Graphical Electrical Circuit Theory

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Abstract

Most physical systems are naturally compositional and relational, yet our standard mathematical tools rarely reflect that structure. Standard mathematical notation favors terms and functions over relations, making relational reasoning second-class.

This changed with the recent development of string diagrams. This 2-dimensional syntax provides the extra flexibility needed for a practical calculus of relations. We build specifically on the development of Graphical Linear Algebra (GLA), a relational calculus for finite-dimensional linear algebra.

Based on this we introduce Graphical Electrical Circuit Theory, a string diagrammatic formalism for analyzing electrical circuits. In this framework, we show that we can reproduce standard results practically. We then extend Graphical Linear Algebra itself to go beyond linear relations to piecewise-linear relations. Using this more expressive semantic domain, we add diodes and transistors to our framework.

Acknowledgements

Above all I thank my family. Etienne, Marie, Anne-Claire, Papa et Maman, je vous aime de tout mon coeur. Quelle joie de vous avoir dans ma vie. Votre soutien inébranlable m'a porté jusqu'ici et me porte toujours. Merci tout particulièrement Papa pour m'avoir recueilli quand j'étais au plus bas et que je ne savais pas appeler à l'aide. Tous les cinq, je marche sur le chemin à vos cotés.

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Over the more than 6 years from the start of this journey I have lived in many places and met many people. Warm thoughts and thanks to all my housemates over the years. You were there for my antics and struggles. Vassilios, Hlér and Karl, you became like brothers. Deep gratitude also for St Anne's college MCR, you were my ground. To the innumerable names and faces who would make me feel like I'm home.

These 6 years were not a straight path. I stumbled and fell, I had to fight to keep going. To all of you who held my hand on this journey I owe eternal gratitude. Keziah, the first to see through me and trust me to go on. Talou and Stéphanie, for your patience and wisdom. Irem, for creating a gorgeous container of dance where I found so much beauty in myself. Marion, for giving me space to listen. Cam, for your support in the final rush of writing.

These 6 years were ultimately a time of deep personal transformation. Merci à vous les patients des Pages pour m'avoir accueilli tel que j'étais. Thank you to my tpot friends, your insane ramblings have changed my life, and led me to a path of dance and play. With you I started to find my people. And to all the wonderful friends who watched me grow from close or afar. Obé, Dan, Alex, Fabiana, Elena, Nasta, Lorella, Sílvia, Georgica, Ananta, Iliyana, even you Babar.

I'd finally like to thank the thesis herself, for trusting me to bring her into being. This thesis has been the greatest challenge to my creative process, my self-coercion, my self-trust. I come out a radically different man. I am deeply grateful for the space this thesis afforded me.

I thank all beings for sharing this dance with me. May we all find peace.

Finally, I thank the EPSRC for funding this work under grant OUCS/GB/1034913.

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GAA Graphical Affine Algebra 3, 9, 10, 12–16, 25–33, 35, 36, 38–41, 46, 49, 55, 85, 95, 96, 104, 105

GADA Graphical Differential Affine Algebra 29, 30, 33–36, 102, 104, 105, 108

GPA Graphical Polyhedral Algebra 9, 12–16, 85–90, 104, 106

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Introduction

1.1 Motivation

Functional thinking underpins most scientific models. Nature, however, does not distinguish inputs and outputs—physical systems are governed by laws that merely express *relations* between their observable variables. While influential scientists, like the famous control theorist J. Willems, have pointed out the blind spots of functional thinking [Wil07], it has remained the dominant paradigm in science and engineering. Arguably, our mathematical practice, especially the limitations of standard algebraic syntax, is partially to blame for the persistence of this status quo.

Recent developments especially in category theory open up a new possibility. Formal graphical languages based on *string diagrams* have been developed that can conveniently represent and reason about relations. Along with these, category theory brings powerful tools for compositional thinking, where systems can be composed and decomposed into parts.

In this thesis, we take the compositional and relational approach to heart and apply it to the study of electrical and electronic circuits. The basis of our work is Graphical Linear Algebra (GLA) [BSZ17], a relational calculus for finite-dimensional linear algebra. We use its graphical syntax to write the equations that describe the physical behavior of electrical circuits.

Using the tools of category theory, we build a compositional framework to analyze circuits, in which we explore analogues of standard results of the field. In the process we also extend GLA to suit our expressivity needs.

This is not a new idea: Baez and Fong [BF18] modeled passive networks using decorated cospans, and Baez, Coya and Rebro [BCR18] modeled passive and active networks using props. Bonchi et al. [BPSZ19] later reformulated the latter approach by interpreting into graphical affine algebra. Our approach starts from Bonchi et al.'s basic idea and extends it in several directions.

1.2 Roadmap and Original Contributions

In Chapter 2 we define the basic structures and theorems we will use throughout the thesis. None of it is original work.

In Chapter 3 we first repurpose GAA to be able to talk about linear differential equations. We then define a prop of electric circuits and build some reasoning tools around it.

- In Theorems 3.1 and 3.2 we show that non-trivial semantics for GLA and GAA are necessarily complete. There are new results;
- In Theorems 3.5 and 3.8 we interpret GLA and GAA into a general vector space and prove completeness of this interpretation. Though it is likely already known, this is the first time it appears on paper;

- In Section 3.1.2 we use $\mathsf{GLA}_{\mathbb{R}(s)}$ to describe linear differential equations and prove its completeness using Laplace transforms. The idea of using $\mathsf{GLA}_{\mathbb{R}(s)}$ with Laplace transforms for differential equations is not new, see for example [BE15]. Our approach is different however, as we provide it we a clear functional semantics which we then prove complete;
- The definition of ECirc (Section 3.2) is based on Bonchi et al.'s [BPSZ19];
- The extension of ECirc to a colored prop is original, as well as all but the most basic of the remaining results of this chapter;
- We note nonetheless that many of these results are inspired by their known analogues in standard electrical circuit theory, as can be found in a standard textbook [DK69];
- We note and appreciate Paweł Sobociński's help with this work.
- Some of the results of this chapter and the next have already been published in [BS21].

In Chapter 4, we derive global properties of ECirc. Most of them are inspired by their classical analogues but this chapter is otherwise fully original work.

- In Section 4.2 we completely analyze circuits made of impedances and wiring;
- In Section 4.3 we tackle the problem that our framework contains circuits with short-circuits, which are not physically sensible;
- Finally in Section 4.4 we derive important standard theorems including the superposition theorem and Thévenin's theorem.

In Chapter 5, we extend the graphical language and our electrical circuits to be able to describe diodes and other electronic components. All of it is original work. Completeness for GPLA was derived in collaboration with Robin Piedeleu as well as various Twitter and Zulip users, most notably Reid Barton. Some of the results of this chapter have already been published in [BP21].

The many diagrams in this thesis were typeset with the help of TikZIt [Kis20].

1.3 Prerequisites

This thesis makes heavy use of string diagrams in props and monoidal mappings between categories. We also assume some familiarity with the setting of GLA and common graphical reasoning techniques. A good introduction to props and their presentations, as well as Graphical Linear Algebra (GLA), can be found in [FS18, Chapter 5].

2

Background

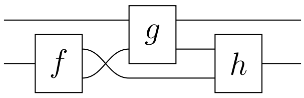
2.1 Props and Their Presentations

The mathematical backbone of our approach is the notion of product and permutations category (prop), a structure which generalises standard algebraic theories [BSZ18]. We will use two variants of props in this thesis. We briefly recall their definitions.

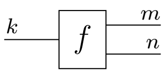
Definition 2.1 ([Mac65])**.** A *prop* (products and permutations category) is a strict symmetric monoidal category with objects freely generated from a singleton. A morphism of props is a strict symmetric monoidal functor that preserves the generating object.

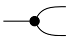
We draw arrows of a prop using *string diagrams*. The specificity of props is that every wire is made of n copies of a same generating wire type, and all crossings and permutations are allowed. In fact we exclusively work with *compact closed* props in this thesis, where wires can additionally be bent 180°.

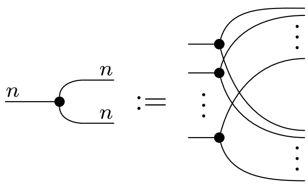
Example 2.2. An arrow may look like:



Remark 2.3. A wire in a picture may stand for any number of copies of the generating wire. When it is not clear from context, we specify the "thickness" of the wire like such:



When a generator such as  is present on a wire of thickness n, this is shorthand for n copies of the generator:



We note 0 copies of the wire by omitting a wire altogether. We draw the empty diagram (the identity on zero copies of the generating wire) as .

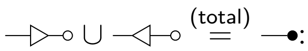
Like standard algebraic theories, a prop can be described by *generators and relations*, i.e. as the free prop generated by a chosen set of arrows, quotiented by a chosen set of equations between two diagrams built from these generators. Most of our studies of props will consist in showing equalities between diagrams, derived from the generating equations of the relevant prop.

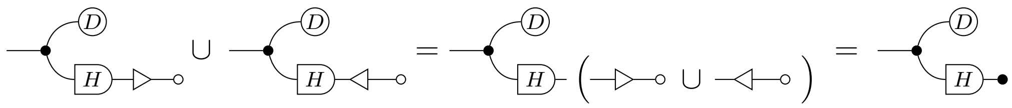
In this thesis we use two extensions of props: ordered props and ∪-props.

Definition 2.4 ([BHPS17, Definition 12])**.** An *ordered prop* is a prop enriched over the category of posets, i.e. a prop whose arrows of a given type can be compared via a partial order. A morphism of ordered props is a monotone morphism of props.

In ordered props, on top of the usual reasoning on equalities between arrows, we can talk about inclusions between arrows.

Definition 2.5 ([BP21, Section 3])**.** A ∪-*prop* is an ordered prop enriched over the monoidal category of join-semi-lattices (partially-ordered sets with least upper bounds for any finite subset) and join-preserving maps, with the Cartesian product as monoidal product. In other words a ∪-prop is a prop with a notion of *union* of arrows. A morphism of ∪-props is a join-preserving morphism of ordered props.

In ∪-props we also have inclusions, and moreover diagrams can include a term-like ∪ operation. A typical reasoning sequence looks like the following, where the last step applies the equality :



The structure of ∪-props ensures that ∪ distributes with composition and tensor as expected.

Just like props, an ordered prop and a ∪-prop can be described by generators and relations. The difference being that these equations can be inequalities, and in the case of ∪-props can use ∪ on either side of the equation.

Definition 2.6. A prop presented by generators and equations is called *sound* for a given mapping of its generators if that mapping respects the equations. It is called *complete* if the functor resulting from this mapping is faithful.

Definition 2.7. A *colored prop* is a strict symmetric monoidal category with objects freely generated from a finite set, called its *colors*. A morphism of colored props is a strict symmetric monoidal functor that maps generating objects to generating objects.

All the above definitions readily extend to colored analogues.

2.2 Graphical Linear Algebra And Its Extensions

We will calculate with graphical equations throughout this thesis. We present here the graphical settings in which we will be working.

2.2.1 The Graphical Algebra Family

Graphical Algebra is a family of graphical theories that provide complete axiomatizations of some classes of relations. It is the base on which we build this thesis. In this section we present $\mathsf{GLA}_{\mathbb{K}}$, $\mathsf{GAA}_{\mathbb{K}}$ and $\mathsf{GPA}_{\mathbb{K}}$. In Chapter 5 we add a fourth member, $\mathsf{GPLA}_{\mathbb{K}}$, to this family.

Definition 2.8. Given a set X, Rel_X is the $\cup$-prop

- whose arrows $n \to m$ are subsets $R \subseteq X^n \times X^m$,
- with identity $n \to n$ given by the diagonal $\{(x, x) \mid x \in X^n\}$,
- symmetry $n + m \to m + n$ given by the relation

$$\left\{ \left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} y \\ x \end{pmatrix} \right) \mid (x, y) \in X^n \times X^m \right\}$$

- composition given by $R; S = \{(x, z) \mid \exists y.\ (x, y) \in R \wedge (y, z) \in S\}$, for $R : n \to m$, $S : m \to l$,
- monoidal product given by

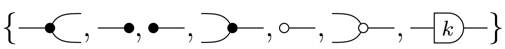
$$R_1 \oplus R_2 = \left\{ \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \mid (x_1, y_1) \in R_1 \wedge (x_2, y_2) \in R_2 \right\}$$

  for $R_1 : n_1 \to m_1$ and $R_2 : n_2 \to m_2$,
- and join given by set union.

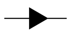
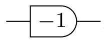
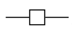
Remark 2.9. X^0 is a singleton set, whose only element we note $\bullet$.

X will typically be a field or vector space. In what follows we fix a field $\mathbb{K}$.

Definition 2.10. $\mathsf{GLA}_{\mathbb{K}}$ (Graphical Linear Algebra) is the ordered prop generated by the generators:



for $k \in \mathbb{K}$, modulo the axioms of Figure 2.1.

Remark 2.11. The original $\mathsf{GLA}_{\mathbb{K}}$ paper [BSZ17] uses  as shorthand for . We prefer to draw it as . Moreover, what we call $\mathsf{GLA}_{\mathbb{K}}$ they call $\mathbb{IH}_{\mathbb{K}}$.

Definition 2.12. $[\![\cdot]\!]_{\mathsf{GLA}_{\mathbb{K}}} : \mathsf{GLA}_{\mathbb{K}} \to \mathsf{Rel}_{\mathbb{K}}$ is the functor of ordered props defined on generators by Figure 2.2. The fact that $[\![\cdot]\!]_{\mathsf{GLA}_{\mathbb{K}}}$ respects the axioms of $\mathsf{GLA}_{\mathbb{K}}$ is shown throughout [BSZ17].

Theorem 2.13 ([BHPS17, Theorem 15])**.** $[\![\cdot]\!]_{\mathsf{GLA}_{\mathbb{K}}}$ *is faithful and its image is the wide subprop of* $\mathsf{Rel}_{\mathbb{K}}$ *on those subsets* R *that are linear subspaces (defined as* $\mathsf{LinRel}_{\mathbb{K}}$ *in [BHPS17, Section 4]). In other words,* $\mathsf{GLA}_{\mathbb{K}}$ *can represent any linear relation and completely reason about their equivalence.*

Proof. The original proof of completeness can be found in [BSZ17, Theorem 6.4] but only applies to $\mathsf{GLA}_{\mathbb{K}}$ as a non-ordered prop. [BHPS17, Theorem 15] proves completeness as an ordered prop. □

Definition 2.14. $\mathsf{GAA}_{\mathbb{K}}$ (Graphical Affine Algebra) is the ordered prop generated by the generators of $\mathsf{GLA}_{\mathbb{K}}$ and  modulo the axioms of Figures 2.1 and 2.3.

Remark 2.15. What we call $\mathsf{GAA}_{\mathbb{K}}$ is called $\mathrm{AIH}_{\mathbb{K}}$ in the original paper [BPSZ19].

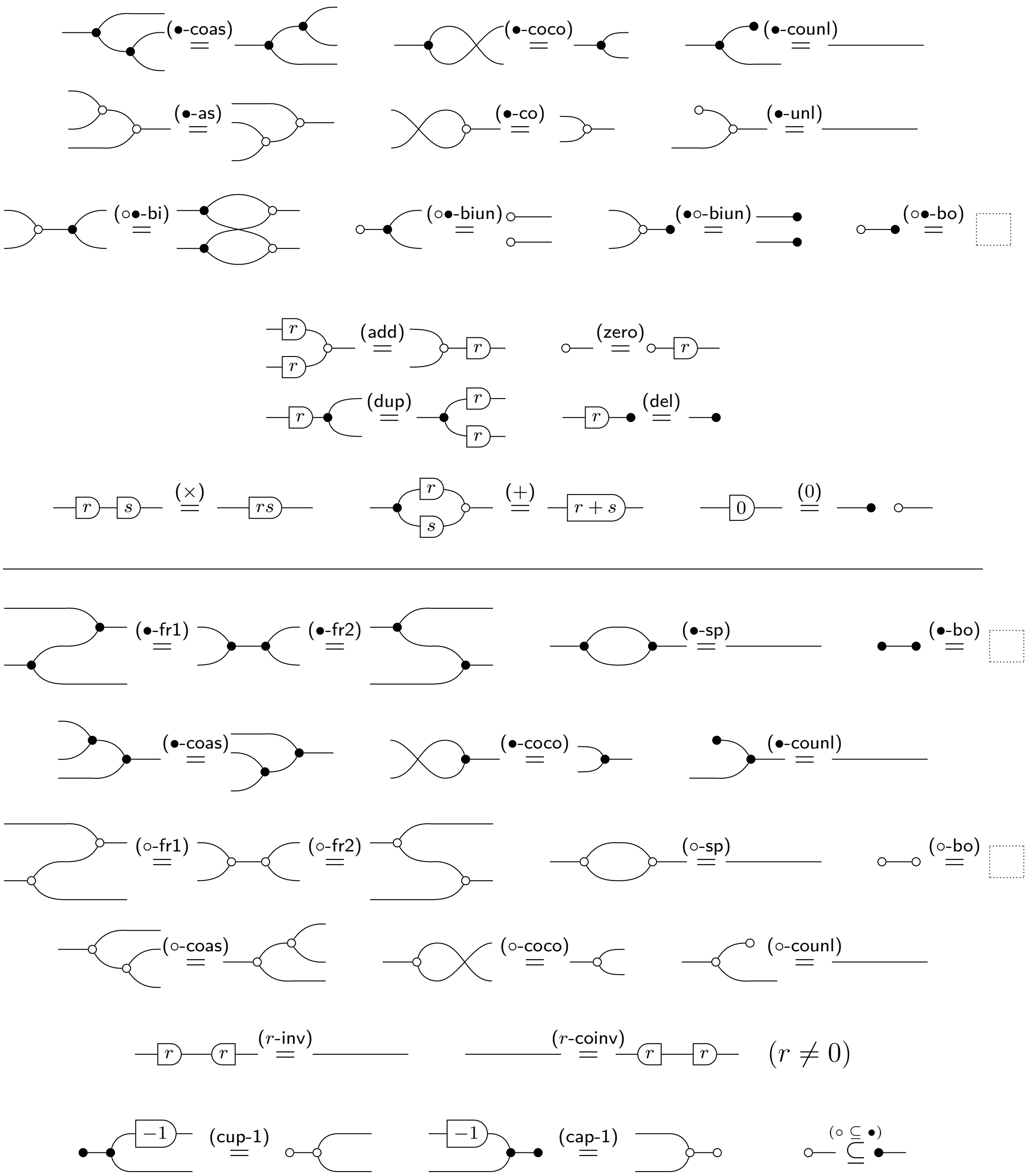


Figure 2.1: Axioms of GLA$_{\mathbb{K}}$.

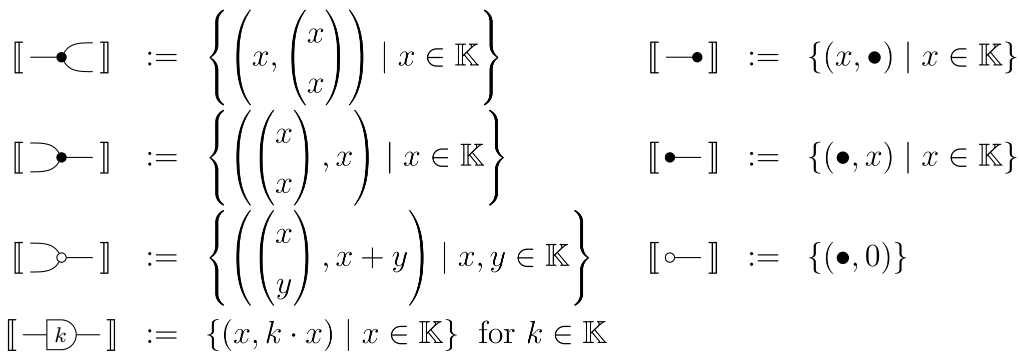


Figure 2.2: Semantics of $\mathsf{GLA}_{\mathbb{K}}$.

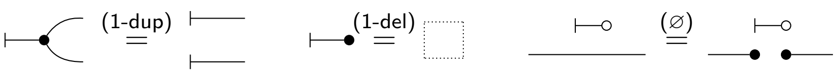


Figure 2.3: Additional axioms of $\mathsf{GAA}_{\mathbb{K}}$.

Definition 2.16. $[\![\cdot]\!]_{\mathsf{GAA}_{\mathbb{K}}} : \mathsf{GAA}_{\mathbb{K}} \to \mathsf{Rel}_{\mathbb{K}}$ is the functor of ordered props that extends $[\![\cdot]\!]_{\mathsf{GLA}_{\mathbb{K}}}$ with the additional equation

$$[\![\vdash\!\!-]\!] := \{(\bullet, 1)\}$$

The fact that $[\![\cdot]\!]_{\mathsf{GAA}_{\mathbb{K}}}$ respects the axioms of $\mathsf{GAA}_{\mathbb{K}}$ is shown in [BPSZ19, Section IV.A].

Theorem 2.17 ([BPSZ19, Theorem 18])**.** $[\![\cdot]\!]_{\mathsf{GAA}_{\mathbb{K}}}$ is faithful and its image is the wide subprop of $\mathsf{Rel}_{\mathbb{K}}$ on those subsets R that are affine subspaces (defined as $\mathsf{AffRel}_{\mathbb{K}}$ in [BPSZ19, Definition 4]). In other words, $\mathsf{GAA}_{\mathbb{K}}$ can represent any affine relation and completely reason about their equivalence.

When discussing $\mathsf{GPA}_{\mathbb{K}}$ we will from now on assume that $\mathbb{K}$ is an ordered field, as defined in [BDGS21, Section 4].

Definition 2.18. $\mathsf{GPA}_{\mathbb{K}}$ (Graphical Polyhedral Algebra) is the ordered prop generated by the generators of $\mathsf{GAA}_{\mathbb{K}}$ and $-\!\triangleright\!-$ modulo the axioms of Figures 2.1, 2.3 and 2.4.

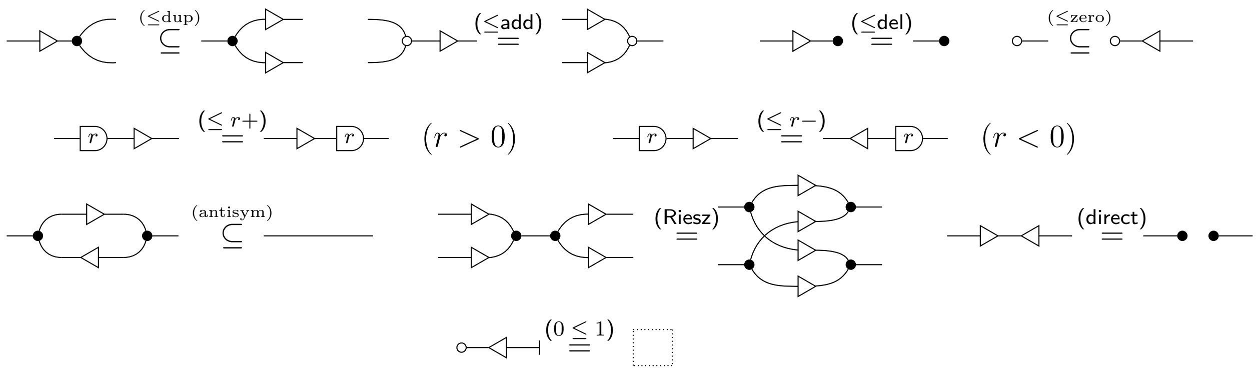


Figure 2.4: Additional axioms of $\mathsf{GPA}_{\mathbb{K}}$.

Remark 2.19. The original GPA paper [BDGS21] uses —[≥⟩— for the comparison generator; we prefer to draw it as —⊳—. Moreover, what we call $\mathsf{GPA}_{\mathbb{K}}$ they call $\mathsf{a}\mathbb{IH}^{\geq}_{\mathbb{K}}$.

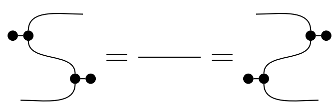
Definition 2.20. $[\![\cdot]\!]_{\mathsf{GPA}_{\mathbb{K}}} : \mathsf{GPA}_{\mathbb{K}} \to \mathsf{Rel}_{\mathbb{K}}$ is the functor of ordered props that extends $[\![\cdot]\!]_{\mathsf{GAA}_{\mathbb{K}}}$ with the additional equation

$$[\![\text{—⊳—}]\!] := \{(x, y) \in \mathbb{K} \times \mathbb{K} \mid x \geq y\}$$

The fact that $[\![\cdot]\!]_{\mathsf{GPA}_{\mathbb{K}}}$ respects the axioms of $\mathsf{GPA}_{\mathbb{K}}$ is shown throughout [BDGS21].

Theorem 2.21 ([BDGS21, Corollary 27])**.** $[\![\cdot]\!]_{\mathsf{GPA}_{\mathbb{K}}}$ is faithful and its image is the wide subprop of $\mathsf{Rel}_{\mathbb{K}}$ on those subsets R that are polyhedra (defined as $\mathrm{P}_{\mathbb{K}}$ in [BDGS21, Definition 2]). In other words, $\mathsf{GPA}_{\mathbb{K}}$ can represent any polyhedral relation and completely reason about their equivalence.

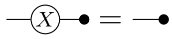
Theorem 2.22. $\mathsf{GLA}_{\mathbb{K}}$, $\mathsf{GAA}_{\mathbb{K}}$, $\mathsf{GPA}_{\mathbb{K}}$ and Rel_{X} are compact closed.



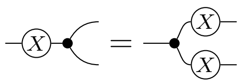
2.2.2 Maps In Graphical Algebra

$\mathsf{GLA}_{\mathbb{K}}$, $\mathsf{GAA}_{\mathbb{K}}$, $\mathsf{GPA}_{\mathbb{K}}$ and Rel_X all have the structure of a cartesian bicategory of relations [CW87], meaning that they feature a black (co)monoid structure that behaves nicely. In such a setting, one can talk about *maps*.

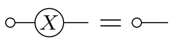
Definition 2.23. A morphism in a cartesian bicategory of relations is *total* if



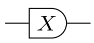
Definition 2.24. A morphism in a cartesian bicategory of relations is *single-valued* if



Proposition 2.25. In $\mathsf{GLA}_{\mathbb{K}}$, single-valuedness is equivalently stated as:

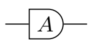
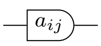


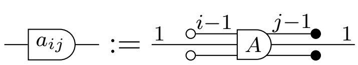
Proof. This is [PS20, Theorem 9] modulo the symmetry of [PS20, Theorem 8]. □

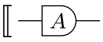
Definition 2.26. A morphism in a cartesian bicategory of relations is a *map* if it is single-valued and total. We use the special notation  to denote maps.

In our relational settings, these correspond to the standard notion of a map: a mapping that takes each value from its codomain to a value in its domain. More specifically, maps in GLA are the linear maps, and maps in GAA and GPA are the affine maps.

As we well know, a linear map (between finite-dimensional spaces as is the case here) can be described by its coefficients and represented as a matrix.

