$, $\min\{p, r\} \leq q \leq \infty$. Then*

$$\|u\|_{L^q(\hat{\sigma})} \leq C \sum_{i=1,2} [u]_{W^{1,r}(\hat{\mathbb{S}}^n)}^{a_i} \|u\|_{L^p(\hat{\mu}_n)}^{1-a_i}, \quad (4.3.19)$$

where

$$\begin{aligned} a_1 &= \left[\frac{1/p - 1/(q\delta)}{1/p - 1/r + [1/(r'\delta_s) + 1/r']/n} \right]^+, \\ a_2 &= \left[\frac{1/p - 1/(q\bar{\delta})}{1/p - 1/r + [1/(r'\delta_s) + 1/r]/n} \right]^+, \end{aligned} \quad (4.3.20)$$

and $C > 0$ is a constant depending only on the constant $C_{\hat{\sigma}}$ in (N).

Proof. For any $\mathbf{i} \in \mathbb{Z}^n$, $m \in \mathbb{Z}$, let $S_{\mathbf{i},m} = \tau_{\mathbf{i}} \circ \mathbf{F}_{(1,\dots,1)}^{-m}(\mathbb{S}^n)$. Then $S_{\mathbf{i},m}$ are dyadic triangles with $\text{diam}(S_{\mathbf{i},m}) = 2^m$, and $\hat{\mathbb{S}}^n$ can be written as the union $\hat{\mathbb{S}}^n = \bigcup_{\mathbf{i} \in \mathbb{Z}^n} S_{\mathbf{i},m}$ of sets with disjoint interiors. Denote

$$\hat{\mu}_{nm} = \hat{\mu}_n(S_{\mathbf{i},m}) = 3^{mn}, \quad [u]_{S_{\mathbf{i},m}} = \frac{1}{\hat{\mu}_{nm}} \int_{S_{\mathbf{i},m}} u d\hat{\mu}_n, \quad \mathbf{i} \in \mathbb{Z}^n, m \in \mathbb{Z}.$$

For $m \geq 0$, by Proposition 4.3.16 and (N),

$$\begin{aligned}
& \int_{\hat{\mathbb{S}}^n} |u|^q d\hat{\sigma} \leq 2^{q-1} \sum_{i \in \mathbb{Z}^n} \left[\int_{S_{i,m}} |u - [u]_{S_{i,m}}|^q d\hat{\sigma} + \hat{\sigma}(S_{i,m}) |[u]_{S_{i,m}}|^q \right] \\
& \leq 2^{q-1} \sum_{i \in \mathbb{Z}^n} \left[\hat{\sigma}(S_{i,m}) \hat{\mu}_{nm}^{\beta_r q/n} \llbracket u \rrbracket_{W^{1,r}(S_{i,m})}^q + \hat{\sigma}(S_{i,m}) \hat{\mu}_{nm}^{-q/p} \left(\int_{S_{i,m}} |u|^p d\hat{\mu}_n \right)^{q/p} \right] \\
& \leq C_\sigma 2^{q-1} \sum_{i \in \mathbb{Z}^n} \left[\hat{\mu}_{nm}^{1/\delta + \beta_r q/n} \llbracket u \rrbracket_{W^{1,r}(S_{i,m})}^q + \hat{\mu}_{nm}^{1/\delta - q/p} \left(\int_{S_{i,m}} |u|^p d\hat{\mu}_n \right)^{q/p} \right] \\
& \leq C_\sigma 2^{q-1} \left[\hat{\mu}_{nm}^{1/\delta + \beta_r q/n} \left(\sum_{i \in \mathbb{Z}^n} \llbracket u \rrbracket_{W^{1,r}(S_{i,m})}^r \right)^{q/r} + \hat{\mu}_{nm}^{1/\delta - q/p} \left(\sum_{i \in \mathbb{Z}^n} \int_{S_{i,m}} |u|^p d\hat{\mu}_n \right)^{q/p} \right] \\
& \leq C_\sigma 2^{q-1} \left[\hat{\mu}_{nm}^{1/\delta + \beta_r q/n} \left(\sum_{i \in \mathbb{Z}^n} \llbracket u \rrbracket_{W^{1,r}(S_{i,m})}^r \right)^{q/r} + \hat{\mu}_{nm}^{1/\delta - q/p} \left(\sum_{i \in \mathbb{Z}^n} \int_{S_{i,m}} |u|^p d\hat{\mu}_n \right)^{q/p} \right] \\
& \leq C_\sigma 2^{q-1} \left[\hat{\mu}_{nm}^{1/\delta + \beta_r q/n} \llbracket u \rrbracket_{W^{1,r}(\hat{\mathbb{S}}^n)}^q + \hat{\mu}_{nm}^{1/\delta - q/p} \|u\|_{L^p(\hat{\mu}_n)}^q \right],
\end{aligned}$$

where α_r, β_r be the exponents given by (4.3.13) and (4.3.18) respectively. Therefore,

$$\|u\|_{L^q(\hat{\sigma})} \leq C \left[\hat{\mu}_{nm}^{1/(q\delta) + \beta_r/n} \llbracket u \rrbracket_{W^{1,r}(\hat{\mathbb{S}}^n)} + \hat{\mu}_{nm}^{1/(q\delta) - 1/p} \|u\|_{L^p(\hat{\mu}_n)} \right], \quad (4.3.21)$$

where $C > 0$ is a constant depending only on $C_{\hat{\sigma}}$ (the constant C can chosen to be independent of q as $q > 1$). Similarly, for $m \leq 0$, we have

$$\|u\|_{L^q(\hat{\sigma})} \leq C \left[\hat{\mu}_{nm}^{1/(q\bar{\delta}) + \alpha_r/n} \llbracket u \rrbracket_{W^{1,r}(\hat{\mathbb{S}}^n)} + \hat{\mu}_{nm}^{1/(q\bar{\delta}) - 1/p} \|u\|_{L^p(\hat{\mu}_n)} \right]. \quad (4.3.22)$$

Without loss of generality, we may assume that $\llbracket u \rrbracket_{W^{1,r}(\hat{\mathbb{S}}^n)} > 0$. For the case $\llbracket u \rrbracket_{W^{1,r}(\hat{\mathbb{S}}^n)} \leq \|u\|_{L^p(\hat{\mu}_n)}$, setting

$$m = \inf \left\{ m \geq 0 : \hat{\mu}_{nm}^{\beta_r/n + 1/p} \geq \|u\|_{L^p(\hat{\mu}_n)} / \llbracket u \rrbracket_{W^{1,r}(\hat{\mathbb{S}}^n)} \right\}$$

in (4.3.21) gives that

$$\|u\|_{L^q(\hat{\sigma})} \leq C \llbracket u \rrbracket_{W^{1,r}(\hat{\mathbb{S}}^n)}^{a_1} \|u\|_{L^p(\hat{\mu}_n)}^{1-a_1}.$$

For the case $\llbracket u \rrbracket_{W^{1,r}(\hat{\mathbb{S}}^n)} > \|u\|_{L^p(\hat{\mu}_n)}$, setting

$$m = \sup \left\{ m \leq 0 : \hat{\mu}_{nm}^{\beta_r/n+1/p} \leq \|u\|_{L^p(\hat{\mu}_n)} / \llbracket u \rrbracket_{W^{1,r}(\hat{\mathbb{S}}^n)} \right\}$$

in (4.3.22) gives that

$$\|u\|_{L^q(\hat{\sigma})} \leq C \llbracket u \rrbracket_{W^{1,r}(\hat{\mathbb{S}}^n)}^{a_2} \|u\|_{L^p(\hat{\mu}_n)}^{1-a_2}.$$

This completes the proof. \square

Remark 4.3.20. (i) Recall that the Sobolev inequality on \mathbb{R}^n takes the form

$$\|u\|_{L^q(\mathbb{R}^n)} \leq \|\nabla u\|_{L^r(\mathbb{R}^n)}^a \|u\|_{L^p(\mathbb{R}^n)}^{1-a}, \quad (4.3.23)$$

where $a \in [0, 1]$ is given by

$$a = \frac{1/p - 1/q}{1/p - 1/r + 1/n}. \quad (4.3.24)$$

Let us compare (4.3.19) and (4.3.23). That the exponent q in (4.3.24) is changed to the two exponents $q\underline{\delta}$ and $q\bar{\delta}$. This is due to the different scaling rates of the measures $\hat{\nu}_n$ and $\hat{\sigma}$, and the inhomogeneity of the scaling rate of $\hat{\sigma}$ under shrinkage and expansion. The factor before $1/n$ in (4.3.24) is changed from 1 to the pair $1/(r'\delta_s) + 1/r'$ and $1/(r\delta_s) + 1/r'$. When $r = 2$, these numbers are both equal to $1/(2\delta_s) + 1/2$, which is the natural factor for $\hat{\mathbb{S}}$ as $\delta_s = 1$ if the spectral dimension were $d_s = 1$. When $r > 2$, these numbers depend on the exponent r . This suggests that $r - 2$, the extra part of r , has a distorting effect on the dimension n . Such a distorting effect can also be seen from (4.3.1) and (4.3.3).

(ii) Only the term corresponding to a_1 is needed on the right hand side of (4.3.19) when $\hat{\mathbb{S}}^n$ is replaced by \mathbb{S}^n , since only the first part of (N) is involved. The proof of this is similar to that of Theorem 4.3.19 and hence omitted.

We now show that the condition (N) is also necessary for the Sobolev inequality (4.3.19) to hold.

Theorem 4.3.21. *Suppose that $\hat{\sigma}$ is a positive Radon measure on $\hat{\mathbb{S}}^n$, and there exist constants $p \geq 1$, $1 \leq q < \infty$, $r > 1 + (n-1)\delta_s$, $r \geq 2$, $a_i \in [0, 1]$, $1 \leq i \leq k$ and $C > 0$ such that*

$$\|u\|_{L^q(\hat{\sigma})} \leq C \sum_{i=1}^k \llbracket u \rrbracket_{W^{1,r}(\hat{\mathbb{S}}^n)}^{a_i} \|u\|_{L^p(\hat{\mu}_n)}^{1-a_i}, \quad \text{for all } u \in W^{1,r}(\hat{\mathbb{S}}^n). \quad (4.3.25)$$

Then σ satisfies the condition (N) for some $0 < \underline{\delta} \leq \bar{\delta} \leq \infty$ with $\bar{\delta} \geq 1$ and $C_{\hat{\sigma}} > 0$.

Proof. For the first part of (N), it suffices to show that $\sup_{i \in \mathbb{Z}^n} \hat{\sigma} \circ \tau_i(\mathbb{S}^n) < \infty$ and take $\bar{\delta} = \infty$. (Note that $\bar{\delta} = \infty$ is the only valid value when σ is a Dirac measure.) Let ϕ_0 be the 1-harmonic function in \mathbb{S} with boundary value $\phi_0|_{V_1} = 1_{\mathbf{F}_2(p_1)}$. Clearly, $\phi_0|_{V_0} = 0$. Therefore, setting $\phi_0 = 0$ on $\hat{\mathbb{S}} \setminus \mathbb{S}$ gives a function $\phi_0 \in \mathcal{F}(\hat{\mathbb{S}})$. It is easily seen that

$$\frac{2}{5} \leq \phi_0 \leq 1 \text{ on } \mathbf{F}_1 \circ \mathbf{F}_2(\mathbb{S}), \quad \text{supp}(\phi_0) \subseteq \mathbb{S}.$$

Note that $(\phi_0 \circ \mathbf{F}_1)|_{\mathbb{S}}$ is the harmonic function in \mathbb{S} with boundary value $(\phi_0 \circ \mathbf{F}_1)|_{V_0} = 1_{\{p_2\}}$.

By (4.3.6), (4.3.1) and (4.3.16),

$$\|\nabla \phi_0\|_{L^\infty(\hat{\mathbb{S}}; \nu)} = \|\nabla \phi_0 \circ \mathbf{F}_1\|_{L^\infty(\hat{\mathbb{S}}; \nu)} \leq \sqrt{\frac{5}{3}} \|\nabla(\phi_0 \circ \mathbf{F}_1)\|_{L^\infty(\mathbb{S}; \nu)} \left\| \frac{d\hat{\nu}}{d(\hat{\nu} \circ \mathbf{F}_1)} \right\|_{L^\infty(\hat{\mathbb{S}}; \nu)}^{1/2} \leq 5\sqrt{2}.$$

Therefore,

$$\llbracket \nabla \phi_0 \rrbracket_{W^{1,r}(\hat{\mathbb{S}})} \leq 5\sqrt{2}.$$

For any $j \in \mathbb{Z}$, let $\phi_j \in \mathcal{F}(\hat{\mathbb{S}})$ be the translation $\phi_j = \phi_0 \circ \tau_j$. For each $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}^n$, let

$$u_{\mathbf{i}}(\mathbf{x}) = \phi_{i_1}(x_1) \cdots \phi_{i_n}(x_n), \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{S}^n.$$

Then $u_{\mathbf{i}} \in W^{1,r}(\hat{\mathbb{S}}^n)$, $\text{supp}(u_{\mathbf{i}}) \subseteq \tau_{\mathbf{i}}(\mathbb{S}^n)$. Let

$$S_{\mathbf{i}} = \tau_{\mathbf{i}} \circ \mathbf{F}_{(1, \dots, 1)} \circ \mathbf{F}_{(2, \dots, 2)}(\mathbb{S}^n).$$

Then

$$\left(\frac{2}{5}\right)^n \leq u_{\mathbf{i}} \leq 1 \text{ on } S_{\mathbf{i}}, \quad \llbracket u_{\mathbf{i}} \rrbracket_{W^{1,r}(\hat{\mathbb{S}}^n)} \leq 5\sqrt{2} n^{1/r}.$$

Setting $u = u_{\mathbf{i}}$ in (4.3.25) gives $\sup_{\mathbf{i} \in \mathbb{Z}^n} \hat{\sigma}(S_{\mathbf{i}})^{1/q} < \infty$. Since $q < \infty$, we see that $\sup_{\mathbf{i} \in \mathbb{Z}^n} \hat{\sigma}(S_{\mathbf{i}}) < \infty$. Clearly, this implies $\sup_{\mathbf{i} \in \mathbb{Z}^n} \hat{\sigma} \circ \tau_{\mathbf{i}}(\mathbb{S}^n) < \infty$, as we may change the boundary value of ϕ_0 to $\phi_0|_{V_1} = 1_{\mathbf{F}_3(p_2)}$ or $\phi_0|_{V_1} = 1_{\mathbf{F}_1(p_3)}$ accordingly, for which similar results still hold.

We now prove the second part of (N). Let $u_{\mathbf{i}}$ and $S_{\mathbf{i}}$ be the functions and triangles defined above. It suffices to consider triangles S of the form $S = \mathbf{F}_{(1,\dots,1)}^{-k}(S_{\mathbf{i}})$, $k \in \mathbb{N}_+$, $\mathbf{i} \in \mathbb{Z}^n$. The desired conclusion follows readily by changing the boundary value of functions.

By (4.3.3),

$$\llbracket u_{\mathbf{i}} \circ \mathbf{F}_{(1,\dots,1)}^k \rrbracket_{W^{1,r}(\hat{\mathbb{S}}^n)} \leq 3^{k\beta_r} \llbracket u_{\mathbf{i}} \rrbracket_{W^{1,r}(\hat{\mathbb{S}}^n)} \leq 3^{k\beta_r} \cdot 5\sqrt{2} n^{1/2} \text{ for all } k \in \mathbb{N}_+, \quad (4.3.26)$$

where $\beta_r > 0$ is the constant given by (4.3.18).

Let $S \subseteq \hat{\mathbb{S}}^n$ be a triangle of the form $S = \mathbf{F}_{(1,\dots,1)}^{-k}(S_{\mathbf{i}})$, $k \in \mathbb{N}_+$, $\mathbf{i} \in \mathbb{Z}^n$. By (4.3.26), setting $u = u_{\mathbf{i}} \circ \mathbf{F}_{(1,\dots,1)}^k$ in (4.3.25) gives

$$\hat{\sigma}(S)^{1/q} \leq c_n 3^{k(n/p+\beta_r)} = c_n \hat{\mu}_n(S)^{1/p+\beta_r/n},$$

where $c_n > 0$ is a constant depending only on n and the constant C in (4.3.25). Therefore, the second part of (N) holds with $\underline{\delta} = q(1/p + \beta_r/n) > 0$. This completes the proof. \square

We may exchange the positions of $\hat{\sigma}$ and $\hat{\mu}_n$ in (4.3.19). Let $\hat{\sigma}$ be a Radon measure satisfying the following condition: there exist constants $0 < \underline{\delta} \leq \bar{\delta} \leq \infty$ with $\underline{\delta} \leq 1$ and $C_{\sigma} > 0$ such that

$$\begin{cases} C_{\hat{\sigma}}^{-1} \hat{\mu}_n(S)^{1/\underline{\delta}} \leq \hat{\sigma}(S), & \text{if } 0 < \text{diam}(S) \leq 1, \\ C_{\hat{\sigma}}^{-1} \hat{\mu}_n(S)^{1/\bar{\delta}} \leq \hat{\sigma}(S), & \text{if } \text{diam}(S) > 1, \end{cases} \quad (\text{N}')$$

for all dyadic triangles $S \subseteq \hat{\mathbb{S}}^n$. Note that, as in (N), the restriction $\underline{\delta} \leq 1$ is necessary in view of the countable additivity of measures.

Remark 4.3.22. The sharp constants for (N') when $\hat{\sigma}$ is the product Kusuoka measure $\hat{\nu}_n = \hat{\nu} \times \cdots \times \hat{\nu}$ will be given later (see Corollary 4.3.26(b)).

For such Radon measures, we have the following two theorems, of which the proofs are similar to those of Theorem 4.3.19 and Theorem 4.3.21, and hence will be omitted.

Theorem 4.3.23. *Suppose $\hat{\sigma}$ is a positive Radon measure on $\hat{\mathbb{S}}^n$ satisfying the condition (N'). Let $r > 1 + (n-1)\delta_s$, $r \geq 2$, $p \geq 1$, $\min\{p, r\} \leq q \leq \infty$. Then*

$$\|u\|_{L^q(\hat{\mu}_n)} \leq C \sum_{i=1,2} \|u\|_{W^{1,r}(\hat{\mathbb{S}}^n)}^{a_i} \|u\|_{L^p(\hat{\sigma})}^{1-a_i},$$

where

$$a_1 = \left[\frac{1/(p\bar{\delta}) - 1/q}{1/(p\bar{\delta}) - 1/r + [1/(r'\delta_s) + 1/r']/n} \right]^+, \quad a_2 = \left[\frac{1/(p\underline{\delta}) - 1/q}{1/(p\underline{\delta}) - 1/r + [1/(r'\delta_s) + 1/r]/n} \right]^+,$$

and $C > 0$ is a constant depending only on the constant $C_{\hat{\sigma}}$ in (N').

Theorem 4.3.24. *Suppose $\hat{\sigma}$ is a positive Radon measure on $\hat{\mathbb{S}}^n$, and there exist constants $p \geq 1$, $1 \leq q < \infty$, $r > 1 + (n-1)\delta_s$, $r \geq 2$, $a_i \in [0, 1]$, $1 \leq i \leq k$ and $C > 0$ such that*

$$\|u\|_{L^q(\hat{\mu}_n)} \leq C \sum_{i=1}^k \|u\|_{W^{1,r}(\hat{\mathbb{S}}^n)}^{a_i} \|u\|_{L^p(\hat{\sigma})}^{1-a_i}, \quad \text{for all } u \in W^{1,r}(\hat{\mathbb{S}}^n).$$

Then $\hat{\sigma}$ satisfies the condition (N') for some $0 < \underline{\delta} \leq \bar{\delta} \leq \infty$ with $\underline{\delta} \leq 1$ and $C_{\hat{\sigma}} > 0$.

