

The Internal Structure of Irreducible Continua

With a Focus on Local Connectedness and Monotone Maps



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Abstract

This thesis is an examination of the structure of irreducible continua, with a particular emphasis on local connectedness and monotone maps. A continuum is irreducible if there exist a pair of points such that no proper subcontinuum contains both, with the arc being the most basic example. Being irreducible has a number of interesting implications for a continuum, both locally and globally, and it is these consequences we shall focus on.

As mentioned above, the arc is the most straightforward example of an irreducible continuum. Indeed, an intuitive understanding of an irreducible continuum would be that it is structured like an arc, with the points of irreducibility at either end joined by a subspace with no loops or offshoots. In Chapter 2 we will see that for a certain class of continua this intuition is well founded by constructing a monotone map from an irreducible continuum onto an arc. This monotone map will preserve much of the structure of our continuum and as such will provide an insight into that structure.

We will next examine a generalisation of irreducibility which considers finite sets of points rather than just pairs. A number of classical results will be re-examined in this light in Chapter 3. While the majority of these theorems will be shown to have close parallels in higher finite and infinite irreducibility there will be several which do not hold without further conditions on the continuum. Such anomalies will be particularly prevalent in continua which have indecomposable subcontinua dominating their structure. In Chapter 4 monotone maps will be constructed for finitely irreducible continua similar to the map to an arc mentioned previously. Chapters 7 and 8 will generalise irreducibility further to the infinite case and we will again construct monotone maps preserving the structure of our continuum.

Along with the arc, another highly significant irreducible continuum is the $\sin \frac{1}{x}$ continuum. Chapter 5 will focus on this continuum, which will be the basis for a nested sequence of continua. A number of results concerning continuous images of these continua will be presented before using the sequence of continua to define an indecomposable continuum. This continuum will be investigated, and it will be shown that the union of our nested continua form a composant of the indecomposable continuum.

In Chapter 6 we will turn to the question of compactifications. If a space X is connected then any metric compactification of X will be a continuum. This chapter will answer the question of when a compactification is an irreducible continuum, with the remainder of the compactification consisting of all of the irreducible points. A list of properties will be given such that a continuum has such a compactification if and only if it has each property on the list. It will also be demonstrated that each of these properties is independent of the others.

Finally, in Chapter 9 we will revisit the idea of structure-preserving monotone maps, but this time in continua which are not irreducible. Motivated by the fibres of the maps in previous chapters, we will introduce two categories of subcontinua of a continuum X . The first will be nowhere dense subcontinua which are maximal with this property and the second will be subcontinua about which X is locally connected and which are minimal with this property. Continua in which every point lies in a maximal nowhere dense subcontinuum will be examined, as well as spaces in which every point lies in a unique minimal subcontinuum about which X is locally connected. We will also look at the properties of monotone maps arising from partitions of X into such subcontinua, and will prove that if every point of X lies in a maximal nowhere dense subcontinuum then the resulting quotient space will be one dimensional.

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Chapter 1

Introduction

We will begin this thesis by studying a range of classic results in the field of continuum theory and irreducibility. These results will give an overview of the subject and should help familiarise the reader with a number of underlying concepts integral to the thesis. Indeed, a number of the theorems presented here are so fundamental to the study of irreducibility that later chapters would be unrecognisable without them, in particular Theorem 1.1.7. These results will focus mainly on points of irreducibility and their composants. The notation that will be used throughout this thesis will also be laid out in this section.

In their paper, [BF77], Bennett and Fugate gave three definitions of subcontinua which could be thought of as lying at the end of a decomposable continuum. All three of these, but particularly end continua and E-continua, are abstractions of the notion of a point of irreducibility to apply to subcontinua. Studying the irreducible points of a continuum can provide valuable insight into the whole space, as well as maps between continua, and the same is true of these analogous subcontinua. In this section a number of definitions and classic theorems will be set out which explore the interactions between these various types of subcontinua and their properties.

We will then briefly discuss monotone maps and quotient maps on continua. Monotone maps will be central to much of the thesis and this section will include some highly useful results that highlight the link between irreducibility and monotone maps which motivates so much of the later chapters. Quotient maps are similarly important as they can be used to reduce the complexity of a continuum while preserving relevant properties and indeed the construction of a quotient map is the main focus of Chapters 2, 3 and 4.

We will end this chapter by examining which pairs of connected topological spaces can occur as the composant complements of a continuum. It will be proved that there

are no restrictions on continua, so any pair of continua can exist as E-continua of a continuum.

1.1 Basic Definitions and Results

A continuum will be defined as a compact connected metric space. A decomposition of a continuum X is a pair of proper subcontinua A, B such that $X = A \cup B$. A continuum is decomposable if a decomposition of it exists, and indecomposable otherwise. It is irreducible if there exist a pair of points $p, q \in X$ such that no proper subcontinuum of X contains both points, in which case we write $X = \text{irr}(p, q)$. Both p and q are called points of irreducibility of X . If no such pair of points exist then X is called reducible. A Peano continuum is a continuum which is locally connected.

Definition 1.1.1. The composant of a point $x \in X$ is the set

$$\kappa(x) := \{y \in X \mid X \neq \text{irr}(x, y)\}$$

An equivalent definition is that $\kappa(x)$ is the union of all proper subcontinua containing x . It is often the case that lots of points in X have the same composants, and as such we will usually talk about the composants of a continuum. The composants of a continuum are the set of all distinct composants of points in the continuum.

The complements of composants are as important as the composants themselves, particularly in decomposable continua. As such we will use the following notation.

Definition 1.1.2. For a non-degenerate continuum X with $X = \text{irr}(p, q)$ let

$$\lambda(p) = X \setminus \kappa(q)$$

and let

$$\lambda(X) = \{X \setminus \kappa(x) \mid x \in X, \kappa(x) \neq X\}$$

be the set of non-trivial composant complements of X .

Our focus will be on the complements of composants in decomposable spaces. The usual notation for a decomposition of a space X is to label the subcontinua as A and B . We will later see that an irreducible, decomposable continuum has exactly two non-empty composant complements, and in such a decomposition one will lie in $A \setminus B$ and one will lie in $B \setminus A$. Thus, they will usually be denoted by a and b , even though they are subsets and not points in X .

Theorem 1.1.3. [Nad92, 5.4] *Let X be a continuum and let U be a nonempty, proper, open subset of X . If K is a component of $\overline{U}$, then $K \cap \partial U \neq \emptyset$. Equivalently, $K \cap (X \setminus U) \neq \emptyset$.*

Theorem 1.1.4. [Nad92, 5.20] *For all continua X and points $x \in X$ the composant $\kappa(x)$ is connected and dense.*

Proof. Since it is a union of connected sets which all intersect at x , $\kappa(x)$ must be connected. Its closure is therefore a subcontinuum of X . Suppose that $\kappa(x)$ is not dense, so there exists $y \in X \setminus \overline{\kappa(x)}$. Since X is regular there exists an open set U with

$$\overline{\kappa(x)} \subseteq U \subseteq \overline{U} \subseteq X \setminus \{y\}$$

Let K be the component of $\overline{\kappa(x)}$ in $\overline{U}$. By Theorem 1.1.3 we have that $K \setminus U$ is non-empty, so $K \neq \overline{\kappa(x)}$. As y is not a point of K we have that K is a proper subcontinuum of X containing x but is not contained in the composant of x . This is a contradiction, and as such $\kappa(x)$ must be dense. $\square$

Theorem 1.1.5. [Nad92, 11.4] *For all continua X and $x \in X$, $X \setminus \kappa(x)$ is connected.*

While we know that the complement of a composant is connected, it may not be a continuum. Theorems 1.1.7 and 1.1.8 will give us an easy counterexample, but first a theorem giving a sufficient condition for compact complements of composants.

Theorem 1.1.6. [Nad92, 11.40] *If X is a hereditarily decomposable continuum and $x \in X$ is a point of irreducibility then $\lambda(x)$ is compact, and therefore a continuum.*

Proof. Suppose $X = \text{irr}(x, x')$ and consider $\overline{\lambda(x)}$. This is a subcontinuum of X so it is decomposable. If it is a singleton then the same must be true of $\lambda(x)$, so $\lambda(x)$ is compact. Otherwise let $\overline{\lambda(x)} = A \cup B$ be a decomposition. As both A and B are proper subcontinua of $\overline{\lambda(x)}$ it must be that both intersect $\lambda(x)$ non-trivially. This is because if, say, $A \cap \lambda(x) = \emptyset$ then $\lambda(x) \subseteq B$ and the same is true of the closure. If $\lambda(x) \neq \overline{\lambda(x)}$ then there exists a point $y \in \overline{\lambda(x)} \cap \kappa(x')$. Suppose without loss of generality that $y \in A$. Then there exists a proper subcontinuum $X' \subsetneq X$ which contains y and x' , so $X' \cup A$ is a proper subcontinuum of X which intersects $\lambda(x)$ and contains x' , which is a contradiction. Thus $\lambda(x) = \overline{\lambda(x)}$, meaning it is compact. $\square$

The next theorem is a fundamental result in continuum theory, which will inform much of what is to come. It categorises continua based on the number of composants, and the subsequent theorems add more detail to what these composants will be.

Theorem 1.1.7. [Nad92, 11.6] *A non-degenerate, decomposable, reducible continuum has precisely one composant. A non-degenerate, decomposable, irreducible continuum has precisely three composants. A non-degenerate indecomposable continuum has uncountably many composants.*

For a decomposable, irreducible continuum the three composants will all intersect each other. For example, the unit interval $[0, 1]$ is a decomposable continuum irreducible between 0 and 1. The composant of 0 is $[0, 1)$ and the composant of 1 is $(0, 1]$. The composant of any other point is the whole of $[0, 1]$.

Theorem 1.1.8. [Nad92, 11.17] *Let X be a continuum. If X is non-degenerate and indecomposable then the composants of X are mutually disjoint.*

Proof. Take two intersecting composants, $K = \kappa(p)$ and $L = \kappa(q)$, with $z \in K \cap L$. Taking $x \in K$ we will show that $x \in L$ to give $K \subseteq L$ and by symmetry $K = L$. Let A, B and C be proper subcontinua of X with $p, x \in A, p, z \in B, q, z \in C$. Let $D = A \cup B \cup C$. Then D is a continuum, and as it is clearly decomposable we have that D is a proper subcontinuum of X . Thus $x \in L$, which completes the proof. $\square$

Example 1.1.9. Using Theorems 1.1.7 and 1.1.8 it is easy to see that if X is irreducible and indecomposable then the complement of a composant is not a continuum. Let $K = X \setminus \kappa(x)$ for some x . Since the composants are mutually disjoint, K contains each composant except $\kappa(x)$ which means K is dense. Since it is a proper subset of X , K cannot be closed and is therefore not compact and not a continuum.

Theorem 1.1.10. [Nad92, 11.2] *Let X be a continuum. If $x \in X$ and X is not irreducible between x and some other point then $\kappa(x) = X$.*

Proof. Let $y \in X$. Since $X \neq \text{irr}(x, y)$ we know that there is a proper subcontinuum $Y \subsetneq X$ containing x and y . Thus $y \in \kappa(x)$. $\square$

Theorem 1.1.11. [Nad92, 11.10] *Let X be a continuum. If X is decomposable then there exists $x \in X$ such that $\kappa(x) = X$.*

Proof. Let $X = A \cup B$ for proper subcontinua A and B . Since X is connected $A \cap B \neq \emptyset$. Take a point x in this intersection. X is not irreducible between x and any other point, as witnessed by A and B , so $\kappa(x) = X$. $\square$

Lemma 1.1.12. *Let X be an irreducible, decomposable continuum with $\lambda(X) = \{a, b\}$. Then $\bar{a} \neq X \neq \bar{b}$.*

Proof. Take a decomposition $X = A \cup B$. Since A and B are both proper subcontinua of X it cannot be that either one of them intersects both a and b . Therefore without loss of generality we can assume that $a \subseteq A$ and $b \subseteq B$. Since A is a subcontinuum it is a closed subset of X , which implies that $\bar{a} \subseteq A$. Since it forms part of a decomposition it cannot be that $A = X$, which proves a is not a dense subset of X , and an identical proof shows the same is true of b . $\square$

Theorem 1.1.13. *Let X be a non-degenerate irreducible continuum. Let $p, q \in X$ such that $X = \text{irr}(p, q)$, which implies $X \neq \kappa(p) \neq \kappa(q) \neq X$. If $x \in \lambda(p), y \in \lambda(q)$ then $X = \text{irr}(x, y)$.*

Proof. Suppose $C \subseteq X$ is a subcontinuum containing x, y . If either p or q is in C then by the definition of composants $C = X$. Let $D = \overline{\lambda(p)}$. This is a subcontinuum, and as $x \in C \cap D$, $C \cup D$ is a subcontinuum. We have $p, y \in C \cup D$ so $C \cup D = X$. Applying Lemma 1.1.12 gives us that D is a proper subcontinuum so $q \notin D$. This implies $q \in C$, which proves $C = X$ and $X = \text{irr}(x, y)$. $\square$

1.2 Continua Lying at the Edge

We will now turn to definitions of various subcontinua which lie, in a way, at either end of an irreducible decomposable continuum, along with a number of theorems connecting them. Of course lying at the end does not have a strict definition, but you would expect a subcontinuum described like this to be non-separating (its complement is connected) and to contain some points of irreducibility if any exist. The subcontinua we will look at do indeed meet these criteria. The three types of subcontinua have progressively stricter conditions, so all end continua are terminal continua and all E-continua are end continua. Naturally more can be said about E-continua than about end or terminal continua. However, E-continua do not always exist, whereas every irreducible decomposable continuum has end and terminal subcontinua. It is also often easier to prove a subcontinuum is terminal or an end continuum, and then use some of the theorems below to show it is an E-continuum than to prove this from the definition of an E-continuum. Proof outlines to some of the theorems are included for the sake of completeness.

Definition 1.2.1. If X is a continuum and K is a subcontinuum, we call K a terminal subcontinuum provided that for all decompositions $X = A \cup B$ if $A \cap K \neq \emptyset \neq B \cap K$ then $X = A \cup K$ or $X = B \cup K$.

Example 1.2.2. The subarc $[0, x] \subseteq [0, 1]$ or two legs of a triod are examples of terminal continua. Another would be any subcontinuum of the $\sin(\frac{1}{x})$ continuum which intersects $\{0\} \times [0, 1]$. It is easy to see from this definition that for any continuum X , X is a terminal subcontinuum of itself.

Theorem 1.2.3. [BF77, 1.5] *Let X be a continuum. If $K \subseteq X$ is a terminal subcontinuum and L is a subcontinuum of X containing K then L is also a terminal subcontinuum.*

Proof. Let $X = A \cup B$ be a decomposition of X with $A \cap L \neq \emptyset \neq B \cap L$. Let $A' = A \cup L, B' = B \cup L$. Then A' and B' are continua, and $X = A' \cup B'$. Now $A' \cap K = K = B' \cap K$ since $K \subseteq L$. Because K is terminal there cannot be a decomposition of X with K a subset of both subcontinua, so it must be that one of A' or B' is equal to X . Thus L is a terminal subcontinuum. $\square$

The following two theorems best show how a terminal subcontinuum can be thought of as being at the end of the continuum it is contained in. First however will be a proposition which will be used both in the proof of Theorem 1.2.5 and throughout the thesis.

Proposition 1.2.4. [Nad92, 6.3] *Let T be a connected topological space and let C be a connected subset of T such that $T \setminus C$ is disconnected and can be expressed as $T \setminus C = A \cup B$. Then $A \cup C$ and $B \cup C$ are connected. Hence, if T and C are continua, $A \cup C$ and $B \cup C$ are continua.*

Theorem 1.2.5. [BF77, 1.3] *Let X be a continuum and K a subcontinuum. If K is a terminal subcontinuum then it is non-separating in X , i.e. $X \setminus K$ is connected.*

Proof. Suppose $X \setminus K$ is not connected. Let U, V be disjoint open subsets of X such that $X \setminus K = U \cup V$. Then $K \cup U$ and $K \cup V$ are proper subcontinua of X by Proposition 1.2.4. These subcontinua form a decomposition, but this decomposition contradicts the fact that K is terminal. $\square$

Theorem 1.2.6. [BF77, 1.8] *Let X be a decomposable, irreducible continuum and K a subcontinuum. Then K is terminal if and only if there exists p in K and q in X such that $X = \text{irr}(p, q)$.*

Proof. For the forwards direction, let K be terminal and $X = \text{irr}(p, q)$ for some points $p, q \in X$. Suppose neither p nor q were in K , and X is not irreducible between p or q and a point of K . Let $k \in K$. There exist proper subcontinua A, B of X such that

$p, k \in A$ and $q, k \in B$. As X is irreducible between p and q we have that $A \cup B$ is a decomposition of X . However, $p \notin B \cup K$ and $q \notin A \cup K$, which is a contradiction as K is terminal.

For the other direction, suppose $X = \text{irr}(p, q)$ and $p \in K$. Take a decomposition $X = A \cup B$ with both A and B intersecting $\{p\}$, so $p \in A$ and $p \in B$. Without loss of generality let $q \in A$. Then by irreducibility of X we have that $A = X$ and in particular $A \cup \{p\} = X$. This means that $\{p\}$ is a terminal subcontinuum of X . As $\{p\} \subseteq K$ we can apply Theorem 1.2.3 to see that K is terminal. $\square$

Definition 1.2.7. If X is a continuum and K is a subcontinuum then K is called an end continuum providing there is no decomposition $X = A \cup B$ such that $A \cap K \neq \emptyset \neq B \cap K$.

Example 1.2.8. Both $\{0\}$ and $\{1\}$ are end continua of $[0, 1]$. Any subcontinuum of $\{0\} \times [0, 1]$ in the $\sin \frac{1}{x}$ continuum is an end continuum.

Terminal and end continua are related through the following theorem.

Theorem 1.2.9. [BF77, 1.11] *If X is a continuum and K a subcontinuum then K is an end continuum if and only if K is a terminal continuum with empty interior.*

Proof. Suppose K is an end continuum. It is clear from the definitions that K is also a terminal continuum. Thus $X \setminus K$ is connected by Theorem 1.2.5, and $L = \overline{X \setminus K}$ is a subcontinuum of X . We have that $X = K \cup L$ and that K intersects L by the connectedness of X . As K is an end continuum it cannot be that K and L form a decomposition, which means $L = X$ and therefore K has empty interior.

Now suppose K is a terminal continuum with empty interior. Let $X = A \cup B$ for subcontinua A, B . Suppose $A \cap K \neq \emptyset \neq B \cap K$. We will show that one of A, B must equal X , which implies there are no decompositions of X where both subcontinua intersect K and that K is an end continuum. As K is terminal we can say without loss of generality that $A \cup K = X$. Then $X \setminus A$ is an open set contained in K , but K has empty interior. Thus $A = X$, which completes the proof. $\square$

Theorem 1.2.10. [BF77, 1.16] *Let X be a continuum. A subcontinuum $K \subseteq X$ is an end continuum if and only if there exists $p \in X$ such that for all $q \in K$, $X = \text{irr}(p, q)$.*

Finally, we introduce the E-continuum, which will be central to the rest of the thesis.

Definition 1.2.11. If X is a non-degenerate continuum and K a subcontinuum then K is called an E-continuum if there exists $x \in X$ such that $K = X \setminus \kappa(x)$. Equivalently, if $\lambda(X) = \{a, b\}$ then a and/or b are E-continua if and only if they are compact.

Note that K must be a subcontinuum, not just the complement of a composant. There exist decomposable, irreducible continua with composants whose complements are not compact.

Example 1.2.12. Let Y be any indecomposable continuum, and let X be the continuum formed by taking a quotient of two disjoint copies of Y , Y_1 and Y_2 , which identifies a single point $y \in Y_1$ to the corresponding point in Y_2 , and has no other non-trivial equivalence classes. For any $x \in Y_i$, let $\kappa_i(x)$ be the composant of x in Y_i and $\kappa(x)$ be the composant of x in X . If $x \in Y_1$ and $\kappa_1(x) \neq \kappa_1(y)$ then $\kappa(x) = Y_1 \cup \kappa_2(y)$. Then $X \setminus \kappa(x) = Y_2 \setminus \kappa_2(y)$, which is not compact as we saw in Example 1.1.9. The continuum X is clearly decomposable as $X = Y_1 \cup Y_2$.

The link between E-continua and end continua can be seen in the following two theorems, but first we will need a definition and a lemma. The lemma will be important throughout this thesis as it highlights a link between indecomposable subcontinua and the existence of E-continua, which will come up a number of times.

Definition 1.2.13. A topological space (T, τ) is locally connected at a subspace A if for all $U \in \tau$ with $A \subseteq U$, there exists $V \in \tau$ such that $A \subseteq V \subseteq U$ and V is connected.

Lemma 1.2.14. [BF77, 1.22] *Let X be a continuum and suppose $X = \text{irr}(p, q)$. Then the following are equivalent:*

1. $\lambda(p)$ is a continuum.
2. $\lambda(p)$ is continuumwise connected.
3. For all subcontinua $K \subseteq X$ if $K \cap \kappa(q) \neq \emptyset \neq K \cap \lambda(p)$ then K is decomposable.

Theorem 1.2.15. [BF77, 1.25] *Let X be a continuum with $X = \text{irr}(p, q)$. A subcontinuum $K \subseteq X$ is an E-continuum if and only if it is a maximal end continuum i.e. it is an end continuum and is not properly contained in any other end continuum.*

Proof. Let K be an E-continuum, $K = \lambda(p)$. From Theorem 1.2.10 it is clear that K is an end continuum. Let L be a subcontinuum of X which properly contains K . As $L \cap \kappa(q) \neq \emptyset$ there is a proper subcontinuum $X' \subseteq X$ such that $q \in X'$ and $X' \cap L \neq \emptyset$. Then $X = L \cup X'$ since $q \in L \cup X'$ and this subcontinuum is not contained in $\kappa(q)$. This means that L does not have an empty interior as $X \setminus X' \subseteq L$, so by Theorem 1.2.9 it is not an end continuum. Thus K is a maximal end continuum.

Now let K be a maximal end continuum. By Theorem 1.2.10 there is a point $x \in X$ such that X is irreducible between x and every point of K , so $K \subseteq X \setminus \kappa(x)$. If K is not an E-continuum then $X \setminus \kappa(x)$ cannot be an E-continuum by maximality of K . Thus by Lemma 1.2.14 there exists an indecomposable subcontinuum L with $L \cap \kappa(x) \neq \emptyset \neq L \setminus \kappa(x)$. In fact $X \setminus \kappa(x) \subseteq L$ so $K \subsetneq L$. As it is a proper subcontinuum, K must be contained in a composant of L . The composants of L are not compact as they are all disjoint (Theorem 1.1.8) and dense (Theorem 1.1.4). Thus there exists a point y in the composant and a proper subcontinuum $Y \subseteq L$ containing y and intersecting K . Let $M = K \cup Y$, so M is a proper subcontinuum of L which properly contains K . Since L is indecomposable the composants of L have empty interior, so M has empty interior in L and therefore in X . K is a terminal subcontinuum so M is as well by Theorem 1.2.3, and since it has empty interior Theorem 1.2.9 tells us that M is an end continuum, which contradicts the maximality of K . Thus K must be an E-continuum of X . $\square$

Theorem 1.2.16. [BF77, 1.30] *The E-continua of X are precisely the end continua at which X is locally connected.*

Proof. Suppose K is an end continuum of X and X is locally connected at K . As K is an end continuum we can apply Theorem 1.2.10 to see that there exists a point $x \in X$ with $K \subseteq X \setminus \kappa(x)$. Let $y \in X \setminus K$. Let U be open, $K \subseteq U \subseteq \overline{U} \subseteq X \setminus \{x, y\}$. Without loss of generality we can assume U is connected. By Theorems 1.2.9 and 1.2.3 we know that $\overline{U}$ is a terminal continuum of X , so by Theorem 1.2.5 we know that $V = X \setminus \overline{U}$ is connected. The closure of V is a subcontinuum contained in $X \setminus K$ which contains x, y so $y \in \kappa(x)$. From this we can deduce that $\kappa(x) = X \setminus K$ and that K is an E-continuum.

Now suppose K is an E-continuum. By Theorem 1.2.15 we know that K is an end continuum of X . Let $X = \text{irr}(p, q)$ and $K = \lambda(p)$. Let U be an open subset of X containing K . Let V be an open subset such that $K \subseteq V \subseteq \overline{V} \subseteq U$ and $q \notin \overline{V}$. As K is connected it must lie in a component of V . Let L be the closure of this component. Since $K \subseteq L$ Theorem 1.2.3 tells us that L is a terminal continuum, so by Theorem

1.2.5 $\overline{X \setminus L}$ is a subcontinuum which contains q . It is terminal as $\{q\}$ is terminal (Theorem 1.2.3). Thus $X \setminus \overline{X \setminus L}$ is connected, and $K \subseteq X \setminus \overline{X \setminus L} \subseteq \overline{V} \subseteq U$. This completes the proof. $\square$

The previous theorem, and the definition of being locally connected at a subset, are very different from saying that X is locally connected at every point of the subset. The next theorem demonstrates this.

Theorem 1.2.17. *If X is a decomposable continuum and $a \in \lambda(X)$ is non-degenerate then X is not locally connected at any point in a .*

Proof. Suppose there exists $x \in a$ such that X is locally connected at x . Let y be any other point of a . As X is Hausdorff, we can find disjoint open sets U, V in X such that $x \in U, y \in V$. Since X is locally connected at x , we can assume that U is connected. $X \setminus a$ is a composant of x , so it is dense and therefore intersects U . Let b be the other element of $\lambda(X)$. There exists a proper subcontinuum X' of X which intersects both b and U . $\overline{U}$ is a continuum as U is connected, and $y \notin \overline{U} \cup X'$. Thus we have a proper subcontinuum $\overline{U} \cup X'$ of X which intersects both a (at x) and b . This is a contradiction, which gives us our result. $\square$

Remark 1.2.18. The set a is not necessarily compact in the previous theorem.

1.3 Monotone Maps and Quotient Maps

This thesis will focus extensively on monotone maps and quotient maps and often on monotone quotient maps. The following theorems will be used extensively throughout the thesis with Theorem 1.3.3 in particular being a strong motivating factor for our interest in monotone maps.

Definition 1.3.1. Let S and T be topological spaces and let $f : S \rightarrow T$ be a continuous map. Then f is called monotone if for each point t in T we have that $f^{-1}(t)$ is connected.