Proposition 2.27. In $\mathsf{GLA}_{\mathbb{K}}$, given a map , its (i, j)th coefficient is the $1 \to 1$ map  defined as:

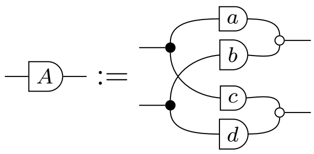


These coefficients correspond to the entries of the matrix $[\![$  $]\!]_{\mathsf{GLA}_{\mathbb{K}}}$.

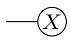
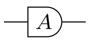
Proposition 2.28. In $\mathsf{GLA}_{\mathbb{K}}$, a map verifies:

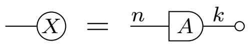
Using Proposition 2.28, a map can be written as a combination of its coefficients.

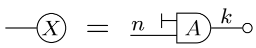
Example 2.29. The matrix $A := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ can be written:

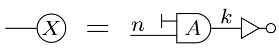


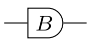
Maps are the basis of normal forms for our various kinds of relations.

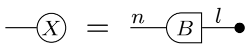
Theorem 2.30 (Normal forms)**.** Given a $n \to 0$ arrow  in $\mathsf{GLA}_{\mathbb{K}}$, $\mathsf{GAA}_{\mathbb{K}}$ or $\mathsf{GPA}_{\mathbb{K}}$, there exists a linear map  such that:

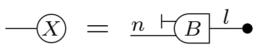
 if $X \in \mathsf{GLA}_{\mathbb{K}}$

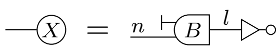
 if $X \in \mathsf{GAA}_{\mathbb{K}}$

 if $X \in \mathsf{GPA}_{\mathbb{K}}$

and a linear map  such that:

 if $X \in \mathsf{GLA}_{\mathbb{K}}$

 if $X \in \mathsf{GAA}_{\mathbb{K}}$

 if $X \in \mathsf{GPA}_{\mathbb{K}}$

Proof. The GLA normal forms are respectively the *cospan* and *span* factorizations in [BSZ17, Theorem 6.2]. We obtain the GAA normal forms by homogenisation [BPSZ19, Definition 5] combined with the GLA normal forms. We obtain the GPA normal forms by homogenisation [BDGS21, Lemma 23] combined with the two normal forms of the cone subset of GPA [BDGS21, Theorems 14 and 18]. □

Corollary 2.31. The only $0 \to 0$ morphisms are:

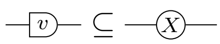
- {⬚} in $\mathsf{GLA}_{\mathbb{K}}$;
- {⬚, ⊢∘} in $\mathsf{GAA}_{\mathbb{K}}$ and $\mathsf{GPA}_{\mathbb{K}}$;

The only $1 \to 0$ morphisms in $\mathsf{GLA}_{\mathbb{K}}$ are {—∘, —•}. $\mathsf{GAA}_{\mathbb{K}}$ and $\mathsf{GPA}_{\mathbb{K}}$ have many more which we will not list here.

Proof. By examining the $n = 0$ and $n = 1$ cases of Theorem 2.30, and simplifying. □

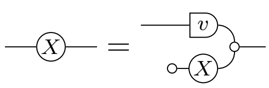
We conclude our exploration of maps with a result linking total relations with maps in GLA.

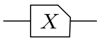
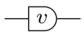
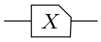
Proposition 2.32 (Choice)**.** If —(X)— : $1 \to n$ is a total relation in $\mathsf{GLA}_{\mathbb{K}}$, there exists a map $v : 1 \to n$ such that

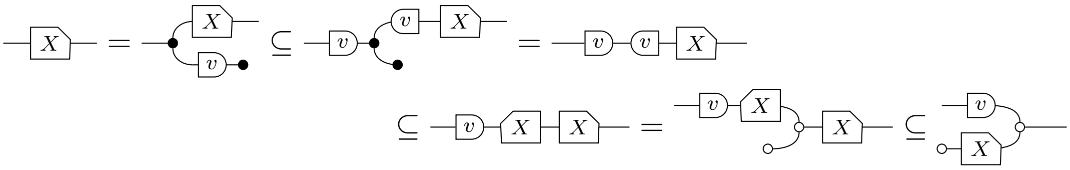


Proof. By [BSS20, Proposition 4.10], it suffices to show that $\mathsf{GLA}_{\mathbb{K}}$ has enough maps and that surjective maps split. Maps in $\mathsf{GLA}_{\mathbb{K}}$ are exactly the linear maps, which are known to split surjectives. [BSZ17, Theorem 6.2] gives a span normal form for every relation, which is equivalent to having enough maps. □

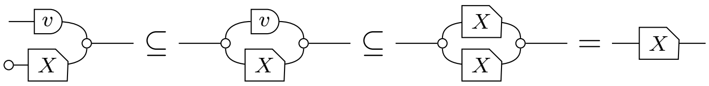
Proposition 2.33. If —(X)— : $1 \to n$ is a total relation in $\mathsf{GLA}_{\mathbb{K}}$, there exists a map $v : 1 \to n$ such that



Proof. Write  for clarity. By the choice theorem, $\exists v$,  $\subseteq$ . Then:



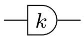
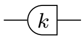
where the second and last step use the same property of cartesian bicategories modulo color-swaping (Definition 2.34). Conversely:



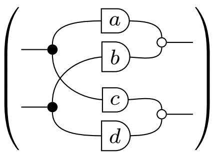
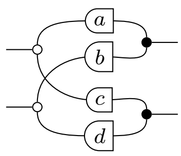
□

2.2.3 The Color-Swap Symmetry

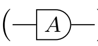
GLA exhibits a noteworthy symmetry: the black and white (co)monoids obey symmetrical rules. This gives rise to an important involution of GLA: *color-swapping*.

Definition 2.34 ([BSZ17, Section 5.3])**.** $\mathit{Colorswap} : \mathsf{GLA}_{\mathbb{K}} \to \mathsf{GLA}_{\mathbb{K}}$ is the contravariant (antimonotone) prop morphism defined on generators by swapping white circles with black circles, and sending  to .

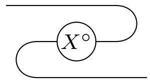
Example 2.35.

Colorswap  = 

Proposition 2.36.

Colorswap() = 

Definition 2.37. The *transpose* of a diagram —(X)— in GLA$_{\mathbb{K}}$ is defined as:



where —(X°)— := *Colorswap* (—(X)—).

Proposition 2.38. On a map, transpose coincides with normal matrix transposition:

$$Transpose(-\boxed{A}\!\!\rangle\!-) = -\boxed{A^T}\!\!\rangle\!-$$

2.3 Rational Fractions

As is standard in electrical circuit theory, we will represent general impedances using rational fractions of a variable s. We will map these onto differential equations in the next section. We focus particularly on positive-real fractions, as these represent impedances that can be constructed using basic elements [Bru31].

Definition 2.39. $\mathbb{R}(s)$ is the field of real rational fractions in the variable s, i.e. quotients of two real polynomials.

Definition 2.40. A fraction $Q \in \mathbb{R}(s)$ is called *positive-real*, noted $Q > 0$, if when seen as a function on complex numbers it has no poles and a positive real part in the right half of the complex plane. We also define $Q < 0 \iff -Q > 0$.

Definition 2.41. A fraction $Q \in \mathbb{R}(s)$ is called *nonnegative-real*, noted $Q \geq 0$, if $Q > 0$ or $Q = 0$. We also define $Q \leq 0 \iff -Q \geq 0$.

Proposition 2.42. The property of being positive-real (resp. nonnegative-real) is preserved by addition, multiplication and inverse. The only fraction that is both nonnegative and nonpositive is the constant zero.

2.4 Laplace Transform for Differential Equations

We present a few results on the Laplace Transform to help with managing differential equations. We use standard results about the transform in this section. We refer the reader to [Sch13] for definitions and details. We call $\mathcal{L}$ the standard unilateral Laplace transform. It maps a sufficiently-well-behaved function f of a real variable to a function $\mathcal{L}(f)$ of a complex variable, conventionally noted s.

We choose a well-understood domain of such sufficiently-well-behaved functions: functions of exponential order. We also impose an initial condition since the unilateral Laplace transform concerns itself only with positive values.

Definition 2.43. A function $f : \mathbb{R} \to \mathbb{R}$ has exponential order [Sch13, Section 1.4] if there exist constants $M > 0$ and α such that for some $t_0 \in \mathbb{R}$,

$$|f(t)| \leq Me^{\alpha t}, \quad \forall t \geq t_0$$

Definition 2.44. Let D be the space of infinitely differentiable $\mathbb{R} \to \mathbb{R}$ functions that are of exponential order and are zero on $(-\infty, 0]$.

Example 2.45. Let Ψ be the bump function $\Psi(t) := e^{-\frac{1}{t(1-t)}}$ restricted to $[0, 1]$ and renormalized. It is infinitely differentiable and of exponential order (since it is bounded), so $\Psi \in D$.

Let u be the integral of Ψ. Ψ is 0 outside $[0, 1]$, so u is a step-like function: $u(t) = 0$ for $t \leq 0$ and $u(t) = 1$ for $t \geq 1$. This makes it of exponential order as well, so $u \in D$.

Using u we can build many elements of D: given an infinitely differentiable function f of exponential order, $f(t)u(t) \in D$. For example: $t^2u(t) \in D$, $sin(t)u(t) \in D$, $e^tu(t) \in D$.

Proposition 2.46. $\mathcal{L}$ is defined and injective on D.

Proof. Functions in D are piecewise continuous and of exponential order, which is sufficient for the existence of their Laplace transform [Sch13, Theorem 1.11]. Two functions have the same Laplace transform if they differ on a set of measure 0 [Sch13, Section 1.7]; since functions in D are continuous, that can only happen if they are equal everywhere. □

Lemma 2.47. If $f \in D$ with derivative f', then $f' \in D$ and $\mathcal{L}(f')(s) = s \cdot \mathcal{L}(f)(s)$

Proof. The derivative of a function of exponential order is also of exponential order [Car34]. Then $\mathcal{L}(f')(s) = s \cdot \mathcal{L}(f)(s) - f(0^+)$ [Sch13, Section 1.7]. We conclude using the fact that f is continuous and $f(0) = 0$. □

Lemma 2.48. $\mathcal{L}(D)$ is closed under multiplication by any $P \in \mathbb{R}[s]$.

Proof. By Lemma 2.47, $\mathcal{L}(D)$ is closed under multiplication by the function $s \mapsto s$. By iterating this, it is closed under multiplication by $s \mapsto s^n$ for any n. Since $\mathcal{L}$ is $\mathbb{R}$-linear, this suffices to allow multiplication by any polynomial. □

Lemma 2.49. $\mathcal{L}(D)$ is closed under division by any $P \in \mathbb{R}[s]$.

Proof. Let $f \in D$. For $a \in \mathbb{R}$, define $g_a(t) := \int_0^t e^{-a(t-u)} f(u) du$. g_a is of exponential order by [Car34, Theorem 2]. It is infinitely differentiable and zero for $t \leq 0$ because f is. So $g_a \in D$. Standard results about $\mathcal{L}$ give us that $\mathcal{L}(g_a)(s) = \frac{1}{s+a}\mathcal{L}(f)(s)$. So $\mathcal{L}(D)$ is closed under division by polynomials of degree 1.

For $a, \omega \in \mathbb{R}$, define $h_{a,\omega}(t) := \frac{1}{\omega} \int_0^t e^{-au} sin(\omega u) f(t-u) du$. $h_{a,\omega}$ is of exponential order by [Car34, Theorem 2] again. It is infinitely differentiable and zero for $t \leq 0$ because f is. So $h_{a,\omega} \in D$. We calculate: $\mathcal{L}(h_{a,\omega})(s) = \frac{1}{(s+a)^2+\omega^2}\mathcal{L}(f)(s)$. Since a polynomial of degree 2 either factors or has form $(s+a)^2 + b$ with $b > 0$, $\mathcal{L}(D)$ is closed under division by polynomials of degree 2.

Finally, any real polynomial can be factored as a product of polynomials of degrees ≤ 2, which concludes our proof. □

Proposition 2.50. $\mathcal{L}(D)$ is a $\mathbb{R}(s)$-vector space where $\mathbb{R}(s)$ acts by pointwise multiplication.

Proof. $\mathcal{L}$ is linear and D is a $\mathbb{R}$-vector space, thus $\mathcal{L}(D)$ is a $\mathbb{R}$-vector space. By Lemmas 2.48 and 2.49, $\mathcal{L}(D)$ is closed under pointwise multiplication by any $P/Q \in \mathbb{R}(s)$. This respects the vector space axioms because of associativity and distributivity of pointwise function multiplication. □

Definition 2.51. For $f \in D$ and $P \in \mathbb{R}(s)$, we define $P \cdot f$ to be the unique function in D such that $\mathcal{L}(P \cdot f)(s) = P(s) \cdot \mathcal{L}(f)(s)$.

Proposition 2.52. $P \cdot f$ is well-defined and unique.

Proof. We know such a function exists from Proposition 2.50. The function is unique by injectivity of $\mathcal{L}$ (Proposition 2.46). □

Theorem 2.53. $P \cdot f$ above gives D the structure of a $\mathbb{R}(s)$-vector space where s acts as differentiation.

Proof. $\mathcal{L} : D \to \mathcal{L}(D)$ is injective (Proposition 2.46) and surjective (by definition of $\mathcal{L}(D)$) thus bijective. $P \cdot f$ is defined as the transport of the $\mathbb{R}(s)$-vector space structure of $\mathcal{L}(D)$ through $\mathcal{L}$. Since $\mathcal{L}$ is bijective, this endows D with the structure of a $\mathbb{R}(s)$-vector space. Since $\mathcal{L}$ is $\mathbb{R}$-linear, this extends the existing $\mathbb{R}$-vector space structure of D. Finally, given $f \in D$, $s \cdot f = \mathcal{L}^{-1}(s \cdot \mathcal{L}(f)(s)) = f'$ by Lemma 2.47. □

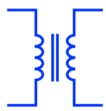
2.5 Standard Electrical Circuit Theory

Electrical circuit theory models physical electrical circuits as an interconnection of idealized elements. We present the simple version of electrical circuit theory we will study in this thesis: the lumped element model. It is wildly used and valid for signals changing slowly enough compared to the size of the circuit.

To provide a basis of understanding we present a simple formalization of circuits (with two-terminal elements only) and circuit semantics here. We will not explain all the standard ideas we use; we refer the unfamiliar reader to a standard textbook like [DK69] for more details.

Definition 2.54. A circuit is a directed multigraph where each edge (called "branch") is annotated with an element type and a corresponding quantity. Allowed element types include resistors (with a corresponding positive resistance) and voltage sources (with a corresponding nonzero voltage), as well as many more we will not detail here. A circuit also has a chosen subset of nodes called *open* that indicate where it might be plugged into other circuits.

Remark 2.55. This simple formulation only allows two-terminal elements. Elements like transformers would require a hypergraph structure to accomodate edges that join more than two vertices:



The setting we define in Chapter 3 naturally admits elements with more than two terminals. For example we construct a (three-terminal) transistor in Section 5.2.2.

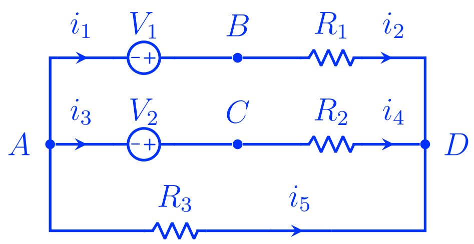
Definition 2.56. A behavior for a circuit with n nodes and e edges is a list of n potentials and e currents, each a smooth function of time, that satisfy the following sets of equations:

- For every node that is not open, the sum of currents on incoming edges equals the sum of currents on outgoing edges (Kirchhoff's current law $\sum_k I_k = 0$);
- For every branch, the characteristic equation of the element (e.g. Ohm's law $\Delta V = RI$ for resistors).

Network analysis is then the set of techniques used to calculate the behaviors of circuits.

Remark 2.57. Kirchhoff's voltage law is implicit in the fact that we assign a potential to each node. The law states that the sum of voltages around any loop in a circuit is zero. This is equivalent to choosing a potential for each node such that the voltage ΔV of each branch is the difference of the potentials at the two nodes of the branch $(V_A - V_B)$. Kirchhoff's voltage law then states that such a choice of potentials can be made consistently.

Example 2.58. A typical circuit is drawn like this:



We labeled the nodes and currents, which shows the straightforward graph structure. Writing out the relevant equations, we get that the behaviors of this circuits are those that respect the following system of equations:

$$
\begin{cases}
v_B - v_D = R_1 i_2 & i_1 + i_3 + i_5 = 0 \\
v_C - v_D = R_2 i_4 & i_2 + i_4 + i_5 = 0 \\
v_A - v_D = R_3 i_5 & i_1 = i_2 \\
v_B - v_A = V_1 & i_3 = i_4 \\
v_C - v_A = V_2 &
\end{cases}
$$

3

The Prop of Generalized Electrical Circuits

Electrical circuit theory models the physical behavior of networks of electrical elements. Circuits are represented by two-dimensional diagrams and are naturally compositional. Classical electrical circuit theory studies circuits using tools from graph theory (incidence matrix, spanning tree) and linear algebra. In this chapter we propose an alternative formalization of circuits based on string diagrams and compositional semantics.

Our approach expands on Bonchi et al.'s [BPSZ19]. Like them, we use diagrammatic tools at two levels: to describe circuits, and, using graphical algebra, to describe their behavior. Unlike previous approaches, we merge the two levels into a multisorted prop and introduce hybrid electric-equational elements, which will prove pivotal in the stating and proving of our theorems.

3.1 Graphical Differential Algebra

Our aim is to represent electrical equations as string diagrams in the style of GLA. Since the equations of electrical circuits are differential equations, we need a graphical language that can express differentiation. In this section we use the Laplace transform (Section 2.4) to turn $\mathsf{GAA}_{\mathbb{R}(s)}$ into a language for linear differential equations.

3.1.1 GLA and GAA on Vector Spaces

In Section 2.2, $\mathsf{GLA}_{\mathbb{K}}$ and $\mathsf{GAA}_{\mathbb{K}}$ are defined with semantics in $\mathsf{Rel}_{\mathbb{K}}$. In this section we show that these semantics can be straightforwardly adapted to Rel_X where X is a $\mathbb{K}$-vector space. Moreover the two theories are still complete for these extensions. We start by showing that all non-trivial semantics for $\mathsf{GLA}_{\mathbb{K}}$ or $\mathsf{GAA}_{\mathbb{K}}$ are complete.

Theorem 3.1. Let P be an ordered prop and let $F : \mathsf{GLA}_{\mathbb{K}} \to P$ be a morphism of ordered props. Then either:

- $F(-\!\circ) = F(-\!\bullet)$, and F is trivial (it equates all morphisms);
- or $F(-\!\circ) \neq F(-\!\bullet)$, and F is faithful.

Proof. First observe that P inherits through F the compact closed structure of $\mathsf{GLA}_{\mathbb{K}}$, so we can restrict ourselves to $n \to 0$ morphisms.

First assume that $F(-\!\circ) = F(-\!\bullet)$. Let R be a $n \to 0$ morphism in $\mathsf{GLA}_{\mathbb{K}}$. By the normal form of Theorem 2.30, we write $-\!\textcircled{R} = -\!\boxed{A}\!-\!\circ$. By functoriality and our assumption:

$$F(-\!\textcircled{R}) = F(-\!\boxed{A}\!-\!\circ) = F(-\!\boxed{A}\!-\!\bullet) = F(-\!\bullet)$$

Hence all morphisms of a given type are made equal by F.

Now assume that $F(-\!\circ) \neq F(-\!\bullet)$. Observe that

$$\vec{v} \in [\![-\!Ⓡ]\!]_{\mathsf{GLA}_{\mathbb{K}}} \iff 1 \in [\![-\!\boxed{v}\!-\!Ⓡ]\!]_{\mathsf{GLA}_{\mathbb{K}}}$$

where $\vec{v}$ is a vector of $\mathbb{K}^n$ and $-\!\boxed{v}\!-$ represents the $1 \to n$ linear map $k \in \mathbb{K} \mapsto k.\vec{v}$. Since the image of $[\![\cdot]\!]_{\mathsf{GLA}_{\mathbb{K}}}$ is always a vector space,

$$[\![-\!Ⓡ]\!]_{\mathsf{GLA}_{\mathbb{K}}} \subseteq [\![-\!Ⓢ]\!]_{\mathsf{GLA}_{\mathbb{K}}} \iff \forall \vec{v} \in \mathbb{K}^n, [\![-\!\boxed{v}\!-\!Ⓡ]\!]_{\mathsf{GLA}_{\mathbb{K}}} \subseteq [\![-\!\boxed{v}\!-\!Ⓢ]\!]_{\mathsf{GLA}_{\mathbb{K}}}$$

By completeness of $[\![\cdot]\!]_{\mathsf{GLA}_{\mathbb{K}}}$,

$$-\!Ⓡ \subseteq_{\mathsf{GLA}_{\mathbb{K}}} -\!Ⓢ \iff \forall \vec{v} \in \mathbb{K}^n, -\!\boxed{v}\!-\!Ⓡ \subseteq_{\mathsf{GLA}_{\mathbb{K}}} -\!\boxed{v}\!-\!Ⓢ$$

Therefore F is faithful if and only if it is faithful on $1 \to 0$ diagrams. We conclude from the fact that the only $1 \to 0$ diagrams are $-\!\circ$ and $-\!\bullet$ (Corollary 2.31), and they are distinct in $\mathsf{GLA}_{\mathbb{K}}$. □

Theorem 3.2. Let P be an ordered prop and let $F : \mathsf{GAA}_{\mathbb{K}} \to P$ be a morphism of ordered props. Then either:

- $F(\vdash\!\!-\!\circ) = F(\,\square\,)$, and F is trivial (it equates all morphisms);
- or $F(\vdash\!\!-\!\circ) \neq F(\,\square\,)$, and F is faithful.

Proof. Like in Theorem 3.1, we can restrict ourselves to $n \to 0$ morphisms. We proceed similarly.

First assume that $F(\vdash\!\!-\!\circ) = F(\,\square\,)$. Let R be a $n \to 0$ morphism in $\mathsf{GAA}_{\mathbb{K}}$. By functoriality and the principle of explosion [BPSZ19, Lemma 16]:

$$F(-\!Ⓡ) = F(-\!Ⓡ\ \square) = F(-\!Ⓡ \vdash\!\!-\!\circ) = F(-\!\bullet \vdash\!\!-\!\circ) = F(-\!\bullet\ \square) = F(-\!\bullet)$$

Hence all morphisms of a given type are made equal by F. Now assume that

$F(\vdash\!\!-\!\circ) \neq F(\,\boxed{}\,)$. Observe that

$$\vec{v} \in [\![-\!\!\!\!-\!\text{Ⓡ}]\!]_{\mathsf{GAA}_{\mathbb{K}}} \iff \bullet \in [\![\vdash\!\!\!-\!\boxed{v}\!-\!\text{Ⓡ}]\!]_{\mathsf{GAA}_{\mathbb{K}}}$$

So

$$[\![-\!\text{Ⓡ}]\!]_{\mathsf{GAA}_{\mathbb{K}}} \subseteq [\![-\!\text{Ⓢ}]\!]_{\mathsf{GAA}_{\mathbb{K}}} \iff \forall \vec{v}, [\![\vdash\!\boxed{v}\!-\!\text{Ⓡ}]\!]_{\mathsf{GAA}_{\mathbb{K}}} \subseteq [\![\vdash\!\boxed{v}\!-\!\text{Ⓢ}]\!]_{\mathsf{GAA}_{\mathbb{K}}}$$

By completeness of $[\![\,\cdot\,]\!]_{\mathsf{GAA}_{\mathbb{K}}}$,

$$-\!\text{Ⓡ} \subseteq_{\mathsf{GAA}_{\mathbb{K}}} -\!\text{Ⓢ} \iff \forall \vec{v}, \vdash\!\boxed{v}\!-\!\text{Ⓡ} \subseteq_{\mathsf{GAA}_{\mathbb{K}}} \vdash\!\boxed{v}\!-\!\text{Ⓢ}$$

Therefore F is faithful if and only if it is faithful on $0 \to 0$ diagrams. The only $0 \to 0$ diagrams are $\vdash\!\!-\!\circ$ and $\boxed{}$ (Corollary 2.31), and they are distinct in $\mathsf{GAA}_{\mathbb{K}}$, so F is faithful iff $F(\vdash\!\!-\!\circ) \neq F(\,\boxed{}\,)$. □

Definition 3.3. Let $\mathbb{K}$ be a field and X be a $\mathbb{K}$-vector space. $[\![\,\cdot\,]\!]_X : \mathsf{GLA}_{\mathbb{K}} \to \mathsf{Rel}_X$ is the functor of ordered props defined on generators by Figure 3.1.

Proposition 3.4. $[\![\,\cdot\,]\!]_X$ is well-defined.

Proof. For this functor to be well-defined, we need to check that it verifies the axioms of Figure 2.1. They are all rather straightforward consequences of the vector space structure:

- The purely black axioms are true in Rel_X regardless of structure;
- The purely white are true from the group structure on X;
- The black and white interactions state that 0 and + are maps;
- The scalar axioms reflect the properties of scalar multiplication in a vector space.

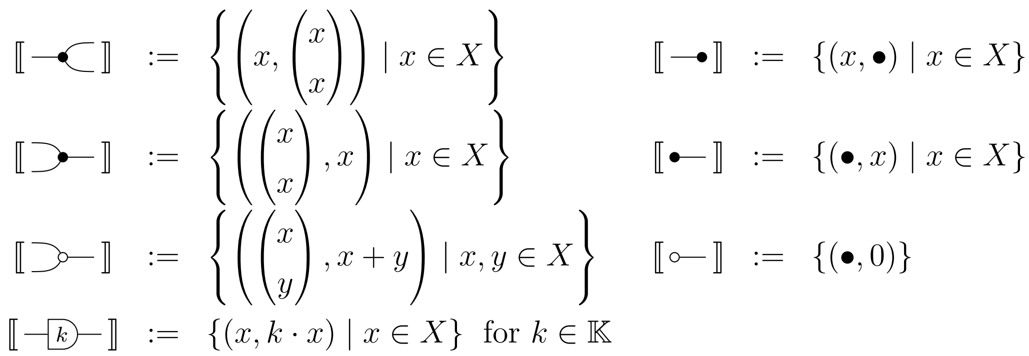


Figure 3.1: Semantics of $[\![\cdot]\!]_X$.

□

Theorem 3.5. Let X be a $\mathbb{K}$-vector space different than $\{0\}$. $\mathsf{GLA}_{\mathbb{K}}$ is complete for $[\![\cdot]\!]_X$.

