

THE MEAN FIELD LIMIT OF STOCHASTIC DIFFERENTIAL EQUATION SYSTEMS MODELING GRID CELLS*

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Abstract. Several differential equation models have been proposed to explain the formation of patterns characteristic of the grid cell network. Understanding the robustness of these patterns with respect to noise is one of the key open questions in computational neuroscience. In the present work, we analyze a family of stochastic differential systems modeling grid cell networks. Furthermore, the well-posedness of the associated McKean–Vlasov and Fokker–Planck equations, describing the average behavior of the networks, is established. Finally, we rigorously prove the mean field limit of these systems and provide a sharp rate of convergence for their empirical measures.

Key words. mean field, Fokker–Planck equations, neuroscience

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1. Introduction. The discovery in 2005 of a type of neurons in the brain named grid cells [15] led to a breakthrough in the understanding of the navigational system in mammalian brains; see [18] for an extensive review. These neurons fire as an animal moves around in an open area, enabling the animal to understand its position in space. The grid cell network has commonly been described by deterministic continuous attractor network dynamics through a system of neural field models [11, 19, 5, 9], which are based on the classical papers [25, 26, 2]. The models can fairly accurately predict what can be observed in experiments. However, the question of how the grid cell network is affected by noise, posed as a challenge in [20], has been left open.

In [6] fundamental limits on how information dissipates in attractor networks of noisy neurons were derived. A different direction pursuing further understanding of the effect of noise on grid cell networks was made in [7] by studying a system of Fokker–Planck-like partial differential equations (PDEs). The system of PDEs was derived by adding noise to the attractor network models in [5, 9] and formally taking the mean field limit. In the present manuscript this limit is rigorously proved. In addition, we derive the limit for more general noise terms, which covers the models considered in [6, 1].

The mean field limit of interacting particle systems has lately received lots of attention in mathematical biology [4, 12, 8]; see [16] for a survey. The closest result

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to the analysis presented in this work shows the mean field limit of a stochastic delayed set of interacting neurons [23]. The system of stochastic differential equations (SDEs) describing interacting grid cells in this work introduces different challenges: boundary conditions imposing positivity of the activity level of the neurons, nonlinearity of the firing rate, and coupling between different families of neurons.

The neural model under consideration, which is based on the model in [5], can be described as follows. Given space points $x_1, \dots, x_N \in Q$ in a region Q of the neural cortex, we will consider the following model for the interaction among NM neurons stacked in N columns at locations x_i with M neurons each, where u_{ik}^β represents the activity level with orientation β of the k th neuron at location x_i :

$$(1.1) \quad u_{ik}^\beta(t)\tau_i^\beta = \tau_i^\beta u_{ik}^\beta(0) + \sigma W_{ik}^\beta(t) - \ell_{ik}^\beta(t) + \int_0^t \left(-u_{ik}^\beta(r) + \phi \left(B^\beta(x_i, r) + \frac{1}{4NM} \sum_{\gamma=1}^4 \sum_{j=1}^N \sum_{m=1}^M K^\gamma(x_i - x_j) u_{jm}^\gamma(r) \right) \right) dr,$$

$$(1.2) \quad \ell_{ik}^\beta(t) = -|\ell_{ik}^\beta|(t), \quad |\ell_{ik}^\beta|(t) = \int_0^t 1_{\{u_{ik}^\beta(r)=0\}} d|\ell_{ik}^\beta|(r) \quad \text{for } \beta = 1, 2, 3, 4.$$

For simplicity, we consider $Q = [0, 1]^d$. The results in this work are easily extended to any bounded open subset $Q \subseteq \mathbb{R}^d$ for any $d \geq 1$. Here, for integers $k = 1, \dots, M$, we have independent and identically distributed (i.i.d.) families of random initial conditions $\{u_{ik}(0)\}_{i=1, \dots, N}$ for each space point x_i in the cortex Q . Moreover, for integers $i = 1, \dots, N$ and $k = 1, \dots, M$, we have four-dimensional Brownian motions $(W_{ik}^\beta)_{\beta=1,2,3,4}$, which can also be correlated.

The nonlinear function $\phi : \mathbb{R} \rightarrow \mathbb{R}$, representing the firing rate of neurons in the network, is globally Lipschitz, whereas the external inputs $B^\beta : Q \times \mathbb{R} \rightarrow \mathbb{R}$ and the interaction kernels $K^\beta : \mathbb{R}^d \rightarrow \mathbb{R}$ for $\beta = 1, 2, 3, 4$ are only required to be locally bounded functions and α -Hölder continuous in the x variable for some $\alpha \in (0, 1]$. The interaction kernels take into account the inhibitory/excitatory effect on nearby neurons. A typical choice of the interaction kernel in computational neuroscience [5] is given by the so-called Mexican hat function. The relaxation times τ_i^β satisfy the condition $0 < \inf_{i,\beta} \tau_i^\beta \leq \sup_{i,\beta} \tau_i^\beta < +\infty$.

Finally, for each i, k , and β , the term ℓ_{ik}^β is a finite variation process defined by (1.2) which prevents the activity level u_{ik}^β from taking negative values. Namely, as we can see in its definition, at each time t this process equals the opposite of its total variation $|\ell_{ik}^\beta|(t) = -|\ell_{ik}^\beta|(t)$. In turn, the total variation stays constant when $u_{ik}^\beta > 0$ and it increases in the form $|\ell_{ik}^\beta|(t) = \int_0^t 1_{\{u_{ik}^\beta(r)=0\}} d|\ell_{ik}^\beta|(r)$ when $u_{ik}^\beta = 0$, so as to push u_{ik}^β away from zero which is being dragged by the other terms at the right-hand side of (1.1). The introduction of such terms and constraints is therefore known as imposing *reflecting boundary conditions* and ℓ_{ik}^β is called a *reflection term*. The existence and uniqueness of such a term need of course to be proved and this process is often referred to as the *Skorokhod problem*. Precise details concerning the well-posedness and the construction of the reflection term in our setting are all presented in the seminal papers [17, 21] by Lions and Sznitman.

Going back to (1.1), we notice that the argument of ϕ in (1.1) can be rewritten as

$$\frac{1}{4NM} \sum_{\gamma=1}^4 \sum_{j=1}^N \sum_{m=1}^M K^\gamma(x_i - x_j) u_{jm}^\gamma(r) = \int_{Q \times \mathbb{R}^4} \frac{1}{4} \sum_{\gamma=1}^4 K^\gamma(x_i - y) u^\gamma f_{N,M}(r, dy, du)$$

by considering the empirical measure associated to these particles, that is,

$$(1.3) \quad f_{N,M}(r, dy, du) = \frac{1}{NM} \sum_{j=1}^N \sum_{m=1}^M \delta_{(x_j, u_{jm}(r))} \quad \text{regarded as a measure on } Q \times \mathbb{R}^4.$$

Concerning the initial conditions and the form of the noise term in (1.1), from a modeling point of view it is reasonable to assume that, for $k \in \mathbb{N}$, we have i.i.d. families of initial conditions $(u_k(x, 0))_{x \in Q}$ for each space point x in the cortex Q . Similarly, we assume that, for $k \in \mathbb{N}$, we have independent four-dimensional space-time white noise terms $(W_k(x, t))_{t \geq 0, x \in Q}$. Naively, $W_k^\beta(x, t)$ is a centered Gaussian random field indexed by $k \in \mathbb{N}$, $\beta = 1, 2, 3, 4$, $x \in Q$, and $t \in [0, \infty)$ with covariance

$$(1.4) \quad \mathbb{E} \left[W_k^\beta(x, t) W_h^\gamma(y, s) \right] = (t \wedge s) \delta_0(k - h) \delta_0(\beta - \gamma) \delta_0(x - y).$$

Then we can just choose points $x_1, \dots, x_N \in Q$ and set $u_{ik}(0) := u_k(x_i, 0)$ and $W_{ik}(t) := W_k(x_i, t)$. As long as we work in a countable setting, this naive construction can be made rigorous upon taking a suitable modification of the W_{ik} 's via the Kolmogorov continuity theorem. We also point out that the way we choose the cloud of points $x_1, \dots, x_N \in Q$ is not that important if we are only concerned with the discrete model for fixed M and N . However, to get a nice limiting behavior as $N, M \rightarrow \infty$, it is useful to take these points to be the nodes of a grid of Q whose mesh tends to zero. Precise details on this are given in section 5.

Remark 1.1. One should not expect the initial data $(u_k(x, 0))_{x \in Q}$ to be independent for different values of x , nor to be equidistributed. Indeed, from the point of view of modeling in neuroscience, $u_k(x, 0)$ should be close to $u_k(y, 0)$ for x close to y . This fact will have consequences both for the exchangeability properties of the particles u_{ik} , which are expected to be exchangeable in the index k only, and for the rate of convergence toward the limiting behavior.

As we let $M, N \rightarrow \infty$ the limiting behavior should be described by independent copies, in the column index k , of solutions to an associated mean field McKean–Vlasov equation. Namely, the activity level of any neuron located at a point $x \in Q$ should satisfy an equation like

$$(1.5) \quad \begin{cases} \bar{u}^\beta(x, t) \tau^\beta(x) = \tau^\beta(x) u^\beta(x, 0) + \sigma W^\beta(x, t) - \bar{\ell}^\beta(x, t) + \int_0^t \left(-\bar{u}^\beta(x, r) + \phi(B^\beta(x, r)) \right. \\ \quad \left. + \frac{1}{4} \sum_{\gamma=1}^4 \int_{Q \times \mathbb{R}^4} K^\gamma(x - y) u^\gamma f(r, y, du) dy \right) dr, \\ \bar{\ell}^\beta(x, t) = -|\bar{\ell}^\beta(x, \cdot)|(t), \quad |\bar{\ell}^\beta(x, \cdot)|(t) = \int_0^t \mathbf{1}_{\{\bar{u}^\beta(x, r)=0\}} d|\bar{\ell}^\beta(x, \cdot)|(r) \text{ for } \beta = 1, 2, 3, 4, \end{cases}$$

where we have set $f(t, y, du) := \text{Law}_{\mathbb{R}^4}(\bar{u}(y, t))$ considered as a measure on \mathbb{R}^4 depending on $t \in [0, \infty)$ and $y \in Q$. Notice that in turn this induces a probability measure $f(t, dx, du)$ on $Q \times \mathbb{R}^4$ defined by integration as

$$(1.6) \quad \int_{Q \times \mathbb{R}^4} \varphi(x, u) f(t, dx, du) := \int_Q \int_{\mathbb{R}^4} \varphi(x, u) f(t, x, du) dx \quad \text{for any } \varphi \in C_b(Q \times \mathbb{R}^4).$$

For each fixed $x \in Q$ and $\beta = 1, 2, 3, 4$, the finite variation process $\bar{\ell}^\beta(x, t)$ is again the reflection term coming from the Skorokhod problem (see the explanation after

(1.1)) and it ensures that $\bar{u}^\beta(x, t) \geq 0$ for every x, t , and β . We refer the reader to [21] for the details about such a process in the context of a classical McKean–Vlasov equation.

Remark 1.2. The McKean–Vlasov equation (1.5) suffers from a major technical issue. Indeed, formula (1.4) does define an \mathbb{R} -valued Gaussian random field. However, it is well-known that such a random field cannot be jointly measurable in the x variable and the sample ω . This causes the lack of x -measurability of the particles $\bar{u}^\beta(x, t)$ and, in turn, of the law $f(t, x, du)$, although we need this to be Lebesgue integrable. In this work, we resolve this issue by considering ϵ -correlated noise.

Another approach, coming from the theory of mean field games, is to address the issue by introducing a “Fubini extension” of the product probability space $Q \times \Omega$. We refer the reader to [3] and the references therein. However, this approach did not seem to fit our modeling purposes. It allows us to regain the x -measurability only with respect to a bigger σ -algebra, *strictly* containing the Lebesgue measurable sets. In turn, the space integral in (1.5) would not be taken with respect to the Lebesgue measure, but instead with respect to some exotic extension of this.

A formal application of the Itô formula shows that f , the joint distribution of the activity levels u^β in the four directions β , satisfies the nonlinear Fokker–Planck equation

$$\begin{aligned}
 (1.7) \quad \partial_t f(t, x, u) &+ \sum_{\beta=1}^4 \frac{1}{\tau^\beta(x)} \partial_{u^\beta} \left(f(t, x, u) \left(-u^\beta + \phi \left(B^\beta(x, t) \right. \right. \right. \\
 &\left. \left. \left. + \frac{1}{4} \sum_{\gamma=1}^4 \int_{Q \times \mathbb{R}^4} K^\gamma(x-y) v^\gamma f(t, y, dv) dy \right) \right) \\
 &= \frac{\sigma^2}{2} \sum_{\beta=1}^4 \frac{1}{\tau^\beta(x)^2} \partial_{u^\beta u^\beta}^2 f(t, x, u),
 \end{aligned}$$

in the weak sense, with initial condition $f(0, x, du) = \text{Law}_{\mathbb{R}^4}(u(x, 0))$ and subjected to the no-flux boundary conditions, for $\beta = 1, 2, 3, 4$,

$$\begin{aligned}
 &\phi \left(B^\beta(x, t) + \frac{1}{4} \sum_{\gamma=1}^4 \int_{Q \times \mathbb{R}^4} K^\gamma(x-y) v^\gamma f(t, y, dv) dy \right) f(t, x, u) \\
 &- \frac{\sigma^2}{2} \frac{1}{\tau^\beta(x)} \frac{\partial}{\partial u^\beta} f(t, x, u) \Big|_{u^\beta=0} = 0,
 \end{aligned}$$

which come from the reflecting boundary conditions at the SDE level.

Remark 1.3. It is worth pointing out that (1.7) would arise as the law of $\bar{u}(x, t)$ even if we set $W(x, t) \equiv B_t$ for every $x \in Q$ for a single Brownian motion B_t , that is, if all the particles were affected by the same noise. The same holds for many other choices of $W(x, t)$ and follows immediately from the Itô formula: the effect of the term $W(x, t)$ is only to generate diffusion in the u variable for fixed x . The choice of noise to consider in (1.1) and (1.5) is therefore dictated by modeling purposes only.

Remark 1.4. We notice that for each $\beta = 1, 2, 3, 4$, integrating (1.7) in \mathbb{R}_+^3 over the remaining variables u^γ for $\gamma \neq \beta$ and exploiting the boundary conditions, we get the equation satisfied by the marginal distribution $f^\beta(r, y, du^\beta) = \text{Law}_{\mathbb{R}}(\bar{u}^\beta(y, r))$. Namely, we obtain

$$\begin{aligned}
 (1.8) \quad & \partial_t f^\beta(t, x, u^\beta) \\
 & + \frac{1}{\tau^\beta(x)} \partial_{u^\beta} \left(f^\beta(t, x, u^\beta) \left(-u^\beta + \phi \left(B^\beta(x, t) + \frac{1}{4} \sum_{\gamma=1}^4 \int_{Q \times \mathbb{R}^4} K^\gamma(x-y) v^\gamma f^\gamma(t, y, dv^\gamma) dy \right) \right) \right) \\
 & = \frac{\sigma^2}{2} \frac{1}{\tau^\beta(x)^2} \frac{\partial^2 f^\beta}{(\partial u^\beta)^2}(t, x, u^\beta).
 \end{aligned}$$

In particular, we stress the fact that each marginal f^β satisfies an equation involving only the other marginals f^γ and not the full joint distribution f . On the other hand, if we sum (1.8) over $\beta = 1, 2, 3, 4$, then we get back (1.7) above for the decoupled distribution $\tilde{f} := \prod_{\beta=1}^4 f^\beta$. Thus (1.7) and the system of equations (1.8) for $\beta = 1, 2, 3, 4$ are completely equivalent, at least for decoupled initial data $f_0 = \prod_{\beta=1}^4 f_0^\beta$. Finally, Theorem 2.5 below asserts we have existence and uniqueness for (1.7). The previous argument then shows that if we start with decoupled initial data, this structure is preserved: the corresponding solution satisfies $f(t) = \prod_{\beta} f^\beta(t)$ for all $t \geq 0$. Notice that (1.8) is the model formally introduced in [7].

The structure of this work is as follows. The next section is devoted to introducing the notation and the setting needed for the results. We finish the section by stating the main theorems concerning the mean field limit of (1.1) and its extensions. Sections 3 and 4 focus on the existence and uniqueness of the particle systems, and the associated McKean–Vlasov equations and Fokker–Planck type PDEs. The main core of this work is found in section 5, where we rigorously prove the mean field limit. Section 6 adapts previous results on empirical measure error estimates [13] to the present setting to provide rates of convergence for the associated empirical measure.

2. Preliminaries and main results.

2.1. Hypotheses and notation. In this section we introduce the hypotheses we assume for our problem. First, we point out that the results of this paper extend to the more general particle system

$$(2.1) \quad \begin{cases} u_{ik}(t) = u_{ik}(0) + \int_0^t b(x_i, r, u_{ik}(r), f_{N,M}(r)) dr \\ \quad + \int_0^t \sigma(x_i, r, u_{ik}(r), f_{N,M}(r)) dW_{ik}(r) - \ell_{ik}(t), \\ \ell_{ik}^\beta(t) = -|\ell_{ik}^\beta|(t), \quad |\ell_{ik}^\beta|(t) = \int_0^t 1_{\{u_{ik}^\beta(r)=0\}} d|\ell_{ik}^\beta|(r) \quad \text{for } \beta = 1, 2, 3, 4, \end{cases}$$

for N columns of M neurons each, located at x_1, \dots, x_N , with general drift term b and diffusion term σ . Here $f_{N,M}(r, dy, du)$ is again the empirical measure associated to the particles (2.1), given by (1.3). As before, ℓ_{ik}^β is the reflection term coming from the Skorokhod problem [17] forcing $u_{ik}^\beta(t) \geq 0$ for every $t \geq 0$.

