Report no. 99/09

A Priori Analysis for the Semi–Discrete Approximation to the Nonlinear Damped Wave Equation

Endre Süli

Oxford University Computing Laboratory, Wolfson Building,
Parks Road, Oxford OX1 3QD, UK

Catherine Wilkins

Department of Earth Sciences, University of Oxford,
Parks Road, Oxford OX1 3PR, UK

We study the second–order nonlinear damped wave equation semi–discretised in space using standard Galerkin finite element methods. Denoting the analytical solution and the corresponding finite element solution to the given problem by u and u_h respectively, we derive an optimal $L_2(\Omega)$ error estimate of the form

$$\max_{t\in[0,T]} \|u(t) - u_h(t)\| \le C(u)\, h^m,$$

for $(\mathbf{x},t) \in \overline{\Omega} \times [0,T]$, where $\Omega \subset \mathbb{R}^d$, C is a positive constant depending on u, h is the grid parameter, and $m > 1 + d/2$, where $m-1$ is the degree of the piecewise polynomials in the finite element test space.

Oxford University Computing Laboratory
Numerical Analysis Group
Wolfson Building
Parks Road
Oxford, England OX1 3QD

January, 2000

1 Introduction

In this paper we will derive an *a priori* error bound for the second–order nonlinear damped wave equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t}\left(a\left(u\right)\right) = \Delta b\left(u\right) + f\left(\mathbf{x}, t\right), \tag{1.1}$$

with given initial and boundary conditions that we will specify in a moment. That is, given a finite element test space V_h (which we will define more specifically later), we consider the semi–discrete analogue of (1.1) on V_h, namely the problem of finding $u_h : (0, T] \to V_h$ such that

$$\left(\frac{\partial^2 u_h}{\partial t^2}, \chi\right) + \left(\frac{\partial}{\partial t}\left(a\left(u_h\right)\right), \chi\right) + \left(\nabla b\left(u_h\right), \nabla \chi\right) = \left(f, \chi\right) \qquad \forall \chi \in V_h, \tag{1.2}$$

given initial values for u_h and $\partial u_h / \partial t$ approximating the initial conditions for u and $\partial u / \partial t$ in V_h.

Our work will show that the problem (1.2) has a locally unique solution, and that, if $m - 1$ is the degree of the piecewise polynomials in V_h, then the optimal $L_2\left(\Omega\right)$ error estimate

$$\max_{t \in [0,T]} \left\| u\left(t\right) - u_h\left(t\right) \right\| \leq C\left(u\right) h^m$$

holds, under the assumption that $m > 1 + d/2$ for $\Omega \subset \mathbb{R}^d$, and for an appropriate choice of initial approximations.

The motivation for studying the nonlinear damped wave equation in the form of (1.1) comes from the nonlinear wave dielectric interaction problem, derived from the full nonlinear Maxwell's equations by assuming linear polarisation (Bloom [3]). This is described by the nonlinearly damped nonlinear wave equation for the electric displacement $\mathbf{D}$:

$$\frac{\partial^2 \mathbf{D}}{\partial t^2} + \frac{\partial}{\partial t}\left(\eta\left(\left|\mathbf{D}\right|\right)\mathbf{D}\right) = \Delta\left(\gamma\left(\left|\mathbf{D}\right|\right)\mathbf{D}\right), \tag{1.3}$$

subject to appropriate initial and boundary conditions, where $\gamma\left(\left|\mathbf{D}\right|\right) = \lambda_0 + \lambda_2 \left|\mathbf{D}\right|^2, \lambda_i > 0$, and $\eta\left(\left|\mathbf{D}\right|\right) = \sigma\left(\mu\left(\left|\mathbf{D}\right|\right)\left|\mathbf{D}\right|\right)\mu\left(\left|\mathbf{D}\right|\right)$, with σ and μ both positive and differentiable on the set of positive real numbers. The approach here follows that of Makridakis, who in [7] studies finite element methods for a class of problems of nonlinear elastodynamics, semi–discretised in space using standard Galerkin methods.

2 Problem Formulation

Let Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz continuous boundary $\partial\Omega$. Then, for $1 \leq p \leq \infty$, $L_p\left(\Omega\right)$ will denote the usual Lebesgue space of real–valued functions with norm $\|\cdot\|_{L_p(\Omega)}$. For $p=2$, we will omit the subscript, writing $\|\cdot\|$ for $\|\cdot\|_{L_2(\Omega)}$, and for $p=\infty$ we will write $\|\cdot\|_\infty$ for $\|\cdot\|_{L_\infty(\Omega)}$. We define the $L_2(\Omega)$ inner product $(\cdot,\cdot)$ by

$$(u,v) := \int_\Omega u(\mathbf{x})\, v(\mathbf{x})\, \mathrm{d}\mathbf{x},$$

for $u, v \in L_2(\Omega)$. Further, for k a non–negative integer, let $W_p^k(\Omega)$ denote the classical Sobolev space equipped with the norm $\|\cdot\|_{W_p^k(\Omega)}$ and the semi–norm $|\cdot|_{W_p^k(\Omega)}$. For $p=2$ we write $H^k(\Omega)$ for $W_2^k(\Omega)$. Also, $H_0^k(\Omega)$ denotes the closure of the space of infinitely smooth functions with compact support in Ω in the norm of $H^k(\Omega)$.

We will make use in this paper of Poincaré's inequality:
There exists a positive constant C_{poin}, depending only on m and on the dimensions of the domain Ω, such that

$$\|v\|^2_{H^m(\Omega)} \le C_{\mathrm{poin}} \sum_{|\alpha|=m} \|D^\alpha v\|^2 = C_{\mathrm{poin}}\, |v|^2_{H^m(\Omega)} \tag{2.1}$$

$\forall v \in H_0^m(\Omega)$.

Proof See Adams [1]. ■

Let $T>0$. In the cylindrical domain $\Omega \times (0,T]$, we consider the second–order nonlinear damped wave equation given by

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t}\left(a(u)\right) = \Delta b(u) + f(\mathbf{x},t), \tag{2.2}$$

subject to the initial conditions

$$u(\cdot,0) = u_0(\cdot) \quad \text{and} \quad \frac{\partial u}{\partial t}(\cdot,0) = u_1(\cdot) \quad \text{on } \Omega, \tag{2.3}$$

and the homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \partial\Omega \times [0,T], \tag{2.4}$$

where $a \in C^3(\mathbb{R})$, $b \in C^4(\mathbb{R})$, $f \in C^2([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$, $u_0 \in H^4(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H^3(\Omega) \cap H_0^1(\Omega)$, and $b'(u) \ge M_0 > 0$, where $b'(u)$ denotes $\mathrm{d}b/\mathrm{d}u$. For an existence and uniqueness theorem for the nonlinear wave equation, of which (2.2)–(2.4) is a special case, see Chen & von Wahl [4].

Let $V_h \subset H_0^1(\Omega)$ denote a finite element test space consisting of piecewise polynomials of degree $k=m-1$ on a quasi–uniform subdivision $\mathcal{T}$ of $\overline{\Omega} \subset \mathbb{R}^d$, where $m \ge 2$. Then there exist positive constants $C_{\mathrm{inv1}}, C_{\mathrm{inv2}}$ and C_{inv3} such that for all $\chi \in V_h$

$$\|\chi\|_\infty \le C_{\mathrm{inv1}} h^{-d/2} \|\chi\|, \tag{2.5a}$$

$$\|\nabla\chi\|_\infty \le C_{\mathrm{inv2}} h^{-1} \|\chi\|_\infty, \tag{2.5b}$$

$$\|\nabla\chi\| \le C_{\mathrm{inv3}} h^{-1} \|\chi\|. \tag{2.5c}$$

Proof See Ciarlet [5], p.142. ∎

Define h by $h = \max_{\kappa \in \mathcal{T}} h_\kappa$, where h_κ is the diameter of κ (for example, the longest side of the triangle if $d = 2$).

We recall that there exists a positive constant C_{*1}, independent of h, such that, for $v \in H_0^1(\Omega) \cap H^m(\Omega)$,

$$\inf_{v_h \in V_h} \|v - v_h\| \leq C_{*1} h^m |v|_{H^m(\Omega)}. \tag{2.6}$$

We also note that, denoting by $\mathcal{I}^H : H_0^1(\Omega) \to V_h$ the standard finite element interpolation operator, there exist positive constants C_{*2} and C_{*3}, independent of h, such that, for $v \in H_0^1(\Omega) \cap H^m(\Omega)$,

$$\|v - \mathcal{I}^h v\|_\infty \leq C_{*2} h^{m-d/2} |v|_{H^m(\Omega)}, \quad m > d/2, \tag{2.7a}$$

$$\|\nabla v - \nabla \mathcal{I}^h v\|_\infty \leq C_{*3} h^{m-d/2-1} |v|_{H^m(\Omega)} \quad m > d/2 + 1; \tag{2.7b}$$

see Brenner and Scott [2] p. 105 for a proof of these results.

We shall assume in the analysis that the solution to (2.2)–(2.4) belongs to the function space $C^2([0,T]; H^m(\Omega) \cap H_0^1(\Omega))$, with $m > d/2 + 1, k = m - 1$.

We approximate (2.2)–(2.4) by a semi–discrete Galerkin finite element method of the following form:

Find $u_h(t) \in V_h,\ 0 < t \leq T$, such that

$$\left(\frac{\partial^2 u_h}{\partial t^2}, \chi\right) + \left(\frac{\partial}{\partial t}(a(u_h)), \chi\right) + (\nabla b(u_h), \nabla \chi) = (f(\mathbf{x}, t), \chi) \tag{2.8}$$

for all $\chi \in V_h$, and

$$u_h(\cdot, 0) = u_h^0(\cdot) \in V_h, \qquad \frac{\partial u_h}{\partial t}(\cdot, 0) = u_h^1(\cdot) \in V_h; \tag{2.9}$$

the precise choice of u_h^0 and u_u^1 in V_h will be given below in Section (4.1).
Throughout the rest of this paper we will denote $\partial u/\partial t$ and $\partial^2 u/\partial t^2$ by $\dot{u}$ and $\ddot{u}$ respectively, etc..

3 Preparation

In this section we recall Banach's Fixed Point Theorem, which is our main tool in both the proof of the existence of a locally unique solution to (2.8)–(2.9) and in the derivation of the *a priori* error estimate. We also define the sets that will be needed in the analysis, and we give some error estimates for a nonlinear projection operator.

First we recall that a mapping $M : X \to X$ is a *contraction* on a metric space $X = (X, \ell)$

if there exists a positive real number $\alpha < 1$ such that, for all $x, y \in X$, $\ell(Mx, My) \leq \alpha\ell(x, y)$.

Theorem 1 (Banach's Fixed Point Theorem) *Consider a metric space $X = (X, \ell)$, where $X \neq \emptyset$. Suppose that X is complete and let $M : X \to X$ be a contraction on X. Then M has precisely one fixed point.*

Proof See Rudin [9]. ■

For V_h as defined above, let $\Pi_h^{nl} w$ be the nonlinear projection of $w \in H_0^1(\Omega)$ onto V_h defined by

$$\left(\nabla b\left(\Pi_h^{nl} w\right) - \nabla b(w), \nabla\chi\right) = 0 \qquad \forall \chi \in V_h. \tag{3.1}$$

We note that Dobrowolski & Rannacher [6] and Rannacher [8] have shown that (3.1) has a locally unique solution $\Pi_h^{nl} w \in V_h$ for $h > 0$ sufficiently small.