Proposition 1.3.2. *Let X and Y be continua and let $\rho : X \rightarrow Y$ be a monotone map. Then for each connected subset C of Y we have that $\rho^{-1}(C)$ is also connected.*

Proof. Let C be a connected subset of Y and express $C' = \rho^{-1}(C)$ as $U \cup V$ for U and V disjoint C' -clopen subsets. For each point $c \in C$ we have that $\rho^{-1}(c)$ is a connected subset of C' so must lie entirely in U or entirely in V . Since X and Y are

compact and Hausdorff we have that $\rho(\overline{U}) = \overline{\rho(U)}$, with the same true for V . Thus $C = \rho(U) \cup \rho(V)$ with these sets disjoint and

$$\overline{\rho(U)}^C = \overline{\rho(U)}^Y \cap C = \rho(\overline{U}^X) \cap \rho(C') = \rho(\overline{U}^X \cap C') = \rho(U)$$

This gives us that $\rho(U)$ is C -closed, and the same argument applies to $\rho(V)$. Since C is connected, one of $\rho(U)$ or $\rho(V)$ must be empty, meaning one of U or V is empty. Hence $\rho^{-1}(C)$ is connected. $\square$

Theorem 1.3.3. *Let X and Y be non-degenerate continua, let $\rho : X \rightarrow Y$ be a surjective monotone map and suppose X is irreducible. Then Y is also irreducible. Further if p and q are points of X with $X = \text{irr}(p, q)$ then $Y = \text{irr}(\rho(p), \rho(q))$.*

Proof. Let $Y' \subseteq Y$ be a subcontinuum of Y containing $\rho(p)$ and $\rho(q)$. By applying Proposition 1.3.2 we have that $X' = \rho^{-1}(Y')$ is connected so it must be a subcontinuum of X . This subcontinuum contains p and q so by the irreducibility of X we have that $X' = X$. Now $Y' = \rho(X')$ and ρ is surjective, so $Y' = Y$. Thus $Y = \text{irr}(\rho(p), \rho(q))$. $\square$

Theorem 1.3.4. *[Nad92, 3.4] Let X be a continuum and let $\mathcal{C}$ be a partition of X . The quotient space induced from this partition is a continuum if and only if it is Hausdorff.*

1.4 Constructing Continua with given Composant Complements

This section will present several theorems concerning which spaces can appear in the set $\lambda(X)$ for a decomposable, irreducible continuum X . By Theorem 1.1.7, the set will consist of two connected subspaces of X , but other than that we do not yet have much information about them, or whether certain combinations are possible. The following theorems will provide answers to these questions, with Theorem 1.4.3 settling the question of combinations and Theorem 1.4.4 giving us that every continuum can appear as an E-continuum.

Theorem 1.4.1. *[Nad92, 3.20] Let X and Y be disjoint continua, let A be a subset of X and let $f : A \rightarrow Y$ be a continuous map. Let Z be the quotient space of $X \cup Y$ in which every point $x \in A$ is identified with its image $f(x)$ in Y . Then Z is a continuum.*

Proposition 1.4.2. *Let X and Y be disjoint irreducible, decomposable continua and let $\lambda(X) = \{a, b\}$ and $\lambda(Y) = \{b', c\}$, with b and b' homeomorphic. Suppose b is compact. Then there exists a decomposable continuum Z such that $\lambda(Z) = \{a, c\}$.*

Proof. Define Z as the quotient of $X \cup Y$ where the points in $b \subseteq X$ are identified with the corresponding point in $b \subseteq Y$. Then Z is a continuum by Theorem 1.4.1, and it is clearly decomposable, for example into the images of X and Y . Let $x \in a$ and $y \in c$. Let $Z' \subseteq Z$ be a continuum containing x and y . If $Z' \cap b = \emptyset$ then $Z' \cap X = Z' \cap (X \setminus b)$ which is therefore a clopen subset of Z' . Since Z' is connected this cannot happen, thus $Z' \cap b \neq \emptyset$.

Now consider the quotient space obtained from Z by mapping Y to a point, and let q be the quotient map. As the map is continuous and the image is clearly Hausdorff we can apply Theorem 1.3.4 to see that the image of Z is a continuum. Let $r : X \rightarrow q(Z)$ be the restriction of q to X , taking $b = X \cap Y$ to a point. Since X is irreducible and r is monotone we can apply Theorem 1.3.3 to show that $r(X)$ is irreducible between any point of a and the image of b . Note that $r(X) = q(Z)$. Since $q(Z')$ is a subcontinuum containing x and the image of b it must be the whole of $r(X)$. Thus $Z' \cap X \supseteq X \setminus b$, and as this is dense in X and $Z \cap X$ is closed, $X = Z' \cap X$. Similarly, $Z' \cap Y = Y$ so $Z' = Z$. From this we can see that $\kappa(x) \subseteq Z \setminus c$.

For the other inclusion, let $z \in Z \setminus c$. If $z \in X$ then X is a proper subcontinuum of Z containing x and z . Otherwise let $z' \in b$, and let Y' be any proper subcontinuum of Y containing z and z' . Then $X \cup Y'$ is a proper subcontinuum of Z containing x and z . Thus we have that $\kappa(x) = Z \setminus c$. By symmetry we can see that $\lambda(Z) = \{a, c\}$. $\square$

Theorem 1.4.3. *Suppose for topological spaces a, b there exists decomposable, irreducible continua X, Y such that $a \in \lambda(X), b \in \lambda(Y)$. Then there exists a decomposable continuum Z such that $\lambda(Z) = \{a, b\}$*

Proof. Let $X = \kappa(x) \cup a$. Consider the continuum X' created by attaching a unit interval $\mathbb{I}$ to X by identifying 0 and x . Then we claim that $\lambda(X') = \{a, 1\}$. Clearly X' is decomposable. If A is a subcontinuum containing 1 and intersecting a then $A \cap \mathbb{I}$ is a subcontinuum of $\mathbb{I}$ containing $0, 1$ so equals $\mathbb{I}$. Similarly $A \cap X = X$, so we have that $A = X'$. Thus X' is irreducible between 1 and any point of a .

If $x' \in X' \setminus a$ then there are two options. If $x' \in \mathbb{I}$ then $\mathbb{I}$ is a proper subcontinuum of X' containing $x', 1$. If $x' \in X$ then pick any proper subcontinuum of X containing x', x and take the union of this with $\mathbb{I}$ to get a proper subcontinuum of X' containing $x', 1$. Thus $\kappa(1) = X' \setminus a$.

Let x' now be an element of $X' \setminus \{1\}$. If $x' \in X$ then X witnesses that x' is in the composant of any element of a , else $X \cup [0, x']$ witnesses this. Thus the composant of any element of a is $X' \setminus \{1\}$.

Applying the same to Y and b gives us a pair of decomposable continua X', Y' with $\kappa^c(X') = \{a, 1\}$ and $\lambda(Y') = \{b, 1\}$. Then by applying Proposition 1.4.2 we have our desired result. $\square$

We now know that any combination is possible as long as the two spaces in question appear as the complement of a composant in some continuum. This is obviously also a necessary condition. It makes sense to next try to find out what such spaces could be. Fortunately the following theorem of Martinez-de-la-Vega and Minc gives us our answer as far as compact spaces are concerned. Since we will be focusing on E-continua this result is sufficient.

Theorem 1.4.4. *[MndlVM14, Theorem 1] For each non-degenerate continuum P there are uncountably many topologically distinct compactifications of $[0, \infty)$ each with P as the remainder.*

Corollary 1.4.5. *Let P be a non-degenerate continuum and let X be a compactification of $[0, \infty)$ as in Theorem 1.4.4. Then $\lambda(X) = \{0, P\}$.*

Proof. Let $q : X \rightarrow X/P$ be a quotient map whose only non-degenerate fibre is P . Then X/P is a one point compactification of $[0, \infty)$, so it is homeomorphic to the unit interval. Let $f : X/P \rightarrow [0, 1]$ be a homeomorphism, with $f \circ q(0) = 0$ and $f \circ q(P) = \{1\}$. Let $g = f \circ q$. The map g is monotone between continua so by Proposition 1.3.2 given any subcontinuum C of $[0, 1]$ we have that $g^{-1}(C)$ is also a continuum.

Let $X' \subseteq X$ be a subcontinuum containing 0 and intersecting P . Then $g(X')$ is also a continuum, containing 0 and 1, so must be the whole of $[0, 1]$. This implies that $g^{-1}([0, 1]) = [0, \infty) \subseteq X'$. This set is dense so it follows that $X' = X$. Thus for any point $p \in P$ we have that $X = \text{irr}(0, p)$.

Now let $x \in X \setminus (\{0\} \cup P) = (0, \infty)$. The interval $[0, x]$ shows that $x \in \kappa(0)$. Let $y = g(x)$ and let $C = [y, 1]$. Then $g^{-1}(C)$ is a subcontinuum of X containing P and x , so given $p \in P$ we have that $x \in \kappa(p)$. Thus $\kappa(0) = [0, \infty)$ and $\kappa(p) = X \setminus \{0\}$. From this we can conclude that $\lambda(X) = \{0, P\}$. $\square$

Theorem 1.4.6. *Let a and b be a pair of continua. There exists a continuum X such that $\lambda(X) = \{a, b\}$.*

Proof. If both a and b are non-degenerate then this is a simple corollary to Theorems 1.4.3 and 1.4.4 as well as Corollary 1.4.5. If a is a singleton then take a compactification X with b an E-continuum as in Theorem 1.4.4, then a is homeomorphic to $\{0\} \in \lambda(X)$. The unit interval proves the case where both a and b are singletons. $\square$

Chapter 2

A Monotone Map To An Arc

This chapter begins the study of local connectedness and monotone maps as related to irreducible continua. It will introduce a class of continua called almost hereditarily decomposable continua which will be the focus of this chapter, while links to hereditarily decomposable continua will be highlighted throughout. The main motivation behind this definition is Proposition 2.1.11. Our interest is in irreducible continua and their E-continua, which this property is clearly closely linked to.

Definition. A continuum X is almost hereditarily decomposable if every indecomposable subcontinuum of X has empty interior in X .

It is clear from this definition that any hereditarily decomposable continuum will also be almost hereditarily decomposable as, if X is a hereditarily decomposable continuum then it is vacuously true that each indecomposable subcontinuum has empty interior. As such the majority of statements about almost hereditarily decomposable continua immediately hold for decomposable continua. Attention will be drawn whenever this is not the case, or whenever the proof for the almost hereditarily decomposable case is significantly different to the hereditarily decomposable one.

For every non-irreducible point x in a continuum X a pair of subcontinua will be defined which can be thought of as being the two halves of the continuum split by x . Properties of these subcontinua will briefly be examined to categorise whether x is a cut point and whether X is locally connected at x . Next these subcontinua will be used to define an order on X , which will in turn lead to a monotone map onto another continuum. This map and its image form the major result of the chapter, namely the following theorems.

Theorem. *Let X be an almost hereditarily decomposable, irreducible continuum. Then there exists a monotone map $\pi : X \mapsto \mathbb{I}$ onto an arc.*

Theorem. *The map in the previous theorem is universal amongst monotone maps to arcs in that for each monotone map $\rho : X \mapsto \mathbb{I}$ there exists a continuous map $f : \mathbb{I} \mapsto \mathbb{I}$ such that $f \circ \pi = \rho$.*

2.1 Referenced Theorems

The following results from other authors will be used in this chapter.

Sam B. Nadler Jr, Continuum Theory, an Introduction

These results can be found in [Nad92].

Lemma 2.1.1 (3.2). *If a Hausdorff space is a continuous image of a compact metric space, then it is metrizable.*

Proposition 2.1.2 (6.3). *Let T be a connected topological space and let C be a connected subset of T such that $T \setminus C$ is disconnected, $T \setminus C = A \sqcup B$. Then $A \cup C$ and $B \cup C$ are connected. Hence, if T and C are continua, $A \cup C$ and $B \cup C$ are continua.*

Theorem 2.1.3 (8.23). *Every (non-degenerate) Peano continuum is arcwise connected.*

Theorem 2.1.4 (11.8). *Let X be a non-degenerate continuum such that X is irreducible between p and q . If A and B are subcontinua of X such that $p \in A$ and $q \in B$ then $X \setminus (A \cup B)$ is connected.*

Proposition 2.1.5 (11.9). *If X is a continuum which is irreducible between p and q , then, for any subcontinuum Y of X , the interior of Y in X is connected.*

Theorem 2.1.6 (11.17). *If X is a nondegenerate indecomposable continuum then the composants of X are mutually disjoint.*

Proposition 2.1.7 (11.20). *A nondegenerate continuum X is indecomposable if and only if there are three points of X such that X is irreducible between each two of these three points.*

D.E. Bennett and J.B. Fugate, Continua and their non-separating Subcontinua

These results can be found in [BF77].

Theorem 2.1.8 (1.3). *Each terminal subcontinuum of a continuum X is non-separating in X .*

Theorem 2.1.9 (1.8). *Suppose that X is irreducible and K is a subcontinuum of X . Then K is terminal if and only if X is irreducible between a pair of points, one of which belongs to K .*

Theorem 2.1.10 (1.22). *Suppose that X is a continuum and $p \in X$. The following are equivalent:*

- $X \setminus \kappa(p)$ is closed;
- $X \setminus \kappa(p)$ is a continuum;
- $X \setminus \kappa(p)$ is continuum-wise connected;
- If K is a subcontinuum of X such that $K \cap (X \setminus \kappa(p)) \neq \emptyset$ and $K \cap \kappa(p) \neq \emptyset$, then K is decomposable.

Proposition 2.1.11 (1.23). *Suppose that M is a continuum such that each indecomposable subcontinuum has empty interior. Then for each $p \in M$, $M \setminus \kappa(p)$ is a continuum.*

Theorem 2.1.12 (1.30). *The E -continua of X are exactly those end continua at which X is locally connected.*

Ryszard Engelking, General Topology

These results can be found in [Eng89].

Theorem 2.1.13 (Kuratowski-Zorn Lemma, page 8). *In a partially ordered set in which every chain has an upper bound, every element has a maximal element above it.*

Theorem 2.1.14 (3.1.13). *Every continuous one to one mapping of a compact space onto a Hausdorff space is a homeomorphism.*

Theorem 2.1.15 (3.2.B). *The map*

$$f(x) = \sum_{i=1}^{\infty} \frac{x_i}{2^i} \text{ where } x = \{x_i\}$$

defines a continuous map from the Cantor set onto the unit interval.

Proposition 2.1.16 (6.1.11). *Let T be a topological space and let C be a connected subspace of T . For each subspace A satisfying $C \subseteq A \subseteq \overline{C}$, the subspace A is connected.*

Theorem 2.1.17 (6.1.19). *The intersection of a decreasing sequence of continua is a continuum.*

2.2 Characterising Local Connectedness

In this section we will introduce two sets of subcontinua $\mathcal{P}_x$ and $\mathcal{Q}_x$ for any point x of X which is not a point of irreducibility, and look at whether x is a cut point or a point of local connectedness using these subcontinua. The sets of subcontinua cannot be defined usefully for points of irreducibility but fortunately these questions can be answered anyway. The answers to these questions for irreducible points will be presented first before moving on to non-irreducible points. A number of lemmas proved in this section will be important throughout the chapter.

Definition 2.2.1. A point $p \in X$ is a necessary point of irreducibility if there exists $q \in X$ such that $X = \text{irr}(p, q)$ and for all $x \in X \setminus \{p\}$, $X \neq \text{irr}(x, q)$. This is equivalent to the existence of a point q of X such that $\kappa(q) = X \setminus \{p\}$ or to $\{p\}$ being an E-continuum of X .

Proposition 2.2.2. *Let p be a point of irreducibility of a decomposable, irreducible continuum X . Then X is locally connected at p if and only if p is a necessary point of irreducibility of X .*

Proof. If p is a necessary point of irreducibility of X then $\{p\}$ is an E-continuum. Since X is locally connected at its E-continua (Theorem 2.1.12) it must be locally connected at p .

Now suppose p is not a necessary point of irreducibility of X . Let $q \in X$ such that $X = \text{irr}(p, q)$ and let $r \in X \setminus \{p\}$ such that $X = \text{irr}(r, q)$. Since X is Hausdorff there exists an open set U with $p \in U, r \notin \overline{U}$. Suppose there exists a connected open set V such that $p \in V \subseteq U$. Since $\kappa(q)$ is dense it intersects V . Thus there is a proper

subcontinuum $C \subseteq X$ such that $q \in C$ and $C \cap V \neq \emptyset$. Since $X = \text{irr}(r, q)$ it cannot be that $r \in C$. Then $C \cup \overline{V}$ is a subcontinuum of X which contains p and q but not r . This contradicts the irreducibility of X between p and q . Thus no such V exists, and X is not locally connected at p . $\square$

Definition 2.2.3. Let T be a connected topological space and $x \in T$. Then x is a cut point of T if $T \setminus \{x\}$ is disconnected.

Proposition 2.2.4. Let X be an irreducible continuum and suppose that p is a point of irreducibility of X . Then p is not a cut point of X .

Proof. There exists a point $q \in X$ with $X = \text{irr}(p, q)$. Then $\kappa(q) \subseteq X \setminus \{p\} \subseteq X$. Since $\kappa(q)$ is connected and $X = \overline{\kappa(q)}$, $X \setminus \{p\}$ is also connected by Proposition 2.1.16. $\square$

We now introduce $\mathcal{P}_x$ and $\mathcal{Q}_x$, based on the following theorem.

Theorem 2.2.5. [Nad92, 4.35] Let X be a continuum and $x, y \in X$ be distinct points. There exists a subcontinuum $Y \subseteq X$ such that $Y = \text{irr}(x, y)$.

Proof. Let $\mathcal{C}(x, y)$ be the set of all subcontinua of X containing x and y . We have that $X \in \mathcal{C}(x, y)$ so in particular $\mathcal{C}(x, y)$ is not empty. The set $\mathcal{C}(x, y)$ is partially ordered by reverse inclusion, so if K and L are elements of $\mathcal{C}(x, y)$ then $K \leq L$ if $L \subseteq K$. Given any chain K_i in $\mathcal{C}(x, y)$ we have that $K = \bigcap K_i$ is a continuum by Theorem 2.1.17, which implies that every chain has a maximal element above it. By applying Lemma 2.1.13 we have that there must be some maximal element M of $\mathcal{C}(x, y)$ which will be a continuum with $M = \text{irr}(x, y)$. $\square$

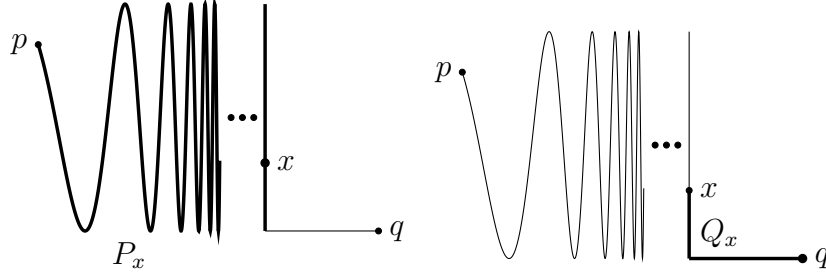
Definition 2.2.6. Let X be a decomposable, irreducible continuum. Let $p, q \in X$ such that $X = \text{irr}(p, q)$. Let $x \in X$ not be a point of irreducibility. Define sets of subcontinua of X as follows.

$$\mathcal{P}_x = \{C \subseteq X \mid C \text{ is a subcontinuum of } X, C = \text{irr}(p, x)\}$$

$$\mathcal{Q}_x = \{D \subseteq X \mid D \text{ is a subcontinuum of } X, D = \text{irr}(x, q)\}$$

By Proposition 2.2.5 there exist subcontinua $C, D \subseteq X$ such that $C = \text{irr}(p, x)$ and $D = \text{irr}(x, q)$ so neither $\mathcal{P}_x$ nor $\mathcal{Q}_x$ are empty sets. We will denote typical elements of these sets P_x and Q_x respectively.

Example 2.2.7. Here we have examples of P_x and Q_x in bold. Note that in this case $\mathcal{P}_x$ and $\mathcal{Q}_x$ both consist of a single subcontinuum.



It is worth noting that for any x , any $P_x \in \mathcal{P}_x$ and any $Q_x \in \mathcal{Q}_x$ we have $X = P_x \cup Q_x$. This is because $P_x \cup Q_x$ is the union of two continua which intersect at x , so is a continuum. It contains p and q so since $X = \text{irr}(p, q)$ we have that $X = P_x \cup Q_x$.

Although it is the case in Example 2.2.7, it is not always the case that p and q are necessary points of irreducibility. Since these two points are used in the definition of $\mathcal{P}_x$ and $\mathcal{Q}_x$ respectively it is important to check whether a different choice of, say, p would lead to a different definition of $\mathcal{P}_x$.

Proposition 2.2.8. *The sets $\mathcal{P}_x$ and $\mathcal{Q}_x$ do not depend on the choice of p and q . If $p' \in \lambda(p)$ and $q' \in \lambda(q)$ were used in Definition 2.2.6 this would produce the same sets $\mathcal{P}_x$ and $\mathcal{Q}_x$.*

Proof. Suppose there exists points $p, p', q \in X$ such that $X = \text{irr}(p, q)$ and also $X = \text{irr}(p', q)$, with $p \neq p'$. Let $P_x \subseteq X$ be a subcontinuum with $P_x = \text{irr}(p, x)$, we will show that $P_x = \text{irr}(p', x)$. Since $Q_x \neq X$ we have that $p' \notin Q_x$ which implies that $p' \in P_x$ as $X = P_x \cup Q_x$. Take a subcontinuum $C \subseteq P_x$ containing p' and x . Then as $X = \text{irr}(p', q)$ and $x \in C \cap Q_x$ we have that $C \cup Q_x$ is a continuum, and is therefore the whole of X . By the irreducibility of X it cannot be that $p \in Q_x$ so $p \in C$. Since $P_x = \text{irr}(p, x)$ it must be that $C = P_x$ meaning $P_x = \text{irr}(p', x)$. This proves that a different choice of p will produce the same set of subcontinua $\mathcal{P}_x$, and the same is true of q . $\square$

We shall now use $\mathcal{P}_x$ and $\mathcal{Q}_x$ to characterise the points of X at which X is locally connected.

Proposition 2.2.9. *Let $x \in X$ not be a point of irreducibility and suppose there exist subcontinua $P_x \in \mathcal{P}_x$ and $Q_x \in \mathcal{Q}_x$ such that x is a necessary point of irreducibility of P_x and of Q_x . Then X is locally connected at x .*

Proof. First we will show that $P_x \cap Q_x = \{x\}$. Suppose otherwise, with y witnessing this. Then $P_x \neq \text{irr}(p, y)$ so there exists a proper subcontinuum $Y_1 \subseteq P_x$ containing p and y . As it is a proper subcontinuum of P_x it cannot contain both p and x , so $x \notin Y_1$. Similarly, there exists $Y_2 \subseteq Q_x$ containing q and y but not x . Then $Y_1 \cup Y_2$ is a subcontinuum of X containing p and q but not containing x . This contradicts the irreducibility of X .

Now let $U \subseteq X$ be an open set with $x \in U$. As x is a necessary point of irreducibility of P_x and Q_x both subcontinua are locally connected at x by Theorem 2.1.12. Since $U \cap P_x$ is a P_x -open set containing x there exists a connected P_x -open subset U_p satisfying $x \in U_p \subseteq U \cap P_x$. Define U_q similarly. As $x \in U_p \cap U_q$ we have that $V := U_p \cup U_q$ is connected. Since $P_x \cap Q_x = \{x\}$ we have

$$X \setminus V = (P_x \setminus V) \cup (Q_x \setminus V) = (P_x \setminus U_p) \cup (Q_x \setminus U_q)$$

which is the union of two compact sets, so is compact. Thus $X \setminus V$ is closed, and V is open, connected, contains x and is contained in U . Thus X is locally connected at x . $\square$

Lemma 2.2.10. *Let X be an almost hereditarily decomposable continuum irreducible between points p and q . Let x be a point of X which is not a point of irreducibility and let $P_x \in \mathcal{P}_x, Q_x \in \mathcal{Q}_x$. One of the following holds.*

- *For each $y \in P_x \cap Q_x$ we have $P_x = \text{irr}(p, y)$.*
- *For each $y \in P_x \cap Q_x$ we have $Q_x = \text{irr}(y, q)$.*

Proof. Suppose neither condition holds. Let $Y \subsetneq P_x$ be a proper subcontinuum with $Y \cap Q_x \neq \emptyset, p \in Y$. Let $Z \subsetneq Q_x$ also be a proper subcontinuum, $Z \cap P_x \neq \emptyset, q \in Z$. Since $x \notin Y \cup Z$ we have that $Y \cup Z \neq X$, so it cannot be a subcontinuum as it contains p and q . Thus $Y \cap Z = \emptyset$. By Theorem 2.1.4, $X \setminus (Y \cup Z)$ is connected, so $C := \overline{X \setminus (Y \cup Z)}$ is a continuum. We know $Y \cap Q_x \neq \emptyset$ so $Y \cup Q_x$ is a continuum. It contains p and q so must be the whole of X , and similarly $X = Z \cup P_x$. From this we have that $X \setminus Z \subseteq P_x$ and $X \setminus Y \subseteq Q_x$, implying $X \setminus (Y \cup Z) \subseteq P_x \cap Q_x$ and therefore $C \subseteq P_x \cap Q_x$.

From the definition of C we have $X = Y \cup C \cup Z$ and it has already been shown that $Y \cap Z = \emptyset$, so since X is connected and all three sets are closed, we have that $C \cap Y \neq \emptyset \neq C \cap Z$. Let $y \in C \cap Y, z \in C \cap Z$. We know that $x \in C$.

Suppose $C \neq \text{irr}(x, y)$ and let $C' \subsetneq C$ witness this. Then $Y \cup C'$ is a subcontinuum of P_x containing p and x so $Y \cup C' = P_x$. Since $C \subseteq P_x$ we have that $C \setminus C' \subseteq Y$. This set is C -open so there exists X -open U such that $C \cap U = C \setminus C'$. As C is the closure of $X \setminus (Y \cup Z)$ it must be that $U \cap (X \setminus (Y \cup Z)) \neq \emptyset$. This is a subset of $U \cap C \subseteq Y$ which is a contradiction. Thus $C = \text{irr}(x, y)$. By symmetry, $C = \text{irr}(x, z)$.

