ABSTRACT

First passage percolation theory in its most general form is the randomised version of the well-known shortest route problem. It thus has several important physical applications. Ordinary percolation theory is but a special case of this more general problem which we formulate as follows.

To each arc of an arbitrary, countably infinite, connected graph g we independently assign a non-negative random variable called the *time coordinate* of that arc. This assignment of random variables induces a *time state* 'w' on g. The 'length' of any path of g is the sum of the time coordinates of its component arcs.

Let P,Q, be two nodes of g. When g has time state 'w', t(P,Q;w) denotes the infimum of the 'lengths' of the paths linking P and Q. t(P,Q;w) is the *first passage time* between P and Q for the time state w. We show that it is a random variable on a well defined probability space and notice that if R is any other node of g

$$t(P,Q;w) + t(Q,R;w) \geq t(P,R;w).$$

This relationship leads us to the following definition. If $(\Omega F P)$ is a probability space, and T is the set of non-negative integers, the 2-parametered sequence of non-negative

random variables $\left[x_{mn}(w) : m < n, \quad m,n \in T, \ w \in \Omega \right]$ is a <u>subadditive stochastic process</u> if

a) $x_{mn}(w) + x_{np}(w) \geq x_{mp}(w) \qquad (m < n \leq p)$

b) The distribution of $x_{mn}(w)$ depends only on the difference $(m-n)$

c) The expected value $E\, x_{mn}(w)$ exists for all m,n.

Consider now first passage percolation on the lattice of integer points (x.y), the arcs (all of unit length) being parallel to the x- and y- axes, and unoriented, while the time coordinates of the arcs have a <u>common</u> distribution U. Let $a_{mn}(w)$ be the first passage time between the nodes $(m,0)$, $(n,0)$ when the lattice has time state w. We prove that provided U has finite mean $\bar{u}$, $\{a_{mn}(w)\}$ is a subadditive stochastic process on a probability space.

Also, the theorems we have proved about subadditive processes enable us to show

$$\lim_{n \to \infty} E\, a_{on}(w)/n = \mu \leq \bar{u}$$

while if the U distribution has a finite r^{th} moment for some $r > 1$, then as $n \to \infty$, $a_{on}(w)/n$ converges to μ with probability 1. $\mu = \mu(U)$ is called the <u>time constant</u>; it depends only on the distribution U. If the second moment of the U distribution exists, then $a_{on}(w)/n$ converges in mean square as $n \rightarrow \infty$, which implies that var $a_{on}(w)$ is $o(n^2)$ as $n \rightarrow \infty$.

If $t_{on}(w)$ is the first passage time under w between the origin and (n,o) over paths which lie inside the strip bounded by $x = 0$, $x = n$, then we prove quantitatively the intuitively appealing idea that there is little difference between $a_{on}(w)$ and $t_{on}(w)$, or indeed between these times and the first passage time $s_{on}(w)$ from the origin to the ordinate $X = n$, over the same restricted set of paths. $s_{on}(w)$ is the minimum of the first passage times from the origin to those nodes of the lattice which lie on $X = n$ and we make a detailed study of these and other first passage times which can be formulated in terms of subadditive stochastic theory.

Classical renewal theory suggests a different approach to the problem. For any time state w, define the integer valued random variable $x_t(w)$ by $x_t(w) = \sup n : t_{on}(w) \leq t$, $t \geq 0$. Thus $(x_t(w), 0)$ is the lattice point on the non-negative x-axis, furthest from the origin, which can be reached in time t. When the underlying U distribution is bounded away from zero and infinity we prove an "elementary renewal theorem"

$$\lim_{t \to \infty} E\, x_t(w)/t = \mu^{-1}.$$

where μ is the time constant associated with U. We show also that as $t \to \infty$, $t^{-1}x_t(w)$ converges in mean square to μ.

The fundamental importance of the time constant $\mu \equiv \mu(U)$ is obvious. We explore its functional dependence on U and determine an upper bound which shows that when U is the standardised rectangular distribution $\mu \leqslant \cdot 425$.

This upper bound is improved by a direct simulation of first passage percolation which shows that with 95% confidence, $\mu(U) \leqslant \cdot 328$ for the same standardised rectangular distribution. We also obtain estimates of the mean value functions and variances of the above first passage times which suggest several further problems. For example, they strongly suggest that $\lim\limits_{n \to \infty} n^{-1} \text{ var } t_{0n}(w) < \infty$, which is a much stronger result than we have been able to show theoretically.

The problems studied in this thesis have not been studied previously.

TOPICS IN STOCHASTIC PROCESSES
with special reference to
FIRST PASSAGE PERCOLATION THEORY

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CONTENTS

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CHAPTER I

The Problem, Its Origin, and Applications.

1.1. The Origin of First Passage Percolation Theory

In 1957, Broadbent and Hammersley presented the first mathematical formulation of percolation theory. Since then much work has been done in this field and it was from this that the problem of first passage percolation theory first arose. Before presenting the latter in its most general form, its mode of growth from ordinary percolation theory is illustrated by examples (1.1.1), (1.1.2).

Example 1.1.1. Suppose the trees in a large orchard are planted at the intersections of a square lattice, that the distance between trees is such as to make it possible for a diseased tree to infect only its 4 nearest neighbours, and that moreover, an infected tree has (independently for each neighbour) a probability p of infecting that neighbour. Percolation theory is concerned with the probability $P_N(p)$, that under these conditions disease from 1 tree will spread to more than N trees.

An extension of the above example is

Example 1.1.2. Suppose that instead of there being a probability structure on the spread of infection there is a random process governing the time of infection. That is, a given tree once infected will not infect a neighbouring tree until time u has elapsed, where u is a non-negative random variable independently but identically distributed for each neighbouring tree of the diseased tree. First passage percolation theory considers the time at which infection first spreads outside a given region.

More generally, first passage percolation theory considers the following problem. Let g be a connected graph with a countable set of arcs $\{l_i\}_{i=0}^{\infty}$ and nodes $\{P_{ij}\}$, where P_{ij} is the intersection of l_i, l_j. For given i, j, P_{ij} may or may not exist. To each arc l_i which does exist we assign a non-negative random variable u_i drawn from a distribution U_i and we may regard 'u_i' as the time taken for a particle to travel along the arc l_i. In its most general form, first passage percolation theory is concerned with the shortest time of travel between two distinguished nodes of the graph say P, Q. This shortest time is called the <u>first passage time</u> between P and Q and is denoted by t(P, Q). That t(P, Q) is a random variable will be shown rigorously in chapter 2. Obviously the distribution of t(P,Q) will depend on which distributions

U_i are used. The bulk of the work below is devoted to determining properties of these first passage times in certain idealised situations.

1.2. The Shortest Route Problem

A well known problem in graph theory is

Example 1.2.1. The Shortest Route Problem

"Given a connected graph g with nodes $\{x_i\}_{i=0}^{\infty}$ and arcs $\{l_{ij}\}$ which may or may not exist we assign to each existing arc a number $c(l_{ij}) \geq 0$ which is called the "length" of l_{ij}; find a path μ from a vertex x_o to vertex x_n such that the total length $\sum_{\ell \varepsilon \mu} c(\ell)$ should be as small as possible."

This problem has many applications, for example let X be a set of localities and U a set of roads connecting these localities. We shall suppose that all road intersections have been included in X. For a given road l, the number c(l) may signify its length in miles, the cost of travelling over it, or the time taken to travel this road (with due allowance made for traffic conditions). Then we can look for the shortest, cheapest, or quickest means of travelling between two towns. In each of these problems we would have to solve (1.2.1).

First passage percolation problems on a graph are merely the extension of (1.2.1) to include the case where c(1) may be a random variable. Many algorithms for selecting the shortest route in example (1.2.1) have been obtained. These are discussed in a review article by Pollack and Wiebenson (1960). It is an interesting and much more difficult problem to decide a) if there exists an optimal general strategy when the "lengths" are random variables and b) what is this strategy?

An extension of (1.2.1) is the determination of the k^{th} best route through a network. Solutions of this problem have important applications. If for any reason the shortest (best) route is unavailable, then alternate routes are desirable. Or, it may be acceptable in some problems to use any route whose "length" is within say 10% of the shortest route. Such alternate routes are useful in road traffic studies, to determine the routes that motorists may use, another use is in determining alternate message routes in communication networks. This problem is reviewed by Pollack (1961). The problem of k^{th} passage percolation theory is however treated only briefly below but the techniques used in considering first passage times are easily extended to k^{th} passage times.

It will be noticed that many of the applications for which the shortest route problem is a model are more closely approximated to by first passage theory. For example, in traffic problems the time to travel a given road is more accurately described by a random variable drawn from a sample distribution than by a constant.

This is mentioned by Fulkerson (1962) and he points out that this randomised version of the shortest route problem is a model for the warehousing problem with random costs, and also has several other dynamic programming applications.

1.3. PERT Networks and Critical Paths.

In 1959, Malcolm (et al) introduced the concept of a PERT network, as a technique for measuring and controlling development progress for the Polaris Fleet Ballistic Missile Programme. Since then similar techniques have been introduced by many organisations, and a vast literature of a not very mathematical nature has sprung up (see Bigelow (1962) for a bibliography).

A PERT network is a directed, acyclic graph. In the PERT model such a network is viewed as representing a partial ordering of the many individual 'jobs' (arcs of the network) that together comprise some 'project' (the complete network), the partial ordering coming from the requirement that all inward pointing jobs at a node must be completed before any outward pointing job at the node can be started. If jobs are assigned duration times (i.e. arcs are assigned lengths), the length of the longest path from the origin to the terminal represents the duration time of the entire project. Such a path is called a critical path.

One of the fundamental problems in this work is the estimation of the expected critical path length. The usual practice in this situation is to assume that the random variable 'arc-lengths' are drawn from a Beta distribution, defined by a range, (difference between the most optimistic and pessimistic estimates of a job's duration time) and a mode (most likely job time). For further details see Clark (1962).

The estimation of the expected critical path length is then carried out by replacing the random path length by its expected value and thus reducing the problem to a 'static', "longest route problem". Provided the graph is directed and acyclic, (which is the case here) many algorithms for this problem exist. This, however, produces an estimate 'g' of the critical path length e which in most cases is highly optimistic ($g \leqslant e$). Fulkerson (1962) has devised a better approximation f, satisfying $g \leqslant f \leqslant e$.

The relation between this problem and first passage percolation is as follows. Firstly most of the algorithms for solving the shortest route problem (example 1.2.1) can be readily adapted to the solution of a longest route problem on a directed acyclic graph. Hence if a general strategy could be formulated for the randomised shortest route problem it is reasonable to expect that an adaptation of this strategy would solve the PERT network problem. Secondly many of the techniques used in first passage percolation theory are applicable to the critical path problem, provided the graph is directed and acyclic.

1.4. The Relation between Ordinary and First Passage Percolation

Ordinary percolation theory as defined by Broadbent and Hammersley (1957) may be defined as follows. Let g be an arbitrary connected graph, let x_0 be a specified node (the origin), let each arc of g be closed or open with probability 1-p, p respectively, independently for each arc, where p is a fixed constant $0 \leqslant p \leqslant 1$. Fluid is supplied at the origin and can only 'flow' along open arcs. The basic problem of percolation theory is to determine $P_N(p)$, the probability that fluid will spread from the origin to at least N nodes of the graph g. Obviously $P_N(p)$ is a non-decreasing function of p such that for all N,

$$(1.4.1) \qquad 0 = P_N(0) \leqslant P_N(p) \leqslant P_N(p+) \leqslant P_N(1) = 1$$

A closely linked problem to the above is to obtain the probability $P_p(x_i)$ which is that under the same conditions fluid will spread from the origin to the node x_i of g. This problem may easily be framed in terms of first passage percolation theory.

Let the arcs of g have 'length' 0 or 1 with probability p, and 1-p respectively. Then the expected first passage time between x_0 and x_i is the expected minimum number of arcs of g which have to be 'opened' for fluid from x_0 to reach x_i. Alternatively $P_p(x_i)$ is the probability that the first passage time between x_0 and x_i is zero.

For a good review of ordinary percolation theory and its many physical applications see Frisch and Hammersley (1963). One refinement of ordinary percolation which may be more tractable when studied by first passage methods is the case where the probability 'p' varies, according as the arc is 'horizontal' or 'vertical'. This very difficult problem has so far only been studied by Mauldon (1960)

1.5. First Passage Percolation Theory and its Relation to Renewal Theory

In example (1.1.2) instead of studying the first passage time of infection between two trees of the orchard we could equally well consider the 'inverse' problem: "what is the maximum distance infection will have spread in time $\leqslant t$". This "maximum distance" problem on a graph is a generalisation of ordinary renewal theory; for let the underlying graph g have nodes at the non-negative integer points of the real line, while its arcs are merely directed lines connecting $(i,0)$ to $(i+1,0)$. If these are given non-negative lengths independently from a fixed distribution $\mathcal{U}$ then the expected maximum distance travelled in time $\leqslant t$ is exactly the renewal function associated with $\mathcal{U}$ (see W. L. Smith 1958).

This problem is treated in chapter 5 and it will be seen that "renewal type " theorems hold for various regular graphs.

1.6. Epidemics

Yet another application of first passage percolation theory is as a model for the spread of epidemics. It is obvious that because of the 'static' nature of the nodes of the graph first passage percolation is not a model for the spread of local epidemics. However, by considering each node of the graph as a town in a given country and the rate of spread of infection as some measure of the communication between the towns it would seem that the problem discussed below might give some worthwhile results concerning the rate of widespread infection.

1.7. Growth Processes

The general one-dimensional growth process where, for example, infection spreads along a straight line population from neighbour to neighbour with the probability of a new infection in time δt given by $\lambda \delta t$ (Poisson Process) is a well-known and much discussed problem. Very little is known however about the corresponding two dimensional problem where infection spreads under the same conditions through the points of a plane lattice.

It is well known that in the one-dimensional case described above, the inter-infection times have a related exponential distribution. It would therefore be reasonable to expect the spread of infection under the Poisson laws over a plane lattice to be equivalent to the problem of first passage percolation when the arcs of the lattice have time coordinates drawn from some related exponential distribution.

CHAPTER 2

Mathematical Formulation of the Problem

In this chapter first passage percolation theory for a general connected graph 'g' is formulated mathematically, and some general theorems are proved.

2.1. The Phase Space (Ω_g, F , P)

Consider a countably infinite connected graph g with arcs $\{l_i\}_{i=1}^{\infty}$ and nodes $\{P_{ij}\}$. The node P_{ij}, if it exists, is the intersection of l_i, l_j. To each arc l_i independently assign a non negative random variable u_i drawn from an associated distribution U_i. This u_i is called the time coordinate of l_i. The sequence $\{u_i\}_{i=1}^{\infty}$ defines a configuration of time coordinates on g which may be represented by an infinite vector $w \equiv (u_1 u_2 \dots, u_n \dots)$, called the time state of g. The set of all possible time states of g forms a set $\Omega_g(U_1 U_2, \dots U_n, \dots)$ which we call the phase space and usually denote by Ω_g.

If $r \equiv l_{i_1} l_{i_2} \dots l_{i_n}$ is any connected path of g (such a path is connected if and only if $P_{i_j i_{j+1}}$ exists for all $j = 1, 2, \dots, n-1$), then the time coordinate of r under w is defined by.

$$t(r,w) = u_{i_1} + u_{i_2} + \dots + u_{i_n} \qquad (2.1.1.)$$

Hereafter, path will always mean a connected path.

Further if R is any non-null set of paths on g, the _first passage time_ of R under w is defined by

(2.1.2) $$t_R(w) = \inf_{r \varepsilon R} t(r,w)$$

If there exist an $r_0 \varepsilon R$ such that $t(r_0,w) = t_R(w)$ the first passage time $t_R(w)$ is said to have a _route_ and this route is r_0. To prove the existence of a route is in general a difficult unsolved problem. However in the majority of practical cases, the route can be shown either to exist absolutely or to exist with probability 1. In the latter case Ω_g can be replaced by the relevant subset of probability 1. Accordingly throughout the remainder of this section the existence of the route of all first passage times considered will be assumed.

When the time coordinates u_i are drawn from probability spaces $U_i = (\Omega_i, F_i, P_i)$ then any time state w can be regarded as an elementary sample point of the probability space (Ω_g, F, P) where Ω_g is the countably infinite cartesian product of the Ω_i, F is the σ-field of subsets of Ω_g derived from the σ-fields F_i, and P is the induced probability measure.

THEOREM 2.1.3. _$t_R(w)$ is a measurable function on (Ω_g, F, P)_

Proof. Let r be any path of R: $r \equiv (l_{i_1}, l_{i_2}, \dots l_{i_k})$

Then $t(r,w) = u_{i_1} + u_{i_2} + \dots + u_{i_k}$ is measurable on (Ω_g, F, P). Since $\{i_i\}$ is a countable set, the set of paths with exactly n arcs is a countable set R_n, and hence R, a subset of the union of R_1, R_2, ... is also countable. Hence $t_R(w) = \inf t(r,w)$ is also measurable on (Ω_g, F, P) which proves that the first passage time $t_R(w)$ is a random variable on (Ω_g, F, P).

2.2. Equivalence under Lateral Shift.

Two sets of paths R_1 and R_2 are said to be equivalent under lateral shift if there exists a one-one mapping f of R_1 onto R_2 such that for every subset of paths $r_1, r_2 \dots r_k$ belonging to R_1, the joint distribution of $t(r_1,w)$, $t(r_2,w), \dots t(r_k,w)$ is identical with the joint distribution of $t(f(r_1),w), \cdots t(f(r_k), w)$.

THEOREM 2.2.1. If R_1 and R_2 are sets of paths equivalent under lateral shift, $t_{R_1}(w)$ and $t_{R_2}(w)$ are identically distributed random variables.

Proof. Let $r_1, r_2, \dots$ be an enumeration of R_1. For prescribed $u \geq 0$, define

$$\Omega_k = \left[w : \inf_{1 \leq i \leq k} t(r_i, w) \leqslant u \right]$$

$$\Omega_k^f = \left[w : \inf_{1 \leq i \leq k} t(f(r_i), w) \leqslant u \right]$$

Then Ω_k and Ω^f_k are non-decreasing set functions of k for which $P(\Omega_k) = P(\Omega^f_k)$ by the equivalence hypothesis.

Hence

$$P(\lim_{k\to\infty}\Omega_k) = P(\lim_{k\to\infty}\Omega_k^f)$$

and as u varies these two quantities are the distribution functions of $t_{R_1}(w)$ and $t_{R_2}(w)$.

EXAMPLE 2.2.2. Let g be the square lattice on the Euclidean plane, R_1 the set of paths connecting the origin to (m, o), and R_2, the set of paths connecting (a,o) to (a+ m, o). Suppose that the U_i - distribution associated with any arc is independent of that arc. Then R_1 and R_2 are equivalent under lateral shift.

2.3. This section contains some rather obvious lemmas used in the comparison of first passage times over different subsets of paths of g.

2.3.1. The Inclusion Lemma.

If R_1 and R_2 are two sets of paths on g such that $R_1 \subset R_2$, where $\subset$ has its usual set theoretic meaning, then $t_{R_1}(w) \geq t_{R_2}(w)$

Proof. When $R_1 \subset R_2$, $\inf\limits_{r \varepsilon R_1} t(r,w) \geq \inf\limits_{r \varepsilon R_2} t(r,w)$

2.3.2. The Connection Lemma

Two paths r_1, r_2 are said to be connected if they have a common end point. If $r_1 \equiv (P_1, P_2, \ldots$ $\ldots P_n = Q)$ (i.e. if $P_1, P_2, \ldots P_n$, are the successive nodes on r_1) and $r_2 \equiv (Q = Q_1, Q_2, \ldots Q_m)$ are connected, their connection $r_1 * r_2$ is defined to be the path

$(P_1, P_2, \ldots P_n = Q = Q_1, Q_2, \ldots Q_m)$. Similarly, two sets of paths R_1, R_2, are said to be connected, with connection $R_3 \equiv R_1 * R_2$ if for arbitrary $r_1 \varepsilon R_1$, $r_2 \varepsilon R_2$ r_1, r_2 are connected with connection $r_3 \varepsilon R_3$ and R_3 is made up only of paths of form $r_1 * r_2$. The lemma then states:

<u>If R_1, R_2, are connected sets of paths, then</u>

$$t_{R_1}(w) + t_{R_2}(w) \geq t_{R_1 * R_2}(w)$$

The proof of the lemma is quite trivial; for, if r^1 and r^2 are the routes of $t_{R_1}(w)$ and $t_{R_2}(w)$ then $r^1 * r^2 \varepsilon R_1 * R_2$ and hence

$$\begin{aligned} t_{R_1 * R_2}(w) \leq t(r^1 * r^2, w) &= t(r^1, w) + t(r^2, w) \\ &= t_{R_1}(w) + t_{R_2}(w) \end{aligned}$$

This lemma is used mostly where $R_1 * R_2 \subset R_3$, for then by a combination of the above two lemmas.

2.3.3. $t_{R_1}(w) + t_{R_2}(w) \geq t_{R_3}(w)$

<u>2.4.</u> Thus the problem of first passage theory is formulated for an arbitrary connected graph g and for arbitrary choice of U_i distributions. As may be realised from the sparsity of general results further results are obtained only at the expense of considering rather idealised situations. The main two restrictions imposed below are that the U_i distributions are all identical, i.e. $U_i = U$ for all i, and the problem is then studied with g, a fixed graph - the square lattice. This problem is considered in chapters 4 et. seq.

It is hoped that the techniques used there will prove useful in more general situations.

In chapter three a study is made of subadditive stochastic processes. This study it is emphasized is by no means comprehensive. The idea of studying such processes sprang from first passage theory and the results obtained are in the main abstract analogies of results previously obtained for first passage times on the square lattice. However, there do exist several other stochastic processes which may be put into subadditive form and it remains to be seen if a more detailed study of subadditive processes is of much practical importance.

CHAPTER 3

Subadditive Stochastic Processes

This chapter contains some results concerning a new concept - subadditive stochastic processes. The results obtained are then applied to first passage theory in chapter 4.

3.1. Let T be the set of non-negative integers. Let $x_{st}(\omega)$, with $s, t, \in T$ and $\omega \in \Omega$ be a two-parameter family of random variables defined on a probability space, (Ω, F, P). We say that $x_{st}(\omega)$ is a subadditive process if conditions (3.1.1.) through (3.1.4.) hold

(3.1.1.) $x_{rt}(\omega) \leq x_{rs}(\omega) + x_{st}(\omega)$ for $r \leq s \leq t$

(3.1.2.) $x_{st}(\omega) \geq 0$

(3.1.3.) $x_{st}(\omega)$ is stationary: that is to say, its distribution depends only on $t - s$.

(3.1.4.) $E\, x_{0t}(\omega) = g(t)$ is finite.

THEOREM 3.1.5. Associated with any subadditive process $\{x_{st}(\omega)\}$, is a time constant γ, such that

$$g(t)/t \geq \gamma = \lim_{t\to\infty} g(t)/t.$$

In view of the stationarity hypothesis (3.1.3.), this implies that for any $s \in T$

(3.1.6.) $$\lim_{t\to\infty} E x_{st}(\omega)/(t-s) = \gamma$$

Proof: Taking expected values of (3.1.1.) and using (3.1.3.) we have

$$g(t-r) \leq g(s-r) + g(t-s) \qquad (r \leq s \leq t)$$

Thus $g(t)$ is a subadditive function on T and the result follows from the fundamental theorem on subadditive functions (Hille. Chapter 6).

3.2. A trivial example of a subadditive stochastic process is the following:-

(3.2.1.) Let $\left\{X_i\right\}_{i=1}^{\infty}$ be a sequence of independent, identically distributed random variables, which are non-negative, and have finite mean. Let S_{mn} be defined for m, n, integers, by

(3.2.2.) $S_{mn} = X_{m+1} + X_{m+2} + \dots + X_n \qquad (m \leqslant n)$

Then S_{mn} is a subadditive stochastic process. Obviously the time constant γ of this process is EX_1.

Another, less trivial subadditive process is derived from a car parking problem of Renyi.

(3.2.3.) Let cars of fixed length M be parked along the real axis. The method of parking is completely random. Define n_{st} to be the number of cars, completely contained in the interval $\left[s,t\right]$. Then it is trivial that for r, s, t, integers such that $r \leqslant s \leqslant t$, we either have

$$n_{rs} + n_{st} = n_{rt}$$

or

$$n_{rs} + n_{st} + 1 = n_{rt}$$

Thus in general we have

(3.2.4.) $n_{rt} \leqslant n_{rs} + n_{st} + 1$

and thus $\left\{x_{rt}\right\} = \left\{(n_{rt} + 1)\right\}$ is a subadditive stochastic process.