Then define the space Y by

$$Y := C\left([0, T]; H_0^1(\Omega)\right) \cap C^1\left([0, T]; L_2(\Omega)\right).$$

Also, let $\mathcal{J} \subset Y$ denote the set defined by

$$\mathcal{J} := \left\{ v : [0, T] \to V_h : \max_{t\in[0,T]} \left(\|\dot{v}(t) - \dot{\Pi}_h^{nl} u(t)\| \right.\right.$$

$$\left.\left. + \|v(t) - \Pi_h^{nl} u(t)\|_{H^1(\Omega)} \right) \leq C_*(u) h^m \right\}. \tag{3.2}$$

In (3.2), $C_*(u) = C_*$ is a positive constant independent of h, and $h \leq h_0$; both C_* and h_0 will be specified later.

We note that the set $\mathcal{J}$ is not empty, since $\Pi_h^{nl} u \in \mathcal{J}$. Also, if a sequence

$$\{v_n\}_{n=1}^{\infty} \subset \mathcal{J}$$

converges in Y to $v \in Y$, we note that $v \in \mathcal{J}$. Hence, $\mathcal{J}$ is a closed subset of Y.

Finally, for $\phi, \psi \in \mathcal{J}$, let ℓ be defined by

$$\ell(\phi, \psi) := \max_{t\in[0,T]} \left(\|\dot{\phi}(t) - \dot{\psi}(t)\| + \|\phi(t) - \psi(t)\|_{H^1(\Omega)} \right). \tag{3.3}$$

We note that then $Y = (Y, \ell)$ is a complete metric space.

In the *a priori* analysis we will make use of the following results:

Lemma 2 (Projection Estimate) *For $m \geq 2$ and V_h as given above, there exists a positive constant $C_{\mathrm{proj1}}(w)$ such that the following inequality holds for $w \in H_0^1(\Omega) \cap H^m(\Omega)$:*

$$\|w - \Pi_h^{nl} w\| \leq C_{\mathrm{proj1}}(w) h^m. \tag{3.4}$$

Proof This result is proved by Dobrowolski & Rannacher [6] and Rannacher [8]; it follows from (2.6). ■

Further, for the analytical solution u such that $u(t), \dot{u}(t), \ddot{u}(t) \in H_0^1(\Omega) \cap H^m(\Omega)$, we put

$$C_{\text{proj1}} = \max_{t\in[0,T]} C_{\text{proj1}}(u(t)), \tag{3.5a}$$

$$C_{\text{proj2}} = \max_{t\in[0,T]} C_{\text{proj1}}(\dot{u}(t)), \tag{3.5b}$$

$$C_{\text{proj3}} = \max_{t\in[0,T]} C_{\text{proj1}}(\ddot{u}(t)). \tag{3.5c}$$

4 A Priori Error Analysis

In this section we prove the *a priori* error bound for $\|u(t) - u_h(t)\|$ as given in Theorem 3 below. In the process of deriving this bound, we establish the existence of a locally unique solution to (2.8)–(2.9).

Theorem 3 *Let u be the solution of (2.2)–(2.4) (such that the constants $C_{\text{proj1}}, C_{\text{proj2}}$ and C_{proj3} defined above exist, i.e. $u \in C^2([0,T]; H_0^1(\Omega) \cap H^m(\Omega))$). Assume also that $m > 1 + d/2$, where $m = k+1$, with k the degree of the piecewise polynomial functions in V_h. Assume also that the initial values $u_h^0, u_h^1 \in V_h$ have been chosen such that*

$$\left\|u_h^0(\cdot) - \Pi_h^{nl} u(\cdot,0)\right\|_{H^1(\Omega)} + \left\|u_h^1(\cdot) - \Pi_h^{nl}\dot{u}(\cdot,0)\right\| \le C_{\text{ini}} h^m, \tag{4.1}$$

where $\Pi_h^{nl}u$ is the solution to (3.1). Then the semi-discrete problem (2.8)–(2.9) has a locally unique solution that satisfies

$$\max_{t\in[0,T]} \|u(t) - u_h(t)\| \le \widetilde{C}_*(u)\, h^m, \tag{4.2}$$

where $\widetilde{C}_*(u) = \widetilde{C}_*$ is a positive constant and $h \le h_0$; both $\widetilde{C}_*$ and h_0 will be fixed later. We note that, in general, the condition

$$\left\|u_h^0(\cdot) - \Pi_h^{nl} u(\cdot,0)\right\|_{H^1(\Omega)} \le C_{\text{ini}} h^m$$

can only be satisfied by choosing $u_h^0 = \Pi_h^{nl} u(\cdot,0)$.

Proof First we consider the weak form of (2.2), so that

$$(\ddot{u},\chi) + (\dot{a}(u),\chi) + (\nabla b(u), \nabla\chi) = (f,\chi) \qquad \forall\chi \in V_h; \tag{4.3}$$

subtracting (2.8) from this gives

$$(\ddot{u} - \ddot{u}_h,\chi) + (\dot{a}(u) - \dot{a}(u_h),\chi) + (\nabla b(u) - \nabla b(u_h), \nabla\chi) = 0 \quad \forall\chi \in V_h.$$

We then use (3.1), the definition of the nonlinear projection operator $\Pi_h^{nl}u$, to give

$$(\ddot{u}-\ddot{u}_h,\chi)+(\dot{a}(u)-\dot{a}(u_h),\chi)+\left(\nabla b\left(\Pi_h^{nl}u\right)-\nabla b(u_h),\nabla\chi\right)=0 \quad \forall\chi\in V_h. \tag{4.4}$$

Now, we can simplify the nonlinear terms in the following way:

$$\begin{aligned}\dot{a}(u)-\dot{a}(u_h) &= \frac{\partial}{\partial t}[a(u)-a(u_h)]\\ &= \frac{\partial}{\partial t}\left[(u-u_h)\int_0^1 a'(\theta u_h+(1-\theta)u)\,\mathrm{d}\theta\right],\end{aligned}$$

and similarly

$$\nabla b\left(\Pi_h^{nl}u\right)-\nabla b(u_h)=\nabla\left[\left(\Pi_h^{nl}u-u_h\right)\int_0^1 b'\left(\theta u_h+(1-\theta)\Pi_h^{nl}u\right)\mathrm{d}\theta\right].$$

Also, let $\eta=u-\Pi_h^{nl}u$ and $\xi=\Pi_h^{nl}u-u_h$; then (4.4) becomes

$$\begin{aligned}\left(\ddot{\eta}+\ddot{\xi},\chi\right) \quad &+ \quad \left(\frac{\partial}{\partial t}\left[(\eta+\xi)\int_0^1 a'(\theta u_h+(1-\theta)u)\,\mathrm{d}\theta\right],\chi\right)\\ &+\left(\nabla\left[\xi\int_0^1 b'\left(\theta u_h+(1-\theta)\Pi_h^{nl}u\right)\mathrm{d}\theta\right],\nabla\chi\right)=0 \quad \forall\chi\in V_h.\end{aligned}$$

We now define a mapping $\mathcal{N}$ on the set $\mathcal{J}$ as follows:

If $\phi\in\mathcal{J}$, then the image $u_\phi:=\mathcal{N}(\phi)$ is given by the relations

$$u_\phi(\cdot,0)=u_h^0(\cdot) \quad \text{and} \quad \dot{u}_\phi(\cdot,0)=u_h^1(\cdot), \tag{4.5}$$

and $u_\phi(t)\in V_h$ for $0<t\le T$ such that

$$\begin{aligned}\left(\ddot{\eta}+\ddot{\xi}_\phi,\chi\right)+&\left(\frac{\partial}{\partial t}\left[(\eta+\xi_\phi)\int_0^1 a'(\theta\phi+(1-\theta)u)\,\mathrm{d}\theta\right],\chi\right)\\ &+\left(\nabla\left[\xi_\phi\int_0^1 b'\left(\theta\phi+(1-\theta)\Pi_h^{nl}u\right)\mathrm{d}\theta\right],\nabla\chi\right)=0\end{aligned} \tag{4.6}$$

$\forall\chi\in V_h$, where $\xi_\phi=\Pi_h^{nl}u-u_\phi$.

In order to complete the proof of Theorem 3, it suffices to show that $\mathcal{N}$ has a unique fixed point in $\mathcal{J}$. Indeed, if v_h is this fixed point, then v_h satisfies (2.8)–(2.9), and as $v_h\in\mathcal{J}$, the approximation property (3.4) implies that

$$\max_{t\in[0,T]}\|u(t)-v_h(t)\|\le C(u)h^m.$$

We will establish the existence of a unique fixed point in $\mathcal{J}$ by showing that the pair $\mathcal{N},\mathcal{J}$ satisfies the assumptions of Banach's Fixed Point Theorem, given above in Theorem 1. That is, we will show that

(a) $\mathcal{N}(\mathcal{J})\subset\mathcal{J}$,

(b) $\mathcal{N}$ is a contraction with respect to $\ell(\cdot,\cdot)$, where, for $\phi,\psi\in\mathcal{J}$, ℓ is defined by (3.3).

In the next two sections we will establish (a) and (b) above in turn.

4.1 Existence of a Fixed Point of $\mathcal{N}$ in $\mathcal{J}$

In this first section, we establish (a). That is, we show that, given $\phi \in \mathcal{J}$, then $u_\phi \in \mathcal{J}$, where $u_\phi := \mathcal{N}(\phi)$, for $\mathcal{N}$ the mapping defined by (4.5) and (4.6) above.

We firstly note that, provided certain Lipschitz conditions are satisfied, $\mathcal{N}$ is a well–defined mapping; the relations (4.5) and (4.6) define $u_\phi : [0,T] \to V_h$ uniquely as the solution of an initial value problem for a second–order system of linear ODEs.

To simplify the presentation of the nonlinear terms, we introduce the following notation:

$$b'\left(\Pi_h^{nl} u, \phi\right) := \int_0^1 b'\left(\theta\phi + (1-\theta)\,\Pi_h^{nl} u\right) \mathrm{d}\theta, \tag{4.7a}$$

and

$$a'(u,\phi) := \int_0^1 a'\left(\theta\phi + (1-\theta)\,u\right) \mathrm{d}\theta. \tag{4.7b}$$

Then we choose $\chi = \dot{\xi}_\phi$ in (4.6), giving

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\xi}_\phi\|^2 = &-\left(\ddot{\eta}, \dot{\xi}_\phi\right) - \left(\nabla b'\left(\Pi_h^{nl} u, \phi\right)\xi_\phi, \nabla\dot{\xi}_\phi\right) - \left(\frac{\partial}{\partial t} a'(u,\phi)\,\eta, \dot{\xi}_\phi\right) \\ &- \left(a'(u,\phi)\,\dot{\eta}, \dot{\xi}_\phi\right) - \left(\frac{\partial}{\partial t} a'(u,\phi)\,\xi_\phi, \dot{\xi}_\phi\right) - \left(a'(u,\phi)\,\dot{\xi}_\phi, \dot{\xi}_\phi\right) \\ &- \left(b'\left(\Pi_h^{nl} u, \phi\right)\nabla\xi_\phi, \nabla\dot{\xi}_\phi\right).\end{aligned} \tag{4.8}$$

We now integrate in time from $\tau = 0$ to t. In particular, the final term in (4.8) can be simplified as follows:

$$\begin{aligned}\int_0^t \left(b'\left(\Pi_h^{nl} u, \phi\right)\nabla\xi_\phi, \nabla\dot{\xi}_\phi\right)\mathrm{d}\tau &= \frac{1}{2}\int_0^t \left(b'\left(\Pi_h^{nl} u, \phi\right), \frac{\partial}{\partial t}|\nabla\xi_\phi|^2\right)\mathrm{d}\tau \\ &= \frac{1}{2}\left(b'\left(\Pi_h^{nl} u, \phi\right), |\nabla\xi_\phi|^2\right)\Big|_0^t \\ &\quad -\frac{1}{2}\int_0^t \left(\frac{\partial}{\partial t} b'\left(\Pi_h^{nl} u, \phi\right), |\nabla\xi_\phi|^2\right)\mathrm{d}\tau.\end{aligned}$$