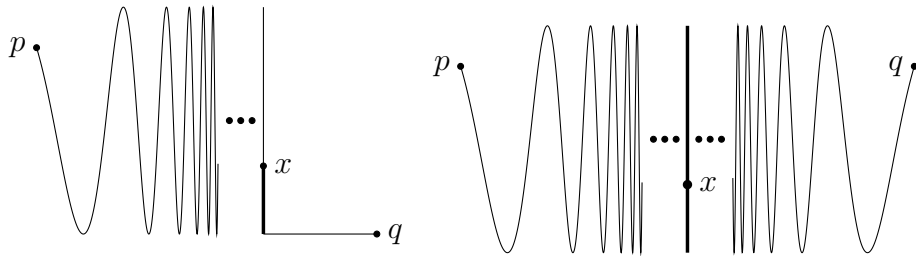
X is almost hereditarily decomposable and C contains the non-empty open set $X \setminus (Y \cup Z)$, so C is decomposable. This means that $C \neq \text{irr}(y, z)$ by Proposition 2.1.7. Let $K \subsetneq C$ witness this and note that as $C = \text{irr}(x, y)$ it must be that $x \notin K$. Then $Y \cup K \cup Z$ is a subcontinuum of X containing p and q but not x , which contradicts the fact that X is irreducible between p and q .

Therefore one of P_x and Q_x is irreducible between p or q respectively and each point of $P_x \cap Q_x$. $\square$

Lemma 2.2.10 introduced a property which will be used extensively throughout this chapter, so we shall give it a name.

Definition 2.2.11. Let $x \in X$ not be a point of irreducibility of x . We say that x is p -sided if there exists $P_x \in \mathcal{P}_x$ and $Q_x \in \mathcal{Q}_x$ such that the first condition of Lemma 2.2.10 holds. We say that x is q -sided if there exists $P_x \in \mathcal{P}_x$ and $Q_x \in \mathcal{Q}_x$ such that the second condition of Lemma 2.2.10 holds.

Example 2.2.12. The following continua give an example of x which is p -sided but not q -sided, and of x as both p -sided and q -sided. The intersection $P_x \cap Q_x$ is in bold.



In the previous examples both $\mathcal{P}_x$ and $\mathcal{Q}_x$ consist of only a single set. Definition 2.2.11 leaves open the possibility that there might exist two pairs P_x, Q_x and P'_x, Q'_x such that the first pair fulfil the first condition of Lemma 2.2.10 but not the second,

while P'_x and Q'_x satisfy the second condition but not the first. Fortunately the following two lemmas show that this cannot be the case. The first of these lemmas will be particularly important.

Lemma 2.2.13. *Suppose X is an irreducible continuum, $x \in X$ is not a point of irreducibility and there exist subcontinua $P_x \in \mathcal{P}_x$ and $Q_x \in \mathcal{Q}_x$ witnessing that x is p -sided. Then P_x is unique, in that $\mathcal{P}_x = \{P_x\}$.*

Proof. Suppose $P'_x \in \mathcal{P}_x$ is not equal to P_x to get a contradiction. Since both are irreducible between p and x , it must be that $P_x \not\subseteq P'_x \not\subseteq P_x$. As $P_x \cup Q_x = X$ we have that $P'_x \setminus P_x \subseteq Q_x$. Similarly, $P_x \setminus P'_x \subseteq Q_x$, so $(P_x \cap P'_x) \cup Q_x = X$. As P_x is irreducible between p and each point of $P_x \cap Q_x$ we have that $P_x \setminus P'_x \subseteq \lambda_{P_x}(p)$, and therefore $P_x \cap P'_x \supseteq \kappa_{P_x}(p)$. Since this is a dense subset of P_x and is contained in the closed subset $P_x \cap P'_x$ of P'_x , we have that $P_x \subseteq P'_x$. This contradiction gives us the result we need. $\square$

Lemma 2.2.14. *Suppose X is an irreducible continuum, $x \in X$ is not a point of irreducibility and there exist subcontinua $P_x \in \mathcal{P}_x$ and $Q_x \in \mathcal{Q}_x$ witnessing that x is p -sided. Then for each $Q'_x \in \mathcal{Q}_x$ we have that P_x and Q'_x satisfy the first condition of Lemma 2.2.10.*

Proof. Suppose there exists $Q'_x \in \mathcal{Q}_x$ contradicting this. Then $Q'_x \cap \kappa_{P_x}(p) \neq \emptyset$ so there exists $C \subsetneq P_x$ such that $C \cap Q'_x \neq \emptyset$ and $p \in C$. It follows from this that $C \cup Q'_x = X$ as the left hand side is a continuum containing p and q . Now $Q_x \subseteq X$ but $C \cap Q_x = \emptyset$ so $Q_x \subsetneq Q'_x$. This contradicts the fact that both Q_x and Q'_x lie in $\mathcal{Q}_x$. $\square$

The next three lemmas will be used to prove that the converse of Proposition 2.2.9 holds, that if X is locally connected at x then x is a necessary point of irreducibility of P_x and Q_x .

Lemma 2.2.15. *Let X be an irreducible, almost hereditarily decomposable continuum and let $x \in X$ not be a point of irreducibility. If x is p -sided and $\mathcal{P}_x = \{P_x\}$ then $\lambda_{P_x}(x)$ is compact.*

Proof. Suppose for a contradiction that $\lambda_{P_x}(x)$ is not compact. Then Theorem 2.1.10 gives us that there exists an indecomposable subcontinuum $C \subseteq P_x$ such that $\lambda_{P_x}(x) \subseteq C$ and $C \cap \kappa_{P_x}(p) \neq \emptyset$. There exists a proper subcontinuum $D \subseteq P_x$ which contains p and intersects C , so $U := P_x \setminus D$ is a P_x -open set contained in C . Then

there exists U' an X -open set with $U = U' \cap P_x$. As x is p -sided and $U \not\subseteq \lambda_{P_x}(x)$ it must be that $U' \setminus Q_x = U \setminus Q_x$ is non empty. Since $U' \setminus Q_x$ is X -open, and contained in C this implies that C has non-empty interior. This contradicts that X is almost hereditarily decomposable. $\square$

Remark 2.2.16. This proof is much simpler for a hereditarily decomposable continuum. If X were hereditarily decomposable then P_x would be as well, and the compactness of $\lambda_{P_x}(x)$ would follow immediately from Theorem 2.1.10. This argument would not actually use the fact that x is p -sided, so would also apply to Q_x .

Lemma 2.2.17. *Let Y be an irreducible continuum with $Y = \text{irr}(p, q)$. Let C be a subcontinuum of Y which intersects $\lambda(p)$. Then either $\lambda(p) \subseteq C$ or $C \subseteq \lambda(p)$. If $\lambda(p)$ is a proper subset of C then it lies in the interior of C .*

Proof. Suppose $C \not\subseteq \lambda(p)$. There exists a point y in $C \cap \kappa(q)$. Take a proper subcontinuum $K \subseteq Y$ containing q and y . As it is a proper subcontinuum we have that $K \cap \lambda(p) = \emptyset$. However, $C \cup K$ is a subcontinuum of Y containing p and q , so $Y = C \cup K$. Thus $\lambda(p) \subseteq C$.

To see that $\lambda(p)$ lies in the interior of C note that $\lambda(p) \subseteq Y \setminus K \subseteq C$ and that $Y \setminus K$ is an open set. $\square$

Lemma 2.2.18. *Suppose X is an almost hereditarily decomposable continuum, $x \in X$ is not a point of irreducibility and x is p -sided. Let $\mathcal{P}_x = \{P_x\}$ and let A be a subset of P_x with $A \not\subseteq \lambda_{P_x}(x)$. If A is disconnected then so is $A \cup \lambda_{P_x}(x)$.*

Proof. Suppose $A \cup \lambda_{P_x}(x)$ is connected but $B = A \setminus \lambda_{P_x}(x)$ is disconnected. Let $B = U \cup V$ for disjoint B -clopen U, V . Since $A \cup \lambda_{P_x}(x)$ and $\lambda_{P_x}(x)$ are connected we have that $U \cup \lambda_{P_x}(x)$ and $V \cup \lambda_{P_x}(x)$ are also connected by Proposition 2.1.2. As $U \subseteq B \subseteq \kappa_{P_x}(p)$ we have that there exists a proper subcontinuum $Y \subseteq P_x$ which contains p and intersects U . Then $Y \cup \overline{U \cup \lambda_{P_x}(x)} = Y \cup \overline{U} \cup \lambda_{P_x}(x)$ as by Lemma 2.2.15 we have that $\lambda_{P_x}(x)$ is compact, and therefore closed. Since this is a subcontinuum of P_x containing p and x , we have that $P_x = Y \cup \overline{U} \cup \lambda_{P_x}(x)$. Thus $V \subseteq Y$, and $P_x = Y \cup \overline{V} \cup \lambda_{P_x}(x) = Y \cup \lambda_{P_x}(x)$. However, this gives a contradiction as Y is a proper subcontinuum containing $\kappa_{P_x}(p)$, which is dense in P_x .

Thus we have that B is connected. If $\overline{B} \cap \lambda_{P_x}(x) \neq \emptyset$ then $\lambda_{P_x}(x) \subseteq \overline{B}$ by Lemma 2.2.17. This would imply $B \subseteq A \subseteq \overline{B}$, which gives A as connected by Proposition 2.1.16. If instead $\overline{B} \cap \lambda_{P_x}(x) = \emptyset$ then B and $\lambda_{P_x}(x)$ are clopen subsets of $A \cup \lambda_{P_x}(x)$, which is a contradiction. Thus we have that if A is disconnected then $A \cup \lambda_{P_x}(x)$ is as well. $\square$

Whether a point is p -sided or q -sided is not particularly interesting in its own right, but the property is very useful for proving other results as it gives a strong assumption to start from. The following proposition, giving the converse of Proposition 2.2.9, is an example of such a proof.

Proposition 2.2.19. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p, q)$. Suppose x is not a necessary point of irreducibility of at least one of P_x and Q_x . Then X is not locally connected at x .*

Proof. We know by Lemma 2.2.10 that x must be either p -sided or q -sided. If x were p -sided and a necessary point of irreducibility of P_x then $P_x \cap Q_x = \{x\}$. Thus x is also q -sided, and from the conditions of the theorem not a necessary point of irreducibility of Q_x . It must therefore be the case that for at least one of p or q , x is not a necessary point of irreducibility of the corresponding P_x or Q_x but is correspondingly p -sided or q -sided. Without loss of generality we will assume that x is p -sided and not a necessary point of irreducibility of P_x .

Let $x, y \in \lambda_{P_x}(p)$ be distinct. Let U be a P_x -open set with $x \in U, y \notin \bar{U}$. We know from the proof of Proposition 2.2.2 that U witnesses that P_x is not locally connected at x . Let V be an X -open set with $U = V \cap P_x$. For any X -open $V' \subseteq V$ containing x we have that $U' = V' \cap P_x$ is disconnected, so $U' \cup \lambda_{P_x}(x)$ is also disconnected by Lemma 2.2.18. Let $U' \cup \lambda_{P_x}(p) = C \cup D$ with $\lambda_{P_x}(p) \subseteq D$, as $\lambda_{P_x}(p)$ is connected. Then $V' = C \cup (D \cap U') \cup (V' \cap Q_x)$. All three of these are closed subsets of V' and $C \cap (D \cap U') = \emptyset = C \cap (V' \cap Q_x)$, as $C \subseteq P_x$ and $P_x \cap Q_x \subseteq \lambda_{P_x}(p) \subseteq D$. Thus V' is disconnected, which proves that X is not locally connected at x . $\square$

We shall now use the subcontinua P_x and Q_x to prove two results related to the cut points of X .

Proposition 2.2.20. *Let X be a continuum, $X = \text{irr}(p, q)$ and let $x \in X$ not be a point of irreducibility of X . Then x is a cut point of X if and only if $P_x \cap Q_x = \{x\}$.*

Proof. If $P_x \cap Q_x = \{x\}$ then $X \setminus \{x\} = (P_x \setminus \{x\}) \cup (Q_x \setminus \{x\})$, with the terms on the right hand side disjoint and closed in $X \setminus \{x\}$. Thus x is a cut point of X .

Suppose $y \in (P_x \cap Q_x) \setminus \{x\}$. Since X is a point of irreducibility of P_x and Q_x it is not a cut point of either by Proposition 2.2.4. Thus $P_x \setminus \{x\}$ and $Q_x \setminus \{x\}$ are connected, intersect at y , and have as their union $X \setminus \{x\}$. Hence $X \setminus \{x\}$ is connected, and x is not a cut point of X . $\square$

Corollary 2.2.21. *If x is not a cut point of X then either x is a necessary point of irreducibility of X , or X is not locally connected at x .*

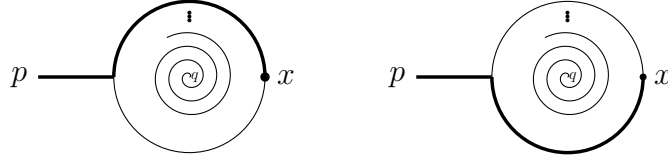
Proof. If x is a point of irreducibility then Proposition 2.2.2 tells us that x is either a necessary point of irreducibility or X is not locally connected at x . Otherwise Proposition 2.2.20 tells us that $P_x \cap Q_x \neq \{x\}$, which means that x is not a unique point of irreducibility of either P_x or Q_x depending on whether x is p -sided or q -sided. Then Proposition 2.2.19 tells us that X is not locally connected at x . $\square$

2.3 An Order on an Irreducible Continuum

In this section we will consider an almost hereditarily decomposable, irreducible continuum X . Using the sets of subcontinua $\mathcal{P}_x$ and $\mathcal{Q}_x$ defined above, an order on X will be constructed which will be the basis of the proof of the central result of this chapter, that a universal monotone map exists from X to an arc.

We will start by proving a number of further results concerning subcontinua P_x in $\mathcal{P}_x$ and Q_x in $\mathcal{Q}_x$. We have already seen in Proposition 2.2.8 that the subcontinua P_x and Q_x do not depend on the choice of p and q . It is unfortunately not the case that they are unique as the following example shows, but throughout this section we will see that they are unique enough for our purposes.

Example 2.3.1. The two subcontinua highlighted in bold below are the two distinct elements of $\mathcal{P}_x$.



Lemma 2.3.2. *Let X be an irreducible continuum and let $x, y \in X$ not be points of irreducibility. If x is p -sided, $\mathcal{P}_x = \{P_x\}$ and $y \in P_x$, then for each $P_y \in \mathcal{P}_y$ we have $P_y \subseteq P_x$.*

Proof. Let $Q_x \in \mathcal{Q}_x$ and suppose $P_y \cap Q_x \neq \emptyset$. Then $P_y \cup Q_x$ is a continuum containing p and q , so it is equal to X . Thus $P_x \setminus Q_x \subseteq P_y$, implying $\kappa_{P_x}(p) \subseteq P_y$. This is a dense subset of P_x and P_y is compact so this implies $P_x \subseteq P_y$. Since $p, y \in P_x$ we have that $P_x = P_y$.

If on the other hand $P_y \cap Q_x = \emptyset$ then as $P_x \cup Q_x = X$ we have that $P_y \subseteq P_x$. $\square$

Lemma 2.3.3. *Let X be an irreducible continuum and let $x \in X$ not be a point of irreducibility. If x is p -sided with $\mathcal{P}_x = \{P_x\}$ and there exists a subcontinuum $C \subseteq X$ with $x, p \in C$ then $P_x \subseteq C$.*

Proof. By Theorem 2.2.5 there exists a subcontinuum $C' \subseteq C$ which is irreducible between p and x . Clearly $C' \in \mathcal{P}_x$ and from Lemma 2.2.13 we have that P_x is unique, so $P_x = C' \subseteq C$. $\square$

Lemma 2.3.4. *Let X be an almost hereditarily decomposable, irreducible continuum and let $x \in X$ not be a point of irreducibility. Suppose x is p -sided but not q -sided, with $\mathcal{P}_x = \{P_x\}$. Let $Q_x \in \mathcal{Q}_x$ and suppose $\lambda_{Q_x}(x)$ is not compact. Then $\lambda_{Q_x}(x) \subseteq \lambda_{P_x}(x)$.*

Proof. As in Lemma 2.2.15, apply Theorem 2.1.10 to get an indecomposable subcontinuum $D \subseteq Q_x$ with $\lambda_{Q_x}(x) \subseteq D$ and let U be the interior of D in Q_x . We have that U is non-empty and from Proposition 2.1.5 we know that U is connected. Thus $\overline{U}$ is a subcontinuum of D intersecting every composant, so $\overline{U} = D$ by Theorem 2.1.6. Now let $U = U' \cap Q_x$ for some X -open set U' . As D has empty interior in X and $U' \setminus P_x \subseteq U \subseteq D$ it must be that $U' \setminus P_x = \emptyset$. From this we have that $U \subseteq P_x$, so $\overline{U} = D \subseteq P_x$. By the definition of D we know $\lambda_{Q_x}(x) \subseteq D \subseteq P_x$ and as x is p -sided it follows that $\lambda_{Q_x}(x) \subseteq \lambda_{P_x}(x)$. $\square$

Note that in all of the preceding lemmas, P_x could be replaced by Q_x and the result would still hold.

We will now define an order on our continuum X and prove various results related to it.

Definition 2.3.5. Let X be an irreducible continuum and let p, q be points of X with $X = \text{irr}(p, q)$. Given $x, y \in X$ define an order as follows; $x \leq y$ if and only if $x \in \lambda(p), y \in \lambda(q)$ or for some $P_y \in \mathcal{P}_y$ and $Q_x \in \mathcal{Q}_x$ we have $P_y \cup Q_x = X$. The third condition is equivalent to the requirement that $P_y \cap Q_x \neq \emptyset$.

The first thing that must be checked is whether this order changes based on which subcontinua P_x and Q_x are chosen for each x . The following proposition shows that this is not the case, so this order is truly a property of X , and not an arbitrary pattern.

Proposition 2.3.6. *The order defined in Definition 2.3.5 does not depend on which P_x, P_y, Q_x and Q_y are chosen.*

Proof. Suppose $P_x, P'_x \in \mathcal{P}_x$ are two distinct subcontinua. As both are irreducible and they are not equal to each other, we have that $P_x \not\subseteq P'_x \not\subseteq P_x$. Let $y \in X$ and $Q_y \in \mathcal{Q}_y$ be such that $P_x \cup Q_y = X$, which would imply $y \leq x$. Then $\emptyset \neq P'_x \setminus P_x \subseteq Q_y$ which means $P'_x \cap Q_y \neq \emptyset$ and $P'_x \cup Q_y = X$. Thus given any y for which P_x witnesses $x \geq y$, P'_x witnesses the same. By symmetry, this proves the result. $\square$

We will now prove a number of lemmas concerning this order.

Lemma 2.3.7. *Let X be an almost hereditarily decomposable, irreducible continuum and $x \in X$. Then $x \leq x$, i.e. the order is reflexive.*

Proof. If $x \in \lambda(p)$ or $\lambda(q)$ then this is immediate, otherwise it is clear that for any P_x in $\mathcal{P}_x$ and Q_x in $\mathcal{Q}_x$ we have $P_x \cap Q_x \neq \emptyset$ so $x \leq x$. $\square$

Lemma 2.3.8. *Let X be an almost hereditarily decomposable, irreducible continuum. If $x, y \in X$ then at least one of $x \leq y$ and $y \leq x$ holds.*

Proof. If either of x or y are points of irreducibility of X then this is clear. Otherwise take appropriate subcontinua P_x, Q_x, P_y and Q_y . $X = P_x \cup Q_x$ so $y \in P_x$ or $y \in Q_x$. If $y \in P_x$ then $P_x \cup Q_y = X$ so $y \leq x$. If $y \in Q_x$ then $P_y \cup Q_x = X$ so $x \leq y$. $\square$

Lemma 2.3.9. *Let X be an almost hereditarily decomposable, irreducible continuum and $x, y, z \in X$. The order is transitive, so if $x \leq y \leq z$ then $x \leq z$.*

Proof. We shall assume $x \leq y$ and $y \leq z$, and seek to show that $x \leq z$. If $x \in \lambda(p)$ or $z \in \lambda(q)$ then this clearly holds, so consider when neither x nor z are points of irreducibility. Take $P_z \in \mathcal{P}_z, P_y \in \mathcal{P}_y, Q_y \in \mathcal{Q}_y$ and $Q_x \in \mathcal{Q}_x$. If $z \in Q_x$ or $x \in P_z$ then $P_z \cap Q_x \neq \emptyset$ so $x \leq z$. Suppose neither of these things are true. We know that $P_y \cup Q_x = X = P_z \cup Q_y$, so $x \in Q_y$ and $z \in P_y$.

Suppose y is p -sided. Since $z \in P_y, P_z \subseteq P_y$ by Lemma 2.3.2. As X is connected, $P_z \cap Q_y \neq \emptyset$, and $P_z \cap Q_y \subseteq P_y \cap Q_y$ which is a subset of $\lambda_{P_y}(y)$ as y is p -sided. This implies that $P_y = P_z$ so $P_z \cup Q_x = X$.

If instead y is q -sided then we can repeat the same argument but with x in place of z and q in place of p . $\square$

We have shown that our order fulfils almost all of the requirements for it to be called a total order. It is not a total order however as it may not be antisymmetric, as can be seen in Example 2.2.7. Take the point x as well as any other point on the same vertical line to be y . Then these are two distinct points with $x \leq y$ and $y \leq x$.

Lemma 2.3.10. *Let X be an almost hereditarily decomposable, irreducible continuum. Suppose $x, y \in X$ are not points of irreducibility, and $x \leq y$ and $y \leq x$. If x is p -sided with $\mathcal{P}_x = \{P_x\}$ then for each $Q_y \in \mathcal{Q}_y$ we have $\lambda_{P_x}(x) \cap \lambda_{Q_y}(y) \neq \emptyset$. If x is q -sided with $\mathcal{Q}_x = \{Q_x\}$ then for each $P_y \in \mathcal{P}_y$ we have $\lambda_{P_y}(y) \cap \lambda_{Q_x}(x) \neq \emptyset$.*

Proof. We only need to prove the case where x is p -sided, as the symmetry of the problem means an identical proof will work for the other case. We have that

$$X = P_x \cup Q_x = P_x \cup Q_y = P_y \cup Q_y = P_y \cup Q_x$$

which implies

$$X = (P_x \cap P_y) \cup (Q_x \cap Q_y)$$

Thus $A = P_x \cap P_y \cap Q_x \cap Q_y \neq \emptyset$ by the connectedness of X . We will consider two cases, y is p -sided and y is q -sided. As we are assuming x is p -sided we must properly check each case, and cannot further appeal to the symmetry of the situation.

Suppose y is p -sided. Then $A \subseteq \lambda_{P_x}(x) \cap \lambda_{P_y}(y) \subseteq \lambda_{P_x}(x) \cap P_y$ so $C = P_y \cup \lambda_{P_x}(x)$ is connected. We are working under the assumption that x is p -sided so from Lemma 2.2.15 we know that $\lambda_{P_x}(x)$ is compact, and therefore C is a continuum containing p and x . Thus $P_x \subseteq C$ by Lemma 2.3.3. This implies $\kappa_{P_x}(p) \subseteq P_y$, which is dense in P_x so $P_x \subseteq P_y$. The same argument gives us that $P_y \subseteq P_x$ which implies $P_x = P_y$. Thus $y \in \lambda_{P_x}(x) \cap \lambda_{Q_y}(y)$.

Suppose now that y is q -sided. Then $A \subseteq \lambda_{P_x}(x) \cap \lambda_{Q_y}(y)$. This completes the proof. $\square$

We will now show that a weak converse is also true.

Lemma 2.3.11. *Let X be an almost hereditarily decomposable, irreducible continuum. Let $x, y \in X$ not be points of irreducibility. If x is p -sided with $\mathcal{P}_x = \{P_x\}$ and for some $Q_y \in \mathcal{Q}_y$ we have $\lambda_{P_x}(x) \cap \lambda_{Q_y}(y) \neq \emptyset$ then $x \leq y$ and $y \leq x$. If x is q -sided with $\mathcal{Q}_x = \{Q_x\}$ and for some $P_y \in \mathcal{P}_y$ we have $\lambda_{P_y}(y) \cap \lambda_{Q_x}(x) \neq \emptyset$ then $x \leq y$ and $y \leq x$.*

Proof. Again we only need to show that this holds for the case where x is p -sided. It is clear in this case that $P_x \cap Q_y \neq \emptyset$ so $P_x \cup Q_y = X$ and $y \leq x$. We will again consider the two cases where y is p -sided or q -sided, but in this instance this reduces to a single case.

Suppose y is p -sided, we will see that we can assume it is q -sided. Take a subcontinuum P_y in $\mathcal{P}_y$. Then $P_y \subseteq P_x \cup Q_y$ so $P_y \setminus Q_y \subseteq P_x$. This implies $\kappa_{P_y}(p) \subseteq P_x$, which in turn implies $P_y \subseteq P_x$. If $P_y = P_x$ then $P_y \cup Q_x = X$ so $x \leq y$ and we are done. Otherwise $P_y \cap \lambda_{P_x}(x) = \emptyset$. As $X = P_y \cup Q_y$ we have that $\lambda_{P_x}(x) \subseteq Q_y$. Since x is p -sided we have by Lemma 2.2.15 that $\lambda_{P_x}(x)$ is a continuum. Now $\lambda_{P_x}(x)$ does not contain y and intersects $\lambda_{Q_y}(y)$ which means that $\lambda_{P_x}(x) \subseteq \lambda_{Q_y}(y)$ by Lemma 2.2.17. Thus for any proper subcontinuum $C \subsetneq Q_y$ containing q we have that $x \notin P_y \cup C$ so

$P_y \cup C \subseteq X \setminus \{x\} \neq X$. This in turn implies that $P_y \cap C = \emptyset$ so $P_y \cap Q_y \subseteq \lambda_{Q_y}(y)$. Thus y is q -sided.