The precise details on the shape and hypotheses on b and σ are given here below and they are simply deduced from the properties of the concrete model (1.1).

Let $\mathcal{P}(Q \times \mathbb{R}^4)$ denote the set of probability measures on $Q \times \mathbb{R}^4$; for $\beta = 1, 2, 3, 4$ we assume that $b_\beta, \sigma_\beta : Q \times \mathbb{R}^+ \times \mathbb{R}^4 \times \mathcal{P}(Q \times \mathbb{R}^4) \rightarrow \mathbb{R}$ take the forms

$$(2.2) \quad b_\beta(x, r, u, f) = b_0^\beta(x, r, u) + \phi_{b_\beta} \left(\int_{Q \times \mathbb{R}^4} b_1^\beta(x, y, r, u, v) f(dy, dv) \right),$$

$$(2.3) \quad \sigma_\beta(x, r, u, f) = \sigma_0^\beta(x, r, u) + \phi_{\sigma_\beta} \left(\int_{Q \times \mathbb{R}^4} \sigma_1^\beta(x, y, r, u, v) f(dy, dv) \right).$$

Having in mind the concrete model (1.1a), we suppose $b_0^\beta, \sigma_0^\beta : Q \times \mathbb{R}^+ \times \mathbb{R}^4 \rightarrow \mathbb{R}$ are measurable, locally bounded, Lipschitz in $u \in \mathbb{R}^4$ uniformly in $x, r \in Q \times \mathbb{R}^+$, and α -Hölder in $x \in Q$ uniformly in $u, r \in \mathbb{R}^4 \times \mathbb{R}^+$. That is,

$$(2.4) \quad |b_0^\beta(x, r, u) - b_0^\beta(x', r, u')| + |\sigma_0^\beta(x, r, u) - \sigma_0^\beta(x', r, u')| \leq L(|x - x'|^\alpha + |u - u'|),$$

$$(2.5) \quad |b_0^\beta(x, r, u)| + |\sigma_0^\beta(x, r, u)| \leq C(1 + |u|)$$

for all $x, x', u, u', r \in Q^2 \times (\mathbb{R}^4)^2 \times \mathbb{R}^+$, for suitable constants L and C . Furthermore we take the functions $\phi_{b_\beta}, \phi_{\sigma_\beta} : \mathbb{R} \rightarrow \mathbb{R}$ to be globally Lipschitz functions, and thus with sublinear growth. Similarly, the mappings $b_1^\beta, \sigma_1^\beta : Q \times Q \times \mathbb{R}^+ \times \mathbb{R}^4 \times \mathbb{R}^4 \rightarrow \mathbb{R}$ are measurable, locally bounded, Lipschitz in $u, v \in \mathbb{R}^4$ uniformly in $x, y, r \in Q^2 \times \mathbb{R}^+$, and α -Hölder in $x, y \in Q$ uniformly in $u, v, r \in (\mathbb{R}^4)^2 \times \mathbb{R}^+$. That is,

$$(2.6) \quad |b_1^\beta(x, y, r, u, v) - b_1^\beta(x, y, r, u', v')| + |\sigma_1^\beta(x, y, r, u, v) - \sigma_1^\beta(x, y, r, u', v')| \leq L(|x - x'|^\alpha + |y - y'|^\alpha + |u - u'| + |v - v'|),$$

$$(2.7) \quad |b_1^\beta(x, y, r, u, v)| + |\sigma_1^\beta(x, y, r, u, v)| \leq C(1 + |u| + |v|)$$

for all $x, y, x', y', u, v, u', v', r \in Q^4 \times (\mathbb{R}^4)^4 \times \mathbb{R}^+$, for suitable constants L and C .

Remark 2.1. With the notation just introduced, the starting model (1.1a) is recovered by setting

$$\tau^\beta(x)b^\beta(x, r, u, f) = -u^\beta + \phi \left(B^\beta(x, r) + \frac{1}{4} \sum_{\gamma=1}^4 \int_{Q \times \mathbb{R}^4} K^\gamma(x - y) u^\gamma f(dy, du) \right), \quad \text{and} \quad \tau^\beta(x)\sigma(x, r, u, f) \equiv \sigma.$$

We now consider the limiting McKean–Vlasov system. Taking into account the measurability issues pointed out in Remark 1.2, we consider instead (1.5) with a suitably rescaled ϵ -correlated noise, for some $\epsilon > 0$. In the setting of the general particle system (2.1), the equation reads

$$(2.8) \quad \begin{cases} \bar{u}^\epsilon(x, t) = u(x, 0) + \int_0^t b(x, r, \bar{u}^\epsilon(x, r), f(r)) dr \\ \quad + \int_0^t \sigma(x, r, \bar{u}^\epsilon(x, r), f(r)) dW^\epsilon(x, r) - \bar{\ell}(x, t), \\ \bar{\ell}^\beta(x, t) = -|\bar{\ell}^\beta(x, \cdot)|(t), \quad |\bar{\ell}^\beta(x, \cdot)|(t) \\ \quad = \int_0^t 1_{\{\bar{u}^\epsilon(x, r)=0\}} d|\bar{\ell}^\beta(x, \cdot)|(r) \quad \text{for } \beta = 1, 2, 3, 4, \end{cases}$$

where $f(r, y, du) = \text{Law}_{\mathbb{R}^4}(\bar{u}^\epsilon(y, r))$ is viewed as a measure on \mathbb{R}^4 , and $f(r) = f(r, dx, du)$ the induced probability measure defined by (1.6) on $Q \times \mathbb{R}^4$. Similarly, the reflection term $\bar{\ell}(x, t)$ still ensures $(\bar{u}^\epsilon)^\beta(x, t) \geq 0$ for each x, t , and β (see again

[21]). Here $W^\epsilon : \Omega \times \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^4$ is a four-dimensional Gaussian random field with independent components $\beta = 1, 2, 3, 4$, zero mean, and covariance

$$(2.9) \quad \begin{aligned} & \mathbb{E} [W^{\epsilon, \beta}(x, t) W^{\epsilon, \beta}(y, s)] \\ & = (t \wedge s) C_\rho \epsilon^d \int_{\mathbb{R}^d} \rho_\epsilon(z-x) \rho_\epsilon(z-y) dz, \quad \text{for } C_\rho = \left(\int_{\mathbb{R}^d} \rho(z)^2 dz \right)^{-1}, \end{aligned}$$

where $\rho : \mathbb{R}^d \rightarrow [0, 1]$ is a radial mollifier supported in the unitary ball and ρ_ϵ the ϵ -rescaled version. Such a process $W^{\epsilon, \beta}$ can, for example, be obtained by convolution and rescaling from a “mathematically rigorous” space-time white noise (see, e.g., [10]). That is, a distribution valued process $W : \Omega \times \mathbb{R}^+ \rightarrow \mathcal{S}'(\mathbb{R}^d)$ such that, for $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the processes $\langle W_t, \varphi \rangle$ are jointly Gaussian with covariance

$$\mathbb{E} [\langle W_t, \varphi \rangle \langle W_s, \psi \rangle] = (t \wedge s) \int_{\mathbb{R}^d} \varphi(z) \psi(z) dz.$$

Then, for $\beta = 1, 2, 3, 4$ and independent copies of such a white noise, one defines

$$(2.10) \quad W^{\epsilon, \beta}(x, t) := C_\rho^{\frac{1}{2}} \epsilon^{\frac{d}{2}} \langle W_t, \rho_\epsilon(\cdot - x) \rangle.$$

For future reference, we highlight some of the properties of W^ϵ . First, from (2.9) we have that $\mathbb{E}[W^{\epsilon, \beta}(x, t) W^{\epsilon, \gamma}(x, s)] = \delta_0(\beta - \gamma) t \wedge s$. Thus, for fixed x , the process $t \mapsto W^\epsilon(x, t)$ is a four-dimensional Brownian motion. Similarly, from $\text{supp}(\rho) \subseteq B(0, 1)$ it follows that

$$\mathbb{E}[W^\epsilon(x, t) W^\epsilon(y, s)] = 0 \quad \text{if } |x - y| > 2\epsilon.$$

Hence the processes $W^\epsilon(x, t)$ and $W^\epsilon(y, t)$ are independent for $|x - y| > 2\epsilon$. Furthermore, using (2.9) one computes

$$\begin{aligned} \mathbb{E} [|W^\epsilon(x, t) - W^\epsilon(y, s)|^2] & \leq C \mathbb{E} [|W^\epsilon(x, t) - W^\epsilon(x, s)|^2 + |W^\epsilon(x, s) - W^\epsilon(y, s)|^2] \\ & \leq C \left(|t - s| + s C_\rho \epsilon^d \int_{\mathbb{R}^d} (\rho_\epsilon(z-x) - \rho_\epsilon(z-y))^2 dz \right) \\ & \leq C \left(|t - s| + \frac{|x - y|^2}{\epsilon^2} \right) \end{aligned}$$

for a constant $C = C(\rho)$. Similar estimates hold for any higher moment $p \geq 2$ and the Kolmogorov continuity theorem ensures the existence of a suitable modification of W^ϵ with continuous trajectories in both x and t . In particular, we have that W^ϵ is jointly measurable in $(x, t) \in \mathbb{R}^d \times \mathbb{R}^+$ and in the sample path $\omega \in \Omega$. Finally, for any $x, y \in \mathbb{R}^d$, a direct computation shows that the quadratic variation of the martingale $W^{\epsilon, \beta}(x, t) - W^{\epsilon, \beta}(y, t)$ satisfies

$$(2.11) \quad \begin{aligned} [W^{\epsilon, \beta}(x, \cdot) - W^{\epsilon, \beta}(y, \cdot)]_t & = t C_\rho \epsilon^d \int_{\mathbb{R}^d} (\rho_\epsilon(z-x) - \rho_\epsilon(z-y))^2 dz \\ & \leq t C \frac{|x - y|^2}{\epsilon^2} \end{aligned}$$

for a constant $C = C(\rho)$.

Finally, $f(r, y, du) = \text{Law}_{\mathbb{R}^4}(\bar{u}^\epsilon(y, r))$ should solve the associated nonlinear Fokker-Planck equation with no-flux boundary conditions,

$$(2.12) \quad \begin{cases} \partial_t f(t, x, u) + \nabla_u \cdot (b(x, t, u, f(t))f(t, x, u)) = \frac{1}{2} \sum_{\beta=1}^4 \frac{\partial^2}{\partial u^\beta \partial u^\beta} (\sigma_\beta(x, t, u, f(t))^2 f(t, x, u)), \\ b_\beta(t, x, u, f(t))f(t, x, u) - \frac{1}{2} \frac{\partial}{\partial u^\beta} (\sigma_\beta(x, t, u, f(t))^2 f(t, x, u)) \Big|_{u^\beta=0} = 0 \text{ for } \beta = 1, 2, 3, 4, \end{cases}$$

in the weak sense.

Let us now see how (2.2)–(2.3) and the assumptions (2.4)–(2.7) translate into Hölder, Lipschitz, and sublinear growth properties of the actual drift and diffusion terms. First, notice that for any fixed $f \in \mathcal{P}(Q \times \mathbb{R}^4)$ the mappings

$$(2.13) \quad Q \times \mathbb{R}^+ \times \mathbb{R}^4 \rightarrow \mathbb{R} \quad | \quad x, r, u \mapsto b_\beta(x, r, u, f), \sigma_\beta(x, r, u, f)$$

are easily seen to be α -Hölder in x , Lipschitz, and with sublinear growth in u , uniformly in r . Next, given a Banach space X with norm $|\cdot|_X$ and a positive integer $m \in \mathbb{N}$, let us denote by $\mathcal{P}_m(X)$ the space of probability measures on X with finite m th moments, endowed with the m th order Wasserstein distance (see, e.g., [24]),

$$(2.14) \quad \mathcal{W}_m(X)(P_1, P_2) := \inf_{\pi \in \Pi(P_1, P_2)} \left\{ \int_X |x - y|_X^m \pi(dx, dy) \right\}^{\frac{1}{m}},$$

where $\Pi(P_1, P_2)$ denotes the set of probability measures on $X \times X$ with marginals P_1 and P_2 . When X is clear from the context we write $\mathcal{W}_m(P_1, P_2)$. Let $L^\infty(Q; \mathcal{P}_m(X))$ be the space of measurable functions $f : Q \rightarrow \mathcal{P}_m(X)$ such that

$$|f|_{L^\infty(Q; \mathcal{P}_m(X))} = \sup_{y \in Q} \left(\int_X |x|^m f(y, dx) \right)^{\frac{1}{m}} < \infty,$$

endowed with the distance

$$d_{L^\infty(Q; \mathcal{P}_m(X))}(f, g) = \sup_{y \in Q} \mathcal{W}_m(X)(f(y, dx), g(y, dx)).$$

Assume $f, g \in L^\infty(Q; \mathcal{P}_m(\mathbb{R}^4))$. Then we can identify them as elements in $\mathcal{P}(Q \times \mathbb{R}^4)$ by their actions on test functions

$$\int_{Q \times \mathbb{R}^4} \psi(y, s) f(dy, ds) := \int_Q \int_{\mathbb{R}^4} \psi(y, s) f(y, ds) dy \quad \forall \psi \in C_b(Q \times \mathbb{R}^4),$$

and similarly for g . Now, using the structure (2.2)–(2.3), the Hölder, Lipschitz, and sublinear growth properties of b_i^β and σ_i^β for $i = 0, 1$, and Hölder’s inequality, it is straightforward to prove the following lemma.

LEMMA 2.2. *In the setting outlined above, and under the assumptions on b and σ ,*

$$\begin{aligned} & |b_\beta(x, r, u, f) - b_\beta(x', r, u', f')| + |\sigma_\beta(x, r, u, f) - \sigma_\beta(x', r, u', f')| \\ & \leq L (|x - x'|^\alpha + |u - u'| + d_{L^\infty(Q; \mathcal{P}_m(\mathbb{R}^4))}(f, f')), \\ & |b_\beta(x, r, u, f)| + |\sigma_\beta(x, r, u, f)| \leq C (1 + |u| + |f|_{L^\infty(Q; \mathcal{P}_m(\mathbb{R}^4))}) \end{aligned}$$

for all $x, x', u, u', r \in Q^2 \times (\mathbb{R}^4)^2 \times \mathbb{R}^+$ and all $f, f' \in L^\infty(Q; \mathcal{P}_m(\mathbb{R}^4))$, for suitable constants L and C .

2.2. Main results. We now present the main results of this work. The theorems are stated for the general models (2.1), (2.8), and (2.12). First we present a result on existence and uniqueness of the particle systems, which is proved in section 3.

THEOREM 2.3 (strong existence and uniqueness for the particle systems). *Assume that the initial data satisfies $\sup_{1 \leq i \leq N} \sup_{1 \leq k \leq M} \mathbb{E}[|u_{ik}(0)|^2] < +\infty$. Then, under assumptions (2.2)–(2.7) on the coefficients, there exists a pathwise unique solution of the particle system (2.1).*

Next we state the theorems on well-posedness of the McKean–Vlasov equations and the associated PDE. The following two results are proved in section 4.

THEOREM 2.4 (strong existence and uniqueness of the McKean–Vlasov equation). *Under the assumptions (2.2)–(2.7) on the coefficients, for any initial data $u(\cdot, 0) \in L^\infty(Q; L^2(\Omega))$, and for any $\epsilon > 0$, there exists a pathwise unique solution $u^\epsilon \in L^\infty(Q; L^2(\Omega; C([0, T]; \mathbb{R}^4)))$ of the McKean–Vlasov equation (2.8). Moreover, for every $T < \infty$,*

$$(2.15) \quad \sup_{x \in Q} \mathbb{E} \left[\sup_{t \in [0, T]} |u^\epsilon(x, t)|^2 \right] \leq C \left(1 + \sup_{x \in Q} \mathbb{E}[|u(x, 0)|^2] \right)$$

for a constant $C = C(T, b, \sigma)$. Finally, if the initial data satisfies $u(\cdot, 0) \in C^\alpha(Q; L^2(\Omega))$ for some $\alpha \in (0, 1)$, then $u^\epsilon \in C^\alpha(Q; L^2(\Omega; C([0, T]; \mathbb{R}^4)))$.

THEOREM 2.5 (well-posedness of the nonlinear Fokker–Planck equation). *Under the assumptions (2.2)–(2.7) on the coefficients, for any initial data $f_0(x, du) \in L^\infty(Q; \mathcal{P}_2(\mathbb{R}^4))$, there exists a weak solution $f(x, t, du) \in L^\infty(Q; C([0, \infty); \mathcal{P}_2(\mathbb{R}^4)))$ of the nonlinear Fokker–Planck equation (2.12). If $|\sigma(x, t, u, g)| \geq c > 0$ for every x, t, u , and g , the solution is also unique. The map f is uniquely characterized as $f(t, x, du) = \text{Law}_{\mathbb{R}^4}(\bar{u}^\epsilon(x, t))$ for any arbitrary $\epsilon > 0$. Moreover, for each fixed $x \in Q$ and for any time $T > 0$, the restriction $f(x, t, du)|_{t \in [0, T]}$ can be seen as a probability measure on the space $C([0, T]; \mathbb{R}^4)$ of continuous paths, and it satisfies*

$$(2.16) \quad \sup_{x \in Q} \int_{C([0, T]; \mathbb{R}^4)} \sup_{t \in [0, T]} |v(t)|^2 f(x, dv) \leq C \left(1 + \sup_{x \in Q} \mathbb{E}[|u(x, 0)|^2] \right)$$

for a constant $C = C(T, b, \sigma)$, where $v \in C([0, T]; \mathbb{R}^4)$. Finally, if $f_0 \in C^\alpha(Q; \mathcal{P}_2(\mathbb{R}^4))$, then $f \in C^\alpha(Q; \mathcal{P}_2(C([0, T]; \mathbb{R}^4)))$, that is to say,

$$(2.17) \quad \mathcal{W}_2(C([0, T]; \mathbb{R}^4))(f(x, \cdot), f(y, \cdot)) \leq C |x - y|^\alpha \quad \forall x, y \in Q,$$

for a constant $C = C(T, b, \sigma, f_0)$.