Then, noting that $b'(\cdot) \geq M_0 > 0$ for $t \in [0,T]$, we see that (4.8) becomes

$$\|\dot{\xi}_\phi\|^2 + M_0\|\nabla\xi_\phi\|^2 \leq |\mathrm{I}| + |\mathrm{II}| + \ldots + |\mathrm{VIII}| + |\mathrm{VII}|, \tag{4.9}$$

where the I, II,...,VII terms are defined and simplified below:

$$\mathrm{I} = -2\int_0^t \left(\ddot{\eta}, \dot{\xi}_\phi\right)\mathrm{d}\tau \leq \int_0^t \left[\|\ddot{\eta}\|^2 + \|\dot{\xi}_\phi\|^2\right]\mathrm{d}\tau;$$

$$\mathrm{II} = -2\int_0^t \left(\nabla b'\left(\Pi_h^{nl} u, \phi\right)\xi_\phi, \nabla\dot{\xi}_\phi\right)\mathrm{d}\tau$$

$$\begin{aligned}
&= -2\left(\nabla b'\left(\Pi_h^{nl}u,\phi\right)\xi_\phi,\nabla\xi_\phi\right)\Big|_0^t + 2\int_0^t\left(\frac{\partial}{\partial t}\left[\nabla b'\left(\Pi_h^{nl}u,\phi\right)\xi_\phi\right]\nabla\xi_\phi\right)\mathrm{d}\tau \\
&= 2\left(\nabla\, b'\left(\Pi_h^{nl}u,\phi\right)\Big|_{t=0}\xi_\phi(0),\nabla\xi_\phi(0)\right) - 2\left(\nabla b'\left(\Pi_h^{nl}u,\phi\right)\xi_\phi,\nabla\xi_\phi\right) \\
&\quad +2\int_0^t\left(\nabla b'\left(\Pi_h^{nl}u,\phi\right)\dot{\xi}_\phi,\nabla\xi_\phi\right)\mathrm{d}\tau + 2\int_0^t\left(\frac{\partial}{\partial t}\nabla b'\left(\Pi_h^{nl}u,\phi\right)\xi_\phi,\nabla\xi_\phi\right)\mathrm{d}\tau \\
&\le 2C_1\|\xi_\phi(0)\|\ \|\nabla\xi_\phi(0)\| + 2C_1\|\xi_\phi\|\ \|\nabla\xi_\phi\| \\
&\quad +C_1\int_0^t\left[\|\dot{\xi}_\phi\|^2+\|\nabla\xi_\phi\|^2\right]\mathrm{d}\tau + C_2\int_0^t\left[\|\xi_\phi\|^2+\|\nabla\xi_\phi\|^2\right]\mathrm{d}\tau,
\end{aligned}$$

where

$$\begin{aligned}
C_i &= \max_{\theta\in[0,1]} C_i(\theta),\ \ i=1 \text{ and } 2, \text{ with} \\
C_1(\theta) &= \max_{t\in[0,T]}\|b''\left(\theta\phi+(1-\theta)\Pi_h^{nl}u\right)\|_\infty \max_{t\in[0,T]}\|\theta\nabla\phi+(1-\theta)\nabla\Pi_h^{nl}u\|_\infty
\end{aligned}$$

and

$$\begin{aligned}
C_2(\theta) &= \max_{t\in[0,T]}\|b'''\left(\theta\phi+(1-\theta)\Pi_h^{nl}u\right)\|_\infty \max_{t\in[0,T]}\|\theta\dot{\phi}+(1-\theta)\Pi_h^{nl}\dot{u}\|_\infty \\
&\quad\times\max_{t\in[0,T]}\|\theta\nabla\phi+(1-\theta)\nabla\Pi_h^{nl}u\|_\infty \\
&\quad+\max_{t\in[0,T]}\|b''\left(\theta\phi+(1-\theta)\Pi_h^{nl}u\right)\|_\infty \max_{t\in[0,T]}\|\theta\nabla\dot{\phi}+(1-\theta)\nabla\Pi_h^{nl}\dot{u}\|_\infty.
\end{aligned}$$

We will estimate the values of these constants, and the others found below, later on. Next we bound $\|\xi_\phi\|^2$ as follows:

$$\|\xi_\phi\|^2 = \|\xi_\phi(0)+\int_0^t\dot{\xi}_\phi\ \mathrm{d}\tau\|^2 \le 2\left[\|\xi_\phi(0)\|^2+\|\int_0^t\dot{\xi}_\phi\ \mathrm{d}\tau\|^2\right];$$

then

$$\begin{aligned}
\|\xi_\phi(0)\|^2 &= \left\|\Pi_h^{nl}u(0)-u_\phi(0)\right\|^2 \\
&= \left\|\Pi_h^{nl}u(0)-u_h^0\right\|^2, \quad \text{from (4.5)}, \\
&\le C_{\text{ini}}^2 h^{2m}, \quad \text{from (4.1)},
\end{aligned}$$

and

$$\left\|\int_0^t\dot{\xi}_\phi\ \mathrm{d}\tau\right\|^2 \le \left(\int_0^t\|\dot{\xi}_\phi\|\mathrm{d}\tau\right)^2 \le t\int_0^t\|\dot{\xi}_\phi\|^2\mathrm{d}\tau \le T\int_0^t\|\dot{\xi}_\phi\|^2\mathrm{d}\tau.$$

Hence, using (4.1) to bound $\|\nabla\xi_\phi(0)\|$ too, we see that

$$\begin{aligned} \text{II} \;\le\;& 2C_1C_{\text{ini}}^2h^{2m} + \frac{2C_1^2T}{\delta}\int_0^t \|\dot{\xi}_\phi\|^2 \mathrm{d}\tau + \frac{2C_1^2C_{\text{ini}}^2}{\delta}h^{2m} + \delta\|\nabla\xi_\phi\|^2 \\ & + C_1\int_0^t \left[\|\dot{\xi}_\phi\|^2 + \|\nabla\xi_\phi\|^2\right] \mathrm{d}\tau + C_2\int_0^t \left[\|\xi_\phi\|^2 + \|\nabla\xi_\phi\|^2\right] \mathrm{d}\tau, \end{aligned}$$

for some $\delta > 0$ that will be specified later, and for all $t \in [0,T]$.
Next,

$$\text{III} = -2\int_0^t \left(\frac{\partial}{\partial t}a'(u,\phi)\,\eta, \dot{\xi}_\phi\right) \mathrm{d}\tau \le \int_0^t \left[\|\eta\|^2 + C_3^2\|\dot{\xi}_\phi\|^2\right] \mathrm{d}\tau,$$

where

$$\begin{aligned} C_3 &= \max_{\theta\in[0,1]} C_3(\theta), \text{ with} \\ C_3(\theta) &= \max_{t\in[0,T]} \|a''(\theta\phi + (1-\theta)\,u)\|_\infty \max_{t\in[0,T]} \|\theta\dot{\phi} + (1-\theta)\,\dot{u}\|_\infty; \\ \text{IV} &= -2\int_0^t \left(a'(u,\phi)\,\dot{\eta}, \dot{\xi}_\phi\right) \mathrm{d}\tau \le \int_0^t \left[\|\dot{\eta}\|^2 + C_4^2\|\dot{\xi}_\phi\|^2\right] \mathrm{d}\tau, \end{aligned}$$

where

$$\begin{aligned} C_4 &= \max_{\theta\in[0,1]} \max_{t\in[0,T]} \|a'(\theta\phi + (1-\theta)\,u)\|_\infty; \\ \text{V} &= -2\int_0^t \left(\frac{\partial}{\partial t}a'(u,\phi)\xi_\phi, \dot{\xi}_\phi\right) \mathrm{d}\tau \le C_3\int_0^t \left[\|\xi_\phi\|^2 + \|\dot{\xi}_\phi\|^2\right] \mathrm{d}\tau; \\ \text{VI} &= -2\int_0^t \left(a'(u,\phi)\dot{\xi}_\phi, \dot{\xi}_\phi\right) \mathrm{d}\tau \le 2C_4\int_0^t \|\dot{\xi}_\phi\|^2 \mathrm{d}\tau; \\ \text{VII} &= \left(b'\left(\Pi_h^{nl}u,\phi\right)\Big|_{t=0}, |\nabla\xi_\phi(0)|^2\right) + \int_0^t \left(\frac{\partial}{\partial t}b'\left(\Pi_h^{nl}u,\phi\right), |\nabla\xi_\phi|^2\right) \mathrm{d}\tau \\ &\le C_5C_{\text{ini}}^2h^{2m} + C_6\int_0^t \|\nabla\xi_\phi\|^2 \mathrm{d}\tau \end{aligned}$$

where

$$C_5 = \max_{\theta\in[0,1]} \left\| b'\left(\theta\phi + (1-\theta)\,\Pi_h^{nl}u\right)\Big|_{t=0}\right\|_\infty$$

and

$$C_6 = \max_{\theta\in[0,1]} C_6(\theta), \text{ with}$$

$$C_6(\theta) \;=\; \max_{t\in[0,T]} \|b''\left(\theta\phi + (1-\theta)\,\Pi_h^{nl}u\right)\|_\infty \max_{t\in[0,T]} \|\theta\dot{\phi} + (1-\theta)\,\Pi_h^{nl}\dot{u}\|_\infty .$$

Before we proceed further, let us estimate the values of the constants C_1– C_6; this gives us the restrictions on h and m that are required by Theorem 3.

Let $C_* = C_*(u)$ be defined by

$$C_*^2 = 2C_7\left[3C_{\text{proj}}^2 T + C_{\text{ini}}^2\left(2C_1 + \frac{2C_1^2}{\delta} + C_5\right)\right], \tag{4.10}$$

where the constant C_7 is specified later. Also, define C_{inv} by

$$C_{\text{inv}} := \max\{C_{\text{inv1}}, C_{\text{inv1}}C_{\text{inv2}}, C_{\text{inv1}}C_{\text{inv3}}\}, \tag{4.11}$$

where $C_{\text{inv1}}, C_{\text{inv2}}$ and C_{inv3} are as given in (2.5), and define C_{proj} by

$$C_{\text{proj}} := \max\{C_{\text{proj1}}, C_{\text{proj2}}, C_{\text{proj3}}\}, \tag{4.12}$$

where $C_{\text{proj}i},\ i = 1, 2, 3$, are as given in (3.4) and (3.5).

