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Uncertainty, Investment and Productivity with Relational Contracts

James M. Malcomson

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James M. Malcomson*
University of Oxford, UK

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Abstract

This paper shows that, in the presence of relational contracts, an increase in uncertainty with no change in factor prices reduces capital investment and productivity in the long run even if the parties are otherwise risk neutral. Recent literature on the effect of uncertainty on investment has focused on the option value arising from irreversibility of investment and on the impact of financial risk on the cost of capital. For the option value model, however, Bloom et al. (*Econometrica*, 2018) find that a negative aggregate shock to total factor productivity, not just an increase in its uncertainty, is required to capture the drop in investment and productivity associated with the increase in uncertainty in recessions. With the capital cost model, the reduction in investment lasts only as long as the higher capital cost is sustained. The current paper first develops a relational contract model to demonstrate the impact of uncertainty on investment and how the effect depends on whether the investment is general or specific. To illustrate how taking account of relational contracts can help capture the effect of uncertainty on investment empirically, it then uses a specification calibrated with parameters from Bloom et al. (*Econometrica*, 2018) to show that this model can generate effects on investment of the magnitude of the negative aggregate shock in that paper purely with an increase in uncertainty with no change in factor prices.

Keywords: Relational contracts, risk, investment, general capital, specific capital

JEL classification: C73, D82, D86

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1 Introduction

Relational contracts, arrangements between parties for which the ongoing relationship between them plays an essential role in determining outcomes, have proved an insightful way to view a variety of economic relationships.¹ This paper investigates what they can contribute to understanding the relationship between uncertainty and capital investment. It first shows that, even with risk-neutral parties and no change in factor prices, uncertainty affects long-run equilibrium investment. It then uses a calibrated specification based on Bloom et al. (2018) to illustrate how relational contracts can help resolve empirical issues in the relationship between them.

Capital investment falls in recessions. Figure 1, in which shaded areas show designated periods of recession, illustrates this. Bloom (2009) provided evidence that uncertainty has an impact on production and employment; Bloom (2014) documents the evidence that uncertainty is higher in recessions. Two recent approaches to understanding these relationships are based on the option value of delaying irreversible investment and on the impact of financial risk on the cost of capital. For examples, see Bloom et al. (2018) and Fernández-Villaverde and Guerrón-Quintana (2020) for the former, Caballero and Simsek (2020) and Pflueger et al. (2020) for the latter. In the option value approach in Bloom et al. (2018), firms are risk neutral and so unconcerned about the extent of productivity and demand uncertainty in the long run. However, with irreversible investment, the option value to delay increases with uncertainty, as demonstrated by Dixit and Pindyck (1994). In the cost of capital approach, perception of high financial risk results in a low value for risky assets and thus a high cost of capital for risky firms.

The option value approach is essentially one of adjustment costs. With risk neutral firms, investment eventually returns to the same share of GDP even if increased uncertainty persists. With the calibration in Bloom (2009), nearly all the adjustment is completed after 36 months. However, in a general equilibrium setting, Bloom et al. (2018) show that, on data up to 2010, the increase in uncertainty itself is insufficient to capture the drop in investment and productivity in recessions. A negative aggregate shock to total factor productivity (a first-moment shock, not just a second-moment shock) is required for that, which Bloom et al. (2018) describe (p. 2) as “controversial, as it suggests that recessions are times of technological regress.” This is illustrated for the 2008-09 recession in Figure 1. Although the investment share increased after 2009, it has remained below its average for 1995 to 2007 right up to 2020 which, with a pure option value effect, would require continually increasing uncertainty. The data for the US for 2010 to 2017 in Ahir et al. (2019) and for 2017 to 2019 in the Atlanta Fed’s Survey

¹See Malcomson (1999) for applications of relational contracts to employment and Malcomson (2013) for applications to supply relationships.

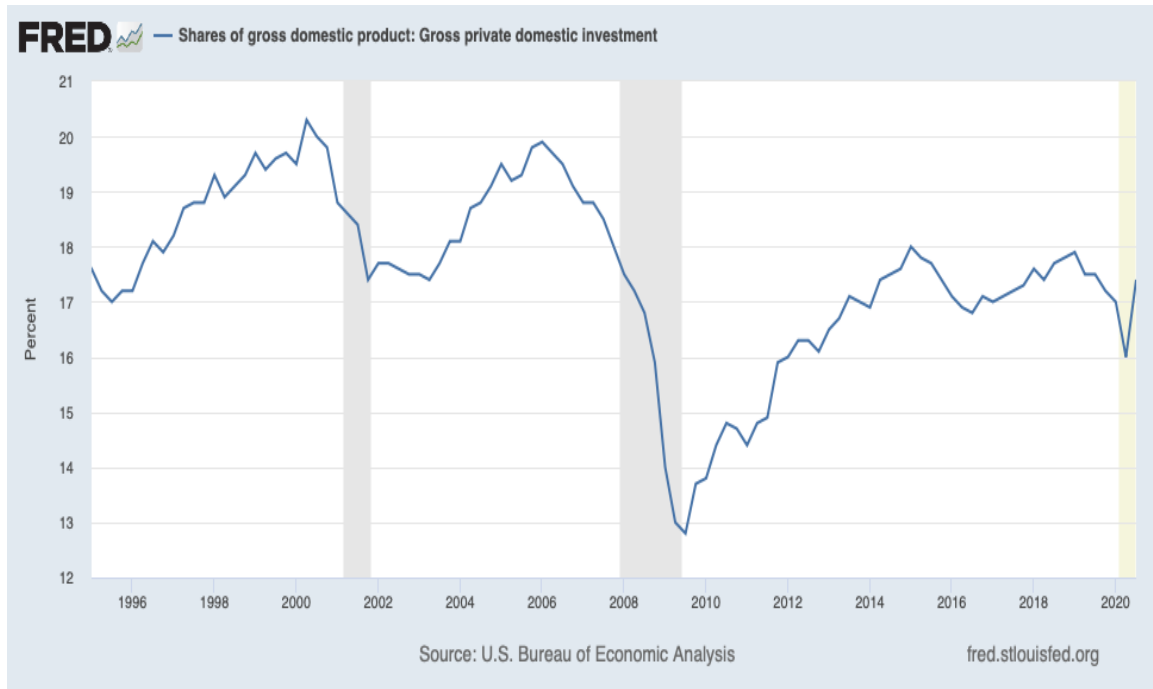


Figure 1: US real gross private domestic investment, percentage of GDP

of Business Expectations² do not indicate that.

Pflueger et al. (2020) measure perceptions of financial risk by the inverse of the price of volatile stocks (PVS), defined as the book-to-market ratio of low-volatility stocks minus the book-to-market ratio of high-volatility stocks, and show that this is inversely related to real investment. But, on the surface at least for the aftermath of the 2008-09 recession, this seems not the whole story. Risk perceptions as measured by PVS increase dramatically from 2007 to 2010 but revert to a level similar to the average of the preceding non-recession periods after 2011, see Pflueger et al. (2020, Figure I) but, as Figure 1 illustrates, investment does not.

What the present paper adds is to show that, in the presence of relational contracts, greater output or revenue uncertainty *per se* (that is, with no first-moment shock and no change in the cost of capital) affects long-run equilibrium investment even when the parties are jointly risk neutral. Moreover, with a production framework similar to Bloom et al. (2018) but with the option value of delaying investment replaced by the constraints arising from a relational contract and the calibrated parameters in that paper, it is straightforward to generate effects on general investment of the magnitude of the negative first-moment shock in that paper purely with greater uncertainty. This raises the possibility that these real business cycle models could be freed from unattractive negative aggregate technology shocks.

In the standard model of relational incentive contracts between a principal and an

²<https://www.frbatlanta.org/research/surveys/business-uncertainty.aspx>

agent, for example MacLeod and Malcomson (1989), the agent chooses a level of performance each period (typically referred to as *effort* in the literature) that affects the payoff to the principal. But agent performance is non-contractible in the sense that payment cannot be conditioned on performance in an enforceable legal agreement. This may be either because performance cannot be verified by third parties in court or because it is too complicated to describe in a legally enforceable way for that to be worthwhile. Whatever the reason, conventional performance-related payments are not available. Instead, incentives for performance are provided by a combination of bonuses related to output that, while not legally enforceable, the principal nevertheless finds it worthwhile to pay and the future payoff from continuation of the relationship. A central question in the literature concerns what payment and performance the parties will deliver even though they are not legally obliged to. To that standard model, the present paper adds two things. The first is that the value of the agent's performance to the principal is uncertain because it is affected by an *iid* shock each period. The second is that the parties may invest in capital that enhances the productivity of the relationship.

The main theoretical results in the paper are the following. The non-contractibility of performance, and the consequent use of a relational contract, places an upper limit on the agent's performance which is more restrictive with favourable shocks that increase the productivity of effort. Under plausible conditions, an increase in risk as measured by second-order stochastic dominance lowers this upper limit and thus, even with risk-neutral parties, affects the expected return on capital investment. With investments that are general in the sense of Becker (1975), this unambiguously reduces investment. Investments that are specific in the sense of Becker (1975), however, increase the upper limit on performance, which mitigates the negative impact of an increase in risk. It may even increase investment, a result of particular interest given the observation in Bloom (2014) that uncertainty stimulates some types of investments.

The paper is organized as follows. The next section discusses related literature. Section 3 sets out the model and the assumptions used for the analysis. Section 4 analyses optimal effort in a relational contract in this setting. Section 5 studies the effect of risk on the returns to the relationship for given capital stock. Section 6 studies the effect of risk on optimal investment in capital. Section 7 develops a specification of the model suitable for calibration and provides results from that. Section 8 contains concluding remarks. Proofs of results are in Appendix A. Other appendices provide details of derivations and calculations for the calibration exercise.

2 Related literature

The literature on the option value associated with uncertainty and irreversible investment stems from Dixit and Pindyck (1994). The theoretical implications for aggreg-

ate investment are set out in Abel and Eberly (1996). The empirical implementations closest to the present paper are in Bloom et al. (2007), Bloom (2009) and Bloom et al. (2018). Many other contributions on capital adjustment costs are discussed in the survey by Bloom (2014). Fernández-Villaverde and Guerrón-Quintana (2020) provide a more recent review of the mechanisms that researchers have postulated to link uncertainty shocks and business cycles. Among the papers not discussed there are Bond et al. (2011), Sedláček (2020) and Berger et al. (2020). The last of these argues that innovations in realized stock market volatility, but not in forward-looking uncertainty, are robustly followed by economic contractions but these are not distinguishable in the model used here. Angeletos et al. (2020) appraise different business-cycle drivers.

In the relational contracts literature, Malcomson (2015b) analyses the effect of productivity shocks, and Li and Matouschek (2013) the effect of shocks to the opportunity cost to the principal, but the models used there have no investment. Malcomson (2015a) studies investment but without productivity shocks, as does Garicano and Rayo (2017) for investment in human capital in the form of knowledge transfer. Fahn et al. (2017) and Fahn et al. (2019) consider the implications of relational contracts for firm capital structure. Englmaier and Fahn (2017) consider their implications for investments in liquidity-generating capital on the ability of firms to meet their financial commitments. None of these investigate the impact of uncertainty on investment of the type illustrated in Figure 1.

3 The model

The model used here is that of MacLeod and Malcomson (1989) with the two additions specified in the Introduction. First, to allow for risk, the productivity of the agent's effort is subject to an *iid* shock each period. Second, the parties may make a capital investment that enhances the productivity of the relationship.

A principal uses an agent to perform a specified task each period. Both are risk neutral and discount the future with the same discount factor $\delta \in (0, 1)$. The relationship between the two can, in principle, continue indefinitely. The value of output from the match in period t is $y(e_t, K, \theta_t)$, where $e_t \in [0, \bar{e}]$ is the agent's effort at t , $K \in [0, \bar{K}]$ is the capital stock, and $\theta_t \in [\underline{\theta}, \bar{\theta}]$ is an *iid* random shock distributed $F(\theta, \sigma)$ that admits an everywhere positive density, with σ a parameter that determines its riskiness. Both parties observe the shock θ_t at the start of period t , as well as observing e_t and K ; there is no asymmetric information. The shock can be to productivity or to the revenue from given output. Effort e_t in period t is chosen at cost $c(e_t)$ to the agent after θ_t is revealed and so can be conditioned on the shock. Neither the value of output nor effort is contractible in the sense that payment can be conditioned on performance in a formal legal agreement. This may be either because they cannot be verified by third parties or because it is too complicated to describe them in a legally enforceable way

to be worthwhile. Effort can be thought of as anything unverifiable the agent may do that affects the payoff to the principal. In the context of employment, it could be literal effort. In the context of a supply chain, it could be the quality of the intermediate products supplied. In principle, it can be multidimensional.

For reasons given in the adjustment cost literature, it is implausible that capital can be fully adjusted to shocks in the short run so, because the concern here is with long-run equilibrium properties, investment in capital is assumed to take place at the beginning of the relationship, before any shock has been revealed, at a one-off cost $C(K)$. This cost is to be thought of as the present discounted cost of using capital K , including replacement investment.³

Assumption 1 *The functions y , c and C have the following conventional properties:*

1. $y(e, K, \theta)$ is strictly concave in (e, K) for all $\theta \in [\underline{\theta}, \bar{\theta}]$, is three times differentiable in its arguments with $y_1(e, K, \theta) > 0$, $y_2(e, K, \theta) > 0$, $y_3(e, K, \theta) > 0$, $y_{12}(e, K, \theta) > 0$ and $y_{13}(e, K, \theta) > 0$ for all $(e, K, \theta) \in (0, \bar{e}] \times [0, \bar{K}] \times [\underline{\theta}, \bar{\theta}]$; it also has $y(0, K, \theta) = 0$ for all $(K, \theta) \in [0, \bar{K}] \times [\underline{\theta}, \bar{\theta}]$ and $y(e, 0, \theta) \geq 0$ for all $(e, \theta) \in [0, \bar{e}] \times [\underline{\theta}, \bar{\theta}]$;
2. $c(e)$ is twice differentiable, with $c'(e) > 0$ and $c''(e) \geq 0$ for $e \in [0, \bar{e}]$ and $c(0) = 0$;
3. $C(K)$ is twice differentiable, with $C'(K) > 0$ and $C''(K) \geq 0$ for $K \in [0, \bar{K}]$ and $C(0) = 0$;
4. $y(e, K, \theta) - c(e) - C(K)$ is strictly increasing in e for $e > 0$ sufficiently small and strictly decreasing in e for $e = \bar{e}$ for $(K, \theta) \in (0, \bar{K}] \times [\underline{\theta}, \bar{\theta}]$; it is also strictly increasing in K for $K > 0$ sufficiently small and strictly decreasing in K for $K = \bar{K}$ for all $(e, \theta) \in (0, \bar{e}] \times [\underline{\theta}, \bar{\theta}]$.

Part 4 of Assumption 1 is a formal statement of conditions for effort and capital that maximize the joint payoff to principal and agent to be interior for any realisation of the shock. The payoff to the principal in period t of the match is $y(e_t, K, \theta) - W_t$, where W_t is the payment to the agent in period t . The payoff to the agent is $W_t - c(e_t)$. Because both are risk neutral, the joint payoff to them both from being matched in a period is just the sum of their individual payoffs. For a period in which the principal and agent are not matched, the principal's payoff is $\underline{v}(K, \sigma) \geq 0$ and the agent's payoff $\underline{u}(K, \sigma) \geq 0$, with $\underline{s}(K, \sigma) := \underline{u}(K, \sigma) + \underline{v}(K, \sigma) > 0$, the inequalities holding for all $K \in [0, \bar{K}]$. Conditional on (e, K, θ) , the joint payoff to principal and agent from being matched in a period is $s(e, K, \theta) := y(e, K, \theta) - c(e)$. As a benchmark, if effort were contractible it would be set at the first-best level for given (K, θ) , denoted $e^*(K, \theta)$, that

³For the calibration in Section 7, $C(K)$ is taken to be linear so, even if capital can be added each period, it is not optimal to build it up slowly over time.

maximizes this joint payoff and is given by

$$y_1(e^*(K, \theta), K, \theta) = c'(e^*(K, \theta)). \quad (1)$$

Because investment in capital is decided at the beginning of the relationship before any shock is revealed, the first-best benchmark for capital stock if effort were contractible is

$$K^*(\sigma) \in \arg \max_{K \in [0, \bar{K}]} \frac{1}{1 - \delta} \int_{\underline{\theta}}^{\bar{\theta}} s(e^*(K, \theta), K, \theta) dF(\theta, \sigma) - C(K). \quad (2)$$

If $y(e_t, K, \theta_t)$ or e_t were contractible, it would be straightforward for the parties to agree a contract that would deliver effort schedule $e^*(K, \theta)$ and investment $K^*(\sigma)$. However, with neither contractible, effort above the minimum level $e = 0$ can be sustained only by a relational contract. Following MacLeod and Malcomson (1989), payment W_t to the agent has two components, a fixed component w_t that is guaranteed independent of performance in period t and a bonus component b_t that can be conditional on performance in period t . The bonus cannot be legally enforceable because performance is not contractible, so the relational contract must ensure that it is in the principal's interest to pay it.

4 Optimal effort

A relational contract carried out even though not legally enforceable is referred to in the literature as *self-enforcing*. The set of self-enforcing contracts is largest when the punishments for deviation are, as in Abreu (1988), the most severe available, which here corresponds to the deviating party receiving the future payoff that would result from the relationship ending. Because the parties are risk neutral and use the same discount factor, they can redistribute the joint gains from their relationship in any way they choose by an upfront payment at its start. It is thus optimal for them to select an equilibrium contract that maximises those joint gains at the start of the relationship. Moreover, by an argument in Levin (2003, Theorem 2) that applies to the model here, if an optimal contract exists, there are stationary contracts that are optimal. An optimal stationary contract depends only on current payoff-relevant information, so optimal effort and payment have the form $e_t = e(K, \theta_t, \sigma)$, $w_t = w(K, \theta_t, \sigma)$ and $b_t = b(e_t, K, \theta_t, \sigma)$.

Proposition 1 *An effort schedule $e(K, \theta, \sigma)$ that generates expected joint payoff $S(K, \sigma)$ each period with capital stock K can be implemented by a stationary contract if and only if*

$$\frac{\delta}{1 - \delta} [S(K, \sigma) - \underline{s}(K, \sigma)] \geq c(e(K, \theta, \sigma)), \quad \text{for all } \theta \in [\underline{\theta}, \bar{\theta}]. \quad (3)$$

In Proposition 1, $S(K, \sigma)$ is the expected joint payoff from one period of the relationship, given investment in capital K at the start, before the realization for that period of the shock θ from a distribution parameterized by σ . Thus the left-hand side of (3) is the joint payoff *gain* to both parties from continuing the relationship in future periods. So (3) states that, for effort $e(K, \theta, \sigma)$ to be implementable, this joint gain must be at least as great as the cost to the agent of delivering effort $e(K, \theta, \sigma)$ in a period. This is a standard result for relational contracts with risk-neutral parties originally derived in MacLeod and Malcomson (1989). The essential intuition is as follows. If the agent were to receive no bonus but instead all the joint payoff gain from continuing the relationship in the future, it would be worth incurring cost of effort up to that amount to keep the relationship going. A positive bonus reduces by the amount of the bonus the amount of the future joint gain the agent must receive to be worth incurring given effort. But, because payment of the bonus cannot be enforced legally, the principal must receive future gain of at least the amount of the bonus to make it worthwhile paying that bonus to keep the relationship going. So the total of the future joint payoff gain required to sustain the relationship is unaffected by having a positive bonus. All the size of the bonus does is affect the amounts of the minimum future joint gain that must go to the principal and to the agent. Thus, provided (3) is satisfied, there are different combinations of the fixed payment w_t and bonus b_t that will induce effort schedule $e(K, \theta, \sigma)$. Payments affect how the joint gain from the relationship is distributed between principal and agent. But no combination of these can induce effort schedule $e(K, \theta, \sigma)$ if (3) is not satisfied. The following corollary is a straightforward consequence of Proposition 1.

Corollary 1 *Suppose, for given $S(K, \sigma)$, first-best effort $e^*(K, \theta)$ does not satisfy (3) for some $\theta' \in [\underline{\theta}, \bar{\theta}]$. Then first-best effort $e^*(K, \theta)$ does not satisfy (3) for any $\theta \in [\theta', \bar{\theta}]$.*

Corollary 1 follows from Proposition 1 because first-best effort given by (1) is increasing in θ and, hence, so is $c(e^*(K, \theta))$. But, because θ is an *iid* shock, the left-hand side of (3) is independent of θ because it is an expectation over future θ . So, with the left-hand side of (3) independent of θ and the right-hand side increasing in θ when effort is first best, the result in the corollary follows directly. That provides an important step in determining the optimal effort schedule. The following assumption, used for the rest of the paper, avoids the uninteresting case in which no relational contract can sustain positive effort.

Assumption 2 *There exists an effort schedule with $e(K, \theta, \sigma) \in (0, \bar{e})$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$ for which (3) is satisfied with strict inequality for some $K \in (0, \bar{K}]$.*

Proposition 2 *If there exists an effort schedule $e(K, \theta, \sigma)$ that implements $S(K, \sigma)$ satisfying (3) for given K , an optimal stationary effort schedule for that K takes one of three forms⁴:*

⁴It can be shown that the results in Proposition 2 hold even if only the principal observes θ . In

1. *First best: if*

$$\frac{\delta}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} [s(e^*(K, \theta), K, \theta) - \underline{s}(K, \sigma)] dF(\theta, \sigma) \geq c(e^*(K, \bar{\theta})),$$

then $e(K, \theta, \sigma) = e^*(K, \theta)$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$.