Now we can simply assume that y is q -sided. Take a subcontinuum Q_x in $\mathcal{Q}_x$. If $x \in P_y$ then $P_y \cup Q_x = X$ and we are done. If on the other hand $x \in Q_y$ then $Q_x \subseteq Q_y$ by Lemma 2.3.2. From our initial assumption we have that $\lambda_{P_x}(x) \cap \lambda_{Q_y}(y) \neq \emptyset$ and x is p -sided while y is q -sided. By applying Lemma 2.2.15 again we have that $C = Q_x \cup \lambda_{P_x}(x) \cup \lambda_{Q_y}(y)$ is a continuum. C contains q and y so $Q_y \subseteq C$ by Lemma 2.3.3. Further, $\kappa_{Q_y}(q) \subseteq Q_x \cup \lambda_{P_x}(x) \subseteq C$ so $Q_y \subseteq Q_x \cup \lambda_{P_x}(x)$. If $Q_x = Q_y$ then we are done. If not then since $Q_x \subsetneq Q_y$ it must be that $\lambda_{Q_y}(y) \subseteq \lambda_{P_x}(x)$. Thus $y \in P_x$ so $P_y \subseteq P_x$, again by Lemma 2.3.2, and as it intersects $\lambda_{P_x}(x)$ we have that $P_y = P_x$, so $P_y \cup Q_x = X$ and $x \leq y$. This completes the proof. $\square$

Lemma 2.3.12. *Let X be an almost hereditarily decomposable, irreducible continuum. Let $x, y, z \in X$ not be points of irreducibility. Suppose $x \leq y, z$ and $y, z \leq x$, and that both y and z are p -sided. Let $\mathcal{P}_y = \{P_y\}$ and $\mathcal{P}_z = \{P_z\}$ and let $Q_x \in \mathcal{Q}_x$. Then $y \in \lambda_{P_z}(z) \cup \lambda_{Q_x}(x)$. If both are q -sided then $y \in \lambda_{P_x}(x) \cup \lambda_{Q_z}(z)$.*

Proof. We will again just prove the case where both points y and z are p -sided. Applying Lemma 2.3.10 to z and x gives us that $\lambda_{Q_x}(x) \cap \lambda_{P_z}(z) \neq \emptyset$. Applying the same lemma to y and x gives us that $\emptyset \neq \lambda_{Q_x}(x) \cap \lambda_{P_y}(y) \subseteq \lambda_{Q_x}(x) \cap P_y$. Let $C = P_y \cup \overline{\lambda_{Q_x}(x)} \cup \lambda_{P_z}(z)$, then as z is p -sided we have that C is a continuum and contains p and z . By Lemma 2.3.3 it follows that $P_z \subseteq C$. Further, $P_z \subseteq P_y \cup \overline{\lambda_{Q_x}(x)}$ as the dense subset $\kappa_{P_z}(p)$ must lie in the right hand side. Similarly, $P_y \subseteq P_z \cup \overline{\lambda_{Q_x}(x)}$.

It is clear that if $y \in \lambda_{Q_x}(x)$ then the lemma holds. Suppose $y \notin \overline{\lambda_{Q_x}(x)}$. Then $y \in P_z$ and $P_y \subseteq P_z$ by Lemma 2.3.2. Since the relation $\leq$ is transitive (Lemma 2.3.9), $z \leq x$ and $x \leq y$ implies that $z \leq y$ so for any $Q_z \in \mathcal{Q}_z$ we have

$$\emptyset \neq P_y \cap Q_z = P_y \cap P_z \cap Q_z \subseteq P_y \cap \lambda_{P_z}(z)$$

Thus $P_y = P_z$ so $y \in \lambda_{P_z}(z)$.

We must finally consider the case where $y \in \overline{\lambda_{Q_x}(x)} \setminus \lambda_{Q_x}(x)$. This of course implies that $\lambda_{Q_x}(x)$ is not compact, so by Theorem 2.1.10 there exists $D \subseteq Q_x$ which is indecomposable and has $\lambda_{Q_x}(x) \subsetneq D$. D has non-empty interior in Q_x by Lemma 2.2.17. Let U be the interior of D in Q_x . As X is almost hereditarily decomposable and $X = P_z \cup Q_x$ it must be that $U \subseteq P_z$. From Proposition 2.1.5 we have that U is connected as Q_x is irreducible. Thus $\overline{U}$ is a subcontinuum of D . It cannot be contained in any single composant of D as it has a non-empty interior and each composant is dense. Since each proper subcontinuum is contained in a composant, it

must therefore be that $\overline{U} = D$, which implies $D \subseteq P_z$ and in turn $y \in P_z$. We can then apply the same argument as for the case where $y \notin \overline{\lambda_{Q_x}(x)}$. $\square$

Remark 2.3.13. Lemma 2.3.12 is less complicated to prove for hereditarily decomposable continua as all of the $\lambda_{P_x}(x)$ are already closed. This removes the need to consider the case $y \in \overline{\lambda_{Q_x}(x)} \setminus \lambda_{Q_x}(x)$.

Note that in the previous lemma the choice of x was arbitrary and any point satisfying $x' \leq y, z$ and $y, z \leq x'$ will do, including y or z .

Lemma 2.3.14. *Let X be an almost hereditarily decomposable, irreducible continuum. Let $x, y \in X$ not be points of irreducibility. Suppose x is p -sided with $\mathcal{P}_x = \{P_x\}$. Suppose further that $P_x \in \mathcal{P}_y$. Then y is p -sided.*

Proof. Suppose y is not p -sided. Take $Q_y \in \mathcal{Q}_y$ and z witness this i.e. $z \in P_x \cap Q_y$ but $P_x \neq \text{irr}(p, z)$. Then each $P_z \in \mathcal{P}_z$ is a subset of P_x and $P_z \cup Q_y = X$. Since $\lambda_{P_x}(x) \subseteq P_x \setminus P_z \subseteq Q_y$ it must be that $Q_x \subseteq Q_y$ for each choice of $Q_x \in \mathcal{Q}_x$. This is because $P_z \cap Q_x \subseteq P_z \cap \lambda_{P_x}(x) = \emptyset$ and $X = P_z \cup Q_y$. We therefore have that $Q_x \cup \lambda_{P_x}(x)$ is a subcontinuum of Q_y . It contains both q and y so it is equal to Q_y . Thus $P_z \cup \lambda_{P_x}(x) \cup Q_x = X$ so $P_z \cup \lambda_{P_x}(x) = P_x$. As $\lambda_{P_x}(x)$ has empty interior in P_x it must be that $P_x = P_z$ and this is a contradiction. $\square$

2.4 Constructing The Monotone Map

In this section we will create an equivalence relation from the order defined in Section 2.3 and will show that the corresponding quotient map is a universal monotone map from our continuum X onto an arc. This will involve proving that each equivalence class is a continuum and that the image of the quotient map is locally connected.

Definition 2.4.1. Let X be an almost hereditarily decomposable, irreducible continuum. Let $\leq$ be the order on X defined in Definition 2.3.5 and let $x, y \in X$. Then we say $x \sim y$ if $x \leq y \leq x$, and denote $x_\sim := \{y \in X \mid x \sim y\}$.

Proposition 2.4.2. *Let X be an almost hereditarily decomposable, irreducible continuum and let $\sim$ be as in Definition 2.4.1. Then $\sim$ is an equivalence relation.*

Proof. From Lemma 2.3.7 we have that $\sim$ is reflexive. From Lemma 2.3.9 we have that $\sim$ is transitive. It is clear from the definition that $\sim$ is symmetric. Thus $\sim$ is an equivalence relation. $\square$

Proposition 2.4.3. *Let X be an almost hereditarily decomposable, irreducible continuum and let $\sim$ be as in Definition 2.4.1. Let $x \in X$. Then $x_\sim$ is a subcontinuum of X .*

Proof. If $x \in \lambda(p)$ or $x \in \lambda(q)$ then the set $x_\sim$ is just the E-continuum of X containing x .

If x is not a point of irreducibility then from Lemmas 2.3.10 and 2.3.11 we have that

$$x_\sim = \{y \in X \mid y \text{ is } p\text{-sided, there exists } Q_x \in \mathcal{Q}_x \text{ with } \lambda_{P_y}(y) \cap \lambda_{Q_x}(x) \neq \emptyset\} \\ \cup \{y \in X \mid y \text{ is } q\text{-sided, there exists } P_x \in \mathcal{P}_x \text{ with } \lambda_{P_x}(x) \cap \lambda_{Q_y}(y) \neq \emptyset\}$$

From this and Lemma 2.3.12 it is clear to see that

$$x_\sim = \begin{cases} \lambda_{P_x}(x) \cup \lambda_{Q_x}(x) \cup \lambda_{P_y}(y) & \begin{array}{l} \exists y \notin \lambda_{Q_x}(x), y \text{ is } p\text{-sided} \\ \lambda_{P_y}(y) \cap \lambda_{Q_x}(x) \neq \emptyset \end{array} \\ \lambda_{P_x}(x) \cup \lambda_{Q_x}(x) \cup \lambda_{Q_z}(z) & \begin{array}{l} \exists z \notin \lambda_{P_x}(x), z \text{ is } q\text{-sided} \\ \lambda_{P_x}(x) \cap \lambda_{Q_z}(z) \neq \emptyset \end{array} \\ \lambda_{P_x}(p) \cup \lambda_{Q_x}(x) \cup \lambda_{P_y}(y) \cup \lambda_{Q_z}(z) & \text{both} \\ \lambda_{P_x}(x) \cup \lambda_{Q_x}(x) & \text{neither} \end{cases}$$

In the first of these four cases $x \in \lambda_{P_x}(x) \cap \lambda_{Q_x}(x)$ and $\lambda_{P_y}(y) \cap \lambda_{Q_x}(x) \neq \emptyset$ so this union is connected. The other cases can be similarly shown to be connected as well. We know that $\lambda_{P_y}(y)$ and $\lambda_{Q_z}(z)$ will be compact whenever they appear in these unions, as y is p -sided and z is q -sided (Lemma 2.2.15). If both $\lambda_{P_x}(x)$ and $\lambda_{Q_x}(x)$ are compact then $x_\sim$ is the union of finitely many compact sets, so is compact. If one of $\lambda_{P_x}(x)$ or $\lambda_{Q_x}(x)$ is not compact then applying Lemma 2.3.4 gives us that $\lambda_{P_x}(x) \cup \lambda_{Q_x}(x)$ is equal to the compact set $\lambda_{P_x}(x)$ if x is p -sided and $\lambda_{Q_x}(x)$ if x is q -sided. Either way, $x_\sim$ is compact and therefore a continuum. $\square$

Definition 2.4.4. Let X be an almost hereditarily decomposable, irreducible continuum and let $\sim$ be as in Definition 2.4.1. Define $\pi : X \mapsto X/\sim$ to be the quotient map corresponding to the equivalence relation $\sim$. Define the open interval between $x_\sim$ and $y_\sim$ as follows.

$$(x_\sim, y_\sim) = \{z_\sim \in X/\sim \mid x \leq z \leq y, z_\sim \neq x_\sim, y_\sim\}$$

Define closed and half open intervals similarly.

Note that as a direct corollary of Proposition 2.4.3 we have that the map π is monotone, as the pre-image of a point is an equivalence class $x_\sim$. We shall now investigate the structure of $X/\sim$, starting by looking at the order induced from X and $\leq$.

Proposition 2.4.5. *Let X be an almost hereditarily decomposable, irreducible continuum, let $\sim$ be as in Definition 2.4.1 and let π be as in Definition 2.4.4. We say $x_\sim \leq y_\sim$ if and only if $x \leq y$. This is a well defined total order.*

Proof. The order is well defined as if $x' \in x_\sim$ and $x \leq y$ then as $x' \leq x$ and $\leq$ is transitive (Lemma 2.3.9) we know $x' \leq y$. It is clear the order on $X/\sim$ is antisymmetric as if $x_\sim \leq y_\sim \leq x_\sim$ then $x \leq y \leq x$ so $x \sim y$ and $x_\sim = y_\sim$. We can then apply Lemmas 2.3.7, 2.3.8 and 2.3.9 to complete the proof. $\square$

We can now consider the two natural topologies on $X/\sim$, the quotient topology and the order topology. We will show that these topologies are in fact the same, and that $X/\sim$ is a continuum.

Lemma 2.4.6. *Let X be an almost hereditarily decomposable, irreducible continuum. For any $x_\sim \neq p_\sim, \pi^{-1}([p_\sim, x_\sim])$ is open in X . Similarly, for any $x_\sim \neq q_\sim, \pi^{-1}((x_\sim, q_\sim])$ is open in X .*

Proof. We will show that $\pi^{-1}([p_\sim, x_\sim]) = X \setminus (Q_x \cup x_\sim)$. If $y \in \pi^{-1}([p_\sim, x_\sim])$ then $y \not\leq x$ and $P_y \cap Q_x = \emptyset$. This means $y \notin Q_x$ and clearly $y \notin x_\sim$.

Now suppose $y \in X \setminus (Q_x \cup x_\sim)$. As $y \notin x_\sim$ we have by Lemma 2.3.8 that precisely one of $x \leq y$ and $y \leq x$ holds. Since $P_x \cup Q_x = X$ and $y \notin Q_x$ we have that $y \in P_x$ which means that $P_x \cap Q_y \neq \emptyset$, so $y \leq x$ and $x \not\leq y$, meaning $y \in \pi^{-1}([p_\sim, x_\sim])$.

An identical argument shows that $\pi^{-1}((x_\sim, q_\sim]) = X \setminus (P_x \cup x_\sim)$. $\square$

Note that by applying Theorems 2.1.8 and 2.1.9 we have that the pre-images of these sets are connected.

Proposition 2.4.7. *The space $X/\sim$ endowed with the order topology is Hausdorff.*

Proof. Let $x_\sim, y_\sim \in X/\sim$ be distinct points. Without loss of generality we can suppose that $x_\sim \leq y_\sim$. Now $\pi^{-1}([p_\sim, x_\sim]) \cup \pi^{-1}([y_\sim, q_\sim])$ is the disjoint union of two closed subsets of X and as X is a continuum there must therefore be some point z not in either pre-image. This implies $x \not\leq z \not\leq y$. Consequently, $x_\sim \in U = [p_\sim, z_\sim)$ and $y_\sim \in V = (z_\sim, q_\sim]$. Both U and V are order-open subsets of $X/\sim$ and they are disjoint, so this completes the proof. $\square$

Corollary 2.4.8. *The quotient topology on $X/\sim$ is the same as the order topology.*

Proof. The sets $[p_\sim, x_\sim)$ and $(x_\sim, q_\sim]$ form a subbasis for the order topology on $X/\sim$ and Lemma 2.4.6 implies they are both open in the quotient topology. Let Y be the space $X/\sim$ endowed with the quotient topology and let Z be $X/\sim$ endowed with the order topology. Lemma 2.4.6 implies that the identity map $id: Y \mapsto Z$ is continuous, and id is clearly a bijection. Since Y is compact and Z is Hausdorff we have from Theorem 2.1.14 that id is in fact a homeomorphism. $\square$

Having shown that these two topologies are the same they will now be used interchangeably. This will make it much easier to work with $X/\sim$ as the open sets for the order topology are far more straightforward with a well understood basis.

Lemma 2.4.9. *Let $x_\sim, y_\sim \in X/\sim$ be distinct points with $x_\sim \leq y_\sim$. Then $\pi^{-1}((x_\sim, y_\sim))$ is connected.*

Proof. Consider first the case where neither $x_\sim$ nor $y_\sim$ are equal to $p_\sim$ or $q_\sim$. Then $\pi^{-1}((x_\sim, y_\sim)) = (X \setminus (Q_x \cup x_\sim)) \cap (X \setminus (P_y \cup y_\sim))$. This is the complement in X of a pair of continua, one containing p and the other containing q , so by Theorem 2.1.4 it is connected. If on the other hand one of $x_\sim, y_\sim$ is $p_\sim$ or $q_\sim$ then the pre-image is also connected by the same argument but with an E-continua in place of say $P_y \cup y_\sim$. $\square$

Proposition 2.4.10. *The space $X/\sim$ is a continuum.*

Proof. This result is an immediate consequence of Lemma 2.1.1 and Proposition 2.4.7. $\square$

We are now in a position to prove the first of this section's two major results.

Theorem 2.4.11. *The space $X/\sim$ is an arc.*

Proof. We have already seen that the basic open sets $(x_\sim, y_\sim)$, $[p_\sim, x_\sim)$ and $(x_\sim, q_\sim]$ have connected inverse images under π in Lemma 2.4.9. Since π is continuous we have that $(x_\sim, y_\sim) = \pi(\pi^{-1}((x_\sim, y_\sim)))$ is also connected. Any space with a basis of connected sets is locally connected, which means that $X/\sim$ is a Peano continuum. Specifically $X/\sim$ is arcwise connected by Theorem 2.1.3. As π is monotone, Theorem 1.3.3 implies $X/\sim$ is irreducible between $p_\sim$ and $q_\sim$. There exists an arc $\mathbb{I} \subseteq X/\sim$ which contains these two points, and by irreducibility we have that $X/\sim = \mathbb{I}$. $\square$

This section began with the claim that we would construct a universal monotone map from our hereditarily decomposable continuum X onto an arc. We have constructed a monotone map from X to an arc, so must now show that it is universal.

Definition 2.4.12. Let $f : S \mapsto T$ be a continuous map between topological spaces. Let $\mathcal{F}$ be a collection of continuous maps $g : S \mapsto T_g$ with $f \in \mathcal{F}$. Then f is universal amongst $\mathcal{F}$ maps if for each $g \in \mathcal{F}$ there exists a continuous $h : T \mapsto T_g$ such that $h \circ f = g$.

In order to show that π is universal amongst monotone maps onto arcs we must first prove the following lemmas about the equivalence classes of $\sim$.

Lemma 2.4.13. *Let T be a topological space and let $K_1, \dots, K_n$ be closed subsets of T with empty interiors. Then $K_1 \cup \dots \cup K_n$ has empty interior in T .*

Proof. We will prove this by induction on n . The base case $n = 1$ is trivial. Suppose this is true for $n = m$ and let $K_1, \dots, K_{m+1}$ be closed, nowhere dense subsets of T . Then $K' = K_1 \cup \dots \cup K_m$ has empty interior and is closed. Let U be a T -open set with $U \subseteq K \cup K_{m+1}$. Then $U \setminus K \subseteq K_{m+1}$ so $U \setminus K = \emptyset$. Thus $U \subseteq K$, implying $U = \emptyset$. Thus $K_1 \cup \dots \cup K_{m+1}$ has empty interior. $\square$

Lemma 2.4.14. *Each equivalence class $x_\sim$ has empty interior.*

Proof. The result holds for $x_\sim = p_\sim$ or $q_\sim$ as these are E-continua of X . From the proof of Proposition 2.4.3 we know that if x is not a point of irreducibility then $x_\sim$ is the union of finitely many subcontinua of X of the form $\lambda_{P_y}(y)$. Each such set has empty interior in their respective P_y or Q_y , so must have empty interior in X . By Lemma 2.4.13 it follows that $x_\sim$ has empty interior. $\square$

Theorem 2.4.15. *The map $\pi : X \mapsto X/\sim$ is a universal monotone map from X to the unit interval.*

Proof. Let $\rho : X \mapsto \mathbb{I}$ be another monotone map. If the image of ρ is a singleton then there clearly exists a map $f : \mathbb{I} \mapsto \mathbb{I}$ such that $f \circ \pi = \rho$ so suppose this is not the case. Without loss of generality ρ is surjective. We will first show that ρ preserves equivalence classes. Let $x \in X$ and suppose $\rho(x_\sim)$ is non-degenerate. Then as $\rho(x_\sim)$ is an interval we can say $\rho(x_\sim) = [\alpha, \beta]$ for some $0 \leq \alpha < \beta \leq 1$. As ρ is a monotone map between continua we have by Proposition 1.3.2 that $\rho^{-1}([0, \alpha])$ and $\rho^{-1}([\beta, 1])$ are both subcontinua of X . One will contain p and the other q , depending on which was mapped to 0 by ρ and which to 1. Then $\rho^{-1}([0, \alpha]) \cup x_\sim \cup \rho^{-1}([\beta, 1])$ is a subcontinuum of X containing p and q , so must be equal to X . Therefore the open set $\rho^{-1}((\alpha, \beta))$ lies in $x_\sim$, which contradicts Lemma 2.4.14. From this we can deduce that for each $x \in X$ the subcontinuum $\rho(x_\sim) \subseteq [0, 1]$ is a singleton.

Now define a map $f : X/\sim \mapsto \mathbb{I}$ by $f(x_\sim) = \rho(x)$. We have just shown that this map is well defined, and it is clear that $f \circ \pi = \rho$. Now let $U \subseteq \mathbb{I}$ be an open set. Since $\rho^{-1}(U)$ is open and consists of whole equivalence classes we have that $\pi^{-1}(\pi(\rho^{-1}(U))) = \rho^{-1}(U)$, which implies $\pi(\rho^{-1}(U))$ is open in the quotient topology of $X/\sim$. As this is equal to $f^{-1}(U)$ we have that f is continuous, which completes our proof. $\square$

Remark 2.4.16. It is worth pointing out that f is monotone. Given any $y \in \mathbb{I}$ we have that $f^{-1}(y) = \pi(\rho^{-1}(y))$. Since ρ is monotone and π is continuous, $f^{-1}(y)$ must be connected.

2.5 The Fibres of π

We will now turn our attention back to the equivalence classes of $\sim$, which are also the fibres of π . We will see that π can be thought of as collapsing sets of non-locally connected points of X to singletons while leaving the locally connected points alone. First we will look at the link between these fibres and local connectedness and then a continuum will be constructed such that every fibre is non-degenerate. The existence of such a continuum negatively answers whether each equivalence class is a component of the set of non-locally connected points or whether the set of locally connected points is dense in X , both of which have been the case in every example we have seen so far.

Proposition 2.5.1. *Let $\rho : X \mapsto Y$ be a surjective monotone map between continua. Y is locally connected if and only if X is locally connected about each fibre of ρ .*

Proof. We will first prove the forwards direction, so suppose Y is locally connected and let $y \in Y$. Let $U \subseteq X$ be an open set with $\rho^{-1}(y) \subseteq U$. Consider the subspace $A = \{z \in Y \mid \rho^{-1}(z) \subseteq U\}$ and suppose that A is not open. Then there exists an injective sequence $(z_n) \in Y \setminus A$ such that (z_n) converges to some $z \in A$. There also exists a sequence $(x_n) \in X \setminus U$ such that $\rho(x_n) = z_n$ for all n . Since X is a compact metric space this sequence must have a convergent subsequence (x_{n_k}) converging to some point x . Since ρ is continuous and $\rho(x_{n_k}) = z_{n_k}$ converges to z it must be that $x \in \rho^{-1}(z) \subseteq U$, which is a contradiction as U is open. Thus we have that A is open, so there exists a connected open set V such that $y \in V \subseteq A$. Then $\rho^{-1}(V)$ is a connected open set with $\rho^{-1}(y) \subseteq \rho^{-1}(V) \subseteq U$, which proves that X is locally connected at $\rho^{-1}(y)$.

We will now prove the reverse implication, so will assume that X is locally connected about each fibre $\rho^{-1}(y)$. Let $y \in Y$ and let $U \subseteq Y$ be an open set containing y . Since $\rho^{-1}(y)$ is connected it must be contained in a component of $V = \rho^{-1}(U)$. Let C be this component. Given any $x \in C$, $\rho^{-1}(\rho(x))$ is a connected subset of V so lies in C , which means that $C = \rho^{-1}(\rho(C))$. As X is locally connected at each $\rho^{-1}(\rho(x))$ there exists connected open V_x such that $\rho^{-1}(\rho(x)) \subseteq V_x \subseteq V$, and as V_x is connected if $x \in C$ then $V_x \subseteq C$. This implies that $C = \bigcup_{x \in C} V_x$, so C is open.

$$\begin{aligned} Y \setminus \rho(X \setminus C) &= \{z \in Y \mid z \notin \rho(X \setminus C)\} \\ &= \{z \in Y \mid \rho^{-1}(z) \cap (X \setminus C) = \emptyset\} \\ &= \{z \in Y \mid \rho^{-1}(z) \subseteq C\} \\ &= \rho(C) \end{aligned}$$

This implies that $\rho(C)$ is an open set, which means it is a connected open set satisfying $y \in \rho(C) \subseteq U$ and thus Y is locally connected at y . $\square$

Corollary 2.5.2. *Let X be an almost hereditarily decomposable, irreducible continuum, and let $\sim$ and π be as in Definitions 2.4.1 and 2.4.4 respectively. Then X is locally connected about each equivalence class of $\sim$.*

Proof. We have from Theorem 2.4.11 that π is a monotone map onto a locally connected continuum, namely an arc. By applying Proposition 2.5.1 it follows that X is locally connected about each of the fibres of π , which are the equivalence classes of $\sim$. $\square$

Lemma 2.5.3. *Let X be a continuum, $C \subseteq X$ a subcontinuum and let $\rho : X \mapsto X/C$ be the natural quotient map to a quotient space. Then X/C is locally connected at the point $\rho(C)$ if and only if X is locally connected about C .*

Proof. First note that X/C is a continuum by Lemma 2.1.1. The proof of this lemma is almost identical to Proposition 2.5.1, indeed the forwards direction is identical. For the reverse direction take U an open subset of X/C containing $\rho(C)$, and let $V = \rho^{-1}(U)$. There exists connected open W with $C \subseteq W \subseteq V$. Since the only non-degenerate fibre of ρ is C we immediately have that $\rho^{-1}(\rho(W)) = W$, meaning $\rho(W)$ is an open set. Since W is connected and ρ is continuous we have that $\rho(W)$ is also connected, and clearly $\rho(C) \in \rho(W) \subseteq U$. This completes the proof. $\square$

Proposition 2.5.4. *Let X be an almost hereditarily decomposable, irreducible continuum and let $\sim$ be as in Definition 2.4.1. Then X is locally connected at x if and only if $x_{\sim} = \{x\}$.*

Proof. One direction is simple, if $x_\sim = \{x\}$ then by Corollary 2.5.2 we have that X is locally connected at x . For the other direction, suppose X is locally connected at x . If x is a point of irreducibility then by Proposition 2.2.2 x is a necessary point of irreducibility of X , and is therefore an E-continuum. Since both E-continua of X are equivalence classes it follows that $x_\sim = \{x\}$.