Let

$$\widehat{h} \;=\; \left(\frac{\max_{t\in[0,T]}\|u\|_\infty}{(C_{*2} + C_{\text{inv}}C_{*1})\max_{t\in[0,T]}|u|_{H^m(\Omega)} + C_{\text{inv}}C_{\text{proj}} + C_{\text{inv}}C_*}\right)^{1/(m-d/2)}, \tag{4.13}$$

$$\widehat{h}^* \;=\; \left(\frac{\max_{t\in[0,T]}\|u\|_\infty}{(C_{*3} + C_{\text{inv}}C_{*1})\max_{t\in[0,T]}|u|_{H^m(\Omega)} + C_{\text{inv}}C_{\text{proj}} + C_{\text{inv}}C_*}\right)^{1/(m-d/2-1)}, \tag{4.14}$$

$$\widetilde{h} \;=\; \left(\frac{\max_{t\in[0,T]}\|\dot{u}\|_\infty}{(C_{*2} + C_{\text{inv}}C_{*1})\max_{t\in[0,T]}|\dot{u}|_{H^m(\Omega)} + C_{\text{inv}}C_{\text{proj}} + C_{\text{inv}}C_*}\right)^{1/(m-d/2)}, \tag{4.15}$$

$$\widetilde{h}^* \;=\; \left(\frac{\max_{t\in[0,T]}\|\dot{u}\|_\infty}{(C_{*3} + C_{\text{inv}}C_{*1})\max_{t\in[0,T]}|\dot{u}|_{H^m(\Omega)} + C_{\text{inv}}C_{\text{proj}} + C_{\text{inv}}C_*}\right)^{1/(m-d/2-1)}, \tag{4.16}$$

and

$$h_1 \;=\; \min\left\{\widehat{h}, \widehat{h}^*, \widetilde{h}, \widetilde{h}^*, 1\right\}. \tag{4.17}$$

In the rest of this paper, we will assume that $h \le h_1$ and $m > 1 + d/2$. Then we can bound the various components that make up the constants $C_i,\ i = 1, \ldots, 6$, as follows:
Looking firstly at the components of C_1, C_2, C_5 and C_6, namely those involving the nonlinear projection $\Pi_h^{nl}u$, we have

$$\begin{aligned}
\|\theta\phi + (1-\theta)\,\Pi_h^{nl}u\|_\infty &= \|\Pi_h^{nl}u + \theta(\phi - \Pi_h^{nl}u)\|_\infty \\
&\le \|\Pi_h^{nl}u\|_\infty + \|\phi - \Pi_h^{nl}u\|_\infty, \quad \text{as } \theta\in[0,1], \\
&\le \|u\|_\infty + \|\eta\|_\infty + \|\phi - \Pi_h^{nl}u\|_\infty,
\end{aligned}$$

as $\eta = u - \Pi_h^{nl} u$.
Now, incorporating $\mathcal{I}^h u$, the interpolant of u as defined above, we have

$$\begin{aligned}
\|\eta\|_\infty &= \|u - \Pi_h^{nl} u\|_\infty \le \|u - \mathcal{I}^h u\|_\infty + \|\mathcal{I}^h u - \Pi_h^{nl} u\|_\infty \\
&\le \|u - \mathcal{I}^h u\|_\infty + C_{\text{inv1}} h^{-d/2} \|\mathcal{I}^h u - \Pi_h^{nl} u\|, \quad \text{by (2.5a)}, \\
&\le \|u - \mathcal{I}^h u\|_\infty + C_{\text{inv1}} \left(\|u - \mathcal{I}^h u\| + \|u - \Pi_h^{nl} u\| \right) h^{-d/2} \\
&\le \left((C_{*2} + C_{\text{inv1}} C_{*1}) \, |u|_{H^m(\Omega)} + C_{\text{inv1}} C_{\text{proj1}} \right) h^{m-d/2},
\end{aligned}$$

by (2.6), (2.7a) and (3.4).
Also,

$$\|\phi - \Pi_h^{nl} u\|_\infty \le C_{\text{inv1}} h^{-d/2} \|\phi - \Pi_h^{nl} u\| \le C_{\text{inv1}} C_* h^{m-d/2},$$

from (3.2), as $\phi \in \mathcal{J}$. Hence we see that

$$\begin{aligned}
\|\theta\phi + (1-\theta)\, \Pi_h^{nl} u\|_\infty &\le \|u\|_\infty + \Big((C_{*2} + C_{\text{inv1}} C_{*1}) \, |u|_{H^m(\Omega)} \\
&\quad + C_{\text{inv1}} C_{\text{proj1}} + C_{\text{inv1}} C_* \Big) h^{m-d/2} \\
&\le \max_{t \in [0,T]} \|u\|_\infty + \Big((C_{*2} + C_{\text{inv}} C_{*1}) \, |u|_{H^m(\Omega)} \\
&\quad + C_{\text{inv}} C_{\text{proj}} + C_{\text{inv}} C_* \Big) h^{m-d/2}, \quad \text{from (4.11) and (4.12)}, \\
&= \max_{t \in [0,T]} \|u\|_\infty + \left(\frac{h}{\widehat{h}} \right)^{m-d/2} \max_{t \in [0,T]} \|u\|_\infty, \quad \text{from (4.13)}, \\
&\le 2 \max_{t \in [0,T]} \|u\|_\infty,
\end{aligned}$$

as $h \le \widehat{h}$ and $m > 1 + d/2$.

Working now on the second component of C_1, a similar argument means that

$$\|\theta\nabla\phi + (1-\theta)\, \nabla\Pi_h^{nl} u\|_\infty \le \|\nabla u\|_\infty + \|\nabla\eta\|_\infty + \|\nabla\phi - \nabla\Pi_h^{nl} u\|_\infty;$$

then, as before, we have

$$\begin{aligned}
\|\nabla\eta\|_\infty &\le \|\nabla u - \nabla\mathcal{I}^h u\|_\infty + \|\nabla\mathcal{I}^h u - \nabla\Pi_h^{nl} u\|_\infty \\
&\le \|\nabla u - \nabla\mathcal{I}^h u\|_\infty + C_{\text{inv1}} C_{\text{inv2}} \left(\|u - \mathcal{I}^h u\| + \|u - \Pi_h^{nl} u\| \right) h^{-d/2-1} \\
&\le \left(C_{*3} \, |u|_{H^m(\Omega)} + C_{\text{inv1}} C_{\text{inv2}} \left(C_{*1} \, |u|_{H^m(\Omega)} + C_{\text{proj1}} \right) \right) h^{m-d/2-1},
\end{aligned}$$

by (2.6), (2.7b) and (3.4).
Also, (2.5), (3.2a) and (3.2b) mean that

$$\left\| \nabla\phi - \nabla\Pi_h^{nl} u \right\|_\infty \le C_{\text{inv1}} C_{\text{inv2}} h^{m-d/2-1} C_*.$$

Hence

$$\begin{aligned}\|\theta\nabla\phi+(1-\theta)\,\nabla\Pi_h^{nl}u\|_\infty &\le \|\nabla u\|_\infty+\Big(C_{*3}\,|u|_{H^m(\Omega)}\\ &\quad +C_{\rm inv}\Big(C_{*1}\,|u|_{H^m(\Omega)}+C_{\rm proj}\Big)+C_{\rm inv}C_*\Big)\,h^{m-d/2-1},\end{aligned}$$

so that

$$\|\theta\nabla\phi+(1-\theta)\,\nabla\Pi_h^{nl}u\|_\infty\le\max_{t\in[0,T]}\left[\|u\|_\infty+\|\nabla u\|_\infty\right],$$

as before, now using (4.14), the definition of $\widehat{h}^*$.

In exactly the same way, we can show using (3.5a) and $\widetilde{h}$, defined by (4.15), that

$$\begin{aligned}\|\theta\dot\phi+(1-\theta)\,\Pi_h^{nl}\dot u\|_\infty &\le \max_{t\in[0,T]}\|\dot u\|_\infty+\Big((C_{*2}+C_{\rm inv}C_{*1})\,|\dot u|_{H^m(\Omega)}\\ &\quad +C_{\rm inv}C_{\rm proj}+C_{\rm inv}C_*\Big)\,h^{m-d/2}\\ &= \max_{t\in[0,T]}\|\dot u\|_\infty+\left(\frac{h}{\tilde{\tilde h}}\right)^{m-d/2}\max_{t\in[0,T]}\|\dot u\|_\infty\\ &\le 2\max_{t\in[0,T]}\|\dot u\|_\infty.\end{aligned}$$

For the final component of C_1, C_2, C_5 and C_6, we have

$$\begin{aligned}\|\theta\nabla\dot\phi+(1-\theta)\,\nabla\Pi_h^{nl}\dot u\|_\infty &\le \|\nabla\dot u\|_\infty+\|\nabla\dot\eta\|_\infty+\|\nabla\dot\phi-\nabla\Pi_h\dot u\|_\infty\\ &\le \|\nabla\dot u\|_\infty+\Big((C_{*3}+C_{\rm inv}C_{*1})\,|\dot u|_{H^m(\Omega)}\\ &\quad +C_{\rm inv}C_{\rm proj2}+C_{\rm inv}C_*\Big)\,h^{m-d/2-1},\end{aligned}$$

from (2.5a), (2.5b), (3.2) and (3.5a). As $h\le\widetilde{h}^*$, we have

$$\|\theta\nabla\dot\phi+(1-\theta)\,\nabla\Pi_h^{nl}\dot u\|_\infty\le\max_{t\in[0,T]}\left[\|\dot u\|_\infty+\|\nabla\dot u\|_\infty\right].$$

The components of C_3 and C_4 can be bounded in exactly the same way, as

$$\begin{aligned}\|\theta\phi+(1-\theta)\,u\|_\infty &= \|u+\theta\,(\phi-u)\|_\infty\le\|u\|_\infty+\|\phi-u\|_\infty,\quad \text{as } \theta\in[0,1],\\ &\le \|u\|_\infty+\|\eta\|_\infty+\left\|\phi-\Pi_h^{nl}u\right\|_\infty,\end{aligned}$$

etc., as before.

Overall then, we can write $C_i,\ i=1,\ldots,6$, as follows:

$$
\begin{aligned}
C_1 &= \max_\zeta |b''(\zeta)| \max_{t\in[0,T]} [\|u\|_\infty + \|\nabla u\|_\infty], \\
C_2 &= 2\max_\zeta |b'''(\zeta)| \max_{t\in[0,T]} \|\dot{u}\|_\infty \max_{t\in[0,T]} [\|u\|_\infty + \|\nabla u\|_\infty] \\
&\quad + \max_\zeta |b''(\zeta)| \max_{t\in[0,T]} [\|\dot{u}\|_\infty + \|\nabla \dot{u}\|_\infty], \\
C_3 &= 2\max_\zeta |a''(\zeta)| \max_{t\in[0,T]} \|\dot{u}\|_\infty, \\
C_4 &= \max_\zeta |a'(\zeta)|, \\
C_5 &= \max_\zeta |b'(\zeta)|, \\
C_6 &= 2\max_\zeta |b''(\zeta)| \max_{t\in[0,T]} \|\dot{u}\|_\infty,
\end{aligned}
$$

where

$$
|\zeta| \leq 2 \max_{t\in[0,T]} \|u\|_\infty.
$$

We now put all of the expressions for I, II,...,VII above into (4.9) to give

$$
\begin{aligned}
\|\dot{\xi}_\phi\|^2 + (M_0 - \delta)\|\nabla\xi_\phi\|^2 \quad &\leq \quad T \max_{t\in[0,T]} \left[\|\eta\|^2 + \|\dot{\eta}\|^2 + \|\ddot{\eta}\|^2\right] \\
&\quad + C_{\text{ini}}^2 \left(2C_1 + \frac{2C_1^2}{\delta} + C_5\right) h^{2m} \\
&\quad + \int_0^t \left[(C_2 + C_3)\|\xi_\phi\|^2 + (C_1 + C_2 + C_6)\|\nabla\xi_\phi\|^2\right. \\
&\quad \left. + \left(1 + \frac{2C_1^2 T}{\delta} + C_1 + C_3 + C_3^2 + 2C_4 + C_4^2\right)\|\dot{\xi}_\phi\|^2\right] \mathrm{d}\tau.
\end{aligned}
$$