2. *Full pooling: if*

$$\frac{\delta}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} [s(e^*(K, \underline{\theta}), K, \theta) - \underline{s}(K, \sigma)] dF(\theta, \sigma) \leq c(e^*(K, \underline{\theta})),$$

then $e(K, \theta, \sigma) = \tilde{e}(K, \sigma)$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$, where $\tilde{e}(K, \sigma)$ is given by

$$\frac{\delta}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} [s(\tilde{e}(K, \sigma), K, \theta) - \underline{s}(K, \sigma)] dF(\theta, \sigma) - c(\tilde{e}(K, \sigma)) = 0. \quad (4)$$

3. *Partial pooling: otherwise, there exists $\hat{\theta}(K, \sigma) \in (\underline{\theta}, \bar{\theta})$ such that*

$$e(K, \theta, \sigma) = \begin{cases} e^*(K, \theta), & \text{for } \theta \in [\underline{\theta}, \hat{\theta}(K, \sigma)), \\ e^*(K, \hat{\theta}(K, \sigma)), & \text{for } \theta \in [\hat{\theta}(K, \sigma), \bar{\theta}], \end{cases}$$

with $\hat{\theta}(K, \sigma)$ the highest $\tilde{\theta} \in (\underline{\theta}, \bar{\theta})$ satisfying

$$\begin{aligned} & \frac{\delta}{1-\delta} \left\{ \int_{\underline{\theta}}^{\tilde{\theta}} [s(e^*(K, \theta), K, \theta) - \underline{s}(K, \sigma)] dF(\theta, \sigma) \right. \\ & \left. + \int_{\tilde{\theta}}^{\bar{\theta}} [s(e^*(K, \tilde{\theta}), K, \theta) - \underline{s}(K, \sigma)] dF(\theta, \sigma) \right\} - c(e^*(K, \tilde{\theta})) = 0. \quad (5) \end{aligned}$$

Proposition 2 establishes that optimal effort is first-best for all θ , or the same for all θ , or the same for all θ above some cutoff value $\hat{\theta}(K, \sigma)$ and first-best for all θ below $\hat{\theta}(K, \sigma)$. The reasoning behind this result is as follows. By Proposition 1, (3) is a necessary and sufficient condition for an effort schedule to be implementable. Thus optimal effort maximizes the joint payoff gain subject to that constraint. By definition, first-best effort maximizes the joint payoff when the constraint (3) is not imposed. It also maximizes the contribution effort for that θ makes to $S(K, \sigma)$, which weakens the constraint (3) for all other θ . So, if first-best effort satisfies (3) for some θ , it remains optimal for that θ when (3) is added as a constraint. When this property holds for

that case, the principal will reveal θ truthfully provided the expected payoff from doing so is non-decreasing in θ and this property is satisfied by the optimal effort in the proposition. To show that formally, however, complicates the exposition and so is not done here.

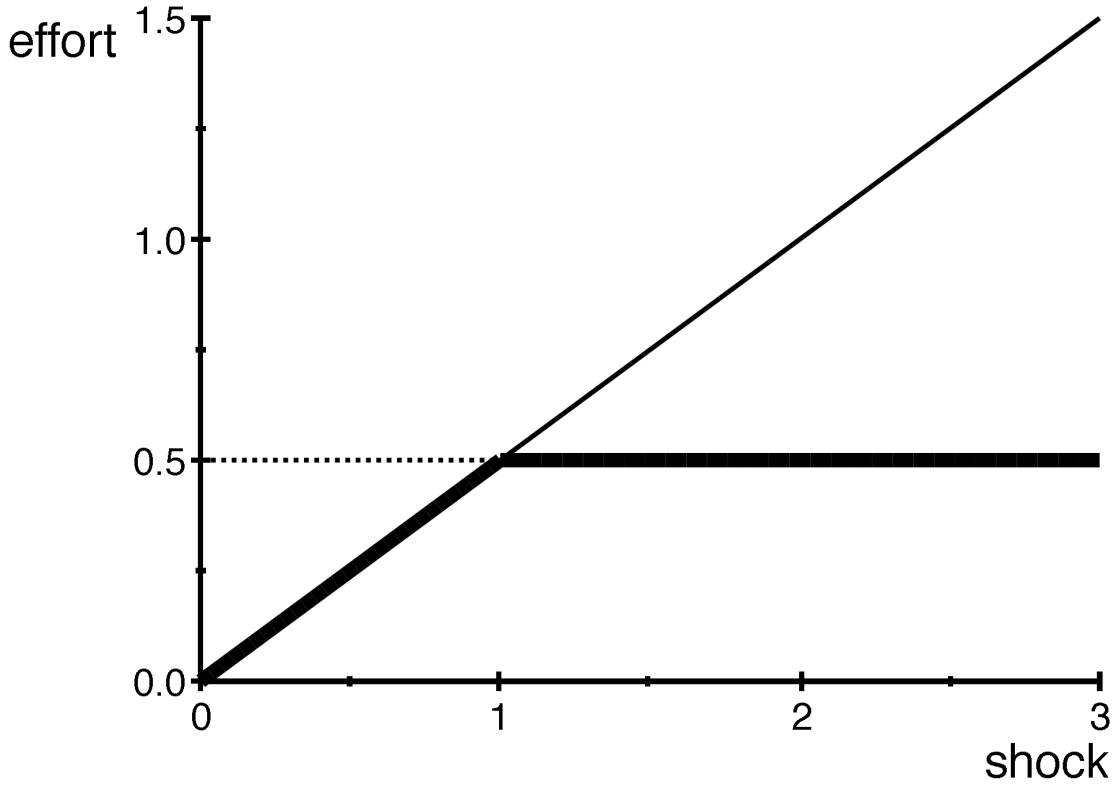


Figure 2: Effect of shock on effort

all θ , as in Part 1, the outcome is exactly the same as if effort were contractible. Of course, (3) may not be satisfied for all θ with the first-best effort schedule. Indeed, it may not be satisfied with the first-best effort for *any* θ . In that case, effort will have to be below first-best for all θ . For each θ individually, it is then optimal to have effort as high as possible while satisfying the constraint (3) so that it is as close to first-best as possible. That does not conflict with getting the highest effort possible for any other θ because, with effort below first-best, increasing effort for some θ increases the joint payoff $s(e, K, \theta)$, which relaxes the constraint (3) for all θ . Because the left-hand side of (3) is independent of θ , having (3) hold with equality for all θ implies the same effort for all θ , so there is full pooling as in Part 2 of the proposition. To have the constraint (3) bind for all θ then implies that $e(K, \theta, \sigma) = \tilde{e}(K, \sigma)$ for all θ , where $\tilde{e}(K, \sigma)$ is given by (4).

The remaining possibility is that (3) is satisfied with first-best effort for some, but not all, θ . It follows from Corollary 1 that, if it is satisfied by first-best effort for some θ , it will also be satisfied for all lower θ . As in Part 2, for all θ for which (3) is not satisfied for first-best effort, it is optimal to have effort at the highest level that satisfies (3) and that level is the same for all such θ . This implies that there is a critical shock $\hat{\theta}(K, \sigma)$ such that effort is at the first-best level for all $\theta \leq \hat{\theta}(K, \sigma)$ and is independent of θ for all $\theta > \hat{\theta}(K, \sigma)$ at $e^*(\hat{\theta}(K, \sigma), K)$. It gives the form of the optimal schedule in Part 3 of

the proposition. (Because it is optimal to have first-best effort for as many shocks as possible, $\hat{\theta}(K, \sigma)$ is the highest θ that satisfies (5) if there is more than one.) Figure 2 illustrates this case. In that, the thin solid line corresponds to first-best effort for given capital stock, the dashed line to the highest effort sustainable given total future payoff gain. So optimal effort is given by the bold line, with $\hat{\theta}(K, \sigma) = 1$. It follows from (5) that $\hat{\theta}(K, \sigma)$ satisfies

$$\frac{\delta}{1-\delta} \left\{ \int_{\underline{\theta}}^{\bar{\theta}} s(e^*(K, \theta), K, \theta) dF(\theta, \sigma) - \underline{s}(K, \sigma) - \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} [s(e^*(K, \theta), K, \theta) - s(e^*(K, \hat{\theta}(K, \sigma)), K, \theta)] dF(\theta, \sigma) \right\} = c(e^*(K, \hat{\theta}(K, \sigma))). \quad (6)$$

The second integral in (6) is strictly positive because $e^*(K, \theta)$ uniquely maximizes $s(e, K, \theta)$.

5 The effect of risk on effort

Critical to the impact of risk on investment is its effect on effort. A conventional way to measure riskiness is in terms of second-order stochastic dominance for distributions. The standard definition of second-order stochastic dominance specifies the sign of the integral of the difference between two distributions for all values of the *upper* limit of integration. But comparison of the expression in (6) for different σ requires comparison of the second integral for variable *lower* limit of integration. This requires a modified definition of second-order stochastic dominance.

Definition 1 $F(\theta, \sigma_L)$ dominates $F(\theta, \sigma_H)$ in the second-order stochastic sense for $\tilde{\theta} \in [\underline{\theta}, \bar{\theta}]$ if

$$\int_{\theta}^{\tilde{\theta}} [F(x, \sigma_L) - F(x, \sigma_H)] dx \geq 0, \quad \text{for all } \theta \in [\underline{\theta}, \tilde{\theta}],$$

with strict inequality for a set of values of $x \in [\tilde{\theta}, \bar{\theta}]$ with positive probability.

This definition corresponds to the standard definition when $F(\theta, \sigma_L)$ has the same mean as $F(\theta, \sigma_H)$ and $\tilde{\theta} = \underline{\theta}$ (in which case the integral in Definition 1 equals zero for $\theta = \underline{\theta}$, see Laffont (1989, p. 25).) It then corresponds to a mean-preserving spread in the sense of Rothschild and Stiglitz (1970). The extension to $\tilde{\theta} > \underline{\theta}$ is used for handling the second integral in (6) that has lower limit of integration greater than $\underline{\theta}$.

The first result here is a straightforward adaptation to Definition 1 of the result for the standard definition of second-order stochastic dominance that a risk-averse agent prefers $F(\theta, \sigma_L)$ to $F(\theta, \sigma_H)$ if the former stochastically dominates the latter in the

second-order sense. The proof is a straightforward adaptation of the standard proof for the case $\tilde{\theta} = \underline{\theta}$ in Laffont (1989, p. 32-33).

Lemma 1 Suppose a twice-differentiable function $g(\theta)$ is non-increasing and concave on $[\tilde{\theta}, \bar{\theta}]$ for $\tilde{\theta} \in [\underline{\theta}, \bar{\theta})$ and $g(\tilde{\theta}) = 0$ if $\tilde{\theta} > \underline{\theta}$. Then, if $F(\theta, \sigma_L)$ dominates $F(\theta, \sigma_H)$ in the second-order stochastic sense for $\tilde{\theta} \in [\underline{\theta}, \bar{\theta}]$ and the two distributions have the same mean,

$$\int_{\tilde{\theta}}^{\bar{\theta}} g(\theta) dF(\theta, \sigma_L) \geq \int_{\tilde{\theta}}^{\bar{\theta}} g(\theta) dF(\theta, \sigma_H), \quad \text{for all } \tilde{\theta} \in [\underline{\theta}, \bar{\theta}],$$

with the inequality strict if $g(\theta)$ is strictly concave on $[\tilde{\theta}, \bar{\theta}]$.

With relational contracts, it is important to distinguish between specific risk that affects only a particular relationship and systemic risk that also affects the payoffs if the parties separate. To define these formally, let $S^*(K, \sigma)$ be the expected joint payoff each period before the realization for that period of the shock θ from adopting the optimal effort schedule in Proposition 2.

Definition 2 Risk is specific if $\underline{s}(K, \sigma)$ is independent of σ . Risk is systemic if $S^*(K, \sigma) - \underline{s}(K, \sigma)$ is independent of σ .

There are obviously partially specific intermediates between these two extremes in which neither $\underline{s}(K, \sigma)$ nor $S^*(K, \sigma) - \underline{s}(K, \sigma)$ is independent of σ but the essential insights are illustrated by those two cases.

To apply the result in Lemma 1, define

$$h(e, K, \theta) := s(e, K, \theta) - s(e^*(K, \theta), K, \theta), \quad \text{for } e \in [0, e^*(K, \theta)]. \quad (7)$$

This is the difference in the joint payoff in a period for effort e less that first-best and that for first-best effort for given K and θ , the negative of the integrand in the second integral in (6).

Lemma 2 $h(e, K, \theta)$:

1. is decreasing in θ for all $(e, K, \theta) \in (0, e^*(K, \theta)] \times (0, \bar{K}] \times [\underline{\theta}, \bar{\theta}]$;
2. is strictly concave in θ for all $(e, K, \theta) \in (0, e^*(K, \theta)] \times (0, \bar{K}] \times [\underline{\theta}, \bar{\theta}]$ if $y_{133}(e, K, \theta) \geq 0$ for $e \leq e^*(K, \theta)$;
3. has $h_2(e, K, \theta)$ strictly decreasing in θ for all $(e, K, \theta) \in (0, e^*(K, \theta)] \times (0, \bar{K}] \times [\underline{\theta}, \bar{\theta}]$ if $y_{123}(e, K, \theta) \geq 0$ for $e \leq e^*(K, \theta)$.

The conditions in Parts 2 and 3 of Lemma 2 are sufficient but not necessary. In view of Part 1, Lemma 1 can be used to compare the second integral in (6) under σ_H with that under σ_L whenever $h(e, K, \theta)$ is concave because certainly $h(e, K, \theta)$ is zero for $e = e^*(K, \theta)$. Since comparison of the first integral gives the effect when effort is contractible, this provides a measure of the difference in the response to risk when effort is not contractible from when it is.

Proposition 3 *Suppose $F(\theta, \sigma_L)$ dominates $F(\theta, \sigma_H)$ in the second-order stochastic sense for $\tilde{\theta} \in [\underline{\theta}, \bar{\theta}]$ and the two distributions have the same mean.*

1. *For systemic risk, $e(K, \theta, \sigma_H) = e(K, \theta, \sigma_L)$ for all $(K, \theta) \in [0, \bar{K}] \times [\underline{\theta}, \bar{\theta}]$.*
2. *For specific risk and an optimal effort schedule with full pooling (the case in Part 2 of Proposition 2) for both $F(\theta, \sigma_H)$ and $F(\theta, \sigma_L)$ and $\tilde{\theta} = \underline{\theta}$: for all $K \in [0, \bar{K}]$ such that $s(\tilde{e}(K, \sigma_H), K, \theta)$ is strictly concave in θ for $\theta \in [\underline{\theta}, \bar{\theta}]$, effort is lower for all θ , and the joint payoff is also lower, for σ_H than for σ_L ; moreover, there is a larger difference for σ_H than for σ_L between the joint payoff when effort is contractible and when it is not.*
3. *For specific risk and an optimal effort schedule with partial pooling (the case in Part 3 of Proposition 2) for both $F(\theta, \sigma_H)$ and $F(\theta, \sigma_L)$ and $\tilde{\theta} = \hat{\theta}(K, \sigma_H)$: for all $K \in [0, \bar{K}]$ such that $h(e^*(K, \hat{\theta}(K, \sigma_H)), K, \theta)$ is strictly concave in θ for $\theta \in [\hat{\theta}(K, \sigma_H), \bar{\theta}]$ and*

$$\int_{\underline{\theta}}^{\bar{\theta}} s(e^*(K, \theta), K, \theta) dF(\theta, \sigma_L) \geq \int_{\underline{\theta}}^{\bar{\theta}} s(e^*(K, \theta), K, \theta) dF(\theta, \sigma_H), \quad (8)$$

then $\hat{\theta}(K, \sigma_H) < \hat{\theta}(K, \sigma_L)$ and the joint payoff is lower for σ_H than for σ_L ; moreover, there is a larger difference for σ_H than for σ_L between the joint payoff when effort is contractible and when it is not.

Proposition 3 establishes that systemic risk has no impact on optimal effort for given capital stock. For specific risk this is not the case even when the parties are jointly risk neutral (in which case equality applies in (8)). The constraints required to make the contract self-enforcing then result in the parties receiving a higher joint payoff from a less risky distribution. In this respect, it is as if they were risk averse. Because with specific risk and partial pooling of efforts $\hat{\theta}(K, \sigma_H) < \hat{\theta}(K, \sigma_L)$, effort is at first-best level for a larger interval of θ for σ_L than σ_H . Moreover, because first-best effort is strictly increasing in θ , $e^*(\hat{\theta}(K, \sigma_L), K) > e^*(\hat{\theta}(K, \sigma_H), K)$, so those θ for which effort is below first-best under σ_L have higher effort than under σ_H . Thus, for all θ with effort below first-best under σ_H , effort is strictly higher under σ_L .

6 Optimal capital

Optimal capital when effort is contractible is $K^*(\sigma)$ given by (2). When effort is non-contractible but involves no pooling (Part 1 in Proposition 2), optimal capital is also $K^*(\sigma)$ because the relational contract constraint (3) is not binding. For partial pooling of efforts (Part 3 in Proposition 2), optimal capital is the solution to

$$\begin{aligned} \max_{K \in [0, \bar{K}]} \frac{1}{1 - \delta} \left\{ \int_{\underline{\theta}}^{\hat{\theta}(K, \sigma)} s(e^*(K, \theta), K, \theta) dF(\theta, \sigma) \right. \\ \left. + \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} s(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) dF(\theta, \sigma) \right\} - C(K) \\ \text{subject to } \hat{\theta}(K, \sigma) \text{ the highest value of } \tilde{\theta} \text{ satisfying (5).} \quad (9) \end{aligned}$$

For full pooling of efforts (Part 2 in Proposition 2), optimal capital is the solution to (9) with $\hat{\theta}(K, \sigma) = \underline{\theta}$ and $e^*(K, \hat{\theta}(K, \sigma))$ replaced by $\tilde{e}(K, \sigma)$ defined by (4). For the cases with pooling of efforts, denote optimal capital by $\hat{K}(\sigma)$.

Becker (1975) made the distinction between investment in specific capital that is valuable only with a specific agent and investment in general capital that is equally valuable with other agents. In the context of relational contracts, this distinction has the additional importance, noted by Klein and Leffler (1981), that investment in specific capital has the effect of relaxing the relational contract constraint (3). For the model here, the distinction is captured by the following assumption.

Definition 3 *Capital is general if the optimal effort schedule in Proposition 2 generates $S^*(K, \sigma)$ such that $S^*(K, \sigma) - \underline{s}(K, \sigma)$ is independent of K . Capital is specific if $\underline{s}(K, \sigma)$ is independent of K .*

There are, of course, intermediates between these two extremes in which $\underline{s}(K, \sigma)$ increases with K but not by as much as $S^*(K, \sigma)$. The present paper follows the literature in focussing on the two cases identified by Becker (1975).

6.1 General capital

For general capital, the next proposition shows that optimal capital stock with non-contractible effort, denoted $\hat{K}^G(\sigma)$, is lower than with contractible effort whenever there is some pooling.

Proposition 4 *When capital is general and the optimal effort schedule exhibits some pooling (Part 2 or Part 3 of Proposition 2 applies), $\hat{K}^G(\sigma) < K^*(\sigma)$.*

This result is perhaps not surprising. When effort is not contractible to the extent that makes a difference, it is constrained below the first-best schedule for given capital

for some values of θ . Thus, given the complementarity of effort and capital in producing output, the marginal product of capital averaged over θ is lower than when effort is contractible and investment is, therefore, less valuable. The next result concerns the effect of risk on general capital.

Proposition 5 *Suppose capital is general, the optimal effort schedule has either full or partial pooling for both σ_H and σ_L (that is, Part 2 or Part 3 of Proposition 2 applies) and the conditions of Proposition 3 hold in each case. Suppose also $h_2(e^*(K, \hat{\theta}(K, \sigma_H)), K, \theta)$ is decreasing and strictly concave in θ for $\theta \in [\hat{\theta}(K, \sigma_H), \bar{\theta}]$ in the case of partial pooling and $h_2(\bar{e}(K, \sigma_H), K, \theta)$ is decreasing and strictly concave in θ for $\theta \in [\underline{\theta}, \bar{\theta}]$ in the case of full pooling. Then with either systemic or specific risk:*

1. $K^*(\sigma_H) - \hat{K}^G(\sigma_H) > K^*(\sigma_L) - \hat{K}^G(\sigma_L)$;
2. *when the parties are jointly risk neutral ((8) holds with equality), $\hat{K}^G(\sigma_H) < \hat{K}^G(\sigma_L)$.*

This result establishes that, under the conditions specified, a more risky distribution increases the difference between optimal general capital when effort is contractible and when effort is not contractible. This applies even to increases in systemic risk which, by Proposition 3, do not affect effort for given capital stock. The intuition for the case of partial effort pooling can be seen from Figure 2, in which the thin solid line corresponds to first-best effort and the bold line to optimal effort with a relational contract. For given capital, the latter results in a lower return to investment for shocks $\theta > 1$ and greater risk corresponds to a higher probability of such shocks. That lowers the joint payoff to continuation of the relationship in the future which, with the relational contract constraint (3), lowers the highest effort sustainable in the present and thus moves the kink point in the bold line in Figure 2 down the thin solid line. With effort and capital complementary in producing output, this reduces the return to investment. Formally, the second integral in the maximand in (9) makes the objective function more concave than when effort choice is unconstrained, which makes the parties jointly more risk averse. This effect is exacerbated with increases in specific risk that, again by Proposition 3, also lower $\hat{\theta}(K, \sigma)$ so that the interval of θ for which effort is first best is reduced. Lemma 2 gives a sufficient condition for $h_2(e, K, \theta)$ everywhere decreasing in θ . This condition holds for the specification in the calibration in Section 7.

6.2 Specific capital

By Definition 3, additional specific capital increases $S(K, \sigma)$ while leaving $\underline{s}(K, \sigma)$ unchanged for any given effort schedule. It thus relaxes the relational contract constraint (3). Malcomson (2015a) showed that, in the absence of productivity shocks, the optimal

level of capital is higher when capital is specific than when it is general whenever the relational contract constraint is binding. The formal result for the present model with productivity shocks, with $\hat{K}^S(\sigma)$ denoting optimal specific capital, is as follows.

Proposition 6 *For an optimal effort schedule exhibiting some pooling (Part 2 or Part 3 of Proposition 2) for both general and specific capital, $\hat{K}^S(\sigma) > \hat{K}^G(\sigma)$ and, for y_{12} sufficiently small, $\hat{K}^S(\sigma) > K^*(\sigma)$.*

Proposition 6 establishes not only that, with non-contractible effort, optimal specific capital is higher than optimal general capital but also that, under certain circumstances, it is higher even than the optimal level when effort is contractible. This occurs when the reduced return that results from the non-contractibility of effort is more than offset by the relaxation of the relational contract constraint. The next result concerns the effects of riskiness on specific capital.