3.3. We first prove the result

THEOREM 3.3.1. *If r is a fixed integer, then the subadditive process $x_{st}(w)$ satisfies*

$$P\left[\lim_{s\to\infty} s^{-1}x_{s,s+r}(w) = 0\right] = 1$$

Proof: Prescribe $\varepsilon > 0$. Define Q_s by

$$Q_s = P\left[x_{s,s+r}(w) \geqslant s\varepsilon\right]$$

By the stationary hypothesis

$$Q_s = P\left[x_{or}(w) \geqslant s\varepsilon\right]$$

Let $F(x)$ be the cumulative distribution function of $x_{or}(w)$

Then

$$\begin{aligned} g(r) &= E\, x_{or}(w) = \int_0^\infty x dF(x) \\ &= \sum_{n=0}^{s} \int_{n\varepsilon}^{(n+1)\varepsilon} x dF(x) + \int_{(s+1)\varepsilon}^{\infty} x dF(x) \\ &\geqslant \sum_{n=0}^{s} n\varepsilon \int_{n\varepsilon}^{(n+1)\varepsilon} dF(x) + \varepsilon(s+1)\int_{(s+1)\varepsilon}^{\infty} dF(x). \qquad (1) \end{aligned}$$

But $\displaystyle\int_{n\varepsilon}^{(n+1)\varepsilon} dF(x) = (Q_n - Q_{n+1})$

Hence (1) may be written

$$g(r) \geqslant \sum_{n=0}^{s} n\varepsilon(Q_n - Q_{n+1}) + (s+1)\varepsilon \quad Q_{s+1}$$

$$= \sum_{n=0}^{s} \varepsilon\, \varphi_{n+1}$$

Since each φ_s is non negative and the above holds for all s, the finiteness of g(r) implies

$$\sum_{n=0}^{\infty} \varphi_n < \infty$$

Hence by the Borel Cantelli Lemma (Halmos P.201)

$$P\left[\limsup_{s \to \infty} s^{-1} x_{s,s+r}(w) \geq \varepsilon\right] = 0$$

Theorem (3.3.1.) now follows since ε is arbitrary.

We say that the subadditive process $x_{st}(w)$ has a δ-blanket if, for some positive integer n, there exists a sequence $\{y_i(w)\}_{i=1}^{\infty}$, of mutually independent, identically distributed, non-negative random variables with mean $E\, y_i(w) = n\delta$ such that with probability 1

(3.3.2.) $x_{0,jn}(w) \leq y_1(w) + y_2(w) + \dots + y_j(w) \qquad (j \geq j_0(w))$

THEOREM 3.3.3. If the subadditive process $x_{st}(w)$ has a δ-blanket, then

(3.3.3.) $$P\left[\limsup_{t \to \infty} t^{-1} x_{0t}(w) \leq \delta\right] = 1$$

Proof: Let n, $\{y_j(w)\}$ be defined as in (3.3.2.). Write $t = jn + r$ where $0 \leq r < n$. By (3.1.1.)

$$t^{-1}\, x_{ot}(w) \leqslant t^{-1}\, x_{o,jn}(w) + t^{-1}\, x_{jn,jn+r}(w) \qquad (1)$$

And since $t \geqslant jn$ (1) gives

$$t^{-1}\, x_{ot}(w) \leqslant (jn)^{-1}\, x_{o,jn}(w) + (jn)^{-1} x_{jn,jn+r}(w) \qquad (2)$$

Also since $\{x_{mn}(w)\}$ has a δ-blanket

$$(jn)^{-1}\, x_{o,jn}(w) \leqslant (jn)^{-1} \sum_{i=1}^{j} y_i(w) \qquad (3)$$

where the $y_i(w)$ are defined as in 3.3.2., with mean $n\delta$. Thus by the strong law of large numbers

$$P\left[\lim_{j\to\infty} (jn)^{-1} \sum_{i=1}^{j} y_i(w) = \delta\right] = 1 \qquad (4)$$

By Theorem 3.3.1.

$$P\left[\lim_{j\to\infty} (jn)^{-1}\, x_{jn,jn+r}(w) = 0\right] = 1 \qquad (5)$$

Since n is fixed, as $t \to \infty$, j and jn both $\to \infty$ and thus from (2), through (5)

$$P\left[\limsup_{t\to\infty} t^{-1}\, x_{ot}(w) \leqslant \delta\right] = 1$$

which completes the proof of Theorem 3.3.3.

We now use the definition of a δ-blanket to define a more useful concept - smotherability.

Definition. A subadditive process $x_{st}(w)$ is said to be smotherable if it possesses a sequence of δ_i-blankets such that $\inf_i \delta_i = \gamma$ where γ is the time constant of the x_{st} process.

THEOREM 3.3.4. *If the subadditive process $x_{st}(w)$ has time constant γ and is smotherable then*

$$(3.3.4.) \quad P\left[\limsup_{t\to\infty} t^{-1} x_{ot}(w) \leqslant \gamma\right] = 1$$

Proof: Since the process is smotherable, (3.3.3.) holds for every $\delta > \gamma$. Hence 3.3.4. follows.

The usefulness of smotherability is not too apparent from the above but it will be seen in the next section that it is essential when studying the convergence of subadditive processes, and more especially it will be shown that it is needed for the study of absolute first passage times in chapter 4.

Since $\liminf E\, x_{on}(w)/n = \gamma$, we could replace inequality by equality in (3.3.4.) if we could show that

$$(3.3.5.) \quad \liminf E\, x_{on}(w)/n \leqslant E \limsup x_{on}(w)/n$$

In the case where $x_{o1}(w) \leqslant M$ for all $w \varepsilon \Omega$ this would be true. However it does not appear easy to prove equality in (3.3.4.) without some such restriction. We shall go into this more closely in § 3.4.

THEOREM 3.3.7 A necessary and sufficient condition that a subadditive process $x_{st}(\omega)$ with time constant γ be smotherable is that

$$(3.3.7.) \qquad P\left[\limsup_{n \to \infty} x_{on}(\omega)/n \leq \gamma\right] = 1$$

Proof: The necessity follows at once from the definition of smotherability, as in the proof of (3.3.4.). The sufficiency follows from the fact that if $\delta > \gamma$ and (3.3.7.) holds we may take $n = 1$, and $y_j(\omega) = \delta$ identically in (3.3.2.)

It can easily be seen that both examples of § 3.2. are smotherable processes. However consider Example 3.3.8. If Ω consists of two points ω_1 and ω_2 and $P(\omega_1) = P(\omega_2) = 1/2$, then define

$$x_{st}(\omega_1) = t - s$$

$$x_{st}(\omega_2) = 0$$

It can be seen that $x_{st}(\omega)$ is a subadditive stochastic process. However, in view of Theorem 3.3.7. this process is not smotherable.

3.4. The Convergence of Subadditive Stochastic Processes.

In this section we shall determine conditions under which the process $\left\{x_{on}(\omega)/n\right\}_{n=1}^{\infty}$ converges with probability 1 to the time constant γ, of the process $\left\{x_{st}(\omega)\right\}$. Obviously by the stationarity hypothesis (3.1.3.), any result true for $x_{on}(\omega)/n$ will hold for $x_{s,\, s+n}(\omega)/n$ (s any integer).

THEOREM 3.4.1. If $x_{st}(w)$ is a smotherable subadditive stochastic process with time constant γ then provided $E\,x_{o1}^{a}(w) < \infty$ for some $a > 1$

$$P\left[\lim_{n\to\infty} x_{on}(w)/n = \gamma\right] = 1 \tag{3.4.1.}$$

Proof: By theorem (3.3.4.)

$$P\left[\limsup_{n\to\infty} x_{on}(w)/n \leqslant \gamma\right] = 1 \tag{1}$$

Hence it is sufficient to prove

$$P\left[\liminf_{n\to\infty} x_{on}(w)/n = \gamma\right] = 1 \tag{2}$$

Define $f(w)$ by

$$f(w) = \gamma - \liminf_{n\to\infty} x_{on}(w)/n \tag{3}$$

Because of (1) it is trivially true that

$$P\left[f(w) \geqslant 0\right] = 1 \tag{4}$$

By subadditivity, for any $a > 1$

$$E\,x_{on}^{a}(w) \leqslant E(x_{o1}(w) + x_{12}(w) + \dots + x_{n-1,n}(w))^{a} \tag{5}$$

By Minkowski's inequality.

$$(E(x_{o1}(w) + x_{12}(w) + \dots + x_{n-1,n}(w))^{a})^{1/a} \leqslant \sum_{i=0}^{n-1} (E\,x_{i,i+1}^{a}(w))^{1/a} \tag{6}$$

If $E\,x_{i,i+1}^{a}(w) = A < \infty$

Then (6) and (5) give the result that

$$E\,x_{on}^{a}(w) \leqslant n^{a} A \tag{7}$$

Hence $$E\left[\frac{x_{on}(\omega)}{n}\right]^{a} \leq A \qquad (8)$$

which proves that $x_{on}(\omega)/n$ is uniformly integrable (Doob P.629) and hence applying Fatou's Lemma for uniformly integrable sequences since $x_{on}(\omega)/n \geqslant 0$

$$\begin{aligned} E\,(\liminf x_{on}(\omega)/n\,) &= \liminf E\, x_{on}(\omega)/n. \\ &= \liminf g(n)/n \\ &= \gamma \qquad (9) \end{aligned}$$

Hence $E\, f\,(\omega) = 0$ (10)

But from (4) $f(\omega)$ is non-negative a.e. and hence (Halmos P.104)

$$P\,(f(\omega) = 0) = 1 \qquad (11)$$

which proves the required result (3.4.1.).

Theorem 3.4.1. gives a sufficient condition for convergence with probability 1 of the sequence $x_{on}(\omega)/n$.

THEOREM 3.4.2. A necessary and sufficient condition that the subadditive process $\{x_{st}(\omega)\}$ satisfies (3.4.1.) is that it is smotherable and in addition $x_{on}(\omega)/n$ is uniformly integrable

Proof: The sufficiency of these conditions has been shown in the proof of Theorem 3.4.1.

Hence let $f(\omega) = \gamma - \liminf x_{on}(\omega)/n.$ (1)

$h(\omega) = \gamma - \limsup x_{on}(\omega)/n.$ (2)

if (3.4.1.) is satisfied

$$P\left[f(\omega) = h(\omega) = 0\right] = 1 \qquad (3)$$

Hence $P\left[\limsup_{n \to \infty} x_{on}(\omega)/n \leq \gamma\right] = 1$ (4)

This in conjunction with Theorem 3.3.7. implies that $x_{st}(\omega)$ is a smotherable process.

Also by Halmos (P.104), since $f(\omega) = 0$ a.e.

$$E\, f(\omega) = 0 \tag{5}$$

Hence

$$\begin{aligned} E \liminf x_{on}(\omega)/n &= \gamma \\ &= \liminf E\, x_{on}(\omega)/n \end{aligned} \tag{6}$$

A necessary and sufficient condition for (6) to hold is (Doob P.629) that $x_{on}(\omega)/n$ be uniformly integrable in n. This completes the proof of 3.4.2.

3.5. The Variance of $x_{0n}(\omega)$

In § 3.4. we gave conditions for the convergence almost everywhere of the subadditive process $x_{st}(\omega)$ as the parameter $t \to \infty$. Before studying conditions for a subadditive process to converge in quadratic mean we need some knowledge of the variance of $x_{0n}(\omega)$ as $n \to \infty$. For example

THEOREM 3.5.1. A necessary and sufficient condition that $x_{0n}(\omega)/n$ converges in mean square to the time constant γ as $n \to \infty$ is that

(3.5.1.) $\lim_{n \to \infty} \operatorname{var} x_{0n}(\omega)/n^2 = 0$

Proof: $x_{0n}(\omega)/n$ will converge in mean square to γ as $n \to \infty$ if and only if

(1) $\lim_{n \to \infty} E\left[\frac{x_{0n}(\omega)}{n} - \gamma\right]^2 = 0$

Now (1) implies that

$$(2) \qquad \lim_{n \to \infty} \left[\frac{Ex_{0n}^{\;2}(\omega)}{n^2} - 2\gamma \frac{Ex_{0n}(\omega)}{n} + \gamma^2 \right] = 0$$

or

$$(3) \qquad \lim_{n \to \infty} \left[\frac{\operatorname{var} x_{0n}}{n^2} + \left(\frac{g(n)}{n} \right)^2 - 2\gamma \frac{g(n)}{n} + \gamma^2 \right] = 0$$

Since by Theorem 3.1.5. $\lim_{n \to \infty} g(n)/n = \gamma$ we see that the theorem is proved.

In many of the practical applications of subadditive stochastic processes we shall be dealing with processes $\{x_{rs}(\omega)\}$ which are such that, if $r \leq s \leq t$, (integers) $x_{rs}(\omega)$ and $x_{st}(\omega)$ are independent random variables. Such processes are called *independent subadditive processes.*

It is obvious from the definitions that *every independent subadditive process is smotherable.* The converse however is not true. An example will be given in Chapter 4. We now state

THEOREM 3.5.2. *If $x_{st}(\omega)$ is an independent subadditive process and the second moment of $x_{01}(\omega)$ exists, then $\operatorname{var}(x_{0n}(\omega))$ is $o(n^2)$ as $n \to \infty$.*

Proof: Since Ex_{01}^2 exists, by the independence hypothesis

$$(3.5.3.) \quad E(x_{0n}^2) \leqslant E(x_{01} + x_{12} + \dots + x_{n-1,n})^2$$
$$= n\sigma^2 + n(n-1)g^2(1) + ng^2(1)$$

where σ^2 is $\operatorname{var} x_{01}(\omega)$.

Now let k be any positive integer,

$$E\left[x_{0,nk}(\omega)\right]^2 \leqslant E\left[x_{0,k} + x_{k,2k} + x_{2k,3k} + \cdots + x_{(n-1)k,nk}\right]^2$$

$$= nEx_{0k}^2 + (n^2 - n)g^2(k)$$

$$= n \text{ var } x_{0k} + n^2 g^2(k)$$

Hence

(3.5.4.) $\frac{1}{n^2k^2} \text{ var } x_{0,nk}(\omega) \leqslant \frac{1}{nk^2} \text{ var } x_{0k} + \left(\frac{g(k)}{k}\right)^2 - \frac{g^2(nk)}{n^2k^2}$

By (3.5.3.) $\lim_{n\to\infty} \overset{\text{var}}{\cancel{\text{var}}} x_{on}(\omega)/n^2 = \theta < \infty$. Hence, allowing $k \to \infty$ in (3.5.4.) and keeping n, fixed, we have

(3.5.5.) $\theta \leqslant \theta/n + \gamma^2 - \gamma^2$.

Hence if n is chosen > 1, we have the required result that $\theta = 0$. Furthermore by Theorem 3.5.1.

<u>CORROLLARY 3.5.6.</u> <u>If $x_{st}(\omega)$ is an independent, subadditive stochastic process and the second moment of $x_{01}(\omega)$ exists, then $n^{-1}x_{0n}(\omega)$ converges in mean square to the time constant γ as $n \to \infty$.</u>

<u>3.6</u> It can be seen that there is much room for further work on subadditive processes. The above results are merely an introduction to the subject, and it will be seen that the results and methods used in obtaining them have been influenced to a large extent by the practical use we make of them in Chapter 4 onwards.

The extension of these results to cover the case of a continous parameter set is one obvious development. This extension we suggest will present greater difficulty than one would imagine at first glance.

CHAPTER 4

First Passage Theory on the Square Lattice

In chapter 2 we outlined the general theory of first passage theory on an arbitrary graph g. Here we shall study the problem in detail for the case where g is the square lattice, i.e. the lattice of integer points (x,y), the arcs (all of unit length) being parallel to the x- and y-axes. The phase space (Ω, F, P) on this lattice is induced by a distribution U which has finite mean $\bar{u}$.

A standard principle in first passage theory is "the more restricted the set of paths R, the more tractable is $t_R(\omega)$". Therefore in 4.1 through 4.3 we study first passage times between nodes of the lattice over paths which are subject to a cylinder restriction (which will be specified below). Then in subsequent sections we use the results obtained to determine the first passage times between nodes of the lattice over paths which are subject to no restrictions whatsoever. Such first passage times are termed absolute first passage times.

The main results of this section will be showing that these first passage times are subadditive stochastic processes with a time constant $\mu = \mu(U)$, which is the same for both cylinder and absolute times. The results of chapter 3 may then be applied to these processes.

4.1. The Cylinder Process $t_{mn}(\omega)$ - An Independent Subadditive Process.

The cylinder defined by two nodes (m_1, m_2), (n_1, n_2) of the lattice is the strip enclosed between the lines $x = m_1$, and $x = m_2$. $t[(m_1,m_2), (n_1,n_2); \omega]$ is defined to be the first passage time under ω between (m_1,m_2), (n_1,n_2) over paths on the lattice lying strictly (save for their endpoints) inside the cylinder defined by these two points. Such a first passage time is called a cylinder time.

By Theorem 2.1.3 $t[(m_1,m_2), (n_1,n_2); \omega]$ is a random variable on (Ω, F, P). We denote $t[(m,0), (n,0); \omega]$ by $t_{mn}(\omega)$ where $m \leq n$ and now we may state

THEOREM 4.1.1. $\{t_{mn}(\omega)\}$ is an independent, subadditive, stochastic process on (Ω, F, P)

Proof: Let m be integer $\leq n$. Since the time coordinates of the arcs of the lattice are non-negative,

$$(4.1.2) \qquad t_{mn}(\omega) \geq 0 \qquad\qquad (m \leq n, \ \omega \varepsilon \Omega)$$

It is a stochastic process by Theorem 2.1.3. and by a simple application of the connection lemma (2.3.2.) we have

$$(4.1.3.) \qquad t_{mn}(\omega) + t_{np}(\omega) \geq t_{mp}(\omega) \qquad (m \leq n \leq p)$$

Also if l_{k_1}, l_{k_2}, ..., $l_{k_{n-m}}$ are the arcs of the lattice which make up the straight line path from $(m,0)$ to $(n,0)$ and u_i is the time co-ordinate of l_i under ω

$$(4.1.4.) \qquad t_{mn}(\omega) \leq u_{k_1} + u_{k_2} + .. + u_{k_{n-m}}$$

so that taking expectations of (4.1.4)

(4.1.5.) $\quad T(m,n) = Et_{mn}(\omega) \leqslant (n-m)\bar{u} \qquad (m \leqslant n)$

Also (Example 2.2.2.), for 'a' any integer the set of cylinder paths from (m,0) to (n,0) is equivalent under lateral shift with the set of paths from (m+a,0) to (n+a,0). Hence the distribution of $t_{mn}(\omega)$ depends only on the difference (n-m).

Thus we see that $\{t_{mn}(\omega)\}$ is a subadditive stochastic process. Also since the distribution of $t_{mn}(\omega)$ depends <u>only</u> on the time co-ordinates of the arcs of the lattice which lie strictly inside the ordinates x = n, x = m, we see that $\{t_{mn}(\omega)\}$ is an independent subadditive stochastic process.

Hence, applying the results of chapter 3, by Theorem 3.1.5. there exists a constant $\mu = \mu(U)$ such that

(4.1.6.) $\quad T(m,n)/(n-m) \geqslant \mu = \lim\limits_{n\to\infty} T(m,n)/(n-m)$

Notice that by (4.1.5.), the time constant $\mu(U)$ satisfies

(4.1.7.) $\quad 0 \leqslant \mu(U) \leqslant \bar{u}$

That strict inequality does not always hold in (4.1.7.) is seen by:-

<u>Example 4.1.8.</u> Let the distribution U be such that each arc of the lattice has time co-ordinate k with probability 1. Then for this the <u>constant distribution</u> we see that $\mu(U) = k = \bar{u}$.

A further enquiry into the value of $\mu(U)$ is postponed until chapters 6 and 9.

The Convergence of $t_{mn}(\omega)$ as $n \to \infty$.

Since $\{t_{mn}(\omega)\}$ is an independent subadditive process and hence a fortiori smotherable, by Theorem 3.3.4. we have

THEOREM 4.1.10. For m any fixed integer

$$(4.1.10.) \quad P\left[\limsup_{n \to \infty} t_{m,n}(\omega)/(n-m) \leq \mu\right] = 1.$$

Further remarks may be made if it is stipulated that the underlying U distribution has finite r^{th} moment for some $r > 1$.

THEOREM 4.1.11. A necessary and sufficient condition that

$$(4.1.11.) \quad P\left[\lim_{n \to \infty} t_{on}(\omega)/n = \mu(U)\right] = 1$$

is that $\left\{t_{on}(\omega)/n\right\}_{n=1}^{\infty}$ be a uniformly integrable sequence of random variables.

COROLLARY 4.1.12. A sufficient condition for (4.1.11.) to hold is that the underlying U distribution has a finite r^{th} moment for some $r > 1$.

Proof: The proof of (4.1.11.) is immediate from Theorem 3.4.2. If the U distribution has an r^{th} moment for some $r > 1$, then since $t_{01}(\omega) \leq$ the time coordinate of the arc linking to origin to (1,0), $t_{01}(\omega)$ has a finite r^{th} moment. Hence by Theorem 3.4.1. Corollary 4.1.12. is true.

For most practical purposes Corollary 4.1.12. is the more important of the above results. The restriction on the U distribution which it imposes is not too severe. A stronger restriction is necessary before convergence in mean square can be proved.

THEOREM 4.1.13. If the U distribution has a finite second moment, the random variable $t_{mn}(\omega)$ satisfies

(4.1.13.)
$$\lim_{n\to\infty} \frac{\operatorname{var} t_{mn}(\omega)}{(n-m)^2} = 0 \qquad (m \text{ fixed})$$

and $t_{mn}(\omega)/(n-m)$ converges in mean square to the time constant $\mu(U)$ as $n \to \infty$

Proof: $\{t_{mn}(\omega)\}$ is an independent process. Since $t_{01}(\omega) \leq u$, the time coordinate of the arc from the origin to (1,0), by hypothesis

$$E\, t_{01}^2(\omega) \leq E\, u^2 < \infty.$$

Thus the conditions of Theorem 3.5.2. are satisfied and

$$\lim_{n\to\infty} n^{-2} \operatorname{var} t_{0n}(\omega) = 0.$$

The convergence in mean square of $t_{mn}(\omega)/(n-m)$ is now a trivial consequence of Corollary 3.5.6.

Thus it may be seen that the cylinder process $\{t_{mn}(\omega)\}$ is a comparatively well behaved subadditive process. In order that the reader may not be lulled into a false sense of security we present

CONJECTURE 4.1.14. It is easily realised that many distributions U exist which induce time states ω on the lattice for which $t_{mn}(\omega)$ is not monotonic in n for fixed m, ω. However, it seems a reasonable conjecture (or even intuitively obvious), that

$$T(0,n) \leqslant T(0,n+1) \qquad (n > 0) \tag{4.1.14}$$

This result we cannot prove.

This closes for the moment our discussion of $\{t_{mn}(\omega)\}$. In later chapters we shall study the process more closely for specified U distributions (the uniform rectangular, the exponential, etc.).

4.2. The Cylinder Process $s_{mn}(\omega)$.

Apart from its own intrinsic interest, the study of the stochastic process $\{s_{mn}(\omega)\}$ is essential if the problem of absolute first passage theory is to be solved. It will be seen that the process although not subadditive, possesses properties of a subadditive nature, which lead one to believe that the definition of subadditive processes (3.1.1. - 3.1.4.) could well be extended so as to include processes such as $\{s_{mn}(\omega)\}$.

The cylinder time $s[(m,y),\ X = n;\ \omega]$ is defined for y any positive, negative or zero integer by

$$(4.2.1.) \qquad s\left[(m,y),\ X = n;\ \omega\right] = \inf_{k} t\left[(m,y),(n,k);\ \omega\right]$$

where k runs through the integers $(-\infty, \infty)$. More loosely, $s\left[(m,y),\ X = n;\ \omega\right]$ is the cylinder time between (m,y) and the line $X = n$. By Theorem 2.1.3. this cylinder time is a random variable on (Ω, F, P).

