

On the locality of indistinguishable quantum systems

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A thesis presented for the degree of

Doctor of Philosophy in Atomic and Laser Physics

Supervised by: *Vlatko Vedral*

Trinity 2023

To She

Acknowledgements

I would like to express my deepest gratitude towards my supervisors and collaborators, Vlatko Vedral and Chiara Marletto. They have encouraged me to flourish, allowing me to unleash all my creativity in the way I saw fit. Meanwhile, they have provided valuable suggestions on the research content produced. Moreover, I have learned a lot of great lessons on how to conduct research and navigate the academic world in a very graceful, down to earth and humane way. Chiara has been a very important figure for me. She has encouraged, cared for, and provided key insights, ideas, points of view and important contentions. Neither this thesis nor I would not be the same without her.

Segundo, quiero extender también mis más sinceros agradecimientos a Lucia Vilchez-Estevez. Compañera y colaboradora, Lucia ha sido un gran apoyo para desarrollar las ideas relacionadas con los anyones. Su impacto en mi trabajo ha sido directo, generando figuras, haciendo cálculos, contrastando ideas, recursos y aplicaciones; pero también indirecto, generando un ambiente ideal en la oficina y siendo en definitiva una gran compañera. Ha sido un absoluto placer compartir estos años con ella.

It has been a pleasure to share adventures, research ideas and life lessons with all the people who have been in the 241.8 office. First, my collaborators in different projects, both finished and ongoing: Aditya Iyer, David Felce, Eduardo Dias, Lucia Vilchez-Estevez, Maria Violaris and Simone Rijavec. Second, the people with who I had key discussions that have shaped my views on physics: Sam Kuypers, Christian Schilling, Lodovico Scarpa, and Felix Tennie. Also, the people who, despite working in parallel, have made the office a great place to be and do research and pub trips exceptionally fun: Giuseppe di Pietra, Amy Searle, Abdulla Alhajri,

Tian Zhang, Jinzhao Sun, Mattheus Burkhard, and Michele. I can say I love my colleagues very much. Finally, I do not want to forget Tristan. I have learned a lot from him, he is always ready to provide good advice and calm me down when being stressed, despite not being his responsibility. His presence is very felt and appreciated.

On the scientific front, I would also like to use this opportunity to thank several fellow academics that have had a great impact on my work and the course of my DPhil. I want to thank for fruitful discussions and inspiration Chiara Marletto, Vlatko Vedral, David Deutsch, Joseph Conlon, Charles Alexandre Bédard, Paul Raymond-Robichaud, Steven Simon, Patrick Fraser, Nesta van der Schaaf, Ryszard Paweł Kostecki, Manabendra Nath Bera, Arnau Riera, Ved Kunte, Maciej Lewenstein, Nitica Sakharwade, Dominic Mayers, Caslav Brukner, Robert Spekkens, Lucien Hardy, Giulio Chiribella, Anne-Catherine de la Hammette, Lorenzo Giannelli, Ali Akil, Laia Xiao Planas Toro, Vincent Wang, Stefano Gogioso, Dominic Verdon, Stefan Lukas Ludescher, Carlo Rovelli, Richard Howl, Nicholas Ormrod, Hlér Kristjánsson, Augustin Vanrietvelde, Andrea di Biagio, Emili Bagan Capella, Ramón Munoz Tapia and John Calsamiglia.

On the personal side, I would not have been able to develop this thesis without a strong social network of support. Allow me to thank all the important people who have made me grow during this period and have supported me throughout.

Primer de tot, vull donarli totes ses gracis del món a sa meu dona, sa meua nena, sa meua esposa. Marta Soria Heredia, no puc expressar quant afortunat sóc per haver-te trobat i tenir-te cada dia as meu costat. Sa meua companyia constant, sa meua companya d'equip, la que em fa costat, la que em fa ser milllor, la que és la millor sense creure-s'ho i la que s'ho mereix absolutament tot. Tu ja saps que no podria haver fet aquesta tesi sense tu, però vull que tothom ho sàpiga. T'estimi

amb tot es meu cor i ànima. Gracis per tot, però sobretot per existir i ser com ets. No puc esperar més en veure com serà tota una vida as teu costat, creixent i aprenent amb tu.

També vull agraïr tot l'amor, força i suport de sa meu meravellosa familia. Des primer a l'últim, encara que no sigueu aquí vos senti molt aprop i vos pensi cada dia. Moltes gracis per fer-me millor persona i valorar el que és important. Us estimi molt Mare, Leonor, Dionís, Júlia, Àvia Júlia, Àvia Leonor, Avi Joan, Avi Moisès i Anna Maria. Moltes gracis de debó per deixar-me perseguir es meu somni i confiar sempre en mi.

També a sa meva nova família, sa meu família política, que m'han rebut amb es braços ben oberts i mh́an donat un suport desmesurat. Moltes gracis families Heredia i Soria. Una menció molt especial a la Mireia, en Ricardo, en Tomàs, en Jaume, la Pili, els avis Maria Trini i Miquel, l'àvia Maria Rosa i en David. I la Rasta també. Moltes gracis per donar-me tant suport tractant-me com a un dels vostres.

Vull donar ses gràcies a tots es amics que durant tots aquests anys m'han donat tants somriures, festes, alegries, plors, penúries, i en definitiva, hem compartit aquest procés vital que és la transcisió a ser adults. Moltíssimes gracis per ser-hi malgrat jo a vegades no hi sóc tant present. Muchas gracias por estos dos últimos años, por ser mi pequeña-gran família en Oxford. Por acojerme, enseñarme a ser mejor persona y tener un vínculo de amistad tan tan fuerte. Muchas gracias Laura, César, Fernando, Alícia y Edgar. Os quiero.

Moltes gràcies Laia, per ser com una germana bessona per mi, per compartir viatje des dels 14 anys i donar-me sempre el punt de vista adequat, seré, responsable i brillant. Jo només aspiro a poder arribar a ser com ets tu. Moltes gràcies al Marc per sempre ser-hi i donar-me tants bons moments d'alegria i felicitat. Gràcies per

cuidar-la i ajudar-la a que continuï siguent com és.

Guillermo y Paloma, sois lo más parecido a dos hermanos mayores que he tenido. Muchísimas gracias por las charlas interminables, los debates apasionantes y vuestros puntos de vista ponderados que me han abierto un mundo de nuevas perspectivas. Siempre querré volver en esos momentos y alargarlos para siempre en el timepo. Hicisteis de Oxford un sition donde querría estar. Os echo muchísimo de menos.

Moltes gracis Albert, Eudald, Maria i Paula. Merci per fer de Cadaqués un lloc tant agradable i fer que me'l pugui sentir encara com casa meua. Merci per interessar-vos tant per sa meu vida aquí fora i per aguantar-me queixar-me i renegar tot lo dia. Us trobi a faltar molt, esperi poder tornar ben aviat i passar moltes tardes al Boia i as Café des Cantó parlant de tot i de res, gaudint des petits moments amb vosaltros.

As nens i nenes de sa carrera, Clàudia, Andreu, Gerard, Llorenç, Biel, Pau, Joan, Sara, Ángel, Eneko, Àngels, Fran, Nil, Carlos, Irina, David, Jaume, i un llarg etcétra. Moltes gràcies per ajudar-me tant durant aquests anys. Gràcies per compartir el procés de créixer peronal i acadèmicament plegats. Sou molt grans i espero veure-us molt més en els propers 5 anys del que us he vist aquests últims.

Y una muy larga lista de gente que ha formado parte de mi vida estos años, que con los pequeños momentos me han dado el disfrutar de la vida. A los de la Caixa: Óscar, Ana Isabel, Laura, Neus, Roberto, Marina, Mireia. Javi, Mar, Marcel, Anna, Gavi, Gabri y Berna. Flatmates: Titiksha & Maria. Muchas gracias también a: Alba i Leire. Una abraçada als Matefísics d'Oxford, en Carles, la Gala, la Kim, el Sergio i l'Adrián. Moltes gracis als del DLS Pere, Josep, Marcos, Sílvia i Mariona. Merci a la Laura per mantenir el contacte durant tants anys. I ja per acabar merci per acollir-me amb els braços oberts Raquel, Sara, Xavi, Torres, Gil, Porta, Postils, Judith, Pau, Toni, Paloma, Javi, Fabio, Deniz, Kaan, Maira, Isidre i

Peter. Moltíssimes gracis a tots perquè em feu ser millor persona i gaudir molt més de la vida.

Per últim, i no menys important, vull donar ses gracis a tot es poble de Cadaqués, as cadaquesencs passats i presents i a sa Tramuntana. Ells m'han criat i és en gran part a sa nostra idiosincràcia de tot es poble que jo sóc com sóc. Encara que sigui lluny us recordi cada dia i us porti a dins el cor.

Tot el que segueix és gràcies a s'esforç de tota aquesta gent. Gracis a ells puc dedicar-me a investigar i ser de sa manera que sóc.

Abstract

This thesis investigates local realism in quantum indistinguishable particle systems, focusing on bosonic, fermionic, and $2D$ non-abelian anyonic systems. The local realism of quantum indistinguishable particle systems is asserted. It proves annihilation operators represent the local ontic states in these systems. It closes the literature gap on obtaining Deutsch-Hayden descriptors in indistinguishable particle systems. The prima facie paradox of action at a distance using fermionic annihilation operators as descriptors is resolved. The work provides examples of using and interpreting the annihilation operators as local ontic states. It contains the novel construction and characterisation of the annihilation operators for 2 D non-abelian anyonic systems. The explicit form of Fibonacci anyon annihilation operators is provided, and their usefulness is shown in expressing the anyonic Hubbard model Hamiltonian algebraically. By studying the indistinguishable particle systems' local realistic structure, the thesis showcases the relevance of the choice of subsystem lattice and exotic possible compositions of subsystems.

***Keywords*—** Local realism - Fermions - Anyons - Ontic states

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List of Abbreviations and Notation

EPR ..... Einstein-Podolsky-Rosen

PR ..... Popescu-Rohrlich

SSR ..... Superselection rule

RR ..... Raymond-Robichaud

+ h.c. ..... And adding the Hermitian conjugation of the previous terms

QFT ..... Quantum field theory

$2+1$ **D** ..... Two spatial dimensions and one temporal dimension.

The following is the standard notation we use:

$\mathbf{AB}$ ..... Composite system of the two subsystems A and B

$\mathbf{A|B}$ ..... Set bipartition, where A and B are disjoint subsets

$\hat{\boldsymbol{\sigma}}_{\boldsymbol{\mu}}$ ..... Pauli matrices $\hat{\sigma}_0 = \mathbb{I}$, $\hat{\sigma}_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\hat{\sigma}_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ & $\hat{\sigma}_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$

$\hat{\mathbf{q}}_{\mathbf{j}}$ ..... Lowering ladder operator for the j'th qubit in a qubit network.

$|\boldsymbol{\Omega}\rangle$ ..... Fermionic vacuum state

$|\mathbf{0}\rangle$ ..... Bosonic vacuum state

$\hat{\boldsymbol{f}}_{\boldsymbol{j}}$ ..... Fermionic annihilation operator in mode j

$\hat{\boldsymbol{b}}_{\boldsymbol{j}}$ ..... Bosonic annihilation operator in mode j

$\hat{\boldsymbol{U}}$ ..... Unitary operator or the set of unitary operators $\{e^{i\phi}\hat{U}\}_{\phi\in\mathbb{R}}$

$\mathbf{F}_{\mathbf{g}}^{\mathbf{abc}}$ ..... F-matrices of an anyon theory.

$\mathbf{R}_{\mathbf{c}}^{\mathbf{ab}}$ . . . . . R-matrix terms of a counterclockwise exchange in an anyon theory.

1 | Introduction

The sole path to attain the universal lies within the realms of the ultra-local.
L'única manera d'arribar a allò universal és a través de l'ultra local.

—Salvador Dalí

1.1 Motivation

Locality, particularly local realism, is a highly desired feature of any physical theory. The ability to explain complex global behaviour in terms of the individual behaviour of the subparts of physical reality is incredibly advantageous and intuitive. The program of classical field theory for electromagnetism and the theory of general relativity are perfect examples of the success and beauty of local theories. They provide excellent and accurate explanations of the physical phenomena they characterise.

However, most physicists agree that John S. Bell [1, 2] proved quantum theory cannot possess this quality. Some suggest that the lack of local realism in quantum theory prevents us from having a successful complete theory of quantum gravity. Having a local-realistic picture of quantum mechanics would improve our understanding and explanations of the quantum phenomena we observe.

Such a picture does exist. Deutsch-Hayden and Raymond-Robichaud [3–5] provide a framework on which quantum theory can be seen as a local-realistic theory. However, their work focuses on distinguishable quantum systems, where a tensor product structure is used for subsystem composition. We aim to extend their analysis to indistinguishable particle systems.

We want to do so for several reasons. First and foremost, indistinguishable particle systems are the basis of our fundamental quantum theories of reality. A local-realistic structure would improve our explanations of the physical behaviour of the fundamental theories of reality.

The second is to understand the Deutsch-Hayden and Raymon-Robichaud constructions better. Applying their ideas to new domains forces us to scrutinise their assumptions and deeply understand their arguments' underlying reasons and mechanics. Understanding these constructions better is important to construct better explanations when using them.

The third reason is that indistinguishable quantum particle systems are more nuanced, in general, than distinguishable systems. They display interesting features not showcased in the usual simple quantum distinguishable scenarios. We want to use the exploration of the local realism of these theories to dive deep into these nuances and understand why they arise and how to treat them. We desire to explore all the capabilities, structures and behaviour quantum theory offers rather than stay at the surface level of toy simple theories.

We believe that exploring the full range of theories the quantum framework can offer is necessary to ground the expected principles of any general theory of physics. Only by fully exploring the known theories can we assert what behaviours and structures we should expect in the general case.

As part of the quantum foundations program, we are interested in learning about other possible physics theories. Ideally, we would have a general faithful characterisation of all possible physical theories. Then, we could explore the landscape, search for new physical theories, and then find experimental regimes where these theories may apply.

In our analysis, we explore the fermionic and bosonic $3+1$ D indistinguishable

particle systems and the local realism properties of $2+1$ D anyons. There are several reasons for this choice. First, they are a valid fundamental theory of indistinguishable particles. Second, they have very unusual, interesting properties regarding subsystem composition. Third, their topological nature poses unique problems to the local-realistic program. Fourth, they are the basis of topological quantum computing. We believe that studying a physical system used as the basis of fault-tolerant quantum computation from a quantum foundations perspective is of the utmost importance. The quantum foundations perspective may add elements to advance our understanding of these systems.

All these reasons are fundamental to our personal motivation to undergo the presented analysis. Moreover, the possibility to meaningfully contribute to such relevant fields is a great opportunity. We hope to close literature gaps with our analysis and increase the overall physics knowledge of reality a tiny bit.

1.2 Aim and Objectives

The main objective of the thesis is to analyse the quantum systems of indistinguishable particles' local realism properties. Concretely, to identify their local realistic structure and express it conveniently and effectively.

We want to navigate the subtleties of quantum indistinguishable particle systems. We aim to provide an exhaustive analysis of their notions of locality and establish the connections and differences among them and quantum distinguishable systems.

To do so, we introduce the different particle formalisms, and we study them under the connection of the Raymond-Robichaud formalism [6] with the Deutsch-Hayden descriptors [3].

The main claim of this thesis is that annihilation operators can be used as represen-

tations of the local ontic states of all quantum indistinguishable particle systems.

To prove such a strong claim, we must navigate the nuances of superselection rules. We resolve the prima facie paradox in fermionic theory where an a priori violation of the no action at a distance principle appears when considering fermionic annihilation operators as representatives of the fermionic local ontic states.

In the anyonic case, we derive the existence of 2 D non-abelian anyonic annihilation operators to prove our thesis. We characterise their properties and behaviour. We also aim to showcase the usefulness of these annihilation operators for the study of anyonic systems per se.

The second main goal of this thesis is to bring closer the nuances of quantum indistinguishable particle systems to the quantum foundations' community. We want to remark on the striking properties some of these systems portray, being the perfect test ground for general axiom testing and a source of inspiration for possible generalisations on the behaviour of general physical theories.

1.3 Thesis Outline

This thesis is organised into three main chapters, a discussion chapter and the appendices.

Chapter 2 introduces the relevant literature on the different notions of local realism in the context of quantum mechanics. We introduce RR's formalism [4] as a formalisation of Einstein's local realism that we use throughout the thesis. Moreover, we explain the established connection between RR's formalism and the notion of descriptors introduced in [3]. We reproduce the qubit networks case analysis, concluding that we can use the qubit annihilation operators as qubit descriptors.

In Chapter 3, we study local realism for bosonic and fermionic systems. In the

fermionic case, we have resolved the prima facie paradox of having no action at a distance when the fermionic annihilation operators represent the local ontic states. We conclude that bosons and fermions are local-realistic, and their annihilation operators represent their local-realistic structure. We exemplify these properties by analysing the bosonic and fermionic Mach-Zehnder interferometers in Section 3.4.

After analysing local realism for $3+1$ D indistinguishable particles, Chapter 4 examines $2+1$ D anyonic particle systems. We prove that anyonic annihilation operators can represent the local ontic states of the local-realistic structure of anyonic systems.

To derive this result, we first discover the existence of anyonic annihilation operators. We characterise their construction and behaviour. We express the 2 D Fibonacci Hubbard model Hamiltonian using the anyonic creation and annihilation operators.

Finally, in the discussion, Chapter 5, we tie up some loose ends, provide an extensive review of the obtained results throughout the thesis, and comment on their significance and future directions.

The relevant original contributions by the author are found in Subsubsection 4.1.4.3, Subsections 2.4.3, 2.4.4, 4.1.6 & 4.1.7, Sections 3.2, 3.4, 4.2 & 4.3, and Chapters 5, A, B & C. The last three are the appendices.

2 | Local realism

Locality is a crucial concept for physics. This notion has played a central role in several physics revolutions. For example, Newton's gravitational laws appeared to be non-local [7]. This tension grew until classical physics developed the notion of field, which, after the developments of Maxwell [8], was incorporated into the representation of physical reality, providing a clean solution to the problem of locality in the law of gravitation.

The current state of affairs in physics echoes Newton's case. Quantum theory and general relativity are the best theories in their respective domains. While general relativity is generally understood as a theory that is local [9], the general perception of quantum theory is that it is non-local [10, 11]. Some have suggested that reconciling general relativity and quantum theory requires addressing the issue of non-locality [12].

We should investigate the role of locality in quantum theory for more reasons than the compatibility of quantum theory with general relativity. Similar to Newton's example, research into the role of locality can deepen our understanding of the theory as a whole. It can bring forward new ideas, methods and concepts that enrich our understanding and add new observations, phenomena, and explanations of the physical world around us.

One of the problems with the treatment of locality in physics is that it is an incredibly intuitive concept used as a term for different but related properties of a physical theory. The intuition of locality comes from macroscopic classical physics, where we perceive common objects existing in space. However, the classical characterisation makes many assumptions about the physical world. When we turn to quantum theory, some assumptions no longer hold. As a result, various

characterisations of this intuition are proposed and given the same label: locality.

This chapter examines how quantum theory can be regarded as a local theory. More precisely, as a local realistic theory. That is, identifying elements of the physical theory with the outside objective physical reality locally and respecting locality. Section 2.1 introduces Bell's notion of local realism [1, 2] and critically examines its assumptions. It is important to note that most authors adopt this notion when referring to quantum theory as a non-local theory. In Section 2.2, we present an alternative formulation of local realism given by Einstein [13, 14]. Section 2.3 examines the development of Einstein's local realism in the Raymond-Robichaud equivalence class formalism [4, 5]. Section 2.4 presents Deutsch and Hayden's notion of descriptors in quantum theory [3] for qubit networks. Lastly, Section 2.5 examines the relationship between descriptors and the notion of local realism proposed by Einstein and modelled by Raymond-Robichaud, while highlighting their convenience as a tool to engage with local realism in quantum theory.

Throughout the chapter, we provide arguments in favour of adopting Einstein's notion of local realism and its adaptation to the quantum theory proposed by Raymond-Robichaud, as opposed to Bell's notion. We introduce the idea of local elements of reality and their relevance for using a physical theory that respects locality and realism. We explain how descriptors are a compact way of representing quantum theory's local elements of reality.

On this note, we motivate the remaining chapters of this thesis, which focus on identifying and analysing the properties of descriptors—those compact representations of the local elements of reality—for quantum systems of indistinguishable particles.

2.1 Bell's notion of local realism

Alain Aspect, John F. Clauser and Anton Zeilinger received the Nobel Prize in Physics in 2022 "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science" [15]. Such is the scope and importance of the work by John Bell [1, 2]. His work has profoundly influenced the development of quantum information and quantum foundations. Moreover, the testing of Bell inequalities has brought about a wide range of experimental techniques [10, 16–19].

Such a profound effect is justified: Bell's was a deep conceptual non-trivial idea. As such, Bell's theorem has elicited extensive discussion on its assumptions, meaning, significance and implications [20–27]. Although there is no consensus in the literature on the misconceptions surrounding Bell's theorem, the common understanding is that it proves that quantum mechanics is not a local-realistic theory and that a local hidden variable model for quantum theory cannot exist.

This section aims to present the notion of local realism employed by Bell [1] in a simple, concise way. The sources cited throughout the section provide great insight into other aspects of Bell's work, the discussion of which would be out of the scope of this thesis.[1]

One cannot fully grasp Bell's theorem without understanding the 1935 paper by Einstein-Podolsky-Rosen (EPR) [28]. Arguably, Bell's argument is a continuation of EPR [23]. EPR argue that quantum mechanics is incomplete and suggest including additional, local deterministic hidden variables to complete it. Their argument establishes a contradiction between the local realism of quantum mechanics and its completeness. They argue that any reasonable theory should be local-realistic,

[1] We especially recommend [23].

therefore discarding the completeness of quantum theory.

Bell expands on the results from EPR. Bell's theorem models all local deterministic hidden variable models that can complete quantum mechanics and deduces a contradiction with the observations of quantum theory. Bell's theorem gives bounds from the conditional probabilities of measurement outcomes and settings of local deterministic hidden variable models. Bell observes that such bounds are violated with quantum mechanical predictions. The quantum mechanical predictions have been successfully verified through the so-called tests of Bell inequalities [10, 16–19].

In refuting the possibility of a local hidden variable model for quantum theory, Bell thought he disproved local realism in quantum theory because of the EPR argument. The impossibility of completing quantum theory with local hidden variables implies that the false condition in the EPR contradiction should be the local realism of quantum mechanics assumption.

Let us present this notion of local realism that Bell disproved and inherited from the EPR paper. In EPR, there is a distinction between the elements in the physical theory and the elements of physical reality. This alludes to an objective reality of the physical world, thus realism. To provide a working basis, one needs a criterion of what constitutes an element of such physical reality. The EPR criterion [28] is the following:

> "If, without in any way disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."

After establishing this sufficient condition for a physical quantity having a corresponding element of physical reality, EPR describes the physical quantities and

their values for quantum theory.

EPR claims that, in quantum theory, the physical quantities correspond to the observables $\hat{A}$ given by Hermitian linear operators over the Hilbert space of states ψ. They say such quantities take definite real values a when the state of the system ψ is an eigenstate of the associated observable $\hat{A}$, so we have $\hat{A}\psi = a\psi$. When ψ is not an eigenstate of $\hat{A}$, they claim one can no longer speak of the physical quantity $\hat{A}$ having a particular value.

Crucially, they consider the eigenvalues of observables as the possible values of the physical quantities. Therefore, these physical quantities taking these real values are susceptible to having an element of physical reality associated with them according to the EPR criterion.

For them, the elements of reality should specify the values of physical quantities. Specifically, the elements of reality should specify which eigenvalue the observable is taking. In other words, the outcomes of measurements should have a corresponding element of reality that specifies it.

Having clarified the notion of reality, let us focus on the concept of locality Bell uses. Bell illustrates his reasoning using the simple system of two spin-$\frac{1}{2}$ particles, acting as two qubits, sent away from each other towards Stern-Gerlach magnets oriented at directions $\vec{a}, \vec{b}$, respectively measuring the quantum observables A and B for the first and second particle. The eigenvalues of the relevant observables are $A = \pm 1 = B$. Thus, there must be elements of physical reality λ that complete quantum theory in determining the physical reality of the system, such that, together with $\vec{a}, \vec{b}$ can determine the value of the physical quantities A, B.

Regarding locality, Bell focuses on the condition that:

> " (...) if two measurements are made at places remote from one another

> the orientation of one magnet does not influence the result obtained with the other."

Bell models this requirement, that we name no action at a distance, for the hidden variables as the measurement settings for one particle cannot affect the other, which is remotely far away. Therefore $A(\vec{a}, \lambda) = \pm 1$ and $B(\vec{b}, \lambda) = \pm 1$. Bell also mentions that one could demand λ to be split between variables associated with the first particle and the second but does not insist on doing so out of lack of necessity.

In summary, Bell's notion of local realism is based on the EPR criterion of reality, the position that physical quantities in quantum theory are observables that can take as values their eigenvalues and that the measurement settings of 1 cannot influence the individual measurement outcomes of 2.

After the schematic presentation of local realism in Bell's theorem, let us make relevant comments that will be useful in the following section. In EPR and Bell's framework, albeit not explicit, there is a subtle distinction between reality and measurable properties. The completing variables λ are precisely named *hidden* by some authors due to the impossibility of measuring their value. However, their existence in physical reality is still affirmed.

One of the key features of EPR and Bell's local realism is that it assumes only one of the measurement values as real or realised. Such condition precludes [23] this notion of local realism to be satisfied in Everett's interpretation of quantum mechanics [29], where all values of a measurement are realised by coupling to the quantum mechanical behaviour of the measuring devices.

Instead of reconsidering the reality of the single-valued measurement processes, Bell prefers to discard the principle of locality and, thus, to allow remote measurement settings to affect each other's measurement outcomes. Other resolutions to save the locality principle have been proposed upon posterior analysis, such as

superdeterminism [24].

As we have explained, Bell models the condition of locality as the impossibility that the measurement settings of 1 can directly influence the reality of 2 instantaneously. We refer to this characterisation as the condition of no action at a distance. One must distinguish such a condition from the no-signalling condition, which is known to be satisfied in quantum theory [30] (see Section 2.2 for an explicit comparison).

Bell's notion of no action at a distance, in terms of measurement settings and outcomes, is relatively theory-independent. Although presented for quantum mechanical systems of distinguishable particles, this notion directly applies to quantum mechanical systems of indistinguishable particles. Since the algebraic properties of observables would change, the bounds obtained in Bell inequalities might also change. Bell's formalisation of no action at a distance would still directly apply to theories with composite systems, measurement settings and outcomes, even non-quantum theories.

Furthermore, Bell's formalisation of the principle of no action at a distance is general enough to be applied abstractly even in physical systems where the different subsystems A, B are not separated in space but are still independent. We end up having a notion that refers to the dependency of measurement outcomes with respect to measurement settings and state specifications. Such generality allows us to discover configurations that, without violating the no-signalling principle, are maximally non-local and do not have a quantum or classical theory behind them, like Popescu-Rohrlich (PR) boxes [31].

2.2 Einstein's principle of locality

Despite Einstein being one of the three authors of the EPR paper, he was not satisfied with how the publication turned out. Over the following years, he refined a different argument for the incompleteness of quantum mechanics specifying in detail the notions of local realism he uses [13, 14, 32].

In this section, we summarise the main arguments of various analyses of Einstein's notion of local realism [9, 33–36]. We broaden and generalise this notion of local realism to encompass general physical theories. In doing so, we pay close attention to [33]. We advise the reader to consult the references cited throughout this section for in-depth analysis and faultless exposition.

We address Einstein's notion of locality more methodically than Bell's 2.1. We follow a more general approach and establish the principles of local realism without discussing their application to the incompleteness of quantum mechanics.

First, let us introduce the idea of ontic states. The ontic, real or noumenal states in a realistic theory are the states that fully characterise all aspects of the constitution of the reality of the system. Such states specify all the properties that make the system be and constitute it as it is. This concept is distinct from the notion of a phenomenal or observable state of a system. The phenomenal state of a system is the configuration of all the system's observable properties. Most physics is done assuming Leibniz's rule of the identity of indiscernibles [37], which imposes that the ontic states are the phenomenal states. In other words, since only observations should allow us to discern differences between physical systems, all the constitutive properties of a system must be observable.

According to [33], Einstein's notion of local realism is based on two principles

> "The first, which I call the 'separability principle', asserts that any two spatially separated systems possess their own separate real states. The second, the 'locality principle' asserts that all physical effects are propagated with finite, subluminal velocities, so that no effects can be communicated between systems separated by a space-like interval."

The locality principle in Einstein's local realism, also known as no action at a distance, corresponds to Bell's notion of locality. It is the impossibility of immediate influence by distant physical systems. The principle of no action at a distance can be demanded at both ontic and phenomenal levels. When required at both, no action occurring at system A can immediately affect any feature of the physical description of the remote system B, even if such feature is observable or not. Even at the phenomenal level, it is a stronger requirement than the no-signalling principle.

The no-signalling principle establishes the impossibility of local observations in the system A being able to discern if an action has occurred in the remote independent system B. Notice that this principle allows B to influence A as long as the action in B is not discernable by A. Therefore we can see how the requirement of no action at a distance is stronger than the no-signalling principle.

Within Bell's scenario, Bell would claim that the no-signalling condition is satisfied, but no action at a distance is not. The maximal characterisation of A one can do using local observations in quantum mechanics is to establish the reduced density operator ρ_A. It is known that ρ_A captures the expected values of the local observables of A and that it is left invariant under any remote action in B [30]. Nevertheless, Bell would say no action at a distance is violated since he would expect, in general, that the action of changing the measurement settings in B would influence the individual measurement outcome in A. However, due to the stochastic

nature of the collapse, no average effect would be witnessed, even though remote influence would have occurred in each individual measurement.

Let us now focus on the separability condition, which, according to Einstein, was the main source of contradiction with quantum theory. The separability condition introduces a desired structure at the level of ontic states. If one follows the Leibniz rule in assigning a one-to-one correspondence between ontic states and phenomenal states, and if the phenomenal states do not possess such structure, then the separability condition fails. If that were the case, one might want to drop the Leibniz rule to retain the separability principle and local realism by selecting a new set of ontic states with the desired structure.

The separability condition stated in the quote above is less strong than the version referred to by Einstein and us. The stronger separability condition states, in addition, that the parts' local ontic states can recover the composite system's complete ontic state. Not only is the whole separable into independent parts, but the combined parts also make the whole. The whole is not more than the union of the parts.

The separability principle is a natural principle to demand for several reasons. It is natural to demand that physical subsystems may be isolated and treated independently of the larger environment, so they may be considered to have their own independent reality. To fully state that A is a subsystem of AB, we must first consider A to be a system in and of itself, independent from the rest of AB.

Because it captures qualitative features from field theory and the theories of special and general relativity, Einstein's notion of locality is deeply rooted in the notion of space. Spatial separation appears in both no action at a distance and separability principles. We can, however, extend these two principles to general subsystems. We want to consider any two independent subsystems A, B that together form

a composite system AB and apply the separability and no action at a distance principles to them without referring to their position in space in any way. For A and B to be proper subsystems, they must be independent. Given this independence, it is natural to demand the separability of the local ontic states. It is also natural to infer that no action at a distance should apply; since a local action in one such independent system should not influence or be influenced by a different independent system. Any interaction between such independent parts must occur necessarily at the global level of composition AB.

Similarly to Bell's locality, with this extension to general subsystems, it is clear that Einstein's locality applies not only to quantum mechanical systems but to any physical theory with notions of independent subsystems. Nevertheless, Einstein's local realism is conditional on the structure of the ontic states and transformations that the theory allows. It is less focused than Bell's on measurements, preparations, settings and outcomes. It enables one to work fully within the abstract framework of the physical theory, expanding the range of possible descriptions of the reality of physical systems.

Precisely because of this freedom, we can consider Everettian quantum mechanics [29] within Einstein's local realism. We can model measurement apparatuses getting coupled with the measured physical systems, and all values of the measurements being part of the extended reality.

As we have seen, Bell's local realism excludes this possibility. Such flexibility results from not committing to a particular criterion of what the elements of physical reality must be and, instead, only specifying some structures they must have on top of describing the constituting properties of the theory's observable physical situations.

Works that discuss the notion of separability and composition [38–43] in physics

are particularly relevant to apply and understand Einstein's notion, providing new tools and perspectives that can be used in the formalisation of Einstein's local realism. Although Bell's locality is still preferred by the communities of quantum information and foundations, Einstein's ideas are hinted at, for example, in the research programme of algebraic quantum field theory [44, 45].

Subsystem locality refers to Einstein's position. Meanwhile, Bell or quantum non-locality refers to the failure to satisfy Bell's notion. We adhere to Einstein's notion throughout the thesis for its flexibility, generality, and compatibility with field theory frameworks; thus, when we refer to local realism in the thesis, we refer to Einstein's local realism.

2.3 Local realism formalised

This section concisely explains the formalisation of local-realistic theories developed in [4, 5]. This formalisation of local-realistic theories provides a useful conceptual and mathematical framework to study Einstein's local realism for general physical theories. In particular, we can see how the no action at a distance, separability and no-signalling principles are mathematically expressed for general physical theories.

We remind the reader that our main goal is to discuss whether indistinguishable quantum systems are Einstein local-realistic and, if so, identify the best way to represent the local elements of reality for such systems. The Raymond-Robichaud (RR) formalism enables us to achieve this goal.

The RR construction includes an important and beneficial result. In [4], the theorem that every theory with reversible dynamics that respects the no-signalling principle is local-realistic is stated and proved constructively.

The following subsections present the theorem and various elements of the RR formalism that are required to state the theorem and work with it. This material is used throughout the thesis.

2.3.1 Realistic theories

To define a realistic physical theory, we use a triad of sets $(\mathcal{P}, \mathcal{R}, \mathcal{T})$ that have additional structure.

The set $\mathcal{P}$ is the set of elements of the physical theory that describe the phenomenal properties of the theory - i.e., the properties that are empirically accessible. In other words, an element $\rho \in \mathcal{P}$ describes all the properties that can be observed for a specific configuration of a physical system.

The set $\mathcal{T}$ is the set of operations that describe the allowed physical transformations. The theorem focuses on theories in which the set of physical transformations forms a group under composition $\circ$. The group structure provides the physical theory with reversible dynamics. The physical theories under consideration satisfy the property that any transformation of the physical system must be reversible by applying a different allowed physical transformation. The group of transformations $\mathcal{T}$ acts on the phenomenal state space $\mathcal{P}$, defining a group action $\cdot$ given by

$$\begin{aligned} \mathcal{T} \times \mathcal{P} &\rightarrow \mathcal{P} \\ (U, \rho) &\longmapsto U \cdot \rho \end{aligned} \tag{2.1}$$

For clarity of exposition, we use the concept of "operational theory" employed in the first version of the RR construction [6]. The group $\mathcal{T}$ under $\circ$, the set $\mathcal{P}$ and the group action $\cdot$ compose an operational theory. These constituents are the minimal ingredients required by all physical theories, the characterisation of the

observational properties of the system and the possible transformations it may undergo.

Further structure is required for an operational theory to be a realistic theory. So far, the system has been described at the observational level. A realistic theory should link the observational features of a physical system with some underlying constituent features of the physical reality. It should also describe the elements that intrinsically constitute the reality of the existence of the physical system.

The set $\mathcal{R}$ is the set of states of a physical system that describe all properties that entirely specify the configuration of that physical system rather than just its observational properties. The set $\mathcal{R}$ is the ontic state space. The ontic states $r \in \mathcal{R}$ can also be named real or noumenal states, referring to the Kantian notion [46].

The distinction between phenomenal and ontic states may be challenging to grasp intuitively. One of the reasons may be the widely assumed by physicists realism under Leibniz's rule of the identity of indiscernibles [37, 47]. This principle can be summarised as follows: ontic states coincide with phenomenal states. In other words, all existence-defining properties of a physical system are given by their observable properties. According to the Leibniz rule, constitutive properties not discernable by observation cannot exist.

Despite this rule, exploring the full range of structures that a clear distinction between the ontic and the phenomenal state spaces provide may be interesting, especially for expressing Einstein's principle of locality.

The ontic state space $\mathcal{R}$ requires further structure. The group of transformations $\mathcal{T}$ must also act on $\mathcal{R}$. A physical transformation can alter the constituent properties of the system. Since $\mathcal{P}$ and $\mathcal{R}$ may be completely different sets, the group action

from $\mathcal{T}$ to $\mathcal{R}$ may also be different. We denote the ontic group action with $\star$.

$$\begin{aligned} \mathcal{T} \times \mathcal{R} &\rightarrow \mathcal{R} \\ (U, r) &\longmapsto U \star r \end{aligned} \tag{2.2}$$

Let us consider a familiar example to illustrate the meaning of the ontic state. Consider a two-qubit composite system in the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. We apply a local rotation applied to one of the qubits. We already know that the Schrödinger state $|\psi\rangle$ will remain unchanged. In this case, one can imagine that the phenomenal state is $|\psi\rangle$ since it provides the observational properties of the system. However, the fact that we have applied a rotation is relevant to the constitutive properties of the system. Despite this rotation having no observational effect, it may have altered the physical reality of the qubit. In that case, the Leibniz rule does not apply.

Given the definitions of ontic and phenomenal, the phenomenal state must be completely characterised by the ontic state of the system. In the RR formalism, this observation is enforced by the requirement of an epimorphism φ from $\mathcal{R}$ to $\mathcal{P}$.

We require $\varphi(\mathcal{R}) = \mathcal{P}$, where for every $\rho \in \mathcal{P}$ exists an $r \in \mathcal{R}$ such that $\varphi(r) = \rho$. Moreover, φ has to be faithful with respect to the two group actions $\cdot$ and $\star$, having $\varphi(U \star r) = U \cdot \varphi(r)$.

2.3.2 Local-realistic theories

Having presented the notions and formalisation of realistic theories in the RR construction, we now introduce the formalisation of local-realistic theories.

Local-realistic theories are a subset of realistic theories. In order to specify if a theory is local-realistic, we need to introduce further structure to the formalism.

We require tools to capture the notion of locality.

The original notion of Einstein's local realism, as presented in Section 2.2, refers directly to regions of space. The notion of locality in the RR construction is a generalisation of Einstein's local realism. Its defining characteristics are the no action at a distance and separability principles. However, it applies to general partitions into subsystems and has spacetime regions as a particular choice of partition. Despite coinciding in using agents or subsystems, the notion of locality in the RR construction is not an extension of Bell's. Opposite to Bell's, no conditional probabilities, measurement outcomes, preparations, or random variables are invoked. Instead, RR focuses on the structural properties of the physical theory. It can be considered a general formulation of the concept of locality implemented in algebraic quantum field theory [44, 45]. From now on, when we talk about locality, we mean being localised in a subsystem of the global system in consideration.

To capture locality, we need a lattice of subsystems. We must specify which systems our states and transformations refer to and act on. To specify the lattice, we consider a maximal system $\mathbb{S}$ and the set of all subsystems $\mathcal{S}$ of the maximal system $\mathbb{S}$. By denoting $AB \in \mathcal{S}$, we denote the subsystem that is the composition of the disjoint subsystems $A, B \in \mathcal{S}$. We also denote $A \sqsubseteq C$ to say the subsystem $A \in \mathcal{S}$ is also a subsystem of $C \in \mathcal{S}$. We always have $A \sqsubseteq AB$. Every subsystem $A \in \mathcal{S}$ is a physical system in itself. Thus, it has a phenomenal state space $\mathcal{P}_A$ and a group of transformations $\mathcal{T}_A$ associated with it.

Throughout this thesis, we concentrate on systems where the maximal system is $\mathbb{S} = A_1 A_2 \dots A_N$. This notation indicates that the order of composition of the elemental subsystems is irrelevant and that the maximal system is the composition of a finite set of elemental subsystems A_j. In such cases, studying the bipartite case is enough. We need mathematical structural properties that establish the relation of

$A, B \sqsubseteq AB$. The first important relations are the group monomorphisms of the local transformations.

$$\begin{aligned} \phi_A : \mathcal{T}_A \to \mathcal{T}_{AB} \qquad & \qquad \phi_B : \mathcal{T}_B \to \mathcal{T}_{AB} \\ U_A \mapsto U_A^{ext} \qquad & \qquad U_B \mapsto U_B^{ext} \end{aligned} \tag{2.3}$$

The global set of transformations $\mathcal{T}_{AB}$ has two distinct subgroups $\mathcal{T}_A^{ext}, \mathcal{T}_B^{ext}$ that are isomorphic to the groups of local transformations in A and B. Physically, this means that the structure of local operations in subsystems A, B must be the same when they are regarded as global transformations in AB.

The second structural feature is the projections of phenomenal global states to phenomenal local states. Intuitively, this relation determines how from all the observational properties of the composite system AB, we can deduce the observational properties of the subsystems A or B alone.

$$\begin{aligned} \pi_A^{\mathcal{P}} : \mathcal{P}_{AB} \to \mathcal{P}_A \qquad & \qquad \pi_B^{\mathcal{P}} : \mathcal{P}_{AB} \to \mathcal{P}_B \\ \rho_{AB} \to \rho_A \qquad & \qquad \rho_{AB} \to \rho_B \end{aligned} \tag{2.4}$$

The phenomenal projections $\pi_A^{\mathcal{P}}, \pi_B^{\mathcal{P}}$ need to be faithful with respect to the group of local transformations monomorphisms. Given ρ_{AB} such that $\pi_A^{\mathcal{P}}(\rho_{AB}) = \rho_A$ and $\pi_B^{\mathcal{P}}(\rho_{AB}) = \rho_B$, then $\pi_A^{\mathcal{P}}\left(U_A^{ext} \cdot \rho_{AB}\right) = U_A \cdot \rho_A$ and $\pi_B^{\mathcal{P}}\left(U_B^{ext} \cdot \rho_{AB}\right) = U_B \cdot \rho_B$.

We have all the primary ingredients to describe subsystem locality in an operational theory. We want to specify the notion of a local-realistic theory. First, we need a realistic theory. Then, for a realistic theory to be local for a lattice of subsystems, it must satisfy the two Einstein locality conditions presented in Section 2.2. The group actions on the ontic state spaces $\mathcal{R}_S$ of the subsystems are also required to be faithful with respect to the monomorphisms ϕ_S.

To formalise Einstein's local realism principles, we need to introduce projection maps at the level of the ontic state spaces, such as $\pi_A^{\mathcal{R}} : \mathcal{R}_{AB} \to \mathcal{R}_A$ and $\pi_B^{\mathcal{R}} : \mathcal{R}_{AB} \to \mathcal{R}_B$. The projection map sends composite ontic states to local ontic states. We require that the ontic projections and the phenomenal projections act in parallel. Moreover, we require the ontic projections to be faithful to the group of local transformations monomorphism. In other words, we need the following diagrams to commute.

$$\begin{array}{ccc} \mathcal{R}_{AB} & \xrightarrow{\phi_A(U_A)} & \mathcal{R}_{AB} \\ \downarrow{\scriptstyle \pi_A^{\mathcal{R}}} & & \downarrow{\scriptstyle \pi_A^{\mathcal{R}}} \\ \mathcal{R}_A & \xrightarrow{U_A} & \mathcal{R}_A \end{array} \qquad \begin{array}{ccc} \mathcal{R}_{AB} & \xrightarrow{\varphi_{AB}} & \mathcal{P}_{AB} \\ \downarrow{\scriptstyle \pi_A^{\mathcal{R}}} & & \downarrow{\scriptstyle \pi_A^{\mathcal{P}}} \\ \mathcal{R}_A & \xrightarrow{\varphi_A} & \mathcal{P}_A \end{array} \tag{2.5}$$

Under these well-behaved projections, we can consider local ontic state spaces. We are in a position to formalise the separability principle. The separability principle can be broadly stated as independent local ontic states exist, and knowing the collection of ontic states of the parts is the ontic state of the whole. More specifically, together with the ontic projection map, we want to be able to define the map $\odot : \mathcal{R}_A \times \mathcal{R}_B \to \mathcal{R}_{AB}$ that assigns the global ontic state from its local counterparts. Consider a phenomenal state $\rho_{AB} \in \mathcal{P}_{AB}$ such that it has an underlying ontic state $r_{AB} \in \mathcal{R}_{AB}$, with $\varphi(r_{AB}) = \rho_{AB}$. Having the local ontic states $r_A = \pi_A^{\mathcal{R}}(r_{AB}), r_B = \pi_B^{\mathcal{R}}(r_{AB})$, we require that we can assign $r_A \odot r_B = r_{AB}$ uniquely. Intuitively, this condition tells us that the whole is not more than the parts at the ontic state level. There is enough information in the local ontic states to recover the global ones. It states that the global physical reality is just the collection of the local physical realities.

Furthermore, we need to formalise the principles of no-signalling and no action at a distance. No-signalling states that given a global system AB, a transformation

local on a subsystem A cannot immediately affect the local observational properties of B. Since the condition is on observable properties, it constrains the structure of phenomenal state spaces. Thus, we can have operational theories that satisfy the no-signalling principle; there is no need to refer to realism. Concretely, the no-signalling condition is formalised as follows:

$$\forall\, \rho_{AB} \in \mathcal{P}_{AB},\ \ \forall\, U_B \in \mathcal{T}_B \qquad \pi_A^{\mathcal{P}}\left(U_B^{ext} \cdot \rho_{AB}\right) = \pi_A^{\mathcal{P}}\left(\rho_{AB}\right) \tag{2.6}$$

Let us consider the ontic no action at a distance principle, which applies to the ontic structures. It can be expressed as, given a composite system AB, under a local transformation in B, the elements of the physical reality (the ontic states) of A should be left unchanged. The following expression gives the natural formalisation in the RR construction:

$$\forall\, r_{AB} \in \mathcal{R}_{AB},\ \ \forall\, U_B \in \mathcal{T}_B \qquad \pi_A^{\mathcal{R}}\left(U_B^{ext} \star r_{AB}\right) = \pi_A^{\mathcal{R}}\left(r_{AB}\right) \tag{2.7}$$

Given such formal definitions, we can see the following interesting proposition holds

Proposition 1. *In a realistic physical theory with a lattice of subsystems, the no-signalling condition follows from the no action at a distance principle.*

Proof. If we have a realistic physical theory with a lattice of subsystems that satisfies the no action at a distance principle, applying to Equation 2.7 the epimorphism φ, we obtain

$$\begin{aligned} &\forall\, r_{AB} \in \mathcal{R}_{AB},\ \ \forall\, U_B \in \mathcal{T}_B \qquad \varphi\left(\pi_A^{\mathcal{R}}\left(U_B^{ext} \star r_{AB}\right)\right) = \varphi\left(\pi_A^{\mathcal{R}}\left(r_{AB}\right)\right) \quad \Rightarrow \\ &\forall\, r_{AB} \in \mathcal{R}_{AB},\ \ \forall\, U_B \in \mathcal{T}_B \qquad \pi_A^{\mathcal{P}}\left(U_B^{ext} \cdot \varphi\left(r_{AB}\right)\right) = \pi_A^{\mathcal{P}}\left(\varphi\left(r_{AB}\right)\right) \end{aligned} \tag{2.8}$$

by applying the faithfulness of φ with respect to the group actions and the projection maps $\pi_A^{\mathcal{R}}$ & $\pi_A^{\mathcal{P}}$ (Equation 2.5) we deduce the implication above. Given that the conditions hold $\forall r_{AB} \in \mathcal{R}_{AB}$ and φ is an epimorphism, then all $\rho_{AB} \in \mathcal{P}_{AB}$ can be written as $\rho_{AB} = \varphi(r_{AB})$ for some r_{AB}. Therefore, the conditions hold $\forall\ \rho_{AB} \in \mathcal{P}_{AB}$, thus recovering the expression of the no-signalling condition in Equation 2.6. □

Subsequent works [39, 40] to Raymond-Robichaud's paper [4] have pointed out that the construction of the lattice of subsystems may already imply the condition of no-signalling. In [39], a deep analysis is done to discern what a subsystem is and what structural properties a lattice of subsystems may have. In [40], the author recovers the tensor product structure from structural properties of the lattice of subsystems, making use of the diagrammatic formulation of quantum theory [41, 48].

The demanded structural condition that the embedded groups $\mathcal{T}_A^{ext}, \mathcal{T}_B^{ext}$ commute with each other implies the no-signalling condition. This commutation property, however, can be justified only on the ground of requiring that the subsystems are independent of each other, thus entering the argumentation of why it is reasonable to demand the no-signalling condition. For our purposes, which assumption is first or primordial is quite irrelevant. We are interested in theories that satisfy both the structural properties of the lattices of independent subsystems and the phenomenal no-signalling principle.

Thus, in order to see an operational theory with a lattice of subsystems such is local-realistic, we need to check that it satisfies the no-signalling principle and find ontic state spaces $(\mathcal{R}_S, \star)$, ontic projectors $\pi_S^{\mathcal{R}}$, epimorphisms φ that satisfy the faithfulness conditions, be able to define a unique join operation $\odot$ for local ontic states that satisfy the no action at a distance condition.

2.3.3 Local ontic states

The RR construction is a compelling framework not only because it can formalise Einstein's notion of local realism but also because of the central theorem of the construction.

Theorem 2. *Every no-signalling operational theory with reversible dynamics is local-realistic.*

The proof that any no-signalling operational theory is a local-realistic theory is constructive [4]. However, first, let us define what a no-signalling operational theory is.

A no-signalling operational theory is given by a lattice of systems, with associated operational theories at each subsystem, with transformations groups $\mathcal{T}_S$, phenomenal state spaces $\mathcal{P}_S$, with the associated group actions $\cdot$ and projection operators $\pi_S^{\mathcal{P}}$. On top of these structures, two key conditions must be met.

The first is that the no-signalling principle is satisfied for every bipartite composite system in the no-signalling operational theory. The second is that if $W_{AC}^{ext} = T_{BC}^{ext}$ in the system ABC, then $W_{AC}^{ext} = T_{BC}^{ext} = S_C^{ext}$. In other words, if a transformation is local in AC and BC, it is necessarily a local transformation in C. This structural property of the lattice of subsystems is directly demanded in the RR construction via Postulate 4.1 (Separation), so the notion of disjoint subsystems is well-represented.

Raymond-Robichaud constructs the ontic state spaces $\mathcal{R}_S$, the ontic group action $\star$, the epimorphism φ, the ontic projection operators $\pi_S^{\mathcal{R}}$ and the join product $\odot$.

For convenience, let us restrict ourselves to the case where we consider the maximal system $\mathcal{S}$ to be $\mathcal{S} = A_1 A_2 \dots A_N$ and the lattice of subsystems all the different compositions one can do with the elemental subsystems A_j. In this scenario, a

subsystem A is given by the collection of elemental subsystems A_j. The complement of the subsystem A will be a disjoint subsystem $\bar{A}$ given by collecting all the elemental systems that are part of $\mathcal{S}$ but not part of A.

Raymond-Robichaud constructs the local-realistic structures by defining an equivalence relation $\sim_A$ on the group of transformations of the maximal system $\mathcal{S}$. Two transformations $U, V \in \mathcal{T}_{\mathcal{S}}$ are equivalent in a subsystem A, $U \sim_A V$ if and only if exists a transformation $W_{\bar{A}}$ local on the subsystem $\bar{A}$, such that $U = W_{\bar{A}}V$.

The equivalence classes $[U]_A$ under this equivalence relation together with a reference phenomenal state ρ_0, which specifies the orbit or sector to which the system's phenomenal state is associated, are the ontic states of the subsystem A. ρ_0 is fixed by convention for each orbit of the group action of $\mathcal{T}_{\mathcal{S}}$ on $\mathcal{P}_{\mathcal{S}}$.