Proposition 7 *Suppose the optimal effort schedule has either full or partial pooling for both σ_H and σ_L (that is, Part 2 or Part 3 of Proposition 2 applies) and the conditions of Proposition 3 hold in each case. Then with either systemic or specific risk, for $y_{133}(e, K, \theta) \geq 0$ for all (e, K, θ) (with strict inequality for full pooling), $\hat{K}^S(\sigma_H) - \hat{K}^G(\sigma_H) > \hat{K}^S(\sigma_L) - \hat{K}^G(\sigma_L)$ and, for y_{12} also sufficiently small, $\hat{K}^S(\sigma_H) - K^*(\sigma_H) > \hat{K}^S(\sigma_L) - K^*(\sigma_L)$.*

Proposition 7 gives conditions under which a more risky distribution results in a greater difference between optimal specific and optimal general capital. It may even result in a greater difference between optimal specific capital when effort is not contractible than when effort is contractible, in which case greater uncertainty results in more investment when the parties are jointly risk neutral. As with Proposition 6, the reason is that additional specific capital relaxes the relational contract constraint (3). When a change in σ reduces the joint payoff $S^*(K, \sigma)$ as in Proposition 3, it tightens the constraint (3) and so increases the return to specific capital that relaxes that constraint. It thus results in higher specific capital. The condition $y_{133}(e, K, \theta) \geq 0$ is sufficient but not necessary. It holds for the example $y(e, K, \theta) = \theta \hat{y}(e, K)$.

The overall conclusions from this section are as follows. With general capital, optimal capital when effort is non-contractible is less than when effort is contractible. Moreover, increased risk increases the difference. But with specific capital, increased capital relaxes the relational contract constraint and so optimal capital is higher than if capital were general and, in some cases, higher even than if effort were contractible. Moreover, the difference can actually be greater with a more risky distribution than with a less risky one.

7 A calibrated model

To illustrate how taking account of relational contracts can help capture the effect of uncertainty on investment empirically, consider the specification

$$y(e, K, \theta) = \theta^\gamma K^\alpha e^\beta, \quad \alpha, \beta, \gamma > 0, \quad \alpha + \beta \leq 1; \quad (10)$$

$$c(e) = ce^n, \quad c > 0, n \geq 1; \quad (11)$$

$$C(K) = CK^k, \quad C > 0, k \geq 1. \quad (12)$$

The value of output function in (10) is that in Bloom et al. (2018) with the labour hours input replaced by the agent's effort. Bloom (2009) shows how the exponents γ, α and β relate to those in an underlying Cobb-Douglas production function and an isoelastic demand curve for the product. Bloom et al. (2018) use a labour cost function that is linear in labour hours and a capital cost function that is also linear, both with the addition of adjustment costs. Here the former is replaced by the effort cost function (11), the latter by (12), which allow for convex costs. Because the model here is stationary and capital is chosen before shocks are revealed, there is no role for capital adjustment costs.

For this specification, first-best effort conditional on K and θ is

$$e^*(K, \theta) = \left(\frac{\beta}{nc} \theta^\gamma K^\alpha \right)^{\frac{1}{n-\beta}}. \quad (13)$$

(Derivations for this section are given in Appendix B.) Then

$$s(e^*(K, \theta), K, \theta) = \left(1 - \frac{\beta}{n}\right) \left(\frac{\beta}{nc}\right)^{\frac{\beta/n}{1-\beta/n}} \theta^{\frac{\gamma}{1-\beta/n}} K^{\frac{\alpha}{1-\beta/n}}, \quad (14)$$

which is concave in θ for $\gamma \leq 1 - \beta/n$, and

$$s(e^*(K, \tilde{\theta}), K, \theta) = \left(\frac{\beta}{nc}\right)^{\frac{\beta/n}{1-\beta/n}} \tilde{\theta}^{\frac{\gamma}{1-\beta/n}} K^{\frac{\alpha}{1-\beta/n}} \left[\left(\frac{\theta}{\tilde{\theta}}\right)^\gamma - \frac{\beta}{n} \right]. \quad (15)$$

Expected revenue per agent (labour productivity in the case of employment) is

$$K^{\frac{\alpha}{1-\beta/n}} \left(\frac{\beta}{nc}\right)^{\frac{\beta/n}{1-\beta/n}} \left[\int_{\underline{\theta}}^{\tilde{\theta}} \theta^{\frac{\gamma}{1-\beta/n}} dF(\theta, \sigma) + \tilde{\theta}^{\frac{\gamma}{1-\beta/n}} \int_{\tilde{\theta}}^{\bar{\theta}} \theta^{\frac{\gamma}{1-\beta/n}} dF(\theta, \sigma) \right]. \quad (16)$$

Bloom (2009, footnote 17), following a discussion in Abel and Eberly (1996), argues that it is appropriate to calibrate the model to avoid long-run effects of uncertainty reducing or increasing output, which corresponds here to setting $\gamma = 1 - \beta/n$ in (14). That makes the parties jointly risk-neutral in the absence of a relational contract, a

good benchmark for determining the effect of a relational contract.

For the calibrations, it fits with Bloom et al. (2018) to take $\theta/E(\theta|\sigma)$ as log-normally distributed with risk parameterized by σ the standard deviation of $\ln \theta/E(\theta|\sigma)$. (For more on this, see below.) For this distribution, full pooling is never an optimal relational contract because it would require $\tilde{e}(K, \sigma) \leq e^*(K, 0) = 0$, so continuing the relationship would not be worthwhile. The same applies to an optimal relational contract with first-best effort for all θ as long as the future joint payoff is finite because then, with first-best effort given by (13), (3) cannot be satisfied as θ goes to infinity. Thus partial pooling is the only relevant case. Rather than solve directly for optimal capital, $\hat{K}^G(\sigma)$ and $\hat{K}^S(\sigma)$, it is convenient to solve for the optimal cutoff values $\hat{\theta}(K, \sigma)$ below which effort is first-best when capital is chosen optimally. Denote these cutoff values by $\hat{\theta}^G(\sigma)$ and $\hat{\theta}^S(\sigma)$ for general and specific capital respectively.

When capital is general, $S^*(K, \sigma) - \underline{s}(K, \sigma)$ in Proposition 1 is by Definition 3 independent of K , so it is convenient to define

$$\hat{S}(\sigma) = \frac{\delta}{1-\delta} [S^*(K, \sigma) - \underline{s}(K, \sigma)]. \quad (17)$$

It can then be shown that the first-order condition for optimal capital yields that

$$\hat{S}(\sigma)^{1-(1-\frac{\beta}{n})\frac{k}{\alpha}} \hat{\theta}^G(\sigma)^{(1-\frac{\beta}{n})\frac{k}{\alpha}-1} \left\{ E(\theta | \sigma) - \int_{\hat{\theta}^G(\sigma)}^{\bar{\theta}} \left[\theta - \hat{\theta}^G(\sigma)^{\frac{\beta}{n}} \theta^{1-\frac{\beta}{n}} \right] dF(\theta, \sigma) \right\} \quad (18)$$

is independent of σ . There exist values of the parameters c and C (neither of which enter (18)) for which this first-order condition corresponds to a maximum as long as the expression in (18) is positive and increasing in $\hat{\theta}^G(\sigma)$. It can also be shown that

$$\frac{\hat{K}^G(\sigma_H)}{\hat{K}^G(\sigma_L)} = \left[\frac{\hat{S}(\sigma_H) \hat{\theta}^G(\sigma_L)}{\hat{S}(\sigma_L) \hat{\theta}^G(\sigma_H)} \right]^{\frac{1-\beta/n}{\alpha}}. \quad (19)$$

For systemic risk, $\hat{S}(\sigma)$ is also independent of σ . For given parameter values and distributions $F(\theta, \sigma_H)$ and $F(\theta, \sigma_L)$, one can then use (18) to solve for $\hat{\theta}^G(\sigma_H)$ for any given value $\hat{\theta}^G(\sigma_L)$, and (19) to solve for the ratio $\hat{K}^G(\sigma_H) / \hat{K}^G(\sigma_L)$, independently of $\hat{S}(\sigma)$. Systemic risk is the natural interpretation of aggregate shocks, so this provides a way to assess the model for general capital against the data underlying Figure 1.

A similar approach can be used to calibrate the model when capital is specific, in which case $\underline{s}(K, \sigma)$ is by Definition 3 independent of K . When the change in risk is also specific, $\underline{s}(K, \sigma)$ is independent of σ . An important qualitative difference in this case is that an increase in uncertainty may increase optimal capital when the parties are jointly risk neutral, an interesting result in the light of the comment in Bloom (2014, p. 153) that uncertainty appears to stimulate some investments. But specific capital

may be too small a part of total capital for the effect of an increase in risk on that to be apparent in the aggregate data used in Bloom et al. (2018). For that reason, the specific capital case is not pursued in this section.

Equation (19) is for differences in *capital*, whereas the data in Figure 1 are for *gross investment*. In comparisons of stationary equilibria, differences in capital result in differences in investment for two reasons. The first is in investment to replace depreciation of capital stocks in continuing firms. The second is in investment by new firms that replace those firms going out of business whose capital becomes valueless. In Bloom et al. (2018), depreciation is assumed to be a constant proportion of capital, so the percentage change between two stationary equilibria in investment to replace depreciation is the same as the percentage change in capital. The same applies to investment by new firms if these are a constant proportion of the total.

The formulation in this section is calibrated using parameters based on those in Bloom et al. (2018). As already noted, Bloom et al. (2018) uses a Cobb-Douglas revenue function like that in (10) but with e being the total labour input measured in hours. The long-run costs of both capital and labour (that is, excluding adjustment costs) are the going market rates, which correspond to $n = k = 1$ in (11) and (12). The parameters α and β are calibrated from factor shares adjusted for an isoelastic product demand function with a 33% markup. It is, however, worth noting that results calculated from (18) and (19) depend on β and n only through their ratio β/n , so they are robust to changes in those parameters that leave this ratio unchanged. Because Bloom et al. (2018) are concerned with business cycles, total factor productivity in their model (corresponding to θ^γ) is time varying. It is the product of separate aggregate (A) and firm (Z) components that switch between two regimes, low risk (L) and high risk (H), all of which follow autoregressive processes with normally distributed logs of innovations. The counterpart to these processes in the stationary long-run framework in the model used here is that θ is log-normally distributed, with the low risk and high risk regimes the two different long-run equilibria corresponding to σ_L and σ_H respectively in the theoretical model of the previous sections. However, the estimated standard deviations in Bloom et al. (2018) apply to θ^γ , not θ itself, and so need to be adjusted correspondingly. Moreover, for the purpose of showing the effects of changes in long-run uncertainty, the aggregate and firm components are combined using the standard formula for the product of log-normal distributions to give the low risk and high risk parameters for θ^γ for σ_L and σ_H respectively. The resulting low and high risk distributions for θ are specified to have the same mean to ensure that the parties are jointly risk neutral in the absence of the relational contract constraint (which corresponds to the expression in (14) being linear in θ with the calibration $\gamma = 1 - \beta/n$). The resulting parameter values are given in Table 1. (Details of calculations for this section are given in Appendix C.) It is worth noting that the aggregate uncertainty change is essentially swamped by the firm uncertainty change because $\sigma_H/\sigma_L \approx \sigma_H^Z/\sigma_L^Z$.

Parameter	Value	Source
δ	$0.95^{1/4}$	Bloom et al (2018), annual discount factor of 95%
α	0.25	Bloom et al (2018), factor share with isoelastic demand, 33% markup
β	0.5	As α with labour share 2/3, capital share 1/3
n	1	Implied by Bloom et al (2018) model
k	1	Implied by Bloom et al (2018) model
σ_L^A	0.67	Bloom et al (2018) estimate, %
σ_H^A / σ_L^A	1.6	Bloom et al (2018) estimate
σ_L^Z	5.1	Bloom et al (2018) estimate, %
σ_H^Z / σ_L^Z	4.1	Bloom et al (2018) estimate
σ_L	0.10	Calculated combined σ_L^A and σ_L^Z for θ
σ_H / σ_L	4.07	Calculated from combined σ_H^A and σ_H^Z for θ

Table 1: Parameter values for calibration

To fully calibrate the relational contract model requires a specification for $\underline{s}(K, \sigma)$, which has no counterpart in Bloom et al. (2018). There also seems no obvious way to derive it from data. But, given the other parameters, each $\underline{s}(K, \sigma)$ implies a cutoff $\hat{\theta}^G(\sigma)$ for which effort for $\theta > \hat{\theta}^G(\sigma)$ is constrained to $e^*(K, \hat{\theta}^G(\sigma))$, that is below first best. So, to illustrate the effect of a change in risk, the effects of the change are calculated for different possible values of $\hat{\theta}^G(\sigma_L)$, the cutoff value of θ above which the relational contract constraint binds with the low risk distribution.

Table 2 gives the effect of the change from σ_L to σ_H in Table 1 with the mean of θ unchanged for the specified values of $\hat{\theta}^G(\sigma_L)$ (relative to the mean of θ) in the top row, on the assumption that all capital is general and all risk systemic. The second row gives the joint payoff gain to principal and agent from continuation of the relationship in the future over what they would get by ending it, $\hat{S}(\sigma_L)$, as a percentage of revenue per period, implied by each value of $\hat{\theta}^G(\sigma_L)$. This joint gain is a present value discounted back to the current period; to express it as a percentage of one period's revenue, it has to be multiplied by $(1 - \delta) / \delta$. The third row gives $\hat{\theta}^G(\sigma_H)$ (again relative to the mean θ), the cutoff θ above which the relational contract constraint binds with the high risk distribution σ_H . This is calculated from (18). The remaining rows give the differences in capital and productivity in changing from the σ_L to σ_H for the specified $\hat{\theta}^G(\sigma_L) / E(\theta)$. The capital differences are calculated from (19). The productivity differences are the expected changes in revenue per agent calculated from (16). For the specification used here, these are the same. It can be shown (see Appendix B) that the impact of an increase in specific risk on general capital is greater in magnitude (that is, with negative numbers more negative) than that of an increase in systemic risk. Thus the numbers in Table 2 can be taken as lower bounds on the magnitudes of the effects on capital if some risk is specific.

In Table 2, $\hat{\theta}^G(\sigma_H) > \hat{\theta}^G(\sigma_L)$ so the cutoff point in the distribution of θ above which the relational contract constraint binds is higher with the higher risk distribution. But the higher risk gives more weight to θ above this cutoff point, which results in less cap-

$\hat{\theta}^G(\sigma_L) / E(\theta)$	0.50	0.75	1.00	1.25	1.50	1.81	2.00
$\hat{S}(\sigma_L)$ as % productivity	35.4	43.4	51.1	62.5	75.0	90.5	100.0
$\hat{\theta}^G(\sigma_H) / E(\theta)$	0.51	0.78	1.06	1.30	1.53	1.83	2.01
Capital change (%)	-3.27	-6.80	-11.55	-8.11	-4.42	-2.01	-1.24
Productivity change (%)	-3.27	-6.80	-11.55	-8.11	-4.42	-2.01	-1.24

Table 2: Effect of increase in systemic risk with general capital for given $\hat{\theta}^G(\sigma_L) / E(\theta)$

ital and lower productivity. The effects of the change in risk on capital and productivity are larger for $\hat{\theta}^G(\sigma_L)$ nearer the mean of the distribution of θ . For $\hat{\theta}^G(\sigma_L)$ equal to that mean, capital and productivity fall by more than 11%. But even for $\hat{\theta}^G(\sigma_L)$ at 50% above or below the mean, the effects are substantial. Recall that these effects would be precisely zero in the absence of the relational contract constraint (which corresponds to the limit as $\hat{\theta}^G(\sigma_L)$ goes to infinity).

Bloom et al. (2018) conclude that, even with adjustment costs, a negative aggregate (first moment) total factor productivity shock is necessary to match the data on recessions in the absence of relational contracts. They use a negative 2% exogenous first moment aggregate shock alongside the second moment shock in a numerical experiment to fit their model to the data. In their model, that results after 12 quarters in an overall approximately 2% drop in output with no change in labour input, a fall in labour productivity of approximately 2% — see Figure 8 in Bloom et al. (2018). It is clear from Table 2 that the relational contract model used here can generate drops in labour productivity as large as 2% with just the second moment shock. For some falls in productivity, there are two values of $\hat{\theta}^G(\sigma_L)$ in Table 2, one each side of the mean, that generate that fall. However, there is no value of $\hat{\theta}^G(\sigma_L)$ below the mean that generates a productivity fall below 2.7%, which is the limit as $\hat{\theta}^G(\sigma_L)$ goes to zero. That limit is greater than zero because, even though the probability density goes to zero as $\hat{\theta}^G(\sigma_L)$ goes to zero, a change in $\hat{\theta}^G(\sigma_L)$ affects the effort, and hence productivity, for all higher θ .

The implication is that there is a unique value of $\hat{\theta}^G(\sigma_L)$, approximately 1.81 times the mean of θ , that generates a fall in productivity of approximately 2% in the model when all capital is general and shocks are systemic. It is only for shocks above this level that the relational contract constraint binds, so this applies to only a very small proportion of shocks. But shocks above the cutoff are those for which an agent would incur particularly high effort in the absence of the relational contract constraint, which is why the 2% fall in productivity is so much larger than the proportion of shocks to which the constraint applies. (The shape of the upper tail of the lognormal distribution is important for this. For distributions with finite support, the proportion of shocks for which the relational contract constraint binds that generates a 2% fall in productivity could be significantly larger.) This value of $\hat{\theta}^G(\sigma_L)$ implies that the joint payoff gain to principal and agent, discounted back to the current period, from continuation of

the relationship in the future over what they would get by ending it, $\hat{S}(\sigma_L)$, is approximately 90% of the revenue per period under the less risky distribution, σ_L . As noted above, to express that joint gain in per period terms, $\hat{S}(\sigma_L)$ has to be multiplied by $(1 - \delta) / \delta$. Bloom et al. (2018) assume a quarterly discount factor of $0.95^{1/4} \approx 0.98726$. For that discount factor, the joint gain per quarter would, therefore, correspond to approximately 1.2% of revenue per quarter. That is a relatively small, and certainly not implausible, joint gain or quasi-rent from the relationship. This relatively small departure from a perfectly competitive environment is sufficient to remove the need for a negative first moment shock to total factor productivity to fit the data in the absence of a relational contract.

The specific value of $\hat{\theta}^G(\sigma_L)$ in the previous paragraph that generates a fall in productivity of approximately 2% is based on all firms using relational contracts, which will presumably not apply to the whole US economy. But in Table 2, higher $\hat{\theta}^G(\sigma_L)$ can result in a fall in productivity of over $11\frac{1}{2}\%$, which is more than five times as much. So even if only one-fifth of general investment is under relational contracts and the rest is unaffected by risk, an overall fall in productivity of 2% could still result from the use of relational contracts.

8 Conclusion

This paper has analysed the effect of changes in the riskiness of the distribution of output on capital investment in a relational contract in which agent performance is not contractible. Unlike when performance is contractible, the effect on investment depends on whether capital is general or specific in the sense of Becker (1975). Investment in general capital is lower with a relational contract than with contractible performance and, under plausible conditions, is more adversely affected by an increase in uncertainty. Investment in specific capital is less adversely affected by reliance on a relational contract and may even increase with an increase in uncertainty. The reason is that, as noted by Klein and Leffler (1981), specific capital relaxes the incentive constraint in a relational contract because it increases the payoff to the parties staying together relative to the payoff from separating and it is this difference which constrains performance.

An implication of these theoretical results is that, in the presence of relational contracts, an increase in uncertainty alone reduces long-run equilibrium investment in general capital and productivity when it would not otherwise do so — even, that is, if the parties are risk neutral. This is potentially important given that relational contracts are widely seen as an insightful way to view economic relationships in a variety of contexts.

To illustrate the potential empirical significance, the paper uses a calibrated production framework similar to that in the real business cycle model in Bloom et al.

(2018) but with the option value of delaying investment replaced by the constraints arising from a relational contract. Bloom et al. (2018) find that, without relational contracts, a negative first moment shock to aggregate total factor productivity seems necessary to match the data on recessions even with allowance for multiple adjustment costs. They comment (p. 2) that “(t)he reliance on negative technology shocks has proven to be controversial, as it suggests that recessions are times of technological regress.” The calibration exercise here shows that the same impact as their negative first moment shock is well within the range that a higher second moment alone could plausibly deliver as an equilibrium effect on general investments in the presence of relational contracts. This is suggestive that recessions of the magnitudes in the data might arise solely from a second moment shock to aggregate total factor productivity, with no negative first moment shock. It is also suggestive that adding relational contracts to perception of capital risk models might account for movements that cannot be accounted for by the cost of high-risk capital alone. To verify these obviously requires a relational contract model that, unlike the one used here, takes full account of dynamics — a far from straightforward task. But if verified in that way, it would open up attractive possibilities.

Appendix A Proofs

Proof of Proposition 1. To simplify notation in the proof, drop the arguments (K, σ) because these are predetermined at the time effort decisions are made and let $u(\theta)$ denote the payoff to the agent in each period of a stationary relational contract in which θ is realized. Then, if both parties stick to the contract,

$$u(\theta) = w(\theta) + b(e(\theta), \theta) - c(e(\theta)). \quad (20)$$

So, having observed θ , the payoff to the agent from sticking to the contract at every date t , conditional on the principal doing so, is

$$u(\theta) + \frac{\delta}{1-\delta} E_{\theta'} u(\theta') \quad (21)$$

because θ is an *iid* draw each period. For the principal, let $v(\theta)$ denote the payoff in each period of a stationary relational contract in which θ is realized. Then, if both parties stick to the contract,

$$v(\theta) = y(e(\theta), \theta) - w(\theta) - b(e(\theta), \theta). \quad (22)$$

So, having observed θ , the payoff to the principal from sticking to the contract at every date t , conditional on the agent doing so, is

$$v(\theta) + \frac{\delta}{1-\delta} E_{\theta'} v(\theta'). \quad (23)$$

Necessity. By setting $e_t = 0$ and quitting the following period, the agent can, even with no bonus, guarantee a payoff of

$$w(\theta) + \frac{\delta}{1-\delta} \underline{u}. \quad (24)$$

A necessary condition for the agent to stick to the contract for all θ is that the payoff in (21) is no less than that in (24) for all θ . With the use of (20), that condition can be written

$$\frac{\delta}{1-\delta} [E_{\theta'} u(\theta') - \underline{u}] \geq c(e(\theta)) - b(e(\theta), \theta), \text{ for all } \theta \in [\underline{\theta}, \bar{\theta}]. \quad (25)$$

Once the agent has incurred effort $e(\theta)$, the principal can, by setting $b_t = 0$ and quitting the following period, guarantee a payoff of

$$y(e(\theta), \theta) - w(\theta) + \frac{\delta}{1-\delta} \underline{v}. \quad (26)$$

A necessary condition for the principal to stick to the contract for all θ is that the payoff in (23) is no less than that in (26) for all θ . With the use of (22), that condition can be written

$$\frac{\delta}{1-\delta} [E_{\theta'} v(\theta') - \underline{v}] \geq b(e(\theta), \theta), \text{ for all } \theta \in [\underline{\theta}, \bar{\theta}]. \quad (27)$$

As both (25) and (27) are necessary, so is their sum, which gives the combined necessary condition

$$\frac{\delta}{1-\delta} [E_{\theta'} u(\theta') + E_{\theta'} v(\theta') - (\underline{u} + \underline{v})] \geq c(e(\theta)), \text{ for all } \theta \in [\underline{\theta}, \bar{\theta}]. \quad (28)$$

But, by definition, $S(K, r) = E_{\theta'} u(\theta') + E_{\theta'} v(\theta')$ and $\underline{s}(K, r) = \underline{u} + \underline{v}$. So that (28) is necessary implies that (3) is necessary.