If x is not a point of irreducibility then by Proposition 2.2.19 we have that x is a necessary point of irreducibility of some P_x in $\mathcal{P}_x$ and Q_x in $\mathcal{Q}_x$. Recall that x must be p -sided or q -sided, so $P_x \cap Q_x$ must lie in either $\lambda_{P_x}(x)$ or $\lambda_{Q_x}(x)$. Both of these sets are equal to $\{x\}$, so it follows not only that $P_x \cap Q_x = \{x\}$ but also that x is both p and q -sided. Now let $y \in X \setminus \{x\}$. Since $X = P_x \cup Q_x$ we can say without loss of generality that $y \in P_x$. Then by Lemma 2.3.2 we have that $P_y \subseteq P_x$ and as $P_x \neq \text{irr}(p, y)$, $P_y \neq P_x$. Thus $x \notin P_y$ so $P_y \cap Q_x \subseteq (P_x \setminus \{x\}) \cap Q_x = \emptyset$. Therefore $y < x$ so $y \approx x$. Applying the same argument to a point $z \in Q_x$ gives us that $x_\sim = \{x\}$. $\square$

Lemma 2.5.5. *Let Y be an almost hereditarily decomposable continuum and let Q be a subcontinuum of Y . Let $\rho : Y \mapsto Y/Q$ be the natural quotient map. Then Y/Q is almost hereditarily decomposable.*

Proof. Let $C \subseteq Y/Q$ be a subcontinuum with non-empty interior. Let U be this interior. It follows that $U \setminus \rho(Q)$ is also open, non-empty and that $\rho^{-1}(U \setminus \rho(Q))$ is a subset of $\rho^{-1}(C)$. We will define two transfinite sequences of subcontinua of Y to show that C is decomposable. These sequences will be denoted A_α and B_α , and will be chosen such that for any α , $\rho(A_\alpha) \cup \rho(B_\alpha) = C$. Let $A_0 = B_0 = \rho^{-1}(C)$. Suppose the sequences have been defined for all ordinals up to and including α . If both $\rho(A_\alpha) \neq C$ and $\rho(B_\alpha) \neq C$ then these two continua prove that C is decomposable. Otherwise relabel them so that $\rho(A_\alpha) = C$ and take a proper decomposition of A_α into $A_{\alpha+1} \cup B_{\alpha+1}$. If γ is a successor ordinal define $A_\gamma = \bigcap_{\alpha < \gamma} A_\alpha$. This will be a continuum by Theorem 2.1.17 and as $\rho(A_\gamma) = C$ there is no need to define B_γ .

Note that if $\alpha < \beta$ then $A_\alpha \supsetneq A_\beta$ so all of these subcontinua are distinct. Each is a closed subset of Y , of which there are at the most 2^ω . This means the sequence must terminate at some point so there is an α such that $\rho(A_\alpha) = C$ but $\rho(A_{\alpha+1}) \neq C$ and $\rho(B_{\alpha+1}) \neq C$. Then $C = \rho(A_{\alpha+1}) \cup \rho(B_{\alpha+1})$ is a proper decomposition of C which proves that Y/Q is almost hereditarily decomposable. $\square$

Proposition 2.5.6. *Let X be an almost hereditarily decomposable, irreducible continuum and let $\sim$ be as in Definition 2.4.1. Let $A \subsetneq x_\sim$ be a proper subcontinuum of $x_\sim$. Then X is not locally connected about A .*

Proof. Suppose it were the case that X is locally connected about A . Let $Y := X/A$ and let $\rho : X \mapsto Y$ be the quotient map with only one non-degenerate fibre, namely A . Then Y is almost hereditarily decomposable by Lemma 2.5.5 and irreducible by Theorem 1.3.3 so Theorem 2.4.11 gives us that there is a monotone map $\pi_Y : Y \mapsto \mathbb{I}$. From Lemma 2.5.3 we know that Y is locally connected at the point $\rho(A)$ and by Proposition 2.5.4 we know that the point $\rho(A)$ is also an equivalence class. We also have that $\pi_Y \circ \rho : X \mapsto \mathbb{I}$ is a monotone map onto an arc, as the composition of monotone maps between continua is monotone (Proposition 1.3.2). This means by Theorem 2.4.15 that there exists a continuous map $f : \mathbb{I} \mapsto \mathbb{I}$ such that $\pi_Y \circ \rho = f \circ \pi_X$, where $\pi_X : X \mapsto \mathbb{I}$ is the universal monotone map for X from Theorems 2.4.11 and 2.4.15. Let $\alpha = \pi_Y(\rho(A))$ and consider the pre-images of α . We know that the pre-image under $\pi_Y \circ \rho$ is A . However, $(f \circ \pi_X)^{-1}(\alpha)$ intersects $x_\sim$ and since $x_\sim$ is a fibre of π_X it must be that $x_\sim \subseteq (f \circ \pi_X)^{-1}(\alpha)$. Thus it follows that $x_\sim \subseteq A \subseteq x_\sim$. This means A was not a proper subcontinuum of $x_\sim$, so X is not locally connected about any proper subcontinua of the equivalence classes of $\sim$. $\square$

The points of local connectedness are clearly important to the structure of X , so it is natural to wonder if they are, for example, a dense subset or whether the non-trivial fibres of π are the components of the set of non-locally connected points. This is not the case however, as irreducible, almost hereditarily decomposable continua exist which are not locally connected at any of their points, as the following example shows. The continuum constructed will in fact be hereditarily decomposable. We will first need to prove a lemma, which will be very similar to Lemma 2.5.5.

Lemma 2.5.7. *Let Y be a hereditarily decomposable continuum and let $\rho : Y \mapsto Z$ be a monotone map onto a continuum Z . Then Z is hereditarily decomposable.*

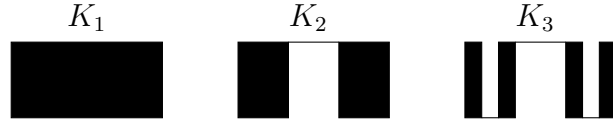
Proof. Let $C \subseteq Z$ be a subcontinuum. We will define two transfinite sequences of subcontinua of Y to show that C is decomposable. These sequences will be denoted A_α and B_α , and will be chosen such that for any α , $\rho(A_\alpha) \cup \rho(B_\alpha) = C$. Define both A_0 and B_0 to be equal to $\rho^{-1}(C)$. Suppose the sequences have been defined for all ordinals up to and including α . If $\rho(A_\alpha) \neq C \neq \rho(B_\alpha)$ then these two continua prove that C is decomposable. Otherwise relabel them so that $\rho(A_\alpha) = C$ and take a proper decomposition of A_α into $A_{\alpha+1} \cup B_{\alpha+1}$. If γ is a successor ordinal define $A_\gamma = \bigcap_{\alpha < \gamma} A_\alpha$. This will be a continuum and as $\rho(A_\gamma) = C$ there is no need to define B_γ .

Note that if $\alpha < \beta$ then $A_\alpha \supsetneq A_\beta$ so all of these are distinct. Each is a closed subset of Y , of which there are at the most 2^ω . This means the sequence must terminate

at some point so there is an α such that $\rho(A_\alpha) = C$ but $\rho(A_{\alpha+1}) \neq C \neq \rho(B_{\alpha+1})$. Then $C = \rho(A_{\alpha+1}) \cup \rho(B_{\alpha+1})$ is a proper decomposition of C which proves that Z is hereditarily decomposable. $\square$

Example 2.5.8. This continuum will be constructed in a similar way to the Cantor set, or rather the product of a Cantor set with the unit interval. We will define a sequence of nested continua K_n and take their intersection to get K , then take a quotient of K to get the continuum we want which we will denote X .

Let K_1 be a unit square $[0, 1] \times [0, 1]$. Form K_2 by deleting the middle third of K_1 except for the horizontal line $[\frac{1}{3}, \frac{2}{3}] \times \{1\}$. So $K_2 = [0, 1] \times [0, 1] \setminus (\frac{1}{3}, \frac{2}{3}) \times [0, 1]$. Form K_3 similarly by deleting the middle third from each of the two rectangles in K_2 and leaving a horizontal line, except this time the horizontal line has 2nd coordinate 0. Keep going in this fashion to define each of the K_n , alternating whether the horizontal lines are at the top or the bottom.



Now by Theorem 2.1.17 $K := \bigcap_{n \in \omega} K_n$ is a continuum as it is the intersection of nested continua. Define an equivalence relation on K by taking as the non-degenerate equivalence classes the horizontal lines. Let X be the corresponding quotient space and $f : K \mapsto X$ the quotient map. All of the equivalence classes are closed, and any pair is contained in disjoint open sets consisting of whole equivalence classes so X is a continuum (Lemma 2.1.1). Since they are all connected, f is monotone.

We will show that K is irreducible between any point with 1st coordinate 0 and any with 1st coordinate 1. Since f is monotone this will prove that X is also irreducible. Let $L \subseteq K$ be a subcontinuum intersecting $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$. If ρ is the projection onto the 1st coordinate then as $\rho(L)$ is connected, $\rho(L) = [0, 1]$. This implies that every horizontal line must lie in L as ρ maps injectively onto their image. Since no two horizontal lines intersect it must be that every vertical line must also lie in L with the possible exceptions of the ones at 0 and 1. However since L is a closed subset of K these must also lie in L , as they are in the closure of the other vertical lines. Thus $L = K$ which proves that K , and by extension X are irreducible.

We will show that K is hereditarily decomposable. Let $Y \subseteq K$ be a non-degenerate subcontinuum and consider $\rho|_Y : Y \mapsto \mathbb{I}$. If the image of Y is a singleton then Y

must be contained in one of the vertical lines, which means that Y is an arc so is decomposable. If $\rho(Y) = [\alpha, \beta]$ for $\alpha < \beta$ then let $\gamma \in (\alpha, \beta)$. From the irreducibility of K we have that $\overline{\rho^{-1}((\alpha, \beta))} \subseteq Y \subseteq \rho^{-1}([\alpha, \beta])$, and it easily follows from this that $Y \cap \rho^{-1}([\alpha, \gamma])$ and $Y \cap \rho^{-1}([\gamma, \beta])$ are both subcontinua of Y forming a decomposition of Y . Thus K is hereditarily decomposable, and by Lemma 2.5.7 so is X .

Finally, we will show that X is not locally connected at any point $x \in X$ by showing K is not locally connected about any of the horizontal lines and applying Proposition 2.5.1. The only non-degenerate fibres of f are the horizontal lines. Let $x \in X$ and let U be the ball of radius $1/3$ around $f^{-1}(x)$. Then as $f^{-1}(x)$ is either a point or a horizontal line this ball cannot meet both the horizontal lines with 2nd coordinate 0 and the ones with 2nd coordinate 1. Without loss of generality say that U misses the lines of 2nd coordinate 1. Take any open subset V of U , then consider $\rho(V)$. Given any horizontal line, ρ maps injectively onto the image so $\rho(V)$ cannot contain the image of any horizontal line with 2nd coordinate 1. These missing line segments mean that $\rho(V)$ cannot be connected, meaning V also cannot be connected. Thus K is not locally connected about any fibre of f , so X is not locally connected anywhere.

It seems worth considering the map $\pi : X \mapsto \mathbb{I}$ where π is monotone universal and X is as in the previous example. Let $p, q \in X$ be points of irreducibility, p the image of something in K with 1st coordinate 0 and q of something with 1st coordinate 1. Given $x \in X$ if $x = f(x')$ and x' is an end point of the Cantor set then depending on whether x' is approximated from the left or the right by the rest of the Cantor set there are two options for $P_{x'}$ and $Q_{x'}$. If it is approximated from the left then $P_{x'}$ consists of all of K to the left of x' including the vertical line containing x' , and $Q_{x'}$ consists of everything to the right and a line segment of the vertical line containing x' between x' and a horizontal line. If x' is approximated from the right then these continua are the other way around. From this we can see that $x_{\sim}$ is the union of two vertical lines which in K are joined by a horizontal line. These end points are a dense set so their image defines the map π . Each vertical line is mapped to a single point, they all map in order and two lines either side of a “gap” both map to the same place. If $\phi : C \mapsto \mathbb{I}$ is the standard map from Theorem 2.1.15 of the Cantor set onto $\mathbb{I}$, $\phi(\sum \frac{a_n}{3^n}) = \sum \frac{a_n/2}{2^n}$ and $\rho : C \times \mathbb{I} \mapsto C$ a projection then π can be thought of like $\phi \circ \rho$, mapping the horizontal lines to the obvious places.

The final result of this section concerns unicoherence, and generalises a result from a paper by Miller (Corollary 2.4 in [Mil50]). In this paper Miller took a very different approach in constructing a partition of a hereditarily decomposable continuum, but

the resulting partition is the same as the one arising from $\sim$. Miller proved the result for hereditarily decomposable continua, I will do so for almost hereditarily decomposable continua.

Definition 2.5.9. Let X be a decomposable continuum. Then X is unicoherent if whenever A and B are proper subcontinua of X with $X = A \cup B$, then $A \cap B$ is connected.

Theorem 2.5.10. *Let X be an almost hereditarily decomposable, irreducible continuum. Let $\sim$ and π be as in Definitions 2.4.1 and 2.4.4 respectively. If each fibre of π is unicoherent then so is X .*

Proof. First let $C \subseteq X$ be a subcontinuum with $\pi(C) = [0, \alpha]$. Let $U = \pi^{-1}([0, \alpha))$. Then U is an open set contained in C . $\bar{U}$ is a decomposable, irreducible continuum with $\kappa_{\bar{U}}(p) = \pi^{-1}((0, \alpha))$, so $\bar{U} \cap \pi^{-1}(\alpha)$ is connected and in fact an E-continuum.

Suppose $C \cap \pi^{-1}(\alpha)$ is disconnected, into say M and N . Since $\bar{U} \subseteq C$ we have that $\bar{U} \cap \pi^{-1}(\alpha) \subseteq M$ or N , so without loss of generality say it lies in M . As $C \subseteq U \cup \pi^{-1}(\alpha)$ we have that $C = (U \cup M) \cup N$ which is a decomposition of C into disjoint closed sets. This contradiction gives us that $C \cap \pi^{-1}(\alpha)$ is connected. It is also a continuum.

Now let $X = A \cup B$ for proper subcontinua A and B . Let $\pi(A) = [0, \alpha]$ and $\pi(B) = [\beta, 1]$. If $\alpha \neq \beta$ then $A \cap B = \pi^{-1}(\beta, \alpha) \cup (A \cap \pi^{-1}(\alpha)) \cup (B \cap \pi^{-1}(\beta))$ which is clearly connected by a similar argument to the previous paragraph. If $\alpha = \beta$ then $A \cap B \subseteq \pi^{-1}(\alpha)$. Now $\pi^{-1}(\alpha) = (A \cap \pi^{-1}(\alpha)) \cup (B \cap \pi^{-1}(\alpha))$ and each of these are continua, so this is a decomposition of $\pi^{-1}(\alpha)$. As $\pi^{-1}(\alpha)$ is unicoherent it must be that $A \cap B = (A \cap \pi^{-1}(\alpha)) \cap (B \cap \pi^{-1}(\alpha))$ is connected. Thus X is unicoherent. $\square$

Chapter 3

Finite Irreducibility

3.1 Referenced Theorems

The following results from other authors will be used in this chapter.

Sam B. Nadler Jr, Continuum Theory, an Introduction

These results can be found in [Nad92].

Lemma 3.1.1 (3.2). *If a Hausdorff space is a continuous image of a compact metric space, then it is metrizable.*

Theorem 3.1.2 (5.4). *Let X be a continuum and let U be a nonempty, proper, open subset of X . If K is a component of $\overline{U}$, then $K \cap \partial U \neq \emptyset$. Equivalently, $K \cap (X \setminus U) \neq \emptyset$.*

Proposition 3.1.3 (6.3). *Let T be a connected topological space and let C be a connected subset of T such that $T \setminus C$ is disconnected, say $T \setminus C = A \cup B$. Then $A \cup C$ and $B \cup C$ are connected. Hence, if T and C are continua, $A \cup C$ and $B \cup C$ are continua.*

Theorem 3.1.4 (11.8). *Let X be a non-degenerate continuum such that X is irreducible between p and q . If A and B are subcontinua of X such that $p \in A$ and $q \in B$ then $X \setminus (A \cup B)$ is connected.*

Theorem 3.1.5 (11.17). *If X is a nondegenerate indecomposable continuum then the composants of X are mutually disjoint.*

D.E. Bennett and J.B. Fugate, Continua and their non-separating Subcontinua

These results can be found in [BF77].

Theorem 3.1.6 (1.3). *Each terminal subcontinuum of a continuum X is non-separating in X .*

Theorem 3.1.7 (1.8). *Suppose that X is irreducible and K is a subcontinuum of X . Then K is terminal if and only if X is irreducible between a pair of points, one of which belongs to K .*

Theorem 3.1.8 (1.22). *Suppose that X is a continuum and $p \in X$. The following are equivalent:*

- $X \setminus \kappa(p)$ is closed;
- $X \setminus \kappa(p)$ is a continuum;
- $X \setminus \kappa(p)$ is continuum-wise connected;
- If K is a subcontinuum of X such that $K \cap (X \setminus \kappa(p)) \neq \emptyset$ and $K \cap \kappa(p) \neq \emptyset$, then K is decomposable.

Theorem 3.1.9 (1.30). *The E -continua of X are exactly those end continua at which X is locally connected.*

Ryszard Engelking, General Topology

These results can be found in [Eng89].

Lemma 3.1.10 (Kuratowski-Zorn Lemma, page 8). *In a partially ordered set in which every chain has an upper bound, every element has a maximal element above it.*

Theorem 3.1.11 (2.1.7). *A space X is hereditarily normal if and only if for every pair of separated subsets A, B of X there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$. A pair of subsets are separated if $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.*

Theorem 3.1.12 (3.1.1). *A Hausdorff space is compact if and only if every family of closed subsets of X which has the finite intersection property has non-empty intersection.*

Theorem 3.1.13 (6.1.19). *The intersection of a decreasing sequence of continua is a continuum.*

Theorem 3.1.14 (6.1.23). *In a compact Hausdorff space X the component of a point $x \in X$ coincides with the quasicomponent of the point x .*

3.2 Introducing n -irreducibility

The following is a generalisation of the idea of irreducibility. For a continuum X classical irreducibility considers whether there exists a pair of points $p, q \in X$ such that no proper subcontinuum contains both. We will consider whether there exists a finite collection of points $p_1, \dots, p_n \in X$ such that no proper subcontinuum contains all n . Generalisations of related concepts, such as terminal or E-continua will also be introduced and a number of classic results about irreducible continua will be adapted to n -irreducible continua.

When defining what it means for a continuum to be irreducible about finitely many points the most obvious thing to consider is whether a proper subcontinuum contains all of them. This is of course necessary, but it is not enough for a useful definition. Consider for example the unit interval $[0, 1]$ and let $x \in (0, 1)$. Then no proper subcontinuum of $[0, 1]$ contains the trio of points $0, x$ and 1 , but it's clear that the point x is contributing nothing to this situation. A generalisation of irreducibility ought to depend equally on each point and should lead itself to a definition of composants, with the interaction of points of irreducibility and composants being as close as possible to the classical case. To achieve this and to avoid situations like the unit interval we will use the following definition of finite irreducibility.

Definition 3.2.1. Let $n \geq 3$ and let $p_1, \dots, p_n \in X$. We say that X is n -irreducible about $p_1, \dots, p_n$ if and only if no proper subcontinuum of X contains all of the points, but for all $1 \leq i \leq n$ there exists a proper subcontinuum $C_i \subsetneq X$ such that $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n \in C_i$. This is written $X = \text{irr}(p_1, \dots, p_n)$. We say that X is n -irreducible if there exists $p_1, \dots, p_n \in X$ such that $X = \text{irr}(p_1, \dots, p_n)$ and we say that X is finitely irreducible if there exists some n such that X is n -irreducible.

If there exist points $p_1, \dots, p_n \in X$ such that no proper subcontinuum of X contains all of these points we write $X = \min(p_1, \dots, p_n)$ and say that X is minimal about the points $p_1, \dots, p_n$. There may or may not be proper subcontinua of X containing each set of $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$.

Under this definition $[0, 1] = \min(0, x, 1)$ but $[0, 1] \neq \text{irr}(0, x, 1)$ as no proper subcontinuum contains 0 and 1. Definition 3.2.1 is also compatible with the classical definition of irreducibility as if $X = \text{irr}(p, q)$ then $\{p\}$ is a proper subcontinuum containing p but not q , with $\{q\}$ acting similarly.

It is clear that if a continuum X is irreducible in the classical sense about points p and q then it satisfies every condition of finite irreducibility except for the requirement that $n \geq 3$. The definition insists that $n \geq 3$ partly to differentiate finite irreducibility from ordinary irreducibility and partly because this is the only case for which the subcontinua C_i can be singletons. We would have to handle the case $n = 2$ separately in the proof of a number of results, many of which have already been proved by other authors or in earlier chapters for this case. The case $n = 2$ would also contradict the following theorem.

Theorem 3.2.2. *Let X be an n -irreducible continuum. Then X is decomposable.*

Proof. Let $X = \text{irr}(p_1, \dots, p_n)$. Note that $n \geq 3$. Let A be a proper subcontinuum of X containing $p_2, \dots, p_n$ but not p_1 and let B be a proper subcontinuum of X containing $p_1, p_3, \dots, p_n$ but not p_2 . Since $p_3 \in A \cap B$ we have that $A \cup B$ is a continuum, and as $p_1, \dots, p_n \in A \cup B$ it follows that $X = A \cup B$. $\square$

When studying this newly defined property, we will first show the link between $X = \min(p_1, \dots, p_n)$ and finite irreducibility and then move on to define composants for a finite number of points.

Proposition 3.2.3. *Let X be a continuum with $X = \min(p_1, \dots, p_n)$. Then there exists integers $k_1, \dots, k_m$ such that $X = \text{irr}(p_{k_1}, \dots, p_{k_m})$.*

Proof. Let $A_0 = \{p_1, \dots, p_n\}$ and define sets $A_1, \dots, A_n$ inductively such that the points $p_{i+1}, \dots, p_n$ lie in A_i and no proper subcontinuum of X contains A_i . Suppose A_i has been defined. If a proper subcontinuum of X exists containing $A_i \setminus \{p_{i+1}\}$ then let $A_{i+1} = A_i$, otherwise let $A_{i+1} = A_i \setminus \{p_{i+1}\}$. No proper subcontinuum of X can contain A_n and for any $p_i \in A_n$ we have that $p_i \in A_i$ which means that there exists a proper subcontinuum of X containing $A_{i-1} \setminus \{p_i\} \supseteq A_n \setminus \{p_i\}$. Thus X is irreducible about the set A_n . The subsequence of points can be constructed from here. $\square$

Definition 3.2.4. Let $x_1, \dots, x_m \in X$. Then define

$$\kappa(x_1, \dots, x_m) = \{x \in X \mid \exists C \subsetneq X \text{ a subcontinuum s.t. } x, x_1, \dots, x_m \in C\}$$

The subset $\kappa(x_1, \dots, x_m)$ is called the composant of $x_1, \dots, x_m$.

Having defined composants for finitely many points, rather than just one, we will now show that a number of results about composants in the classical sense remain true when considering the finite case.

Proposition 3.2.5. *Let X be a continuum. For any collection of points $x_1, \dots, x_m \in X$ the set $\kappa(x_1, \dots, x_m)$ is either empty or connected and dense.*

Proof. Suppose that $\kappa(x_1, \dots, x_m) \neq \emptyset$. It is clear then that $\kappa(x_1, \dots, x_m)$ is the union of all proper subcontinua of X which contain $x_1, \dots, x_m$, so it is connected. If it were not dense then it must be closed, else the closure would be a proper subcontinuum containing $x_1, \dots, x_m$. Let Y be a point not in $\kappa(x_1, \dots, x_m)$ and U be an open set with

$$\kappa(x_1, \dots, x_m) \subseteq U \subseteq \overline{U} \subseteq X \setminus \{y\}$$

Let K be the component of $\overline{U}$ containing $\kappa(x_1, \dots, x_m)$. By Theorem 3.1.2 we have that $K \cap X \setminus U$ is non-empty, so in particular K is not a subset of $\kappa(x_1, \dots, x_m)$. But $K \subseteq X \setminus \{y\}$ so K is a proper subcontinuum of X with $x_1, \dots, x_m \in K$. This contradiction implies $\kappa(x_1, \dots, x_m)$ must be dense. $\square$

Definition 3.2.6. Suppose X is a continuum and $X = \text{irr}(p_1, \dots, p_n)$. Then for $1 \leq i \leq n$ define the set $\lambda(p_i) = X \setminus \kappa(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$. Note that $p_i \in \lambda(p_i)$ and $p_j \notin \lambda(p_i)$ for any $j \neq i$. In cases where it would otherwise be ambiguous which points define the composant, this set will be denoted $\lambda(p_i | p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$. In cases where it would otherwise be ambiguous which continuum is being considered, this set will be denoted $\lambda_X(p_i)$.

Proposition 3.2.7. *Let X be a continuum and $X = \text{irr}(p_1, \dots, p_n)$. Then for each $1 \leq i \leq n$, the set $\lambda(p_i)$ is connected*

Proof. Let $\lambda(p_i) = M \cup N$ for disjoint subsets M, N clopen in $\lambda(p_i)$. Without loss of generality let $p_i \in M$. By Theorem 3.1.11 there exist disjoint X -open sets U and V containing M and N respectively. Let P be the component of p_i in U . By Theorem 3.1.2 we have that $\overline{P} \setminus U \neq \emptyset$ and as $N \subseteq V$ we know $\overline{P} \cap N = \emptyset$ so $\overline{P} \cap \kappa(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n) \neq \emptyset$. From this we have that there exists a subcontinuum $Y \subsetneq X$ containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ and intersects $\overline{P}$. Thus $Y \cup \overline{P}$ is a subcontinuum of X , containing $p_1, \dots, p_n$ and not intersecting N . Since $X = \text{irr}(p_1, \dots, p_n)$ it must be that $X = Y \cup \overline{P}$, so $N = \emptyset$. From this we have that $\lambda(p_i)$ is connected. $\square$

We will now show that the counterpart to Lemma 2.2.17 also holds for the finite case. The proof is nearly identical to the case with just two points of irreducibility. This proposition has a pair of corollaries which begin to show that Definition 3.2.1 is an accurate generalisation of irreducibility, as we see it preserves some of the basic notions of irreducible points and composants.