We now give the sharp values of the exponents $\underline{\delta}$ and $\bar{\delta}$ in (N) and (N') for the product Kusuoka measure $\hat{\nu}_n$.

Proposition 4.3.25. *The following holds*

$$\inf_{\omega \in W_*} \liminf_{m \rightarrow \infty} [\text{tr}(\mathbf{Y}_{\omega_1}^t \cdots \mathbf{Y}_{\omega_m}^t \mathbf{Y}_{\omega_m} \cdots \mathbf{Y}_{\omega_1})]^{1/m} = \frac{3}{25}. \quad (4.3.27)$$

Moreover, the infimum (4.3.27) can be achieved by some $\omega \in W_*$.

Proof. Let \mathbf{M}_i , $i = 1, 2, 3$ be the 2×2 matrices given by (4.3.15). As seen in the proof of Lemma 4.3.14, it suffices to prove the lemma for \mathbf{M}_i , $i = 1, 2, 3$. Notice that

$$\det(\mathbf{M}_1) = \det(\mathbf{M}_2) = \det(\mathbf{M}_3) = \frac{3}{25}.$$

We see that

$$\det(\mathbf{M}_{\omega_1}^t \cdots \mathbf{M}_{\omega_m}^t \mathbf{M}_{\omega_m} \cdots \mathbf{M}_{\omega_1}) = \prod_{i=1}^m \det(\mathbf{M}_{\omega_i})^2 = \left(\frac{3}{25}\right)^{2m}.$$

Since $\mathbf{M}_{\omega_1}^t \cdots \mathbf{M}_{\omega_m}^t \mathbf{M}_{\omega_m} \cdots \mathbf{M}_{\omega_1}$ is non-negative definite, by the arithmetic-geometric mean inequality,

$$\frac{1}{2} \operatorname{tr}(\mathbf{M}_{\omega_1}^t \cdots \mathbf{M}_{\omega_m}^t \mathbf{M}_{\omega_m} \cdots \mathbf{M}_{\omega_1}) \geq [\det(\mathbf{M}_{\omega_1}^t \cdots \mathbf{M}_{\omega_m}^t \mathbf{M}_{\omega_m} \cdots \mathbf{M}_{\omega_1})]^{1/2} = \left(\frac{3}{25}\right)^m.$$

This implies that

$$\inf_{\omega \in W_*} \liminf_{m \rightarrow \infty} [\operatorname{tr}(\mathbf{M}_{\omega_1}^t \cdots \mathbf{M}_{\omega_m}^t \mathbf{M}_{\omega_m} \cdots \mathbf{M}_{\omega_1})]^{1/m} \geq \frac{3}{25}.$$

To show the reverse, we construct a finite sequence $(\omega_1 \dots \omega_r) \in \{1, 2, 3\}^r$ such that $\mathbf{M}_{\omega_r} \cdots \mathbf{M}_{\omega_1}$ has no real eigenvalues. Once such a finite sequence can be found, since \mathbf{M}_i , $i = 1, 2, 3$ are 2×2 matrices, the spectrum of $\mathbf{M}_{\omega_r} \cdots \mathbf{M}_{\omega_1}$ must consist of a pair of conjugate eigenvalues $\lambda, \bar{\lambda} \in \mathbb{C}$. Therefore, $\mathbf{M}_{\omega_r} \cdots \mathbf{M}_{\omega_1}$ is similar to the diagonal matrix

$$\begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}.$$

Moreover, we have

$$|\lambda| = \left(\frac{3}{25}\right)^{r/2}$$

in view of

$$\det (\mathbf{M}_{\omega_r} \cdots \mathbf{M}_{\omega_1}) = \prod_{i=1}^r \det (\mathbf{M}_{\omega_i}) = \left(\frac{3}{25}\right)^r.$$

Now set ω to be the repetition

$$\omega = (\omega_1 \dots \omega_r)(\omega_1 \dots \omega_r) \dots \in W_*.$$

Then, with $m = rk$, $k \in \mathbb{N}_+$,

$$\operatorname{tr}(\mathbf{M}_{\omega_1}^t \cdots \mathbf{M}_{\omega_m}^t \mathbf{M}_{\omega_m} \cdots \mathbf{M}_{\omega_1}) = |\lambda|^{2k} + |\bar{\lambda}|^{2k} = 2 \cdot \left(\frac{3}{25}\right)^{rk} = 2 \cdot \left(\frac{3}{25}\right)^m, \quad k \in \mathbb{N}_+.$$

This implies that

$$\lim_{m \rightarrow \infty} \left[\operatorname{tr}(\mathbf{M}_{\omega_1}^t \cdots \mathbf{M}_{\omega_m}^t \mathbf{M}_{\omega_m} \cdots \mathbf{M}_{\omega_1}) \right]^{1/m} = \frac{3}{25}.$$

By a direct search, it can be seen that the least possible value of r is $r = 3$, and accordingly, (312) is a finite sequence satisfying the desired property. In particular, the product

$$\mathbf{M}_2 \mathbf{M}_1 \mathbf{M}_3 = \begin{bmatrix} \frac{6}{125} & \frac{\sqrt{3}}{125} \\ -\frac{\sqrt{3}}{125} & \frac{4}{125} \end{bmatrix}$$

is a such product having eigenvalues $\frac{1}{25}(1 \pm \frac{\sqrt{2}}{5}i)$. We may now take $\omega = (312)(312) \dots \in W_*$ and complete the proof. \square

Corollary 4.3.26. (a) The measure $\hat{\sigma} = \hat{\nu}_n$ satisfies the condition (N) with $\underline{\delta} = 1$ and $\bar{\delta} = \delta_s$. Conversely, if $\hat{\nu}_n$ satisfies (N) for some $\underline{\delta}$ and $\bar{\delta}$, then $\underline{\delta} \leq 1$, $\bar{\delta} \geq \delta_s$.

(b) The measure $\hat{\sigma} = \hat{\nu}_n$ satisfies the condition (N') with $\underline{\delta} = (1 + 1/\delta_s)^{-1}$ and $\bar{\delta} = 1$. Conversely, if $\hat{\nu}_n$ satisfies (N') for some $\underline{\delta}$ and $\bar{\delta}$, then $\underline{\delta} \leq (1 + 1/\delta_s)^{-1}$, $\bar{\delta} \geq 1$.

Proof. (a) We only need to prove the statements on $\bar{\delta}$ for ν_n , as $\hat{\nu}_n(S) = \hat{\mu}_n(S) = 3^k$ for

dyadic triangles $S \subseteq \hat{\mathbb{S}}^n$ with $\text{diam}(S) = 2^k$, $k \in \mathbb{N}$. Recall that the spectrum of \mathbf{Y}_i are $\text{spec}(\mathbf{Y}_i) = \{0, \frac{1}{5}, \frac{3}{5}\}$, $i = 1, 2, 3$. We deduce that

$$\nu(\mathbf{F}_{[\omega]_m}(\mathbb{S})) \leq \frac{1}{2} \left(\frac{5}{3}\right)^m \left[2 \cdot \left(\frac{3}{5}\right)^{2m}\right] = 3^{m/\delta_s} = \mu(\mathbf{F}_{[\omega]_m}(\mathbb{S}))^{1/\delta_s}$$

for all $\omega \in W_*$, $m \in \mathbb{N}$. Therefore,

$$\nu_n(\mathbf{F}_{[\omega]_m}(\mathbb{S})) \leq \mu_n(\mathbf{F}_{[\omega]_m}(\mathbb{S}))^{1/\delta_s},$$

which shows that the first part of (N) with $\bar{\delta} = \delta_s$.

Conversely, suppose ν_n satisfies the first part of (N) for some $\bar{\delta}$. Let $S_m = \mathbf{F}_{(1, \dots, 1)}^m(\mathbb{S}^n)$, $m \in \mathbb{N}$. Since

$$\nu(\mathbf{F}_1^m(\mathbb{S})) = \frac{1}{2} \left(\frac{5}{3}\right)^m \left[\left(\frac{3}{5}\right)^{2m} + \left(\frac{1}{5}\right)^{2m}\right] \geq \frac{3^{m/\delta_s}}{2},$$

we see that

$$1/\bar{\delta} \leq \lim_{m \rightarrow \infty} \frac{\log \nu_n(S_m)}{\log \mu_n(S_m)} = 1/\delta_s.$$

Therefore, $\bar{\delta} \geq \delta_s$.

(b) As in the proof of (a), we only need to prove the statements on $\underline{\delta}$ for ν_n . By Proposition 4.3.25,

$$\nu(\mathbf{F}_{[\omega]_m}(\mathbb{S})) \geq \frac{1}{2} \left(\frac{5}{3}\right)^m \left(\frac{3}{25}\right)^m = \frac{1}{2} 3^{m(1+1/\delta_s)} = \frac{1}{2} \mu(\mathbf{F}_{[\omega]_m}(\mathbb{S}))^{1+1/\delta_s}$$

for all $\omega \in W_*$, $m \in \mathbb{N}$. Therefore,

$$\nu_n(\mathbf{F}_{[\omega]_m}(\mathbb{S}^n)) \geq \frac{1}{2^n} \mu_n(\mathbf{F}_{[\omega]_m}(\mathbb{S}^n))^{1+1/\delta_s}, \text{ for all } \omega \in W_*.$$

This shows the first part of (N') holds with $\underline{\delta} = (1 + 1/\delta_s)^{-1}$.

Conversely, suppose ν_n satisfies the first part of (N') for some $\underline{\delta}$. By Proposition 4.3.25,

there exists an $\omega \in W_*$ such that

$$\lim_{m \rightarrow \infty} [\operatorname{tr}(\mathbf{Y}_{[\omega]_m}^t \mathbf{Y}_{[\omega]_m})]^{1/m} = \frac{3}{25}.$$

Therefore,

$$\lim_{m \rightarrow \infty} \nu(\mathbf{F}_{[\omega]_m}(\mathbb{S}))^{1/m} = \frac{5}{3} \lim_{m \rightarrow \infty} [\operatorname{tr}(\mathbf{Y}_{[\omega]_m}^t \mathbf{Y}_{[\omega]_m})]^{1/m} = \frac{1}{5},$$

Let $S_m = \mathbf{F}_{[\omega]_m} \times \cdots \times \mathbf{F}_{[\omega]_m}(\mathbb{S}^n)$. The above and (N') give

$$1/\underline{\delta} \geq \lim_{m \rightarrow \infty} \frac{\log \nu_n(S_m)}{\log \mu_n(S_m)} = \frac{\log 5}{\log 3} = 1 + 1/\delta_s.$$

This completes the proof. □

4.4 Semi-linear parabolic PDEs

In this section, we study a type of semi-linear parabolic equations on \mathbb{S} , for which energy estimates and existence and uniqueness of solutions are established (Theorem 4.4.14). Moreover, the regularity of solutions to these PDEs is derived under additional conditions.

We consider the following initial–boundary value problem for semi-linear parabolic PDEs (see Definition 4.4.11 for a precise interpretation)

$$\begin{cases} \partial_t u \, d\mu = \mathcal{L}u \, d\mu + f(t, x, u, \nabla u) \, d\nu, & \text{in } (0, T] \times (\mathbb{S} \setminus V_0), \\ u = 0 & \text{on } (0, T] \times V_0, \quad u(0) = \psi, \end{cases} \quad (4.4.1)$$

where V_0 is the boundary of \mathbb{S} (cf. Section 1.2.1), $\psi \in L^2(\mu)$, and the coefficient $f : [0, T] \times \mathbb{S} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies the following:

- (i) There exists a constant $K > 0$ such that

$$|f(t, x, y, z) - f(t, x, \bar{y}, \bar{z})| \leq K (|y - \bar{y}| + |z - \bar{z}|), \quad (\text{A.1})$$

for all $(t, x) \in [0, T] \times \mathbb{S}$, $(y, z), (\bar{y}, \bar{z}) \in \mathbb{R}^2$;

(ii) $f(\cdot, 0, 0) \in L^2(0, T; L^2(\nu))$, that is,

$$\|f(\cdot, 0, 0)\|_{L^2(0, T; L^2(\nu))}^2 = \int_0^T \int_{\mathbb{S}} f(t, x, 0, 0)^2 \nu(dx) dt < \infty. \quad (\text{A.2})$$

Remark 4.4.1. There exist different formulations of non-linear PDEs on fractals. For example, a type of non-linear equations on fractals was considered in [28], where the non-linearity $f(\nabla u)$ is a bounded mapping $f : L^2(\nu) \rightarrow L^2(\mu)$. The equations studied there are essentially defined via a single measure (the Hausdorff measure μ). Therefore, the PDEs studied in this paper are different in essence from those considered in [28] in the way the gradients interact with the equations.

From now on, we shall use the notation $\langle f, g \rangle_\lambda = \int_{\mathbb{S}} fg d\lambda$ for any Borel measure λ on \mathbb{S} and any λ -a.e. defined functions f, g on \mathbb{S} , whenever the integral is well-defined. As in the previous section, we denote by C_* a generic universal constant which may vary on different occasions.

Let $p^0(t, x, y)$ be the transition kernel associated with the killed Brownian motion $\{X_t^0\}_{t \geq 0}$. The following result on resolvent kernel estimate was first proved in [4, Theorem 1.5 and Theorem 1.8].

Lemma 4.4.2. *Let ρ_α^0 , $\alpha > 0$ be the α -resolvent kernel of $\{X_t^0\}_{t \geq 0}$, that is,*

$$\rho_\alpha^0(x, y) = \int_0^\infty e^{-\alpha t} p^0(t, x, y) dt, \quad x, y \in \mathbb{S}.$$

Then $\rho_\alpha^0(\cdot, \cdot)$ is Lipschitz continuous with respect to the resistance metric (cf. Definition 1.2.22), i.e.

$$|\rho_\alpha^0(x, z) - \rho_\alpha^0(y, z)| \leq C_\alpha R(x, y), \quad x, y, z \in \mathbb{S}, \quad (4.4.2)$$

for some constant $C_\alpha > 0$ depending only on α .

Remark 4.4.3. The formulation of Lemma 4.4.2 is different from the original form of [4, Theorem 1.5 and Theorem 1.8]. Here we have used (1.2.16) to arrive at (4.4.2).

In view of the joint continuity of $p^0(t, x, y)$, the definition below is legitimate.

Definition 4.4.4. For any Radon measure λ on \mathbb{S} , we define $P_t^0 \lambda(x) = \int_{\mathbb{S}} p^0(t, x, y) \lambda(dy)$, $x \in \mathbb{S}$, $t \in (0, \infty)$.

Remark 4.4.5. (i) Let λ be a Radon measure on \mathbb{S} . By the symmetry of $p^0(t, \cdot, \cdot)$, it is easy to see that $\langle P_t^0(g\lambda), f \rangle_\mu = \langle g, P_t^0 f \rangle_\lambda$ for all $f \in L^2(\mu)$, $g \in L^1(\lambda)$.

(ii) For any Radon measure λ on \mathbb{S} , we have $P_t^0 \lambda \in \text{Dom}(\mathcal{L})$ for $t > 0$. In fact, since $p^0(t, x, y) \in C((0, \infty) \times \mathbb{S} \times \mathbb{S})$, we have $P_{t/2}^0 \lambda \in C(\mathbb{S})$, which implies that $P_t^0 \lambda = P_{t/2}^0(P_{t/2}^0 \lambda) \in \text{Dom}(\mathcal{L})$. Moreover, $P_t^0 \lambda \in C^1(0, \infty; L^2(\mu))$ and $(d/dt)P_t^0 \lambda = \mathcal{L}P_t^0 \lambda$.

(iii) Notice that, due to the singularity between μ and ν , the contractivity $\|P_t^0(g\nu)\|_{L^2(\mu)} \leq \|g\|_{L^2(\nu)}$, $t > 0$ is in general no longer valid. In fact, for $g \in L^2(\nu)$, $g \neq 0$, we have

$$\lim_{t \rightarrow 0} \|P_t^0(g\nu)\|_{L^2(\mu)} = \infty.$$

To see this, suppose, for a contradiction, that $\lim_{t \rightarrow 0} \|P_t^0(g\nu)\|_{L^2(\mu)} = \sup_{t > 0} \|P_t^0(g\nu)\|_{L^2(\mu)} < \infty$. Then there exists a unique $g_0 \in L^2(\mu)$ such that $\lim_{t \rightarrow 0} P_t^0(g\nu) = g_0$ weakly in $L^2(\mu)$.

On the other hand, for any $v \in \mathcal{F}(\mathbb{S} \setminus V_0)$, we have

$$\langle g_0, v \rangle_\mu = \lim_{t \rightarrow 0} \langle P_t^0(g\nu), v \rangle_\mu = \lim_{t \rightarrow 0} \langle g, P_t^0 v \rangle_\nu = \langle g, v \rangle_\nu,$$

where the last equality follows from the uniform convergence $\lim_{t \rightarrow 0} P_t^0 v = v$ as a consequence of the convergence in $\mathcal{F}(\mathbb{S} \setminus V_0)$. By the density of $\mathcal{F}(\mathbb{S} \setminus V_0)$ in $C(\mathbb{S})$, it is seen that $g\nu = g_0\mu$, which contradicts the fact that μ and ν are mutually singular.

To study the semi-linear parabolic PDEs (4.4.1), let us first investigate the formal in-

tegral

$$\int_0^t P_{t-s}^0(g(s)\nu) ds, \quad (4.4.3)$$

which is the formal solution to the equation $\partial_t u d\mu = \mathcal{L}u d\mu + g(t) d\nu$. Since P_t is not bounded from $L^2(\nu)$ to $L^2(\mu)$ (cf. Remark 4.4.5(iii)), there is a singularity in the integrand of (4.4.3) at $s = t$. In the following lemmas, we shall show that (4.4.3) is a well-defined function in the space $L^\infty(0, T; L^2(\mu)) \cap L^2(0, T; \mathcal{F}(\mathbb{S} \setminus V_0))$, and is jointly Hölder continuous if $g(t)$ is uniformly bounded in $L^2(\nu)$.