Sufficiency. For (3) and hence (28) satisfied, there certainly exists a $b(e(\theta), \theta) \in [0, c(e(\theta))]$ for each θ such that (25) and (27) are both satisfied. For such $b(e(\theta), \theta)$, the individual rationality conditions

$$E_{\theta} u(\theta) - \underline{u} \geq 0 \quad (29)$$

$$E_{\theta} v(\theta) - \underline{v} \geq 0 \quad (30)$$

are also satisfied. To establish that sticking to the contract is a best response for both

parties, it remains only to show that the payoffs following deviation specified in (24) and (26), $\delta \underline{u} / (1 - \delta)$ and $\delta \underline{v} / (1 - \delta)$, are equilibrium payoffs. That is immediate if the strategies for the two parties specify that play following deviation by either is for both to end the relationship because, if the strategy for either is to end the relationship it is a best response of the other also to end the relationship given $\underline{u}, \underline{v} \geq 0$. ■

Proof of Corollary 1. If first-best effort $e^*(K, \theta)$ is unattainable for θ' for given $S(K, \sigma)$, it must be because (3) is a binding constraint for θ' . The left-hand side of (3) is independent of θ . Moreover, $e^*(K, \theta)$ is increasing in θ and $c(e)$ is increasing in e . Thus the right-hand side of (3) is increasing in θ for $e(K, \theta, \sigma) = e^*(K, \theta)$. So if (3) is not satisfied for $e(K, \theta, \sigma) = e^*(K, \theta')$, it is not satisfied for $e(K, \theta, \sigma) = e^*(K, \theta)$ for any $\theta \in (\theta', \bar{\theta}]$. ■

Proof of Proposition 2. By Proposition 1, (3) is necessary and sufficient for an effort schedule to be implementable. Thus an optimal effort schedule for given K is a solution to

$$\begin{aligned} \max_{e(K, \cdot, \sigma) \in [0, \bar{e}]} & \frac{1}{1 - \delta} \int_{\underline{\theta}}^{\bar{\theta}} s(e(K, \tilde{\theta}, \sigma), K, \tilde{\theta}) - \underline{s}(K, \sigma) dF(\tilde{\theta}, \sigma) \quad \text{subject to} \\ & \frac{\delta}{1 - \delta} \int_{\underline{\theta}}^{\bar{\theta}} [s(e(K, \tilde{\theta}, \sigma), K, \tilde{\theta}) - \underline{s}(K, \sigma)] dF(\tilde{\theta}, \sigma) \geq c(e(K, \theta)), \quad \text{for all } \theta \in [\underline{\theta}, \bar{\theta}]. \end{aligned}$$

By Assumptions 1 and 2, optimal effort for all θ is interior to $[0, \bar{e}]$. There are three cases to consider: (1) the constraint is not binding for any $\theta \in [\underline{\theta}, \bar{\theta}]$; (2) the constraint is binding for all $\theta \in [\underline{\theta}, \bar{\theta}]$; and (3) the constraint is not binding for some $\theta \in [\underline{\theta}, \bar{\theta}]$ but is binding for other $\theta \in [\underline{\theta}, \bar{\theta}]$. For any $\tilde{\theta} \in [\underline{\theta}, \bar{\theta}]$ for which the constraint is not binding, it is immediate that it is optimal to set $e(K, \tilde{\theta}, \sigma) = e^*(K, \tilde{\theta})$, the first-best effort for which $s_1(e^*(K, \tilde{\theta}), K, \tilde{\theta}) = 0$, because that both maximizes the objective function for $\tilde{\theta}$ and relaxes the constraint most for $\theta \neq \tilde{\theta}$. So, for case 1, in which the constraint does not bind for any θ (for which a necessary and sufficient condition is that it does not bind for $\bar{\theta}$ because $e^*(\theta, K)$ is increasing in θ), optimal effort is first best for all θ , as in Part 1 of the proposition. Next note that the integral in the constraint is independent of θ . So, for case 2, in which the constraint binds for all θ (for which a necessary and, by Corollary 1, sufficient condition is that it binds for $\underline{\theta}$), $c(e(K, \theta, \sigma))$ and hence $e(K, \theta, \sigma)$ must be the same for all θ . In that case, optimal effort is the highest effort independent of θ that satisfies the constraint, so $e(K, \theta, \sigma) = \tilde{e}(K, \sigma)$ given by (4), as in Part 2 of the proposition. For case 3 by the argument just given, for the set of θ for which the constraint does not bind, optimal effort must be first best and, for the set of θ for which the constraint binds, $e(K, \theta, \sigma)$ must be independent of θ . But, from Corollary

1, if the latter set contains θ' , it also contains all $\theta > \theta'$. That implies there is a cutoff type $\hat{\theta}(K, \sigma)$ such that, for all $\theta \leq \hat{\theta}(K, \sigma)$, effort is at the first-best level and, for all $\theta > \hat{\theta}(K, \sigma)$, effort is constant at $e^*(K, \hat{\theta}(K, \sigma))$, as specified in Part 3 of the proposition. ■

Proof of Lemma 1. Integration by parts gives

$$\begin{aligned} & \int_{\tilde{\theta}}^{\bar{\theta}} g(\theta) dF(\theta, \sigma_L) - \int_{\tilde{\theta}}^{\bar{\theta}} g(\theta) dF(\theta, \sigma_H) \\ &= [g(\theta) F(\theta, \sigma_L)]_{\tilde{\theta}}^{\bar{\theta}} - \int_{\tilde{\theta}}^{\bar{\theta}} g'(\theta) F(\theta, \sigma_L) d\theta \\ & \quad - \left[[g(\theta) F(\theta, \sigma_H)]_{\tilde{\theta}}^{\bar{\theta}} - \int_{\tilde{\theta}}^{\bar{\theta}} g'(\theta) F(\theta, \sigma_H) d\theta \right] \\ &= - \int_{\tilde{\theta}}^{\bar{\theta}} [F(\theta, \sigma_L) - F(\theta, \sigma_H)] g'(\theta) d\theta, \end{aligned}$$

the last line following because $F(\bar{\theta}, \sigma_H) = F(\bar{\theta}, \sigma_L) = 1$, $F(\underline{\theta}, \sigma_H) = F(\underline{\theta}, \sigma_L) = 0$ and the statement of the lemma specifies that $g(\tilde{\theta}) = 0$ if $\tilde{\theta} > \underline{\theta}$. Integration of that last line by parts gives

$$\begin{aligned} & \int_{\tilde{\theta}}^{\bar{\theta}} g(\theta) dF(\theta, \sigma_L) - \int_{\tilde{\theta}}^{\bar{\theta}} g(\theta) dF(\theta, \sigma_H) \\ &= - \left[g'(\theta) \int_{\underline{\theta}}^{\theta} [F(x, \sigma_L) - F(x, \sigma_H)] dx \right]_{\tilde{\theta}}^{\bar{\theta}} \\ & \quad + \int_{\tilde{\theta}}^{\bar{\theta}} \left[\int_{\underline{\theta}}^{\theta} [F(x, \sigma_L) - F(x, \sigma_H)] dx \right] g''(\theta) d\theta. \end{aligned} \tag{31}$$

Now,

$$\int_{\underline{\theta}}^{\theta} [F(x, \sigma_L) - F(x, \sigma_H)] dx = \int_{\underline{\theta}}^{\bar{\theta}} [F(x, \sigma_L) - F(x, \sigma_H)] dx - \int_{\theta}^{\bar{\theta}} [F(x, \sigma_L) - F(x, \sigma_H)] dx$$

and, with $F(\theta, \sigma_L)$ and $F(\theta, \sigma_H)$ having the same mean,

$$\int_{\underline{\theta}}^{\bar{\theta}} [F(x, \sigma_L) - F(x, \sigma_H)] dx = 0,$$

see Laffont (1989, p. 25). Use of these in (31) gives that

$$\begin{aligned} & \int_{\tilde{\theta}}^{\bar{\theta}} g(\theta) dF(\theta, \sigma_L) - \int_{\tilde{\theta}}^{\bar{\theta}} g(\theta) dF(\theta, \sigma_H) \\ &= -g'(\tilde{\theta}) \int_{\tilde{\theta}}^{\bar{\theta}} [F(x, \sigma_L) - F(x, \sigma_H)] dx \\ & \quad - \int_{\tilde{\theta}}^{\bar{\theta}} \left[\int_{\theta}^{\bar{\theta}} [F(x, \sigma_L) - F(x, \sigma_H)] dx \right] g''(\theta) d\theta \geq 0, \end{aligned}$$

the sign following from $g'(\theta) \leq 0, g''(\theta) \leq 0$ and $\int_{\theta}^{\bar{\theta}} [F(x, \sigma_L) - F(x, \sigma_H)] dx \geq 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$, with strict inequality for a set of values of $x \in [\tilde{\theta}, \bar{\theta}]$ with positive probability, as specified in Definition 1. If $g''(\theta) < 0$, the inequality is strict. ■

Proof of Lemma 2. Part 1. From the definition of h in (7),

$$h_3(e, K, \theta) = s_3(e, K, \theta) - s_1(e^*(K, \theta), K, \theta) e_2^*(K, \theta) - s_3(e^*(K, \theta), K, \theta). \quad (32)$$

The definition of $e^*(K, \theta)$ in (1) implies

$$s_1(e^*(K, \theta), K, \theta) = 0, \quad \text{for all } \theta \in [\underline{\theta}, \bar{\theta}], \quad (33)$$

so

$$h_3(e, K, \theta) = s_3(e, K, \theta) - s_3(e^*(K, \theta), K, \theta) \leq 0 \text{ for } e \leq e^*(K, \theta),$$

the sign following from $s_{13} = y_{13} > 0$ by Assumption 1, so h is decreasing in θ for $e \leq e^*(K, \theta)$.

Part 2. From (32),

$$\begin{aligned} & h_{33}(e, K, \theta) \\ &= s_{33}(e, K, \theta) - s_{11}(e^*(K, \theta), K, \theta) e_2^*(K, \theta)^2 - s_{13}(e^*(K, \theta), K, \theta) e_2^*(K, \theta) \\ & \quad - s_1(e^*(K, \theta), K, \theta) e_{22}^*(K, \theta) - s_{13}(e^*(K, \theta), K, \theta) e_2^*(K, \theta) - s_{33}(e^*(K, \theta), K, \theta) \\ &= s_{33}(e, K, \theta) - s_{33}(e^*(K, \theta), K, \theta) \\ & \quad - s_{11}(e^*(K, \theta), K, \theta) e_2^*(K, \theta)^2 - 2s_{13}(e^*(K, \theta), K, \theta) e_2^*(K, \theta), \end{aligned}$$

the last equality making use of (33). Differentiation of (33) gives

$$e_2^*(K, \theta) = -\frac{s_{13}(e^*(K, \theta), K, \theta)}{s_{11}(e^*(K, \theta), K, \theta)}.$$

Thus

$$\begin{aligned}
h_{33}(e, K, \theta) &= s_{33}(e, K, \theta) - s_{33}(e^*(K, \theta), K, \theta) \\
&\quad - s_{11}(e^*(K, \theta), K, \theta) \left(-\frac{s_{13}(e^*(K, \theta), K, \theta)}{s_{11}(e^*(K, \theta), K, \theta)} \right)^2 \\
&\quad - 2s_{13}(e^*(K, \theta), K, \theta) \left(-\frac{s_{13}(e^*(K, \theta), K, \theta)}{s_{11}(e^*(K, \theta), K, \theta)} \right) \\
&= s_{33}(e, K, \theta) - s_{33}(e^*(K, \theta), K, \theta) + \frac{s_{13}(e^*(K, \theta), K, \theta)^2}{s_{11}(e^*(K, \theta), K, \theta)}.
\end{aligned}$$

By the definition of $s(e, K, \theta)$, $s_{ij}(e, K, \theta) = y_{ij}(e, K, \theta)$ for $i, j = 1, 2$, so the first two terms in this are non-positive for $e \leq e^*(K, \theta)$, as specified in the definition of $h(e, K, \theta)$ in (7), if $y_{133}(e, K, \theta) \geq 0$ and the final term is strictly negative because, by Assumption 1, $y_{11}(e, K, \theta) < 0$ and $y_{13}(e, K, \theta) > 0$. This implies $h_{33}(e, K, \theta) < 0$ and thus $h(e, K, \theta)$ strictly concave in θ for given (e, K) for $y_{133}(e, K, \theta) \geq 0$.

Part 3. From the definition of h in (7),

$$h_2(e, K, \theta) = s_2(e, K, \theta) - s_2(e^*(K, \theta), K, \theta) - s_1(e^*(K, \theta), K, \theta)e_1^*(K, \theta),$$

so

$$\begin{aligned}
h_{23}(e, K, \theta) &= s_{23}(e, K, \theta) - s_{23}(e^*(K, \theta), K, \theta) - s_{12}(e^*(K, \theta), K, \theta)e_2^*(K, \theta) \\
&\quad - s_{13}(e^*(K, \theta), K, \theta)e_1^*(K, \theta) - s_1(e^*(K, \theta), K, \theta)e_{12}^*(K, \theta) \\
&\quad - s_{11}(e^*(K, \theta), K, \theta)e_1^*(K, \theta)e_2^*(K, \theta).
\end{aligned}$$

With the use of (33), this can be written

$$\begin{aligned}
h_{23}(e, K, \theta) &= s_{23}(e, K, \theta) - s_{23}(e^*(K, \theta), K, \theta) - s_{13}(e^*(K, \theta), K, \theta)e_1^*(K, \theta) \\
&\quad - [s_{12}(e^*(K, \theta), K, \theta) + s_{11}(e^*(K, \theta), K, \theta)e_1^*(K, \theta)]e_2^*(K, \theta). \quad (34)
\end{aligned}$$

Differentiation of (33) gives

$$e_1^*(K, \theta) = -\frac{s_{12}(e^*(K, \theta), K, \theta)}{s_{11}(e^*(K, \theta), K, \theta)} > 0,$$

from which the expression in square brackets in (34) is zero so

$$h_{23}(e, K, \theta) = s_{23}(e, K, \theta) - s_{23}(e^*(K, \theta), K, \theta) - s_{13}(e^*(K, \theta), K, \theta)e_1^*(K, \theta).$$

Because $s_{23}(e, K, \theta) = y_{23}(e, K, \theta)$ and $s_{123}(e, K, \theta) = y_{123}(e, K, \theta)$, the first two terms in this are non-positive for $e \leq e^*(K, \theta)$, as specified in the definition of $h(e, K, \theta)$ in (7), if $y_{123}(e, K, \theta) > 0$ and the final term is strictly negative because $e_1^*(K, \theta) > 0$ and,

by Assumption 1, $y_{13}(e, K, \theta) > 0$. ■

Proof of Proposition 3. Part 1. For systemic risk, the constraint (3) is by Definition 2 unaffected by changes in σ for given K , so the optimal effort schedule in Proposition 2 is also unaffected by changes in σ for given K .

Part 2. For specific risk and full pooling optimal for both σ_L and σ_H , the binding constraint is (4) for both. Thus, by Lemma 1, for $s(\tilde{e}(K, \sigma_L), K, \theta)$ strictly concave in θ for all $\theta \in [\underline{\theta}, \bar{\theta}]$, $F(\theta, \sigma_L)$ stochastically dominating $F(\theta, \sigma_H)$ in the second-order sense for $\tilde{\theta} = \underline{\theta}$ implies

$$\int_{\underline{\theta}}^{\bar{\theta}} s(\tilde{e}(K, \sigma_L), K, \theta) dF(\theta, \sigma_L) > \int_{\underline{\theta}}^{\bar{\theta}} s(\tilde{e}(K, \sigma_L), K, \theta) dF(\theta, \sigma_H).$$

For specific risk, $\underline{s}(K, \sigma)$ is by Definition 2 unaffected by changes in σ for given K . Thus, for (4) satisfied for σ_L , its left-hand side would be strictly negative for σ_H if $\tilde{e}(K, \sigma_H) = \tilde{e}(K, \sigma_L)$. To satisfy (4) for σ_H , therefore, requires $\tilde{e}(K, \sigma_H) < \tilde{e}(K, \sigma_L)$ which, being further below first-best effort for every θ , implies a lower joint payoff for σ_H than for σ_L for given K . It also implies a larger difference for σ_H than for σ_L between the joint payoff when effort is contractible and when it is not.

Part 3. For specific risk and partial pooling optimal for both σ_L and σ_H (5), and thus (6), must hold for both. From (7), $h(e^*(K, \theta), K, \theta) = 0$ and, from Lemma 2, $h(e, K, \theta)$ is decreasing in θ . Thus, by Lemma 1 for $h(e^*(K, \hat{\theta}(K, \sigma_H)), K, \theta)$ strictly concave in θ for $\theta \in [\hat{\theta}(K, \sigma_H), \bar{\theta}]$, $F(\theta, \sigma_L)$ stochastically dominating $F(\theta, \sigma_H)$ in the second-order sense for $\tilde{\theta} = \hat{\theta}(K, \sigma_H)$ implies

$$\int_{\hat{\theta}(K, \sigma_H)}^{\bar{\theta}} h(e^*(K, \hat{\theta}(K, \sigma_H)), K, \theta) dF(\theta, \sigma_L) > \int_{\hat{\theta}(K, \sigma_H)}^{\bar{\theta}} h(e^*(K, \hat{\theta}(K, \sigma_H)), K, \theta) dF(\theta, \sigma_H).$$

Because h is the negative of the integrand in the second integral in (6) and $\underline{s}(K, \sigma)$ is by Definition 2 unaffected by changes in σ for given K for specific risk, the left-hand side of (6) would be lower for σ_H than for σ_L if $\hat{\theta}(K, \sigma_L) = \hat{\theta}(K, \sigma_H)$ when (8) holds. So, with (6) holding for σ_H , its left-hand side would be greater than its right-hand side for σ_L if $\hat{\theta}(K, \sigma_L) = \hat{\theta}(K, \sigma_H)$, which implies that the left-hand side of (5) would be positive for σ_L if $\tilde{\theta} = \hat{\theta}(K, \sigma_H)$. But, the left-hand side of (5) is continuous in $\tilde{\theta}$ and negative for $\tilde{\theta} = \bar{\theta}$ because otherwise Case 1 of Proposition 2 would apply. Because, by definition, $\hat{\theta}(K, \sigma_L)$ is the highest value of $\tilde{\theta}$ that satisfies (5) for σ_L , the left-hand side of (5) must be lower for all $\tilde{\theta} > \hat{\theta}(K, \sigma_L)$ than for $\tilde{\theta} = \hat{\theta}(K, \sigma_L)$ — otherwise, by continuity, there would be a $\tilde{\theta} > \hat{\theta}(K, \sigma_L)$ for which equality holds in (5), which is a contradiction. So it cannot be that $\hat{\theta}(K, \sigma_L) \leq \hat{\theta}(K, \sigma_H)$. It must, therefore, be that $\hat{\theta}(K, \sigma_L) > \hat{\theta}(K, \sigma_H)$. From (6), this implies a larger difference for σ_H than for σ_L between the joint payoff when effort is contractible and when it is not. When (8) holds, it also implies a lower

joint payoff for σ_H than for σ_L for given K . ■

Lemma 3 *When capital is general, optimal capital $\hat{K}^G(\sigma)$ satisfies the following.*

1. *For no pooling of efforts (Part 1 of Proposition 2), $\hat{K}^G(\sigma) = K^*(\sigma)$ given by*

$$\frac{1}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_2(e^*(K^*(\sigma), \theta), K^*(\sigma), \theta) dF(\theta, \sigma) - C'(K^*(\sigma)) = 0. \quad (35)$$

2. *For full pooling of efforts (Part 2 of Proposition 2), $\hat{K}^G(\sigma)$ satisfies*

$$\begin{aligned} & \frac{1}{1-\delta} \left\{ \int_{\underline{\theta}}^{\bar{\theta}} s_2(e^*(\hat{K}^G(\sigma), \theta), \hat{K}^G(\sigma), \theta) dF(\theta, \sigma) \right. \\ & \left. + \int_{\underline{\theta}}^{\bar{\theta}} [s_2(\tilde{e}(\hat{K}^G(\sigma), \sigma), \hat{K}^G(\sigma), \theta) - s_2(e^*(\hat{K}^G(\sigma), \theta), \hat{K}^G(\sigma), \theta)] dF(\theta, \sigma) \right\} \\ & - C'(\hat{K}^G(\sigma)) = 0. \quad (36) \end{aligned}$$

3. *For partial pooling of efforts (Part 3 of Proposition 2), $\hat{K}^G(\sigma)$ satisfies*

$$\begin{aligned} & \frac{1}{1-\delta} \left\{ \int_{\underline{\theta}}^{\bar{\theta}} s_2(e^*(\hat{K}^G(\sigma), \theta), \hat{K}^G(\sigma), \theta) dF(\theta, \sigma) \right. \\ & \quad + \int_{\hat{\theta}(\hat{K}^G(\sigma), \sigma)}^{\bar{\theta}} [s_2(e^*(\hat{K}^G(\sigma), \hat{\theta}(\hat{K}^G(\sigma), \sigma)), \hat{K}^G(\sigma), \theta) \\ & \quad \left. - s_2(e^*(\hat{K}^G(\sigma), \theta), \hat{K}^G(\sigma), \theta)] dF(\theta, \sigma) \right\} - C'(\hat{K}^G(\sigma)) = 0. \quad (37) \end{aligned}$$

The left-hand sides of (35), (36) and (37) are everywhere strictly decreasing in $\hat{K}^G(\sigma)$.

Proof. For no effort pooling (Part 1 of Proposition 2), the relational contract constraint (3) is not binding, so $\hat{K}^G(\sigma) = K^*(\sigma)$. Assumption 1 ensures optimal capital is then interior to $[0, \bar{K}]$ and so must satisfy the first-order condition for the maximisation in (2) which, given the definition of s , is just (35). Strict concavity of y in Assumption 1 ensures that the left-hand side of (35) is everywhere strictly decreasing in $\hat{K}^G(\sigma) = K^*(\sigma)$.