Proof. $[\![-\!\bullet]\!]_X = X \neq \{0\} = [\![-\!\circ]\!]_X$. Thus by Theorem 3.1, $[\![\cdot]\!]_X$ is faithful. □

Definition 3.6. Let $\mathbb{K}$ be a field, X be a $\mathbb{K}$-vector space, and $x_1 \in X$ be nonzero. $[\![\cdot]\!]_{X,x_1} : \mathsf{GAA}_{\mathbb{K}} \to \mathsf{Rel}_X$ is the functor of ordered props defined on generators by Figure 3.1 and $[\![\vdash\!-]\!]_{X,x_1} := \{(\bullet, x_1)\}$.

Proposition 3.7. $[\![\cdot]\!]_{X,x_1}$ is well-defined.

Proof. For this functor to be well-defined, we need to check that it verifies the axioms of Figures 2.1 and 2.3. It is clearly an extension of $[\![\cdot]\!]_X$ so it respects Figure 2.1 for the same reasons. Remain the three axioms of Figure 2.3. The first two state that $[\![\vdash\!-]\!]_{X,x_1}$ is a singleton set, which it is. The third is a consequence of $x_1 \neq 0$. □

Theorem 3.8. Let $\mathbb{K}$ be a field, X be a $\mathbb{K}$-vector space, and $x_1 \in X$ be nonzero. $\mathsf{GAA}_{\mathbb{K}}$ is complete for $[\![\cdot]\!]_{X,x_1}$.

Proof. $[\![\vdash\!\!-\!\circ]\!]_{X,x_1} = \emptyset \neq \{\bullet\} = [\![\quad]\!]_{X,x_1}$. Thus by Theorem 3.2, $[\![\cdot]\!]_{X,x_1}$ is faithful.

□

3.1.2 Graphical Affine Differential Algebra

We can now define a graphical language for linear differential equations. Recall the domain D of smooth functions we defined in Definition 2.44.

Definition 3.9. Let u be a nonzero nonnegative function in D, such as the step-like function u defined in Example 2.45. The specific choice of step function is inconsequential for our purposes.

Definition 3.10. Define $[\![\cdot]\!]_{\mathsf{GADA}} : \mathsf{GAA}_{\mathbb{R}(s)} \to \mathsf{Rel}_D$ (GADA stands for Graphical Affine Differential Algebra) to be $[\![\cdot]\!]_{D,u}$ (as defined in Definition 3.6) where D is seen as a $\mathbb{R}(s)$-vector space (Theorem 2.53).

Proposition 3.11. $[\![\cdot]\!]_{\mathsf{GADA}} : \mathsf{GAA}_{\mathbb{R}(s)} \to \mathsf{Rel}_D$ is the unique prop functor that extends $[\![\cdot]\!]_D : \mathsf{GLA}_{\mathbb{R}} \to \mathsf{Rel}_D$ (Definition 3.3) with the additional equations:

$$[\![\vdash\!\!-]\!]_{\mathsf{GADA}} := \{(\bullet, u)\}$$

$$[\![-\!\boxed{s}\!-]\!]_{\mathsf{GADA}} := \{(f, f') \mid f \in D\}$$

Proof. The equation on $\vdash\!\!-$ is verified by definition of $[\![\cdot]\!]_{D,u}$. The equation on $-\!\boxed{s}\!-$ is verified because s acts by differentiation in D (Theorem 2.53).

Now to show uniqueness. Consider the subprop of $\mathsf{GAA}_{\mathbb{R}(s)}$ generated by $\mathsf{GLA}_{\mathbb{R}}$, $\vdash\!\!-$ and $-\!\boxed{s}\!-$. It contains all $-\!\boxed{P(s)}\!-$ where $P(s)$ is a polynomial in $\mathbb{R}[s]$. Hence it also contains polynomial fractions $-\!\boxed{R(s)}\!- = -\!\boxed{P(s)}\!-\!\boxed{Q(s)}\!-$ where $R(s) = P(s)/Q(s)$ is a polynomial fraction in $\mathbb{R}(s)$. Therefore it contains all the generators of $\mathsf{GAA}_{\mathbb{R}(s)}$ i.e. coincides with the whole of $\mathsf{GAA}_{\mathbb{R}(s)}$. Therefore the definition above defines $[\![\cdot]\!]_{\mathsf{GADA}}$ uniquely. □

Theorem 3.12. $\mathsf{GAA}_{\mathbb{R}(s)}$ is complete for $[\![\cdot]\!]_{\mathsf{GADA}}$.

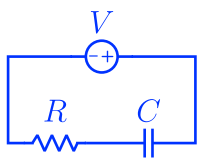
Proof. This is simply Theorem 3.8. □

Using $[\![\cdot]\!]_{\mathsf{GADA}}$ we can thus use the great flexibility of $\mathsf{GAA}_{\mathbb{R}(s)}$ to reason about systems of linear differential equations. This is the graphical counterpart of the ubiquitous use of Laplace transforms in standard electrical circuit theory.

Remark 3.13. While both our setting and *differential categories* [BCS06] are categorical settings that capture differentiation, they are conceptually very different. GADA aims to reason about systems of linear differential equations on functions of a chosen domain D: our objects are powers of D and our arrows are systems of equations. Comparatively, in differential categories the objects are spaces and arrows are smooth maps between these spaces. Differentiation is an arrow in our setting, and an operator on arrows in differential categories; the kind of reasoning we do in GADA would have to be done at the level of operators in a differential category, which is not usually the focus of such theories.

3.2 Modeling Electrical Circuits Graphically

As we saw in Section 2.5, we describe electrical circuits as an interconnection of basic elements such as the following:

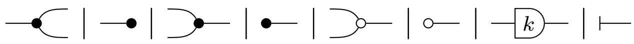


Syntactically, circuits are made by plugging together some basic elements and wiring. Whereas standard formulations tend to use graphs, we choose to make this formal using a prop. The specificity of our approach lies in the mixing of circuits and equations: a single diagram includes both electrical elements (in thick blue) and equational elements (in thin black) as well as hybrid elements where the two interact.

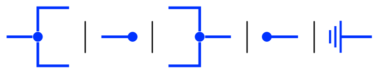
3.2.1 The syntax of circuits and equations: the prop SynECirc

Definition 3.14. The prop SynECirc is the free colored prop over the colors $\{\bullet, \bullet\}$ generated by the following generators:

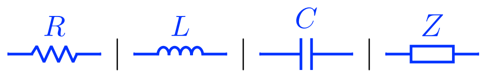
Generators of $\mathsf{GAA}_{\mathbb{R}(s)}$:



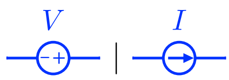
Electrical wiring:



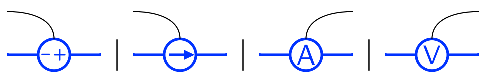
Passive elements:



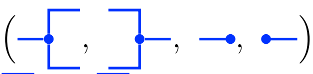
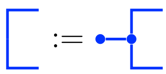
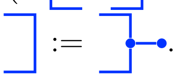
Independent sources:

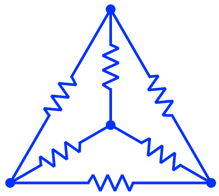
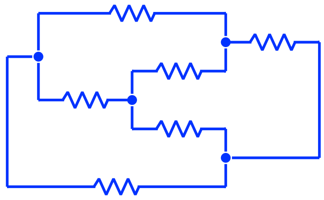


Hybrid elements:



for $R, L, C \in \mathbb{R}_+$, $Z \in \mathbb{R}(s)_+$, and $k, V, I \in \mathbb{R}(s)$. Starting from the electrical (blue) wiring, these generators are called respectively *cojunction*, *counit*, *junction*, *unit*, *ground*, *resistor*, *inductor*, *capacitor*, *impedance*, *independent voltage source*, *independent current source*, *controlled voltage source*, *controlled current source*, *ammeter*, *voltmeter*.

The signature includes generators for standard circuit elements like resistors and capacitors, for which we use the standard symbols. To model electrical wiring, as is standard in string diagrammatic settings, we use a monoid/comonoid pair  along with the standard syntactic sugar  and . This is sufficient to express any connectivity: any circuit can be deformed into a form that fits in the prop syntax. For example:

 can be written: 

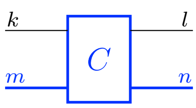
In what follows we occasionally take liberties with the orientation of elements for clarity, as long as it does not introduce any ambiguity.

Beyond electrical elements and wiring (in thick blue), SynECirc includes so-called *information wires* (in thin black). Just like in standard electrical circuits theory one can assign a name to a current or voltage of interest, in our diagrammatic setting we can represent such a variable by a $\mathsf{GAA}_{\mathbb{R}(s)}$-wire, and use hybrid elements to link the variable with the electrical quantity of interest:

This makes the variable available for graphical equations, which we can write alongside the circuit using the generators of $\mathsf{GAA}_{\mathbb{R}(s)}$.

Remark 3.15. This idea of making variables explicit in a compositional setting is not new. It is used for example for bayesian networks [JKZ19], where the *observed* variables of a causal model are made an output of the overall diagram by copying them out, while *latent* variables are not.

Remark 3.16. By convention we group information wires and electric wires separately. The generic circuit thus looks like:



3.2.2 Interpreting circuits

Once we have a circuit diagram, we want to reason about its physical behavior. The behavior of a circuit is concerned with two quantities at each electrical wire: current and potential. Every electrical element imposes constraints on the currents and potentials on its open wires. In this chapter and the following, we work with standard idealized linear elements, whose behaviors are described by simple affine differential equations between quantities that are smooth functions of time.

As can be expected, we write circuit equations graphically. We use $\mathsf{GAA}_{\mathbb{R}(s)}$, with s representing differentiation as formalized through $[\![\cdot]\!]_{\mathsf{GADA}}$ (Definition 3.10). We follow the idea of [BPSZ19], with extensions to suit our development in subsequent sections. We thus interpret an electrical wire as a pair of $\mathsf{GAA}_{\mathbb{R}(s)}$ wires: one for potential and one for current. Each element is then mapped to the equation it imposes on these quantities.

The crux of this approach is that the interpretation is functorial: interpreting elements individually and wiring them together correctly constructs an interpretation for the whole circuit.

Definition 3.17. We define the interpretation functor $\mathcal{I} : \mathsf{SynECirc} \to \mathsf{GAA}_{\mathbb{R}(s)}$ as follows: it is the monoidal functor defined on objects by $\mathcal{I}(\bullet) := 2$ and $\mathcal{I}(\bullet) := 1$, that acts on the generators of $\mathsf{GAA}_{\mathbb{R}(s)}$ as the identity, and on the rest of the generators as described in Figure 3.2.

Example 3.18. Let us illustrate how this works on the resistor. Its behavior is given by Ohm's law: $\Delta V = RI$ where ΔV is the difference in potential across the resistor, and I is the current through it (in the associated direction). If we make explicit the currents and potentials at the input and output nodes, we get the following system of equations:

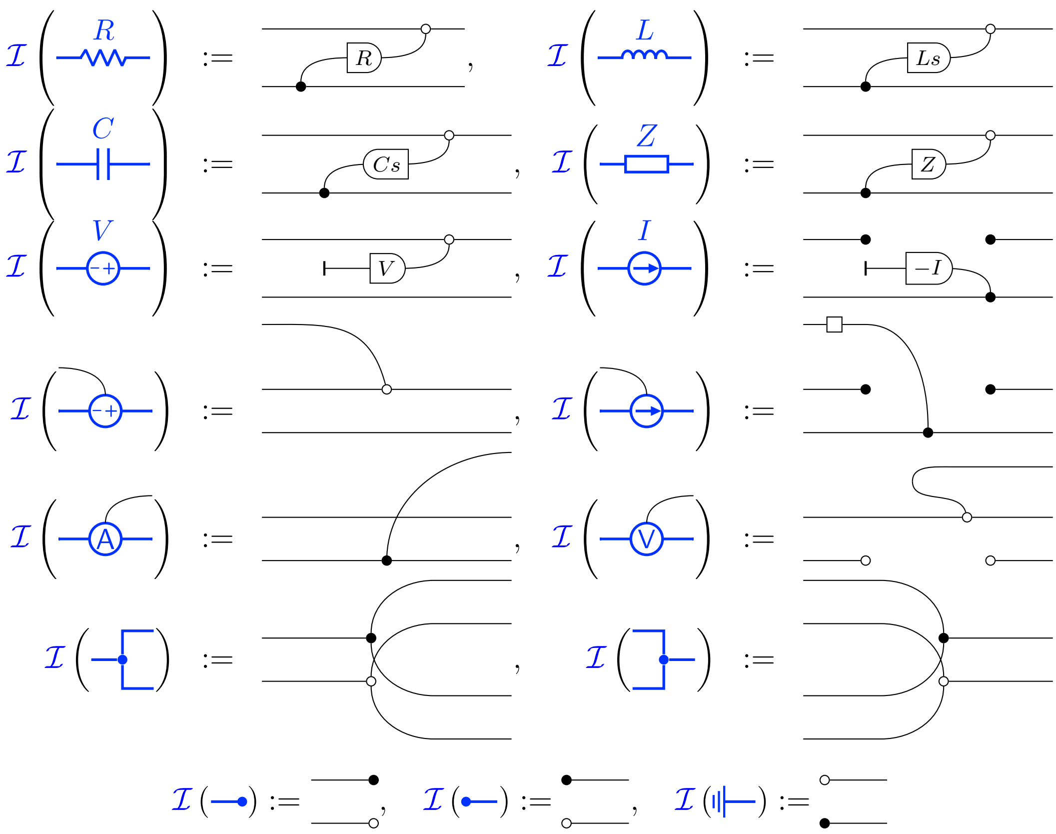
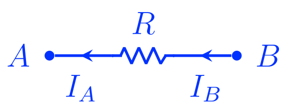
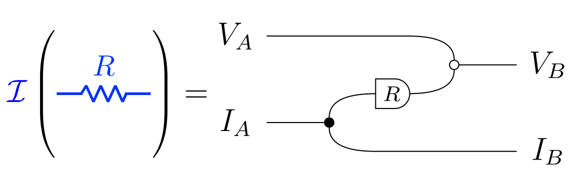


Figure 3.2: Compositional interpretation of circuits.

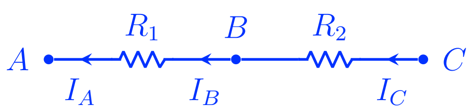


$$\begin{cases} V_B - V_A = RI_A \\ I_A = I_B \end{cases}$$

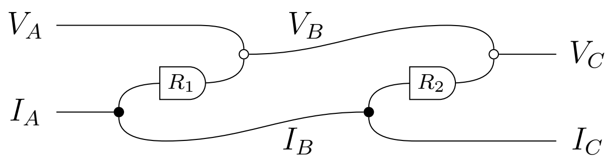
which is exactly the semantics via $[\![\cdot]\!]_{\mathsf{GADA}}$ of:



Example 3.19. To illustrate functoriality, take the series connection of two resistors:



Its interpretation is obtained by interpreting the two resistors and connecting the resulting $\mathsf{GAA}_{\mathbb{R}(s)}$ diagrams:



Of note is that by the structure of $[\![\cdot]\!]_{\mathsf{GADA}}$, the quantities on node B are existentially quantified: the semantics of this diagram only relate the quantities at nodes A and C.

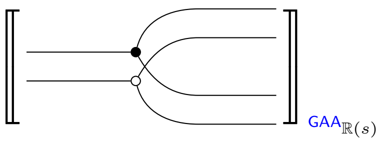
Remark 3.20. This layered structure of a prop being interpreted into another by doubling each wire is very similar to the *doubling construction* of categorical quantum mechanics [CK16]. Our hybrid elements are parallel to their classical-quantum processes. This similarity is made explicit in the setting of *Lagrangian relations* [CK21], which describe both the interpretation of our basic elements and a specific class of quantum circuits, using such a doubling construction.

Remark 3.21. We made an implicit choice of convention for the currents: we always consider currents going from right to left. This choice makes composition straightforward. Incidentally, this is a very convenient consequence of having only horizontal wires. Allowing arbitrary-direction wires would prevent us from choosing a single current convention and we would be forced, like electrical engineers, to annotate diagrams with current directions and take good care to make no sign errors in tracking them.

Theorem 3.22. Given a circuit C in $\mathsf{SynECirc}$, $[\![\mathcal{I}(C)]\!]_{\mathsf{GADA}}$ represents a system of linear differential equations on quantities for potential and current. This system is equivalent to the one derived using standard circuit theory for the electrical part of the circuit C subject to the additional equations imposed by the equational parts of C.

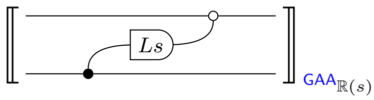
Proof. [DK69, Chap. 10] provides a systematic method to determine the equations for an electrical circuit. It consists of two parts: writing the characteristic laws (Kirchhoff, Ohm's, etc), and applying techniques to reduce the resulting equations. The second part is handled in our case by the functor $[\![\cdot]\!]_{\mathsf{GADA}}$. Remains to show we do the first part correctly.

The two Kirchhoff laws are encoded in how we model wiring: the wiring (co)monoid pair constrains the currents incoming to a node to sum to zero (Kirchhoff's Current Law) and gives each node a single potential (equivalent to Kirchhoff's Voltage Law):



$$ = \left\{ \left(\begin{pmatrix} v \\ i_1 \end{pmatrix}, \begin{pmatrix} v \\ i_2 \\ v \\ i_3 \end{pmatrix} \right) \;\middle|\; i_1 = i_2 + i_3 \right\}$$

The characteristic laws of each elements are straightforwardly imposed by their interpretation. For example, we saw above that the interpretation of a resistor imposes Ohm's law. Similarly, the characteristic equation of the inductor is, as required:



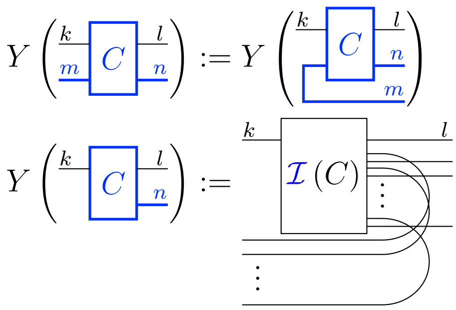
$$ = \left\{ \left(\begin{pmatrix} v_1 \\ i \end{pmatrix}, \begin{pmatrix} v_2 \\ i \end{pmatrix} \right) \;\middle|\; v_2 - v_1 = L\frac{di}{dt} \right\}$$

□

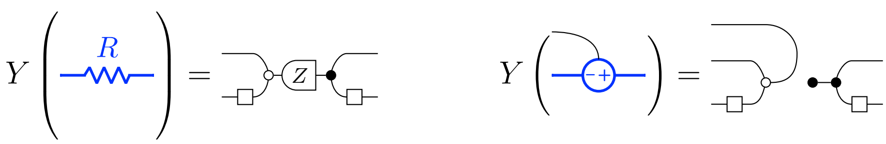
Remark 3.23. Information wires and hybrid elements are crucial for the analysis of closed circuits (circuits with no open electrical wires) like the above. Indeed, our semantics concerns the behavior of a circuit seen from its open wires (so-called *black-boxing* [BF18]). This poses a problem for closed circuits: a circuit with no open wires is mapped to an equation with no variables, and only two of these exist — the true equation and the false equation. In other words, the semantics would only tell us whether such a circuit has a solution. Using information wires we can observe the behavior of a closed circuit.

The interpretation of a circuit alternates wires that correspond to currents and wires that correspond to potentials. It is sometimes more convenient to group separately the current wires and the potential wires; for this we define the admittance of a circuit as a sibling of its interpretation.

Definition 3.24. The admittance $Y(C)$ of a circuit C is defined by taking the interpretation $\mathcal{I}(C)$ of C and bending all the potential wires to one side and all the current wires to the other, leaving the information wires untouched:



Example 3.25.



Remark 3.26. This generalizes the standard notion of the admittance matrix of a circuit, which has the characteristic property of expressing the behavior of a circuit as a map from potentials to currents. Our version of admittance need not be a map, but when it is it mirrors the standard notion.

3.2.3 Reasoning about circuits and equations: the prop ECirc

The prop ECirc equates two circuits when they are semantically equivalent. This is the setting in which we will reason about circuits.

Definition 3.27. The ordered prop ECirc is the quotient of SynECirc by the relation:

$$C \subseteq_{\mathsf{ECirc}} D \quad \text{whenever} \quad \mathcal{I}(C) \subseteq_{\mathsf{GAA}} \mathcal{I}(D)$$

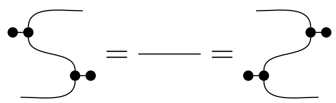
for arbitrary circuits C and D. This in particular implies:

$$C =_{\mathsf{ECirc}} D \quad \text{whenever} \quad \mathcal{I}(C) =_{\mathsf{GAA}} \mathcal{I}(D)$$

The functor $\mathcal{I}$ naturally restricts to a functor of order-enriched monoidal categories: $\mathcal{I} : \mathsf{ECirc} \to \mathsf{GAA}_{\mathbb{R}(s)}$.

A major property of ECirc is that the direction of wires is unimportant. Wires can be crossed and bent at will.

Theorem 3.28. ECirc is compact closed, i.e.



Proof. A direct consequence of the fact that $\mathsf{GAA}_{\mathbb{R}(s)}$ is compact closed via both the black and white (co)monoids. □

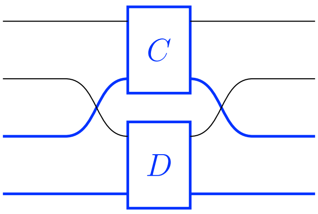
This means we can restrict ourselves circuits that have no wires coming from the left without loss of generality.

3.2.4 Describing Physical Circuits: Controlled Circuits

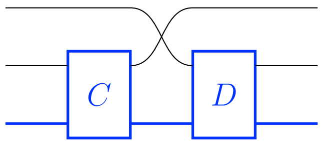
The prop ECirc contains more than just circuits. Since we want to prove general results about circuits we need to identify diagrams that represent a bona fide circuit. For this we must restrict the use of information wires.

Definition 3.29. A circuit is in *controlled form* if it is written with no non-trivial interactions between information wires. A circuit that admits a controlled form is called a *controlled circuit*. The controlled form is defined inductively as follows:

- Electrical identity and swap are in controlled form;
- Any non-GAA generator of ECirc is in controlled form;
- If C and D are in controlled form, then so are:



and

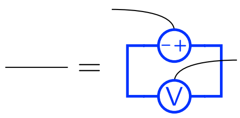


- If C is in controlled form, then a permutation of its information inputs or outputs is also in controlled form.

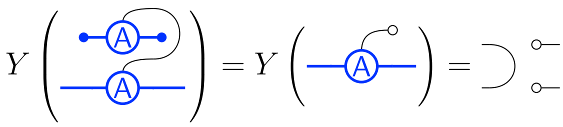
Note how composition works normally for electrical wires, but information wires never connect.

Example 3.30. The following two circuits are not in controlled form:

The first one is however a controlled circuit:

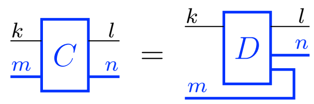


The second one is not, because as we will see in Section 4.2, if it was controlled its admittance would be a map, but we calculate that it is not:

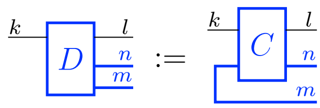


Controlled circuits complicate compact closure: they forbid connecting information wires internally, which prevents bending them. This is a desired property as it allows distinguishing inputs from outputs (which we formalize in Proposition 4.15). We can still bend electric wires.

Proposition 3.31. Given a controlled circuit $C : k + m \to l + n$, there exists a controlled circuit $D : k \to l + n + m$ such that:



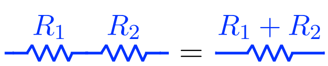
Proof.



D is visibly a controlled circuit, and we conclude by compact closure (Theorem 3.28). □

3.3 Reasoning About Circuits

We are interested in reasoning about circuits. The standard way to do this is to study their semantics, but our hybrid approach makes it natural to reason by circuit equivalence. Specifically, two circuits are considered equal in ECirc whenever they have identical semantics. This will be our main way to state and derive results. For example, we will see that:



The fact that ECirc is a prop means that we can reason locally about circuit equivalence.

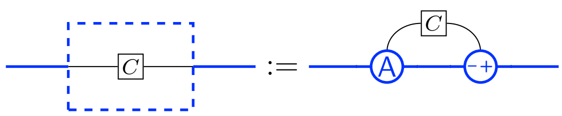
Remark 3.32. This reasoning includes equivalences between the $\mathsf{GAA}_{\mathbb{R}(s)}$-equations that can coexist with our circuits (on the information wires).

Remark 3.33. ECirc is also an ordered prop, which means that on top of comparing circuits for equality, it makes sense to compare them for inclusion. We will see examples of this in Theorems 4.51 and 4.53 and example 4.52.

3.3.1 Impedance Boxes

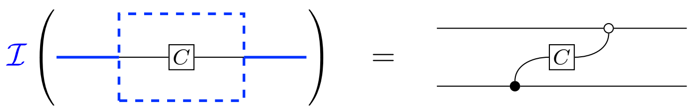
To help us derive circuit equivalences, we introduce a final tool to our framework. Notice that the basic elements all have the same shape, whereby the current is conserved and the voltage (potential difference) is constrained. We introduce syntactic sugar for this common case, it will turn out to be convenient for calculations.

We extend the syntax of SynECirc and ECirc with *impedance boxes* —illustrated below left— defined as a syntactic sugar parametrised w.r.t. arbitrary GAA circuits of type (1, 1): that is, with one wire on the left and one on the right. Impedance boxes are an example of a *window* in a layered prop [LZ23], as they allow us to peek into the lower level of interpretation and calculate at that level before going back to the higher level of electrical elements. They are in this sense the dual of a functor box (which correspond to *cowindows*).



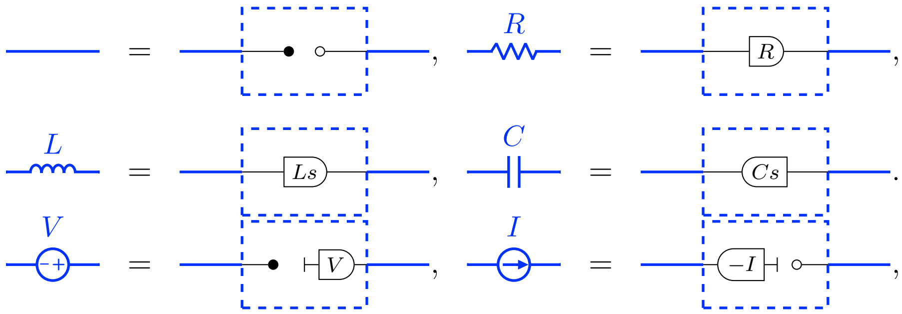
(3.1)

Their interpretation captures this common shape of elements:



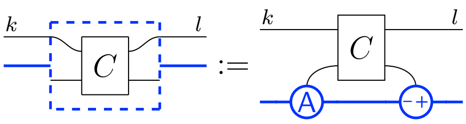
(3.2)

The following are easy observations in the equational theory of GAA:

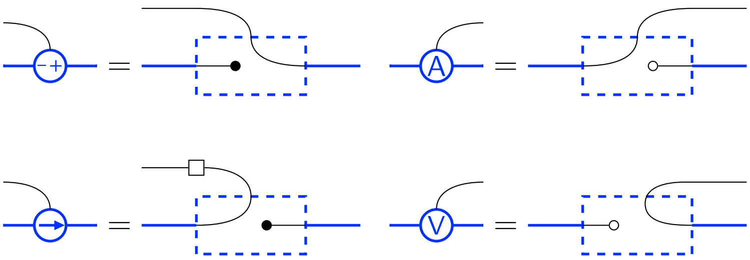


The middle three examples illustrate why we use the name “impedance”: this generalizes the standard notion of impedance. The standard notion is a (complex or Laplace-domain) quantity Z, and applies to an element whose behavior equation is $\Delta V = ZI$. Impedance boxes generalize this idea to an arbitrary affine relation between the current and the voltage, which allows in particular to describe voltage and current sources. Like its classical counterpart, this notion considerably simplifies calculations.