Lemma 4.4.6. *Let $g \in L^2(0, T; L^2(\nu))$. For each $\delta \in (0, T)$, let*

$$u_\delta(t) = \int_0^{(t-\delta)^+} P_{t-s}^0(g(s)\nu) ds, \quad t \in [0, T]. \quad (4.4.4)$$

Then $u_\delta \in L^\infty(0, T; L^2(\mu)) \cap L^2(0, T; \mathcal{F}(\mathbb{S} \setminus V_0))$, and

$$\|u_\delta\|_{L^\infty(0, T; L^2(\mu))}^2 + \int_0^T e^{C_\epsilon(T-s)} \mathcal{E}(u_\delta(s)) ds \leq \epsilon \int_0^T e^{C_\epsilon(T-s)} \|g(s)\|_{L^2(\nu)}^2 ds,$$

for any $\epsilon > 0$, where $C_\epsilon > 0$ is a constant depending only on ϵ . Moreover,

$$\|\partial_t u_\delta\|_{L^2(0, T; \mathcal{F}^{-1})} \leq C_* \|g\|_{L^2(0, T; L^2(\nu))}.$$

Proof. It is convenient to set $g(t) = 0$ for $t < 0$. Clearly, $u_\delta(t) \in \text{Dom}(\mathcal{L})$, $t \in [0, T]$. For each $s \in (0, T)$, since the function

$$t \mapsto P_{t-s}^0(g(s)\nu) = P_{t-s-\delta}^0[P_\delta^0(g(s)\nu)], \quad t \in (s + \delta, T)$$

is a continuously differentiable $L^2(\mu)$ -valued function, we see that $u_\delta \in C^1(\delta, T; L^2(\mu))$

and

$$\partial_t u_\delta(t) = P_\delta^0(g(t-\delta)\nu) + \int_0^{t-\delta} \mathcal{L}P_{t-s}^0(g(s)\nu) ds = \mathcal{L}u_\delta(t) + P_\delta^0(g(t-\delta)\nu). \quad (4.4.5)$$

For any $\epsilon > 0$ and each $t \in (0, T)$, testing (4.4.5) against u_δ , and applying Corollary 4.2.18 and Young's inequality gives that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_\delta(t)\|_{L^2(\mu)}^2 &= \langle P_\delta^0(g(t-\delta)\nu), u_\delta(t) \rangle_\mu - \mathcal{E}(u_\delta(t)) \\ &= \langle g(t-\delta), P_\delta^0(u_\delta(t)) \rangle_\nu - \mathcal{E}(u_\delta(t)) \\ &\leq C_\epsilon \mathcal{E}[P_\delta^0(u_\delta(t))]^{d_s-1} \|P_\delta^0(u_\delta(t))\|_{L^2(\mu)}^{2(2-d_s)} - \mathcal{E}(u_\delta(t)) + \epsilon \|g(t-\delta)\|_{L^2(\nu)}^2 \\ &\leq C_\epsilon \mathcal{E}(u_\delta(t))^{d_s-1} \|u_\delta(t)\|_{L^2(\mu)}^{2(2-d_s)} - \mathcal{E}(u_\delta(t)) + \epsilon \|g(t-\delta)\|_{L^2(\nu)}^2 \\ &\leq C_\epsilon \|u_\delta(t)\|_{L^2(\mu)}^2 - \frac{1}{2} \mathcal{E}(u_\delta(t)) + \epsilon \|g(t-\delta)\|_{L^2(\nu)}^2, \end{aligned}$$

where $C_\epsilon > 0$ denotes a generic constant depending only on ϵ which may vary on different occasions. By Grönwall's inequality and the fact that $u_\delta(t) = 0$, $t \in [0, \delta]$, we deduce

$$\|u_\delta(t)\|_{L^2(\mu)}^2 + \int_0^t e^{C_\epsilon(t-s)} \mathcal{E}(u_\delta(s)) ds \leq \epsilon \int_0^{(t-\delta)^+} e^{C_\epsilon(t-s)} \|g(s)\|_{L^2(\nu)}^2 ds. \quad (4.4.6)$$

By (4.4.5) and Young's inequality, for any $v \in \mathcal{F}(\mathbb{S} \setminus V_0)$,

$$\begin{aligned} |\langle \partial_t u_\delta(t), v \rangle_\mu| &\leq |\langle g(t-\delta), P_\delta^0 v \rangle_\nu| + \mathcal{E}(u_\delta(t))^{1/2} \mathcal{E}(v)^{1/2} \\ &\leq C_* \|g(t-\delta)\|_{L^2(\nu)} \mathcal{E}(P_\delta^0 v)^{1/2} + \mathcal{E}(u_\delta(t))^{1/2} \mathcal{E}(v)^{1/2} \\ &\leq C_* [\|g(t-\delta)\|_{L^2(\nu)} + \mathcal{E}(u_\delta(t))^{1/2}] \|v\|_{\mathcal{F}}. \end{aligned}$$

The above inequality also holds for $v \in \mathcal{F}(\mathbb{S})$. This can be seen by considering the \mathcal{F} -orthogonal projection of v on $\mathcal{F}(\mathbb{S} \setminus V_0)$. Therefore,

$$\|\partial_t u_\delta(t)\|_{\mathcal{F}^{-1}} \leq C_* [\|g(t-\delta)\|_{L^2(\nu)} + \mathcal{E}(u_\delta(t))^{1/2}], \quad t \in (\delta, T],$$

which, together with (4.4.6), implies the desired estimate for $\|\partial_t u_\delta\|_{L^2(0,T;\mathcal{F}^{-1})}$. \square

Lemma 4.4.7. *The limit*

$$u(t) = \lim_{\delta \rightarrow 0} \int_0^{(t-\delta)^+} P_{t-s}^0(g(s)\nu) ds, \quad (4.4.7)$$

exists with respect to the norm $\|\cdot\|_{L^\infty(0,T;L^2(\mu))} + \|\cdot\|_{L^2(0,T;\mathcal{F})}$, and satisfies

$$\|u\|_{L^\infty(0,T;L^2(\mu))}^2 + \int_0^T e^{C_\epsilon(T-s)} \mathcal{E}(u(s)) ds \leq \epsilon \int_0^T e^{C_\epsilon(T-s)} \|g(s)\|_{L^2(\nu)}^2 ds.$$

Moreover, $u(t)$ has a weak derivative $\partial_t u$ in $L^2(0,T;\mathcal{F}^{-1})$, and

$$\|\partial_t u\|_{L^2(0,T;\mathcal{F}^{-1})} \leq C_* \|g\|_{L^2(0,T;L^2(\nu))}.$$

Proof. As before, we set $g(t) = 0$ for $t < 0$. Let $\delta, \delta' \in (0, T)$ and $w = u_\delta - u_{\delta'}$, where u_δ are the functions defined by (4.4.4). By (4.4.5), we have

$$\partial_t w = \mathcal{L}w + P_\delta^0[(g(t-\delta) - g(t-\delta'))\nu] + (P_\delta^0 - P_{\delta'}^0)(g(t-\delta')\nu),$$

from which it follows that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w(t)\|_{L^2(\mu)}^2 &= -\mathcal{E}(w(t)) + \langle P_\delta^0[(g(t-\delta) - g(t-\delta'))\nu], w(t) \rangle_\mu \\ &\quad + \langle (P_\delta^0 - P_{\delta'}^0)(g(t-\delta')\nu), w(t) \rangle_\mu. \end{aligned} \quad (4.4.8)$$

The first term on the right hand side of (4.4.8) can be estimated in the same way as in the proof of Lemma 4.4.6, which yields that

$$\begin{aligned} &\langle P_\delta^0[(g(t-\delta) - g(t-\delta'))\nu], w(t) \rangle_\mu \\ &\leq \frac{1}{2} \|g(t-\delta) - g(t-\delta')\|_{L^2(\nu)}^2 + \frac{1}{2} \mathcal{E}(w(t))^{d_s-1} \|w(t)\|_{L^2(\mu)}^{2(2-d_s)}. \end{aligned}$$

For the second term on the right hand side of (4.4.8), we have

$$\langle (P_\delta^0 - P_{\delta'}^0)(g(t - \delta')\nu), w(t) \rangle_\mu \leq C_* \mathcal{E}((P_\delta^0 - P_{\delta'}^0)w(t))^{(d_s-1)/2} \|(P_\delta^0 - P_{\delta'}^0)w(t)\|_{L^2(\mu)}^{2-d_s} \|g(t - \delta')\|_{L^2(\nu)}.$$

Let $-\mathcal{L} = \int_0^\infty \lambda dE_\lambda$ be the spectral decomposition. We have

$$\begin{aligned} \|(P_\delta^0 - P_{\delta'}^0)w(t)\|_{L^2(\mu)}^2 &= \int_0^\infty (e^{-\lambda\delta} - e^{-\lambda\delta'})^2 d\|E_\lambda w(t)\|_{L^2(\mu)}^2 \\ &\leq \int_0^\infty (1 - e^{-\lambda|\delta - \delta'|})^2 d\|E_\lambda w(t)\|_{L^2(\mu)}^2 \\ &\leq |\delta - \delta'| \int_0^\infty \lambda d\|E_\lambda w(t)\|_{L^2(\mu)}^2 \\ &= |\delta - \delta'| \mathcal{E}(w(t)), \end{aligned}$$

which, together with the fact that $\mathcal{E}((P_\delta^0 - P_{\delta'}^0)w(t)) \leq \mathcal{E}(w(t))$, implies that

$$\langle (P_\delta^0 - P_{\delta'}^0)(g(t - \delta')\nu), w(t) \rangle_\mu \leq C_* |\delta - \delta'|^{1-d_s/2} \mathcal{E}(w(t))^{1/2} \|g(t - \delta')\|_{L^2(\nu)}.$$

Therefore, we deduce from (4.4.8) that

$$\begin{aligned} \frac{d}{dt} \|w(t)\|_{L^2(\mu)}^2 &\leq -\mathcal{E}(w(t)) + C_* \|w(t)\|_{L^2(\mu)}^2 + C_* \|g(t - \delta) - g(t - \delta')\|_{L^2(\nu)}^2 \\ &\quad + C_* |\delta - \delta'|^{2-d_s} \|g(t - \delta')\|_{L^2(\nu)}^2. \end{aligned}$$

It follows from the above inequality and Grönwall's inequality that

$$\begin{aligned} \|w\|_{L^\infty(0,T;L^2(\mu))} + \|w\|_{L^2(0,T;\mathcal{F})} &\leq C_* \left[|\delta - \delta'|^{2-d_s} \|g\|_{L^2(0,T;L^2(\nu))} \right. \\ &\quad \left. + \int_0^T \|g(t - \delta) - g(t - \delta')\|_{L^2(\nu)}^2 dt \right]. \end{aligned}$$

Therefore, $\{u_\delta\}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{L^\infty(0,T;L^2(\mu))} + \|\cdot\|_{L^2(0,T;\mathcal{F})}$, which proves the convergence of (4.4.7). Moreover, the desired estimates for u follows readily from the similar estimates for u_δ . \square

Definition 4.4.8. By virtue of Lemma 4.4.7, the convolution $\int_0^t P_{t-s}^0(g(s)\nu) ds$ can be defined to be the limit in (4.4.7).

Lemma 4.4.9. *If $g \in L^\infty(0, T; L^2(\nu))$, then the convolution u defined by (4.4.7) is jointly continuous in $[0, T] \times \mathbb{S}$. Moreover, for any $0 < \theta < \frac{3}{2}(1 - d_s/2)$,*

$$|u(t, x) - u(s, y)| \leq \|g\|_{L^\infty(0, T; L^2(\nu))} [C_\theta |t - s|^\theta + C_T R(x, y)^{1/2}], \quad (4.4.9)$$

where $C_\theta > 0$ is a constant depending only on θ , and $C_T > 0$ one depending only on T .

Remark 4.4.10. The author believes that $\frac{1}{2}$ is the correct Hölder exponent in $x \in \mathbb{S}$ for (4.4.7) in general, which is suggested by the fact that a generic $u \in \mathcal{F}(\mathbb{S})$ has only $\frac{1}{2}$ -Hölder continuity (cf. (A'.3)). As a matter of fact, the convolution (4.4.7) only has mild regularity in general due to the singularity between μ and ν (cf. Remark 4.4.12(ii)).

Proof. Let $g(t) = 0$ for $t < 0$. We first show that

$$|u(t, x) - u(t, y)| \leq C_T R(x, y)^{1/2}, \quad x, y \in \mathbb{S}, \quad (4.4.10)$$

where $C_T > 0$ is a constant depending only on T . Denote $p_{s,x}^0(y) = p^0(s, x, y)$. By the definition of $u(t)$, we have

$$\begin{aligned} |u(t, x) - u(t, y)| &= \left| \int_0^t \langle g(t-s), p_{s,x}^0 - p_{s,y}^0 \rangle_\nu ds \right| \\ &\leq \|g\|_{L^\infty(0, T; L^2(\nu))} \int_0^t \|p_{s,x}^0 - p_{s,y}^0\|_{L^2(\nu)} ds. \end{aligned} \quad (4.4.11)$$

By the Sobolev inequality (4.2.22),

$$\|p_{s,x}^0 - p_{s,y}^0\|_{L^2(\nu)} \leq C [\mathcal{E}(p_{s,x}^0 - p_{s,y}^0)^{(d_s-1)/2} \|p_{s,x}^0 - p_{s,y}^0\|_{L^2(\mu)}^{2-d_s} + \|p_{s,x}^0 - p_{s,y}^0\|_{L^2(\mu)}].$$

Let $-\mathcal{L} = \int_0^\infty \lambda dE_\lambda$ be the spectral representation. Then

$$\begin{aligned}
\mathcal{E}(p_{s,x}^0 - p_{s,y}^0) &= \mathcal{E}(P_{s/2}^0(p_{s/2,x}^0 - p_{s/2,y}^0)) \\
&= \int_0^\infty \lambda e^{-\lambda s} d\|E_\lambda(p_{s/2,x}^0 - p_{s/2,y}^0)\|_{L^2(\mu)}^2 \\
&\leq s^{-1} \|p_{s/2,x}^0 - p_{s/2,y}^0\|_{L^2(\mu)}^2,
\end{aligned} \tag{4.4.12}$$

Therefore,

$$\|p_{s,x}^0 - p_{s,y}^0\|_{L^2(\nu)} \leq s^{-(d_s-1)/2} \|p_{s/2,x}^0 - p_{s/2,y}^0\|_{L^2(\mu)}^{d_s-1} \|p_{s,x}^0 - p_{s,y}^0\|_{L^2(\mu)}^{2-d_s} + \|p_{s,x}^0 - p_{s,y}^0\|_{L^2(\mu)},$$

By the inequality above and Hölder's inequality,

$$\begin{aligned}
\int_0^t \|p_{s,x}^0 - p_{s,y}^0\|_{L^2(\nu)} ds &\leq \left(\int_0^t s^{1-d_s} ds \right)^{1/2} \left(\int_0^t \|p_{s/2,x}^0 - p_{s/2,y}^0\|_{L^2(\mu)}^2 ds \right)^{(d_s-1)/2} \\
&\quad \times \left(\int_0^t \|p_{s,x}^0 - p_{s,y}^0\|_{L^2(\mu)}^2 ds \right)^{1-d_s/2} + t^{1/2} \left(\int_0^t \|p_{s,x}^0 - p_{s,y}^0\|_{L^2(\mu)}^2 ds \right)^{1/2} \\
&\leq C_T \left(\int_0^t \|p_{s/2,x}^0 - p_{s/2,y}^0\|_{L^2(\mu)}^2 ds \right)^{1/2},
\end{aligned}$$

where we have used that $p_{s,x}^0 - p_{s,y}^0 = P_{s/2}^0(p_{s/2,x}^0 - p_{s/2,y}^0)$ and the $L^2(\mu)$ -contractivity of $P_{s/2}^0$ for the last inequality.

Let $\rho_\alpha^0(\cdot, \cdot)$ be the α -resolvent kernel. By the Chapman–Kolmogorov equation,

$$\begin{aligned}
\int_0^t \|p_{s/2,x}^0 - p_{s/2,y}^0\|_{L^2(\mu)}^2 ds &\leq e^{\alpha t} \int_0^\infty e^{-\alpha s} \|p_{s/2,x}^0 - p_{s/2,y}^0\|_{L^2(\mu)}^2 ds \\
&= e^{\alpha t} \int_0^\infty e^{-\alpha s} [p^0(s, x, x) - 2p^0(s, x, y) + p^0(s, y, y)] ds \\
&= e^{\alpha t} [\rho_\alpha^0(x, x) - 2\rho_\alpha^0(x, y) + \rho_\alpha^0(y, y)],
\end{aligned}$$

which, together with Lemma 4.4.2, implies that

$$\int_0^t \|p_{s/2,x}^0 - p_{s/2,y}^0\|_{L^2(\mu)}^2 ds \leq C_\alpha R(x, y).$$

Therefore, we deduce that

$$\int_0^t \|p_{s,x}^0 - p_{s,y}^0\|_{L^2(\nu)} ds \leq C_T R(x, y)^{1/2}. \quad (4.4.13)$$

Now the Hölder continuity (4.4.10) follows readily from (4.4.11) and (4.4.13).

Next, we turn to the Hölder continuity of $u(t, x)$ in t . Let $t \geq 0$, $\delta > 0$. By the definition of u ,

$$\begin{aligned} & u(t + \delta, x) - u(t, x) \\ &= \int_t^{t+\delta} P_{t+\delta-s}^0(g(s)\nu)(x) ds + \int_0^t [P_{t+\delta-s}^0(g(s)\nu)(x) - P_{t-s}^0(g(s)\nu)(x)] ds \\ &= I_1(\delta) + I_2(\delta). \end{aligned}$$

For $I_1(\delta)$, in the same way as (4.4.11), we have

$$\begin{aligned} |I_1(\delta)| &= \left| \int_0^\delta P_s^0(g(t-s+\delta)\nu)(x) ds \right| \\ &\leq \|g\|_{L^\infty(0,T;L^2(\nu))} \int_0^\delta \|p_{s,x}^0\|_{L^2(\nu)} ds. \end{aligned}$$

By the Sobolev inequality (4.2.22),

$$\|p_{s,x}^0\|_{L^2(\nu)} \leq \mathcal{E}(p_{s,x}^0)^{(d_s-1)/2} \|p_{s,x}^0\|_{L^2(\mu)}^{2-d_s}.$$

It follows from an argument similar to (4.4.12) that $\mathcal{E}(p_{s,x}^0) \leq s^{-1} \|p_{s/2,x}^0\|_{L^2(\mu)}^2$. Therefore,

$$\|p_{s,x}^0\|_{L^2(\nu)} \leq s^{-(d_s-1)/2} \|p_{s/2,x}^0\|_{L^2(\mu)}^{d_s-1} \|p_{s,x}^0\|_{L^2(\mu)}^{2-d_s}.$$

Using the Chapman–Kolmogorov equation and the estimate $p^0(t, x, y) \leq C_* t^{-d_s/2}$, we deduce from the above inequality that

$$\begin{aligned} \int_0^\delta \|p_{s,x}^0\|_{L^2(\nu)} ds &\leq \int_0^\delta s^{-(d_s-1)/2} p^0(s, x, x)^{(d_s-1)/2} p^0(2s, x, x)^{1-d_s/2} ds \\ &\leq C_* \int_0^\delta s^{-3d_s/4+1/2} ds = C_* \delta^{\frac{3}{2}(1-d_s/2)}. \end{aligned}$$

Thus

$$|I_1(\delta)| \leq C_* \|g\|_{L^\infty(0,T;L^2(\nu))} \delta^{\frac{3}{2}(1-d_s/2)}. \quad (4.4.14)$$

For $I_2(\delta)$, by the same argument as in the estimate of $|I_1(\delta)|$, we have

$$\begin{aligned} |I_2(\delta)| &\leq \|g\|_{L^\infty(0,T;L^2(\nu))} \int_0^t \|p_{s+\delta,x}^0 - p_{s,x}^0\|_{L^2(\nu)} ds \\ &\leq C_* \|g\|_{L^\infty(0,T;L^2(\nu))} \int_0^t \|p_{s/2+\delta,x}^0 - p_{s/2,x}^0\|_{L^2(\mu)}^{d_s-1} \|p_{s+\delta,x}^0 - p_{s,x}^0\|_{L^2(\mu)}^{2-d_s} \frac{ds}{s^{(d_s-1)/2}} \\ &\leq C_* \|g\|_{L^\infty(0,T;L^2(\nu))} \int_0^t \|p_{s/2+\delta,x}^0 - p_{s/2,x}^0\|_{L^2(\mu)} \frac{ds}{s^{(d_s-1)/2}}. \end{aligned} \quad (4.4.15)$$

For any $\theta \in [0, 1]$, by the spectral representation,

$$\begin{aligned} \|p_{s/2+\delta,x}^0 - p_{s/2,x}^0\|_{L^2(\mu)}^2 &= \|(P_{s/4+\delta}^0 - P_{s/4}^0)p_{s/4,x}^0\|_{L^2(\mu)}^2 \\ &= \int_0^\infty (1 - e^{-\delta\lambda})^2 e^{-s\lambda/2} d\|E_\lambda p_{s/4,x}^0\|_{L^2(\mu)}^2 \\ &\leq \delta^{2\theta} \int_0^\infty \lambda^{2\theta} e^{-s\lambda/2} d\|E_\lambda p_{s/4,x}^0\|_{L^2(\mu)}^2 \\ &\leq C_* (\delta/s)^{2\theta} \|p_{s/4,x}^0\|_{L^2(\mu)}^2 = C_* (\delta/s)^{2\theta} p^0(s/2, x, x) \leq C_* \delta^{2\theta} s^{-2\theta-d_s/2}, \end{aligned}$$

which, together with (4.4.15), implies that

$$|I_2(\delta)| \leq C_* \|g\|_{L^\infty(0,T;L^2(\nu))} \delta^\theta \int_0^t s^{\frac{3}{2}(1-d_s/2)-\theta} \frac{ds}{s}.$$

Therefore, for any $\theta < \frac{3}{2}(1 - d_s/2)$, the estimate

$$|I_2(\delta)| \leq C_\theta \|g\|_{L^\infty(0,T;L^2(\nu))} \delta^\theta, \quad (4.4.16)$$

is valid. Combining (4.4.14) and (4.4.16), we deduce that

$$|u(t, x) - u(s, x)| \leq C_\theta \|g\|_{L^\infty(0,T;L^2(\nu))} |t - s|^\theta, \quad (4.4.17)$$

for all $0 < \theta < \frac{3}{2}(1 - d_s/2)$.