For full pooling of efforts (Part 2 of Proposition 2), optimal capital is given by

$$\max_{K \in [0, \bar{K}]} \frac{1}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s(\tilde{e}(K, \sigma), K, \theta) dF(\theta, \sigma) - C(K), \text{ subject to } \tilde{e}(K, \sigma) \text{ satisfying (4).} \quad (38)$$

The integral term in (4) is just $S^*(K, \sigma) - \underline{s}(K, \sigma)$. For general capital, Definition 3 specifies that the optimal effort schedule in Proposition 2 results in $S^*(K, \sigma) - \underline{s}(K, \sigma)$ independent of K , so (4) implies that $\tilde{e}(K, \sigma)$ is independent of K . The derivative of the maximand in (38) with respect to K subject to the constraint is thus

$$\frac{1}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_2(\tilde{e}(K, \sigma), K, \theta) dF(\theta, \sigma) - C'(K).$$

Assumption 1 ensures that $\hat{K}^G(\sigma)$ is then interior to $[0, \bar{K}]$ and so satisfies the first-order condition that can be written as (36). The second derivative is

$$\frac{1}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_{22}(\tilde{e}(K, \sigma), K, \theta) dF(\theta, \sigma) - C''(K),$$

which is strictly negative by Assumption 1, so the first derivative is everywhere strictly decreasing in K and thus the left-hand side of (36) is everywhere strictly decreasing in $\hat{K}^G(\sigma)$.

For partial pooling of efforts (Part 3 of Proposition 2), optimal capital is given by (9) with $\hat{\theta}(K, \sigma)$ the highest $\tilde{\theta}$ satisfying the constraint (5) whatever K is chosen. The integral terms in the constraint (5) are just $S^*(K, \sigma) - \underline{s}(K, \sigma)$ which is independent of K for general capital by Definition 3. So total differentiation of (5) with respect to K for $\tilde{\theta} = \hat{\theta}(K, \sigma)$ gives

$$-c'(e^*(K, \hat{\theta}(K, \sigma))) \frac{de^*(K, \hat{\theta}(K, \sigma))}{dK} = 0$$

which, with $c'(e) > 0$ for all e , implies $de^*(K, \hat{\theta}(K, \sigma))/dK = 0$. Because the two integrands in (9) take the same value for $\theta = \hat{\theta}(K, \sigma)$, the derivative of the maximand in (9) with respect to K subject to the constraint is

$$\begin{aligned} \frac{1}{1-\delta} \left\{ \int_{\underline{\theta}}^{\hat{\theta}(K, \sigma)} \left[s_1(e^*(K, \theta), K, \theta) \frac{\partial e^*(K, \theta)}{\partial K} + s_2(e^*(K, \theta), K, \theta) \right] dF(\theta, \sigma) \right. \\ \left. + \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} \left[s_1(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) \frac{de^*(K, \hat{\theta}(K, \sigma))}{dK} \right. \right. \\ \left. \left. + s_2(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) \right] dF(\theta, \sigma) \right\} - C'(K). \quad (39) \end{aligned}$$

Because $de^*(K, \hat{\theta}(K, \sigma))/dK = 0$ and (1) implies $s_1(e^*(K, \theta), K, \theta) = 0$, (39) can be rearranged to give the first-order condition (37) for $\hat{K}^G(\sigma)$. Because $s_1(e^*(K, \theta), K, \theta) = 0$, the two integrands in (39) are the same for $\theta = \hat{\theta}(K, \sigma)$, so the second derivative of

the maximand in (9) with respect to K is

$$\begin{aligned} & \frac{1}{1-\delta} \left\{ \int_{\underline{\theta}}^{\hat{\theta}(K,\sigma)} \left[s_{11}(e^*(K,\theta), K, \theta) \left(\frac{\partial e^*(K,\theta)}{\partial K} \right)^2 + s_1(e^*(K,\theta), K, \theta) \frac{\partial^2 e^*(K,\theta)}{\partial K^2} \right. \right. \\ & \quad \left. \left. + 2s_{12}(e^*(K,\theta), K, \theta) \frac{\partial e^*(K,\theta)}{\partial K} + s_{22}(e^*(K,\theta), K, \theta) \right] dF(\theta, \sigma) \right. \\ & \quad \left. + \int_{\hat{\theta}(K,\sigma)}^{\bar{\theta}} \left[s_{11}(e^*(K, \hat{\theta}(K,\sigma)), K, \theta) \left(\frac{de^*(K, \hat{\theta}(K,\sigma))}{dK} \right)^2 \right. \right. \\ & \quad \left. \left. + s_1(e^*(K, \hat{\theta}(K,\sigma)), K, \theta) \frac{d^2 e^*(K, \hat{\theta}(K,\sigma))}{dK^2} + 2s_{12}(e^*(K, \hat{\theta}(K,\sigma)), K, \theta) \frac{de^*(K, \hat{\theta}(K,\sigma))}{dK} \right. \right. \\ & \quad \left. \left. + s_{22}(e^*(K, \hat{\theta}(K,\sigma)), K, \theta) \right] dF(\theta, \sigma) \right\} - C''(K). \end{aligned}$$

From the definition of $e^*(K, \theta)$ in (1),

$$\frac{\partial e^*(K, \theta)}{\partial K} = -\frac{s_{12}(e^*(K, \theta), K, \theta)}{s_{11}(e^*(K, \theta), K, \theta)}.$$

With this, $s_1(e^*(K, \theta), K, \theta) = 0$ and $de^*(K, \hat{\theta}(K, \sigma))/dK = d^2 e^*(K, \hat{\theta}(K, \sigma))/dK^2 = 0$, the second derivative can be written

$$\begin{aligned} & \frac{1}{1-\delta} \left\{ \int_{\underline{\theta}}^{\hat{\theta}(K,\sigma)} \left[s_{11}(e^*(K, \theta), K, \theta) \left(-\frac{s_{12}(e^*(K, \theta), K, \theta)}{s_{11}(e^*(K, \theta), K, \theta)} \right)^2 \right. \right. \\ & \quad \left. \left. - 2s_{12}(e^*(K, \theta), K, \theta) \frac{s_{12}(e^*(K, \theta), K, \theta)}{s_{11}(e^*(K, \theta), K, \theta)} + s_{22}(e^*(K, \theta), K, \theta) \right] dF(\theta, \sigma) \right. \\ & \quad \left. + \int_{\hat{\theta}(K,\sigma)}^{\bar{\theta}} s_{22}(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) dF(\theta, \sigma) \right\} - C''(K) \end{aligned}$$

or

$$\begin{aligned} & \frac{1}{1-\delta} \left\{ \int_{\underline{\theta}}^{\hat{\theta}(K,\sigma)} \left[s_{22}(e^*(K, \theta), K, \theta) - \frac{s_{12}(e^*(K, \theta), K, \theta)^2}{s_{11}(e^*(K, \theta), K, \theta)} \right] dF(\theta, \sigma) \right. \\ & \quad \left. + \int_{\hat{\theta}(K,\sigma)}^{\bar{\theta}} s_{22}(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) dF(\theta, \sigma) \right\} - C''(K) \end{aligned}$$

or

$$\begin{aligned} \frac{1}{1-\delta} \left\{ \int_{\underline{\theta}}^{\hat{\theta}(K,\sigma)} \frac{1}{s_{11}(e^*(K,\theta), K, \theta)} [s_{11}(e^*(K,\theta), K, \theta) s_{22}(e^*(K,\theta), K, \theta) \right. \\ \left. - s_{12}(e^*(K,\theta), K, \theta)^2] dF(\theta, \sigma) \right. \\ \left. + \int_{\hat{\theta}(K,\sigma)}^{\bar{\theta}} s_{22}(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) dF(\theta, \sigma) \right\} - C''(K). \end{aligned}$$

By the definition of $s(e, K, \theta)$, $s_{ij}(e, K, \theta) = y_{ij}(e, K, \theta)$ for $i, j = 1, 2$ so, by the strict concavity of y from Assumption 1,

$$s_{11}(e, K, \theta), s_{22}(e, K, \theta) < 0, s_{11}(e, K, \theta)s_{22}(e, K, \theta) - s_{12}(e, K, \theta)^2 > 0.$$

Thus the second derivative is everywhere strictly negative, and the first derivative is everywhere strictly decreasing in K , so the left-hand side of (37) is also everywhere strictly decreasing $\hat{K}^G(\sigma)$.

Finally, note that the left-hand sides of (35), (36) and (37) are continuous in $\hat{K}^G(\sigma)$ so one of these conditions must hold. ■

Proof of Proposition 4. From Lemma 3, with partial pooling of efforts (Part 3 of Proposition 2), $\hat{K}^G(\sigma)$ is uniquely determined by (37). To have $\hat{K}^G(\sigma) = K^*(\sigma)$, the second integral would have to be zero in view of (35). But $s_{12} = y_{12} > 0$ by Assumption 1, so the second integral in (37) is negative because $e^*(K, \theta) > e^*(K, \hat{\theta}(K, \sigma))$ for $\theta > \hat{\theta}(K, \sigma)$. By Lemma 3, the left-hand side of (37) is strictly decreasing in K , which implies $\hat{K}^G(\sigma) < K^*(\sigma)$. For full pooling of efforts (Part 2 of Proposition 2), $\hat{K}^G(\sigma)$ is uniquely determined by (36) to which the same argument applies. ■

Proof of Proposition 5. Part 1. From Lemma 3, with partial pooling of efforts (Part 3 of Proposition 2), $\hat{K}^G(\sigma)$ is uniquely determined by (37). With the definition of h in (7), (37) can be written

$$\begin{aligned} \frac{1}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_2(e^*(\hat{K}^G(\sigma), \theta), \hat{K}^G(\sigma), \theta) dF(\theta, \sigma) \\ + \frac{1}{1-\delta} \int_{\hat{\theta}(\hat{K}^G(\sigma), \sigma)}^{\bar{\theta}} h_2(e^*(\hat{K}^G(\sigma), \hat{\theta}(\hat{K}^G(\sigma), \sigma)), \hat{K}^G(\sigma), \theta) dF(\theta, \sigma) - C'(\hat{K}^G(\sigma)) \\ = 0. \quad (40) \end{aligned}$$

In view of (35), the second integral term would have to be zero to have $\hat{K}(\sigma) = K^*(\sigma)$. By Lemma 1, with $h_2(e^*(K, \hat{\theta}(K, \sigma)), K, \theta)$ decreasing and strictly concave in θ for

$$\theta \in [\hat{\theta}(K, \sigma), \bar{\theta}],$$

$$\int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} h_2(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) dF(\theta, \sigma_L) > \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} h_2(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) dF(\theta, \sigma_H)$$

for given K and $\hat{\theta}(K, \sigma)$. From Proposition 3 with partial pooling of efforts for both σ_H and σ_L , $\hat{\theta}(K, \sigma_H) \leq \hat{\theta}(K, \sigma_L)$ for given K for both systemic and specific risk. By Assumption 1, $s_{12} > 0$ which implies $h_{12} > 0$ and h_2 negative for $e < e^*(K, \theta)$ implies $h_2(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) < 0$ for $\theta > \hat{\theta}(K, \sigma)$. Moreover, from (1), $e^*(K, \theta)$ is increasing in θ , so $e^*(K, \hat{\theta}(K, \sigma_L)) \geq e^*(K, \hat{\theta}(K, \sigma_H))$ and thus, with $h_{12} > 0$,

$$\int_{\hat{\theta}(K, \sigma_L)}^{\bar{\theta}} h_2(e^*(K, \hat{\theta}(K, \sigma_L)), K, \theta) dF(\theta, \sigma_L) > \int_{\hat{\theta}(K, \sigma_H)}^{\bar{\theta}} h_2(e^*(K, \hat{\theta}(K, \sigma_H)), K, \theta) dF(\theta, \sigma_H).$$

From (35), (40) and, by Assumption 1, the strict concavity of $s(e, K, \theta) - C(K)$, this implies $K^*(\sigma_H) - \hat{K}^G(\sigma_H) > K^*(\sigma_L) - \hat{K}^G(\sigma_L)$. For full pooling of efforts (Part 2 of Proposition 2) for both σ_H and σ_L , for which $\hat{K}^G(\sigma)$ is uniquely determined by (36), the result follows by replacing $\hat{\theta}(K, \sigma)$ in the above by $\underline{\theta}$.

Part 2. With the parties jointly risk neutral, (8) holds with equality. It then follows from (2) that $K^*(\sigma_H) = K^*(\sigma_L)$, so the result follows from the argument in the previous paragraph. ■

Lemma 4 When capital is specific, optimal capital $\hat{K}^S(\sigma)$, if interior to $[0, \bar{K}]$, satisfies:

1. for no pooling of efforts (Part 1 of Proposition 2), $\hat{K}^S(\sigma) = K^*(\sigma)$ given by (35);
2. for full pooling of efforts (Part 2 of Proposition 2),

$$\begin{aligned} & \frac{1}{1-\delta} \left\{ \int_{\underline{\theta}}^{\bar{\theta}} s_2(e^*(\hat{K}^S(\sigma), \theta), \hat{K}^S(\sigma), \theta) dF(\theta, \sigma) \right. \\ & \left. - \int_{\underline{\theta}}^{\bar{\theta}} \left[s_2(e^*(\hat{K}^S(\sigma), \theta), \hat{K}^S(\sigma), \theta) - s_2(\tilde{e}(\hat{K}^S(\sigma), \sigma), \hat{K}^S(\sigma), \theta) \right] dF(\theta, \sigma) \right\} \\ & \times \frac{1}{1 - \frac{\delta}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_1(\tilde{e}(\hat{K}^S(\sigma), \sigma), \hat{K}^S(\sigma), \theta) dF(\theta, \sigma)} - C'(\hat{K}^S(\sigma)) = 0, \quad (41) \\ & \quad \quad \quad c'(\tilde{e}(\hat{K}^S(\sigma), \sigma)) \end{aligned}$$

in which $c'(\tilde{e}(\hat{K}^S(\sigma), \sigma)) > \frac{\delta}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_1(\tilde{e}(\hat{K}^S(\sigma), \sigma), \hat{K}^S(\sigma), \theta) dF(\theta, \sigma);$

3. for partial pooling of efforts (Part 3 of Proposition 2),

$$\begin{aligned}
& \frac{1}{1-\delta} \left\{ \int_{\underline{\theta}}^{\bar{\theta}} s_2(e^*(\hat{K}^S(\sigma), \theta), \hat{K}^S(\sigma), \theta) dF(\theta, \sigma) \right. \\
& \quad + \int_{\hat{\theta}(\hat{K}^S(\sigma), \sigma)}^{\bar{\theta}} \left[s_2\left(e^*\left(\hat{K}^S(\sigma), \hat{\theta}(\hat{K}^S(\sigma), \sigma)\right), \hat{K}^S(\sigma), \theta\right) \right. \\
& \quad \left. \left. - s_2\left(e^*\left(\hat{K}^S(\sigma), \theta\right), \hat{K}^S(\sigma), \theta\right) \right] dF(\theta, \sigma) \right\} \\
& \quad \times \frac{1}{1 - \frac{\delta}{1-\delta} \frac{\int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} s_1\left(e^*\left(\hat{K}^S(\sigma), \hat{\theta}(\hat{K}^S(\sigma), \sigma)\right), \hat{K}^S(\sigma), \theta\right) dF(\theta, \sigma)}{c'\left(e^*\left(\hat{K}^S(\sigma), \hat{\theta}(\hat{K}^S(\sigma), \sigma)\right)\right)}} \\
& \quad - C'(\hat{K}^S(\sigma)) = 0, \quad (42)
\end{aligned}$$

in which

$$\begin{aligned}
& c'\left(e^*\left(\hat{K}^S(\sigma), \hat{\theta}(\hat{K}^S(\sigma), \sigma)\right)\right) \\
& > \frac{\delta}{1-\delta} \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} s_1\left(e^*\left(\hat{K}^S(\sigma), \hat{\theta}(\hat{K}^S(\sigma), \sigma)\right), \hat{K}^S(\sigma), \theta\right) dF(\theta, \sigma).
\end{aligned}$$

The left-hand sides of (35), (41) and (42) are everywhere strictly decreasing in $\hat{K}^S(\sigma)$.

Proof. For no effort pooling (Part 1 of Proposition 2), the relational contract constraint (3) is not binding. Assumption 1 ensures optimal capital is then interior to $[0, \bar{K}]$ and so must satisfy the first-order condition (35).

For full pooling of efforts (Part 2 of Proposition 2), optimal capital is given by

$$\max_{K \in [0, \bar{K}]} \frac{1}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s(\tilde{e}(K, \sigma), K, \theta) dF(\theta, \sigma) - C(K), \text{ subject to } \tilde{e}(K, \sigma) \text{ satisfying (4)}. \quad (43)$$

For specific capital, Definition 3 specifies that $\underline{s}(K, \sigma)$ is independent of K . Differentiation of (4) with respect to K thus gives

$$\frac{\delta}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} [s_1(\tilde{e}(K, \sigma), K, \theta) \tilde{e}_1(K, \sigma) + s_2(\tilde{e}(K, \sigma), K, \theta)] dF(\theta, \sigma) - c'(\tilde{e}(K, \sigma)) \tilde{e}_1(K, \sigma) = 0.$$

From this,

$$\tilde{e}_1(K, \theta) = \frac{\frac{\delta}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_2(\tilde{e}(K, \sigma), K, \theta) dF(\theta, \sigma)}{c'(\tilde{e}(K, \sigma)) - \frac{\delta}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_1(\tilde{e}(K, \sigma), K, \theta) dF(\theta, \sigma)} > 0, \quad (44)$$

the sign following because $s_2 > 0$ everywhere and, if the denominator were not positive, $\tilde{e}(K, \sigma)$ would not be the highest effort satisfying the relational contract constraint (3). The derivative of the maximand in (43) with respect to K is

$$\frac{1}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} \{s_1(\tilde{e}(K, \sigma), K, \theta) \tilde{e}_1(K, \sigma) + s_2(\tilde{e}(K, \sigma), K, \theta)\} dF(\theta, \sigma) - C'(K).$$

Use of (44) to substitute for $\tilde{e}_1(K, \sigma)$ in this gives

$$\begin{aligned} & \frac{1}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_2(\tilde{e}(K, \sigma), K, \theta) dF(\theta, \sigma) \\ & \times \left[\frac{\frac{\delta}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_1(\tilde{e}(K, \sigma), K, \theta) dF(\theta, \sigma)}{c'(\tilde{e}(K, \sigma)) - \frac{\delta}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_1(\tilde{e}(K, \sigma), K, \theta) dF(\theta, \sigma)} + 1 \right] - C'(K), \end{aligned}$$

which can be simplified to

$$\begin{aligned} & \frac{1}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_2(\tilde{e}(K, \sigma), K, \theta) dF(\theta, \sigma) \\ & \times \frac{c'(\tilde{e}(K, \sigma))}{c'(\tilde{e}(K, \sigma)) - \frac{\delta}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_1(\tilde{e}(K, \sigma), K, \theta) dF(\theta, \sigma)} - C'(K). \quad (45) \end{aligned}$$

From this, the first-order condition for interior $\hat{K}^S(\sigma)$ can be written as (41).

To show that the left-hand side of (41) is strictly decreasing in $\hat{K}^S(\sigma)$, differentiate (45) with respect to K . For notational convenience in evaluating this derivative, let

$$x(K, \sigma) = c'(\tilde{e}(K, \sigma)) - \frac{\delta}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_1(\tilde{e}(K, \sigma), K, \theta) dF(\theta, \sigma).$$

The derivative of (45) with respect to K , with arguments suppressed to simplify notation, can then be written

$$\frac{1}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} [s_{12}\tilde{e}_1 + s_{22}] dF \frac{c'}{x} + \frac{1}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_2 dF \frac{1}{x^2} [xc''\tilde{e}_1 - c'x_1] - C''. \quad (46)$$

Note that

$$x_1 = c''\tilde{e}_1 - \frac{\delta}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} [s_{11}\tilde{e}_1 + s_{12}]dF$$

so

$$\begin{aligned} xc''\tilde{e}_1 - c'x_1 &= \left[c' - \frac{\delta}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_1 dF \right] c''\tilde{e}_1 - c' \left[c''\tilde{e}_1 - \frac{\delta}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} [s_{11}\tilde{e}_1 + s_{12}]dF \right] \\ &= \frac{\delta}{1-\delta} \left[c' \int_{\underline{\theta}}^{\bar{\theta}} [s_{11}\tilde{e}_1 + s_{12}]dF - c''\tilde{e}_1 \int_{\underline{\theta}}^{\bar{\theta}} s_1 dF \right]. \end{aligned}$$

With this, (46) becomes

$$\begin{aligned} &\frac{1}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} [s_{12}\tilde{e}_1 + s_{22}] dF \frac{c'}{x} \\ &\quad + \frac{1}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_2 dF \frac{1}{x^2} \frac{\delta}{1-\delta} \left[c' \int_{\underline{\theta}}^{\bar{\theta}} [s_{11}\tilde{e}_1 + s_{12}]dF - c''\tilde{e}_1 \int_{\underline{\theta}}^{\bar{\theta}} s_1 dF \right] - C'' \end{aligned}$$

or, with the substitution $\tilde{e}_1 = \frac{\delta}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} s_2 dF / x$ from (44),

$$\frac{1}{1-\delta} \int_{\underline{\theta}}^{\bar{\theta}} [s_{12}\tilde{e}_1 + s_{22}] dF \frac{c'}{x} + \frac{1}{1-\delta} \frac{1}{x} \tilde{e}_1 \left[c' \int_{\underline{\theta}}^{\bar{\theta}} [s_{11}\tilde{e}_1 + s_{12}]dF - c''\tilde{e}_1 \int_{\underline{\theta}}^{\bar{\theta}} s_1 dF \right] - C''$$

or

$$\frac{1}{1-\delta} \frac{c'}{x} \int_{\underline{\theta}}^{\bar{\theta}} \left\{ s_{11}\tilde{e}_1^2 + 2s_{12}\tilde{e}_1 + s_{22} - \frac{c''}{c'} s_1 \tilde{e}_1 \right\} dF - C''. \quad (47)$$

Strict concavity of y from Assumption 1 implies strict concavity of s , which implies $s_{11}, s_{22} < 0$ and $s_{11}s_{22} > s_{12}^2$. Moreover, with $\sqrt[+]{r}$ used to indicate the larger square root of r for any r ,

$$s_{11}s_{22} > s_{12}^2 \implies s_{12} < \sqrt[+]{s_{11}s_{22}} = \sqrt[+]{(-s_{11})(-s_{22})} = \sqrt[+]{(-s_{11})} \sqrt[+]{(-s_{22})}. \quad (48)$$

Consider the expression $(\sqrt[+]{s_{11}}\tilde{e}_1 - \sqrt[+]{s_{22}})^2$, which is negative because it is the square of an imaginary number.

$$\begin{aligned} (\sqrt[+]{s_{11}}\tilde{e}_1 - \sqrt[+]{s_{22}})^2 &= s_{11}(\tilde{e}_1)^2 - 2\sqrt[+]{s_{11}}\sqrt[+]{s_{22}}\tilde{e}_1 + s_{22} \\ &= s_{11}(\tilde{e}_1)^2 - 2i^2\sqrt[+]{-s_{11}}\sqrt[+]{-s_{22}}\tilde{e}_1 + s_{22} \\ &= s_{11}(\tilde{e}_1)^2 + 2\sqrt[+]{-s_{11}}\sqrt[+]{-s_{22}}\tilde{e}_1 + s_{22}, \end{aligned}$$

the final line following because $i^2 = -1$ by definition. With $s_{12} < \sqrt[+]{(-s_{11})} \sqrt[+]{(-s_{22})}$

from (48) and $\tilde{e}_1 > 0$, this implies

$$s_{11} (\tilde{e}_1)^2 + 2s_{12}\tilde{e}_1 + s_{22} < s_{11} (\tilde{e}_1)^2 + 2\sqrt{-s_{11}}\sqrt{-s_{22}}\tilde{e}_1 + s_{22} < 0.$$

Because $s_1, c', c'', x, C'' > 0$, that implies (47) negative. Thus, the left-hand side of (41) is everywhere strictly decreasing in K .