Define $s_{mn}(\omega)$ to be $s\left[(m,0),\ X = n;\ \omega\right]$. Then $\left[s_{mn}(\omega) : \omega \varepsilon \Omega\right]$ is a 2-parameter stochastic process on (Ω, F, P). By definition, for all $\omega \varepsilon \Omega$

$$(4.2.2.) \qquad 0 \leqslant s_{mn}(\omega) \leqslant t_{mn}(\omega) \qquad (m \leqslant n)$$

Thus $S(m,n) = Es_{mn}(\omega)$ exists, and satisfies

$$(4.2.3.) \qquad 0 \leqslant S(m,n) \leqslant T(m,n) \leqslant (n-m)\bar{u}.$$

By the principle of equivalence under lateral shift (2.2.1.), we see that $s_{mn}(\omega)$ and $s\left[(m,y),\ X = n;\ \omega\right]$ are identically distributed. In particular the distribution of $s_{mn}(\omega)$ depends only on $(n-m)$. However, although conditions (3.1.2. - 4.) are satisfied, it is *not* possible to say in general that

$$(4.2.4.) \qquad s_{mn}(\omega) + s_{np}(\omega) \geqslant s_{m,p}(\omega)$$

and hence $\left\{s_{mn}(\omega)\right\}$ is *not* a subadditive process. However, we do have

THEOREM 4.2.5. For any distribution U, the function $S(m,n)$ satisfies

$$(4.2.5.) \qquad S(m,n) + S(n,p) \geqslant S(m,p) \qquad (m \leqslant n \leqslant p)$$

Proof: Let r_1, the route of $s_{mn}(\omega)$ meet $X = n$ at $P \equiv (n, y_1)$. [The existence of r_1 will be proved in

Chapter 8]. Let $f(\omega)$ be the first passage time from P to $X = p$ over cylinder paths whose first arc is from (n, y_1) to $(n+1, y_1)$. $f(\omega)$ is a random variable on (Ω, F, P). Its distribution depends on the distribution of time co-ordinates of the arcs in the strip bounded by $X = n$, $X = p$.

Hence $f(\omega)$ has the distribution of $s_{np}(\omega)$. If r_2 is the route of $f(\omega)$, by a simple application of the connection lemma (2.3.2.)

(4.2.6.) $$s_{mn}(\omega) + f(\omega) \geqslant s_{mp}(\omega) \qquad (m \leqslant n \leqslant p)$$

Hence taking expected values of (4.2.6.), since $Ef(\omega) = S(n,p)$, we have the required result (4.2.5.)

Since $\{s_{mn}(\omega)\}$ is a stationary process, $S(m,n)$ may be written in the form $h(n-m)$. By Theorem 4.2.5. $h(t)$ is a subadditive function on the integers and hence (Hille. Ch.6)

(4.2.7.) $$\inf_{n} \frac{S(m,n)}{(n-m)} = \mu^*(U) = \lim_{n\to\infty} \frac{S(m,n)}{(n-m)} \qquad (m, \text{fixed})$$

The time constant $\mu^*(U)$ is dependent only on U. From (4.2.3.) it satisfies

(4.2.8.) $$0 \leqslant \mu^*(U) \leqslant \mu(U) \leqslant \bar{u} < \infty$$

The main result of this section is

THEOREM 4.2.9. For any distribution U, the time constants $\mu(U)$, $\mu^*(U)$ are equal

Theorem 4.2.9. has very important mathematical consequences in this work. Physically it may be interpreted as follows. If fluid is supplied at r collinear nodes of the lattice and the fluid can only flow along arcs of the lattice, the time of flow along any arc being a random variable, then the expected time to 'wet' a specified node is asymptotically independent of r.

Proof of Theorem 4.2.9.

Let x_0 be a prescribed integer. Let $s^1_{x_0}(\omega)$ be $s_{0,x_0}(\omega)$. Let its route r_1 meet the line $X = x_0$ at $P_1 \equiv (x_0, h_1(\omega))$. Let $s^2_{x_0}(\omega)$ be the cylinder first passage time from P_1 to the line $X = 2x_0$ over paths whose first arc links P_1 to $(x_0+1, h_1(\omega))$.

Let r_2 be the route of $s^2_{x_0}(\omega)$, and let r_2 meet $X = 2x_0$ at $P_2 \equiv (2x_0, h_1(\omega) + h_2(\omega))$. Similarly define $s^3_{x_0}(\omega)$ to be the first passage time, under the same conditions, from P_2 to $X = 3x_0$.

Continuing in this way it is possible to define sequences

$$\left\{s^i_{x_0}(\omega)\right\}^n_{i=1}, \quad \left\{r_i\right\}^n_{i=1}, \quad \left\{h_i(\omega)\right\}^n_{i=1}$$

for n, any positive integer, such that

a) $\left\{s^i_{x_0}(\omega)\right\}^n_{i=1}$ forms a sequence of independent, identically distributed random variables having the distribution of $s_{0,x_0}(\omega)$.

b) $\left\{r_i\right\}^n_{i=1}$ is a sequence of paths on the lattice such that $r_1 * r_2 * \ldots * r_n$ is a connected path from the origin to $X = nx_0$

c) $\{h_i(\omega)\}_{i=1}^{n}$ is a sequence of integer valued, symetric, identically distributed random variables.

Define $N(\omega)$ by $N(\omega) = \sum_{i=1}^{h} h_i(\omega)$.

Let r_o be the straight line path on the lattice from $P \equiv (nx_o, N(\omega))$ to $(nx_o, 0)$. Let l_o be the arc from $(nx_o, 0)$ to $(nx_o+1, 0)$.

Now $r_1 * r_2 * \ldots * r_n$ connects the origin to P, r_o connects P to $(nx_o, 0)$ and l_o connects $(nx_o, 0)$ to $(nx_o+1, 0)$. Hence $r_1 * r_2 * \ldots * r_n * r_o * l_o$ is a connected cylinder path from the origin to the point $(nx_o+1, 0)$. Hence by the inclusion lemma

$$(1) \quad t_{o,nx_o+1}(\omega) \leqslant t(r_1 * r_{*} \ldots * r_n * r_o * l_o, \omega)$$

$$= \sum_{i=1}^{n} t(r_i, \omega) + t(r_o, \omega) + t(l_o, \omega)$$

$$= \sum_{i=1}^{n} s_{x_o}^{i}(\omega) + t(r_o, \omega) + t(l_o, \omega).$$

Consider now $t(r_o, \omega)$. It is the sum of $|N(\omega)|$ independent random variables, each with mean $\bar{u}$, and they are all independent of $N(\omega)$.

In Theorem 8.2.15. we prove

$$(2) \qquad E\,|h_1(\omega)| < \infty$$

and in corollary 8.2.25. show that

(3) $$E\, h_i^2(\omega) < \infty .$$

Hence

(4) $$\begin{aligned} E\,|N(\omega)| &= E\left|\sum_{i=1}^{n} h_i(\omega)\right| \\ &\leq E \sum_{i=1}^{n} |h_i(\omega)| \\ &= n\, E\,|h_1(\omega)| < \infty . \end{aligned}$$

Hence by the theory of a random number of random variables $Et(r_0, \omega)$ exists and satisfies

(5) $$E\, t(r_0, \omega) = \bar{u}\, E\,|N(\omega)| .$$

Consider now the quantity $E\,|N(\omega)|$. By Schwartz's Inequality

(6) $$\begin{aligned} \left[E\,|N(\omega)|\right]^2 &\leq E(|N(\omega)|^2) \\ &= E\, N^2(\omega) \\ &= E\,\left(\sum_{i=1}^{n} h_i(\omega)\right)^2 \end{aligned}$$

Since each $h_i(\omega)$ has zero mean and $h_i(\omega)$, $h_j(\omega)$ are independent $(i \neq j)$,

(7) $$\left[E\,|N(\omega)|\right]^2 \leq E \sum_{i=1}^{n} h_i^2(\omega) = n\, Eh_1^2(\omega)$$

∴ (8) $$E\, t(r_0, \omega) \leq n^{\frac{1}{2}}\, \bar{u} \left[Eh_1^2(\omega)\right]^{1/2}$$

Now taking expectations of (1) we get since $E\, t(l_0, \omega) = \bar{u}$

$$(9) \quad T(0,\ nx_o+1) \leqslant n\, S(x_o) + \bar{u}n^{\frac{1}{2}} \left[Eh_1{}^2(\omega) \right]^{\frac{1}{2}} + \bar{u}.$$

Dividing (9) by nx_o and taking the limit as $n \to \infty$ with x_o fixed, we now get

$$(10) \qquad \mu(U) \leqslant S(0,x_o) / x_o$$

For any $\varepsilon > 0$, an x_o can be originally chosen such that

$$(11) \qquad \mu^*(U) \leqslant S(0,\ x_o) / x_o \leqslant \mu^*(U) + \varepsilon$$

Hence by the arbitrariness of ε, (11) and (10) show that $\mu(U) \leqslant \mu^*(U)$, which with (4.2.8.) proves the required result (4.2.9.)

<u>The Convergence as $n \to \infty$ of the Random Variable $s_{0n}(\omega)/n$</u>

That $\{s_{0n}(\omega)\}$ is <u>not</u> a subadditive process prevents us making use of the theorems of chapter 3 when studying the convergence as $n \to \infty$ of $s_{0n}(\omega)/n$. Nevertheless results analogous to those obtained for $t_{mn}(\omega)$ are true for the $s_{mn}(\omega)$ process. The methods of proof are very similar to those used in chapter 3, and as a result will be postponed until Appendix I.

Since $s_{0n}(\omega) \leqslant t_{0n}(\omega)$ for all ω, we have in view of Theorem 4.1.10. the immediate result

$$(4.2.10.) \qquad P\left[\limsup_{n \to \infty} s_{0n}(\omega)/n \leqslant \mu(U) \right] = 1$$

By methods similar to that used to prove Theorem 3.41. we have

<u>THEOREM 4.2.11.</u> <u>As $n \to \infty$ the random variable $s_{0n}(\omega)/n$ converges with probability 1 to the time constant</u> μ

provided the U distribution has finite r^{th} moment for some $r > 1$

THEOREM 4.2.12. *If the U distribution has, in addition to a finite mean $\bar{u}$, a finite variance σ^2 then* var $s_{0n}(\omega)$ *exists and satisfies*

$$(4.2.13.) \qquad \lim_{n\to\infty} \text{var } s_{0n}(\omega)/n^2 = 0$$

and $s_{0n}(\omega)/n$ *converges to* μ *in mean square as* $n \to \infty$.

Because of stationarity the results (4.2.10.) through (4.2.13.) may be extended to $s_{mn}(\omega)/(n-m)$.

From the results of this section the close relationship between the t- and s-processes can be readily appreciated. This relationship will be explored further in later chapters.

4.3. *The Absolute First Passage Time $a_{mn}(\omega)$:- A Smotherable but not Independent Subadditive Process.*

The previous sections have dealt with first passage times on the square lattice over cylinder paths. For most practical purposes, more important quantities are *absolute first passage times* i.e. first passage times over paths which are subject to no restriction whatsoever. $a_{mn}(\omega)$ denotes the absolute first passage time between (m,o) and (n,o) under ω. In this section we shall show that $a_{mn}(\omega)$ is "asymptotically equivalent" to the cylinder process $t_{mn}(\omega)$, or in other words, we shall prove quantitatively the intuitively appealing idea that "the average time spent outside the fundamental cylinder when travelling as quickly as possible from (m,o) to (n,o) is relatively small."

From Theorem 2.1.3. we have that $\left[a_{mn}(\omega),\ \omega \varepsilon \Omega\right.$, m,n, integers $\left.\right]$ is a 2-parameter stochastic process. By the inclusion lemma (2.3.1.) it is immediate that

(4.3.1.) $\quad 0 \leqslant a_{mn}(\omega) \leqslant t_{mn}(\omega)$

<u>THEOREM 4.3.2.</u> $\{a_{mn}(\omega)\}$ <u>is a subadditive stochastic process on (Ω. F. P.)</u>

<u>Proof</u>: Trivially from (4.3.1.), $a_{mn}(\omega) \geqslant 0$, while $A(m,n) = Ea_{mn}(\omega)$ exists and satisfies.

(4.3.2.) $\quad 0 \leqslant A(m,n) \leqslant T(m,n) \leqslant (n-m)\bar{u} \quad (n \geqslant m)$

By the principle of equivalence under lateral shift (2.2.3), the distribution of $a_{mn}(\omega)$ depends only on the difference (n-m), while the connection lemma (2.3.2.) proves that

(4.3.3.) $\quad a_{mn}(\omega) + a_{np}(\omega) \geq a_{mp}(\omega) \qquad (m \leqslant n \leqslant p)$

Hence $\{a_{mn}(\omega)\}$ is a subadditive stochastic process. However, since the paths over which these first passage times are taken are not restricted to being inside a cylinder it is not true that $a_{mn}(\omega)$ and $a_{np}(\omega)$ are independent random variables. Hence, unlike $\{t_{mn}(\omega)\}$ $\{a_{mn}(\omega)\}$ is not an independent subadditive process.

Applying the results of chapter 3. By Theorem 3.1.5. there exists a constant $\mu_A(U)$ such that

(4.3.4.) $$A(m,n)/(n-m) \geq \mu_A(U) = \lim_{n\to\infty} A(m,n)/(n-m)$$

From (4.3.2.) $\mu_A(U)$ satisfies

(4.3.4.) $$0 \leq \mu_A(U) \leq \mu(U)$$

Intuitively one would expect the difference $T(m,n) - A(m,n)$ to be relatively small. In the case where U is the constant distribution (Example 4.1.8.) it is obvious that for all m,n $T(m,n)$ and $A(m,n)$ are equal.

THEOREM 4.3.6. The time constants $\mu_A(U)$ and $\mu(U)$ are equal for any distribution U.

Proof: Define $q^k_{mn}(\omega)$, for m,n and k a positive integer, to be the first passage time between (m,o) and (n,o) under ω over paths which lie strictly inside the strip bounded by $X = m-k$, $X = n+k$, $q^k_{mn}(\omega)$ is a non-negative random variable on $(\Omega, F.P.)$. By the principle of equivalence under lateral shift, $q^k_{mn}(\omega)$ has a distribution which depends only on $(n-m)$ for fixed k, By the connection lemma (2.3.2.)

(1) $$q^k_{mn}(\omega) + q^k_{np}(\omega) \geq q^k_{mp}(\omega) \qquad (m \leq n \leq p)$$

Hence for fixed k, $\left\{q^k_{mn}(\omega)\right\}$ is a subadditive stochastic process on $(\Omega . F.P.)$

Hence by Theorem 3.1.5. there exists a constant $\mu_k(U)$ such that $Q_k(n) = Eq^k_{on}(\omega)$ satisfies

(2) $$\frac{Q_k(n)}{n} \geq \mu_k(U) = \lim_{n\to\infty} \frac{Q_k(n)}{n}$$

Now by the inclusion lemma (2.3.1.)

(3) $a_{on}(\omega) \leqslant q^{k}_{on}(\omega) \leqslant q^{k-1}_{on}(\omega) \leqslant t_{on}(\omega)$ $(k \geqslant 2)$

Hence we have

(4) $\mu_{A}(U) \leqslant \mu_{k}(U) \leqslant \mu(U)$ $(k \geqslant 2)$

Let r_o be the route of $t_{-k,o}(\omega)$. Let r_1 be the route of $q^{k}_{on}(\omega)$. Let r_2 be the route of $t_{n,n+k}(\omega)$.

Then $r_o{}^{*}\, r_1{}^{*}\, r_2$ is a connected cylinder path from $(-k,o)$ to $(n+k,o)$ and hence

(5) $$\begin{aligned} t_{-k,n+k}(\omega) &\leqslant t(r_0{}^{*}r_1{}^{*}r_2, \omega) \\ &= t(r_o, \omega) + t(r_1, \omega) + t(r_2, \omega) \\ &= t_{-k,o}(\omega) + q^{k}_{on}(\omega) + t_{n,n+k}(\omega) \end{aligned}$$

Hence taking expected values of (5), by stationarity we have

(6) $T(o, n+2k) \leqslant 2T(k) + q_{k}(n)$

Dividing (6) by n and taking the limit as $n \to \infty$ with k fixed we therefore get

(7) $\mu(U) \leqslant \mu_{k}(U)$

which together with (2) implies that for all fixed k

(8) $\mu(U) = \mu_{k}(U)$

Now consider the random variable $q^{k}_{on}(\omega)$. This is monotonic decreasing in k for fixed n, and

(9) $\lim\limits_{k \to \infty} q^{k}_{on}(\omega) = a_{on}(\omega)$ (n, fixed)

Hence by the Monotone Convergence Theorem

(10) $\lim_{k \to \infty} G_k(n) = A(o,n)$ (n fixed)

Now by (2) and (8)

(11) $G_k(n)/n \geqslant \mu(U)$ for all k,n

Hence by (10), (11)

(12) $A(o,n)/n \geqslant \mu(U)$

Since $\lim_{n \to \infty} A(o,n)/n = \mu_A(U)$ (12) implies the result that $\mu_A(U) = \mu(U)$ and completes the proof of Theorem 4.3.6.

Apart from its physical significance Theorem 4.3.6. enables us to consider the convergence of $a_{on}(\omega)/n$ as $n \to \infty$. By an extension of the subadditive property (4.4.3.) for any j,n, integers

$$(4.3.7.) \quad a_{o,jn}(\omega) \leqslant a_{o,n}(\omega) + a_{n,2n}(\omega) + \dots + a_{(j-1)n,jn}(\omega)$$

$$\leqslant t_{o,n}(\omega) + t_{n,2n}(\omega) + \dots + t_{(j-1)n,jn}(\omega)$$

since $a_{mn}(\omega) \leqslant t_{mn}(\omega)$ for any m,n .

If we let $t_{(i-1)n,in}(\omega) = y_i(\omega)$ then we see by comparing (4.3.7.) with (3.3.2.) that $\{a_{mn}(\omega)\}$ has a δ_n-blanket where δ_n is given by

$$(4.3.8.) \quad n\delta_n = \Sigma t_{(i-1)n,in}(\omega) = T(o,n)$$

since $\lim_{n \to \infty} T(o,n)/n = \inf_n T(o,n)/n = \mu(U)$, by 3.4.4.

$\{a_{mn}(\omega)\}$ is a <u>smotherable subadditive process</u> and hence by

Theorem 3.3.4. we have the result.

THEOREM 4.3.9. The absolute first passage time $a_{on}(\omega)$ satisfies.

$$\text{(4.3.9.)} \qquad P\left[\limsup_{n \to \infty} a_{on}(\omega)/n \le \mu(U)\right] = 1$$

Furthermore by Theorem 3.4.2., since $\left\{a_{mn}(\omega)\right\}$ is a smotherable subadditive process

THEOREM 4.3.10. The uniform integrability of the sequence $\left\{a_{on}(\omega)/n\right\}_{n=1}^{\infty}$ is a necessary and sufficient condition for

$$P\left[\lim_{n \to \infty} a_{on}(\omega)/n = \mu(U)\right] = 1.$$

Corollary 4.3.11. If the U distribution has a finite r^{th} moment for some $r > 1$, then

$$P\left[\lim_{n \to \infty} a_{on}(\omega)/n = \mu(U)\right] = 1.$$

Proof: If u is the time coordinate of the arc linking the origin to (1,0) then trivially

$$0 \leqslant a_{o1}(\omega) \leqslant u$$

Hence the existence of Eu^r, implies the existence of an r^{th} moment of $a_{o1}(\omega)$ which by Theorem 3.4.1. proves corollary 4.3.11.

Despite the fact that $\left\{a_{mn}(\omega)\right\}$ is not independent and hence we cannot apply Theorem 3.5.2. we can prove

THEOREM 4.3.12. If the U distribution has finite variance then

$$\text{(4.3.12.)} \qquad \lim_{n \to \infty} \operatorname{var} a_{on}(\omega)/n^2 = 0$$

Proof: $\operatorname{var} a_{on}(\omega) = Ea^2_{on}(\omega) - [A(o,n)]^2$

By (4.3.7.) we have

$$(1)\quad \operatorname{var} a_{o,nj}(\omega) \leq E\left[t_{o,n}(\omega) + t_{n,2n}(\omega) + \ldots + t_{(j-1)n,jn}(\omega)\right]^2 - A^2(o,nj)$$

$$= j \operatorname{var} t_{o,n}(\omega) + j^2\left[T(o,n)\right]^2 - A^2(o,jn)$$

Hence

$$(2)\quad \frac{\operatorname{var} a_{o,nj}(\omega)}{n^2j^2} \leq \frac{1}{j} \operatorname{var} \frac{t_{on}(\omega)}{n^2} + T^2\frac{(o,n)}{n^2} - A^2\frac{(o,jn)}{n^2j^2}$$

By Theorems 4.1.13. 4.3.6. the limit as $n \to \infty$ of the r.h.s. of (2) is zero

Hence

$$(3)\quad \lim_{n\to\infty} \operatorname{var} a_{o,nj}(\omega)/n^2j^2 = 0$$

Since this holds for any j we have proved the required results. Notice that the stipulation that the U distribution has a finite variance is necessary for $Et^2_{on}(\omega)$ to exist.

An immediate corollary to this result is

COROLLARY 4.3.11. Provided the U distribution has finite variance, the random variable $a_{on}(\omega)/n$ converges in mean square to $\mu(U)$ as $n \to \infty$

Proof: Follows from Theorem 3.5.1.

This completes our study of $a_{mn}(\omega)$ for the time being. The close relationship between $a_{mn}(\omega)$ and $t_{mn}(\omega)$ is evident. In the next section we shall study the absolute analogue of $s_{mn}(\omega)$. This it will be seen is a much more difficult process to handle.

4.4. The Absolute First Passage Times between a Point and a Line.

For many practical purposes an important quantity is not the absolute first passage time between two nodes of the lattice, but the first passage times between a specified node and some linear barrier. This problem was first tackled in 4.2. when, however, we restricted ourselves to considering first passage time over cylinder paths. This problem was not too difficult to deal with, even though it was not subadditive. The corresponding analogue however is so difficult that our results are sparse and consist mainly of conjectures based on quite strong heuristic evidence.

Let $b_{mn}(\omega)$ be the absolute first passage time from $(m,0)$ to the line $X = n$. $(m \leqslant n)$. From 2.1.3 $b_{mn}(\omega)$ is a 2-parametered stochastic process on the phase space $(\Omega.F.\mu.)$. Its expected value $B(m,n)$ exists, and the following inequalities are true for all m,n,ω.

(4.4.1.) $0 \leqslant b_{mn}(\omega) \leqslant a_{mn}(\omega)$

(4.4.2.) $0 \leqslant b_{mn}(\omega) \leqslant s_{mn}(\omega)$

Physically, $b_{mn}(\omega)$ bears the same relationship to $s_{mn}(\omega)$ as does $a_{mn}(\omega)$ to $t_{mn}(\omega)$. It is tempting to <u>conjecture</u> that

(4.4.3.) $B(m,n) + B(n,p) \geqslant B(m,p) \qquad (m \leqslant n \leqslant p)$

This cannot be proved by a straightforward application of the connection lemma.

We further conjecture that $\lim_{n \to \infty} B(o,n)/n$ exists and and satisfies

(4.4.4.) $\lim_{n \to \infty} B(o,n)/n = \mu(U)$

where $\mu(U)$ is the time constant of the a,- and t-processes. We sketch our reasons for such conjectures:-

Let $s^{k}_{mn}(\omega)$ be defined for $k \geqslant o$ as the first passage time from (m,o) to $X = n$ over paths which lie strictly between the lines $X = m-k$, $X = n$. Then if $S_k(n) = Es^{k}_{on}(\omega)$ it can be shown that

(4.4.5.) $\lim_{n \to \infty} S_k(n)/n = \mu(U)$ for any k

By the inclusion lemma (2.3.1.)

(4.4.6.) $B(o,n) \leqslant S_{k+1}(n) \leqslant S_k(n) \leqslant S(o,n) \qquad (k \geqslant o)$

Also by an application of the Monotone Convergence Theorem

(4.4.7.) $\lim_{k \to \infty} S_k(n) = B(o,n)$

<u>If</u> it could be shown that $S_k(n)/n \geqslant \mu(U)$ (which would

be the case if $s^k_{mn}(\omega)$ were a subadditive process for fixed k) (4.4.4.) would follow. However, we see no way of proving this result at the moment.

Further enquiries, such as the convergence with n of $b_{0n}(\omega)/n$ await the proof of (4.4.4.)

This completes our study of first passage theory for the time being. We shall return to these processes in chapters 6 - 8.

CHAPTER 5

The Relation Between First Passage Theory and Renewal Theory.

In this chapter we point out a relationship between first passage percolation theory and renewal theory. Throughout this chapter, the underlying graph g will be the square lattice, with phase space (Ω, F, μ) induced by a distribution U. We introduce reach functions on the space Ω which are random variables, the expected values of which possess properties analogous to those of the renewal function. The reach functions have an inverse relationship with first passage times. This relation is explored below.