Now the joint Hölder continuity (4.4.9) follows readily from (4.4.10) and (4.4.17). \square

Let us introduce the following definition of weak solutions, which seems weaker than the one used in Section 2.4 (cf. Definition 2.4.3). However, as we shall see, these definitions are in fact equivalent.

Definition 4.4.11. A function u is called a *weak solution* to the PDE (4.4.1) if:

(WS'.1) $u \in L^2(0, T; \mathcal{F}(\mathbb{S} \setminus V_0))$ and u has a weak derivative $\partial_t u$ in $L^2(0, T; \mathcal{F}^{-1}(\mathbb{S}))$;

(WS'.2) For any $v \in \mathcal{F}(\mathbb{S} \setminus V_0)$,

$$\langle \partial_t u(t), v \rangle_\mu = -\mathcal{E}(u(t), v) + \langle f(t, u(t), \nabla u(t)), v \rangle_\nu, \quad \text{a.e. } t \in [0, T];$$

(WS'.3) $\lim_{t \rightarrow 0} u(t) = \psi$ in $L^2(\mu)$.

Remark 4.4.12. (i) The term $\langle f(t, u(t), \nabla u(t)), v \rangle_\nu$ in (WS'.2) is legitimate since ∇u is ν -a.e. defined and $u \in \mathcal{F}(\mathbb{S}) \subseteq C(\mathbb{S})$.

(ii) Notice that, in general, the equation (4.4.1) does not admit a solution u such that $u(t) \in \text{Dom}(\mathcal{L})$ and $\partial_t u(t) \in L^2(\mu)$ for a.e. $t \in [0, T]$. Otherwise, the functional $v \mapsto \langle f(t, u, \nabla u), v \rangle_\nu$ will be $L^2(\mu)$ -bounded, which contradicts the singularity between μ and ν . Therefore, solutions to non-linear parabolic PDEs on \mathbb{S} can only have mild regularity in general. This is a remarkable feature of non-linear PDEs on \mathbb{S} , which suggests a significant distinction between the PDE theory on Euclidean spaces and that on fractals.

(iii) We shall show that if u is a weak solution to (4.4.1) then $u \in C((0, T] \times \mathbb{S})$ (see Theorem 4.4.14). Therefore, Definition 4.4.11 coincides with Definition 2.4.3. The joint continuity of solutions is needed for the validity of the Feynman–Kac representation given by Theorem 2.4.5, which will be crucial in the study of the Burgers equations on \mathbb{S} (see Section 4.5).

Proposition 4.4.13. *Suppose that $g \in L^2(0, T; L^2(\nu))$. Then the initial and boundary problem to the PDE*

$$\begin{cases} \partial_t u \, d\mu = \mathcal{L}u \, d\mu + g(t, x) \, d\nu, & \text{in } (0, T] \times (\mathbb{S} \setminus V_0), \\ u = 0 & \text{on } (0, T] \times V_0, \quad u(0) = \psi \end{cases} \quad (4.4.18)$$

admits a unique weak solution u given by

$$u(t) = P_t^0 \psi + \int_0^t P_{t-s}^0 (g(s) \nu) \, ds, \quad t \in [0, T].$$

Moreover

$$\begin{aligned} \|u\|_{L^\infty(0, T; L^2(\mu))} + \|u\|_{L^2(0, T; \mathcal{F})} + \|\partial_t u\|_{L^2(0, T; \mathcal{F}^{-1})} \\ \leq C_* (\|\psi\|_{L^2(\mu)}^2 + \|g\|_{L^2(0, T; L^2(\nu))}). \end{aligned} \quad (4.4.19)$$

Proof. Clearly, we only need to prove for the case when $\psi = 0$. Let u_δ be the truncated convolution defined by (4.4.4), and let u be the convolution given by (4.4.7). For any $v \in \mathcal{F}(\mathbb{S} \setminus V_0)$, by (4.4.5),

$$\langle \partial_t u_\delta(t), v \rangle_\mu = -\mathcal{E}(u_\delta(t), v) + \langle P_\delta^0(g(t-\delta)\mu), v \rangle_\mu = -\mathcal{E}(u_\delta(t), v) + \langle g(t-\delta), P_\delta^0 v \rangle_\nu, \quad \text{a.e. } t \in (0, T].$$

Since $\lim_{\delta \rightarrow 0} P_\delta^0 v = v$ uniformly, by considering a subsequence if necessary and setting $\delta \rightarrow 0$, we deduce that

$$\langle \partial_t u(t), v \rangle_\mu = -\mathcal{E}(u(t), v) + \langle g(t), v \rangle_\nu, \quad \text{a.e. } t \in (0, T].$$

Therefore, u is a weak solution to (4.4.18).

The estimate (4.4.19) follows readily from Lemma 4.4.7, and the uniqueness of solutions is an immediate consequence of (4.4.19). \square

We are now in a position to state and give the proof of the main result of this section.

Theorem 4.4.14. *Suppose that (A.1) and (A.2) hold. Then (4.4.1) admits a unique weak solution u satisfying the following estimate*

$$\begin{aligned} & \|u\|_{L^\infty(0,T;L^2(\mu))} + \|u\|_{L^2(0,T;\mathcal{F})} + \|\partial_t u\|_{L^2(0,T;\mathcal{F}^{-1})} \\ & \leq C_{K,T} \left(\|\psi\|_{L^2(\mu)} + \|f(\cdot, 0, 0)\|_{L^2(0,T;L^2(\nu))} \right), \end{aligned} \quad (4.4.20)$$

where $C_{K,T} > 0$ is a constant depending only on T and the Lipschitz constant K in (A.1).

Moreover, if \tilde{u} is the weak solution to (4.4.1) with initial value $\tilde{\psi} \in L^2(\mu)$, then

$$\begin{aligned} & \|u - \tilde{u}\|_{L^\infty(0,T;L^2(\mu))} + \|u - \tilde{u}\|_{L^2(0,T;\mathcal{F})} \\ & + \|\partial_t u - \partial_t \tilde{u}\|_{L^2(0,T;\mathcal{F}^{-1})} \leq C_{K,T} \|\psi - \tilde{\psi}\|_{L^2(\mu)}. \end{aligned} \quad (4.4.21)$$

Suppose, in addition, that $\psi \in \mathcal{F}(\mathbb{S} \setminus V_0)$ and $f(\cdot, 0, 0) = 0$. Then

$$\|u\|_{L^\infty(0,T;\mathcal{F})} \leq C_{K,T} \mathcal{E}(\psi)^{1/2}. \quad (4.4.22)$$

Moreover, $u(t, x)$ is jointly continuous in $(0, T] \times \mathbb{S}$, with θ -Hölder continuity in $t \in (0, T]$ for any $\theta < \frac{3}{2}(1 - d_s/2)$ and $\frac{1}{2}$ -Hölder continuity in $x \in \mathbb{S}$ with respect to the resistance metric.

Proof. We first prove the existence. Let $u^0(t) = P_t^0 \psi$, $t \in [0, T]$. By Proposition 4.4.13, we may define a sequence $\{u^n\}_{n \in \mathbb{N}_+}$ in $L^2(0, T; \mathcal{F}(\mathbb{S} \setminus V_0))$ inductively by

$$\begin{cases} \partial_t u^n d\mu = \mathcal{L}u^n d\mu + f^{n-1}(t) d\nu, \\ u^n|_{V_0} = 0, \quad u^n(0) = \psi, \end{cases} \quad (4.4.23)$$

where $f^{n-1}(t, x) = f(t, x, u^{n-1}(t, x), \nabla u^{n-1}(t, x))$. By Proposition 4.4.13, $u^n \in L^2(0, T; \mathcal{F}(\mathbb{S} \setminus V_0))$, $\partial_t u^n \in L^2(0, T; \mathcal{F}^{-1}(\mathbb{S}))$. Denote $w^n = u^n - u^{n-1}$, $n \in \mathbb{N}_+$. By (4.4.23),

w^{n+1} , $n \in \mathbb{N}_+$ is the solution to

$$\begin{cases} \partial_t w^{n+1} d\mu = \mathcal{L}w^{n+1} d\mu + [f^n(t) - f^{n-1}(t)] d\nu, \\ w^{n+1}|_{V_0} = 0, \quad w^{n+1}(0) = 0. \end{cases} \quad (4.4.24)$$

By Lemma 2.2.7(b), testing (4.4.24) against $w^{n+1}(t)$ gives that

$$\frac{1}{2} \frac{d}{dt} \|w^{n+1}(t)\|_{L^2(\mu)}^2 \leq -\mathcal{E}(w^{n+1}(t)) + \frac{1}{16} \mathcal{E}(w^n(t)) + C_K \|w^{n+1}(t)\|_{L^2(\nu)}^2,$$

where, and in the rest of the proof, $C_K > 0$ denotes a generic constant depending only on K which may vary on different occasions. For any $\epsilon \in (0, 1)$, by Corollary 4.2.18,

$$\frac{d}{dt} \|w^{n+1}(t)\|_{L^2(\mu)}^2 \leq -(1 - C_K \epsilon^2) \mathcal{E}(w^{n+1}(t)) + \frac{1}{8} \mathcal{E}(w^n(t)) + C_K C_\epsilon^2 \|w^{n+1}(t)\|_{L^2(\mu)}^2,$$

where $C_\epsilon > 0$ is a constant depending only on ϵ . By choosing $\epsilon > 0$ sufficiently small, we have that

$$\frac{d}{dt} \|w^{n+1}(t)\|_{L^2(\mu)}^2 \leq -\mathcal{E}(w^{n+1}(t)) + \frac{1}{8} \mathcal{E}(w^n(t)) + C_K \|w^{n+1}(t)\|_{L^2(\mu)}^2, \quad (4.4.25)$$

By the above and Grönwall's inequality,

$$\|w^{n+1}(t)\|_{L^2(\mu)}^2 + \int_0^t e^{C_K(t-s)} \mathcal{E}(w^{n+1}(s)) ds \leq \frac{1}{8} \int_0^t e^{C_K(t-s)} \mathcal{E}(w^n(s)) ds, \quad (4.4.26)$$

which implies that

$$\begin{aligned} & \|u^m - u^{m-1}\|_{L^\infty(0,T;L^2(\mu))} + \left(\int_0^T e^{C_K(T-s)} \mathcal{E}(u^m(s) - u^{m-1}(s)) ds \right)^{1/2} \\ & \leq 2^{-m+n} \left[\|u^n - u^{n-1}\|_{L^\infty(0,T;L^2(\mu))} + \left(\int_0^T e^{C_K(T-s)} \mathcal{E}(u^n(s) - u^{n-1}(s)) ds \right)^{1/2} \right], \end{aligned} \quad (4.4.27)$$

for all $m \geq n$, and that

$$\begin{aligned} \|u^n\|_{L^\infty(0,T;L^2(\mu))} + \left(\int_0^T e^{C_K(T-s)} \mathcal{E}(u^n(s)) ds \right)^{1/2} \\ \leq \|u^1\|_{L^\infty(0,T;L^2(\mu))} + \left(\int_0^T e^{C_K(T-s)} \mathcal{E}(u^1(s)) ds \right)^{1/2}. \end{aligned}$$

Moreover, by Proposition 4.4.13, we have

$$\begin{aligned} \|u^1\|_{L^\infty(0,T;L^2(\mu))} + \left(\int_0^T e^{C_K(T-s)} \mathcal{E}(u^1(s)) ds \right)^{1/2} \\ \leq C_* e^{C_K T} (\|\psi\|_{L^2(\mu)} + \|f(\cdot, u^0, \nabla u^0)\|_{L^2(0,T;L^2(\nu))}) \\ \leq C_* e^{C_K T} (\|\psi\|_{L^2(\mu)} + \|u^0\|_{L^2(0,T;\mathcal{F})} + \|f(\cdot, 0, 0)\|_{L^2(0,T;L^2(\nu))}). \end{aligned}$$

Let $-\mathcal{L} = \int_0^\infty \lambda dE_\lambda$ be the spectral decomposition. Then

$$\begin{aligned} \|u^0\|_{L^2(0,T;\mathcal{F})}^2 &\leq T\|\psi\|_{L^2(\mu)}^2 + \int_0^T \mathcal{E}(P_t \psi) dt \\ &= T\|\psi\|_{L^2(\mu)}^2 + \int_0^T \int_0^\infty \lambda e^{-2t\lambda} d\|E_\lambda \psi\|_{L^2(\mu)}^2 dt \\ &= T\|\psi\|_{L^2(\mu)}^2 + \frac{1}{2} \int_0^\infty (1 - e^{-2T\lambda}) d\|E_\lambda \psi\|_{L^2(\mu)}^2 \\ &\leq (T + 1)\|\psi\|_{L^2(\mu)}^2. \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} \|u^n\|_{L^\infty(0,T;L^2(\mu))} + \left(\int_0^T e^{C_K(T-s)} \mathcal{E}(u^n(s)) ds \right)^{1/2} \\ \leq C_* e^{C_K T} (\|\psi\|_{L^2(\mu)} + \|f(\cdot, 0, 0)\|_{L^2(0,T;L^2(\nu))}), \quad n \in \mathbb{N}_+. \end{aligned} \tag{4.4.28}$$

Furthermore, by (4.4.23), $u^m - u^n$ is the solution to

$$\begin{cases} \partial_t(u^m - u^n) d\mu = \mathcal{L}(u^m - u^n) d\mu + [f^{m-1}(t) - f^{n-1}(t)] d\nu, \\ (u^m - u^n)|_{V_0} = 0, \quad (u^m - u^n)(0) = 0. \end{cases}$$

For any $v \in \mathcal{F}(\mathbb{S} \setminus V_0)$, by the above equation,

$$|\langle \partial_t(u^m - u^n), v \rangle_\mu| \leq [\mathcal{E}(u^m - u^n)^{1/2} + C_K \mathcal{E}(u^{m-1} - u^{n-1})^{1/2}] \mathcal{E}(v)^{1/2},$$

which implies that

$$\|\partial_t(u^m - u^n)\|_{\mathcal{F}^{-1}} \leq \mathcal{E}(u^m - u^n)^{1/2} + C_K \mathcal{E}(u^{m-1} - u^{n-1})^{1/2}. \quad (4.4.29)$$

By (4.4.27) and (4.4.28), we see that

$$\|\partial_t(u^m - u^n)\|_{L^2(0,T;\mathcal{F}^{-1})} \leq C_* e^{C_K T} 2^{-(m-n)} (\|\psi\|_{L^2(\mu)} + \|f(\cdot, 0, 0)\|_{L^2(0,T;L^2(\nu))}).$$

Therefore, $\{u^n\}$ is a $\|\cdot\|_*$ -Cauchy sequence satisfying

$$\|u^n\|_* \leq C_* e^{C_K T} (\|\psi\|_{L^2(\mu)} + \|f(\cdot, 0, 0)\|_{L^2(0,T;L^2(\nu))}), \quad n \in \mathbb{N}_+,$$

where

$$\|u\|_* = \|u\|_{L^\infty(0,T;L^2(\mu))} + \|u\|_{L^2(0,T;\mathcal{F})} + \|\partial_t u\|_{L^2(0,T;\mathcal{F}^{-1})}.$$

Therefore, there exists a $u \in L^2(0, T; \mathcal{F}(\mathbb{S} \setminus V_0))$ such that $\lim_{n \rightarrow \infty} \|u^n - u\|_* = 0$. It is clear that u is a weak solution to (4.4.1), and the estimate (4.4.20) holds as $\mathcal{E}^{1/2}(\cdot)$ and $\|\cdot\|_{\mathcal{F}}$ are equivalent on $\mathcal{F}(\mathbb{S} \setminus V_0)$. This proves the existence.

Suppose that \tilde{u} is a weak solution to (4.4.1) with initial value $\tilde{\psi}$. By an argument similar to (4.4.25) and (4.4.29), it can be shown that

$$\frac{d}{dt} \|u(t) - \tilde{u}(t)\|_{L^2(\mu)}^2 \leq -\frac{1}{2} \mathcal{E}(u(t) - \tilde{u}(t)) + C_K \|u(t) - \tilde{u}(t)\|_{L^2(\mu)}^2,$$

and that

$$\|\partial_t(u - \tilde{u})\|_{\mathcal{F}^{-1}} \leq C_K \mathcal{E}(u - \tilde{u})^{1/2}.$$

The estimate (4.4.21) follows readily from the above two inequalities. The uniqueness of

solutions is now an immediate consequence of (4.4.21).

Suppose, in addition, that $\psi \in \mathcal{F}(\mathbb{S} \setminus V_0)$ and $f(\cdot, 0, 0) = 0$. Then (4.4.26) also holds for $n = 0$ with $u^{-1} = 0$. Therefore,

$$\int_0^T \mathcal{E}(u^m(t)) dt \leq e^{C_K T} \int_0^T \mathcal{E}(u^0(t)) dt = e^{C_K T} \int_0^T \mathcal{E}(P_t \psi) dt \leq T e^{C_K T} \mathcal{E}(\psi),$$

which implies that

$$\int_0^T \mathcal{E}(u(t)) dt \leq T e^{C_K T} \mathcal{E}(\psi). \quad (4.4.30)$$

Now for any $\delta \in (0, T)$, u is the solution to

$$\begin{cases} \partial_t u d\mu = \mathcal{L}u d\mu + f(t, x, u, \nabla u) d\nu, & t \in (t_0, t_0 + \delta], \\ u|_{V_0} = 0, \quad u|_{t=t_0} = u(t_0). \end{cases}$$

Applying (4.4.30) to the above PDE and using $\|u\|_{L^2(\mu)}^2 \leq C_* \mathcal{E}(u)$ gives that

$$\frac{1}{\delta} \int_{t_0}^{t_0+\delta} \mathcal{E}(u(t)) dt \leq e^{C_K \delta} \mathcal{E}(u(t_0)), \quad \text{a.e. } t_0 \in [0, T - \delta] \text{ and any } \delta > 0. \quad (4.4.31)$$

We claim that (4.4.31) implies (4.4.22). We first show the following lemma.