One of the postulates chosen in [4], Postulate 4.4, is the global transitivity of the phenomenal state space. It posits that the phenomenal space has a single orbit under the action of the group of transformations. However, we have found such postulates to fail in some examples we present in the thesis (see Subsections 3.1.2 & 4.1.2). We present the reformulated RR construction relaxing this postulate. The only effect it plays is that ρ_0 has more physical content indicating among which orbit the observable properties of the systems are and that it allows ρ_0 not to be a pure state in quantum theories. A convention can be chosen to state the representative for each orbit, but it cannot be said, as in the original construction, that ρ_0 can be fixed by convention only. Nevertheless, ρ_0 remains a fixed property of the system's ontic state and needs to be fixed by convention.

The elements $([U]_A, \rho_0)$ form the ontic state space $\mathcal{R}_A$ with all the desired properties. It is enough to specify the bipartite case, which generalises naturally.

The group action of $\mathcal{T}_A$ on $\mathcal{R}_A$ is given by $V_A \star ([U]_A, \rho_0) = ([V_A^{ext} \circ U]_A, \rho_0)$ for $V_A \in \mathcal{T}_A$ and $([U]_A, \rho_0) \in \mathcal{R}_A$.

The ontic projections $\pi_A^{\mathcal{R}}$ from $\mathcal{R}_{AB}$ to $\mathcal{R}_A$ are given by $\pi_A^{\mathcal{R}}\left(([U]_{AB}, \rho_0)\right) = ([U]_A, \rho_0)$. Projections from $\mathcal{R}_{ABC}$ to $\mathcal{R}_A$ can be seen as a composition of two bipartite projections, for example, the first being from $(AB)C$ to AB and the second from AB to A.

Notice that the ontic state space of the maximal system $\mathcal{R}_{\mathcal{S}}$ is given by the elements (U, ρ_0). The epimorphism φ from $\mathcal{R}_{\mathcal{S}}$ to $\mathcal{P}_{\mathcal{S}}$ is given by $\varphi\left((U, \rho_0)\right) = U \cdot \rho_0$. The subsequent epimorphisms φ_A for strict subsystems A between the local ontic state spaces $\mathcal{R}_A$ and the local phenomenal state spaces $\mathcal{P}_A$ are given by $\varphi_A\left(([U]_A, \rho_0)\right) = \pi_A^{\mathcal{P}}\left(U \cdot \rho_0\right)$. It is precisely the fact that the no-signalling principle is satisfied that ensures that the above definition is independent of the representative of the equivalence class.

The join product $\odot$ is defined for compatible states only. That means we need to have a global ontic state that we separate in two to join it. Consider a global ontic state for a bipartite composite system, $([U]_{AB}, \rho_0)$, the projections to the local ontic states are $([U]_A, \rho_0)$ and $([U]_B, \rho_0)$. It is possible to define the join product uniquely $([U]_A, \rho_0) \odot ([U]_B, \rho_0) = ([U]_{AB}, \rho_0)$ uniquely and independently of the representative of the equivalence classes.

The independence of the representative of the equivalence classes follows from the transitive property of the equivalence relation. The uniqueness condition is guaranteed by seeing that if $U \sim_A V$ and $U \sim_B V$, then $U \sim_{AB} V$. One can prove that is the case for no-signalling operational theories due to the condition that if $W_{AC}^{ext} = T_{BC}^{ext}$ in the system ABC, then $W_{AC}^{ext} = T_{BC}^{ext} = S_C^{ext}$.

The local realistic theory given by the RR equivalence class construction for any no-signalling operational theory satisfies the no action at a distance principle. For a bipartite composite system AB, the ontic states are $[U]_{AB}$. Their local projection on A is $\pi_A^{\mathcal{R}}\left([U]_{AB}\right) = [U]_A$. Acting on the ontic state with a local unitary on B, V_B,

and then projecting on A is given by $\pi_A^{\mathcal{R}}\left(V_B^{ext} \star [U]_{AB}\right) = \pi_A^{\mathcal{R}}\left([V_B^{ext} \circ U]_{AB}\right) = [V_B^{ext} \circ U]_A$. By the definition of the equivalence relation on A, $[V_B^{ext} \circ U]_A = [U]_A$. Therefore, the no action at a distance principle is satisfied by any no-signalling operational theory.

2.4 Descriptors

In this section, we explain the notion of descriptors present in the literature [3, 49–54], and we explain the connection with the equivalence class formalism for ontic states that we have explained in Section 2.3. This section selects the relevant notions from the literature that will be expanded on and used throughout the following thesis chapters.

The notion of descriptors is built within the usual quantum formalism by exploring the foundational consequences of the Heisenberg picture of quantum mechanics.

All the literature refers to distinguishable quantum mechanics, where the tensor products give system composition. Descriptors are defined in [3] for finite-dimensional qubit networks. In this section, we follow the review presented in [51] that establishes the connection between the equivalence class formalism of ontic states and descriptors for the qubit networks scenario.

2.4.1 Qubit networks as a no-signalling operational theory

Consider a qubit network system. In particular, an ordered lattice of N qubits. We name the set of all lattice sites as $\mathcal{N} = \{1, \ldots, N\}$. The global observables of the system $\mathcal{O}$ are Hermitian operators that act on the complex Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \ldots, \mathcal{H}_N = \mathbb{C}^{2^{\otimes N}}$.

The subsets of qubit lattice sites give the lattice of subsystems. The local observ-

ables for a subset of lattice sites $\mathcal{M} = \{m_1, \ldots, m_M\} \subseteq \mathcal{M}$ are given by Hermitian operators that act on the Hilbert space $\mathcal{H}_{\mathcal{M}} = \mathcal{H}_{m_1} \otimes \cdots \otimes \mathcal{H}_{m_M} = \mathbb{C}^{2^{\otimes M}}$. The embedding of the local observables $\hat{A}_{\mathcal{M}} \in \mathcal{O}_{\mathcal{M}}$ to the global observables $\mathcal{O}$ is the canonical extension, by setting $\hat{A}_{\mathcal{M}} \otimes \mathbb{I}_{\mathcal{N}\setminus\mathcal{M}} = \hat{A}^{ext}_{\mathcal{M}} \in \mathcal{O}$.

This lattice of the local algebras of observables provides the required lattice of subsystems we need in order to define a local realistic theory. We will see how descriptors focus on the minimal elements to track the evolution of the local algebra instead of focusing on the whole algebra.

The sets of local phenomenal states are given by $\mathcal{P}_{\mathcal{M}} = \{\rho \in \mathcal{O}_{\mathcal{M}} | \operatorname{Tr}(\rho) = 1, \rho \geq \hat{0}\}$, where the trace is the usual one for operators acting on a Hilbert space. The groups of local transformations are given by $\mathcal{T}_{\mathcal{M}} = \{e^{i\hat{A}} | \hat{A} \in \mathcal{O}_{\mathcal{M}}\}/U(1)$. The quotient on $U(1)$ refers to the global phase redundancy of quantum mechanics. [2] Throughout the thesis, we use $\hat{U} \in \mathcal{T}_{\mathcal{M}}$ as a shortcut for saying that $\hat{U}$ is one of the representatives of its associated equivalence class in $\mathcal{T}_{\mathcal{M}}$. We deal with the global phase redundancy of the representatives a posteriori.

In general, in a quantum mechanical system, fixing the lattice of the algebras of observables with their embedding is enough to provide the local phenomenal state spaces and the local transformation groups. Notice that the local observables embedding guarantees that the local groups of transformations are subgroups of the global transformation group.

It is also necessary for a lattice of subsystems to have a projection operation $\pi^{\mathcal{P}}_{\mathcal{M}}$. In the case of a quantum mechanical system, such projection is given by the partial tracing operation. Concretely, we see that projecting at the phenomenal level to a subsystem given by the lattice sites $\mathcal{M}$ corresponds to partial tracing the modes

[2] We need to make such redundancy explicit here, so later we can apply the RR construction and Postulate 4.3 of [4] is satisfied.

in $\mathcal{N}\backslash\mathcal{M}$. The operation of partial tracing is also inherited from the local net of observables by requiring that the embedded local observables on the global state and the local observables on the reduced state give the same observational properties. In the case of quantum mechanics, this corresponds to having the same expectation values, which are given by the total trace of the observables and the state.

$$\mathrm{Tr}\left(\hat{A}^{ext}_{\mathcal{M}}\rho\right) = \mathrm{Tr}\left(\hat{A}_{\mathcal{M}}\pi^{\mathcal{P}}_{\mathcal{M}}(\rho)\right) \qquad \forall \hat{A}_{\mathcal{M}} \in \mathcal{O}_{\mathcal{M}} \tag{2.9}$$

The partial trace is the unique linear operation that satisfies these equations. Associating the partial trace to the algebra of observables is a common way to analyse constrained quantum systems [55, 56].

2.4.2 Descriptors of a qubit network

The notion of descriptors arises from recognising the importance and the central role that the local algebras of observables play in describing quantum mechanics. In particular, we have already said that the observable content of quantum mechanical systems is in the expectation values of the observables given by a state, provided by the expression $\mathrm{Tr}\left(\hat{O}\rho\right)$.

The usual picture of quantum mechanical systems in the context of quantum foundations and information is the Schrödinger picture. That is, it is interpreted that ρ corresponds to the system's state, which provides the system's configuration in the current instant. However, it may change; it may evolve with time. An evolution in time is described by a unitary $\hat{U} \in \mathcal{T}$ and the Schrödinger state has evolved to $\hat{U}\cdot\rho\cdot\hat{U}^{\dagger}$. In the Schrödinger picture we take the observables $\hat{O}$ not to evolve. They act like references that we use to calculate the updated expectation values by the time evolution $\mathrm{Tr}\left(\hat{O}\cdot\hat{U}\cdot\rho\cdot\hat{U}^{\dagger}\right)$.

However, we can use the Heisenberg picture. The state ρ remains invariant under

the transformation $\hat{U}$ and acts as a reference. The observables $\hat{O}$ evolve under the transformation given by the unitary $\hat{U}$ as $\hat{O}' = \hat{U}^\dagger \cdot \hat{O} \cdot \hat{U}$. Since the observables of the system have evolved and ρ has not, it is reasonable to assume that we could find a framework where the observables are the ones that determine the configuration of the system and not ρ.

In order to do so, it is helpful to find which observables we can track to determine any $\hat{U}^\dagger \cdot \hat{O} \cdot \hat{U}$. Remember that the product of Pauli observables $\hat{\sigma}_{\mu_1} \otimes \cdots \otimes \hat{\sigma}_{\mu_N}$ is a basis of the global algebra of operators that act on the Hilbert space $\mathbb{C}^{2^{\otimes N}}$. Since any global observable $\hat{O}$ is a global operator, it is possible to decompose any observable in the Pauli basis:

$$\hat{O} = \sum_{k_1,\ldots,k_N} \alpha_{k_1,\ldots,k_N} \hat{\sigma}_{k_1} \otimes \cdots \otimes \hat{\sigma}_{k_N} \tag{2.10}$$

where $\alpha_{k_1,\ldots,k_N} \in \mathbb{R}$. We use this decomposition to notice that since the action of time evolution is linear on the observables, given the action of the unitary on the product of Pauli observables, one can know the action of the unitary on any observable. Thus, it is enough to track the products of Pauli observables to keep track of the whole set of observables.

$$\begin{aligned} \hat{U}^\dagger \cdot \hat{O} \cdot \hat{U} &= \sum_{k_1,\ldots,k_N} \alpha_{k_1,\ldots,k_N} \hat{U}^\dagger \cdot (\hat{\sigma}_{k_1} \otimes \cdots \otimes \hat{\sigma}_{k_N}) \cdot \hat{U} = \\ &= \sum_{k_1,\ldots,k_N} \alpha_{k_1,\ldots,k_N} \left(\hat{U}^\dagger \cdot \hat{\sigma}^{ext}_{k_1} \cdot \hat{U}\right) \cdot \cdots \cdot \left(\hat{U}^\dagger \cdot \hat{\sigma}^{ext}_{k_N} \cdot \hat{U}\right) \end{aligned} \tag{2.11}$$

On the second line, we have used the canonical extension of observables applied to Pauli observables as $\hat{\sigma}^{ext}_{k_j} = \bigotimes_{l=1}^{j-1} \mathbb{I} \otimes \hat{\sigma}_{k_j} \bigotimes_{r=j+1}^{N} \mathbb{I}$, and introduced identities of the form $\mathbb{I} = \hat{U}\hat{U}^\dagger$ between each lattice site contribution. From the second line of the equation, we can read that by using unitarity to know the unitary evolution of

all the products of Pauli operators. Knowing the evolution of the extended Pauli operators individually is enough to deduce the evolution of any global observable.

This point is exciting. The mathematical treatment is simple; it is used to prove that qubit networks satisfy the notion of local tomography. There is enough information on the evolution of the extension of the local Pauli operators to deduce the evolution of any global observables. A different way to phrase it might be that we have seen that specifying the evolution of all local observables of the lattice sites is enough to specify the evolution of all global observables. We can interpret that in the Heisenberg picture, the time evolution of observables gives the properties of a quantum system. We have seen that specifying all the local properties of a quantum mechanical system is enough to specify all the global properties exactly what one would expect in a local-realistic theory.

The qubit network descriptors are usually given as the extended Pauli observables on each lattice site. Notice that the first Pauli operator, the identity, remains invariant under unitary evolution. Thus, it is not necessary to specify its evolution. In most literature qubit descriptors are denoted by $\{\left(q_j^x, q_j^y, q_j^z\right)\}_{j=1}^N$. Such notation indicates that these operators satisfy the Pauli algebra but are global operators and do not necessarily have a fixed matrix representation since these can evolve and different basis may be chosen. We did the decomposition in the concrete Pauli basis, but any set of local observables that satisfies the Pauli algebra could have done the same trick. This representation would give us $3N$ descriptors for an N qubit network.

Nevertheless, there is still some redundancy. Given that the individual qubit descriptors are generators of the lie algebra $\mathfrak{su}(2)$ and are given by the algebraic relations

$$q_j^r \cdot q_j^s = i\epsilon_{rst} q_j^s + \delta_{rs}\mathbb{I} \tag{2.12}$$

Given these algebraic relations, deducing one of the three lattice site descriptors is always possible using the other two. Without loss of generality, we exemplify it by considering the q_j^y descriptor. We can see that $i\, q_j^x \cdot q_j^z = q_j^y$. Given the unitarity of evolution, indeed $\hat{U}^\dagger \cdot q_j^y \cdot \hat{U} = i \left(\hat{U}^\dagger \cdot q_j^x \cdot \hat{U}\right) \cdot \left(\hat{U}^\dagger \cdot q_j^z \cdot \hat{U}\right)$. So, we can say that q_j^y is redundant in our description. This redundancy is disposed of in some parts of the literature but not others. When disposed of, we require $2N$ descriptors for a system of N qubits, two descriptors per each qubit.

2.4.3 One descriptor per qubit

This subsection contains the main original idea of this Section 2.4. We show that we can reduce the number of descriptors even further.

We need to relax the requirement that descriptors are themselves observables. We can have a single extended local operator to deduce each lattice site's extended local observables q_j^x, q_j^z. Mathematically, this local operator can be written as $\hat{q}_j = \frac{1}{2}\left(q_j^x + i q_j^y\right)$.

First, let us observe that we can deduce the evolution of $\hat{q}_j^\dagger$ from the evolution of $\hat{q}_j$ by taking the Hermitian conjugation. Indeed, $\hat{U}^\dagger \cdot \hat{q}_j^\dagger \cdot \hat{U} = \left(\hat{U}^\dagger \cdot \hat{q}_j \cdot \hat{U}\right)^\dagger$. Furthermore, we can deduce q_j^x and q_j^z from $\hat{q}_j$ and $\hat{q}_j^\dagger$.

$$q_j^x = \hat{q}_j + \hat{q}_j^\dagger \qquad\qquad q_j^z = \hat{q}_j \cdot \hat{q}_j^\dagger - \hat{q}_j^\dagger \cdot \hat{q}_j \tag{2.13}$$

Since these relations are polynomial on $\hat{q}_j, \hat{q}_j^\dagger$ it is straightforward to observe that the unitary evolution of $\hat{q}_j$ specifies the unitary evolution of both q_j^x and q_j^z. Therefore, the unitary evolutions of the N elements $\hat{q}_j$ specify the unitary evolution of any observable of the N-qubit network.

By knowing the initial state of the system ρ_0 and knowing the expression of

$\left(\hat{U}^\dagger \cdot \hat{q}_1 \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{q}_N \cdot \hat{U}\right)$ one can recover all the expectation values of any observable in the current state of the system. In the Schrödinger picture, such phenomenal state would be given by $\hat{U}\rho_0\hat{U}^\dagger$. A few things to note are that each $\hat{U}^\dagger \cdot \hat{q}_j \cdot \hat{U}$ is a global N qubit operator so that it can have contributions in its expression from other lattice sites $j \neq i$. Nevertheless, the "updated" descriptors satisfy the qubit algebra of observables. In the sense of Equation 2.12.

However, even though having a single descriptor per qubit site is mathematically elegant, we may wonder if we are physically justified in using the operators $\hat{q}_j$ them not being physical observables of the qubit network.

We are for two main reasons. First, the goal of descriptors is to describe, minimally, the evolution of the algebra of observables referring to the local properties of the qubit network. Though they are not observables, each $\hat{q}_j$ is an element of each qubit's local algebra of operators extended to the global algebra of operators. In particular, when fixing the Pauli basis for q_j^x and q_j^z , the descriptors $\hat{q}_j$ become

$$\hat{q}_j = \bigotimes_{k=1}^{j-1} \mathbb{I} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \bigotimes_{r=j+1}^{N} \mathbb{I} \tag{2.14}$$

2.4.4 Physicality of qubit descriptors

Using $\hat{q}_j$, we can describe the global system referring to local elements in a minimal way, needing only one single algebraic element per qubit site. Physically, it is very relevant that the local observables of site j can be deduced using only the descriptors j as we have seen in Eq.2.13. It is straightforward to see that, in fact, the local observables of any subset of sites $\mathcal{M}$ can be specified by the collection $\{\hat{q}_j\}_{j\in\mathcal{M}}$. Another factor that may convince the reader that we should allow operators that are not observables to be descriptors is the fact that such operators

can be expressed as a complex linear combination of observables, as we have seen in the definition of $\hat{q}_j = \frac{1}{2}\left(q_j^x + iq_j^y\right)$.

In order to motivate more the physical relevance and nature of $\hat{q}_j$, we want to point out to the reader that such operators are nothing else than the lowering ladder operators for a qubit system. In other words, they correspond to the annihilation-like operators for a qubit network of the mode associated with the lattice site j. One can note this from the definition and Equation 2.14. Such operators are widely used in qubit network systems. They allow us to interpret a qubit network system from a perspective more in connection with condensed matter and systems of indistinguishable particles. Therefore, such a view is precious for the interests and goals of this thesis.

We want to note that the commutation relations of the qubit network annihilation operators $\hat{q}_j$ are exactly the same as an N mode strict hard-core boson lattice [57].

$$[\hat{q}_j, \hat{q}_k] = 0 \qquad [\hat{q}_j^\dagger, \hat{q}_k] = \delta_{jk}\left(2\hat{q}_j^\dagger\hat{q}_j - \mathbb{I}\right) \qquad (\hat{q}_j)^2 = 0 \tag{2.15}$$

Such observation provides a stronger case for using the $\hat{q}_j$ as descriptors as physically relevant. The point becomes that tracking the local annihilation operators for sites j of the strict hard-core boson system is enough to specify all the properties of the global system. This perspective lays the foundation for the whole thesis, where we will find matching results for bosons, fermions and 2D anyons.

2.5 Descriptors as local elements of reality

The curious reader may wonder how descriptors relate to the notion of local realism we have introduced in Section 2.3. The answer is inspired by the work of Bédard [51]. We can apply the RR construction specified in Subsection 2.3.3 to

the no-signalling operational theory framework for a qubit network described in Subsection 2.4.1. Two conditions must be checked that the operational theory with the subsystem lattice meets. The first is the no-signalling principle. It has been long known [30, 58] that the no-signalling principle is satisfied for finite-dimensional quantum mechanical systems bipartite by a tensor product. Such property is critical in understanding the EPR paper [13, 28, 59, 60] and for any analysis of quantum communications limitations [61].

The second is the well-posedness of the lattice of subsystems. The group of transformations must satisfy that if $\hat{U} \in \mathcal{ABC}$ is local on the subsystem AC and is local on BC, then it is necessarily local on C alone. Note that $\hat{U}$ is an operator. We can decompose $\hat{U}$ in terms of the extended generators of the local algebras of observables, the local descriptors $\hat{q}_A, \hat{q}_B, \hat{q}_C$ and their Hermitian conjugates. We note that $\hat{U}$ being local on AC is equivalent to saying that there is a decomposition where only $\hat{q}_A, \hat{q}_C, \hat{q}_A^\dagger, \hat{q}_C^\dagger$ appear. Similarly, being local on BC is equivalent to saying that there is a decomposition where only $\hat{q}_B, \hat{q}_C, \hat{q}_B^\dagger, \hat{q}_C^\dagger$ appear. Thus, if both claims are true, only the terms containing $\hat{q}_C, \hat{q}_C^\dagger$ can have non-zero weights. Therefore, it does imply that $\hat{U}$ is local on C, as desired.

Having both conditions met, we can apply the construction of the local ontic states or local elements of reality. Therefore doting qubit networks with a local-realistic structure. The local ontic states on the qubit j are given by $\left([\hat{U}]_j, \rho_0\right)$. One wonders, though, how does this structure relate to descriptors?

We claim that the set of evolved descriptors together with the initial Heisenberg state $\left(\left\{\hat{U}^\dagger \cdot \hat{q}_j \cdot \hat{U}\right\}_{j\in\mathcal{M}}, \rho_0\right)$ is a compact way to represent the local ontic states. Theorem 3 gives the crucial connection between qubit descriptors and their local ontic states provided by the equivalence classes.

Theorem 3. *The following equivalence holds for any subset of lattice sites $\mathcal{M}$ of*

an N qubit network.

$$\hat{U} \sim_{\mathcal{M}} \hat{V} \qquad \Longleftrightarrow \qquad \hat{U}^\dagger \cdot \hat{q}_j \cdot \hat{U} = \hat{V}^\dagger \cdot \hat{q}_j \cdot \hat{V} \quad \forall j \in \mathcal{M} \tag{2.16}$$

Thus, $[\hat{U}]_{\mathcal{M}} = \left\{ \hat{V} \in \mathcal{T}_{1,\ldots,N} \;\middle|\; \hat{U}^\dagger \cdot \hat{q}_j \cdot \hat{U} = \hat{V}^\dagger \cdot \hat{q}_j \cdot \hat{V} \quad \forall j \in \mathcal{M} \right\}$.

Proof. The last statement follows directly from the definition of an equivalence class, so the equation that needs to be proven is Equation 2.16:

"⇒": Remember $\mathcal{N} = \{1, \ldots, N\}$. $\hat{U} \sim_{\mathcal{M}} \hat{V}$ implies $\hat{U} = \hat{W}_{\mathcal{N}\setminus\mathcal{M}} \cdot \hat{V}$ for some $\hat{W}_{\mathcal{N}\setminus\mathcal{M}}$ being a unitary, local on the set of lattice sites that excludes all the sites $j \in \mathcal{M}$. Thus, since $\hat{W}_{\mathcal{M}\setminus\mathcal{M}}$ is an operator that does not contain any terms involving $\hat{q}_j, \hat{q}_j^\dagger$ for $j \in \mathcal{M}$, is straightforward to check that $[\hat{W}_{\mathcal{N}\setminus\mathcal{M}}, \hat{q}_j] = 0$ for all $j \in \mathcal{M}$. Therefore: $\hat{U}^\dagger \cdot \hat{q}_j \cdot \hat{U} = \hat{V}^\dagger \cdot \hat{W}^\dagger_{\mathcal{N}\setminus\mathcal{M}} \cdot \hat{q}_j \cdot \hat{W}_{\mathcal{N}\setminus\mathcal{M}} \cdot \hat{V} = \hat{V}^\dagger \cdot \hat{q}_j \cdot \hat{W}^\dagger_{\mathcal{N}\setminus\mathcal{M}} \cdot \hat{W}_{\mathcal{N}\setminus\mathcal{M}} \cdot \hat{V} = \hat{V}^\dagger \cdot \hat{q}_j \cdot \hat{V}$.

"⇐": We have that $\hat{U}^\dagger \cdot \hat{q}_j \cdot \hat{U} = \hat{V}^\dagger \cdot \hat{q}_j \cdot \hat{V}$ for all $j \in \mathcal{M}$. To see that $\hat{U} \sim_{\mathcal{M}} \hat{V}$ we need to see that $\hat{U} = \hat{W}_{\mathcal{N}\setminus\mathcal{M}} \cdot \hat{V}$. Equivalently, since we have a group structure where transformations are unitaries, proving that $\hat{U} \cdot \hat{V}^\dagger = \hat{W}_{\mathcal{N}\setminus\mathcal{M}}$ is enough to prove that $\hat{U} \sim_{\mathcal{M}} \hat{V}$.

From $\hat{U}^\dagger \cdot \hat{q}_j \cdot \hat{U} = \hat{V}^\dagger \cdot \hat{q}_j \cdot \hat{V}$ for all $j \in \mathcal{M}$ is straightforward to deduce that then $\hat{q}_j \cdot (\hat{U} \cdot \hat{V}^\dagger) = (\hat{U} \cdot \hat{V}^\dagger) \cdot \hat{q}_j$ for all $j \in \mathcal{M}$. Naming $\hat{U} \cdot \hat{V}^\dagger = \hat{W}$, noticing $\hat{W}$ is a unitary and taking the dagger of the previous equation, we have that the two following equalities hold:

$$\hat{W} \cdot \hat{q}_j = \hat{q}_j \cdot \hat{W} \qquad \hat{W} \cdot \hat{q}_j^\dagger = \hat{q}_j^\dagger \cdot \hat{W} \qquad \forall j \in \mathcal{M} \tag{2.17}$$

Moreover, now since $\hat{W}$ is a priori a general unitary, it is not difficult to see that we can decompose it as $\hat{W} = \hat{O}_0 + \hat{q}_{j_1}\hat{O}_1 + \hat{q}^\dagger_{j_1}\hat{O}_2 + \hat{q}_{j_1}\hat{q}^\dagger_{j_1}\hat{O}_3$ for $j_1 \in \mathcal{M}$. Where

$\hat{O}_0, \hat{O}_3, \hat{O}_1, \hat{O}_2$ are local operators on the set of lattice sites $\mathcal{N}\backslash\{j_1\}$. Using this decomposition of $\hat{W}$ in the first condition of Equation 2.17 and commuting the $\hat{q}_{j_1}, \hat{q}_{j_1}^\dagger$ terms with the $\hat{O}_k$ operators we obtain that:

$$\hat{q}_{j_1}(\hat{O}_0 + \hat{O}_3) - \hat{q}_{j_1}\hat{q}_{j_1}^\dagger\hat{O}_2 + \hat{O}_2 = \hat{q}_{j_1}\hat{O}_0 + \hat{q}_{j_1}\hat{q}_{j_1}^\dagger\hat{O}_2$$

Implying that $\hat{O}_2 = \hat{0}$ and $\hat{O}_3 = \hat{0}$. Then, using that $\hat{W} = \hat{O}_0 + \hat{f}_{j_1}\hat{O}_1$ and replacing in the second condition of Equation 2.17 we obtain:

$$\hat{q}_{j_1}^\dagger\hat{O}_0 + \hat{q}_{j_1}\hat{q}_{j_1}^\dagger\hat{O}_1 = \hat{q}_{j_1}^\dagger\hat{O}_0 + \hat{O}_1 - \hat{q}_{j_1}\hat{q}_{j_1}^\dagger\hat{O}_1$$

Therefore, $\hat{O}_1 = \hat{0}$. Thus, we have seen that the conditions imply that $\hat{W} = \hat{O}_0$, thus being a local unitary on the set of modes $\mathcal{N}\backslash\{j_1\}$. Because each of the conditions of Equation 2.17 for each $j \in \mathcal{M}$ is independent, the same reasoning can be followed exactly with the other lattice sites in $\mathcal{M}$ that are not j_1. Therefore the conditions imply that none of the sites in $\mathcal{M}$ appears in the decomposition of $\hat{W}$ in terms of qubit creation and annihilation operators. Therefore $\hat{W} = \hat{W}_{\mathcal{N}\backslash\mathcal{M}}$ is a local operator in the lattice sites $\mathcal{N}\backslash\mathcal{M}$, and therefore we have proven that $\hat{U} \sim_{\mathcal{M}} \hat{V}$. □

With Theorem 3, we have obtained a direct connection between the qubit descriptors and the equivalence classes of the local ontic states. We can define the equivalence classes that are the local elements of reality in terms of properties satisfied by the descriptors. Moreover, we can determine the equivalence class that gives the local ontic state in a subsystem in terms of the descriptors associated with that subsystem and the Heisenberg state, which plays the same reference role as the one in the local ontic states.

2.5.1 Ontic operations for descriptors

We want to identify the qubit descriptors with the Heisenberg state as an equivalent representation of the local ontic states. Such a presentation may be more convenient since it uses the evolution of the qubit annihilation operators as part of the local ontic states instead of abstract equivalence classes. This gives the local-realistic structure a more manageable and natural way to interpret local ontic states. Using descriptors, one can draw connections to quantum field theory, as we do in Chapter 3, and interpret the local ontic states as positing the realism of the creation operators of a quantum field.

To fully see how qubit descriptors and the Heisenberg state can represent local ontic states, we need to specify how the several operations at the ontic state space level operate on them.

2.5.1.1 Ontic group action $\star$

The action $\star$ of the groups of transformations $\mathcal{T}_{\mathcal{M}}$ on the ontic state spaces $\mathcal{R}_{\mathcal{M}}$, $\hat{W}_{\mathcal{M}} \star ([U]_{\mathcal{M}}, \rho_0)$ from 2.3 in the descriptor representation is given by

$$\hat{W}_{\mathcal{M}} \star \left(\left\{ \hat{U}^\dagger \cdot \hat{q}_j \cdot \hat{U} \right\}_{j \in \mathcal{M}}, \rho_0 \right) =$$
$$= \left(\left\{ \left(\mathbb{I}_{\mathcal{N} \setminus \mathcal{M}} \otimes \hat{W}^\dagger_{\mathcal{M}} \right) \cdot \hat{U}^\dagger \cdot \hat{q}_j \cdot \hat{U} \cdot \left(\mathbb{I}_{\mathcal{N} \setminus \mathcal{M}} \otimes \hat{W}_{\mathcal{M}} \right) \right\}_{j \in \mathcal{M}}, \rho_0 \right) \tag{2.18}$$

2.5.1.2 Ontic-phenomenal epimorphisms $\varphi_{\mathcal{M}}$

In order to analyse how the epimorphisms $\varphi_{\mathcal{M}}$ act on descriptors, it is instructive to see what are the orbits of the group action of $\mathcal{T}_{\mathcal{N}}$ on $\mathcal{P}_{\mathcal{N}}$. Since any unitary acting on the qubit network is a possible transformation of the system, given any phenomenal state ρ can be transformed to its diagonal form $\sigma_\rho = \hat{B}_\rho \cdot \rho \cdot \hat{B}^\dagger_\rho$ by

using physical transformations $\hat{B}_\rho \in \mathcal{T}_\mathcal{N}$.

Thus, we can fix the convention for the representatives of the orbits. Given the canonical computational basis $\{|0\cdots00\rangle, |0\cdots01\rangle, |0\cdots10\rangle, \dots\}$ of a qubit network, the representatives of the orbits are diagonal mixtures in the computational basis with non-increasing eigenvalues. They are given by the following matrices, which are associated with the computational basis:

$$\rho_0 = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{2^N} \end{pmatrix} \qquad \lambda_{j-1} \geq \lambda_j \ \ \forall j \in \{2, \dots, 2^N\} \tag{2.19}$$

Consider the critical case where the global state of the system is pure. Then, the pure states have a single orbit, and the representative is the first element of the computational basis, the phenomenal state $\rho_0 = |0\cdots0\rangle\langle0\cdots0|$. We see that for qubits, considering the global state of the network being pure allows us to completely fix by convention the reference state ρ_0 in the ontic states. The exact same reasoning is applied when using the regular Heisenberg picture in qubit networks [3, 51, 53, 62].

Being aware of the orbit structure of the phenomenal state space allows us to present Theorem 4, which is vital to understand the completeness of the descriptor picture and its relationship with Theorem 3, the group of transformations $\mathcal{T}_\mathcal{N}$ and its equivalence classes. Theorem 4 also helps to understand the definition of the epimorphism $\varphi_\mathcal{N}$ for a complete set of descriptors.

Theorem 4. *Using the complete set of descriptors* $\left(\hat{U}^\dagger \cdot \hat{q}_1 \cdot \hat{U}, \dots, \hat{U}^\dagger \cdot \hat{q}_N \cdot \hat{U}\right)$ *it is possible to uniquely find the* $\hat{U} \in \mathcal{T}_\mathcal{N}$ *that has evolved them from their canonical form* $(\hat{q}_1, \dots, \hat{q}_N)$.

Notice that the result is consistent with Theorem 3, considering that the equivalence

class $[\hat{U}]_{\mathcal{N}}$ consists of the single transformation group element with representative $\hat{U}$. The proof of this result is entirely algebraic. The idea is to decompose the unitary on an operator basis and see each basis element as a polynomial of the qubit descriptors and their adjoints. The complete proof is in Appendix A.

The construction of the global epimorphism $\varphi_{\mathcal{N}}$ is straightforward from this theorem.

$$\varphi_{\mathcal{N}}\left(\left(\hat{U}^{\dagger}\cdot\hat{q}_1\cdot\hat{U},\ldots,\hat{U}^{\dagger}\cdot\hat{q}_N\cdot\hat{U}\right),\rho_0\right)=\hat{U}\cdot\rho_0\cdot\hat{U}^{\dagger} \tag{2.20}$$

Consider a strict subsystem of the N qubit network, given by a subset of lattice sites $\mathcal{M}=\{j_1,\ldots,j_M\}$. We now present how from the ontic local state $\left(\left(\hat{U}^{\dagger}\hat{q}_{j_1}\hat{U}\ldots,\hat{U}^{\dagger}\hat{q}_{j_M}\hat{U}\right),\rho_0\right)$ we associate the phenomenal state $\rho_{\mathcal{M}}$, which is local density operator on $\mathcal{M}$. The idea is the same as for the proof of Theorem 4, which uses the local map between the Heisenberg and Schrödinger pictures. We can see that the local state $\rho_{\mathcal{M}}$ can be written as:

$$\begin{aligned}\rho_{\mathcal{M}}&=\sum_k \mathrm{Tr}\left(\hat{O}^{(k)}_{\mathcal{M}}\cdot\rho_{\mathcal{M}}\right)\ \hat{O}^{(k)}_{\mathcal{M}}=\\&=\sum_k \mathrm{Tr}\left(\hat{U}^{\dagger}\cdot\hat{O}^{(k)}_{\mathcal{M}}\otimes\mathbb{I}_{\mathcal{N}\setminus\mathcal{M}}\cdot\hat{U}\cdot\rho_0\right)\ \hat{O}^{(k)}_{\mathcal{M}}\end{aligned} \tag{2.21}$$

Since, $\hat{U}^{\dagger}\cdot\hat{O}^{(k)}_{\mathcal{M}}\otimes\mathbb{I}_{\mathcal{N}\setminus\mathcal{M}}\cdot\hat{U}$ can be calculated from the set of local descriptors $\left(\left(\hat{U}^{\dagger}\cdot\hat{q}_{j_1}\cdot\hat{U},\ldots,\hat{U}^{\dagger}\cdot\hat{q}_{j_M}\cdot\hat{U}\right),\rho_0\right)$, we can assign from our ontic local states the phenomenal local states through the epimorphism $\varphi_{\mathcal{M}}$ given by:

$$\begin{aligned}&\varphi_{\mathcal{M}}\left(\left(\hat{U}^{\dagger}\cdot\hat{q}_{j_1}\cdot\hat{U},\ldots,\hat{U}^{\dagger}\cdot\hat{q}_{j_M}\cdot\hat{U}\right),\rho_0\right)=\\&\quad=\sum_k \mathrm{Tr}\left(\hat{U}^{\dagger}\cdot\hat{O}^{(k)}_{\mathcal{M}}\otimes\mathbb{I}_{\mathcal{N}\setminus\mathcal{M}}\cdot\hat{U}\rho_0\right)\hat{O}^{(k)}_{\mathcal{M}}\end{aligned} \tag{2.22}$$

The complete details of the decomposition we use are in the proof of Theorem 4 in Appendix A.

2.5.1.3 Ontic projections $\pi^{\mathcal{R}}_{\mathcal{M}}$

At the ontic state space in the RR formalism to define the ontic projection mappings of the equivalence classes, the ontic projections are given by $\pi^{\mathcal{R}}_{A}\left([U]_{AB}\right) = [U]_{A}$. It is straightforward to see that it follows from theorem 3 that the ontic projections on a subsystem given by $\mathcal{M} = \{j_1, \ldots, j_M\}$ for qubit descriptors are:

$$\begin{aligned}\pi^{\mathcal{R}}_{\mathcal{M}}\left(\left(\hat{U}^{\dagger}\cdot\hat{q}_1\cdot\hat{U},\ldots,\hat{U}^{\dagger}\cdot\hat{q}_N\cdot\hat{U}\right),\rho_0\right) = \\ = \left(\left(\hat{U}^{\dagger}\cdot\hat{q}_{j_1}\cdot\hat{U},\ldots,\hat{U}^{\dagger}\cdot\hat{q}_{j_M}\cdot\hat{U}\right),\rho_0\right)\end{aligned} \tag{2.23}$$

2.5.1.4 Ontic join product $\odot$

Consider $A \subset \mathcal{M}$ and the associated bipartition $A|B$, with $B = \mathcal{M}\backslash A$, of the qubit network subsystem. Consider an ontic state of subsystem A represented with qubit descriptors, $\left(\left(\hat{U}^{\dagger}\cdot\hat{q}_{a_1}\cdot\hat{U},\ldots,\hat{U}^{\dagger}\cdot\hat{q}_{a_S}\cdot\hat{U}\right),\rho_0\right)$ and an ontic state of the disjoint subsystem B $\left(\left(\hat{V}^{\dagger}\cdot\hat{q}_{b_1}\cdot\hat{V},\ldots,\hat{V}^{\dagger}\cdot\hat{q}_{b_{M-S}}\cdot\hat{V}\right),\sigma_0\right)$

Then, we say that such a pair of local states are compatible if a global state $\left(\left(\hat{W}^{\dagger}\cdot\hat{q}_{j_1}\cdot\hat{W},\ldots,\hat{W}^{\dagger}\cdot\hat{q}_{j_M}\cdot\hat{W}\right),\tau_0\right)$ of the system $\mathcal{M} = AB$ exists, such that, when projected to subsystems A and B, it equals the states mentioned above.

Note that this means that for two states to be compatible, they have to have the same Heisenberg state and that there must exist a unitary transformation $\hat{W}$ such that $[\hat{W}]_A = [\hat{U}]_A$ and $[\hat{W}]_B = [\hat{V}]_B$. The ontic join product $\odot$ is defined for compatible local ontic states. For the descriptor form, the ontic join product looks like this:

$$\begin{aligned}\left(\hat{U}^{\dagger}\cdot(\hat{q}_{a_1},\ldots,\hat{q}_{a_S})\cdot\hat{U},\rho_0\right)\odot\left(\hat{V}^{\dagger}\cdot(\hat{q}_{b_1},\ldots,\hat{q}_{b_{M\text{-}S}})\cdot\hat{V},\rho_0\right) = \\ = \left(\left(\hat{W}^{\dagger}\hat{q}_{a_1}\hat{W},\ldots,\hat{W}^{\dagger}\hat{q}_{a_S}\hat{W},\hat{W}^{\dagger}\hat{q}_{b_1}\hat{W},\ldots,\hat{W}^{\dagger}\hat{q}_{b_{M\text{-}S}}\hat{W}\right),\rho_0\right)\end{aligned} \tag{2.24}$$

Of course, we can repeat the bipartition process until we reach the point of individual lattice sites forming subsystems. This ontic join product can be defined properly due to the demanded Separation property of the group of transformations (see Subsection 2.3.3 for more details).

Subsystem projections and the ontic-phenomenal epimorphism commute, see Figure 2.1. This is a particularly interesting result obtained from the representation of ontic states of a qubit network with descriptors.

Theorem 5. *The diagram of Figure 2.1 commutes. In other words:*

$$\begin{aligned}\pi_A^{\mathcal{P}}\left(\varphi_{\mathcal{M}}\left(\left(\left(\hat{U}^\dagger\cdot\hat{q}_{j_1}\cdot\hat{U},\ldots,\hat{U}^\dagger\cdot\hat{q}_{j_M}\cdot\hat{U}\right),\rho_0\right)\right)\right)=\\ =\varphi_A\left(\pi_A^{\mathcal{R}}\left(\left(\left(\hat{U}^\dagger\cdot\hat{q}_{j_1}\cdot\hat{U},\ldots,\hat{U}^\dagger\cdot\hat{q}_{j_M}\cdot\hat{U}\right),\rho_0\right)\right)\right)\end{aligned} \tag{2.25}$$

The proof of the theorem is algebraic. We apply the form of the epimorphism $\varphi_{\mathcal{M}}$ for descriptors that we have on Equation 2.22. We also use the definition of ontic projection $\pi_{\mathcal{M}}^{\mathcal{R}}$ that consists in forgetting the descriptors for the lattice sites that are not in $\mathcal{M}$. Furthermore, finally, we use the properties of $\pi_{\mathcal{M}}^{\mathcal{P}}$ being the usual partial trace for tensor product systems. The complete proof is in Appendix A.

2.5.2 Example

We consider the following circuits to exemplify the use of qubit descriptors and their interpretation. We have a 3-qubit network initialised at the reference phenomenal state $|000\rangle\langle 000|$. Such a state is unitarily evolved in two different ways that yield the same phenomenal output $\frac{1}{\sqrt{2}}\left(|00\rangle\,|\phi\rangle+|10\rangle\,|-\phi\rangle\right)$, see Figure 2.2. The single qubit pure states $|\phi\rangle\,,|-\phi\rangle$ stand for $\frac{1}{\sqrt{2}}\left(|0\rangle\pm e^{i\phi}\,|1\rangle\right)$ respectively. The phase ϕ is arbitrary, and we use it to track how the information of which phase has been applied flows and spreads throughout the network.

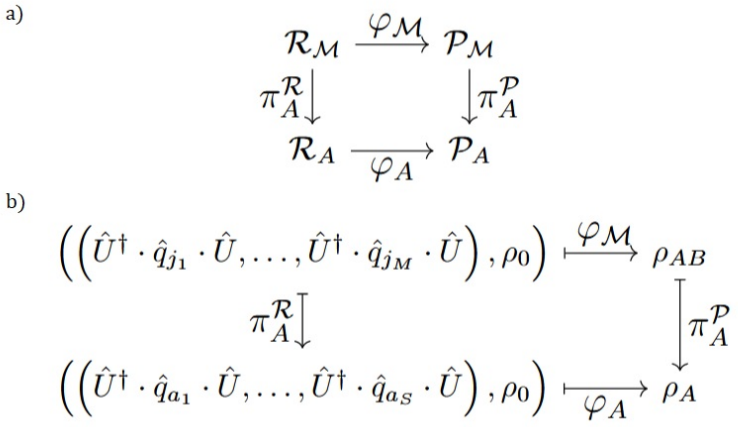


Figure 2.1: Commuting diagram that represents taking the projection into subsystems and the ontic-phenomenal epimorphism. Diagram a) represents the spaces, and diagram b) represents the action of the mappings in the descriptor picture.

Since the phenomenal state is the same at the end, all measurable physical system properties are identical for both situations. Nevertheless, we show how these correspond to two different ontic states. The fact that the phase gate $R_Z(\phi) = \left(\begin{smallmatrix}1 & 0\\ 0 & e^{i\phi}\end{smallmatrix}\right)$ has been applied to two different qubits ensures that the constitutional properties of these two setups might differ.

We can give a local-realistic account of the physical behaviour encoded in these two circuits. We use the Heisenberg picture and interpret the descriptors as local ontic states.

We first need to fix a system of references [63]. First, we need to fix a point in the phenomenal state space. For us, this is the state $|000\rangle\langle 000|$, which is the initial phenomenal state of the circuit. For pure states, it can be chosen by convention to be always $|000\rangle\langle 000|$. However, if we have a circuit initialised differently, we need to make an arbitrary choice of how $|000\rangle\langle 000|$ is unitarily evolved into such phenomenal state.

This choice would then be reflected in the second element of our system of ref-

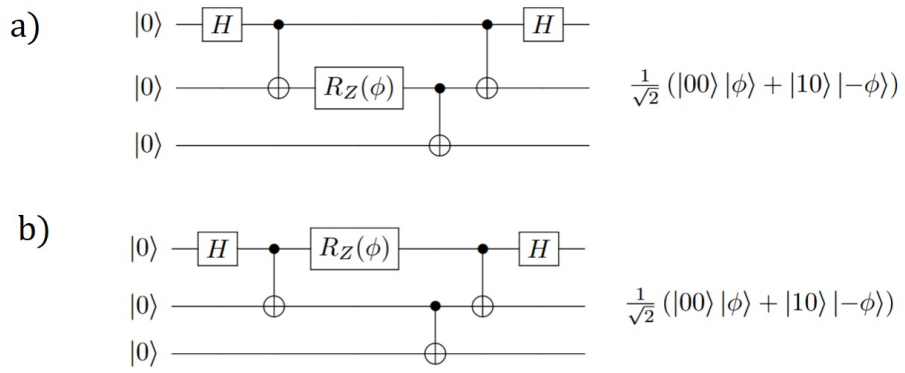


Figure 2.2: Circuit diagrams that yield the same pure entangled state in a 3-qubit network. The phase gate is applied in diagram $a)$ to the second qubit and in diagram $b)$ to the first.

erences, the representation of our initial descriptors. For convenience, we fix the initial representation of the three qubit descriptors to be the operators $\hat{q}_1, \hat{q}_2, \hat{q}_3$ in the computational basis. However, any three operators that satisfy the same algebraic relations of generating the local and global qubit algebra can be chosen as initial representations.

For the situations described in Figure 2.2, our systems of reference are the specification of the origin point $|000\rangle\langle 000|$ and the initial values of the descriptors $\hat{q}_1, \hat{q}_2, \hat{q}_3$. Their matrix representation in the computational basis is:

$$\mathcal{R}|_{\mathcal{C}} = \{\hat{q}_1, \hat{q}_2, \hat{q}_3, |000\rangle\langle 000|\} =$$
$$\left\{ \begin{pmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \right\} \tag{2.26}$$

These can be loosely imagined as fixing an origin and the three axes of rotation for three-dimensional space. However, we also choose the computational basis to

represent them in matrix form.

Once we have set up the reference system, we can see how the initial descriptors evolve and compare them to their original form. We remind the reader that the origin phenomenal state reference $|000\rangle\langle 000|$ remains invariant under time evolution. The chosen reference descriptors evolve as $\left(\hat{U}^\dagger \hat{q}_1 \hat{U}, \hat{U}^\dagger \hat{q}_2 \hat{U}, \hat{U}^\dagger \hat{q}_3 \hat{U}\right)$.

By construction, we see that local unitaries only affect the form of the local descriptors. One might be worried by the order of application of the unitaries. Nevertheless, the issue is resolved when writing the unitaries as functions of the descriptors. See [51] for the detailed practicalities of using descriptors.

Considering that the unitaries evolve the descriptors locally, we now restrict our attention to the final form of the evolved qubit descriptors. We are interested in seeing if the form of the descriptors is meaningfully different and how the parameter ϕ has spread over the system.

For circuits $a)$ and $b)$ from Figure 2.2, the ontic state of the systems with the descriptor formalism using the initial reference system is given by the following expressions, respectively. For diagram $a)$:

$$\left\{ \frac{1}{2} \begin{pmatrix} 0 & -i\sin(\phi) & 0 & 0 & 1 & \cos(\phi) & 0 & 0 \\ -i\sin(\phi) & 0 & 0 & 0 & \cos(\phi) & 1 & 0 & 0 \\ 0 & 0 & 0 & i\sin(\phi) & 0 & 0 & 1 & \cos(\phi) \\ 0 & 0 & i\sin(\phi) & 0 & 0 & 0 & \cos(\phi) & 1 \\ 1 & -\cos(\phi) & 0 & 0 & 0 & i\sin(\phi) & 0 & 0 \\ -\cos(\phi) & 1 & 0 & 0 & i\sin(\phi) & 0 & 0 & 0 \\ 0 & 0 & 1 & -\cos(\phi) & 0 & 0 & 0 & -i\sin(\phi) \\ 0 & 0 & -\cos(\phi) & 1 & 0 & 0 & -i\sin(\phi) & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0\,0 & 0 & \cos(\phi) & 0\,0 & 0 & i\sin(\phi) \\ 0\,0 & \cos(\phi) & 0 & 0\,0 & i\sin(\phi) & 0 \\ 0\,0 & 0 & 0 & 0\,0 & 0 & 0 \\ 0\,0 & 0 & 0 & 0\,0 & 0 & 0 \\ 0\,0 & 0 & i\sin(\phi) & 0\,0 & 0 & \cos(\phi) \\ 0\,0 & i\sin(\phi) & 0 & 0\,0 & \cos(\phi) & 0 \\ 0\,0 & 0 & 0 & 0\,0 & 0 & 0 \\ 0\,0 & 0 & 0 & 0\,0 & 0 & 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 0 & 1\,0 & 0 & 0 & 1\,0 & 0 \\ 1 & 0\,0 & 0 & -1\,0 & 0 & 0 \\ 0 & 0\,0 & 1 & 0 & 0\,0 & -1 \\ 0 & 0\,1 & 0 & 0 & 0\,1 & 0 \\ 0 & 1\,0 & 0 & 0 & 1\,0 & 0 \\ -1 & 0\,0 & 0 & 1 & 0\,0 & 0 \\ 0 & 0\,0 & -1 & 0 & 0\,0 & 1 \\ 0 & 0\,1 & 0 & 0 & 0\,1 & 0 \end{pmatrix}, \begin{pmatrix} 1\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0 \end{pmatrix} \right\} \tag{2.27}$$

For diagram b):

$$\left\{ \frac{1}{2}\begin{pmatrix} 0 & -i\sin(\phi) & 0 & 0 & 1 & \cos(\phi) & 0 & 0 \\ -i\sin(\phi) & 0 & 0 & 0 & \cos(\phi) & 1 & 0 & 0 \\ 0 & 0 & 0 & -i\sin(\phi) & 0 & 0 & 1 & \cos(\phi) \\ 0 & 0 & -i\sin(\phi) & 0 & 0 & 0 & \cos(\phi) & 1 \\ 1 & -\cos(\phi) & 0 & 0 & 0 & i\sin(\phi) & 0 & 0 \\ -\cos(\phi) & 1 & 0 & 0 & i\sin(\phi) & 0 & 0 & 0 \\ 0 & 0 & 1 & -\cos(\phi) & 0 & 0 & 0 & i\sin(\phi) \\ 0 & 0 & -\cos(\phi) & 1 & 0 & 0 & i\sin(\phi) & 0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \right.$$

$$\left. \frac{1}{2}\begin{pmatrix} 0&1&0&0&0&1&0&0 \\ 1&0&0&0&-1&0&0&0 \\ 0&0&0&1&0&0&0&-1 \\ 0&0&1&0&0&0&1&0 \\ 0&1&0&0&0&1&0&0 \\ -1&0&0&0&1&0&0&0 \\ 0&0&0&-1&0&0&0&1 \\ 0&0&1&0&0&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix} \right\} \tag{2.28}$$

We can observe how the descriptors of the third qubit have evolved in the same way for the two situations. We also observe that such evolution is not trivial since it involves factors present in the sector of qubits 1 and 2. By commuting the phase gate $R_Z(\phi)$ in a) with the $CNOT$ between qubits 2 and 3, we can see that up to the $CNOT$ that interacts with the third qubit, both circuits are the same. All posterior evolution from that point on is local on qubits 1 and 2. Thus, the elements of reality of qubit 3 are left invariant.