We finally present two statements concerning the convergence of the particle system toward the limiting model as $M, N \rightarrow \infty$. The setting is the following. For $k \in \mathbb{N}$, let $(W_k(x, t))_{x \in Q, t \geq 0}$ be independent four-dimensional space-time white noise terms over $Q \times [0, \infty)$, which we then convolve and rescale to obtain W_k^ϵ as in formula (2.10). Similarly, let $u_k(\cdot, 0) \in C^\alpha(Q; L^2(\Omega))$ be i.i.d. families of random initial conditions along the cortex Q , and let them be independent of all the white noise terms. Finally, let X_i for $i = 1, \dots, N$ be points on an equispaced grid on Q , with squares of side length $N^{\frac{1}{d}}$. More details on the setting and the proofs of the results are given in sections 5 and 6.

THEOREM 2.6 (mean squared error estimates for actual particles versus McKean–Vlasov particles). *In the setting outlined above and in Theorems 2.3 and 2.4, for any*

$N, M \in \mathbb{N}$, let $u_{ik}^\epsilon(\cdot)$ be the solution of the particle system (2.1) with initial data $u_{ik}(0) := u_k(X_i, 0)$ and noise terms $W_{ik}(t) := W_k^\epsilon(X_i, t)$. For each $k \in \mathbb{N}$, let $\bar{u}_k^\epsilon(x, t)$ be the solution of the McKean–Vlasov equation (2.8) with initial data $(u_k(x, 0))_{x \in Q}$ and rescaled ϵ -correlated noise $(W_k^\epsilon(x, t))_{x \in Q, t \geq 0}$. For each $i \in \mathbb{N}$, denote $\bar{u}_{ik}^\epsilon(t) := \bar{u}_k^\epsilon(X_i, t)$. Then, for any $T > 0$,

$$(2.18) \quad \mathbb{E} \left[\sup_{r \in [0, t]} |u_{ik}^\epsilon(r) - \bar{u}_{ik}^\epsilon(r)|^2 \right]^{1/2} \leq C t \left(\frac{1}{N^{\frac{\alpha}{d}}} + \frac{1}{M^{\frac{1}{2}}} \right) \left(1 + \sup_{x \in Q} \mathbb{E} [|u_k(x, 0)|^2]^{1/2} \right)$$

for any $i = 1, \dots, N$, $k = 1, \dots, M$, and $t \in [0, T]$, where $C = C(T, \rho, b, \sigma, [u(\cdot, 0)]_\alpha)$ and $[u(\cdot, 0)]_\alpha$ denotes the Hölder seminorm of $u(\cdot, 0)$.

We notice that the decay has rate $(\frac{1}{N^{\frac{\alpha}{d}}} + \frac{1}{M^{\frac{1}{2}}})$ instead of the usual $\frac{1}{(MN)^{1/2}}$, as we might expect according to classical results in mean field theory [22] since we have MN particles. As anticipated in Remark 1.1, this phenomenon goes back to the fact that the particles u_{ik} are exchangeable in the second index only. Hence, what we will get is a mean field limit in the column index k but a Riemann sum type convergence in the space index i . This phenomenon will be made clear when we perform the computations in section 5.

We also remark that the ratio between ϵ and $N^{-\gamma_d}$ in Theorem 2.6, that is, between the correlation radius of the noise and the distance among the neuron locations x_i , is completely arbitrary and the decay rate in (2.18) is independent of this. The choice of this ratio is purely dictated by modeling arguments, namely by the correlation strength we want for the noise sensed by two nearby neurons, which, for example, can be taken to be zero.

Finally we translate the previous result about convergence of particles to the level of laws.

THEOREM 2.7 (rate of convergence for the empirical measure). *In the setting of Theorem 2.6, let*

$$f_{N,M}^\epsilon(t, dx, du) = \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M \delta_{(X_j, u_{jm}^\epsilon(t))}$$

be the empirical measure on $Q \times \mathbb{R}^4$ associated with the particle system (2.1). Let $f(t, x, du)$ be the unique solution of the Fokker–Planck equation (2.12) and consider the induced probability measure $f(t, dx, du)$ on $Q \times \mathbb{R}^4$ given by (1.6). Then, as $M, N \rightarrow \infty$, and possibly but not necessarily as $\epsilon \rightarrow 0$, $f_{N,M}^\epsilon(t, dx, du)$ converges to $f(t, dx, du)$ in the Wasserstein distance in the sense

$$\sup_{t \in [0, T]} \mathbb{E} [\mathcal{W}_1(Q \times \mathbb{R}^4)(f_{N,M}^\epsilon(t), f(t))] \leq C \left(1 + \sup_{x \in Q} \mathbb{E} [|u_k(x, 0)|^2]^{1/2} \right) \left(\frac{1}{N^{\frac{\alpha}{d}}} + \frac{1}{M^{\frac{1}{2}}} + \frac{1}{M^{\frac{1}{4}}} \right)$$

for any $T > 0$, for $C = C(T, \rho, b, \sigma, [u(\cdot, 0)]_\alpha, Q)$.

3. Strong existence and uniqueness for the particle systems. In this section we establish strong existence and uniqueness for the particle system (2.1), thus proving Theorem 2.3. The proof is based on a classical contraction argument and a crucial observation about the reflection term ℓ_{ik} .

Proof of Theorem 2.3. Fix $N, M \in \mathbb{N}$. Take a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting the initial conditions $u_{ik}(0) : \Omega \rightarrow \mathbb{R}^4$ and the four-dimensional Brownian motions $W_{ik}(t)$ for $i = 1, \dots, N$ and $k = 1, \dots, M$. Suppose $\mathbb{E}[|u_{ik}(0)|^2] < \infty$ for all i and k . For any $T > 0$ let us define the Banach space

$$(3.1) \quad H_T^2 := \left\{ \text{continuous adapted processes } Y_t : \Omega \rightarrow \mathbb{R}^4 \quad \text{with} \quad \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t|^2 \right]^{\frac{1}{2}} < \infty \right\},$$

endowed with the norm $\|Y\| := \mathbb{E}[\sup_{t \in [0, T]} |Y_t|^2]^{\frac{1}{2}}$, and then consider the product space $(H_T^2)^{NM}$ equipped with the product norm.

Define $F : (H_T^2)^{NM} \rightarrow (H_T^2)^{NM}$ by sending an element $v_{ik} \in (H_T^2)^{NM}$ to the pathwise solutions \tilde{v}_{ik} of the SDEs with reflecting boundary conditions, for $i = 1, \dots, N$ and $k = 1, \dots, M$,

$$(3.2) \quad \begin{cases} \tilde{v}_{ik}(t) = u_{ik}(0) + \int_0^t b(x_i, r, v_{ik}(r), f_{N, M}^v(r)) dr \\ \quad + \int_0^t \sigma(x_i, r, v_{ik}(r), f_{N, M}^v(r)) dW_{ik}(r) - \ell_{ik}^v(t), \\ \ell_{ik}^{v, \beta}(t) = -|\ell_{ik}^{v, \beta}|(t), \quad |\ell_{ik}^{v, \beta}|(t) = \int_0^t 1_{\{\tilde{v}_{ik}(r)=0\}} d|\ell_{ik}^{v, \beta}|(r) \quad \text{for } \beta = 1, 2, 3, 4, \end{cases}$$

where we define

$$f_{N, M}^v(t) = \frac{1}{NM} \sum_{j=1}^N \sum_{m=1}^M \delta_{(x_j, v_{jm}(t))}.$$

Under the hypotheses (2.2)–(2.7) on b and σ , and by straightforward modifications of the setting and the proofs in [21, 17], strong existence and uniqueness can be established for the SDEs (3.2) with initial data with bounded second moments. Moreover, for initial data with $\mathbb{E}[|u_{ik}(0)|^2] < \infty$ and data $v_{ik} \in H_T^2$, the solutions \tilde{v}_{ik} belong to H_T^2 .

We want to find T small enough so that the map F is a contraction. Take two elements $u_{ik}, v_{ik} \in (H_T^2)^{NM}$, and consider $\tilde{u}_{ik} = F(u_{ik})$ and $\tilde{v}_{ik} = F(v_{ik})$. We apply the Itô formula to $|\tilde{u}_{ik} - \tilde{v}_{ik}|^2$ and exploit the respective equations (3.2) to get

$$(3.3) \quad \begin{aligned} |\tilde{u}_{ik}(t) - \tilde{v}_{ik}(t)|^2 &= 2 \int_0^t (\tilde{u}_{ik}(r) - \tilde{v}_{ik}(r)) (b(x_i, r, u_{ik}(r), f_{N, M}^u(r)) \\ &\quad - b(x_i, r, v_{ik}(r), f_{N, M}^v(r))) dr \\ &\quad + 2 \int_0^t (\tilde{u}_{ik}(r) - \tilde{v}_{ik}(r)) (\sigma(x_i, r, u_{ik}(r), f_{N, M}^u(r)) \\ &\quad - \sigma(x_i, r, v_{ik}(r), f_{N, M}^v(r))) dW_{ik}(r) \\ &\quad + 2 \int_0^t (\tilde{u}_{ik} - \tilde{v}_{ik}) (d\ell_{ik}^v(r) - d\ell_{ik}^u(r)) \\ &\quad + \int_0^t (\sigma(x_i, r, u_{ik}, f_{N, M}^u) - \sigma(x_i, r, v_{ik}, f_{N, M}^v))^2 dr. \end{aligned}$$

Exploiting the very definition of the reflection terms ℓ_{ik}^u and ℓ_{ik}^v shows that the third term on the right-hand side of (3.3) is negative. Indeed, we use the second line in (3.2) to expand this term as

$$\int_0^t (\tilde{u}_{ik} - \tilde{v}_{ik})(d\ell_{ik}^v(r) - d\ell_{ik}^u(r)) = \sum_{\beta=1}^4 \left(\int_0^t (\tilde{u}_{ik}^\beta - \tilde{v}_{ik}^\beta) 1_{\{\tilde{u}_{ik}(r)=0\}} d|\ell_{ik}^{u,\beta}|(r) + \int_0^t (\tilde{v}_{ik}^\beta - \tilde{u}_{ik}^\beta) 1_{\{\tilde{v}_{ik}(r)=0\}} d|\ell_{ik}^{v,\beta}|(r) \right).$$

Since the reflecting boundary conditions ensure that $\tilde{u}_{ik}^\beta, \tilde{v}_{ik}^\beta \geq 0$, we see that all the integrals in the sum on the right-hand side are negative, since the integrands are.

Now we drop the third term in (3.3), take the supremum in time, and apply the expectation to get

$$\begin{aligned} (3.4) \quad & \mathbb{E} \left[\sup_{t \in [0, T]} |\tilde{u}_{ik}(t) - \tilde{v}_{ik}(t)|^2 \right] \\ & \leq \left(\int_0^T \mathbb{E} [|\tilde{u}_{ik} - \tilde{v}_{ik}| |b(x_i, r, u_{ik}(r), f_{N,M}^u(r)) - b(x_i, r, v_{ik}(r), f_{N,M}^v(r))|] dr \right. \\ & \quad + \mathbb{E} \left[\sup_{t \in [0, T]} \left(\int_0^t (\tilde{u}_{ik} - \tilde{v}_{ik})(\sigma(x_i, r, u_{ik}(r), f_{N,M}^u(r)) \right. \right. \\ & \quad \left. \left. - \sigma(x_i, r, v_{ik}(r), f_{N,M}^v(r))) dW_{ik}(r) \right) \right] \\ & \quad \left. + \int_0^T \mathbb{E} [|\sigma(x_i, r, u_{ik}(r), f_{N,M}^u(r)) - \sigma(x_i, r, v_{ik}(r), f_{N,M}^v(r))|^2] dr \right). \end{aligned}$$

The second term on the right-hand side is handled with the Burkholder–Davis–Gundy inequality and with Hölder’s inequality:

$$\begin{aligned} (3.5) \quad & \mathbb{E} \left[\sup_{t \in [0, T]} \left(\int_0^t (\tilde{u}_{ik} - \tilde{v}_{ik})(\sigma(x_i, r, u_{ik}(r), f_{N,M}^u(r)) - \sigma(x_i, r, v_{ik}(r), f_{N,M}^v(r))) dW_{ik}(r) \right) \right] \\ & \leq \mathbb{E} \left[\left(\int_0^T |\tilde{u}_{ik} - \tilde{v}_{ik}|^2 |\sigma(x_i, r, u_{ik}(r), f_{N,M}^u(r)) - \sigma(x_i, r, v_{ik}(r), f_{N,M}^v(r))|^2 dr \right)^{\frac{1}{2}} \right] \\ & \leq \mathbb{E} \left[\sup_{t \in [0, T]} |\tilde{u}_{ik} - \tilde{v}_{ik}| \left(\int_0^T |\sigma(x_i, r, u_{ik}(r), f_{N,M}^u(r)) - \sigma(x_i, r, v_{ik}(r), f_{N,M}^v(r))|^2 dr \right)^{\frac{1}{2}} \right] \\ & \leq \frac{1}{2} \mathbb{E} \left[\sup_{t \in [0, T]} |\tilde{u}_{ik} - \tilde{v}_{ik}|^2 \right] \\ & \quad + \frac{1}{2} \mathbb{E} \left[\int_0^T |\sigma(x_i, r, u_{ik}(r), f_{N,M}^u(r)) - \sigma(x_i, r, v_{ik}(r), f_{N,M}^v(r))|^2 dr \right]. \end{aligned}$$

Then we absorb the first term on the right-hand side of (3.5) into the left-hand side of (3.4) to get, for C a numeric constant,

$$\begin{aligned} (3.6) \quad & \mathbb{E} \left[\sup_{t \in [0, T]} |\tilde{u}_{ik}(t) - \tilde{v}_{ik}(t)|^2 \right] \\ & \leq C \left(\int_0^T \mathbb{E} [|\tilde{u}_{ik} - \tilde{v}_{ik}| |b(x_i, r, u_{ik}(r), f_{N,M}^u(r)) - b(x_i, r, v_{ik}(r), f_{N,M}^v(r))|] dr \right. \\ & \quad \left. + \int_0^T \mathbb{E} [|\sigma(x_i, r, u_{ik}(r), f_{N,M}^u(r)) - \sigma(x_i, r, v_{ik}(r), f_{N,M}^v(r))|^2] dr \right). \end{aligned}$$

Now, we use the structure (2.2)–(2.3) and the Lipschitz properties (2.4)–(2.6) of b and σ , the definition of f_{MN}^u and f_{MN}^v , and applications of Hölder's inequality to get, for $C = C(b, \sigma)$,

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\tilde{u}_{ik}(t) - \tilde{v}_{ik}(t)|^2 \right] \leq C \left(\int_0^T \mathbb{E} [|\tilde{u}_{ik}(r) - \tilde{v}_{ik}(r)|^2] dr + \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M \int_0^T \mathbb{E} [|u_{jm}(r) - v_{jm}(r)|^2] dr \right).$$

Then, we exploit Grönwall's lemma to get, for $C = C(T, b, \sigma)$,

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |\tilde{u}_{ik}(t) - \tilde{v}_{ik}(t)|^2 \right] &\leq C \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M \int_0^T \mathbb{E} [|u_{jm}(r) - v_{jm}(r)|^2] dr \\ &\leq TC \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M \mathbb{E} \left[\sup_{t \in [0, T]} |u_{jm}(t) - v_{jm}(t)|^2 \right]. \end{aligned}$$

Finally we sum over $i = 1, \dots, N$ and $k = 1, \dots, M$. In conclusion, by taking another time $T^* < T$ small enough with respect to $C = C(T, b, \sigma)$, we find that the map $F : (H_{T^*}^2)^{NM} \rightarrow (H_{T^*}^2)^{NM}$ is indeed a contraction. The unique fixed point $u_{ik} = F(u_{ik}) \in H_{T^*}^2$ is then the (pathwise unique) solution on $[0, T^*]$. We conclude by gluing solutions on subsequent intervals $[nT^*, (n+1)T^*]$ up to $[0, \infty)$. \square

4. Well-posedness of the limiting McKean–Vlasov SDEs and PDE. In this section we analyze the limiting model for the particle system (2.1), that is, the McKean–Vlasov equation (2.8) and the nonlinear Fokker–Planck equation (2.12). In particular, Theorems 2.4 and 2.5 about existence and uniqueness for these equations will be proved using a contraction argument.