For partial pooling of efforts (Part 3 of Proposition 2), optimal capital is given by (9) with $\hat{\theta}(K, \sigma)$ the highest $\tilde{\theta}$ satisfying the constraint (5) whatever K is chosen. With specific capital, $\underline{s}(K, \sigma)$ is independent of K by Definition 3. So, because the two integrands in (5) take the same value for $\theta = \hat{\theta}(K, \sigma)$, total differentiation of (5) with respect to K for $\tilde{\theta} = \hat{\theta}(K, \sigma)$ gives

$$\begin{aligned} \frac{\delta}{1-\delta} \left\{ \int_{\underline{\theta}}^{\hat{\theta}(K, \sigma)} \left[s_1(e^*(K, \theta), K, \theta) \frac{\partial e^*(K, \theta)}{\partial K} + s_2(e^*(K, \theta), K, \theta) \right] dF(\theta, \sigma) \right. \\ \left. + \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} \left[s_1(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) \frac{de^*(K, \hat{\theta}(K, \sigma))}{dK} \right. \right. \\ \left. \left. + s_2(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) \right] dF(\theta, \sigma) \right\} - c'(e^*(K, \hat{\theta}(K, \sigma))) \frac{de^*(K, \hat{\theta}(K, \sigma))}{dK} = 0. \end{aligned} \quad (49)$$

Because $s_1(e^*(K, \theta), K, \theta) = 0$ for all (K, θ) by definition of $e^*(K, \theta)$ in (1), this gives

$$\begin{aligned} \frac{de^*(K, \hat{\theta}(K, \sigma))}{dK} &= \frac{\delta}{1-\delta} \left\{ \int_{\underline{\theta}}^{\hat{\theta}(K, \sigma)} s_2(e^*(K, \theta), K, \theta) dF(\theta, \sigma) \right. \\ &\quad \left. + \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} s_2(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) dF(\theta, \sigma) \right\} \\ &\div \left\{ c'(e^*(K, \hat{\theta}(K, \sigma))) - \frac{\delta}{1-\delta} \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} s_1(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) dF(\theta, \sigma) \right\} > 0, \end{aligned} \quad (50)$$

the sign following because $s_2 > 0$ everywhere and, if the denominator were not positive, $e^*(K, \hat{\theta}(K, \sigma))$ would not be the highest effort satisfying the relational contract constraint (3). Because the two integrands in (9) are the same for $\theta = \hat{\theta}(K, \sigma)$, the

derivative of the maximand in (9) with respect to K is

$$\begin{aligned} \frac{1}{1-\delta} \left\{ \int_{\underline{\theta}}^{\hat{\theta}(K,\sigma)} \left[s_1(e^*(K,\theta), K, \theta) \frac{\partial e^*(K,\theta)}{\partial K} + s_2(e^*(K,\theta), K, \theta) \right] dF(\theta, \sigma) \right. \\ \left. + \int_{\hat{\theta}(K,\sigma)}^{\bar{\theta}} \left[s_1(e^*(K, \hat{\theta}(K,\sigma)), K, \theta) \frac{de^*(K, \hat{\theta}(K,\sigma))}{dK} \right. \right. \\ \left. \left. + s_2(e^*(K, \hat{\theta}(K,\sigma)), K, \theta) \right] dF(\theta, \sigma) \right\} - C'(K). \end{aligned}$$

Because $s_1(e^*(K,\theta), K, \theta) = 0$ by definition of $e^*(K,\theta)$ in (1), this can, with the use of (50), be re-arranged as

$$\begin{aligned} \frac{1}{1-\delta} \left[\int_{\underline{\theta}}^{\hat{\theta}(K,\sigma)} s_2(e^*(K,\theta), K, \theta) dF(\theta, \sigma) + \int_{\hat{\theta}(K,\sigma)}^{\bar{\theta}} s_2(e^*(K, \hat{\theta}(K,\sigma)), K, \theta) dF(\theta, \sigma) \right] \\ \times \frac{c'(e^*(K, \hat{\theta}(K,\sigma)))}{c'(e^*(K, \hat{\theta}(K,\sigma))) - \frac{\delta}{1-\delta} \int_{\hat{\theta}(K,\sigma)}^{\bar{\theta}} s_1(e^*(K, \hat{\theta}(K,\sigma)), K, \theta) dF(\theta, \sigma)} - C'(K), \quad (51) \end{aligned}$$

which can be re-written to give the first-order condition (42) for interior $\hat{K}^S(\sigma)$.

To show that the left-hand side of (42) is strictly decreasing in $\hat{K}^S(\sigma)$, differentiate (51) with respect to K . For notational convenience in evaluating this, let

$$z(K, \sigma) = c'(e^*(K, \hat{\theta}(K, \sigma))) - \frac{\delta}{1-\delta} \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} s_1(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) dF(\theta, \sigma). \quad (52)$$

Because the two integrands in (51) are the same for $\theta = \hat{\theta}(K, \sigma)$, the derivative can then be written

$$\begin{aligned} \frac{1}{1-\delta} \left\{ \int_{\underline{\theta}}^{\hat{\theta}(K,\sigma)} \left[s_{12}(e^*(K,\theta), K, \theta) \frac{\partial e^*(K,\theta)}{\partial K} + s_{22}(e^*(K,\theta), K, \theta) \right] dF(\theta, \sigma) \right. \\ \left. + \int_{\hat{\theta}(K,\sigma)}^{\bar{\theta}} \left[s_{12}(e^*(K, \hat{\theta}(K,\sigma)), K, \theta) \frac{de^*(K, \hat{\theta}(K,\sigma))}{dK} \right. \right. \\ \left. \left. + s_{22}(e^*(K, \hat{\theta}(K,\sigma)), K, \theta) \right] dF(\theta, \sigma) \right\} \frac{c'(e^*(K, \hat{\theta}(K,\sigma)))}{z(K, \sigma)} \\ + \frac{1}{1-\delta} \left[\int_{\underline{\theta}}^{\hat{\theta}(K,\sigma)} s_2(e^*(K,\theta), K, \theta) dF(\theta, \sigma) + \int_{\hat{\theta}(K,\sigma)}^{\bar{\theta}} s_2(e^*(K, \hat{\theta}(K,\sigma)), K, \theta) dF(\theta, \sigma) \right] \\ \times \frac{1}{z(K, \sigma)^2} \left[z(K, \sigma) c''(e^*(K, \hat{\theta}(K,\sigma))) \frac{de^*(K, \hat{\theta}(K,\sigma))}{dK} - c'(e^*(K, \hat{\theta}(K,\sigma))) z_1(K, \sigma) \right] \\ - C''(K). \quad (53) \end{aligned}$$

From (52), with $s_1(e^*(K, \hat{\theta}(K, \sigma)), K, \hat{\theta}(K, \sigma)) = 0$ and arguments suppressed to simplify notation,

$$z_1 = c'' \frac{de^*}{dK} - \frac{\delta}{1-\delta} \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} \left[s_{11} \frac{de^*}{dK} + s_{12} \right] dF$$

so

$$\begin{aligned} z c'' \frac{de^*}{dK} - c' z_1 &= \left[c' - \frac{\delta}{1-\delta} \int_{\hat{\theta}}^{\bar{\theta}} s_1 dF \right] c'' \frac{de^*}{dK} \\ &\quad - c' \left[c'' \frac{de^*}{dK} - \frac{\delta}{1-\delta} \int_{\hat{\theta}}^{\bar{\theta}} \left(s_{11} \frac{de^*}{dK} + s_{12} \right) dF \right] \\ &= \frac{\delta}{1-\delta} \left[c' \int_{\hat{\theta}}^{\bar{\theta}} \left(s_{11} \frac{de^*}{dK} + s_{12} \right) dF - \int_{\hat{\theta}}^{\bar{\theta}} s_1 dF c'' \frac{de^*}{dK} \right] \end{aligned}$$

and, from the definition of $e^*(K, \theta)$ in (1),

$$\frac{\partial e^*(K, \theta)}{\partial K} = - \frac{s_{12}(e^*(K, \theta), K, \theta)}{s_{11}(e^*(K, \theta), K, \theta)}.$$

With substitution for these, (53) can be written

$$\begin{aligned} &\frac{1}{1-\delta} \left\{ \int_{\underline{\theta}}^{\hat{\theta}(K, \sigma)} \left[- \frac{s_{12}(e^*(K, \theta), K, \theta)^2}{s_{11}(e^*(K, \theta), K, \theta)} + s_{22}(e^*(K, \theta), K, \theta) \right] dF(\theta, \sigma) \right. \\ &\quad \left. + \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} \left[s_{12}(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) \frac{de^*(K, \hat{\theta}(K, \sigma))}{dK} \right. \right. \\ &\quad \left. \left. + s_{22}(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) \right] dF(\theta, \sigma) \right\} \frac{c'(e^*(K, \hat{\theta}(K, \sigma)))}{z(K, \sigma)} \\ &+ \frac{1}{1-\delta} \left[\int_{\underline{\theta}}^{\hat{\theta}(K, \sigma)} s_2(e^*(K, \theta), K, \theta) dF(\theta, \sigma) + \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} s_2(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) dF(\theta, \sigma) \right] \\ &\times \frac{1}{z(K, \sigma)^2} \frac{\delta}{1-\delta} \left\{ c'(e^*(K, \hat{\theta}(K, \sigma))) \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} \left[s_{11}(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) \frac{de^*(K, \hat{\theta}(K, \sigma))}{dK} \right. \right. \\ &\quad \left. \left. + s_{12}(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) \right] dF(\theta, \sigma) \right. \\ &\quad \left. - \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} s_1(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) dF(\theta, \sigma) c''(e^*(K, \hat{\theta}(K, \sigma))) \frac{de^*(K, \hat{\theta}(K, \sigma))}{dK} \right\} \\ &\quad - C''(K). \quad (54) \end{aligned}$$

But, with the expressions for z in (52) and $\frac{de^*(K, \hat{\theta}(K, \sigma))}{dK}$ in (50),

$$\begin{aligned} \frac{1}{z(K, \sigma)^2} \frac{\delta}{1-\delta} \frac{1}{1-\delta} \left[\int_{\underline{\theta}}^{\hat{\theta}(K, \sigma)} s_2(e^*(K, \theta), K, \theta) dF(\theta, \sigma) \right. \\ \left. + \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} s_2(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) dF(\theta, \sigma) \right] = \frac{z(K, \sigma)}{\delta} \frac{de^*(K, \hat{\theta}(K, \sigma))}{dK}, \end{aligned}$$

so (54) can be written

$$\begin{aligned} \frac{1}{1-\delta} \left\{ \int_{\underline{\theta}}^{\hat{\theta}(K, \sigma)} \left[-\frac{s_{12}(e^*(K, \theta), K, \theta)^2}{s_{11}(e^*(K, \theta), K, \theta)} + s_{22}(e^*(K, \theta), K, \theta) \right] dF(\theta, \sigma) \right. \\ \left. + \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} \left[s_{12}(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) \frac{de^*(K, \hat{\theta}(K, \sigma))}{dK} \right. \right. \\ \left. \left. + s_{22}(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) \right] dF(\theta, \sigma) \right\} \frac{c'(e^*(K, \hat{\theta}(K, \sigma)))}{z(K, \sigma)} \\ + \frac{1}{z(K, \sigma)} \frac{1}{1-\delta} \frac{de^*(K, \hat{\theta}(K, \sigma))}{dK} \left\{ c'(e^*(K, \hat{\theta}(K, \sigma))) \right. \\ \left. \times \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} \left[s_{11}(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) \frac{de^*(K, \hat{\theta}(K, \sigma))}{dK} + s_{12}(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) \right] dF(\theta, \sigma) \right. \\ \left. - \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} s_1(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) dF(\theta, \sigma) c''(e^*(K, \hat{\theta}(K, \sigma))) \frac{de^*(K, \hat{\theta}(K, \sigma))}{dK} \right\} - C''(K). \end{aligned}$$

Collection of terms in the integration interval $[\hat{\theta}(K, \sigma), \bar{\theta}]$ gives

$$\begin{aligned} \frac{1}{1-\delta} \frac{c'(e^*(K, \hat{\theta}(K, \sigma)))}{z(K, \sigma)} \int_{\underline{\theta}}^{\hat{\theta}(K, \sigma)} \left[-\frac{s_{12}(e^*(K, \theta), K, \theta)^2}{s_{11}(e^*(K, \theta), K, \theta)} + s_{22}(e^*(K, \theta), K, \theta) \right] dF(\theta, \sigma) \\ + \frac{1}{1-\delta} \frac{c'(e^*(K, \hat{\theta}(K, \sigma)))}{z(K, \sigma)} \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} \left\{ 2s_{12}(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) \frac{de^*(K, \hat{\theta}(K, \sigma))}{dK} \right. \\ \left. + s_{22}(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) \right. \\ \left. + s_{11}(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) \left(\frac{de^*(K, \hat{\theta}(K, \sigma))}{dK} \right)^2 \right. \\ \left. - s_1(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) \frac{c''(e^*(K, \hat{\theta}(K, \sigma)))}{c'(e^*(K, \hat{\theta}(K, \sigma)))} \left(\frac{de^*(K, \hat{\theta}(K, \sigma))}{dK} \right)^2 \right\} dF(\theta, \sigma) - C''(K). \end{aligned} \tag{55}$$

Strict concavity of y from Assumption 1 implies strict concavity of s , which in turn

implies $s_{11}, s_{22} < 0$ and $s_{11}s_{22} > s_{12}^2$. Moreover, with $s_{11} < 0$,

$$s_{11}s_{22} > s_{12}^2 \implies s_{11}s_{22} - s_{12}^2 > 0 \implies s_{22} - \frac{s_{12}^2}{s_{11}} < 0.$$

Thus the integrand in the first integral in (55) is negative and, because c' and z are strictly positive, the whole term involving this integral is also negative. Moreover, with $\sqrt[+]{r}$ used to indicate the larger square root of r for any r ,

$$s_{11}s_{22} > s_{12}^2 \implies s_{12} < \sqrt[+]{s_{11}s_{22}} = \sqrt[+]{(-s_{11})(-s_{22})} = \sqrt[+]{(-s_{11})} \sqrt[+]{(-s_{22})}. \quad (56)$$

Consider the expression $\left(\sqrt[+]{s_{11}} \frac{de^*}{dK} - \sqrt[+]{s_{22}}\right)^2$, which is negative because it is the square of an imaginary number.

$$\begin{aligned} \left(\sqrt[+]{s_{11}} \frac{de^*}{dK} - \sqrt[+]{s_{22}}\right)^2 &= s_{11} \left(\frac{de^*}{dK}\right)^2 - 2 \sqrt[+]{s_{11}} \sqrt[+]{s_{22}} \frac{de^*}{dK} + s_{22} \\ &= s_{11} \left(\frac{de^*}{dK}\right)^2 - 2i^2 \sqrt[+]{-s_{11}} \sqrt[+]{-s_{22}} \frac{de^*}{dK} + s_{22} \\ &= s_{11} \left(\frac{de^*}{dK}\right)^2 + 2 \sqrt[+]{-s_{11}} \sqrt[+]{-s_{22}} \frac{de^*}{dK} + s_{22} \end{aligned}$$

the final line following because $i^2 = -1$ by definition. With $s_{12} < \sqrt[+]{(-s_{11})} \sqrt[+]{(-s_{22})}$ from (56) and $\frac{de^*}{dK} > 0$, this implies

$$s_{11} \left(\frac{de^*}{dK}\right)^2 + 2s_{12} \frac{de^*}{dK} + s_{22} < s_{11} \left(\frac{de^*}{dK}\right)^2 + 2 \sqrt[+]{-s_{11}} \sqrt[+]{-s_{22}} \frac{de^*}{dK} + s_{22} < 0.$$

Because $s_1, c', c'', z > 0$, that implies that the whole term involving the second integral in (55) is negative and, with $C'' > 0$, so is the whole expression in (55). Thus, the left-hand side of (42) is everywhere strictly decreasing in $\hat{K}^S(\sigma)$.

Finally, note that the left-hand sides of (35), (41) and (42) are continuous in $\hat{K}^S(\sigma)$ so one of these conditions must hold. ■

Proof of Proposition 6. For specific capital, Lemma 4 applies. Assumption 1 does not rule out $\hat{K}^S(\sigma) = \bar{K}$ but, since it ensures both $\hat{K}^G(\sigma)$ and $K^*(\sigma)$ are strictly less than \bar{K} , the proposition certainly holds if $\hat{K}^S(\sigma) = \bar{K}$.

For partial pooling of efforts (Part 3 of Proposition 2), with specific capital $\hat{K}^S(\sigma)$ satisfies (42), whereas with general capital $\hat{K}^G(\sigma)$ satisfies (37), which would imply $\hat{K}^S(\sigma) = \hat{K}^G(\sigma)$ if the fraction multiplying the term in braces in (42) were equal to 1. But it follows from Lemma 4 that this fraction is strictly greater than 1. Moreover, because specific capital relaxes the constraint (3), $\hat{\theta}(K, \sigma)$ is higher for given (K, σ) when capital is specific than when it is general. Furthermore, $s_{12} = y_{12} > 0$ from Assump-

tion 1, which implies that the second integral in (42) is positive given $e^*(K, \hat{\theta}(K, \sigma)) < e^*(K, \theta)$. Thus, the left-hand side of (42) is larger for given K when capital is specific than when it is general so, because that left-hand side is everywhere strictly decreasing in K , it must be that $\hat{K}^S(\sigma) > \hat{K}^G(\sigma)$. For contractible effort, $K^*(\sigma)$ satisfies (35), so (42) implies that $\hat{K}^S(\sigma)$ would converge to $K^*(\sigma)$ as $s_{12} = y_{12}$ converges to zero if the fraction multiplying the term in braces in (42) were equal to 1. Because from Lemma 4 this fraction is strictly greater than 1, this implies $\hat{K}^S(\sigma) > K^*(\sigma)$ for y_{12} sufficiently small.

For full pooling of efforts (Part 2 of Proposition 2), with specific capital, $\hat{K}^S(\sigma)$ satisfies (41), whereas with general capital $\hat{K}^G(\sigma)$ satisfies (36). These have exactly the same forms as (42) and (37) except that $e^*(\hat{K}^i(\sigma), \hat{\theta}(\hat{K}^i(\sigma), \sigma))$ is replaced by $\tilde{e}(\hat{K}^i(\sigma), \sigma)$, and the limits of integration $\hat{\theta}(\hat{K}^i(\sigma), \sigma)$ by $\underline{\theta}$, for $i \in \{S, G\}$ respectively. These differences make no difference to the argument that follows, so the same conclusion applies.