This picture allows us to understand the flow of interactions within the circuits. Both qubits 1 and 2 influence qubit 3. The descriptors of qubit 3 are the same for both circuits. It is not the case for the descriptors of qubits 1 and 2. Qualitatively, the two circuits are meaningfully different at the ontic level. We see how the phase gate parameter ϕ does not influence the elements of reality of qubit 2 in diagram b). However, it does so in diagram a).

Such considerations and analyses are impossible if we stay at the phenomenal level, where both final states are identical. Moreover, we see that the representation of the final Schrödinger state could suggest that the phase information encoded in ϕ has been transferred to the third qubit, from which it can be retrieved. Nevertheless, using the descriptor representation of the local-realistic structure, we have seen

that the phase information in both diagrams has not flown towards qubit 3. Instead, it has remained within qubits one and two, localised in qubit 1 for circuit b).

This example showcases the use of descriptors as ontic states, providing key insights and further understanding of the physical reality of quantum theory.

3 | Fermions and bosons

The contents of this Chapter 3 are an explanation of the publications [64, 65] done in collaboration with Chiara Marletto and Vlatko Vedral.

The standard model of particle physics consists of two main particle types, bosons and fermions. Fermions constitute matter, and bosons are force carriers. Notorious fermions include electrons and quarks, the matter constituents of atoms. Notorious bosons include the photon or the Higgs boson. In three spatial dimensions, boson and fermion statistics are the only particle statistics possible [66].

Quantum field theory formalises the notion of fermion and boson as a type of quantum field [67]. The quantum field is decomposed in creation and annihilation operators that satisfy either completely anticommuting or completely commuting commutation relations, thus discerning between fermions and bosons.

Fermions and bosons are also characterised as fundamental indistinguishable particles that have strict semi-integer and integer spin, respectively. The notorious spin-statistics theorem [68] gives the connection between the two notions. However, the characterisation of the commutation properties is better for our interests.

In this thesis, we analyse what representation the local elements of reality can have for indistinguishable particle systems and how to interpret and manipulate them. We see in Chapter 2 how these local elements can be represented by the qubit annihilation operators for qubit networks. Qubit networks are systems with very nice algebraic properties: finite-dimensional spaces, tensor products for composition and commutativity between lattice sites. We wonder if the excellent result is a consequence of these nice algebraic properties or has a deeper cause present in more exotic systems of indistinguishable particles.

In this chapter, we show how fermion and boson annihilation operators represent their respective local elements of reality. We do so by showing that fermionic and bosonic annihilation operators can be considered the fermionic and bosonic descriptors, respectively. Then we link the descriptors with the local ontic states of the RR formalism.

We find it crucial to study the local realism within these systems since fermionic and bosonic theories are the fundamental theories that explain phenomena in the laboratory. Concretely they are widely used in quantum optics to describe photons and in quantum chemistry and atomic physics to explain the electronic levels and interactions. These models deserve to showcase their local-realistic structure.

Furthermore, many results where Bell non-locality is shown [10, 16–19] are performed with single photons, whose description is given by the bosonic formalism we will treat. Thus, if we provide an Einstein local account of fermions and bosons, we will provide an Einstein local account of the Bell non-locality experiments.

Even though the conclusion is essentially the same for fermions and bosons, the path to reaching this significant result is quite different. Because fermions are more nuanced algebraically than bosons, we must tread carefully through the arguments, and we focus more on them throughout the chapter.

We study fermions and bosons in the context of quantum field theory in a discrete finite lattice. For fermions, their anticommutation relations have significant physical consequences. The most basic is that when two fermions are exchanged, a phase π is gathered, resulting in a $e^{i\pi} = -1$ factor. This has important repercussions for locality.

In particular, a paradox appears: when applying a local transformation in A, $U_A = \hat{f}_A + \hat{f}_A^\dagger$, to a local annihilation operator in B, it does not leave it invariant, $U_A^\dagger \hat{f}_B U_A = -\hat{f}_B \neq \hat{f}_B$. Therefore, in fermions, a change would happen in Bob's

local operators by Alice applying a local transformation. We would have action at a distance.

To resolve this paradox, we explain that to satisfy the no-signalling principle, it is necessary to impose the fermionic parity superselection rule (SSR) [69]. In Section 3.1, we showcase the fermionic formalism as an operational no-signalling theory from Subsection 2.3.1. We show how the parity SSR restricts the fermionic algebra of observables, the group of physical unitaries and the set of physical states of all fermionic systems.

These restrictions force us to delve deeper into the concepts of kinematical and physical spaces. In Section 3.2, we resolve the paradox and prove that fermionic annihilation operators represent the local elements of reality of fermionic systems. Interestingly, we conclude that the descriptors cannot be obtained through fermionic observables, which sparks a debate.

In Section 3.3, we briefly present the boson formalism. Without entering the mathematical details due to pedagogical reasons, we provide the rationale for why their annihilation operators represent their local elements of reality.

In Section 3.4, we use fermionic descriptors to explain interferometry through a fully local mechanism. We explain how phase acquisition can be fully tracked and described locally. We also show how, in the fermionic case, the full Dirac field observables can be used in an interferometer to represent the local features of phase acquisition.

3.1 Fermions as an operational theory

There are several formalisms for fermionic systems, which depend on one's area of work within theoretical physics. There is the Grassman variable approach [70]

for quantum field theory methods that use a path-integral approach. The canonical quantum field theoretic approach imposes conditions on quantum fields that solve the Dirac equation. Anticommutation relations are imposed on the Dirac field, and spinors are introduced and discussed together with the γ_μ matrices to obtain a relativistic fermionic field [71, 72].

In this work, we adopt the general information-theoretic formalism for fermions. This formalism is a simplified version of the canonical quantum field theoretic formalism but without considering the spinor component, the continuous space and the relativistic nature of fermions. This formalism captures all the relevant structural aspects of fermionic systems. Thus, it is ideal to discuss the information-theoretic and structural properties of the system. The formalism is used when discussing fermions in the fields of atomic physics, quantum many-body physics, quantum information and quantum foundations [56, 64, 73–96]

3.1.1 Kinematical space

In this thesis, we use the general information-theoretic formalism for fermions. We adhere to the position of studying locality in fermionic systems in terms of modes [56, 73–83], not particles [89–96]. We choose the mode perspective because it generalises the particle picture. It is based on the second quantisation and quantum field theory, generalising the first quantisation approach.

Within this approach, a fermionic system consists of a set of modes I. For simplicity, we take $I = \{1, \ldots, N\}$. Each mode $i \in I$ has fermionic creation and annihilation operators $\hat{f}_i^\dagger, \hat{f}_i$. The vacuum state $|\Omega\rangle$ is defined by the requirement that $\hat{f}_i |\Omega\rangle = 0$. The fermionic operator algebra is given by the creation and annihilation operators of all modes $i \in I$ obeying the following anticommutation

relations:

$$\{\hat{f}_i, \hat{f}_j\} = \hat{0} \qquad \{\hat{f}_i, \hat{f}_j^\dagger\} = \delta_{ij}\hat{\mathbb{I}} \qquad \text{where } \{\hat{A}, \hat{B}\} := \hat{A}\cdot\hat{B} + \hat{B}\cdot\hat{A} \tag{3.1}$$

To define a quantum system as a no-signalling operational theory, as in Subsection 2.4.1, we first need to identify the set of allowed physical observables $\mathcal{O}$. The set of allowed reversible physical transformations (unitaries) and the set of phenomenal states can be generated from $\mathcal{O}$. The form they take is $\mathcal{T} = \{e^{i\hat{A}} | \hat{A} \in \mathcal{O}\}/U(1)$ and $\mathcal{P} = \{\rho \in \mathcal{O} | \operatorname{Tr}(\rho) = 1, \rho \geq \hat{0}\}$. The quotient of $U(1)$ is a consequence of the global phase redundancy of quantum mechanics. This will be disregarded throughout the thesis because, for all practical purposes, we can work as if we have fixed a representative of the equivalence classes.

The usual presentation of fermionic systems relies on using the Fock space [75, 97]. The fermionic Fock space is generated by acting with all the possible combinations of creation operators on the vacuum state $|\Omega\rangle$. Given that the anticommutation relations for fermionic modes imply $(\hat{f}_i^\dagger)^2 = 0$ and $\hat{f}_i^\dagger \hat{f}_j^\dagger = -\hat{f}_j^\dagger \hat{f}_i^\dagger$, for a system of N fermionic modes, we can find a fermionic Fock space spanned by 2^N number states:

$$\mathcal{F}_I = \left\langle \hat{f}_{i_1}^\dagger \dots \hat{f}_{i_n}^\dagger |\Omega\rangle \right\rangle_{n=0, i_k \in I}^{N} \quad \text{such that} \quad i_1 < \dots < i_n, \tag{3.2}$$

Such Fock space is constructed as the direct sum of the fixed number subspaces, $\mathcal{F}_I = \bigoplus_{n=0}^N \mathcal{F}_I^{(n)}$. The spaces $\mathcal{F}_I^{(n)} = \left\langle \hat{f}_{i_1}^\dagger \dots \hat{f}_{i_n}^\dagger |\Omega\rangle \right\rangle_{i_k \in I}$ are of dimension $\binom{n}{N}$ and correspond to the spaces spanned by the states that contain n fermions. It is straightforward to check that all the fermionic Fock states are normalised using the anticommutation relations. We can easily see that if we construct the number operator $\hat{N} = \sum_{i \in I} \hat{f}_i^\dagger \hat{f}_i$, we obtain that $\hat{N}|\psi_n\rangle = n|\psi_n\rangle$ for all $|\psi_n\rangle \in \mathcal{F}_I^{(n)}$. We want the reader to notice that the Fock space is a Hilbert space, where the field over which it is a vector space is the complex numbers; thus, it contains any

superposition of the 2^N number states, which form an orthonormal basis of the Hilbert space.

We can identify the set of all linear operators that are endomorphisms of $\mathcal{F}_I$ as the set of linear operators that are sums and products of the creation and annihilation operators $\hat{f}_i^\dagger, \hat{f}_i$. This claim becomes straightforward once one notices that the linear map on $\mathcal{F}_I$, $|\Omega\rangle\langle\Omega|$, can be expressed as $|\Omega\rangle\langle\Omega| = \hat{f}_N \dots \hat{f}_1 \hat{f}_1^\dagger \dots \hat{f}_N^\dagger$.

Let us name the algebra of linear operators spanned by the creation and annihilation operators as $\mathcal{A}_I$. We know $\mathcal{A}_I$ is the set of linear endomorphisms in our Hilbert space. The algebra of operators $\mathcal{A}_I$ is in itself a Hilbert space, with the scalar product $\langle \hat{A}, \hat{B} \rangle = \mathrm{Tr}\left(\hat{A}^\dagger \cdot \hat{B}\right)$. The trace is derived from the scalar product of the original Hilbert space by $\mathrm{Tr}\left(|a\rangle\langle b| \cdot |c\rangle\langle d|\right) = \langle b|c\rangle \langle d|a\rangle$.

As we explain in Chapter 2, RR considers a lattice of subsystems as part of a no-signalling operational theory. A subsystem in our N mode fermionic lattice corresponds to choosing a subset $\mathcal{M}$ of M out of the N modes. A fermionic system is considered over the modes $\mathcal{M}$. Therefore, we construct the local algebra of operators $\mathcal{A}_\mathcal{M}$ for the set of modes $\mathcal{M}$ as the algebra over $\mathbb{C}$ spanned from polynomials of the creation and annihilation operators $\{\hat{f}_i, \hat{f}_i^\dagger\}_{i \in \mathcal{M}}$.

From the definition of $\mathcal{A}_\mathcal{M}$, we can choose the lattice of local algebras of observables $\mathcal{O}_\mathcal{M}$ in $\mathcal{M}$, being a local observable a local operator $\hat{O}_\mathcal{M} \in \mathcal{A}_\mathcal{M}$ such that $\hat{O}_\mathcal{M} = \hat{O}_\mathcal{M}^\dagger$. We identify this trivial choice as the notion of the kinematical space of observables, where every Hermitian local operator is considered a physical observable.

Under this choice, the set $\mathcal{T}_\mathcal{M}$ of local transformations in $\mathcal{M}$ is given by the local operators $\hat{U}_\mathcal{M} \in \mathcal{A}_\mathcal{M}$ such that $\hat{U}_\mathcal{M} \cdot \hat{U}_\mathcal{M}^\dagger = \mathbb{I}_\mathcal{M}$ up to a global complex phase $e^{i\phi}$, where $\mathbb{I}_\mathcal{M}$ is the product identity of the local algebra of operators $\mathcal{A}_\mathcal{M}$.

The set $\mathcal{P}_{\mathcal{M}}$ of local states in $\mathcal{M}$ is given by the local kinematical observables $\hat{\rho}_{\mathcal{M}} \in \mathcal{O}_{\mathcal{M}}$ such that $\hat{\rho}_{\mathcal{M}} \geq 0$ and $\mathrm{Tr}(\hat{\rho}_{\mathcal{M}}) = 1$. Thus, every unitary transformation is a valid physical transformation. Every density operator is an allowed physical phenomenal state.

These sets $\mathcal{P}_{\mathcal{M}}, \mathcal{T}_{\mathcal{M}}$, the associated usual group action of the unitaries on density operators $\hat{U} \cdot \rho \cdot \hat{U}^{\dagger}$, the fermionic partial trace (explained in Subsection 3.1.4) and the embedding of local observables (see Subsection 3.1.3) constitute the kinematical operational theory with a subsystem lattice for fermions.

3.1.2 Parity SSR

The kinematical operational theory would be a sensible choice for a no-signalling operational theory of fermions. Unfortunately, such a theory violates the no-signalling principle. The fermionic violation of the no-signalling principle was first pointed out by Wick and Wigner [69] and subsequently studied in the literature [56, 75, 76, 89]. See Subsection 3.1.5 for an explanation of why the kinematical observables may be used to violate no-signalling.

A minimal constraint is imposed on the fermionic observables to have a no-signalling operational theory of fermions. The restriction of observables then affects the physically allowed states and transformations. Such constraint is the parity superselection rule (SSR).

There are several superselection rules applied to different areas of physics. The name derives from the concept of selection rules that forbid transitions between energy levels in atomic theory [98]. The term "super" is added to emphasise that the constraint imposed is not due to the specific dynamics of the model under consideration (e.g. the system having a specific Hamiltonian that forbids states). It is a structural a priori constraint independent of specific dynamics. It is

necessary to define the structural features of the system being studied. In this case, the superselection rule is introduced to have fermions satisfy the no-signalling principle.

First introduced by Wick and Wigner [69], we apply the fermionic parity superselection rule. Parity does not refer to the parity symmetry (P symmetry) involving left and right-handedness that is usually discussed in quantum field theory. Rather, it refers to the parity of the number of fermions in a phenomenal state (if the number of fermions is even or odd). As an example, the vacuum state $|\Omega\rangle$ has the same parity as any $2k$ particle states such as $\hat{f}_1^\dagger \hat{f}_2^\dagger |\Omega\rangle$ or $\hat{f}_1^\dagger \hat{f}_3^\dagger \hat{f}_4^\dagger \hat{f}_5^\dagger |\Omega\rangle$. However, a different parity than any $2k+1$ particle state such as $\frac{1}{\sqrt{2}}(\hat{f}_1^\dagger + \hat{f}_3^\dagger) |\Omega\rangle$ or $\hat{f}_1^\dagger \hat{f}_2^\dagger \hat{f}_3^\dagger |\Omega\rangle$.

We introduce the parity observable $\hat{\mathbb{P}} = e^{i\pi \hat{N}}$ and unitary. $\hat{\mathbb{P}}$ is a Hermitian operator that can be diagonalised by the number states basis, with eigenvalue $+1$ for number states with an even number of particles and -1 for number states with an odd number of particles.

The parity SSR is implemented by restricting the set of states, observables and unitaries to produce the corresponding set of physically allowed states, observables and unitaries for a given subsystem:

$$\mathcal{O}_{\mathcal{M}}^{phys} = \left\{ \hat{O} \,\middle|\, \hat{O} \in \mathcal{A}_{\mathcal{M}},\ \hat{O} = \hat{O}^\dagger,\ \left[\hat{\mathbb{P}}, \hat{O}\right] = 0 \right\} \subset \mathcal{O}_{\mathcal{M}} \tag{3.3}$$

$$\begin{aligned} \mathcal{T}_{\mathcal{M}}^{phys} &= \left\{ \hat{U} \,\middle|\, \hat{U} \in \mathcal{A}_{\mathcal{M}},\ \hat{U} \cdot \hat{U}^\dagger = \mathbb{I},\ \left[\hat{\mathbb{P}}, \hat{U}\right] = 0 \right\} / U(1) = \\ &= \left\{ e^{i\hat{O}} \,\middle|\, \hat{O} \in \mathcal{O}_{\mathcal{M}}^{phys} \right\} / U(1) \subset \mathcal{T}_{\mathcal{M}} \end{aligned} \tag{3.4}$$

$$\mathcal{P}_{\mathcal{M}}^{phys} = \left\{ \rho \,\middle|\, \rho \in \mathcal{O}_{\mathcal{M}}^{phys},\ \mathrm{Tr}(\rho) = 1,\ \rho \geq 0 \right\} \subset \mathcal{P}_{\mathcal{M}} \tag{3.5}$$

Using the physical observables to generate the set of local transformations and phenomenal states yields the expressions above. Notice that the resulting sets of states, observables and transformations are strict subsets of the kinematical sets

we previously defined with the usual definitions in standard quantum mechanics. One could check that $\mathcal{T}_{\mathcal{M}}^{phys}$ is a subgroup of $\mathcal{T}_{\mathcal{M}}$ and $\mathcal{O}_{\mathcal{M}}^{phys}$ is a subalgebra of $\mathcal{O}_{\mathcal{M}}$. Nevertheless, $\mathcal{P}_{\mathcal{M}}^{phys}$ is just a subset of its kinematical counterpart. The boundary of the convex sets $\mathcal{P}_{\mathcal{M}}^{phys}$ are pure states that are rays of the Hilbert space $\mathcal{F}_{\mathcal{M}}$. However, it is worth noting that the physically allowed pure states that are the boundary of $\mathcal{P}_{\mathcal{M}}^{phys}$ are not the rays of a subvector space of the kinematical Hilbert space. Throughout the chapter, we use $\hat{U} \in \mathcal{T}_{\mathcal{M}}^{phys}$ as a shortcut for saying that $\hat{U}$ is one of the representatives of its associated equivalence class in $\mathcal{T}_{\mathcal{M}}^{phys}$. We deal with the global phase redundancy of the representatives a posteriori.

A physically allowed pure state $|\psi\rangle$ under the parity SSR takes the form of either being a normalised linear combination of even number states $\left\{\hat{f}_{k_1}^{\dagger} \dots \hat{f}_{k_{2r}}^{\dagger} |\Omega\rangle\right\}$ or odd number states $\left\{\hat{f}_{k_1}^{\dagger} \dots \hat{f}_{k_{2t+1}}^{\dagger} |\Omega\rangle\right\}$ (where $k_l \in \mathcal{M}$, $0 \leq 2r \leq M$, $1 \leq 2t+1 \leq M$ with $r, t \in \mathbb{N}$). The pure physical states are either even states or odd states. The physical transformations $\mathcal{T}_{\mathcal{M}}^{phys}$ cannot change such property. An even state remains even, and an odd state remains odd. The conservation of parity has the effect that the pure states in a fermionic physical theory have two orbits. This is a significant difference from the qubit case in Section 2.4. Because of this fermionic property, we dropped the RR postulate of global phenomenal transitivity in Subsection 2.3.3.

The physical observables, transformations and phenomenal states are local algebraic operators. They can be expressed as a polynomial of the creation and annihilation operators. The effect of the parity SSR is to label as non-physical the polynomial expressions with any monomial with an odd degree. As a result, all physical operators are polynomials with all monomials of an even degree.

We define the reordered number state basis as the number state basis with the even states shifted. The first 2^{N-1} elements are even states, and the last 2^{N-1} are odd.

For a single mode i, in the reordered number state basis $\{|\Omega\rangle\,, \hat{f}_i^\dagger\,|\Omega\rangle\}$ the local matrix representations[1] of the parity and even operators is

$$\hat{\mathbb{P}} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \qquad \hat{A} = \begin{pmatrix} a & \\ & b \end{pmatrix} \qquad \text{where } a, b \in \mathbb{C} \tag{3.6}$$

For two modes, i, j, in the reordered number state basis $\{|\Omega\rangle\,, \hat{f}_i^\dagger \hat{f}_j^\dagger\,|\Omega\rangle\,, \hat{f}_i^\dagger\,|\Omega\rangle\,,$ $\hat{f}_j^\dagger\,|\Omega\rangle\}$ the local matrix representations of the parity and even operators is

$$\hat{\mathbb{P}} = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & -1 & 0 \\ & & 0 & -1 \end{pmatrix} \quad \hat{A} = \begin{pmatrix} a_1 & a_2 & & \\ a_3 & a_4 & & \\ & & b_1 & b_2 \\ & & b_3 & b_4 \end{pmatrix} \qquad \text{where } a_l, b_t \in \mathbb{C} \tag{3.7}$$

This block-diagonal structure appears for any mode number. Although the kinematical Hilbert space of an N mode fermionic system is isomorphic to the Hilbert space of an N qubit network, the spaces of physical operators differ significantly, with the block diagonal structure appearing. Imposing the parity SSR so that fermions satisfy the no-signalling principle has profound structural consequences.

Physically, the meaning of the parity SSR is that it only allows superpositions of states with the same number parity. Let us provide more intuition for the parity SSR. Recall that fermions are systems of particles of semi-integer spin; in other words, the spin of a fermion is $\frac{2m-1}{2}$. Consider a number of states of n fermions. We can now wonder what type of effective particle this global system would behave as if we looked from very far away. If n is even, the spin of the global configuration of fermions will be an integer, behaving like a boson. Nevertheless, if n is odd, the total spin of the global configuration will be a semi-integer, thus behaving like a fermionic particle.

In this context, the parity SSR may be interpreted in the sense that superpositions

[1] All empty entries are zeros

between bosons and fermions cannot exist. Moreover, it precludes the transformation of a globally bosonic fermion configuration to a globally fermionic fermion configuration and vice-versa. This perspective shows that the parity SSR is natural since we only forbid coherence and transmutation between different particle types. This view also allows us to see the fermionic parity SSR as a particular case of the anyonic SSR introduced in Section 4.1.

The action of $\mathcal{T}_{\mathcal{M}}^{phys}$ onto $\mathcal{P}_{\mathcal{M}}^{phys}$ is the usual $\hat{U} \cdot \rho \cdot \hat{U}^{\dagger}$. A no-signalling operational theory requires that the notion of phenomenal projection, the embedding of the local transformations onto the global transformation space, the no-signalling principle and the separation property be satisfied. After describing fermions' structural properties, we will see how SSR observables allow the no-signalling principle to be satisfied, whereas the kinematical observables do not.

3.1.3 Embedding of local observables

The lack of a tensor product structure for fermions raises a question. How are the local observables and local transformations of $\mathcal{M}$ embedded into a larger subsystem $\mathcal{M}'$ that contains it $\mathcal{M} \subset \mathcal{M}'$?

Let us remind ourselves that $\mathcal{A}_{\mathcal{M}}$ is the algebra over $\mathbb{C}$ generated by the creation and annihilation operators $\{\hat{f}_j, \hat{f}_j^{\dagger}\}_{j \in \mathcal{M}}$. A local operator in $\mathcal{M}$ is a polynomial $\hat{A}_{\mathcal{M}} = p\left(\{\hat{f}_j, \hat{f}_j^{\dagger}\}\right)$ for $j \in \mathcal{M}$. The same polynomial $p(\{\hat{f}_j, \hat{f}_j^{\dagger}\})$ also represents an operator local in the larger system of modes $\mathcal{M}'$. Thus, we have $\hat{A}_{\mathcal{M}}^{ext} = p\left(\{\hat{f}_j, \hat{f}_j^{\dagger}\}\right)$ since all $j \in \mathcal{M}$ are in $\mathcal{M}'$ also. Therefore, this is the natural embedding of local operators and, thus, of observables and transformations for both the kinematical and SSR choice of observables.

Despite seeming a remarkably trivial embedding, it is not so much when operators are expressed as linear maps of the kinematical Hilbert space. Let us show a local

operator in $\mathcal{M}$ and its extension to $\mathcal{M}' = \mathcal{M} \cup \{j_\star\}$, decomposed in the reordered number state basis:

$$\hat{A}_{\mathcal{M}} = \sum_{\vec{r},\vec{s}} a_{\vec{r},\vec{s}} \left(\hat{f}^\dagger_{m_1}\right)^{s_1} \dots \left(\hat{f}^\dagger_{m_M}\right)^{s_M} |\Omega\rangle\langle\Omega| \, \hat{f}^{r_M}_{m_M} \dots \hat{f}^{r_1}_{m_1} \tag{3.8}$$

$$\begin{aligned}\hat{A}^{ext}_{\mathcal{M}} = &\sum_{\vec{r},\vec{s}} a_{\vec{r},\vec{s}} \left(\hat{f}^\dagger_{m_1}\right)^{s_1} \dots \left(\hat{f}^\dagger_{m_M}\right)^{s_M} |\Omega\rangle\langle\Omega| \, \hat{f}^{r_M}_{m_M} \dots \hat{f}^{r_1}_{m_1} + \\ &+ \sum_{\vec{r},\vec{s}} a_{\vec{r},\vec{s}} \left(\hat{f}^\dagger_{m_1}\right)^{s_1} \dots \left(\hat{f}^\dagger_{m_M}\right)^{s_M} \hat{f}^\dagger_{j_\star} |\Omega\rangle\langle\Omega| \, \hat{f}_{j_\star} \hat{f}^{r_M}_{m_M} \dots \hat{f}^{r_1}_{m_1}\end{aligned} \tag{3.9}$$

where $s_i, r_j \in \{0, 1\}$ and $a_{\vec{r},\vec{s}} \in \mathbb{C}$.

The expression looks contradictory with the trivial extension until one notices that the terms $|\Omega\rangle\langle\Omega|$ are not the same in both expressions 3.8 & 3.9. The operator $|\Omega\rangle\langle\Omega|$ contains the information of the modes that are part of its system. In the first expression $|\Omega\rangle\langle\Omega| = \hat{f}_{m_M} \dots \hat{f}_{m_1} \hat{f}^\dagger_{m_1} \dots \hat{f}^\dagger_{m_M}$. Meanwhile, in the second $|\Omega\rangle\langle\Omega| = \hat{f}_{j_\star} \hat{f}_{m_M} \dots \hat{f}_{m_1} \hat{f}^\dagger_{m_1} \dots \hat{f}^\dagger_{m_M} \hat{f}^\dagger_{j_\star}$. These operators differ since one is local in $\mathcal{M}$ and the other is not.

Equation 3.9 can also be related to the usual extension given in systems with a tensor product. The extension can be seen as the sum of the local operator tensor[2] each orthonormal basis state density operator of the ancillary system (i.e. $\hat{A}^{ext}_{\mathcal{M}} = \sum_i \hat{A} \otimes |i\rangle\langle i|$). This comparison shows that the trivial extension makes sense physically.

3.1.4 Partial trace

For any no-signalling operational theory, we need phenomenal projection maps $\pi^{\mathcal{P}}_{\mathcal{M}} : \mathcal{P}_{\mathcal{M}'} \to \mathcal{P}_{\mathcal{M}}$ for $\mathcal{M} \subset \mathcal{M}'$. In quantum theories, these maps are given by partial tracing the subsystems $\mathcal{M}' \backslash \mathcal{M}$.

[2]For fermions there is a different, antisymmetrised tensor product [75]

Given the kinematical and SSR choices of sets of local observables, states and transformations, we could expect two different partial tracing procedures. The partial trace can be defined uniquely as the unique linear map $\mathrm{Tr}_{\mathcal{M}'\setminus\mathcal{M}} : \mathcal{P}^{phys}_{\mathcal{M}'} \to \mathcal{P}^{phys}_{\mathcal{M}}$ such that for any local observable $\hat{O}_{\mathcal{M}} \in \mathcal{O}^{phys}_{\mathcal{M}}$ and its unique extension to the global space $\hat{O}^{ext}_{\mathcal{M}} \in \mathcal{O}^{phys}_{\mathcal{M}'}$ and for any global state $\rho \in \mathcal{P}^{phys}_{\mathcal{M}'}$ it obeys the equation

$$\mathrm{Tr}\left(\hat{O}^{ext}_{\mathcal{M}} \cdot \rho\right) = \mathrm{Tr}\left(\hat{O}_{\mathcal{M}} \cdot \mathrm{Tr}_{\mathcal{M}'\setminus\mathcal{M}}\left(\rho\right)\right) \tag{3.10}$$

In this case, it has been shown [75] that for both fermionic SSR and kinematical choices, the partial tracing procedure is the same. The partial trace is given by tracing out each mode individually, and for each fermionic mode, the procedure is as follows:

$$\begin{aligned}\mathrm{Tr}_{m_i}\left(\left(\hat{f}^{\dagger}_1\right)^{s_1} \dots \left(\hat{f}^{\dagger}_{m_i}\right)^{s_{m_i}} \dots \left(\hat{f}^{\dagger}_N\right)^{s_N} |\Omega\rangle\langle\Omega|\, \hat{f}^{r_N}_N \dots \hat{f}^{r_{m_i}}_{m_i} \dots \hat{f}^{r_1}_1\right) = \\ = \delta_{s_{m_i} r_{m_i}} (-1)^k \left(\hat{f}^{\dagger}_1\right)^{s_1} \dots \left(\hat{f}^{\dagger}_N\right)^{s_N} |\Omega\rangle\langle\Omega|\, \hat{f}^{r_N}_N \dots \hat{f}^{r_1}_1,\end{aligned} \tag{3.11}$$

where $k = \sum_{j=m_i}^{N-1} s_j s_{j+1} + r_j r_{j+1}$ and s_i, r_j take the value 0 or 1. In the second line, the creation and annihilation operators of the mode m_i do not appear.

Notice that due to the anticommutation of the fermionic operators, some terms gather a -1 phase. Such a phase has serious consequences. It is shown in [56] that despite an N fermionic kinematical Hilbert space being isomorphic to a N qubit network, the isomorphism cannot be faithful with respect to the partial tracing procedures. Therefore, fermions and qubits are truly different systems. Fermionic systems must be considered in their own right, not within a qubit network system.

3.1.5 No-signalling

We are ready to showcase that the choice of kinematical observables violates the no-signalling principle. We need to showcase an example where signalling can occur by using fermionic kinematical transformations, observables, and states.

Consider a two-mode fermionic system 12. Consider the phenomenal pure global state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(\mathbb{I} + \hat{f}_1^\dagger\right)|\Omega\rangle$. We apply the fermionic partial trace of mode 2 to this pure phenomenal state $|\psi\rangle$. We obtain the local phenomenal state for mode 1 being the pure state $\frac{1}{\sqrt{2}}\left(\mathbb{I} + \hat{f}_1^\dagger\right)|\Omega\rangle$. Consider that in mode 2 the local kinematical unitary $\hat{U} = \hat{f}_2 + \hat{f}_2^\dagger$ is applied. We obtain the pure phenomenal global transformed state $\hat{U}|\psi\rangle = \frac{1}{\sqrt{2}}\left(\hat{f}_2^\dagger + \hat{f}_2^\dagger\hat{f}_1^\dagger\right)|\Omega\rangle$. The local state for mode 1 is the pure state $\frac{1}{\sqrt{2}}\left(\mathbb{I} - \hat{f}_1^\dagger\right)|\Omega\rangle$, taking now the fermionic partial trace over mode 2 . This local state is orthogonal to the original local state for mode 1 by performing only a local transformation on mode 2. Thus, mode 2 could signal to mode 1 by performing only a local transformation. Information could be transmitted instantaneously without mode 1 and mode 2 interacting.

The above is a kinematical fermionic protocol that violates the no-signalling principle. It shows that the following equations do not hold when using kinematical fermionic observables, transformations and phenomenal states. Consider a subsystem of a set of fermionic modes $\mathcal{M} \subset \mathcal{M}'$, the fermionic no-signalling principle is that the following equations are satisfied:

$$\mathrm{Tr}_{\mathcal{M}}\left(\hat{U}_{\mathcal{M}} \cdot \rho_{\mathcal{M}'} \cdot \hat{U}_{\mathcal{M}}^\dagger\right) = \mathrm{Tr}_{\mathcal{M}}\left(\rho_{\mathcal{M}'}\right) \tag{3.12}$$

Due to the failure to satisfy the above equations, the kinematical choice of observables cannot provide a no-signalling operational theory for fermions.

It has been shown [75] the necessity to restrict the set of physically allowed

observables, states and transformations to the parity SSR-respecting operators to avoid violating the no-signalling principle. If we choose parity SSR observables, we can provide a no-signalling operational theory of fermions.

Observe that SSR observables that are local in disjoint sets of modes A and B, $\hat{O}_A, \hat{L}_B$, commute. The observables can be expressed as polynomials on the creation and annihilation operators of the disjoint sets of modes, with all the monomial terms of even degrees.

Thus, when applying the anticommutation relations for disjoint modes, we always get an even number of accumulated -1 phases due to the anticommutation of the individual creation and annihilation operators. An even number of accumulated -1 phases is always a total phase of $+1$, thus making the disjoint observables commute.

Having all disjoint observables commute (i.e. microcausality in field theory [88]) is the reason behind the no-signalling principle being satisfied for parity SSR observables [75]. The detailed proof that parity SSR observables lead to the satisfaction of the no-signalling principle is in Appendix B.

The second crucial condition to be met is the Separation property of the lattice of subsystems transformation groups $\mathcal{T}^{phys}_{\mathcal{M}}$. It is required that for any three disjoint composable subsystems A, B, C, the local transformations and their embeddings respect that if $V = U_{AC}$ is a local physical transformation on AC and $V = W_{BC}$ is also a local physical transformation on BC, then necessarily $V = T_C$ is a local transformation in C. Fermionic transformations in mode subsystems satisfy this property.

A transformation $\hat{V}$ in ABC is local on AB if it can be expressed as a polynomial of creation and annihilation operators of A and B alone. Therefore the conditions of

being local in AC and in BC imply that $\hat{V} = p\left(\{\hat{f}_{a_i}, \hat{f}^\dagger_{a_i}\}_i, \{\hat{f}_{c_k}, \hat{f}^\dagger_{c_k}\}_k\right)$ and $\hat{V} = q\left(\{\hat{f}_{b_j}, \hat{f}^\dagger_{b_j}\}_j, \{\hat{f}_{c_k}, \hat{f}^\dagger_{c_k}\}_k\right)$. Equating the two and considering that the annihilation and creation operators are independent of each other algebraically, one concludes that necessarily the components in $p()$ that contain A terms must vanish or can be grouped to form the identity operator $\mathbb{I}$. $\mathbb{I}$ can be expressed in terms of modes in C alone via $\hat{f}_{c_k}\hat{f}^\dagger_{c_k} + \hat{f}^\dagger_{c_k}\hat{f}_{c_k} = \mathbb{I}$.

Therefore, we have seen how choosing the parity SSR observables for fermionic theory can be regarded as a no-signalling operational theory. Thus, we can apply the RR formalism described in Chapter 2 to find the local ontic states that make fermions a local-realistic theory.

3.1.6 Local-tomography

By imposing the parity SSR, we have succeeded in describing fermionic systems as no-signalling operational theories. Nevertheless, such constraint breaks some properties we are used to having in distinguishable quantum mechanics.

One such property is local tomography. Local tomography has been regarded as a key property of quantum systems [43, 99, 100] in the program of reconstructing quantum theory from informational principles. Local tomography is the property by which measuring only local observables in coordination between the parties is enough to characterise the global phenomenal state of the system fully. In other words, given an N-partite quantum system, knowing all $\langle \hat{O}_1 \dots \hat{O}_N \rangle$ for all the $\hat{O}_j$ local observables, it is enough to characterise the phenomenal state completely.

For fermionic systems under the parity SSR, local tomography is not satisfied. A simple example of this failure is a two-mode fermionic system 1 2.

Consider the global phenomenal pure states $|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|\Omega\rangle + \hat{f}^\dagger_1\hat{f}^\dagger_2|\Omega\rangle\right)$ and

$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|\Omega\rangle - \hat{f}_1^\dagger \hat{f}_2^\dagger |\Omega\rangle\right)$. The SSR local observables can all be written as $\hat{A} = a_1 \hat{f}_1 \hat{f}_1^\dagger + a_2 \hat{f}_1^\dagger \hat{f}_1$ and $\hat{B} = b_1 \hat{f}_2 \hat{f}_2^\dagger + b_2 \hat{f}_2^\dagger \hat{f}_2$; as seen in Equation 3.7. Calculating $\langle \hat{A} \cdot \hat{B} \rangle_{\psi_1} = \frac{a_1 b_1 + a_2 b_2}{2}$ and $\langle \hat{A} \cdot \hat{B} \rangle_{\psi_2} = \frac{a_1 b_1 + a_2 b_2}{2}$. Since all expectation values are the same for all local observables, it is impossible to locally distinguish the states $|\psi_1\rangle$ and $|\psi_2\rangle$. This violates local tomography.

In the following sections, we explore how to define fermionic descriptors considering the superselected fermionic system, and we explore the interpretation subtleties that arise due to the parity SSR. In particular, we will see how the violation of local tomography will play an important role in interpreting fermionic descriptors in Subsection 3.2.6.

3.2 Fermionic descriptors

In this Section 3.2, we analyse what mathematical objects represent the fermionic descriptors and how they are representatives of the local ontic states of the RR construction [4]. We obtain results analogous to the qubit case described in Section 2.5. *In this section 3.2, we adapt the main results published in [65] in collaboration with Chiara Marletto and Vlatko Vedral.*

The first goal of the section is to find the notion of fermionic descriptors with the same equivalences to the RR formalism. The other goals consist in characterising the behaviour and properties of fermionic descriptors.

In the previous Section 3.1, we have seen how fermions are a no-signalling operational theory when applying the parity SSR. The transformations groups are $\mathcal{T}_{\mathcal{M}}^{phys}$. The phenomenal state spaces are $\mathcal{P}_{\mathcal{M}}^{phys}$. The group action is the usual $\hat{U} \cdot \rho \cdot \hat{U}^\dagger$. The phenomenal projections are the fermionic partial traces. The extension of local transformations in global spaces is the trivial expression when expressing the

transformations as polynomials of creation and annihilation operators.

As we have explained in Section 2.5, the paper [51] shows how for qubit networks, descriptors from [3] and the equivalence class formalism from [4, 5] are equivalent.

Section 2.3 shows that the equivalence classes $[U]_A$ are the core of the local ontic states. The equivalence relation $\sim_A$ is over $\mathcal{T}_{\mathcal{S}}$ and given by $U \sim_A V$ if and only if exists a transformation $W_{\bar{A}}$ local on the subsystem $\bar{A} = \mathcal{S} \backslash A$, such that $U = W_{\bar{A}} V$. The set of all the equivalence classes $[U]_A$ is the local ontic state space $\mathcal{R}_A$, with the maximal space $\mathcal{R}_{\mathcal{S}} = \mathcal{T}_{\mathcal{S}}$.

Section 2.5 shows that qubit descriptors $\{\hat{q}_j\}_{j=1}^N$ satisfy that $\hat{U} \sim_j \hat{V}$ iff $\hat{U}^\dagger \cdot \hat{q}_j \cdot \hat{U} = \hat{V}^\dagger \cdot \hat{q}_j \cdot \hat{V}$. Such a property provides a different characterisation of the equivalence classes since it provides a different mechanism to understand that U and V are equivalent in j.

The key reason that qubit ladder operators are qubit descriptors is that $\hat{q}_j, \hat{q}_j^\dagger$ are the generators of the local operator algebra. Following the exact same argument, the fermionic annihilation operators are fermionic descriptors (see Appendix B for the full details). Knowing the unitary evolution of $\{\hat{f}_j\}_{j \in \mathcal{M}}$ guarantees to know the evolution of any SSR observable within $\mathcal{M}$.

We claim that, similarly to the qubit network case, the set of evolved descriptors together with the initial Heisenberg state $\left(\left\{ \hat{U}^\dagger \cdot \hat{f}_j \cdot \hat{U} \right\}_{j \in \mathcal{M}}, \rho_0 \right)$ is a compact way to represent the local ontic states.

The fermionic analogue of Theorem 3, Theorem 6 gives us the same crucial connection between fermionic descriptors and fermionic local ontic states.

Theorem 6. *The following equivalence holds for any subset of modes $\mathcal{M}$ of an N*

mode fermionic system.

$$\hat{U} \sim_{\mathcal{M}} \hat{V} \quad \Longleftrightarrow \quad \hat{U}^\dagger \cdot \hat{f}_j \cdot \hat{U} = \hat{V}^\dagger \cdot \hat{f}_j \cdot \hat{V} \quad \forall j \in \mathcal{M},\ \hat{U}, \hat{V} \in \mathcal{T}_{\mathcal{N}}^{phys} \quad (3.13)$$

Thus, $[\hat{U}]_{\mathcal{M}} = \left\{ \hat{V} \in \mathcal{T}_{\mathcal{N}}^{phys} \;\middle|\; \hat{U}^\dagger \cdot \hat{f}_j \cdot \hat{U} = \hat{V}^\dagger \cdot \hat{f}_j \cdot \hat{V} \quad \forall j \in \mathcal{M} \right\}$.

Given the commuting properties the parity SSR dotes disjoint observables, the proof of Theorem 6 is analogous to the proof of Theorem 3, up to some very minor subtleties. The complete fermionic proof is in Appendix B.

With Theorem 6, we obtain a direct connection between the fermionic descriptors and the ontic states of the equivalence class formalism. The set of evolved descriptors with the initial Heisenberg state $\left(\left\{ \hat{U}^\dagger \cdot \hat{f}_j \cdot \hat{U} \right\}_{j \in \mathcal{M}}, \rho_0 \right)$ is a compact representation of the local ontic states. We express all the ontic operations of Subsection 2.3.3 in terms of the fermionic descriptors.

3.2.1 Ontic group action $\star$

The action $\star$ of the groups of transformations $\mathcal{T}_{\mathcal{M}}^{phys}$ on the ontic state spaces $\mathcal{R}_{\mathcal{M}}^{phys}$, in the fermionic descriptor representation is the following.

$$\begin{aligned} \hat{W}_{\mathcal{M}} \star \left(\left(\hat{U}^\dagger \hat{f}_{j_1} \hat{U}, \ldots, \hat{U}^\dagger \hat{f}_{j_M} \hat{U} \right), \rho_0 \right) = \\ = \left(\left(\hat{W}_{\mathcal{M}}^\dagger \hat{U}^\dagger \hat{f}_{j_1} \hat{U} \hat{W}_{\mathcal{M}}, \ldots, \hat{W}_{\mathcal{M}}^\dagger \hat{U}^\dagger \hat{f}_{j_M} \hat{U} \hat{W}_{\mathcal{M}} \right), \rho_0 \right) \end{aligned} \quad (3.14)$$

where $\hat{W}_{\mathcal{M}}$ in the second line is the extended local fermionic operator on the maximal system $\mathcal{N}$. We remind that such extension is trivial when expressing the operator as a polynomial of fermionic annihilation and creation operators.

3.2.2 Ontic-phenomenal epimorphisms $\varphi_{\mathcal{M}}$

The phenomenal state space's orbit representatives can be diagonal in the reordered number state basis. Exactly as in the qubit networks case exposed in Subsubsection 2.5.1.2. In matrix representation, the representatives are of the form:

$$\rho_0 = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_{2^{N-1}} & & & \\ & & & \mu_1 & & \\ & & & & \ddots & \\ & & & & & \mu_{2^{N-1}} \end{pmatrix} \quad \lambda_{j-1} \geq \lambda_j,\ \mu_{j-1} \geq \mu_j \ \ \forall j \in \{2, \ldots, 2^{N-1}\} \tag{3.15}$$

Nevertheless, the orbit structure is significantly different from the qubit network scenario. The parity SSR restriction of unitaries does not allow transformations from the even sector to the odd sector. Concretely, this implies we have two distinct orbits in the pure case. We can fix the convention up to the global sector. Thus, by convention, when the global phenomenal state is pure, it is either $|\Omega\rangle\langle\Omega|$ if even, or $\hat{f}_1^\dagger |\Omega\rangle\langle\Omega| \hat{f}_1$ if odd.

After discussing the orbit structure of the phenomenal state space, we are ready to introduce the fermionic analogue of Theorem 4: Theorem 7. Theorem 7 allows one to understand better the definition of the fermionic epimorphisms $\varphi_{\mathcal{M}}$.

Theorem 7. *Using* $\left(\hat{U}^\dagger \cdot \hat{f}_1 \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{f}_N \cdot \hat{U}\right)$*, the complete set of fermionic descriptors, it is possible to uniquely find the transformation* $\hat{U} \in \mathcal{T}_{\mathcal{N}}^{phys}$ *that has evolved them from their canonical form* $\left(\hat{f}_1, \ldots, \hat{f}_N\right)$.

Notice that the result is consistent with Theorem 6, considering that the equivalence class consists of the single element $\hat{U}$. The proof of this result is entirely algebraic and analogous to the qubit case, up to the fermionic and parity SSR subtleties. The complete proof is in Appendix B.

From Theorem 7, the definition of $\varphi_{\mathcal{N}}$ follows naturally.

$$\varphi_{\mathcal{N}}\left(\left(\hat{U}^\dagger\cdot\hat{f}_1\cdot\hat{U},\ldots,\hat{U}^\dagger\cdot\hat{f}_N\cdot\hat{U}\right),\rho_0\right)=\hat{U}\cdot\rho_0\cdot\hat{U}^\dagger \tag{3.16}$$

Let us focus on the strict subsystem mappings $\varphi_{\mathcal{M}}$. The strategy is to use the local map between the Heisenberg and Schrödinger pictures. We use the decomposition of the Schrödinger state $\rho_{\mathcal{M}}$:

$$\rho_{\mathcal{M}}=\sum_{\vec{r},\vec{s}}c_{\vec{r},\vec{s}}\left(\hat{f}_{j_1}^\dagger\right)^{r_1}\ldots\left(\hat{f}_{j_M}^\dagger\right)^{r_M}|\Omega\rangle\langle\Omega|\left(\hat{f}_{j_M}\right)^{s_M}\ldots\left(\hat{f}_{j_1}\right)^{s_1} \tag{3.17}$$

$$c_{\vec{r},\vec{s}}=\mathrm{Tr}\left(\hat{U}^\dagger\cdot\left(\hat{f}_{j_1}^\dagger\right)^{r_1}\ldots\left(\hat{f}_{j_M}^\dagger\right)^{r_M}\hat{f}_{j_M}\ldots\hat{f}_{j_1}\hat{f}_{j_1}^\dagger\ldots\hat{f}_{j_M}^\dagger\left(\hat{f}_{j_M}\right)^{s_M}\ldots\left(\hat{f}_{j_1}\right)^{s_1}\cdot\hat{U}\cdot\rho_0\right)$$

where the components of $\vec{r},\vec{s}$ are either 0 or 1. The number of ones in $\vec{r}$ equals the number in $\vec{s}$ modulus 2, summing over even operators only. In the first line, $|\Omega\rangle\langle\Omega|=\hat{f}_{j_M}\ldots\hat{f}_{j_1}\hat{f}_{j_1}^\dagger\ldots\hat{f}_{j_M}^\dagger$. It is precisely this decomposition that ensures that from the descriptor evolution, one can deduce the coefficients $c_{\vec{r},\vec{s}}$. Since the ontic state representation also holds ρ_0, we can make the assignment:

$$\begin{aligned}&\varphi_{\mathcal{M}}\left(\left(\hat{U}^\dagger\cdot\hat{f}_{j_1}\cdot\hat{U},\ldots,\hat{U}^\dagger\cdot\hat{f}_{j_M}\cdot\hat{U}\right),\rho_0\right)=\\&=\sum_{\vec{r},\vec{s}}c_{\vec{r},\vec{s}}\left(\hat{f}_{j_1}^\dagger\right)^{r_1}\ldots\left(\hat{f}_{j_M}^\dagger\right)^{r_M}|\Omega\rangle\langle\Omega|\left(\hat{f}_{j_M}\right)^{s_M}\ldots\left(\hat{f}_{j_1}\right)^{s_1}\end{aligned} \tag{3.18}$$

where again

$$c_{\vec{r},\vec{s}}=\mathrm{Tr}\left(\hat{U}^\dagger\cdot\left(\hat{f}_{j_1}^\dagger\right)^{r_1}\ldots\left(\hat{f}_{j_M}^\dagger\right)^{r_M}\hat{f}_{j_M}\ldots\hat{f}_{j_1}\hat{f}_{j_1}^\dagger\ldots\hat{f}_{j_M}^\dagger\left(\hat{f}_{j_M}\right)^{s_M}\ldots\left(\hat{f}_{j_1}\right)^{s_1}\cdot\hat{U}\cdot\rho_0\right)$$

3.2.3 Ontic projections $\pi_{\mathcal{M}}^{\mathcal{R}}$

It follows from Theorem 6 that the ontic projections for fermionic descriptors are:

$$\pi_{\mathcal{M}}^{\mathcal{R}}\left(\left(\hat{U}^\dagger\cdot\hat{f}_1\cdot\hat{U},\ldots,\hat{U}^\dagger\cdot\hat{f}_N\cdot\hat{U}\right),\rho_0\right)=$$

$$= \left(\left(\hat{U}^\dagger \cdot \hat{f}_{j_1} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{f}_{j_M} \cdot \hat{U} \right), \rho_0 \right) \tag{3.19}$$

Completely equivalent to the qubit case in Subsubsection 2.5.1.3.

3.2.4 Ontic join product $\odot$

Let us consider $A \subset \mathcal{M}$ and the associated bipartition $A|B$ of the fermionic subsystem with $B = \mathcal{M} \backslash A$. Consider the two compatible fermionic ontic states of subsystems A and B:

$$\left(\left(\hat{U}^\dagger \cdot \hat{f}_{a_1} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{f}_{a_S} \cdot \hat{U} \right), \rho_0 \right) \quad \left(\left(\hat{V}^\dagger \cdot \hat{f}_{b_1} \cdot \hat{V}, \ldots, \hat{V}^\dagger \cdot \hat{f}_{b_{M\text{-}S}} \cdot \hat{V} \right), \rho_0 \right) \tag{3.20}$$

Since they are compatible local ontic states, there must be a maximal unitary $\hat{W} \in \mathcal{T}_{\mathcal{N}}^{phys}$ such that $\hat{W} = \hat{R}^{ext}_{\mathcal{N}\backslash A} \hat{U}$ and $\hat{W} = \hat{T}^{ext}_{\mathcal{N}\backslash B} \hat{U}$. For two compatible ontic states, we define the fermionic ontic join product $\odot$ as:

$$\begin{aligned} &\left(\hat{U}^\dagger \cdot \left(\hat{f}_{a_1}, \ldots, \hat{f}_{a_S} \right) \cdot \hat{U}, \rho_0 \right) \odot \left(\hat{V}^\dagger \cdot \left(\hat{f}_{b_1}, \ldots, \hat{f}_{b_{M\text{-}S}} \right) \cdot \hat{V}, \rho_0 \right) = \\ &= \left(\left(\hat{W}^\dagger \hat{f}_{a_1} \hat{W}, \ldots, \hat{W}^\dagger \hat{f}_{a_S} \hat{W}, \hat{W}^\dagger \hat{f}_{b_1} \hat{W}, \ldots, \hat{W}^\dagger \hat{f}_{b_{M\text{-}S}} \hat{W} \right), \rho_0 \right) \end{aligned} \tag{3.21}$$

The uniqueness of this product is guaranteed by the Separation property satisfied by the groups of transformations $\mathcal{T}_S$. Fermionic unitaries under the parity superselection rule satisfy the Separation property, as it is seen in Subsection 3.1.5.