Let us define the functional setting for the contraction argument. The Banach space H_T^2 is defined as in (3.1); for any $T > 0$ we shall also consider the complete metric space $C_T^2 := C([0, T]; \mathcal{P}_2(\mathbb{R}^4))$ of continuous functions with values in the complete metric space $(\mathcal{P}_2(\mathbb{R}^4), \mathcal{W}_2(\mathbb{R}^4))$, where \mathcal{W}_2 is the Wasserstein distance (2.14), endowed with the supremum distance $d_{C_T^2}(f, g) = \sup_{t \in [0, T]} \mathcal{W}_2(f(t), g(t))$. Finally, we will employ the Banach space $L^\infty(Q; H_T^2)$ of bounded measurable maps $Q \rightarrow H_T^2$ endowed with the norm

$$|u|_{L^\infty(Q; H_T^2)} := \sup_{x \in Q} |u(x, \cdot)|_{H_T^2} = \sup_{x \in Q} \mathbb{E} \left[\sup_{t \in [0, T]} |u(x, t)|^2 \right]^{\frac{1}{2}}.$$

Similarly, we also make use of the space $L^\infty(Q; C_T^2)$. Notice that despite $C_T^2 = C([0, T]; \mathcal{P}_2(\mathbb{R}^4))$ not being a vector space, it still makes sense to say that a function $f : Q \rightarrow C_T^2$ is bounded by taking an arbitrary point $P_0 \in C_T^2$ and imposing $\sup_{x \in Q} d_{C_T^2}(f(x), P_0) < \infty$. For simplicity, we take $P_0(t) \equiv \delta_0$ the function (in t) identically equal to $\delta_0 \in \mathcal{P}_2(\mathbb{R}^4)$ —the Dirac mass centered at zero. With abuse of notation, we denote

$$|f|_{L^\infty(Q; C_T^2)} := \sup_{x \in Q} d_{C_T^2}(f(x), \delta_0) = \sup_{x \in Q} \sup_{t \in [0, T]} \left(\int_{\mathbb{R}^4} |v|^2 f(t, x, dv) \right)^{\frac{1}{2}}.$$

Then $L^\infty(Q; C_T^2)$ is a complete metric space with the distance $d_{L^\infty(Q; C_T^2)}(f, g) := \sup_{x \in Q} d_{C_T^2}(f(x), g(x))$.

Let us now introduce the maps yielding the contraction. We are interested in the composition

$$(4.1) \quad L^\infty(Q; C_T^2) \xrightarrow{S^\epsilon} L^\infty(Q; H_T^2) \xrightarrow{L} L^\infty(Q; C_T^2).$$

The map L sends an element $u \in L^\infty(Q; H_T^2)$ to its bounded-in-space and continuous-in-time law on \mathbb{R}^4 . That is to say, $L[u](x, \cdot) \in C([0, T]; \mathcal{P}_2(\mathbb{R}^4))$ is given by $L[u](x, t) = \text{Law}_{\mathbb{R}^4}(u(x, t))$ for each $x \in Q$ and $t \in [0, T]$. A direct computation gives

$$\begin{aligned} \sup_{x \in Q} d_{C_T^2}(L[u](x), \delta_0) &= \sup_{x \in Q} \sup_{t \in [0, T]} W_2(L[u](x, t), \delta_0) \\ &= \sup_{x \in Q} \sup_{t \in [0, T]} \mathbb{E} [|u(x, t)|^2]^{\frac{1}{2}} \\ &\leq \sup_{x \in Q} \mathbb{E} \left[\sup_{t \in [0, T]} |u(x, t)|^2 \right]^{\frac{1}{2}} = |u|_{L^\infty(Q; H_T^2)} < \infty, \end{aligned}$$

and $L[u]$ is indeed an element of $L^\infty(Q; C_T^2)$.

The map S^ϵ is defined by sending an element $f \in L^\infty(Q; C_T^2)$ to the solutions $(S^\epsilon[f](x, t))_{t \geq 0}$ of the following SDEs with reflecting boundary conditions: for each fixed $x \in Q$

$$(4.2) \quad \begin{cases} S^\epsilon[f](x, t) &= u(x, 0) + \int_0^t b(x, r, S^\epsilon[f](x, r), f(r)) dr \\ &\quad + \int_0^t \sigma(x, r, S^\epsilon[f](x, r), f(r)) dW^\epsilon(x, t) - \ell[f](x, t), \\ \ell^\beta[f](x, t) &= -|\ell^\beta[f](x, \cdot)|(t), \\ |\ell^\beta[f](x, \cdot)|(t) &= \int_0^t \mathbf{1}_{\{S^\epsilon[f]^\beta(x, r)=0\}} d|\ell^\beta[f](x, \cdot)|(r), \quad \beta = 1, 2, 3, 4, \end{cases}$$

where $u(\cdot, 0) \in L^\infty(Q; L^2(\Omega))$ is the initial condition for the McKean–Vlasov equation. Notice that we slightly abuse notation since we identify an element $f \in L^\infty(Q; C_T^2)$ with the time dependent probability measure $f(t, dx, du)$ on $Q \times \mathbb{R}^4$ defined by

$$\int_{Q \times \mathbb{R}^4} \varphi(x, u) f(t, dx, du) := \int_Q \int_{\mathbb{R}^4} \varphi(x, u) f(t, x, du) dx \quad \text{for any } \varphi \in C_b(Q \times \mathbb{R}^4).$$

Standard theory of SDEs with reflecting boundary conditions [21] ensures that, for each fixed $x \in Q$, (4.2) has a pathwise unique solution. Indeed, owing to the conditions (2.2)–(2.7) on b and σ , for fixed $x \in Q$ and $f \in L^\infty(Q; C_T^2)$ the drift $\tilde{b}(r, u) := b(x, r, u, f(r))$ and diffusion $\tilde{\sigma}(r, u) := \sigma(x, r, u, f(r))$ terms can be verified to satisfy the needed assumptions. The measurability of $x \mapsto S^\epsilon[f](x, \cdot) \in H_T^2$ then immediately follows from that of the initial data $u(x, 0)$ and of the noise $W^\epsilon(x, t)$, using a Picard iteration representation of the solution of the SDEs (4.2). The fact that the map S is well defined, i.e., that $S^\epsilon[f](x, \cdot)$ is indeed an element of H_T^2 uniformly bounded in $x \in Q$, is the subject of Lemma 4.1.

By definition, for every $f \in L^\infty(Q; C_T^2)$ we have $(L \circ S^\epsilon)[f](x, t) = \text{Law}_{\mathbb{R}^4}(S^\epsilon[f](x, t))$ for all $x \in Q$ and $t \in [0, T]$. In fact, since f is the law of the SDE 4.2, we can also say that $(L \circ S^\epsilon)[f](x, \cdot) = \text{Law}_{C([0, T]; \mathbb{R}^4)}(S^\epsilon[f](x, \cdot))$. (4.2) That is, $(L \circ S^\epsilon)[f](x, \cdot)$ can be seen as a probability measure on the space of continuous paths $C([0, T]; \mathbb{R}^4)$. Furthermore, if $f \in L^\infty(Q; C_T^2)$ is a fixed point of $L \circ S^\epsilon$, namely $f(x, t) = \text{Law}_{\mathbb{R}^4}(S^\epsilon[f](x, t))$ for all $x \in Q$ and $t \in [0, T]$, then $f(x, \cdot) = \text{Law}_{C([0, T]; \mathbb{R}^4)}(S^\epsilon[f](x, \cdot))$.

LEMMA 4.1 (a priori estimates on moments). *Given $f \in L^\infty(Q; C_T^2)$, the pathwise unique solution $(S^\epsilon[f](x, t))_{x \in Q, t \geq 0}$ to (4.2) satisfies*

$$\begin{aligned}
 |S^\epsilon[f]|_{L^\infty(Q; H_T^2)}^2 &= \sup_{x \in Q} \mathbb{E} \left[\sup_{t \in [0, T]} |S^\epsilon[f](x, t)|^2 \right] \\
 (4.3) \quad &\leq C \left(1 + \sup_{x \in Q} \mathbb{E} [|u(x, 0)|^2] + \int_0^T \sup_{x \in Q} \int_{\mathbb{R}^4} |u|^2 f(t, x, du) dt \right) \\
 &\leq C \left(1 + \sup_{x \in Q} \mathbb{E} [|u(x, 0)|^2] + |f|_{L^\infty(Q; C_T^2)}^2 \right)
 \end{aligned}$$

for a constant $C = C(T, b, \sigma)$. In particular, $S^\epsilon[f] \in L^\infty(Q; H_T^2)$, and the map S^ϵ and the composition $L \circ S^\epsilon$ are well defined. Moreover, if $f \in L^\infty(Q; C_T^2)$ is a fixed point of $L \circ S^\epsilon$, then $f(x, \cdot)$, as a probability measure on the space of continuous paths $C([0, T]; \mathbb{R}^4)$, satisfies the stronger bound

$$(4.4) \quad \sup_{x \in Q} \int_{C([0, T]; \mathbb{R}^4)} \sup_{t \in [0, T]} |v(t)|^2 f(x, dv) = |S^\epsilon[f]|_{L^\infty(Q; H_T^2)}^2 \leq C \left(1 + \sup_{x \in Q} \mathbb{E} [|u(x, 0)|^2] \right).$$

Proof. Fix any $f \in L^\infty(Q; C_T^2)$; we want to estimate $|S^\epsilon[f](x, t)|^2$. Owing to the structure (2.2)–(2.3) and the sublinear growth properties (2.5)–(2.7) of the drift and diffusion terms, we have

$$(4.5) \quad |b(x, r, u, f(r))| + |\sigma(x, r, u, f(r))| \leq C \left(1 + |u| + \sup_{y \in Q} \int_{\mathbb{R}^4} |v| f(r, y, dv) \right).$$

Then, using Hölder’s inequality one gets

$$\begin{aligned}
 \sup_{y \in Q} \int_{\mathbb{R}^4} |v| f(r, y, dv) &\leq \sup_{y \in Q} \left(\int_{\mathbb{R}^4} |v|^2 f(r, y, dv) \right)^{\frac{1}{2}} \\
 (4.6) \quad &\leq \sup_{y \in Q} \sup_{r \in [0, T]} \left(\int_{\mathbb{R}^4} |v|^2 f(r, y, dv) \right)^{\frac{1}{2}} = |f|_{L^\infty(Q; C_T^2)}.
 \end{aligned}$$

Moreover, the explicit details in [21] on the construction of the reflection term ℓ in the SDE (4.2) imply that we can control it as follows:

$$\begin{aligned}
 |\ell[f](x, t)| &\leq \sup_{\tau \in [0, t]} \left| u(x, 0) + \int_0^\tau b(x, r, S^\epsilon[f](x, r), f(r)) dr \right. \\
 (4.7) \quad &\quad \left. + \int_0^\tau \sigma(x, r, S^\epsilon[f](x, r), f(r)) dW^\epsilon(x, r) \right|.
 \end{aligned}$$

Squaring both sides of the SDE (4.2), controlling the reflection term with the estimate (4.7), applying convexity inequalities, taking the supremum over $t \in [0, T]$ and then the expectation, and finally handling the deterministic integral with Hölder’s inequality and the stochastic integral with Itô isometry, we obtain

$$\begin{aligned}
 \mathbb{E} \left[\sup_{t \in [0, T]} |S^\epsilon[f](x, t)|^2 \right] &\leq C \left(\mathbb{E} [u(x, 0)^2] + \mathbb{E} \left[\int_0^T b(x, r, S^\epsilon[f](x, r), f(r))^2 dr \right] \right. \\
 &\quad \left. + \mathbb{E} \left[\int_0^T \sigma(x, r, S^\epsilon[f](x, r), f(r))^2 dr \right] \right)
 \end{aligned}$$

for a numeric constant C . In turn, using the sublinear growth estimates (4.5)–(4.6), we get

$$(4.8) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |S^\epsilon[f](x, t)|^2 \right] \leq C \left(1 + \mathbb{E} [u(x, 0)^2] + \int_0^T \mathbb{E} \left[\sup_{r \in [0, t]} |S^\epsilon[f](x, r)|^2 \right] dt + \int_0^T \sup_{y \in Q} \int_{\mathbb{R}^4} |u|^2 f(t, y, du) dt \right)$$

for a constant $C = C(T, b, \sigma)$. Then we exploit Grönwall’s lemma to get rid of the third term on the right-hand side of the inequality in (4.8). Eventually, by taking the supremum over $x \in Q$ and using (4.6) again, we deduce the inequalities (4.3).

Suppose now that f is a fixed point of $L \circ S^\epsilon$. Since $f(t, y) = \text{Law}_{\mathbb{R}^4}(S^\epsilon[f](y, t))$, we readily verify that

$$\begin{aligned} \int_0^T \sup_{y \in Q} \int_{\mathbb{R}^4} |u|^2 f(t, y, du) dt &= \int_0^T \sup_{y \in Q} \mathbb{E} [|S^\epsilon[f](y, t)|^2] dt \\ &\leq \int_0^T \sup_{y \in Q} \mathbb{E} \left[\sup_{r \in [0, t]} |S^\epsilon[f](y, r)|^2 \right] dt. \end{aligned}$$

Then we exploit this bound and again use Grönwall’s lemma in the first inequality (4.3) to get rid of the third term at the right-hand side and obtain (4.4) in the statement. \square

Now, assuming that the composition $L \circ S^\epsilon$ is a contraction in $L^\infty(Q; C_T^2)$, we first show how to conclude the strong existence and uniqueness for the McKean–Vlasov equation (2.8). Let $f \in L^\infty(Q; C_T^2)$ be the unique fixed point of $L \circ S^\epsilon$: since $S^\epsilon[f]$ solves (4.2) and $L \circ S^\epsilon[f] = f$, we obtain that $S^\epsilon[f]$ solves the McKean–Vlasov equation (2.8) on our stochastic basis with initial data $(u(x, 0))_{x \in Q}$. Conversely, let $(u(x, t))_{x \in Q, t \geq 0}$ be a strong solution of (2.8) on our stochastic basis with these initial data; then $L[u] \in L^\infty(Q; C_T^2)$ is a fixed point of $L \circ S^\epsilon$ and thus we must have $L[u] = f$, the unique fixed point. But then, since we have strong uniqueness for the SDEs (4.2) defining the map S^ϵ and since $u(x, t)$ solves these SDEs with this data f , we conclude that $(u(x, t))_{x \in Q, t \geq 0} = S^\epsilon[f]$.

Proof of Theorem 2.4. We show that the mapping $L \circ S^\epsilon$ is a strict contraction and then apply the Banach fixed point theorem. Take $f, g \in L^\infty(Q; C_T^2)$. By definition of the map S^ϵ we have

$$(4.9) \quad S^\epsilon[f](x, t) = u(x, 0) + \int_0^t b(x, r, S^\epsilon[f], f) dr + \int_0^t \sigma(x, r, S^\epsilon[f], f) dW^\epsilon(x, r) - \ell[f](x, t),$$

$$(4.10) \quad S^\epsilon[g](x, t) = u(x, 0) + \int_0^t b(x, r, S^\epsilon[g], g) dr + \int_0^t \sigma(x, r, S^\epsilon[g], g) dW^\epsilon(x, r) - \ell[g](x, t).$$

Then, by definition of the map L , we have that $(S^\epsilon[f](x, t), S^\epsilon[g](x, t))$ is an admissible coupling for $(L \circ S^\epsilon[f](x, t), L \circ S^\epsilon[g](x, t))$ and we can use it to estimate $\mathcal{W}_2(L \circ S^\epsilon[f](x, t), L \circ S^\epsilon[g](x, t))$.

We take the difference of (4.9) and (4.10) and use the Itô formula to get

$$\begin{aligned}
 (4.11) \quad & |S^\epsilon[f](x, t) - S^\epsilon[g](x, t)|^2 \\
 &= 2 \int_0^t (S^\epsilon[f](x, r) - S^\epsilon[g](x, r)) (b(x, r, S^\epsilon[f], f) - b(x, r, S^\epsilon[g], g)) dr \\
 &\quad + 2 \int_0^t (S^\epsilon[f](x, r) - S^\epsilon[g](x, r)) (\sigma(x, r, S^\epsilon[f], f) - \sigma(x, r, S^\epsilon[g], g)) dW^\epsilon(x, r) \\
 &\quad + 2 \int_0^t (S^\epsilon[f](x, r) - S^\epsilon[g](x, r)) (d\ell[g](x, r) - d\ell[f](x, r)) \\
 &\quad + \int_0^t (\sigma(x, r, S^\epsilon[f](x, r), f(r)) - \sigma(x, r, S^\epsilon[g](x, r), g(r)))^2 dr.
 \end{aligned}$$

We now argue analogously to (3.3)–(3.6) in the proof of Theorem 2.3. First, as in (3.4), the third term on the right-hand side of (4.11) is negative, and we drop it. Then we take the supremum in time and apply the expectation, we control the first deterministic integral with Hölder's inequalities and the stochastic integral with the Burkholder–Davis–Gundy and Hölder's inequality, and finally we absorb the necessary terms on the left-hand side of (4.11) to get

$$\begin{aligned}
 \mathbb{E} \left[\sup_{r \in [0, t]} |S^\epsilon[f](x, r) - S^\epsilon[g](x, r)|^2 \right] &\leq C \mathbb{E} \left[\int_0^t (b(x, r, S^\epsilon[f], f) - b(x, r, S^\epsilon[g], g))^2 dr \right. \\
 &\quad \left. + \int_0^t (\sigma(x, r, S^\epsilon[f], f) - \sigma(x, r, S^\epsilon[g], g))^2 dr \right]
 \end{aligned}$$

for a numeric constant C . Now we exploit the Lipschitz properties of the drift and diffusion terms stated in Lemma 2.2 and we obtain, for $C = C(b, \sigma)$,

$$\begin{aligned}
 \mathbb{E} \left[\sup_{r \in [0, t]} |S^\epsilon[f](x, r) - S^\epsilon[g](x, r)|^2 \right] &\leq C \left(\int_0^t \mathbb{E} [|S^\epsilon[f](x, r) - S^\epsilon[g](x, r)|^2] dr \right. \\
 &\quad \left. + \int_0^t \sup_{y \in Q} \mathcal{W}_2(f(r, y, du), g(r, y, du))^2 dr \right).
 \end{aligned}$$

Using Grönwall's lemma we get rid of the first term on the right-hand side at the expense of a larger constant $C = C(T, b, \sigma)$. Moreover, we have $\mathcal{W}_2(f(r, y, du), g(r, y, du)) \leq \sup_{r \in [0, T]} \mathcal{W}_2(f(r, y, du), g(r, y, du))$ for any $r \in [0, T]$, and we conclude that

$$(4.12) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |S^\epsilon[f](x, t) - S^\epsilon[g](x, t)|^2 \right] \leq CT \sup_{y \in Q} \sup_{r \in [0, T]} \mathcal{W}_2(f(r, y, du), g(r, y, du))^2$$

for a constant $C = C(T, b, \sigma)$. Finally, since the right-hand side is independent of x , we take the supremum over $x \in Q$ on the left-hand side of (4.12).