Lemma. *Let $h(t)$ be a locally integrable function on $[0, \infty)$ satisfying*

$$\frac{1}{\delta} \int_t^{t+\delta} h(s) ds \leq L\delta + h(t), \quad \text{a.e. } t \in [0, \infty) \text{ and any } \delta > 0, \quad (4.4.32)$$

for some constant $L > 0$. Then $h(t) - h(s) \leq 6L(t - s)$, a.e. $0 < s \leq t < \infty$.

Proof of the lemma. Suppose first that h is differentiable on $(0, \infty)$. Suppose the contrary that $h(t) - h(s) > 3L(t - s)$ for some $0 < s < t < \infty$. Then there exists a $t_0 \in (s, t)$ such that $h'(t_0) > 3L$. Moreover, $h(r) - h(t_0) > 3L(r - t_0)$, $r \in [t_0, t_0 + \delta]$ for $\delta > 0$ sufficiently small. This implies that $\frac{1}{\delta} \int_{t_0}^{t_0+\delta} h(r) dr > 3L\delta/2 + h(t_0)$, which contradicts (4.4.32). This proves the lemma for differentiable functions h .

For general h , let $h_\epsilon(t) = \frac{1}{\epsilon} \int_t^{t+\epsilon} h(s) ds$, $\epsilon > 0$. Then h_ϵ is differentiable and satisfies (4.4.32) with L replaced by $2L$. The above case gives that $h_\epsilon(t) - h_\epsilon(s) \leq 6L(t-s)$. It remains to apply the Lebesgue differentiation theorem to complete the proof of the lemma.

Now by (4.4.31) and Jensen's inequality, the function $h(t) = \log[\mathcal{E}(u(t))]$ satisfies (4.4.32) with $L = C_K$. It follows from the previous lemma that

$$\mathcal{E}(u(t)) \leq e^{C_K(t-s)} \mathcal{E}(u(s)), \text{ a.e. } 0 < s \leq t \leq T.$$

Using the above inequality and (4.4.30) again, we deduce that

$$\mathcal{E}(u(t)) \leq \frac{1}{t} \int_0^t e^{C_K(t-s)} \mathcal{E}(u(s)) ds \leq e^{C_K t} \mathcal{E}(\psi), \text{ a.e. } t \in (0, T],$$

which implies (4.4.22).

We now prove the joint Hölder continuity. Let $g(t, x) = f(t, x, u(t, x), \nabla u(t, x))$. Then u is the solution to the PDE

$$\partial_t u d\mu = \mathcal{L}u d\mu + g(t) d\nu.$$

By Proposition 4.4.13,

$$u(t) = P_t^0 \psi + \int_0^t P_{t-s}^0 (g(s) \nu) ds.$$

By (4.4.22) and the Sobolev inequality (4.2.22), it is easily seen that

$$\|g\|_{L^\infty(0, T; L^2(\nu))} < \infty.$$

We can now apply Lemma 4.4.9 and Proposition 4.4.13 and to deduce the desired joint Hölder continuity. □

4.5 Burgers equations

As an application of Theorem 4.4.14 and the Feynman–Kac representation for (backward) parabolic PDEs on \mathbb{S} (Theorem 2.4.5), we study the initial–boundary value problem for the following analogue on \mathbb{S} of the Burgers equations on \mathbb{R}

$$\begin{cases} \partial_t u \, d\mu = \mathcal{L}u \, d\mu + u \nabla u \, d\nu, & \text{in } (0, T] \times (\mathbb{S} \setminus V_0), \\ u = 0 & \text{on } (0, T] \times V_0, \quad u(0) = \psi, \end{cases} \quad (4.5.1)$$

where $\psi \in \mathcal{F}(\mathbb{S} \setminus V_0)$. We shall prove the existence and uniqueness of solutions to the equation (4.5.1), and derive the regularity of the solutions.

Remark 4.5.1. We would like to point out a difference between the Burgers equations on \mathbb{S} and those on \mathbb{R} . The Burgers equations on \mathbb{R} can be exactly solved with an explicit formula for the solutions via the Cole–Hopf transformation, and properties of solutions can be derived using the explicit formula. However, this Cole–Hopf type of transformation is not available on \mathbb{S} . The Cole–Hopf transformation reduces the Burgers equation on \mathbb{R} for u to a heat equation for $-\nabla(\log u)$. In contrast, on \mathbb{S} , the formal expression $\mathcal{L}[\nabla(\log u)]$ is not well-defined, since the gradient $\nabla(\log u)$ is only ν -a.e. defined and therefore $\nabla(\log u) \notin \mathcal{F}(\mathbb{S})$ due to the singularity between μ and ν . Hence, different approaches must be employed for the study of (4.5.1). However, the reader should be advised of a related work [27] on Burgers equations, where the Cole–Hopf transformation remains valid. As pointed out in Section 4.1, the main difference between our setting and that of [27] lies in the difference in definitions of Laplacians.

Recall that the representation in Theorem 2.4.5 is given for backward parabolic equations. For the convenience of later use, we reformulate the result for forward equations.

Theorem 4.5.2. *If the PDE (4.4.1) admits a weak solution u jointly continuous in $(0, T] \times \mathbb{S}$, then*

$$(Y_t, Z_t) = (u(T - t, X_t), \nabla u(T - t, X_t))$$

is the unique solution to the BSDE

$$\begin{cases} dY_t = -f(T-t, X_t, Y_t, Z_t)d\langle W \rangle_t + Z_t dW_t, & t \in [0, \sigma^{(T)}), \\ Y_{\sigma^{(T)}} = \Psi(\sigma^{(T)}, X_{\sigma^{(T)}}), \end{cases}$$

on (Ω, \mathbb{P}_x) for each $x \in \mathbb{S}$, where $\sigma^{(T)} = T \wedge \inf\{t > 0 : X_t \in V_0\}$, and

$$\Psi(t, x) = \begin{cases} 0, & \text{if } (t, x) \in [0, T) \times V_0, \\ \psi(x), & \text{if } (t, x) \in \{T\} \times \mathbb{S} \setminus V_0. \end{cases}$$

Moreover, the solution to (4.4.1) has the representation $u(T, x) = Y_0 = \mathbb{E}_x(Y_0)$ for all $x \in \mathbb{S}$.

Proposition 4.5.3. *The Burgers equation (4.5.1) admits a unique weak solution u satisfying the maximal principle below*

$$\|u\|_{L^\infty(0, T; L^\infty)} \leq \|\psi\|_{L^\infty}.$$

Moreover,

$$\|u\|_{L^\infty(0, T; L^2(\mu))} + \|u\|_{L^2(0, T; \mathcal{F})} + \|\partial_t u\|_{L^2(0, T; \mathcal{F}^{-1})} \leq C, \quad (4.5.2)$$

for some constant $C > 0$ depending only on $\|\psi\|_{L^\infty}$ and T . The solution u is jointly continuous in $(0, T] \times \mathbb{S}$, with θ -Hölder continuity in $t \in (0, T]$ for any $\theta < \frac{3}{2}(1 - d_s/2)$ and $\frac{1}{2}$ -Hölder continuity in $x \in \mathbb{S}$ with respect to the resistance metric.

Proof. Existence. We define the sequence $\{u^n\}_{n \in \mathbb{N}} \subseteq L^2(0, T; \mathcal{F})$ by induction as follows. Let $u^0(t) = P_t \psi$. Then $\|u^0\|_{L^\infty(0, T; L^\infty)} \leq \|\psi\|_{L^\infty}$. Suppose that u^{n-1} with $\|u^{n-1}\|_{L^\infty(0, T; L^\infty)} \leq \|\psi\|_{L^\infty}$ has been defined. The function u^n is defined to be the unique weak solution to the

PDE (cf. Theorem 4.4.14)

$$\begin{cases} \partial_t u^n d\mu = \mathcal{L}u^n d\mu + u^{n-1} \nabla u^n d\nu, & \text{in } (0, T] \times (\mathbb{S} \setminus V_0), \\ u^n = 0 & \text{on } (0, T] \times V_0, \quad u^n(0) = \psi. \end{cases}$$

To verify the definition of $\{u^n\}$, we must show that $\|u^n\|_{L^\infty(0, T; L^\infty)} \leq \|\psi\|_{L^\infty}$. Without loss of generality, we only need to show that $\|u^n(T)\|_{L^\infty} \leq \|\psi\|_{L^\infty}$. By Theorem 4.5.2, $(Y_t, Z_t) = (u^n(T-t, X_t), \nabla u^n(T-t, X_t))$ is the unique solution to the BSDE

$$\begin{cases} dY_t = -u^{n-1}(T-t, X_t) Z_t d\langle W \rangle_t + Z_t dW_t, & t \in [0, \sigma^{(T)}), \\ Y_{\sigma^{(T)}} = \Psi(\sigma^{(T)}, X_{\sigma^{(T)}}), \end{cases} \quad (4.5.3)$$

where $\sigma^{(T)} = T \wedge \inf\{t > 0 : X_t \in V_0\}$, and

$$\Psi(t, x) = \begin{cases} 0, & \text{if } (t, x) \in [0, T) \times V_0, \\ \psi(x), & \text{if } (t, x) \in \{T\} \times \mathbb{S} \setminus V_0. \end{cases}$$

For each $x \in \mathbb{S} \setminus V_0$, we define a measure $\tilde{\mathbb{P}}_x$ by

$$\frac{d\tilde{\mathbb{P}}_x}{d\mathbb{P}_x} = \exp \left[\int_0^{\sigma^{(T)}} u^{n-1}(T-r, X_r) dW_r - \frac{1}{2} \int_0^{\sigma^{(T)}} u^{n-1}(T-r, X_r)^2 d\langle W \rangle_r \right].$$

The measure $\tilde{\mathbb{P}}_x$ is a probability measure. In fact, by Corollary 2.2.16,

$$\sup_{x \in \mathbb{S}} \mathbb{E}_x[\exp(\beta \langle W \rangle_T)] < \infty,$$

for all $\beta, T > 0$. Hence, in view of the uniform boundedness $\|u^{n-1}\|_{L^\infty(0, T; L^\infty)} \leq \|\psi\|_{L^\infty}$, we see that the Novikov condition is satisfied and therefore $\tilde{\mathbb{P}}_x$ is a probability measure. By

(4.5.3),

$$Y_t = Y_0 + \int_0^t Z_r dW_r - \left\langle \int_0^t Z_r dW_r, \int_0^t u^{n-1}(T-r, X_r) dW_r \right\rangle_t.$$

Notice that

$$\mathbb{E}_\nu \left(\int_0^T Z_r^2 d\langle W \rangle_r \right) = \int_0^T \|\nabla u^n(T-r)\|_{L^2(\nu)}^2 dr \leq \|u^n\|_{L^2(0,T;\mathcal{F})}^2 < \infty,$$

which implies that $\mathbb{E}_x \left(\int_0^T Z_r^2 d\langle W \rangle_r \right) < \infty$ for μ -a.e. $x \in \mathbb{S}$ and therefore, for all $x \in \mathbb{S}$ in view of the quasi-continuity of the function $x \mapsto \mathbb{E}_x \left(\int_0^T Z_r^2 d\langle W \rangle_r \right)$ and the fact that the empty set is the only subset of \mathbb{S} having zero capacity since $\mathcal{F}(\mathbb{S}) \subseteq C(\mathbb{S})$. Hence, $\int Z_r dW_r$ is a \mathbb{P}_x -martingale for all $x \in \mathbb{S}$. Moreover, it follows from the Girsanov theorem that $\{Y_t\}_{t \geq 0}$ is a $\tilde{\mathbb{P}}_x$ -martingale, and therefore,

$$u^n(T, x) = Y_0 = \tilde{\mathbb{E}}_x(Y_0) = \tilde{\mathbb{E}}_x(Y_{\sigma(T)}) = \tilde{\mathbb{E}}_x(\Psi(\sigma^{(T)}, X_{\sigma(T)})),$$

which, together with the fact that $|\Psi| \leq \|\psi\|_{L^\infty}$, implies that $\|u^n(T)\|_{L^\infty} \leq \|\psi\|_{L^\infty}$. Hence, we conclude that $\|u^n\|_{L^\infty(0,T;L^\infty)} \leq \|\psi\|_{L^\infty}$, and that the sequence $\{u^n\}$ is well-defined.

Now, by Theorem 4.4.14,

$$\|u^n\|_{L^2(0,T;\mathcal{F})} + \|\partial_t u^n\|_{L^2(0,T;\mathcal{F}^{-1})} \leq CT, \quad n \in \mathbb{N},$$

where $C > 0$ is a generic constant depending only on $\|\psi\|_{L^\infty}$ which may vary on different occasions. Therefore, there exists a subsequence $\{u^{n_k}\}$ and a $u \in L^2(0, T; \mathcal{F}(\mathbb{S} \setminus V_0))$ such that $\partial_t u \in L^2(0, T; \mathcal{F}^{-1}(\mathbb{S}))$, and

$$\lim_{k \rightarrow \infty} u^{n_k} = u, \quad \text{weakly in } L^2(0, T; \mathcal{F}(\mathbb{S} \setminus V_0)), \quad (4.5.4)$$

$$\lim_{k \rightarrow \infty} \partial_t u^{n_k} = \partial_t u, \quad \text{weakly in } L^2(0, T; \mathcal{F}^{-1}(\mathbb{S})). \quad (4.5.5)$$

Since $\|u^n\|_{L^\infty(0,T;L^\infty)} \leq \|\psi\|_{L^\infty}$, the sequence $\{u^n \nabla u^n\}_{n \in \mathbb{N}_+}$ is bounded in $L^2(0, T; L^2(\nu))$. By considering a subsequence of $\{u^{n_k}\}$ if necessary, we may assume that $\{u^{n_k} \nabla u^{n_k}\}$ is weakly convergent in $L^2(0, T; L^2(\nu))$. By the uniqueness of weak limits,

$$\lim_{k \rightarrow \infty} u^{n_k} \nabla u^{n_k} = u \nabla u, \quad \text{weakly in } L^2(0, T; L^2(\nu)). \quad (4.5.6)$$

Thus, it follows readily from (4.5.4)–(4.5.6) that u is a weak solution to (4.5.1). Moreover, the estimate $\|u\|_{L^\infty(0,T;L^\infty)} \leq \|\psi\|_{L^\infty}$ follows as a corollary of the inequalities $\|u^n\|_{L^\infty(0,T;L^\infty)} \leq \|\psi\|_{L^\infty}$.

Testing (4.5.1) against $u(t)$ and using the Sobolev inequality (4.2.22) gives that for any $\epsilon \in (0, 1)$ and a.e. $t \in [0, T]$,

$$\frac{d}{dt} \|u(t)\|_{L^2(\mu)}^2 \leq -\mathcal{E}(u(t)) + \|\psi\|_{L^\infty} [\epsilon \mathcal{E}(u(t))^{1/2} + C_\epsilon \|u(t)\|_{L^2(\mu)}] \mathcal{E}(u(t))^{1/2}.$$

Choosing $\epsilon > 0$ sufficiently small gives that

$$\frac{d}{dt} \|u(t)\|_{L^2(\mu)}^2 \leq -\frac{1}{2} \mathcal{E}(u(t)) + C \|u(t)\|_{L^2(\mu)}^2, \quad \text{a.e. } t \in [0, T],$$

from which the estimate (4.5.2) follows readily.

Uniqueness. Suppose that \bar{u} is also a weak solution to (4.5.1). Then $\|u\|_{L^\infty(0,T;L^\infty)} + \|\bar{u}\|_{L^\infty(0,T;L^\infty)} \leq 2\|\psi\|_{L^\infty}$. For any $\epsilon > 0$, testing the equation for $u(t) - \bar{u}(t)$ against $u(t) - \bar{u}(t)$ itself gives that

$$\frac{d}{dt} \|u(t) - \bar{u}(t)\|_{L^2(\mu)}^2 \leq -\mathcal{E}(u(t) - \bar{u}(t)) + C \|u(t) - \bar{u}(t)\|_{L^2(\nu)} [\mathcal{E}(u(t))^{1/2} + \mathcal{E}(u(t) - \bar{u}(t))^{1/2}],$$

where, as before, $C > 0$ is a generic constant depending only on $\|\psi\|_{L^\infty}$. For any $\epsilon \in (0, 1)$, using the Sobolev inequality (4.2.22), we deduce that

$$\frac{d}{dt} \|u(t) - \bar{u}(t)\|_{L^2(\mu)}^2 \leq \frac{C}{\epsilon} \|u(t) - \bar{u}(t)\|_{L^2(\mu)}^2 + C(1 + \epsilon) [\mathcal{E}(u(t)) + \mathcal{E}(\bar{u}(t))].$$

Therefore,

$$\|u(t) - \bar{u}(t)\|_{L^2(\mu)}^2 \leq C(1 + \epsilon) \int_0^t e^{-C(t-s)/\epsilon} [\mathcal{E}(u(s)) + \mathcal{E}(\bar{u}(s))] ds.$$

By the dominated convergence theorem, setting $\epsilon \rightarrow 0$ in the above gives that $\|u(t) - \bar{u}(t)\|_{L^2(\mu)} = 0$, $t \in [0, T]$, which proves the uniqueness.

We now turn to the proof of the joint Hölder continuity. Let $g(t) = u(t)\nabla u(t)$. Then $|g(t)| \leq \|\psi\|_{L^\infty} |\nabla u(t)|$. By an argument similar to the proof of (4.4.22) in Theorem 4.4.14, we may show that $\|g\|_{L^\infty(0,T;L^2(\nu))} < \infty$. Since u is the unique solution to

$$\begin{cases} \partial_t u \, d\mu = \mathcal{L}u \, d\mu + g(t) \, d\nu, & \text{in } (0, T] \times (\mathbb{S} \setminus V_0), \\ u = 0 & \text{on } (0, T] \times V_0, \quad u(0) = \psi, \end{cases}$$

we may now apply Lemma 4.4.9 and Proposition 4.4.13 to obtain the desired joint Hölder continuity. □

Chapter 5

Stochastic Processes with Controlled Conditional Increments

5.1 Introduction

In this chapter, we shall study tail asymptotics of suprema of a class of stochastic processes. The materials in Section 5.2 and Section 5.3 are based on the work [44] by the current author, and those in Section 5.4 are based on an ongoing joint project with G. Xi.

Let $\{\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F}, \mathbb{P}\}$ be a filtered probability space satisfying the usual conditions, i.e. \mathcal{F}_0 is \mathbb{P} -complete and $\{\mathcal{F}_t\}_{t \geq 0}$ is right continuous. Let $\{X_t\}_{t \in [0, T]}$ be an $\{\mathcal{F}_t\}$ -adapted stochastic process. It is of great mathematical and practical significance to investigate the following problem:

Given knowledge of the marginal distributions of $\{X_t\}_{t \in [0, T]}$, what can be asserted about the distribution of the supremum $\sup_{t \in [0, T]} |X_t|$?