3.2.4.1 Faithfulness of splitting operation

Of course, we can repeat the bipartition process until we reach the point of individual modes forming subsystems. We see how the local and global states are on the same footing. A particular excellent result that one can obtain from the representation of ontic states with descriptors is that splitting into subsystems is a

faithful operation. Concretely, the diagram in Figure 3.1 commutes. Theorem 8 condenses the information in Figure 3.1.

a)

$$\begin{array}{ccc} \mathcal{R}_{\mathcal{M}} & \xrightarrow{\varphi_{\mathcal{M}}} & \mathcal{P}_{\mathcal{M}}^{phys} \\ \pi_A^{\mathcal{R}} \downarrow & & \downarrow \pi_A^{\mathcal{P}} \\ \mathcal{R}_A & \xrightarrow[\varphi_A]{} & \mathcal{P}_A^{phys} \end{array}$$

b)

$$\begin{array}{ccc} \left(\left(\hat{U}^\dagger \cdot \hat{f}_{j_1} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{f}_{j_M} \cdot \hat{U}\right), \rho_0\right) & \xmapsto{\varphi_{\mathcal{M}}} & \rho_{AB} \\ \pi_A^{\mathcal{R}} \downarrow & & \downarrow \pi_A^{\mathcal{P}} \\ \left(\left(\hat{U}^\dagger \cdot \hat{f}_{a_1} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{f}_{a_S} \cdot \hat{U}\right), \rho_0\right) & \xmapsto[\varphi_A]{} & \rho_A \end{array}$$

Figure 3.1: Commuting diagram that represents taking the projection into subsystems and the ontic-phenomenal epimorphism. Diagram a) represents the spaces, and diagram b) represents the action of the mappings in the descriptor picture.

Theorem 8. *The diagram of Figure 3.1 commutes. In other words:*

$$\begin{aligned} &\pi_A^{\mathcal{P}}\left(\varphi_{\mathcal{M}}\left(\left(\left(\hat{U}^\dagger \cdot \hat{f}_{j_1} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{f}_{j_M} \cdot \hat{U}\right), \rho_0\right)\right)\right) = \\ &= \varphi_A\left(\pi_A^{\mathcal{R}}\left(\left(\left(\hat{U}^\dagger \cdot \hat{f}_{j_1} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{f}_{j_M} \cdot \hat{U}\right), \rho_0\right)\right)\right) \end{aligned} \tag{3.22}$$

The proof of the theorem is algebraic. We apply the form of the epimorphism φ_A for descriptors that we have on Equation 3.18. We also use the definition of ontic projection $\pi_A^{\mathcal{R}}$ that consists in forgetting the descriptors for the modes that are not in A. Furthermore, finally, we use the properties of $\pi_A^{\mathcal{P}}$ being the fermionic partial trace. Moreover, we use some algebraic properties of fermionic systems under the parity superselection rule derived in previous works [75]. The complete proof is in Appendix B.

Having seen all the ontic operations, we can claim that fermions under the parity

SSR have a local-realistic structure. We have found a neat representation of the fermionic ontic states explicitly using the widely known Heisenberg picture of quantum mechanics. Furthermore, we have seen that tracking the creation or annihilation operators is enough to have a complete description of the ontic states of the fermionic system.

3.2.5 The fermionic action at a distance paradox

We have shown that fermionic annihilation operators are the fermionic descriptors that can represent the local ontic states, seeing that fermionic theory is local-realistic.

If one is not careful, though, when presented with this claim may come up with the following prima facie paradox:

Paradox. ***Fermionic action at a distance.*** *Consider the fermionic annihilation as the fermionic descriptors that represent the local ontic states (or local elements of reality). Consider two fermionic position modes A and B representing two distant space points. The local ontic state in A is given by $\left(\hat{f}_A, \rho_0\right)$. We see that if in mode B the local unitary $\hat{U}_B = \hat{f}_B + \hat{f}_B^\dagger$ is applied, then the local ontic state in A becomes*

$$\left(\hat{U}_B^\dagger \cdot \hat{f}_A \cdot \hat{U}_B, \rho_0\right) = \left(\left(\hat{f}_B + \hat{f}_B^\dagger\right) \cdot \hat{f}_A \cdot \left(\hat{f}_B + \hat{f}_B^\dagger\right), \rho_0\right) = \left(-\hat{f}_A, \rho_0\right) \tag{3.23}$$

Therefore, the local elements of reality in A have changed instantaneously when applying a local unitary in B. Therefore we have a local-realistic theory that portrays action at a distance.

This prima facie paradox is natural to appear when just analysing the statement that fermionic annihilation operators act as descriptors and represent the local ontic states.

Let us introduce two points to resolve this confusion. First, a fermionic theory is only expected to be local-realistic if it is no-signalling. RR's theorem (explained in 2.3.3) applies only to no-signalling operational theories. It does not apply to theories that portray signalling.

The second and final point is to observe that the local unitary $\hat{U}_B = \hat{f}_B + \hat{f}_B^\dagger$ in B is a unitary that violates the parity SSR restriction. It is straightforward to check since, in its polynomial decomposition, the monomials are all of degree 1. Therefore it is, in fact, an odd unitary and not an even unitary.

The fact that $\hat{U}_B$ is not an allowed physical transformation under the parity SSR is relevant since we had to impose the parity SSR to guarantee that fermions satisfy the no-signalling principle. Thus, using $\hat{U}_B$ to show no action at a distance is not surprising since we expect that we can have signalling by using it.

Therefore the prima facie paradox resolves by simply stating that the unitary used is not physically allowed since it allows fermions to signal. Moreover, in the following lines, we show that if one uses any allowed local physical transformation in B, the fermionic descriptors in A are left invariant, as expected.

The initial local ontic state in A is $\left(\hat{f}_A, \rho_0\right)$, applying a parity SSR unitary $\hat{V}_B$ local in B, we obtain the updated local ontic states in A being $\left(\hat{V}_B^\dagger \cdot \hat{f}_A \cdot \hat{V}_B, \rho_0\right)$. However, $\hat{V}_B$ and $\hat{f}_A$ commute. Since $\hat{V}_B$ is even, the accumulated phase of anticommuting $\hat{f}_A$ throughout all the terms of $\hat{V}_B$ is always $+1$.

Thus, the updated local ontic state in A becomes $\left(\hat{V}_B^\dagger \cdot \hat{V}_B \cdot \hat{f}_A, \rho_0\right) = \left(\hat{f}_A, \rho_0\right)$. Proving that no action at a distance can occur under the parity SSR and considering fermionic annihilation operators as descriptors and representatives of the local ontic states.

3.2.6 Are fermionic descriptors physical?

This chapter shows how fermionic annihilation operators can be used as representatives of the local ontic states, doting fermionic theory of a convenient local-realistic structure. The use of the annihilation operators is inspired by the use of qubit ladder operators as qubit network descriptors.

To have no-signalling and thus local realism in fermions, we have seen the importance of imposing the parity SSR. This rule restricts being physical or physically allowed to even operators. Nevertheless, we observe that the fermionic annihilation operators are not even operators. They are odd fermionic operators. Therefore, due to the parity SSR, we cannot find the annihilation operators as a physical fermionic observable, any linear combination of them, or any physical unitary or phenomenal state.

Considering this impossibility, one can question whether fermionic annihilation operators are physical. We discuss our thoughts on this challenging question in the fermionic case.

The first point we want to raise is the impossibility of any set of even observables to be fermionic descriptors, thus representing local ontic states. Consider two sets of local operators $\{\hat{d}_A^{(j)}\}_j, \{\hat{d}_B^{(k)}\}_k$ in A and B, being disjoint subsystems. To be descriptors, they need to fulfil the condition that knowing their evolution in time is enough to retrieve the evolution of any global observable. This requirement is to encode global observations as features of the local parts.

Such a simple condition cannot be met. The violation of local tomography (seen in Subsection 3.1.6) in fermionic SSR systems impedes any subsets of local observables from tracking all global observables. The parity SSR restricts the physicality of local observables too much, so there are global features that coordinated local

measurements cannot capture. In any theory that violates local tomography, the descriptors cannot be sets of local observables.

The second point is that the term physical may have two different meanings when applying the Leibniz rule [37, 47] become one. The term "physical" may refer to the ontic level, as there is a physical embodiment in the objective outside reality of the mathematical element being considered. The second meaning may refer to the observable properties in the physical objective outside reality of the mathematical element being considered. If one assumes the Leibniz rule to be true, it is not meaningful to discern the two since one expects all physical embodiment to have an observable effect.

If we wonder whether fermionic annihilation operators are physical in the observable sense, the answer is no. These operators are not needed in the operational theory, so they are considered non-physical. The observables $(\hat{f}_j + \hat{f}_j^\dagger), i(\hat{f}_j - \hat{f}_j^\dagger)$ that could retrieve the annihilation operators are also deemed unphysical by the parity SSR. Nevertheless, this does not prevent us from thinking there is an ontic embodiment of the annihilation operator in physical reality. Such embodiment happens when referring to the reality of the fermionic Dirac field and its conjugate.

By believing that quantum fields exist in physical reality, one could hold the position that annihilation operators have an indirect physical embodiment; moreover, they embody the local-realistic structure of fermionic theory in physical reality. This is a reasonable possibility, allowing for unification when considering the status of the quantum electromagnetic fields.

These arguments and claims might not convince the reader. The reader could contend that to consider annihilation operators physical, they need a corresponding element in the operational theory. In such a case, we could point out that one needs fermionic annihilation operators in fermionic information theory to act as Kraus

operators [75]. Kraus operators could not be applied to all parity SSR-allowed quantum channels without fermionic annihilation operators.

The reader could reply that interpreting Kraus operators in correspondence with physical reality poses too many challenges. Then, the reader can take the local ontic state descriptor representation as a convenient compact mathematical representation of the local ontic states given by the equivalence classes in the RR formalism.

The equivalence classes are defined only in terms of physically allowed parity SSR operators, concretely parity SSR unitaries. The grouping of physical elements should be deemed as physical. Thus, despite not being convinced that fermionic annihilation operators are physical, we hope to convince the reader that, at minimum are mathematical representations of a structure of physical operators.

3.3 Bosons

After thoroughly examining the fermionic case, let us briefly introduce the boson formalism. We do not conduct the extensive analysis we did in the fermionic case. The main reason is that the bosonic case has no substantial complications. It is equivalent to the qubit network scenario explained in detail in Sections 2.4 & 2.5.

We introduce bosons in a finite lattice setting. This setting is commonly used in quantum many-body systems and quantum information. Similarly to fermions, we follow the mode perspective on bosonic systems and work within second quantisation.

Consider a bosonic system with a finite set of modes $I = \{1, \ldots, N\}$. Each mode j has associated bosonic creation and annihilation operators $\hat{b}_j^\dagger, \hat{b}_j$. There is a bosonic vacuum state $|0\rangle$, defined as the only state that all the annihilation operators annihilate $\hat{b}_j |0\rangle = 0$.

The defining commutation relations of bosonic creation and annihilation operators are

$$[\hat{b}_j, \hat{b}_k] = \hat{0} \qquad [\hat{b}_j, \hat{b}_k^\dagger] = \delta_{jk}\mathbb{I} \tag{3.24}$$

where $[\hat{A}, \hat{B}] = \hat{A} \cdot \hat{B} - \hat{B} \cdot \hat{A}$. These commutation relations are similar to those in the qubit network case and correspond with hard-core bosons in Equation 2.15. The only difference is that bosons do not satisfy $\left(\hat{b}_j\right)^2 = \hat{0}$ nor $[\hat{b}_j^\dagger, \hat{b}_j] = 2\hat{b}_j^\dagger\hat{b}_j - \mathbb{I}$. These two conditions are added to bosons to transform them into hard-core bosons. This is done in order to have a bosonic system that portrays a Pauli exclusion principle.

Such constraints are sometimes desirable to model more realistically massive bosonic systems. Another good reason is to avoid working with an infinite dimensional Hilbert space. The infinite dimensionality is a consequence of being able to generate orthogonal states by applying $\hat{b}_j^\dagger$ indefinitely. More precisely, the bosonic Hilbert space is given by the bosonic Fock space.

The bosonic Fock space is the span of the states obtained by applying the bosonic creation operators to the vacuum state $|0\rangle$. Following this construction, the commutation relations give [3]

$$\mathcal{F}_I = \left\langle \frac{\left(\hat{b}_1^\dagger\right)^{n_1} \cdots \left(\hat{b}_N^\dagger\right)^{n_N}}{\sqrt{n_1!} \cdots \sqrt{n_N!}} |0\rangle \right\rangle_{n_1 \ldots n_N \in \mathbb{N}} \tag{3.25}$$

These Fock states constitute an orthonormal basis of the Hilbert space. $N_T = n_1 + \cdots + n_N$ is the number of bosons in each Fock basis state.

Because of the infinite dimensionality of the Hilbert space, there are some mathematical subtleties in the scalar's product convergence. We will not concentrate on

[3] Here, we consider $\mathbb{N}$ to have 0 as an element.

them, as they are not interesting for our purposes. Instead, we shall only consider states that are finite sums of elements in the Fock space. This allows us to work with systems with a finite maximum total number of particles.

We can now focus on the boson operators that provide the bosonic observables and unitaries. The bosonic operator algebra on the set of modes I, $\mathcal{A}_I$, can be seen as generated by the bosonic creation and annihilation bosonic operators. The bosonic algebra of observables $\mathcal{O}_I$ can be defined as the self-adjoint elements of $\mathcal{A}_I$. Correspondingly, the group of transformations $\mathcal{T}_I$ is given by [4] the operators $\mathcal{A}_I$ that are unitary. The system's phenomenal states $\mathcal{P}_I$ can be considered finite probabilistic mixtures of finite superpositions of elements in the bosonic Fock basis.

Extending the bosonic operators into a larger set of modes is trivial. They retain their form when expressed in terms of the bosonic creation and annihilation operators. Similarly, the associated partial tracing procedure has no complications, as indicated by the equation:

$$\mathrm{Tr}_j\left(\frac{\left(\hat{b}_1\right)^{n_1}\cdots\left(\hat{b}_N\right)^{n_N}|0\rangle\langle 0|\,\hat{b}_N^{m_N}\cdots\hat{b}_1^{m_1}}{\sqrt{n_1!\cdots n_N!m_1!\cdots m_M!}}\right)=$$
$$=\mathrm{Tr}\left(\frac{\left(\hat{b}_j\right)^{n_j}|0\rangle\langle 0|\,\hat{b}_j^{m_j}}{\sqrt{n_j!m_j!}}\right)\frac{\left(\hat{b}_1\right)^{n_1}\cdots\left(\hat{b}_N\right)^{n_N}|0\rangle\langle 0|\,\hat{b}_N^{m_N}\cdots\hat{b}_1^{m_1}}{\sqrt{n_1!\cdots n_N!m_1!\cdots m_M!}} \tag{3.26}$$

On the right-hand side of the equality, the terms j have disappeared. To trace out a given set of modes, we can apply the formula above to trace out each individual mode from the set in succession.

It follows from the boson commuting properties at different modes that any local

[4] Up to a global phase factor $e^{i\phi}$.

unitary in $\mathcal{M}$ commutes with any local observable in $\mathcal{M}'$ if $\mathcal{M} \cap \mathcal{M}' = \emptyset$. As we have mentioned in the fermionic case, this is why the no-signalling condition is satisfied.

The Separation property is also satisfied. The reason is the algebraic independence of the bosonic annihilation operators. Being a local operator on AC requires being generated solely by the creation and annihilation operators of A and C. If the operator is local on BC, then the creation and annihilation operators of B and C generate it. Since the generators are independent algebraic entities, it follows that elements of C alone generate the operator. This makes it a local operator on C.

Therefore, bosons per se are a no-signalling operational theory. According then to the reasoning in Subsection 2.3.3, bosons are a local-realistic theory. We need not restrict the bosonic algebra of observables in any way. The lack of restrictions in the set of kinematical transformations ensures the global transitivity of the pure phenomenal state space. Starting from any pure phenomenal state $|\psi\rangle$, one can evolve to an arbitrary $|\eta\rangle$ by performing an allowed transformation $\hat{U}_{\psi,\eta}$. Therefore the pure phenomenal state space has a single orbit. We choose the canonical representative to be the pure state $\hat{b}_1^\dagger |0\rangle$.

Because the bosonic creation and annihilation operators are generators of the local and global algebras of observables, the bosonic annihilation operators can be considered the bosonic descriptors. The reasoning is that bosonic creation and annihilation operators generate the local and global algebra of observables.

Therefore, knowing the evolution of $\{\hat{b}_j\}_{j\in\mathcal{M}}$ is enough to know the evolution of any bosonic observable local in $\mathcal{M}$. We can see from the trick described in Subsection 2.4.3 that knowing the evolution of $\hat{b}_j$ is enough to know the evolution of $\hat{b}_j^\dagger$. The trick consists in pointing out that $\hat{U}^\dagger \cdot \hat{b}_j^\dagger \cdot \hat{U} = \left(\hat{U}^\dagger \cdot \hat{b}_j \cdot \hat{U}\right)^\dagger$.

These properties ensure that the bosonic analogues of Theorems 6, 7 & 8 hold. As

a result, we can consider bosonic annihilation operators as the representations of the local ontic states of a bosonic subsystem $\mathcal{M}$. We denote them as:

$$\left(\left(\hat{U}^\dagger \cdot \hat{b}_{j_1} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{b}_{j_M} \cdot \hat{U}\right), \rho_0\right) \tag{3.27}$$

The ontic operations are analogous to the fermionic case. In the epimorphisms $\varphi_{\mathcal{M}}$, one just needs to substitute the fermionic Fock states with the bosonic Fock states.

We have seen that the commutation properties of bosonic systems do not pose any problems when analysing locality. One could say that it is precisely because of the commuting properties that such systems do not pose a problem. The commutation negates the necessity of imposing any superselection rule in order to satisfy the no-signalling principle. Therefore, the kinematical space can be used directly. Its structure resembles a lot the structure from $(\ell^2)^{\otimes N}$. Then it is natural to expect the same behaviour as distinguishable quantum systems regarding local realism.

We conclude by reiterating that bosons are local-realistic and that the bosonic annihilation operators can represent their local ontic states. We obtain the same result as in the fermionic case without the nuance of including the parity SSR. Since bosons satisfy local tomography, we expect the bosonic descriptors to be able to be expressed in terms of bosonic observables.

3.4 Mach-Zehnder interferometer

This section reproduces an adaptation of the main findings published in [64] in collaboration with Chiara Marletto and Vlatko Vedral.

In this section, we examine the physical scenario of single-particle interference. We consider bosonic and fermionic Mach-Zehnder interferometers. Without loss of generality, we consider the self-interfered fermion a spinless electron and the

boson a structureless photon, i.e. we disregard their spinorial and polarisation components. We identify two separated extended spatial regions as two bosonic or fermionic modes. The two regions are the Mach-Zehnder interferometer's left and right arms. For a more detailed and accurate modelling of the physical situation, we should have discretised space and consider the regions L and R as sets of discrete position points. However, such a level of detail would only muddle the analysis without adding additional insight.

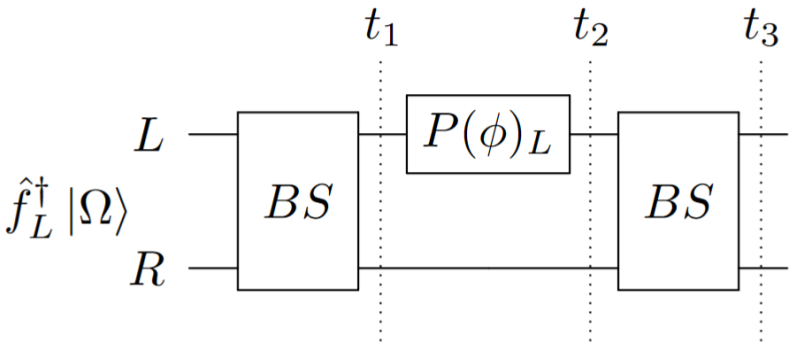


Figure 3.2: Circuit diagram of a single-particle Mach-Zender interferometer.

3.4.1 Bosonic Mach-Zehnder

Let us first consider the bosonic interferometer. A single-photon passing through a Mach-Zehnder interferometer has been a popular way of thinking about interference in quantum information and computation [61]. It is analogous to the double-slit experiment, which, in the words of Feynman, contains "the only mystery" in quantum physics [101].

We consider the standard quantisation procedures of the electromagnetic field $\hat{A}_\mu(x)$, which leads to the introduction of two bosonic annihilation operators $\hat{a}_L, \hat{a}_R$ [67]. The $\hat{a}_j$ satisfy the properties for bosonic annihilation operators $\hat{b}_j$ described in Section 3.3.

The evolution under consideration is depicted in Figure 3.2, except for the initial state, where the fermionic operator $\hat{f}_L^\dagger$ needs to be replaced by the bosonic creation operator $\hat{a}_L^\dagger$.

In the Schrödinger picture, the photon's quantum state evolves after the first beam splitter (BS). Then, it acquires the additional phase in the left arm via $P(\phi)_L$. Lastly, it undergoes another transformation at the final beamsplitter (BS). The dynamical evolution of the photon is given by:

$$\begin{aligned}\hat{a}_L^\dagger |0\rangle \xrightarrow{BS} \frac{1}{\sqrt{2}}\left(\hat{a}_L^\dagger + \hat{a}_R^\dagger\right)|0\rangle \xrightarrow{P(\phi)_L} \frac{1}{\sqrt{2}}\left(\hat{a}_R^\dagger + e^{i\phi}\hat{a}_L^\dagger\right) \xrightarrow{BS} \\ \xrightarrow{BS} \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\phi}|-\rangle\right) = \left(\sin\left(\frac{\phi}{2}\right)\hat{a}_L^\dagger + \cos\left(\frac{\phi}{2}\right)\hat{a}_R^\dagger\right)|0\rangle \end{aligned} \tag{3.28}$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}(\hat{a}_L^\dagger \pm \hat{a}_R^\dagger)|0\rangle$ are equally weighted superpositions of the left and right path.

We can locally describe the interferometry process using the Heisenberg picture and the bosonic annihilation operators as boson descriptors.

At the start, let the photon descriptors be

$$t_0 : ((\hat{a}_L, \hat{a}_R), |\Psi_0\rangle\langle\Psi_0|) \tag{3.29}$$

where the Heisenberg state is the pure state $|\Psi_0\rangle = \hat{a}_L^\dagger |0\rangle$.

The unitary beam splitter BS applied at time t acts as Bogoliubov transformations on the creation and annihilation operators evolving them as $\hat{a}_L \xrightarrow{BS} \frac{1}{\sqrt{2}}(\hat{a}_L + \hat{a}_R)$ and $\hat{a}_R \xrightarrow{BS} \frac{1}{\sqrt{2}}(\hat{a}_L - \hat{a}_R)$.

The photon field operator descriptors after the first beam splitter, expressed as functions of the initial descriptors, are:

$$t_1 : \left(\left(\frac{1}{\sqrt{2}} \left(\hat{a}_L + \hat{a}_R \right), \frac{1}{\sqrt{2}} \left(\hat{a}_L - \hat{a}_R \right) \right), |\Psi_0\rangle\langle\Psi_0| \right) \tag{3.30}$$

The phase shift $P(\phi)_L(t_1)$ only acts on the left arm. That is, it is a local operator, and thus a function of the local operators $\hat{a}_L(t_1)$ at the time it is applied. Hence, it only affects the left modes: $\hat{a}_L \xrightarrow{P(\phi)_L} e^{i\phi}\hat{a}_L$ and $\hat{a}_R \xrightarrow{P(\phi)_L} \hat{a}_R$. Given all of the above, the new field operators after the phase shift are:

$$t_2 : \left(\left(\frac{e^{i\phi}}{\sqrt{2}} \left(\hat{a}_L + \hat{a}_R \right), \frac{1}{\sqrt{2}} \left(\hat{a}_L - \hat{a}_R \right) \right), |\Psi_0\rangle\langle\Psi_0| \right) \tag{3.31}$$

The property of no-action at a distance is the crux of quantum field theory in the Heisenberg picture: changes caused by a phase shift acting locally on one mode do not affect operators of other modes. In our example, only the left mode descriptors contain the phase, while the right mode field operators do not. As a result, we can determine where the phase shift was applied by inspecting the descriptors of the two modes.

Note also that a state-tomography of the left mode would not at this stage reveal the phase (the expected value of the number operator of the left mode does not depend on the phase). At this stage, the phase is encoded in the left mode, but it is locally inaccessible.

The final beam splitter combines the modes' descriptors making the phase accessible. The descriptors at the output of the interferometer are as follows:

$$t_3 : \left(\left(\cos\left(\frac{\phi}{2}\right)\hat{a}_L + i\sin\left(\frac{\phi}{2}\right)\hat{a}_R, \cos\left(\frac{\phi}{2}\right)\hat{a}_R - i\sin\left(\frac{\phi}{2}\right)\hat{a}_L \right), |\Psi_0\rangle\langle\Psi_0| \right) \tag{3.32}$$

The interference is manifested when we take the expected value of the local number operator $\hat{N}_x = \hat{a}_x^\dagger \hat{a}_x$ at time t_3, using the descriptors to express the evolved number operators and the Heisenberg state $|\Psi_0\rangle = \hat{a}_L^\dagger |0\rangle$. For the output left mode, we

obtain the following:

$$\langle \hat{N}_L(t_3) \rangle_{\Psi_0} = \cos^2 \frac{\phi}{2} \tag{3.33}$$

The expected value of the output mode R could be calculated similarly. It would yield the value of $\sin^2 \phi/2$. The expected values at the end of the interferometry are empirically equivalent and thus the same in the Heisenberg and Schrödinger pictures.

The key difference in the explanation for the interference is that, in using descriptors, the phase introduced by the phase shift on one mode is only locally manifested in that mode and not others. This would not be the case in the Schrödinger picture: the wavefunction does not allow for a separable description, and the phase difference due to the beam splitter acting on mode L could just as well have been introduced by a beam splitter acting on mode R.

In the Heisenberg picture, any bosonic field (e.g. a field of Bose condensate of atoms) has the same description as above. We can interfere condensates by applying this Mach-Zehnder interferometer implementation, and the operator description of this interference would be identical to the one presented above.

We have used the annihilation operators as descriptors. However, in the bosonic case, the abovementioned situation can be described using two observables per mode instead. The vector potential operator $\hat{A}_x = \hat{a}_x + \hat{a}_x^\dagger$ and the conjugate field $\hat{\pi}_x = i\left(\hat{a}_x - \hat{a}_x^\dagger\right)$ can be used as bosonic descriptors. It is clear from the fact that bosonic annihilation operators are a linear combination of the two: $\hat{a}_x = \frac{\hat{A}_x - i\hat{\pi}_x}{2}$.

We can use two observables per mode as bosonic descriptors. This property is possible because bosons satisfy local tomography. It contrasts with the fermionic context, where such a possibility is ruled out, as seen in Subsection 3.1.6.

3.4.2 Fermionic Mach-Zehnder

We have a system with two electron annihilation operators $\hat{f}_L$ and $\hat{f}_R$. The first annihilates an electron in the left region, and the second in the right region. The effect of an electron evolving freely through the localised regions of the interferometer is modelled as the identity evolution.

Consider the basis for this two-mode system: $|\Omega\rangle\,, \hat{f}_L^\dagger \hat{f}_R^\dagger\,|\Omega\rangle\,, \hat{f}_L^\dagger\,|\Omega\rangle\,, \hat{f}_R^\dagger\,|\Omega\rangle$. The beam splitters in the interferometer are modelled in our setup as the gate BS. The phase gate on the left arm is denoted by $P(\phi)_L$. Their matrix representations and algebraic effects on annihilation operators are as follows:

$$BS = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & & \\ 0 & \sqrt{2} & & \\ & & 1 & 1 \\ & & 1 & -1 \end{pmatrix} \qquad P(\phi)_L = \begin{pmatrix} 1 & 0 & & \\ 0 & e^{i\phi} & & \\ & & e^{i\phi} & 0 \\ & & 0 & 1 \end{pmatrix}$$

$$BS\left(\hat{f}_L, \hat{f}_R\right) BS = \frac{1}{\sqrt{2}}\left(\hat{f}_L[\hat{f}_R, \hat{f}_R^\dagger] + \hat{f}_R[\hat{f}_L, \hat{f}_L^\dagger]\,,\ \hat{f}_L[\hat{f}_R, \hat{f}_R^\dagger] - \hat{f}_R[\hat{f}_L, \hat{f}_L^\dagger]\right)$$

$$P(\phi)_L^\dagger \left(\hat{f}_L\,,\ \hat{f}_R\right) P(\phi)_L = \left(e^{i\phi}\hat{f}_L\,,\ \hat{f}_R\right) \tag{3.34}$$

The initial global phenomenal state is the pure state $|\psi_0\rangle = \hat{f}_L^\dagger\,|\Omega\rangle$. The evolution it undergoes in the Mach-Zehnder interferometer according to the Schrödinger picture is $BS\,|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(\hat{f}_L^\dagger\,|\Omega\rangle + \hat{f}_R^\dagger\,|\Omega\rangle\right)$, then applying the phase gate we obtain $P(\phi)_L \cdot BS\,|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(e^{i\phi}\hat{f}_L^\dagger\,|\Omega\rangle + \hat{f}_R^\dagger\,|\Omega\rangle\right)$. Finally, recombining the two interferometer arms up to a global phase results in $BS \cdot P(\phi)_L \cdot BS\,|\psi_0\rangle = \cos\left(\frac{\phi}{2}\right)\hat{f}_L^\dagger\,|\Omega\rangle + i\sin\left(\frac{\phi}{2}\right)\hat{f}_R^\dagger\,|\Omega\rangle$.

We can set the phenomenal reference state as the conventional $|\psi_0\rangle = \hat{f}_L^\dagger\,|\Omega\rangle$ for the orbit of odd pure states. Then we set $\rho_0 = |\psi_0\rangle\langle\psi_0|$. The expressions give the

evolution of the ontic state in the descriptor representation:

$$t_0 : \left(\hat{f}_L, \hat{f}_R, \rho_0\right) \tag{3.35}$$

$$t_1 : \left(\frac{1}{\sqrt{2}}\left(\hat{f}_L[\hat{f}_R, \hat{f}_R^\dagger] + \hat{f}_R[\hat{f}_L, \hat{f}_L^\dagger]\right), \frac{1}{\sqrt{2}}\left(\hat{f}_L[\hat{f}_R, \hat{f}_R^\dagger] \text{ - } \hat{f}_R[\hat{f}_L, \hat{f}_L^\dagger]\right), \rho_0\right) \tag{3.36}$$

$$t_2 : \left(\frac{e^{i\phi}}{\sqrt{2}}\left(\hat{f}_L[\hat{f}_R, \hat{f}_R^\dagger] + \hat{f}_R[\hat{f}_L, \hat{f}_L^\dagger]\right), \frac{1}{\sqrt{2}}\left(\hat{f}_L[\hat{f}_R, \hat{f}_R^\dagger] \text{ - } \hat{f}_R[\hat{f}_L, \hat{f}_L^\dagger]\right), \rho_0\right) \tag{3.37}$$

$$t_3 : \left(\frac{e^{i\phi}+1}{2}\hat{f}_L + \frac{e^{i\phi}-1}{2}\hat{f}_R\ ,\ \frac{e^{i\phi}-1}{2}\hat{f}_L + \frac{e^{i\phi}+1}{2}\hat{f}_R\ ,\ \rho_0\right) \tag{3.38}$$

Algebraically specifying the evolution allows us to study the dependency of the local ontic states on the information about the local phase without referring to their matrix representations individually. The qubit network example in Subsection 2.5.2 specifies the computational basis and displays the evolution in matrix form. We could specify the qubit case algebraically, but we opt for presenting both possibilities.

We observe from the algebraic evolution that when the local phase is introduced, the parameter ϕ appears only in the descriptor of the left interferometer arm. When both arms interact with each other again by the recombining beam-splitter, the dependency on the parameter ϕ flows to the right arm. We know that, at t_2, the parameter ϕ has no locally observable effect. However, in the later stage t_3, ϕ has locally observable consequences in both the left and right arms.

To demonstrate that these two claims are true, we calculate if ϕ appears in the expression of the evolved physical fermionic local observables at t_2 and t_3. The local observables in a single mode j all have the form $\hat{O}_j = a\hat{f}_j\hat{f}_j^\dagger + b\hat{f}_j^\dagger\hat{f}_j$ with $a, b \in \mathbb{R}$. It is clear that at t_2, no dependency on ϕ can appear on any local

observable. On the left arm, the exponentials cancel in each term. On the right arm, the parameter does not appear at all.

In contrast, the expressions of the local observables at t_3 are of the form:

$$\hat{O}_L = a\,\hat{f}_L\hat{f}_L^\dagger\hat{f}_R\hat{f}_R^\dagger + b\,\hat{f}_L^\dagger\hat{f}_L\hat{f}_R^\dagger\hat{f}_R + \left(a\sin^2\left(\frac{\phi}{2}\right) + b\cos^2\left(\frac{\phi}{2}\right)\right)\hat{f}_L^\dagger\hat{f}_L\hat{f}_R\hat{f}_R^\dagger + \\ + \left(b\sin^2\left(\frac{\phi}{2}\right) + a\cos^2\left(\frac{\phi}{2}\right)\right)\hat{f}_L\hat{f}_L^\dagger\hat{f}_R^\dagger\hat{f}_R + \frac{i}{2}\sin(\phi)(b-a)\left(\hat{f}_L^\dagger\hat{f}_R + \hat{f}_L\hat{f}_R^\dagger\right) \tag{3.39}$$

$$\hat{O}_R = a\,\hat{f}_L\hat{f}_L^\dagger\hat{f}_R\hat{f}_R^\dagger + b\,\hat{f}_L^\dagger\hat{f}_L\hat{f}_R^\dagger\hat{f}_R + \left(b\sin^2\left(\frac{\phi}{2}\right) + a\cos^2\left(\frac{\phi}{2}\right)\right)\hat{f}_L^\dagger\hat{f}_L\hat{f}_R\hat{f}_R^\dagger + \\ + \left(a\sin^2\left(\frac{\phi}{2}\right) + b\cos^2\left(\frac{\phi}{2}\right)\right)\hat{f}_L\hat{f}_L^\dagger\hat{f}_R^\dagger\hat{f}_R + \frac{i}{2}\sin(\phi)(a-b)\left(\hat{f}_L^\dagger\hat{f}_R + \hat{f}_L\hat{f}_R^\dagger\right) \tag{3.40}$$

We see that they are highly dependent on the parameter ϕ. Therefore, ϕ has local observable effects in t_3. It is well known that the relative frequency of electron counts in the local arms is given by $\cos^2\left(\frac{\phi}{2}\right)$ and $\sin^2\left(\frac{\phi}{2}\right)$ respectively.

Notice that despite having different algebraic evolutions, qualitatively, the underlying locality explanation of the fermionic and bosonic cases is the same. This should not be surprising since we have used the same circuit, the same model, just interpreted in the bosonic or fermionic setting.

In this example, we have seen that using electron physical observables makes locating and tracking the phase ϕ impossible. We can conveniently express the ontic states using fermionic annihilation operators as fermionic descriptors. This allows us to localise and track the dependency of the phase parameter ϕ within the interferometric system. This property is not unique to the fermionic Mach-Zehnder. The fermionic descriptor picture can be used to analyse any fermionic physical process, including those that do not conserve the particle number.

3.4.3 Phase localisation using the current density observable

In this subsection, we highlight how observables can be used to locate the phase and track some of its effects in the specific setup of the fermionic Mach-Zehnder. The trick is to use Dirac field physical observables. The Dirac field contains both electron and positron modes. The positron vacuum is intended to act as a phase reference for the electrons.

Let us focus on a more quantum field theoretic (QFT) approach to fermionic theory. The proper second-quantised Dirac field is described by the four-spinor operator (see [67])

$$\hat{\psi}(x) = \hat{f}_x + \hat{p}_x^\dagger \, . \tag{3.41}$$

This field involves the electron annihilation operator $\hat{f}_x$ and the positron creation operator $d_x^\dagger$ at point x. We deliberately omit the spinor details and momentum representation, as they are irrelevant to the following argument – see e.g.[102]. This fermionic Dirac field operator is not Hermitian; thus, it is not an observable. Furthermore, the superselection rules prohibit odd operators from being observables. Therefore, no linear combination of creation and annihilation operators is allowed to represent a physical variable.

However, consider the charge density observable of each arm mode $\hat{j}_0(L)$ $\hat{j}_0(R)$, where $\hat{j}_0(x) = -e : \hat{\psi}(x)^\dagger \hat{\psi}(x) := -e(\hat{f}_x^\dagger \hat{f}_x - \hat{f}_x \hat{p}_x + \hat{f}_x^\dagger \hat{p}_x^\dagger - \hat{p}_x^\dagger \hat{p}_x)$ (we would, in general, have to use the 4-vector also including the current density, but, in this case, the other 3 components do not add to our analysis). The normal ordering of the fermionic operators $\hat{A}$ and $\hat{B}$ is represented by $: \hat{A}\hat{B} :$. Even though $\hat{p}_x, \hat{f}_x$ are spinors with some orthogonality properties imposed, all four terms are generally non-zero. For the sake of simplicity, we also assume that the Heisenberg state is $|\Psi\rangle_{ep} = \frac{1}{\sqrt{2}} \left(\hat{f}_R + \hat{f}_L \right) |\Omega\rangle$, so we describe the interferometry just after applying

the first beam splitter. $|\Omega\rangle$ is the vacuum for the two electron and two positron modes.

We follow the time-evolution of the local density operators $\hat{j}_0(L), \hat{j}_0(R)$ in the Heisenberg picture. We can see that a phase rotation applied on the left arm mode now manifests itself in the quantum observables of the Dirac field by modifying the charge density operator!

Consider that the Dirac field at t_1 is given by $\hat{f}_x + \hat{p}_x^\dagger$. The Dirac field transforms under the phase rotation $P(\phi)_L$ at time t_2 as $e^{i\phi}\hat{b}_L + e^{-i\phi}\hat{p}_L^\dagger$ and $\hat{f}_R + \hat{p}_R^\dagger$. After applying the phase rotation $P(\phi)_L$, at t_2, the charge density of the Dirac field in the left mode is:

$$\hat{j}_0(L) = -e\left(\hat{f}_L^\dagger \hat{f}_L + e^{-2i\phi}\hat{f}_L^\dagger \hat{p}_L^\dagger - e^{2i\phi}\hat{f}_L \hat{p}_L - \hat{p}_L^\dagger \hat{p}_L\right) , \tag{3.42}$$

while the right mode charge density remains unchanged.

After applying the phase shift, we see that the phase is present in the charge density operator of the left mode. Therefore, even under superselection rules, a perfectly valid observable can keep track of the phase locally to each mode of the Dirac field. The positronic component of the Dirac field provides a phase reference for the electron field. The phase shift's different action on the electron and positron field operators allows us to keep track of the phase. It accomplishes this by providing a local phase reference to the left arm between the fermionic and positronic fields.

We emphasise that if, as in the case of the interferometer under consideration, the Heisenberg state consists of an electron superposed across the left and right modes and no positrons, then the expected value of $\hat{j}_0$ will still be phase independent. That is, the phase will be locally inaccessible via empirical observation. In the absence of superposed positrons or another superposed electron acting as a reference, the

phase is unobservable (all we can observe is whether the electron is in the mode L or mode R).

However, when considering the Dirac field as a q-number [103], the local picture of quantum field theory in the Heisenberg picture reveals that the phase has been applied on one mode and not the other. Moreover, we can see this in the Mach-Zehnder case by tracking a physical observable rather than the Dirac field itself.

4 | Anyons

The contents of this Chapter 4 contain an explanation of the publication [104] written with the collaboration of Lucia Vilchez-Estevez.

In regular 3+1 D, bosons and fermions are the only indistinguishable particles possible [66]. However, in 2+1 D, exotic types of indistinguishable particles can exist: anyons. Anyons are indistinguishable particles with fractional non-semi-integer spin value [105]. They are generalisations of bosons and fermions with more complex factors appearing as a result of particle type exchange. Anyons are classified into two types: abelian and non-abelian. Non-abelian anyons are more exotic and much harder to realise in the lab. Nevertheless, their fascinating properties deserve our attention.

Anyon properties are inextricably linked to topology. Anyonic particle systems emerge from the non-trivial topological properties of the configuration space of the particles [106].

Anyon theory has a long history and has evolved in several directions. Following their initial proposal in the 1980s [105], some works focused on analysing the possible values of spin particles with methods based on group representation theory and topology. Most research used the first quantisation perspective. The wave-functions over configuration spaces of particles were analysed under the braiding and permutation groups. Concrete models in the path-integral formalism were developed where anyons could emerge, making use of Chern-Simons potentials [107–110]. These allowed for regime candidates in condensed matter systems where anyons could be detected. The most promising were the fractional quantum hall effect [111] and Josephson junctions [112–114].

Recent claims have been made on having detected non-abelian anyons, specifically

Majorana fermions [115–122]. Nonetheless, as of July 2023, there has yet to be a consensus on whether non-abelian anyons have ever been detected.

In the 1990s, there was a shift in how anyon theories were described. The notion of exclusion statistics developed by Haldane [123], mathematical physicists' use of category theory, and the explosion of quantum computation and information all contributed to this reorientation. The diagrammatic categorical approach was introduced [124] to understand better a quantum computer's computational capabilities based on non-abelian anyons [125]. The promise of naturally occurring fault tolerance [125] due to topological protection in these topological quantum computers sparked much interest, which prompted the expression of this area of condensed matter in quantum information language [126–128].

It is an illuminating lesson on how careful study of a fundamental concept may open further avenues for fundamental research. In this case, it led to substantial advances in condensed matter and quantum computation, concretely in quantum error correction.

Our primary goal in including anyons in this thesis is to study local realism in a relevant type of indistinguishable quantum system. Our secondary goal is to divulge the theory to a quantum foundations audience. We believe that quantum foundations would benefit from considering nuanced condensed matter systems to test some of its assumptions and understandings of quantum theory.

The anyonic formalism in $2 + 1$ dimensions lacks the notion of creation and annihilation operators. Instead, it is studied using either representation theory or a categorical diagrammatic approach. This thesis employs the latter since it is complete and facilitates the identification of structural features of subsystems.

The concept of superselection rules is naturally incorporated into the formalism. Moreover, it treats the abelian and non-abelian cases in a unified form. We are

more interested in the more challenging non-abelian case, in which embedding the non-physical operators in a composite system is difficult to define consistently. The complications arise from the possibility that different compositions of systems are not given by a unique tensor product, as in distinguishable quantum systems.

This chapter aims to use Einstein's notion of local realism introduced in Section 2.3 and used throughout this thesis to identify the descriptors of non-abelian anyons. In particular, we seek to identify descriptors we can consider annihilation operators. In the bosonic and fermionic cases, descriptors could be identified as annihilation operators. We must now reverse the process. In wondering about local realism in anyons, we obtain an excellent result: identifying the anyonic creation and annihilation operators.

Since the treatment of anyon theory tends to be focused on its topological properties, having a clear notion of the anyonic local elements of reality may be useful. Exploring this different perspective of locality could lead to advances in finding these elusive particles, better understanding how they form, and how to manipulate them. It might also help study the computational advantages of using these particles as the building blocks of a quantum computer.

In Section 4.1, we present the diagrammatic anyon formalism in detail. In Section 4.3, we are able to describe the local-realistic structure of anyonic systems using the anyonic annihilation operators we discover in Section 4.2. We discuss the model of Fibonacci anyons to exemplify the concepts throughout the chapter.

4.1 Anyon formalism

Some parts of this section are adapted from the publication [104], written with the collaboration of Lucia Vilchez-Estevez.

Anyons are postulated quasiparticle excitations in two-dimensional systems [105, 129]. Their topological nature and exotic exchange statistics [105, 106, 123, 129, 130] differentiate them from bosons and fermions. They are referred to as topological particles because the geometry of space-time or the distance between them does not affect the result of the relevant operations. These topological properties make anyon systems a promising platform for quantum information processing [124, 125, 127, 131, 132]. Topological quantum computing seeks to exploit these features to attain a robust computation against errors produced by local perturbations and environment noise.

Information processing with topological systems has been one of the main attractions to the study of anyonic theories. We build on recent information-theoretic perspectives on anyons [126, 128, 133–137].

On the other hand, anyons can be very intriguing from a more foundational standpoint. The notion of subsystems and locality in quantum information theory is crucial to understand interactions between systems. As an example, in a qubit network, we use the tensor product structure to describe systems composed of multiple subsystems. Two non-abelian anyons can merge (fuse) together to different anyonic charges depending on the fusion channel. Therefore, to describe an anyonic quantum system completely, we need to know all the charges that make up the system and how they fuse. Since we need that extra bit of information on the overall charge of the composed system, there is no such thing as a tensor product between two subsystems.

There is a gap in the literature regarding creation and annihilation operator algebra for non-abelian anyons in 2D. Bosons and fermions have well-defined annihilation operators, so it is natural to look for them in anyon theories too. For anyons in one spatial dimension, the creation and annihilation operators have already been found

[138–140]. The main reasons for its absence in the 2D case may be the difficulty in defining modes (or subsystems) and the topological charge superselection rule. The latter is an interesting characteristic of anyon theories that ensures operators will only be physical observables when the total topological charge is conserved.

The formalism we consider is diagrammatic. Category theory notions heavily influence it. It is similar to the applied category theory formalism of resource theories and the ZX calculus but with twists [41, 48, 141]. Concretely, an anyon theory is given by a modular tensor category [124, 134]. We follow the formalisation, convention and notation of [126, 142]. Following the physics' normalisation rather than the isotopy invariant normalisation of the diagrams is preferable for our purposes.

4.1.1 Fusion, splitting, F-moves and exchanging

To describe the anyonic formalism, let us imagine anyons as particles that live in a 2-dimensional manifold. There are different particle types in an anyon theory. We label the anyon particle types as $a, b, c, \ldots$.

Two anyons can be combined to form a new particle. This process is known as fusion. Two particles a and b can fuse to produce particle c. We can describe this process by writing $a \times b = b \times a = c$. However, in the case of non-abelian anyons, two anyons a, b can fuse to multiple types of particles. In this case, we write:

$$a \times b = \ b \times a = \ \sum_c N_{ab}^c \, c \tag{4.1}$$

where N_{ab}^c are the fusion multiplicities. They indicate the different ways in which a and b can fuse to c. There is a trivial anyon e known as the vacuum or the identity. This particle satisfies the property $N_{ea}^b = \delta_{ab}$ for any particle type. Every particle a

also has its own unique antiparticle $\bar{a}$ such that $N_{ab}^{e} = \delta_{b\bar{a}}$. The fusion rules dictate which particles make up the anyon theory and which fusions (fusion channels) are physically allowed within the theory.

Splitting is the reverse process (going from one particle to two). Both processes are shown diagrammatically [126, 128, 142] in Figure 4.1.

When employing the diagrammatic algebra, we will always set the time direction vertically and going upwards and assume that all particles move forward in time. We can interpret a particle going back in time as its antiparticle moving forward in time.

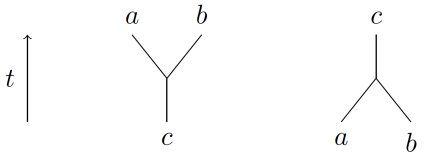


Figure 4.1: Splitting (left) and fusion (right)

If $N_{ab}^{c} > 1$, there are two distinct processes by which a, b fuse to c. Usually, the fusion vertex diagram labels each of these processes by a term μ in the vertex. Throughout this thesis, we assume, for practical purposes, that all fusion multiplicities N_{ab}^{c} are either 0 or 1. Therefore, the label μ is unnecessary. Our procedures and conclusions can be easily extended to anyon theories where fusion multiplicities are larger than one.

We can write an orthonormal complete set of pure phenomenal states for n anyons as a fusion tree as in Figure 4.2. If any of the $N_{a_{i-1}a_i}^{a_{i+1}} = 0$, then the fusion is not allowed, and the diagram is zero. The corresponding bras $\langle\psi_i|$ are obtained with the Hermitian conjugate, equivalent to flipping the diagram along a horizontal axis.

The diagrams in Figure 4.2 represent that in order to specify an n-anyonic state, it is not enough to establish the charges (particle types) of each of the n anyons. Because two anyon charges a, b may fuse to two distinct particle types c or d, $a \times b = c + d + ...$; it is necessary to establish their fusion channel. Also, the fusion channel of the resulting particle type with other remaining unfused anyons in the

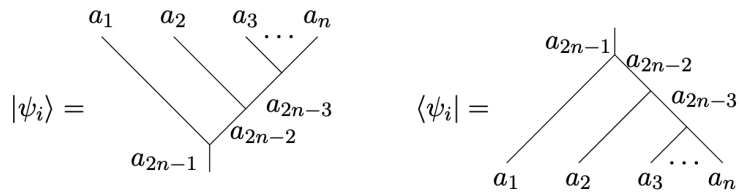


Figure 4.2: Basis $|\psi_i\rangle$ and its conjugate $\langle\psi_i|$ of an n-anyon system. All vertices are allowed fusion channels.

system, successively until all n anyons are split from the global overall anyonic charge. The grouping of the n anyons can be done in different orders. The states where $a \times b = c$ or $a \times b = d$ are considered completely orthogonal. Thus, we need the specification to discriminate the two distinct states.

One can choose a different order to combine the anyons while still specifying the same state. The F-matrices are the unitary matrices that can be used for such a change of basis. Their factors are specified by each anyon theory together with the fusion rules. They determine how to change from the fusion tree given by the grouping $((ab)c)$ to the grouping $(a(bc))$. See Figure 4.3.

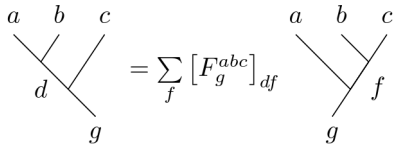


Figure 4.3: The F-matrix defines a change of basis

To define the case where a, c are grouped first, and then with b, we need to establish how c crosses b to get to a. The diagram's line with charge c can go in front or behind line b. For an anyon theory, these are at least two different ways of joining a and c.

As we have explained, two-particle lines can cross over or underneath each other, representing the exchange of two anyons. When working with the exchange of anyons, it is necessary to define the R-matrices components as in Figure 4.4.

As we can see, the information on the phase of exchange of two particle types is encoded in these matrices. In general, the factors R_c^{ab} are phases $e^{i\theta_{ab}^c}$. When $a = e$ or $b = e$, then the phase gathered is 2π, thus $R_a^{ea} = 1$. The R-matrices are the last piece of information an anyonic theory requires. To have a concrete anyonic theory, we need the particle types, the fusion rules, and the F and R matrices. In Figure 4.5, we show the grouping of a, c happening behind b. The factor that appears is $[B_g^{abc}]_{dh} = \sum_f \left[F_g^{bac}\right]^*_{fd} R_f^{ab} \left[F_g^{abc}\right]_{fh}$. We can get it by combining F and R matrices.

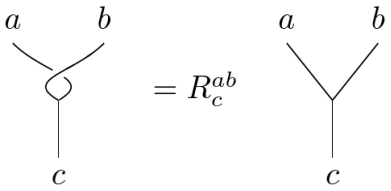


Figure 4.4: The phase resulting after exchanging two anyon particles counterclockwise.

The kinematical Hilbert space $\mathcal{H}_n$ to consider is the span of the orthonormal states of the fusion trees in Figure 4.2, fixing the number of anyonic particles n. Their orthonormality condition induces the scalar product in the vector space. We consider anyon theories with a finite number of particles and fusion rules, thus giving finite-dimensional Hilbert spaces $\mathcal{H}_n$.

We can define $\mathcal{H}_n^g$ as the subspace of $\mathcal{H}_n$ such that the global charge of the n anyon system is g. The n anyons can be seen as having split from a common unique particle type g. We have $\mathcal{H}_n = \bigoplus_g \mathcal{H}_n^g$.