In conclusion, recalling that $(S^\epsilon[f](x, t), S^\epsilon[g](x, t))$ is a coupling for $(L \circ S^\epsilon[f](x, t), L \circ S^\epsilon[g](x, t))$, we obtain

$$\begin{aligned}
 d_{L^\infty(Q; C_T^2)}(L \circ S^\epsilon[f], L \circ S^\epsilon[g])^2 &= \sup_{x \in Q} \sup_{t \in [0, T]} \mathcal{W}_2(L \circ S^\epsilon[f](x, t), L \circ S^\epsilon[g](x, t))^2 \\
 &\leq \sup_{x \in Q} \sup_{t \in [0, T]} \mathbb{E} [|S^\epsilon[f](x, t) - S^\epsilon[g](x, t)|^2]
 \end{aligned}$$

$$\begin{aligned} &\leq \sup_{x \in Q} \mathbb{E} \left[\sup_{t \in [0, T]} |S^\epsilon[f](x, t) - S^\epsilon[g](x, t)|^2 \right] \\ &\leq C T \sup_{x \in Q} \sup_{t \in [0, T]} \mathcal{W}_2(f(x, t), g(x, t))^2 \\ &= C T d_{L^\infty(Q; C_T^2)}(f, g)^2 \end{aligned}$$

for $C = C(T, b, \sigma)$. This constant C is increasing in T . Therefore, upon possibly working in $L^\infty(Q; C_{T^*}^2)$ for some smaller $T^* < T$, we can assume that $CT < 1$. That is to say, if $T > 0$ is small enough, we have a contraction in $L^\infty(Q; C_T^2)$. In turn this implies that we have a pathwise unique solution to (2.8) over $[0, T]$. Repeating the same argument over $[T, 2T]$, $[2T, 3T]$, and so on, and exploiting the uniqueness, we can show there exists a pathwise unique solution defined over all $[0, \infty)$.

Now, assume in addition that $u(x, 0) \in C^\alpha(Q; L^2(\Omega))$. The following argument proves that in this case, for any $f \in L^\infty(Q; C_T^2)$, we have $S^\epsilon[f] \in C^\alpha(Q; L^2(\Omega; C([0, T]; \mathbb{R}^4)))$. In particular, the solution of the McKean–Vlasov equation (2.8) satisfies $u^\epsilon(x, t) \in C^\alpha(Q; L^2(\Omega; C([0, T]; \mathbb{R}^4)))$.

Given $x, y \in Q$, we manipulate the equations (4.2) for $S^\epsilon[f](x)$ and $S^\epsilon[f](y)$ to write

$$\begin{aligned} S^\epsilon[f](x, t) - S^\epsilon[f](y, t) &= (u(x, 0) - u(y, 0)) \\ &\quad + \int_0^t (b(x, r, S^\epsilon[f](x, r), f(r)) - b(x, r, S^\epsilon[f](y, r), f(r))) dr \\ &\quad + \int_0^t (\sigma(x, r, S^\epsilon[f](x, r), f(r)) \\ &\quad \quad - \sigma(y, r, S^\epsilon[f](y, r), f(r))) dW^\epsilon(x, r) \\ &\quad + \int_0^t \sigma(y, r, S^\epsilon[f](y, r), f(r)) (dW^\epsilon(x, r) - dW^\epsilon(y, r)) \\ &\quad + (\ell[f](y, r) - \ell[f](x, r)). \end{aligned}$$

Applying the Itô formula to the squared power yields

$$\begin{aligned} (4.13) \quad &|S^\epsilon[f](x, t) - S^\epsilon[f](y, t)|^2 \\ &= (u(x, 0) - u(y, 0))^2 \\ &\quad + 2 \int_0^t (S^\epsilon[f](x, r) - S^\epsilon[f](y, r)) (b(x, r, S^\epsilon[f], f) - b(x, r, S^\epsilon[f], f)) dr \\ &\quad + 2 \int_0^t (S^\epsilon[f](x, r) - S^\epsilon[f](y, r)) \\ &\quad \quad (\sigma(x, r, S^\epsilon[f], f) - \sigma(y, r, S^\epsilon[f], f)) dW^\epsilon(x, r) \\ &\quad + 2 \int_0^t (S^\epsilon[f](x, r) - S^\epsilon[f](y, r)) \sigma(y, r, S^\epsilon[f], f) (dW^\epsilon(x, r) - dW^\epsilon(y, r)) \\ &\quad + 2 \int_0^t (S^\epsilon[f](x, r) - S^\epsilon[f](y, r)) (d\ell[f](y, r) - d\ell[f](x, r)) \\ &\quad + \int_0^t (\sigma(x, r, S^\epsilon[f](x, r), f(r)) - \sigma(y, r, S^\epsilon[f](y, r), f(r)))^2 dr \\ &\quad + \int_0^t \sigma(y, r, S^\epsilon[f](y, r), f(r))^2 d[W^\epsilon(x, r) - W^\epsilon(y, r)]. \end{aligned}$$

For the first stochastic integral, the Burkholder–Davis–Gundy inequality and Hölder’s inequality yield

$$\begin{aligned}
 (4.14) \quad & \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t (S^\epsilon[f](x, r) - S^\epsilon[f](y, r)) (\sigma(x, r, S^\epsilon[f], f) - \sigma(y, r, S^\epsilon[f], f)) dW^\epsilon(x, r) \right| \right] \\
 & \leq \mathbb{E} \left[\left(\int_0^T (S^\epsilon[f](x, r) - S^\epsilon[f](y, r))^2 (\sigma(x, r, S^\epsilon[f], f) - \sigma(y, r, S^\epsilon[f], f))^2 dr \right)^{\frac{1}{2}} \right] \\
 & \leq \mathbb{E} \left[\sup_{t \in [0, T]} |S^\epsilon[f](x, t) - S^\epsilon[f](y, t)| \left(\int_0^T (\sigma(x, r, S^\epsilon[f], f) - \sigma(y, r, S^\epsilon[f], f))^2 dr \right)^{\frac{1}{2}} \right] \\
 & \leq \delta \mathbb{E} \left[\sup_{t \in [0, T]} |S^\epsilon[f](x, t) - S^\epsilon[f](y, t)|^2 \right] \\
 & \quad + \frac{1}{\delta} \mathbb{E} \left[\int_0^T (\sigma(x, r, S^\epsilon[f], f) - \sigma(y, r, S^\epsilon[f], f))^2 dr \right],
 \end{aligned}$$

where $\delta > 0$ shall be chosen small enough so as to absorb the first term on the right-hand side. Similarly, for the second stochastic integral we find

$$\begin{aligned}
 (4.15) \quad & \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t (S^\epsilon[f](x, r) - S^\epsilon[f](y, r)) \sigma(y, r, S^\epsilon[f](y, r), f(r)) (dW^\epsilon(x, r) - dW^\epsilon(y, r)) \right| \right] \\
 & \leq \mathbb{E} \left[\left(\int_0^T (S^\epsilon[f](x, r) - S^\epsilon[f](y, r))^2 \sigma(y, r, S^\epsilon[f], f)^2 d[W^\epsilon(x, r) - W^\epsilon(y, r)] \right)^{\frac{1}{2}} \right] \\
 & \leq \delta \mathbb{E} \left[\sup_{t \in [0, T]} |S^\epsilon[f](x, t) - S^\epsilon[f](y, t)|^2 \right] \\
 & \quad + \frac{1}{\delta} \mathbb{E} \left[\int_0^T \sigma(y, r, S^\epsilon[f], f)^2 d[W^\epsilon(x, r) - W^\epsilon(y, r)] \right],
 \end{aligned}$$

where again $\delta > 0$ shall be chosen small enough to absorb the first term on the right-hand side.

We now go back to (4.13). As in (3.3), the third term on the right-hand side is always negative and we drop it. Then we take the supremum in time and we apply the expectation, we handle the first deterministic integral with Hölder’s inequality, and we use estimates (4.14) and (4.15) for the stochastic integrals, absorbing the necessary terms on the left-hand side by choosing δ small enough. We obtain, for a numeric constant C ,

$$\begin{aligned}
 & \mathbb{E} \left[\sup_{t \in [0, T]} |S^\epsilon[f](x, t) - S^\epsilon[f](y, t)|^2 \right] \\
 & \leq C \left(\mathbb{E} [|u(x, 0) - u(y, 0)|^2] \right. \\
 & \quad \left. + \int_0^T \mathbb{E} [|S^\epsilon[f](x, r) - S^\epsilon[f](y, r)|^2] dr \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \int_0^T \mathbb{E} \left[|b(x, r, S^\epsilon[f], f) - b(y, r, S^\epsilon[f], f)|^2 \right] dr \\
 & + \int_0^T \mathbb{E} \left[|\sigma(x, r, S^\epsilon[f], f) - \sigma(y, r, S^\epsilon[f], f)|^2 \right] dr \\
 & + \mathbb{E} \left[\int_0^T \sigma(y, r, S^\epsilon[f](y, r), f(r))^2 d[W^\epsilon(x, r) - W^\epsilon(y, r)] \right].
 \end{aligned}$$

Now we recall formula (2.11) for the quadratic variation of $W^\epsilon(x, t) - W^\epsilon(y, t)$, and we use the Lipschitz and Hölder properties (2.13) of b and σ and convexity inequalities to get

$$\begin{aligned}
 (4.16) \quad & \mathbb{E} \left[\sup_{t \in [0, T]} |S^\epsilon[f](x, t) - S^\epsilon[f](y, t)|^2 \right] \\
 & \leq C \left(\mathbb{E} [|u(x, 0) - u(y, 0)|^2] + \int_0^T \mathbb{E} [|S^\epsilon[f](x, r) - S^\epsilon[f](y, r)|^2] dr + |x - y|^{2\alpha} \right. \\
 & \quad \left. + \frac{|x - y|^2}{\epsilon^2} \int_0^T \mathbb{E} [|\sigma(y, r, S^\epsilon[f], f)|^2] dr \right)
 \end{aligned}$$

for a constant $C = C(T, b, \sigma, \rho)$. We get rid of the second term on the right-hand side of (4.16) with Grönwall’s lemma, at the price of a larger constant $C = C(T, b, \sigma)$. The first term is handled with the assumption $u(\cdot, 0) \in C^\alpha(Q; L^2(\Omega))$. We control the last term with the sublinear growth property (2.13) of σ and the a priori estimate (4.3). In conclusion we obtain

$$\begin{aligned}
 (4.17) \quad & \mathbb{E} \left[\sup_{t \in [0, T]} |S^\epsilon[f](x, t) - S^\epsilon[f](y, t)|^2 \right] \\
 & \leq C \left(|x - y|^{2\alpha} + \frac{|x - y|^2}{\epsilon^2} \left(1 + \sup_{z \in Q} \mathbb{E} [|u(z, 0)|^2] + |f|_{L^\infty(Q; C_T^2)}^2 \right) \right)
 \end{aligned}$$

for a constant $C = C(T, b, \sigma, \rho, [u(\cdot, 0)]_\alpha)$. Since $x, y \in Q$ are arbitrary, this concludes the proof that $S^\epsilon[f] \in C^\alpha(Q; L^2(\Omega; C([0, T]; \mathbb{R}^4)))$. \square

We end this section by proving the existence and uniqueness of solutions to the associated Fokker–Planck equation.

Proof of Theorem 2.5. The result is a consequence of the Itô formula, the same fixed point argument as for the McKean–Vlasov equation, and the uniqueness statement for the linear version of the Fokker–Planck type equation. Given any admissible initial condition $f_0(x, du) \in L^\infty(Q; \mathcal{P}_2(\mathbb{R}^4))$, standard probability theory ensures that we can find a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a four-dimensional space-time white noise $(W(x, t))_{x \in Q, t \geq 0}$ and a family of random variables $u(x, 0) \in L^\infty(Q; L^2(\Omega))$ independent of the noise $W(x, t)$ with $\text{Law}_{\mathbb{R}^4}(u(x, 0)) = f_0(x, du)$ for every $x \in Q$. Given any $\epsilon > 0$, we convolve and rescale the white noise to obtain $W^\epsilon(x, t)$ as in (2.10). With this stochastic basis and initial data, let $(\bar{u}^\epsilon(x, t) \in L^\infty(Q; H_T^2))$ be the solution of the ϵ -correlated McKean–Vlasov equation (2.8), whose existence is guaranteed by Theorem 2.4, and let us denote $f^\epsilon(x, t, du) = \text{Law}_{\mathbb{R}^4}(\bar{u}^\epsilon(x, t)) \in L^\infty(Q; C_T^2)$. We claim that f^ϵ is a weak solution of (2.12).

Take any $\phi \in C_c^2(\mathbb{R}_+ \times \mathbb{R}^4)$ satisfying the Neumann boundary condition

$$(4.18) \quad \nabla_u \phi(t, u) \cdot n_{\partial(\mathbb{R}_+^4)}(u) = 0 \quad \forall t, u \in \mathbb{R}^+ \times \partial(\mathbb{R}_+^4),$$

where $n_{\partial(\mathbb{R}_+^4)}(u)$ denotes the unit outward normal at u . An application of the Itô formula yields

$$(4.19) \quad \begin{aligned} \phi(t, \bar{u}^\epsilon(x, t)) &= \phi(0, \bar{u}(x, 0)) + \int_0^t \partial_t \phi(r, \bar{u}^\epsilon(x, r)) dr \\ &+ \int_0^t \nabla_u \phi(r, \bar{u}^\epsilon(x, r)) \cdot b(x, r, \bar{u}^\epsilon(x, r), f^\epsilon(r)) dr \\ &+ \int_0^t \nabla_u \phi(r, \bar{u}^\epsilon(x, r)) \cdot \sigma(x, r, \bar{u}^\epsilon(x, r), f^\epsilon(r)) dW^\epsilon(x, r) \\ &+ \int_0^t \sum_{\beta=1}^4 \partial_{u^\beta} \phi(r, \bar{u}^\epsilon(x, r)) \mathbf{1}_{\{\bar{u}^{\epsilon, \beta}(x, r)=0\}} d|\ell^\beta(x, \cdot)|(r) \\ &+ \int_0^t \frac{1}{2} \sum_{\beta=1}^4 \partial_{u^\beta u^\beta}^2 \phi(r, \bar{u}^\epsilon(x, r)) (\sigma_\beta(x, r, \bar{u}^\epsilon(x, r), f^\epsilon(r)))^2 dr. \end{aligned}$$

The fifth term on the right-hand side is identically zero thanks to the condition (4.18) on ϕ . Now we apply the expectation on both sides. The fourth term on the right-hand side vanishes by the martingale property of the stochastic integral. Recalling that $\bar{u}^\epsilon(x, t)$ takes values in \mathbb{R}_+^4 only, we get

$$(4.20) \quad \begin{aligned} \int_{\mathbb{R}_+^4} \phi(t, u) f^\epsilon(t, x, du) &= \int_{\mathbb{R}_+^4} \phi(0, u) f_0(x, du) + \int_0^t \int_{\mathbb{R}_+^4} \partial_t \phi(r, u) f^\epsilon(r, x, du) dr \\ &+ \int_0^t \int_{\mathbb{R}_+^4} \nabla_u \phi(r, u) \cdot b(x, r, u, f^\epsilon(r)) f^\epsilon(r, x, du) dr \\ &+ \int_0^t \int_{\mathbb{R}_+^4} \frac{1}{2} \sum_{\beta=1}^4 \partial_{u^\beta u^\beta}^2 \phi(r, u) (\sigma_\beta(x, r, u, f^\epsilon(r)))^2 f^\epsilon(r, x, du) dr. \end{aligned}$$

This is nothing but the weak formulation of (2.12) subjected to the no-flux boundary conditions. Since for every $T > 0$ we have $\bar{u}^\epsilon(x, \cdot) \in H_T^2$ and since it satisfies the bound (2.15), we conclude that $f^\epsilon(x, t) = \text{Law}_{\mathbb{R}^4}(\bar{u}^\epsilon(x, t))$ is a weak solution of (2.12) with initial condition $f_0(x, du)$, that it lies in the space $L^\infty(Q; C([0, \infty); \mathcal{P}_2(\mathbb{R}^4)))$, and that it actually satisfies the stronger bound (2.16).