Distributions of suprema $B^* = \sup_{t \in [0, T]} B_t$ of Brownian motions $\{B_t\}_{t \geq 0}$ can be derived analytically by utilizing the reflection principle. One of the important applications of the distribution of B^* is the derivation of a closed form pricing formula for European up-and-out options and their counterparts. However, it has been suggested by empirical data that

financial time series possess properties such as being autocorrelated and non-Markov (cf. [1, 52, 11] and references therein). Hence these time series may not be well modelled by Brownian motions. Non-Markovian processes such as fractional Brownian motions have been proposed for financial modelling as alternatives. However, once we go beyond Markovian settings, exact formula for distributions of suprema become intractable. It is then desirable to obtain good estimates for the asymptotic behavior of suprema, which will provide useful bounds for up-and-out option prices. Besides option pricing theory, knowledge of tail decays for suprema can also be applied to the study of extreme/rare events and related industries (e.g. insurance industries). There has been abundant research on properties of supremum of stochastic processes. In particular, for Gaussian processes, many profound results have been established in terms of entropy condition, majorizing measures, and generic chaining. See, for example, [42, 63, 7, 43, 64, 65, 66]. A crucial property of Gaussian processes is that they are completely determined by their covariance functions, which turn out to provide a suitable topology on the parameter space for Gaussian processes.

In this chapter, we consider a different setting. We shall give an investigation into supremum tail decay for stochastic processes satisfying conditional increment controls (see Definition 5.1.1), of which marginal tail decays are available. The assumption of conditional increment control is mild enough to include all continuous martingales and many non-Markov processes as examples to which other estimates may apply (see Remark 5.1.2). The virtue of our results is that only conditional increment controls and knowledge of margins (snapshots) are needed, which can be extracted from empirical snapshot data using statistical analysis. In particular, no specific dynamic/structural assumptions (such as differential equations, autoregressive moving average dynamics, jointly Gaussian, etc.) are required. Therefore, our results are robust to bias in model preferences.

Definition 5.1.1. Let $\{X_t\}_{t \in [0, T]}$ be a continuous $\{\mathcal{F}_t\}$ -adapted stochastic process, and let $p > 1, h \in (0, 1], ph > 1$. $\{X_t\}_{t \in [0, T]}$ is said to satisfy the *conditional increment control*

with parameter (p, h) if there exists a constant $A_{p,h} \geq 0$ independent of t such that

$$\mathbb{E}\left[|\mathbb{E}(X_t|\mathcal{F}_s) - X_s|^p\right] \leq A_{p,h}|t - s|^{ph}, \quad \text{for all } 0 \leq s < t \leq T. \quad (5.1.1)$$

Remark 5.1.2. We provide below several frequently encountered examples of processes which satisfy conditional increment controls.

(i) Any continuous martingale. Then the conditional increment control is satisfied for any (p, h) , $ph > 1$ with $A_{p,h} = 0$.

(ii) Fractional Brownian motions with Hurst parameters $h \in (0, 1)$.

(iii) Let $\{X_t\}_{t \in [0, T]}$ be a continuous stochastic process satisfying

$$\mathbb{E}(|X_t - X_s|^p) \leq A_{p,h}|t - s|^{ph}, \quad \text{for all } 0 \leq s < t \leq T. \quad (5.1.2)$$

In fact, by Jensen's inequality, it is easily seen that the conditional increment control is satisfied with the same parameter (p, h) and the same constant $A_{p,h}$. Many Gaussian processes arising from practice satisfy this condition. Condition (5.1.2) is also satisfied by many diffusions governed by SDEs with minor regularity assumptions on the coefficients (e.g. the stochastic DiPerna–Lions flow considered in Section 5.4). Processes satisfying (5.1.2) are archetypal examples considered and well studied in rough path theory as their associated geometric rough paths can be constructed canonically (see [47, 19] etc.).

Definition 5.1.3. Let $\alpha > 0$. A continuous stochastic process $\{X_t\}_{t \in [0, T]}$ is said to have *uniform α -exponential marginal decays* if there exist constants $C_1, C_2 > 0$, $M \geq 0$ such that

$$\mathbb{P}(|X_t| \geq \lambda) \leq C_2 \exp(-C_1 \lambda^\alpha), \quad \text{for all } \lambda > M \text{ and all } t \in [0, T]. \quad (5.1.3)$$

Let $\{X_t\}_{t \in [0, T]}$ be a process having uniform α -exponential marginal decay for some $\alpha > 0$. We are interested in the decay of its supremum $\sup_{t \in [0, T]} |X_t|$, i.e. the asymptotic

behavior of

$$\mathbb{P}\left(\sup_{t \in [0, T]} |X_t| \geq \lambda\right) \quad (5.1.4)$$

for large $\lambda > 0$. Our main result states that if a continuous process $\{X_t\}_{t \in [0, T]}$ has uniform α -exponential marginal decays and satisfies a conditional increment control with (p, h) , $ph > 1$, then its supremum $\sup_{t \in [0, T]} |X_t|$ decays similarly to its margins X_t , $t \in [0, T]$ in distribution.

5.2 A Doob type maximal inequality

In this section, we shall prove a Doob type inequality (see Theorem 5.2.3) for processes satisfying the conditional increment control (5.1.1). Throughout Section 5.2 to Section 5.4, $\{X_t\}_{t \in [0, T]}$ will be a continuous stochastic process satisfying the conditional increment control (5.1.1).

Lemma 5.2.1. *For any $0 \leq s_0 < t_0 \leq T$, there holds that*

$$\mathbb{E}\left(\sup_{s_0 \leq s < t \leq t_0} |\mathbb{E}(X_t | \mathcal{F}_s) - X_s|^p\right) \leq C_{p,h} A_{p,h} |t_0 - s_0|^{ph},$$

where

$$C_{p,h} = [2\zeta(\theta)]^{p-1} \left(\frac{p}{p-1}\right)^p \left(\frac{4}{ph-1}\right)^{\theta(p-1)+1} \Gamma[\theta(p-1)+1],$$

with $\theta > 1$ being an arbitrary constant, $\zeta(\theta) = \sum_{m=1}^{\infty} m^{-\theta}$ is the Riemann zeta function, and $\Gamma(z)$ is the Gamma function.

Proof. Let $s, t \in [s_0, t_0]$, $s < t$ be fixed temporarily. Denote

$$I_l^m = [t_{l-1}^m, t_l^m] = s_0 + (t_0 - s_0) \times \left[\frac{l-1}{2^m}, \frac{l}{2^m}\right].$$

Then there exists a sequence $\{J_k\}_k \subseteq \{I_l^m : 1 \leq l \leq 2^m, m \geq 0\}$ such that:

(i) J_k , $k = 1, 2, \dots$ are mutually disjoint;

(ii) For any $m \geq 1$, there are at most two elements of $\{J_k\}$ with length $(t_0 - s_0)2^{-m}$;

(iii) $[s, t] = \cup_{k=1}^{\infty} J_k$.

Let $J_k = [u_{k-1}, u_k]$. Then

$$\begin{aligned} |\mathbb{E}(X_t | \mathcal{F}_s) - X_s| &= \left| \sum_{k=1}^{\infty} \mathbb{E}(\Delta X_{J_k} | \mathcal{F}_s) \right| \\ &\leq \sum_{k=1}^{\infty} \mathbb{E} \left[\left| \mathbb{E}(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}}) \right| \middle| \mathcal{F}_s \right] \\ &= \sum_{m=1}^{\infty} \sum_{\{J_k: |J_k|=(t_0-s_0)2^{-m}\}} \mathbb{E} \left[\left| \mathbb{E}(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}}) \right| \middle| \mathcal{F}_s \right], \end{aligned}$$

where $\Delta X_{J_k} = X_{u_k} - X_{u_{k-1}}$. Let $\xi_l^m = \mathbb{E}(\Delta X_{I_l^m} | \mathcal{F}_{t_{l-1}^m})$, $1 \leq l \leq 2^m$, $m = 1, 2, \dots$. For an arbitrary $\theta > 1$, by Jensen's inequality,

$$\begin{aligned} |\mathbb{E}(X_t | \mathcal{F}_s) - X_s|^p &\leq \left(\sum_{m=1}^{\infty} \frac{1}{\zeta(\theta)m^\theta} \cdot \zeta(\theta)m^\theta \sum_{\{J_k: |J_k|=(t_0-s_0)2^{-m}\}} \mathbb{E} \left[\left| \mathbb{E}(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}}) \right| \middle| \mathcal{F}_s \right] \right)^p \\ &\leq \sum_{m=0}^{\infty} \frac{1}{\zeta(\theta)m^\theta} \left(\zeta(\theta)m^\theta \sum_{\{J_k: |J_k|=(t_0-s_0)2^{-m}\}} \mathbb{E} \left[\left| \mathbb{E}(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}}) \right| \middle| \mathcal{F}_s \right] \right)^p \\ &= \zeta(\theta)^{p-1} \sum_{m=0}^{\infty} m^{\theta(p-1)} \left(\sum_{\{J_k: |J_k|=(t_0-s_0)2^{-m}\}} \mathbb{E} \left[\left| \mathbb{E}(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}}) \right| \middle| \mathcal{F}_s \right] \right)^p \\ &\leq [2\zeta(\theta)]^{p-1} \sum_{m=0}^{\infty} m^{\theta(p-1)} \sum_{\{J_k: |J_k|=(t_0-s_0)2^{-m}\}} \left(\mathbb{E} \left[\left| \mathbb{E}(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}}) \right| \middle| \mathcal{F}_s \right] \right)^p \\ &\leq [2\zeta(\theta)]^{p-1} \sum_{m=0}^{\infty} m^{\theta(p-1)} \sum_{l=1}^{2^m} \sup_{r \in [s_0, t_0]} [\mathbb{E}(|\xi_l^m| | \mathcal{F}_r)]^p, \end{aligned}$$

where the inequality in the fourth line is due to the property (ii). Hence,

$$\sup_{s_0 \leq s < t \leq t_0} |\mathbb{E}(X_t | \mathcal{F}_s) - X_s|^p \leq [2\zeta(\theta)]^{p-1} \sum_{m=0}^{\infty} m^{\theta(p-1)} \sum_{l=1}^{2^m} \sup_{r \in [s_0, t_0]} [\mathbb{E}(|\xi_l^m| | \mathcal{F}_r)]^p.$$

By Doob's maximal inequality applied to the martingales $\{\mathbb{E}(|\xi_l^m| | \mathcal{F}_r)\}_r$,

$$\begin{aligned}
& \mathbb{E} \left(\sup_{s_0 \leq s < t \leq t_0} |\mathbb{E}(X_t | \mathcal{F}_s) - X_s|^p \right) \\
& \leq [2\zeta(\theta)]^{p-1} \sum_{m=1}^{\infty} m^{\theta(p-1)} \sum_{l=1}^{2^m} \mathbb{E} \left(\sup_{r \in [s_0, t_0]} [\mathbb{E}(|\xi_l^m| | \mathcal{F}_r)]^p \right) \\
& \leq [2\zeta(\theta)]^{p-1} \sum_{m=1}^{\infty} m^{\theta(p-1)} \sum_{l=1}^{2^m} \left(\frac{p}{p-1} \right)^p \mathbb{E} \left([\mathbb{E}(|\xi_l^m| | \mathcal{F}_r)]^p \right) \\
& \leq [2\zeta(\theta)]^{p-1} \left(\frac{p}{p-1} \right)^p \sum_{m=1}^{\infty} m^{\theta(p-1)} \sum_{l=1}^{2^m} \mathbb{E}(|\xi_l^m|^p) \\
& \leq A_{p,h} [2\zeta(\theta)]^{p-1} \left(\frac{p}{p-1} \right)^p \sum_{m=1}^{\infty} m^{\theta(p-1)} \cdot 2^m \cdot \left(\frac{|t_0 - s_0|}{2^m} \right)^{ph} \\
& = C_{p,h} A_{p,h} |t_0 - s_0|^{ph},
\end{aligned}$$

where

$$C_{p,h} = [2\zeta(\theta)]^{p-1} \left(\frac{p}{p-1} \right)^p \sum_{m=1}^{\infty} m^{\theta(p-1)} 2^{-m(ph-1)}.$$

Notice that

$$\begin{aligned}
\sum_{m=1}^{\infty} m^{\theta(p-1)} \cdot 2^{-m(ph-1)} & \leq \sum_{m=1}^{\infty} 2^{\theta(p-1)} \int_{m-1}^m r^{\theta(p-1)} e^{-r(ph-1) \log 2} dr \\
& \leq \left(\frac{4}{ph-1} \right)^{\theta(p-1)+1} \int_0^{\infty} r^{\theta(p-1)} e^{-r} dr \\
& = \left(\frac{4}{ph-1} \right)^{\theta(p-1)+1} \Gamma[\theta(p-1)+1].
\end{aligned}$$

The proof is therefore completed. \square

As an application of Lemma 5.2.1, we show that a Doob type inequality for processes satisfying the condition (5.1.1). To this end, we shall need the following elementary result.

Lemma 5.2.2. *Let $\{Y_t\}_{t \in [0, T]}$ be a continuous stochastic process such that $\mathbb{E}(|Y_t|) < \infty$ for all $t \in [0, T]$. Let $0 \leq s_0 < t_0 \leq T$.*

(a) For any stopping time τ with $s_0 \leq \tau \leq t_0$, we have

$$|\mathbb{E}(Y_{t_0} | \mathcal{F}_\tau) - Y_\tau| \leq \mathbb{E} \left[\sup_{u \in [s_0, t_0]} |\mathbb{E}(Y_{t_0} | \mathcal{F}_\tau) - Y_u| \mid \mathcal{F}_\tau \right]. \quad (5.2.1)$$

(b) For any $\lambda > 0$,

$$\mathbb{P} \left(\sup_{u \in [s_0, t_0]} Y_u \geq \lambda \right) \leq \frac{1}{\lambda} \int_{\{\sup_{u \in [s_0, t_0]} Y_u \geq \lambda\}} \left[\sup_{u \in [s_0, t_0]} |\mathbb{E}(Y_{t_0} | \mathcal{F}_u) - Y_u| + Y_{t_0} \right] d\mathbb{P}. \quad (5.2.2)$$

Proof. (a) By the continuity of Y_t , we may assume that τ takes only countably many values $\{u_k : k = 1, 2, \dots\} \subseteq [s_0, t_0]$. Then

$$\begin{aligned} |\mathbb{E}(Y_{t_0} | \mathcal{F}_\tau) - Y_\tau| &= \sum_{k=1}^{\infty} |\mathbb{E}(Y_{t_0} | \mathcal{F}_\tau) - Y_\tau| 1_{\{\tau=u_k\}} \\ &= \sum_{k=1}^{\infty} \left| \mathbb{E}[(Y_{t_0} - Y_\tau) 1_{\{\tau=u_k\}} \mid \sigma(\mathcal{F}_\tau \cap \{\tau = u_k\})] \right| \\ &= \sum_{k=1}^{\infty} \left| \mathbb{E} \left[\mathbb{E}[(Y_{t_0} - Y_{u_k}) \mid \mathcal{F}_{u_k}] 1_{\{\tau=u_k\}} \mid \sigma(\mathcal{F}_\tau \cap \{\tau = u_k\}) \right] \right| \\ &\leq \sum_{k=1}^{\infty} \mathbb{E} \left[\left(\sup_{u \in [s_0, t_0]} |\mathbb{E}(Y_{t_0} | \mathcal{F}_u) - Y_u| \right) 1_{\{\tau=u_k\}} \mid \sigma(\mathcal{F}_\tau \cap \{\tau = u_k\}) \right] \\ &= \sum_{k=1}^{\infty} \mathbb{E} \left[\sup_{u \in [s_0, t_0]} |\mathbb{E}(Y_{t_0} | \mathcal{F}_u) - Y_u| \mid \mathcal{F}_\tau \right] 1_{\{\tau=u_k\}} \\ &= \mathbb{E} \left[\sup_{u \in [s_0, t_0]} |\mathbb{E}(Y_{t_0} | \mathcal{F}_u) - Y_u| \mid \mathcal{F}_\tau \right]. \end{aligned}$$

This proves (a).

(b) Let $\tau = \inf \{u \in [s_0, t_0] : Y_u \geq \lambda\} \wedge t_0$. Then $\{\sup_{t \in [s_0, t_0]} Y_t \geq \lambda\} = \{Y_\tau \geq \lambda\} \in \mathcal{F}_\tau$.

Therefore, by (5.2.1),

$$\begin{aligned} \int_{\{\sup_{u \in [s_0, t_0]} Y_u \geq \lambda\}} Y_\tau d\mathbb{P} &= - \int_{\{\sup_{u \in [s_0, t_0]} Y_u \geq \lambda\}} [\mathbb{E}(Y_{t_0} | \mathcal{F}_\tau) - Y_\tau] d\mathbb{P} \\ &\quad + \int_{\{\sup_{u \in [s_0, t_0]} Y_u \geq \lambda\}} Y_{t_0} d\mathbb{P} \end{aligned}$$

$$\leq \int_{\{\sup_{u \in [s_0, t_0]} Y_u \geq \lambda\}} \left[\sup_{u \in [s_0, t_0]} |\mathbb{E}(Y_{t_0} | \mathcal{F}_u) - Y_u| + Y_{t_0} \right] d\mathbb{P}.$$

The inequality (5.2.2) follows readily from the above inequality. \square

We are now in a position to state and prove a Doob type maximal inequality.

Proposition 5.2.3. *Let $0 \leq s_0 < t_0 \leq T$, and $X^* = \sup_{u \in [s_0, t_0]} |X_u|$. Then for any $1 < q \leq p$,*

$$\|X^*\|_{L^q} \leq \frac{q}{q-1} [C_{p,h}^{1/p} A_{p,h}^{1/p} |t_0 - s_0|^h + \|X_{t_0}\|_{L^q}]. \quad (5.2.3)$$

where $C_{p,h}$ is a constant depending only on p, h (more precisely, a constant which differs from the $C_{p,h}$ in Lemma 5.2.1 by a multiplicative constant depending only on p).

Proof. Denote $Y = \sup_{u \in [s_0, t_0]} |\mathbb{E}(X_{t_0} | \mathcal{F}_u) - X_u| + |X_{t_0}|$. Then $\{X^* \geq \lambda\} \subseteq \{\sup_{u \in [s_0, t_0]} X_u \geq \lambda\}$. By Lemma 5.2.2(b) and Lemma 5.2.1

$$\begin{aligned} \|X^*\|_{L^q}^q &= q \int_0^\infty \lambda^{q-1} \mathbb{P}(X^* \geq \lambda) d\lambda \\ &\leq q \int_0^\infty \lambda^{q-2} \int_{\{\sup_{u \in [s_0, t_0]} X_u \geq \lambda\}} Y d\mathbb{P} d\lambda \\ &\leq q \int_0^\infty \lambda^{q-2} \int_{\{X^* \geq \lambda\}} Y d\mathbb{P} d\lambda \\ &= q \int_\Omega \left(\int_0^{X^*} \lambda^{q-2} d\lambda \right) Y d\mathbb{P} \\ &= \frac{q}{q-1} \int_\Omega |X^*|^{q-1} Y d\mathbb{P} \\ &\leq \frac{q}{q-1} \|X^*\|_{L^q}^{q-1} \|Y\|_{L^q}, \end{aligned}$$

Therefore

$$\begin{aligned} \|X^*\|_{L^q} &\leq \frac{q}{q-1} \|Y\|_{L^q} \\ &\leq \frac{q}{q-1} \left(\left\| \sup_{u \in [s_0, t_0]} |\mathbb{E}(X_{t_0} | \mathcal{F}_u) - X_u| \right\|_{L^q} + \|X_{t_0}\|_{L^q} \right) \end{aligned}$$

$$\leq \frac{q}{q-1} [C_{p,h}^{1/p} A_{p,h}^{1/p} |t_0 - s_0|^h + \|X_{t_0}\|_{L^q}].$$

This completes the proof. □

5.3 Tail estimate for the suprema

This section is devoted to the statement and the proof of the main result (Theorem 5.3.2) of this chapter, which in plain words states that the supremum $\sup_{t \in [0, T]} |X_t|$ has α -exponential tail decay, if and only if the margins of X_t have uniform α -exponential decay.