5.1. The Reach Functions.

Following the notation of W.L.Smith (1958), let $\left\{X_i\right\}_{i=1}^{\infty}$ be a sequence of non-negative, independent, identically distributed random variables with finite mean. Renewal theory is concerned with the distribution of the n^{th} partial sum $S_n \equiv (S_n = X_1 + X_2 + \ldots + X_n)$. More especially it considers the distribution of $N_t \equiv$ maximum n such that $S_n \leq t$. The expected value of the random variable N_t is called the renewal function and is denoted by H(t).

If the time coordinate u_i of the arc l_i of the square lattice is infinity with probability 1 when l_i is not on the x-axis and are drawn independently from a fixed distribution U when l_i is part of the x-axis, then $t_{on}(\omega) = S_n$

More generally let (Ω, F, P) be the phase space of the lattice and define the reach function $x_t(\omega)$ for all $t \geq 0$, all integer m by

(5.1.1.) $x_t(\omega) \;=\; \sup n : t_{on}(\omega) \leq t$

Similarly we may define another reach function $y_t(\omega)$ for all $t \geq 0$ by

(5.1.2.) $y_t(\omega) \;=\; \sup n : s_{on}(\omega) \leq t$

THEOREM 5.1.3. The x and y reach functions are measurable functions on the phase space (Ω, F, P)

The proof of this theorem is not difficult and will be found in appendix 2.

Thus $\{x_t(\omega)\}$, $\{y_t(\omega)\}$ are integer valued, non decreasing stochastic processes on (Ω, F, P) with the interval $(0,\infty)$ as parameter sets.

As their names suggest the reach functions may be loosely interpreted as the X coordinates of points furthermost 'east' of the origin which are attainable by cylinder paths in time $\leq t$, when the time state of the lattice is ω. In a certain sense, therefore, the reach functions are two dimensional analogues of N(t). This analogy will be heightened by some of the results of this section. To prove these results however, we have to impose a rather heavy bounding restriction on the U-distribution. At the moment we see no way of removing this restriction and obtaining comparable results.

5.2. The Reach Functions for Bounded U.

Henceforth the phase space (Ω, F, P) is derived from a distribution U which is bounded i.e. the time coordinate u_i of l_i satisfies for all i

(5.2.1) $$0 < U_0 \leq u_i \leq U_1 < \infty$$

With this restriction it is easy to see that for all

(5.2.2) $$\frac{t}{U_1} \leq x_t(\omega) \leq y_t(\omega) \leq t/U_0 \qquad (t \geq 0)$$

We now state an extension of theorem 5.1.3.

THEOREM 5.2.3. The reach functions $x_t(\omega)$, $y_t(\omega)$ are measurable functions on the product space $\Omega \times T$ where T is the interval $(0,\infty)$ of the real line.

The proof of this is also in appendix 2. As a corollary to (5.2.3), by Doob (1952, P.62)

COROLLARY 5.2.4 $X(t) \equiv Ex_t(\omega)$, $Y(t) = Ey_t(\omega)$, both exist and are Lebesgue measurable, functions on the positive real line

Trivially in view of (5.2.2)

(5.2.4.) $$t/U_1 \leq X(t) \leq Y(t) \leq t/U_0 \qquad (t \geq 0)$$

Feller (1941) proved the elementary renewal theorem which in the notation of (5.1.) stated

(5.2.5.) $$\lim_{t \to \infty} H(t)/t = \bar{u}^{-1} \qquad (\bar{u} = EX_1)$$

For the analogy between reach functions and renewal processes to be of any standing we would expect an "elementary reach theorem":-

THEOREM 5.2.6. *As* $t \to \infty$, $Y(t)$ *satisfies*

$$\lim_{t \to \infty} Y(t)/t = \lambda(U) < \infty \tag{5.2.6.}$$

where $\lambda(U)$ *is a constant depending only on the distribution U.*

The proof of Theorem 5.2.6. will follow from some lemmas which we shall prove below. First, however, for each node A of the lattice define a y-reach function $y_t(A, \omega)$. In other words $y_t(A, \omega)$ has the value $y_t(\omega)$ would take if the lattice were shifted by a horizontal then vertical translation so that A became the origin. By the principle of equivalence under lateral shift therefore we have

LEMMA 5.2.7. *For fixed* A, $y_t(A, \omega)$ *obeys the same probability law as* $y_t(\omega)$.

LEMMA 5.2.8. *The function* $Y(t)$ *satisfies*

$$Y(t_1) + Y(t_2) + 1 \leqslant Y(t_1 + t_2 + U_1) \tag{5.2.8.}$$

$$(t_1, t_2 \geqslant 0)$$

Proof. Let t_1, t_2 be fixed. Let $y_{t_i}(\omega) = m_1$ Then by definiton

$$\left.\begin{aligned} s_{o,m_1}(\omega) &\leqslant t_1 \\ s_{o,m_1+1}(\omega) &> t_1 \end{aligned}\right\} \quad \ldots\ldots \tag{1}$$

Let r_1 be the route of $s_{o,m_1}(\omega)$; (r_1 must exist since the

U distribution is bounded). Let the endpoint of r_1 be $P \equiv (m_1, z)$. Now it will be noticed that $y_{t_1}(\omega)$ is determined only by the time coordinates of the arcs lying inside the strip bounded by $X = 0$, $X = m_1 + 1$. Let 1 be the horizontal arc linking P to $P' \equiv (m_1 + 1, z)$. Since the time coordinates of the arcs of the lattice are $\leqslant U_1$, we may write

$$\begin{aligned} t(r_1 * l, \omega) &= t(r_1, \omega) + t(l, \omega) \\ &\leqslant t_1 + U_1 \qquad (2) \end{aligned}$$

Consider now the random variable $y_{t_2}(P', \omega)$. Let its value be m_2. Then there must exist a cylinder path r_2 from P' to $X = m_1 + m_2 + 1$, such that

$$t(r_2, \omega) \leqslant t_2 \qquad \ldots \qquad (3)$$

Hence, consider the connected cylinder path $r_1 * l * r_2$ which links the origin to $X = m_1 + m_2 + 1$. Then

$$t(r_1 * l * r_2, \omega) \leqq t_1 + t_2 + U_1 \qquad (4)$$

Therefore

$$\begin{aligned} y_{t_1 + t_2 + U_1}(\omega) &\geqslant m_1 + m_2 + 1 \\ &= y_{t_1}(\omega) + y_{t_2}(P', \omega) + 1 \qquad (5) \end{aligned}$$

Consider now the expected value of $y_{t_2}(P', \omega)$. P' is a random node. Since the U distribution is bounded, for given t_2, P' can be only one of a finite set of nodes

$\{A_i\}_{i=1}^{k}$. Define sequence of random variables $\{Z(A_i,w)\}_{i=1}^{k}$ by

$$\begin{aligned} Z(A_i,w) &= 1 \quad \text{if } F' = A_i \\ &= 0 \quad \text{otherwise}. \end{aligned} \tag{6}$$

Now $y_{t_2}(A_i,w)$ is independent of $Z(A_i,w)$ since the values of these two random variables depend on disjoint sets of arcs. Hence by Lemma 5.2.7.

$$\begin{aligned} E\, y_{t_2}(F',w) &= E\left(\sum_i y_{t_2}(A_i,w)\, Z(A_i,w)\right) \\ &= \sum_i \left(E\, y_{t_2}(A_i,w)\right)\left(E\, Z(A_i,w)\right) \\ &= Y(t_2)\, E\left(\sum_i Z(A_i,w)\right) = Y(t_2) \end{aligned} \tag{7}$$

Hence taking expected values of (5) and using (7) we get the required result 5.2.8. Notice that we cannot prove a similar result for $X(t)$ since $t_{on}(w)$ is not necessarily increasing and hence independence is not present in (7).

<u>Proof of Theorem 5.2.6.</u>

From (5.2.8.) we see that $Z(t) \equiv Y(t - U_1)$ satisfies

$$z(t_1) + Z(t_2) + 1 \leqslant Z(t_1 + t_2). \qquad (t_i \geqslant U_1)$$

Hence a fortiori $Z(t)$ is a measurable superadditive function of t which is bounded below (5.2.2.) and hence by (Hille Chap.6)

$$\lim_{t\to\infty} Z(t)/t = \lambda(U)$$

which proves Theorem 5.2.6.

Finally we state, though deferring the proof till the next section

THEOREM 5.2.9. *Provided the U distribution is bounded*

(5.2.9.) $$\lim_{t\to\infty} X(t)/t = \lambda(U)$$

5.3. *The Relation between $\lambda(U)$ and $\mu(U)$.*

Intuitively one would expect some form of an inverse relationship between the two constants $\lambda(U)$ and $\mu(U)$. This is proved in

THEOREM 5.3.1. *For any bounded distributions U the constants $\lambda(U)$ and $\mu(U)$ satisfy*

(5.3.1.) $$\mu(U)\,\lambda(U) = 1$$

The proof of this fundamental result follows from the following lemmas.

LEMMA 5.3.2. *Y(t) satisfies for all bounded distributions U.*

(5.3.2.) $$\lim_{t\to\infty} \frac{Y(t)}{t} = \lambda \leq [\mu]^{-1}$$

Proof: Consider the random variable $y_{s_{om}(\omega)}(\omega)$. Since the time coordinate of an arc is strictly positive and hence $s_{om}(\omega)$ is strictly increasing it follows that

(1) $$y_{s_{om}(\omega)}(\omega) = Ey_{s_{om}(\omega)}(\omega) = m \quad (m \text{ any integer})$$

Since the U distribution is bounded and hence has an r^{th} moment for some $r > 1$, by Theorem 4.2.11. $s_{om}(\omega)/m$ converges with probability 1 to $\mu(U)$ as $m \to \infty$.

Define $\Omega_m \equiv [\omega : s_{om}(\omega) \leq (\mu - \varepsilon)m]$

Then, $\varepsilon, \eta, > 0$, $\exists$ $m_o(\varepsilon, \eta)$ such that

(2) $$P(\Omega_m) \leq \eta \qquad (m \geq m_o)$$

Since $y_t(\omega)$ is non decreasing in t for fixed ω

(3) $$y_{(\mu - \varepsilon)m}(\omega) \leq y_{s_{om}(\omega)}(\omega) = m \qquad (\omega \varepsilon \Omega - \Omega_m)$$

Since $y_t(\omega) \leq t/U_o$ for all $\omega \varepsilon \Omega$

(4) $$y_{(\mu - \varepsilon)m}(\omega) \leq (\mu - \varepsilon)m/U_o \qquad (\omega \varepsilon \Omega_m)$$

Considering the expected value of $y_{(\mu - \varepsilon)m}(\omega)$ we therefore have

(5) $$Y((\mu - \varepsilon)m) \leq \frac{(\mu - \varepsilon)m}{U_o} P(\Omega_m) + mP(\Omega - \Omega_m)$$

By (2)

(6) $$Y((\mu - \varepsilon)m) \leq \frac{m(\mu - \varepsilon)}{U_o}\eta + m \qquad (m \geq m_o(\varepsilon, \eta))$$

Dividing by m, and taking the limit as $m \to \infty$

(7) $$(\mu - \varepsilon)\lambda \leq 1 + \eta\mu U_o^{-1} - \eta\varepsilon U_o^{-1}$$

Since ε, η can be taken arbitrarily small we therefore have

(8) $$\mu\lambda \leq 1$$

This completes the proof of Lemma 5.3.2.

LEMMA 5.3.3. As $t \to \infty$ the function $X(t)$ satisfies

$$\limsup_{t \to \infty} X(t)/t \leq [\mu(U)]^{-1}$$

Proof: This follows immediately from the fact that $X(t) \leq Y(t)$ for all t, and lemma 5.3.2.

LEMMA 5.3.4. As $t \to \infty$ the function $X(t)$ satisfies

$$\liminf_{t \to \infty} X(t)/t \geq \left[\mu(U)\right]^{-1}$$

Proof: Define, $S^n_m(\omega)$ for m,n, positive integers by

$$(1) \quad S^n_m(\omega) = t_{0,n}(\omega) + t_{n,2n}(\omega) + \dots + t_{(m-1)n,mn}(\omega)$$

Then $S^n_m(\omega)$ is the m^{th} partial sum of independent, identically distributed random variables each having the distribution of $t_{0,n}(\omega)$.

Define $N^n_t(\omega)$ by

$$(2) \quad N^n_t(\omega) = \sup m : S^n_m(\omega) \leq t$$

and let $H_n(t) \equiv E\, N^n_t(\omega)$.

Then for each fixed n, $H_n(t)$ is the renewal function of the sequence $\left\{ t_{(i-1)n,in}(\omega) \right\}_{i=1}^{\infty}$

Hence by the Elementary Renewal Theorem, (Feller 1941) for each fixed n

$$\lim_{t \to \infty} \frac{H_n(t)}{t} = (T(0,n))^{-1} \tag{3}$$

Suppose that $N^n_t(\omega) = k$, then by definiton

$$S^n_k(\omega) \leq t \tag{4}$$

But by subadditivity

$$t_{0,rn}(\omega) \leq S^n_r(\omega) \qquad (r \text{ any integer} \geq 0)$$

In particular $t_{o,kn}(\omega) \leqslant S_k^n(\omega)$ (5)

This implies by (4) that

$$t_{o,kn}(\omega) \leqslant t$$

which in turn implies

$$x_t(\omega) \geqslant kn = n N_t^n(\omega) \qquad (6)$$

Taking expected values of (6)

$$X(t) \geqslant n H_n(t) \qquad \text{(all integer } n) \qquad (7)$$

and hence by (3)

$$\liminf_{t \to \infty} \frac{X(t)}{t} \geqslant n/T(0,n) \qquad (8)$$

Since (8) holds for any integer n and

$$\lim_{n \to \infty} T(0,n)/n = \mu(U)$$

we have the result that

$$\liminf_{t \to \infty} X(t)/t \geqslant (\mu(U))^{-1} \qquad (9)$$

which completes the proof of Lemma 5.3.4.

<u>LEMMA 5.3.5.</u> <u>Y(t) satisfies</u>

$$\liminf_{t \to \infty} \frac{Y(t)}{t} \geqslant [\mu(U)]^{-1}$$

<u>Proof</u>. Immediate from (5.3.4.) since $Y(t) \geqslant X(t)$.

The proofs of Theorems 5.3.1. and 5.2.10. now follow trivially from the above lemmas. since

$$\mu(U)^{-1} \leqslant \liminf X(t)/t \leqslant \limsup X(t)/t \leqslant \lim Y(t)/t \leqslant \mu(U)^{-1}.$$

5.4. The Convergence of the Reach Functions.

Since the U distribution is bounded Theorems 4.1.11., 4.2.11., show that as $n \to \infty$ the random variables $t_{on}(\omega)/n$, $s_{on}(\omega)/n$ both converge with probability 1 to the time constant $\mu(U)$. Correspondingly for the reach functions $x_t(\omega)$, $y_t(\omega)$ we have

THEOREM 5.4.1. *As the parameter* $t \to \infty$, $x_t(\omega)/t$ *and* $y_t(\omega)/t$ *both converge in mean square to the constant* $\lambda(U) = (\mu(U))^{-1}$.

COROLLARY 5.4.2. *As the parameter* $t \to \infty$, $x_t(\omega)/t$ *and* $y_t(\omega)/t$ *both converge in probability to the constant* $\lambda(U) = (\mu(U))^{-1}$.

Proof of Theorem 5.4.1.

Consider the random variable $y_t(\omega)$

Prescribe $\varepsilon, \eta, > 0$ and define an integer m by

$$m(\mu - \varepsilon) > t \geq (m-1)(\mu - \varepsilon) \qquad (1)$$

Such a definition is meaningful since $t > 0$.

Define $$\Omega_m \equiv (\omega : s_{om}(\omega) \leq (\mu - \varepsilon)m) \qquad (2)$$

By Theorem 4.2.11. there exists $m_o = m_o(\varepsilon, \eta)$ such that

$$P(\Omega_m) \leq \eta \quad \text{for } m \geq m_o(\varepsilon, \eta) \qquad (3)$$

Now
$$y_{m(\mu-\varepsilon)}(\omega) \leq y_{s_{om}(\omega)}(\omega) \quad (\omega \in \Omega - \Omega_m)$$

$$= m \quad (\omega \in \Omega - \Omega_m) \qquad (4)$$

and $y_{m(\mu-\varepsilon)}(\omega) \leqslant m(\mu-\varepsilon)/U_o \qquad (\omega \varepsilon \Omega_m) \qquad (5)$

Hence taking expectations

$$E\left[y_{m(\mu-\varepsilon)}(\omega)\right]^2 \leqslant m^2 P(\Omega-\Omega_m) + \frac{m^2(\mu-\varepsilon)^2}{U_o^2} P(\Omega_m)$$

$$\leqslant m^2 + m^2(\mu-\varepsilon)^2\eta/U_o^2 \qquad (m \geqslant m_o) \quad (6)$$

Let $t_o = (m_o-1)(\mu-\varepsilon)$. Let $t_1 \geqslant t_o$. Then let m_1 be defined in terms of t_1 by equation (1). Then since $y_t(\omega)$ is increasing in t for fixed ω.

$$y_{t_1}(\omega) \leqslant y_{m_1(\mu-\varepsilon)}(\omega).$$

Since $t_1 \geqslant t_o$ implies $m_1 \geqslant m_o$, by (6)

$$E\, y^2_{t_1}(\omega) \leqslant E\left[y_{m_1(\mu-\varepsilon)}(\omega)\right]^2$$

$$\leqslant m_1^2 + m_1^2(\mu-\varepsilon)^2\eta/U_o^2$$

Since by (1) $t_1 \geqslant (m_1-1)(\mu-\varepsilon)$

$$\frac{E\, y^2_{t_1}(\omega)}{t_1^2} \leqslant \frac{m_1^2\left[1+(\mu-\varepsilon)^2\eta/U_o^2\right]}{(m_1-1)^2(\mu-\varepsilon)^2} \qquad (t_1 \geqslant t_o)$$

$$\leqslant \mu^{-2} + O(\varepsilon,\eta).$$

Taking the limit as $t_1 \to \infty$, since ε, η, can be made arbitrarily small we get

$$\lim_{t\to\infty} E\, y_t^2(\omega)/t^2 \leqslant \mu^{-2}$$

Since $\lim_{t\to\infty} Y(t)/t = \mu^{-1}$ we have the result that

(5.4.3.) $\quad \lim_{t\to\infty} \operatorname{var}(y_t(\omega)/t) = 0$

Using (5.4.3.) and remembering that $x_t(\omega) \leqslant y_t(\omega)$ while $\lim_{t \to \infty} X(t)/t = \lambda$ it is not difficult to prove

$$(5.4.4.) \quad \lim_{t \to \infty} \operatorname{var} (x_t(\omega)/t) = 0$$

The proof of Theorem 5.4.1. now follows for

$$E\left[\frac{y_t(\omega)}{t} - \lambda\right]^2 = \frac{\operatorname{var} y_t(\omega)}{t^2} + \frac{Y^2(t)}{t^2} - \frac{2\,Y(t)\lambda}{t} + \lambda^2 \qquad (10)$$

Taking the limit in (10) as $t \to \infty$ we have the result that $y_t(\omega)/t$ converges in mean square to the time constant λ. A similar proof holds for $x_t(\omega)/t$.

The proof of Corollary 5.4.2. is a trivial consequence of Theorem 5.4.1.

5.5. Further Conjectures Concerning $x_t(\omega)$, $y_t(\omega)$.

Below are stated some conjectures concerning $x_t(\omega)$. They apply equally well to $y_t(\omega)$.

A comparison of (5.4.3.) and (5.4.4.) with the corresponding result in renewal theory (Smith 1958) that $\operatorname{var} N_t \sim kt$ (k a constant) leads one to suspect that (5.4.3.-4.) are rather weak results and that in actual fact

$$(5.5.1.) \quad \operatorname{var} x_t(\omega) \sim k_1 t \qquad (k_1 \text{ a constant})$$

A conjecture of considerable intuitive appeal is

$$(5.5.2.) \qquad \lim_{t \to \infty} X(t + a) - X(t) = \lambda a$$

This conjecture is derived from Blackwell's Renewal Theorem. However, the intricacy of Blackwell's proof is such as to intimidate attacks on the even more difficult problem (5.5.2.)

It seems reasonable to expect the results obtained when the U distribution is bounded also to hold in the unbounded case. The replacement of U_0 by zero should not present too great a difficulty. However I see no way of replacing U_1 by $+\infty$ and obtaining even the fundamental result (5.2.8.)

Similarly if the reach functions are defined in terms of the absolute first passage times, there seems no way of proving (5.2.8.). This is because the introduction of absolute first passage times destroys a great deal of the existing independence.

CHAPTER 6

The Time Constant of the Square Lattice

The fundamental importance of the time constant $\mu(U)$ is evident from chapters 4, 5. In this chapter we explore the functional relationship between $\mu(U)$ and the underlying distribution U for the case of the square lattice. It is hoped that the extension of these results, or rather the techniques used in obtaining them will prove useful in other more difficult problems.

6.1. The Theoretical Estimation of $\mu(U)$

In chapter 4 we stated the obvious inequality

(6.1.1.) $0 \leqslant \mu(U) \leqslant \bar{u}$

where $\bar{u}$ is the mean of the U distribution. Example (4.1.8.) showed that distributions U exist for which equality holds in (6.1.1.). This section is devoted to a search for better upper bounds for $\mu(U)$.

6.1.2. Consider the algorithm for the travel by an abstract particle between the origin and X = m, which has the following simple rule. At each node of the lattice (m_1, n_1) at which the particle arrives, it subsequently takes one of three paths

$$r_1 \equiv (m_1,n_1) \to (m_1, n_1+1) \to (m_1+1,n_1+1)$$
$$r_2 \equiv (m_1,n_1) \longrightarrow (m_1+1, n_1)$$
$$r_3 \equiv (m_1,n_1) \longrightarrow (m_1,n_1-1) \longrightarrow (m_1+1,n_1-1)$$

The choice of path is made by considering the time coordinates of r_1, r_2, r_3, and travelling along that r_i which has the minimum time coordinate. This is an algorithm for travel between the origin and $X = m$ in at most $2m$ steps and if r_o is the path taken by this method, then obviously

$$(6.1.3.) \qquad Et(r_o, \omega) = m\, E \min \begin{pmatrix} u_1 + u_2 \\ u_3 \\ u_4 + u_5 \end{pmatrix}$$

where u_i $(i = 1 \ldots 5)$ are independent random variables each having the distribution U.

Clearly from (6.1.3.) we have

$$(6.1.4.) \qquad \mu(U) \leqslant \lim_{m \to \infty} Et(r_o, \omega)/m$$

$$= E \min \begin{pmatrix} u_1 + u_2 \\ u_3 \\ u_4 + u_5 \end{pmatrix}$$

Applying (6.1.4.) in two simple cases

$$(6.1.5.) \qquad \mu(U) \leqslant \cdot 425$$ when U is the uniform rectangular distribution on (0.1)

$$(6.1.6.) \qquad \mu(U) \leqslant \cdot 629$$ when U is the exponential distribution with parameter 1.

The above results (6.1.5. - 6.) represent an expected 'saving' of 15% and 37% respectively on results obtained from (6.1.1.).

6.2. The case when U is the Bernoulli Distribution.

Let the time coordinates of the arcs of the lattice be zero with probability $\frac{1}{2}$ and 1 with probability $\frac{1}{2}$. The problem of first passage percolation is now very similar to that of ordinary percolation (see chapter 1). For this rather simple distribution the inequality (6.1.4.) gives for U this Bernoulli distribution

(6.2.1.) $\mu(U) \leq \cdot 2813$

However (6.2.1.) may be improved by the following method. From (4.4.5) it may be seen that $\mu(U)$ is bounded above by $g(U)$ where $g(U)$ is the expected first passage time from the origin to the ordinate $X = 1$ over paths which lie strictly inside the strip bounded by $X = -1$, $X = +2$. For U the Bernouli distribution ($p = \frac{1}{2}$), $g(U)$ can (with some labour) be shown to equal $\cdot 167$. Hence it is true that for U the Bernoulli distribution

(6.2.2.) $\mu(U) \leq \cdot 167$

The extension of this method to more complex distributions U seems too laborious to be justifiable.

6.3. A Lower Bound for $\mu(U)$.

The problem of determining non-trivial lower bounds for $\mu(U)$ is at the moment completely unsolved. At one stage it was conjectured that

$$\mu(U) \geq E \min (u_1, u_2)$$

where u_1, u_2 are independent random variables drawn from the U distribution. This is false, a counter example is the case where U is the Bernoulli distribution above.