Conversely, let $g \in L^\infty(Q; C([0, \infty); \mathcal{P}_2(\mathbb{R}^4)))$ be a weak solution of the nonlinear Fokker–Planck equation (2.12) with the same initial data f_0 . We claim that $g = f^\epsilon$. First, we can solve the family of standard SDEs with reflecting boundary conditions for the chosen g , for $x \in Q$:

$$\begin{cases} S^\epsilon[g](x, t) &= u(x, 0) + \int_0^t b(r, x, S^\epsilon[g](x, r), g(r)) dr \\ &+ \int_0^t \sigma(r, x, S^\epsilon[g](x, r), g(r)) dW^\epsilon(x, r) - \ell[g](x, t), \\ \ell^\beta[g](x, t) &= -|\ell^\beta[g](x, \cdot)|(t) \\ |\ell^\beta[g](x, \cdot)|(t) &= \int_0^t \mathbf{1}_{\{S^\epsilon[g]^\beta(x, r)=0\}} d|\ell^\beta[g](x, \cdot)|(r) \quad \text{for } \beta = 1, 2, 3, 4. \end{cases}$$

Arguing as in (4.19)–(4.20), we see that $h := L \circ S^\epsilon[g]$ now solves the *linear* Fokker–Planck equation with this fixed g and with the same initial data f_0 :

$$(4.21) \quad \begin{cases} \partial_t h(t, x, u) + \nabla_u \cdot (b(x, t, u, g(t))h(t, x, u)) \\ \quad = \frac{1}{2} \sum_{\beta=1}^4 \partial_{u^\beta u^\beta}^2 (\sigma_\beta(x, t, u, g(t))^2 h(t, x, u)), \\ b^\beta(x, t, u, g(t))h(t, x, u) - \frac{1}{2} \frac{\partial}{\partial u^\beta} (\sigma_\beta(x, t, u, g(t))^2 h(t, x, u)) \Big|_{u^\beta=0} \\ \quad = 0 \quad \text{for } \beta = 1, 2, 3, 4. \end{cases}$$

This linear equation is readily verified to satisfy uniqueness by a duality argument: indeed, for fixed $x \in Q$, it suffices to test it against arbitrary functions $\varphi(t, u)$ satisfying the so-called backward Kolmogorov equation with Neumann boundary conditions on \mathbb{R}_+^4 . That is,

$$\begin{cases} \partial_t \varphi + \nabla_u \varphi \cdot b(x, t, u, g(t)) + \frac{1}{2} \sum_{\beta=1}^4 (\sigma_\beta(x, t, u, g(t)))^2 \partial_{u^\beta u^\beta}^2 \varphi = 0 & \text{on } (0, t_0) \times \mathbb{R}_+^4, \\ \nabla_u \varphi(t, u) \cdot n_{\partial(\mathbb{R}_+^4)}(u) = 0 & \text{on } (0, t_0) \times \partial(\mathbb{R}_+^4), \\ \varphi(t_0, u) = \Phi(u) & \text{on } \{t_0\} \times \mathbb{R}_+^4, \end{cases}$$

where we let $t_0 \in \mathbb{R}^+$ and $\Phi \in C_c^2(\mathbb{R}_+^4)$ be arbitrary. Such an equation is always solvable since we have the right sign of the diffusion term (see [14] for details). Going back to (4.21), we know that g as well is a solution of this equation and thus we must conclude that $g = L \circ S^\epsilon[g]$. Now let $T > 0$ be small enough so that the composition map $L \circ S^\epsilon$ is a contraction in $L^\infty(Q; C_T^2)$. This implies that $g \in L^\infty(Q; C_T^2)$ is a fixed point of $L \circ S^\epsilon$ and hence it must coincide with f^ϵ over $[0, T]$. Applying the same argument over subsequent intervals $[T, 2T]$, $[2T, 3T]$, and so forth proves the uniqueness statement.

In particular, given any two $\epsilon, \tilde{\epsilon} > 0$, we take $g = f^\epsilon$ and we conclude that $f^\epsilon = f^{\tilde{\epsilon}}$. That is to say, $f(x, t) := \text{Law}_{\mathbb{R}^4}(\bar{u}^\epsilon(x, t))$ is independent of ϵ and is the unique solution of the nonlinear Fokker–Planck equation.

Finally, we assume that $f_0 \in C^\alpha(Q; \mathcal{P}_2(\mathbb{R}^4))$ and we show that the corresponding solution satisfies $f \in C^\alpha(Q; \mathcal{P}_2(C[0, T]; \mathbb{R}^4))$. The theory of Wasserstein distances (see, e.g., [24]) ensures that we can find a stochastic basis supporting the white noise W and random variables $u(x, 0) \in C^\alpha(Q; L^2(\Omega))$ such that $\text{Law}_{\mathbb{R}^4}(u(x, 0)) = f_0(x, du)$ for every $x \in Q$. We fix $\epsilon = 1$ and we consider the iteration maps (4.1) defined via this stochastic basis and with these initial data. In particular we have $L \circ S^\epsilon[f] = f$, and thus for any $x, y \in Q$ we obtain

$$\mathcal{W}_2(C([0, T]; \mathbb{R}^4))(f(x, \cdot), f(y, \cdot)) \leq \mathbb{E} \left[\sup_{t \in [0, T]} |S^\epsilon[f](x, t) - S^\epsilon[f](y, t)|^2 \right].$$

This and formula (4.17) with $\epsilon = 1$ show that $f \in C^\alpha(Q; \mathcal{P}_2(C[0, T]; \mathbb{R}^4))$. □

5. Error estimates between the particle system and the limiting model.

In this section we rigorously show that the limiting behavior of the particle system (2.1) as $M, N \rightarrow \infty$ is described by the McKean–Vlasov equation (2.8) as stated in Theorem 2.6 by obtaining an error estimate. We will use the so-called Sznitman coupling method (cf. [22]).

First, we lay out the right setting so as to get the convergence result. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume it supports all the random variables listed below. First, for each $k \in \mathbb{N}$, let $\{W_k(x, t)\}_{k \in \mathbb{N}}$ be independent four-dimensional space-time white noise terms over $Q \times [0, \infty)$.

For any $\epsilon > 0$ we then convolve and rescale the noise terms to obtain the ϵ -correlated noise W_k^ϵ as in formula (2.9). For $h \in \mathbb{N}$, we assume i.i.d. families of

random initial conditions $u_h(x, 0) \in C^\alpha(Q; L^2(\Omega))$ on the sheet Q . Moreover, we require them to be independent of the white noise terms $\{W_k(x, t)\}_{k \in \mathbb{N}}$. Finally, as noted in section 1, we take points $X_1, \dots, X_N \in Q$ in the center of the squares of an equispaced grid on $Q = [0, 1]^d$ with side length $N^{-\frac{1}{d}}$. We denote by Q_i^N the square with center X_i , and we notice that $\text{meas}(Q_i^N) = \frac{1}{N}$ and $\text{diam}(Q_i^N) = \sqrt{d}N^{-\frac{1}{d}}$.

We finally introduce the particles for the coupling method. For $i = 1, \dots, N$ and $k = 1, \dots, M$, let $u_{ik}^\epsilon(t)$ be the solution of the particle system (2.1) with initial data $u_{ik}^\epsilon(0) := u_k(X_i, 0)$ and Brownian motions $W_{ik}^\epsilon(t) := W_k^\epsilon(X_i, t)$. Let $\bar{u}_k^\epsilon(x, t)$ be the solution of the McKean–Vlasov equation with initial data $u_k(x, 0)$ and correlated noise $W_k^\epsilon(x, t)$, and for $i = 1, \dots, N$ define $\bar{u}_{ik}^\epsilon(x, t) := \bar{u}_k^\epsilon(X_i, t)$.

Owing the i.i.d. properties of the initial data and the noise terms, we have the following.

LEMMA 5.1. *For fixed i , the particles $u_{ik}^\epsilon(t)$ are exchangeable for $k = 1, \dots, M$. Moreover, for fixed i , the particles $\bar{u}_{ik}^\epsilon(t)$ are i.i.d. for $k \in \mathbb{N}$.*

We point out that this is not the case for the index i , both for the particles u_{ik}^ϵ and \bar{u}_{ik}^ϵ . Indeed, the laws of u_{ik}^ϵ and u_{jk}^ϵ , or \bar{u}_{ik}^ϵ and \bar{u}_{jk}^ϵ , respectively, might differ as a result of the x dependence of their defining equations. Furthermore, even if the points $X_i, X_j \in Q$ are far from each other, namely if $|X_i - X_j| > 2\epsilon$, so that their noise terms $W_k^\epsilon(X_i, t)$ and $W_k^\epsilon(X_j, t)$ are independent, the particles might still be correlated as a result of their initial data. In fact, from the point of view of modeling in neuroscience, we expect $u_k(x, 0)$ to be close to $u_k(y, 0)$ for x close to y .

We are finally ready to prove the convergence result of Theorem 2.6. We first stress the following.

Remark 5.2. As mentioned in section 2.2, we point out that we do not need to impose any constraint on the ratio between the correlation radius ϵ of the noise and the minimum distance $\sqrt{d}N^{-\frac{1}{d}}$ between two grid points $X_i, X_j \in Q$. The choice of the scaling regime (ϵ, N) , with $N \rightarrow \infty$ and $\epsilon \rightarrow 0$ or possibly also $\epsilon \equiv \epsilon_0$ a constant, is purely arbitrary and dictated by modeling arguments only. One might impose $\epsilon N^{\frac{1}{d}} < \sqrt{d}$, so that all the particles sense independent noise, or choose to impose a certain ratio $\epsilon N^{\frac{1}{d}} > \sqrt{d}$, so that neurons at locations close enough to each other sense correlated noise. The results and the proof of Theorem 2.6 are unchanged.

Proof of Theorem 2.6. For any $i = 1, \dots, N$ and $k = 1, \dots, M$, take the difference $|u_{ik}^\epsilon - \bar{u}_{ik}^\epsilon|$ between actual particles and McKean–Vlasov particles. Applying the Itô formula and exploiting the respective equations (2.1) and (2.8), we get

$$\begin{aligned}
 |u_{ik}^\epsilon(t) - \bar{u}_{ik}^\epsilon(t)|^2 &= 2 \int_0^t (u_{ik}^\epsilon(r) - \bar{u}_{ik}^\epsilon(r)) \\
 &\quad (b(X_i, r, u_{ik}^\epsilon(r), f_{MN}^\epsilon(r)) - b(X_i, r, \bar{u}_{ik}^\epsilon(r), f(r))) dr \\
 &+ 2 \int_0^t (u_{ik}^\epsilon(r) - \bar{u}_{ik}^\epsilon(r)) \\
 &\quad (\sigma(X_i, r, u_{ik}^\epsilon(r), f_{MN}^\epsilon) - \sigma(X_i, r, \bar{u}_{ik}^\epsilon(r), f)) dW^\epsilon(X_i, r) \\
 &+ 2 \int_0^t (u_{ik}^\epsilon(r) - \bar{u}_{ik}^\epsilon(r)) (d\bar{\ell}_{ik}^\epsilon(r) - d\ell_{ik}^\epsilon(r)) \\
 &+ \int_0^t (\sigma(X_i, r, u_{ik}^\epsilon(r), f_{MN}^\epsilon(r)) - \sigma(X_i, r, \bar{u}_{ik}^\epsilon(r), f(r)))^2 dr.
 \end{aligned}
 \tag{5.1}$$

Now we argue as in (3.3)–(3.6). First we drop the third term in (5.1), which is always negative owing to the definition of the reflection terms ℓ_{ik} and $\bar{\ell}_{ik}$. Then we take the

supremum in $t \in [0, \tau]$ and apply the expectation. Next we use the Burkholder–Davis–Gundy and Hölder’s inequalities, we absorb the necessary terms into the left-hand side, and finally we exploit Grönwall’s lemma. We eventually obtain, for $C = C(T)$,

$$(5.2) \quad \mathbb{E} \left[\sup_{t \in [0, \tau]} |u_{ik}^\epsilon(t) - \bar{u}_{ik}^\epsilon(t)|^2 \right] \leq C \left(\int_0^\tau \mathbb{E} [|b(X_i, r, u_{ik}^\epsilon, f_{MN}^\epsilon) - b(X_i, r, \bar{u}_{ik}^\epsilon, f)|^2] dr + \int_0^\tau \mathbb{E} [|\sigma(X_i, r, u_{ik}^\epsilon, f_{MN}^\epsilon) - \sigma(X_i, r, \bar{u}_{ik}^\epsilon, f)|^2] dr \right).$$

In order to split the terms on the right-hand side of the inequality (5.2) and exploit the particular structure of the drift and diffusion terms, we introduce the following probability measure on $Q \times \mathbb{R}^4$:

$$(5.3) \quad \bar{f}_{MN}^\epsilon(t, dy, dv) = \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M \delta_{(X_j, \bar{u}_{jm}^\epsilon(t))} \in \mathcal{P}(Q \times \mathbb{R}^4).$$

This measure is just the empirical measure associated to the collection of McKean–Vlasov particles $(X_j, \bar{u}_{jm}^\epsilon(t))$. We have

$$(5.4) \quad \begin{aligned} |b(X_i, r, u_{ik}^\epsilon, f_{MN}^\epsilon) - b(X_i, r, \bar{u}_{ik}^\epsilon, f)| &\leq |b(X_i, r, u_{ik}^\epsilon, f_{MN}^\epsilon) - b(X_i, r, \bar{u}_{ik}^\epsilon, f_{MN}^\epsilon)| \\ &\quad + |b(X_i, r, \bar{u}_{ik}^\epsilon, f_{MN}^\epsilon) - b(X_i, r, \bar{u}_{ik}^\epsilon, \bar{f}_{MN}^\epsilon)| \\ &\quad + |b(X_i, r, \bar{u}_{ik}^\epsilon, \bar{f}_{MN}^\epsilon) - b(X_i, r, \bar{u}_{ik}^\epsilon, f)|. \end{aligned}$$

Due to the structure of the drift term (2.2) and its Lipschitz properties (2.4)–(2.6), and owing to the definition of $\bar{f}_{MN}^\epsilon(r)$ in (5.3), we get the following estimates for terms on the right-hand side of (5.4):

$$(5.5) \quad \begin{aligned} |b(X_i, r, u_{ik}^\epsilon, f_{MN}^\epsilon) - b(X_i, r, \bar{u}_{ik}^\epsilon, f_{MN}^\epsilon)| &\leq C |u_{ik}^\epsilon(r) - \bar{u}_{ik}^\epsilon(r)|, \\ |b(X_i, r, \bar{u}_{ik}^\epsilon, f_{MN}^\epsilon) - b(X_i, r, \bar{u}_{ik}^\epsilon, \bar{f}_{MN}^\epsilon)| &\leq C \left| \int_{Q \times \mathbb{R}^4} b_1(X_i, y, r, \bar{u}_{ik}^\epsilon, v) f_{MN}^\epsilon(r, dy, dv) - \int_{Q \times \mathbb{R}^4} b_1(X_i, y, r, \bar{u}_{ik}^\epsilon, v) \bar{f}_{MN}^\epsilon(r, dy, dv) \right| \\ &\leq C \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M |u_{jm}^\epsilon(r) - \bar{u}_{jm}^\epsilon(r)|, \\ |b(X_i, r, \bar{u}_{ik}^\epsilon, \bar{f}_{MN}^\epsilon) - b(X_i, r, \bar{u}_{ik}^\epsilon, f)| &\leq C \left| \int_{Q \times \mathbb{R}^4} b_1(X_i, y, r, \bar{u}_{ik}^\epsilon, v) \bar{f}_{MN}^\epsilon(r, dy, dv) - \int_{Q \times \mathbb{R}^4} b_1(X_i, y, r, \bar{u}_{ik}^\epsilon, v) f(r, dy, dv) \right| \\ &= C \left| \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M \left(b_1(X_i, X_j, r, \bar{u}_{ik}^\epsilon, \bar{u}_{jm}^\epsilon(r)) - \int_{Q \times \mathbb{R}^4} b_1(X_i, y, r, \bar{u}_{ik}^\epsilon, v) f(r, dy, dv) \right) \right| \end{aligned}$$

for a constant $C = C(b)$ only depending on the Lipschitz constants of b . An identical splitting (5.4) holds for the term $(\sigma(X_i, r, u_{ik}^\epsilon, f_{MN}^\epsilon) - \sigma(X_i, r, \bar{u}_{ik}^\epsilon, f))$ and using the Lipschitz properties (2.4)–(2.6) of σ we obtain analogous estimates to (5.5).