Proposition 3.2.8. *Let X be a continuum, $C \subseteq X$ be a subcontinuum and suppose $X = \text{irr}(p_1, \dots, p_n)$. If $C \cap \lambda(p_i) \neq \emptyset$ and $C \not\subseteq \lambda(p_i)$ then $\lambda(p_i) \subseteq \text{int}(C)$.*

Proof. Let $x \in C \setminus \lambda(p_i)$. Since $x \in X \setminus \lambda(p_i) = \kappa(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ there exists $D \subsetneq X$ a subcontinuum containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ and x . As it is a proper subcontinuum, $D \cap \lambda(p_i) = \emptyset$ and $C \cup D$ is a continuum containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ and intersecting $\lambda(p_i)$. This implies that $C \cup D = X$, and therefore $\lambda(p_i) \subseteq X \setminus D \subseteq \text{int}(C)$. $\square$

Corollary 3.2.9. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$ and suppose $i \neq j$. Then $\lambda(p_i) \cap \lambda(p_j) = \emptyset$.*

Proof. Suppose for contradiction that $\lambda(p_i) \cap \lambda(p_j) \neq \emptyset$. Since $X = \text{irr}(p_1, \dots, p_n)$ there exists a proper subcontinuum $C \subsetneq X$ containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$. Then C intersects $\lambda(p_j)$ so contains it by Proposition 3.2.8, which means it intersects $\lambda(p_i)$ and therefore contains that. This is a contradiction as C therefore would contain $p_1, \dots, p_n$ but is a proper subcontinuum of X . $\square$

Remark 3.2.10. It is clear to see from the proof of this Corollary that for distinct p_i, p_j we have $\lambda(p_i) \cap \overline{\lambda(p_j)} = \emptyset$. This is because $\lambda(p_j) \subseteq C$ and C is a closed subset of X , so $\overline{\lambda(p_j)} \subseteq C$. Then if $\overline{\lambda(p_j)} \cap \lambda(p_i) \neq \emptyset$ we would be able to apply Proposition 3.2.8 and reach the same contradiction.

Corollary 3.2.11. *Let X be a continuum and let $X = \text{irr}(p_1, \dots, p_n)$. Let $q_1, \dots, q_n$ be points of X and suppose for all $1 \leq i \leq n$ we have $q_i \in \lambda(p_i)$. Then it must be that $X = \text{irr}(q_1, \dots, q_n)$.*

Proof. Any continuum containing each q_i will contain $\lambda(p_i)$ by Proposition 3.2.8 so will contain p_i . Thus no proper subcontinuum of X contains $q_1, \dots, q_n$. Given $C_i \subsetneq X$ containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ we again have that for $j \neq i$, $\lambda(p_j) \subseteq C_i$ and $C_i \cap \lambda(p_i) = \emptyset$. Thus $q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n \in C_i$ and $q_i \notin C_i$. This proves that $X = \text{irr}(q_1, \dots, q_n)$. $\square$

Proposition 3.2.12. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$. Then the subset $X \setminus \bigcup_{i=1}^n \lambda(p_i)$ is dense in X .*

Proof. Let $U \subseteq \bigcup \lambda(p_i)$ be an X -open set. Since each $\lambda(p_i)$ does not intersect the closures of the other $\lambda(p_j)$ from Corollary 3.2.9 we have that $U \cap \lambda(p_i) = U \setminus \bigcup_{i \neq j} \overline{\lambda(p_j)}$. This means that $U \cap \lambda(p_i)$ is also an X -open set, so it must be empty by Proposition 3.2.5. This is true of every $1 \leq i \leq n$ so $U = \emptyset$. $\square$

Next we will prove a theorem which is vital if we wish to regard n -irreducibility as an intrinsic property of a continuum. If Theorem 3.2.13 did not hold then any discussion of the composants or the E-continua of a continuum X would be impossible, as X would have different sets of composants or E-continua depending on whether you were considering it as n irreducible or m irreducible.

Theorem 3.2.13. *Let X be a continuum and suppose both that $X = \text{irr}(p_1, \dots, p_n)$ and that $X = \text{irr}(q_1, \dots, q_m)$. Then $n = m$.*

Proof. For each $1 \leq i \leq n$ let $J_i = \{j | 1 \leq j \leq m, q_j \in \kappa(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)\}$. For each $j \in J_i$ there exists a proper subcontinuum $C_{i,j}$ of X containing q_j and $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$. If $C = \bigcup_{j \in J_i} C_{i,j}$ then C is a proper subcontinuum of X , as it does not contain p_i , so C cannot contain $q_1, \dots, q_m$. Thus each $\lambda(p_i | p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ must contain some q_j . All of the $\lambda(p_i | p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ are disjoint by Corollary 3.2.9, so each q_j lies in at most one $\lambda(p_i)$. This gives us that $n \leq m$ and by symmetry $n = m$. $\square$

The proof of this theorem leads to the following obvious corollary.

Corollary 3.2.14. *If $X = \text{irr}(p_1, \dots, p_n)$ and $X = \text{irr}(q_1, \dots, q_n)$ then there exists $\sigma \in S_n$ such that $\lambda(q_i) = \lambda(p_{\sigma(i)})$.*

We are now in a position to prove the following proposition, showing that the two intuitive definitions of a composant are one and the same, namely Definition 3.2.4 and the set of points x with $X \neq \text{irr}(x, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$.

Proposition 3.2.15. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$. Then for each i we have $\lambda(p_i) = \{x \in X | X = \text{irr}(x, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)\}$*

Proof. Let $A = \{x \in X | X = \text{irr}(x, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)\}$. It is clear that $\lambda(p_i)$ contains A so we only need to show that $\lambda(p_i)$ is a subset of A . Suppose there exists $x \in \lambda(p_i) \setminus A$. As $x \in \lambda(p_i)$ we can apply Proposition 3.2.8 to see that no proper subcontinuum of X contains x and $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ and therefore $X = \text{min}(x, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$. As $x \notin A$ it must be that there is some j such that no proper subcontinuum of X contains $x, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{j-1}, p_{j+1}, \dots, p_n$. Thus

$X = \min(x, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{j-1}, p_{j+1}, \dots, p_n)$. Applying Proposition 3.2.3 implies X is k -irreducible for some $k < n$. This contradicts Theorem 3.2.13. $\square$

Throughout this section we have seen that there can only be one $n \in \mathbb{N}$ such that X is n -irreducible, and that X is n -irreducible if and only if there is a unique collection of n disjoint connected subsets of X such that no proper subcontinuum intersects them all, with $X = \text{irr}(p_1, \dots, p_n)$ if and only if the set $\{p_1, \dots, p_n\}$ contains one point from each of these disjoint connected subsets. All of this portrays Definition 3.2.1 as a true generalisation of irreducibility, carrying over the fundamental aspects of irreducibility which make it an interesting property of continua.

Monotone and Quotient Maps

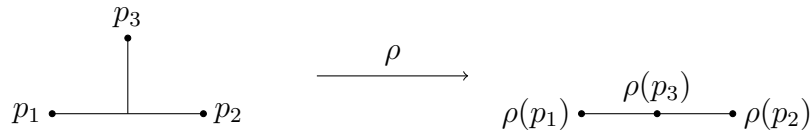
We will end this section with a number of results concerning monotone maps or quotient maps and how they interact with n -irreducibility.

Proposition 3.2.16. *Let X and Y be continua and let $X = \text{irr}(p_1, \dots, p_n)$. Suppose there exists a monotone surjection $\rho : X \mapsto Y$. Then $Y = \min(\rho(p_1), \dots, \rho(p_n))$.*

Proof. Suppose $C \subseteq Y$ is a subcontinuum of Y containing $\rho(p_1), \dots, \rho(p_n)$. Then $\rho^{-1}(C)$ is a subcontinuum of X by Proposition 1.3.2 and contains $p_1, \dots, p_n$. Thus $\rho^{-1}(C) = X$ and as ρ is surjective we have that $C = Y$. $\square$

Unlike in Proposition 1.3.2 we cannot say that Y is irreducible about the images of the p_i , merely that it is minimal. The following example demonstrates this.

Example 3.2.17. Let X be the subset $\mathbb{I} \times \{0\} \cup \{1/2\} \times \mathbb{I}$ of $\mathbb{I}^2$. Consider the points $p_1 = (0, 0)$, $p_2 = (0, 1)$ and $p_3 = (1/2, 1)$ and let $\rho : X \mapsto \mathbb{I}$ be the projection onto the first coordinate.



While $X = \text{irr}(p_1, p_2, p_3)$ we have that the image of X is an arc, irreducible between $\rho(p_1)$ and $\rho(p_2)$ but not about the images of all three points.

The following two lemmas prove parallel results about hereditarily decomposable and almost hereditarily decomposable continua. It is not possible to simply apply the almost hereditarily decomposable result to hereditarily decomposable continua as it would not guarantee that the image would be hereditarily decomposable, merely almost hereditarily decomposable. Additionally, the result for hereditarily decomposable continua is stronger than the almost hereditarily decomposable version.

Lemma 3.2.18. *Let Y be a hereditarily decomposable continuum and let $\rho : Y \mapsto Z$ be a weakly confluent map onto a continuum Z , meaning each subcontinuum of Z is the image under ρ of a subcontinuum of Y . Then Z is hereditarily decomposable.*

Proof. Let C be a non-degenerate subcontinuum of Z . Take $\mathcal{C}$ to be the collection of subcontinua K of Y with $\rho(K) = C$ and order $\mathcal{C}$ by reverse inclusion. Since ρ is weakly confluent we have that $\mathcal{C}$ is not empty. Given any chain K_i of continua in $\mathcal{C}$ let $K = \bigcap K_i$, then by Theorem 3.1.13 we have that K is a continuum. Since $\mathcal{C}$ is ordered by reverse inclusion we can deduce that $\rho(K) = \bigcap \rho(K_i) = C$, so $K \in \mathcal{C}$ and every chain lies under a maximal element. By Lemma 3.1.10 there exists a maximal element $M \in \mathcal{C}$. As Y is hereditarily decomposable we have that there exist proper subcontinua A and B of M with $M = A \cup B$. As M is maximal under reverse inclusion, neither A nor B lie in $\mathcal{C}$ which means $\rho(A)$ and $\rho(B)$ are proper subcontinua of C . As $\rho(M) = C$ we have that $\rho(A) \cup \rho(B) = C$, meaning C is decomposable. $\square$

Corollary 3.2.19. *Lemma 3.2.18 applies to monotone maps.*

Proof. This follows immediately from the fact that every monotone map is weakly confluent. $\square$

Lemma 3.2.20. *Let Y be an almost hereditarily decomposable continuum and let $Q \subseteq X$ be a subcontinuum. Let $\rho : Y \mapsto Y/Q$ be the natural quotient map. Then Y/Q is almost hereditarily decomposable.*

Proof. Let $C \subseteq Y/Q$ be a subcontinuum with non-empty interior. Let U be this interior. It follows that $U \setminus \rho(Q)$ is also open, non-empty and $\rho^{-1}(U \setminus \rho(Q)) \subseteq \rho^{-1}(C)$. The proof now proceeds exactly as for Lemma 3.2.18. At any stage in the induction if $\rho(A_\alpha) = C$ then $\rho^{-1}(U \setminus \rho(Q)) \subseteq A_\alpha$ meaning A_α has non-empty interior, so it is decomposable. This is the only adaptation required, so it follows that C is decomposable and Y/Q is almost hereditarily decomposable. $\square$

This final lemma will be used numerous times to reduce the relatively complicated case of finite irreducibility to the far more straightforward case of a continuum irreducible between a pair of points.

Lemma 3.2.21. *Let X be an n -irreducible continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let $C, D \subseteq X$ be disjoint proper subcontinua with $p_1, \dots, p_n \in C \cup D$ and let $\rho : X \mapsto X/C, D$ be the quotient map to the quotient space $X/C, D$, so the only two non-degenerate fibres of ρ are C and D . Then $X/C, D$ is a continuum with $X/C, D = \text{irr}(\rho(C), \rho(D))$. If $D = \{p_i\}$ for some $1 \leq i \leq n$ then $\rho(\lambda_X(p_i)) = \lambda_{X/C}(\rho(p_i))$.*

Proof. To see that $X/C, D$ is a continuum we only need to show it is Hausdorff and apply Lemma 3.1.1. However since there are only two non-degenerate fibres of ρ and X is normal it follows immediately that $X/C, D$ is Hausdorff.

From Proposition 3.2.16 we have that $X/C, D = \min(\rho(p_1), \dots, \rho(p_n))$. There are only two distinct points in the set $\{\rho(p_1), \dots, \rho(p_n)\}$, so we can instead express this as $X/C, D = \min(\rho(C), \rho(D))$. There is no difference between being minimal about a pair of points and being irreducible between them, so $X/C, D = \text{irr}(\rho(C), \rho(D))$.

To show that $\rho(\lambda_X(p_i)) = \lambda_{X/C}(\rho(p_i))$ first let $x \in \kappa_X(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ and take $K \subsetneq X$ witnessing this. Then $\rho(K)$ is a proper subcontinuum of X/C which contains $\rho(C)$ and $\rho(x)$, which implies $\rho(x) \notin \lambda_{X/C}(\rho(p_i))$. From this we can conclude that $\rho(\lambda_X(p_i)) \supseteq \lambda_{X/C}(\rho(p_i))$.

For the reverse inclusion, let $y \in \kappa_{X/C}(\rho(C))$ and let L be a proper subcontinuum of X/C witnessing this. We have that $\rho^{-1}(L)$ is a proper subcontinuum of X which contains C and $\rho^{-1}(y)$. Thus $\rho^{-1}(y) \cap \lambda_X(p_i) = \emptyset$ and as such $y \notin \rho(\lambda_X(p_i))$. This gives us that $\rho(\lambda_X(p_i)) \subseteq \lambda_{X/C}(\rho(p_i))$, so these two sets are equal. $\square$

3.3 Terminal and End Continua

In their paper [BF77] Bennett and Fugate worked extensively with terminal continua, end continua and E-continua. The ideas of terminal and end continua are closely related to irreducibility, and E-continua are directly defined by irreducibility. It therefore makes sense to investigate these notions in the new setting of n -irreducibility, adapting the definitions suitably. A number of the results from [BF77] will also be proved in this context, particularly ones which relate to whether or not a subcontinuum is terminal.

The definitions of terminal and end continua are based around decompositions of a continuum X into proper subcontinua A and B . As with irreducibility, a new

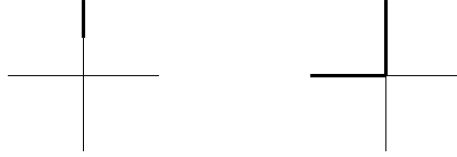
condition must be added when switching from two subcontinua to n subcontinua which would always be satisfied for just two.

Definition 3.3.1. Let X be a continuum. The subcontinua $A_1, \dots, A_n$ form an n -decomposition of X providing X is an essential sum of the A_i 's, i.e. $X = A_1 \cup \dots \cup A_n$ and for all $1 \leq i \leq n$, $X \neq A_1 \cup \dots \cup A_{i-1} \cup A_{i+1} \cup \dots \cup A_n$.

Definition 3.3.2. Let K be a subcontinuum of a continuum X . K is n -terminal if whenever $A_1, \dots, A_n$ is an n -decomposition of X with $A_i \cap K \neq \emptyset$ for all i then there exists $1 \leq j \leq n$ such that $X = K \cup \bigcup_{i \neq j} A_i$.

Note that it is possible for a subcontinuum K to be n -terminal for multiple values of n . We will primarily be interested in the minimum value of n for which K is n -terminal, but several of the following results involve non-minimal values of n .

Example 3.3.3. The following continuum gives an example of a 4-terminal subcontinuum on the left in bold and a 3-terminal subcontinuum on the right in bold.



The following proposition provides an equivalent definition of an n -terminal continuum which will be useful in proving further results, as seen in the corollary.

Proposition 3.3.4. A subcontinuum K of X is not n -terminal if and only if there exists an n -decomposition $A_1, \dots, A_n$ of X with $K \subseteq \bigcap_{i=1}^n A_i$.

Proof. Only the forward direction needs to be proved, so let $B_1, \dots, B_n$ be an n -decomposition of X witnessing that K is not n -terminal. Define $A_i = K \cup B_i$. Then it is clear to see that these A_i 's satisfy the conditions in the statement of the proposition. $\square$

Corollary 3.3.5. If K is an n -terminal subcontinuum of a continuum X and L is a subcontinuum of X containing K then L is n -terminal.

Proof. We will instead prove the contrapositive, that if L is not n -terminal then neither are any of the subcontinua of L . Let A_i be an n -decomposition as in Proposition 3.3.4, witnessing that L is not n -terminal. Then $K \subseteq L \subseteq \bigcap_{i=1}^n A_i$. This implies that K is not n -terminal, again by Proposition 3.3.4. $\square$

Since a continuum X can contain multiple n -terminal subcontinua for various different values of n (see Example 3.3.3) we should investigate what range these values of n can take. While the question of minimal n -values will be answered in a later section we are in a position to cover maximum n -values here.

Proposition 3.3.6. *Let X be a continuum and $K \subseteq X$ be a subcontinuum. If K is n -terminal then K is m -terminal for all $m \geq n$.*

Proof. Let $m > n$ and suppose for contradiction that K is not m -terminal. By Proposition 3.3.4 there exists a decomposition $A_1, \dots, A_m$ with $K \subseteq \bigcap_{i=1}^m A_i$. Define $A'_1 = A_1, \dots, A'_{n-1} = A_{n-1}$ and let $A'_n = \bigcup_{i=n}^m A_i$. Now A'_n is a continuum as it is the union of continua A_i each of which intersect at K . The subcontinua $A'_1, \dots, A'_n$ form an n -decomposition of X with $K \subseteq \bigcap_{i=1}^n A'_i$, so K is not n -terminal by Proposition 3.3.4. This contradiction completes the proof. $\square$

We will now introduce n -end continua and briefly examine their relation with n -terminal continua. In Section 3.5 n -end continua will be studied more exhaustively.

Definition 3.3.7. A subcontinuum $K \subseteq X$ is an n -end continuum if and only if no n -decomposition $A_1, \dots, A_n$ of X exists such that for all $1 \leq i \leq n$, $A_i \cap K \neq \emptyset$.

Proposition 3.3.8. *Let X be a continuum and $K \subseteq X$ a subcontinuum. If K is n -end it is n -terminal.*

Proof. If K is n -end then the condition to be n -terminal is vacuously met, as there are no decompositions $A_1, \dots, A_n$ which each intersect K . $\square$

Example 3.3.9. Any subcontinuum of a simple closed curve is a 3-terminal subcontinuum. Any point in a simple closed curve is a 3-end subcontinuum. Note that a simple closed curve is not n -irreducible for any n .

Theorem 3.3.10. *If X is a continuum with $X = \text{irr}(p_1, \dots, p_n)$ then each $\{p_k\}$ is an n -end continuum.*

Proof. Suppose $X = A_1 \cup \dots \cup A_n$ such that each A_i is a continuum and for some k each A_i contains p_k . Every p_i must lie in some A_{j_i} so by relabelling we have that $p_1, \dots, p_n \in A := \bigcup_{i=1}^{n-1} A_i$. Since each A_i contains p_k they must all intersect non-trivially and it follows that A is a continuum. As $X = \text{irr}(p_1, \dots, p_n)$ we have that $A = X$. Thus $A_1, \dots, A_n$ was not an n -decomposition of X . $\square$

Corollary 3.3.11. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$ and let K be a subcontinuum of X . If for any $1 \leq i \leq n$ we have $p_i \in K$ then K is n -terminal.*

Proof. By Theorem 3.3.10 we have that $\{p_i\}$ is an n -end continuum, so by Proposition 3.3.8 we have that $\{p_i\}$ is an n -terminal continuum. Finally by Corollary 3.3.5 we have that K is also n -terminal. $\square$

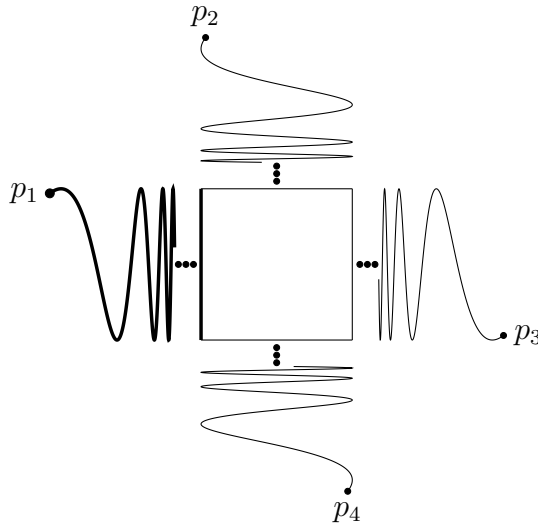
In their paper [BF77] Bennett and Fugate placed a lot of emphasis on non-separating and non-cutting subcontinua as well as the link to terminal and end continua. This relationship becomes more complicated when abstracting to the finite case, and will be covered in some detail in the next section. For now we will prove the following result which helps to characterise n -terminal continua.

Proposition 3.3.12. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$. If K is a non-cutting (complement is continuumwise connected) subcontinuum of X it must contain some p_i and is therefore n -terminal.*

Proof. Let $P = \{i \mid 1 \leq i \leq n, p_i \notin K\}$, let $k \in P$ and for any $i \in P$ let $X_i \subseteq X \setminus K$ be a continuum containing p_i, p_k . Let $X' = \bigcup_{i \in P} X_i$, so X' is a continuum. Since X' is contained in $X \setminus K$ it cannot be the whole of X so cannot contain all of the p_i . This means $P \neq \{1, \dots, n\}$. Therefore some p_i lies in K , which in turn implies that K is n -terminal by applying Corollary 3.3.11. $\square$

Note that just as with standard irreducibility, there exist n -terminal continua that are cutting as the following example shows. In this example K is non-separating.

Example 3.3.13. Let X be the continuum below, with K highlighted in bold.



Now $X = \text{irr}(p_1, p_2, p_3, p_4)$ and as $p_1 \in K$ we can apply Corollary 3.3.11 to see that K is 4-terminal. The subspace $X \setminus K$ is not continuumwise connected however as any subcontinuum of X containing p_2 and p_4 would need to contain the top left and bottom left corners of the central square, both of which lie in K .

It has been mentioned before that there can be multiple m -terminal subcontinua with different values of m in the same continuum, and in Proposition 3.3.6 we saw that a subcontinuum can be m -terminal for multiple values of m . The next proposition will put a cap on the size of m for which it is noteworthy to say that a subcontinuum is m -terminal.

Proposition 3.3.14. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$ and let K be a subcontinuum of X . Then K is $(n + 1)$ -terminal.*

Proof. Let $A_1, \dots, A_{n+1}$ be an $(n+1)$ -decomposition of X such that each A_i intersects K . There exists a map $\sigma : \{1, \dots, n\} \mapsto \{1, \dots, n+1\}$ such that $p_i \in A_{\sigma(i)}$. Then $X' = K \cup \bigcup_{i=1}^n A_{\sigma(i)}$ is a continuum containing each p_i , which means $X' = X$. Thus K is $(n+1)$ -terminal. $\square$

3.4 Terminal Subcontinua and Complements

As has previously been mentioned there are a number of results in the paper [BF77] which explore the relationship between subcontinua such as terminal or end continua and whether or not their complements are connected. In this section we will explore analogous results for n -terminal continua. In order to do so we will also prove a number of lemmas that will be useful throughout the discussion of n -irreducibility.

Proposition 3.4.1. *Let X be a continuum. If K is an n -terminal subcontinuum of X then $X \setminus K$ has at most $n - 1$ components.*

Proof. Suppose $X \setminus K$ has n components and let $X \setminus K = U_1 \cup \dots \cup U_n$ for each U_i open and pairwise-disjoint. Then each $K \cup U_i$ is a continuum by Proposition 3.1.3 so $X = \bigcup_{i=1}^n (K \cup U_i)$ is an n -decomposition of X which contradicts Proposition 3.3.4. Thus no such expression for $X \setminus K$ exists, which in turn implies that $X \setminus K$ has at most $n - 1$ components. $\square$

In Chapter 2 extensive use was made of Theorem 2.2.5, which states that given any continuum X and any two distinct points x and y in X there exists a subcontinuum $Y \subseteq X$ with $Y = \text{irr}(x, y)$. We shall now prove an analogous statement for finitely many points.

Lemma 3.4.2. *Let X be a continuum with $x_1, \dots, x_m \in X$. Then there exists a subcontinuum $Y \subseteq X$ such that $Y = \min(x_1, \dots, x_m)$.*

Proof. Let $\mathcal{C}(x_1, \dots, x_m)$ be the set of subcontinua of X containing $x_1, \dots, x_m$ and order $\mathcal{C}(x_1, \dots, x_m)$ by reverse inclusion. This set is non-empty as it contains X . Given a chain of continua $K_i \in \mathcal{C}(x_1, \dots, x_m)$ let $K = \bigcap K_i$. Then by Theorem 3.1.13 we have that K is a continuum and clearly $x_1, \dots, x_m \in K$, so $K \in \mathcal{C}(x_1, \dots, x_m)$. Thus every chain lies below a maximal element, and by Lemma 3.1.10 there exists a maximal element M of $\mathcal{C}(x_1, \dots, x_m)$. This continuum must have $M = \min(x_1, \dots, x_m)$ as it is maximal under reverse inclusion. $\square$

Remark 3.4.3. Lemma 3.4.2 is weaker than Theorem 2.2.5 as we cannot say that $Y = \text{irr}(x_1, \dots, x_n)$. This is because Y may not have proper subcontinua containing each set $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$. For example, take any three points in a simple closed curve. Any subcontinuum is an arc, so will not be irreducible about all three points. However given any pair of these three there exists an arc irreducible between the pair of points and containing the third and so minimal about the trio.

Although Lemma 3.4.2 is weaker, it can be strengthened under an additional assumption as will now be shown.

Lemma 3.4.4. *Let X be a continuum. If $X = \text{irr}(p_1, \dots, p_n)$, each $\lambda(p_i)$ is compact and Y is a subcontinuum of X with $Y = \min(p_1, \dots, p_m)$ then $Y = \text{irr}(p_1, \dots, p_m)$.*

Proof. We need to show that for any $1 \leq i \leq m$ there exists a proper subcontinuum of Y containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_m$. It is only necessary to prove this for $i = m$. Let $Z \subsetneq X$ be a subcontinuum containing $p_1, \dots, p_{m-1}, p_{m+1}, \dots, p_n$. Let $U_Z = X \setminus Z$ and note that as $X = \text{irr}(p_1, \dots, p_n)$ and $p_1, \dots, p_n \in Y \cup Z$ we have that $X = Y \cup Z$ and $U_Z \subseteq Y$. This open set is connected, as if $U_Z = V \cup W$ with V and W disjoint open and $p_m \in V$ then $Z \cup V$ is a continuum by Proposition 3.1.3 and contains $p_1, \dots, p_n$, so $Z \cup V = X$ and $W = \emptyset$. As $X = Y \cup Z$ we have that $U_Z \subseteq Y$.