6.4. $\mu(U)$ as a Functional of U

In this section an attempt is made to establish a 'calculus' between U and $\mu(U)$. More precisely if U_1 and U_2 are two distributions and f is any relation connecting U_1 and U_2, then we shall look for the existence or non existence of a relation between $\mu(U_1)$ and $\mu(U_2)$.

Notice first that if $\bar{u}_1$ and $\bar{u}_2$ are the means of U_1 and U_2 respectively then $\bar{u}_1 \leqslant \bar{u}_2$ does not imply $\mu(U_1) \leqslant \mu(U_2)$. For, let U_1 be the constant distribution with $\bar{u}_1 = \cdot 45$ while U_2 is the uniform rectangular distribution on (0.1), so that $\bar{u}_2 = \cdot 5$. $\mu(U_1) = \bar{u}_1 = \cdot 45$ but by (6.1.5.) $\mu(U_2) \leqslant \cdot 425$. One result which is true however is

THEOREM 6.4.1. If $F_1(x)$, $F_2(x)$ are the cumulative distribution functions of U_1 and U_2 and if

$$F_1(x) \leqslant F_2(x) \qquad \text{(all x)}$$

then

$$\mu(U_1) \geqslant \mu(U_2) \tag{6.4.1.}$$

Before proving this theorem we introduce an inverse function $T_F^{-1}(.)$ associated with each distribution function

$F(x)$. Precisely if $F(x)$ is any distribution function define

(6.4.2.) $$T_F^{-1}(\xi) = \inf x : F(x) > \xi \qquad (0 \leq \xi \leq 1)$$

Then it is well known that

(6.4.3.) $$F(T_F^{-1}(\xi)) \geq \xi$$

Also we have

LEMMA 6.4.4. *If ξ is uniformly distributed in (0.1) then*

(6.4.4.) $$P\left[T_F^{-1}(\xi) \leq y\right] = F(y)$$

for any distribution function $F(x)$

LEMMA 6.4.5. *If $F_1(x)$ and $F_2(x)$ are cumulative distribution functions such that for all x*

$$F_1(x) \leq F_2(x)$$

then for all ξ in (0.1)

(6.4.5.) $$T_{F_1}^{-1}(\xi) \geq T_{F_2}^{-1}(\xi)$$

Proof of Theorem 6.4.1.

Let (Ω_o, F_o, P_o) be the phase space induced on the lattice by U_o - the uniform rectangular distribution on (0.1). For ω any element of Ω_o, and $F(x)$ any cumulative distribution function, define $T_F^{-1}(\omega)$ as follows. Let u_i be the time coordinate of l_i under ω. Define $T_F^{-1}(u_i)$ by (6.4.2.) and let it be the time coordinate of l_i under $T_F^{-1}(\omega)$. Thus corresponding to any time state $\omega \varepsilon \Omega_o$ we have a time state $T_F^{-1}(\omega)$. By Lemma (6.4.4.)

$\left\{ T_F^{-1}(\omega) \right\}_{\omega \in \Omega_0}$ is the phase space on the lattice induced by an underlying distribution U which has distribution function F(x).

Now let U_1, U_2 be two distributions with distribution functions $F_1(x)$ and $F_2(x)$ such that for all x, $F_1(x) \leq F_2(x)$. Then by Lemma (6.4.5.) if r is any path on the lattice

$$(6.4.6.) \qquad t(r, T_{F_1}^{-1}(\omega)) \geq t(r, T_{F_2}^{-1}(\omega)) \qquad (\text{all } \omega \in \Omega_0)$$

Hence

$$(6.4.7.) \qquad t_{on}(T_{F_1}^{-1}(\omega)) \geq t_{on}(T_{F_2}^{-1}(\omega)) \qquad (\text{all } \omega \in \Omega_0)$$

Taking expected values of (6.4.7.), dividing by n and taking the limit as $n \to \infty$ we therefore get the required result

$$(6.4.8.) \qquad \mu(U_1) \geq \mu(U_2)$$

6.5. The Effect of Elementary Operations on U

In several practical examples one can easily envisage a uniform change in the time state 'ω' of the lattice, due to some change in the environment. For example, the time coordinate of each arc may be multiplied by 3 or divided by 2. This section considers the effect such changes have on the functional $\mu(U)$.

If the lattice has time state ω, denote by '$k\omega$' the corresponding time state when the time coordinate of each arc is multiplied by $k \geq 0$. Then since for any path r,

$t(r, k\omega) = kt(r, \omega)$ we have the trivial result

$$(6.5.1.) \quad kt_{mn}(\omega) = t_{mn}(k\omega)$$

which implies that the new time coordinate is $k\,\mu(U)$. Notice also that for this case the route of the first passage time between any two points is unchanged.

This is not the case however when the time coordinate of each arc l_i of the lattice is changed from u_i to $u_i + k$ (k some non-negative constant). Denote this new time state of the lattice by '$\omega + k$'. Then if $n(r)$ is the number of arcs in the lattice path r it is easily seen that

$$(6.5.2.) \quad t(r, \omega + k) = t(r, \omega) + kn(r)$$

From (6.5.2.) it can be appreciated that equality <u>cannot</u> replace inequality in

$$(6.5.3.) \quad t_{mn}(\omega + k) \leqslant t_{mn}(\omega) + kn_{mn}(\omega)$$

where $n_{mn}(\omega)$ represents the number of arcs in the route of $t_{mn}(\omega)$.

The above relation will be explored further in chapter 8. The effect of further operations of this nature on the time state 'ω' may be obtained by similar methods, however they do not appear to give very interesting results. The main purpose of this section has been to show the care that has to be taken when performing seemingly elementary operations on the phase space of the lattice.

We return to a further examination of $\mu(U)$ in chapter 9 where some numerical results are compared with our theoretical bounds.

CHAPTER 7

Two Other Subadditive Percolation Processes.

For many physical problems it is not necessary when dealing with first passage times to consider the complete set of paths joining the points under consideration. For example, when considering the first passage time between the origin and (n,o) on the square lattice, we could, with some justification, consider only those paths which contained less than n^2 arcs. The error induced by such a restriction would in all probability, be negligible for large n.

The study of first passage times over such restricted sets of paths is obviously less difficult. In this chapter we shall study a few problems in first passage theory which are closely related to the main problems of the thesis but which are simplified by some restriction or other. These problems are of importance for two reasons :-

a) They have a certain physical significance of their own right.

b) They serve as showrooms for some techniques which it is hoped might be of use in more complex problems.

Throughout this chapter it is assumed that the underlying U distribution is fixed, has a finite mean $\bar{u}$ and gives rise to a phase space (Ω, F, μ) on the square lattice. Also, because of the similarity between the processes discussed and the fundamental processes studied in detail in chapter 4, many of the proofs will be only sketched and several results, which are common to these new processes and the t, s, processes of chapter 4, may be omitted.

7.1. First Passage Theory in the Upper Half Plane

In this section we consider an independent, subadditive process which is particularly amenable to the intersection technique (to be explained below). Precisely, define $q_{mn}(\omega)$ to be the first passage time between (m,o) and (n,o) when the lattice has time state ω, over cylinder paths which have the additional restriction that they pass through no node of the lattice which has negative y-coordinate.

By similar methods as in 4.1. it can be shown that $\{q_{mn}(\omega)\}$ is an independent, subadditive stochastic process and all the theorems of chapter 3 can be applied to this process.

The expected value of $q_{mn}(\omega)$ exists, is denoted by Q(m,n), and trivially by the inclusion lemma satisfies

$$(7.1.1.) \qquad \frac{T(m,n)}{(n-m)} \leq \frac{Q(m,n)}{(n-m)} \leq \bar{u}$$

As an example of the use of the intersection technique we prove

THEOREM 7.1.2. For fixed m, Q(m,n) satisfies

(7.1.2.) $Q(m,n) \leqslant Q(m,n+1) \qquad (m \leqslant n)$

This result should be compared with conjecture 4.1.14. concerning a similar result for the t-process. Unfortunately we see no means of extending this proof to a proof of conjecture 4.1.14. Before proving this theorem we introduce the first passage stochastic process $z_{mn}(\omega)$ defined as the first passage time under ω between (m,o) and the line X = n over cylinder paths which lie in the upper half plane. $z_{mn}(\omega)$ is the half plane analogue of $s_{mn}(\omega)$. The important property to note is

(7.1.3.) $z_{mn}(\omega) \leqslant z_{m,n+1}(\omega) \qquad (m \leqslant n)$

Proof of Theorem 7.1.2.

It will be shown in chapter 8 that the routes of all first passage times discussed here exist. Let r_1 be the route of $q_{on}(\omega)$. Let r_2 be the route of $z_{1,n}(\omega)$. Inspection will show that r_1 and r_2 must intersect at at least one node. Let P be any node common to both r_1 and r_2. Let r_{11} be the set of arcs of r_1 which link the origin to P. Let r_{12} be the set of arcs of r_1 which link P to (n,o). Similarly let r_{21} be the set of arcs of r_2 which link

(1,0) to P and let r_{22} be the set of arcs belonging to r_2 but not to r_{21}.

It is obvious that all four paths r_{ij} have the point P in common. Moreover $r_{11} * r_{22}$ is a connected path from the origin to the line $\lambda = n$ while $r_{21} * r_{12}$ is a connected path from (1,o) to (n,o).

Hence, since both these paths are contained in the upper half plane

(7.1.4.) $\quad t(r_{11} * r_{22}, \omega) \geqslant z_{on}(\omega)$

(7.1.5.) $\quad t(r_{21} * r_{12}, \omega) \geqslant q_{1,n}(\omega)$.

But since $r_{11} * r_{22}$ and $r_{21} * r_{12}$ contain the same arcs as r_1, r_2

(7.1.6.) $\quad t(r_{11} * r_{22}, \omega) + t(r_{21} * r_{12}, \omega) = q_{on}(\omega) + z_{1,n}(\omega)$

Hence combining (7.1.4. - 6.)

(7.1.7.) $\quad q_{on}(\omega) + z_{1,n}(\omega) \geqslant z_{o,n}(\omega) + q_{1,n}(\omega)$.

Taking expected values of (7.1.7.) and remembering the principle of equivalence under lateral shift

(7.1.8.) $\quad Q(o,n) - Q(o,n-1) \geqslant Z(o,n) - Z(o,n-1)$

where $\quad Z(o,n) = Ez_{on}(\omega)$.

But by (7.1.3.) the right hand side of (7.1.8.) is always non-negative.

Hence

(7.1.9.) $\quad Q(o,n) - Q(o,n-1) \geqslant 0$

The result (7.1.2.) now follows by the principle of equivalence under lateral shift.

Further results concerning $Q(o,n)$ may be proved by similar methods. However because of space we shall only state two of the more important properties

(7.1.10.) $$\lim_{n \to \infty} Q(o,n)/n = \lim_{n \to \infty} T(o,n)/n$$

(7.1.11.) $Q(o,n)/n$ is decreasing as n increases.

7.2. A Subadditive Process with Superadditive Properties

In this section we study the problem of first passage percolation along a pipe or strip of the square lattice of width 2k. This is a model (idealised somewhat) for the flow of an abstract fluid along a porous pipe when the radius of the pore, and hence the rate of flow along individual pores, is a random variable.

Define $p^k_{mn}(\omega)$ to be the first passage time between $(m,0)$ and $(n,0)$ over paths which lie strictly inside the rectangle bounded by the lines $Y = \pm k$, $X = m$, $X = n$. Then for k fixed, $p^k_{mn}(\omega)$ is a 2-parameter stochastic process on (Ω, F, P). By the inclusion lemma it is obvious that

(7.2.1.) $$p^k_{mn}(\omega) \geqslant p^{k+1}_{mn}(\omega) \geqslant t_{mn}(\omega) \qquad (\omega \varepsilon \Omega, k \geqslant 0)$$

The expected value $P_k(m,n)$ of $p^k_{mn}(\omega)$ exists and satisfies

(7.2.2.) $\bar{u}(n-m) \geqslant P_k(m,n) \geqslant P_{k+1}(m,n) \geqslant T(m,n) \qquad (k \geqslant 0)$

Also it is not difficult to see that when k is fixed

(7.2.3.) $p^k_{mn}(\omega) + p^k_{nq}(\omega) \geqslant p^k_{mq}(\omega) \qquad (m \leqslant n \leqslant q)$

and in particular $\left\{p^k_{mn}(\omega)\right\}$ is, for fixed k, an independent subadditive stochastic process.

Hence by Theorem 3.1.5. there exists a time constant $\mu_k(U)$ such that for k fixed

(7.2.4.) $P_k(0,n)/n \geqslant \mu_k(U) = \lim_{n\to\infty} P_k(0,n)/n$

and by (7.2.2.)

(7.2.5.) $\bar{u} \geqslant \mu_k(U) \geqslant \mu_{k+1}(U) \geqslant \mu(U) \qquad (k \geqslant 0)$

Also since $p^k_{0n}(\omega)$ is non-increasing for fixed n and ω, and

(7.2.6.) $\lim_{k\to\infty} p^k_{0n}(\omega) \overset{\downarrow}{=} t_{0n}(\omega)$

By the Monotone Convergence Theorem

(7.2.7.) $\lim_{k\to\infty} P_k(m,n) = T(m,n) \qquad$ (fixed m,n)

and it is not difficult to show

(7.2.8.) $\lim_{k\to\infty} \mu_k(U) = \mu(U)$

Up till now we have not used the fact that $Y = \pm k$ are bounds on the geometrical extent of the route of $p^k_{mn}(\omega)$

THEOREM 7.2.9. The subadditive stochastic process $p^k_{mn}(\omega)$ satisfies the superadditive relation

$$(7.2.9.)\qquad P_k(m,n) + P_k(n,q) \leq P_k(m,q) + 2\bar{u}(k+1) \qquad (m \leq n \leq q)$$

and hence for all n

$$(7.2.10.)\qquad P_k(0,n) \leq n\mu_k(U) + 2(k+1)\bar{u} \qquad (n \geq 0)$$

Proof of Theorem 7.2.9.

Let r, the route of $p^k_{mq}(\omega)$ meet $X = n-1$, $X = n+1$, at the points P_1, P_2, respectively. Let r_1, r_2, be the straight line paths from P_1, P_2, to the X-axis. Let r_3, r_4, be the arcs from $(n-1,0)$ to $(n,0)$ and from $(n,0)$ to $(n+1,0)$ respectively. Then it is not difficult to see that two disjoint sections of the arcs making up the paths r, r_1, r_2, r_3, r_4 are paths from $(m,0)$ to $(n,0)$ and from $(n,0)$ to $(q,0)$. Hence

$$(1)\qquad t(r,\omega) + \sum_{i=1}^{4} t(r_i,\omega) \geq p^k_{mn}(\omega) + p^k_{nq}(\omega)$$

There are at most k arcs in r_1, r_2, and since the 'positions' of these arcs are fixed we may write

$$(2)\qquad \operatorname{Max}\left[E\,t(r_1,\omega),\ E\,t(r_2,\omega)\right] \leq k\bar{u}$$

and also

$$(3)\qquad Et(r_3,\omega) = E\,t(r_4,\omega) = \bar{u}$$

Hence taking expected values of (1) we have for fixed k

(7.2.9'.) $\quad P_k(m,q) + 2k\bar{u} + 2\bar{u} \geqslant P_k(m,n) + P_k(n,q)$

writing (7.2.9'.) in the form

(7.2.9".) $\quad \left[P_k(m,q) - 2(k+1)\bar{u}\right] \geqslant$

$$\geqslant \left[P_k(m,n) - 2(k+1)\bar{u}\right] + \left[P_k(n,q) - 2(k+1)\bar{u}\right]$$

Hence $P_k(0,n) - 2(k+1)\bar{u}$ is a <u>superadditive</u> function of n for fixed k and hence (Hille. Chapter 6) (7.2.10.) follows. Combining (7.2.10.) and (7.2.4.) we have

<u>Corollary 7.2.11.</u> <u>For fixed k, $P_k(0,n)$ satisfies</u>

(7.2.11.) $\quad 0 \leqslant n\,\mu_k(U) \leqslant P_k(0,n) \leqslant n\,\mu_k(U) + 2(k+1)\bar{u}$

Before closing this section we notice that $\mu_0(U) = \bar{u}$ for all U distributions. To obtain $\mu_1(U)$ however is so difficult that it destroys any hope of using extrapolation on the sequence $\left\{\mu_i(U)\right\}_{i=1}^{\infty}$ as a means of estimating $\mu(U)$.

We now consider the convergence as $n \to \infty$ of the process $p^k_{on}(\omega)/n$.

The importance of this superadditive property of $p^k_{mn}(\omega)$ is better exemplified here. It will be seen that a strong law of convergence holds for $p^k_{mn}(\omega)$ <u>without</u> any stipulation concerning the higher moments of the U distribution as was necessary to prove a similar result for $t_{mn}(\omega)$ which was merely an independent subadditive process.

THEOREM 7.2.12. As $n \to \infty$ the random variable $p^k_{on}(\omega)/n$ converges with probability 1 to the time constant $\mu_k(U)$.

Proof of Theorem 7.2.12.

Whilst proving Theorem 7.1.9. we showed (4) that

$$(1) \quad p^k_{mn}(\omega) + p^k_{nq}(\omega) \leqslant p^k_{mq}(\omega) + f(\omega) \qquad (m \leqslant n \leqslant q)$$

where $f(\omega)$ was a random variable on (Ω, F, P) such that

$$(2) \quad 0 \leqslant E\, f(\omega) \leqslant 2(k+1)\bar{u}$$

Let q_i be the connected sets of arcs connecting $(in-1,-k)$ to $(in-1,+k)$, connecting $(in-1,0)$ to $(in+1,0)$, and connecting $(in+1,-k)$ to $(in+1,+k)$. q_i therefore has the 'shape' of the letter 'H'.

Since each q_i $(i = 1, \ldots, j-1)$ intersects the route of $p^k_{o,jn}(\omega)$ and also passes through the point (in,o) by the Connection lemma (2.3.2.)

$$(3) \quad p^k_{o,jn}(\omega) + \sum_{i=1}^{j-1} t(q_i,\omega) \geqslant \sum_{i=0}^{j-1} p^k_{in,(i+1)n}(\omega)$$

Now the sequence $\left\{t(q_i,\omega)\right\}_{i=1}^{j-1}$ is a sequence of independent, identically distributed random variables with finite mean $\bar{u}(4k+2)$. Hence Halmos (P.205 Ex.7)

$$(4) \quad P\left[\lim_{j\to\infty} j^{-1}\sum_{i=1}^{j-1} t(q_i,\omega) - \bar{u}(4k+2) = 0\right] = 1$$

and since the r.h.s. of (3) is a sum of independent random variables, Halmos (P.205) again gives

(5) $$P\left[\lim_{j\to\infty} j^{-1}\sum_{i=0}^{j-1} p^k_{in,(i+1)n}(\omega) = P_k(0,n)\right] = 1$$

Hence by (3), (4), (5) we have

(6) $$P\left[\liminf_{j\to\infty} j^{-1} p^k_{o,jn}(\omega) \geqslant P_k(0,n) - \bar{u}(4k+2)\right] = 1$$

Given arbitrary $\varepsilon > 0$, n can be chosen so that $n^{-1}\bar{u}(4k+2) \leqslant \varepsilon$ and since $n^{-1} P_k(0,n) \geqslant \mu_k(U)$ (6) becomes

(7) $$P\left[\liminf_{j\to\infty} (jn)^{-1} p^k_{o,jn}(\omega) \geqslant \mu_k(U)\right] = 1$$

Let now $m = jn + r$, where $0 \leqslant r < n$. By subadditivity

(8) $$p^k_{o,m}(\omega) + p^k_{m,(j+1)n}(\omega) \geqslant p^k_{o,(j+1)n}(\omega).$$

or

(9) $$\liminf_{m\to\infty} \frac{p^k_{om}(\omega)}{m} \geqslant \liminf_{j\to\infty} \frac{p^{\kappa}_{o,(j+1)n}(\omega)}{(j+1)n} - \limsup_{m\to\infty} \frac{p^{\kappa}_{m,(j+1)n}(\omega)}{m}$$

By Theorem 3.3.2. $\limsup_{m\to\infty} p^{\kappa}_{m,(j+1)n}(\omega)/m = 0$ with probability 1.

Hence by (7) and (9)

(10) $$P\left[\liminf_{m\to\infty} p^k_{om}(\omega)/m \geqslant \mu_{\kappa}(U)\right] = 1$$

But since $p^{\kappa}_{mn}(\omega)$ is an independent subadditive process we know

(11) $$P\left[\limsup_{m\to\infty} p^k_{om}(\omega)/m \leqslant \mu_k(U)\right] = 1$$

Theorem 7.2.12. follows from (11) and (10).

which completes the proof of Theorem 7.2.12.

Many other subadditive first passage percolation processes exist which have properties of a superadditive nature. However lack of space prevents their inclusion here but usually they can be solved by a combination of the techniques shown in chapters 4, 7.

CHAPTER 8

Problems in Geometrical Probability Associated with First Passage Percolation

Fundamental problems associated with the study of first passage theory on a graph g are the following. Does there exist an algorithm for determining the route of any first passage time? Does this optimal route exist? These problems are the main problems associated with PERT networks (Chapter 1). In Chapter 6 we gave an algorithm for travel between a point and a line on the square lattice, which reduced the expected time of travel considerably. However, we make no claim that this is a good algorithm,(it is just the best we can do). The magnitude of the reduction in expected travel time produced, however, is sufficient to justify an enquiry into this problem.

In Chapter 7 we considered the effect of limitations on the set of paths over which first passage times are taken. Before limitations such as are imposed in Chapter 7 are justified we need to find out more about the nature of the route of these first passage times. This is the purpose of this chapter.

Our first problem is to consider the existence of the route of the first passage times. Then we shall study an important geometrical problem associated with these routes.

8.1. The Existence of a Route of a First Passage Time

Before we can logically discuss properties of the route of a first passage time it is obviously necessary to consider whether or not this route exists. Trivially when we are dealing with a first passage time over a finite set of paths on any graph g the route of this first passage time must exist. The route is a 'random connected path' on the graph g. To show that the route of $t_R(\omega)$ exists when R is an infinite set of paths is however quite difficult. Here we shall let our graph g be the square lattice and consider the existence of the routes of the first passage times $t_{On}(\omega)$, $a_{On}(\omega)$ and $b_{On}(\omega)$.

First notice that when the time co-ordinates u_i of the arcs l_i of the lattice satisfy the bounding restriction

(8.1.1.) $0 < U_0 \leq u_i \leq U_1 < \infty$

then since the first passage times satisfy

(8.1.2.) $nU_0 \leq a_{On}(\omega),\ b_{On}(\omega),\ t_{On}(\omega),\ s_{On}(\omega) \leq nU_1$

we need only consider the first passage times over paths with not more than m arcs, where

(8.1.3.) $mU_0 \leq nU_1$

Since we are now considering first passage times over only a finite set of paths, the route of each first passage time exists for all ω.

If, however, the distribution U is not bounded as in (8.1.1.) we can only partly solve the problem.

THEOREM 8.1.4. The routes of $t_{On}(\omega)$, $s_{On}(\omega)$ exist with probability 1.

CONJECTURE 8.1.5. The routes of $a_{On}(\omega)$, $b_{On}(\omega)$ also exist with probability 1 even when U is an unbounded distribution.

Intuitively, Conjecture 8.1.5. is appealing. However it will probably need a 'new' idea to prove it as there seems to be no way of extending the proof of (8.1.4.) to cover (8.1.5.) also. That Conjecture 8.1.5. holds for bounded U is yet another example of the simplifying effect of bounding the distribution U.

Before proving Theorem 8.1.4. we introduce the concept of a barrier. Let $\{u_i\}_{i=1}^{n}$ be the time coordinates of the arcs of the straight line path r_o which links the origin to $(n,0)$. Define V by

(8.1.6.) $V = \max u_i \qquad (1 \leqslant i \leqslant n)$

Then trivially

(8.1.7.) $t(r_o, \omega) \leqslant nV$

Let m be any positive integer. We say there is a barrier at $Y = m$ if each of the $2n(n+1)$ arcs in the rectangle defined by $Y = m$, $Y = m+2n$, $X = 0$, $X = n$ has time coordinate not less than $V/2$. Similarly there is a barrier at $Y = -m$ if each of the $2n(n+1)$ arcs in the rectangle defined by $Y = -m$, $Y = -m-2n$, $X = 0$, $X = n$ has time coordinate not less than $V/2$. Let the event E_m denote the presence of a barrier at $Y = m$. Then since

(8.1.8.) P (a given arc has time coordinate $\geqslant V/2) = k > 0$

where k is a constant depending on the distribution U.