We also extend impedance boxes to allow information wires to exit:



Then controlled sources and meters become instances of impedance boxes as well:



3.3.2 Simple Circuit Equivalences and Impedance Calculus

Let us go back to the case of two resistors in series:

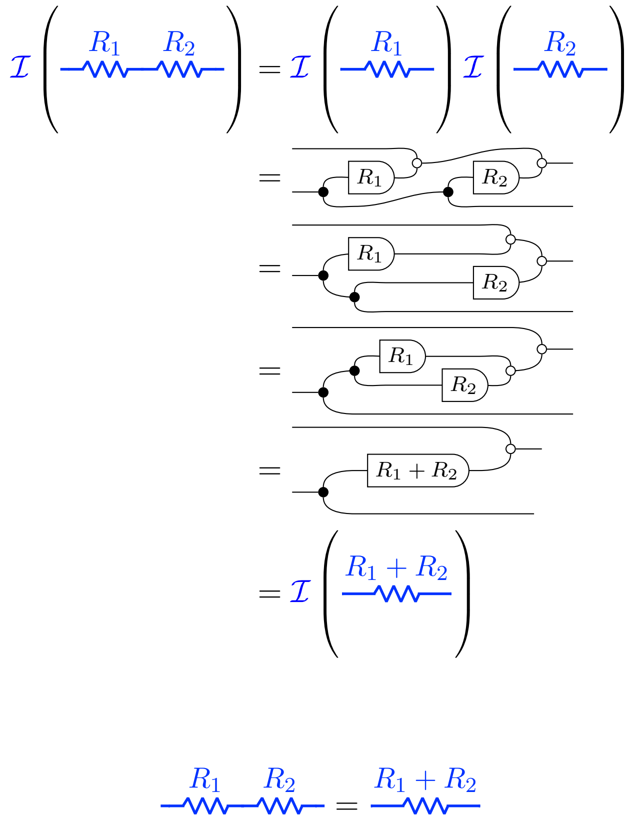
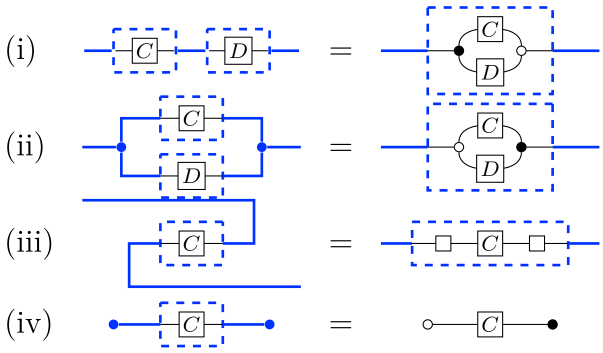


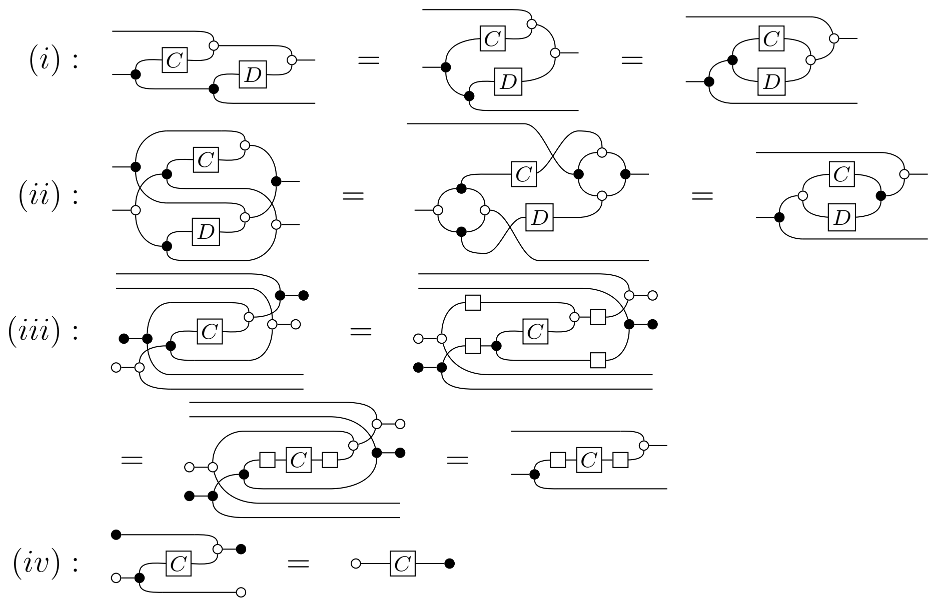
Figure 3.3: Typical circuit reasoning

If we compose other elements in series and parallel, we observe very similar calculations. We can generalize them by reasoning with arbitrary impedance boxes. We dub these results "impedance calculus", and we will use them throughout this thesis to simplify circuit calculations.

Lemma 3.34.



Proof.



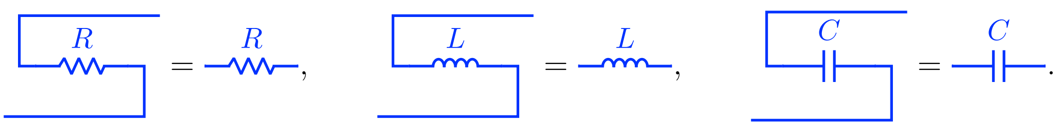
□

These all generalize straightforwardly to the case where the boxes have extra information input and outputs. Note the elegant symmetry between series and parallel composition.

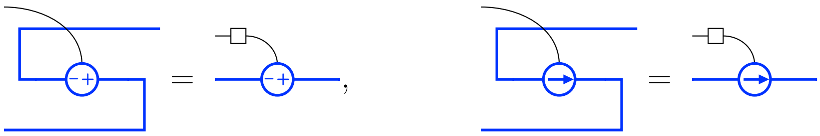
Using Lemma 3.34 we can immediately derive several useful properties of circuits:

Corollary 3.35.

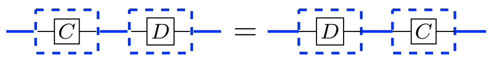
(i) Resistors, inductors and capacitors are "directionless":



(ii) Reversing the direction of voltage and current sources negates their value:

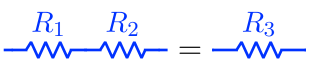
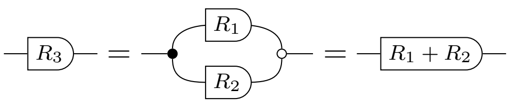


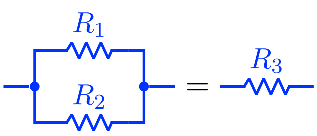
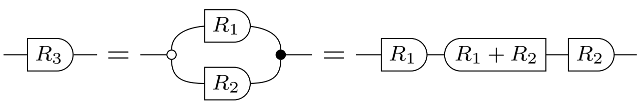
(iii) The order of elements plugged in series does not matter

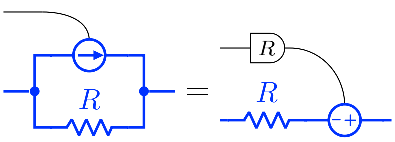


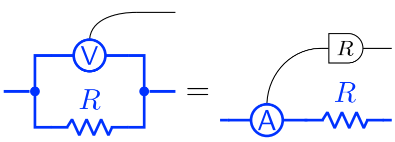
The impedance calculus is useful for proving circuit equivalences. The first 3 of the following equivalences would be found in any textbook.

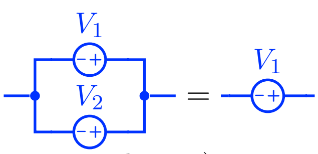
Proposition 3.36.

(i)  where 

(ii)  where 

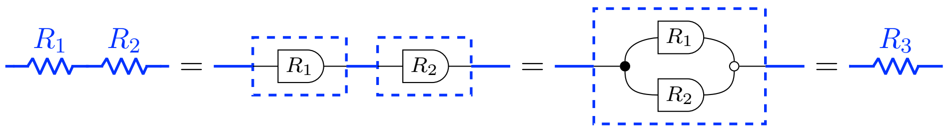
(iii)  (known as *Thévenin-Norton equivalence*)

(iv) 

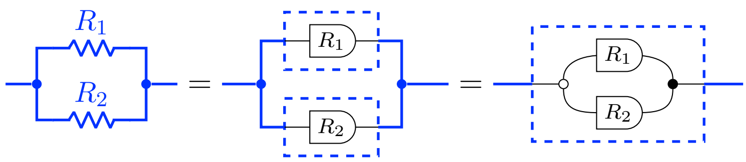
(v)  if $V1 = V2$, otherwise its interpretation is $\vdash\!\circ$ (the empty relation)

It is useful to contrast our treatment with the classical approach. (i), (ii) and (iii) are standard and often-used equivalences. (ii) is known classically, but R_3 is typically given a formula like $R_1R_2/(R_1 + R_2)$. The graphical formula is its graphical equivalent, with the added benefit that it is still valid for $R_1 = R_2 = 0$. The graphical calculus also neatly emphasizes the similarity between parallel sum and series sum. (iv) exemplifies a beautiful symmetry between sources and meters what we will explore further in Section 4.4.4. Finally, in (v), the empty case is usually excluded by classical treatments. A textbook deems such a circuit *degenerate* and ignores that case when proving theorems. In GAA however, the empty relation is first-class and our semantics uniformly includes the empty case.

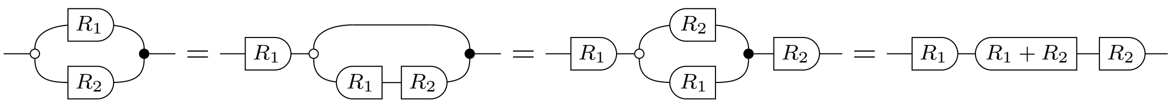
Proof. (i) is a simple exercise in the use of the impedance calculus:



For (ii), we have

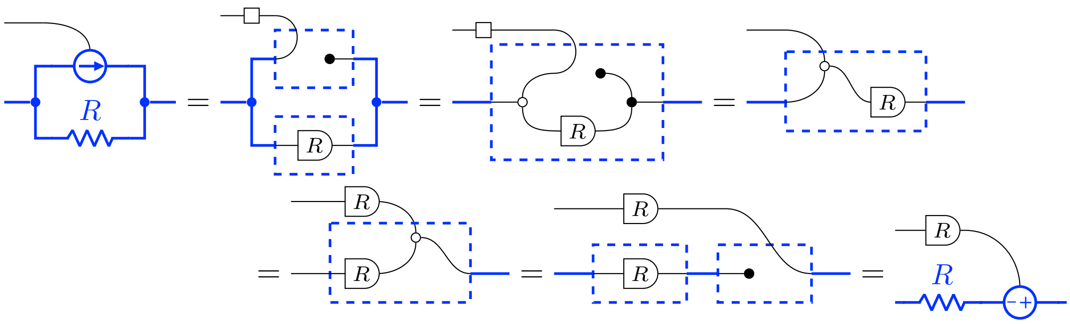


. Now



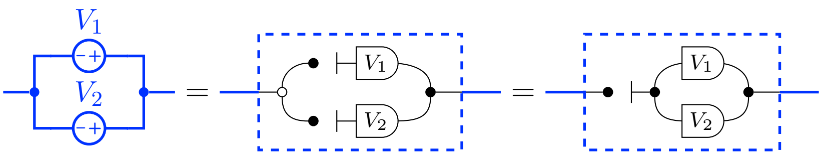
extracts the classical formula: because R_1 and R_2 are nonnegative, either $R_1 + R_2 \neq 0$, and $-\!\!\boxed{R_1 + R_2}\!\!-$ is a scalar, or $R_1 = R_2 = 0$, and the formula is equal to 0. In both cases the result is a nonnegative scalar.

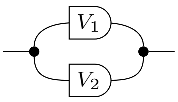
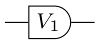
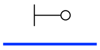
(iii) is another simple calculation:



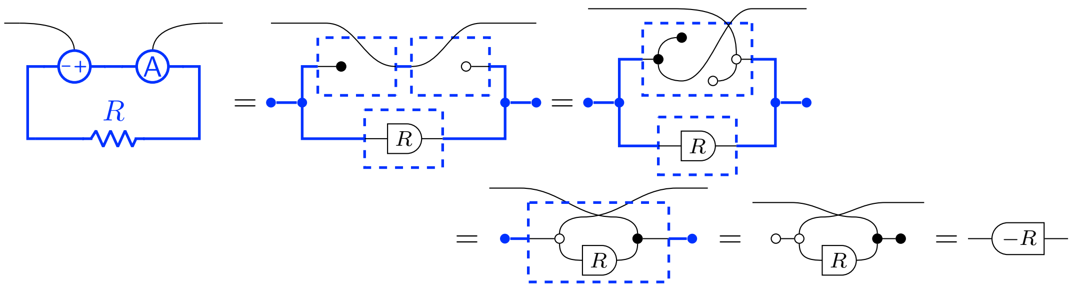
(iv) proceeds exactly like (iii) but color-swapped (Definition 2.34).

For (v), we can simplify using the impedance calculus as follows:



Now  is just  if $V_1 = V_2$, and otherwise. In that case, the circuit is equivalent to  which is denoted by the empty relation. □

Example 3.37. We can analyze simple circuits using this new technology:



4

Graphical Electrical Circuit Theory

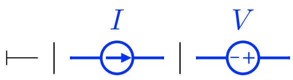
Equipped with a flexible framework to reason about electrical circuits, we now reproduce a number of results from classical electrical circuit theory, among which the superposition theorem and Thévenin's theorem. Our versions of these theorems differ in interesting ways from the standard ones, due to the relational nature of our framework. Along the way we will analyze passive circuits exhaustively and study the issue of short-circuited circuits.

4.1 Simplifying Analyses

Before we dive into the meat of this chapter, we prepare the field by naming interesting subprops of ECirc to which we can often restrict our analyses.

4.1.1 The Linear Subprop

Definition 4.1. The prop LinECirc is the subprop of ECirc generated by the generators of ECirc (Definition 3.14) whose interpretation is linear (i.e. lands in $\mathsf{GLA}_{\mathbb{R}(s)}$). That is, all generators except:

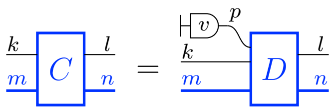


Proposition 4.2. The image of LinECirc through $\mathcal{I}$ lands in $\mathsf{GLA}_{\mathbb{R}(s)}$.

Proof. The generators of LinECirc were selected to all have interpretation in $\mathsf{GLA}_{\mathbb{R}(s)}$. □

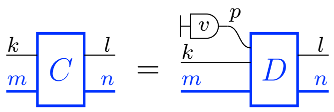
The following theorem allows us to work in LinECirc without loss of generality.

Theorem 4.3 (Homogenisation)**.** Given a circuit $C : k + m \to l + n$, there exists a circuit $D : p + k + m \to l + n$ in the linear subset and a linear vector $v : 1 \to p$ such that:



If moreover C is in controlled form, then so is the corresponding D.

Proof. We construct a mapping *homog* from C to (v, D) inductively as in Figure 4.1, where $(v, D) := \mathit{homog}\,(C)$ and $(v', D') := \mathit{homog}\,(C')$. This mapping reflects a well-known property of GAA that diagrams can be homogenized in a similar fashion (see [BPSZ19, Theorem 17]). *homog* preserves the following invariant: if $\mathit{homog}(C) = (v, D)$, then $D \in$ LinECirc and



Finally a simple induction shows that if C is a controlled circuit, then so is the corresponding D. □

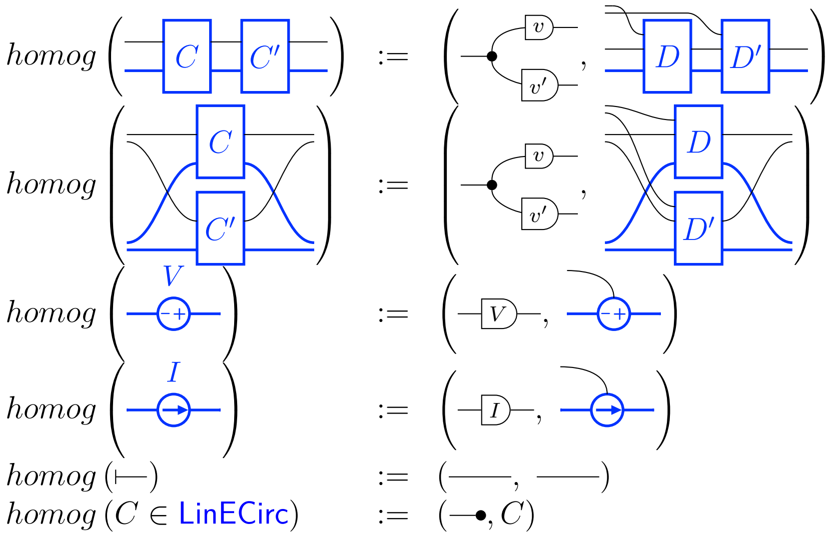


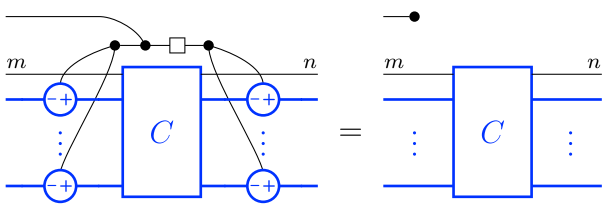
Figure 4.1: Inductive definition of *homog*

4.1.2 The Groundless Subprop

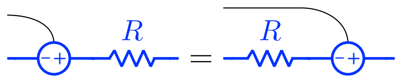
Definition 4.4. The prop GroundlessECirc is the subprop of ECirc generated by the generators of ECirc (Definition 3.14) except the ground ⫴—.

Kirchhoff's laws imply two global invariants that are satisfied by ungrounded circuits. They can be elegantly stated and proved graphically.

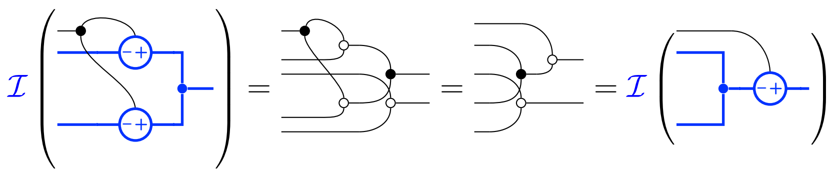
Proposition 4.5 (Relativity of potentials)**.** An ungrounded circuit constrains voltages (differences in potential), not absolute potentials. That is, adding the same voltage to all open wires of a circuit does not change its behavior:



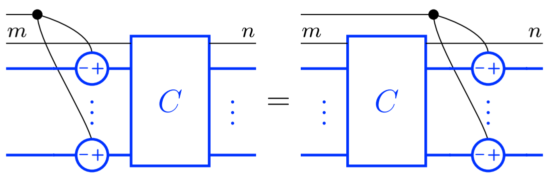
Proof. We first show that voltage sources commute with everything. By Corollary 3.35, voltage sources commute with any impedance box, which covers all basic elements. For example for resistors:



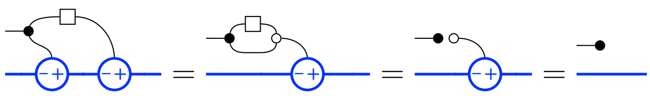
Voltage sources commute with junctions in the following sense:



A short induction then shows that voltage sources with a common input commute with arbitrary ungrounded circuits:



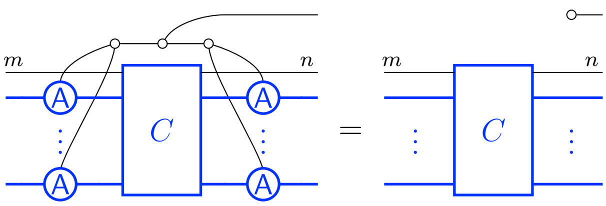
Now:



which concludes the proof.

□

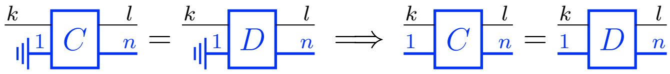
Proposition 4.6 (Conservation of currents)**.** The sum of the currents going into an ungrounded circuit is equal to the sum of the outgoing currents.



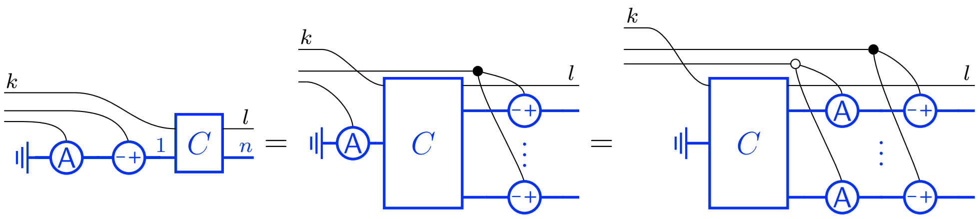
Proof. The proof proceeds exactly like the proof of Proposition 4.5. □

These two theorems allow us to work in GroundlessECirc without loss of generality.

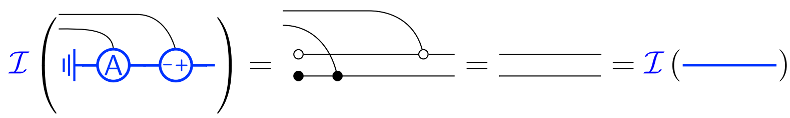
Proposition 4.7. If C and D are ungrounded circuits, then:

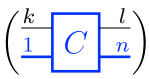
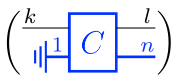


Proof. Let C be an ungrounded circuit. Recall from the proof of Proposition 4.5 that voltage sources with a common input commute with ungrounded circuits, and similarly for ammeters with a summed output. Then:

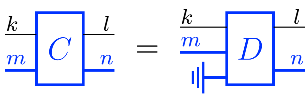


Noticing that:



we conclude that we can compute $\mathcal{I}$  from $\mathcal{I}$ , which proves the theorem. □

Theorem 4.8 (Ungrounding)**.** Given a circuit $C : k + m \to l + n$, there exists a unique circuit $D : k + m + 1 \to l + n$ in the groundless subset such that:



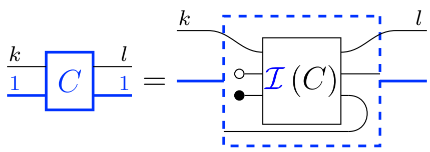
If moreover C is in controlled form, then so is the corresponding D.

Proof.  behaves in a similar fashion to ⊢:

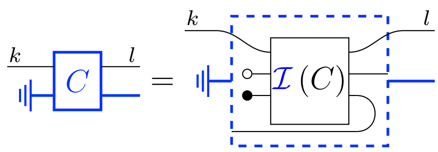
We can thus similarly group any n uses of  into a single one, which proves the existence of D. Since this operation does not touch information wires, it preserves the controlled form. We get unicity from Proposition 4.7. □

This theorem has a useful corollary for one-port circuits (those with one electrical input and one electrical output): all ungrounded one-port circuits are representable by an impedance box. Thus the impedance calculus can be used with any ungrounded one-port circuit that we encounter.

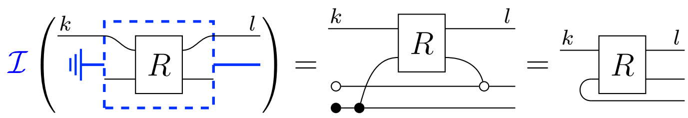
Theorem 4.9 (Representation theorem for one-ports)**.** Any ungrounded one-port circuit C is representable by an impedance box:



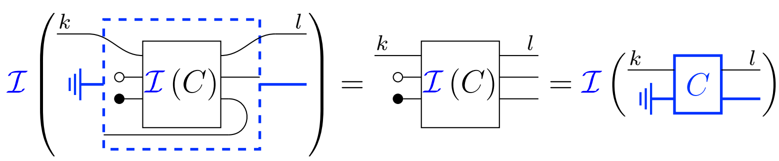
Proof. Using Proposition 4.7 it suffices to prove that:



For a general impedance box:



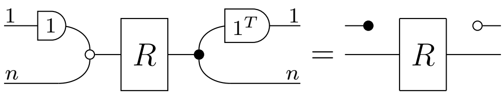
Hence:

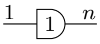
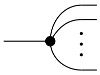


□

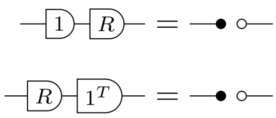
The Kirchhoff invariants of Propositions 4.5 and 4.6 are moreover neatly expressed in terms of the admittance.

Definition 4.10. We call a relation *Kirchhoff* if it has the following property:



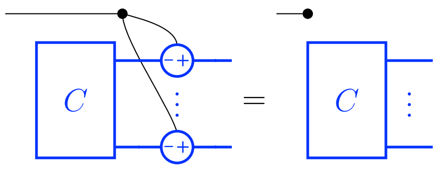
where  :=  represents the vector with all ones.

If R is a map, this is equivalent to the rows and columns of R each summing to 0:

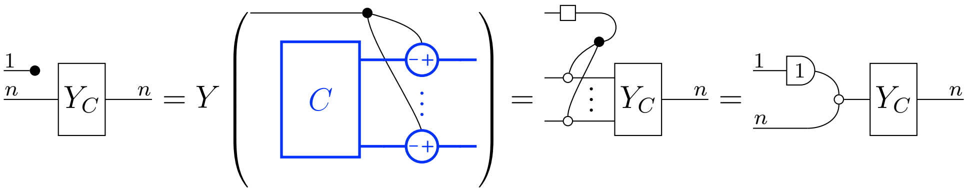


Proposition 4.11. The admittance of an ungrounded circuit with no open information wires is Kirchhoff.