Going back to (5.2), we exploit (5.4) and (5.5). After standard convexity inequalities we obtain, for $C = C(T, b, \sigma)$,

$$\begin{aligned}
 \mathbb{E} \left[\sup_{t \in [0, \tau]} |u_{ik}^\epsilon(t) - \bar{u}_{ik}^\epsilon(t)|^2 \right] &\leq C \left\{ \int_0^\tau \mathbb{E} [|u_{ik}^\epsilon(r) - \bar{u}_{ik}^\epsilon(r)|^2] dr \right. \\
 &\quad + \int_0^\tau \frac{1}{MN} \sum_{j,m=1}^{N,M} \mathbb{E} [|u_{jm}^\epsilon(r) - \bar{u}_{jm}^\epsilon(r)|^2] dr \\
 (5.6) \quad &\quad + \int_0^\tau \mathbb{E} \left[\left| \frac{1}{MN} \sum_{j,m=1}^{N,M} \left(b_1(X_i, X_j, r, \bar{u}_{ik}^\epsilon, \bar{u}_{jm}^\epsilon) \right. \right. \right. \\
 &\quad \quad \left. \left. \left. - \int_{Q \times \mathbb{R}^4} b_1(X_i, y, r, \bar{u}_{ik}^\epsilon, v) f(r, dy, dv) \right) \right|^2 \right] dr \Big\} \\
 &\quad + \int_0^\tau \mathbb{E} \left[\left| \frac{1}{MN} \sum_{j,m=1}^{N,M} \left(\sigma_1(X_i, X_j, r, \bar{u}_{ik}^\epsilon, \bar{u}_{jm}^\epsilon) \right. \right. \right. \\
 &\quad \quad \left. \left. \left. - \int_{Q \times \mathbb{R}^4} \sigma_1(X_i, y, r, \bar{u}_{ik}^\epsilon, v) f(r, dy, dv) \right) \right|^2 \right] dr \Big\}.
 \end{aligned}$$

Averaging (5.6) over $i = 1, \dots, N$ and $k = 1, \dots, M$, and then using Grönwall's lemma to get rid of the first two terms on the right-hand side, we obtain

$$(5.7) \quad \frac{1}{MN} \sum_{i=1}^N \sum_{k=1}^M \mathbb{E} \left[\sup_{t \in [0, \tau]} |u_{ik}^\epsilon(t) - \bar{u}_{ik}^\epsilon(t)|^2 \right] \leq C \frac{1}{MN} \sum_{i=1}^N \sum_{k=1}^M \int_0^\tau R_{ik}^b(t) + R_{ik}^\sigma(t) dt$$

for another constant $C = C(T, b, \sigma)$. Here we have defined

$$\begin{aligned}
 R_{ik}^b(t) &= \mathbb{E} \left[\left| \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M \left(b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon(t), \bar{u}_{jm}^\epsilon(t)) \right. \right. \right. \\
 &\quad \left. \left. - \int_{Q \times \mathbb{R}^4} b_1(X_i, y, t, \bar{u}_{ik}^\epsilon(t), v) f(t, y, dv) dy \right) \right|^2 \Big], \\
 R_{ik}^\sigma(t) &= \mathbb{E} \left[\left| \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M \left(\sigma_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon(t), \bar{u}_{jm}^\epsilon(t)) \right. \right. \right. \\
 &\quad \left. \left. - \int_{Q \times \mathbb{R}^4} \sigma_1(X_i, y, t, \bar{u}_{ik}^\epsilon(t), v) f(t, y, dv) dy \right) \right|^2 \Big],
 \end{aligned}$$

which are the arguments of the last two integrals on the right-hand side of (5.6). Heuristically, the error terms R_{ik}^b and R_{ik}^σ should be small in view of the weak law of large numbers. Indeed, upon conditioning on \bar{u}_{ik}^ϵ , for each fixed $j = 1, \dots, N$, we are essentially taking the average of the i.i.d. terms $b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon(t), \bar{u}_{jm}^\epsilon(t))$ for $m = 1, \dots, M$, and then subtracting their common expectation, given by

$$\int_{Q \times \mathbb{R}^4} b_1(X_i, y, r, \bar{u}_{ik}^\epsilon(t), v) f(t, y, dv) dy.$$

In order to control the term to the right in (5.7), we need the following estimate, whose proof is postponed for the sake of the reader. For any $T > 0$, we have

$$(5.8) \quad \sup_{t \in [0, T]} |R_{ik}^b(t)| + |R_{ik}^\sigma(t)| \leq C \left(1 + \sup_{x \in Q} \mathbb{E} [|u_k(x, 0)|^2] \right) \left(\frac{1}{M} + \frac{1}{N^{\frac{\alpha}{d}}} \right)$$

for a constant $C = C(T, b, \sigma, \rho, [u(\cdot, 0)]_\alpha)$, for every $i = 1, \dots, N$ and $k = 1, \dots, M$. Plugging (5.8) into (5.7) we obtain, for $C = C(T, b, \sigma, \rho, [u(\cdot, 0)]_\alpha)$,

$$(5.9) \quad \frac{1}{MN} \sum_{i=1}^N \sum_{k=1}^M \mathbb{E} \left[\sup_{t \in [0, \tau]} |u_{ik}^\epsilon(t) - \bar{u}_{ik}^\epsilon(t)|^2 \right] \leq C \left(1 + \sup_{x \in Q} \mathbb{E} [|u_k(x, 0)|^2] \right) \left(\frac{1}{M} + \frac{1}{N^{\frac{\alpha}{d}}} \right).$$

We can now finally prove Theorem 2.6. We go back to (5.6), and get rid of the first term on the right-hand side with Grönwall’s lemma. We control the second term on the right-hand side with (5.9) and the last two terms with (5.8). This yields formula (2.18) and concludes the proof. \square

Proof of estimate (5.8). We prove the estimate for R_{ik}^b . Identical computations replacing b with σ prove the analogous result for R_{ik}^σ . Recalling that $\text{meas}(Q_j^N) = \frac{1}{N}$, we split the term as

$$(5.10) \quad \begin{aligned} R_{ik}^b(t) = & \mathbb{E} \left[\left| \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M \left(b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon(t), \bar{u}_{jm}^\epsilon(t)) \right. \right. \right. \\ & \left. \left. - \int_{Q \times \mathbb{R}^d} b_1(X_i, y, t, \bar{u}_{ik}^\epsilon(t), v) f(t, y, dv) dy \right) \right|^2 \Big] \\ \leq & \mathbb{E} \left[\left| \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M \left(b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon(t), \bar{u}_{jm}^\epsilon(t)) \right. \right. \right. \\ & \left. \left. - \int_{\mathbb{R}^d} b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon(t), v) f(t, X_j, dv) \right) \right|^2 \Big] \\ + & \mathbb{E} \left[\left| \sum_{j=1}^N \left(\int_{Q_j^N} \int_{\mathbb{R}^d} b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon(t), v) f(t, X_j, dv) \right. \right. \right. \\ & \left. \left. - \int_{\mathbb{R}^d} b_1(X_i, y, t, \bar{u}_{ik}^\epsilon(t), v) f(t, y, dv) dy \right) \right|^2 \Big]. \end{aligned}$$

For the first term of (5.10), the estimate is proved similarly to the weak law of large numbers. Indeed, for $C = C(T, b, \sigma)$, we compute

$$\begin{aligned} & \mathbb{E} \left[\left| \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M \left(b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon(t), \bar{u}_{jm}^\epsilon(t)) - \int_{\mathbb{R}^d} b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon(t), v) f(t, X_j, dv) \right) \right|^2 \right] \\ & \leq \frac{1}{N} \sum_{j=1}^N \mathbb{E} \left[\left| \frac{1}{M} \sum_{m=1}^M \left(b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon(t), \bar{u}_{jm}^\epsilon(t)) \right. \right. \right. \\ & \quad \left. \left. - \int_{\mathbb{R}^d} b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon(t), v) f(t, X_j, dv) \right) \right|^2 \right] \end{aligned}$$

$$\begin{aligned}
(5.11) \quad &= \frac{1}{N} \sum_{j=1}^N \frac{1}{M^2} \sum_{\substack{m_1=1 \\ m_2=1}}^M \mathbb{E} \left[\left(b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon, \bar{u}_{jm_1}^\epsilon) - \int_{\mathbb{R}^4} b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon, v) f(t, X_j, dv) \right) \right. \\
&\quad \left. \cdot \left(b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon, \bar{u}_{jm_2}^\epsilon) - \int_{\mathbb{R}^4} b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon, v) f(t, X_j, dv) \right) \right] \\
&= \frac{1}{N} \sum_{j=1}^N \frac{1}{M^2} \sum_{m=1}^M \mathbb{E} \left[\left(b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon, \bar{u}_{jm}^\epsilon) - \int_{\mathbb{R}^4} b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon, v) f(t, X_j, dv) \right)^2 \right] \\
&\leq \frac{1}{M} C \left(1 + \sup_{x \in Q} \mathbb{E} [|u_k(x, 0)|^2] \right).
\end{aligned}$$

In the first passage we used a convexity inequality. In the last passage we used the sublinear growth properties (2.7) of b_1 and the a priori estimate (2.15) for McKean–Vlasov particles. In the second passage we unfolded the square, and in the third we noticed that, after conditioning with respect to $\bar{u}_{ik}^\epsilon(t)$, only the “diagonal terms” survive in the sum, i.e., those with $m_1 = m_2$. Namely, when $m_1 \neq m_2$ the corresponding term in (5.11) is identically zero. Indeed, under this condition, assuming by symmetry $m_1 \neq k$, we have that $\bar{u}_{jm_1}^\epsilon(t)$ is independent of $\bar{u}_{jm_2}^\epsilon(t)$ and $\bar{u}_{ik}^\epsilon(t)$. Hence we compute

$$\begin{aligned}
(5.12) \quad &\mathbb{E} \left[\left(b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon, \bar{u}_{jm_1}^\epsilon) - \int_{\mathbb{R}^4} b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon, v) f(t, X_j, dv) \right) \right. \\
&\quad \left. \cdot \left(b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon, \bar{u}_{jm_2}^\epsilon) - \int_{\mathbb{R}^4} b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon, v) f(t, X_j, dv) \right) \right] \\
&= \mathbb{E} \left[\mathbb{E} \left[\left(b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon, \bar{u}_{jm_1}^\epsilon) - \int_{\mathbb{R}^4} b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon, v) f(t, X_j, dv) \right) \right. \right. \\
&\quad \left. \left. \cdot \left(b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon, \bar{u}_{jm_2}^\epsilon) - \int_{\mathbb{R}^4} b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon, v) f(t, X_j, dv) \right) \middle| \bar{u}_{ik}^\epsilon \right] \right] \\
&= \mathbb{E} \left[\mathbb{E} \left[\left(b_1(X_i, X_j, t, u, \bar{u}_{jm_1}^\epsilon) - \int_{\mathbb{R}^4} b_1(X_i, X_j, t, u, v) f(t, X_j, dv) \right) \right]_{u=\bar{u}_{ik}^\epsilon} \right. \\
&\quad \left. \cdot \mathbb{E} \left[\left(b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon, \bar{u}_{jm_2}^\epsilon) - \int_{\mathbb{R}^4} b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon, v) f(t, X_j, dv) \right) \middle| \bar{u}_{ik}^\epsilon \right] \right] \\
&= 0.
\end{aligned}$$

In the second passage we conditioned on \bar{u}_{ik}^ϵ and in the third passage we used standard properties of the conditional expectation (see, e.g., [10, Chapter 2]). Finally we used that $E[b_1(X_i, X_j, t, u, \bar{u}_{jm}^\epsilon)] = \int_{\mathbb{R}^4} b_1(X_i, X_j, t, u, v) f(t, X_j, dv)$ by definition of $f(t, X_j, dv)$.

For the second term on the right-hand side of (5.10), we first compute

$$\begin{aligned}
&\left| \int_{\mathbb{R}^4} b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon(t), v) f(t, X_j, dv) - \int_{\mathbb{R}^4} b_1(X_i, y, t, \bar{u}_{ik}^\epsilon(t), v) f(t, y, dv) \right| \\
&\leq \int_{(\mathbb{R}^4)^2} |b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon(t), v) - b_1(X_i, y, t, \bar{u}_{ik}^\epsilon(t), w)| \pi_0(X_j, y, dv, dw)
\end{aligned}$$

$$\begin{aligned}
 (5.13) \quad &\leq C \int_{(\mathbb{R}^4)^2} |X_j - y|^\alpha + |v - w| \pi_0(X_j, y, dv, dw) \\
 &\leq C |X_j - y|^\alpha + \mathcal{W}_1(\mathbb{R}^4)(f(t, X_j, dv), f(t, y, dv)) \\
 &\leq C |X_j - y|^\alpha
 \end{aligned}$$

for a constant $C = C(T, b, \sigma, \rho, [u(\cdot, 0)]_\alpha)$. In the second passage we took any optimal pairing $\pi_0(X_j, y, dv, dw)$ for $\mathcal{W}_1(f(t, X_j, dv), f(t, y, dv))$, in the third we used the Lipschitz and Hölder properties (2.6) of b_1 , and in the last we used the ordering $\mathcal{W}_1 \leq \mathcal{W}_2$ of Wasserstein distances and the Hölder continuity (2.17) of f in \mathcal{W}_2 . Then, using (5.13) and recalling that $\text{meas}(Q_j^N) = \frac{1}{N}$ and $\text{diam}(Q_j^N) = N^{-\frac{1}{d}}$, we compute

$$\begin{aligned}
 (5.14) \quad &\mathbb{E} \left[\left| \sum_{j=1}^N \left(\int_{Q_j^N} \int_{\mathbb{R}^4} b_1(X_i, X_j, t, \bar{u}_{ik}^\epsilon(t), v) f(t, X_j, dv) \right. \right. \right. \\
 &\quad \left. \left. \left. - \int_{\mathbb{R}^4} b_1(X_i, y, t, \bar{u}_{ik}^\epsilon(t), v) f(t, y, dv) dy \right) \right|^2 \right] \\
 &\leq \mathbb{E} \left[\left| \sum_{j=1}^N \int_{Q_j^N} C |X_j - y|^\alpha dy \right|^2 \right] \\
 &\leq C \text{diam}(Q_j^N)^{2\alpha} \\
 &= C N^{-\frac{2\alpha}{d}}
 \end{aligned}$$

for a constant $C = C(T, b, \sigma, \rho, [u(\cdot, 0)]_\alpha)$.

In conclusion, combining (5.10) with estimates (5.12) and (5.14) we obtain the estimate (5.8). \square

6. Convergence of empirical measures. In this last section, we further analyze the limiting behavior of the particle system as we let $M, N \rightarrow \infty$ and prove Theorem 2.7. In the same setting outlined in section 5, we show that the time dependent empirical measure

$$f_{MN}^\epsilon(t, dx, du) = \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M \delta_{(X_j, u_{jm}^\epsilon(t))} \in \mathcal{P}(Q \times \mathbb{R}^4),$$

associated to the particle system (2.1), located at the grid points X_1, \dots, X_N , converges in Wasserstein distance $\mathcal{W}_1(Q \times \mathbb{R}^4)$ to the measure $f(t, dx, du)$, obtained from the solution of the Fokker–Planck equation (2.12) via formula (1.6). The key step toward the result is to split the Wasserstein distance:

$$\begin{aligned}
 (6.1) \quad \mathcal{W}_1(Q \times \mathbb{R}^4)(f_{MN}^\epsilon(t), f(t)) &\leq \mathcal{W}_1(f_{MN}^\epsilon(t), \bar{f}_{MN}^\epsilon(t)) + \mathcal{W}_1(\bar{f}_{MN}^\epsilon(t), \bar{f}_N(t)) \\
 &\quad + \mathcal{W}_1(\bar{f}_N(t), f(t)).
 \end{aligned}$$

Here $\bar{f}_{MN}^\epsilon = \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M \delta_{(X_j, \bar{u}_{jm}^\epsilon(t))}$ is the empirical measure of the associated McKean–Vlasov particles as in section 5, and

$$\bar{f}_N(t, dx, du) := \frac{1}{N} \sum_{j=1}^N \delta_{X_j} \otimes f(t, X_j, du),$$

an auxiliary measure, can be viewed as a Riemann sum approximation for the measure $f(t, dx, du)$. Then Theorem 2.7 is an immediate consequence of the splitting (6.1) and Lemmas 6.1, 6.4, and 6.5 below. The first term in (6.1) is readily handled with Theorem 2.6 as follows.

LEMMA 6.1. *In the setting above, for any $T > 0$ we have*

$$\sup_{t \in [0, T]} \mathbb{E} [\mathcal{W}_1(Q \times \mathbb{R}^4)(f_{MN}^\epsilon(t), \bar{f}_{MN}^\epsilon(t))] \leq C \left(1 + \sup_{x \in Q} \mathbb{E} [|u_k(x, 0)|^2]^{\frac{1}{2}} \right) \left(\frac{1}{M^{\frac{1}{2}}} + \frac{1}{N^{\frac{\alpha}{d}}} \right)$$

for a constant $C = C(T, \rho, b, \sigma, [u(\cdot, 0)]_\alpha)$.

Proof. It suffices to notice that

$$\pi_0 = \frac{1}{MN} \sum_{j=1}^N \sum_{k=1}^M \delta_{(X_j, u_{jk}^\epsilon(t)), (X_j, \bar{u}_{jk}^\epsilon(t))}$$

is an admissible pairing for $f_{MN}^\epsilon(t)$ and $\bar{f}_{MN}^\epsilon(t)$. Then, by definition of the Wasserstein distance,

$$\mathcal{W}_1(f_{MN}^\epsilon(t), \bar{f}_{MN}^\epsilon(t)) \leq \int_{(Q \times \mathbb{R}^4)^2} |x - y| + |u - v| d\pi_0 = \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M |u_{jm}^\epsilon(t) - \bar{u}_{jm}^\epsilon(t)|.$$

Next we apply \mathbb{E} at both sides of the inequality and use Hölder's inequality to get

$$\begin{aligned} \mathbb{E} [\mathcal{W}_1(f_{MN}^\epsilon(t), \bar{f}_{MN}^\epsilon(t))] &\leq \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M \mathbb{E} [|u_{jm}^\epsilon(t) - \bar{u}_{jm}^\epsilon(t)|] \\ (6.2) \qquad \qquad \qquad &\leq \frac{1}{MN} \sum_{j=1}^N \sum_{m=1}^M \mathbb{E} [|u_{jm}^\epsilon(t) - \bar{u}_{jm}^\epsilon(t)|^2]^{\frac{1}{2}}. \end{aligned}$$

Now we conclude by plugging (2.18) into (6.2). \square

Let us now turn to the second term in the splitting (6.1). To start with, in a weak law of large numbers manner, we get the following lemma.