(8.1.9.) $P(E_m) = k^{2n(n+1)} = \theta > 0$ (for all m)

Notice that if E_m occurs ($m > 0$), any path r which crosses $Y = m$, $Y = m+2n$ must satisfy

(8.1.10.) $t(r,w) \geqslant 2nV/2 = nV \geqslant t(r_0,\omega)$

Proof of Theorem 8.1.4.

Since r_0 is a path linking the origin to $(n,0)$ the event $E_m \cap E_{-m}$ implies that the routes of $t_{on}(\omega)$, and $s_{on}(\omega)$ exist and lie inside the rectangle bounded by $Y = \pm (m+2n)$. Hence if $D_m \equiv E_m \cap E_{-m}$, for any positive integers M, M', $(M=m+2nM')$

(8.1.11.) P (route of $t_{on}(\omega)$ exists) $\geqslant P(\bigcup_{m=1}^{M} D_m)$

$$\geqslant P(\bigcup_{r=0}^{M'} D_{m+2rn})$$

Now from (8.1.9.) since E_m, E_{-m} are independent events

(8.1.12.) $P(D_m) = \theta^2 > 0$ (for any integer m)

Denote by $\Omega - D_m$ the event complementary to D_m. Then

(8.1.13.) $P(\bigcup_{r=0}^{M'} D_{m+2rn}) = P(\bigcup_{r=0}^{M'}(\Omega-\Omega+D_{m+2rn}))$

$$= 1 - P\bigcap_{r=0}^{M'} (\Omega - D_{m+2rn}))$$

Since the events on the right hand side of (8.1.13.) are mutually independent inasmuch as they depend on the time coordinates of disjoint sets of arcs, by (8.1.11. - 13.)

(8.1.14.) P (route of $t_{on}(\omega)$ exists) $\geqslant 1 - (1-\theta^2)^{M'}$

and as $M' \to \infty$ the right hand side of (8.1.14.) tends to 1 since by (8.1.12.) $\theta^2 > 0$. This completes the proof for $t_{on}(\omega)$. The adjustment necessary to include $s_{on}(\omega)$ is trivial.

8.2. The Height Problem

If r is the route of $t_{0n}(\omega)$ define $h_n(\omega)$, to be the maximum deviation of r from the x-axis. Precisely $h_n(\omega) = \max|y|$ such that $P \equiv (x,y)$ belongs to the route r. The determination of any property of $h_n(\omega)$ is called the <u>height problem</u>. It is almost completely unsolved.

$h_n(\omega)$ is a non-negative integer valued random variable on (Ω, F, P). We will illustrate its mathematical importance below. First we mention a relation between the t- and s-processes.

THEOREM 8.2.1. The expected values $T(0,n)$, $S(0,n)$ satisfy for any m,n, positive integers

$$T(0,m+n) \geq S(0,n) + S(0,m) \tag{8.2.1.}$$

Proof: Let r be the route of $t_{0,m+n}(\omega)$ and let r meet the line $X = n$ at a point P. Let r_1, r_2 be the portions of r which run from (0,0) to P and from (m+n,0) to P respectively. Then

$$\begin{aligned} t_{0,m+n}(\omega) &= t(r, \omega) \\ &= t(r_1, \omega) + t(r_2, \omega) \end{aligned} \tag{8.2.2.}$$

Now inspection instantly shows that $t(r_1, \omega) \geq s_{0n}(\omega)$

while $t(r_2,\omega) \geq s_{m+n,n}(\omega)$. By stationarity $s_{m+n,n}(\omega)$ has the distribution of $s_{0,m}(\omega)$ and hence taking expected values of (8.2.2.) we have

(8.2.3.) $\quad T(0,m+n) \geq S(0,n) + S(0,m)$

which completes the proof of the theorem.

COROLLARY 8.2.4. For m,n, any integers ≥ 0, the absolute first passage times $a_{on}(\omega)$, $b_{on}(\omega)$ satisfy for all n

(8.2.4.) $\quad A(0,m+n) \geq B(0,m) + B(0,n)$

Proof: This is exactly the same as in (8.2.1.) if we everywhere replace t, s, by a, b, respectively.

Even though the 't' and 's' processes have the same time constant $\mu(U)$, Theorem 8.2.1. does not give us any more information about the error term $T(0,n) - n\mu(U)$. If however it is known with probability 1 that $h_n(\omega) \leq f(n)$ where f(n) is a function only of n, then it is easily proved that

(8.2.5.) $\quad T(0,n) + T(0,m) \leq T(0,m+n) + \bar{u}f(n+m)$.

Hence if we could show that f(n) was $O(n^{3/4})$ for example (which intuitively is a fair conjecture), then we could apply the theory of generalised superadditive functions (Hammersley 1962) to obtain

(8.2.6.) $\quad T(0,n) \leq \mu n + \bar{u}f(n) = \mu n + O(n^{3/4})$

Such a result as (8.2.6.) would be a major step forward. It would enable us to prove a strong law of convergence for $t_{0n}(\omega)/n$ without making any stipulations about the higher moments of the U distribution and would improve most of our past results.

Although we state

CONJECTURE 8.2.7. $h_n(\omega) \leq f(n) = O(n^{3/4})$ with probability 1 as $n \to \infty$

the complete lack of success we have had in attacking this height problem prompts the statement of a much weaker

CONJECTURE 8.2.8. $H(n) = Eh_n(\omega)$ satisfies for all U distributions

$$\lim_{n \to \infty} H(n)/n = 0 \tag{8.2.8.}$$

An attack on the height problem through the method of considering first passage times in pipes of restricted width as in (7.2.) has so far proved fruitless. One relation at our disposal concerns $n_m(\omega)$, the number of arcs in the route of $t_{0m}(\omega)$. This is

$$m + 2h_m(\omega) \leq n_m(\omega) \qquad (\omega \varepsilon \Omega) \tag{8.2.9.}$$

The truth of (8.2.9.) is not hard to see. Since, however, it seems highly doubtful that $En_m(\omega) \sim m$, while (8.2.8.) is intuitively appealing, the inequality (8.2.9.) we suggest is very weak.

Another 'height problem' is to determine $h_n^*(\omega)$, which is defined to be $|y|$ where $P = (n,y)$ is the point where the route of $s_{0n}(\omega)$ meets the ordinate $x = n$. We will not go into the details of the importance of $h_n^*(\omega)$. It suffices to say that its relationship to $h_n(\omega)$ is so close that any result proved for $h^*_n(\omega)$ would produce a correspondingly similar result for $h_n(\omega)$. $h^*_n(\omega)$ is a measure of the difference between $t_{0n}(\omega)$ and $s_{0n}(\omega)$. By the inclusion lemma, if r_0 is the path from $(n,h^*_n(\omega))$ to $(n,0)$ and $\left\{u_i\right\}_{i=1}^{h^*_n(\omega)}$ are the time coordinates of the arcs of r_0.

(8.2.10.) $$s_{0n}(\omega) \leq t_{0n}(\omega) \leq s_{0n}(\omega) + \sum_{i=1}^{h^*_n(\omega)} u_i$$

and taking expected values of (8.2.10.) by (Feller (1957) p.268) we have

(8.2.11.) $$S(0,n) \leq T(0,n) \leq S(0,n) + \bar{u}H^*(n)$$

where $H^*(n) = Eh^*_n(\omega)$

Another result which would follow from a partial solution of the height problem is Conjecture 4.4.4. For let the route of $b_{0n}(\omega)$ meet $X = n$ at $P = (n,y)$ and define $h^b_n(\omega) \equiv |y|$. Then by dropping a perpendicular from P to $(n,0)$ we have by a simple combination of the connection and inclusion lemmas, that

(8.2.12.) $$B(0,n) + \bar{u}Eh^b_n(\omega) \geq A(0,n)$$

Since $\lim_{n\to\infty} A(0,n)/n = \mu$, if we could show that $\lim_{n\to\infty} Eh^b_n(\omega)/n = 0$ we would prove that $\lim_{n\to\infty} B(0,n)/n$ also equals μ.

One result which has been tacitly assumed so far is that the expected value of the height function exists. We close this section by proving this result which is essential for the proof of Theorem 4.2.9.

Notice that if the U distribution is bounded as in (8.1.1.) the variables $h_n(\omega)$, $h_n^*(\omega)$ satisfy for all ω, n,

(8.2.13.) $U_o(n + h_n^*(\omega)) \leq nU_1$

(8.2.14.) $U_o(n + 2h_n(\omega)) \leq nU_1$

and hence $h_n(\omega)$, $h_n^*(\omega)$ are bounded random variables whose expectation necessarily exists.

The truth of (8.2.13.) follows by letting r be the route of $s_{on}(\omega)$. Then obviously the number of arcs $n_n^*(\omega)$ in r satisfies

$$n_n^*(\omega) \leqq n + h_n^*(\omega).$$

Hence in the extreme case $t(r,\omega) = s_{on}(\omega) \geq U_o(n+h_n^*(\omega))$ and (8.2.13.) now follows. Similarly for (8.2.14.).

<u>THEOREM 8.2.15</u>. <u>For all distributions U which have a finite mean the expected values of $h_n(\omega)$ and $h_n^*(\omega)$ exist.</u>

<u>Proof</u>: We prove the theorem for $h_n(\omega)$ with n fixed. The modification necessary to prove the result for $h_n^*(\omega)$ is trivial.

(8.2.16.) $$Eh_n(\omega) = \sum_r P\left[h_n(\omega) > r\right]$$

$$= \sum_{r=o}^{2n} P\left[h_n(\omega) > r\right] + \sum_{r=2n+1}^{\infty} P\left[h_n(\omega) > r\right]$$

It is obvious that

(8.2.17.) $$P\left[h_n(\omega) > r+1\right] \leq P\left[h_n(\omega) > r\right] \qquad (r \geq o)$$

Hence applying this to the expression on the right hand side of (8.2.16.) we obtain

$$(8.2.18.)\qquad \sum_{r=2n+1}^{\infty} P\left[h_n(\omega) > r\right] \leqslant 2n\sum_{j=1}^{\infty} P\left[h_n(\omega) > 2jn+1\right]$$

Let A_j be the event that there is no barrier at each of $Y = 1$, $Y = 2n+1$, $Y = 4n+1$, ..., ..., $Y = 2(j-1)n+1$

Let A_{-j} be the event that there is no barrier at each of $Y = -1$, $Y = -(2n+1)$, $Y = -(4n+1)$, ..., $Y = -2(j-1)n-1$. That is for j any positive integer define A_j, A_{-j} by

$$(8.2.19.)\qquad A_j = \bigcap_{r=0}^{j-1} (\Omega - E_{2nr+1})$$

$$(8.2.20.)\qquad A_{-j} = \bigcap_{r=0}^{j-1} (\Omega - E_{-2nr-1})$$

where E_i is the event defined in the proof of Theorem 8.1.4. Now if $h_n(\omega) > 2jn+1$ either A_j or A_{-j} must occur hence

$$(8.2.21.)\qquad \left[\omega : h_n(\omega) > 2jn+1\right] \subset A_j \cup A_{-j}$$

Now by (8.1.9.) we have

$$(8.2.22.)\qquad P(A_j) = P(A_{-j}) = (1-\theta)^j \qquad (\theta > 0)$$

Since A_j and A_{-j} depend on the time coordinates of disjoint sets of arcs they are independent and

$$(8.2.23.)\qquad P(h_n(\omega) > 2jn+1) \leqslant P(A_j \cup A_{-j}) \leqslant 2(1-\theta)^j$$

Hence applying this to (8.2.18.)

$$(8.2.24.)\qquad \sum_{r=2n+1}^{\infty} P(h_n(\omega) > r) \leqslant 2n\sum_{j=1}^{\infty} 2(1-\theta)^j$$

and since $\delta > 0$, the right hand side of (8.2.24.) converges so that by (8.2.16.) the expected value of $h_n(\omega)$ exists and thus the proof of the theorem is complete.

<u>COROLLARY 8.2.26.</u> <u>Under the same conditions as in 8.2.15.</u> <u>the expected values of</u> $h_n^2(\omega)$ <u>and</u> $(h_n^*(\omega))^2$ <u>exist</u>

<u>Proof</u>. We prove the result for $h_n^2(\omega)$.

Since $h_n(\omega)$ and $h_n^2(\omega)$ are integer valued

$$(1)\quad Eh_n^2(\omega) = \sum_r P\left[h_n^2(\omega) > r\right] = \sum_s (2s+1)P\left[h_n(\omega) > s\right]$$

$$= \sum_{s=0}^{2n} (2s+1)P\left[h_n(\omega) > s\right] + \sum_{s=2n+1}^{\infty} (2s+1)P\left[h_n(\omega) > s\right]$$

Grouping this last expression as in (8.2.18.)

$$(2)\quad \sum_{s=2n+1}^{\infty} (2s+1)P\left[h_n(\omega) > s\right] \leqslant \sum_{j=1}^{\infty} f(j,n)P\left[h_n(\omega) > 2jn+1\right]$$

where $f(j,n) = \displaystyle\sum_{s=2jn+1}^{2(j+1)n} (2s+1)$

$$\leqslant 2n(4n(j+1)+1)$$

By (8.2.23.), the right hand side of (2) is convergent, and hence by (1), $E\, h_n^2(\omega) < \infty$ which proves (8.2.25.)

CHAPTER 9

Some Conjectures from a Simulation of First Passage Percolation

Here we discuss results obtained from an experiment on a Ferranti Mercury Computer in which we simulated first passage percolation on the square lattice. Because of the magnitude of the problem we could only work on a comparatively small section of the lattice and only for the case when the underlying U distribution is the standardised rectangular distribution.

9.1. Details of Method

Words of the computing store represent each node and arc of the lattice strip bounded by $x = 0$, $x = 39$, $y = \pm 16$. Let $\left\{ l_i \right\}_{i=1}^{3}$ be the arcs of the lattice section which link the origin to $P_1 \equiv (0,1)$, $P_2 \equiv (1,0)$, $P_3 \equiv (0,-1)$ respectively. To each l_i we assign a pseudo-random number u_i in the range (0.1.). This represents the time coordinate of l_i. The node P_i is now attainable in time u_i, in the sense that the first passage time to it from the origin is certainly $\leqslant u_i$. We now insert the cartesian coordinates of P_i together with its attainable time u_i in an attainable list.

Let u_k be the minimum attainable time in the attainable list. Then the first passage time from the origin to P_k

under this configuration of time coordinates must be u_k and we say that P_k is an attained node. Remove the entry (P_k, u_k) from the attainable list.

Let $\left\{l_j\right\}_{j=1}^{n}$ be the arcs of the lattice section which have P_k as an endpoint and which have not yet been assigned time coordinates. Assign time coordinates $\left\{v_j\right\}_{j=1}^{n}$ to these arcs. Then if l_j links P_k with Q_j, Q_j is attainable in time $u_k + v_j$. Insert (Q_j, u_k+v_j) in the attainable list. Search this list for the node R which has minimum attainable time t_o, say, and remove any entries of the form (R,t). Then t_o must be the first passage time from the origin to R and the first passage time from the origin to any other node of the lattice section cannot be less than t_o. R is our new attained node. We assign time coordinates to those arcs from R which lead to nodes which are not yet attained nodes, insert these new attainable nodes in the attainable list and repeat the process with our next new attained node until each node of the lattice is attained. This gives us first passage times from the origin to each point of the lattice for a configuration of time coordinates induced by the standardised rectangular distribution.

Notice that for a node to be an attainable node it must number among its neighbours at least one attained node. Thus the attainable list contains up to 4 entries corresponding to a single node A (4 is unusual but feasible).

If A is the new attained node at any stage we must remove all entries of the form (A,t) from the attainable list. This involves yet another search of the attainable list. Since the size of this list quickly becomes so large that searching takes up the bulk of the time we introduce a time saving stratified search routine.

This divides the attainable list into 40 subsets B_i. These subsets B_i are called <u>compartments</u>. An entry of the attainable list is a member of B_i if it corresponds to a node of the lattice which lies on $X = i$. Associated with each compartment B_i is a computer word C_i, called the <u>index</u> of B_i. C_i contains the y-coordinate and attainable time of that node belonging to the set B_i which has minimum attainable time. If there is no attainable node on $X = i$ (that is, B_i is empty), then C_i indicates this.

This means that when searching we have to examine only the 40 words C_i, select that C_{i_o} which contains the minimum attainable time and then search the compartment B_{i_o} removing any entries which have the same y-coordinate as the newly selected attained node and at the same time search for a new entry to insert in C_{i_o}. Since any compartment usually contains at most 5 entries, this means that a double search of the whole attainable list is reduced to a single search through approximately 45 words.

For example, suppose that at a particular time there are exactly 3 entries in each of the compartments B_i. This means that the attainable list contains 120 entries. A double search would involve 240 comparisons. By the above method we need only look at 43 words.

This method has the further advantage that it is now very easy to find the first passage time from the origin to each of the lines $X = i$.

When all nodes have been attained, we store the required first passage times, and repeat the process 100 times using different random number generators.

Even with this stratified search routine the time and storage requirements of this problem are excessive. To conduct the experiment on a larger scale in a reasonable time it is necessary to discover some ingenious technique similar to that discovered by Hammersley (1963) for ordinary percolation probabilities.

9.2. The Output

From this simulation we obtain realisations of the processes $\left\{t_n(\omega)\right\}_{n=1}^{39}$, $\left\{s_n(\omega)\right\}_{n=1}^{39}$, where $t_n(\omega)$ and $s_n(\omega)$ are the first passage times from the origin to $(n,0)$ and $X = n$ respectively. $t_n(\omega)$ differs very slightly from $t_{on}(\omega)$ inasmuch as we do not demand that the first arc of the route of $t_n(\omega)$ is horizontal. It can easily be seen that

$$(9.3.1.) \qquad T(0,n) \geqslant T(n) = E\, t_n(\omega) \geqslant A(0,n).$$

and that

(9.3.2.) $S(0,n) \leqslant T(n) \leqslant T(0,n) \leqslant \bar{u} + T(n-1)$

so that

(9.3.3.) $\inf T(n)/n = \lim_{n \to \infty} T(n)/n = \mu(U)$

Similar results hold for $S(n) = E\, s_n(\omega)$. A discussion of these processes and proofs of results analagous to those of chapter 4 for $t_n(\omega)$ and $s_n(\omega)$ is contained in Appendix 3. Henceforth we quote any theorem of chapter 4 in terms of t_n and s_n. (For justification see Appendix 3).

9.3. A Possible Source of Error

We state in 9.2. that we obtain realisations of the processes $t_n(\omega)$, $s_n(\omega)$. Clearly this is impossible on finite sections of the lattice. More precisely the actual processes are of the processes giving first passage times along a pipe (see chapter 7). To check that this width limitation is negligible we printed out values of $h_n^*(\omega)$, the divergence of $s_n(\omega)$ from the x-axis.

9.4. The Time Constant $\mu(U)$.

When U is the standardised rectangular distribution, (6.1.5.) gives

(9.4.1.) $\mu(U) \leqslant \cdot 425$

In (4.2.11.) we proved that for n any integer and U any distribution $\mu(U) \leqslant S(n)/n$. The simulation gave $\cdot 323$ as an estimate of $S(39)/39$, with standard error $\cdot 003$ so

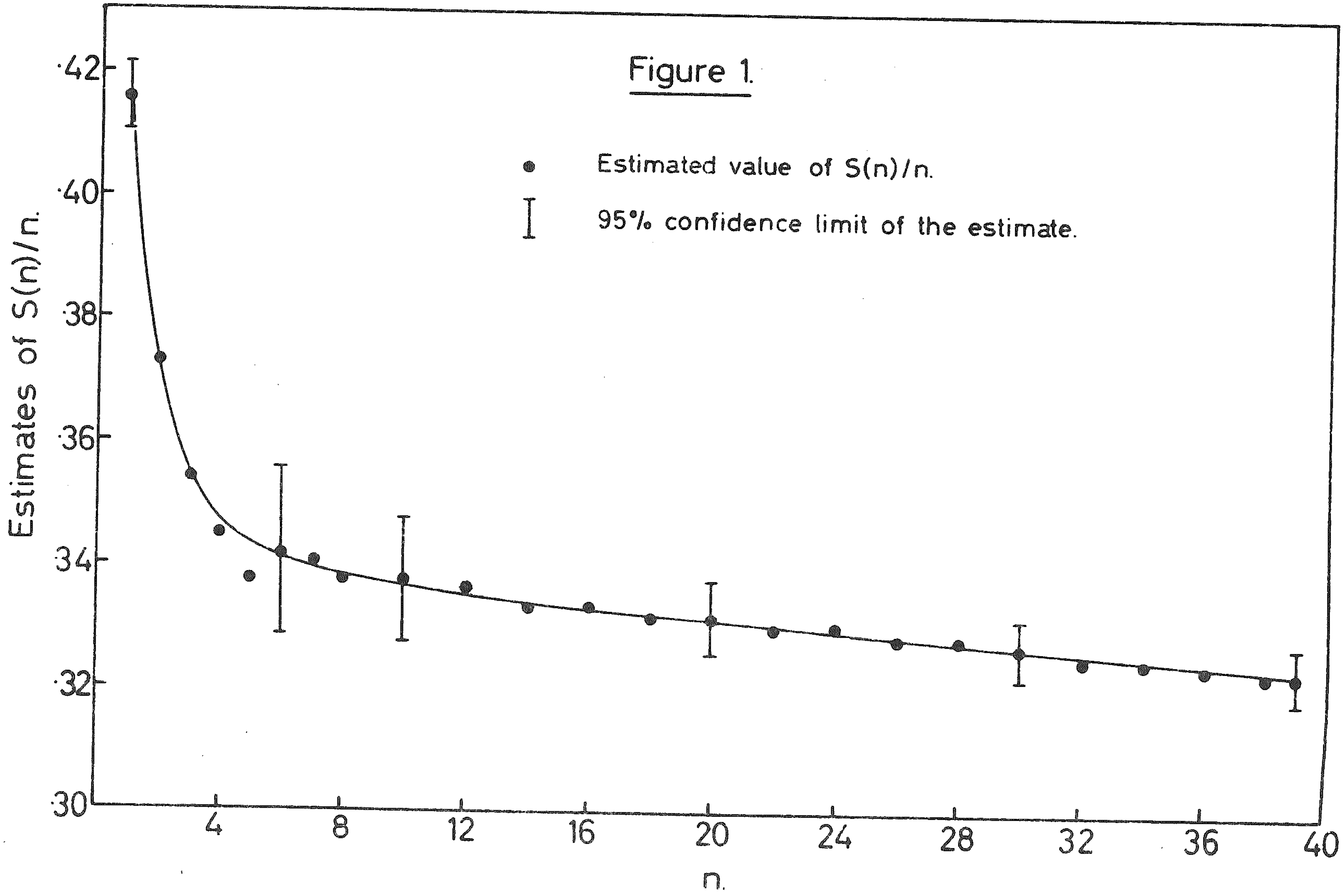


Figure 1.

● Estimated value of S(n)/n.
I 95% confidence limit of the estimate.
Estimates of S(n)/n.
·42
·40
·38
·36
·34
·32
·30
4
8
12
16
20
24
28
32
36
40
n.

that with 95% confidence

(9.4.2.) $\mu(U) \leqslant \cdot 323 + \cdot 005$

This is a big improvement on the best theoretical upper bound (9.4.1.). It shows that the algorithm described in (6.1.) for the travel of a particle from a point to a line is much further from optimal, (if an optimum strategy exists), than we originally suspected.