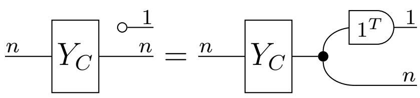
Proof. Without loss of generality, C has type $0 \to n$. C is groundless, so by Proposition 4.5,



Therefore

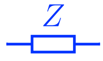


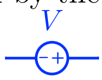
By Proposition 4.6 we similarly obtain:

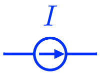


∎

4.1.3 The Infoless Subprop

Definition 4.12. The prop InfolessECirc is the subprop of ECirc generated by the elements with no information wires, namely: electrical wiring, , 

, 

and 

.

Proposition 4.13. Every circuit in InfolessECirc is a controlled circuit.

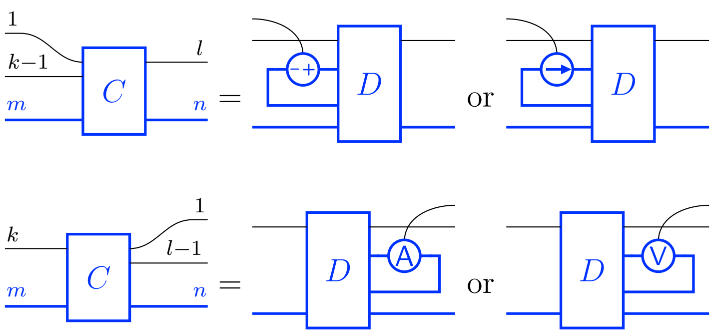
Proof. Immediate since InfolessECirc contains no circuits with information wires. □

Proposition 4.14. A controlled circuit with no open information wires lives in InfolessECirc.

Proof. A simple induction on the structure of controlled circuits shows that a controlled circuit that contains a hybrid element or a $\mathsf{GAA}_{\mathbb{R}(s)}$ generator must have a corresponding open information wire. □

The restriction on connectivity of controlled circuits means that every information input connects to exactly one source, and every information output to exactly one meter, as the following propositions show. Thus a controlled circuit is essentially a circuit in InfolessECirc probed by a few sources and meters.

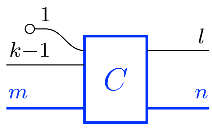
Proposition 4.15. If C is a controlled circuit:



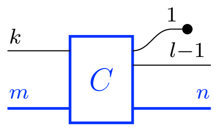
where D is another controlled circuit.

Proof. By induction on the structure of controlled circuits. □

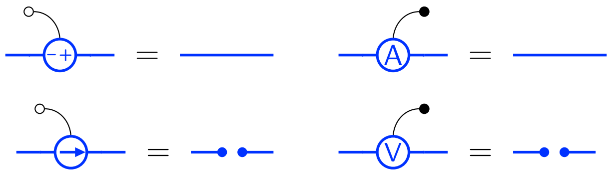
Proposition 4.16. If C is a controlled circuit, then so are



and

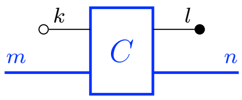


Proof. By Proposition 4.15, and the fact that discarding a meter or source turns it into wiring:



□

Corollary 4.17. If C is a controlled circuit,

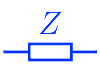


$\in$ InfolessECirc

Proof. By induction on the number of information wires, using Proposition 4.14 as the base case and Proposition 4.16 as the inductive case. □

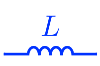
4.1.4 The Passive Subprop

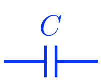
Voltage and current sources are usually called *active* because they supply power to the circuit. Conversely, resistors, capacitors and general impedances draw power from the circuit and are called *passive*.

Definition 4.18. The prop PassiveECirc is the subprop of ECirc generated by 

, , and electrical wiring.

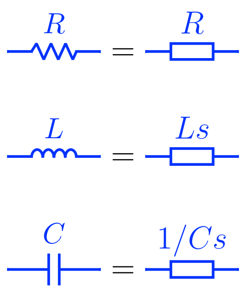
Proposition 4.19. 

, 

, 

$\in$ PassiveECirc

Proof.



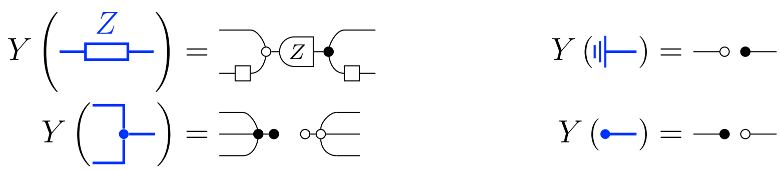
□

Proposition 4.20. PassiveECirc is the intersection of LinECirc and InfolessECirc.

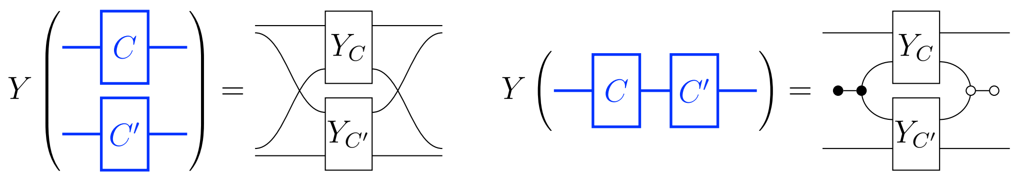
Proof. Taking only the linear elements in the definition of InfolessECirc gives the definition of PassiveECirc. □

Proposition 4.21. The admittance of a passive circuit is self-transpose.

Proof. By induction, with base cases:



and inductive cases:



which are all visibly self-transpose. □

4.2 Passive Circuits and Meshes

To get started on global properties of circuits, we study a nicely behaved subset of passive circuits: meshes.

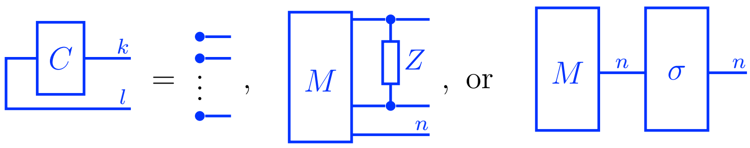
4.2.1 Mesh Circuits

A mesh is a circuit that, in graph terms, has no internal node. In other words, every element connects two of the open wires of the circuit (modulo wiring). They provide a good basis for discussing global properties of circuits.

(a) A mesh **(b)** Not a mesh

Figure 4.2: Meshes vs non-meshes

Definition 4.22. A circuit C is a *mesh* if it can be inductively written, modulo compact closure, as either:



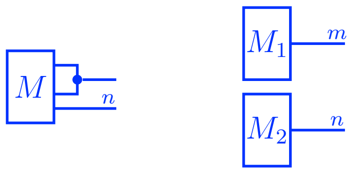
where M is another mesh, $Z \geq 0$, and σ is a permutation on the wires. The vertical element is shorthand but unambiguous.

Remark 4.23. Meshes are reminiscent of *graph states* in categorical quantum mechanics [CK17, Def. 9.121], which are similarly made exclusively from gates placed between the open wires of a quantum circuit, and which provide a basis for reasoning about general quantum circuits.

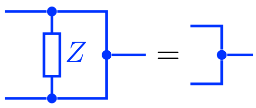
Definition 4.24. A *strict mesh* is a circuit that can be written as a mesh above using only impedances with $Z \neq 0$.

Example 4.25. An example of a mesh that is not strict is the plain wire ———.

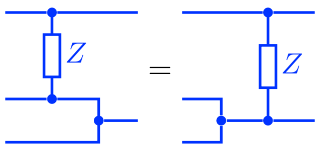
Proposition 4.26. Meshes (resp. strict meshes) are closed under tensoring and merging two nodes: if M_1, M_2 and M are meshes (resp. strict meshes), then so are the following two circuits:



Proof. By straightforward induction, noting that

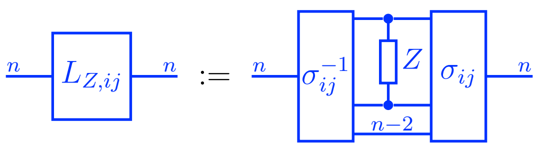


and



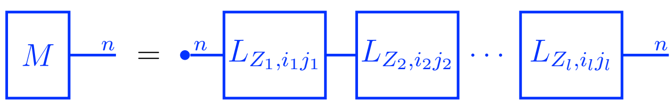
□

Definition 4.27. Given an impedance $Z \geq 0$ and $1 \leq i < j \leq n$, a (Z, ij)-*layer*, noted $L_{Z,ij}$, is the following circuit that places an impedance Z between wires i and j:



where σ_{ij} is the permutation that swaps wire 1 with wire i and wire 2 with wire j.

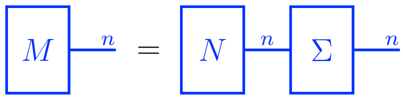
Proposition 4.28. A mesh M can be written as



Moreover M is strict if and only if $\forall k, Z_k \neq 0$.

Proof. By induction on the structure of meshes, the only subtlety being the precise tracking of permutations. The particular cases $n = 0$ and $n = 1$ are also included in this definition, they simply use no layers. □

Proposition 4.29. A mesh M can be written as

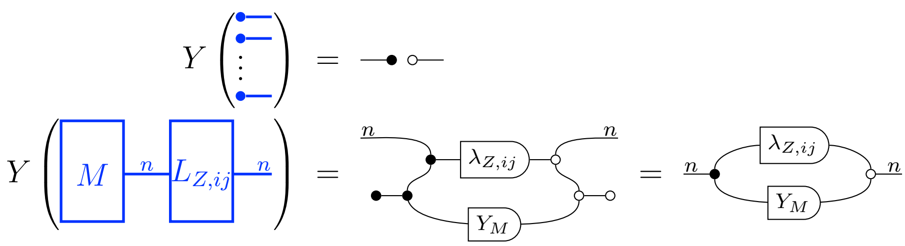


where N is a strict mesh and Σ is a composite of layers $L_{0,ij}$ i.e. with $Z = 0$.

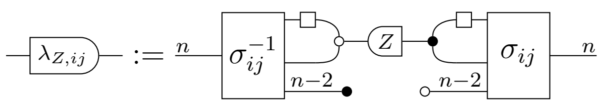
Proof. A $L_{Z,ij}$ layer places an element between wires i and j without shuffling them, so two such layers commute. In Proposition 4.28, we can therefore without loss of generality sort the layers such that the layers with $Z \neq 0$ are at the beginning. Together with •— they form the strict mesh N, and the remaining layers form Σ. □

Proposition 4.30. The admittance of a strict mesh is a self-transpose Kirchhoff map whose off-diagonal coefficients (Proposition 2.27) are nonpositive (in the sense of Definition 2.41).

Proof. The admittance is self-transpose and Kirchhoff by Propositions 4.11 and 4.21. We prove the other two properties inductively on the decomposition as layers of Proposition 4.28:



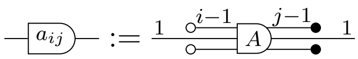
where



Since $Z \neq 0$, $\lambda_{Z,ij}$ is a map. The new admittance is the sum of two maps so is a map. Moreover both maps have nonpositive off-diagonal coefficients, so their sum does too by Proposition 2.42. □

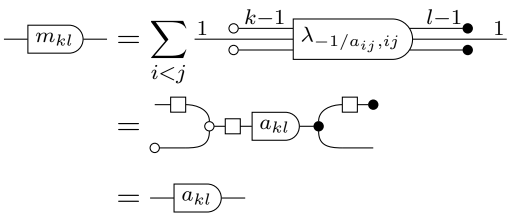
Proposition 4.31. If A is a self-transpose Kirchhoff map whose off-diagonal coefficients are nonpositive, then there exists a strict mesh with admittance A.

Proof. Let A be such a map. Let a_{ij} be the i, jth coefficient of A; diagrammatically:



Since A is self-transpose, $a_{ij} = a_{ji}$.

Let M be the strict mesh circuit with n wires made (as in Proposition 4.28) from the composition of the layers $L_{-1/a_{ij},ij}$ for each $1 \leq i < j \leq n$ such that $a_{ij} \neq 0$. Remembering the proof of Proposition 4.30, we see that Y_M is the sum of $\lambda_{-1/a_{ij},ij}$. We inspect the coefficients m_{kl} of Y_M: for $k < l$:



So A and Y_M coincide on their off-diagonal coefficients. Thus $A - Y_M$ is diagonal. Since both are Kirchhoff, so is $A - Y_M$. As we saw in Definition 4.10, this means its columns sum to 0. Since it is diagonal, $A - Y_M = 0$ and M is a strict mesh with admittance $Y_M = A$ as required. □

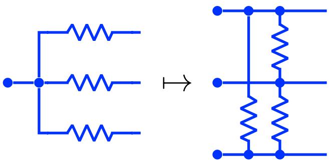
These two theorems put the layers of Proposition 4.28 and the coefficients of the admittance in direct correspondance.

Corollary 4.32. A mesh M can be written in layers form (Proposition 4.28) such that no two layers use the same indices.

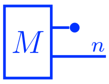
Proof. The mesh constructed in Proposition 4.31 has the desired property. By Proposition 4.30, every mesh can be put in this form. □

4.2.2 Star-Mesh Transform

The core theorem that makes meshes general is the star-mesh transform. It states that a circuit with n admittances connected to a central node can be transformed into a circuit with admittances only between the open nodes, i.e. a mesh.



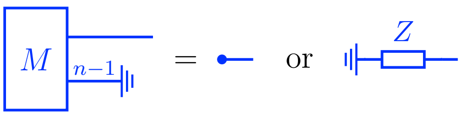
The general theorem is as follows:

Theorem 4.33. Meshes are closed under discarding nodes. Formally, if M is a (strict) mesh, then the circuit 

obtained by plugging a single —• into M is also a (strict) mesh.

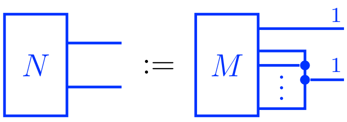
We need two lemmas before we can prove this.

Lemma 4.34. If M is a mesh and $n \geq 1$:

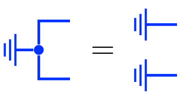


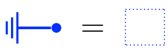
where $Z \geq 0$. If moreover M is strict then $Z \neq 0$.

Proof. Define:

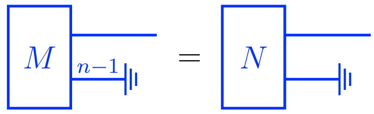


Knowing that:

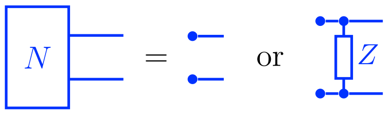




we deduce that

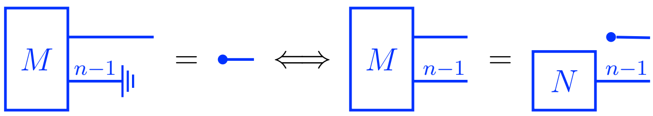


By Proposition 4.26, meshes are closed under merging nodes and tensoring with •—, so N is a mesh, and is strict if M is. By Corollary 4.32, N can be written in layer form with at most one layer. In other words:



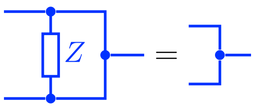
Which concludes our proof. □

Lemma 4.35. If M is a mesh and $n \geq 1$:

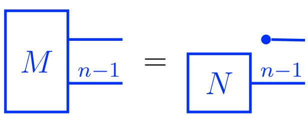


where N is another mesh. If moreover M is strict then so is N.

Proof. Using the layer form (Proposition 4.28), observe the N that we constructed in the proof of Lemma 4.34. It was obtained by merging all the nodes of M with index $j \geq 2$. Since

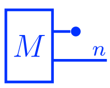
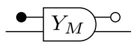


layers that do not use the first wire are irrelevant. Then N has a layer $L_{Z,12}$ if and only if M has at least one layer $L_{Z,1j}$. We conclude by noting that



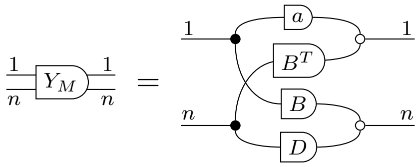
(for some other mesh N) if and only if M has no layer of the form $L_{Z,1j}$. □

We can now prove the general star-mesh transform theorem.

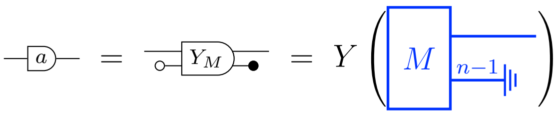
Proof of Theorem 4.33 in the strict case. The admittance of  is .

By Proposition 4.31, we only need to show that this admittance is a self-transpose Kirchhoff map with nonpositive off-diagonal coefficients. By Propositions 4.11 and 4.21 we only need to show that it is a map with nonpositive off-diagonal coefficients.

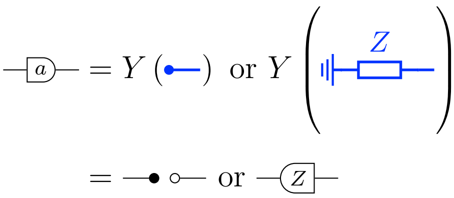
By Proposition 4.30, Y_M is a self-transpose Kirchhoff map, that we decompose as follows:



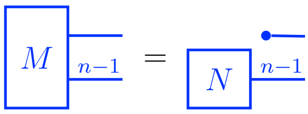
From the number of wires, we get that a is a scalar, B is a column vector, and D is a $n \to n$ square matrix. We identify a:



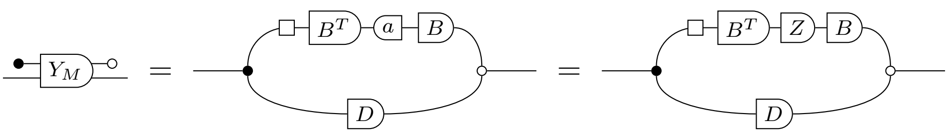
Using Lemma 4.34,

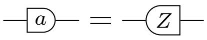


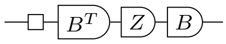
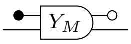
where $Z > 0$. If $a = 0$, by Lemma 4.35



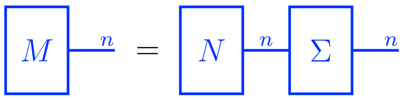
and the theorem is immediate. We therefore assume $a \neq 0$. Then:



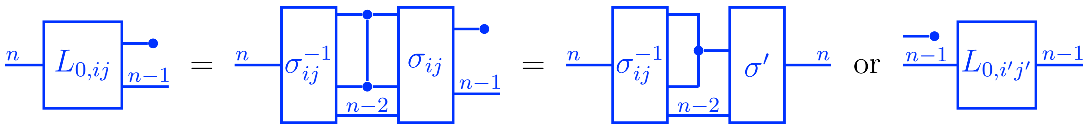
This is the diagrammatic expression of the *Schur complement*. While in general it need not be a map,  makes it so in our case.

Only remains to show that its off-diagonal coefficients are nonpositive. Because the off-diagonal coefficients of Y_M are nonpositive, this is also the case for D, and moreover the coefficients of B are all nonpositive. Since also $Z > 0$, the coefficients of  are all nonpositive. Summing this with D gives  nonpositive off-diagonal coefficients overall. □

Proof of Theorem 4.33 in the non-strict case. By Proposition 4.29, we write

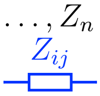


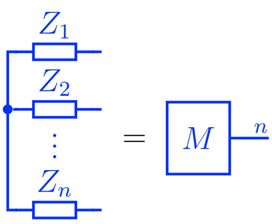
where N is strict and Σ is made up of layers with $Z = 0$. We then proceed inductively on the number of layers that make up Σ. If there are no such layers, M is strict so we conclude by the strict case of the theorem. Otherwise, depending on i and j,



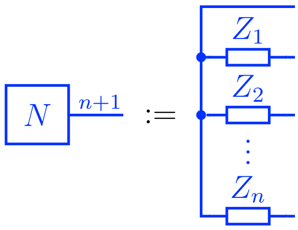
Since meshes are closed under permuting wires (by definition), merging two wires (by Proposition 4.26), and discarding with (by inductive hypothesis), the resulting circuit is a mesh. □

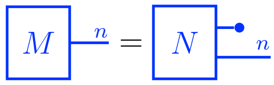
We recover the simpler star-mesh transform from the general theorem.

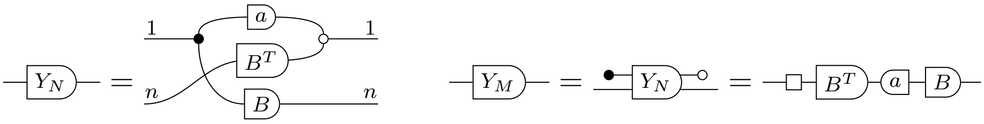
Corollary 4.36. Let $Z_1, \ldots, Z_n$ be positive rational fractions. Let M be the strict mesh that has an element  between node i and node j with $Z_{ij} = Z_i Z_j \sum_k \frac{1}{Z_k}$. Then



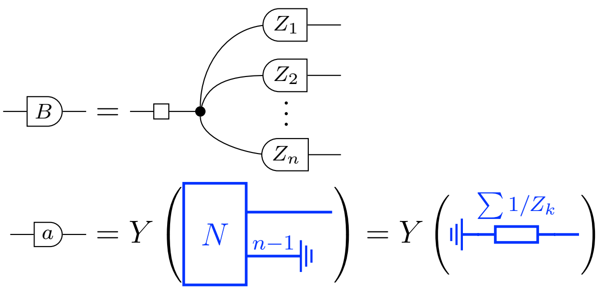
Proof. Let



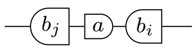
N is visibly a strict mesh, so by Theorem 4.33,  is one too. Following the proof of Theorem 4.33, we find:



where:



Finally

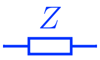
$$Z_{ij} = -1/Y_{Mij} =$$  $$= Z_i Z_j \sum_k \frac{1}{Z_k}$$

□

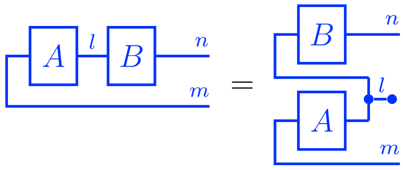
4.2.3 Representing Circuits as Meshes

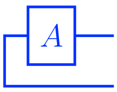
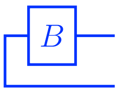
Using the previous section, we can transform any ungrounded passive circuit into a mesh, by repeated application of the star-mesh transform. This gives rise to a general theorem about the shape of controlled circuits.

Theorem 4.37. Any ungrounded passive circuit is a mesh.

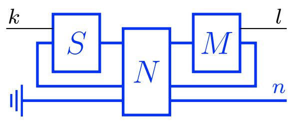
Proof. We proceed inductively. Groundless passive circuits are generated by impedances  and wiring. and are trivially meshes. For the remaining generators:

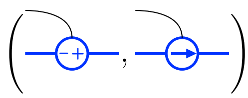
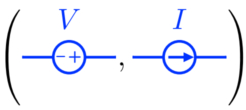
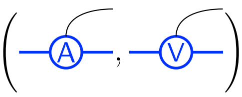
For the induction step, parallel composition is covered in Proposition 4.26. For sequential composition, if A and B are meshes:



where  and  are meshes. By proposition 4.26 and theorem 4.33 the resulting composition is a mesh too. □

Theorem 4.38 (Representation theorem for controlled circuits)**.** An arbitrary controlled circuit C can be written as:



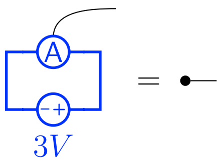
where N is a mesh, S is made by tensoring k controlled sources  and a number of independent sources , and M is made by tensoring l meters .

Proof. By ungrounding (Theorem 4.8) and homogenisation (Theorem 4.3), we restrict ourselves to the case of a linear ungrounded controlled circuit. By repeated application of Proposition 4.15, we rewrite C as a circuit N probed by some sources and meters. By Proposition 4.14, $N \in$ InfolessECirc, and since N is linear

$N \in$ PassiveECirc. Finally using Theorem 4.37, N is a mesh. Those of the sources that are connected to the information inputs constructed during homogenisation become independent sources in the final circuit. □

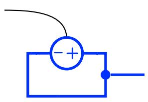
4.3 Short-Circuits

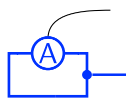
Our framework still contains circuits that do not correspond to physical systems. For example here is a meter whose output is not a single value:



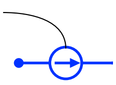
The source of these issues is short-circuits: they are non-physical and they complicate theorems. Classical theorems implicitly only apply to circuits with no short-circuits. In this section we define what short-circuits are in our framework and show that excluding them recovers the expected behavior.

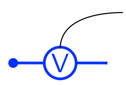
Definition 4.39 (Short-Circuited Element)**.** We call the following two circuits respectively *short-circuited voltage source* and *short-circuited ammeter*:



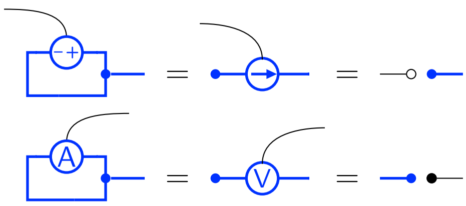


We call the following two circuits respectively *short-circuited current source* and *short-circuited voltmeter*:

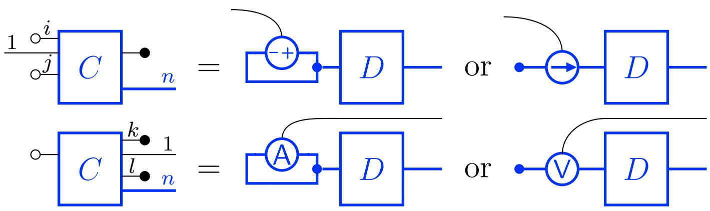




Remark 4.40. Semantically:

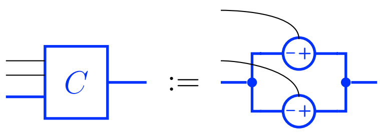


Definition 4.41 (Short-Circuit)**.** A linear controlled circuit C is said to have a short-circuit if, when we discard every information wire but one, the resulting circuit contains any of the above short-circuited elements connected to the remaining information wire.

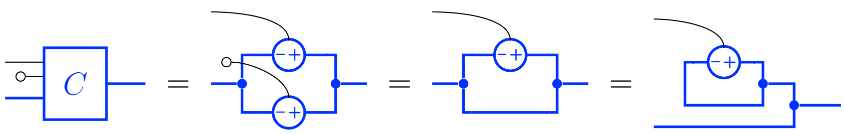


A non-linear controlled circuit has a short circuit if all of its possible homogenizations (Theorem 4.3) have a short-circuit.

Example 4.42. The simplest non-trivial short-circuit is two voltage sources in parallel:

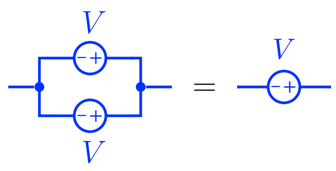


This is indeed a short-circuit according to our definition:

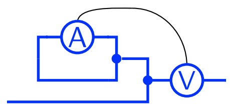


Remark 4.43. To illustrate why the affine case considers all possible homogenizations, observe that the following two circuits are equivalent yet the obvious homogenization of the left one has a short-circuit. Our definition makes it so this

circuit is not considered to have a short-circuit.