Let us now point out the dimensions of the kinematical Hilbert spaces. $\dim(\mathcal{H}_1) = n_T$ being the total number of particle types or anyon charges in our theory. $\dim(\mathcal{H}_2) = \sum_{abc=1}^{n_T} N_{ab}^c$, is natural since we have an element basis per each fusion channel. $\dim(\mathcal{H}_3) = \sum_{abcdg=1}^{n_T} N_{bc}^d N_{ad}^g$. Notice that the growth of the

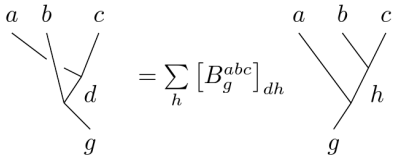


Figure 4.5: A change of basis behind b.

kinematical size depends on each anyon theory; generally, it is not strictly exponential.

We can naturally consider the linear operators that map the kinematical Hilbert space $\mathcal{H}_n$ to $\mathcal{H}_m$. A diagram with kets stacked on bras can represent a linear operator. A general linear anyonic operator is written as in Figure 4.6. All operators we will consider satisfy $m = n$, thus kinematical operators of the kinematical Hilbert space $\mathcal{H}_n$. For a fixed n, we denote the algebra of all anyonic linear operators as $\mathcal{A}_n$.

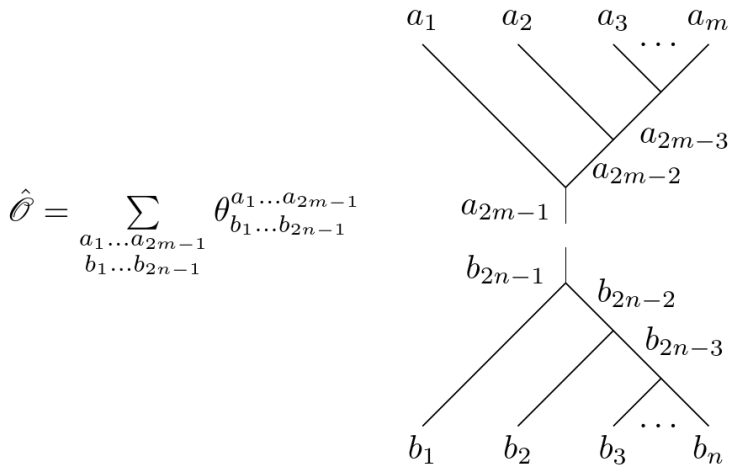


Figure 4.6: General operator with n inputs and m outputs. $\theta^{a_1 \dots a_{2m-1}}_{b_1 \dots b_{2n-1}} \in \mathbb{C}$.

4.1.1.1 Example: Fibonacci anyons

Let us exemplify the following formalism by focusing on a non-abelian anyon family: Fibonacci anyons [143]. The Fibonacci model is perhaps the simplest non-abelian example and has only two particle types, the vacuum or trivial anyon e and the Fibonacci anyon τ. The only non-trivial fusion rule of this theory reads

$$\tau \times \tau = e + \tau. \tag{4.2}$$

One can convert between bases associated with different fusion trees using the F-matrices shown in Figure 4.3. In the Fibonacci theory, the only nontrivial F-matrix is $F_\tau^{\tau\tau\tau} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix}$. ϕ^{-1} is the inverse golden ratio, $\phi^{-1} = \frac{\sqrt{5}-1}{2}$.

We have explained that exchanging two anyons results in a phase factor that depends on the overall charge. For Fibonacci anyons, there are two non-trivial R-matrix factors: (i) when two τ anyons fuse to the identity, and (ii) when two τ anyons fuse to τ. Their respective phases are $R_e^{\tau\tau} = e^{-4\pi i/5}$ and $R_\tau^{\tau\tau} = e^{3\pi i/5}$.

The name of Fibonacci anyons comes from the fact that the dimensions of the subspaces of global charge e and τ are Fibonacci numbers F_k. Thus, the total dimension of the kinematical Hilbert space is always a Fibonacci number. We have $\dim(\mathcal{H}_1^e) = \dim(\mathcal{H}_1^\tau) = 1$, so $\dim(\mathcal{H}_1) = 2$. From the fusion rules, $\dim(\mathcal{H}_2^e) = 2$ and $\dim(\mathcal{H}_2^\tau) = 3$, so $\dim(\mathcal{H}_2) = 5$. Observe $5 > 2^2$. In general, we have $\dim(\mathcal{H}_n^e) = \dim\left(\mathcal{H}_{n-1}^e\right) + \dim\left(\mathcal{H}_{n-1}^\tau\right)$ and $\dim(\mathcal{H}_n^\tau) = \dim\left(\mathcal{H}_{n-1}^e\right) + 2 \cdot \dim\left(\mathcal{H}_{n-1}^\tau\right)$. This ensures that all the relevant dimensions are always Fibonacci numbers. Obtaining $\dim(\mathcal{H}_n^e) = F_{2n\text{-}1}$ and $\dim(\mathcal{H}_n^\tau) = F_{2n}$.

4.1.2 Superselection rule

The general kinematical operators we have defined in Figure 4.6 are rarely defined or used in anyon theory. The reason is that anyon theory includes an anyonic superselection rule (SSR). It can be seen as a generalisation of the fermionic parity SSR seen in Subsection 3.1.2. Usually, only the physical operators are included when presenting the anyon formalism.

The anyonic SSR can be stated as follows: operators will only be physical when the total anyonic charge is conserved. In Figure 4.6, this would mean that $a_{2m-1} = b_{2n-1}$. Therefore, we could connect the ket with the bra in the diagram [144].

The super selection rule is that it is not possible to implement an operator that changes the overall topological charge of the system. When looking at topological charges as particle types, the anyonic SSR rule impedes the transmutation of a single particle type on its own. If one has a single anyonic particle, there is no operation to change its particle type. We show that the parity SSR for fermionic systems can be interpreted in a similar fashion in Subsection 3.1.2.

Given the anyon superselection rule, the physical operators for an arbitrary 4-anyon system are shown in Figure 4.7. The physicality condition can be seen in the diagram as there is no break in the middle, indicating that the overall topological charge is conserved. In the usual Dirac vector notation, we would construct a charge observable $\hat{\mathbf{C}}$ to provide the anyon SSR. Its eigenvectors are the elements of the orthonormal tree fusion basis. The associated eigenvalues change only and necessarily with the overall global charge g of the eigenvectors. With such observable, the condition for a linear operator $\hat{O}$ to be a physical operator is that $[\hat{O}, \hat{\mathbf{C}}] = 0$. We denote the algebra of all physical operators for a fixed number of anyons n as $\mathcal{A}_n^{phys}$.

$$\hat{O}^{phys} = \sum_{i,j} \alpha_j^i \left|\psi_i\right\rangle \left\langle\psi_j\right| = \sum_{\substack{a,b,c,d,h,f,g,\\ a',b',c',d',h',f'}} o^{abcdhfg}_{a'b'c'd'h'f'}$$

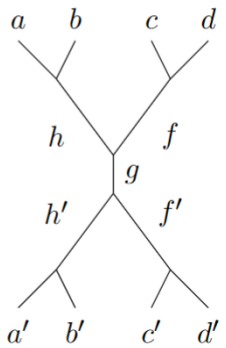


Figure 4.7: Physical operators of a four-anyon system.

Such a condition implies that, when representing a physical operator with a matrix representation on the global charge-ordered orthonormal fusion tree basis, it has a

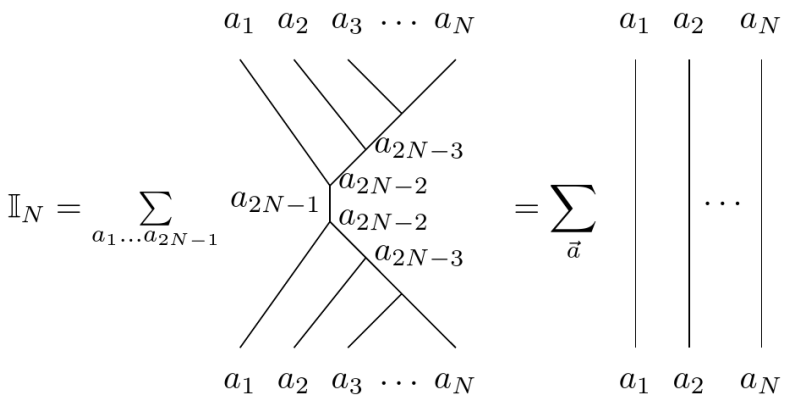


Figure 4.8: Identity operator diagram

block diagonal form. The blocks separate the subspaces of the partition given by $\mathcal{H}_n = \bigoplus_{g=1}^{n_T} \mathcal{H}_n^g$.

Under the anyon SSR, we can already consider our set of physical observables and unitaries.

$$\mathcal{O}_n^{phys} = \left\{ \hat{O} \middle| \ \hat{O} \in \mathcal{A}_n^{phys} \ \ \& \ \ \hat{O} = \hat{O}^\dagger \right\} \tag{4.3}$$

$$\begin{aligned} \mathcal{T}_n^{phys} &= \left\{ e^{i\hat{O}} \middle| \ \hat{O} \in \mathcal{A}_n^{phys} \ \ \& \ \ \hat{O} = \hat{O}^\dagger \right\} / U(1) = \\ &= \left\{ \hat{U} \middle| \ \hat{U} \in \mathcal{A}_n^{phys} \ \ \& \ \ \hat{U}\hat{U}^\dagger = \mathbb{I}_n \right\} / U(1) \end{aligned} \tag{4.4}$$

The identity operator $\mathbb{I}_n$ is, of course, the identity matrix in the matrix representation of an orthonormal basis. The diagrammatic form is shown in Figure 4.8. It is both a physical unitary and observable.

4.1.2.1 Fibonacci SSR

Let us consider Fibonacci anyons presented in Subsubsection 4.1.1.1. For $n = 1$ we choose the basis $\mathcal{B}_1 = \{|e\rangle, |\tau\rangle\}$. For $n = 2$ we choose the basis $\mathcal{B}_2 = \{|e, e; e\rangle, |\tau, \tau; e\rangle, |\tau, e; \tau\rangle, |e, \tau; \tau\rangle, |\tau, \tau; \tau\rangle\}$. Given this basis, the matrix repre-

sentation of the Fibonacci physical operators is the following.

$$\hat{O}_1 = \begin{pmatrix} a_1 & \\ & b_1 \end{pmatrix} \qquad \hat{O}_2 = \begin{pmatrix} a_1 & a_2 & & & \\ a_3 & a_4 & & & \\ & & b_1 & b_2 & b_3 \\ & & b_4 & b_5 & b_6 \\ & & b_7 & b_8 & b_9 \end{pmatrix} \tag{4.5}$$

4.1.3 Trace

We need a trace on the physical operators to define the phenomenal states as density operators. We also need the notion of trace to take expectation values of physical observables.

As usual, the trace in the kinematical operator space is inherited from the scalar product of the kinematical Hilbert space $\mathcal{H}_n$. Remember that we have defined the fusion tree diagram basis as orthonormal. Let us label the fusion tree diagram basis as $\{|i\rangle\}_{i=1}^{\dim(\mathcal{H}_n)}$. The trace in $\mathcal{A}_n$ is given by the rule $\mathrm{Tr}(|i\rangle\langle j|) = \langle j|i\rangle = \delta_{ij}$.

Diagrammatically, the trace is represented by joining all external anyon lines, as shown in Figure 4.9. Notice that the global charges' external lines have also been connected. These closed loops can be evaluated in terms of Kronecker deltas on the particle types a_j, a'_k. The evaluation is just a diagrammatic representation of the orthonormality of the fusion tree diagram basis. Sometimes, F and R matrices must be applied to untangle the diagram. The result will then depend on these F and R matrices.

The global trace we use for physical operators is the same as the kinematical trace. Therefore, we can now define the physical density operators of an n-anyon system using the trace operation. These are the phenomenal states we consider in the operational anyonic setting.

$$\mathcal{P}_n^{phys} = \left\{ \rho \;\middle|\; \rho \in \mathcal{O}_n^{phys} \,,\; \rho \geq \hat{0} \,,\; \mathrm{Tr}(\rho) = 1 \right\} \tag{4.6}$$

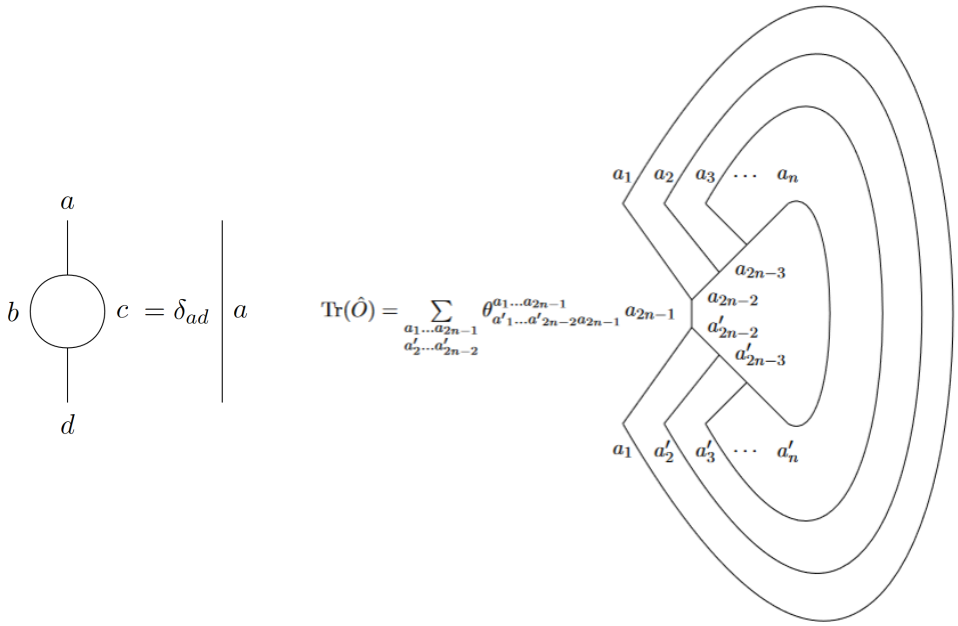


Figure 4.9: Left: Evaluation of a bubble diagram. Right: Diagrammatic trace.

4.1.4 Local operators

In this subsection, we start to address the concept of subsystem locality in anyonic theories. Firstly, we must clarify the notion of subsystem we will use for any anyon theory.

There is a strong debate in the literature on fermionic correlations regarding alternative notions of subsystem. There are proponents of understanding the subsystems as sets of fermionic modes [56, 73–83]. In this picture, the minimal subsystem is a single mode. This choice falls naturally from the second quantisation perspective of fermions and can describe any fermionic configuration and evolution. On the other hand, there are proponents of the particle picture, in which the minimal subsystem is a fermionic particle [89–96]. This formulation falls naturally from the description of fermions in the first quantisation. It provides exciting results for configurations and evolutions of a fixed number of particles. These results are

concrete and can be tested and interpreted in the lab.

For anyons, a similar debate can be presented; however, given the fusion rules, particles can change particle type by fusing and not preserving the number of particles. Arguably, the mode perspective is more general and fundamental than the particle picture. We believe that it is closer to the essence of quantum field theory and provides necessary fundamental insights. When the number and type of anyonic particles are fixed and only exchanging operations are performed, a particle picture may be insightful. Nevertheless, we have the intuition from fermions that the particle perspective can be deduced from the mode perspective results [80].

Let us now focus on the mode perspective for anyon theories. In $2+1$ D anyon theories, anyons can be at any point of the two-dimensional manifold. The changes in their configurations are tracked in the time direction. The set of anyonic modes that we will refer to throughout the paper are the positions of the modes in the two-dimensional space.

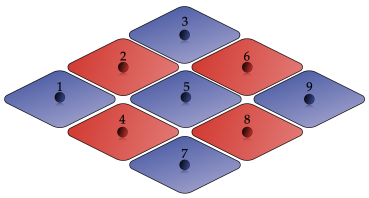


Figure 4.10: Partition of the plane in mode subregions. The union of regions $1, 2, 3, 4, 6, 7, 8$ & 9 is not simply-connected. It is not a valid subsystem.

We identify a simply connected sub-region with boundaries of our 2D space as a single mode where the different anyon types can be excited. The complete system consists of a finite number N of regions glued along their boundaries; see Figure 4.10. We prefer to keep the number of anyonic modes finite. Therefore, we use a finite 2D lattice populated by the different anyon particle types of the theory.

It is worth noting that the mode picture is justified by the presence of particle type e in all anyon theories. e can be interpreted as the vacuum particle. If a mode has the associated charge e, we claim no relevant anyons are populating such mode.

Within the mode anyon picture, we are enforcing a Pauli exclusion principle. We require that any mode is only populated by a single anyon particle type. If we wish to fine-tune the model and allow more anyons to populate for a certain region, we will change the partition of the manifold in terms of more modes.

We want to understand how to map the subsystem structure at the level of simply connected regions in the 2D manifold to planar diagrams. Notice that there are different ways to glue the boundaries between the regions to compose them into larger, simply connected regions. Even given a canonical ordering of the modes to represent them in planar diagrams, each glueing scheme has an associated planar representation (see Figure 4.11). These different planar representations correspond to different partitions of the systems given by the planar canonical basis of the anyon theory.

When defining a partition into subsystems, we are not only specifying the subregions of each mode. We are also required to show how they are connected. In any of our decompositions, there will be the elemental modes as subsystems and the collection of all modes as the global unique system. However, the lattice of subsystems we consider only has some of the possible unions of modes. We fix the ordering of the modes. We establish that all the possible unions of modes happen either in front of all the modes that are not being joined or completely behind them. This choice allows us to work within the RR formalism. In Chapter 5, we discuss the possibility of relaxing this constraint and include alternative paths.

Having identified the anyonic mode subsystems we will work with, we must assign them an algebra of physical observables. The total system of N anyonic modes is $\mathcal{N}$. We fix the order of the N anyon modes. Given a subset of in-front or behind joined modes $\mathcal{M} \subset \mathcal{N}$, their local associated algebra of observables is $\mathcal{O}^{phys}_{\mathcal{M}} = \mathcal{O}^{phys}_{|\mathcal{M}|}$. With this association, the local transformations and phenomenal

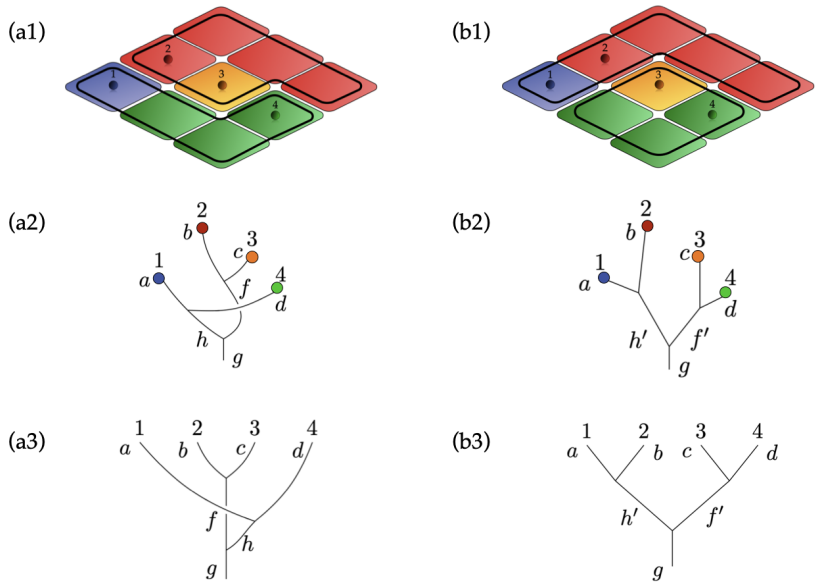


Figure 4.11: Planar representations for different compositions of regions. Sub-figure (a1) indicates that we are first fusing anyon 1 (blue) with anyon 4 (green) and anyon 2 (red) with anyon 3 (orange). In (a2), we express such a system in the diagrammatic form. The diagrammatic form is equivalent to the planar representation in sub-figure (a3). In the right column, we maintain everything but fuse anyon 1 with anyon 2 and anyon 3 with anyon 4.

states are straightforward. We have $\mathcal{T}^{phys}_{\mathcal{M}} = \mathcal{T}^{phys}_{|\mathcal{M}|}$ and $\mathcal{P}^{phys}_{\mathcal{M}} = \mathcal{P}^{phys}_{|\mathcal{M}|}$. Similarly, all local physical operators can be set $\mathcal{A}^{phys}_{\mathcal{M}} = \mathcal{A}^{phys}_{|\mathcal{M}|}$.

We want to distinguish the cases where the subsystem $\mathcal{M}$ is considered to be joined in front or behind the rest of the modes. To do so, we introduce the behindness property. We say an operator has behindness in-front or behind, referring to which direction its modes are joined with respect to others. Behindness is a property that the local observables inherit from the subsystem notion. We may introduce some concepts using behindness in-front only, but one needs to keep in mind that behind is also a possibility.

4.1.4.1 Embedding of physical local operators

The local physical operators of $\mathcal{M}$ must be embedded in the local physical operators of $\mathcal{M}'$ if $\mathcal{M} \subset \mathcal{M}'$. Let us do it in two steps.

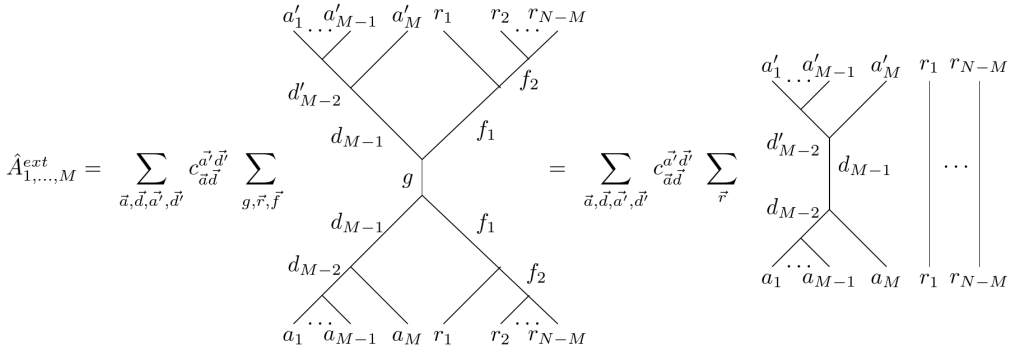


Figure 4.12: Embedding of an arbitrary physical operator local in $1, \dots, M$.

First, consider a local physical operator $\hat{A}_{1\dots M}$ on modes $1, \dots, M$, and we want to embed it to $\mathcal{N}$. The embedding of $\hat{A}_{1\dots M}$ is shown in Figure 4.12. It can be seen as the way to embed is to split $\hat{A}_{1\dots M}$ into its ket and bra part, then fuse the overall local charge of $\hat{A}_{1\dots M}$ with any possible state of the rest of the $N - M$ modes, and sum over all the possibilities keeping the same coefficients.

This procedure is analogous to extending a local operator $\hat{O}_A$ on $\mathcal{H}_A$ to $\sum_i \hat{O} \otimes |i\rangle\langle i|$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ in distinguishable quantum systems. However, in anyonic systems, it is only defined consistently for physical operators.

Once the embedding from local operators on modes $1, \dots, M$ is clear; we need to explain the general case of any subset of modes $\mathcal{M} = \{s_1, \dots, s_M\}$. We consider the subsystems where the joining of all the anyonic modes happens in front of the non-joined modes. Thus, by only exchanging the anyonic modes counterclockwise, we can bring a local observable in $1, \dots, M$ to a local observable in $s_1, \dots, s_M$. Concretely, we use the unitary $U = \prod_{i=0}^{M-1} \prod_{j=M-i+1}^{s_{M-i}} R^\dagger_{j-1\ j}$. Where $R_{j-1\ j}$ is the counterclockwise exchange of the anyons in modes j - 1 and j, given by the R-

matrix [126, 142]. Using the following transformation, we transform any extended local observable in $1 \dots M$ to a local observable in $s_1, \dots, s_M$.

$$\hat{A}_{s_1,\dots,s_M} = U^\dagger \cdot \hat{A}_{1,\dots,M} \cdot U \tag{4.7}$$

It is straightforward to check that $s_1, \dots, s_M$ have been joined in front of the modes $\mathcal{N}\backslash\mathcal{M}$. The explained procedure describes the extension of a local observable in $s_1, \dots, s_M$. First, extend its local representation as it was local on $1, \dots, M$. Then, exchange the modes to 'put the observable to the right place'. This process gives us the necessary embeddings of the local unitaries, providing us with the mechanism that allows us to see $\mathcal{T}^{phys}_{\mathcal{M}}$ in correspondence with a subgroup of $\mathcal{T}^{phys}_{\mathcal{N}}$.

We can embed the local physical operator to be a behind operator as well. We do so by constructing a similar $\hat{\tilde{U}}$ using $R^\dagger_{j\text{-}1\ j}$ instead.

4.1.4.2 Fibonacci local unitaries

Let us provide a simple example of how embedded simple local unitaries look in the matrix representation of Fibonacci anyons. Imagine a simple two-mode Fibonacci anyon system. The local unitaries in mode 1 and mode 2 are extremely simple. We use the matrix representations described in Subsubsection 4.1.2.1. The anyon SSR restrictions impose that the most general form they take is: [1]

$$\hat{U}_1 = \begin{pmatrix} 1 & \\ & e^{i\phi_{\tau,1}} \end{pmatrix} \qquad \hat{U}_2 = \begin{pmatrix} 1 & \\ & e^{i\phi_{\tau,2}} \end{pmatrix} \tag{4.8}$$

And their embedding into the two-mode Fibonacci system is:

$$\hat{U}^{ext}_1 = \begin{pmatrix} 1 & 0 & & & \\ 0 & e^{i\phi_{\tau,1}} & & & \\ & & e^{i\phi_{\tau,1}} & 0 & 0 \\ & & 0 & 1 & 0 \\ & & 0 & 0 & e^{i\phi_{\tau,1}} \end{pmatrix} \qquad \hat{U}^{ext}_2 = \begin{pmatrix} 1 & 0 & & & \\ 0 & e^{i\phi_{\tau,2}} & & & \\ & & 1 & 0 & 0 \\ & & 0 & e^{i\phi_{\tau,2}} & 0 \\ & & 0 & 0 & e^{i\phi_{\tau,2}} \end{pmatrix} \tag{4.9}$$

[1] We have fixed the global phase redundancy to cancel the e phase.

4.1.4.3 Extended local operators?

The reader might wonder why we have restricted the extension to physical operators. Why not use the same strategy for embedding all local operators?

First, notice that the extension procedure shown in Figure 4.12 necessarily provides a global physical operator. Provided a local non-physical operator cannot be represented by an extended physical operator, we would need to modify the procedure per se. Let us focus on a natural modification one could do. Let us extend any local linear operator by fusing their tree components with any possible ancillary state, summing over all the possibilities. Given the usual tensor product procedure and our interpretation of the anyonic physical embedding, such a procedure is natural.

In this case, important problems emerge. Consider the non-physical Fibonacci local unitary in mode 1 given by the creation of a τ anyon from the vacuum and the annihilation of a τ anyon to the vacuum. Its local matrix representation and the matrix representation of its extension are:

$$\hat{U}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{U}_1^{ext} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{4.10}$$

We can see that the extended operator is not unitary. Thus, the local structure is not faithfully represented. Trying to renormalise the extension does not help. We need to find a deeper, non-trivial algebraic structure to capture the local non-physical properties on the global space.

There is no known extension of local non-physical operators, rendering the use of the kinematical space practically useless. Nevertheless, restricting to physical operators is enough for the development of topological quantum computing and anyonic theory since only observational features are important.

4.1.5 Partial trace

We define the partial trace of a subsystem of anyons only for physical operators. The diagrammatic procedure is similar to the total trace. To partial trace a set of anyonic modes, we connect the outgoing and incoming lines corresponding to such anyons. Given the choice of in-front observables, the associated partial tracing procedure is to loop the traced modes behind the non-traced modes.

The procedure is exemplified in Figure 4.13. The procedure is justified by expressing the consistency conditions of the partial trace in diagrammatic form. The consistency conditions define the partial tracing operation as the unique linear operation that satisfies the following equations for all $\hat{O}_{\mathcal{M}}$ local physical observables and all physical states $\rho_{\mathcal{N}}$:

$$\mathrm{Tr}\Big(\hat{O}^{ext}_{\mathcal{M}}\rho_{\mathcal{N}}\Big) = \mathrm{Tr}\Big(\hat{O}_{\mathcal{M}}\,\mathrm{Tr}_{\mathcal{N}\setminus\mathcal{M}}\left(\rho_{\mathcal{N}}\right)\Big) \tag{4.11}$$

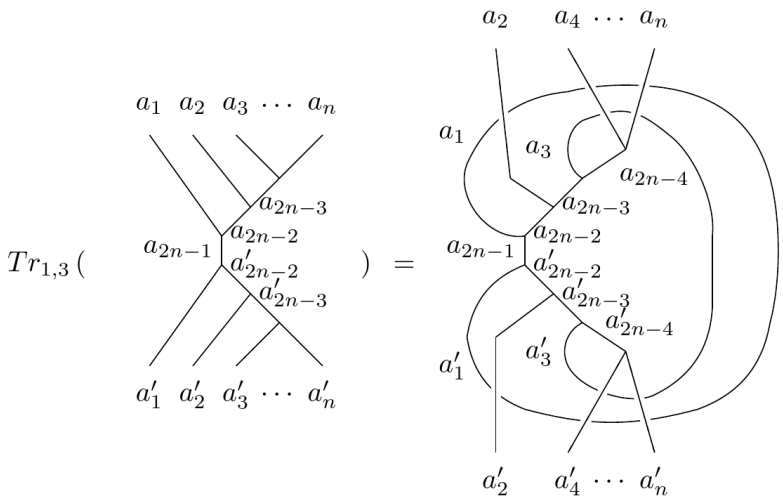


Figure 4.13: Behind partial trace of modes $1, 3$ of an n-anyon physical operator component.

Figure 4.14 shows how the consistency conditions lead to the diagrammatic defini-

tion of the anyonic partial trace for in-front physical operators. The local operator given by behind partial tracing modes 2 & 4 satisfies the equality with the elements in the right-hand side of the figure.

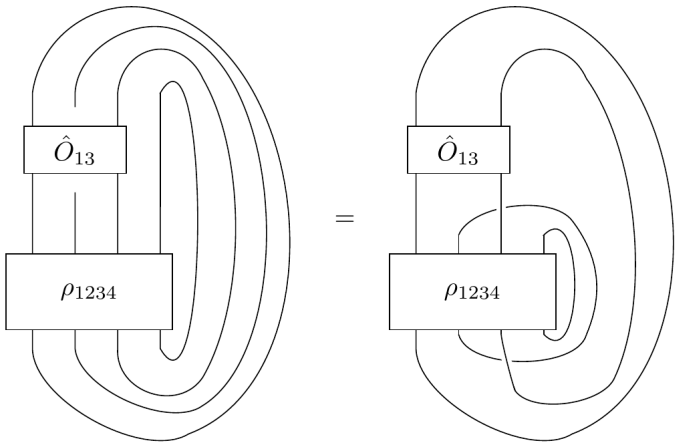


Figure 4.14: Left-hand side expression of the consistency condition of the in-front partial trace of anyonic modes $2, 4$ of a 4-anyon system.

4.1.6 No-signalling

Physical operators, the operators under the SSR, satisfy that local operators in disjoint subsystems commute. Commuting disjoint local operators under the SSR is also a feature in fermionic systems, as we show in Subsection 3.1.2. Proposition 9 formalises this commuting property.

Proposition 9. *For any two disjoint sets of modes $\mathcal{M} \cap \mathcal{M}' = \emptyset$, their respective in-front and behind physical anyon local operators commute.*

Proof. The proof follows easily from the diagrammatic rules of representing physical local operator extensions. It can be summarised by the diagram equality of Figure 4.15. One can also painfully calculate the algebraic equality using a fusion tree basis decomposition of the embedded local operators. □

In Figure 4.16, we introduce handy notation for physical operators in bipartite systems. A and B denote sets of disjoint modes. The vector notation $\vec{a}, \vec{b}$ takes care of all the charge specifications for these subsets of modes. Moreover, the modes A and B are not necessarily ordered.

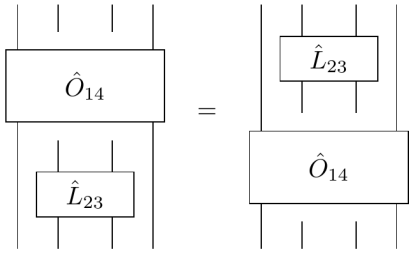


Figure 4.15: diagram showing the commutation of local disjoint physical operators.

Since we can assume all modes of one subset always cross behind the modes in the other subset, we can denote them separately. One should imagine that the vector represents a fusion tree on its own. Such fusion trees may exchange positions of the strands, always one behind the other in the same direction in both the top and bottom parts of the diagram.

Proposition 9 leads to the no-signalling principle being satisfied for anyon theories. Consider a system being the composition of two disjoint sets of anyonic modes $\mathcal{M} \cup \mathcal{M}'$. The no-signalling principle for anyon systems is that the following equations are satisfied for any $\hat{U}_{\mathcal{M}} \in \mathcal{T}^{phys}_{\mathcal{M}}$ and any $\rho_{\mathcal{M}\cup\mathcal{M}'} \in \mathcal{P}^{phys}_{\mathcal{M}\cup\mathcal{M}'}$.

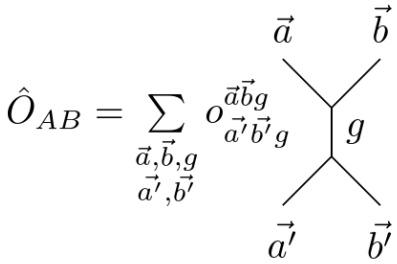


Figure 4.16: notation for operators in bipartite systems.

$$\mathrm{Tr}_{\mathcal{M}}\left(\hat{U}^{ext}_{\mathcal{M}} \cdot \rho_{\mathcal{M}\cup\mathcal{M}'} \cdot \hat{U}^{\dagger,\,ext}_{\mathcal{M}}\right) = \mathrm{Tr}_{\mathcal{M}}\left(\rho_{\mathcal{M}\cup\mathcal{M}'}\right) \tag{4.12}$$

Proof. Notice that the partial trace being considered must be associated with the behindness of the physical operators in $\mathcal{M}$. In other words, if the physical operators in $\mathcal{M}$ are all behind operators, the physical operators in $\mathcal{M}'$ are all in-front.

Therefore, if we express the consistency conditions of the partial trace, we obtain

that the no-signalling conditions are equivalent to the following set of equations.

$$\forall \hat{O}_{\mathcal{M}'} \in \mathcal{O}^{phys}_{\mathcal{M}'} : \ \mathrm{Tr}\Big(\hat{O}^{ext}_{\mathcal{M}'} \cdot \hat{U}^{ext}_{\mathcal{M}} \cdot \rho_{\mathcal{M}\cup\mathcal{M}'} \cdot \hat{U}^{\dagger,ext}_{\mathcal{M}}\Big) = \mathrm{Tr}\Big(\hat{O}^{ext}_{\mathcal{M}'}\rho_{\mathcal{M}\cup\mathcal{M}'}\Big) \quad (4.13)$$

Using the cyclic nature of the total trace and the commutation properties of Proposition 9, we obtain that; indeed, the conditions hold.

$$\begin{aligned}&\mathrm{Tr}\Big(\hat{O}^{ext}_{\mathcal{M}'} \cdot \hat{U}^{ext}_{\mathcal{M}} \cdot \rho_{\mathcal{M}\cup\mathcal{M}'} \cdot \hat{U}^{\dagger,\ ext}_{\mathcal{M}}\Big) = \mathrm{Tr}\Big(\hat{U}^{\dagger,\ ext}_{\mathcal{M}} \cdot \hat{O}^{ext}_{\mathcal{M}'} \cdot \hat{U}^{ext}_{\mathcal{M}} \cdot \rho_{\mathcal{M}\cup\mathcal{M}'}\Big) = \\ &= \mathrm{Tr}\Big(\hat{O}^{ext}_{\mathcal{M}'} \cdot \hat{U}^{\dagger,\ ext}_{\mathcal{M}} \cdot \hat{U}^{ext}_{\mathcal{M}} \cdot \rho_{\mathcal{M}\cup\mathcal{M}'}\Big) = \mathrm{Tr}\Big(\hat{O}^{ext}_{\mathcal{M}'}\rho_{\mathcal{M}\cup\mathcal{M}'}\Big) \ \ \forall \hat{O}_{\mathcal{M}'} \in \mathcal{O}^{phys}_{\mathcal{M}'}\end{aligned} \quad (4.14)$$

Therefore, anyonic systems satisfy the no-signalling principle. □

We conjecture that, similarly to the fermionic case, the anyonic SSR is necessary to satisfy the no-signalling principle. Unfortunately, since we do not know of a faithful embedding of local non-physical operators to larger mode systems, we cannot postulate the unique partial tracing procedure for general anyonic operators. Henceforth, we cannot establish the violation or not of the no-signalling principle in these unrestricted systems.

4.1.7 Local tomography

Our main goal in this chapter is to identify which mathematical elements can be considered anyonic descriptors. The second is to expose how these can be used to represent the local-realistic structure of anyonic theory.

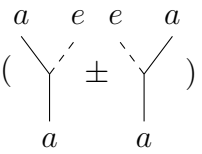
We have seen how anyonic systems have a diagrammatic formulation which inherently incorporates a particle-type superselection rule. The first place to look for anyonic descriptors would be the anyonic physical local observables.

In Section 3.1.6, we show we cannot use local physical observables as descriptors in fermionic systems. One hopes that the extra fusion structure of non-abelian anyons will remedy such a pathological case. In the following lines, we show that this is not the case. As fermions, any general anyonic theory does not satisfy local tomography.

The principle of local tomography is widely used in the reconstructing program of quantum theory [99, 100]. It is one of its axioms to derive quantum theory from first principles. Nevertheless, this property is not generally satisfied in constrained quantum systems or physical indistinguishable quantum particle systems.

A theory satisfies local tomography if the coordinated measurement of local observables can fully describe the global phenomenal state of the system. In quantum theories, the condition is given by stating that in a bipartite system AB, ρ_{AB} can be deduced from $\mathrm{Tr}\left(\hat{O}_A^{ext} \cdot \hat{O}_B^{ext} \cdot \rho_{AB}\right)$ for any $\hat{O}_A \in \mathcal{O}_A^{phys}$ and any $\hat{O}_B \in \mathcal{O}_B^{phys}$.

Let us show that any anyon theory violates local tomography. We introduce the standard notation where a dashed anyon line in a diagram represents that its associated particle type is the identity particle e. In a bipartite 2-mode anyon system, AB, consider the following two orthogonal distinguishable physical pure anyon states:

$|\psi\rangle_\pm = \frac{1}{\sqrt{2}}($  $)$ (4.15)

where $a \neq e$. Building the general form of the embedded local observables in

modes 1 and 2 respectively, yields:

$$\hat{O}_A^{ext} = \sum_{a,b,c} o_a \;\; , \;\; \hat{O}_B^{ext} = \sum_{a,b,c} o_b \;\; , \;\; \hat{O}_A^{ext} \cdot \hat{O}_B^{ext} = \sum_{a,b,c} o_a o_b \tag{4.16}$$

Calculating $\mathrm{Tr}\Big(\hat{O}_A^{ext} \cdot \hat{O}_B^{ext} \cdot |\psi_\pm\rangle\langle\psi_\pm|\Big)$, we obtain:

$$\sum_{a',b,c} \frac{o_{a'} o_b}{2} \, Tr(\;\; (\;\; + \;\; \pm \;\; \pm \;\;)) \; =$$

$$= \frac{o_{a'} o_b}{2} \left(\delta_{aa'}\delta_{be} + \delta_{ab}\delta_{a'e}\right) \tag{4.17}$$

All the obtained values are the same without depending on the $\pm$ factor. Therefore, local observations cannot discriminate the two orthogonal states $|\psi_\pm\rangle$. Henceforth, local tomography is not satisfied in any anyonic system.

The culprit is again the superselection rule. It restricts the local observables to such an extent that there are not enough parameters in all the physical local observables to recover the parameters from the global physical observables.

The violation of local tomography impedes us from taking the anyonic descriptors to be any set of local observables. Thus, we need to find the mathematical structure that can represent anyonic descriptors elsewhere.

4.2 Anyonic annihilation operators

This section finds the mathematical expressions for annihilation operators for any 2+1 D anyon theory, even non-abelian anyons. *The contents of this section are original work presented in [104] written in collaboration with Lucia Vilchez-Estevez.*

We want anyonic annihilation operators to use them as anyonic descriptors. Unfortunately, the current anyonic formalism does not include annihilation operators. The main reason is the lack of a faithful embedding of local non-physical operators. Drawing connections with the fermionic case in Section 3.2, we expect anyonic annihilation operators to be local non-physical operators.

We use the properties of local realism to find candidates for anyonic annihilation operators. We expect the annihilation operator of mode i to be invariant under local transformations on $\mathcal{N}\backslash\{i\}$. We want such property because we expect no action at a distance. We expect the anyonic annihilation operator to be an element of the local reality of i, and these, if no action at a distance holds, should not vary under a local remote transformation.

If our system consists of modes $\mathcal{M} = \{1, \dots, m\}$, we can say that an in-front candidate local operator in mode $i \in \mathcal{M}$ is an extended operator $\hat{O} \in \mathcal{A}_{\mathcal{M}}$ such that is invariant under the action of all behind local unitaries in the modes $\mathcal{M}\backslash\{i\}$. In equation form that reads as: $\hat{O}$ is an in-front candidate local operator on mode $i \in \mathcal{M}$ if and only if

$$\hat{U}^{\dagger,\,ext}_{\mathcal{M}\backslash\{i\}} \cdot \hat{O} \cdot \hat{U}^{ext}_{\mathcal{M}\backslash\{i\}} = \hat{O} \tag{4.18}$$

for all $\hat{U}_{M\backslash\{i\}} \in \mathcal{T}^{phys}_{\mathcal{M}}$ being behind.

It is not difficult to check that the conditions in Figure 4.18 give that the collection of all in-front candidate extended local operators in i form an algebra under the usual sum and operator multiplication, and $\mathbb{C}$ as scalars. So, we can say that we have an abstract definition of the algebra of in-front candidate extended local operators in mode i.

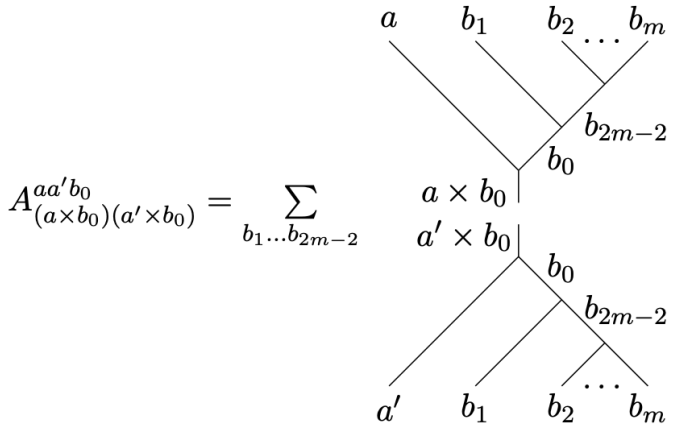


Figure 4.17: Basis elements of the in-front local operator algebra for the first mode.

Using the diagrammatic approach for anyons, we can characterise the allowed local unitaries and explore the candidate local operators for any given mode and behindness. We solve Equation 4.18 that defines candidate local operators using the diagrammatic formalism, and we find the general form of a candidate local operator on an anyonic mode. We show it here for the first mode [2]. We express the general form of a candidate local operator on mode 1 in terms of linear combinations of the elements of a canonical basis:

$$\hat{O}_1 = \sum_{\substack{a,a',b_0 \\ d=a\times b_0, d'=a'\times b_0}} c_{a,a',b_0,d,d'}\ A^{aa'b_0}_{dd'} \tag{4.19}$$

where $c_{a,a',b_0,d,d'} \in \mathbb{C}$ and the canonical basis of the candidate local operator algebra

[2] In mode 1 both behindness yield the same expression

for mode 1 given by the terms $A_{dd'}^{aa'b_0}$ can be seen in Figure 4.17 as planar diagrams.

We use these basis elements to identify components where the first mode is transformed to the vacuum, as an annihilation operator component would. Only one anyon type can be in the same mode in anyon diagrams. Therefore, the components of the anyonic annihilation operators should consist only of terms that send anyon particle types to the vacuum and not any other particle type. In Figure 4.18, one can observe that if we fix the particle type $a \neq e$ in mode 1 bra and the vacuum e in the mode 1 ket, the basis components then depend only on the global charge of the rest of the system b_0 and the term $a \times b_0$, since e is an abelian particle and then $e \times b_0$ is always b_0.

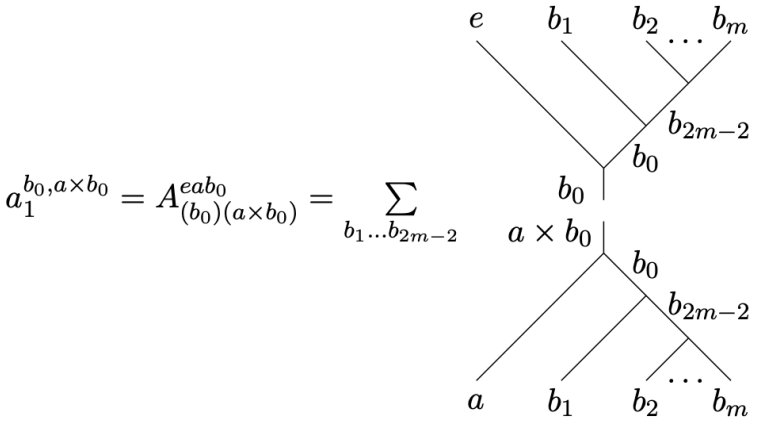


Figure 4.18: Annihilating elements of the basis of local operators for mode 1.

Thus, we realise that the number of annihilation elements that a particle type a has associated in a mode is the number of fusion channels that such particle type has associated with it. This result comes directly from the explicit dependency of having the different annihilating components from $a' \times b_0$, being b_0 any particle type. Thus, all fusion channels of a' will have an associated annihilating element.

For notation, we label each of these annihilation elements of the canonical basis $a_1^{b_0,a\times b_0} = A_{b_0 a\times b_0}^{eab_0}$ (where 1 expresses the fact they are annihilating on the first

mode, b_0 and $a \times b_0$ specify the fusion channel and annihilating term, and a is the particle type being annihilated). In all the above and the following expressions, one must remember that $a \neq e$.

We will refer to the Hermitian conjugate of such annihilating elements as the creating elements. By direct calculation, we find two exciting results. First, the annihilating and creating elements of mode j with some behindness are generators of the candidate local algebra of mode j for such behindness. It follows from the fact that any element of the basis in Figure 4.17 can be written in terms of the local creating and annihilating elements. The second is that the collection of all in-front or behind annihilating and creating elements are generators of the global kinematical operator algebra.

Let us remark on this crucial point. We have seen that the annihilating elements of Figure 4.18, together with their adjoints, are generators of the candidate local operator algebra. Having obtained these results, we now naturally wonder if the annihilation operators we seek are these annihilating elements.

We think they are not. However, annihilation operators must be concrete linear combinations of these annihilating elements. In other words, we find that the annihilating elements are components of the annihilation operators, and now we have to decide the right way to combine them.

We have these insights by analysing the annihilation operators of spinless fermionic theory in a finite lattice presented in Section 3.1. Let us fix the simple setting of having two spinless fermionic modes in $2 + 1$ D.

We have a vacuum $|\Omega\rangle$ and two annihilation operators $\hat{f}_1, \hat{f}_2$ that anticommute. We can represent this theory as an abelian anyon theory with two particle types: a fermion ψ and the vacuum e [126]. The non-trivial fusion rule is to specify the fermion as its antiparticle type $\psi \times \psi = e$. One needs to understand these particle

types as the fermionic parity charge. e represents the system being even and ψ being odd. The only non-trivial R-matrix component is $R_e^{\psi\psi} = -1$.

It is straightforward to see that if we associate each annihilating element with an annihilation operator, we find that instead of a single annihilation operator $\hat{f}_i$ per mode, we have two annihilation operators per mode: $\psi_i^{e,\psi}$ and $\psi_i^{\psi,e}$ (see Figure 4.18 when replacing $a = \psi$ and summing over the two particle types e and ψ). Therefore, this assignment cannot be the correct one. However, by using matrix representations, we can observe that

$$\hat{f}_1 = \psi_1^{e,\psi} + \psi_1^{\psi,e} \qquad \hat{f}_2 = \psi_2^{e,\psi} - \psi_2^{\psi,e} \tag{4.20}$$

These relations imply that the fermionic annihilation operators are linear combinations of the annihilation components. In the following lines, we derive which exact linear combinations can be taken to get the annihilation operators.

Concretely, we are proposing that the annihilation operators will be of the form:

$$\alpha_k^{(j)} = \sum_{b_0,c_0=a\times b_0} C^{(j)}_{b_0,c_0,k}\, a_k^{b_0,c_0} \tag{4.21}$$

where $C^{(j)}_{b_0,c_0,k} \in \mathbb{C}$. The term α refers to the fact of being the annihilation operator of the particle type a. The label (j) indicates that we may need more than one annihilation operator per particle type, given that we have an annihilating element per fusion channel.

To constrain the coefficients $C^{(j)}_{b_0,c_0,k}$ we consider three conditions that the annihilation operators $\alpha_k^{(j)}$ need to satisfy. The first is that $\{\alpha_{k_1}^{(j)}, \ldots, \alpha_{k_m}^{(j)}\}_{j,\alpha}$ and their adjoints generate the local algebra of observables in the modes $k_1, \ldots, k_m$ with heir associated behindness.

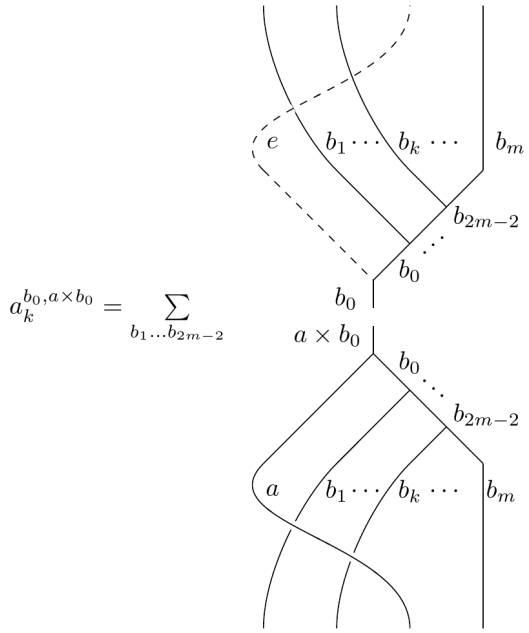


Figure 4.19: In-front annihilating elements of the basis of local operators for the $k+1$th-mode.

Second, we require that to obtain $\alpha_k^{(j)}$ we only need to know $\alpha_1^{(j)}$ and exchange our way through to k. This requirement comes from the intuition that if one wants to annihilate a particle in k, it should be equivalent to bringing that particle to 1, annihilating it there and then undoing the path we have taken. Figure 4.19 shows that the concrete path we take is the chain of simple counterclockwise exchanges. One could pose different paths giving different annihilation operators. The annihilation operators under this in-front path are obtained when imposing the condition of Equation 4.18, for behind unitaries in $\mathcal{M}\backslash\{k\}$.

Concretely, the path taken guarantees that the physical observables we will obtain will all be in-front local physical observables. We can obtain different annihilation operators that give behind local observables by exchanging clockwise instead. Each notion of subsystem lattice will have an associated different set of local annihilation operators. We specify the in-front case in the following lines.

The braiding condition imposes the following recursive relation to constrain the coefficients $C^{(j)}_{b_0,c_0,k}$

$$\alpha^{(j)}_k = R_{k-1k} \cdot \alpha^{(j)}_{k-1} \cdot R^{\dagger}_{k-1k} \tag{4.22}$$

In the fermionic example that we pose in Equation 4.20, we can see how the -1 term in $\hat{f}_2$ arises from the R-matrix element $\left(R^{\psi\psi}_e\right)^{-1}$.