We then choose $\delta < M_0$, and use Poincaré's inequality (2.1) to give, for $0 \leq t \leq T$,

$$
\begin{aligned}
\|\dot{\xi}_\phi\|^2 + \|\xi_\phi\|^2_{H^1(\Omega)} \quad &\leq \quad C_8 T \max_{t\in[0,T]} \left[\|\eta\|^2 + \|\dot{\eta}\|^2 + \|\ddot{\eta}\|^2\right] \\
&\quad + C_8 C_{\text{ini}}^2 \left(2C_1 + \frac{2C_1^2}{\delta} + C_5\right) h^{2m} \\
&\quad + C_8 \int_0^t \left[(C_2 + C_3)\|\xi_\phi\|^2 + (C_1 + C_2 + C_6)\|\nabla\xi_\phi\|^2\right. \\
&\quad \left. + \left(1 + \frac{2C_1^2 T}{\delta} + C_1 + C_3 + C_3^2 + 2C_4 + C_4^2\right)\|\dot{\xi}_\phi\|^2\right] \mathrm{d}\tau,
\end{aligned}
$$

where

$$C_8 = \frac{1}{\min\left\{1, \dfrac{M_0 - \delta}{C_{\mathrm{poin}}}\right\}}.$$

Gronwall's inequality gives

$$\begin{aligned}\|\dot{\xi}_\phi\|^2 + \|\xi_\phi\|^2_{H^1(\Omega)} &\le C_7 T \max_{t\in[0,T]} \left[\|\eta\|^2 + \|\dot{\eta}\|^2 + \|\ddot{\eta}\|^2\right] \\ &\quad + C_7 C_{\mathrm{ini}}^2 \left(2C_1 + \frac{2C_1^2}{\delta} + C_5\right) h^{2m},\end{aligned}$$

where

$$C_7 = C_8 \exp\left(C_8 C_9 T\right),$$

for

$$C_9 = \max\left\{C_2 + C_3, 1 + \frac{2C_1^2 T}{\delta} + C_1 + C_3 + C_3^2 + 2C_4 + C_4^2, C_1 + C_2 + C_6\right\}.$$

Finally, as $\eta = u - \Pi_h^{nl} u$, we can use (3.4) and (3.5) to obtain

$$\left\|\dot{\xi}_\phi\right\|^2 + \|\xi_\phi\|^2_{H^1(\Omega)} \le C_7 \left[3C_{\mathrm{proj}}^2 T + C_{\mathrm{ini}}^2 \left(2C_1 + \frac{2C_1^2}{\delta} + C_5\right)\right] h^{2m},$$

so that we obtain our required result

$$\|\dot{\xi}_\phi\| + \|\xi_\phi\|_{H^1(\Omega)} \le C_* h^m,$$

using (4.10), the definition of C_*.

That is, we have shown that $\mathcal{N}(\mathcal{J}) \subset \mathcal{J}$, for $h \le h_1$, defined by (4.17), and for $m > 1 + d/2$.

4.2 Establishment that $\mathcal{N}$ is a Contraction Mapping

In this second section we establish (b), the contractivity of $\mathcal{N}$, as detailed on p. 7. That is, we show that there exists a positive real number $\alpha < 1$ such that, for $\phi,\ \psi \in \mathcal{J}$,

$$\ell\left(\mathcal{N}(\phi), \mathcal{N}(\psi)\right) \le \alpha \ell(\phi, \psi),$$

with ℓ as defined by (3.3).
Before we continue, we note that, since a', a'', k', k'' and k''' are Lipschitz–continuous, there exist positive constants C_{L1}– C_{L5} such that

$$|a'(u,\phi) - a'(u,\psi)| \le C_{L1}|\phi - \psi|, \tag{4.18a}$$

$$|a''(u,\phi) - a''(u,\psi)| \le C_{L2}|\phi - \psi|, \tag{4.18b}$$

$$|b'\left(\Pi_h^{nl}u, \phi\right) - b'\left(\Pi_h^{nl}u, \psi\right)| \le C_{L3}|\phi - \psi|, \tag{4.18c}$$

$$|b''\left(\Pi_h^{nl}u,\phi\right)-b''\left(\Pi_h^{nl}u,\psi\right)| \quad\leq\quad C_{L4}|\phi-\psi|, \tag{4.18d}$$

and

$$|b'''\left(\Pi_h^{nl}u,\phi\right)-b'''\left(\Pi_h^{nl}u,\psi\right)| \quad\leq\quad C_{L5}|\phi-\psi|, \tag{4.18e}$$

where $b'\left(\Pi_h^{nl}u,\phi\right)$ and $a'\left(u,\phi\right)$ are as defined before by (4.7), and we define the other terms as follows:

$$a''\left(u,\phi\right) \quad:=\quad \int_0^1 a''\left(\theta\phi+\left(1-\theta\right)u\right)\mathrm{d}\theta,$$

$$b''\left(\Pi_h^{nl}u,\phi\right) \quad:=\quad \int_0^1 b''\left(\theta\phi+\left(1-\theta\right)\Pi_h^{nl}u\right)\mathrm{d}\theta,$$

and

$$b'''\left(\Pi_h^{nl}u,\phi\right) \quad:=\quad \int_0^1 b'''\left(\theta\phi+\left(1-\theta\right)\Pi_h^{nl}u\right)\mathrm{d}\theta.$$

We prove the first inequality above, namely (4.18a); the others follow in exactly the same manner.

We firstly note that, by definition,

$$\begin{aligned} a'\left(u,\phi\right)-a'\left(u,\psi\right) &= \int_0^1\left[a'\left(\theta\phi+\left(1-\theta\right)u\right)-a'\left(\theta\psi+\left(1-\theta\right)u\right)\right]\mathrm{d}\theta \\ &= \int_0^1\int_0^1\left(\theta\phi-\theta\psi\right)a''\left(\theta\left[\tau\phi+\left(1-\tau\right)\psi\right]+\left(1-\theta\right)u\right)\mathrm{d}\tau\mathrm{d}\theta, \end{aligned}$$

so that

$$|a'\left(u,\phi\right)-a'\left(u,\psi\right)|\leq\frac{1}{2}\left|\phi-\psi\right|\max_{\theta,\tau\in[0,1]}\max_{t\in[0,T]}\|a''\left(\theta\left[\tau\phi+\left(1-\tau\right)\psi\right]+\left(1-\theta\right)u\right)\|_\infty.$$

Working on the argument of a'' as before, we have

$$\begin{aligned} \|\theta\left[\tau\phi+\left(1-\tau\right)\psi\right]+\left(1-\theta\right)u\|_\infty &\leq \|u\|_\infty+\|\tau\phi+\left(1-\tau\right)\psi-u\|_\infty \\ &\leq \|u\|_\infty+\|\psi-u\|_\infty+\|\phi-\psi\|_\infty \\ &\leq \|u\|_\infty+\|\Pi_h^{nl}u-u\|_\infty+\|\psi-\Pi_h^{nl}u\|_\infty \\ &\quad +\|\phi-\Pi_h^{nl}u\|_\infty+\|\psi-\Pi_h^{nl}u\|_\infty. \end{aligned}$$

Now, as in the previous section,

$$\|\Pi_h^{nl}u-u\|_\infty\leq\left(\left(C_{*2}+C_{\mathrm{inv1}}C_{*1}\right)|u|_{H^m(\Omega)}+C_{\mathrm{inv1}}C_{\mathrm{proj1}}\right)h^{m-d/2},$$

and we also have that, from (2.5a),

$$\begin{aligned} \|\phi-\Pi_h^{nl}u\|_\infty+2\|\psi-\Pi_h^{nl}u\|_\infty &\leq C_{\mathrm{inv1}}\left(\|\phi-\Pi_h^{nl}u\|+2\|\psi-\Pi_h^{nl}u\|\right)h^{m-d/2} \\ &\leq 3C_{\mathrm{inv1}}C_*h^{m-d/2} \end{aligned}$$

by (3.2), as $\phi, \psi \in \mathcal{J}$.
Hence overall we have

$$\begin{aligned}\|\theta\,[\tau\phi + (1-\tau)\,\psi] + (1-\theta)\,u\|_\infty &\le \|u\|_\infty + \big((C_{*2} + C_{\text{inv1}}C_{*1})\,|u|_{H^m(\Omega)} \\ &\quad + C_{\text{inv1}}C_{\text{proj1}} + 3C_{\text{inv1}}C_*\big)\,h^{m-d/2},\end{aligned}$$

Now define $\bar{h}$ and h_2 by

$$\bar{h} := \left(\frac{\max_{t\in[0,T]}\|u\|_\infty}{(C_{*2} + C_{\text{inv1}}C_{*1})\max_{t\in[0,T]}|u|_{H^m(\Omega)} + C_{\text{inv1}}C_{\text{proj1}} + 3C_{\text{inv1}}C_*}\right)^{1/(m-d/2)}$$

and

$$h_2 := \min\{\bar{h}, 1\}. \tag{4.19}$$

Throughout the rest of this paper we will assume that $h \le \min\{h_1, h_2\}$. Then

$$\|\theta\,[\tau\phi + (1-\tau)\,\psi] + (1-\theta)\,u\|_\infty \le 2\,\|u\|_\infty\,,$$

so that

$$|a'(u,\phi) - a'(u,\psi)| \le \frac{1}{2}\max_\zeta |a''(\zeta)|\,|\phi - \psi|\,,$$

where $|\zeta| \le 2\max_{t\in[0,T]}\|u\|_\infty$, as before.
Hence we see that (4.18a) is proved, with

$$C_{L1} = \frac{1}{2}\max_\zeta |a''(\zeta)|\,.$$

The proof of (4.18b) follows in exactly the same way, with the same restrictions on h and m, so we do not give it here.

The proofs of (4.18c)–(4.18e) also follow immediately, as

$$\begin{aligned}\left\|\theta\,[\tau\phi + (1-\tau)\,\psi] + (1-\theta)\,\Pi_h^{nl}u\right\|_\infty &\le \|u\|_\infty + \left\|\Pi_h^{nl}u - u\right\|_\infty + \left\|\psi - \Pi_h^{nl}u\right\|_\infty \\ &\quad + \left\|\phi - \Pi_h^{nl}u\right\|_\infty + \left\|\psi - \Pi_h^{nl}u\right\|_\infty, \quad \text{etc..}\end{aligned}$$

Moving on to the proof of the contraction itself, we firstly differentiate out some of the terms in (4.6) to give

$$\begin{aligned}&\left(\ddot\eta + \ddot\xi_\phi, \chi\right) + \left(a'(u,\phi)\left(\dot\eta + \dot\xi_\phi\right), \chi\right) + \left(\frac{\partial}{\partial t}a'(u,\phi)\,(\eta + \xi_\phi), \chi\right) \\ &\quad + \left(b'\left(\Pi_h^{nl}u, \phi\right)\nabla\xi_\phi, \nabla\chi\right) + \left(\nabla b'\left(\Pi_h^{nl}u, \phi\right)\xi_\phi, \nabla\chi\right) = 0\end{aligned} \tag{4.20}$$

$\forall\chi \in V_h$.