9.5. The Mean Value Function S(n)

Figure 1 shows the estimates of $S(n)/n$ plotted against n. Notice that the range of values of n is too small for $S(n)/n$ to settle to its asymptotic value μ. Theory says nothing about the functional form of $S(n)/n - \mu$ and it was difficult to fit the data of Figure 1. Plotting $y = S(n)/n$ against $1/n$ and taking successive differences suggested that a polynomial in $1/n$ was a more likely form for $S(n)/n$ than any other well known form such as the negative exponential. However we used the regression method of Aitken (1934) for correlated quantities to obtain an unbiased and very nearly minimum-variance estimate of the vector (A, B, C), on the assumption that

(9.5.1.) $S(n)/n = A + Bn^{-1} + Cn^{-2}$

The result of this calculation gives $A = \cdot 327$, $C = \cdot 1195$ while B did not differ significantly from zero. Admittedly

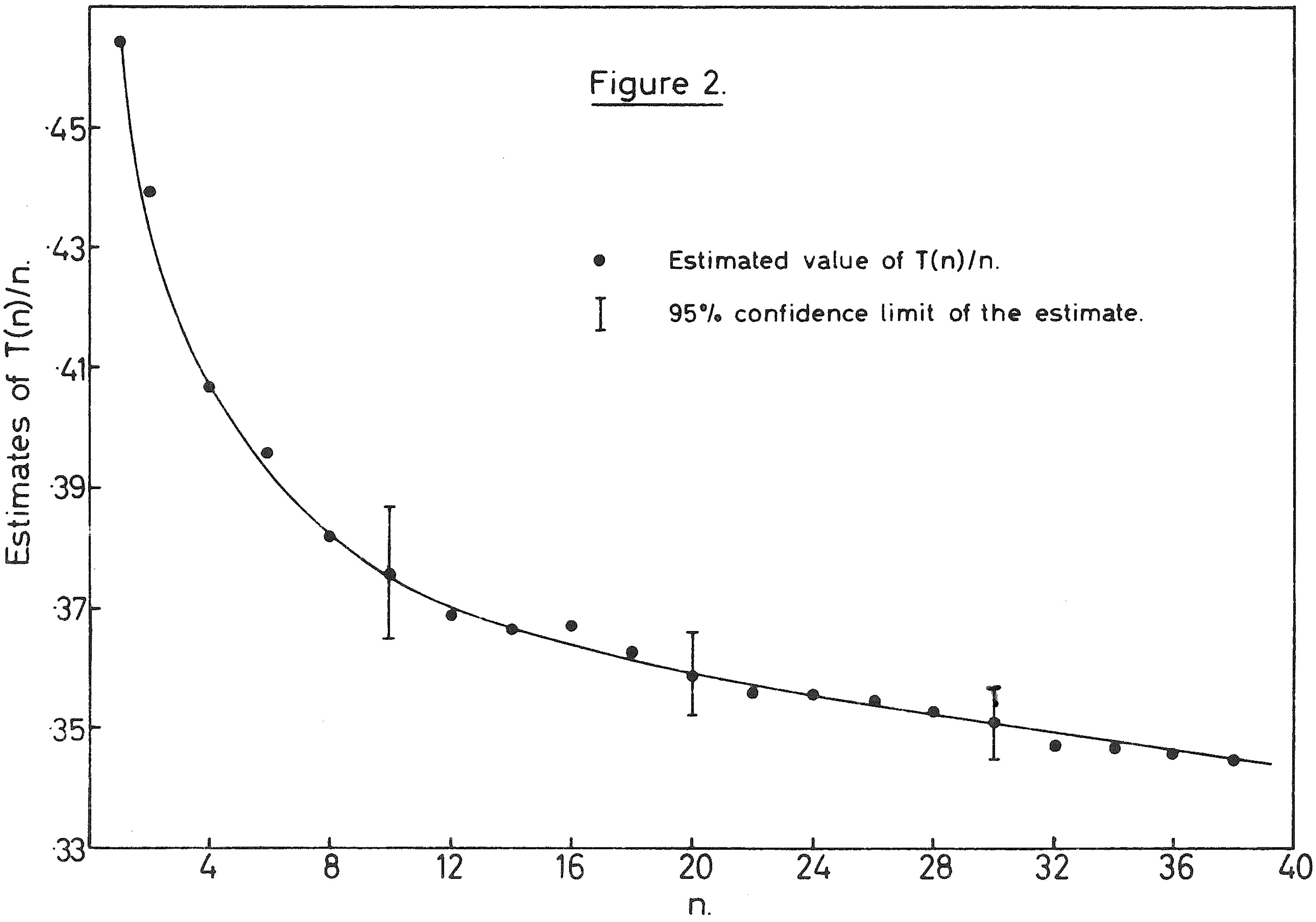
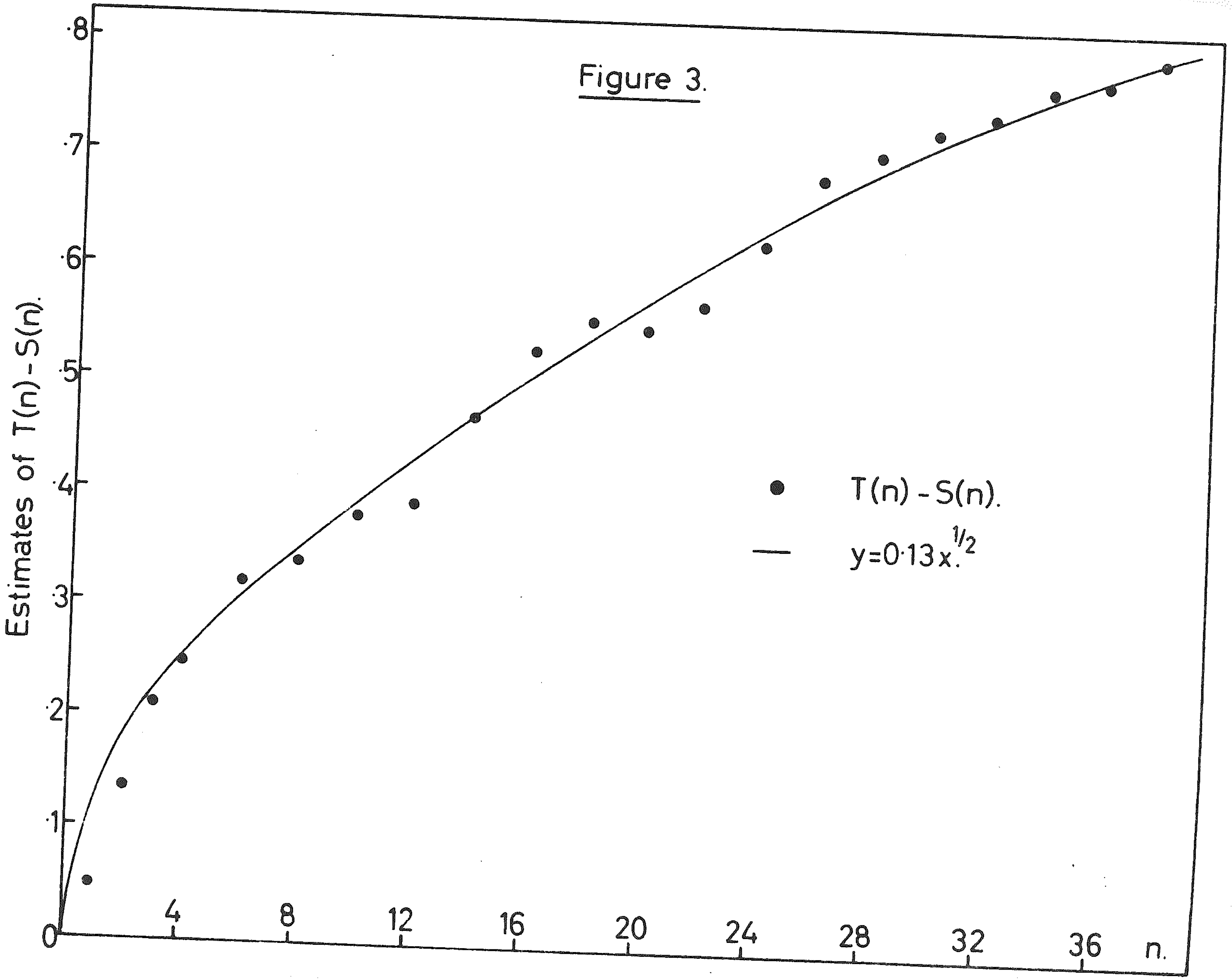


Figure 2.

● Estimated value of T(n)/n.
I 95% confidence limit of the estimate.
Estimates of T(n)/n.
·45
·43
·41
·39
·37
·35
·33
4
8
12
16
20
24
28
32
36
40
n.

Figure 3.


Estimates of T(n) - S(n).
·8
·7
·6
·5
·4
·3
·2
·1
0
4
8
12
16
20
24
28
32
36
n.
T(n) - S(n).
y=0·13x.^1/2

this gives the best representation in the form (9.5.1.) of the data at hand, but it does not give a very realistic representation of the process, inasmuch as it does not consider the constraint that the subadditive nature of $S(n)$ demands that $S(n)/n$ is never less than its limiting value.

Figure 1 of Appendix 4 shows the functions $S(n) - kn$, ($k = \cdot 32$ and $\cdot 315$). These decrease significantly for large n ($\geqslant 28$), which suggests that $S(n) - kn$ is decreasing linearly in n, and hence that

(9.5.2.) $$\mu \leqslant \cdot 315$$

An alternative form for $S(n)/n$ which is suggested by the above is

(9.5.3.) $$S(n)/n = f(n) = A + Bn^{-\frac{1}{2}} + Cn^{-1}$$

A plot of the data against $f(n)$ for suitably chosen A, B, C, is shown in Figure 2 of Appendix 4.

The attraction of a form such as (9.5.3.) is that it demands values of the constant $A (\equiv \mu)$ which satisfy (9.5.2.). However it does not seem profitable to carry out a large scale analysis until either closer estimates of $S(n)$ are obtained or better still, we know from theory the form of the error function $S(n)/n - \mu$

9.6. The Mean Value Function $T(n)$

From Theorem 4.2.9. we know that $T(n)/n$ and $S(n)/n$

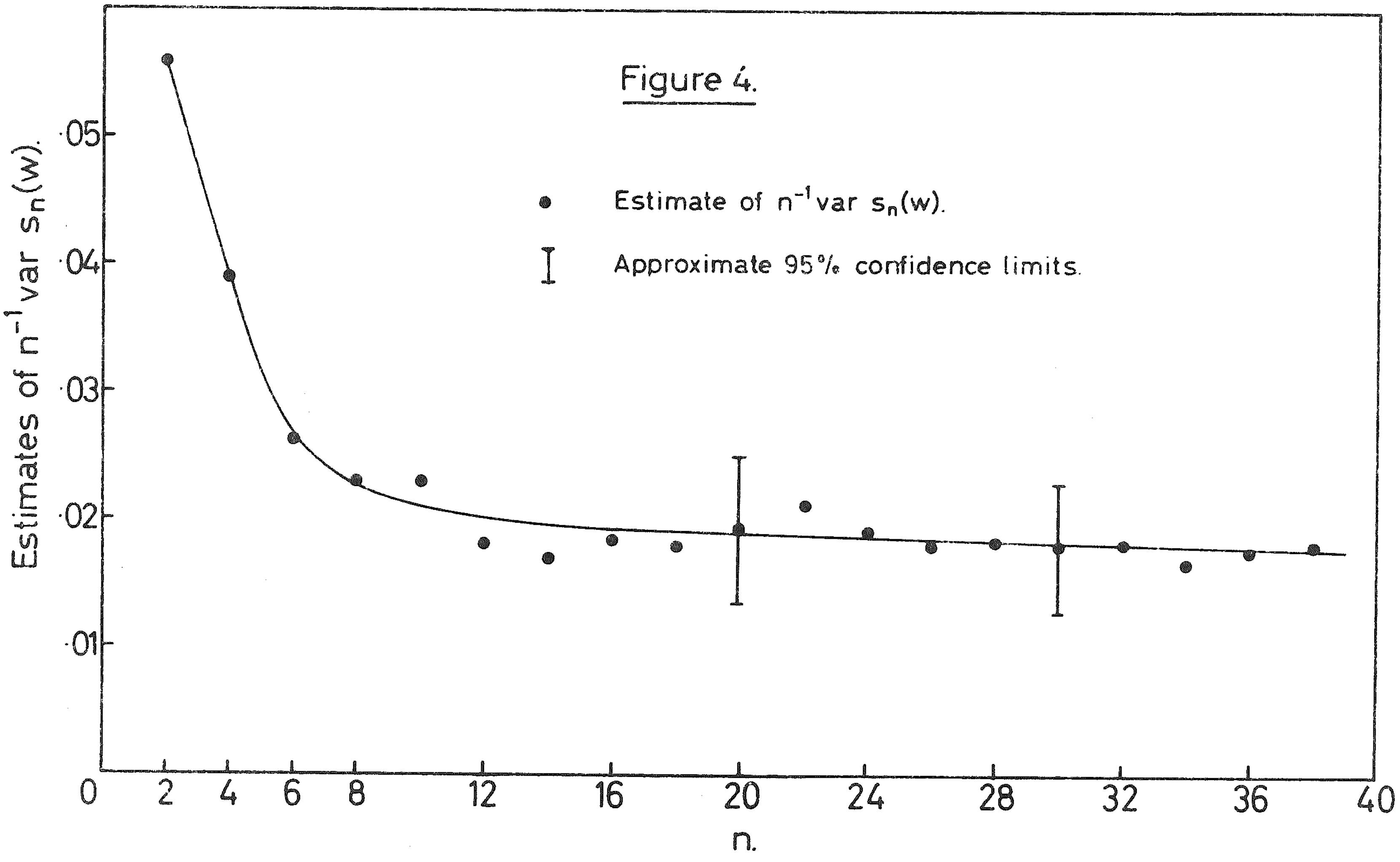
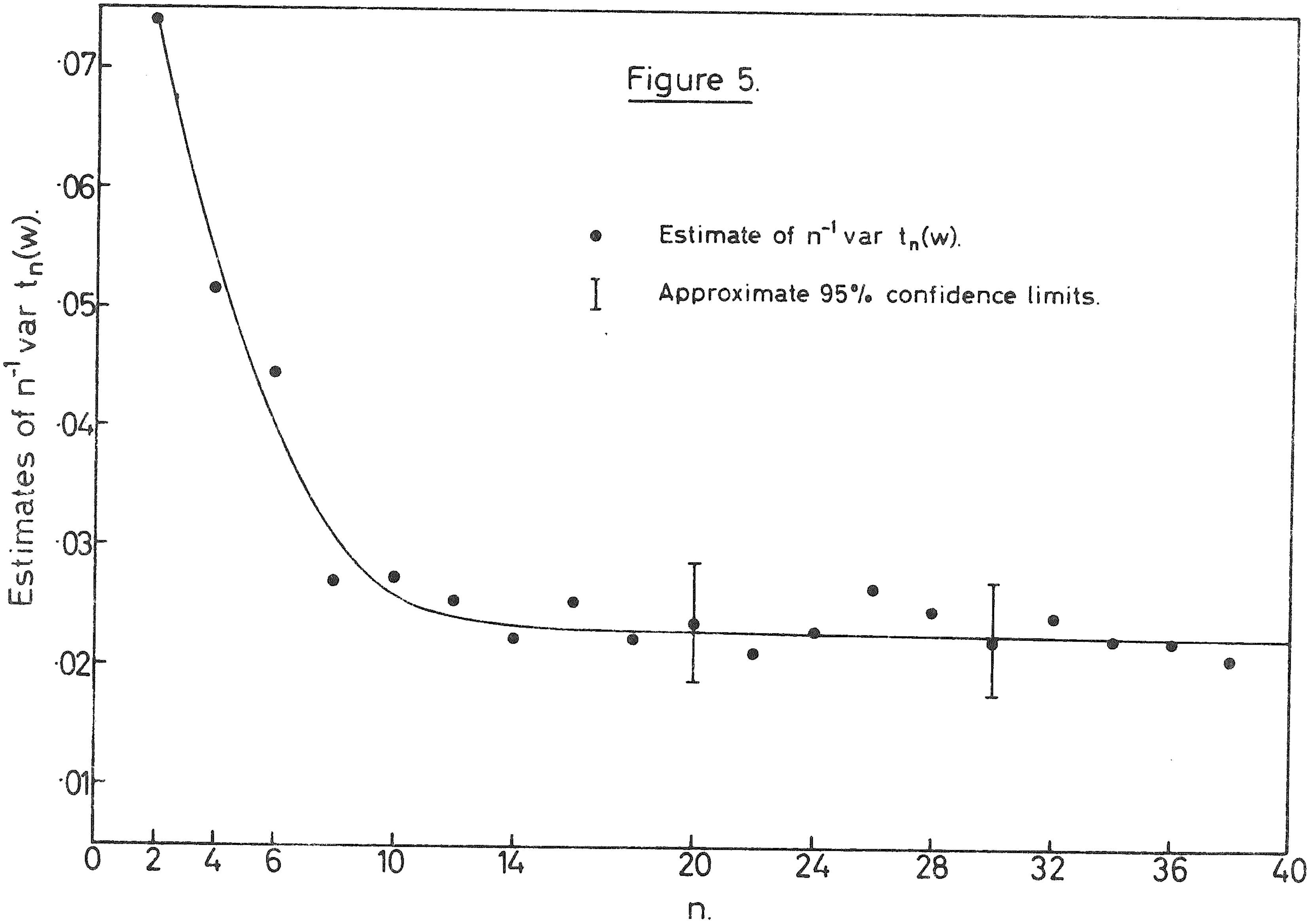


Figure 4.

Estimate of n^{-1} var $s_n(w)$.
Approximate 95% confidence limits.
Estimates of n^{-1} var $s_n(w)$.
·05
·04
·03
·02
·01
0
2
4
6
8
12
16
20
24
28
32
36
40
n.

Figure 5.


Estimate of n^{-1} var $t_n(w)$.
Approximate 95% confidence limits.
Estimates of n^{-1} var $t_n(w)$.
·07
·06
·05
·04
·03
·02
·01
0
2
4
6
10
14
20
24
28
32
36
40
n.

have a common asymptote. Figure 2 shows estimates of $T(n)/n$ plotted against n. The difference between our estimate of $T(39)/39$ and (9.4.2.) is not alarming if one remembers from (8.2.3.) that

$$(9.6.1.) \qquad T(2n)/2n \geqslant S(n)/n \geqslant \mu(U)$$

Define $G(n)$ by

$$(9.6.2.) \qquad G(n) = T(n) - S(n) \qquad (1 \leqslant n \leqslant 39)$$

Figure 3 of Appendix 4 shows log 10 $G(n)$ against log n and suggests that $G(n)$ has the form $An^{\frac{1}{2}}$. Figure 3 shows the scatter of $G(n)$ about the curve $y = 0.13x^{\frac{1}{2}}$.

Since $S(n) \geqslant \mu n$ for all n this suggests that for all distributions U with finite mean, $T(n)$ satisfies

$$(9.6.3.) \qquad T(n)/n \geqslant \mu + An^{-\frac{1}{2}} \qquad (A \text{ constant})$$

Finally notice that the results obtained strongly support Conjecture 4.1.14 that $T(n)$ is an increasing function of n.

9.7. The Variance of $t_n(w)$ and $s_n(w)$.

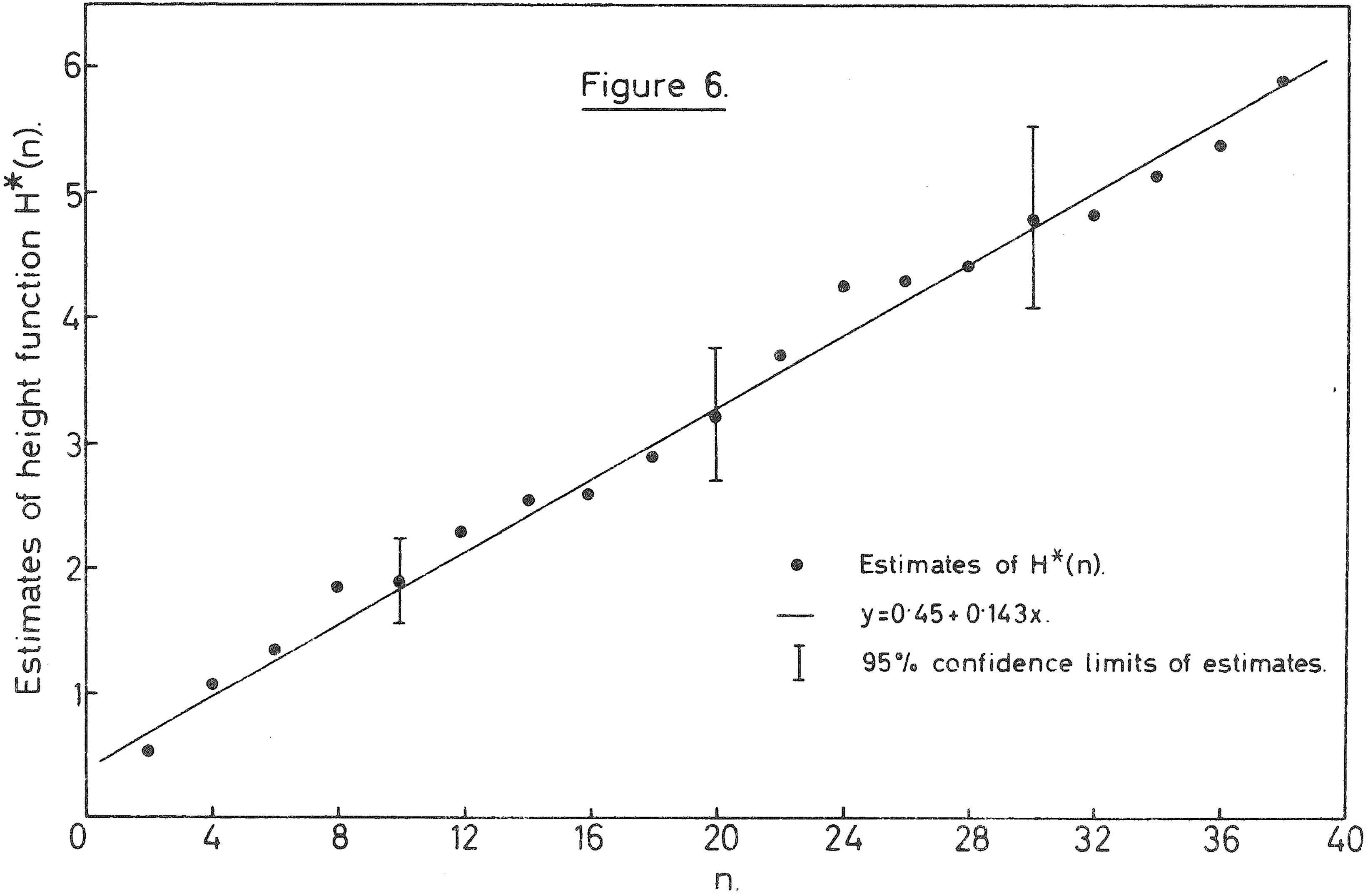
In chapter 4 we proved that

$$\lim_{n\to\infty} n^{-2} \operatorname{var} t_n(w) = \lim_{n\to\infty} n^{-2} \operatorname{var} s_n(w) = 0.$$

but suspected that this was rather a weak result. This is confirmed by Figures 4, 5, which suggest

CONJECTURE 9.7.1. Provided the underlying U distribution has finite variance

Figure 6.


Estimates of height function $H^*(n)$.
0
1
2
3
4
5
6
0
4
8
12
16
20
24
28
32
36
40
n.
Estimates of $H^*(n)$.
$y = 0{\cdot}45 + 0{\cdot}143x$.
95% confidence limits of estimates.

$$\lim_{n\to\infty} n^{-1} \text{var } t_n(w) = V_1 \geq V_2 = \lim_{n\to\infty} n^{-1} \text{var } s_n(w).$$

That strict inequality cannot hold generally above is seen by letting U be the constant distribution.

Figures 4, 5, explain the decreasing nature of the standard errors in Figures 1. 2. The confidence limits in Figures 4, 5, are obtained by assuming approximate normality of the parent distribution. The strength of this approximation may be judged from Figure 4 of Appendix 4.

9.8. The Height Function $H^*(n)$

Estimates of the height function $H^*(n)$ studied in chapter 8, were a byproduct of the check made on the width limitation. As was stated in chapter 8, incomplete theoretical grounds suggest that $H^*(n)$ is $o(n)$ as $n \to \infty$. Figure 6 suggests a linear form for $H^*(n)$. It would be interesting to have estimates of the function for larger values of n.

9.9. Conclusion

This thesis contain a number of conjectures supported by varying amounts of evidence. Proofs or counterproofs of these would be nice. However there are many other problems in this field which have not yet been attacked.

The extension of the above theory to other regular lattice structures should not be too difficult while very good general results may even be obtained for random graphs as defined by Gilbert (1959). Also it seems quite straightforward to extend the techniques and proofs above, to the case where the random time mechanism is attached to the nodes and not the arcs of the lattice.

The basic tool in handling these problems would be the theory of subadditive processes as developed in chapter 3. However, the definition of a subadditive process has obviously been fashioned to fit first passage percolation theory, and probably the most interesting problem is "how far can this definition be extended without too much loss?"

APPENDIX I

Proof of Some Theorems on the Convergence as $n \to \infty$ of the Stochastic Process $s_{on}(\omega)/n$

To Prove Theorem 4.2.11.

Define $f(\omega)$ by

(1) $f(\omega) = \mu - \liminf_{n \to \infty} s_{on}(\omega)/n$

By (4.2.10.)