The third requirement is that for every b_0, j, k, there is at least one term $C^{(j)}_{b_0,c_0,k}$ that is non-zero. It ensures that the annihilation operators $\alpha^{(j)}_k$ have support on any total charge value for modes other than k. It explicitly prevents situations where the annihilating terms can be considered annihilation operators and have redundancy.

We have found a solution to these three constraints. Thus we have found a way to define annihilation operators in anyonic systems. For the solution we propose, the $C^{(j)}_{b_0,c_0,1} \in \mathbb{C}$ we set them to be either 0 or 1. However, one could modify our presented solution, including different non-zero factors to the terms that are 1.

The number of annihilating elements in a mode for the anyon type a is $n_a = \sum_{bc=1}^{n} N^c_{ab}$. Following our general construction, the number of annihilation operators associated with this anyon type a for a given mode will be $J = n_a - n + 1$, where n is the total number of particle types in the theory. Notice that with this scheme, we find that for an abelian anyon particle type a, there is a single annihilation operator, since for abelian anyon types $n_a = n$ because there are no multiplicities in the fusion channels associated with a.

4.2.1 Fibonacci annihilation operators

We show how to construct the J annihilation operators for any anyon theory in Appendix C.1. To make the main text of the thesis not extremely lengthy, we show here the construction for the simplest non-abelian case: Fibonacci anyons.

We order the Fibonacci particle types as e, τ of the different allowed fusion channels. We label $c_{b_0,j}$ the j'th particle type such that $c_{b_0,j} = \tau \times b_0$. For the first annihilation operator of τ, we set the terms $C^{(0)}_{b_0,c_{b_0,1},1} = 1$ and the rest, $C^{(0)}_{b_0,c_{b_0,j},1}$, vanish. This implies that $\alpha^{(0)}_1$ is given by the coefficients being $C^{(0)}_{e,\tau,1} = 1$, $C^{(0)}_{\tau,e,1} = 1$, and $C^{(0)}_{\tau,\tau,1} = 0$.

To define $\alpha^{(1)}_1$, we look at the first b_0 with more than one compatible c_0. In this case, this is $b_0 = \tau$. Now all coefficients remain the same as in $\alpha^{(0)}_1$ except for setting $C^{(1)}_{\tau,c_{b_0,2},1} = 1$ and $C^{(1)}_{\tau,c_{b_0,1},1} = 0$. Implying that $\alpha^{(1)}_1$ is given by the coefficients being $C^{(1)}_{e,\tau,1} = 1$, $C^{(1)}_{\tau,e,1} = 0$, and $C^{(1)}_{\tau,\tau,1} = 1$.

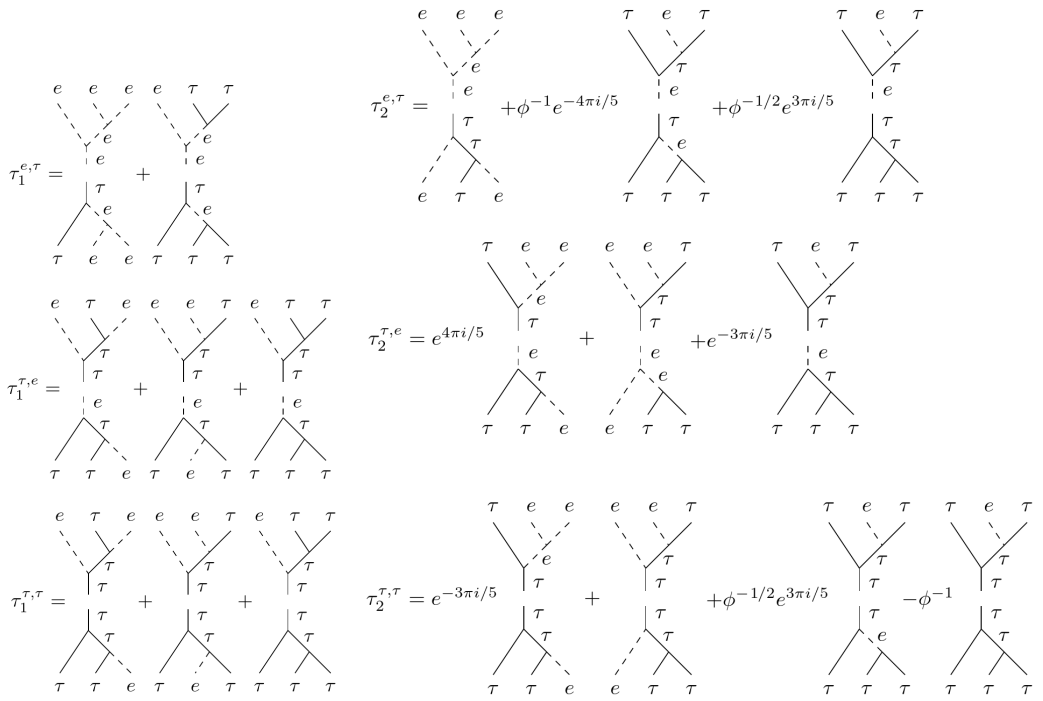


Figure 4.20: In-front annihilation elements acting on the first ($\tau_1^{b_0,c_0}$) and second mode ($\tau_2^{b_0,c_0}$) of a three Fibonacci anyon system.

We would follow the construction to find $\alpha^{(2)}_1$ by applying the same changes but with $c_{\tau,3}$. However, there is no such valid fusion channel. Then we would proceed to the next b_0 following the ordering for which $c_{b_0,2}$ exists and follow the same

procedure. In the Fibonacci case, there is no next b_0. Thus the construction has been completed.

We obtain for the Fibonacci case that τ has $J = 2$ annihilation operators. See Figure 4.20 for a diagrammatic representation of the Fibonacci annihilating elements for a three-anyon Fibonacci space. The two Fibonacci annihilation operators are:

$$\alpha_k^{(0)} = \tau_k^{e,\tau} + \tau_k^{\tau,e} \qquad \alpha_k^{(1)} = \tau_k^{e,\tau} + \tau_k^{\tau,\tau} \tag{4.23}$$

Theorem 10 shows the desired properties of anyonic annihilation operators. From it, we obtain that the annihilation and creation operators for a set of modes generate the local algebra of in-front observables for such a set of modes. As a corollary, we obtain that collecting all in-front annihilation and creation operators for all modes can generate the global algebra of operators and observables in particular.[3]

Theorem 10. *Consider a general anyon theory with n particle types and N lattice sites. Consider a set of lattice sites $\mathcal{M} = \{s_1, \ldots, s_M\}$ and the subsystem bipartition where the selected sites are always in front of the other $N - M$ sites. Under this bipartition, any local observable in these M sites can be written as a polynomial of these lattice sites' creation and annihilation operators.*

The complete general proof can be found in Appendix C.2. Its idea is first to prove the case $\mathcal{M} = \{1, \ldots, M\}$ using the following equality:

$$\hat{O}_{\vec{a},\vec{d},g} = \prod_{j=2}^{M} \left(\sum_{\substack{b_{M\text{-}j+2} \\ c_{M\text{-}j+2}}} \left[F_g^{d_{M-j} a_{M\text{-}j+2} b_{M\text{-}j+2}}\right]^*_{d_{M\text{-}j+1} c_{M\text{-}j+2}} (a_{M\text{-}j+2})^{b_{M\text{-}j+2}, c_{M\text{-}j+2}}_{M\text{-}j+2} \right) \cdot \sum_{b_1} (a_1)^{b_1, g}_1 \tag{4.24}$$

[3]There is a corresponding theorem for behind operators as well.

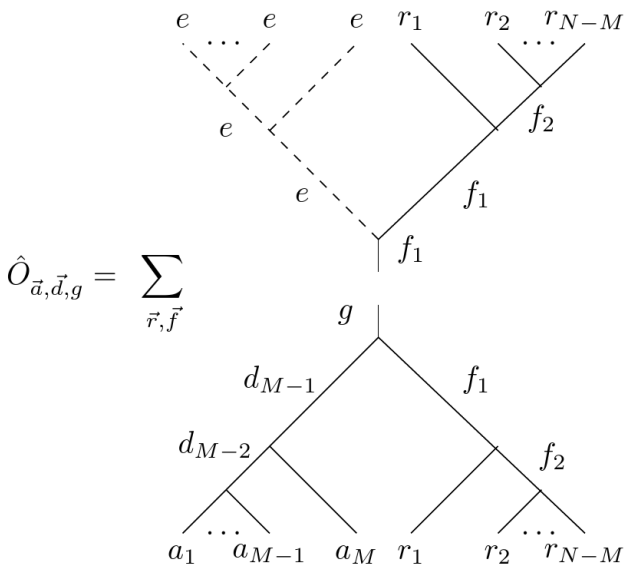


Figure 4.21: Operators that generate the local observables in $1, \ldots, M$

where $\hat{O}_{\vec{a},\vec{d},g}$ are shown in Figure 4.21 and generate the local algebra of physical observables. Then, the proof uses the in-front exchange R-matrix properties to extend it to a general subset of modes $\mathcal{M}$.

4.2.2 Fibonacci physical observables in terms of the anyonic creation and annihilation operators

We have defined the anyonic creation and annihilation operators. We are ready to showcase their uses. We focus on the Fibonacci example for pedagogical convenience.

We start by looking at the annihilating elements for three Fibonacci anyons. Figure 4.20 shows the three in-front annihilating elements for a Fibonacci anyon τ in the left lattice site 1 and central lattice site 2. Note that the operators acting on the site 2, $\tau_2^{b_0,c_0}$, can be obtained from $\tau_1^{b_0,c_0}$ by counterclockwise exchanging the anyons on modes 1 and 2. We express all the operators in the canonical basis by using the F and R matrices.

In Fibonacci anyons, we have to define two in-front annihilation operators $\alpha_k^{(1)}$,$\alpha_k^{(0)}$ for the Fibonacci τ particle type. Both operators use the term $\tau_k^{e,\tau}$. To have better algebraic properties, we choose to add a factor of $\frac{1}{\sqrt{2}}$ in front of the repeated terms. We call these two unnormalised annihilation operators: α_k and β_k. We use them throughout the rest of the text.

$$\alpha_k = \frac{1}{\sqrt{2}}\tau_k^{e,\tau} + \tau_k^{\tau,e}, \qquad \beta_k = \frac{1}{\sqrt{2}}\tau_k^{e,\tau} + \tau_k^{\tau,\tau}. \tag{4.25}$$

In Figure 4.22, we see how some in-front local observables in modes 1 & 2 can be expressed in terms of the in-front local creation and annihilation operators of such modes. Appendix C.3 contains an exhaustive list of all observable terms.

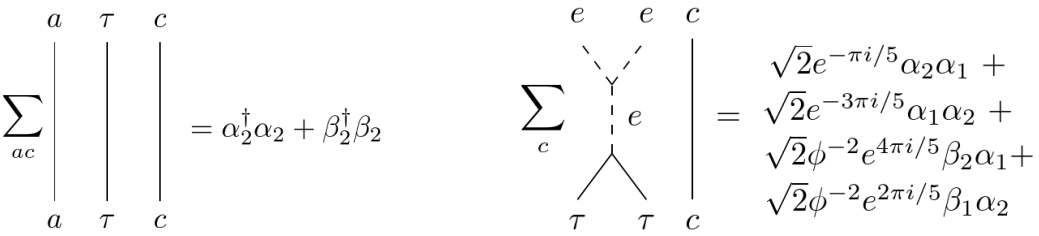


Figure 4.22: Expression of Fibonacci observables regarding anyonic creation and annihilation operators.

A straightforward application of anyonic creation and annihilation operators is to express Hamiltonians more conveniently. By expressing Hamiltonians using annihilation operators, we hope to showcase the similarities and differences between Fibonacci anyons and other particle types such as fermions and bosons. Second, we hope to provide tools for the simulation of such Hamiltonian systems, allowing the application of tensor-networks methods [145], explore mapping for applying the Bethe ansatz [146] and other methods already used in the $1+1$ D case where the notion of annihilation operators is exploited [138–140].

4.2.3 Hubbard anyon model

We focus on the Hubbard anyon Hamiltonian described in [147]. We have a $2 \times N$ square lattice with the ordering shown in Figure 4.23. This ordering is chosen to guarantee that physical in-front operators can describe all the two-mode interactions.

The Hamiltonian has two contributions. First, a hopping contribution between nearest neighbours, where a τ-anyon can jump to the nearest neighbour if it is unoccupied. The second term is a self-energy term for when there is a τ in some site. For simplicity and conciseness, we take the same coupling strength for longitudinal and transverse hopping $t_\perp = t_\parallel = t$ [147].

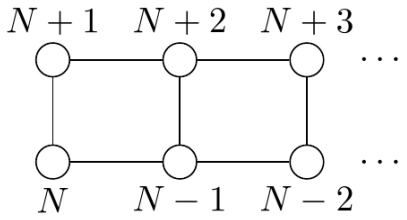
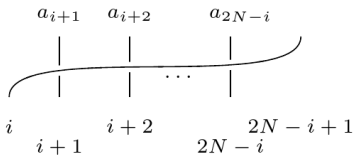
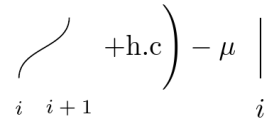


Figure 4.23: Lattice of model and chosen ordering for a $2 \times N$ lattice.

$$\hat{H} = -t \sum_{i=1}^{N-1} \sum_{a_{i+1} \ldots a_{2N-i}}$$



$$+ \sum_{i=1}^{2N-1}$$



$$- \mu$$

Figure 4.24: Hubbard Hamiltonian for Fibonacci anyons.

The unnormalised annihilation operators α_k, β_k allow us to express the Hamiltonian more compactly. It can be expressed without using the unnormalised annihilation operators, but the expression obtained is not as clean and clear as the one obtained

using them, which is:

$$\hat{H} = -t \sum_{i=1}^{2N-1} \left(\alpha_{i+1}^{\dagger} \alpha_i + \beta_{i+1}^{\dagger} \beta_i \right) - t \sum_{i=1}^{N-1} \left(\alpha_{2N-i+1}^{\dagger} \alpha_i + \beta_{2N-i+1}^{\dagger} \beta_i \right) + \text{h.c.}$$
$$-\mu \sum_{i=1}^{2N} \left(\alpha_i^{\dagger} \alpha_i + \beta_i^{\dagger} \beta_i \right) \tag{4.26}$$

We can see how the Hamiltonian has the same terms as in the 2D Fermi-Hubbard model with the same lattice ordering but with two different types of annihilation operators. This expression was not found by directly replacing the fermionic annihilation operators with anyonic annihilation operators. It was found by expressing the Hamiltonian in diagrammatic form in Figure 4.24 and expressing the diagrammatic observables in terms of the unnormalised anyonic creation and annihilation operators we defined.

We want to remark that there is nothing in particular of the Hamiltonian in Figure 4.24 which makes it expressable in terms of the creation and annihilation operators. Any physically allowed Hamiltonian can be expressed as a polynomial of the creation and annihilation operators we have defined. It is a matter of convenience to use the unnormalised annihilation operators. These can be described in terms of the original normalised annihilation operators as $\alpha_j = \frac{1}{\sqrt{2}} \alpha_j^{(1)} \alpha_j^{(0)\dagger} \alpha_j^{(0)} + \alpha_j^{(0)} - \alpha_j^{(1)} \alpha_j^{(0)\dagger} \alpha_j^{(0)}$ and $\beta_j = \alpha_j^{(1)} \alpha_j^{(0)\dagger} \alpha_j^{(0)} + \alpha_j^{(1)} - \alpha_j^{(1)} \alpha_j^{(0)\dagger} \alpha_j^{(0)}$.

Nevertheless, there is an important subtlety. One needs to pick specific lattices and orderings to express the desired notion of subsystem locality. In order to express the correct notion of nearest neighbour locality in terms of the in-front annihilation operators we defined, we need to pick the ordering such that the connection happens in front of all the in-between modes of the connection.

We want to explore this further in future works and be able to prove the conjecture

that for any 2 D lattice, one can find an ordering such that all the two-mode interactions that form a planar graph can be made to happen either completely behind the in-between modes or completely in front. Thus, it would make any nearest neighbour Hamiltonian expressable with creation and annihilation operators of the neighbouring terms alone.

4.3 Anyonic descriptors

This section analyses the use of the anyonic annihilation operators we define in Section 4.2 as anyonic descriptors. It showcases how the anyonic annihilation operators can represent the local ontic states of anyonic theories, thus clearly representing local realism in anyon systems. *The results presented in this section are original and unpublished.*

We want to express anyon theory as a no-signalling operational theory, as defined in Section 2.3. We have identified in Section 4.1 all the necessary elements: the subsystem lattice with their associated $\mathcal{P}_{\mathcal{M}}^{phys}$ and $\mathcal{T}_{\mathcal{M}}^{phys}$, the embeddings of local transformations, and the phenomenal projections as the anyonic behind partial trace. We have shown in Subsection 4.1.6 that anyons satisfy the no-signalling principle. All that is left is to check that the Separation property of the local groups of transformations is satisfied.

The Separation property requirement is that in any tripartite system ABC, if a transformation $\hat{U}_{ABC}$ is local in AC and BC, then it is necessarily local on C alone[4]. To be local in AC and in BC, $\hat{U}_{ABC}$ needs to equal extended local unitaries V_{AC}^{ext} and W_{BC}^{ext}.

The proof of the Separation property does not require the existence of anyonic

[4]The global phase factor is irrelevant, here.

annihilation operators, as it is a property of the physical anyon transformations. Nevertheless, we use the annihilation operators to have a simpler proof, analogous to the proof of the fermionic case, in Subsection 3.1.5.

A physical transformation $\hat{V}$ in ABC is local on AB if and only if it can be expressed as a polynomial of creation and annihilation operators of A and B alone. We need to take the collection of all multiple annihilation operators for the multiple particle types.

Therefore, the condition of the physical operator $\hat{V}$ being local in AC is equivalent to $\hat{V} = p\left(\left\{\alpha_{a_i}^{(j)}, \left(\alpha_{a_i}^{(j)}\right)^\dagger\right\}_{ij\alpha}, \left\{\alpha_{c_k}^{(j)}, \left(\alpha_{c_k}^{(j)}\right)^\dagger\right\}_{kj\alpha}\right)$ and for being local in BC is equivalent to $\hat{V} = q\left(\left\{\alpha_{b_{j'}}^{(j)}, \left(\alpha_{b_{j'}}^{(j)}\right)^\dagger\right\}_{j'j\alpha}, \left\{\alpha_{c_k}^{(j)}, \left(\alpha_{c_k}^{(j)}\right)^\dagger\right\}_{kj\alpha}\right)$. Equating the two and considering that the annihilation and creation operators are independent of each other algebraically, one concludes that necessarily the components in $p()$ that contain A terms must vanish or can be grouped to form the identity operator $\mathbb{I}$. $\mathbb{I}$ can always be expressed in terms of modes in C alone, see Appendix C.2. Thus $\hat{V}$ ends up containing C terms alone.

Therefore, we obtain that anyons are an operational no-signalling theory. Henceforth, we can apply the RR formalism to anyonic theory. Similarly to the fermionic case shown in Section 3.2, we want to find anyonic descriptors to represent the local ontic states, showcasing the local-realistic structure of anyons.

We have seen that qubit ladder operators are qubit descriptors (in Section 2.5) and fermionic annihilation operators are fermionic descriptors (in Section 3.2). The key reason is that they generate, together with their Hermitian conjugates, the local and global algebras of observables. Since the anyonic annihilation operators satisfy these properties, we know they can be considered anyonic descriptors. Knowing the unitary evolution of $\{\alpha_j^{(j')}\}_{j\in\mathcal{M},\alpha,j'}$ guarantees to know the evolution of any

anyonic physical observable within $\mathcal{M}$.

For anyons, we must consider the subtlety of the in-front and behind possible system partitions. Each partition has different associated equivalence relations, as the partition in position or momentum modes would. The partitions we consider rely only on having in the subsets either completely in-front or completely behind physical operators.

Thus, $\hat{U} \sim_{\mathcal{M}} \hat{V}$ is equivalent to $\hat{U} = \hat{W}_{\mathcal{N}\setminus\mathcal{M}} \cdot \hat{V}$ where $\hat{W}_{\mathcal{N}\setminus\mathcal{M}}$ is a completely in-front or completely behind local unitary in $\mathcal{N}\setminus\mathcal{M}$. It has the opposite behindness than the observables in $\mathcal{M}$ in the subsystem chosen partition. In the subsequent analysis, we refer to $\alpha_j^{(j')}$ as the normalised anyonic j'th annihilation operator of the particle type a (associated with its Greek counterpart α) in the j'th mode. The behindness of $\alpha_j^{(j')}$ depends on the subsystem partition chosen and which role plays j in the subsystem $\mathcal{M}$ in consideration. In all cases, the behindness of operators associated with $\mathcal{N}\setminus\{j\}$ is the opposite of j. We consider that both orientations of $\alpha_j^{(j')}$ can be chosen. In the discussion chapter, Section 5.3, we navigate the behindness subtleties of the subsystem lattice.

Similarly to the qubit network and fermionic cases, the set of evolved descriptors together with the initial Heisenberg state $\left(\left\{\hat{U}^\dagger \cdot \alpha_j^{(j')} \cdot \hat{U}\right\}_{j\in\mathcal{M},\alpha,j'}, \rho_0\right)$ is a compact way to represent the local ontic states.

We obtain Theorem 11, the anyonic analogue of Theorems 3 & 6. This theorem ensures the representation of local ontic states by anyonic descriptors.

Theorem 11. *The following equivalence holds for any subset of in-front joining modes $\mathcal{M}$ of an N mode anyonic system of n particle types.*

$$\hat{U} \sim_{\mathcal{M}} \hat{V} \quad \Longleftrightarrow \quad \hat{U}^\dagger \cdot \alpha_j^{(j')} \cdot \hat{U} = \hat{V}^\dagger \cdot \alpha_j^{(j')} \cdot \hat{V} \tag{4.27}$$

$\forall j \in \mathcal{M},\ j' \in \{1, \ldots, J_\alpha\},\ \alpha$ *and* $\forall \hat{U}, \hat{V} \in \mathcal{T}_{\mathcal{N}}^{phys}$ *where* $J_\alpha = \sum_{b,c} N_{ab}^c - n + 1$ *and* α *is associated to the particle type* a*.*

Thus,

$$[\hat{U}]_{\mathcal{M}} = \left\{ \hat{V} \in \mathcal{T}_{\mathcal{N}}^{phys} \;\middle|\; \hat{U}^\dagger \cdot \alpha_j^{(j')} \cdot \hat{U} = \hat{V}^\dagger \cdot \alpha_j^{(j')} \cdot \hat{V}\ \forall j \in \mathcal{M},\ \alpha, j' \in \{1, \ldots, J_\alpha\}, \right\} \tag{4.28}$$

The proof of Theorem 11 consists in applying the commutation properties inherited from the definition of the annihilation operators and direct calculation of the local properties of a unitary given it commutes with local annihilation operators. The proof is in Appendix C.6.

Theorem 11 establishes a direct connection between the anyonic annihilation operators and the local ontic states of the RR construction. The set of evolved anyonic annihilation operators with the initial Heisenberg state $\left(\left\{ \hat{U}^\dagger \cdot \alpha_j^{(j')} \cdot \hat{U} \right\}_{j \in \mathcal{M}, j', \alpha}, \rho_0 \right)$ is a compact representation of the local ontic states. We describe the ontic operations using this representation in the rest of this section. We use the notation $\left(\left(\hat{U}^\dagger \alpha_{j_1}^{(j')} \hat{U}, \ldots, \hat{U}^\dagger \alpha_{j_M}^{(j')} \hat{U} \right), \rho_0 \right)$ to indicate the representations. The notation also represents the collection over all particle types α and over all annihilation operators j'.

4.3.1 Ontic group action $\star$

In the anyonic descriptor representation, the action $\star$ of the groups of transformations $\mathcal{T}_{\mathcal{M}}^{phys}$ on the ontic state spaces $\mathcal{R}_{\mathcal{M}}^{phys}$ is given by:

$$\begin{aligned} \hat{W}_{\mathcal{M}} \star \left(\left(\hat{U}^\dagger \cdot \alpha_{j_1}^{(j')} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \alpha_{j_M}^{(j')} \cdot \hat{U} \right), \rho_0 \right) = \\ = \left(\left(\hat{W}_{\mathcal{M}}^{\dagger\, ext} \cdot \hat{U}^\dagger \cdot \alpha_{j_1}^{(j')} \cdot \hat{U} \cdot \hat{W}_{\mathcal{M}}^{ext}, \ldots, \hat{W}_{\mathcal{M}}^{\dagger\, ext} \cdot \hat{U}^\dagger \alpha_{j_M}^{(j')} \cdot \hat{U} \hat{W}_{\mathcal{M}}^{ext} \right), \rho_0 \right) \end{aligned} \tag{4.29}$$

4.3.2 Ontic-phenomenal epimorphisms $\varphi_{\mathcal{M}}$

The global phenomenal state space's orbit representatives can be diagonal in the canonical fusion tree basis. Exactly as in the qubit networks and fermion cases exposed in Subsubsection 2.5.1.2 & Subsection 3.2.2. By ordering the fusion tree basis to showcase the block-diagonal form given by the global charge SSR, the matrix representations of the phenomenal orbit representatives are of the form:

$$\rho_0 = \begin{pmatrix} \lambda_1^{a_1} & & & & & & \\ & \ddots & & & & & \\ & & \lambda^{a_1}_{\dim(\mathcal{H}_N^{a_1})} & & & & \\ & & & \ddots & & & \\ & & & & \lambda_1^{a_n} & & \\ & & & & & \ddots & \\ & & & & & & \lambda^{a_n}_{\dim(\mathcal{H}_N^{a_n})} \end{pmatrix} \tag{4.30}$$

where $a_1, \ldots, a_n$ are the n particle types in the theory and $\lambda_{j-1}^{a_k} \geq \lambda_j^{a_k} \quad \forall j \in \{2, \ldots, \dim(\mathcal{H}_N^{a_k})\}$ and $\forall k \in \{1 \ldots, n\}$.

The orbit structure is like the fermionic case but with n sectors instead of 2. We have n different orbits corresponding to each block in the pure case. We can fix the convention up to the global sector. By convention, when the global phenomenal state is pure, the orbit representative corresponds to mode 1 having a particle type a_k and the other $N-1$ modes the vacuum particle type e. These choices fix the state uniquely due to the trivial fusion rules for the vacuum.

We are ready to introduce the ontic-phenomenal epimorphisms $\varphi_{\mathcal{M}}$. We use the local map between the Heisenberg and Schrödinger pictures. We use the decomposition of the Schrödinger state $\rho_{\mathcal{M}}$:

$$\rho_{\mathcal{M}} = \sum_k \mathrm{Tr}\Big(\hat{U}^\dagger \cdot \hat{O}_{\mathcal{M}}^{(k)\ ext} \cdot \hat{U} \cdot \rho_0\Big)\ \hat{O}_{\mathcal{M}}^{(k)} \tag{4.31}$$

where $\hat{O}^{(k)}_{\mathcal{M}}$ are the local physical operator components of $\mathcal{M}$. The chosen order is left to right and up to down in the matrix representation in the local canonical fusion tree basis. These operators can be expressed as polynomials of all the anyonic annihilation operators associated with the modes $j \in \mathcal{M}$. We use the physical operator extension presented in Subsection 4.1.4.

It is precisely the polynomial decomposition that ensures that from the descriptor evolution, one can deduce the coefficients $\mathrm{Tr}\Big(\hat{U}^\dagger \cdot \hat{O}^{(k)\ ext}_{\mathcal{M}} \cdot \hat{U} \cdot \rho_0\Big)$. Since the ontic state representation also holds ρ_0, we can make the assignment:

$$\begin{aligned} \varphi_{\mathcal{M}} \left(\left(\hat{U}^\dagger \cdot \alpha^{(j')}_{j_1} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \alpha^{(j')}_{j_M} \cdot \hat{U} \right), \rho_0 \right) = \\ = \sum_k \mathrm{Tr}\Big(\hat{U}^\dagger \cdot \hat{O}^{(k)\ ext}_{\mathcal{M}} \cdot \hat{U} \cdot \rho_0\Big)\, \hat{O}^{(k)}_{\mathcal{M}} \end{aligned} \tag{4.32}$$

We point out that, to implement the epimorphism, we need some canonical reference of the annihilation operators; we need to specify which are the original operators we are tracking as descriptors, not only their evolution.

4.3.3 Ontic projections $\pi^{\mathcal{R}}_{\mathcal{M}}$

Theorem 11 implies that the ontic projections for anyonic descriptors are:

$$\begin{aligned} \pi^{\mathcal{R}}_{\mathcal{M}} \left(\left(\hat{U}^\dagger \cdot \alpha^{(j')}_{1} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \alpha^{(j')}_{N} \cdot \hat{U} \right), \rho_0 \right) = \\ = \left(\left(\hat{U}^\dagger \cdot \alpha^{(j')}_{j_1} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \alpha^{(j')}_{j_M} \cdot \hat{U} \right), \rho_0 \right) \end{aligned} \tag{4.33}$$

Completely equivalent to the qubit and fermionic cases in Subsubsection 2.5.1.3 and Subsection 3.2.3.

4.3.4 Ontic join product $\odot$

Consider $A \subset \mathcal{M}$ and a valid associated bipartition $A|B$ of the anyonic subsystem, being part of the anyonic subsystem lattice, thus being well-behaved in behindness. We denote $B = \mathcal{M}\backslash A$. Consider the following two compatible anyonic ontic states of subsystems A and B: [5]

$$\left(\left(\hat{U}^\dagger \cdot \alpha_{a_1}^{(j')} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \alpha_{a_S}^{(j')} \cdot \hat{U}\right), \rho_0\right) \quad \left(\left(\hat{V}^\dagger \cdot \alpha_{b_1}^{(j')} \cdot \hat{V}, \ldots, \hat{V}^\dagger \cdot \alpha_{b_{M\text{-}S}}^{(j')} \cdot \hat{V}\right), \rho_0\right) \tag{4.34}$$

Being compatible local ontic states implies there must be a maximal unitary $\hat{W} \in \mathcal{T}_{\mathcal{N}}^{phys}$ such that $\hat{W} = \hat{R}_{\mathcal{N}\backslash A}^{ext}\hat{U}$ and $\hat{W} = \hat{T}_{\mathcal{N}\backslash B}^{ext}\hat{U}$. Under these circumstances, we define the anyonic ontic join product $\odot$ as:

$$\begin{aligned}&\left(\hat{U}^\dagger \cdot \left(\alpha_{a_1}^{(j')}, \ldots, \alpha_{a_S}^{(j')}\right) \cdot \hat{U}, \rho_0\right) \odot \left(\hat{V}^\dagger \cdot \left(\alpha_{b_1}^{(j')}, \ldots, \alpha_{b_{M\text{-}S}}^{(j')}\right) \cdot \hat{V}, \rho_0\right) = \\ &= \left(\left(\hat{W}^\dagger \alpha_{a_1}^{(j')} \hat{W}, \ldots, \hat{W}^\dagger \alpha_{a_S}^{(j')} \hat{W}, \hat{W}^\dagger \alpha_{b_1}^{(j')} \hat{W}, \ldots, \hat{W}^\dagger \alpha_{b_{M\text{-}S}}^{(j')} \hat{W}\right), \rho_0\right)\end{aligned} \tag{4.35}$$

The uniqueness of this product is guaranteed by the Separation property satisfied by the groups of transformations $\mathcal{T}_S$, seen at the beginning of this section.

4.3.4.1 Faithfulness of splitting operation

When the anyonic subsystem lattice allows it, we can repeat the bipartition process until we reach the point of individual modes forming subsystems. For anyonic systems, the diagram in Figure 4.25 commutes. Theorem 12 establishes this fact.

Theorem 12. *The diagram of Figure 3.1 commutes. In other words:*

$$\pi_A^{\mathcal{P}}\left(\varphi_{\mathcal{M}}\left(\left(\left(\hat{U}^\dagger \cdot \alpha_{j_1}^{(j')} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \alpha_{j_M}^{(j')} \cdot \hat{U}\right), \rho_0\right)\right)\right) =$$

[5] Here, the a_j are mode labels, not the particle types of the theory.

$$= \varphi_A \left(\pi_A^{\mathcal{R}} \left(\left(\left(\hat{U}^\dagger \cdot \alpha_{j_1}^{(j')} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \alpha_{j_M}^{(j')} \cdot \hat{U} \right), \rho_0 \right) \right) \right) \tag{4.36}$$

a)

$$\begin{array}{ccc} \mathcal{R}_{\mathcal{M}} & \xrightarrow{\varphi_{\mathcal{M}}} & \mathcal{P}_{\mathcal{M}}^{phys} \\ \pi_A^{\mathcal{R}} \downarrow & & \downarrow \pi_A^{\mathcal{P}} \\ \mathcal{R}_A & \xrightarrow[\varphi_A]{} & \mathcal{P}_A^{phys} \end{array}$$

b)

$$\begin{array}{ccc} \left(\left(\hat{U}^\dagger \cdot \alpha_{j_1}^{(j')} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \alpha_{j_M}^{(j')} \cdot \hat{U} \right), \rho_0 \right) & \xmapsto{\varphi_{\mathcal{M}}} & \rho_{AB} \\ \pi_A^{\mathcal{R}} \downarrow & & \downarrow \pi_A^{\mathcal{P}} \\ \left(\left(\hat{U}^\dagger \cdot \alpha_{a_1}^{(j')} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \alpha_{a_S}^{(j')} \cdot \hat{U} \right), \rho_0 \right) & \xmapsto[\varphi_A]{} & \rho_A \end{array}$$

Figure 4.25: Commuting diagram that represents taking the projection into subsystems and the ontic-phenomenal epimorphism. Diagram a) represents the spaces, and diagram b) represents the action of the mappings in the descriptor picture.

The complete proof is in Appendix C.6. The idea is to use the consistency conditions that define the anyonic partial trace.

After seeing all the ontic operations, we can assert that anyons have a local-realistic structure. We have found anyonic annihilation operators that act as anyonic descriptors. Thus, annihilation operators can explicitly represent the anyonic ontic states using the Heisenberg picture. We have seen how the topological nature of non-abelian anyonic systems poses some hurdles, but these can be overcome to expose the inherent local-realistic nature of the theory.

5 | Discussion

We have thoroughly examined quantum indistinguishable particle systems satisfy Einstein's notion of local realism. We now delve deeper into discussing the implications of our findings and tie up any loose ends we may have left in previous chapters.

In Chapter 2, we have introduced relevant literature on the different notions of local realism in the context of quantum mechanics. We have justified our preference for Einstein's general formulation of local realism over Bell's. We have introduced RR's formalism [4] as a formalisation of Einstein's local realism and its connection with the concept of descriptors introduced in [3]. Lastly, we have explored the case of qubit networks to uncover how qubit annihilation operators can be used as qubit descriptors. Therefore, we may use the qubit annihilation operators to represent the local ontic states of a qubit network.

In Chapter 3, we have found analogous results for bosonic and fermionic systems, revealing their local-realistic structure. In the fermionic case, we have resolved the prima facie paradox of having no action at a distance when the fermionic annihilation operators represent the local ontic states. We have done so by pointing out the need to restrict the allowed transformations using the parity SSR to satisfy the no-signalling principle. In Section 3.4, we have provided the local-realistic interpretation of the bosonic and fermionic Mach-Zehnder interferometers by using the annihilation operators of the system.

After analysing local realism for $3 + 1$ D indistinguishable particles, we have considered $2 + 1$ D anyonic particle systems in Chapter 4. We have proved that anyonic annihilation operators can represent the local ontic states of the local-realistic structure of anyonic systems, which is a fundamental point of this thesis. To

derive this result, we first discovered the existence of anyonic annihilation operators. We have characterised the construction and behaviour of these key elements within anyonic theory. Subsequently, we have expressed the 2 D Fibonacci Hubbard model Hamiltonian in a compact expression (Equation 4.26) using the anyonic creation and annihilation operators.

This collection of results provides an in-depth and comprehensive analysis of the local realism of quantum indistinguishable particle systems. We believe that this work significantly contributes to determining whether quantum indistinguishable particle systems are local-realistic and how to realise their local-realistic structure. Furthermore, it fills a gap in the literature regarding the existence and definition of non-abelian anyonic annihilation operators. In doing so, it opens up new avenues for further research into these exotic indistinguishable particle systems, which are the basis of the topological quantum computing scheme for fault-tolerant quantum computation.

In the following sections, we discuss our results' significance, consequences, limitations and future directions for further research.

5.1 Heisenberg and Leibniz

One criticism that could be levelled at this thesis is that we are just using the well-known and widely used Heisenberg model. Although we are indeed using the Heisenberg picture, there is a conceptual leap in how we interpret the Heisenberg picture.

First, we emphasise the relevance and necessity of having local objects associated with each indistinguishable particle mode. Second, we propose interpreting the dynamical evolution of the annihilation operators as a representation of local ontic

states in our system. This consistent interpretation is a novel element with respect to the usual Heisenberg picture.

Moreover, the widespread use of the Heisenberg picture does not include the operators in the theory's ontology. Instead, it is generally used only for calculation purposes. When using it, the general belief appears to be that the ontology of quantum mechanics is dictated by the quantum state and its measurable properties. Generally, the theory's observables are associated with measurement apparatus rather than intrinsic, real properties attached to the system.

Having clarified this first point, let us now discuss whether our description satisfies Leibniz's rule of the identity of indiscernibles [37]. In a formal sense, this rule is violated. The ontic states we consider are not in isomorphism with the phenomenal states. Nevertheless, the global ontic state is given by the unitary that has been applied to the system. A physical process gives such a unitary. Given enough measurements and the ability to modify the initial state, one could perform process tomography on such a physical process and retrieve the measurement. Therefore, in this broader sense, the global ontic states are discernible by measurement.

This argument should be approached with caution. We require a large supply of initial states undergoing the same process to perform state tomography. However, to perform process tomography, we must also be able to modify the initial state. Thus, we need the ability to modify the overall process. To measure different observables, we must couple the system in different ways to probes, essentially modifying the overall process as well. As a result, we can conclude that it is equally possible for an all-powerful agent to perform process tomography or state tomography.

The crucial point is that we have not considered such a possibility when describing the phenomenal state space. We have not stated explicitly that the unitaries in

quantum theory can, in principle, be retrieved from observation. Thus, we have not considered unitaries as phenomenal. We have specified that the density operator of the system provides all measurable properties of a quantum system. Given that such a possibility was not considered at the outset of the discussion, it would be contradictory to now claim that unitaries are observable.

We can either consider the density operators as the amalgamation of all observable properties in the theory, thus violating the Leibniz rule, or start a new analysis considering the unitaries as phenomenal in the first place. In the latter case, we expect the Leibniz rule to be respected despite having yet to work out the details.

5.2 Are annihilation operators physical?

In Subsection 3.2.6, we have discussed whether fermionic descriptors are physical. We have stressed that we believe they are or, at the very least, should be considered representations of a physical entity: the RR unitary equivalence classes.

Bosonic annihilation operators should be considered on the same footing as the fermionic operators. We have seen how bosonic annihilation operators can be retrieved from bosonic observables $\hat{b}_j = \frac{(\hat{b}_j+\hat{b}_j^\dagger)-i(i(\hat{b}_j-\hat{b}_j^\dagger))}{2}$ in Subsection 3.4.1.

Proposing the physicality of the annihilation operators solves the problem of interpreting the descriptors in a local-realistic setting. Nevertheless, it raises some questions in other regards.

Let us examine this postulate from a quantum field theory perspective. In proposing the physicality of the annihilation operators, we are essentially proposing the physicality of the Dirac and photonic fields $\hat{\psi}(x)$ & $\hat{A}_\mu(x)$.

The position of considering $\hat{\psi}(x)$ & $\hat{A}_\mu(x)$ as physical, i.e. real, can be postulated in the context of the discussions on the Aharonov-Bohm effect [148–152]. In

the Ahoronov-Bohm context, such a choice can be attacked from the perspective of gauge dependence [151]. Without entering this debate, the situation we are analysing substantially differs from the gauge dependence problem in the Aharonov-Bohm effect.

The main difference is that in the case that concerns us, one can discern between two ontic states that yield the same phenomenal state using process tomography. One could a priori characterise the process and the Heisenberg state through process and state tomography. A posteriori, one would be able to perfectly and uniquely describe the system's local and global ontic states, having obtained all the information through direct evidence. Nevertheless, we acknowledge that the philosophical problems raised by the Aharonov-Bohm effect, associated with the physicality of the quantum fields, remain unsolved. More specific, in-depth, and targeted work in this direction is required. We intend to engage with this discussion fully in the not-so-distant future, considering the complete details of the quantum field theory setup.

The anyonic case can be considered to lay on the same grounds as the fermionic case. The similarities with the presence of the superselection rule, the violation of local tomography, and the phenomenal state space orbit structure suggest that the physicality of the anyonic annihilation operators should be at the same level as that of the fermionic case.

Nonetheless, there may be additional caveats. The first is the lack of overwhelming experimental evidence of non-abelian 2 D anyons. One cannot assert the physicality of an element whose theory has not been confirmed experimentally.

Even if we were to consider a scenario in which non-abelian anyons have been experimentally verified, we would still need to address the lack of notion of fields to which such annihilation operators can refer. One could draw connections with

topological quantum field theories. However, that may be a highly speculative program.

Despite these hurdles, we claim the physicality of the annihilation operators in all three cases. Our main justification for making such a claim is that we have shown how these elements showcase the physical structure of local realism. We have seen how natural the behaviour of these ontic states' representations is. It is clear that the local ontic states of the theory are physical. These objects are the best candidates for the local ontic states. Therefore, we should not disregard them as such.

5.2.1 The relevance of local tomography

We want to emphasise the importance of local tomography in the debate about the physicality of annihilation operators.

In Subsections 3.1.6 & 4.1.7, we have seen how fermions and anyons do not satisfy local tomography due to their imposed superselection rules. We have discussed how, in both cases, the failure of local tomography prevents any set of local observables from being descriptors.

As a result, in theories where local tomography is not satisfied but the no-signalling principle is, we must either relinquish the search for elements within the theory that represent the local ontic states or use elements that are not local observables. Both cases imply using elements not canonically agreed to be physical as local ontic states. In such cases, having annihilation operators as local ontic states is reassuring. They are not observables but possess a powerful physical meaning and associated intuition.

5.2.2 Bilocal tomography

It is important to note that, while fermions and anyons are not local tomographic, fermions are bilocal tomographic [153]. Bilocal tomography is the property that any global observable in a tripartite system ABC can be obtained through measuring observables local on pairs of the elemental subsystems. In other words, if K_A is the number of linearly independent observables in subsystem A, we say a theory is bilocal iff [43]

$$K_{ABC} \leq K_{AB}K_C + K_{AC}K_B + K_{BC}K_A - 2K_AK_BK_C \tag{5.1}$$

Real quantum theory is also a superselected theory that is also bilocal and not locally tomographic [153]. Nonetheless, we prove that Fibonacci anyons are not.

Solving the recurrence relations from Subsubsection 4.1.1.1, the number of states of global charge e in an n-mode Fibonacci system is F_{2n-1}. With global charge τ is F_{2n}. Therefore, in an n-mode subsystem there are $K_n = F_{2n-1}^2 + F_{2n}^2 = \frac{\phi^{4n-1}+\phi^{1-4n}}{\sqrt{5}}$ linearly independent observables. Using the K_n formula, it is straightforward to check that in a $3n$-mode Fibonacci tripartite system $K_{3n} > 3K_{2n}K_n - 2\left(K_n\right)^3$. Therefore making Fibonacci anyons not bilocal tomographic.

Some readers may object here. The conditions in Equation 5.1 for bilocal tomography are derived assuming that the individual subsystems can only be paired in one way. However, as previously discussed, in non-abelian anyons, joining the modes in front or behind the others may lead to different subsystems. In fact, some local in-front observables are linearly independent of all the behind local observables for the same pair of subsystems.

Regardless, if we consider three sets of n modes in sequence, then the in-front and behind observables of AB and BC will coincide. This is because they are

not really going either behind or in front of C and A, respectively. Now, even considering AC in-front and behind observables all linearly independent of each other, still the modified[1] bilocal inequality is violated:

$$K_{ABC} \leq K_{AB}K_C + K_{BC}K_A + K_{AC_f}K_B + K_{AC_b}K_B - 3K_AK_BK_C \qquad (5.2)$$

because $K_{3n} > 4K_{2n}K_n - 3\left(K_n\right)^3$. Fibonacci anyons are thus not bilocal tomographic. More sophisticated alternating behind and in-front paths could be considered. However, because we have not considered them throughout the analysis, we feel it would be cheating, and many of the concepts we have discussed would be very unclear. We wonder whether all non-abelian anyon systems are not bilocal tomographic. We hope such characterisation is provided in the future.

5.3 The importance of subsystem lattices

In this section, we emphasise that the lattice of subsystems of a theory is an extra layer of the structure that the particle system observable algebra alone does not determine. Chiribella has also pointed this out [39].

More concretely, in the qubit, bosonic and fermionic case, the lattice of systems is determined by the chosen notion of modes. In non-abelian anyons, the extra structure of in-front and behind observables must be specified.

Moreover, the local ontic states can be assigned under the system mode partition. Yet, the choice of modes is not unique. For fermions and bosons, we could perfectly define a different set of modes by applying a unitary Bogoliubov transformation $\hat{a}'_k = \sum_j u_{jk}\hat{a}_j$, where u_{jk} are the components of an $N \times N$ unitary matrix. This would define a different partitioning of the global system in terms of subsystems.

[1] AC_f for in-front observables of A and C. AC_b for behind.

In this other partition, we would be able to identify the associated local ontic states. Therefore, even though we can uniquely identify the local elements of reality given a notion of subsystem locality, the subsystem locality is not unique for a group of physical transformations.

The specification of the lattice of subsystems becomes even more crucial in the anyonic scenario. We have seen how we can specify different ways to fuse the anyonic modes depending on their paths relative to the non-fused modes.

In the planar representation, we identify two main ways to fuse a given set of modes that do not follow the mode order. The in-front and behind options consist of the chosen modes passing in front of or behind all the non-selected modes to fuse together. With these two minimal options, we can split the set of modes in such a way that in a bipartition $A|B$, we can define in-front observables in A and behind observables in B, ensuring that these will commute. The commutation property is key in order to satisfy no signalling.

One can prove that if such a distinction is not made and one considers any observable local in the modes A (regardless of being in-front or behind), then the partial trace consistency conditions have no solution. Suppose we insist on fixing a single behindness. In that case, the complementary system of A will either not exist or consist in the set of local observables in B that have the same behindness as A. These would not commute. Therefore, the no-signalling principle would not be guaranteed, and one could prove its violation.

Therefore we are forced to consider both possibilities separately. As in $A_f|B_b$ is a different partition of the modes than $A_b|B_f$. [2]

It would be interesting to explore the possibility and necessity of defining more

[2]The f and b subscripts mean in-front and behind, respectively.

enriched subsystem lattices, in which alternating in-front and behind paths and their complements may be considered in their own right. Since we have not needed to do so, we have preferred to keep the minimal choice.

By doing so, we have defined a consistent subsystem lattice where partial tracing can be defined uniquely. It satisfies both the no-signalling and the Separability postulate. Moreover, we have proved that under a consistent subsystem lattice partition, local realism will apply.

Let us provide a minimal example of a consistent subsystem lattice partition. Consider a 3-mode system, the subsystem lattice given by the partitions $ABC, AB|C, A|BC, AC_b|B_f, A|B_f|C$ is well-posed. Where $A \sqsubset AC_b, AB \sqsubset ABC$, and $B \sqsubset AB, BC \sqsubset ABC$, and $C \sqsubset AC_b, BC \sqsubset ABC$. This lattice is not boolean, but it does work for the use we give it in the chapter.

To calculate the local ontic states of the subsystem AC_b, we need the behind annihilation operators for A and C. We will need the in-front annihilation operators if we choose a different subsystem lattice with the subsystem AC_f instead. Therefore, we must consider that each mode holds both the in-front and the behind annihilation operators as descriptors.

We have expressed physically meaningful Hamiltonians as the anyon Hubbard model as a polynomial of in-front only annihilation operators. We are severely restricted if we only use in-front interactions. However, our capabilities increase substantially when using both in-front and behind annihilation operators.

By analysing the graph properties of two-mode interactions in a plane, we introduce the following conjecture, which we expect to hold.

Conjecture 1. *For any 2 D lattice where the two-mode interactions form a planar graph, an ordering of the modes exists such that all two-mode interactions are*

completely in-front or behind.

Proving this conjecture would imply that any reasonable two-mode interaction Hamiltonian may be expressed using the anyonic in-front and behind annihilation operators. For example, the non-abelian anyonic Mach-Zehnder interferometer proposed in [126, 127] could be modelled by a Hamiltonian that includes both interactions that are in front and behind. These are needed to close the loop of one anyon going behind another.

5.4 Anyonic subtleties

This Section contains a detailed discussion of the anyonic results we have obtained. *Some part of this Section is adapted from [104], written with the collaboration of Lucia Vilchez-Estevez.*

For non-abelian anyon particles, we observe that the number of in-front annihilation operators per lattice site is $J = n_a - n + 1$, where n_a is the total number of allowed fusion channels associated with that particle type. In the Fibonacci case, for the τ particle $n_a = 3$, we have $\tau \times e = \tau$ and $\tau \times \tau = e + \tau$, and $n = 2$ because there are two particle types in Fibonacci anyons, therefore $J = 2$. Including in a single mode both in-front and behind annihilation operators, we obtain a total of $2J = 4$ annihilation operators per mode in Fibonacci anyons.

The construction applies to all non-abelian anyon theories. We have exemplified it with Fibonacci anyons to be concise. Still, it is important to note that annihilation operators can be defined for Ising anyons [132] or any other non-abelian anyon theory. In future work, we would like to explore the connections between the annihilation operators defined using this method for Ising anyons and those for Majorana fermions.

5.4.1 Non-physical operator embedding

In Subsection 4.1.4.3, we have introduced the problem that non-abelian non-physical anyonic operators do not have a consistent embedding on larger mode systems. One might hope that by finding the anyonic annihilation operators, this problem might be solved. Unfortunately, this has not been the case, or at least not yet.

Despite identifying the anyonic annihilation operators in a fixed n-mode anyonic system, the definition lacks faithful embedding into larger systems. A canonical extension could be used in which only the Nth mode splits into all the fusion-allowed possibilities.

However, using the Fibonacci anyon example, it is straight-forward to see that such embedding does not work for the in-front mode 2 annihilation operator when extending from a 2-mode system to a 3-mode system. For example, consider the term in modes $1, 2$ where the two τ particles fuse to the vacuum, and then the vacuum splits into two vacuum particles.

This observable term can be expressed using the 2-mode annihilating elements as: $(R_e^{\tau\tau})^* \tau_1^{e,\tau} \tau_2^{\tau,e}$. However, when replacing this expression with the 3-mode annihilating elements, we do not obtain the local observable of two τ annihilating into the vacuums in modes $1, 2$. Instead, we obtain one of the components of such an observable term.

Nonetheless, the opposite process does work. Consider the expression for the local observable embedded in the 3-mode system in terms of the anyonic annihilation operators in the 3-mode system. In the 2-mode system, replacing the expression with the 2-mode anyonic annihilation operators yields the correct local observable.

We hope that a full algebraic characterisation of the anyonic system, using only

algebraic rules involving the creation and annihilation operators, may help with this problem by providing universal expressions for the local observables. Unfortunately, obtaining such characterisation is very challenging.

5.4.2 Anyonic commutation relations and Fock space

This thesis presents annihilation operators in the diagrammatic formalism for non-abelian 2D anyons. We aim to describe the algebraic properties of the anyonic creation and annihilation operators in commutation-like relations to have a complete algebraic characterisation of the anyonic theory. This allows for manipulations using annihilation operators directly without computing any diagram.

A complete characterisation at the algebraic level might be very challenging. Determining algebraically whether a polynomial of creation and annihilation operators is superselection-respecting is quite cumbersome. Check Appendix C.4 for some derived algebraic relations of the Fibonacci creation and annihilation operators.