Now let $\mathcal{N}(\psi) =: u_\psi$ and $\xi_\psi = \Pi_h^{nl}u - u_\psi$. We consider (4.20) with $\phi \in \mathcal{J}$ and $\psi \in \mathcal{J}$, and take the difference. We simplify each of the five resulting terms separately, denoting them by I,...,V:

$$\mathrm{I} = \left(\ddot{\eta} + \ddot{\xi}_\phi, \chi\right) - \left(\ddot{\eta} + \ddot{\xi}_\psi, \chi\right) = (\ddot{u}_\psi - \ddot{u}_\phi, \chi);$$

$$\begin{aligned}
\mathrm{II} &= \left(a'(u,\phi)\left(\dot{\eta} + \dot{\xi}_\phi\right), \chi\right) - \left(a'(u,\psi)\left(\dot{\eta} + \dot{\xi}_\psi\right), \chi\right) \\
&= \left(a'(u,\phi)\left(\dot{\eta} + \dot{\xi}_\phi\right), \chi\right) - \left(a'(u,\phi)\left(\dot{\eta} + \dot{\xi}_\psi\right), \chi\right) \\
&\quad + \left(a'(u,\phi)\left(\dot{\eta} + \dot{\xi}_\psi\right), \chi\right) - \left(a'(u,\psi)\left(\dot{\eta} + \dot{\xi}_\psi\right), \chi\right) \\
&= \left(a'(u,\phi)(\dot{u}_\psi - \dot{u}_\phi), \chi\right) + \left(\left[a'(u,\phi) - a'(u,\psi)\right]\left(\dot{\eta} + \dot{\xi}_\psi\right), \chi\right);
\end{aligned}$$

$$\begin{aligned}
\mathrm{III} &= \left(\frac{\partial}{\partial t} a'(u,\phi)(\eta + \xi_\phi), \chi\right) - \left(\frac{\partial}{\partial t} a'(u,\psi)(\eta + \xi_\psi), \chi\right) \\
&= \left(\frac{\partial}{\partial t} a'(u,\phi)(u_\psi - u_\phi), \chi\right) + \left(\frac{\partial}{\partial t}\left[a'(u,\phi) - a'(u,\psi)\right](\eta + \xi_\psi), \chi\right);
\end{aligned}$$

$$\begin{aligned}
\mathrm{IV} &= \left(b'\left(\Pi_h^{nl}u, \phi\right)\nabla\xi_\phi, \nabla\chi\right) - \left(b'\left(\Pi_h^{nl}u, \psi\right)\nabla\xi_\psi, \nabla\chi\right) \\
&= \left(b'\left(\Pi_h^{nl}u, \phi\right)\nabla(u_\psi - u_\phi), \nabla\chi\right) + \left(\left[b'\left(\Pi_h^{nl}u, \phi\right) - b'\left(\Pi_h^{nl}u, \psi\right)\right]\nabla\xi_\psi, \nabla\chi\right);
\end{aligned}$$

$$\begin{aligned}
\mathrm{V} &= \left(\nabla b'\left(\Pi_h^{nl}u, \phi\right)\xi_\phi, \nabla\chi\right) - \left(\nabla b'\left(\Pi_h^{nl}u, \psi\right)\xi_\psi, \nabla\chi\right) \\
&= \left(\nabla b'\left(\Pi_h^{nl}u, \phi\right)(u_\psi - u_\phi), \nabla\chi\right) + \left(\nabla\left[b'\left(\Pi_h^{nl}u, \phi\right) - b'\left(\Pi_h^{nl}u, \psi\right)\right]\xi_\psi, \nabla\chi\right).
\end{aligned}$$

Amalgamating all these terms gives

$$\begin{aligned}
0 &= (\ddot{u}_\psi - \ddot{u}_\phi, \chi) + \left(a'(u,\phi)(\dot{u}_\psi - \dot{u}_\phi), \chi\right) + \left(\left[a'(u,\phi) - a'(u,\psi)\right]\left(\dot{\eta} + \dot{\xi}_\psi\right), \chi\right) \\
&\quad + \left(\frac{\partial}{\partial t} a'(u,\phi)(u_\psi - u_\phi), \chi\right) + \left(\frac{\partial}{\partial t}\left[a'(u,\phi) - a'(u,\psi)\right](\eta + \xi_\psi), \chi\right) \\
&\quad + \left(b'\left(\Pi_h^{nl}u, \phi\right)\nabla(u_\psi - u_\phi), \nabla\chi\right) + \left(\left[b'\left(\Pi_h^{nl}u, \phi\right) - b'\left(\Pi_h^{nl}u, \psi\right)\right]\nabla\xi_\psi, \nabla\chi\right) \\
&\quad + \left(\nabla b'\left(\Pi_h^{nl}u, \phi\right)(u_\psi - u_\phi), \nabla\chi\right) + \left(\nabla\left[b'\left(\Pi_h^{nl}u, \phi\right) - b'\left(\Pi_h^{nl}u, \psi\right)\right]\xi_\psi, \nabla\chi\right)
\end{aligned}$$

$\forall \chi \in V_h$.

We now choose $\chi = \dot{u}_\psi - \dot{u}_\phi$, and, writing Θ for $u_\psi - u_\phi$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Theta}\|^2 &= -2\left(a'(u,\phi)\dot{\Theta},\dot{\Theta}\right) - 2\left(\left[a'(u,\phi) - a'(u,\psi)\right]\left(\dot{\eta}+\dot{\xi}_\psi\right),\dot{\Theta}\right) \\
&\quad -2\left(\frac{\partial}{\partial t}a'(u,\phi)\,\Theta,\dot{\Theta}\right) - 2\left(\frac{\partial}{\partial t}\left[a'(u,\phi)-a'(u,\psi)\right](\eta+\xi_\psi),\dot{\Theta}\right) \\
&\quad -2\left(\left[b'\left(\Pi_h^{nl}u,\phi\right) - b'\left(\Pi_h^{nl}u,\psi\right)\right]\nabla\xi_\psi,\nabla\dot{\Theta}\right) - 2\left(\nabla b'\left(\Pi_h^{nl}u,\phi\right)\Theta,\nabla\dot{\Theta}\right) \\
&\quad -2\left(\nabla\left[b'\left(\Pi_h^{nl}u,\phi\right) - b'\left(\Pi_h^{nl}u,\psi\right)\right]\xi_\psi,\nabla\dot{\Theta}\right) - 2\left(b'\left(\Pi_h^{nl}u,\phi\right)\nabla\Theta,\nabla\dot{\Theta}\right).
\end{aligned} \tag{4.21}$$

We now integrate (4.21) in time from $\tau = 0$ to t. In particular, the final term of (4.21) can be simplified as follows:

$$\begin{aligned}
\int_0^t 2\left(b'\left(\Pi_h^{nl}u,\phi\right)\nabla\Theta,\nabla\dot{\Theta}\right)\mathrm{d}\tau &= \int_0^t \left(b'\left(\Pi_h^{nl}u,\phi\right), \frac{\partial}{\partial t}|\nabla\Theta|^2\right)\mathrm{d}\tau \\
&= \left(b'\left(\Pi_h^{nl}u,\phi\right),|\nabla\Theta|^2\right)\Big|_0^t \\
&\quad - \int_0^t \left(\frac{\partial}{\partial t}b'\left(\Pi_h^{nl}u,\phi\right),|\nabla\Theta|^2\right)\mathrm{d}\tau.
\end{aligned}$$

Then we note that, from the definition of $\mathcal{N}$,

$$\Theta(\cdot,0) = \dot{\Theta}(\cdot,0) = 0, \tag{4.22}$$

so that, with $b'(\cdot) \geq M_0 > 0$ for $t \in [0,T]$ as before, (4.21) becomes

$$\|\dot{\Theta}\|^2 + M_0\|\nabla\Theta\|^2 \leq |\mathrm{I}| + |\mathrm{II}| + \ldots + |\mathrm{VIII}|, \tag{4.23}$$

where the I, ..., VIII terms are defined and simplified below. Working on each of these terms in turn, we have

$$\mathrm{I} = -2\int_0^t \left(a'(u,\phi)\dot{\Theta},\dot{\Theta}\right)\mathrm{d}\tau \leq C_{10}\int_0^t \|\dot{\Theta}\|^2\mathrm{d}\tau,$$

where

$$\begin{aligned}
C_{10} &= 2\max_{\theta\in[0,1]}\max_{t\in[0,T]}\|a'(\theta\phi + (1-\theta)u)\|_\infty; \\
\mathrm{II} &= -2\int_0^t \left(\left[a'(u,\phi)-a'(u,\psi)\right]\left(\dot{\eta}+\dot{\xi}_\psi\right),\dot{\Theta}\right)\mathrm{d}\tau \\
&\leq 2C_{L1}\int_0^t \|\phi-\psi\|\,\|\dot{\eta}-\dot{\xi}_\psi\|_\infty\|\dot{\Theta}\|\mathrm{d}\tau, \qquad \text{from (4.18a)},
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^t \left[\|\dot\eta+\dot\xi_\psi\|_\infty^2\|\phi-\psi\|^2+C_{L1}^2\|\dot\Theta\|^2\right]\mathrm{d}\tau;\\
\mathrm{III} &= -2\int_0^t \left(\frac{\partial}{\partial t}a'(u,\phi)\,\Theta,\dot\Theta\right)\mathrm{d}\tau\\
&\leq C_{11}\int_0^t \left[\|\Theta\|^2+\|\dot\Theta\|^2\right]\mathrm{d}\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
C_{11} &= \max_{\theta\in[0,1]}\left[\max_{t\in[0,T]}\|a''(\theta\phi+(1-\theta)\,u)\|_\infty \max_{t\in[0,T]}\|\theta\dot\phi+(1-\theta)\,\dot u\|_\infty\right];\\
\mathrm{IV} &= -2\int_0^t \left(\frac{\partial}{\partial t}\left[a'(u,\phi)-a'(u,\psi)\right](\eta+\xi_\psi),\dot\Theta\right)\mathrm{d}\tau\\
&= -2\int_0^t \left(\int_0^1 \left\{\left[a''(u,\phi)-a''(u,\psi)\right]\left(\theta\dot\phi+(1-\theta)\,\dot u\right)\right.\right.\\
&\quad \left.\left.+a''(u,\psi)\,\theta\left(\dot\phi-\dot\psi\right)\right\}\mathrm{d}\theta\,(\eta+\xi_\psi),\dot\Theta\right)\mathrm{d}\tau\\
&\leq \int_0^t \left[\|\eta+\xi_\psi\|_\infty^2\|\phi-\psi\|^2+C_{L2}^2C_{12}^2\|\dot\Theta\|^2\right]\mathrm{d}\tau\\
&\quad +\int_0^t \left[\|\eta+\xi_\psi\|_\infty^2\|\dot\phi-\dot\psi\|^2+C_{13}^2\|\dot\Theta\|^2\right]\mathrm{d}\tau,
\end{aligned}
$$

where

$$
C_{12} = \max_{\theta\in[0,1]}\max_{t\in[0,T]}\|\theta\dot\phi+(1-\theta)\,\dot u\|_\infty
$$