Claim. Each component of $Y \cap Z$ has non-trivial intersection with $\overline{U_Z}$.

Let K be a component of $Y \cap Z$, we will prove that $K \cap \overline{U_Z} \neq \emptyset$. By Theorem 3.1.14 K is a nested intersection of $(Y \cap Z)$ -clopen sets, $K = \bigcap Q_\alpha$. If $K \cap \overline{U_Z} = \emptyset$ then $\bigcap (Q_\alpha \cap \overline{U_Z}) = \emptyset$. Each $(Y \cap Z)$ -clopen set is X -closed, so each $Q_\alpha \cap \overline{U_Z}$ is also closed. X is compact so if the nested intersection of closed sets is empty we can

infer from Theorem 3.1.12 that there is some Q_α with $Q_\alpha \cap \overline{U_Z} = \emptyset$. But then as $Y = (Y \cap Z) \cup U_Z$,

$$\overline{Y \setminus Q_\alpha} = \overline{(Y \cap Z) \setminus Q_\alpha} \cup \overline{U_Z \setminus Q_\alpha} = (Y \cap Z) \setminus Q_\alpha \cup \overline{U_Z}$$

Thus we have that $\overline{Y \setminus Q_\alpha} \cap Q_\alpha = \emptyset$ which means Q_α is Y -open. Q_α is compact so it is also Y -closed, meaning Q_α is a proper non-trivial clopen subset of Y . Y is connected so this contradiction gives us that $K \cap \overline{U_Z} \neq \emptyset$, which proves the claim.

Let $\rho : X \mapsto X/Z$ be the quotient map onto the quotient space X/Z . We can apply Lemma 3.2.21 to show that $X/Z = \text{irr}(\rho(p_m), \rho(Z))$ and that $\rho(\lambda(p_m))$ is an E-continuum of X/Z .

Claim. $\overline{U_Z}$ is irreducible between p_m and any point of $Z \cap \overline{U_Z}$

Let $L \subseteq \overline{U_Z}$ contain p_m and intersect Z . Then $\rho(L) = X/Z$ which in turn implies that $\rho^{-1}(X/Z \setminus \rho(Z)) \subseteq L$. Note that $U_Z = \rho^{-1}(X/Z \setminus \rho(Z))$ and consequently $\overline{U_Z} \subseteq L$. This proves our claim.

Claim. $\overline{U_Z}$ is decomposable.

We have that $\lambda(p_m)$ is an E-continuum of X/Z and thus X/Z is decomposable by Theorem 3.1.8. Let $X/Z = C \cup D$ for proper subcontinua C, D , and without loss of generality let $\rho(Z) \in C$ and $\rho(p_m) \in D$.

We will show that $\rho(Z)$ is not a cut point of C . Suppose $C \setminus \rho(Z) = V \cup W$ for disjoint V, W , each clopen in $C \setminus \rho(Z)$. One of them, say V , must intersect D . Thus $D \cup V \cup \rho(Z)$ is a continuum containing $\rho(Z)$ and $\rho(p_m)$, which by irreducibility is the whole of X/Z . This in turn implies that $W \subseteq D$. Applying the same argument we get that $V \subseteq D$, so $X/Z \setminus D = \rho(Z)$. This is a contradiction as a continuum has no isolated points. Thus $C \setminus \rho(Z)$ is connected. As ρ is monotone we have that $\overline{\rho^{-1}(C \setminus \rho(Z))}$ is a subcontinuum of X . Now $\overline{U_Z} = \overline{\rho^{-1}(C \setminus \rho(Z))} \cup \rho^{-1}(D)$. This is a proper decomposition so $\overline{U_Z}$ is decomposable.

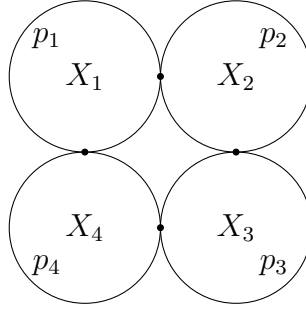
So far we have that each component of $Y \cap Z$ intersects $\overline{U_Z}$ and these intersections lie in the complement of a composant κ of $\overline{U_Z}$. This is a composant in the classic sense, of a continuum irreducible about a pair of points. Thus $(Y \cap Z) \cup \overline{U_Z} \setminus \kappa$ is a continuum contained in Y and containing $p_1, \dots, p_{m-1}$ but not p_m . This completes the proof. $\square$

Corollary 3.4.5. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$ and suppose each $\lambda(p_i)$ is compact. There exists a strictly descending chain $C_i \supsetneq C_{i+1}$ of subcontinua of X such that $C_i = \text{irr}(p_i, \dots, p_n)$.*

Proof. Let $C_1 = X$, then supposing $C_1, \dots, C_k$ are defined, let C_{k+1} be a proper subcontinuum of C_k such that $C_{k+1} = \text{irr}(p_{k+1}, \dots, p_n)$, in accordance with Lemmas 3.4.2 and 3.4.4. $\square$

The following example will crop up throughout this section and the next one. It is an excellent example of the unusual and unexpected behaviour that can arise when a continuum has indecomposable subcontinua which dominate its structure.

Example 3.4.6. The assumption of compactness in Lemma 3.4.4 is essential. As a counterexample let X_1, X_2, X_3 and X_4 be indecomposable continua and let X be a continuum obtained by taking the union of all four such that X_1 has one point in common with X_2 and one in common with X_4 , but none in common with X_3 and similarly with the other three continua to form a sort of square. Let p_i be a point of X_i which is not in the composants of the points X_i shares with its neighbours.



Then $X = \text{irr}(p_1, p_2, p_3, p_4)$. Let $Y \subseteq X$ be a subcontinuum with $Y = \min(p_1, p_2, p_3)$. Then $Y = X_1 \cup X_2 \cup X_3$ so $Y = \text{irr}(p_1, p_3)$.

The last few lemmas and corollaries have depended on $\lambda(p_i)$ being compact. This of course makes $\lambda(p_i)$ an analogy to an E-continuum. E-continua will be discussed in the following section in more detail, but we will look at a few results here related to complements and connectedness.

Theorem 3.4.7. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$ and let U be an open subset of X such that for some i , $\lambda(p_i) \subseteq U$ and $\lambda(p_i)$ is compact. Then there exists a connected open set V such that $\lambda(p_i) \subseteq V \subseteq U$ and $X \setminus \bar{V}$ is connected.*

Proof. Let $C_i \subsetneq X$ be a subcontinuum with $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n \in C_i$ so that $U_i = X \setminus C_i$ contains $\lambda(p_i)$. Let $\rho : X \mapsto Y = X/C_i$ be the quotient map to the quotient space X/C_i . By applying Lemma 3.2.21 we have that $Y = \text{irr}(\rho(p_i), \rho(C_i))$

and $\rho(\lambda(p_i))$ is an E-continuum of Y . Thus Y is locally connected at $\rho(\lambda(p_i))$ so there exists a connected open subset V' of Y such that $\rho(\lambda(p_i)) \subseteq V' \subseteq U \cap U_i$ and $Y \neq \overline{V'}$. If $V = \rho^{-1}(V')$ then

$$\lambda(p_i) \subseteq V \subseteq U \cap U_i \subseteq U$$

and V is a connected open set. Now $X \setminus \overline{V} = \rho^{-1}(Y \setminus \overline{V'})$ and since $\overline{V'}$ contains a point of irreducibility it is a terminal subcontinuum of Y (Theorem 3.1.7) and its complement is connected by Theorem 3.1.6. Consequently, $X \setminus \overline{V}$ is connected. $\square$

Proposition 3.4.8. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$ and suppose each $\lambda(p_i)$ is compact. For $1 \leq i \leq n$ let U_i be an open set containing $\lambda(p_i)$. Then there exist $V_1, \dots, V_n$ connected and open with $X \setminus \bigcup_{i=1}^n \overline{V_i}$ connected and for each $1 \leq i \leq n$ we have $V_i \subseteq U_i$.*

Proof. Shrink the U_i if necessary so that they each have disjoint closures. Take a descending chain $C_1, \dots, C_{n-1}$ of subcontinua of X such that $C_i = \text{irr}(p_i, \dots, p_n)$ as in Corollary 3.4.5. For $1 \leq i \leq n-2$ let $V_i \subseteq C_i$ be an open set with $\lambda(p_i) \subseteq V_i \subseteq U_i \cap C_i$, which is guaranteed to exist by Theorem 3.4.7. In C_{n-1} , take any connected open subsets V_{n-1}, V_n whose existence is guaranteed as C_{n-1} is locally connected about its E-continua $\lambda(p_{n-1}), \lambda(p_n)$ by Theorem 3.1.9. Let $A_i = C_i \setminus (C_{i+1} \cup \overline{V_i})$ for $1 \leq i \leq n-2$. From Lemma 3.2.21 we can see that A_i is the preimage under $\rho_i : C_i \mapsto C_i/C_{i+1}$ of the complement of a pair of disjoint terminal subcontinua, namely $\rho_i(C_{i+1})$ and $\rho(\overline{V_i})$, and is therefore connected by Theorem 3.1.4.

In each C_i there exists a proper subcontinuum containing all but one of the $p_i, \dots, p_n$, for any particular p_j . The image of this under the quotient map ρ_i clearly shows that A_i must lie in this subcontinuum, unless p_i is the point being excluded. Thus $\overline{A_i}$ must also be contained and as a result, $\overline{A_i}$ cannot intersect any $\lambda(p_j)$ for $j \neq i$. Consequently, each V_j can be chosen such that $\overline{A_i} \cap \overline{V_j} = \emptyset$ whenever $i < j$.

Now consider $C = X \setminus \bigcup_{i=1}^n \overline{V_i} = (C_{n-1} \setminus (\overline{V_{n+1}} \cup \overline{V_n})) \cup A_{n-2} \cup \dots \cup A_1$. From the previous paragraph we know that A_{n-2} limits onto $(C_{n-1} \setminus (\overline{V_{n+1}} \cup \overline{V_n}))$ so their union is connected. Said union is $C_{n-2} \setminus (\overline{V_{n-2}} \cup \overline{V_{n-1}} \cup \overline{V_n})$. Again, A_{n-3} must limit onto this, so their union is connected, and so on until we conclude that C is connected. $\square$

Proposition 3.4.9. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$ and suppose each $\lambda(p_i)$ is compact. Then $K = X \setminus \bigcup_{i=1}^n \lambda(p_i)$ is continuumwise connected.*

Proof. Let $x, y \in K$. Let U_i be open sets for $1 \leq i \leq n$ such that $\lambda(p_i) \subseteq U_i$ and $x, y \notin \overline{U_i}$. Take V_i as in Proposition 3.4.8 and let $C = X \setminus \bigcup_{i=1}^n \overline{V_i}$. Then C is a continuum containing x and y . Finally, $C \subseteq K$ as each $\lambda(p_i)$ is separated from $X \setminus \bigcup_{i=1}^n \overline{V_i}$ by the open set V_i , proving it is not in the closure C . $\square$

Example 3.4.10. The space in Example 3.4.6 also shows that the compactness assumption is required for Proposition 3.4.9. In fact if K were defined for the space in Example 3.4.6 it would not even be connected, much less continuumwise connected.

The next proposition seems like something which ought to be obvious and yet it is only now that we find ourselves in a position to prove it. In the classical case of irreducibility this would not even be a question; the very definition of irreducibility and composants makes the whole thing trivial. It is therefore reassuring that we can indeed prove it, at least when the $\lambda(p_i)$'s are compact. Yet again, Example 3.4.6 provides a counterexample if the requirement that the $\lambda(p_i)$'s are compact is dropped.

Proposition 3.4.11. *Let X be a continuum and suppose $X = \text{irr}(p_1, \dots, p_n)$ with each $\lambda(p_i)$ compact. Then given any $x \in X \setminus \bigcup_{i=1}^n \lambda(p_i)$ and any $k \in \{1, \dots, n\}$ there is a subcontinuum $Y \subseteq X$ such that $x \in Y$ and $p_j \in Y$ if and only if $j = k$.*

Proof. Take an open set U containing $\lambda(p_k)$ such that the closure of U does not intersect any other $\lambda(p_i)$. Using Theorem 3.4.7 take $V \subseteq U$ a connected open subset containing $\lambda(p_k)$. Since $X \setminus \bigcup_{i=1}^n \lambda(p_i)$ is dense by Proposition 3.2.12 there exists a point $y \in (X \setminus \bigcup_{i=1}^n \lambda(p_i)) \cap V$ and by Proposition 3.4.9 a subcontinuum $X' \subseteq X \setminus \bigcup_{i=1}^n \lambda(p_i)$ containing x and y . Then $Y = X' \cup \overline{V}$ is the desired continuum. $\square$

We end this section with a characterisation of k -terminal subcontinua of an n -irreducible continuum. This is an extension of Corollary 3.3.11.

Proposition 3.4.12. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$ and let each $\lambda(p_i)$ be compact. Then a subcontinuum $K \subseteq X$ is k -terminal for minimal k if and only if $|\{p_1, \dots, p_n\} \setminus K| = k - 1$.*

Proof. We will prove the reverse direction first. Take an n -decomposition $X = X_1 \cup \dots \cup X_k$ and suppose each $X_i \cap K \neq \emptyset$. By relabelling we can say $p_1, \dots, p_{k-1} \notin K$. Each point p_i for $1 \leq i \leq k - 1$ must lie in some X_j so again with some relabelling we can say that $p_1, \dots, p_{k-1} \in X_1 \cup \dots \cup X_{k-1}$. Then $X_1 \cup \dots \cup X_{k-1} \cup K$ is a subcontinuum containing $p_1, \dots, p_n$ so must be the whole of X , which proves that K is k terminal. For minimality, suppose $k > 1$ and let $x \in K \setminus \bigcup_{i=1}^n \lambda(p_i)$. Take P_i as in Proposition 3.4.11 for $1 \leq i \leq k - 1$. Then $X = \bigcup_{i=1}^{k-1} (K \cup P_i)$ which proves that K is not $(k - 1)$ -terminal.

The forwards direction is now trivial. Let $|\{p_1, \dots, p_n\} \setminus K| = l$. Then by the previous part, K is l -terminal for minimum l which implies that $l = k$. $\square$

It is worth noting that as a consequence of Proposition 3.4.12 we have that every n -irreducible continuum with compact $\lambda(p_i)$'s has a subcontinuum K which is k -terminal for each $k \geq 2$.

3.5 End Continua and E-Continua

In this section we will introduce n -E-continua and investigate the links between terminal, end and E-continua. A number of results will be proved which give criteria for an n -terminal continuum to be n -end, and for an n -end continuum to be an n -E-continuum. A common assumption in the statements of propositions will be that each $\lambda(p_i)$ is compact. Where possible these results will be proved without this assumption because it obviously stifles the use of a theorem proving something is an E-continuum if you must first assume the space has a full set of E-continua.

Proposition 3.5.1. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$ and let $K \subseteq X$ be an n -terminal continuum which has empty interior. Then K is an n -end continuum.*

Proof. Let $X = A_1 \cup \dots \cup A_n$ for subcontinua A_i such that each intersects K . Since K is n -terminal we can say without loss of generality that $X = K \cup A_1 \cup \dots \cup A_{n-1}$. Then $X \setminus (A_1 \cup \dots \cup A_{n-1})$ is an open set which lies in the interior of K , which is empty. Thus $X = A_1 \cup \dots \cup A_{n-1}$. From this we can conclude that no n -decomposition of X exists with every subcontinuum intersecting K , and in turn that K is an n -end continuum. $\square$

Proposition 3.5.2. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$ and let $K \subseteq X$ be an n -end continuum. Suppose each $\lambda(p_i)$ is compact. Then K is an n -terminal continuum with empty interior.*

Proof. It is clear that K is n -terminal from Proposition 3.3.8. If K had non-empty interior then it would intersect $X \setminus \bigcup_{i=1}^n \lambda(p_i)$ by Proposition 3.2.12. Let x witness this and let P_i be subcontinua of X as in Proposition 3.4.11, so for $1 \leq i, j \leq n$ we have $x \in P_i$ and $p_j \in P_i$ if and only if $i = j$. Clearly by irreducibility $X = \bigcup_{i=1}^n P_i$ and each P_i intersects K . The collection of P_i form an n -decomposition of X so this contradicts the fact that K is an n -end continuum. This contradiction gives us that K must have an empty interior. $\square$

We shall see in the following example that the assumption of compactness was essential in the statement of Proposition 3.5.2.

Example 3.5.3. Let X be the continuum defined in Example 3.4.6. We will show that X_1 is a 4-end continuum, despite not having empty interior. Take a decomposition $X = A_1 \cup A_2 \cup A_3 \cup A_4$ with each A_i intersecting X_1 non-trivially. As it is a decomposition we have that $\emptyset \neq X \setminus (A_2 \cup A_3 \cup A_4) \subseteq A_1$, so A_1 does not have an empty interior, and the same is true of each other A_i .

Let $p_3 \in A_i$ for some i . Since $A_i \cap X_1 \neq \emptyset$ it must be that either $X_2 \cup X_3 \subseteq A_i$ or $X_3 \cup X_4 \subseteq A_i$. Without loss of generality suppose it is the first option. Let $p_1 \in A_j$ and $p_4 \in A_k$ for some j, k not necessarily distinct from each other or i . As $A_k \cap X_1 \neq \emptyset$ it must be that $X_4 \subseteq A_k$. As A_j has non-empty interior it cannot be that $A_j \subseteq X_1$ so it follows that $X_1 \subseteq A_j$. Therefore $X = A_i \cup A_j \cup A_k$, which contradicts the fact that $X = A_1 \cup A_2 \cup A_3 \cup A_4$ is a decomposition. We can conclude that X_1 is a 4-end continuum.

Corollary 3.5.4. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$ and $K \subseteq X$ a subcontinuum. If $K \subseteq \lambda(p_i)$ for some $1 \leq i \leq n$ then K is an n -end continuum. If K is an n -end continuum and each $\lambda(p_i)$ is compact then $K \subseteq \lambda(p_i)$ for some i .*

Proof. For the first statement suppose $K \subseteq \lambda(p_i)$. This has empty interior so K does as well. Any $\{x\} \subseteq \lambda(p_i)$ is n -terminal by Proposition 3.3.8 and Theorem 3.3.10 so by Corollary 3.3.11 K is n -terminal. Thus K is an n -end continuum by Proposition 3.5.1.

Suppose now that K is an n -end continuum and each $\lambda(p_i)$ is compact. The proof of Proposition 3.5.2 shows that if K intersects $X \setminus \bigcup_{i=1}^n \lambda(p_i)$ then it cannot be n -end, so K must lie in $\bigcap_{i=1}^n \lambda(p_i)$. Each of the $\lambda(p_i)$ is closed in $\bigcap_{i=1}^n \lambda(p_i)$ so since K is connected it must be contained in one of them.

□

We will now formally introduce n -E-continua and begin investigating the conditions under which a subcontinuum is an n -E-continuum. Bennett and Fugate proved a number of theorems relating to this, and I will generalise their results to the setting of finite irreducibility.

Definition 3.5.5. Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$ and let $K \subseteq X$ be a subcontinuum. Then K is an n -E-continuum provided $K = \lambda(p_i)$ for some $p_i \in X$. Since each X can only be n -irreducible for one $n \in \mathbb{N}$ we shall refer to these continua as E-continua whenever it will not cause confusion.

Lemma 3.5.6. *Let $X = \text{irr}(p_1, \dots, p_n)$. Then $\lambda(p_i)$ is an E-continuum i.e. $\lambda(p_i)$ is compact if and only if for any subcontinuum K such that $K \cap \lambda(p_i) \neq \emptyset$ and $K \not\subseteq \lambda(p_i)$, K is decomposable.*

Proof. For the forwards direction suppose $\lambda(p_i)$ is compact and let K be a counterexample i.e. an indecomposable subcontinuum of X with $K \cap \lambda(p_i) \neq \emptyset$ and $K \not\subseteq \lambda(p_i)$. The composants of K are continuumwise connected, disjoint and dense by Theorem 3.1.5. The first of these properties implies that if one of the composants intersects $\kappa(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ then it must be contained in $\kappa(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$. From Proposition 3.2.8 we have that $\lambda(p_i) \subseteq K$ and as $K \not\subseteq \lambda(p_i)$ we know that this is a strict inclusion. As $\lambda(p_i)$ is a proper subcontinuum of K it is contained in a composant of K , and since it is compact it cannot be the whole of this composant, so the composant containing $\lambda(p_i)$ intersects, and is therefore contained in, $\kappa(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$. This is a contradiction, so no such K can exist.

For the reverse direction suppose the condition holds, but $\lambda(p_i)$ is not compact. Let $K = \overline{\lambda(p_i)}$. We know that K is decomposable so let $K = A \cup B$. Clearly it is not the case that $\lambda(p_i) \subseteq A$ or $\lambda(p_i) \subseteq B$. By our assumption $\lambda(p_i)$ cannot contain both A and B so without loss of generality let $A \not\subseteq \lambda(p_i)$. Since $A \cap \kappa(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n) \neq \emptyset$ there exists a proper subcontinuum $C \subsetneq X$ containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ and intersecting A . Then $A \cup C$ is a proper subcontinuum of X , as it does not contain $B \cap \lambda(p_i)$, with $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n \in A \cup C$ and some point of $\lambda(p_i)$ in $A \cup C$. This is a contradiction, so we have that $\lambda(p_i)$ must be compact. $\square$

We will use this lemma to prove the first of several results linking E-continua and end continua.

Proposition 3.5.7. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$ and let $\lambda(p_i)$ be compact, and therefore an E-continuum of X . Then $\lambda(p_i)$ is a maximal n -end continuum.*

Proof. If $A_1, \dots, A_n$ is an n -decomposition of X with each $A_j \cap \lambda(p_i)$ non-empty then by Proposition 3.2.8 we have that $\lambda(p_i) \subseteq A_j$ for each $1 \leq j \leq n$. For each $k \neq i$, p_k must lie in some A_j . Taking these at most $n - 1$ subcontinua to be $A'_1, \dots, A'_m$ then by the irreducibility of X it follows that $X = \bigcup_{j=1}^m A'_j$. Thus $\lambda(p_i)$ is an n -end continuum.

To see that $\lambda(p_i)$ is maximal, consider a subcontinuum K of X with $\lambda(p_i) \subseteq K$. Let C_i be a proper subcontinuum of X with $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n \in C_i$ and $C_i \cap K \neq \emptyset$. We know from Propositions 3.4.1 and 3.4.12 that $X \setminus C_i$ is connected and it must

contain $\lambda(p_i)$, so by Lemma 3.5.6 it must be that we can decompose $\overline{X \setminus C_i}$ into non-empty proper subcontinua $A_1 \cup B_1$. Suppose $\lambda(p_i) \subseteq A_1$. It cannot be that $X \setminus A_1 \subseteq C_i$ as then $X \setminus C_i \subseteq A_1$ so $\overline{X \setminus C_i} \subseteq A_1$, which contradicts that $A_1 \cup B_1$ is a decomposition of $\overline{X \setminus C_i}$. We further have that $C_i \cap A_1 = \emptyset$, else $X = C_i \cup A_1$ by irreducibility, which would again imply $X \setminus C_i \subseteq A_1$.

Repeat this same argument with $C_i \cup B_1$ in the place of C_i to create a new decomposition $A_2 \cup B_2 \subseteq A_1$, and then finitely many times more until we have $A_n \cup B_n \subseteq A_{n-1}$. Then

$$X = A_n \cup B_n \cup B_{n-1} \cup \cdots \cup B_1 \cup C_i$$

From the construction of each pair A_j and B_j we have that this is an $(n + 1)$ -decomposition. Each subcontinuum intersects K non-trivially, with all except C_i being contained in K . Thus K is not an n -end continuum, so $\lambda(p_i)$ is a maximal n -end continuum. $\square$

Proposition 3.5.8. *Let X be a continuum and $X = \text{irr}(p_1, \dots, p_n)$. Suppose each $\lambda(p_i)$ is compact. Then every maximal n -end continuum in X is an E -continuum.*

Proof. This follows immediately from Corollary 3.5.4. Each n -end continuum K must lie in some $\lambda(p_i)$. By Proposition 3.5.7 we have that $\lambda(p_i)$ is an n -end continuum and K is maximal, so $K = \lambda(p_i)$. $\square$

Example 3.5.9. A continuum similar to the one defined in Example 3.4.6 will again show the necessity of compact $\lambda(p_i)$ in the statement of Proposition 3.5.8. Let X be the continuum defined and let X' be constructed from X by attaching an arc $[0, 1]$ with the only point of intersection being to identify p_1 and 1. It is clear that $X' = \text{irr}(0, p_2, p_3, p_4)$. Let $x = 1/2 \in X'$. We shall show that $\{x\}$ is a maximal 4-end continuum.

To show that it is a 4-end continuum let $X = A_1 \cup A_2 \cup A_3 \cup A_4$ for four subcontinua and suppose $x \in \bigcap_{i=1}^4 A_i$. Without loss of generality let $p_3 \in A_1$. As $x \in A_1$ it follows that $A_1 \cap X_1 \neq \emptyset$, which in turn implies that either $X_2, X_3 \subseteq A_1$ or $X_3, X_4 \subseteq A_1$. Either way, A_1 contains two of the four points of irreducibility. Two of A_2, A_3 and A_4 will contain the others, so the union of this pair with A_1 will be the whole of X . Thus A_1, A_2, A_3 and A_4 do not form a decomposition, so $\{x\}$ is a 4-end continuum.

We must now show that $\{x\}$ is maximal. Let K be a subcontinuum of X' containing x . We can take a subarc of $[0, 1]$ lying in K and containing x , so let this be $[\alpha, \beta]$ with $\alpha \neq 0, \beta \neq 1$. Let $x' \in (\alpha, \beta)$. Then

$$X' = [0, \alpha] \cup [\alpha, x'] \cup [x', \beta] \cup ([\beta, 1] \cup X)$$

This is a decomposition of X' into four subcontinua, each intersecting K so K is not a 4-end continuum.