In the general case, let us refer to the fusion tree where all the components are the vacuum particle type as $|0\rangle$. Take note that $\alpha_k^{(j)} |0\rangle = 0$ for all j and k. $|0\rangle$ is unique under this property. Any canonical fusion tree basis state can be expressed as a well-ordered sequence of creation operators acting on $|0\rangle$. Therefore, we are providing a Fock space construction of anyonic systems. We have calculated concrete expressions for 3-mode Fibonacci anyons, included in Appendix C.5.

These expressions may be useful to find Jordan-Wigner mappings for $2 + 1$ D anyons. Using the algebraic characterisation to explore Bogoliubov-like transformations for the non-abelian anyonic annihilation operators would also be very interesting. Specifically, performing Bogoliubov transformations would allow us to change the mode notion. Seeing and interpreting these new anyonic modes not tied to simply-connected regions of the manifold would be fascinating and facilitate

some tasks.

5.4.3 Anyonic Hamiltonians

The ability to express the Hubbard-like anyonic Hamiltonian in terms of local annihilation operators may have implications in the simulation of the model. Until recently, the community lacked good numerical techniques to simulate non-Abelian anyons systems. The main difficulty arises from the need for a tensor product structure and the growth of the Hilbert space with the number of modes. There have been some recent efforts to generalise the tensor network formalism to anyonic lattices [154–156]. We expect that having access to the local annihilation operators of an anyonic theory will facilitate the numerical simulation in some cases. In this way, we can exploit the parallelism between the anyonic Hubbard Hamiltonian and its bosonic or fermionic counterpart, for instance.

The Hamiltonian in Equation 4.26 contains terms with long-range interactions. These highly long-range terms (with respect to the ordering) can make the simulations time-inefficient. The ordering of Figure 4.23 is deliberately chosen to require only in-front annihilation operators.

Using in-front annihilation operators only, we restrict ourselves to a non-local expression of behind-only interactions. Therefore, we choose the lattice ordering so that no two-mode interaction is a behind term.

Of course, we can think of a more natural (and short-range) ordering, e.g. ladder ordering. In such ordering, the Hamiltonian is either expressed using both in-front and behind annihilation operators, or the Hubbard Hamiltonian will contain products of several operators local in not only the nearest-neighbour interacting sites.

In conclusion, if we want to avoid long-range terms (with respect to the ordering) in

our Hamiltonian, we need to sacrifice the simplicity of the current expression. We believe that the study of the similarities and differences between these approaches is a promising future direction to follow.

We hope that the expressions found for the $2+1$ D non-abelian anyon creation and annihilation operators will advance the study and understanding of this topic, especially by allowing us to apply known techniques to the study of topological quantum computing and the experimental detection of the particles described.

5.5 Possible extensions of the work

Regarding the local realism and the quantum foundation's aspect of the thesis, there are several exciting directions in which this work can be extended.

Firstly, the natural extension to the full quantum field theory case. The mathematical subtleties of infinite modes will have to be taken into account. However, the expected results are the same as the ones presented in this thesis.

Secondly, working out the details of the relative descriptors program [52, 54] for the indistinguishable particle cases. The program consists in defining the relative descriptors upon measuring an outcome in a projective measurement. It can be seen as an Everett relative state construction for descriptors. Such a program is necessary to provide a more in-depth account of the Bell scenario.

Thirdly, considering anyonic systems as a possibility can have significant consequences when modelling general physical theories. Axioms such as bilocal tomography or unique subsystem composition should be abandoned. We want to make sure that general probabilistic theories [39, 141] are not overly restrictive in their assumptions and can include anyonic features.

We intend to study the quantum information aspects of non-abelian anyonic systems

next. The notions of separability and entanglement characterisation. They may display striking behaviour that advances our understanding of information systems and their relation to their physical underpinnings.

We want to establish if anyons are the only systems that exhibit this set of nuanced conditions or if these can also be found in other constrained quantum systems. The latter should be the case. We expect that there are constrained systems without an indistinguishable particle interpretation that behave as strikingly as non-abelian anyons regarding subsystem composition and, thus, information capabilities.

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A | Mathematical details of Chapter 2

In this appendix, we include the mathematical details we do not specify in the main text of Chapter 2. These include the proof of Theorem 4 and Theorem 5.

(Theorem. *4) Using the full set of descriptors* $\left(\hat{U}^\dagger \cdot \hat{q}_1 \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{q}_N \cdot \hat{U}\right)$ *it is possible to uniquely find the* $\hat{U} \in \mathcal{T}_N$ *that has evolved them from their canonical form* $(\hat{q}_1, \ldots, \hat{q}_N)$.

Proof. First, note that the unitary conjugation action is up to a global phase. Keeping this in mind, considering the unitary $\hat{U}$ as an operator, we can regard it as a vector on the vector space of operators. We can consider in the global operator algebra an orthonormal basis, where the scalar product between two operators $\hat{A}, \hat{B}$ is given by $\mathrm{Tr}\left(\hat{A}^\dagger \cdot \hat{B}\right)$. It is straightforward to see that for the qubit network operators if one labels the computational basis as $\{|k\rangle\}_{k=1}^{2^N}$, then $|k\rangle\langle l|$ is an orthonormal basis of the operator vector space with scalar product given by $\mathrm{Tr}\left(\hat{A}^\dagger \cdot \hat{B}\right)$.

Considering this, now $\hat{U}$ can be written as $\hat{U} = \sum_{k,l=1}^{2^N} \mathrm{Tr}\left(\hat{U}\,|k\rangle\langle l|\right)|k\rangle\langle l|$. So, if we know $\mathrm{Tr}\left(\hat{U} \cdot |k\rangle\langle l|\right)$ we know the unitary. $|k\rangle\langle l|$ is a polynomial of $\{\hat{q}_j, \hat{q}_j^\dagger\}_{j=1}^N$. Using this, having at our disposal $(\hat{q}_1, \ldots, \hat{q}_N)$ and $\left(\hat{U}^\dagger \cdot \hat{q}_1 \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{q}_N \cdot \hat{U}\right)$ we can construct $|k\rangle\langle l|$ and $\hat{U}^\dagger \cdot |k\rangle\langle l| \cdot \hat{U}$, where to construct the second we have replaced the $\hat{q}_j, \hat{q}_j^\dagger$ in the decomposition of $|k\rangle\langle l|$ by $\hat{U}^\dagger \cdot \hat{q}_j \cdot \hat{U}, \hat{U}^\dagger \cdot \hat{q}_j^\dagger \cdot \hat{U}$. This works due to the linearity and the fact that in the products, it can be applied that $\hat{U} \cdot \hat{U}^\dagger = \mathbb{I}$. Let us denote $\left|\bar{k}\right\rangle = \hat{U}^\dagger\,|k\rangle$.

We take the scalar product of any two of these objects. In other words, consider $\mathrm{Tr}\left(\left|\bar{k}\right\rangle\!\left\langle\bar{l}\right| \cdot |m\rangle\langle n|\right) = \mathrm{Tr}\left(\hat{U}^\dagger \cdot |k\rangle\langle l| \cdot \hat{U} \cdot |m\rangle\langle n|\right)$. Now, if we insert the decomposition of $\hat{U}$ found above and we use the linearity properties of the trace, we

obtain that

$$\begin{aligned}
&\mathrm{Tr}\left(\left|\bar{k}\right\rangle\!\left\langle\bar{l}\right| |m\rangle\langle n|\right) = \\
&= \sum_{o,p,q,r=1}^{2^N} \mathrm{Tr}\left(\hat{U}^\dagger |o\rangle\langle p|\right) \mathrm{Tr}\left(\hat{U} |q\rangle\langle r|\right) \mathrm{Tr}\left(|p\rangle\langle o| \, |k\rangle\langle l| \, |r\rangle\langle q| \, |m\rangle\langle n|\right) = \\
&= \sum_{o,p,q,r=1}^{2^N} \mathrm{Tr}\left(\hat{U}^\dagger |o\rangle\langle p|\right) \mathrm{Tr}\left(\hat{U} |q\rangle\langle r|\right) \delta_{ok}\delta_{lr}\delta_{qm}\delta_{np} = \\
&= \mathrm{Tr}\left(\hat{U}^\dagger \cdot |k\rangle\langle n|\right) \mathrm{Tr}\left(\hat{U} \cdot |m\rangle\langle l|\right)
\end{aligned}$$

where we have used the orthonormality of the computational basis and the properties of the Kronecker delta. Using the cyclic properties of the trace and complex conjugation, we obtain $\mathrm{Tr}\left(\hat{U}^\dagger |k\rangle\langle n|\right) \mathrm{Tr}\left(\hat{U} |m\rangle\langle l|\right) = \mathrm{Tr}\left(\hat{U} |n\rangle\langle k|\right)^* \mathrm{Tr}\left(\hat{U} |m\rangle\langle l|\right)$.

By knowing all the values of $\mathrm{Tr}\left(\left|\bar{k}\right\rangle\!\left\langle\bar{l}\right| |m\rangle\langle n|\right)$, which we obtain from $(\hat{q}_1, \ldots, \hat{q}_N)$ and $\left(\hat{U}^\dagger \cdot \hat{q}_1 \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{q}_N \cdot \hat{U}\right)$ alone, we can retrieve $\mathrm{Tr}\left(\hat{U} \cdot |m\rangle\langle l|\right)$. Notice that $\mathrm{Tr}\left(\hat{U} \cdot |m\rangle\langle l|\right)$ is a complex number, so knowing its polar form is enough. We see that $\mathrm{Tr}\left(\left|\bar{l}\right\rangle\!\left\langle\bar{l}\right| |m\rangle\langle m|\right) = \left|\mathrm{Tr}\left(\hat{U} \cdot |m\rangle\langle l|\right)\right|^2$. Thus, we obtain the modulus of the complex number.

We can see now that $\mathrm{Tr}\left(\hat{U} \cdot |m\rangle\langle l|\right) = \sqrt{\mathrm{Tr}\left(\left|\bar{l}\right\rangle\!\left\langle\bar{l}\right| \cdot |m\rangle\langle m|\right)} e^{i\phi_{m,l}}$. So only the phases are left to determine. Here is where the issue of the freedom of a global phase intervenes. Since $\hat{U}$ is unitary, we know they must exist m_0, l_0 such that $\left|\mathrm{Tr}\left(\hat{U} \cdot |m_0\rangle\langle l_0|\right)\right|^2 = \mathrm{Tr}\left(\left|\bar{l_0}\right\rangle\!\left\langle\bar{l_0}\right| \cdot |m_0\rangle\langle m_0|\right) > 0$. Due to the overall phase redundancy, we can fix the phase $\phi_{m_0,l_0} = 0$.

In other words, we could always choose a global phase to cancel the phase ϕ_{m_0,l_0} so it is set to 0. Now, we can see that if we consider

$$\mathrm{Tr}\left(\left|\bar{l_0}\right\rangle\!\left\langle\bar{l}\right| \cdot |m\rangle\langle m_0|\right) = \mathrm{Tr}\left(\hat{U} \cdot |m_0\rangle\langle l_0|\right)^* \mathrm{Tr}\left(\hat{U} \cdot |m\rangle\langle l|\right) =$$

$$= \sqrt{\mathrm{Tr}\left(\left|\bar{l_0}\right\rangle\!\left\langle\bar{l_0}\right| \cdot |m_0\rangle\langle m_0|\right)}\sqrt{\mathrm{Tr}\left(\left|\bar{l}\right\rangle\!\left\langle\bar{l}\right| \cdot |m\rangle\langle m|\right)}\ e^{i\phi_{ml}}$$

Thus, we obtain that

$$\mathrm{Tr}\left(\hat{U} \cdot |m\rangle\langle l|\right) = \frac{\mathrm{Tr}\left(\left|\bar{l_0}\right\rangle\!\left\langle\bar{l}\right| \cdot |m\rangle\langle m_0|\right)}{\sqrt{\mathrm{Tr}\left(\left|\bar{l_0}\right\rangle\!\left\langle\bar{l_0}\right| \cdot |m_0\rangle\langle m_0|\right)}}$$

Therefore, indeed we can retrieve the unitary $\hat{U}$. □

The next theorem's proof is Theorem 5.

(Theorem. *5) The diagram of Figure 2.1 commutes. In other words:*

$$\begin{aligned}\pi_A^{\mathcal{P}}\left(\varphi_{\mathcal{M}}\left(\left(\left(\hat{U}^\dagger \cdot \hat{q}_{j_1} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{q}_{j_M} \cdot \hat{U}\right), \rho_0\right)\right)\right) = \\ = \varphi_A\left(\pi_A^{\mathcal{R}}\left(\left(\left(\hat{U}^\dagger \cdot \hat{q}_{j_1} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{q}_{j_M} \cdot \hat{U}\right), \rho_0\right)\right)\right)\end{aligned} \tag{A.1}$$

Proof. We begin by expanding the right-hand side of the equation by applying the definition of the ontic projection operator, obtaining:

$$\varphi_A\left(\left(\left(\hat{U}^\dagger \cdot \hat{q}_{a_1} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{q}_{a_S} \cdot \hat{U}\right), \rho_0\right)\right)$$

Applying now the definition of φ_A of Equation 2.22, we obtain:

$$\frac{1}{2^S}\sum_{\vec{p}} \mathrm{Tr}\left(\hat{U}^\dagger\left(\hat{\sigma}_{a_1}^{p_1} \otimes \cdots \otimes \hat{\sigma}_{a_S}^{p_S} \otimes \mathbb{I}_{\mathcal{N}\setminus A}\right)\hat{U}\rho_0\right)\hat{\sigma}_{a_1}^{p_1} \otimes \cdots \otimes \hat{\sigma}_{a_S}^{p_S}$$

where we have chosen as orthonormal basis for the local operator space of the reduced qubit network system with lattice sites $A = \{a_1, \ldots, a_S\}$ the basis given by the renormalised product of Pauli operators $\{2^{-\frac{S}{2}}\hat{\sigma}_{a_1}^{p_1} \otimes \cdots \otimes \hat{\sigma}_{a_S}^{p_S}\}$, with the labels $p_k \in \{0, x, y, z\}$.

We now turn to expand the left-hand side of the initial equation by applying the definition of the ontic-phenomenal epimorphism and choosing the same basis as before but for the larger set of lattice sites $\mathcal{M} = \{j_1, \ldots, j_M\}$, we obtain:

$$\pi_A^{\mathcal{P}} \left(\varphi_{\mathcal{M}} \left(\left(\left(\hat{U}^\dagger \cdot \hat{q}_{j_1} \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{q}_{j_M} \cdot \hat{U} \right), \rho_0 \right) \right) \right) =$$

$$= \pi_A^{\mathcal{P}} \left(\frac{1}{2^M} \sum_{\vec{d}} \mathrm{Tr} \left(\hat{U}^\dagger \left(\hat{\sigma}_{d_1}^{j_1} \otimes \cdots \otimes \hat{\sigma}_{j_M}^{d_M} \otimes \mathbb{I}_{\mathcal{N} \setminus \mathcal{M}} \right) \hat{U} \rho_0 \right) \; \hat{\sigma}_{j_1}^{d_1} \otimes \cdots \otimes \hat{\sigma}_{d_M}^{d_M} \right)$$

The main difference with the expanded version of the right-hand side is the length of the vectors $\vec{d}$ versus $\vec{p}$ and the normalisation factor.

Now, let's take the partial trace of the lattice sites that are in $\mathcal{M}$ but not in A. Since $\mathrm{Tr}_B \left(\sigma_A^a \otimes \sigma_B^b \right) = 2\delta_{b0} \sigma_A^a$, the expression of the left hand side becomes

$$\mathrm{Tr}_{\mathcal{M} \setminus A} \left(\frac{1}{2^M} \sum_{\vec{d}} \mathrm{Tr} \left(\hat{U}^\dagger \left(\hat{\sigma}_{d_1}^{j_1} \otimes \cdots \otimes \hat{\sigma}_{j_M}^{d_M} \otimes \mathbb{I}_{\mathcal{N} \setminus \mathcal{M}} \right) \hat{U} \rho_0 \right) \hat{\sigma}_{j_1}^{d_1} \otimes \cdots \otimes \hat{\sigma}_{d_M}^{d_M} \right) =$$

$$= \frac{1}{2^M} \sum_{\vec{d}} \mathrm{Tr} \left(\hat{U}^\dagger \left(\hat{\sigma}_{d_1}^{j_1} \otimes \cdots \otimes \hat{\sigma}_{j_M}^{d_M} \otimes \mathbb{I}_{\mathcal{N} \setminus \mathcal{M}} \right) \hat{U} \rho_0 \right)$$

$$\hat{\sigma}_{a_1}^{d_{a_1}} \otimes \cdots \otimes \hat{\sigma}_{d_{a_S}}^{a_S} 2^{M-S} \delta_{b_1 0} \ldots \delta_{b_{M-S} 0} =$$

$$= \frac{1}{2^S} \sum_{\vec{p}} \mathrm{Tr} \left(\hat{U}^\dagger \left(\hat{\sigma}_{a_1}^{p_1} \otimes \cdots \otimes \hat{\sigma}_{a_S}^{p_S} \otimes \mathbb{I}_{\mathcal{N} \setminus A} \right) \hat{U} \rho_0 \right) \; \hat{\sigma}_{a_1}^{p_1} \otimes \cdots \otimes \hat{\sigma}_{p_S}^{a_S}$$

which exactly coincides with the expression on the right-hand side. □

B | Mathematical details of Chapter 3

This appendix includes the mathematical details and laborious proofs of statements we could not show in the main text.

Proposition 13. *Fermionic annihilation operators are fermionic descriptors. From* $\hat{U}^\dagger \hat{f}_j \hat{U}$ *for all* $j \in \mathcal{M}$ *one can get* $\hat{U}^\dagger \hat{O}^{(k)}_{\mathcal{M}} \hat{U}$ *for any local operator* $\hat{O}^{(k)}_{\mathcal{M}} \in \mathcal{A}_{\mathcal{M}}$.

Proof. Given any $\hat{O}^{(k)}_{\mathcal{M}} \in \mathcal{A}_{\mathcal{M}}$, by definition it can be written as a polynomial of the creation and annihilation operators of the modes $j \in \mathcal{M}$. Therefore we can write $\hat{O}^{(k)}_{\mathcal{M}} = p\left(\left\{\hat{f}_j, \hat{f}_j^\dagger\right\}_{j\in\mathcal{M}}\right) = a_1 \hat{f}_{j_1} + a_2 \hat{f}^\dagger_{j_1} + a_3 \hat{f}^\dagger_{j_1}\hat{f}_{j_1} + a_4 \hat{f}_{j_1}\hat{f}^\dagger_{j_1} + ...$, with a maximum of 2^{M+1} monomials. Now, we can see that due to the linearity of the unitary action, from all $\hat{U}^\dagger m\left(\left\{\hat{f}_j, \hat{f}_j^\dagger\right\}_{j\in\mathcal{M}}\right)\hat{U}$ one can retrieve $\hat{U}^\dagger \hat{O}^{(k)}_{\mathcal{M}} \hat{U}$, where $m(-)$ is a monomial of its arguments.

Any monomial of degree bigger than one is either of the form $m\left(\left\{\hat{f}_j, \hat{f}_j^\dagger\right\}_{j\in\mathcal{M}}\right) = \hat{f}_{j\star} m_\star\left(\left\{\hat{f}_j, \hat{f}_j^\dagger\right\}_{j\in\mathcal{M}}\right)$ or $m\left(\left\{\hat{f}_j, \hat{f}_j^\dagger\right\}_{j\in\mathcal{M}}\right) = \hat{f}^\dagger_{j\star} m_\star\left(\left\{\hat{f}_j, \hat{f}_j^\dagger\right\}\right)$ where $j\star \in \mathcal{M}$ and $m_\star\left(\left\{\hat{f}_j, \hat{f}_j^\dagger\right\}\right)$ is a monomial of a lesser degree than the degree of $m()$. Since $\hat{U}$ is unitary, then either

$$\hat{U}^\dagger m\left(\left\{\hat{f}_j, \hat{f}_j^\dagger\right\}_{j\in\mathcal{M}}\right)\hat{U} = \hat{U}^\dagger \hat{f}_{j\star}\hat{U}\cdot\hat{U}^\dagger m_\star\left(\left\{\hat{f}_j, \hat{f}_j^\dagger\right\}_{j\in\mathcal{M}}\right)\hat{U}$$
$$\text{or } \hat{U}^\dagger m\left(\left\{\hat{f}_j, \hat{f}_j^\dagger\right\}_{j\in\mathcal{M}}\right)\hat{U} = \hat{U}^\dagger \hat{f}^\dagger_{j\star}\hat{U}\cdot\hat{U}^\dagger m_\star\left(\left\{\hat{f}_j, \hat{f}_j^\dagger\right\}_{j\in\mathcal{M}}\right)\hat{U}$$

Applying recursively such decomposition and trivially for monomials of degree one, we get that any unitary evolution of any monomial can be retrieved from all the unitary evolutions $\hat{U}^\dagger \hat{f}_j \hat{(U)}$ and $\hat{U}^\dagger \hat{f}_j^\dagger \hat{(U)}$ for all $j \in \mathcal{M}$.

Finally, observe that $\left(\hat{U}^\dagger \hat{f}_j^\dagger \hat{(U)}\right)^\dagger = \hat{U}^\dagger \hat{f}_j \hat{(U)}$. It follows that from all $\hat{U}^\dagger \hat{f}_j \hat{(U)}$

one can retrieve all $\hat{U}^\dagger \hat{O}^{(k)}_{\mathcal{M}} \hat{U}$ $\square$

Theorem 14. *Given the parity SSR applied on observables, states and transformations, the no-signalling principle is satisfied in fermionic systems. Given a bipartition $A|B$ of $\mathcal{M}$, it is satisfied*

$$\mathrm{Tr}_B \left(\hat{U}_B \cdot \rho_{\mathcal{M}} \cdot \hat{U}_B^\dagger \right) = \mathrm{Tr}_B \left(\rho_{\mathcal{M}} \right) \tag{B.1}$$

for any $\hat{U}_B \in \mathcal{T}_B^{phys}$, $\rho_{\mathcal{M}} \in \mathcal{P}_{\mathcal{M}}^{phys}$.

Proof. The crucial property that we will use to prove the statement is that if $\mathrm{Tr}\left(\hat{O}_A \cdot \hat{P}_A\right) = \mathrm{Tr}\left(\hat{O}_A \cdot \hat{Q}_A\right)$ for all local parity SSR observables $\hat{O}_A$, and $\hat{P}_A, \hat{Q}_A$ are local parity SSR observables, then $\hat{P}_A = \hat{Q}_A$. This result follows from seeing the parity SSR observables as a subvector space of the finite Hilbert space of operators under the trace scalar product.

We need to see that for any local even unitary $\hat{U}_B$ and any phenomenal state $\rho_{\mathcal{M}} \in \mathcal{P}_{\mathcal{M}}$ we have $\mathrm{Tr}_B \left(\hat{U}_B \cdot \rho_{\mathcal{M}} \cdot \hat{U}_B^\dagger \right) = \mathrm{Tr}_B \left(\rho_{\mathcal{M}} \right)$.

The property that uniquely identifies the partial trace operation is the fulfilment of the consistency conditions in Equation 3.10. By applying the consistency conditions of the partial trace to the state $\hat{U}_B \cdot \rho_{\mathcal{M}} \cdot \hat{U}_B^\dagger$ we obtain

$$\begin{aligned} \forall \hat{O}_A \in \mathcal{O}_A \qquad \mathrm{Tr}\left(\hat{O}_A \cdot \mathrm{Tr}_B \left(\hat{U}_B \cdot \rho_{\mathcal{M}} \cdot \hat{U}_B^\dagger \right) \right) &= \mathrm{Tr}\left(\hat{O}_A \cdot \hat{U}_B \cdot \rho_{\mathcal{M}} \cdot \hat{U}_B^\dagger \right) = \\ &= \mathrm{Tr}\left(\hat{U}_B^\dagger \cdot \hat{O}_A \cdot \hat{U}_B \cdot \rho_{\mathcal{M}} \right) \end{aligned} \tag{B.2}$$

Now, using that $\hat{U}_B, \hat{O}_A$ are disjoint even fermionic operators, we see that they commute. By applying the anticommutation relations, due to the fact that all monomials have an even degree, the gathered phase in each term is the same and

equal to $+1$, therefore being commuting operators. Therefore

$$\forall \hat{O}_A \in \mathcal{O}_A \qquad \mathrm{Tr}\left(\hat{O}_A \cdot \mathrm{Tr}_B\left(\hat{U}_B \cdot \rho_{\mathcal{M}} \cdot \hat{U}_B^\dagger\right)\right) = \mathrm{Tr}\left(\hat{U}_B^\dagger \cdot \hat{U}_B \cdot \hat{O}_A \cdot \rho_{\mathcal{M}}\right) = \\ = \mathrm{Tr}\left(\hat{O}_A \cdot \rho_{\mathcal{M}}\right) = \mathrm{Tr}\left(\hat{O}_A \cdot \mathrm{Tr}_B\left(\rho_{\mathcal{M}}\right)\right) \tag{B.3}$$

Applying the property described at the beginning of the proof, we obtain that, indeed, $\mathrm{Tr}_B\left(\rho_{\mathcal{M}}\right) = \mathrm{Tr}_B\left(\hat{U}_B \cdot \rho_{\mathcal{M}} \cdot \hat{U}_B^\dagger\right)$. □

(Theorem. *6) The following equivalence holds for any subset of modes $\mathcal{M}$ of an N mode fermionic system.*

$$\hat{U} \sim_{\mathcal{M}} \hat{V} \qquad \Longleftrightarrow \qquad \hat{U}^\dagger \cdot \hat{f}_j \cdot \hat{U} = \hat{V}^\dagger \cdot \hat{f}_j \cdot \hat{V} \quad \forall j \in \mathcal{M},\ \hat{U}, \hat{V} \in \mathcal{T}_{\mathcal{N}}^{phys}$$

Thus, $[\hat{U}]_{\mathcal{M}} = \left\{ \hat{V} \in \mathcal{T}_{\mathcal{N}} \,\middle|\, \hat{U}^\dagger \cdot \hat{f}_j \cdot \hat{U} = \hat{V}^\dagger \cdot \hat{f}_j \cdot \hat{V} \quad \forall j \in \mathcal{M} \right\}$.

Proof. The last statement follows directly from the definition of an equivalence class, so the equation that needs to be proven is Equation 3.13:

"⇒": Remember $\mathcal{N} = \{1, \ldots, N\}$. $\hat{U} \sim_{\mathcal{M}} \hat{V}$ implies $\hat{U} = \hat{W}_{\mathcal{N}\setminus\mathcal{M}} \cdot \hat{V}$ for some $\hat{W}_{\mathcal{N}\setminus\mathcal{M}}$ being a parity SSR unitary, local on the set of lattice sites that excludes all the sites $j \in \mathcal{M}$. Thus, since $\hat{W}_{\mathcal{M}\setminus\mathcal{M}}$ is an even operator that does not contain any terms involving $\hat{q}_j, \hat{q}_j^\dagger$ for $j \in \mathcal{M}$, is straightforward to check that due to having all the monomials of the polynomial expression of the unitary an even degree, $[\hat{W}_{\mathcal{N}\setminus\mathcal{M}}, \hat{f}_j] = 0$ for all $j \in \mathcal{M}$. Therefore: $\hat{U}^\dagger \cdot \hat{f}_j \cdot \hat{U} = \hat{V}^\dagger \cdot \hat{W}_{\mathcal{N}\setminus\mathcal{M}}^\dagger \cdot \hat{f}_j \cdot \hat{W}_{\mathcal{N}\setminus\mathcal{M}} \cdot \hat{V} = \hat{V}^\dagger \cdot \hat{f}_j \cdot \hat{W}_{\mathcal{N}\setminus\mathcal{M}}^\dagger \cdot \hat{W}_{\mathcal{N}\setminus\mathcal{M}} \cdot \hat{V} = \hat{V}^\dagger \cdot \hat{f}_j \cdot \hat{V}$.

"⇐": We have that $\hat{U}^\dagger \cdot \hat{f}_j \cdot \hat{U} = \hat{V}^\dagger \cdot \hat{f}_j \cdot \hat{V}$ for all $j \in \mathcal{M}$. To see that $\hat{U} \sim_{\mathcal{M}} \hat{V}$ we need to see that $\hat{U} = \hat{W}_{\mathcal{N}\setminus\mathcal{M}} \cdot \hat{V}$. Or, equivalently, since we have a group structure, where transformations are parity SSR unitaries, proving that $\hat{U} \cdot \hat{V}^\dagger = \hat{W}_{\mathcal{N}\setminus\mathcal{M}}$ is enough to prove that $\hat{U} \sim_{\mathcal{M}} \hat{V}$.

From $\hat{U}^\dagger \cdot \hat{f}_j \cdot \hat{U} = \hat{V}^\dagger \cdot \hat{f}_j \cdot \hat{V}$ for all $j \in \mathcal{M}$ is straightforward to deduce that then $\hat{f}_j \cdot (\hat{U} \cdot \hat{V}^\dagger) = (\hat{U} \cdot \hat{V}^\dagger) \cdot \hat{f}_j$ for all $j \in \mathcal{M}$. Naming $\hat{U} \cdot \hat{V}^\dagger = \hat{W}$, noticing $\hat{W}$ is a unitary and taking the dagger of the previous equation, we have that the two following equalities hold:

$$\hat{W} \cdot \hat{f}_j = \hat{f}_j \cdot \hat{W} \qquad \hat{W} \cdot \hat{f}_j^\dagger = \hat{f}_j^\dagger \cdot \hat{W} \; \forall j \in \mathcal{M} \tag{B.4}$$

Moreover, now since $\hat{W}$ is a priori a general parity SSR unitary, it is not difficult to see that we can decompose it as $\hat{W} = \hat{O}_0 + \hat{f}_{j_1}\hat{O}_1 + \hat{f}_{j_1}^\dagger \hat{O}_2 + \hat{f}_{j_1}\hat{f}_{j_1}^\dagger \hat{O}_3$ for $j_1 \in \mathcal{M}$. Where $\hat{O}_0, \hat{O}_3$ are even local operators and $\hat{O}_1, \hat{O}_2$ are odd local operators, both on the set of lattice sites $\mathcal{N}\backslash\{j_1\}$. Using this decomposition of $\hat{W}$ in the first condition of Equation B.4 and commuting/anticommuting the $\hat{f}_{j_1}, \hat{f}_{j_1}^\dagger$ terms with the $\hat{O}_k$ operators we obtain that:

$$\hat{f}_{j_1}(\hat{O}_0 + \hat{O}_3) + \hat{f}_{j_1}\hat{f}_{j_1}^\dagger \hat{O}_2 - \hat{O}_2 = \hat{f}_{j_1}\hat{O}_0 + \hat{f}_{j_1}\hat{f}_{j_1}^\dagger \hat{O}_2$$

Implying that $\hat{O}_2 = \hat{0}$ and $\hat{O}_3 = \hat{0}$. Then, using that $\hat{W} = \hat{O}_0 + \hat{f}_{j_1}\hat{O}_1$ and replacing in the second condition of Equation B.4 we obtain:

$$\hat{f}_{j_1}^\dagger \hat{O}_0 - \hat{f}_{j_1}\hat{f}_{j_1}^\dagger \hat{O}_1 = \hat{f}_{j_1}^\dagger \hat{O}_0 + \hat{f}_{j_1}^\dagger \hat{f}_{j_1}\hat{O}_1$$

Therefore, $\hat{O}_1 = \hat{0}$. Thus, we have seen that the conditions imply that $\hat{W} = \hat{O}_0$, thus being a local unitary on the set of modes $\mathcal{N}\backslash\{j_1\}$. Because each of the conditions of Equation B.4 for each $j \in \mathcal{M}$ is independent, the same reasoning can be followed exactly with the other lattice sites in $\mathcal{M}$ that are not j_1. Therefore the conditions imply that none of the sites in $\mathcal{M}$ appears in the decomposition of $\hat{W}$ in terms of fermionic creation and annihilation operators. Therefore $\hat{W} = \hat{W}_{\mathcal{N}\backslash\mathcal{M}}$

is a local operator in the lattice sites $\mathcal{N}\backslash\mathcal{M}$, and therefore we have proven that $\hat{U} \sim_{\mathcal{M}} \hat{V}$. □

(Theorem. *7) Using* $\left(\hat{U}^\dagger \cdot \hat{f}_1 \cdot \hat{U}, \ldots, \hat{U}^\dagger \cdot \hat{f}_N \cdot \hat{U}\right)$*, the complete set of fermionic descriptors, it is possible to uniquely find the transformation* $\hat{U} \in \mathcal{T}_{\mathcal{N}}^{phys}$ *that has evolved them from their canonical form* $\left(\hat{f}_1, \ldots, \hat{f}_N\right)$.

Proof. First, note that the unitary conjugation action is up to a global phase. Keeping this in mind, considering the unitary $\hat{U}$ as an operator, we can regard it as a vector on the vector space of operators. We can consider in the operator algebra an orthonormal basis, where the scalar product between two operators $\hat{A}, \hat{B}$ is given by $\mathrm{Tr}\left(\hat{A}^\dagger \hat{B}\right)$. It is straightforward to see that for the fermionic operators if one labels the orthonormal Fock basis as $\{|k\rangle\}_{k=1}^{2^N}$, then $|k\rangle\langle l|$ is an orthonormal basis of the operator vector space with scalar product given by $\mathrm{Tr}\left\{\hat{A}^\dagger \hat{B}\right\}$. Let us rename $\left|\bar{k}\right\rangle = \hat{U}^\dagger |k\rangle$. Then, we denote $\hat{\bar{f}}_i = \hat{U}^\dagger \hat{f}_i \hat{U}$.

$\hat{U}$ can be written as $\hat{U} = \sum_{k,l=1}^{2^N} \mathrm{Tr}\left(\hat{U} |k\rangle\langle l|\right) |k\rangle\langle l|$. So, if we know $\mathrm{Tr}\left(\hat{U} |k\rangle\langle l|\right)$ we know the unitary. $|k\rangle\langle l|$ is a product of creation and annihilation operators, since $|k\rangle = \hat{f}_{i_1}^\dagger \ldots \hat{f}_{i_n}^\dagger |\Omega\rangle$ and $|\Omega\rangle\langle\Omega| = \hat{f}_N \ldots \hat{f}_1 \hat{f}_1^\dagger \ldots \hat{f}_N^\dagger$. Using this fact, having at our disposal $\left(\hat{f}_1, \ldots, \hat{f}_N\right)$ and $\left(\hat{\bar{f}}_1, \ldots, \hat{\bar{f}}_N\right)$ we can construct $|k\rangle\langle l|$ and $\left|\bar{k}\right\rangle\left\langle\bar{l}\right|$, where to construct the second we have replaced the $\hat{f}_i, \hat{f}_i^\dagger$ in the decomposition of $|k\rangle\langle l|$ by $\hat{U}^\dagger \hat{f}_i \hat{U}$, $\hat{U}^\dagger \hat{f}_i^\dagger \hat{U}$. We can see easily that $\left|\bar{k}\right\rangle\left\langle\bar{l}\right| = \hat{U}^\dagger |k\rangle\langle l| \hat{U}$.

We take the scalar product of any two of these objects. In other words, consider $\mathrm{Tr}\left(\left|\bar{k}\right\rangle\left\langle\bar{l}\right| |m\rangle\langle n|\right) = \mathrm{Tr}\left(\hat{U}^\dagger |k\rangle\langle l| \hat{U} |m\rangle\langle n|\right)$. Now, if we insert the decomposition of $\hat{U}$ found above and we use the linearity properties of the trace, we obtain that $\mathrm{Tr}\left(\left|\bar{k}\right\rangle\left\langle\bar{l}\right| |m\rangle\langle n|\right)$ equals

$$\sum_{o,p,q,r=1}^{2^N} \mathrm{Tr}\left(\hat{U}^\dagger |o\rangle\langle p|\right) \mathrm{Tr}\left(\hat{U} |q\rangle\langle r|\right) \mathrm{Tr}\left(|p\rangle\langle o| |k\rangle\langle l| |r\rangle\langle q| |m\rangle\langle n|\right) =$$

$$\sum_{o,p,q,r=1}^{2^N} \mathrm{Tr}\left(\hat{U}^\dagger \left|o\right\rangle\!\left\langle p\right|\right) \mathrm{Tr}\left(\hat{U} \left|q\right\rangle\!\left\langle r\right|\right) \delta_{ok}\delta_{lr}\delta_{qm}\delta_{np} = \mathrm{Tr}\left(\hat{U}^\dagger \left|k\right\rangle\!\left\langle n\right|\right) \mathrm{Tr}\left(\hat{U} \left|m\right\rangle\!\left\langle l\right|\right)$$

where we have used the orthonormality of the Fock basis and the properties of the Kronecker delta. Using the cyclic properties of the trace and complex conjugation we obtain that $\mathrm{Tr}\left(\hat{U}^\dagger \left|k\right\rangle\!\left\langle n\right|\right) \mathrm{Tr}\left(\hat{U} \left|m\right\rangle\!\left\langle l\right|\right) = \mathrm{Tr}\left(\hat{U} \left|n\right\rangle\!\left\langle k\right|\right)^* \mathrm{Tr}\left(\hat{U} \left|m\right\rangle\!\left\langle l\right|\right)$. The question that now arises is that if knowing all the values of $\mathrm{Tr}\left(\left|\bar{k}\right\rangle\!\left\langle \bar{l}\right| \left|m\right\rangle\!\left\langle n\right|\right)$ (that we can obtain since we only use $(f_1, \ldots, f_N)$ and $(\bar{f}_1, \ldots, \bar{f}_N)$), we can retrieve $\mathrm{Tr}\left(\hat{U} \left|m\right\rangle\!\left\langle l\right|\right)$. We can. We need to notice that $\mathrm{Tr}\left(\hat{U} \left|m\right\rangle\!\left\langle l\right|\right)$ is a complex number, so knowing its polar form is enough. We see that $\mathrm{Tr}\left(\left|\bar{l}\right\rangle\!\left\langle \bar{l}\right| \left|m\right\rangle\!\left\langle m\right|\right) = \left|\mathrm{Tr}\left(\hat{U} \left|m\right\rangle\!\left\langle l\right|\right)\right|^2$. Thus, we obtain the modulus of the complex number.

We can see now that $\mathrm{Tr}\left(\hat{U} \left|m\right\rangle\!\left\langle l\right|\right) = \sqrt{\mathrm{Tr}\left(\left|\bar{l}\right\rangle\!\left\langle \bar{l}\right| \left|m\right\rangle\!\left\langle m\right|\right)} e^{i\phi_{m,l}}$. So only the phases are up to determination. Here is where the issue of the overall phase redundancy intervenes. Since $\hat{U}$ is unitary, we know they must exist m_0, l_0 such that $\left|\mathrm{Tr}\left(\hat{U} \left|m_0\right\rangle\!\left\langle l_0\right|\right)\right|^2 = \mathrm{Tr}\left(\left|\bar{l_0}\right\rangle\!\left\langle \bar{l_0}\right| \left|m_0\right\rangle\!\left\langle m_0\right|\right) > 0$. We have the freedom to fix the phase $\phi_{m_0,l_0} = 0$ due to the overall phase redundancy.

In other words, we could always choose a global phase in $\hat{U}$'s equivalence class in $\mathcal{T}_{\mathcal{N}}^{phys}$ to cancel the phase ϕ_{m_0,l_0} so it is set to 0. Now, we can see that if we consider $\mathrm{Tr}\left(\left|\bar{l_0}\right\rangle\!\left\langle \bar{l}\right| \left|m\right\rangle\!\left\langle m_0\right|\right) = \mathrm{Tr}\left(\hat{U} \left|m_0\right\rangle\!\left\langle l_0\right|\right)^* \mathrm{Tr}\left(\hat{U} \left|m\right\rangle\!\left\langle l\right|\right)$ which equals $\sqrt{\mathrm{Tr}\left(\left|\bar{l_0}\right\rangle\!\left\langle \bar{l_0}\right| \left|m_0\right\rangle\!\left\langle m_0\right|\right)} \sqrt{\mathrm{Tr}\left(\left|\bar{l}\right\rangle\!\left\langle \bar{l}\right| \left|m\right\rangle\!\left\langle m\right|\right)} e^{i\phi_{ml}}$. Thus, we obtain that

$$\mathrm{Tr}\left(\hat{U} \left|m\right\rangle\!\left\langle l\right|\right) = \frac{\mathrm{Tr}\left(\left|\bar{l_0}\right\rangle\!\left\langle \bar{l}\right| \left|m\right\rangle\!\left\langle m_0\right|\right)}{\sqrt{\mathrm{Tr}\left(\left|\bar{l_0}\right\rangle\!\left\langle \bar{l_0}\right| \left|m_0\right\rangle\!\left\langle m_0\right|\right)}}$$

Therefore, indeed we can retrieve uniquely the unitaries that relate the descriptors $\left(\hat{\bar{f}}_1, \ldots, \hat{\bar{f}}_N\right)$ to $\left(\hat{f}_1, \ldots, \hat{f}_N\right)$. □

(Theorem. *8) The diagram of Figure 3.1 commutes. In other words:*

$$\pi_A^{\mathcal{P}}\left(\varphi_{\mathcal{M}}\left(\left(\left(\hat{U}^\dagger\cdot\hat{f}_{j_1}\cdot\hat{U},\ldots,\hat{U}^\dagger\cdot\hat{f}_{j_M}\cdot\hat{U}\right),\rho_0\right)\right)\right) = \\ = \varphi_A\left(\pi_A^{\mathcal{R}}\left(\left(\left(\hat{U}^\dagger\cdot\hat{f}_{j_1}\cdot\hat{U},\ldots,\hat{U}^\dagger\cdot\hat{f}_{j_M}\cdot\hat{U}\right),\rho_0\right)\right)\right)$$

Proof. We begin by expanding the right-hand side of the equation by applying the definition of the ontic projection operator, obtaining:

$$\varphi_A\left(\left(\left(\hat{U}^\dagger\cdot\hat{f}_{a_1}\cdot\hat{U},\ldots,\hat{U}^\dagger\cdot\hat{f}_{a_S}\cdot\hat{U}\right),\rho_0\right)\right)$$

Applying now the definition of φ_A of Equation 3.18, we obtain:

$$\sum_{\vec{r},\vec{s}} c_{\vec{r},\vec{s}}\left(\hat{f}_{a_1}^\dagger\right)^{r_1}\ldots\left(\hat{f}_{a_S}^\dagger\right)^{r_S}|\Omega\rangle\langle\Omega|\left(\hat{f}_{a_S}\right)^{s_S}\ldots\left(\hat{f}_{a_1}\right)^{s_1}$$

where

$$c_{\vec{r},\vec{s}} = \mathrm{Tr}\left(\hat{U}^\dagger\cdot\left(\hat{f}_{a_1}^\dagger\right)^{r_1}\ldots\left(\hat{f}_{a_S}^\dagger\right)^{r_S}\hat{f}_{a_S}\ldots\hat{f}_{a_1}\hat{f}_{a_1}^\dagger\ldots\hat{f}_{a_S}^\dagger\left(\hat{f}_{a_S}\right)^{s_S}\ldots\left(\hat{f}_{a_1}\right)^{s_1}\cdot\hat{U}\cdot\rho_0\right)$$

We now turn to expand the left-hand side of the initial equation by applying the definition of the ontic-phenomenal epimorphism and choosing the same basis as before but for the larger set of lattice sites $\mathcal{M} = \{j_1,\ldots,j_M\}$, we obtain:

$$\pi_A^{\mathcal{P}}\left(\varphi_{\mathcal{M}}\left(\left(\left(\hat{U}^\dagger\cdot\hat{f}_{j_1}\cdot\hat{U},\ldots,\hat{U}^\dagger\cdot\hat{f}_{j_M}\cdot\hat{U}\right),\rho_0\right)\right)\right) = \\ = \pi_A^{\mathcal{P}}\left(\sum_{\vec{q},\vec{t}} c_{\vec{q},\vec{t}}\left(\hat{f}_{j_1}^\dagger\right)^{q_1}\ldots\left(\hat{f}_{j_M}^\dagger\right)^{q_M}|\Omega\rangle\langle\Omega|\left(\hat{f}_{j_M}\right)^{t_M}\ldots\left(\hat{f}_{j_1}\right)^{t_1}\right)$$

where

$$c_{\vec{q},\vec{t}} = \mathrm{Tr}\left(\hat{U}^\dagger\cdot\left(\hat{f}_{j_1}^\dagger\right)^{q_1}\ldots\left(\hat{f}_{j_M}^\dagger\right)^{q_M}\hat{f}_{j_M}\ldots\hat{f}_{j_1}\hat{f}_{j_1}^\dagger\ldots\hat{f}_{j_M}^\dagger\left(\hat{f}_{j_M}\right)^{t_M}\ldots\left(\hat{f}_{j_1}\right)^{t_1}\cdot\hat{U}\cdot\rho_0\right)$$

The main difference with the expanded version of the right-hand side is the lengths of the vectors $\vec{r}, \vec{s}$ versus $\vec{q}, \vec{t}$. Let's take the fermionic partial trace of the fermionic modes that are in $\mathcal{M}$ but not in A. Following the fermionic partial trace procedure in Equation 3.11, the expression of the left-hand side becomes

$$\sum_{\vec{q},\vec{t}} c_{\vec{q},\vec{t}} \mathrm{Tr}_{\mathcal{M}\backslash A}\left(\left(\hat{f}^\dagger_{j_1}\right)^{q_1}\dots\left(\hat{f}^\dagger_{j_M}\right)^{q_M}|\Omega\rangle\langle\Omega|\left(\hat{f}_{j_M}\right)^{t_M}\dots\left(\hat{f}_{j_1}\right)^{t_1}\right) =$$
$$= \sum_{\vec{q},\vec{t}} c_{\vec{q},\vec{t}}\left(\delta\vec{g},\vec{l}(-1)^{k(q,\vec{t})}\right)\left(\hat{f}^\dagger_{a_1}\right)^{r_1}\dots\left(\hat{f}^\dagger_{a_S}\right)^{r_S}|\Omega\rangle\langle\Omega|\left(\hat{f}_{a_S}\right)^{s_S}\dots\left(\hat{f}_{a_1}\right)^{s_1}$$

where $\vec{q}/\vec{t}$ are the concatenation in the mode order of the vectors $\vec{r}\cup\vec{g}/\vec{s}\cup\vec{l}$. $\vec{r},\vec{s}$ refer to the modes in A and $\vec{g},\vec{l}$ refer to the modes in $\mathcal{M}\backslash A$. The factor $k(\vec{q},\vec{t}) = \sum_{n=1}^{M\text{-}S}\sum_{i=b_n}^{M\text{-}1} q_i q_{i+1} + t_i t_{i+1}$ tracks the number of anticommutations that have happened to bring all the creation and annihilation operators of modes in $B = \mathcal{M}\backslash A$ towards the center, near $|\Omega\rangle\langle\Omega|$. We see the operators agree with the right-hand side, we just need to prove the coefficients are the same. Observe that because of the definition of $k(\vec{q},\vec{t})$ we have the property, that if inside the coefficient $c_{\vec{q},\vec{t}}$ we anticommute all the modes of B towards the center as in the partial tracing procedure we cancel the phase since $(-1)^{2k}=1$, leaving:

$$(-1)^{k(q,\vec{t})}\, c_{\vec{q},\vec{t}} = \mathrm{Tr}\left(\hat{U}^\dagger\cdot\left(\hat{f}^\dagger_{a_1}\right)^{r_1}\dots\left(\hat{f}^\dagger_{a_S}\right)^{r_S}\left(\hat{f}^\dagger_{b_1}\right)^{g_1}\dots\left(\hat{f}^\dagger_{b_{M\text{-}S}}\right)^{g_{M\text{-}S}}\cdot\right.$$
$$\left.\cdot\hat{f}_{j_M}\dots\hat{f}_{j_1}\hat{f}^\dagger_{j_1}\dots\hat{f}^\dagger_{j_M}\left(\hat{f}_{b_{M\text{-}S}}\right)^{l_{M\text{-}S}}\dots\left(\hat{f}_{b_1}\right)^{l_1}\left(\hat{f}_{a_S}\right)^{s_S}\dots\left(\hat{f}_{a_1}\right)^{s_1}\cdot\hat{U}\cdot\rho_0\right)$$

The sum $\sum_{\vec{q},\vec{t}}$ can be seen as $\sum_{\vec{r},\vec{s},\vec{g},\vec{l}}$. Incorporating now the other terms, the sums over $\vec{g},\vec{l}$ and the Kronecker delta, we obtain:

$$\sum_{\vec{g},\vec{l}}\delta_{\vec{g},\vec{l}}(-1)^{k(q,\vec{t})}\, c_{\vec{q},\vec{t}} = \sum_{\vec{g}}\mathrm{Tr}\left(\hat{U}^\dagger\cdot\left(\hat{f}^\dagger_{a_1}\right)^{r_1}\dots\left(\hat{f}^\dagger_{a_S}\right)^{r_S}\left(\hat{f}^\dagger_{b_1}\right)^{g_1}\dots\left(\hat{f}^\dagger_{b_{M\text{-}S}}\right)^{g_{M\text{-}S}}\cdot\right.$$
$$\left.\cdot\hat{f}_{j_M}\dots\hat{f}_{j_1}\hat{f}^\dagger_{j_1}\dots\hat{f}^\dagger_{j_M}\left(\hat{f}_{b_{M\text{-}S}}\right)^{g_{M\text{-}S}}\dots\left(\hat{f}_{b_1}\right)^{g_1}\left(\hat{f}_{a_S}\right)^{s_S}\dots\left(\hat{f}_{a_1}\right)^{s_1}\cdot\hat{U}\cdot\rho_0\right) =$$
$$= \mathrm{Tr}\left(\hat{U}^\dagger\cdot\left(\hat{f}^\dagger_{a_1}\right)^{r_1}\dots\left(\hat{f}^\dagger_{a_S}\right)^{r_S}\hat{f}_{a_S}\dots\hat{f}_{a_1}\hat{f}^\dagger_{a_1}\dots\hat{f}^\dagger_{a_S}\left(\hat{f}_{a_S}\right)^{s_S}\dots\left(\hat{f}_{a_1}\right)^{s_1}\cdot\right.$$
$$\left.\cdot\sum_{\vec{g}}\left(\hat{f}^\dagger_{b_1}\right)^{g_1}\dots\left(\hat{f}^\dagger_{b_{M\text{-}S}}\right)^{g_{M\text{-}S}}\hat{f}_{b_{S\text{-}M}}\dots\hat{f}_{b_1}\hat{f}^\dagger_{b_1}\dots\hat{f}^\dagger_{b_{M\text{-}M}}\left(\hat{f}_{b_{M\text{-}S}}\right)^{g_{M\text{-}S}}\dots\left(\hat{f}_{b_1}\right)^{g_1}\cdot\hat{U}\cdot\rho_0\right) =$$

$$= \mathrm{Tr}\left(\hat{U}^\dagger \cdot \left(\hat{f}^\dagger_{a_1}\right)^{r_1} \dots \left(\hat{f}^\dagger_{a_S}\right)^{r_S} \hat{f}_{a_S} \dots \hat{f}_{a_1} \hat{f}^\dagger_{a_1} \dots \hat{f}^\dagger_{a_S} \left(\hat{f}_{a_S}\right)^{s_S} \dots \left(\hat{f}_{a_1}\right)^{s_1} \cdot \hat{U} \cdot \rho_0\right)$$

where in the last equality, we have applied the resolution of the identity operator in terms of local operators on B alone. Notice that the expression of the last line is exactly equal to the coefficient $c_{\vec{r},\vec{s}}$ that we had in the expansion of the right-hand side of the initial equality we wanted to prove. Since the coefficients are equal and the algebraic terms are also, we can conclude the theorem is proven. □

C | Mathematical details of Chapter 4

This Appendix presents the detailed proofs and mathematical subtleties that could not be included in Chapter 4. *Some of the content in this Appendix is adapted contents from the publication [104] done in collaboration with Lucia Vilchez-Estevez.*

C.1 General annihilation operators

We show how to construct the J annihilation operators for any anyon theory.