and

$$
\begin{aligned}
C_{13} &= \max_{\theta\in[0,1]}\max_{t\in[0,T]}\|a''(\theta\psi+(1-\theta)\,u)\|_\infty;\\
\mathrm{V} &= -2\int_0^t \left(\left[b'\left(\Pi_h^{nl}u,\phi\right)-b'\left(\Pi_h^{nl}u,\psi\right)\right]\nabla\xi_\psi,\nabla\dot\Theta\right)\mathrm{d}\tau\\
&= -2\left(\left[b'\left(\Pi_h^{nl}u,\phi\right)-b'\left(\Pi_h^{nl}u,\psi\right)\right]\nabla\xi_\psi,\nabla\Theta\right)\\
&\quad +2\int_0^t \left(\left[b'\left(\Pi_h^{nl}u,\phi\right)-b'\left(\Pi_h^{nl}u,\psi\right)\right]\nabla\dot\xi_\psi,\nabla\Theta\right)\mathrm{d}\tau\\
&\quad +2\int_0^t \left(\int_0^1 \left[b''\left(\Pi_h^{nl}u,\phi\right)\left(\theta\dot\phi+(1-\theta)\,\Pi_h^{nl}\dot u\right)-b''\left(\Pi_h^{nl}u,\psi\right)\left(\theta\dot\psi+(1-\theta)\,\Pi_h^{nl}\dot u\right)\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
& - b''\left(\Pi_h^{nl} u, \psi\right)\left(\theta\dot{\phi} + (1-\theta)\,\Pi_h^{nl}\dot{u}\right) + b''\left(\Pi_h^{nl} u, \psi\right)\left(\theta\dot{\phi} + (1-\theta)\,\Pi_h^{nl}\dot{u}\right)\Big]\,\mathrm{d}\theta \\
& \times \nabla\xi_\psi, \nabla\Theta)\,\mathrm{d}\tau \\
\leq\ & 2C_{L3}\|\phi-\psi\|\ \|\nabla\xi_\psi\|_\infty\|\nabla\Theta\| \\
& +2C_{L3}\int_0^t \|\phi-\psi\|\ \|\nabla\dot{\xi}_\psi\|_\infty\|\nabla\Theta\|\mathrm{d}\tau \\
& +2C_{L4}C_{14}\int_0^t \|\phi-\psi\|\ \|\nabla\xi_\psi\|_\infty\|\nabla\Theta\|\mathrm{d}\tau \\
& +2C_{15}\int_0^t \|\dot{\phi}-\dot{\psi}\|\ \|\nabla\xi_\psi\|_\infty\|\nabla\Theta\|\mathrm{d}\tau \\
\leq\ & \frac{C_{L3}^2}{\delta}\|\phi-\psi\|^2\|\nabla\xi_\psi\|_\infty^2 + \delta\|\nabla\Theta\|^2 \\
& +\int_0^t \left[\|\phi-\psi\|^2\|\nabla\dot{\xi}_\psi\|_\infty^2 + C_{L3}^2\|\nabla\Theta\|^2\right]\mathrm{d}\tau \\
& +\int_0^t \left[\|\phi-\psi\|^2\|\nabla\xi_\psi\|_\infty^2 + C_{14}^2C_{L4}^2\|\nabla\Theta\|^2\right]\mathrm{d}\tau \\
& +\int_0^t \left[\|\dot{\phi}-\dot{\psi}\|^2\|\nabla\xi_\psi\|_\infty^2 + C_{15}^2\|\nabla\Theta\|^2\right]\mathrm{d}\tau,
\end{aligned}
$$

for some $\delta > 0$, where

$$
C_{14} = \max_{\theta\in[0,1]}\max_{t\in[0,T]}\|\theta\dot{\phi} + (1-\theta)\,\Pi_h^{nl}\dot{u}\|_\infty
$$

and

$$
\begin{aligned}
C_{15} &= \max_{\theta\in[0,1]}\max_{t\in[0,T]}\|b''\left(\theta\psi + (1-\theta)\,\Pi_h^{nl} u\right)\|_\infty; \\
\mathrm{VI} &= -2\int_0^t \left(\nabla b'\left(\Pi_h^{nl} u, \phi\right)\Theta, \nabla\dot{\Theta}\right)\mathrm{d}\tau \\
&= -2\left(\nabla b'\left(\Pi_h^{nl} u, \phi\right)\Theta, \nabla\Theta\right) + 2\int_0^t \left(\nabla b'\left(\Pi_h^{nl} u, \phi\right)\dot{\Theta}, \nabla\Theta\right)\mathrm{d}\tau \\
&\quad +2\int_0^t \left(\frac{\partial}{\partial t}\left[\nabla b'\left(\Pi_h^{nl} u, \phi\right)\right]\Theta, \nabla\Theta\right)\mathrm{d}\tau \\
&\leq 2C_{16}\|\Theta\|\ \|\nabla\Theta\| + 2C_{16}\int_0^t \|\dot{\Theta}\|\ \|\nabla\Theta\|\mathrm{d}\tau + 2C_{17}\int_0^t \|\Theta\|\ \|\nabla\Theta\|\mathrm{d}\tau,
\end{aligned}
$$

where

$$C_i = \max_{\theta\in[0,1]} C_i(\theta), \ i = 16, 17, \text{ with}$$

$$C_{16}(\theta) = \max_{t\in[0,T]} \|b''\left(\theta\phi + (1-\theta)\,\Pi_h^{nl} u\right)\|_\infty \max_{t\in[0,T]} \|\theta\nabla\phi + (1-\theta)\,\nabla\Pi_h^{nl} u\|_\infty$$

and

$$\begin{aligned} C_{17}(\theta) &= \max_{t\in[0,T]} \|b'''\left(\theta\phi + (1-\theta)\,\Pi_h^{nl} u\right)\|_\infty \max_{t\in[0,T]} \|\theta\dot\phi + (1-\theta)\,\Pi_h^{nl} \dot u\|_\infty \\ &\quad \times \max_{t\in[0,T]} \|\theta\nabla\phi + (1-\theta)\,\nabla\Pi_h^{nl} u\|_\infty \\ &\quad + \max_{t\in[0,T]} \|b''\left(\theta\phi + (1-\theta)\,\Pi_h^{nl} u\right)\|_\infty \max_{t\in[0,T]} \|\theta\nabla\dot\phi + (1-\theta)\,\nabla\Pi_h^{nl} \dot u\|_\infty \ . \end{aligned}$$

Now, $\Theta(\cdot, 0) = 0$, from (4.22), so that

$$\|\Theta\|^2 = \left\|\int_0^t \dot\Theta \mathrm{d}\tau\right\|^2 \le \left(\int_0^t \left\|\dot\Theta\right\|\right)^2 \le T\int_0^t \|\dot\Theta\|^2 \mathrm{d}\tau,$$

and hence

$$\mathrm{VI} \le \frac{C_{16}^2 T}{\delta}\int_0^t \|\dot\Theta\|^2\mathrm{d}\tau + \delta\|\nabla\Theta\|^2 + \int_0^t \left[C_{16}^2\|\dot\Theta\|^2 + C_{17}^2\|\Theta\|^2 + 2\|\nabla\Theta\|^2\right]\mathrm{d}\tau,$$

for $\delta > 0$ as before. Next,

$$\begin{aligned} \mathrm{VII} &= -2\int_0^t \left(\nabla\left[b'\left(\Pi_h^{nl} u, \phi\right) - b'\left(\Pi_h^{nl} u, \psi\right)\right]\xi_\psi, \nabla\dot\Theta\right)\mathrm{d}\tau \\ &= -2\left(\left[\nabla b'\left(\Pi_h^{nl} u, \phi\right) - \nabla b'\left(\Pi_h^{nl} u, \psi\right)\right]\xi_\psi, \nabla\Theta\right) \\ &\quad +2\int_0^t \left(\left[\nabla b'\left(\Pi_h^{nl} u, \phi\right) - \nabla b'\left(\Pi_h^{nl} u, \psi\right)\right]\dot\xi_\psi, \nabla\Theta\right)\mathrm{d}\tau \\ &\quad +2\int_0^t \left(\frac{\partial}{\partial t}\left[\nabla b'\left(\Pi_h^{nl} u, \phi\right) - \nabla b'\left(\Pi_h^{nl} u, \psi\right)\right]\xi_\psi, \nabla\Theta\right)\mathrm{d}\tau \\ &\le \frac{C_{18}^2 C_{L4}^2}{\delta}\|\phi-\psi\|^2\|\xi_\psi\|_\infty^2 + \delta\|\nabla\Theta\|^2 \\ &\quad + \frac{C_{14}^2}{\delta}\|\nabla\phi - \nabla\psi\|^2\|\xi_\psi\|_\infty^2 + \delta\|\nabla\Theta\|^2 \\ &\quad + \int_0^t \left[\|\phi-\psi\|^2\|\dot\xi_\psi\|_\infty^2 + C_{18}^2 C_{L4}^2\|\nabla\Theta\|^2\right]\mathrm{d}\tau \end{aligned}$$

$$+\int_0^t \left[\|\nabla\phi-\nabla\psi\|^2\|\dot{\xi}_\psi\|_\infty^2+C_{15}^2\|\nabla\Theta\|^2\right]\mathrm{d}\tau$$

$$+\int_0^t \left[\|\phi-\psi\|^2\|\xi_\psi\|_\infty^2+C_{19}^2C_{L5}^2\|\nabla\Theta\|^2\right]\mathrm{d}\tau$$

$$+\int_0^t \left[\|\dot{\phi}-\dot{\psi}\|^2\|\xi_\psi\|_\infty^2+C_{20}^2\|\nabla\Theta\|^2\right]\mathrm{d}\tau$$

$$+\int_0^t \left[\|\phi-\psi\|^2\|\xi_\psi\|_\infty^2+C_{21}^2C_{L4}^2\|\nabla\Theta\|^2\right]\mathrm{d}\tau$$

$$+\int_0^t \left[\|\nabla\dot{\phi}-\nabla\dot{\psi}\|^2\|\xi_\psi\|_\infty^2+C_{15}^2\|\nabla\Theta\|^2\right]\mathrm{d}\tau$$

$$+\int_0^t \left[\|\nabla\phi-\nabla\psi\|^2\|\xi_\psi\|_\infty^2+C_{22}^2\|\nabla\Theta\|^2\right]\mathrm{d}\tau,$$

with $\delta>0$ as before, where

$$C_i = \max_{\theta\in[0,1]} C_i(\theta),\ i=18,\ldots,22, \text{ with}$$

$$C_{18}(\theta) = \max_{t\in[0,T]}\|\theta\nabla\phi+(1-\theta)\nabla\Pi_h^{nl}u\|_\infty,$$

$$C_{19}(\theta) = \max_{t\in[0,T]}\|\theta\nabla\phi+(1-\theta)\nabla\Pi_h^{nl}u\|_\infty \max_{t\in[0,T]}\|\theta\dot{\phi}+(1-\theta)\Pi_h^{nl}\dot{u}\|_\infty,$$

$$C_{20}(\theta) = \max_{t\in[0,T]}\|\theta\nabla\phi+(1-\theta)\nabla\Pi_h^{nl}u\|_\infty \max_{t\in[0,T]}\|b'''\left(\theta\psi+(1-\theta)\Pi_h^{nl}u\right)\|_\infty,$$

$$C_{21}(\theta) = \max_{t\in[0,T]}\|\theta\nabla\dot{\phi}+(1-\theta)\nabla\Pi_h^{nl}\dot{u}\|_\infty,$$

and

$$C_{22}(\theta) = \max_{t\in[0,T]}\|b'''\left(\theta\psi+(1-\theta)\Pi_h^{nl}u\right)\|_\infty \max_{t\in[0,T]}\|\theta\dot{\psi}+(1-\theta)\Pi_h^{nl}\dot{u}\|_\infty.$$

Finally,

$$\mathrm{VIII} = \int_0^t \left(\frac{\partial}{\partial t}b'\left(\Pi_h^{nl}u,\phi\right),|\nabla\Theta|^2\right)\mathrm{d}\tau \le C_{23}\int_0^t\|\nabla\Theta\|^2\mathrm{d}\tau,$$

where

$$C_{23} = \max_{\theta\in[0,1]}\left[\max_{t\in[0,T]}\|b''\left(\theta\phi+(1-\theta)\Pi_h^{nl}u\right)\|_\infty \max_{t\in[0,T]}\|\theta\dot{\phi}+(1-\theta)\Pi_h^{nl}\dot{u}\|_\infty\right].$$

To ensure that the constants C_{10}–C_{23} are independent of h, we put exactly the same restrictions on h and m as we did before in the proof of (a), namely we require $h \le \min\{h_1, h_2\}$ and $m > 1 + d/2$, where h_1 is as defined by (4.17) and h_2 by (4.19). We then note that all the constants C_{10}– C_{23} have the same components as the constants C_1– C_9 found before, and, as $\psi \in \mathcal{J}$, they can therefore be approximated in exactly the same way. We omit the details here.