The final case to consider is when, for either specific or general capital, partial pooling of efforts applies while full pooling applies to the other. In this case, because $\hat{\theta}(K, \sigma)$ is higher for given (K, σ) when capital is specific than when it is general, the only possibility is that the full pooling applies to general, but not to specific, capital. It follows from (42) and (36) that $\hat{K}^S(\sigma) > \hat{K}^G(\sigma)$ in this case too. ■

Proof of Proposition 7. For the purpose of the proof, let

$$g(\theta) = 1 - \frac{y_1(e^*(K, \hat{\theta}(K, \sigma)), K, \theta)}{c'(e^*(K, \hat{\theta}(K, \sigma)))}, \quad \text{for given } K, \hat{\theta}(K, \sigma).$$

From the definition of $e^*(K, \theta)$ in (1), $g(\theta) = 0$ for $\theta = \hat{\theta}(K, \sigma)$ and, with $y_{13} > 0$ everywhere by Assumption 1,

$$\begin{aligned} g'(\theta) &= -\frac{y_{13}(e^*(K, \hat{\theta}(K, \sigma)), K, \theta)}{c'(e^*(K, \hat{\theta}(K, \sigma)))} < 0, \\ g''(\theta) &= -\frac{y_{133}(e^*(K, \hat{\theta}(K, \sigma)), K, \theta)}{c'(e^*(K, \hat{\theta}(K, \sigma)))} \leq 0, \quad \text{for } y_{133} \geq 0. \end{aligned}$$

Thus, under the conditions of the proposition, $g(\theta)$ satisfies the conditions of Lemma 1 and

$$\int_{\tilde{\theta}}^{\bar{\theta}} g(\theta) dF(\theta, \sigma_L) \geq \int_{\tilde{\theta}}^{\bar{\theta}} g(\theta) dF(\theta, \sigma_H), \quad \text{for given } K, \tilde{\theta}. \quad (57)$$

For partial pooling of efforts (Part 3 of Proposition 2) for both σ_H and σ_L , $\tilde{\theta}$ corresponds to $\hat{\theta}(K, \sigma)$. Moreover, $g(\theta) < 0$ for $\theta > \hat{\theta}(K, \sigma)$ and, from Proposition 3, $\hat{\theta}(K, \sigma_L) \geq \hat{\theta}(K, \sigma_H)$ for given K for both systemic and specific risk. Furthermore, from (1), $e^*(K, \theta)$ is increasing in θ so $e^*(K, \hat{\theta}(K, \sigma_L)) \geq e^*(K, \hat{\theta}(K, \sigma_H))$ and it fol-

lows from Assumption 1 that $y_1(e, K, \theta) / c'(e)$ is decreasing in e . Together with (57) these imply

$$\begin{aligned} & \int_{\hat{\theta}(K, \sigma_L)}^{\bar{\theta}} \left[1 - \frac{y_1(e^*(K, \hat{\theta}(K, \sigma_L)), K, \theta)}{c'(e^*(K, \hat{\theta}(K, \sigma_L)))} \right] dF(\theta, \sigma_L) \\ & > \int_{\hat{\theta}(K, \sigma_H)}^{\bar{\theta}} \left[1 - \frac{y_1(e^*(K, \hat{\theta}(K, \sigma_H)), K, \theta)}{c'(e^*(K, \hat{\theta}(K, \sigma_H)))} \right] dF(\theta, \sigma_H), \quad \text{for given } K, \end{aligned}$$

so

$$\begin{aligned} & \int_{\hat{\theta}(K, \sigma_H)}^{\bar{\theta}} \left[\frac{y_1(e^*(K, \hat{\theta}(K, \sigma_H)), K, \theta)}{c'(e^*(K, \hat{\theta}(K, \sigma_H)))} - 1 \right] dF(\theta, \sigma_H) \\ & > \int_{\hat{\theta}(K, \sigma_L)}^{\bar{\theta}} \left[\frac{y_1(e^*(K, \hat{\theta}(K, \sigma_L)), K, \theta)}{c'(e^*(K, \hat{\theta}(K, \sigma_L)))} - 1 \right] dF(\theta, \sigma_L), \quad \text{for given } K. \quad (58) \end{aligned}$$

For partial pooling of efforts, (42) applies with either systemic or specific risk. With $s_1(e, K, \theta) = y_1(e, K, \theta) - c'(e)$,

$$\frac{\int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} s_1(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) dF(\theta, \sigma)}{c'(e^*(K, \hat{\theta}(K, \sigma)))} = \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} \left[\frac{y_1(e^*(K, \hat{\theta}(K, \sigma)), K, \theta)}{c'(e^*(K, \hat{\theta}(K, \sigma)))} - 1 \right] dF(\theta, \sigma),$$

so (58) implies

$$\frac{\int_{\hat{\theta}(K, \sigma_H)}^{\bar{\theta}} s_1(e^*(K, \hat{\theta}(K, \sigma_H)), K, \theta) dF(\theta, \sigma_H)}{c'(e^*(K, \hat{\theta}(K, \sigma_H)))} > \frac{\int_{\hat{\theta}(K, \sigma_L)}^{\bar{\theta}} s_1(e^*(K, \hat{\theta}(K, \sigma_L)), K, \theta) dF(\theta, \sigma_L)}{c'(e^*(K, \hat{\theta}(K, \sigma_L)))}.$$

Thus, for given $\hat{K}^S(\sigma)$, the fraction multiplying the term in braces in (42) is larger for σ_H than for σ_L . If this fraction had been equal to 1, (37) and (42) would have implied $\hat{K}^S(\sigma) = \hat{K}^G(\sigma)$. But this fraction is greater than 1 so that it is larger for σ_H than for σ_L which, along with the left hand side of (42) being strictly decreasing in $\hat{K}^S(\sigma)$, implies $\hat{K}^S(\sigma_H) - \hat{K}^G(\sigma_H) > \hat{K}^S(\sigma_L) - \hat{K}^G(\sigma_L)$. Moreover, (37) in Lemma 3 implies that $\hat{K}^G(\sigma)$ approaches $K^*(\sigma)$ as $s_{12} = y_{12}$ approaches zero. So, for $s_{12} = y_{12}$ sufficiently small, $\hat{K}^S(\sigma_H) - K^*(\sigma_H) > \hat{K}^S(\sigma_L) - K^*(\sigma_L)$.

For full pooling of efforts (Part 2 of Proposition 2) for both σ_H and σ_L , (41) applies with either systemic or specific risk. But (41) has exactly the same form as (42) except that the limit of integration $\hat{\theta}(\hat{K}^S(\sigma), \sigma)$ is replaced by $\underline{\theta}$ and $e^*(\hat{K}^S(\sigma), \hat{\theta}(\hat{K}^S(\sigma), \sigma_H))$ by $\tilde{e}(\hat{K}^S(\sigma), \sigma)$. The only difference that makes to the argument for partial pooling is that to get a strict inequality in Lemma 1 requires $y_{133} > 0$. With that change in assumption, the same conclusions apply. ■

Appendix B Derivations for Section 7 (for online)

Derivation of Equation (13). For the specification in (10) and (11)

$$s(e, K, \theta) = \theta^\gamma K^\alpha e^\beta - ce^n. \quad (59)$$

Thus, the first-order condition for optimal choice of effort conditional on K and θ , $e^*(K, \theta)$, is

$$\beta \theta^\gamma K^\alpha e^{*\beta-1}(K, \theta) - nce^{*n-1}(K, \theta) = 0,$$

which can be written

$$\frac{\beta}{e^*(K, \theta)} y(e^*(K, \theta), K, \theta) = nce^{*n-1}(K, \theta)$$

or

$$e^*(K, \theta)^n = \frac{\beta}{nc} y(e^*(K, \theta), K, \theta). \quad (60)$$

Used in the production function (10), this gives

$$y(e^*(K, \theta), K, \theta) = \theta^\gamma K^\alpha \left(\frac{\beta}{nc}\right)^{\frac{\beta}{n}} y(e^*(K, \theta), K, \theta)^{\frac{\beta}{n}},$$

which can be solved for $y(e^*(K, \theta), K, \theta)$ to give

$$y(e^*(K, \theta), K, \theta)^{1-\frac{\beta}{n}} = \theta^\gamma K^\alpha \left(\frac{\beta}{nc}\right)^{\frac{\beta}{n}}$$

or

$$y(e^*(K, \theta), K, \theta) = \left[\theta^\gamma K^\alpha \left(\frac{\beta}{nc}\right)^{\frac{\beta}{n}} \right]^{\frac{1}{1-\beta/n}}.$$

Thus, from (60),

$$e^*(K, \theta)^n = \frac{\beta}{nc} \left[\theta^\gamma K^\alpha \left(\frac{\beta}{nc}\right)^{\frac{\beta}{n}} \right]^{\frac{1}{1-\beta/n}} = \left(\frac{\beta}{nc} \theta^\gamma K^\alpha \right)^{\frac{1}{1-\beta/n}}, \quad (61)$$

from which (13) follows.

Derivation of Equations (14) and (15). Use of (61) in (59) gives

$$\begin{aligned} s(e^*(K, \theta), K, \theta) &= \theta^\gamma K^\alpha \left(\frac{\beta}{nc} \theta^\gamma K^\alpha \right)^{\frac{\beta/n}{1-\beta/n}} - c \left(\frac{\beta}{nc} \theta^\gamma K^\alpha \right)^{\frac{1}{1-\beta/n}} \\ &= \left(\frac{\beta}{nc} \right)^{\frac{\beta/n}{1-\beta/n}} (\theta^\gamma K^\alpha)^{\frac{1}{1-\beta/n}} - c \frac{\beta}{nc} \left(\frac{\beta}{nc} \right)^{\frac{\beta/n}{1-\beta/n}} (\theta^\gamma K^\alpha)^{\frac{1}{1-\beta/n}}, \end{aligned}$$

from which (14) follows. Moreover, from (59) and (61) for given $\tilde{\theta}$,

$$\begin{aligned} s(e^*(K, \tilde{\theta}), K, \theta) &= \theta^\gamma K^\alpha e^*(K, \tilde{\theta})^\beta - c e^*(K, \tilde{\theta})^n \quad (62) \\ &= \theta^\gamma K^\alpha \left(\frac{\beta}{nc} \tilde{\theta}^\gamma K^\alpha \right)^{\frac{\beta}{n} \frac{1}{1-\beta/n}} - c \left(\frac{\beta}{nc} \tilde{\theta}^\gamma K^\alpha \right)^{\frac{1}{1-\beta/n}} \\ &= \left(\frac{\beta}{nc} \tilde{\theta}^\gamma K^\alpha \right)^{\frac{1}{1-\beta/n}} \left[\theta^\gamma K^\alpha \left(\frac{\beta}{nc} \tilde{\theta}^\gamma K^\alpha \right)^{\left(\frac{\beta}{n}-1\right) \frac{1}{1-\beta/n}} - c \right] \\ &= \left(\frac{\beta}{nc} \tilde{\theta}^\gamma K^\alpha \right)^{\frac{1}{1-\beta/n}} \left[\theta^\gamma K^\alpha \left(\frac{\beta}{nc} \tilde{\theta}^\gamma K^\alpha \right)^{-1} - c \right], \end{aligned}$$

from which (15) follows.

Derivation of Equations (18) and (19). With the definition of $\hat{S}(\sigma)$ in (17) and the specification of the effort cost function in (11), (5) can be written

$$\hat{S}(\sigma) = c e^*(K, \tilde{\theta})^n, \text{ for } \tilde{\theta} = \hat{\theta}(K, \sigma), \quad (63)$$

which is the form taken by the constraint on the choice of optimal capital in (9) when capital is general. Use of (63) in (62) for $\tilde{\theta} = \hat{\theta}(K, \sigma)$ gives

$$\begin{aligned} s(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) &= \theta^\gamma K^\alpha \left(\frac{\hat{S}(\sigma)}{c} \right)^{\frac{\beta}{n}} - c \frac{\hat{S}(\sigma)}{c} \\ &= \theta^\gamma K^\alpha \left(\frac{\hat{S}(\sigma)}{c} \right)^{\frac{\beta}{n}} - \hat{S}(\sigma). \end{aligned}$$

The derivative of this with respect to K is

$$\frac{\partial}{\partial K} s(e^*(K, \hat{\theta}(K, \sigma)), K, \theta) = \theta^\gamma \alpha K^{\alpha-1} \left(\frac{\hat{S}(\sigma)}{c} \right)^{\frac{\beta}{n}}.$$

Moreover, from (14),

$$\begin{aligned}\frac{\partial}{\partial K} s(e^*(K, \theta), K, \theta) &= \left(1 - \frac{\beta}{n}\right) \left(\frac{\beta}{nc}\right)^{\frac{\beta/n}{1-\beta/n}} \theta^{\frac{\gamma}{1-\beta/n}} \frac{\alpha}{1-\beta/n} K^{\frac{\alpha}{1-\beta/n}-1} \\ &= \alpha \left(\frac{\beta}{nc}\right)^{\frac{\beta/n}{1-\beta/n}} \theta^{\frac{\gamma}{1-\beta/n}} K^{\frac{\alpha}{1-\beta/n}-1}.\end{aligned}$$

So the derivative with respect to K of the maximand in (9) with constraint (5) in the form (63) is, because the two integrands take the same value for $\theta = \hat{\theta}(K, \sigma)$,

$$\begin{aligned}\frac{\alpha}{1-\delta} \left\{ \left(\frac{\beta}{nc}\right)^{\frac{\beta/n}{1-\beta/n}} K^{\frac{\alpha}{1-\beta/n}-1} \int_{\underline{\theta}}^{\hat{\theta}(K, \sigma)} \theta^{\frac{\gamma}{1-\beta/n}} dF(\theta, \sigma) \right. \\ \left. + K^{\alpha-1} \left(\frac{\hat{S}(\sigma)}{c}\right)^{\frac{\beta}{n}} \int_{\hat{\theta}(K, \sigma)}^{\bar{\theta}} \theta^{\gamma} dF(\theta, \sigma) \right\} - kCK^{k-1}.\end{aligned}$$

The first-order condition for optimal K is obtained by setting this equal to zero which, divided through by K^{k-1} gives the condition for $\hat{K}^G(\sigma)$

$$\begin{aligned}\frac{\alpha}{1-\delta} \left\{ \left(\frac{\beta}{nc}\right)^{\frac{\beta/n}{1-\beta/n}} \hat{K}^G(\sigma)^{\frac{\alpha}{1-\beta/n}-k} \int_{\underline{\theta}}^{\hat{\theta}(\hat{K}^G(\sigma), \sigma)} \theta^{\frac{\gamma}{1-\beta/n}} dF(\theta, \sigma) \right. \\ \left. + \hat{K}^G(\sigma)^{\alpha-k} \left(\frac{\hat{S}(\sigma)}{c}\right)^{\frac{\beta}{n}} \int_{\hat{\theta}(\hat{K}^G(\sigma), \sigma)}^{\bar{\theta}} \theta^{\gamma} dF(\theta, \sigma) \right\} = kC. \quad (64)\end{aligned}$$

As noted in the main text, it is convenient for calibration to write this in terms of $\hat{\theta}^G(\sigma)$ instead of $\hat{K}^G(\sigma)$. To do this note that, with (13), the constraint (63) can be written

$$\hat{S}(\sigma) = c \left(\frac{\beta}{nc} \hat{\theta}(K, \sigma)^{\gamma} K^{\alpha} \right)^{\frac{1}{1-\beta/n}},$$

or

$$\left(\frac{\hat{S}(\sigma)}{c} \right)^{1-\beta/n} = \frac{\beta}{nc} \hat{\theta}(K, \sigma)^{\gamma} K^{\alpha}.$$

Hence, it must be that

$$\hat{K}^G(\sigma) = \left(\frac{\hat{S}(\sigma)}{c} \right)^{\frac{1-\beta/n}{\alpha}} \left(\frac{\beta}{nc} \right)^{-\frac{1}{\alpha}} \hat{\theta}^G(\sigma)^{-\frac{\gamma}{\alpha}}. \quad (65)$$

Use of this in (64) and multiplication through by $(1 - \delta) / \alpha$ gives the first-order condition for optimal general capital in terms of $\hat{\theta}^G(\sigma)$ as

$$\begin{aligned} & \left(\frac{\beta}{nc} \right)^{\frac{\beta/n}{1-\beta/n}} \left[\left(\frac{\hat{S}(\sigma)}{c} \right)^{\frac{1-\beta/n}{\alpha}} \left(\frac{\beta}{nc} \right)^{-\frac{1}{\alpha}} \hat{\theta}^G(\sigma)^{-\frac{\gamma}{\alpha}} \right]^{\frac{\alpha}{1-\beta/n}-k} \int_{\underline{\theta}}^{\hat{\theta}^G(\sigma)} \theta^{\frac{\gamma}{1-\beta/n}} dF(\theta, \sigma) \\ & + \left[\left(\frac{\hat{S}(\sigma)}{c} \right)^{\frac{1-\beta/n}{\alpha}} \left(\frac{\beta}{nc} \right)^{-\frac{1}{\alpha}} \hat{\theta}^G(\sigma)^{-\frac{\gamma}{\alpha}} \right]^{\alpha-k} \left(\frac{\hat{S}(\sigma)}{c} \right)^{\frac{\beta}{n}} \int_{\hat{\theta}^G(\sigma)}^{\bar{\theta}} \theta^{\gamma} dF(\theta, \sigma) = (1 - \delta) \frac{kC}{\alpha} \end{aligned}$$

or

$$\begin{aligned} & \left(\frac{\beta}{nc} \right)^{\frac{\beta/n}{1-\beta/n}-\frac{1}{\alpha}\left(\frac{\alpha}{1-\beta/n}-k\right)} \left(\frac{\hat{S}(\sigma)}{c} \right)^{\frac{1-\beta/n}{\alpha}\left(\frac{\alpha}{1-\beta/n}-k\right)} \hat{\theta}^G(\sigma)^{-\frac{\gamma}{\alpha}\left(\frac{\alpha}{1-\beta/n}-k\right)} \int_{\underline{\theta}}^{\hat{\theta}^G(\sigma)} \theta^{\frac{\gamma}{1-\beta/n}} dF(\theta, \sigma) \\ & + \left(\frac{\beta}{nc} \right)^{-\frac{1}{\alpha}(\alpha-k)} \left(\frac{\hat{S}(\sigma)}{c} \right)^{\frac{1-\beta/n}{\alpha}(\alpha-k)+\frac{\beta}{n}} \hat{\theta}^G(\sigma)^{-\frac{\gamma}{\alpha}(\alpha-k)} \int_{\hat{\theta}^G(\sigma)}^{\bar{\theta}} \theta^{\gamma} dF(\theta, \sigma) = (1 - \delta) \frac{kC}{\alpha} \end{aligned}$$

or

$$\begin{aligned} & \left(\frac{\beta}{nc} \right)^{\frac{\beta/n-1}{1-\beta/n}+\frac{k}{\alpha}} \left(\frac{\hat{S}(\sigma)}{c} \right)^{1-\left(1-\frac{\beta}{n}\right)\frac{k}{\alpha}} \hat{\theta}^G(\sigma)^{\gamma\left(\frac{k}{\alpha}-\frac{1}{1-\beta/n}\right)} \int_{\underline{\theta}}^{\hat{\theta}^G(\sigma)} \theta^{\frac{\gamma}{1-\beta/n}} dF(\theta, \sigma) \\ & + \left(\frac{\beta}{nc} \right)^{\frac{k}{\alpha}-1} \left(\frac{\hat{S}(\sigma)}{c} \right)^{1-\frac{\beta}{n}-\left(1-\frac{\beta}{n}\right)\frac{k}{\alpha}+\frac{\beta}{n}} \hat{\theta}^G(\sigma)^{\gamma\frac{k}{\alpha}-\gamma} \int_{\hat{\theta}^G(\sigma)}^{\bar{\theta}} \theta^{\gamma} dF(\theta, \sigma) = (1 - \delta) \frac{kC}{\alpha} \end{aligned}$$

or

$$\begin{aligned} & \left(\frac{\beta}{nc} \right)^{\frac{k}{\alpha}-1} \left(\frac{\hat{S}(\sigma)}{c} \right)^{1-\left(1-\frac{\beta}{n}\right)\frac{k}{\alpha}} \hat{\theta}^G(\sigma)^{\gamma\left(\frac{k}{\alpha}-\frac{1}{1-\beta/n}\right)} \int_{\underline{\theta}}^{\hat{\theta}^G(\sigma)} \theta^{\frac{\gamma}{1-\beta/n}} dF(\theta, \sigma) \\ & + \left(\frac{\beta}{nc} \right)^{\frac{k}{\alpha}-1} \left(\frac{\hat{S}(\sigma)}{c} \right)^{1-\left(1-\frac{\beta}{n}\right)\frac{k}{\alpha}} \hat{\theta}^G(\sigma)^{\gamma\frac{k}{\alpha}-\gamma} \int_{\hat{\theta}^G(\sigma)}^{\bar{\theta}} \theta^{\gamma} dF(\theta, \sigma) = (1 - \delta) \frac{kC}{\alpha} \end{aligned}$$

or, multiplied through by $(\beta/nc)^{1-k/\alpha} c^{1-(1-\beta/n)k/\alpha}$,

$$\begin{aligned} & \hat{S}(\sigma)^{1-\left(1-\frac{\beta}{n}\right)\frac{k}{\alpha}} \hat{\theta}^G(\sigma)^{\gamma\left(\frac{k}{\alpha}-\frac{1}{1-\beta/n}\right)} \left\{ \int_{\underline{\theta}}^{\hat{\theta}^G(\sigma)} \theta^{\frac{\gamma}{1-\beta/n}} dF(\theta, \sigma) \right. \\ & \left. + \hat{\theta}^G(\sigma)^{\gamma\left(\frac{1}{1-\beta/n}-1\right)} \int_{\hat{\theta}^G(\sigma)}^{\bar{\theta}} \theta^{\gamma} dF(\theta, \sigma) \right\} = (1 - \delta) \frac{kC}{\alpha} \left(\frac{\beta}{nc} \right)^{1-\frac{k}{\alpha}} c^{1-\left(1-\frac{\beta}{n}\right)\frac{k}{\alpha}}. \end{aligned}$$

With the calibration $\gamma = 1 - \beta/n$, this becomes

$$\hat{S}(\sigma)^{1 - \left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha}} \hat{\theta}^G(\sigma)^{\left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha} - 1} \left\{ \int_{\underline{\theta}}^{\hat{\theta}^G(\sigma)} \theta dF(\theta, \sigma) + \hat{\theta}^G(\sigma)^{\frac{\beta}{n}} \int_{\hat{\theta}^G(\sigma)}^{\bar{\theta}} \theta^{1 - \frac{\beta}{n}} dF(\theta, \sigma) \right\} = (1 - \delta) \frac{kC}{\alpha} \left(\frac{\beta}{n}\right)^{1 - \frac{k}{\alpha}} c^{\frac{\beta}{n} \frac{k}{\alpha}}.$$

Note that the right-hand side of this is independent of σ and that the left-hand side can be written as the expression in (18). (19) follows directly from taking the ratio of the expression in (65) for σ_H to that for σ_L along with the calibration $\gamma = 1 - \beta/n$.