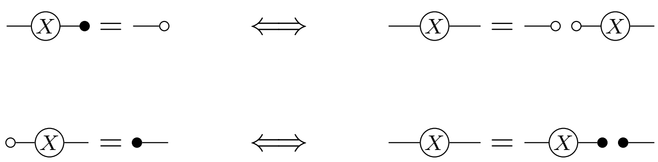


Remark 4.44. This definition does not capture short-circuits if we allow connecting internal information wires. For example in the following, the ammeter is short-circuited but this is not observable from the open information wires. For this reason we consider only controlled circuits.

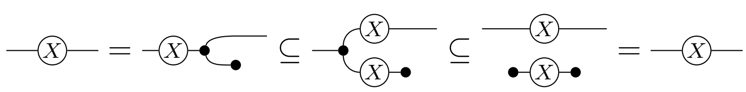


Short-circuits are observable on the admittance as follows:

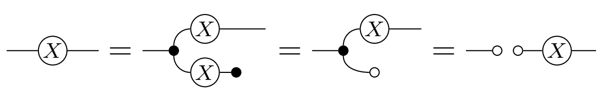
Lemma 4.45. Let —(X)— be a $\mathsf{GLA}_{\mathbb{K}}$ relation.



Proof.

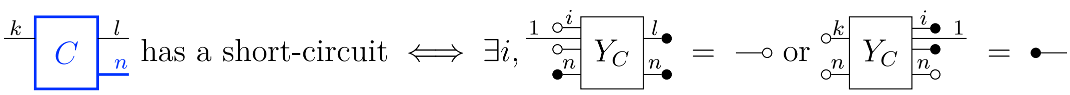


So:

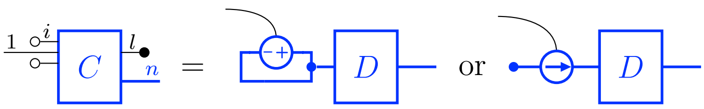


We conclude by —∘ ∘— ⊆ ——. The second statement it then obtained by transposing the first. □

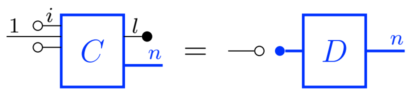
Lemma 4.46. If C is a linear controlled circuit:

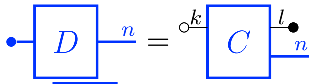
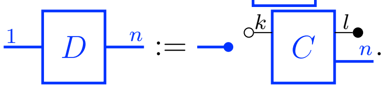


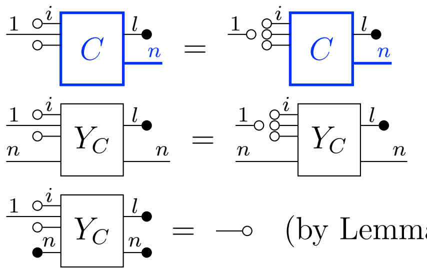
Proof. By definition, C has an input short-circuit if and only if

$\exists i, \exists D,$ 

which is equivalent to

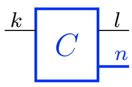
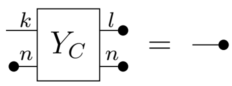
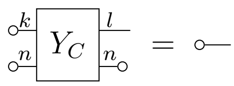
$\exists i, \exists D,$ 

Clearly then , so without loss of generality we can choose . Now:

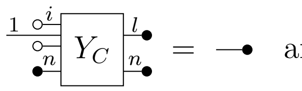
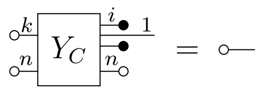
C has an input short-circuit $\iff \exists i,$  (by Lemma 4.45)

The case of output short-circuits is identical modulo transposition. □

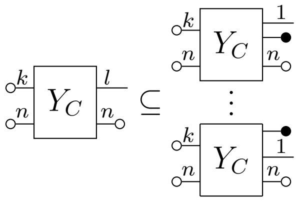
Lemma 4.47. If C is a linear controlled circuit:

 has no short-circuits $\iff$  and 

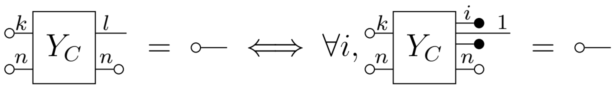
Proof. The only two linear $0 \to 1$ diagrams in $\mathsf{GLA}_{\mathbb{R}(s)}$ are •— and ∘—. So by negating Lemma 4.46, we get:

C has no short-circuits $\iff \forall i,$  and 

Now:



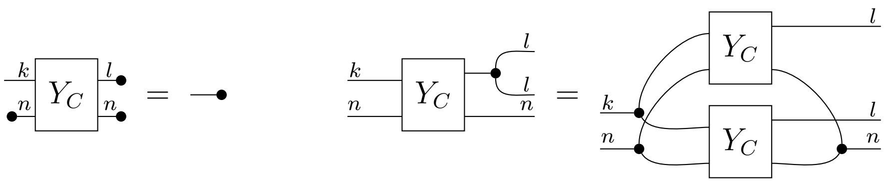
So



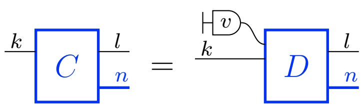
The input case works symmetrically. □

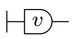
We can now show the main result of this section: in the absence of short-circuits, meters return single values and all source inputs are allowed. Thus it all behaves as would be expected.

Theorem 4.48. If a controlled circuit C has no short-circuits then it is total in its information inputs and single-valued in its information outputs. Formally:



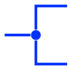
Proof. If C is linear, by Proposition 2.25 the theorem is equivalent to Lemma 4.47. Assume C is not linear. By definition, C has no short-circuits iff it can be written as

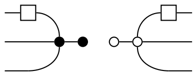
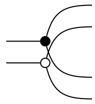


where D is a linear controlled circuit without short-circuits. By the above, Y_D is single-valued and total in the desired way. Plugging in  preserves this property. □

Corollary 4.49. A closed controlled circuit with no short-circuits is a map.

Proof. A closed circuit has no electrical wires, so $\mathcal{I}(C) = Y_C$. By Theorem 4.48, Y_C is total and single-valued, thus is a map. □

Remark 4.50. If the circuit is not closed, then neither its admittance nor its interpretation are necessarily a map. For example, if $C :=$  :

$Y_C =$  $\mathcal{I}(C) =$ 

4.4 Deriving Standard Theorems

We are now equipped to reproduce the major theorems of circuit theory.

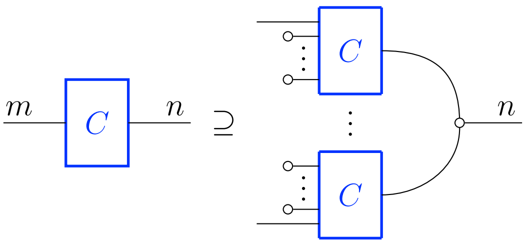
4.4.1 Superposition Theorem

We start with one of the two most useful theorems of classical circuit theory: the Superposition Theorem (the other one being Thévenin's Theorem, see Section 4.4.3).

The superposition theorem ensures that the behavior of a circuit with multiple sources can be calculated from its behaviors with one source turned on at a time. It is used ubiquitously to simplify the analysis of circuits.

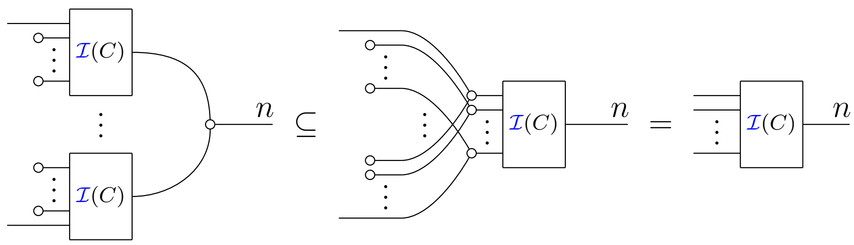
We "turn off" parts of a circuit by plugging ∘— into some information inputs. We can then partially recover the behavior of a circuit from these partially turned off versions, as follows:

Theorem 4.51 (Superposition Theorem)**.** The behavior of a linear closed circuit C with m information inputs is a superset of the sum of its behaviors with one input turned on at a time:



If moreover the circuit is a controlled circuit with no short-circuits, then the inclusion is an equality.

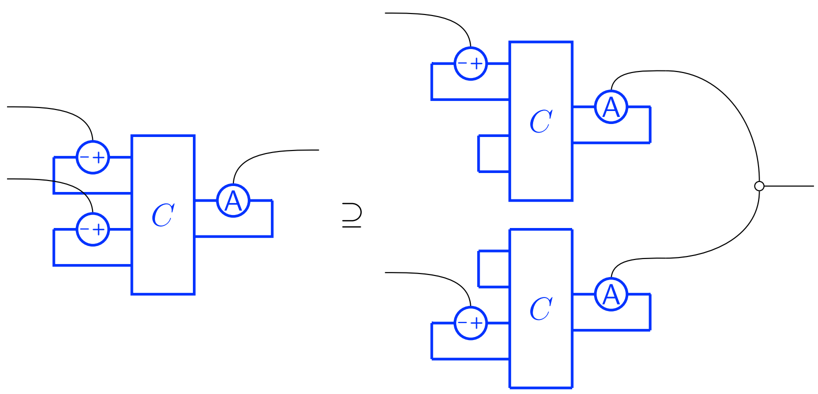
Proof. Since C is in the linear subset, its interpretation lands in GLA. We can then use the fact that GLA is an abelian bicategory of relations [CW87]:



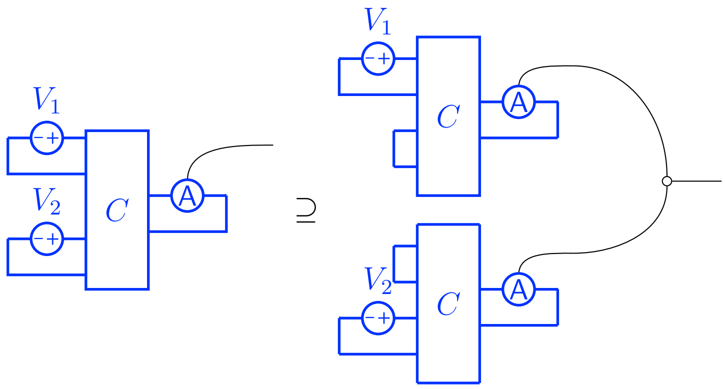
If the circuit is controlled and has no short-circuits, by Corollary 4.49 it is a map. The equality case is the statement that maps are linear. More formally this is a well-known property of maps in abelian bicategories of relations. □

We can recover a more familiar formulation of the superposition theorem by noticing that a voltage source set to zero is equivalent to a plain wire, and a current source set to zero is equivalent to a disconnected circuit:

The theorem above then applies to a circuit with e.g. two voltage sources and one ammeter as:

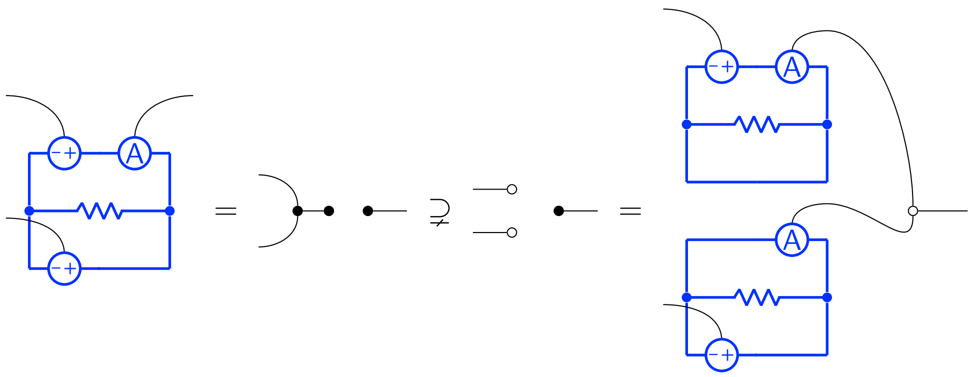


The theorem as stated excludes independent sources, but we can easily recover them using homogenisation:



Still this differs from the standard theorem: the general case of our theorem applies also to circuits that contain short-circuits. As such it is only an inequality in general.

Example 4.52. In the following we have accidentally short-circuited the source, which makes the circuit have no solution if the sources are set to different values. This, indeed, is a degenerate case and we do not have equality.



We recover the familiar equality case in the absence of short-circuits.

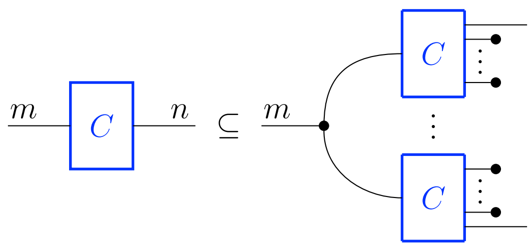
4.4.2 Independent Measurement Theorem

Using the superposition theorem, we can analyse a circuit's information inputs one at a time. What about its information outputs? A very similar theorem applies to them.

This second theorem is quite intuitive: measuring somewhere in a circuit ought to be independent from measurements elsewhere. In other words, we should be able to extract the full behavior of a circuit by considering measurements one at a time. In fact this is so natural as to be assumed true classically, and not mentioned in textbooks. We are being more mathematically precise here: the result is more subtle than one might think.

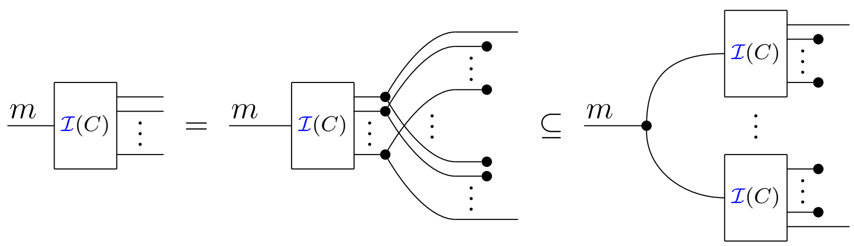
We "discard" parts of a circuit by plugging —• into some information outputs. We can then partially recover the behavior of a circuit from these partially discarded versions, as follows:

Theorem 4.53 (Independent Measurement Theorem)**.** Given a closed circuit C with n information outputs, the behavior of the full circuit is a subset of the behavior of one output at a time, discarding the others.



If moreover the circuit is a controlled circuit with no short-circuits, then the inclusion is an equality.

Proof. The proof is a simple derivation using cartesian bicategory structure:

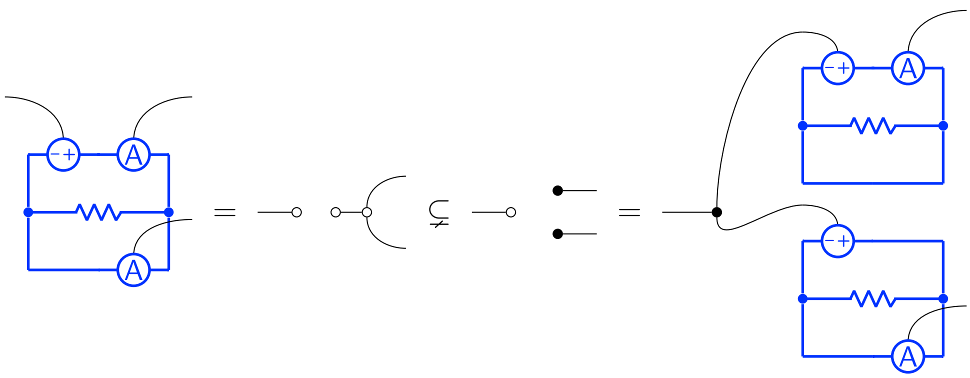


If the circuit is controlled and has no short-circuits, by Corollary 4.49 it is a map. The equality case is then the statement that maps are single-valued, which is part of the definition of a map. □

Like for the superposition theorem, equality holds for most circuits we care about. Similar to setting sources to zero, ignoring a meter turns it into the expected piece of wiring:

Remark 4.54. In a precise sense this is the dual of the superposition theorem (Theorem 4.51): they are related by the colorswap transformation (Definition 2.34). This transformation has more interesting consequences for circuits which we explore in Section 4.4.4.

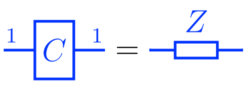
Example 4.55. Cases where we cannot recover the behavior of the circuit from individual measurements are cases of short-circuits, for example:

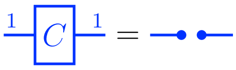


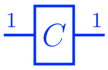
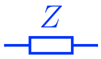
4.4.3 Thévenin's Theorem

We now come to Thévenin's Theorem, a fundamental result in common practice that is also very relevant to compositional reasoning. It allows one to replace a controlled one-port circuit by a much simpler, equivalent one.

Lemma 4.56 (Passive Thévenin Theorem)**.** If C is an ungrounded passive one-port, then one of the following is true:

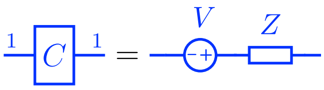
(i)  for some $Z \geq 0$

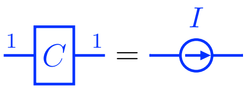
(ii) 

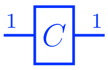
Proof. By Theorem 4.37,  is a mesh. By Corollary 4.32, C can be written in layer form with at most one layer. So C has either no layers and is , or it has a single layer and is  for some $Z \geq 0$. □

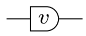
Remark 4.57. Our study of meshes in Section 4.2 is crucial to establish the non-negativity of Z.

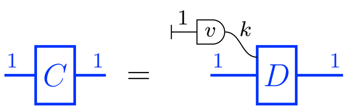
Theorem 4.58 (Thévenin's Theorem)**.** If C is an ungrounded controlled one-port with no open information wires, then one of the following is true:

(i)  for some $V \in \mathbb{R}(s)$ and $Z \geq 0$,

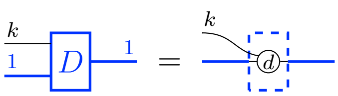
(ii)  for some $I \in \mathbb{R}(s)$,

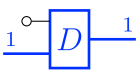
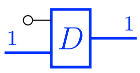
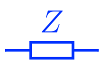
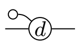
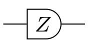
(iii)  denotes the empty relation.

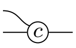
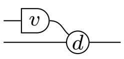
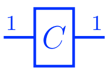
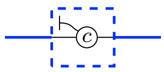
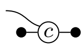
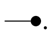
Proof. By homogenisation (Theorem 4.3), there exist a linear $1 \to k$ map  and a linear ungrounded controlled circuit D such that:

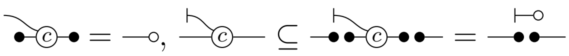
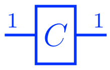


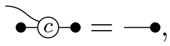
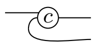
By the representation theorem (Theorem 4.9), D can be written as an impedance box, namely there is d such that

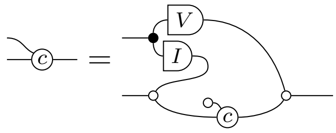


By Corollary 4.17,  $\in$ InfolessECirc. Since it is also linear it is passive, and by Lemma 4.56  = or  for some $Z \geq 0$. Thus  = or  for some $Z \geq 0$.

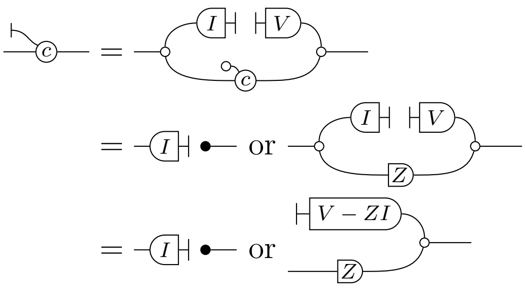
Writing  := , we get  = . Now  is a $1 \to 0$ linear relation, so is either or .

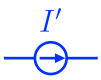
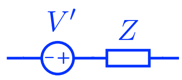
If , so  $= \emptyset$.

Otherwise, , i.e.  is total. By Proposition 2.33,



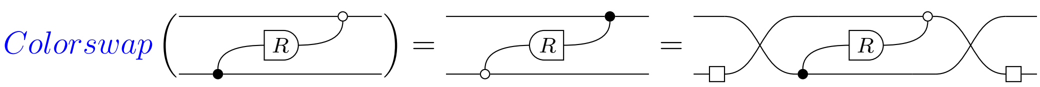
Finally:



which correspond respectively to  with $I' = -I$ and  with $V' = V - ZI$. □

4.4.4 Reciprocity Theorem

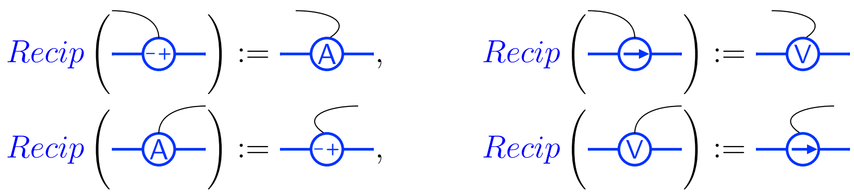
This less common theorem captures a deep symmetry in circuits, related to color-swapping (Definition 2.34) or transposing the underlying linear relations. We start with an observation: color-swapping the interpretation of a resistor has no effect up to a reversible operation on the wires:



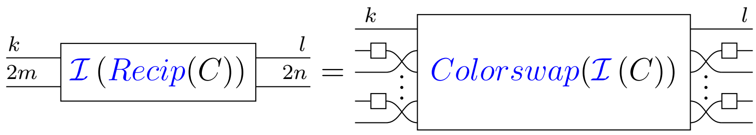
This observation generalizes: we can build an operation on linear circuits that roughly corresponds to color-swapping in the interpretation. We call it reciprocation.

Definition 4.59. *Recip* is the prop functor from the linear subset of SynECirc to itself defined on generators (Definition 3.14) as follows:

- It acts on generators of $\mathsf{GLA}_{\mathbb{R}(s)}$ by color-swapping (Definition 2.34);
- It acts on wiring, ground and passive generators as the identity;
- It acts on the hybrid elements as follows:



Theorem 4.60 (Reciprocity)**.** *Recip* on circuits corresponds to color-swapping in the interpretation, modulo a reversible operation on the image of each electrical wire:



Proof. We saw it for the resistor above. It is similarly immediate on the other generators. We conclude with a straightforward induction. □

Corollary 4.61. *Recip* extends to a prop functor from $\mathsf{LinECirc}$ to itself.

Proof. By Theorem 4.60, for any two linear circuits C and D in $\mathsf{SynECirc}$:

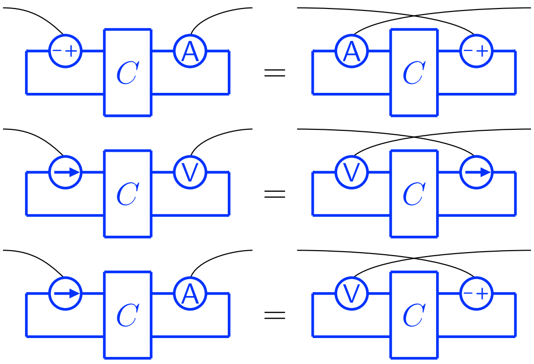
$$\begin{aligned}\mathcal{I}\,(C) = \mathcal{I}\,(D) &\implies Colorswap(\mathcal{I}\,(C)) = Colorswap(\mathcal{I}\,(D)) \\ &\implies \mathcal{I}\,(Recip\,(C)) = \mathcal{I}\,(Recip\,(D))\end{aligned}$$

□

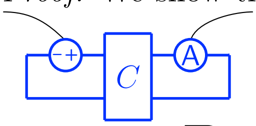
This theorem is particularly interesting when applied to circuits made of elements that are left unchanged by reciprocation (so-called *reciprocal* elements). This includes our passive elements, as well as several elements that we could have included in our framework like mutual inductances and transformers.

The Reciprocity Theorem as conventionally known then applies to such circuits. It states a strong relationships between responses of a $2 \to 2$ circuit fed with various sources and the responses of its left-to-right mirror.

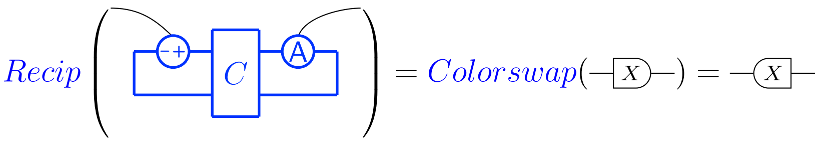
Corollary 4.62 (Textbook Reciprocity Theorem [DK69])**.** For a passive circuit C, each of these equalities holds in the absence of short-circuits:



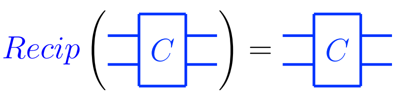
Proof. We show the first equality; the others are proven similarly. Let —X— :=



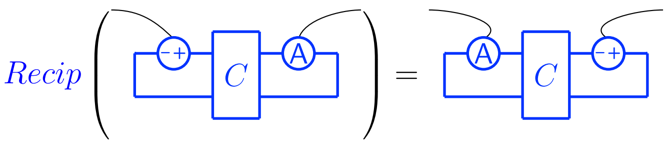
. Since —X— has no short-circuits, by Corollary 4.49 it is a $1 \to 1$ map —X—. By theorem Theorem 4.60:



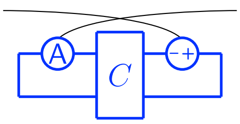
Moreover



, therefore by definition of *Recip*:



So —X— =



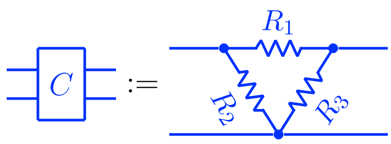
. □

Remark 4.63. There is something important to note here: the theorem could seem to say that a reciprocal $2 \to 2$ circuit behaves the same backwards, but this in incorrect. The relationship is only between specific responses. Note in particular the third correspondence: it relates a current input/current output response with a voltage input/voltage output response, which is not what one might expect at first glance.

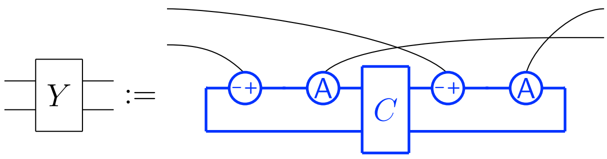
We can see it more concretely with an example.

Example 4.64. Let C be the following circuit made of a triangle of resistors. Let us analyze the following closed circuit Y using impedance calculus.