Let us start with an easy estimate below, which follows simple calculations.

Lemma 5.3.1. *Let $X_t, t \in [0, T]$, be a continuous stochastic process satisfying (5.1.1), and $q > 0$. Then*

$$\mathbb{E}(|X_t|^q) \leq C_2 C_1^{-q/\alpha} \Gamma\left(\frac{q}{\alpha} + 1\right).$$

Theorem 5.3.2. *Let $X_t, t \in [0, T]$ be a continuous stochastic process satisfying (5.1.1). Suppose that there exist constants $\alpha > 0$, $C_1 > 0$, and $M \geq 0$ such that*

$$\mathbb{P}(|X_t| \geq \lambda) \leq C_2 \exp(-C_1 \lambda^\alpha), \text{ for all } \lambda > 0 \text{ and all } t \in [0, T].$$

Then

$$\mathbb{P}\left(\sup_{t \in [0, T]} |X_t| \geq 2\lambda\right) \leq K \lambda^{-1/h} \exp\left[-\left(1 - \frac{1}{ph}\right) C_1 \lambda^\alpha\right], \text{ for all } \lambda \geq \delta_0,$$

where

$$K = 4T [C_{p,h} A_{p,h}]^{1/(ph)} \left[1 + C_2 \left(\frac{p}{p-1}\right)^p\right]^{1-1/(ph)},$$

with $C_{p,h}$ being the same as in Lemma 5.2.1.

Proof. For $N \in \mathbb{N}_+$, let $I_n = [t_{n-1}, t_n] = [(n-1)T/N, nT/N]$. Then

$$\left\{ \sup_{t \in [0, T]} |X_t| \geq 2\lambda \right\} \subseteq \bigcup_{n=1}^N \left\{ \sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t) - X_t| \geq \lambda \right\} \cup \bigcup_{n=0}^{N-1} \left\{ \sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)| \geq \lambda \right\}.$$

Therefore

$$\begin{aligned} \mathbb{P}\left(\sup_{t \in [0, T]} |X_t| \geq 2\lambda \right) &\leq \sum_{n=1}^N \mathbb{P}\left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t) - X_t| \geq \lambda \right) \\ &\quad + \sum_{n=0}^{N-1} \mathbb{P}\left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)| \geq \lambda \right). \end{aligned} \quad (5.3.1)$$

By Lemma 5.2.1, we deduce that

$$\mathbb{P}\left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t) - X_t| \geq \lambda \right) \leq C_{p,h} A_{p,h} \lambda^{-p} \left(\frac{T}{N} \right)^{ph}. \quad (5.3.2)$$

We need to estimate $\mathbb{P}\left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)| \geq \lambda \right)$. If $\alpha \geq p$, by Doob's inequality applied to $\{\mathbb{E}(X_{t_n} | \mathcal{F}_t)\}_t$ and Lemma 5.3.1,

$$\mathbb{E}\left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)|^\alpha \right) \leq \left(\frac{\alpha}{\alpha - 1} \right)^\alpha \mathbb{E}(|X_{t_n}|^\alpha) \leq C_2 C_1^{-1} \left(\frac{p}{p-1} \right)^p.$$

If $\alpha < p$, by Hölder's inequality and the above inequality applied to $\alpha = p$,

$$\mathbb{E}\left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)|^\alpha \right) \leq \left[\mathbb{E}\left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)|^p \right) \right]^{\alpha/p} \leq C_2 C_1^{-1} \left(\frac{p}{p-1} \right)^p.$$

Moreover, for any $q \geq 2$, by a similar argument, we can deduce that

$$\begin{aligned} \mathbb{E}\left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)|^{\alpha q} \right) &\leq \mathbb{E}\left(\sup_{t \in I_n} |\mathbb{E}(|X_{t_n}|^\alpha | \mathcal{F}_t)|^q \right) \\ &\leq \left(\frac{q}{q-1} \right)^q \mathbb{E}(|X_{t_n}|^{\alpha q}) \\ &\leq C_2 \left(\frac{q}{C_1(q-1)} \right)^q \Gamma(q+1) \\ &\leq C_2 (2C_1^{-1})^q \Gamma(q+1). \end{aligned}$$

Therefore

$$\begin{aligned}
\mathbb{E} \left[\exp \left(\frac{C_1}{4} \sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)|^\alpha \right) \right] &= \sum_{q=0}^{\infty} \frac{(C_1/4)^q}{q!} \mathbb{E} \left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)|^{\alpha q} \right) \\
&\leq 1 + \frac{C_2}{4} \left(\frac{p}{p-1} \right)^p + C_2 \sum_{q=2}^{\infty} 2^{-q} \\
&\leq 2 \left[1 + C_2 \left(\frac{p}{p-1} \right)^p \right].
\end{aligned}$$

By Chebyshev's inequality,

$$\mathbb{P} \left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)| \geq \lambda \right) \leq 2 \left[1 + C_2 \left(\frac{p}{p-1} \right)^p \right] \exp \left(- \frac{C_1}{4} \lambda^\alpha \right). \quad (5.3.3)$$

Therefore, by (5.3.1), (5.3.2) and (5.3.3),

$$\mathbb{P} \left(\sup_{t \in [0, T]} |X_t| \geq 2\lambda \right) \leq C_{p,h} A_{p,h} \frac{N}{\lambda^p} \left(\frac{T}{N} \right)^{ph} + 2N \left[1 + C_2 \left(\frac{p}{p-1} \right)^p \right] \exp \left(- \frac{C_1}{4} \lambda^\alpha \right).$$

Setting N to optimize the right hand side of the above inequality gives that

$$\begin{aligned}
&\mathbb{P} \left(\sup_{t \in [0, T]} |X_t| \geq 2\lambda \right) \\
&\leq 4T [C_{p,h} A_{p,h}]^{1/(ph)} \left[1 + C_2 \left(\frac{p}{p-1} \right)^p \right]^{1-1/(ph)} \lambda^{-1/h} \exp \left[- C_1 \left(1 - \frac{1}{ph} \right) \lambda^\alpha \right].
\end{aligned}$$

□

Example 5.3.3. We consider the tail decay of the supremum of a standard fractional Brownian motion B_t^h , $t \in [0, T]$ with Hurst parameter $h \in (0, 1)$, that is, a Gaussian process with $B_0^h = 0$ and covariance function

$$R(t, s) = \frac{1}{2} (|t|^{2h} + |s|^{2h} - |t - s|^{2h}), \quad t, s \in [0, T].$$

For the fractional Brownian motion B^h , we have $B_t^h - B_s^h \sim \mathcal{N}(0, |t - s|^{2h})$, and there-

fore,

$$\mathbb{E}(|B_t^h - B_s^h|^p) = A_p |t - s|^{ph}, \quad t, s \in [0, T],$$

where

$$A_p = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right).$$

For any $t \in [0, 1]$ and any $\lambda \geq 1$,

$$\begin{aligned} \mathbb{P}(|B_t^h| \geq \lambda) &= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2}\lambda/t^h}^{\infty} \exp\left(-\frac{1}{2}u^2\right) du \\ &\leq \frac{t^h}{2\sqrt{\pi}\lambda} \exp\left(-\frac{\lambda^2}{t^{2h}}\right) \leq \frac{1}{2\sqrt{\pi}} \exp(-\lambda^2). \end{aligned}$$

Let $\phi(\lambda) = (2\sqrt{\pi})^{-1} \exp(-\lambda^2)$. For any $p > 1/h$, by Lemma 5.3.2(2),

$$\mathbb{P}\left(\sup_{t \in [0,1]} |B_t^h| \geq 2\lambda\right) \leq 2[C_{p,h}A_p]^{1/(ph)} \lambda^{-1/h} \phi(\lambda)^{1-1/(ph)}, \quad \text{for all } \lambda \geq 1, \quad (5.3.4)$$

where

$$C_{p,h} = [2\zeta(\theta)]^{p-1} \left(\frac{p}{p-1}\right)^p \left(\frac{4}{ph-1}\right)^{\theta(p-1)+1} \Gamma[\theta(p-1)+1]$$

with an arbitrary constant $\theta > 1$. Setting $p = 2/h$, $\theta = p/(p-1)$ in (5.3.4) gives

$$\mathbb{P}\left(\sup_{t \in [0,1]} |B_t^h| \geq 2\lambda\right) \leq C_h \exp\left(-\frac{\lambda^2}{2}\right), \quad \text{for all } \lambda \geq 1,$$

where C_h is a constant depending only on h . By self-similarity, we deduce that, for $T > 0$,

$$\mathbb{P}\left(\sup_{t \in [0,T]} |B_t^h| \geq 2\lambda\right) \leq C_h T^h \exp\left(-\frac{\lambda^2}{2T^{2h}}\right), \quad \text{for all } \lambda \geq T^h.$$

Let $a = (a_1, a_2, \dots)$ be a sequence of real numbers, let $f_k(t)$, $k = 1, 2, \dots$ be a sequence of real functions on $[0, T]$, and let ξ_k , $k = 1, 2, \dots$ be i.i.d. Rademacher random variables¹. R. Paley and A. Zygmund showed in [54, Theorem I, p. 339] that if $\sum_{k=1}^{\infty} a_k^2 <$

¹A random variable ξ is said to have Rademacher distribution if $\mathbb{P}(\xi = 1) = \mathbb{P}(\xi = -1) = 1/2$.

∞ and

$$\int_0^T f_k(t)^2 dt \leq A, \quad k = 1, 2, \dots$$

for some constant $A < \infty$, then for almost all $\omega \in \Omega$, the series $\sum_{k=1}^{\infty} a_k \xi_k(\omega) f_k(t)$ converges in the sense of a.e. $t \in [0, T]$ and in $L^2([0, T])$. As an application of Theorem 5.3.2, we investigate the uniform convergence of the function series $\sum_{k=1}^{\infty} a_k \xi_k(\omega) f_k(t)$ under an additional assumption on the Hölder continuity for f_k , $k = 1, 2, \dots$

Proposition 5.3.4. *Let (a_1, a_2, \dots) be a sequence of real numbers, $f_k(t)$, $k = 1, 2, \dots$ be a sequence of real functions on $[0, T]$, and let ξ_k , $k = 1, 2, \dots$ be i.i.d. Rademacher random variables. Denote by $H \subseteq \Omega$ the set of ω for which*

$$\sum_{k=1}^{\infty} a_k \xi_k(\omega) f_k(t)$$

converges for some $t \in [0, T]$. Then, by [54, Theorem I, p. 339], $\mathbb{P}(H) = 1$. Suppose that $h \in (0, 1)$ and

$$|f_k(t) - f_k(s)| \leq L_k |t - s|^h, \quad \text{for all } s, t \in [0, T], \text{ all } k = 1, 2, \dots,$$

and that

$$\sum_{k=1}^{\infty} a_k^2 [f_k(0)^2 + L_k^2] < \infty.$$

Then, for a.e. $\omega \in \Omega$, the series $\sum_{k=1}^{\infty} a_k \xi_k(\omega) f_k(t)$ converges uniformly in $t \in [0, T]$.

Proof. Let $X(t, \omega) = \sum_{k=1}^{\infty} a_k \xi_k(\omega) f_k(t) 1_H(\omega)$. To simplify notation, we shall refer to $\sum_{k=1}^{\infty} a_k \xi_k(\omega) f_k(t) 1_H(\omega)$ by simply writing $\sum_{k=1}^{\infty} a_k \xi_k(\omega) f_k(t)$. For any $p > 0$, by Khintchine's inequality (cf. [23, p. 586]),

$$\mathbb{E} \left[\left| \sum_{k=1}^{\infty} a_k \xi_k f_k(0) \right|^p \right] \leq C_p \left[\sum_{k=1}^{\infty} a_k^2 f_k(0)^2 \right]^{p/2},$$

where

$$C_p = \max \left\{ \sqrt{\frac{2^p}{\pi}} \Gamma\left(\frac{p+1}{2}\right), 1 \right\}.$$

Similarly,

$$\mathbb{E} \left[\left| \sum_{k=1}^{\infty} a_k \xi_k (f_k(t) - f_k(0)) \right|^p \right] \leq C_p \left[\sum_{k=1}^{\infty} a_k^2 L_k^2 \right]^{p/2} t^{ph}.$$

By the above, we see that

$$\mathbb{E}(|X_t|^p) \leq 2^{p/2} C_p \left[\sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right]^{p/2} \quad (5.3.5)$$

and

$$\mathbb{E}(|X_t - X_s|^p) \leq 2^{p/2} C_p \left[\sum_{k=1}^{\infty} a_k^2 L_k^2 \right]^{p/2} |t - s|^{ph}, \quad \text{for all } p > 0. \quad (5.3.6)$$

Let

$$A_{p,h} = \frac{2^p}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right) \left[\sum_{k=1}^{\infty} a_k^2 L_k^2 \right]^{p/2}. \quad (5.3.7)$$

Then condition (5.1.2) is satisfied for any $p > 1/h$.

Next, we give an estimate for tail decays of the margins X_t . For any $u \geq 0$, by (5.3.5) and Stirling's formula,

$$\begin{aligned} \mathbb{E}[\exp(u|X_t|)] &\leq \sum_{p=0}^{\infty} \frac{2^{p/2} u^p C_p}{p!} \left[\sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right]^{p/2} \\ &\leq K \sum_{p=0}^{\infty} 2^p \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma(p+1)} \left[u^2 \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right]^{p/2} \\ &= K \sum_{p=0}^{\infty} \frac{\sqrt{\pi}}{\Gamma(p/2+1)} \left[u^2 \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right]^{p/2} \\ &\leq K \left(\sum_{p=0}^{\infty} \frac{1}{\Gamma(p/2+1)^2} \left[u^2 \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right]^p \right)^{1/2} \end{aligned}$$

where K is a universal constant which may be different at various occurrences. Since

$$\Gamma\left(\frac{p}{2} + 1\right)^2 \geq \Gamma\left(\frac{p+2}{2}\right) \Gamma\left(\frac{p+1}{2}\right) = 2^{-p} \sqrt{\pi} \Gamma(p+1) = \frac{\sqrt{\pi} p!}{2^p},$$

we obtain that

$$\begin{aligned}\mathbb{E}[\exp(u|X_t|)] &\leq K \left(\sum_{p=0}^{\infty} \frac{1}{p!} \left[2u^2 \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right]^p \right)^{1/2} \\ &= K \exp \left[u^2 \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right].\end{aligned}$$

By Chebyshev's inequality,

$$\mathbb{P}(|X_t| \geq \lambda) \leq K \exp \left[-u\lambda + u^2 \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right].$$

Setting

$$u = \lambda \left[2 \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right]^{-1}$$

gives that

$$\mathbb{P}(|X_t| \geq \lambda) \leq K \exp \left[-\frac{\lambda^2}{2 \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2)} \right], \quad \text{for all } \lambda > 0. \quad (5.3.8)$$

Now by Theorem 5.3.2 (with $M = 0$), we have

$$\mathbb{P} \left(\sup_{t \in [0, T]} \left| \sum_{k=1}^{\infty} a_k \xi_k f_k(t) \right| \geq 2\lambda \right) \leq C_h \exp \left[-\frac{D_h \lambda^2}{\sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2)} \right], \quad (5.3.9)$$

where C_h, D_h are constants depending only on $h \in (0, 1)$. Let $\sigma^2 = \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2)$, and, for any $u \in [0, 1]$, let

$$l(u) = \inf \left\{ l \in \mathbb{R}_+ : \int_0^l \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) 1_{(k-1, k]}(s) ds > u\sigma^2 \right\}. \quad (5.3.10)$$

Then $l(u) \rightarrow \infty$ as $u \rightarrow 1$.

To show the a.s. uniform convergence of $\sum_{k=1}^{\infty} a_k \xi_k f_k(t)$, we need to show that

$$\mathbb{P} \left(\bigcap_{0 < u < 1} \left\{ \sup_{n \geq l(u)} \sup_{t \in [0, T]} \left| \sum_{k=n}^{\infty} a_k \xi_k f_k(t) \right| \geq 2\lambda \right\} \right) = 0, \quad \text{for all } \lambda > 0.$$

Clearly, it suffices to show that

$$\lim_{u \rightarrow 1} \mathbb{P} \left(\sup_{n \geq l(u)} \sup_{t \in [0, T]} \left| \sum_{k=n}^{\infty} a_k \xi_k f_k(t) \right| \geq 2\lambda \right) = 0, \quad \text{for all } \lambda > 0. \quad (5.3.11)$$

Define

$$Y_u = \sup_{t \in [0, T]} \left| \int_{l(u)}^{\infty} \sum_{k=1}^{\infty} a_k \xi_k f_k(t) 1_{(k-1, k]}(s) ds \right|, \quad u \in [0, 1].$$

Then, in order to prove (5.3.11), it suffices to show that

$$\lim_{u \rightarrow 1} \mathbb{P} \left(\sup_{v \in [u, 1]} Y_v \geq 2\lambda \right) = 0. \quad (5.3.12)$$

We show that for any $0 \leq u < v \leq 1$,

$$|Y_v - Y_u| \leq \sup_{t \in [0, T]} \left| \int_{l(u)}^{l(v)} \sum_{k=1}^{\infty} a_k \xi_k f_k(t) 1_{(k-1, k]}(s) ds \right|.$$

In fact, let $t^* \in \arg \max_{t \in [0, T]} \left| \int_{l(u)}^{\infty} \sum_{k=1}^{\infty} a_k \xi_k f_k(t) 1_{(k-1, k]}(s) ds \right|$. Then

$$\begin{aligned} Y_v - Y_u &\leq \left| \int_{l(v)}^{\infty} \sum_{k=1}^{\infty} a_k \xi_k f_k(t^*) 1_{(k-1, k]}(s) ds \right| - \left| \int_{l(u)}^{\infty} \sum_{k=1}^{\infty} a_k \xi_k f_k(t^*) 1_{(k-1, k]}(s) ds \right| \\ &\leq \left| \int_{l(u)}^{l(v)} \sum_{k=1}^{\infty} a_k \xi_k f_k(t^*) 1_{(k-1, k]}(s) ds \right| \\ &\leq \sup_{t \in [0, T]} \left| \int_{l(u)}^{l(v)} \sum_{k=1}^{\infty} a_k \xi_k f_k(t) 1_{(k-1, k]}(s) ds \right|. \end{aligned} \quad (5.3.13)$$

Similarly,

$$Y_u - Y_v \leq \sup_{t \in [0, T]} \left| \int_{l(u)}^{l(v)} \sum_{k=1}^{\infty} a_k \xi_k f_k(t) 1_{(k-1, k]}(s) ds \right|.$$

For any $u < v$, by the definition of $l(u)$,

$$\sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \int_{l(u)}^{l(v)} 1_{(k-1, k]}(s) ds = (v - u) \sigma^2. \quad (5.3.14)$$

Applying (5.3.9) to the sequence (a'_1, a'_2, \dots) with

$$a'_k = a_k \left(\int_{l(u)}^{l(v)} 1_{(k-1, k]}(s) ds \right)^{1/2}, \quad k \geq 1,$$

we obtain that

$$\mathbb{P}(|Y_v - Y_u| \geq 2\lambda) \leq C_h \exp \left[- \frac{D_h \lambda^2}{|v - u| \sigma^2} \right], \quad (5.3.15)$$

where C_h and D_h are constants depending only on h and may be different at various occurrences. In particular, since $Y_1 = 0$,

$$\mathbb{P}(|Y_v| \geq 2\lambda) \leq C_h \exp \left[- \frac{D_h \lambda^2}{(1 - u) \sigma^2} \right], \quad \text{for all } v \in [u, 1]. \quad (5.3.16)$$

Since (5.3.15) implies that

$$\mathbb{E}(|Y_v - Y_u|^p) \leq C_h |v - u|^{p/2}$$

for any $p > 2$. We are now in a position to apply Theorem 5.3.2 again, and deduce that

$$\mathbb{P} \left(\sup_{v \in [u, 1]} Y_v \geq 2\lambda \right) \leq C_h \exp \left[- \frac{D_h \lambda^2}{(1 - u) \sigma^2} \right].$$

Therefore, (5.3.12) follows readily, which completes the proof. \square

5.4 An application to divergence-free stochastic DiPerna–Lions flows

In this section, we apply the tail estimate in Theorem 5.3.2 to the study of existence of strong solutions to stochastic DiPerna–Lions flows with divergence-free drifts

$$dX_t = b(t, X_t) dt + dB_t, \quad X_0 = x, \quad (5.4.1)$$

where $\{B_t\}_t$ is a standard n -dimensional Brownian motion ($n \geq 3$). Materials in this section is based on an ongoing work with G. Xi.