As we discussed in the main text, we have identified the annihilating elements for the first mode $a_1^{b_0,a\times b_0}$ (see Figure 4.18). We have also seen that for a general mode k, we define the in-front annihilating elements according to the notion of mode locality where we exchange the first mode in front of all the $k-1$ others until k (see Figure 4.19).

Now, to make the annihilation operators, we have seen in the text that we need to take linear combinations of the annihilating elements as an analogy to the known fermionic annihilation operators. This is without messing with the properties of spanning the local algebra of observables. We want to construct the normalised annihilation operators by specifying the coefficients $C^{(j)}_{b_0,c_0,k} \in \mathbb{C}$ of the linear combinations of the annihilating terms:

$$\alpha_k^{(j)} = \sum_{b_0,c_0=a\times b_0} C^{(j)}_{b_0,c_0,k}\, a_k^{b_0,c_0} \tag{C.1}$$

As we explain in the text, under braiding in front of the other modes, $C^{(j)}_{b_0,c_0,k}$ is determined by $C^{(j)}_{b_0,c_0,1}$. Without loss of generality, we are interested in the

normalised annihilation operators where $C^{(j)}_{b_0,c_0,1}$ is either 0 or 1; the following arguments can be repeated in general because the relevance is on which $C^{(j)}_{b_0,c_0,1}$ need to vanish.

If we try to have a single annihilation operator, for $\alpha^{(0)}_1$ to be able to generate the whole algebra of observables, no coefficient must vanish. We see here a big difference between abelian and non-abelian particles. For particles that are abelian (there is a single possible value for $c_0 = a \times b_0$ for any b_0), it is straightforward to see that all coefficients $|C^{(0)}_{b_0,c_0,1}| = 1$ then $\alpha^{(0)}_1 \alpha^{(0)\dagger}_1$ and $\alpha^{(0)\dagger}_1 \alpha^{(0)}_1$ generate the local algebra of observables for mode 1.

However, considering that the particle a is not abelian, two fusion channels are compatible for $a \times b_0 = c_0 + c'_0$ for some b_0. Now when taking $\alpha^{(0)\dagger}_1 \alpha^{(0)}_1$, terms that violate the superselection rule appear. Specifically, terms that convert total charge c_0 to total charge c'_0 and vice versa. It is straightforward to observe that one cannot eliminate these terms while keeping the relevant terms necessary to have the local observable, the projector of the a particle type in mode 1, by adding more terms in the monomial. The only way to make these undesired terms go away is by either setting $C^{(0)}_{b_0,c_0,1} = 0$ or $C^{(0)}_{b_0,c'_0,1} = 0$. Let us consider, without loss of generality, we have done the second. We now need at least another annihilation operator with a vanishing $C^{(j)}_{b_0,c_0,1}$ and a non-vanishing $C^{(j)}_{b_0,c'_0,1}$ such that the terms with total charge c'_0 that appear in the projector of particle type a in mode 1 can be generated.

We have seen that for each "extra" fusion channel, we need at least one annihilation operator that contains such a term. The general construction we provide guarantees such property. However, there might be better options. There may be a different grouping of the terms such that the total number of annihilation operators per non-abelian particle is smaller. We know that at least the number of annihilation operators needs to be $J' = \max_k(\sum_l N^{a_l}_{aa_k})$. In the construction we present, we

obtain $J = \sum_{k,l} N_{a_j a_k}^{a_l} - n + 1$ annihilation operators. For the two most relevant non-abelian anyon families, Fibonacci and Ising anyons, we have that $J' = J$; therefore, our construction is optimal for these important cases.

The construction is as follows. First, we fix an order in the particle types of the anyon theory. We choose to bring all abelian particles at the beginning of the ordering. This fixed order defines a preferred basis for the $n \times n$ matrices for each particle type a_j defined as $(A_j)_{kl} = N_{a_j a_k}^{a_l}$. Now to construct the $J = \sum_{k,l} N_{a_j a_k}^{a_l} - n + 1$ annihilation operators for a_j we label $c_{a_k,i}$ the i'th particle type such that $c_{a_l,k} = a_j \times a_k$. For the first annihilation operator of a_j, we set the terms $C^{(0)}_{a_k,c_{a_k,1},1} = 1$ and the rest, $C^{(0)}_{a_k,c_{a_k,i},1}$ for $i > 1$, vanish. This is analogous to selecting the first fusion channel in each row of A_j. The choice of the coefficients defines the first annihilation operator.

If a_j is abelian, our work is over since then $J = 1$, and there are no $C^{(0)}_{a_k,c_{a_k,i},1}$ for $i > 1$. However, if a_j is non-abelian, we must construct the other annihilation operators. To do so, we go to the first a_{k_0} in the ordering such that exists a $c_{a_{k_0},2}$; so, that has more than one allowed fusion channel when fusing a_j with a_{k_0}. Once identified, we set $C^{(1)}_{a_k,c_{a_k,1},1} = 1$ for $k \neq k_0$, $C^{(1)}_{a_{k_0},c_{a_{k_0},2},1} = 1$, and making all others coefficients vanish. This specification would fix the second annihilation operator.

To produce a third annihilation operator for a_j, we would first check if there exists a $c_{a_{k_0},3}$, if it does we would set $C^{(2)}_{a_k,c_{a_k,1},1} = 1$ for $k \neq k_0$, $C^{(2)}_{a_{k_0},c_{a_{k_0},3},1} = 1$, and making all others coefficients vanish. Thus setting the third annihilation operator. If $c_{a_{k_0},3}$ does not exist, though, we then go to the next a_{k_1} in the ordering such that $c_{a_{k_1},2}$ exists, and we would set $C^{(2)}_{a_k,c_{a_k,1},1} = 1$ for $k \neq k_1$, $C^{(2)}_{a_{k_1},c_{a_{k_1},2},1} = 1$, and making all others coefficients vanish. Therefore we would have specified the third annihilation operator.

To produce the mth annihilation operator, one can see the recursive strategy we

are following. Given the special $a_{k'}$ we have identified in the $m-1$th annihilation operator for which we have set $C^{(m-1)}_{a_k,c_{a_{k'},i},1}=1$ for $i>1$, then either $c_{a_{k'},i+1}$ exists or not. If it does we set $C^{(m)}_{a_k,c_{a_k},1,1}=1$ for $k\neq k'$, $C^{(m)}_{a_{k'},c_{a_{k'},i+1},1}=1$, and make all others coefficients zero. However, if it does not exist, we find the next $a_{k''}$ from $a_{k'}$ in the ordering such that $c_{a_{k''},2}$ exists. Then, we specify $C^{(m)}_{a_k,c_{a_k},1,1}=1$ for $k\neq k''$, $C^{(m)}_{a_{k''},c_{a_{k''}},2,1}=1$, and make all others coefficients zero.

The process terminates when $c_{a_{k'},i+1}$ does not exist and there is no $a_{k''}$ further down the ordering than $a_{k'}$ such that $c_{a_{k''},2}$ exists. It is straightforward to check that this will happen for the Jth term.

The above procedure fixes the annihilation operators for each particle type a_j in the first lattice site. For the other lattice sites, we exchange the annihilation operators in position, bringing the first lattice site in front of the others into the kth lattice site.

We want to make a technical remark where the identity particle e can also be considered to have an annihilation operator per mode (being an abelian particle). However, the identity annihilation operator can always be expressed in terms of the other annihilation operators of the theory. Concretely, any $(\alpha_l)^{(j)}_k(\alpha_l)^{(j)\dagger}_k$ will give such annihilation operator.

C.2 Creation and annihilation operators theorem

Using the construction of the anyonic annihilation operators specified in Appendix C.1, we can prove the desirable properties of the anyonic annihilation operators. In particular, we want to ensure that with this construction, it is possible to express any local observable in a set of lattice sites in terms of the creation and annihilation operators of such lattice sites.

Concretely the theorem we prove is the following.

(Theorem. *10) Consider a general anyon theory with n particle types and N lattice sites. Consider a set of lattice sites $\mathcal{M} = \{s_1, \ldots, s_M\}$ and the subsystem bipartition where the selected sites are always in front of the other $N - M$ sites. Under this bipartition, any local observable in these M sites can be written as a polynomial of these lattice sites' creation and annihilation operators.*

Proof. To prove this general theorem, we first prove that the statement is true for the sets of lattice sites $\{1, \ldots, M\}$, and then we prove that that implies the statement holds for any set of lattice sites.

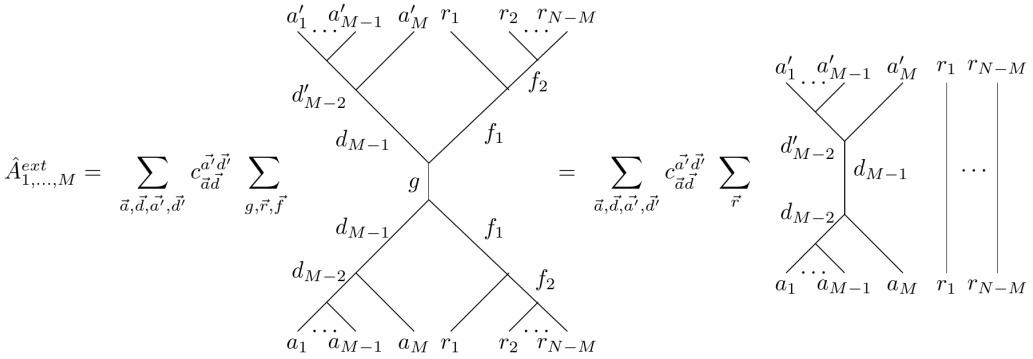


Figure C.1: Local observables in $1, \ldots, M$. With $c^{\vec{a'},\vec{d'}}_{\vec{a},\vec{d}} = c^{\vec{a},\vec{d}\ *}_{\vec{a'},\vec{d'}} \in \mathbb{C}$.

Given the set of lattice sites $\{1, \ldots, M\}$, a local observable for the chosen bipartition has the general form shown in Figure C.1. Under this bipartition of $1 \ldots M | M+1 \ldots N$, we can find the elements of the candidate local algebra of operators following the same procedure as in the main paper. We see that any local observable will be an element of the candidate local algebra of operators since it is left invariant by local unitaries acting on the complement of $1, \ldots, M$. We define the operators $\hat{O}_{\vec{a},\vec{d},g}$ as in Figure C.2. Notice that any local observable in $1, \ldots, M$ can be written as a linear combination of $\hat{O}^{\dagger}_{\vec{a},\vec{d},g} \hat{O}_{\vec{a'},\vec{a'},g}$.

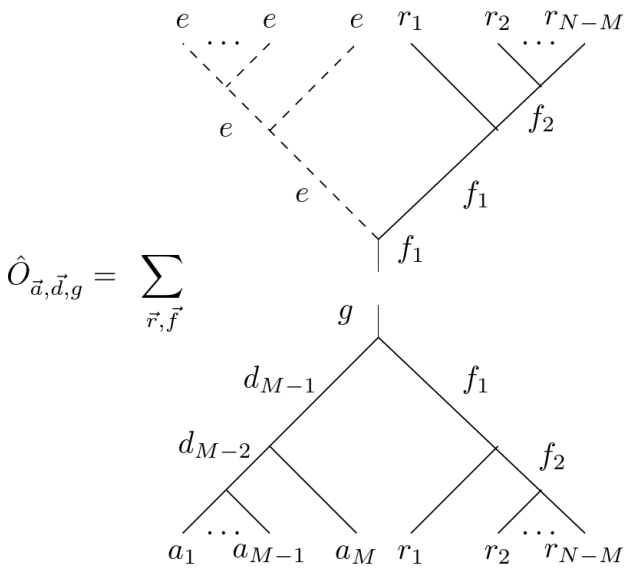


Figure C.2: Operators that generate the local observables in $1, \ldots, M$

Suppose polynomials of the local creation and annihilation operators of the lattice sites $1, \ldots, M$ can express the operators $\hat{O}_{\vec{a},\vec{d},g}$. In such case, they can express any physical local observable in these lattice sites.

Consider the annihilation operators $(\alpha_l)_k^{(j)}$ for one of the non-abelian particle types a_l. Notice that due to the particle type being non-abelian, there is strictly more than one annihilation operator associated with this particle type. Note that we can compute ${(\alpha_l)^{(j)}}_k - {(\alpha_l)^{(j)}}_k {(\alpha_l)^{(0)}}^\dagger_k {(\alpha_l)^{(0)}}_k = (a_l)_k^{b_j,c_j}$, where $c_j = a_l \times b_j$ such that is not the first in the ordering for the fusion of a_l and b_j. With this calculation, we see that we can retrieve from polynomials of the creation and annihilation operators all the annihilating terms that do not appear in ${(\alpha_l)^{(0)}}_k$. Moreover, by now calculating ${(\alpha_l)^{(0)}}_k - {(\alpha_l)^{(j)}}_k + (a_l)_k^{b_j,c_j} = (a_l)_k^{b_j,c_0}$ where $c_0 = a_l \times b_j$ is the first allowed fusion channel between a_l and b_j under the fixed ordering. We can also calculate ${(\alpha_l)^{(0)}}_k - \sum_{b_j} (a_l)_k^{b_j,c_0} = \sum_{b_r} (a_l)_k^{b_r,a_l \times b_r}$ where the sum over b_j is over the particle types b_j that have more than one allowed fusion channel with a_l, and the sum over b_r is over the particle types b_r that have only one allowed fusion

channel with a_l: $a_l \times b_r$.

For every b_r that is not an abelian particle, we have that there will exist some particle type a_s such that there is more than one allowed fusion channel $c_t = a_s \times b_r$. Thus, the terms $(a_s)_k^{b_r,c_t}$ can be expressed as polynomials of the creation annihilation operators for the particle type a_s as we have shown before. It is easy to see that $(a_s)_k^{b_r,c_t}(a_s)_k^{b_r,c_t\,\dagger} \sum_{b_{r'}} (a_l)_k^{b_{r'},a_l\times b_{r'}} = (a_l)_k^{b_r,a_l\times b_r}$.

After all these calculations, we can conclude that we can express any annihilating term $(a_j)_k^{b_0,c_0}$ in terms of local creation and annihilation operators on k, for b_0 being a non-abelian particle type. For the abelian terms, we know we can express $\sum_b (a_l)_k^{b,b\times a_l}$ where the sum runs over b being abelian particle types, in terms of the creation and annihilation operators.

Once we have these results, we are ready to see how we can express $\hat{O}_{\vec{a},\vec{d},g}$ in terms of $(a_j)_k^{b_0,c_0}$ and $\sum_b (a_l)_k^{b,b\times a_l}$, for b_0 non-abelian and b abelian and $k \leq M$. Thus, implying that we can express $\hat{O}_{\vec{a},\vec{d},g}$ in terms of the local creation and annihilation operators in the modes $1, \ldots, M$.

We can see with direct computation the following expression:

$$\hat{O}_{\vec{a},\vec{d},g} = \prod_{j=2}^{M} \left(\sum_{\substack{b_{M\text{-}j+2} \\ c_{M\text{-}j+2}}} \left[F_g^{d_{M-j}a_{M\text{-}j+2}b_{M\text{-}j+2}}\right]^*_{d_{M\text{-}j+1}c_{M\text{-}j+2}} (a_{M\text{-}j+2})_{M\text{-}j+2}^{b_{M\text{-}j+2},c_{M\text{-}j+2}} \right) \cdot \sum_{b_1} (a_1)_1^{b_1,g} \tag{C.2}$$

where $d_0 = a_1$. We can express each term of the product in terms of the local creation and annihilation operators. Let us start with the term $\sum_{b_1} (a_1)_1{}^{b_1,g}$ we can decompose the sum between the sum over abelian particles plus the sum over non-abelian particles. Each term of the non-abelian sum can be expressed in terms

of the creation and annihilation operators in mode 1, so the sum of such terms is also. Moreover, we have seen how the sum over the abelian terms is expressable in terms of the creation and annihilation operators.

Similarly, we do the same decomposition of the sum for the terms in the product involving the F-matrices components. For the non-abelian particles b_{M-j+2}, each term $(a_{M-j+2})_{M-j+2}^{b_{M-j+2},c_{M-j+2}}$ is expressable in terms of the creation and annihilation operators, thus the linear combinations of such terms can be expressed in terms of such local operators. For abelian particles, it should be noted that if b_{M-j+2} is abelian, $[F_g^{d_{M-j}a_{M-j+2}b_{M-j+2}}]^*_{d_{M-j+1}c_{M-j+2}}$ equals $\delta_{c_{M-j+2},a_{M-j+2}\times b_{M-j+2}}$. This follows from the fact that the F-moves for abelian particles are trivial.

Therefore, we obtain that the sum over the abelian particles ends up becoming $\sum_{b_{M-j+2}} (a_{M-j+2})_{M-j+2}^{b_{M-j+2},a_{M-j+2}\times b_{M-j+2}}$. Thus, expressable in terms of the local creation and annihilation operators of the mode $M-j+2$.

This concludes that $\hat{O}_{\vec{a},\vec{d},g}$ can be expressed in terms of the creation and annihilation operators. Therefore, any local observable in $1,\ldots,M$ can be expressed using the $1,\ldots,M$ creation and annihilation operators. Concretely, as a polynomial of such operators. Moreover, note that our proof is constructive and that by using it, one could find a closed expression of any observable in terms of our local creation and annihilation operators.

All we have left now to prove the general theorem is to use the fact that we know the theorem holds for the set of modes $1,\ldots,M$ to extend it to any set of modes $s_1,\ldots,s_M$ ($s_i < s_{i+1}$). Notice that $i \leq s_i$ always. We will see that we can find a good map between the local observables in a general set of modes $s_1,\ldots,s_M$ and the local observables $1,\ldots,M$.

Remember that the notion of locality is such that the relevant modes move in front

of the ancillary modes. It is straightforward to observe that applying the unitary transformation $U = \prod_{i=0}^{M\text{-}1} \prod_{j=M\text{-}i+1}^{s_{M\text{-}i}} R^{\dagger}_{j\text{-}1\ j}$ we transform any local observable in the modes $s_1, \ldots, s_M$ to a local observable in $1 \ldots M$.

$$U \hat{A}_{s_1,\ldots,s_M} U^{\dagger} = \hat{A}_{1,\ldots,M} \tag{C.3}$$

Remember that we proved that $\hat{A}_{1,\ldots,M} = p((\alpha_l)^{(j)}_k, (\alpha_l)^{(j')^{\dagger}}{}_{k'})$, where $p(\cdot)$ is a polynomial, and $k, k' \in \{1, \ldots, M\}$. Hence, by linearity and unitarity, $\hat{A}_{s_1,\ldots,s_M} = U^{\dagger} \hat{A}_{1,\ldots,M} U$ equals $p(U^{\dagger}(\alpha_l)^{(j)}_k U, U^{\dagger}(\alpha_l)^{(j')^{\dagger}}_{k'} U)$. Moreover, notice that $U^{\dagger} = \prod_{i=1}^{M} \prod_{j=i}^{s_i\text{-}1} R_{s_i\text{-}1\text{-}j+i\ s_i\text{-}j+i}$. The annihilation operators in k are left invariant by unitaries local in the set of modes that excludes k, and that $(\alpha_l)^{(j)}{}_k = V(\alpha_l)^{(j)}_1 V^{\dagger}$ by definition, where $V = \prod_{j=0}^{k\text{-}2} R_{k\text{-}1\text{-}j\ k\text{-}j}$. Now, it is easy to see that

$$\begin{aligned} U^{\dagger}(\alpha_l)^{(j)}_k U = & \prod_{i=1}^{k} \prod_{j=i}^{s_i\text{-}1} R_{s_i\text{-}1\text{-}j+i\ s_i\text{-}j+i} \prod_{i=k+1}^{M} \prod_{j=i}^{s_i\text{-}1} R_{s_i\text{-}1\text{-}j+i\ s_i\text{-}j+i} \cdot \left((\alpha_l)^{(j)}_k \right) \cdot \\ & \cdot \left(\prod_{i=k+1}^{M} \prod_{j=i}^{s_i\text{-}1} R_{s_i\text{-}1\text{-}j+i\ s_i\text{-}j+i} \right)^{\dagger} \left(\prod_{i=1}^{k} \prod_{j=i}^{s_i\text{-}1} R_{s_i\text{-}1\text{-}j+i\ s_i\text{-}j+i} \right)^{\dagger}, \end{aligned} \tag{C.4}$$

Moreover, the unitary $\prod_{i=k+1}^{M} \prod_{j=i}^{s_i\text{-}1} R_{s_i\text{-}1\text{-}j+i\ s_i\text{-}j+i}$ is local on the set of modes that do not include k, so it leaves any creation operator invariant. Thus,

$$\begin{aligned} U^{\dagger}(\alpha_l)^{(j)}_k U = & \prod_{i=1}^{k\text{-}1} \prod_{j=i}^{s_i\text{-}1} R_{s_i\text{-}1\text{-}j+i\ s_i\text{-}j+i} \left(\prod_{j=k}^{s_k\text{-}1} R_{s_k\text{-}1\text{-}j+k\ s_k\text{-}j+k} \right) \cdot \left((\alpha_l)^{(j)}_k \right) \cdot \\ & \cdot \left(\prod_{j=k}^{s_k\text{-}1} R_{s_k\text{-}1\text{-}j+k\ s_k\text{-}j+k} \right)^{\dagger} \left(\prod_{i=1}^{k\text{-}1} \prod_{j=i}^{s_i\text{-}1} R_{s_i\text{-}1\text{-}j+i\ s_i\text{-}j+i} \right)^{\dagger} \end{aligned} \tag{C.5}$$

By the definition of $(\alpha_l)^{(j)}_k$, we see that applying the unitary action of the operator

$\left(\prod_{j=k}^{s_k\text{-}1} R_{s_k\text{-}1\text{-}j+k\ s_k\text{-}j+k}\right)$ to it it gives us $(\alpha_l)^{(j)}_{s_k}$, giving:

$$U^\dagger(\alpha_l)^{(j)}_k U = \prod_{i=1}^{k\text{-}1}\prod_{j=i}^{s_i\text{-}1} R_{s_i\text{-}1\text{-}j+i\ s_i\text{-}j+i}\left((\alpha_l)^{(j)}_{s_k}\right)\left(\prod_{i=1}^{k\text{-}1}\prod_{j=i}^{s_i\text{-}1} R_{s_i\text{-}1\text{-}j+i\ s_i\text{-}j+i}\right)^\dagger \quad \text{(C.6)}$$

The only thing that is just left to check is that $\left(\prod_{i=1}^{k\text{-}1}\prod_{j=i}^{s_i\text{-}1} R_{s_i\text{-}1\text{-}j+i\ s_i\text{-}j+i}\right)$ is local on the set of modes that do not include s_k. We observe that the largest mode in the expression is $s_{k\text{-}1}$, which we know is strictly smaller than s_k. Therefore, all modes appearing in the expression are strictly smaller than s_k, making the unitary local on the modes that exclude s_k. Thus the unitary action leaves $\left((\alpha_l)^{(j)}_{s_k}\right)$ invariant, giving $U^\dagger(\alpha_l)^{(j)}_k U = (\alpha_l)^{(j)}_{s_k}$. This also proves it for the creation operators $U^\dagger(\alpha_l)^{(j')^\dagger}_{k'} U = (\alpha_l)^{(j')^\dagger}_{s'_k}$. Therefore, we indeed see that any local observable $\hat{A}_{s_1,\ldots,s_M}$ in any set of modes $s_1,\ldots,s_M$ can be written as a polynomial of the local creation and annihilation operators for such set of modes, since $\hat{A}_{s_1,\ldots,s_M} = \hat{A}_{s_1,\ldots,s_M} = U^\dagger \hat{A}_{1,\ldots,M} U = p(U^\dagger(\alpha_l)^{(j)}_k U, U^\dagger(\alpha_l)^{(j')^\dagger}_{k'} U) = p((\alpha_l)^{(j)}_{s_k}, (\alpha_l)^{(j')^\dagger}_{s'_k})$. This concludes the proof of the theorem. □

A corollary of the theorem is that $\mathbb{I}$ can always be expressed in terms of only creation and annihilation operators of a single anyonic mode. To prove it, we just need to establish that the identity $\mathbb{I}$ is a local physical observable in any mode. This follows from the identity decomposition, using Figure C.2:

$$\mathbb{I} = \sum_{ag} \hat{O}^\dagger_{a\ g} \hat{O}_{a\ g} \quad \text{(C.7)}$$

A consequence of the construction is the algebraic independence of the annihilation operators of different modes. When decomposing any physical operator $\hat{O}_{AB}$ local in AB as a polynomial of creation and annihilation operators, we can always

choose to represent the operator as: $\hat{O}_{AB} = \hat{O}_A + \hat{O}_B + \hat{C}_{AB}$. All three being physical operators, with $\hat{O}_A, \hat{O}_B$ being local on A and B, respectively, and thus polynomials of the creation and annihilation operators of A or B alone. $\hat{C}_{AB}$ is a physical operator neither local on A nor B. $\hat{C}_{AB}$ decomposition includes non-reducible factors involving creation and annihilation operators of both A and B necessarily.

The algebraic independence of the annihilation operators in different modes implies the Separation property. Imagine $\hat{V}$ being local on AC and local on BC. We would have $\hat{V}_{AC} = \hat{U}_A + \hat{U}_C + \hat{C}_{AC} = \hat{O}_{AC} = \hat{V}_B + \hat{V}_C + \hat{L}_{BC}$. Thus, algebraic independence implies necessarily that $\hat{V}_B = \hat{U}_A = \hat{L}_{BC} = \hat{C}_{AC} = \hat{0}$ and $\hat{V}_C = \hat{U}_C = \hat{V}_{AC}$, thus making $\hat{V}_{AC}$ a local physical operator on C.

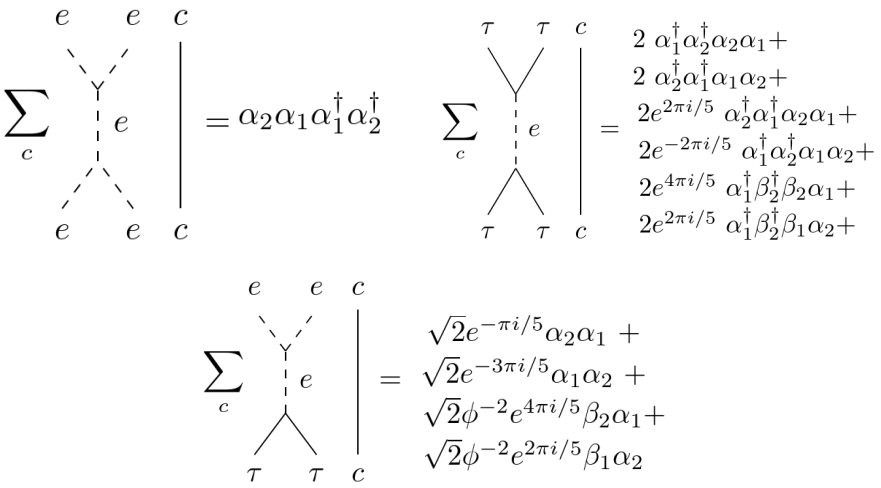


Figure C.3: Local observable terms in $1, 2$ with global charge e, up to hermitian conjugation

C.3 3-anyon Fibonacci observables

We present a complete list of all observable terms local in the $1, 2$ modes of a three-mode Fibonacci anyons model. Up to hermitian conjugation, there are nine

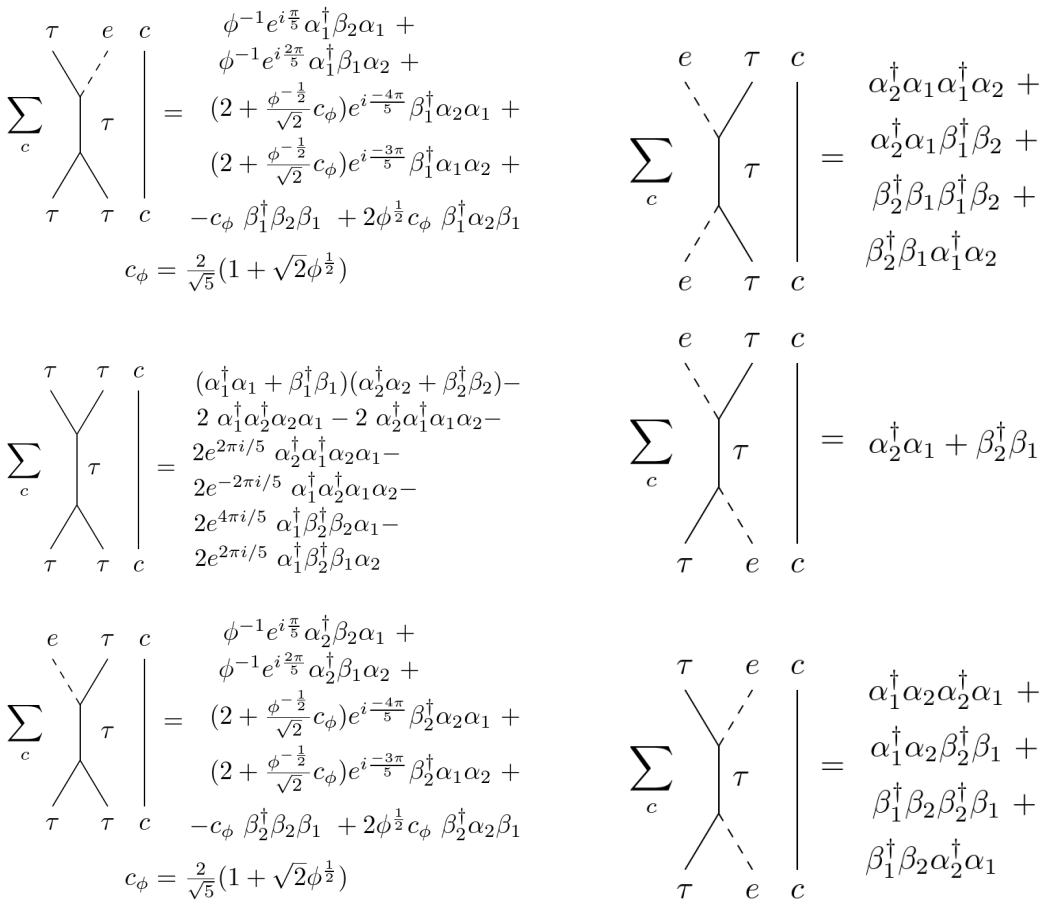


Figure C.4: Local observable terms in $1, 2$ with global charge τ, up to hermitian conjugation

linearly independent terms. We show all of them in Figures C.3 & C.4.

Notice that the expressions we show in Figures C.3 & C.4 are more compressed than the expressions obtained through the reasoning in the proof of the general theorem. We present these expressions because we think it is more convenient to work with them, especially when investigating Hamiltonians.

C.4 Fibonacci commutation relations

We have defined the Fibonacci creation and annihilation operators in terms of the diagrammatic formalism we have for non-abelian anyons. A future avenue for research is to give a completely algebraic characterisation of Fibonacci anyons. To do so, we need to specify the algebraic relations the Fibonacci creation and annihilation operators follow and which operators one can specify as polynomials of the creation and annihilation operators are observables.

Such a goal is quite ambitious and difficult, being out of the scope of this publication. Nevertheless, presenting some algebraic relations satisfied by the Fibonacci creation and annihilation operators may be helpful to provide initial insight into such a task and help in becoming familiar with manipulating expressions where the creation and annihilation operators are present.

We can find the following relations for the operators of a single mode:

$$(\alpha_S)^2 = 0 \quad \alpha_S\beta_S = \beta_S\alpha_S = 0 \quad \alpha_S\alpha_S^\dagger = \beta_S\beta_S^\dagger \tag{C.8}$$

$$\alpha_S\beta_S^\dagger\beta_S = \alpha_S\beta_S^\dagger\alpha_S = \beta_S\alpha_S^\dagger\alpha_S = \beta_S\alpha_S^\dagger\beta_S \tag{C.9}$$

$$\alpha_S\alpha_S^\dagger\alpha_S = \alpha_S - \beta_S\alpha_S^\dagger\alpha_S \tag{C.10}$$

$$\beta_S\beta_S^\dagger\beta_S = \beta_S - \alpha_S\beta_S^\dagger\beta_S \tag{C.11}$$

$$\beta_S^\dagger\beta_S + \alpha_S^\dagger\alpha_S + \alpha_S\alpha_S^\dagger + \alpha_S\beta_S^\dagger\alpha_S\beta_S^\dagger = \mathbb{I} \tag{C.12}$$

Thanks to the relations above, we can see that any single-mode annihilation and creation operator polynomial reduces to a fourth-degree polynomial at most. The algebraic relations between creation and annihilation operators at different lattice sites are much more difficult to express in simple algebraic equations. It is exciting to see that the annihilation operators do not satisfy equations of the form $\alpha_A\alpha_B =$

$q\alpha_B\alpha_A$ where $q \in \mathbb{C}$. In fact, $\alpha_A\alpha_B$ and $\alpha_B\alpha_A$ have disjoint support.

C.5 Fibonacci Fock states

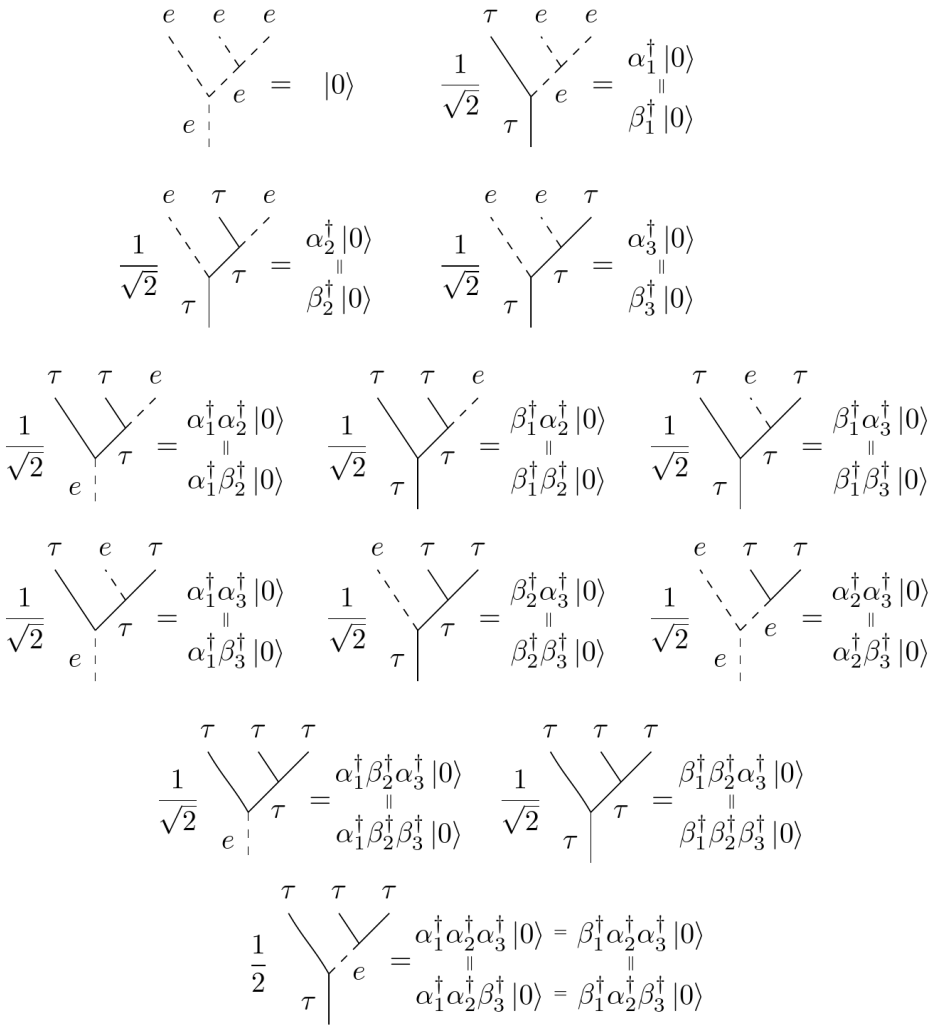


Figure C.5: Canonical basis as a Fock basis, applying the renormalised anyonic creation operators α, β to the vacuum.

We can define the $|0\rangle$ state as the global state where all the anyon particle types are the vacuum in the whole fusion tree. It satisfies the physical notion of being the vacuum state and the mathematical property of being the unique pure state that all

the anyonic annihilation operators annihilate.

We can now express any state of the canonical basis as a well-ordered sequence of creation operators acting on $|0\rangle$. We present the concrete expressions for three-mode Fibonacci anyons. The expressions for the canonical basis are in Figure C.5.

C.6 Local realism proofs

In this section, we prove the theorems and results that lead to the ability to use anyonic annihilation operators as representations of the local ontic states.

(Theorem. *11) The following equivalence holds for any subset of in-front joining modes $\mathcal{M}$ of an N mode anyonic system of n particle types.*

$$\hat{U} \sim_{\mathcal{M}} \hat{V} \quad \Longleftrightarrow \quad \hat{U}^\dagger \cdot \alpha_j^{(j')} \cdot \hat{U} = \hat{V}^\dagger \cdot \alpha_j^{(j')} \cdot \hat{V} \tag{C.13}$$

$\forall j \in \mathcal{M},\ j' \in \{1, \ldots, J_\alpha\},\ \alpha$ *and* $\forall \hat{U}, \hat{V} \in \mathcal{T}_{\mathcal{N}}^{phys}$ *where* $J_\alpha = \sum_{b,c} N_{ab}^c - n + 1$ *and* α *is associated to the particle type* a*.*

Thus,

$$[\hat{U}]_{\mathcal{M}} = \left\{ \hat{V} \in \mathcal{T}_{\mathcal{N}}^{phys} \,\middle|\, \hat{U}^\dagger \cdot \alpha_j^{(j')} \cdot \hat{U} = \hat{V}^\dagger \cdot \alpha_j^{(j')} \cdot \hat{V}\ \forall j \in \mathcal{M},\ \alpha, j' \in \{1, \ldots, J_\alpha\}, \right\} \tag{C.14}$$

Proof. The last statement follows directly from the definition of an equivalence class, so the equation that needs to be proven is Equation 4.27:

"⇒": Remember $\mathcal{N} = \{1, \ldots, N\}$. $\hat{U} \sim_{\mathcal{M}} \hat{V}$ implies $\hat{U} = \hat{W}_{\mathcal{N}\setminus\mathcal{M}} \cdot \hat{V}$ for some $\hat{W}_{\mathcal{N}\setminus\mathcal{M}}$ being a physical in-front or behind unitary, local on the set of lattice sites $\mathcal{N}\setminus\mathcal{M}$ with the opposite behindness than $\mathcal{M}$. Any such $\hat{W}_{\mathcal{N}\setminus\mathcal{M}}$ commutes with the

annihilation operators with opposite behindness $\alpha_j^{(j')}\ \forall j \in \mathcal{M}, \alpha, j' \in \{1, \ldots, J_\alpha\}$. This is a direct consequence of the defining feature of the annihilating terms expressed in Equation 4.18.

Therefore, we obtain $\alpha_j^{(j')}\hat{W}_{\mathcal{N}\setminus\mathcal{M}} = \hat{W}_{\mathcal{N}\setminus\mathcal{M}}\alpha_j^{(j')}$. Using $\hat{W}_{\mathcal{N}\setminus\mathcal{M}} = \hat{U}\cdot\hat{V}^\dagger$, and manipulating the equality we end up obtaining $\forall j \in \mathcal{M}, \alpha, j' \in \{1, \ldots, J_\alpha\}$ that $\hat{U}^\dagger \cdot \alpha_j^{(j')} \cdot \hat{U} = \hat{V}^\dagger \cdot \alpha_j^{(j')} \cdot \hat{V}$ just as desired.

"$\Leftarrow$": We have that $\hat{U}^\dagger \cdot \alpha_j^{(j')} \cdot \hat{U} = \hat{V}^\dagger \cdot \alpha_j^{(j')} \cdot \hat{V}$ for all $j \in \mathcal{M}, \alpha, j' \in \{1, \ldots, J_\alpha\}$. To see that $\hat{U} \sim_{\mathcal{M}} \hat{V}$ we need to see that $\hat{U} = \hat{W}_{\mathcal{N}\setminus\mathcal{M}} \cdot \hat{V}$ and that it is of opposite behindness than the observables in $\mathcal{M}$. Since we have a group structure where transformations behindness definite physical unitaries, proving that $\hat{U}\cdot\hat{V}^\dagger = \hat{W}_{\mathcal{N}\setminus\mathcal{M}}$ with opposite behindness to $\alpha_j^{(j')}$ is enough to prove that $\hat{U} \sim_{\mathcal{M}} \hat{V}$.

From $\hat{U}^\dagger \cdot \alpha_j^{(j')} \cdot \hat{U} = \hat{V}^\dagger \cdot \alpha_j^{(j')} \cdot \hat{V}$ for all $j \in \mathcal{M}$ is straightforward to deduce that then $\alpha_j^{(j')} \cdot (\hat{U}\cdot\hat{V}^\dagger) = (\hat{U}\cdot\hat{V}^\dagger) \cdot \alpha_j^{(j')}$ for all $j \in \mathcal{M}, \alpha, j' \in \{1, \ldots, J_\alpha\}$. Naming $\hat{U}\cdot\hat{V}^\dagger = \hat{W}$, we have that the following equality holds:

$$\hat{W} \cdot \alpha_j^{(j')} = \alpha_j^{(j')} \cdot \hat{W} \tag{C.15}$$

Without loss of generality, let us assume $\mathcal{M}$ has in-front behindness. The proof is analogous in the opposite case.

We use a different basis for each mode $j \in \mathcal{M}$. The matrix basis change is given by the unitary $B = \prod_{i=0}^{M\text{-}1} \prod_{j=M\text{-}i+1}^{s_{M\text{-}i}} R^\dagger_{j\text{-}1\ j}$ defined in Appendix C.2. Using this basis, the unitary $\hat{W}$ takes the form shown in Figure C.6.

We can express Equation C.15 in these diagrammatic bases and see the restriction to the coefficients $w^{a'\vec{b}'c_0}_{a''\vec{b}''c_0}$. The equality implies that the coefficients must satisfy

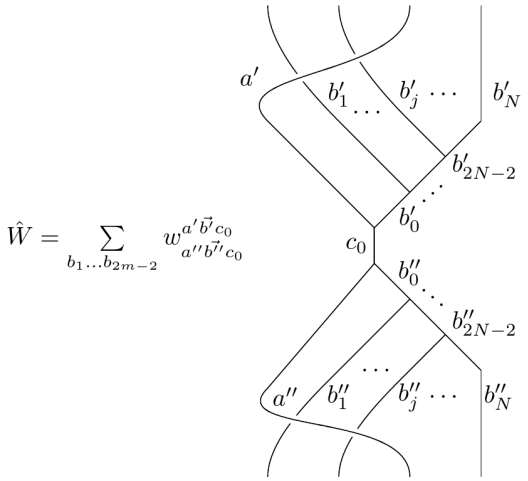


Figure C.6: $\hat{W}$ in the basis associated to j.

the following:

$$w^{a'\vec{b}'(a'\times b'_0)^{j'}}_{ab''(a'\times b'_0)^{j'}} = \delta_{a'a}\delta_{b'_0 b''_0}\tilde{w}^{\vec{b}'}_{\vec{b''}} \tag{C.16}$$

where a is the particle type associated with the annihilation operator label α.

Since these conditions hold for all α & $j' \in \{1, \ldots, J_\alpha\}$, we obtain that it holds for all the terms of $\hat{W}$, thus, we obtain that $\hat{W}$ is a behind only local unitary in $\mathcal{N}\backslash\{j\}$. This is because the Kronecker delta terms in the coefficients imply that $\hat{W}$ has the form of a local unitary as in Figure 4.12, with a single mode and exchanged positions using B.

Given that this holds for all $j \in \mathcal{M}$, we obtain that for each $j \in \mathcal{M}$, $\hat{W}$ is an extended behind only physical unitary local in $\mathcal{N}\backslash\{j\}$. We use the Separation property to see that this implies it must be a behind local unitary on $\mathcal{N}\backslash\mathcal{M}$ necessarily. Therefore, concluding our proof. □

We move to the next and last proof of this Appendix:

(Theorem. *12) The diagram of Figure 3.1 commutes. In other words:*

$$\begin{aligned}\pi_A^{\mathcal{P}}\left(\varphi_{\mathcal{M}}\left(\left(\left(\hat{U}^\dagger\cdot\alpha_{j_1}^{(j')}\cdot\hat{U},\ldots,\hat{U}^\dagger\cdot\alpha_{j_M}^{(j')}\cdot\hat{U}\right),\rho_0\right)\right)\right)=\\ =\varphi_A\left(\pi_A^{\mathcal{R}}\left(\left(\left(\hat{U}^\dagger\cdot\alpha_{j_1}^{(j')}\cdot\hat{U},\ldots,\hat{U}^\dagger\cdot\alpha_{j_M}^{(j')}\cdot\hat{U}\right),\rho_0\right)\right)\right)\end{aligned} \tag{C.17}$$

Proof. Let us first develop the right-hand side of the equation.

$$\begin{aligned}\varphi_A\left(\pi_A^{\mathcal{R}}\left(\left(\left(\hat{U}^\dagger\cdot\alpha_{j_1}^{(j')}\cdot\hat{U},\ldots,\hat{U}^\dagger\cdot\alpha_{j_M}^{(j')}\cdot\hat{U}\right),\rho_0\right)\right)\right)=\\ =\varphi_A\left(\left(\left(\hat{U}^\dagger\cdot\alpha_{a_1}^{(j')}\cdot\hat{U},\ldots,\hat{U}^\dagger\cdot\alpha_{a_S}^{(j')}\cdot\hat{U}\right),\rho_0\right)\right)=\\ =\sum_k \mathrm{Tr}\left(\hat{U}^\dagger\cdot\hat{O}_A^{(k)\ \ ext}\cdot\hat{U}\cdot\rho_0\right)\hat{O}_A^{(k)}\end{aligned} \tag{C.18}$$

where we have applied Equation 4.32 in the second equality.

Let us now develop the left-hand side of the initial equation we want to prove it holds.

$$\begin{aligned}\pi_A^{\mathcal{P}}\left(\varphi_{\mathcal{M}}\left(\left(\left(\hat{U}^\dagger\cdot\alpha_{j_1}^{(j')}\cdot\hat{U},\ldots,\hat{U}^\dagger\cdot\alpha_{j_M}^{(j')}\cdot\hat{U}\right),\rho_0\right)\right)\right)=\\ =\mathrm{Tr}_B\left(\sum_{k'}\mathrm{Tr}\left(\hat{U}^\dagger\cdot\hat{O}_{\mathcal{M}}^{(k')\ \ ext}\cdot\hat{U}\cdot\rho_0\right)\hat{O}_{\mathcal{M}}^{(k')}\right)\end{aligned} \tag{C.19}$$

We can use the consistency conditions that define the partial tracing procedure uniquely. They are presented in the main text in Equation 4.11. Let us name $\hat{L}_A$ the local observable given at the left-hand side of the original equation, and name its right-hand side counterpart as $\hat{R}_A$. $\hat{L}_A = \hat{R}_A$ if and only if, for any physical observable $\hat{P}_A$ local in A, $\mathrm{Tr}\left(\hat{P}_A\hat{L}_A\right) = \mathrm{Tr}\left(\hat{P}_A\hat{R}_A\right)$. Let us choose the canonical orthonormal basis $\{\hat{O}_A^{(k'')}\}_{k''}$ of the local physical operators in A. If we see the equality holds for every element of the basis, it implies that it holds for any local observable $\hat{P}_A$.

We develop $\mathrm{Tr}\Big(\hat{O}_A^{(k'')}\hat{L}_A\Big)$ using the orthonormality of the basis under the trace scalar product in the operator space.

$$\begin{aligned}
&\mathrm{Tr}\left(\hat{O}_A^{(k'')}\sum_k \mathrm{Tr}\Big(\hat{U}^\dagger\cdot\hat{O}_A^{(k)\ \ ext}\cdot\hat{U}\cdot\rho_0\Big)\,\hat{O}_A^{(k)}\right) = \\
&= \sum_k \mathrm{Tr}\Big(\hat{U}^\dagger\cdot\hat{O}_A^{(k)\ \ ext}\cdot\hat{U}\cdot\rho_0\Big)\,\mathrm{Tr}\Big(\hat{O}_A^{(k'')}\hat{O}_A^{(k)}\Big) = \\
&= \sum_k \mathrm{Tr}\Big(\hat{U}^\dagger\cdot\hat{O}_A^{(k)\ \ ext}\cdot\hat{U}\cdot\rho_0\Big)\delta_{kk''} = \\
&= \mathrm{Tr}\Big(\hat{U}^\dagger\cdot\hat{O}_A^{(k'')\ \ ext}\cdot\hat{U}\cdot\rho_0\Big)
\end{aligned} \tag{C.20}$$

To develop the right-hand side, we use the consistency conditions of the partial trace. The extended observables are extended in the canonical behindness of A. We use the notation $\hat{O}_A^{(k)\ \ ext_{\mathcal{M}}}$ for the extension of the local observable in A to $\mathcal{M}$ alone. We use the decomposition $\hat{O}_A^{(k)\ \ ext_{\mathcal{M}}} = \sum_{k'}\mathrm{Tr}\Big(\hat{O}_A^{(k)\ \ ext_{\mathcal{M}}}\hat{O}_{\mathcal{M}}^{(k')}\Big)\hat{O}_{\mathcal{M}}^{(k')}$. Moreover, the global extension of this decomposition, as $\hat{O}_A^{(k)\ \ ext} = \sum_{k'}\mathrm{Tr}\Big(\hat{O}_A^{(k)\ \ ext_{\mathcal{M}}}\hat{O}_{\mathcal{M}}^{(k')}\Big)\hat{O}_{\mathcal{M}}^{(k')\ \ ext}$.

Thus, we prove the theorem since $\mathrm{Tr}\Big(\hat{O}_A^{(k'')}\hat{R}_A\Big)$ is

$$\begin{aligned}
&\mathrm{Tr}\left(\hat{O}_A^{(k'')}\,\mathrm{Tr}_B\left(\sum_{k'}\mathrm{Tr}\Big(\hat{U}^\dagger\cdot\hat{O}_{\mathcal{M}}^{(k')\ \ ext}\cdot\hat{U}\cdot\rho_0\Big)\,\hat{O}_{\mathcal{M}}^{(k')}\right)\right) = \\
&= \mathrm{Tr}\left(\hat{O}_A^{(k'')\ \ ext_{\mathcal{M}}}\sum_{k'}\mathrm{Tr}\Big(\hat{U}^\dagger\cdot\hat{O}_{\mathcal{M}}^{(k')\ \ ext}\cdot\hat{U}\cdot\rho_0\Big)\,\hat{O}_{\mathcal{M}}^{(k')}\right) = \\
&= \sum_{k'}\mathrm{Tr}\Big(\hat{U}^\dagger\cdot\hat{O}_{\mathcal{M}}^{(k')\ \ ext}\cdot\hat{U}\cdot\rho_0\Big)\,\mathrm{Tr}\Big(\hat{O}_A^{(k'')\ \ ext_{\mathcal{M}}}\,\hat{O}_{\mathcal{M}}^{(k')}\Big) = \\
&= \mathrm{Tr}\left(\hat{U}^\dagger\cdot\left(\sum_{k'}\mathrm{Tr}\Big(\hat{O}_A^{(k'')\ \ ext_{\mathcal{M}}}\,\hat{O}_{\mathcal{M}}^{(k')}\Big)\,\hat{O}_{\mathcal{M}}^{(k')\ \ ext}\right)\cdot\hat{U}\cdot\rho_0\right) = \\
&= \mathrm{Tr}\Big(\hat{U}^\dagger\cdot\Big(\hat{O}_A^{(k'')\ \ ext}\Big)\cdot\hat{U}\cdot\rho_0\Big)
\end{aligned} \tag{C.21}$$

□