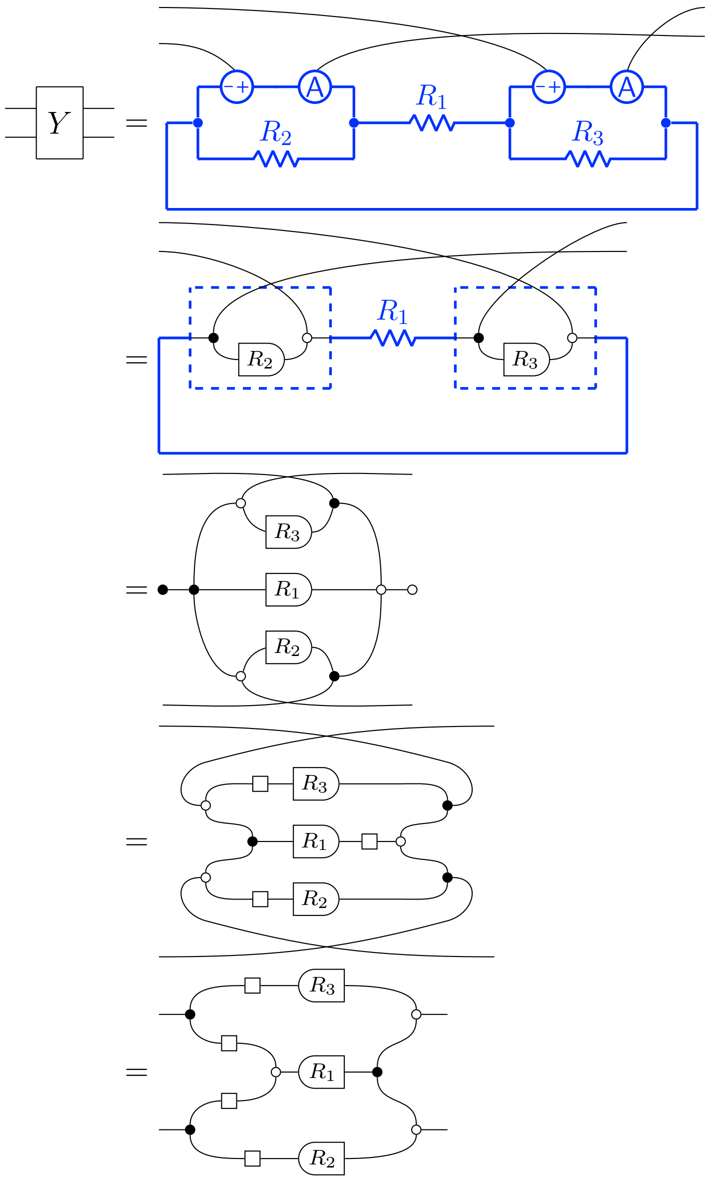
Let



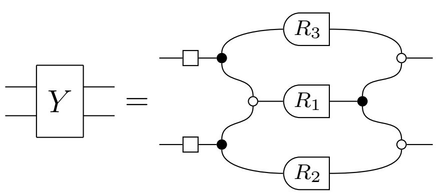
and



Now

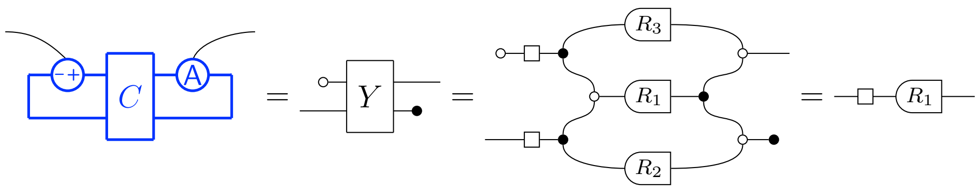


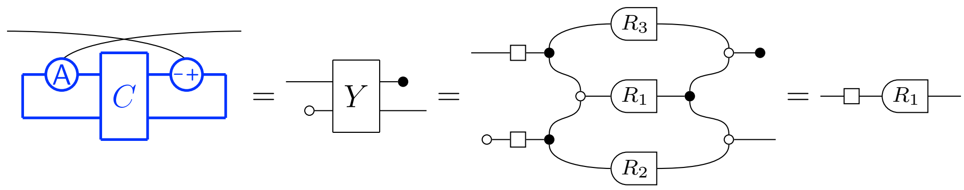
Thus



From Y we see that C does not behave identically when reversed: indeed, reversing C amounts to swapping R_2 and R_3, which gives a different Y in general.

However, as per the theorem, some responses are reversible, e.g.:





5

Graphical Piecewise-Linear Algebra

We now have an effective framework to study linear and affine circuits. It is however not expressive enough to model some basic electronic elements, most notably the diode. The limitation is at the level of semantics: the relational semantics of a diode is not affine and thus cannot be captured by GAA. It is instead piecewise-linear.

In this chapter we define a new graphical language that extends GAA with the ability to talk about piecewise-linear relations. GPA already extends GAA with a generator for comparing numbers ($\geq$). Building on this, we introduce syntax for logical disjunction, which is sufficient to express any piecewise-linear relation.

The core result is a presentation of piecewise-linear relations as a graphical axiomatic theory, that we prove complete. In other words, just like for GAA, we can reason about piecewise-linear relations purely graphically. We then rebuild our prop of electrical circuits on top of this new theory, however without capacitors and inductors.

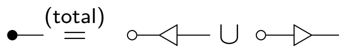


Figure 5.1: Additional axioms of $\mathsf{GPLA}_{\mathbb{K}}$.

5.1 The Theory of Piecewise-Linear Relations

5.1.1 Axiomatic Theory and Semantics

Definition 5.1. $\mathsf{GPLA}_{\mathbb{K}}$ (Graphical Piecewise-Linear Algebra) is the $\cup$-prop generated by the generators of $\mathsf{GPA}_{\mathbb{K}}$ modulo the axioms of Figures 2.1, 2.3, 2.4 and 5.1.

A diagram in $\mathsf{GPLA}_{\mathbb{K}}$ is therefore a finite union of diagrams in $\mathsf{GPA}_{\mathbb{K}}$. The key addition of $\mathsf{GPLA}_{\mathbb{K}}$ over $\mathsf{GPA}_{\mathbb{K}}$ is the axiom of *totality*. This states that any number belongs to the non-negative *or* to the non-positive fragment of $\mathbb{K}$. Remarkably, this simple axiom is the only one we need to add to $\mathsf{GPA}_{\mathbb{K}}$ to obtain a complete theory for pl relations.

Definition 5.2. $[\![\cdot]\!]_{\mathsf{GPLA}_{\mathbb{K}}} : \mathsf{GPLA}_{\mathbb{K}} \to \mathsf{Rel}_{\mathbb{K}}$ is the functor of $\cup$-props that extends $[\![\cdot]\!]_{\mathsf{GPA}_{\mathbb{K}}}$ over joins in the obvious way.

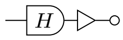
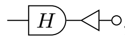
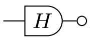
Proposition 5.3. $[\![\cdot]\!]_{\mathsf{GPLA}_{\mathbb{K}}}$ is well-defined.

Proof. For the functor to be well-defined, it must respect the defining axioms of $\mathsf{GPLA}_{\mathbb{K}}$. We know from Definition 2.20 that it respects the axioms of Figures 2.1, 2.3 and 2.4. The remaining axiom (Figure 5.1) is true by totality of the $\geq$ order, which is part of the definition of an ordered field. □

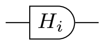
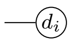
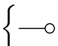
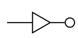
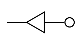
We call piecewise-linear (pl) any relation in the image of this functor, i.e., any relation that is a finite union of polyhedral relations. As far as we know, this is the first time that this notion appears in print. It captures a straightforward notion of piecewise-linearity, namely submanifolds of $\mathbb{K}^n$ that can be subdivided into affine fragments.

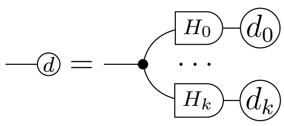
5.1.2 Completeness Theorem

As planned, GPLA$_{\mathbb{K}}$ forms a complete theory for pl relations. We prove that claim in this section. We start by defining appropriate normal forms for polyhedral and pl relations, and then show that every diagram can be reduced to normal form. GPLA$_{\mathbb{K}}$ is inherits compact closure from GPA$_{\mathbb{K}}$, thus without loss of generality we limit ourselves to $n \to 0$ diagrams.

Definition 5.4. We use the name "hyperplane" for a nonzero affine map  : $n \to 1$. A given hyperplane H defines two half-spaces  and , as well as an affine subspace . Since inequality is not strict, the half-spaces include the affine subspace.

[BDGS21, Theorem 14] gives polyhedral relations a normal form as a set of inequations of the form $A_i x + b_i \geq 0$. In other words, the normal form is given by an intersection of half-spaces. For our purposes we define a related but slightly different normal form.

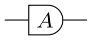
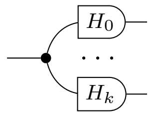
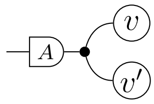
Definition 5.5. A GPA$_{\mathbb{K}}$-diagram $d : n \to 0$ is in polyhedral normal form if there are hyperplanes  and diagrams  $\in$ { , ,  } such that:

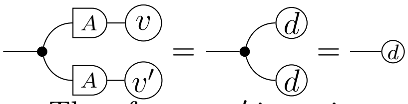


where the d_i are minimal in the following sense: fixing the set of hyperplanes H_i, we consider all choices of d_i that give d when composed as above. We then require the d_i in the normal form to be minimal (wrt the order of GPA$_{\mathbb{K}}$) among those. We call the choice of such d_i a *valuation* for d relative to the hyperplanes H_i.

Definition 5.6. We say that a diagram D of GPLA$_{\mathbb{K}}$ is in pl normal form if it is written as a non-empty union of diagrams d_j each in the language of GPA$_{\mathbb{K}}$ (i.e. without unions), the d_j are in the polyhedral normal form defined in Definition 5.5, and all the normal forms use the same set of hyperplanes.

Lemma 5.7. Every $d : n \to 0$ in GPA$_{\mathbb{K}}$ has a polyhedral normal form.

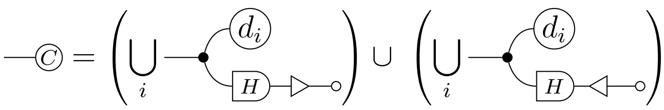
Proof. The normal form from [BDGS21, Theorem 14] already has the right shape. We only need to find a minimal valuation. Observe that the intersection of two valuations for d is again a valuation for d: let v and v' be two valuations for d relative to the hyperplanes H_i. If we write  :=  then  =



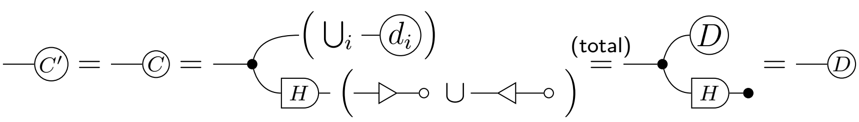
Therefore $v \cap v'$ is again a valuation for d. Since there are finitely many valuations, we construct the minimal one by intersecting them all. □

Lemma 5.8. If a diagram D of GPLA$_{\mathbb{K}}$ is in pl normal form and H is a hyperplane, there exists C in pl normal form such that $D \overset{\mathsf{GPLA}_{\mathbb{K}}}{=} C$ and $\mathsf{Hyperplanes}(C) = \mathsf{Hyperplanes}(D) \cup \{H\}$.

Proof. We write the normal form of D as $D = \bigcup_i d_i$. Define C to be the following morphism:



We transform C into C' by finding minimal valuations as in Lemma 5.7. This makes C' be in pl normal form and keeps the same set of hyperplanes. Since we add the same hyperplane H to all d_i, $\mathsf{Hyperplanes}(C') = \mathsf{Hyperplanes}(D) \cup \{H\}$. Moreover:



□

Theorem 5.9. Every morphism of GPLA$_{\mathbb{K}}$ has a pl normal form.

Proof. Let D be a $n \to 0$ morphism of GPLA$_{\mathbb{K}}$. First using distributivity of the union over sequential and parallel composition, we move all the uses of the union to the top-level.

Thus D is written $\bigcup_i d_i$ where each d_i does not use the union, i.e. is in the language of $\mathsf{GPA}_{\mathbb{K}}$. We then rewrite each d_i into polyhedral normal form using Lemma 5.7.

Each d_i is thus also individually in *pl normal form*, so we can use Lemma 5.8 to add to each d_i all the hyperplanes of the other d_j. For each i we get a new diagram $d'_i \overset{\mathsf{GPLA}_{\mathbb{K}}}{=} d_i$ in pl normal form, and all the d'_i use the same set of hyperplanes. So $\bigcup_i d'_i$ is a pl normal form for D. □

Before we can prove completeness, we need a final notion: the relative interior of a polyhedral relation [Roc97, Chapter II, Section 6], which is the set of its points that do not touch any of its faces.

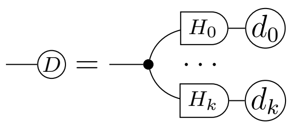
Definition 5.10. Let D be a diagram in polyhedral normal form. We define $\mathsf{Int}(D)$ to be the set of points $x \in [\![D]\!]_{\mathsf{GPLA}_{\mathbb{K}}}$ for which $H_i(x) \neq 0$ when $-\!\!\!\textcircled{\scriptsize d_i} \neq -\!\!\!\circ$. In other words, $H_i(x)$ is nonzero for all i where it can be nonzero without x leaving $[\![D]\!]_{\mathsf{GPLA}_{\mathbb{K}}}$.

Note that we define Int only on polyhedral normal form diagrams. Int appears to be representation-independent at least when $\mathbb{K} = \mathbb{R}$, but we will not try to prove it in the general case as we do not need this here.

Remark 5.11. This is not the topological notion of interior. In particular, this notion is independent from the dimension of the surrounding space: a polyhedron of dimension $0 < k < n$ within $\mathbb{R}^n$ has an empty topological interior but a nonempty relative interior, as we will see in the next theorem. $\mathsf{Int}(D)$ instead coincides with the topological interior of D with the topology of the smallest containing affine space.

Lemma 5.12. Let D be a diagram in polyhedral normal form. If $[\![D]\!]_{\mathsf{GPLA}_{\mathbb{K}}}$ is nonempty, then $\mathsf{Int}(D)$ is nonempty.

Proof. First, write D in polyhedral normal form:



Up to negating some of the H_i, we can assume that none of the $-\!\textcircled{d_i}$ are $-\!\!\lhd\!\!-\!\circ$. If $\forall i. -\!\textcircled{d_i} = -\!\circ$, then by definition $\mathsf{Int}(D) = [\![D]\!]_{\mathsf{GPLA}_{\mathbb{K}}}$ which is nonempty so we are done. Assume then that $-\!\textcircled{d_i} = -\!\!\rhd\!\!-\!\circ$ for at least some i. For each such i, by minimality of the d_i in the normal form there must be a $x_i \in [\![D]\!]_{\mathsf{GPLA}_{\mathbb{K}}}$ such that $H_i(x_i) > 0$. We pick such an x_i for each i, and define $x := \frac{1}{p}\sum_i x_i$ to be their average, where p is the number of chosen x_i. $[\![D]\!]_{\mathsf{GPLA}_{\mathbb{K}}}$ is a polyhedron hence convex, so $x \in [\![D]\!]_{\mathsf{GPLA}_{\mathbb{K}}}$. H_i is an affine map, hence is concave, thus if we had picked an x_i then $H_i(x) \geq \frac{1}{p}\sum_j H_i(x_j) \geq \frac{1}{p}H_i(x_i) > 0$. So for each i either $-\!\textcircled{d_i} = -\!\circ$ or $H_i(x) > 0$, hence $x \in \mathsf{Int}(D)$. □

Theorem 5.13 (Completeness)**.** $[\![D]\!]_{\mathsf{GPLA}_{\mathbb{K}}} \subseteq [\![C]\!]_{\mathsf{GPLA}_{\mathbb{K}}} \implies D \overset{\mathsf{GPLA}_{\mathbb{K}}}{\subseteq} C$

Proof. Using compact closure we assume without loss of generality that D and C have n inputs and 0 outputs. Using Theorem 5.9, we reduce D and C into pl normal form. Using Lemma 5.8, we add each others' hyperplanes to D and C so that they both use the exact same set. So $D = \bigcup_i d_i$ and $C = \bigcup_j c_j$, where the d_i and c_i are in polyhedral normal form and use a same set of hyperplanes $\{H_k\}_k$. Pick one of the d_i in D (since by Definition 5.6 the unions are nonempty).

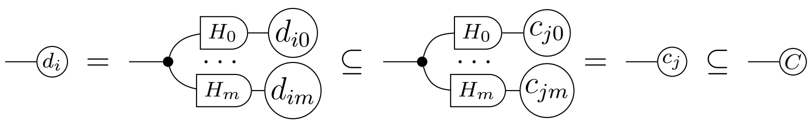
If d_i is the empty polyhedron, we have $[\![d_i]\!]_{\mathsf{GPA}_{\mathbb{K}}} = \emptyset \subseteq [\![c_0]\!]_{\mathsf{GPA}_{\mathbb{K}}}$, so by completeness of $\mathsf{GPA}_{\mathbb{K}}$ we get $d_i \overset{\mathsf{GPA}_{\mathbb{K}}}{\subseteq} c_0$. Thus $d_i \overset{\mathsf{GPLA}_{\mathbb{K}}}{\subseteq} c_0 \overset{\mathsf{GPLA}_{\mathbb{K}}}{\subseteq} C$.

Otherwise d_i is nonempty, and using Lemma 5.12 we pick $x \in \mathsf{Int}(d_i)$. Then:

$$x \in \mathsf{Int}(d_i) \subseteq [\![d_i]\!]_{\mathsf{GPLA}_{\mathbb{K}}} \subseteq [\![D]\!]_{\mathsf{GPLA}_{\mathbb{K}}} \subseteq [\![C]\!]_{\mathsf{GPLA}_{\mathbb{K}}} = \left[\!\!\left[\bigcup_j c_j \right]\!\!\right]_{\mathsf{GPLA}_{\mathbb{K}}} = \bigcup_j [\![c_j]\!]_{\mathsf{GPLA}_{\mathbb{K}}}$$

Thus there is a j such that $x \in [\![c_j]\!]_{\mathsf{GPLA}_{\mathbb{K}}}$. Now pick a k. If $-\!\textcircled{d_{ik}} = -\!\circ$, then

$-\!\!(d_{ik}) \overset{\mathsf{GPLA}_{\mathbb{K}}}{\subseteq} -\!\!(c_{jk})$ regardless of $-\!\!(c_{jk})$. If $-\!\!(d_{ik}) = -\!\!\triangleright\!\!-\!\circ$, then by definition of $\mathsf{Int}(d_i)$, we have $H_k(x) > 0$. Since moreover $x \in [\![c_j]\!]_{\mathsf{GPLA}_{\mathbb{K}}}$, $-\!\!(c_{jk})$ must be $-\!\!\triangleright\!\!-\!\circ$. If $-\!\!(d_{ik}) = -\!\!\triangleleft\!\!-\!\circ$, similarly $-\!\!(c_{jk})$ must be $-\!\!\triangleleft\!\!-\!\circ$. In all three cases, $-\!\!(d_{ik}) \overset{\mathsf{GPLA}_{\mathbb{K}}}{\subseteq} -\!\!(c_{jk})$. This is the case for every k, so:



Finally, since we have $d_i \overset{\mathsf{GPLA}_{\mathbb{K}}}{\subseteq} C$ for all i, we derive $D = \bigcup_i d_i \overset{\mathsf{GPLA}_{\mathbb{K}}}{\subseteq} C$. □

Thus $\mathsf{GPLA}_{\mathbb{K}}$ is complete for its semantics in $\mathsf{Rel}_{\mathbb{K}}$. As a bonus, it is also complete for any other nontrivial semantics.

Theorem 5.14. Let P be a $\cup$-prop. Then any a morphism of $\cup$-props $F : \mathsf{GPLA}_{\mathbb{K}} \to P$ such that $F(\,\boxed{}\,) \neq \emptyset$ is faithful.

Proof. Just like in Theorem 3.2,

$$\vec{v} \in [\![-\!\!(R)]\!]_{\mathsf{GPLA}_{\mathbb{K}}} \iff \bullet \in [\![\vdash\!\!\boxed{v}\!\!-\!\!(R)]\!]_{\mathsf{GPLA}_{\mathbb{K}}}$$

So

$$[\![-\!\!(R)]\!]_{\mathsf{GPLA}_{\mathbb{K}}} \subseteq [\![-\!\!(S)]\!]_{\mathsf{GPLA}_{\mathbb{K}}} \iff \forall \vec{v}, [\![\vdash\!\!\boxed{v}\!\!-\!\!(R)]\!]_{\mathsf{GPLA}_{\mathbb{K}}} \subseteq [\![\vdash\!\!\boxed{v}\!\!-\!\!(S)]\!]_{\mathsf{GPLA}_{\mathbb{K}}}$$

By completeness of $[\![\cdot]\!]_{\mathsf{GPLA}_{\mathbb{K}}}$,

$$-\!\!(R) \subseteq_{\mathsf{GPLA}_{\mathbb{K}}} -\!\!(S) \iff \forall \vec{v}, \vdash\!\!\boxed{v}\!\!-\!\!(R) \subseteq_{\mathsf{GPLA}_{\mathbb{K}}} \vdash\!\!\boxed{v}\!\!-\!\!(S)$$

Therefore F is faithful if and only if it is faithful on $0 \to 0$ diagrams. The only $0 \to 0$ diagrams in $\mathsf{GPLA}_{\mathbb{K}}$ are $\vdash\!\!-\!\circ$ and $\boxed{}$ and they are distinct, so F is faithful iff $F(\vdash\!\!-\!\circ) \neq F(\,\boxed{}\,)$. We conclude by $F(\vdash\!\!-\!\circ) = \emptyset$ which is true because $\emptyset$ is the least element and F preserves the semilattice structure. □

5.1.3 Pointwise GPLA Semantics

The intended semantics for $\cup$ regarding electrical circuits is that $\cup$ acts pointwise, so that the piecewise-linear equations hold true at any given point in time. This requires a different target for our semantics, as the $\cup$-structure of Rel is not pointwise.

Definition 5.15. A partial function from $\mathbb{R}$ to X is a pair (I, f) of $I \subseteq \mathbb{R}$ and $f : I \to X$. I is called the *domain* of f, and noted dom_f.

Definition 5.16. Given two partial functions f and g such that $\text{dom}_f = \text{dom}_g$, their sum $f + g$ is defined pointwise as usual. It is not defined if $\text{dom}_f \neq \text{dom}_g$.

Definition 5.17. Given a partial function f and $I \subseteq \text{dom}_f$, $f|_I$ is called the *restriction of f to I* and is the partial function with domain I defined by $f|_I(x) = f(x)$.

Definition 5.18. $\perp$ is the partial function with empty domain.

Definition 5.19. Given two partial functions f and g, their *juxtaposition*, when it exists, is the unique function $f \cup g$ such that $\text{dom}_{f \cup g} = \text{dom}_f \cup \text{dom}_g$ and both f and g are restrictions of $f \cup g$. It is defined if and only if f and g are equal on $\text{dom}_f \cap \text{dom}_g$.

Proposition 5.20. Juxtaposition is associative and commutative, with $\perp$ as neutral element.

Definition 5.21. Let D be the set of all partial functions into $\mathbb{R}$. Let D_n be the set of all partial functions into $\mathbb{R}^n$. When $\vec{f} \in D_n$ and $\vec{g} \in D_m$ have the same domain, we call $\langle \vec{f}, \vec{g} \rangle$ the obvious element of D_{n+m}. It is undefined if their domain do not match.

Definition 5.22. PointwiseRel is the $\cup$-prop:

- whose arrows $n \to m$ are subsets $R \subseteq D_{n+m}$ closed under juxtaposition and restriction;

- with identity $n \to n$ given by the diagonal $\{\langle \vec{f}, \vec{f} \rangle \mid \vec{f} \in D_n\}$;
- symmetry $n + m \to m + n$ given by the relation

$$\left\{ \left\langle \left\langle \begin{matrix} \vec{f} \\ \vec{g} \end{matrix} \right\rangle, \left\langle \begin{matrix} \vec{g} \\ \vec{f} \end{matrix} \right\rangle \right\rangle \mid \langle \vec{f}, \vec{g} \rangle \in D_{n+m} \right\}$$

- composition given by $R; S = \{\langle \vec{f}, \vec{h} \rangle \mid \exists \vec{g}.\ \langle \vec{f}, \vec{g} \rangle \in R \wedge \langle \vec{g}, \vec{h} \rangle \in S\}$;
- monoidal product given by

$$R_1 \oplus R_2 = \left\{ \left\langle \left\langle \begin{matrix} \vec{f_1} \\ \vec{f_2} \end{matrix} \right\rangle, \left\langle \begin{matrix} \vec{g_1} \\ \vec{g_2} \end{matrix} \right\rangle \right\rangle \mid \langle \vec{f_1}, \vec{g_1} \rangle \in R_1 \wedge \langle \vec{f_2}, \vec{g_2} \rangle \in R_2 \right\}$$

- and union of R and S given by $R \cup S = \{\vec{f} \cup \vec{g} \mid \vec{f} \in R, \vec{g} \in S\}$.

Proposition 5.23. PointwiseRel is well-defined.

Proof. The prop structure is almost identical to Rel_D; it is a prop for the same reasons. The delicate part is the $\cup$-prop structure. The associativity and commutativity of function juxtaposition gives the same for the prop join. Idempotency comes from the requirement that arrows be closed under juxtaposition. The bottom element is the singleton $\{\bot\}$. Thus arrows of a given type form a bounded join-semilattice. Remains to show compatibility of $\cup$ with the prop structure.

$$\begin{aligned}(R \cup S); T =& \{\langle \vec{f_1} \cup \vec{f_2}, \vec{h} \rangle \mid \exists \vec{g_1}, \vec{g_2}, \langle \vec{f_1}, \vec{g_1} \rangle \in R, \\ & \langle \vec{f_2}, \vec{g_2} \rangle \in S, \langle \vec{g_1} \cup \vec{g_2}, \vec{h} \rangle \in T\} \\ (R; T) \cup (S; T) =& \{\langle \vec{f_1} \cup \vec{f_2}, \vec{h_1} \cup \vec{h_2} \rangle \mid \exists \vec{g_1}, \vec{g_2}, \langle \vec{f_1}, \vec{g_1} \rangle \in R, \\ & \langle \vec{f_2}, \vec{g_2} \rangle \in S, \langle \vec{g_1}, \vec{h_1} \rangle \in T, \langle \vec{g_2}, \vec{h_2} \rangle \in T\}\end{aligned}$$

Without loss of generality we can restrict $\vec{f_1}$, $\vec{g_1}$ and $\vec{h_1}$ (who have all the same domain) such that $\mathrm{dom}_{\vec{f_1}} \cap \mathrm{dom}_{\vec{f_2}} = \emptyset$. Now to go from top to bottom we pick

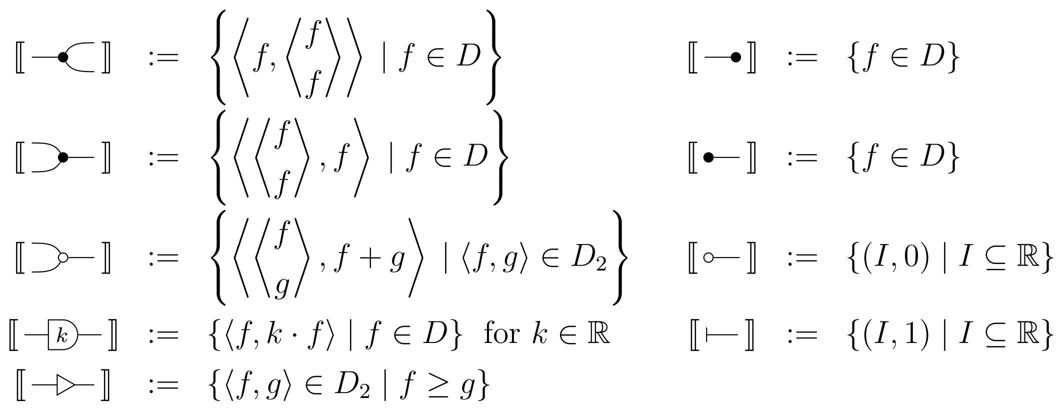


Figure 5.2: Pointwise semantics of $\mathsf{GPLA}_{\mathbb{R}}$.