Effect of change in specific risk on general capital. The following argument shows that the effect of a change in specific risk on general capital is greater in magnitude than that of a change in systemic risk for $\left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha} > 1$ as with the parameter values in Table 1. It follows from (18) that, for two different levels of risk σ_H and σ_L ,

$$\begin{aligned} & \hat{\theta}^G(\sigma_H)^{\left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha} - 1} \left\{ E(\theta | \sigma_H) - \int_{\hat{\theta}^G(\sigma_H)}^{\bar{\theta}} \left[\theta - \hat{\theta}^G(\sigma_H)^{\frac{\beta}{n}} \theta^{1 - \frac{\beta}{n}} \right] dF(\theta, \sigma_H) \right\} \\ &= \left(\frac{\hat{S}(\sigma_H)}{\hat{S}(\sigma_L)} \right)^{\left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha} - 1} \hat{\theta}^G(\sigma_L)^{\left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha} - 1} \left\{ E(\theta | \sigma_L) - \int_{\hat{\theta}^G(\sigma_L)}^{\bar{\theta}} \left[\theta - \hat{\theta}^G(\sigma_L)^{\frac{\beta}{n}} \theta^{1 - \frac{\beta}{n}} \right] dF(\theta, \sigma_L) \right\}. \end{aligned} \quad (66)$$

For $\left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha} > 1$, the total derivative of this with respect to $\hat{S}(\sigma_H) / \hat{S}(\sigma_L)$ for given $\hat{\theta}^G(\sigma_L)$ has the same sign as the total derivative with respect to $x \equiv [\hat{S}(\sigma_H) / \hat{S}(\sigma_L)]^{\left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha} - 1}$ for given $\kappa > 0$ of

$$\hat{\theta}^G(\sigma_H)^{\left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha} - 1} \left\{ E(\theta | \sigma_H) - \int_{\hat{\theta}^G(\sigma_H)}^{\bar{\theta}} \left[\theta - \hat{\theta}^G(\sigma_H)^{\frac{\beta}{n}} \theta^{1 - \frac{\beta}{n}} \right] dF(\theta, \sigma_H) \right\} = \kappa x.$$

Because the integrand in this equals zero for $\theta = \hat{\theta}^G(\sigma_H)$, this total derivative is

$$\begin{aligned} & \left[\left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha} - 1 \right] \hat{\theta}^G(\sigma_H)^{\left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha} - 2} \frac{d\hat{\theta}^G(\sigma_H)}{dx} \left\{ E(\theta, \sigma_H) \right. \\ & \quad \left. - \int_{\hat{\theta}^G(\sigma_H)}^{\bar{\theta}} \left[\theta - \hat{\theta}^G(\sigma_H)^{\frac{\beta}{n}} \theta^{1 - \frac{\beta}{n}} \right] dF(\theta, \sigma_H) \right\} \\ & + \hat{\theta}^G(\sigma_H)^{\left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha} - 1} \left\{ - \int_{\hat{\theta}^G(\sigma_H)}^{\bar{\theta}} \left[-\frac{\beta}{n} \hat{\theta}^G(\sigma_H)^{\frac{\beta}{n} - 1} \theta^{1 - \frac{\beta}{n}} \frac{d\hat{\theta}^G(\sigma_H)}{dx} \right] dF(\theta, \sigma_H) \right\} = \kappa \end{aligned}$$

or

$$\begin{aligned} \frac{d\hat{\theta}^G(\sigma_H)}{dx} \left[\left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha} - 1 \right] \hat{\theta}^G(\sigma_H)^{(1-\frac{\beta}{n})\frac{k}{\alpha}-2} & \left\{ E(\theta, \sigma_H) \right. \\ & \left. - \int_{\hat{\theta}^G(\sigma_H)}^{\bar{\theta}} \theta \left[1 - \left(\frac{\hat{\theta}^G(\sigma_H)}{\theta} \right)^{\frac{\beta}{n}} \right] dF(\theta, \sigma_H) \right\} \\ & + \frac{d\hat{\theta}^G(\sigma_H)}{dx} \frac{\beta}{n} \hat{\theta}^G(\sigma_H)^{(1-\frac{\beta}{n})\frac{k}{\alpha}+\frac{\beta}{n}-2} \int_{\hat{\theta}^G(\sigma_H)}^{\bar{\theta}} \theta^{1-\frac{\beta}{n}} dF(\theta, \sigma_H) = \kappa \end{aligned}$$

or

$$\begin{aligned} \frac{d\hat{\theta}^G(\sigma_H)}{dx} \hat{\theta}^G(\sigma_H)^{(1-\frac{\beta}{n})\frac{k}{\alpha}-2} & \left\{ \left[\left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha} - 1 \right] \left[E(\theta, \sigma_H) \right. \right. \\ & \left. \left. - \int_{\hat{\theta}^G(\sigma_H)}^{\bar{\theta}} \theta \left(1 - \left(\frac{\hat{\theta}^G(\sigma_H)}{\theta} \right)^{\frac{\beta}{n}} \right) dF(\theta, \sigma_H) \right] \right. \\ & \left. + \frac{\beta}{n} \hat{\theta}^G(\sigma_H)^{\frac{\beta}{n}} \int_{\hat{\theta}^G(\sigma_H)}^{\bar{\theta}} \theta^{1-\frac{\beta}{n}} dF(\theta, \sigma_H) \right\} = \kappa. \end{aligned}$$

For $\left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha} > 1$, the terms in square brackets in this are both positive and thus so is the term in braces. Since $\kappa > 0$, this implies $d\hat{\theta}^G(\sigma_H)/dx > 0$. Since for specific risk $\hat{S}(\sigma_H)/\hat{S}(\sigma_L)$ is smaller than for systemic risk when σ_H is more risky than σ_L , this implies that $\hat{\theta}^G(\sigma_H)$ is lower for specific than for systemic risk for given $\hat{\theta}^G(\sigma_L)$. From (19) and (66),

$$\begin{aligned} \left[\frac{\hat{K}^G(\sigma_L)}{\hat{K}^G(\sigma_H)} \right]^{k-\frac{\alpha}{1-\beta/n}} & \left\{ E(\theta, \sigma_H) - \int_{\hat{\theta}^G(\sigma_H)}^{\bar{\theta}} \theta \left[1 - \left(\frac{\hat{\theta}^G(\sigma_H)}{\theta} \right)^{\frac{\beta}{n}} \right] dF(\theta, \sigma_H) \right\} \\ & = E(\theta, \sigma_L) - \int_{\hat{\theta}^G(\sigma_L)}^{\bar{\theta}} \theta \left[1 - \left(\frac{\hat{\theta}^G(\sigma_L)}{\theta} \right)^{\frac{\beta}{n}} \right] dF(\theta, \sigma_L). \end{aligned}$$

The right-hand side of this is independent of $\hat{S}(\sigma_H)/\hat{S}(\sigma_L)$ for given $\hat{\theta}^G(\sigma_L)$, so the left-hand side must be too. Thus, for $\left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha} > 1$, the sign of the effect on $\hat{K}^G(\sigma_L)/\hat{K}^G(\sigma_H)$ of an increase in $\hat{S}(\sigma_H)/\hat{S}(\sigma_L)$ must be the opposite of the sign of the effect of the increase on

$$E(\theta, \sigma_H) - \int_{\hat{\theta}^G(\sigma_H)}^{\bar{\theta}} \theta \left[1 - \left(\frac{\hat{\theta}^G(\sigma_H)}{\theta} \right)^{\frac{\beta}{n}} \right] dF(\theta, \sigma_H),$$

which, given $d\hat{\theta}^G(\sigma_H)/dx > 0$, has the same sign as the effect of an increase in $\hat{\theta}^G(\sigma_H)$ for given $E(\theta, \sigma_H)$. Because the integrand equals zero for $\theta = \hat{\theta}^G(\sigma_H)$, the derivative of this expression with respect to $\hat{\theta}^G(\sigma_H)$ is

$$\int_{\hat{\theta}^G(\sigma_H)}^{\bar{\theta}} \frac{\beta}{n} \left(\frac{\hat{\theta}^G(\sigma_H)}{\theta} \right)^{\frac{\beta}{n}-1} dF(\theta, \sigma_H) > 0.$$

So an increase in $\hat{S}(\sigma_H)/\hat{S}(\sigma_L)$ reduces $\hat{K}^G(\sigma_L)/\hat{K}^G(\sigma_H)$. Thus, since specific risk has a lower value of $\hat{S}(\sigma_H)/\hat{S}(\sigma_L)$ than systemic risk, it has a higher value of $\hat{K}^G(\sigma_L)/\hat{K}^G(\sigma_H)$ and, hence a lower value of $\hat{K}^G(\sigma_H)$ for given $\hat{K}^G(\sigma_L)$.

Concavity of first derivative term. To derive conditions under which $h_2(e, K, \theta)$ is concave in θ , use the simplified notation without arguments for the function s but * to indicate that the first argument is $e^*(K, \theta)$. So, for example,

$$s_{ij} = s_{ij}(e, K, \theta); \quad s_{ij}^* = s_{ij}(e^*(K, \theta), K, \theta).$$

Then

$$\begin{aligned} h_{233}(e, K, \theta) &= s_{233} - s_{223}^* - s_{123}^* e_2^* - s_{123}^* e_2^* - s_{112}^* e_2^{*2} - s_{12}^* e_{22}^* - s_{113}^* e_1^* e_2^* - s_{11}^* e_{12}^* e_2^* - s_{11}^* e_1^* e_{22}^* \\ &\quad - s_{113}^* e_1^* e_2^{*2} - s_{13}^* e_{12}^* - s_{11}^* e_{122}^* - s_{11}^* e_1^* e_{12}^* - s_{133}^* e_1^* - s_{13}^* e_{12}^* - s_{113}^* e_1^{*2} \\ &= s_{233} - s_{223}^* - 2s_{123}^* e_2^* - s_{112}^* e_2^{*2} - s_{12}^* e_{22}^* - s_{113}^* e_1^* e_2^* - s_{11}^* e_{12}^* e_2^* - s_{11}^* e_1^* e_{22}^* \\ &\quad - s_{113}^* e_1^* e_2^{*2} - 2s_{13}^* e_{12}^* - s_{11}^* e_1^* e_{12}^* - s_{133}^* e_1^* - s_{113}^* e_1^{*2} \\ &= s_{233} - s_{223}^* - 2s_{123}^* e_2^* - s_{112}^* e_2^{*2} - s_{113}^* e_1^* e_2^* - (s_{12}^* + s_{11}^* e_1^*) e_{22}^* - s_{113}^* e_1^* e_2^{*2} \\ &\quad - (s_{11}^* e_2^* + 2s_{13}^* + s_{11}^* e_1^*) e_{12}^* - s_{133}^* e_1^* - s_{113}^* e_1^{*2}, \end{aligned}$$

the second equality following because $s_1^* = 0$. Moreover,

$$\begin{aligned} e_1^*(K, \theta) &= -\frac{s_{12}(e^*(K, \theta), K, \theta)}{s_{11}(e^*(K, \theta), K, \theta)} \\ e_2^*(K, \theta) &= -\frac{s_{13}(e^*(K, \theta), K, \theta)}{s_{11}(e^*(K, \theta), K, \theta)} \\ e_{12}^*(K, \theta) &= -\frac{1}{s_{11}(e^*(K, \theta), K, \theta)^2} \{s_{11}(e^*(K, \theta), K, \theta) [s_{113}(e^*(K, \theta), K, \theta) e_2^*(K, \theta) \\ &\quad + s_{133}(e^*(K, \theta), K, \theta)] - s_{13}(e^*(K, \theta), K, \theta) [s_{111}(e^*(K, \theta), K, \theta) e_2^*(K, \theta) \\ &\quad + s_{113}(e^*(K, \theta), K, \theta)]\}. \end{aligned}$$

From these.

$$e_{22}^* = \frac{1}{s_{11}^{*2}} \{s_{11}^* [s_{133}^* + s_{113}^* e_2^*] - s_{13}^* [s_{113}^* + s_{111}^* e_2^*]\}. \quad (67)$$

With the use of these

$$\begin{aligned}
h_{233}(e, K, \theta) = & s_{233} - s_{223}^* + 2s_{123}^* \frac{s_{13}^*}{s_{11}^*} - s_{112}^* \left(-\frac{s_{13}^*}{s_{11}^*} \right)^2 - s_{113}^* \frac{s_{12}^* s_{13}^*}{s_{11}^* s_{11}^*} + s_{113}^* \frac{s_{12}^*}{s_{11}^*} \left(-\frac{s_{13}^*}{s_{11}^*} \right)^2 \\
& - \left(s_{12}^* - s_{11}^* \frac{s_{12}^*}{s_{11}^*} \right) \frac{1}{s_{11}^{*2}} \left\{ s_{11}^* \left[s_{133}^* - s_{113}^* \frac{s_{13}^*}{s_{11}^*} \right] - s_{13}^* \left[s_{113}^* - s_{111}^* \frac{s_{13}^*}{s_{11}^*} \right] \right\} \\
& - \left(-s_{11}^* \frac{s_{13}^*}{s_{11}^*} + 2s_{13}^* - s_{11}^* \frac{s_{12}^*}{s_{11}^*} \right) \frac{1}{s_{11}^{*2}} \left[s_{11}^* \left(-s_{113}^* \frac{s_{13}^*}{s_{11}^*} + s_{133}^* \right) - s_{13}^* \left(-s_{111}^* \frac{s_{13}^*}{s_{11}^*} + s_{113}^* \right) \right] \\
& + s_{133}^* \frac{s_{12}^*}{s_{11}^*} - s_{113}^* \left(-\frac{s_{12}^*}{s_{11}^*} \right)^2
\end{aligned}$$

or

$$\begin{aligned}
h_{233}(e, K, \theta) = & s_{233} - s_{223}^* + 2s_{123}^* \frac{s_{13}^*}{s_{11}^*} - s_{112}^* \left(-\frac{s_{13}^*}{s_{11}^*} \right)^2 - s_{113}^* \frac{s_{12}^* s_{13}^*}{s_{11}^* s_{11}^*} + s_{113}^* \frac{s_{12}^*}{s_{11}^*} \left(-\frac{s_{13}^*}{s_{11}^*} \right)^2 \\
& - (s_{13}^* - s_{12}^*) \frac{1}{s_{11}^{*2}} \left[-2s_{13}^* s_{113}^* + s_{11}^* s_{133}^* + s_{111}^* \frac{s_{13}^{*2}}{s_{11}^*} \right] + s_{133}^* \frac{s_{12}^*}{s_{11}^*} - s_{113}^* \left(-\frac{s_{12}^*}{s_{11}^*} \right)^2
\end{aligned}$$

or

$$\begin{aligned}
h_{233}(e, K, \theta) = & s_{233} - s_{223}^* + 2s_{123}^* \frac{s_{13}^*}{s_{11}^*} - s_{112}^* \left(-\frac{s_{13}^*}{s_{11}^*} \right)^2 - s_{113}^* \frac{s_{12}^* s_{13}^*}{s_{11}^* s_{11}^*} + s_{113}^* \frac{s_{12}^*}{s_{11}^*} \left(-\frac{s_{13}^*}{s_{11}^*} \right)^2 \\
& + \frac{2s_{13}^{*2} s_{113}^*}{s_{11}^{*2}} - \frac{s_{13}^* s_{133}^*}{s_{11}^*} - s_{111}^* \frac{s_{13}^3}{s_{11}^{*3}} \\
& - \frac{2s_{12}^* s_{13}^* s_{113}^*}{s_{11}^{*2}} + \frac{s_{12}^* s_{133}^*}{s_{11}^*} + s_{12}^* s_{111}^* \frac{s_{13}^{*2}}{s_{11}^{*3}} + s_{133}^* \frac{s_{12}^*}{s_{11}^*} - s_{113}^* \left(-\frac{s_{12}^*}{s_{11}^*} \right)^2
\end{aligned}$$

or

$$\begin{aligned}
h_{233}(e, K, \theta) = & s_{233} - s_{223}^* + 2s_{123}^* \frac{s_{13}^*}{s_{11}^*} - s_{112}^* \left(-\frac{s_{13}^*}{s_{11}^*} \right)^2 - 3s_{113}^* \frac{s_{12}^* s_{13}^*}{s_{11}^* s_{11}^*} + s_{113}^* \frac{s_{12}^*}{s_{11}^*} \left(-\frac{s_{13}^*}{s_{11}^*} \right)^2 \\
& + 2 \frac{s_{13}^{*2} s_{113}^*}{s_{11}^{*2}} - \frac{s_{13}^* s_{133}^*}{s_{11}^*} - s_{111}^* \frac{s_{13}^3}{s_{11}^{*3}} + 2 \frac{s_{12}^* s_{133}^*}{s_{11}^*} + s_{12}^* s_{111}^* \frac{s_{13}^{*2}}{s_{11}^{*3}} - s_{113}^* \left(-\frac{s_{12}^*}{s_{11}^*} \right)^2
\end{aligned}$$

or

$$\begin{aligned}
h_{233}(e, K, \theta) = & s_{233} - s_{223}^* + 2s_{123}^* \frac{s_{13}^*}{s_{11}^*} - s_{112}^* \left(-\frac{s_{13}^*}{s_{11}^*} \right)^2 + s_{111}^* \frac{s_{13}^{*2}}{s_{11}^{*3}} [s_{12}^* - s_{13}^*] \\
& + \frac{s_{113}^*}{s_{11}^{*2}} \left[-3s_{12}^* s_{13}^* + \frac{s_{12}^*}{s_{11}^*} s_{13}^{*2} + 2s_{13}^{*2} - s_{12}^{*2} \right] + \frac{s_{133}^*}{s_{11}^*} [2s_{12}^* - s_{13}^*].
\end{aligned}$$

Appendix C Calculations for calibration (for online)

All numerical calculations were done with Scientific WorkPlace 6.0.30.

C.1 Bloom et al. (2018) calibrations

In Bloom et al. (2018), total factor productivity (TFP) is the product of two components, an aggregate component that affects all firms and an idiosyncratic component that affects an individual firm. The log of each component of TFP is assumed to follow a first-order autoregressive process with the variance of innovations constant within each of two regimes with Markov switching between regimes.

In the formulation here with stationary TFP, the autoregressive formulation with innovations normally distributed corresponds to θ being lognormal. Table V in Bloom et al. (2018) gives values of $\sigma_L^A = 0.0067$ for the standard deviation of the aggregate component (L refers to the low-risk regime, H to the high-risk regime) and $\sigma_L^Z = 0.051$ for that of the idiosyncratic component for the log autoregressive process, with $\sigma_H^A/\sigma_L^A = 1.6$ and $\sigma_H^Z/\sigma_L^Z = 4.1$. These apply to TFP with exponent 1 in the revenue function, whereas here TFP is given by θ^γ . The standard errors for $\ln \theta$ in the calibration here need to be adjusted accordingly. For $X \sim N(\mu, \sigma^2)$, $aX \sim N(a\mu, a^2\sigma^2)$ for $a > 0$. For $X_H \sim N(\mu_1, \sigma_H^2)$ and $X_L \sim N(\mu_2, \sigma_L^2)$ independent, $X_H + X_L \sim N(\mu_1 + \mu_2, \sigma_H^2 + \sigma_L^2)$. With these formulae, $\gamma \ln \theta \sim N(\mu_1 + \mu_2, \sigma_j^{A^2} + \sigma_j^{Z^2})$ for $j = L, H$, so $\ln \theta \sim N(\mu_1 + \mu_2, \frac{1}{\gamma^2}(\sigma_j^{A^2} + \sigma_j^{Z^2}))$. The values for σ_L and σ_H in Table 1 are calculated using this formula with $\gamma = 1 - \beta/n$ as used in the calibration.

A calibration for the other parameters corresponding to Bloom et al. (2018, Table IV) is

$$k = 1, \alpha = \frac{1}{4}, \beta = \frac{1}{2}, n = 1 \implies \left(1 - \frac{\beta}{n}\right) \frac{k}{\alpha} = \frac{1}{2} \times 4 = 2.$$

The values of α and β are derived from a labour share of 2/3, a capital share of 1/3 and an isoelastic demand with 33% markup.

C.2 Lognormal distribution

When x is log-normally distributed, $\ln x \sim N(\mu, \sigma^2)$ and the probability density function for x is given by

$$\frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}.$$

The mean and variance are given by

$$E(x) = e^{(\mu + \sigma^2/2)}; \text{var}(x) = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}.$$

Note that changing σ changes the mean unless μ is changed correspondingly. The calibration uses $x = \theta/E(\theta|\sigma)$ lognormal with mean 1, which requires $\mu = -\sigma^2/2$. In that case the variance is just $e^{\sigma^2} - 1$. With that specification, the probability density

function becomes

$$\frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(\ln x + \sigma^2/2)^2}{2\sigma^2}}.$$

C.3 General capital

Let $x^G(\sigma) = \hat{\theta}^G(\sigma) / E(\theta|\sigma)$. For $x = \theta / E(\theta|\sigma)$ lognormal with mean 1, the expression in (18) becomes

$$\begin{aligned} & \hat{S}(\sigma)^{-1} E(\theta|\sigma)^{\left(1-\frac{\beta}{n}\right)\frac{k}{\alpha}-1} x^G(\sigma)^{\left(1-\frac{\beta}{n}\right)\frac{k}{\alpha}-1} \\ & \left\{ E(\theta|\sigma) - E(\theta|\sigma) \int_{x^G(\sigma)}^{\infty} \left[x - x^G(\sigma)^{\frac{\beta}{n}} x^{1-\frac{\beta}{n}} \right] \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x + \sigma^2/2)^2}{2\sigma^2}} dx \right\} \\ & = \hat{S}(\sigma)^{-1} E(\theta|\sigma)^{\left(1-\frac{\beta}{n}\right)\frac{k}{\alpha}} x^G(\sigma)^{\left(1-\frac{\beta}{n}\right)\frac{k}{\alpha}-1} \\ & \left\{ 1 - \int_{x^G(\sigma)}^{\infty} \left(1 - x^G(\sigma)^{\frac{\beta}{n}} x^{-\frac{\beta}{n}} \right) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\ln x + \sigma^2/2)^2}{2\sigma^2}} dx \right\}. \end{aligned}$$

The results in Table 2 for systemic risk with $\hat{S}(\sigma)$ and $E(\theta|\sigma)$ independent of σ and the parameter values in Table 1, are calculated in the following way. $\hat{\theta}^G(\sigma_H)$ for given $\hat{\theta}^G(\sigma_L)$ can be calculated by solving

$$\begin{aligned} & x^G(\sigma_H) - x^G(\sigma_H) \int_{x^G(\sigma_H)}^{\infty} \left(1 - x^G(\sigma_H)^{\frac{1}{2}} x^{-\frac{1}{2}} \right) \frac{1}{4.07\sigma\sqrt{2\pi}} e^{-\frac{\left(\ln x + \frac{(4.07\sigma)^2}{2}\right)^2}{2(4.07\sigma)^2}} dx \\ & = x^G(\sigma_L) - x^G(\sigma_L) \int_{x^G(\sigma_L)}^{\infty} \left(1 - x^G(\sigma_L)^{\frac{1}{2}} x^{-\frac{1}{2}} \right) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\left(\ln x + \frac{\sigma^2}{2}\right)^2}{2\sigma^2}} dx \end{aligned}$$

for $\sigma \equiv \sigma_L = 0.10$. The change in capital can be calculated directly from (19) as

$$\frac{\hat{K}^G(\sigma_H)}{\hat{K}^G(\sigma_L)} = \left[\frac{x^G(\sigma_L)}{x^G(\sigma_H)} \right]^2.$$

The ratio of productivity with σ_H to that with σ_L can then be calculated directly from (16) with this and the specified parameter calibrations as

$$\frac{x^G(\sigma_L)}{x^G(\sigma_H)} \frac{1 - \int_{x^G(\sigma_H)}^{\infty} \left(x - x^G(\sigma_H)^{\frac{\beta}{n}} x^{1-\frac{\beta}{n}} \right) \frac{1}{x4.07\sigma\sqrt{2\pi}} e^{-\frac{\left(\ln x + \frac{(4.07\sigma)^2}{2}\right)^2}{2(4.07\sigma)^2}} dx}{1 - \int_{x^G(\sigma_L)}^{\infty} \left(x - x^G(\sigma_L)^{\frac{\beta}{n}} x^{1-\frac{\beta}{n}} \right) \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{\left(\ln x + \frac{\sigma^2}{2}\right)^2}{2\sigma^2}} dx}.$$

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