LEMMA 6.2. *In the setting above, for every $N \in \mathbb{N}$ and every $\epsilon > 0$, for any $t \geq 0$,*

$$\lim_{M \rightarrow \infty} \mathcal{W}_1(Q \times \mathbb{R}^4)(\bar{f}_{MN}^\epsilon(t), \bar{f}_N(t)) = 0 \quad \text{in probability.}$$

Proof. The key observation is the following: if $\varphi(x, v) \in C(Q \times \mathbb{R}^4)$ has linear growth in v , that is, $|\varphi(x, v)| \leq L_\varphi(1 + |v|)$ for some constant $L_\varphi \geq 0$, then we have $\langle \varphi, \bar{f}_{MN}^\epsilon(t) \rangle \rightarrow \langle \varphi, \bar{f}_N(t) \rangle$ in $L^2(\Omega)$ as $M \rightarrow \infty$, uniformly in $N \in \mathbb{N}$. Indeed, with the same arguments as in the proof of Lemma 5.8, we have, for a constant $C = C(T, b, \sigma)$ independent of φ ,

$$\begin{aligned} &\mathbb{E} [|\langle \varphi, \bar{f}_{MN}^\epsilon(t) \rangle - \langle \varphi, \bar{f}_N(t) \rangle|^2] \\ &\leq \frac{1}{N} \sum_{j=1}^N \mathbb{E} \left[\left| \frac{1}{M} \sum_{m=1}^M \left(\varphi(X_j, \bar{u}_{jm}^\epsilon(t)) - \int_{\mathbb{R}^4} \varphi(X_j, v) f(t, X_j, dv) \right) \right|^2 \right] \\ &= \frac{1}{N} \sum_{j=1}^N \frac{1}{M^2} \sum_{m_1 \neq m_2} \mathbb{E} \left[\left(\varphi(X_j, \bar{u}_{jm_1}^\epsilon(t)) - \int_{\mathbb{R}^4} \varphi(X_j, v) f(t, X_j, dv) \right) \right. \\ (6.3) \qquad \qquad \qquad &\quad \left. \cdot \left(\varphi(X_j, \bar{u}_{jm_2}^\epsilon(t)) - \int_{\mathbb{R}^4} \varphi(X_j, v) f(t, X_j, dv) \right) \right] \\ &\leq \frac{1}{M} L_\varphi^2 C(T, b, \sigma) \left(1 + \sup_{x \in Q} \mathbb{E} [|u(x, 0)|^2] \right). \end{aligned}$$

We now collect some auxiliary facts and then use these observations to complete the proof. Since $Q \times \mathbb{R}^4$ is a Polish space, it embeds continuously in the compact space $[0, 1]^{\mathbb{N}}$ endowed with the distance $\eta(x, y) := \sum_{k=1}^{\infty} \frac{1}{2^k} |x_k - y_k|$. Let $\overline{Q \times \mathbb{R}^4}$ denote the closure of (the image of) $Q \times \mathbb{R}^4$ in $[0, 1]^{\mathbb{N}}$. Let $U_{\eta}(Q \times \mathbb{R}^4)$ denote the space of function $\psi : Q \times \mathbb{R}^4 \rightarrow \mathbb{R}$ which are bounded and uniformly continuous with respect to the distance η restricted to $Q \times \mathbb{R}^4$. By the continuous extension theorem we have that $U_{\eta}(Q \times \mathbb{R}^4) = C_b(\overline{Q \times \mathbb{R}^4})$, that is to say, each bounded uniformly continuous function on $Q \times \mathbb{R}^4$ extends uniquely to a bounded continuous function on $\overline{Q \times \mathbb{R}^4}$ and conversely each such function restricts to a bounded uniformly continuous function on $Q \times \mathbb{R}^4$. The space $U_{\eta}(Q \times \mathbb{R}^4)$ is separable, since $\overline{Q \times \mathbb{R}^4}$ is compact. Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a dense countable subset and set $\psi_n := \frac{1}{\|\varphi_n\|_{\infty} + 1} \varphi_n$ for every n .

Given measures $\mu_j, \mu \in \mathcal{P}(Q \times \mathbb{R}^4)$, the Portmanteau theorem implies that $\mu_j \rightharpoonup \mu$ as $j \rightarrow \infty$ if and only if $\langle \varphi, \mu_j - \mu \rangle \rightarrow 0$ for every $\varphi \in U_{\eta}(Q \times \mathbb{R}^4)$. Defining the distance δ on $\mathcal{P}(Q \times \mathbb{R}^4)$ by

$$\delta(\mu, \nu) := \sum_{n \in \mathbb{N}} \frac{1}{2^n} |\langle \psi_n, \mu - \nu \rangle|,$$

we immediately see that, as $j \rightarrow \infty$,

$$\mu_j \rightharpoonup \mu \iff \delta(\mu_j, \mu) \rightarrow 0.$$

Finally, we recall that the convergence in Wasserstein distance of order 1 is equivalent to weak convergence combined with convergence of first moments (see, e.g., [24, Chapter 7]), i.e.

$$(6.4) \quad \mathcal{W}_1(Q \times \mathbb{R}^4)(\mu_j, \mu) \rightarrow 0 \iff \begin{cases} \delta(\mu_j, \mu) \rightarrow 0, \\ \int_{Q \times \mathbb{R}^4} |x| + |u| d\mu_j \rightarrow \int_{Q \times \mathbb{R}^4} |x| + |u| d\mu. \end{cases}$$

We now conclude the proof. Using the definitions of the measures \bar{f}_{MN} and \bar{f}_N , convexity inequalities and (6.3), we compute, for every $M, N \in \mathbb{N}$ and $\epsilon > 0$,

$$(6.5) \quad \begin{aligned} \mathbb{E} [\delta(\bar{f}_{MN}, \bar{f}_N)^2] &\leq \sum_{k=1}^{\infty} \frac{1}{2^k} \mathbb{E} [|\langle \psi_k, \bar{f}_{MN} - \bar{f}_N \rangle|^2] \\ &\leq \frac{1}{M} C(T, b, \sigma) \left(1 + \sup_{x \in Q} \mathbb{E} [|u(x, 0)|^2] \right). \end{aligned}$$

Analogously we have, for every $M, N \in \mathbb{N}$ and $\epsilon > 0$,

$$(6.6) \quad \mathbb{E} \left[\left| \int_{Q \times \mathbb{R}^4} |x| + |u| d\bar{f}_{MN} - \int_{Q \times \mathbb{R}^4} |x| + |u| d\bar{f}_N \right|^2 \right] \leq \frac{1}{M} C(T, b, \sigma) \left(1 + \sup_{x \in Q} \mathbb{E} [|u(x, 0)|^2] \right).$$

Given any arbitrary subsequence $M_k \rightarrow \infty$, using (6.5)–(6.6) and a diagonal argument, we find a subsequence $M_{k_j} \rightarrow \infty$ such that

$$\forall N \in \mathbb{N} \quad \delta(\bar{f}_{NM_{k_j}}, \bar{f}_N) \rightarrow 0 \quad \text{and} \quad \int_{Q \times \mathbb{R}^4} |x| + |u| (d\bar{f}_{NM_{k_j}} - d\bar{f}_N) \rightarrow 0 \quad \text{almost surely.}$$

The result now follows from (6.4) and the relation between almost sure convergence and convergence in probability. \square

Remark 6.3. It is possible to improve the result of the previous lemma with elementary cut-off techniques and show that

$$\lim_{M \rightarrow \infty} \mathbb{E}[\mathcal{W}_1(Q \times \mathbb{R}^4)(f_{MN}^\epsilon(t), \bar{f}_N(t))] = 0 \quad \text{for every } t \geq 0, N \in \mathbb{N}, \text{ and } \epsilon > 0.$$

However, this method does not retain any information about the precise rate of convergence to 0, which in principle also depends on N . To keep track of this, we need to rely on a more sophisticated result by Fournier and Guillin [13] about the convergence of empirical laws of i.i.d. particles toward their actual law.

LEMMA 6.4. *In the setting above, for any $T > 0$ and any $N \in \mathbb{N}$,*

$$\sup_{t \in [0, T]} \mathbb{E}[\mathcal{W}_1(Q \times \mathbb{R}^4)(\bar{f}_{MN}^\epsilon(t), \bar{f}_N(t))] \leq C \left(1 + \sup_{x \in Q} \mathbb{E}[|u_k(x, 0)|^2] \right)^{\frac{1}{2}} \frac{1}{M^{\frac{1}{4}}}$$

for a constant $C = C(T, b, \sigma, Q)$.

Proof. The explicit expressions for \bar{f}_{MN}^ϵ and \bar{f}_N and the convexity of the Wasserstein distance yield

$$\begin{aligned} & \mathbb{E}[\mathcal{W}_1(Q \times \mathbb{R}^4)(f_{MN}^\epsilon(t), \bar{f}_N(t))] \\ (6.7) \quad & \leq \frac{1}{N} \sum_{j=1}^N \mathbb{E} \left[\mathcal{W}_1(Q \times \mathbb{R}^4) \left(\frac{1}{M} \sum_{m=1}^M \delta_{(X_j, \bar{u}_{jm}^\epsilon)}, \delta_{X_j} \otimes f(t, X_j, dv) \right) \right] \\ & = \frac{1}{N} \sum_{j=1}^N \mathbb{E} \left[\mathcal{W}_1(\mathbb{R}^4) \left(\frac{1}{M} \sum_{m=1}^M \delta_{\bar{u}_{jm}^\epsilon}, f(t, X_j, dv) \right) \right]. \end{aligned}$$

Now, observe that for each fixed j , the particles $\bar{u}_{jm}^\epsilon(t)$ for $m = 1, \dots, M$ are i.i.d. with common law $f(t, X_j, dv)$. A direct application of Theorem 1 in [13], with $p = 1$, $q = 2$, and $d = 4$, implies

$$(6.8) \quad \mathbb{E} \left[\mathcal{W}_1(\mathbb{R}^4) \left(\frac{1}{M} \sum_{m=1}^M \delta_{\bar{u}_{jm}^\epsilon(t)}, f(t, X_j, dv) \right) \right] \leq C \mathbb{E}[|\bar{u}_{jm}^\epsilon(t)|^2] M^{-\frac{1}{4}}.$$

We conclude using the a priori estimates (2.15) and plugging formula (6.8) into (6.7). □

Finally we consider the last term in (6.1). For this *deterministic* term we have the following.

LEMMA 6.5. *In the setting above, for any $T \geq 0$,*

$$\sup_{t \in [0, T]} \mathcal{W}_1(Q \times \mathbb{R}^4)(\bar{f}_N(t), f(t)) \leq C \frac{1}{N^{\frac{\alpha}{d}}}$$

for a constant $C = C(T, \rho, b, \sigma, [u(\cdot, 0)]_\alpha)$.

Proof. For every $t \in [0, T]$, we consider the following pairing $\pi(t)$ defined by integration as

$$\begin{aligned} & \int_{(Q \times \mathbb{R}^4)^2} \varphi(x, y, u, v) \pi(t, dx, dy, du, dv) \\ & = \sum_{j=1}^N \int_{Q_j^N} \int_{(\mathbb{R}^4)^2} \varphi(X_j, y, u, v) \pi_0(t, X_j, y, du, dv) dy \quad \forall \varphi \in C_b((Q \times \mathbb{R}^4)^2), \end{aligned}$$

where $\pi_0(t, X_j, y, du, dv)$ is a chosen optimal pairing for $\mathcal{W}_1(\mathbb{R}^4)(f(t, X_j, du), f(t, y, dv))$. Recalling that $\text{meas}(Q_j^N) = 1/N$, an elementary check shows that $\pi(t)$ is indeed a pairing. Taking $\varphi(x, y, u, v) = |x - y| + |u - v|$, using the definition of $\pi(t)$ and $\pi_0(t, X_j, y)$, and recalling formula (2.17) we compute

$$\begin{aligned} \mathcal{W}_1(\bar{f}_N(t), f(t)) &\leq \int_{(Q \times \mathbb{R}^4)^2} |x - y| + |u - v| \pi(t, dx, dy, du, dv) \\ &= \sum_{j=1}^N \int_{Q_j^N} \int_{(\mathbb{R}^4)^2} |X_j - y| + |u - v| \pi_0(t, X_j, y, du, dv) dy \\ &\leq \text{diam}(Q_j^N) + \sum_{j=1}^N \int_{Q_j^N} \int_{(\mathbb{R}^4)^2} |u - v| \pi_0(t, X_j, y, du, dv) dy \\ &= \text{diam}(Q_j^N) + \sum_{j=1}^N \int_{Q_j^N} \mathcal{W}_1(\mathbb{R}^4)(f(t, X_j, du), f(t, y, dv)) dy \\ &\leq \text{diam}(Q_j^N) + \sum_{j=1}^N \int_{Q_j^N} C |X_j - y|^\alpha dy \\ &\leq \text{diam}(Q_j^N) + C \text{diam}(Q_j^N)^\alpha \end{aligned}$$

for a constant $C = C(T, \rho, b, \sigma, [u(\cdot, 0)]_\alpha)$. Recalling that $\text{diam}(Q_j^N) = N^{-\frac{1}{d}}$ concludes the proof. \square

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REFERENCES

- [1] H. AGAMON AND Y. BURAK, *A theory of joint attractor dynamics in the hippocampus and the entorhinal cortex accounts for artificial remapping and grid cell field-to-field variability*, eLife, 9 (2020), e56894.
- [2] S.-I. AMARI, *Dynamics of pattern formation in lateral-inhibition type neural fields*, Biol. Cybernet., 27 (1977), pp. 77–87, <https://doi.org/10.1007/BF00337259>.
- [3] A. AURELL, R. CARMONA, AND M. LAURIERE, *Stochastic Graphon Games: II. The Linear-Quadratic Case*, preprint, <https://arxiv.org/abs/2105.12320>, 2021.
- [4] F. BOLLEY, J. A. CAÑIZO, AND J. A. CARRILLO, *Stochastic mean-field limit: Non-Lipschitz forces and swarming*, Math. Models Methods Appl. Sci., 21 (2011), pp. 2179–2210.
- [5] Y. BURAK AND I. FIETE, *Accurate path integration in continuous attractor network models of grid cells*, PLoS Comput. Biol., 5 (2009), e1000291.
- [6] Y. BURAK AND I. FIETE, *Fundamental limits on persistent activity in networks of noisy neurons*, Proc. Natl. Acad. Sci. USA, 109 (2012), pp. 17645–17650.
- [7] J. A. CARRILLO, H. HOLDEN, AND S. SOLEM, *Noise-Driven Bifurcations in a Neural Field System Modelling Networks of Grid Cells*, <https://arxiv.org/abs/2109.07936>, 2021.
- [8] Y.-P. CHOI AND S. SALEM, *Propagation of chaos for aggregation equations with no-flux boundary conditions and sharp sensing zones*, Math. Models Methods Appl. Sci., 28 (2018), pp. 223–258, <https://doi.org/10.1142/S0218202518500070>.
- [9] J. J. COUEY, A. WITOELAR, S.-J. ZHANG, K. ZHENG, J. YE, B. DUNN, R. CZAJKOWSKI, M.-B. MOSER, E. I. MOSER, Y. ROUDI, AND M. P. WITTER, *Recurrent inhibitory circuitry as a mechanism for grid formation*, Nat. Neurosci., 16 (2013), pp. 318–324.
- [10] G. DA PRATO AND J. ZABCZYK, *Stochastic Equations in Infinite Dimensions*, Encyclopedia Math. Appl. 152, Cambridge University Press, Cambridge, UK, 1992.
- [11] G. B. ERMENTROUT AND D. H. TERMAN, *Mathematical Foundations of Neuroscience*, Interdiscip. Appl. Math. 35, Springer, New York, 2010, <https://doi.org/10.1007/978-0-387-87708-2>.
- [12] O. FAUGERAS AND J. INGLIS, *Stochastic neural field equations: A rigorous footing*, J. Math. Biol., 71 (2015), pp. 259–300, <https://doi.org/10.1007/s00285-014-0807-6>.

- [13] N. FOURNIER AND A. GUILLIN, *On the rate of convergence in wasserstein distance of the empirical measure*, Probab. Theory Related Fields, 162 (2013), pp. 707–738.
- [14] T. F. H. RISKEN, *The Fokker-Planck Equation: Methods of Solution and Applications*, Springer Ser. Synergetics, 2nd ed., Springer, New York, 1996, <http://gen.lib.rus.ec/book/index.php?md5=dfbf97c53d317f2b955411208c34af48>.
- [15] T. HAFTING, M. FYHN, S. MOLDEN, M.-B. MOSER, AND E. I. MOSER, *Microstructure of a spatial map in the entorhinal cortex*, Nature, 436 (2005), pp. 801–806.
- [16] P.-E. JABIN AND Z. WANG, *Mean field limit for stochastic particle systems*, in Active Particles. Vol. 1. Advances in Theory, Models, and Applications, Model. Simul. Sci. Eng. Technol., Birkhäuser/Springer, Cham, 2017, pp. 379–402.
- [17] P. LIONS AND A. SZNITMAN, *Stochastic differential equations with reflecting boundary conditions*, Commun. Pure Appl. Math., 37 (1984), pp. 511–537.
- [18] B. MCNAUGHTON, E. MOSER, AND M.-B. MOSER, *Spatial representation in the hippocampal formation: A history*, Nat. Neurosci., 20 (2017), pp. 1448–1464.
- [19] B. L. MCNAUGHTON, F. P. BATTAGLIA, O. JENSEN, E. I. MOSER, AND M.-B. MOSER, *Path integration and the neural basis of the 'cognitive map'*, Nat. Rev. Neurosci., 7 (2006), pp. 663–678, <https://doi.org/10.1038/nrn1932>.
- [20] D. C. ROWLAND, Y. ROUDI, M.-B. MOSER, AND E. I. MOSER, *Ten years of grid cells*, Annu. Rev. Neurosci., 39 (2016), pp. 19–40.
- [21] A. SZNITMAN, *Nonlinear reflecting diffusion process, and the propagation of chaos and fluctuations associated*, J. Funct. Anal., 56 (1984), pp. 311–336.
- [22] A.-S. SZNITMAN, *Topics in propagation of chaos*, in École d'Été de Probabilités de Saint-Flour XIX—1989, Lecture Notes in Math. 1464, Springer, Berlin, 1991, pp. 165–251, <https://doi.org/10.1007/BFb0085169>.
- [23] J. TOUBOUL, *Mean-field equations for stochastic firing-rate neural fields with delays: Derivation and noise-induced transitions*, Phys. D, 241 (2012), pp. 1223–1244.
- [24] C. VILLANI, *Topics in Optimal Transportation*, in Grad. Stud. Math. 58, AMS, Providence, RI, 2003.
- [25] H. WILSON AND J. COWAN, *Excitatory and inhibitory interactions in localized populations of model neurons*, Biophys. J., 12 (1972), pp. 1–24.
- [26] H. WILSON AND J. COWAN, *A mathematical theory of the functional dynamics of cortical and thalamic nervous tissue*, Biol. Cybernet., 13 (1973), pp. 55–80.