(2) $P\left[f(\omega) \geqslant 0\right] = 1$

Since for all n, w, $s_{on}(\omega) \leqslant t_{on}(\omega)$

(3) $0 \leqslant E\, s_{on}^{a}(\omega)/n^{a} \leqslant E\, t_{on}^{a}(\omega)/n^{a}$ $(a > 1)$

As was shown in the proof of Theorem 4.1.11. the existence of the right hand side of (3) is implied by the existence of an ath moment of the underlying U distribution. Hence under the hypotheses of Theorem 4.2.11. $s_{on}(\omega)/n$ is a uniformly integrable sequence of random variables and hence applying Fatou's Lemma (Doob P.629)

(4) $E \liminf_{n \to \infty} s_{on}(\omega)/n = \mu(U)$

Hence

(5) $E\, f(\omega) = 0$

By Halmos (P.104), (2) and (5) imply that

(6) $P\left[f(\omega) = 0\right] = 1$

which with (4.2.10.) proves the required result

(7) $P\left[\lim_{n \to \infty} s_{on}(\omega)/n = \mu\right] = 1$

To Prove Theorem 4.2.12.

This follows trivially from the proof of a corresponding result for $t_{on}(\omega)$

$$\begin{aligned}
\text{var } s_{on}(\omega) &= E\, s_{on}^2(\omega) - \left[S(0,n)\right]^2 \\
&\leqslant E\, t_{on}^2(\omega) - \left[S(0,n)\right]^2 \\
&= \text{var } t_{on}(\omega) + \left[T(0,n)\right]^2 - \left[S(0,n)\right]^2
\end{aligned}$$

Hence since $\lim\limits_{n \to \infty} T(0,n)/n = \lim\limits_{n \to \infty} S(0,n)/n = \mu(U)$

$$\lim_{n \to \infty} \text{var } s_{on}(\omega)/n^2 = \lim_{n \to \infty} \text{var } t_{on}(\omega)/n^2 + \mu^2 - \mu^2$$

The result now follows trivially by Theorem 4.1.13.

Since also

$$\begin{aligned}
E\left[\frac{s_{on}(\omega)}{n} - \mu\right]^2 &= \frac{E\, s_{on}^2(\omega)}{n^2} + \mu^2 - 2\mu\frac{S(0,n)}{n} \\
&= \frac{1}{n^2}\text{var } s_{on}(\omega) + \mu^2 - 2\mu\frac{S(0,n)}{n} + \left[\frac{S(0,n)}{n}\right]^2
\end{aligned}$$

it follows that $s_{on}(\omega)/n$ converges in mean square to μ as $n \to \infty$.

APPENDIX 2

To Prove that the Reach Function $x_t(\omega)$ is a Measurable Stochastic Process

By Doob (1953) it is sufficient to show that $x_t(\omega)$ is a measurable function on the product space Ω x T where Ω is the phase space, and T is the interval $(0, \infty)$. Since $x_t(\omega)$ is an integer valued random variable it is sufficient to show that

$$A_c \equiv \left[(\omega, s) \ : \ x_s(\omega) \leqslant c \right]$$

is a measurable subset of Ω x T for any positive integer c.

Now by definition of a reach function

$$(1) \quad A_c = \bigcup_{n=1}^{c} \left[(\omega, s) : t_{on}(\omega) \leqslant s \right] \bigcap \quad \bigcap_{n=c+1}^{\infty} \left[(\omega, s) : t_{on}(\omega) > s \right]$$

Consider the set

$$(2) \quad \left[(\omega, s) : t_{on}(\omega) \leqslant s \right] = \left[(\omega, s) : \inf_{r \varepsilon R} t(r, \omega) \leqslant s \right].$$

where R is the set of cylinder lattice paths which join the origin to (n,0). R is a countable set (since only self avoiding paths may be considered). Hence

$$(3) \quad \left[(\omega, s) : \inf_{r \varepsilon R} t(r, \omega) \leqslant s \right] = \bigcup_{r \varepsilon R} \left[(\omega, s) : t(r, \omega) \leqslant s \right]$$

Let r be any fixed lattice path, define

$$g_1(\omega, s) = t(r, \omega)$$
$$g_2(\omega, s) = -s.$$

Then $g_1(\omega, s)$ is a measurable function on Ω and hence on $\Omega \times T$, since it is independent of s. Trivially $g_2(\omega, s)$ is a measurable function on $\Omega \times T$. Hence, $g_1(\omega, s) + g_2(\omega, s)$ is a measurable function on $\Omega \times T$. In particular

$$\left[(\omega, s) : g_1(\omega, s) + g_2(\omega, s) \leq 0\right]$$

is a measurable sub set of $\Omega \times T$

Hence $\left[(\omega, s) : t(r, \omega) \leq s\right]$ is a measurable subset of $\Omega \times T$ for any r. By (1), (2) and (3) this implies that A_c is a measurable subset of $\Omega \times T$.

This is true for any integer c and thus the required result is proved.

A similar result may be proved for $y_t(\omega)$ by exactly the same method.

By Halmos (P.142) this theorem implies the truth of Theorem 5.1.3.

Appendix 3

In this appendix we prove that any of the conjectures made about the processes $t_n(\omega)$ and $s_n(\omega)$ in the course of chapter 9 hold equally well for the processes $t_{on}(\omega)$ and $s_{on}(\omega)$ studied in chapter 4.

Similarly we prove that any result from chapter 4 concerning $t_{on}(\omega)$, $s_{on}(\omega)$ which is used in chapter 9 holds equally well for $t_n(\omega)$ and $s_n(\omega)$. Except where the extension is not straightforward we prove results in terms of the t-processes.

By the inclusion lemma for all

$$s_{on}(\omega) \leq t_n(\omega) \leq t_{on}(\omega) \qquad (1)$$

Hence taking expected values, dividing by n, and taking the limit as $n \to \infty$, we have

$$\lim_{n \to \infty} \frac{T(n)}{n} = \mu(U) \qquad (2)$$

Since also by a simple application of the Connection lemma (2.3.2.).

$$u + s_n(\omega) \geq s_{-1,n}(\omega) \qquad (3)$$

where u is the time coordinate of the arc linking the origin to (-1,0), we have taking expected values, that

$$\bar{u} + S(n) \geq S(0,n+1) \qquad (4)$$

Dividing by n and taking the limit as $n \to \infty$ we get

$$\inf_n \frac{S(n)}{n} = \lim_{n \to \infty} \frac{S(n)}{n} = \mu(U) \qquad (5)$$

In (9.6.1.) we state that $T(2n)/2n \geqslant S(n)/n$. The proof that $T(0,2n)/2n \geqslant S(0,n)$ is an immediate corollary of Theorem 8.2.3. which states that

$$T(0,m+n) \geqslant S(0,m) + S(0,n) \tag{6}$$

The proof that

$$T(m+n) \geqslant S(m) + S(n) \tag{7}$$

is exactly the same, (replacing t_{on} by t_n etc.) as the proof of Theorem 8.2.3. Hence as an immediate corollary of (7) we get (9.6.1.).

In (9.7.) we state that

$$\lim_{n \to \infty} n^{-2} \operatorname{var} t_n(\omega) = 0 \tag{8}$$

In chapter 4 we proved

$$\lim_{n \to \infty} n^{-2} \operatorname{var} t_{on}(\omega) = 0. \tag{9}$$

The proof of (8) from (9) is as follows.

$$\begin{aligned} \operatorname{var} t_n(\omega) &= E\, t_n^2(\omega) - \left[T(n)\right]^2 \\ &\leqslant E\, t_{on}^2(\omega) - \left[T(n)\right]^2 \\ &= \operatorname{var} t_{on}(\omega) + \left[T(0,n)\right]^2 - \left[T(n)\right]^2 \end{aligned} \tag{10}$$

Dividing (10) by n^2, taking the limit as $n \to \infty$ and using (9) and (2) we have

$$\lim_{n \to \infty} n^{-2} \operatorname{var} t_n(\omega) \leqslant \lim_{n \to \infty} n^{-2} \operatorname{var} t_{on}(\omega) = 0 \tag{11}$$

which proves the required result (8).

Similarly the results of our experiment suggest as a conjecture that

$$\lim_{n\to\infty} n^{-1} \text{ var } t_n(\omega) = V_1 < \infty . \qquad (12)$$

We show that the truth of (12) implies

$$\lim_{n\to\infty} n^{-1} \text{ var } t_{on}(\omega) = V_1^{\,1} < \infty \qquad (13)$$

Proof.

Since $t_{on}(\omega) \leqslant t_{n-1}(\omega) + u$ (14)

where 'u' is the time coordinate of the arc from (n-1,0) to (n,0).

$$\begin{aligned} \text{var } t_{on}(\omega) &= E\, t_{on}^2(\omega) - \left[T(0,n)\right]^2 \\ &\leqslant E\left[t_{n-1}(\omega) + u\right]^2 - \left[T(0,n)\right]^2 \\ &= E\, t_{n-1}^2(\omega) + 2\bar{u}\, T(n-1) + Eu^2 - \left[T(0,n)\right]^2 \\ &= \text{var } t_{n-1}(\omega) + 2\bar{u}\, T(n-1) + \sigma^2 + \bar{u}^2 + \left[T(n-1)\right]^2 \\ &\qquad - \left[T(0,n)\right]^2 . \end{aligned} \qquad (15)$$

where σ^2 is the variance of the U distribution.

Hence dividing by n and taking the limit as $n \to \infty$

$$\begin{aligned} \lim_{n\to\infty} n^{-1} \text{ var } t_{on}(\omega) \leqslant{}& \lim_{n\to\infty} n^{-1} \text{ var } t_{n-1}(\omega) + 2\bar{u}\mu \\ &+ \lim_{n\to\infty} n^{-1}\left[\left(T(n-1)\right)^2 - \left(T(0,n)\right)^2\right] \end{aligned} \qquad (16)$$

Now by (14)

$$\lim_{n\to\infty} n^{-1}\left[\left(T(n-1)\right)^2 - T(0,n)^2\right] \leqslant \lim n^{-1} \bar{u}2\mu = 0 \qquad (17)$$

$$\text{Hence } \lim_{n\to\infty} n^{-1} \text{ var } t_{on}(\omega) \leqslant V_1 + 2\bar{u}\mu$$

$$= V_1^{\,1}$$

which proves the required result (13)

APPENDIX 4

Some Figures from Chapter 9

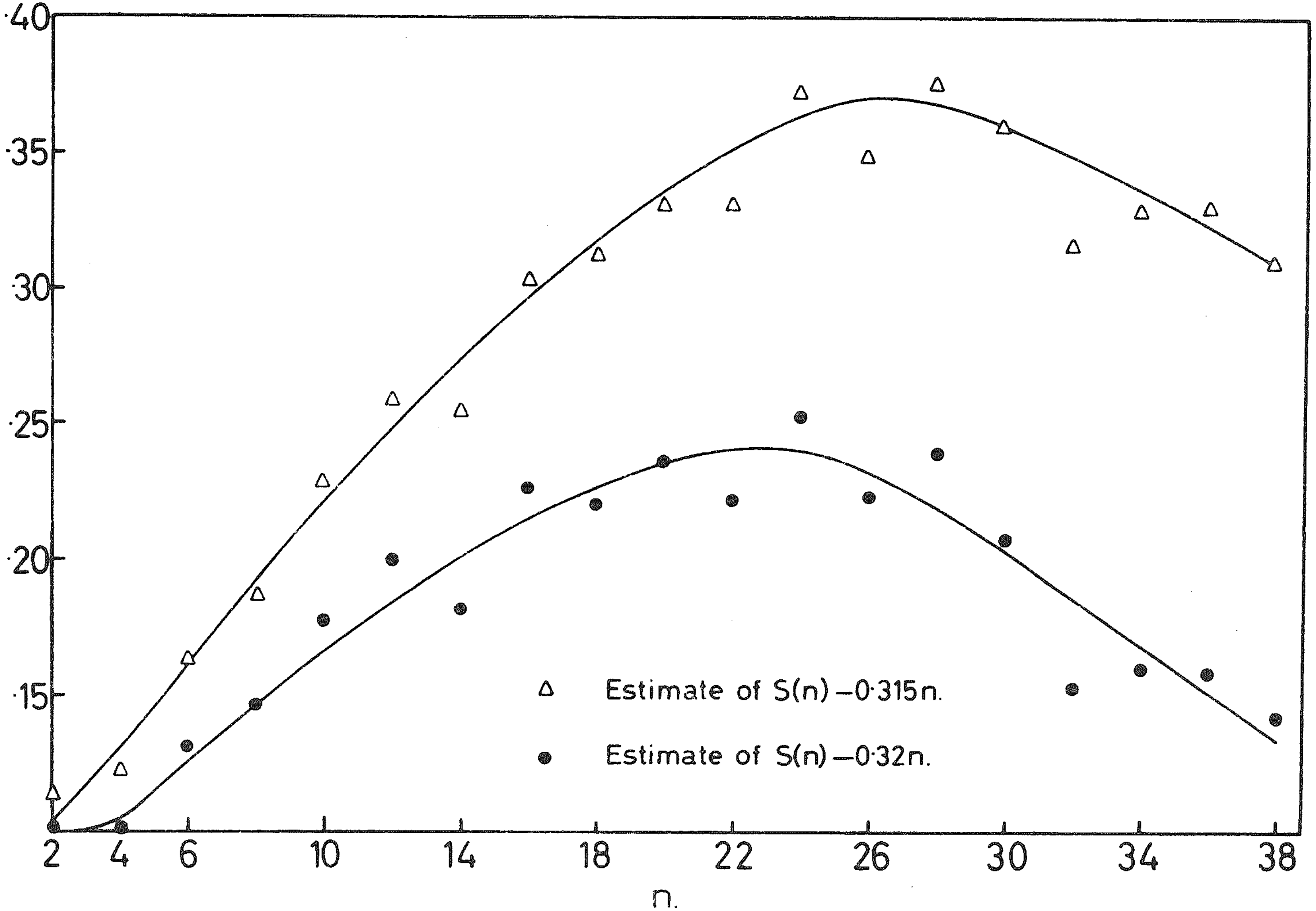


Figure 1. Estimates of S(n) - kn for k=0·32, 0·315.

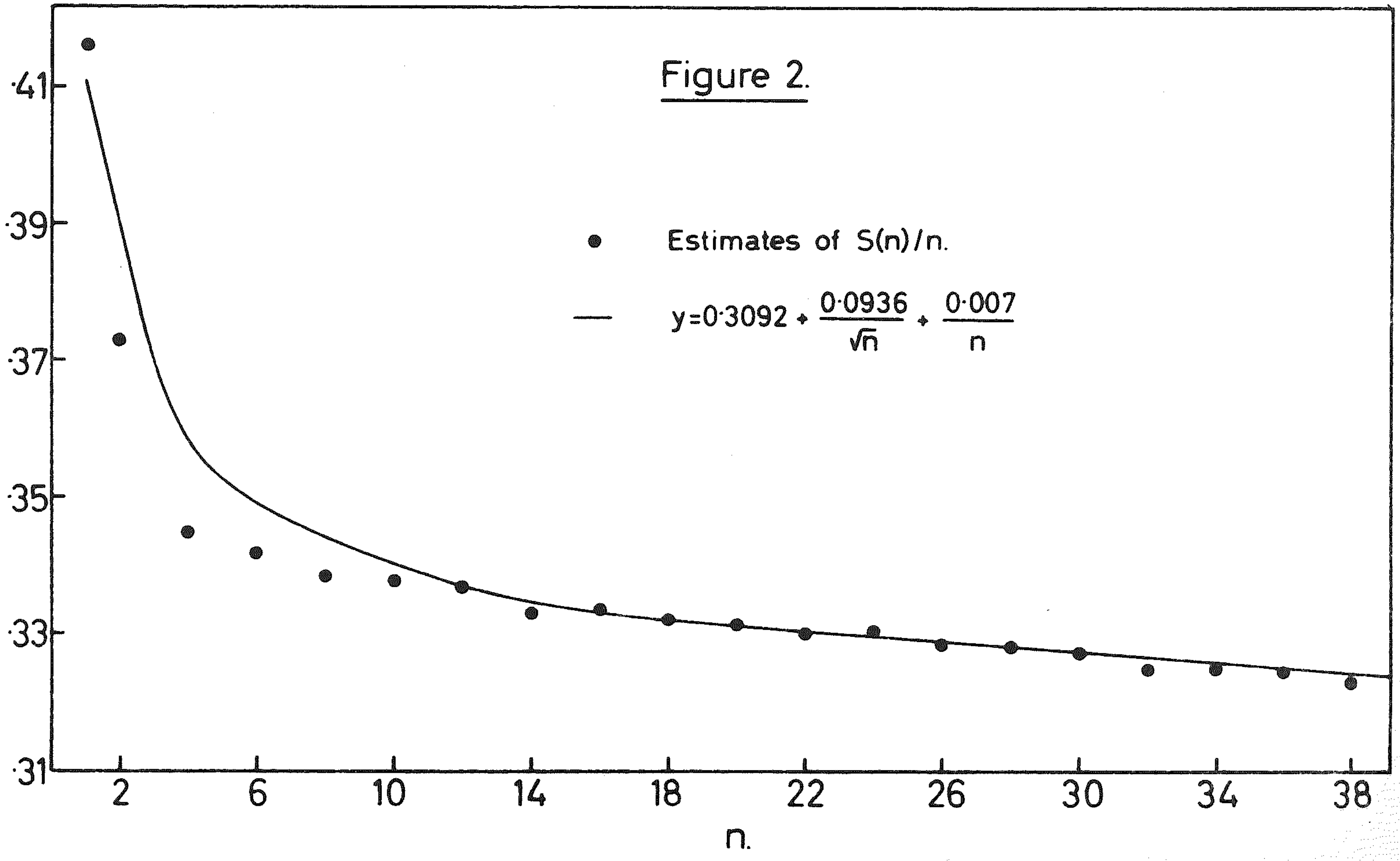


Figure 2.

Estimates of S(n)/n.
$y = 0{\cdot}3092 + \frac{0{\cdot}0936}{\sqrt{n}} + \frac{0{\cdot}007}{n}$
·41
·39
·37
·35
·33
·31
2
6
10
14
18
22
26
30
34
38
n.

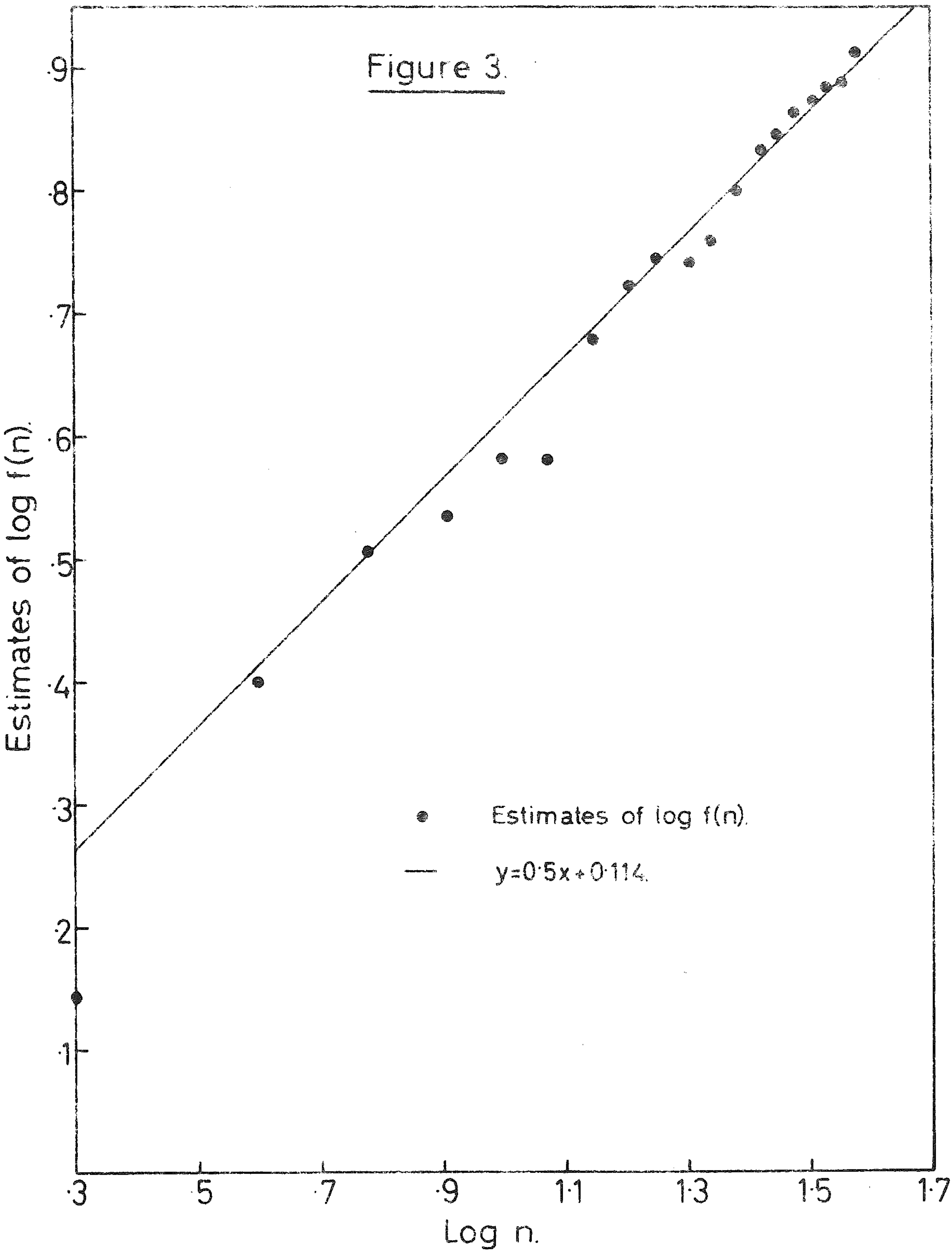


Figure 3. $f(n) = 10\,G(n) = 10\,[T(n) - S(n)]$

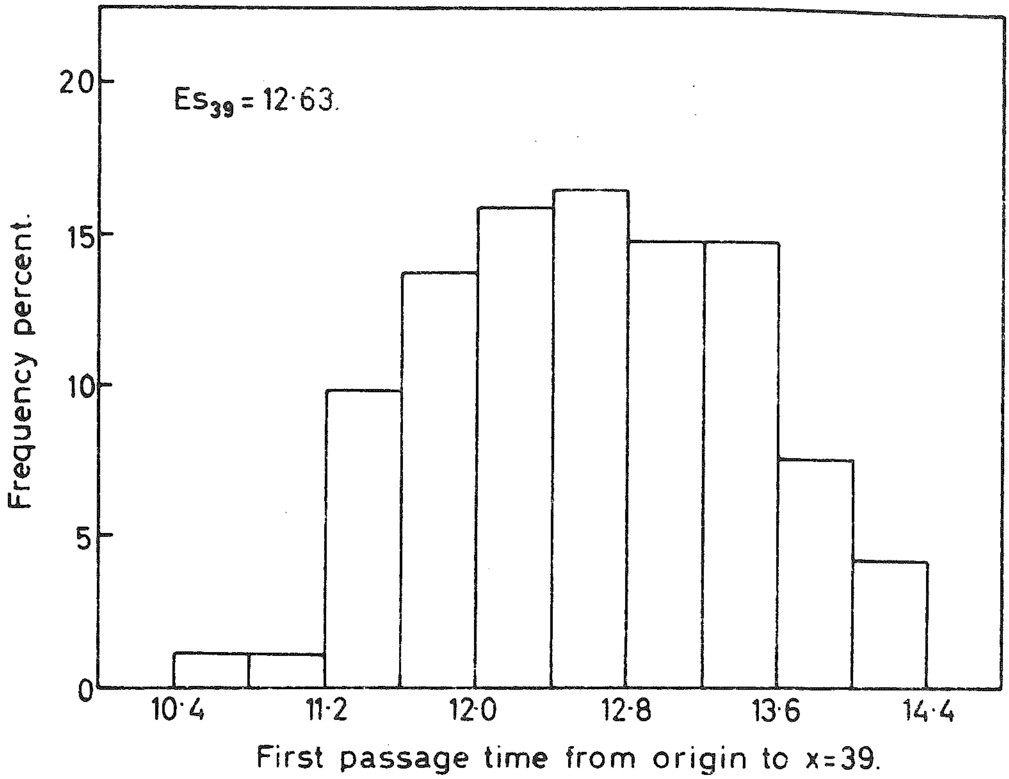


Figure 4. Frequency histogram for the first passage time S_{39}.

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