Throughout this section, we assume the following conditions on the coefficient $b(t, x)$:

(1.4.i) $b \in L^l(0, T; L^q(\mathbb{R}^n)) \cap L^1(0, T; W^{1,p}(\mathbb{R}^n))$ for some $l > 1$, $q > \frac{n}{2}$, $p > 1$ such that

$$\gamma = \frac{2}{l} + \frac{n}{q} \in [1, 2); \quad (5.4.2)$$

(1.4.ii) b is divergence-free, i.e. $\operatorname{div}(b) = \sum_{i=1}^n \partial_i b_i = 0$.

The SDE (5.4.1) attracts researchers' attention because of its close relationship to the Navier–Stokes equation

$$\partial_t u(t, x) = \Delta u(t, x) + (b \cdot \nabla)u(t, x), \quad (5.4.3)$$

which arises from the study of fluid dynamics of incompressible flows. There have been many research works on existence and uniqueness of solutions to the SDE (5.4.1) (e.g. [39, 9, 71, 13, 16]). These works can be divided into three cases according to the value of $\gamma = \frac{2}{l} + \frac{n}{q}$, i.e. the subcritical case $0 < \gamma < 1$, the critical case $\gamma = 1$, and the supercritical case $\gamma > 1$. The constant γ is involved as a renormalization parameter under the scaling $(t, x) \mapsto (\rho^2 t, \rho x)$ for $\rho > 0$. To see this, notice that

$$\|b^\rho\|_{L^l(0,T;L^q(\mathbb{R}^n))} = \rho^{1-\gamma} \|b\|_{L^l(0,T;L^q(\mathbb{R}^n))},$$

where $b^\rho(t, x) = b(\rho^2 t, \rho x)$. The quantity $\|b\|_{L^l(0,T;L^q(\mathbb{R}^n))}$ appears as the multiplicative constant of the parabolic Harnack inequality (cf. [53, Section 3] or [31, Lemma 2.1]). Therefore, γ determines whether or not the parabolic Harnack inequality is locally uniform. For the subcritical case (i.e. $\gamma < 1$), the existence of unique strong solutions to (5.4.1) was proved in [39]. Results on the critical and the supercritical cases (i.e. $\gamma \geq 1$), however, remain quite limited. SDEs with time-inhomogeneous coefficients were studied in [14]

under boundedness condition on b and certain integrability condition on coefficients and their weak derivatives. In [71], existence and uniqueness of strong solutions was obtained for the time-homogeneous case (i.e. $b = b(x)$) under linear growth condition on b and that $\nabla b \in (L \log L)_{\text{loc}}(\mathbb{R}^n)$. In an ongoing work by the current author and G. Xi, we shall investigate, under different conditions on the coefficients, the existence and uniqueness of strong solutions to SDEs when $\gamma \geq 1$ and the coefficients are time-inhomogeneous.

The objective of this section is to present an application of Theorem 5.3.2 to the existence of strong solutions to the SDE (5.4.1) for almost every initial data $x \in \mathbb{R}^n$. Since the argument mainly follows that in [71] with necessary adaptations to the assumptions (1.4.i) and (1.4.ii), we shall only give a sketch of the proof, and indicate necessary modifications and places where Theorem 5.3.2 is applied. The reader is referred to [71, Section 6] for details.

Let $\{b_k\}_k$ be a sequence of smooth functions on $[0, T] \times \mathbb{R}^n$ such that

$$\lim_{k \rightarrow \infty} \|b_k - b\|_{L^1(0,T;L^q(\mathbb{R}^n))} = 0, \quad \sup_k \|\nabla b_k\|_{L^1(0,T;L^p(\mathbb{R}^n))} \leq C \|\nabla b\|_{L^1(0,T;L^p(\mathbb{R}^n))}. \quad (5.4.4)$$

For each b_k , let $X_k(x) = \{X_{k,t}(x)\}_t$ be the unique strong solution to the SDE (5.4.1) with coefficient b_k and initial data x . The objective is to show that, for any $N > 0$ and any $r \in [1, 2)$,

$$\lim_{j,k \rightarrow \infty} \mathbb{E} \left[\int_{|x| \leq N} \sup_{t \in [0, T]} |X_{j,t}(x) - X_{k,t}(x)|^r dx \right] = 0. \quad (5.4.5)$$

Using an argument similar to that of [71, Lemma 6.1], it can be easily shown that, for any $\theta > 0$,

$$\begin{aligned} & \mathbb{E} \left[\int_{|x| \leq N} \log \left(1 + \sup_{t \in [0, T]} \frac{|X_{j,t}(x) - X_{k,t}(x)|^2}{\theta^2} \right) dx \right] \\ & \leq C_N \left(\|\nabla b_j\|_{L^1(0,t;L^p(\mathbb{R}^n))} + \theta^{-1} \|b_j - b_k\|_{L^1(0,t;L^q(\mathbb{R}^n))} \right), \end{aligned} \quad (5.4.6)$$

where C_N is a constant depending only on N, T, n . As demonstrated in [71, Theorem 6.3], the inequality (5.4.6) (with $\theta = \|b_j - b_k\|_{L^1(0,t;L^q(\mathbb{R}^n))}$) implies the convergence in

probability

$$\lim_{j,k \rightarrow \infty} \mathbb{P} \left[\int_{|x| \leq N} \sup_{t \in [0, T]} |X_{j,t}(x) - X_{k,t}(x)|^2 dx \geq \eta \right] = 0, \quad \text{for all } \eta > 0. \quad (5.4.7)$$

With the convergence in probability (5.4.7) in hand, the proof of (5.4.5) reduces to showing the following uniform integrability

$$\sup_k \sup_{|x| \leq N} \mathbb{E} \left(\sup_{t \in [0, T]} |X_{k,t}(x)|^2 \right) < \infty. \quad (5.4.8)$$

In our argument, the main modification made to [71] is the derivation of (5.4.8). For the cases considered in [71], linear growth condition on b_k is exploited to establish (5.4.8) by a standard argument. For our case where linear growth is not assumed, we instead derive (5.4.8) using Theorem 5.3.2 and an upper estimate for the transition kernel, which in fact give a tail estimate for $\sup_{t \in [0, T]} |X_{k,t}|$.

The following transition kernel estimate was proved in [58, Corollary 9].

Theorem 5.4.1. *Suppose that b is a smooth function with bounded derivatives, and b satisfies (1.4.i) and (1.4.ii). Let $\Gamma(t, x; s, y)$ be the fundamental solution to (5.4.3), and let*

$$\mu = \frac{2}{2 - n/q}, \quad \nu = \frac{2 - \gamma}{2 - n/q}, \quad (5.4.9)$$

where γ is the constant defined by (5.4.2).

(a) *If $\mu > 1$, then*

$$\Gamma(t, x; s, y) \leq \begin{cases} \frac{K_2}{(t-s)^{n/2}} \exp \left[-\frac{K_1 |x-y|^2}{t-s} \right], & \text{if } \frac{|x|^{\mu-2}}{t^{\mu-\nu-1}} < 1, \\ \frac{K_2}{(t-s)^{n/2}} \exp \left[-\frac{K_1 |x-y|^{\mu/(\mu-1)}}{|t-s|^{\nu/(\mu-1)}} \right], & \text{if } \frac{|x|^{\mu-2}}{t^{\mu-\nu-1}} \geq 1, \end{cases} \quad (5.4.10)$$

where $K_1 = K_1(l, q, n)$, $K_2 = K_2(l, q, n, \|b\|_{L^1(0, T; L^q(\mathbb{R}^n))})$ are constants depending only on the parameters in the corresponding parentheses.

(b) If $\mu = 1$ (which forces $q = \infty$), then

$$\Gamma(t, x; s, y) \leq \frac{K_2}{(t-s)^{n/2}} \exp \left[-\frac{K_2((t-s)^\nu - |x-y|)^2}{t-s} \right], \quad (5.4.11)$$

where the constant K_2 depends on the same parameters as in (a).

Before we derive the uniform moment estimate (5.4.8) by Theorem 5.3.2 and Theorem 5.4.1, let us point out a special case for which a tail estimate for $\sup_{t \in [0, T]} |X_{k,t}|$ can be obtained without using Theorem 5.3.2. This is the case when the transition kernel $\Gamma(t, x; s, y)$ has both lower and upper Gaussian estimate:

$$\frac{C_4}{(t-s)^{n/2}} \exp \left[-\frac{C_3|x-y|^2}{t-s} \right] \leq \Gamma(t, x; s, y) \leq \frac{C_2}{(t-s)^{n/2}} \exp \left[-\frac{C_1|x-y|^2}{t-s} \right]. \quad (5.4.12)$$

These Gaussian upper and lower bounds are valid when $l = \infty$, $q = n$ (see [58, p. 4] for more details). The crucial point in this case is the exact space–time scaling. To be precise, let $\tau = \inf\{t \in [0, T] : |X_t| \geq \lambda\}$. By definition,

$$\mathbb{P} \left(\sup_{t \in [0, T]} |X_t| \geq \lambda \right) = \mathbb{P}(\tau \leq T).$$

Hence, by Bayes' rule,

$$\mathbb{P} \left(\sup_{t \in [0, T]} |X_t| \geq \lambda \right) = \frac{\mathbb{P}(|X_T| \geq \lambda)}{\mathbb{P}(|X_T| \geq \lambda \mid \tau \leq T)}. \quad (5.4.13)$$

By the Gaussian lower bound in (5.4.12),

$$\begin{aligned} \mathbb{P}(|X_T| \geq \lambda \mid \tau \leq T) &= \int_0^T \mathbb{P}(|X_T| \geq \lambda \mid \tau = t) \mathbb{P}(\tau \in dt) \\ &\geq C \int_0^T \mathbb{P}(X_T - X_t \geq 0) \mathbb{P}(\tau \in dt) \\ &\geq C \int_0^T \int_{\mathbb{R}_+^n} \frac{C_4}{(T-t)^{n/2}} \exp \left[-\frac{C_3|x|^2}{T-t} \right] dx \mathbb{P}(\tau \in dt) \\ &\geq C, \end{aligned} \quad (5.4.14)$$

where $C > 0$ is a constant depending only on n, C_3, C_4 . By (5.4.14), (5.4.13), and the upper Gaussian bound in (5.4.12), we deduce that

$$\mathbb{P}\left(\sup_{t \in [0, T]} |X_t| \geq \lambda\right) \leq C_T \exp(-C_T \lambda^2),$$

which clearly implies (5.4.8). However, this approach is quite limited, since it requires both Gaussian lower and upper bounds.

Let us return to the derivation of (5.4.8). The conditions in Theorem 5.3.2 are justified by the following estimates.

Corollary 5.4.2. *Suppose that b is a smooth function with bounded derivatives, and satisfies (1.4.i) and (1.4.ii). Then the solution to (5.4.1) satisfies the estimate*

$$\mathbb{E}(|X_t - X_s|^r) \leq K|t - s|^{\theta_r}, \text{ for all } 0 \leq s < t \leq T, r \geq 1, \quad (5.4.15)$$

where $\theta_r = (1 - \gamma/2)r - (\gamma - 1)n/2$, and K is a constant depending only on l, q, n and $\|b\|_{L^l(0, T; L^q(\mathbb{R}^n))}$. *Moreover,*

$$\mathbb{P}(|X_t - X_0| \geq \lambda) \leq C_2 \exp(-C_1 \lambda^\alpha) \quad (5.4.16)$$

where $\alpha = \min\{2, \mu/(\mu - 1)\}$ ($\alpha = 2$ when $\mu = 1$), C_1 is a constant depending only on $l, q, n, \|b\|_{L^l(0, T; L^q(\mathbb{R}^n))}$, and C_2 is a constant depending only on $l, q, \|b\|_{L^l(0, T; L^q(\mathbb{R}^n))}, n, T$. Here μ is the constant defined by (5.4.9).

Proof. Without loss of generality, we may assume $s = 0$ and $t \leq 1$. When $\mu > 1$, by (5.4.10) (with $b(t, x)$ replaced by $b(t, x + x_0)$),

$$\begin{aligned} \mathbb{E}(|X_t - X_0|^r) &= \int_{\mathbb{R}^n} |x|^r \Gamma(t, x; 0, 0) dx \\ &\leq \int_{\mathbb{R}^n} |x|^r \frac{K_2}{t^{n/2}} \exp\left(-\frac{K_1|x|^2}{t}\right) dx \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}^n} |x|^r \frac{K_2}{t^{n/2}} \exp \left[-K_1 \left(\frac{|x|^\mu}{t^\nu} \right)^{\frac{1}{\mu-1}} \right] dx \\
& = K \left(t^{\frac{r}{2}} + t^{(2-\gamma)\frac{r}{2} - (\gamma-1)\frac{n}{2}} \right) \leq K t^{\theta r}.
\end{aligned}$$

The proof of (5.4.15) for the case when $\mu = 1$ is similar to the above.

We now turn to the proof of (5.4.16). When $\mu > 1$, by (5.4.10),

$$\mathbb{P}(|X_t - X_0| \geq \lambda) \leq \int_{|x| \geq \lambda} \frac{K_2}{t^{n/2}} \exp \left(-\frac{K_1 |x|^2}{t} \right) dx + \int_{|x| \geq \lambda} \frac{K_2}{t^{n/2}} \exp \left[-K_1 \left(\frac{|x|^\mu}{t^\nu} \right)^{\frac{1}{\mu-1}} \right] dx,$$

where ν is the constant defined by (5.4.9). Clearly,

$$\int_{|x| \geq \lambda} \frac{K_2}{t^{n/2}} \exp \left(-\frac{K_1 |x|^2}{t} \right) dx \leq C_2 \exp(-C_1 \lambda^2), \quad t \in [0, T],$$

and

$$\int_{|x| \geq \lambda} \frac{K_2}{t^{n/2}} \exp \left[-K_1 \left(\frac{|x|^\mu}{t^\nu} \right)^{\frac{1}{\mu-1}} \right] dx \leq C_2 \exp \left(-C_1 \lambda^{\frac{\mu}{\mu-1}} \right), \quad t \in [0, T].$$

This proves (5.4.16) for the case $\mu > 1$. When $\mu = 1$, by (5.4.11),

$$\mathbb{P}(|X_t - X_0| \geq \lambda) \leq \int_{|x| \geq \lambda} \frac{K_2}{t^{n/2}} \exp \left(-\frac{K_2 (t^\nu - |x|)^2}{t} \right) dx \leq C_2 \exp(-C_1 \lambda^2), \quad t \in [0, T].$$

This completes the proof. □

Now we are ready to apply Theorem 5.3.2 to derive the tail estimate and moment estimate for the supremum.

Proposition 5.4.3. *Suppose that b is a smooth function with bounded derivatives, and satisfies (1.4.i) and (1.4.ii). Then the solution to (5.4.1) satisfies*

$$\mathbb{P} \left(\sup_{t \in [0, T]} |X_t - X_0| \geq \lambda \right) \leq C_2 \exp(-C_1 \lambda^\alpha), \quad (5.4.17)$$

with $\alpha = \min\{2, \mu/(\mu - 1)\}$ ($\alpha = 2$ when $\mu = 1$), where μ is the constant defined by (5.4.9); C_1 is a constant depending only on $l, q, n, \|b\|_{L^l(0,T;L^q(\mathbb{R}^n))}$, and C_2 is a constant depending only on $l, q, \|b\|_{L^l(0,T;L^q(\mathbb{R}^n))}, n, T$. Therefore,

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t - X_0|^r \right] \leq C < \infty, \quad \text{for all } r \geq 1, \quad (5.4.18)$$

where C is a constant depending only on $l, q, \|b\|_{L^l(0,T;L^q(\mathbb{R}^n))}, n, T$.

Proof. By Corollary 5.4.2, applying Theorem 5.3.2 to the process $\{X_t - X_0\}_t$ immediately yields (5.4.17). The estimate (5.4.18) is a direct corollary of (5.4.17). \square

Now by (5.4.4) and Proposition 5.4.3 applied to each b_k , the desired uniform moment estimate (5.4.8) follows immediately. As demonstrated earlier, following an argument similar to that in [71, Section 6], the estimate (5.4.8) implies the following almost everywhere existence and uniqueness of strong solutions to stochastic DiPerna–Lions flows with divergence-free drifts.

Theorem 5.4.4. *Under assumptions (1.4.i) and (1.4.ii), the SDE (5.4.1) admits a unique strong solution $\{X_t\}_t$ for almost every initial data $X_0 = x \in \mathbb{R}^n$. More precisely, let $\{B_t\}_t$ be a standard Brownian motion on a filtered probability space, and $\{\mathcal{F}_t\}_t$ be the filtration determined by $\{B_t\}_t$. There exists a measurable \mathbb{R}^n -valued stochastic field $X = X(t, \omega; x)$ on $[0, \infty) \times \Omega \times \mathbb{R}^n$ such that, for almost all $x \in \mathbb{R}^n$, the process $\{X_t(x)\}_t$ is the unique strong solution to (5.4.1), that is, $\{X_t(x)\}_t$ is an $\{\mathcal{F}_t\}$ -adapted process such that*

$$\int_0^t |b(r, X_r(x))| dr < \infty \quad \mathbb{P}\text{-a.s.},$$

and

$$X_t(x) = x + \int_0^t b(r, X_r(x)) dr + B_t, \quad t \geq 0 \quad \mathbb{P}\text{-a.s.}$$

Remark 5.4.5. We should point out that the estimate (5.4.18) is crucial to the argument used to prove Theorem 5.4.4. The tail estimate (5.4.17) is a technical intermediate step.

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