$\vec{h_i} := \vec{h}|_{\mathrm{dom}\,\vec{f_i}}$. And to go from bottom to top we pick $\vec{h} := \vec{h_1} \cup \vec{h_2}$, which is well-defined since their domains are disjoint. The compatibility with the monoidal product involves substantially the same considerations. □

Definition 5.24. $[\![\cdot]\!]_{\mathsf{PwGPLA}} : \mathsf{GPLA}_{\mathbb{R}} \to \mathsf{PointwiseRel}_D$ is the functor of $\cup$-props defined on generators as Figure 5.2.

Proposition 5.25. $[\![\cdot]\!]_{\mathsf{PwGPLA}}$ is well-defined.

Proof. This needs to respect the axioms of Figures 2.1, 2.3, 2.4 and 5.1. This is obviously built to match the semantics of $\mathsf{GPLA}_{\mathbb{R}}$ pointwise. As such, all the axioms that do not involve $\cup$ are immediately true by virtue of being true pointwise. Remains the axiom of totality:

$$[\![-\!\lhd\!-\!\circ \cup -\!\rhd\!-\!\circ]\!]_{\mathsf{PwGPLA}} = \{\vec{f} \cup \vec{g} \mid \vec{f} \leq 0, \vec{g} \geq 0\}$$

Given $\vec{h} \in D_n$, let $I = \{x \in \mathrm{dom}_{\vec{h}} \mid \vec{h} \leq 0\}$ and $J = \mathrm{dom}_{\vec{h}} \setminus I$. Then $\vec{h} = \vec{h}|_I \cup \vec{h}|_J \in [\![-\!\lhd\!-\!\circ \cup -\!\rhd\!-\!\circ]\!]_{\mathsf{PwGPLA}}$. □

Theorem 5.26. $\mathsf{GPLA}_{\mathbb{R}}$ is complete for $[\![\cdot]\!]_{\mathsf{PwGPLA}}$.

Proof. By Theorem 5.14, since indeed $\boxed{} = \{\bullet\} \neq \emptyset$ in $\mathsf{PointwiseRel}$. □

Theorem 5.27. Let $-(R)-$ be a diagram in $\mathsf{GPLA}_{\mathbb{R}}$ and $\vec{f} \in D_{n+m}$.

$$\vec{f} \in [\![-(R)-]\!]_{\mathsf{PwGPLA}} \iff \forall x, \vec{f}(x) \in [\![-(R)-]\!]_{\mathsf{GPLA}_{\mathbb{R}}}$$

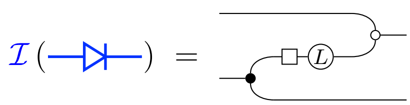
Proof. This is clearly true on generators, and easily preserved by tensor and composition. Remains to show it is inductively preserved by union. One direction is immediate from the definitions of $\cup$ and juxtaposition. For the other direction, assume that $\forall x, \vec{f}(x) \in [\![-(A)- \cup -(B)-]\!]_{\mathsf{GPLA}_{\mathbb{R}}}$. Let $I = \{x \in \mathrm{dom}_{\vec{f}} \mid \vec{f}(x) \in [\![-(A)-]\!]_{\mathsf{GPLA}_{\mathbb{R}}}\}$ and $J = \mathrm{dom}_{\vec{f}} \backslash I$. Since $\forall x, \vec{f}(x) \in [\![-(A)-]\!]_{\mathsf{GPLA}_{\mathbb{R}}} \cup [\![-(B)-]\!]_{\mathsf{GPLA}_{\mathbb{R}}}$, we get on J that $\forall x \in J, \vec{f}(x) \in [\![-(B)-]\!]_{\mathsf{GPLA}_{\mathbb{R}}}$. By inductive hypothesis $\vec{f}|_I \in [\![-(A)-]\!]_{\mathsf{PwGPLA}}$ and $\vec{f}|_J \in [\![-(B)-]\!]_{\mathsf{PwGPLA}}$. Then $\vec{f} = \vec{f}|_I \cup \vec{f}|_J \in [\![-(A)- \cup -(B)-]\!]_{\mathsf{PwGPLA}}$. □

5.2 Electronic Circuits

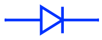
We can finally return to our original motivation: modeling electronic circuits. This is unfortunately more limited than could be hoped, as we do not have a setting that can model both piecewise-linearity and differentiation.

5.2.1 The Prop of Electronic Circuits

Definition 5.28. Let PIECirc be the prop defined just like ECirc (Definition 3.27), with two differences: the interpretation lands in GPLA instead of $\mathsf{GAA}_{\mathbb{R}(s)}$, and we replace the generators that require differentiation (capacitor C, inductor L, impedance Z) with a new basic element (diode) with interpretation:

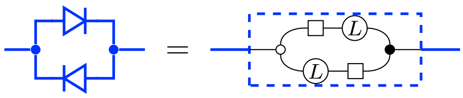


where $-(L)- := -\triangleright\!\circ\ \circ- \cup -\circ\ \circ\!\triangleleft-$. The interpretation functor is otherwise syntactically identical as before, and equality in PIECirc is modulo this functor.

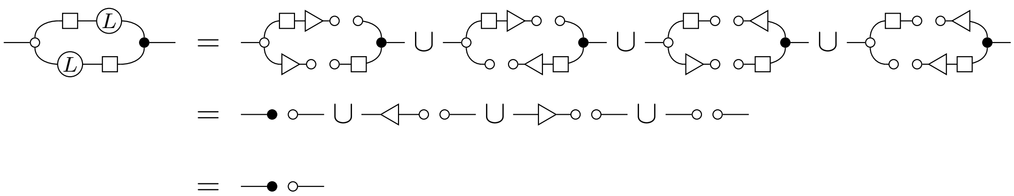
As we hoped to achieve,  models the ideal diode: its semantics allow either a left-to-right current across it with no voltage, or no current across it and a positive voltage.

Remark 5.29. Despite using a different interpretation, many results from the previous two chapters still hold in PIECirc. For example, impedance calculus, homogenisation and ungrounding still apply, and all circuits without diodes obey the same theorems in PIECirc as they did in ECirc. Formally, PIECirc and ECirc have in common the subprop generated by resistors, wiring, hybrid elements and the generators of $\mathsf{GAA}_{\mathbb{R}}$.

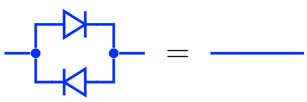
Example 5.30. As an example of calculation in PIECirc, consider two diodes in parallel and opposite directions. We use impedance calculus:



Where:



Hence

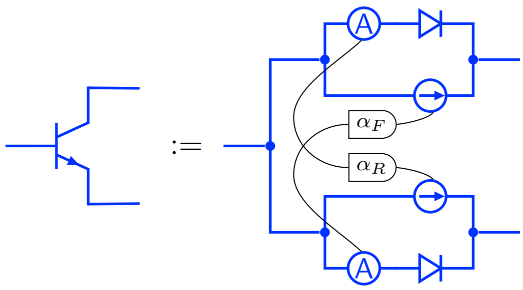


Remark 5.31. This mixed use of graphical and formulaic syntaxes is comparable to the use of sums in categorical quantum mechanics [CK17, Sec. 5.1.3]. Both have this "extra-diagrammatic" feel but are nonetheless more practical than fully graphical alternatives like 3-dimensional sheet diagrams.

5.2.2 A Simple Transistor Model

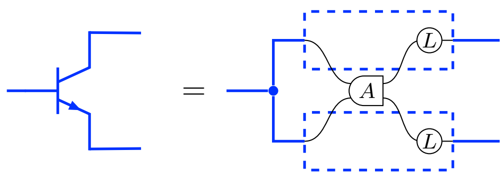
To showcase the expressivity of this framework, we encode transistors using a simple piecewise-linear model based on the Ebers–Moll model [EM54] proposed by Sedra et al. [SS87, p.903].

Definition 5.32. We fix two parameters $\alpha_F, \alpha_R \in [0, 1]$. We define an element called *transistor* as follows:

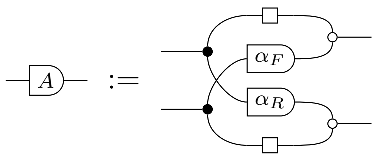


Proposition 5.33. The transistor exhibits the characteristic property of transistors: when current flows into its input, current can pass between its outputs; when no current flows into its input, current is blocked between its outputs.

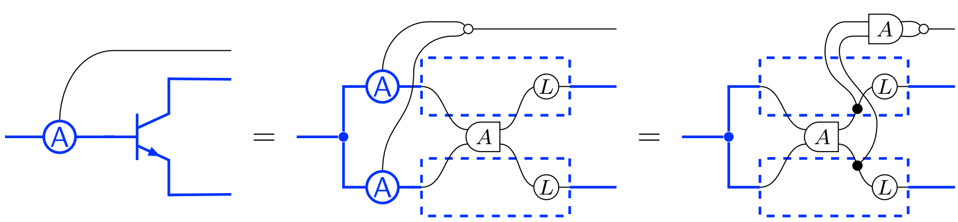
Proof. Without going into excessive details about the calculations, impedance calculus gives us:



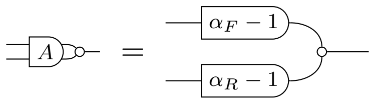
where:



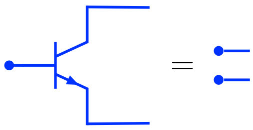
Then:



Since —Ⓛ— ⊆ , the inputs to are nonnegative. Moreover:



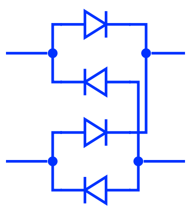
Combined with the fact that both α_R and α_F are below 1, we deduce that the output of is nonpositive, which means that current into the input of the transistor is constrained to flow in the left-to-right direction. Moreover, whenever the input current is 0, this forces the inputs of A to 0 too, which sets all currents to zero. In pictures:



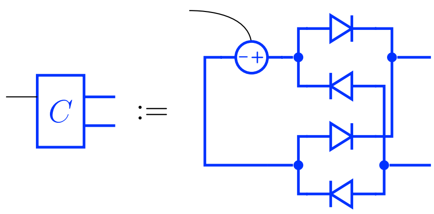
And when the input current is nonzero, the output currents can be nonzero too. □

5.2.3 Example Circuit Analysis

In this section we illustrate reasoning about electronic circuits by studying the full-bridge rectifier [Gra97, Ele15] A full-bridge rectifier is the following circuit, used to convert AC current to DC:

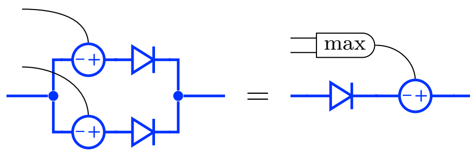


When fed an alternating current on the left, it outputs a pulsating voltage of constant sign, that can then be smoothed to serve as direct current. We want to analyze the behavior of this circuit when fed a voltage:

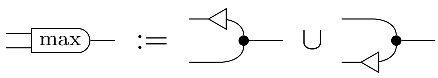


Some things can be said early on from looking at this circuit. For example, the diodes can be seen to force the current to go in a single direction. First let us study a related circuit.

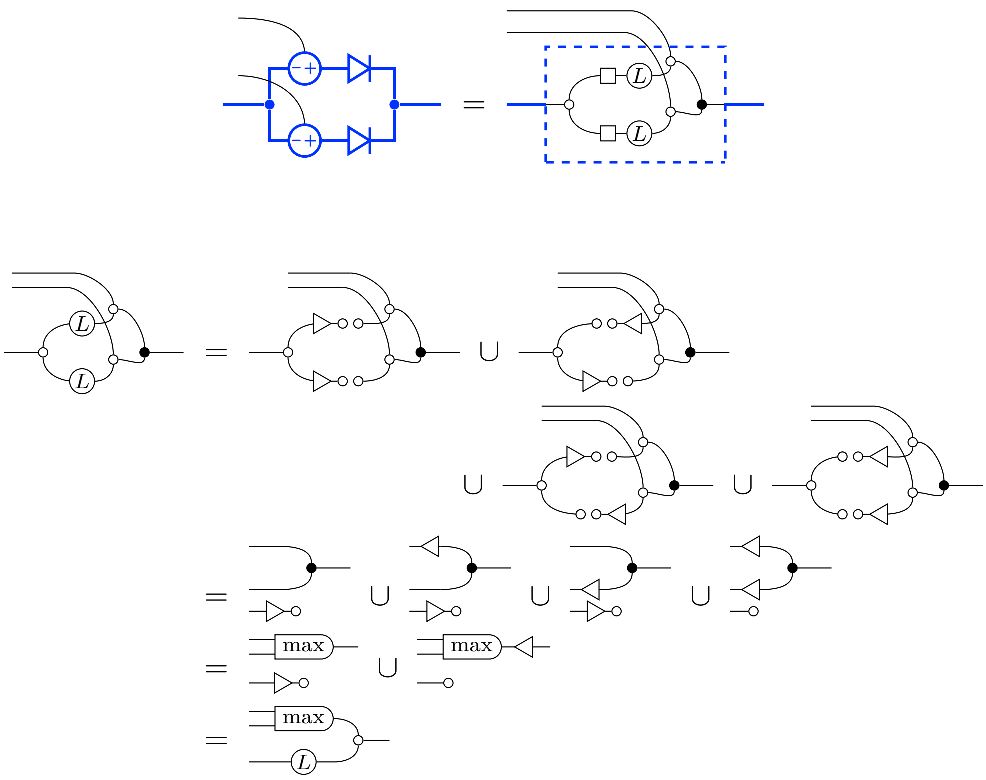
Lemma 5.34.



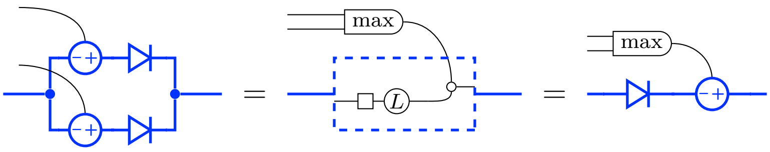
where max is the maximum function, defined as:



Proof. Using impedance calculus:

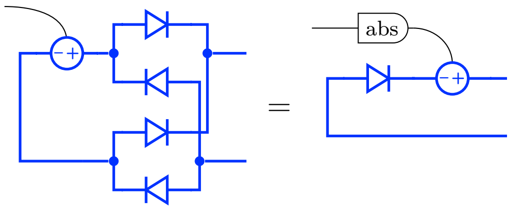


Therefore:

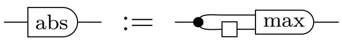


□

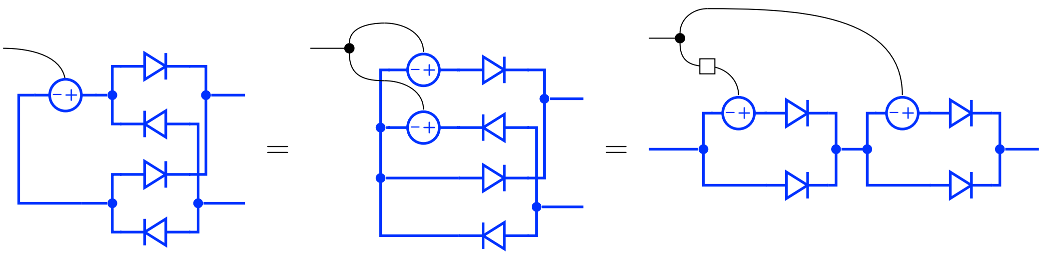
Proposition 5.35.



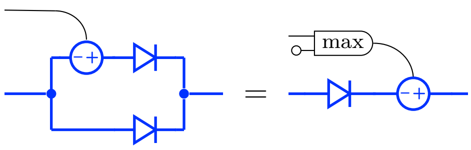
where abs is the absolute value function, defined as:



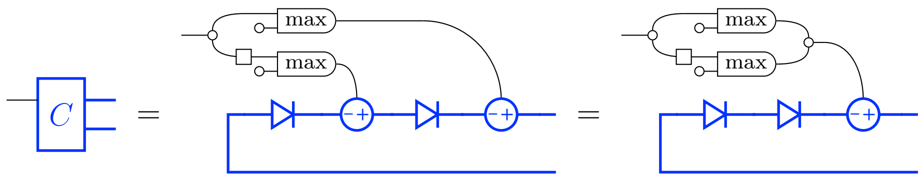
Proof.



By Lemma 5.34,

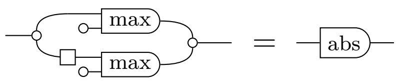


So:



We conclude by noticing that:

and



□

This simplified form clarifies the behavior of the original circuit. We see that the circuit indeed outputs a positive voltage, specifically the absolute value of its input, and forces the current direction. This concludes our analysis of this circuit.

Remark 5.36. The number of alternatives grows exponentially in the number of diodes. If one were to list them all, analysis would become quickly impractical. This is not a specificity of our graphical language; using non-graphical systems of equations presents the same issue. In both cases one must be careful when reasoning to keep the number of cases manageable.

6 Conclusion

We set out to analyze electrical circuits using graphical relations. This led us to develop graphical languages ($[\![\cdot]\!]_X$, GADA, GPLA, and $[\![\cdot]\!]_{\mathsf{PwGPLA}}$) that extend the expressivity of GLA, on top of which we built a graphical setting for idealized electric and electronic circuits (ECirc and PlECirc). Most importantly in my view, we showed that such an approach is practical by using it concretely to reason about circuits.

6.1 Building Compositional Models

The core intention of this thesis was to explore the practicality of modeling a domain using graphical equations. This was a success! We derived many classical results as well as some less standard ones with relative ease. This approach appears practical both for global results like Thévenin's Theorem (Theorem 4.58) and for analyzing individual circuits like in Section 5.2.3.

The compositional framework in itself is not new. A similar compositional circuit framework with graphical semantics was already defined by Coya [Coy18], and an even more similar one by Bonchi et al. [BPSZ19] was the basis of our work. Our main contribution was to take the framework seriously as a reasoning tool.

In fact at the mathematical level even adapting standard theorems is somewhat redundant: they are simple consequences of the completeness of our graphical language. Beyond the exploration of relational semantics, this work was an exploration in style into the practicality of graphical relational languages for the working mathematician. I consider this a resounding success. I found proving general results with graphical equations much more pleasant and practical than the standard alternative of matrices and graphs. Hybrid elements and impedances boxes are a great example of the flexibility afforded by this approach.

So from one angle, what we have built is a new formal setting in which to study circuits, as a relational and graphical alternative to the more common multigraph model (Definition 2.54). I would like to propose a larger view: that this work is an illustrative example of a particular approach to building models. By pushing the complexity down to the semantic language, we have made the modeling step (the definition of ECirc and its interpretation) straightforward. Assuming the availability of graphical languages for various domains in common use, this makes it straightforward to model a known domain; one need only rewrite the equations in a graphical language. Contrast this with e.g. Fong et al.'s compositional semantics for circuits [BF18]: their approach describes a circuit through its power functional; proving that their setup works required a fair amount of technical work.

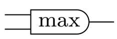
A valuable benefit of this approach is that models built that way are likely to be extensible. We saw it when adding the diode in Definition 5.28: once we figured out the right graphical language, adding the diode simply required specifying its graphical interpretation. Similarly, consider adding an element like a transformer, which connects four nodes instead of two. It is immediate in our framework but would require work to adapt to the standard multigraph model (Definition 2.54).

I hope this work sparks interest in building models that combine composability, convenience and mathematical correctness using graphical languages.

6.2 Graphical Equational Languages

To support our developments around circuits, we built a few different extensions to GLA.

GADA (Definition 3.10) is a formalization of the ubiquitous Laplace trick used by engineers. This has strong similarities with the work of Coya [Coy18]. In both cases we use linear relations on $\mathbb{R}(s)$ to model differential equations. The difference is in the target domain: Coya simply models circuits as such a linear relation, in other words as their transfer function. By contrast, per our principle of "maximally obvious semantics", we chose a domain of smooth functions, and did some work to adapt GAA to it.

GPLA (Definition 5.1) came as a surprise. The informal consensus at the time seemed to be that languages more expressive than GPA were unlikely to be complete. Piecewise-linear spaces also are not a well-known object. Luckily this turned out to work well, and the simplicity of the axiomatization of GPLA was a particularly pleasant surprise. The choice of a term-like $\cup$ syntax strays from the purely-diagrammatic flavor of our approach, but proved much more practical for calculations than purely-graphical alternatives such as generating GPLA from GAA combined with 
.

This thesis emphasized the practical usability of the languages, in particular with the choice of join syntax and the preference for straightforward semantics. All the graphical languages we built have "the obvious semantics"; they aim to reproduce familiar equations with no added mystery. For example, our nonobvious setup of PointwiseRel (Section 5.1.3) was there to obtain the obvious pointwise semantics (Theorem 5.27).

I hope this work sparks more exploration into practical graphical languages for the working mathematician.

6.3 Unifying GADA and GPLA

A big question raised by our work is the following: is there a graphical language that can reason about the semantics of diodes and capacitors together? It so happens that adding the interpretation of diodes to GAA suffices to generate all of GPLA, so the question becomes: is there a version of GPLA that can talk about differentiation?

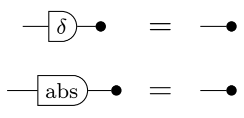
To be more precise, the constraint is that we want functional semantics that are morally pointwise. Namely, the semantic functor $[\![\cdot]\!]$ would verify:

$$\vec{f} \in [\![-\textcircled{A}]\!] \iff \forall t. \vec{f}(t) \in [\![-\textcircled{A}]\!]_{\mathsf{GPLA}}$$

This raises several interesting challenges.

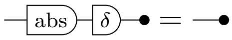
6.3.1 Smooth Functions

The main tension is between two equations we want to be true:



where δ is differentiation and abs is the absolute value function we saw in Proposition 5.35.

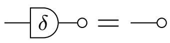
The first of these equations is true in GADA and says that every function is differentiable. The second is true in GPLA, is a direct consequence of the axiom of totality, and says that for every function f in the domain, there exists g that coincides with $\text{abs}(f)$. Combining the two, we get:



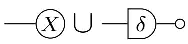
In English: the absolute value of every function is differentiable. This is of course problematic. Solutions could include using distributions, or cleverly managing the range of the partial functions. We tried several approaches but none succeeded.

6.3.2 Injectivity of Differentiation

Even then, a second issue arises from the equation

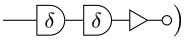
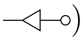
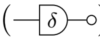


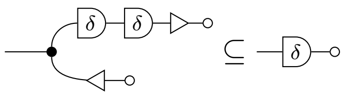
A lot of our developments in Chapter 3 rely on treating differentiation like a plain number. This means that differentiation should be injective. For this to hold we need to rule out constant functions, and more generally any nonzero solution to a simple linear differential equation. We did that in Section 2.4 by forcing functions to be zero on $(-\infty, 0]$. If we want disjunctions however, a function that is sometimes constant could belong to



for some X. Having the derivative behave like a number therefore excludes many more functions than we did in Section 2.4. This exposes a tradeoff between convenience of reasoning and the ability to use standard functions like step functions or sinusoidals.

6.3.3 Completeness

If we do find a sound theory that can express GPLA and derivatives, there remains the big question of completeness. Already the combination of GPA and derivatives is unexplored. An example of the kind of things one might need such a theory to prove is that a convex function () that is also bounded above (e.g. by 0: ) is necessarily constant (). Graphically:



This is an exciting open question.

6.4 Limitations

This work revealed important limitations in our approach. A big source of limitations is the underlying graphical language we use. As we saw we do not have a language that can reason properly about both diodes and capacitors. We also do not yet have a language with binary multiplication. This prevents a number of electrical analyses, notably around electrical power. Beyond the languages themselves, using graphical languages for functions prevents many natural considerations, like “this function reaches a certain value eventually”, or the naming of constants of integration. Whether or not this can be elegantly alleviated is an important open question.

More prosaically, graphical equations get unwieldy when large. This does get easier with practice, but in complex cases I tend to go back and forth with normal equations to solve a problem.

An unexpected realization from this work is that handling generic relations is significantly less convenient than dealing with maps. A lot less can be said about a generic relation than a generic map. A lot of the work of Chapter 4 involved delineating classes of circuits whose semantics are maps. This is a good illustration of why maps are the dominant paradigm.

6.5 Future Work

Our work opens up a number of specific avenues of future work. Of course a lot remains to be explored about the props we defined. Questions abound, like whether all maps in GPLA are continuous, which results of Chapters 3 and 4 still hold in PIECirc, or whether there is a complete graphical language into which we can interpret both diodes and capacitors.

In terms of electric circuits, much can be done in our framework. Notions like connectivity, dual circuits and circuit synthesis seem likely to be well-suited to the graphical approach. More generally, ECirc could become the basis of interactive tools to explore the behavior of circuits. From a teaching perspective in particular, categorical quantum mechanics [CK17] has shown that well-contained graphical theories are potent pedagogical tools.

In terms of electronic circuits, our framework models them from a signals perspective, i.e. with full detail about the dynamics of currents and voltages. This can in theory be used to model stateful electronic circuits (like flip-flops) and other logical notions of bits and computation. Whether this is convenient in practice remains to be explored, as well as whether a bit-oriented framework could be fruitfully linked with this signal-oriented one.

In terms of graphical languages for equations, I see two fruitful directions of investigation. The first is to find a sound unification of GPLA and GADA; we saw in Section 6.3 why this appears tricky. The second is to extend GPLA with a generator for binary multiplication. There is hope for this to be well-behaved because this would describe semialgebraic sets, which are known to be a computable theory. This new language would know about polynomials, square roots, and $\neq$, which would significantly increase what equations can be written graphically.

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