We now put all the above into (4.23), to give

$$
\begin{aligned}
\|\dot\Theta\|^2 + (M_0 - 4\delta)\|\nabla\Theta\|^2 \le\ & \int_0^t \left[C_{24}\|\Theta\|^2 + C_{25}\|\dot\Theta\|^2 + C_{26}\|\nabla\Theta\|^2\right] \mathrm{d}\tau \\
& + \frac{C_{L3}^2}{\delta}\|\phi - \psi\|^2\|\nabla\xi_\psi\|_\infty^2 + \frac{C_{18}^2 C_{L4}^2}{\delta}\|\phi - \psi\|^2\|\xi_\psi\|_\infty^2 \\
& + \frac{C_{15}^2}{\delta}\|\nabla\phi - \nabla\psi\|^2\|\xi_\psi\|_\infty^2 \\
& + \int_0^t \Big[2\|\phi - \psi\|^2\|\xi_\psi\|_\infty^2 + \|\dot\phi - \dot\psi\|^2\|\xi_\psi\|_\infty^2 \\
& + \|\phi - \psi\|^2\|\dot\xi_\psi\|_\infty^2 + \|\phi - \psi\|^2\|\nabla\xi_\psi\|_\infty^2 \\
& + \|\phi - \psi\|^2\|\nabla\dot\xi_\psi\|_\infty^2 + \|\dot\phi - \dot\psi\|^2\|\nabla\xi_\psi\|_\infty^2 \\
& + \|\nabla\phi - \nabla\psi\|^2\|\dot\xi_\psi\|_\infty^2 + \|\nabla\phi - \nabla\psi\|^2\|\xi_\psi\|_\infty^2 \\
& + \|\nabla\dot\phi - \nabla\dot\psi\|^2\|\xi_\psi\|_\infty^2 + \|\eta + \xi_\psi\|_\infty^2\|\phi - \psi\|^2 \\
& + \|\eta + \xi_\psi\|_\infty^2\|\dot\phi - \dot\psi\|^2 + \|\dot\eta + \dot\xi_\psi\|_\infty^2\|\phi - \psi\|^2\Big] \mathrm{d}\tau,
\end{aligned}
\tag{4.24}
$$

where

$$
C_{24} = C_{11} + C_{17}^2,
$$

$$
C_{25} = C_{10} + C_{L1}^2 + C_{11} + C_{L2}^2 C_{12}^2 + C_{13}^2 + C_{16}^2 + \frac{C_{16}^2 T}{\delta},
$$

and

$$
\begin{aligned}
C_{26} =\ & C_{L3}^2 + C_{14}^2 C_{L4}^2 + 3C_{15}^2 + 2 + C_{23} + C_{18}^2 C_{L4}^2 + C_{19}^2 C_{L5}^2 \\
& + C_{20}^2 + C_{21} C_{L4}^2 + C_{22}^2.
\end{aligned}
$$

Looking now at the terms on the right–hand side of (4.24), we note that all but the first three terms there can be bounded by some constant multiplied by

$$
h^{2(m-d/2-1)} \max_{t\in[0,T]} \left[\|\dot\phi - \dot\psi\|^2 + \|\phi - \psi\|_{H^1(\Omega)}^2\right].
$$

For example,

$$\begin{aligned}
\frac{C_{L3}^2}{\delta}\|\phi-\psi\|^2\|\nabla\xi_\psi\|_\infty^2 &\leq \frac{C_{L3}}{\delta}\|\nabla\xi_\psi\|_\infty^2 \max_{t\in[0,T]}\|\phi-\psi\|^2 \\
&\leq \frac{C_{L3}C_{\mathrm{inv1}}^2C_{\mathrm{inv2}}^2}{\delta}h^{2(m-d/2-1)}C_*^2 \max_{t\in[0,T]}\|\phi-\psi\|^2,
\end{aligned}$$

$$\begin{aligned}
\int_0^t \|\nabla\dot{\phi}-\nabla\dot{\psi}\|^2\|\xi_\psi\|_\infty^2 \mathrm{d}\tau &\leq T\max_{t\in[0,T]}\|\xi_\psi\|_\infty^2 \max_{t\in[0,T]}\|\nabla\dot{\phi}-\nabla\dot{\psi}\|^2 \\
&\leq TC_{\mathrm{inv1}}^2C_{\mathrm{inv3}}^2h^{2(m-d/2-1)}C_*^2 \max_{t\in[0,T]}\|\dot{\phi}-\dot{\psi}\|^2,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t \|\eta+\xi_\psi\|_\infty^2\|\phi-\psi\|^2 \mathrm{d}\tau &\leq T\max_{t\in[0,T]}\|\eta+\xi_\psi\|_\infty^2 \max_{t\in[0,T]}\|\phi-\psi\|^2 \\
&\leq T\max_{t\in[0,T]}\left[\|u-\Pi_h^{nl}u\|_\infty+\|\Pi_h^{nl}u-u_\psi\|_\infty\right]^2 \max_{t\in[0,T]}\|\phi-\psi\|^2 \\
&\leq 2T\max_{t\in[0,T]}\left[\|u-\Pi_h^{nl}u\|_\infty^2+\|\Pi_h^{nl}u-u_\psi\|_\infty^2\right] \max_{t\in[0,T]}\|\phi-\psi\|^2 \\
&\leq 2T\Big[\left((C_{*2}+C_{\mathrm{inv1}}C_{*1})\,|u|_{H^m(\Omega)}+C_{\mathrm{inv1}}C_{\mathrm{proj1}}\right)^2 \\
&\quad +C_{\mathrm{inv1}}^2C_*^2\Big]\,h^{2(m-d/2)} \\
&\leq 2T\Big[\left((C_{*2}+C_{\mathrm{inv1}}C_{*1})\,|u|_{H^m(\Omega)}+C_{\mathrm{inv1}}C_{\mathrm{proj1}}\right)^2 \\
&\quad +C_{\mathrm{inv1}}^2C_*^2\Big]\,h^{2(m-d/2-1)},
\end{aligned}$$

where in each case we have used the fact that $u_\psi \in \mathcal{J}$.

Overall, then, we can write (4.24) as

$$\begin{aligned}
\|\dot{\Theta}\|^2+(M_0-4\delta)\,\|\nabla\Theta\|^2 \leq{}& \int_0^t \left[C_{24}\|\Theta\|^2+C_{25}\|\dot{\Theta}\|^2+C_{26}\|\nabla\Theta\|^2\right]\mathrm{d}\tau \\
&+C_{27}h^{2(m-d/2-1)}\left[\|\dot{\phi}-\dot{\psi}\|^2+\|\phi-\psi\|_{H^1(\Omega)}^2\right],
\end{aligned}$$

where C_{27} is a known constant such that

$$C_{27}=C_{27}\,(\delta, C_{15}, C_{18}, C_{L3}, C_{L4}, C_*, C_{\mathrm{proj}}, C_{\mathrm{inv}}, T)\,.$$

Choosing $\delta < M_0/4$ and using Poincaré's inequality (2.1) and Gronwall's inequality gives

$$\|\dot{\Theta}\|^2 + \|\Theta\|^2_{H^1(\Omega)} \leq C_{28} h^{2(m-d/2-1)} \max_{t\in[0,T]} \left[\|\dot{\phi} - \dot{\psi}\|^2 + \|\phi - \psi\|^2_{H^1(\Omega)} \right],$$

where

$$C_{28} = C_{27} C_{29} \exp\left(C_{30} T\right),$$

for

$$C_{29} = \frac{1}{\min\left\{1, \dfrac{M_0 - 4\delta}{C_{\text{poin}}}\right\}}$$

and

$$C_{30} = C_{29} \max\{C_{24}, C_{25}, C_{26}\}.$$

Finally, therefore, we take $h < h_3$ where h_3 is defined by

$$h_3 := \left(\frac{1}{C_{28}}\right)^{\alpha_0},$$

where

$$\alpha_0 = \frac{1}{2\left(m - d/2 - 1\right)};$$

this is, in fact, where our requirement that $m > d/2+1$ rather than $m \geq d/2+1$ (which would have sufficed until now) comes in.
This restriction on h gives the required result, namely that

$$\max_{t\in[0,T]} \left(\|\dot{\Theta}\left(t\right)\| + \|\Theta\left(t\right)\|_{H^1(\Omega)} \right) \leq \alpha \max_{t\in[0,T]} \left(\|\dot{\phi}\left(t\right) - \dot{\psi}\left(t\right)\| + \|\phi\left(t\right) - \psi\left(t\right)\|_{H^1(\Omega)} \right),$$

where

$$\alpha = \left(\frac{h}{h_3}\right)^{2(m-d/2-1)} < 1.$$

Hence, for $h < h_0 := \min\{h_1, h_2, h_3\}$, we see that (a) and (b) hold. Thus, by Banach's Fixed Point Theorem, there exists a unique $u_h \in \mathcal{J}$ such that $\mathcal{N}\left(u_h\right) = u_h$.

This establishes the existence of a unique solution to (2.8)–(2.9). To obtain the $L_2\left(\Omega\right)$ error bound of the form of (4.2), we proceed as follows:

Since $u_h \in \mathcal{J}$, then, from (3.2),

$$\max_{t\in[0,T]} \|u_h - \Pi_h^{nl} u\| \leq C_*\left(u\right) h^m,$$

for $m > d/2+1$ and $h < h_0$, where C_* and h_0 were defined above. Therefore,

$$\begin{aligned} \max_{t\in[0,T]} \|u - u_h\| &\leq \max_{t\in[0,T]} \|u - \Pi_h^{nl} u\| + \max_{t\in[0,T]} \|\Pi_h^{nl} u - u_h\| \\ &\leq C_{\text{proj1}}\left(u\right) h^m + C_*\left(u\right) h^m, \end{aligned}$$

from (3.4). This completes the proof of Theorem 3, by setting $\widetilde{C}_*(u) = C_{\text{proj1}}(u) + C_*(u)$. ∎

This completes the work of this paper, which has been two–fold. Firstly, we have given an existence and uniqueness result for the solution to the semi–discrete analogue of the second–order nonlinear wave equation, as given by (2.8) and (2.9). Secondly, we have derived an optimal *a priori* error estimate, of the form

$$\max_{t \in [0,T]} \| u(t) - u_h(t) \| \leq C(u)\, h^m,$$

under certain restrictions on h and m that have been detailed above. The extension to the fully–discrete approximation to the nonlinear damped wave equation would follow along the same lines as those of Makridakis, who in [7] studies single–step fully–discrete methods that have temporal accuracy of order 2, 3 or 4.

References

[1] R.A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[2] S.C. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods*, volume 15 of *Texts in Applied Mathematics.* Springer-Verlag, 1994.

[3] F. Bloom. *Mathematical Problems of Classical Nonlinear Electromagnetic Theory.* Longman Scientific & Technical, 1993.

[4] V.C. Chen and W. von Wahl. Das Rand-Anfangswertproblem für Quasilineare Wellengleichungen in Sobolevräumen Niedriger Ordnung. *J. Reine Angew. Math.*, 337:77–112, 1982.

[5] P.G. Ciarlet. *The Finite Element Method for Elliptic Problems.* North Holland, Amsterdam, 1978.

[6] M. Dobrowolski and R. Rannacher. Finite element methods for nonlinear elliptic systems of second order. *Math. Nachr.*, 94:155–172, 1980.

[7] C.G. Makridakis. Finite element approximations of nonlinear elastic waves. *Mathematics of Computation*, 61(204):569–594, 1993.

[8] R. Rannacher. On finite element approximation of general boundary value problems in non-linear elasticity. *Calcolo*, 17:175–193, 1980.

[9] W. Rudin. *Functional Analysis.* McGraw–Hill, 1973.