The previous example shows how unusual finite irreducibility can become once indecomposable subcontinua start to arise prominently. The subcontinuum $\{x\}$ is a 4-end continuum which cannot in any way be described as lying at the end of X .

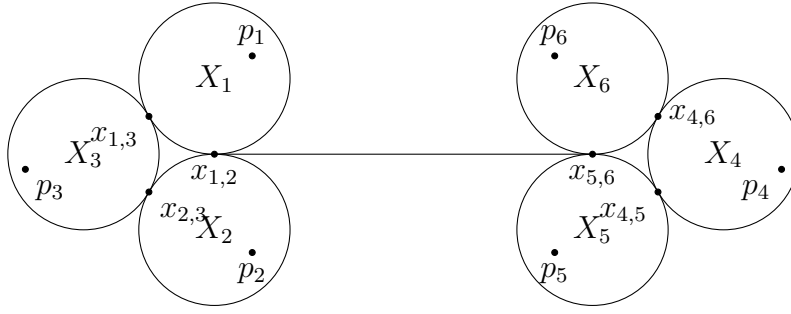
Proposition 3.5.10. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$. Then each E -continuum of X is a non-cutting n -end continuum.*

Proof. Suppose $\lambda(p_i)$ is compact. By Proposition 3.5.7 we have that $\lambda(p_i)$ is an n -end continuum, and it is clear from the definition of $\lambda(p_i)$ that its complement is continuumwise connected. $\square$

Proposition 3.5.11. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$ and suppose each $\lambda(p_i)$ is compact. Let $K \subseteq X$ be a non-cutting n -end continuum. Then K is an E -continuum.*

Proof. From Corollary 3.5.4 we know that $K \subseteq \lambda(p_i)$ for some $1 \leq i \leq n$. If $K \neq \lambda(p_i)$ then take $x \in \lambda(p_i) \setminus K$ and consider $X \setminus K$. Take a subcontinuum $C \subseteq X \setminus K$ containing x and some point of $\kappa(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$. Take a second subcontinuum D intersecting C and containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$. Then $C \cup D \subseteq X \setminus K$ is a proper subcontinuum of X containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ and a point of $\lambda(p_i)$ which is a contradiction. Thus $X \setminus K$ is continuumwise connected if and only if $K = \lambda(p_i)$. $\square$

Example 3.5.12. We shall construct a continuum similar to that of Example 3.4.6 to prove the necessity of the compactness assumption in Corollary 3.5.11. Instead of four indecomposable continua, this time only take three and again take a union of them with each pair intersecting at a single point. Let these indecomposable continua be X_1, X_2 and X_3 and the points of intersection be $x_{1,2}, x_{1,3}$ and $x_{2,3}$, chosen so that $x_{1,2}$ and $x_{1,3}$ lie in different composants of X_1 , and the same for the other pairings. Let $p_i \in X_i$ lie in a third composant. Take two such continua and join them together by an arc as shown below.



Denote this continuum as X and let $X' = X_1 \cup X_2 \cup X_3$. Then $X = \text{irr}(p_1, \dots, p_6)$ and X' is a 6-end continuum. It is not, however, an E-continuum.

Example 3.5.12 also proves that there are n -end continua about which X is locally connected which are not E-continua. This is again in contrast to the classical case of irreducibility.

Proposition 3.5.13. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$ and let $K \subseteq X$ be a subcontinuum. K is an E-continuum if and only if K is non cutting and has empty interior.*

Proof. The forwards direction is obvious as the complement of an E-continuum is a composant, which is continuumwise connected and dense. For the reverse, assume K is non-cutting and has empty interior. If for each $1 \leq i \leq n$ there exists $x_i \in \lambda(p_i) \setminus K$ then take a subcontinuum $C_i \subseteq X \setminus K$ containing x_1 and x_i . The union of these is a proper subcontinuum of X which intersects every $\lambda(p_i)$, contradicting that X is irreducible. Therefore there must be some i with $\lambda(p_i) \subseteq K$.

If $K \cap \kappa(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n) \neq \emptyset$ then there exists a subcontinuum $D \subsetneq X$ intersecting K and containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$. Then $X = K \cup D$ so $X \setminus D \subseteq K$. This is an open set, but K has empty interior. It follows that $K \subseteq \lambda(p_i)$, so $K = \lambda(p_i)$. $\square$

Chapter 4

Finite Irreducibility and Monotone Maps

4.1 Referenced Theorems

The following results from other authors will be used in this chapter.

Sam B. Nadler Jr, Continuum Theory, an Introduction

These results can be found in [Nad92].

Lemma 4.1.1 (3.2). *If a Hausdorff space is a continuous image of a compact metric space, then it is metrizable.*

Proposition 4.1.2 (9.27). *Let X be a non-degenerate tree. Then, a point $p \in X$ is a non-cut point of X if and only if p is an end point of X .*

Theorem 4.1.3 (9.28). *A continuum X is a tree if and only if X has only finitely many non-cut points.*

Theorem 4.1.4 (10.10). *A continuum X is a dendrite if and only if the intersection of any two connected subsets of X is connected. In particular, dendrites are unicoherent.*

Theorem 4.1.5 (11.8). *Let X be a non-degenerate continuum such that X is irreducible between p and q . If A and B are subcontinua of X such that $p \in A$ and $q \in B$ then $X \setminus (A \cup B)$ is connected.*

Corollary 4.1.6 (11.20). *A nondegenerate continuum X is indecomposable if and only if there are three points of X such that X is irreducible between each two of these three points.*

4.2 Monotone Maps to n -ods

In this section we will create a monotone map from an almost hereditarily decomposable n -irreducible continuum onto an n -od much like the monotone map from an almost hereditarily decomposable irreducible continuum onto an arc in Chapter 2. An n -od is the cone over an n -point discrete space, or equivalently the union of n arcs, each intersecting at one end point. For this section we will denote by X an almost hereditarily decomposable continuum with points $p_1, \dots, p_n \in X$ such that $X = \text{irr}(p_1, \dots, p_n)$. We will begin, as previously, by defining relevant irreducible subcontinua of X and proving properties about these subcontinua. This section will again focus on almost hereditarily decomposable continua with the assumption that every result will immediately hold for hereditarily decomposable continua. Where this is not the case the results will be proved separately.

Proposition 4.2.1. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. Then each $\lambda(p_i)$ is compact.*

Proof. Any subcontinuum K properly containing a $\lambda(p_i)$ must have non-empty interior by Proposition 3.2.8. From Lemma 3.5.6 we know that if $\lambda(p_i)$ were not compact then it would lie in an indecomposable subcontinuum, which would not have empty interior. Thus it must be that each $\lambda(p_i)$ is an E-continuum. $\square$

Definition 4.2.2. Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let x lie in X and let x not be a point of irreducibility of X . Define sets of subcontinua of X as follows.

$$\begin{aligned}\mathcal{P}_x^i &= \{C \subseteq X \mid C = \text{irr}(p_i, x), \forall j \neq i \ p_j \notin C\} \\ \mathcal{Q}_x^i &= \{C \subseteq X \mid C = \min(x, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)\}\end{aligned}$$

To see that each $\mathcal{P}_x^i$ is non-empty take subcontinua Y_i as in Proposition 3.4.11 and apply Theorem 2.2.5 to define a subcontinuum P_x^i of Y_i such that $P_x^i = \text{irr}(x, p_i)$ and $p_j \in P_x^i$ if and only if $i = j$. To see that $\mathcal{Q}_x^i$ is non-empty simply apply Lemma 3.4.2.

Proposition 4.2.3. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let $x \in X$ and let x not be a point of irreducibility of X . Let $Q_x^i \in \mathcal{Q}_x^i$. Suppose that $Q_x^i \neq \text{irr}(x, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$. It must be the case that $Q_x^i = \text{irr}(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$.*

Proof. If $Q_x^i \neq \text{irr}(x, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ then as Q_x^i is minimal about this set we must either have that $Q_x^i = \min(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ or that for some $j \neq i$,

$Q_x^i = \min(x, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{j-1}, p_{j+1}, \dots, p_n)$. In the first case we can apply Lemma 3.4.4 to reach our conclusion, so we will show that the second case cannot occur.

Let $C \subseteq X$ be a proper subcontinuum containing $p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_n$ and let $\rho : X \mapsto X/C$ be the quotient map to a quotient space. From Lemma 3.2.21 we have that $X/C = \text{irr}(\rho(C), \rho(p_j))$ and that $\rho(\lambda(p_j))$ is an E-continuum of X/C . It is therefore clear that $\rho(Q_x^i) = X/C$ as it contains both points of irreducibility $\rho(C)$ and $\rho(p_j)$. Apply Lemma 3.2.20 to see that X/C is almost hereditarily decomposable and apply Theorem 2.4.11 to get a monotone map $\pi : X/C \mapsto \mathbb{I}$. Let $\pi(\rho(p_j)) = 0, \pi(\rho(C)) = 1$ and $\pi(\rho(x)) = \alpha$ and let $U = (\pi \circ \rho)^{-1}[0, \alpha/2)$. If $Q_x^i \setminus U$ is disconnected then express it as disjoint compact sets A and B . Let $Y := X/C \setminus (\pi^{-1}([0, \alpha/2)))$ and consider that $Y = \rho(A) \cup \rho(B)$, so $Y \setminus \rho(C) = (\rho(A) \setminus \rho(C)) \cup (\rho(B) \setminus \rho(C))$. This implies that $Y \setminus \rho(C)$ is disconnected, but $Y \setminus \rho(C)$ is the inverse image under π of a connected set, $[\alpha/2, 1)$. Since π is monotone this is a contradiction.

We have shown that $Q_x^i \setminus U$ is connected, so must be a continuum. It contains the points $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{j-1}, p_{j+1}, \dots, p_n$ as well as x . This proves that for any $j \neq i$, $Q_x^i \neq \min(x, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{j-1}, p_{j+1}, \dots, p_n)$, thus completing our proof. $\square$

Proposition 4.2.4. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let $x \in X$ not be a point of irreducibility and let $1 \leq i \leq n$. Let P_x^i be an element of $\mathcal{P}_x^i$ and Q_x^i an element of $\mathcal{Q}_x^i$. Then one of the following holds.*

- $\forall y \in P_x^i \cap Q_x^i, P_x^i = \text{irr}(y, p_i)$
- $\forall y \in P_x^i \cap Q_x^i, Q_x^i = \min(y, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$

Proof. Suppose for contradiction that neither are the case and let subcontinua $Y \subsetneq P_x^i$ and $Z \subsetneq Q_x^i$ witness this i.e. $Y \cap Q_x^i \neq \emptyset \neq Z \cap P_x^i$ but $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n \in Z$ and $p_i \in Y$. Since they are proper subcontinua it must be that $x \notin Y \cup Z$ so $X \neq Y \cup Z$, which in turn implies $Y \cap Z = \emptyset$. Let $\rho : X \mapsto X/Z$ be the natural quotient map. Now from Lemma 3.2.21 we know $X/Z = \text{irr}(\rho(p_i), \rho(Z))$ so by Theorem 4.1.5 we have that $X/Z \setminus (\rho(Y) \cup \rho(Z))$ is connected. Since ρ is monotone we also have that $X \setminus (Y \cup Z)$ is connected. Let $C = \overline{X \setminus (Y \cup Z)}$. C is a continuum. As $Y \cup Q_x^i = X = Z \cup P_x^i$ we have that $X \setminus Y \subseteq Q_x^i$ and $X \setminus Z \subseteq P_x^i$, which in turn implies that $C \subseteq P_x^i \cap Q_x^i$. Since $X = Y \cup Z \cup C$ and Y and Z are disjoint it must be that $C \cap Y \neq \emptyset \neq C \cap Z$. Let y and z lie in these intersections.

Suppose $C \neq \text{irr}(x, y)$ and let $C' \subsetneq C$ witness this. Then $Y \cup C'$ is a subcontinuum of P_x^i containing p_i, x so $Y \cup C' = P_x^i$. Since $C \subseteq P_x^i$ we have that $C \setminus C' \subseteq Y$. This set is C -open so there exists X -open U such that $C \cap U = C \setminus C'$. As C is the closure of $X \setminus (Y \cup Z)$ it must be that $U \cap X \setminus (Y \cup Z) \neq \emptyset$. This is a subset of $U \cap C \subseteq Y$ which is a contradiction. Thus $C = \text{irr}(x, y)$. An identical argument gives that $C = \text{irr}(x, z)$.

Now suppose $C \neq \text{irr}(y, z)$ witnessed by C' . As $C = \text{irr}(x, y)$ it must be that $x \notin C'$. Consequently, $x \notin C' \cup Y \cup Z$ which is a continuum containing $p_1, \dots, p_n$. This contradiction gives us that $C = \text{irr}(y, z)$. Since C is irreducible between each pair of x, y, z it must be indecomposable by Corollary 4.1.6, but X is almost hereditarily decomposable and the open set $X \setminus (Y \cup Z)$ lies in C . Therefore, C ought to be decomposable and this contradiction completes the proof. $\square$

Definition 4.2.5. A point x is p_i -sided if the first option from Proposition 4.2.4 holds for some $P_x^i \in \mathcal{P}_x^i$ and $Q_x^i \in \mathcal{Q}_x^i$, and $\mathcal{Q}_x^i$ -sided if the second option holds. It is possible for a point to be both p_i -sided and $\mathcal{Q}_x^i$ -sided.

Proposition 4.2.6. Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let $x \in X$ not be a point of irreducibility. If x is p_i -sided then the set $\mathcal{P}_x^i$ has only one element.

Proof. Let $P_x^i, \tilde{P}_x^i \in \mathcal{P}_x^i$ and suppose these two continua are distinct and that P_x^i witnesses that x is p_i -sided to get a contradiction. Since both are irreducible, it must be that $P_x^i \not\subseteq \tilde{P}_x^i \not\subseteq P_x^i$. Take $Q_x^i \in \mathcal{Q}_x^i$. As $P_x^i \cup Q_x^i = X$ we have that $\tilde{P}_x^i \setminus P_x^i \subseteq Q_x^i$. Similarly, $P_x^i \setminus \tilde{P}_x^i \subseteq Q_x^i$, so $(P_x^i \cap \tilde{P}_x^i) \cup Q_x^i = X$. As P_x^i is irreducible between p_i and each point of $P_x^i \cap Q_x^i$ we have that $P_x^i \setminus \tilde{P}_x^i \subseteq \lambda_{P_x^i}(x)$, and therefore $P_x^i \cap \tilde{P}_x^i \supseteq \kappa_{P_x^i}(p_i)$. This is a dense subset of P_x^i and is contained in a closed subset $P_x^i \cap \tilde{P}_x^i$, so we have that $P_x^i \subseteq \tilde{P}_x^i$. This contradiction gives us the result we need. $\square$

Lemma 4.2.7. Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let $x \in X$ not be a point of irreducibility. If x is p_i -sided then each $P_x^i \in \mathcal{P}_x^i$ and $Q_x^i \in \mathcal{Q}_x^i$ witness this.

Proof. Suppose $P_x^i \in \mathcal{P}_x^i$ and $Q_x^i \in \mathcal{Q}_x^i$ witness that x is p_i -sided. From Proposition 4.2.6 we have that P_x^i is the only element of $\mathcal{P}_x^i$. Suppose $\tilde{Q}_x^i \in \mathcal{Q}_x^i$ is distinct from Q_x^i . If $P_x^i \cap \tilde{Q}_x^i \not\subseteq \lambda_{P_x^i}(x)$ then there exists a proper subcontinuum $C \subsetneq P_x^i$ with $p_i \in C$ and $C \cap \tilde{Q}_x^i \neq \emptyset$. From this we have that $C \cap Q_x^i = \emptyset$ and that $C \cup \tilde{Q}_x^i = X$, implying that $Q_x^i \subsetneq \tilde{Q}_x^i$. This is a contradiction as $\tilde{Q}_x^i = \min(x, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$, thus proving our result. $\square$

A number of the following propositions parallel similar results from Chapter 2. While they have been split into two to handle the difference between P_x^i and Q_x^i , the proofs are often nearly identical for these two cases. We will start by showing that Definition 4.2.2 does not depend on the choices of p_i .

Proposition 4.2.8. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let $x \in X$ not be a point of irreducibility and let p' be in $\lambda(p_i)$ and $P_x^i \in \mathcal{P}_x^i$. Then $p' \in P_x^i$ and $P_x^i = \text{irr}(p', x)$.*

Proof. Take $Q_x^i \in \mathcal{Q}_x^i$. We know that Q_x^i is a proper subcontinuum of X , so $p' \notin Q_x^i$. As $P_x^i \cup Q_x^i$ is a subcontinuum of X containing $p_1, \dots, p_n$ it follows that $X = P_x^i \cup Q_x^i$ and $p' \in P_x^i$. Take a subcontinuum $C \subseteq P_x^i$ containing x and p' . Then $X = C \cup Q_x^i$ so $p_i \in C$, meaning $C = P_x^i$. Thus $P_x^i = \text{irr}(p', x)$. $\square$

Proposition 4.2.9. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let $x \in X$ not be a point of irreducibility and for each $j \neq i$ let $q_j \in \lambda(p_j)$. Then for each $Q_x^i \in \mathcal{Q}_x^i$ we have that $q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n \in Q_x^i$ and $Q_x^i = \min(x, q_1, \dots, q_n)$.*

Proof. Take $P_x^i \in \mathcal{P}_x^i$. We know that the only point of $\{p_1, \dots, p_n\}$ lying in P_x^i is p_i so by Proposition 3.2.8 none of the q_j lie in P_x^i . As $P_x^i \cup Q_x^i$ is a subcontinuum of X containing $p_1, \dots, p_n$ it follows that $X = P_x^i \cup Q_x^i$ and $q_j \in Q_x^i$ for each $j \neq i$. Take a subcontinuum $C \subseteq Q_x^i$ containing x and $q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n$. Then $X = C \cup P_x^i$ so $p_j \in C$ for each j , meaning $C = Q_x^i$. Thus $Q_x^i = \min(x, q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n)$. $\square$

Corollary 4.2.10. *Let X be an almost hereditarily decomposable continuum and suppose $X = \text{irr}(p_1, \dots, p_n)$. Let $x \in X$ not be a point of irreducibility. If x is p_i -sided and $x, p_i \in C$ for some subcontinuum $C \subseteq X$ then the unique element P_x^i in $\mathcal{P}_x^i$ is a subcontinuum of C .*

Proof. Reapplying Theorem 2.2.5, there must be a subcontinuum of C which is irreducible between x and p_i which must be P_x^i since P_x^i is unique by Proposition 4.2.6. $\square$

Proposition 4.2.11. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let $x \in X$ not be a point of irreducibility. If x is $\mathcal{Q}_x^i$ -sided and there exists Q_x^i in $\mathcal{Q}_x^i$ with $Q_x^i = \text{irr}(x, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ then $\mathcal{Q}_x^i = \{Q_x^i\}$.*

Proof. The proof is identical to the one for Proposition 4.2.6. $\square$

Corollary 4.2.12. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let $x \in X$ not be a point of irreducibility. Suppose x is $\mathcal{Q}_x^i$ -sided and there exists $Q_x^i \in \mathcal{Q}_x^i$ with $Q_x^i = \text{irr}(x, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$. For any subcontinuum $C \subseteq X$ containing $x, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ we have that $Q_x^i \subseteq C$.*

Proof. The proof is identical to the one for Corollary 4.2.10. $\square$

We will now define an equivalence relation on X , again mirroring the approach taken with classical irreducibility. In Chapter 2 this was done by first defining an order with the points of irreducibility as maxima and minima, then taking points x and y to be related if $x \leq y \leq x$. Such an order will not exist for an n -irreducible continuum because every E-continuum ought to lie at an extreme of the order but it is not possible to assign each as a maximum or a minimum. Instead we will define the relation on its own, rather than using an order.

Definition 4.2.13. Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. For two points $x, y \in X$ we have that $x \sim y$ if and only if $x, y \in \lambda(p_i)$ for some i or for all i there exists $P_x^i \in \mathcal{P}_x^i, P_y^i \in \mathcal{P}_y^i, Q_x^i \in \mathcal{Q}_x^i$ and $Q_y^i \in \mathcal{Q}_y^i$ such that $P_x^i \cup Q_y^i = X = P_y^i \cup Q_x^i$.

The requirement that $P_x^i \cup Q_y^i = X$ is equivalent to $P_x^i \cap Q_y^i \neq \emptyset$. This second condition will be very useful in further proofs. The definition is seemingly determined by the choices of P_x^i etc, but as the next proposition will show this is not the case, making $\sim$ a well defined relation.

Proposition 4.2.14. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$ and let $\sim$ be as defined in Definition 4.2.13. This relation does not depend on which elements of $\mathcal{P}_x^i, \mathcal{Q}_x^i, \mathcal{P}_y^i$ or $\mathcal{Q}_y^i$ are chosen.*

Proof. Suppose P_x^i and $\tilde{P}_x^i$ are two distinct elements of $\mathcal{P}_x^i$. As both are irreducible and they are not equal to each other, we have that $P_x^i \not\subseteq \tilde{P}_x^i \not\subseteq P_x^i$. Let $y \in X$ and $Q_y^i \in \mathcal{Q}_y^i$ be such that $P_x^i \cup Q_y^i = X$. Then $\emptyset \neq \tilde{P}_x^i \setminus P_x^i \subseteq Q_y^i$ which means $\tilde{P}_x^i \cap Q_y^i \neq \emptyset$ and $\tilde{P}_x^i \cup Q_y^i = X$. An identical argument gives the same result for $Q_x^i, \tilde{Q}_x^i$ and from this we can conclude that if $P_x^1, \dots, P_x^n, Q_x^1, \dots, Q_x^n$ give $x \sim y$ then so do $\tilde{P}_x^1, \dots, \tilde{P}_x^n, \tilde{Q}_x^1, \dots, \tilde{Q}_x^n$ $\square$

The proof of the next theorem is slightly more complicated than the proof of its counterpart in Chapter 2. This extra complexity arises from the uncertainty of whether $Q_z^i = \text{irr}(z, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ or $Q_z^i = \text{irr}(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ (see Proposition 4.2.3).

Theorem 4.2.15. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$ and let $\sim$ be as defined in Definition 4.2.13. The relation $\sim$ is an equivalence relation.*

Proof. It is clear to see that $\sim$ is reflexive and symmetric. Suppose $x \sim y$ and $y \sim z$ for three distinct points $x, y, z \in X$. If any one of them lies in $\lambda(p_i)$ for some i then all three must do so and $x \sim z$. Suppose none of them lie in an E-continuum of X .

We only need to show that for all i and any $P_x^i \in \mathcal{P}_x^i, Q_z^i \in \mathcal{Q}_z^i$, we have that $P_x^i \cup Q_z^i = X$ because x and z can be interchanged freely. If $z \in P_x^i$ or $x \in Q_z^i$ then this is the case so let us consider when $z \notin P_x^i$ and $x \notin Q_z^i$. Since $x \sim y$ and $y \sim z$ we know for all $P_y^i \in \mathcal{P}_y^i$ and all $Q_y^i \in \mathcal{Q}_y^i$ that $P_y^i \cup Q_y^i = X = P_x^i \cup Q_y^i$. Thus without loss of generality we can say $x \in P_y^i$ and $z \in Q_y^i$.

If y is p_i -sided then $P_x^i \subseteq P_y^i$ by Corollary 4.2.10. Since $X = P_x^i \cup Q_y^i$ we have that

$$\emptyset \neq P_x^i \cap Q_y^i \subseteq P_y^i \cap Q_y^i \subseteq \lambda_{P_y^i}(y)$$

Therefore $P_x^i = P_y^i$ which gives us that $P_x^i \cup Q_z^i = X$.

If y is Q_y^i -sided and $Q_y^i = \text{irr}(y, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ then use the same argument as for y being p_i -sided. If $Q_y^i = \text{irr}(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ then as $z \in Q_y^i$ we have that $Q_y^i \in \mathcal{Q}_z^i$. Proposition 4.2.14 and the fact that $P_x^i \cup Q_y^i = X$ give us that $P_x^i \cup Q_z^i = X$. $\square$

Definition 4.2.16. Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$ and let $\sim$ be as defined in Definition 4.2.13. Let $x \in X$. We denote by $x_\sim$ the equivalence class of x under $\sim$.

In order to study the equivalence classes we will split X into various subsets and consider the equivalence classes of points in each of those subsets. It will turn out that each subset is the union of whole equivalence classes and that the interactions between the subsets will create the n -od structure we are hoping to map to.

Lemma 4.2.17. *Let X be an almost hereditarily decomposable, n -irreducible continuum with $X = \text{irr}(p_1, \dots, p_n)$ and let $x \in X$ not be a point of irreducibility. Suppose there exists $Q_x^i \in \mathcal{Q}_x^i$ with $Q_x^i = \text{irr}(x, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ and this is witnessed by $Q \ni p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ with $Q \cap P_x^i = \emptyset$. Let $\rho : X \mapsto X/Q$ be the natural quotient map. Then X/Q is almost hereditarily decomposable and $X/Q = \text{irr}(\rho(p_i), \rho(Q))$.*

Proof. From Lemma 3.2.20 we know that X/Q is almost hereditarily decomposable. From Lemma 3.2.21 we have that $X/Q = \text{irr}(\rho(p_i), \rho(Q))$. $\square$

Proposition 4.2.18. *Let X, x, Q_x^i, Q and ρ be as in Lemma 4.2.17 and $\sim, x_\sim$ be as in Definitions 4.2.13 and 4.2.16. By applying Theorem 2.4.11 let $\pi : X/Q \mapsto \mathbb{I}$ be a monotone quotient map onto an arc. Then $x_\sim$ is homeomorphic under ρ to the fibre of π containing $\rho(x)$.*

Proof. First suppose y is in Q . Then there exists a choice of $Q_y^i \subseteq Q$ and as $Q \cap P_x^i = \emptyset$ we have that $P_x^i \cup Q_y^i \neq X$ and $x \approx y$. This implies that $x_\sim \subseteq X \setminus Q$ and that $\rho|_{x_\sim}$ is a homeomorphism.

Now take a point $y \notin Q$. Since $X/Q = \rho(Q_x^i) = \rho(Q_y^j)$ for any $j \neq i$ and any $Q_x^j \in \mathcal{Q}_x^j, Q_y^j \in \mathcal{Q}_y^j$ it must be that $x \in Q_y^j, y \in Q_x^j$. Thus we only have to check when $X = P_x^i \cup Q_y^i = P_y^i \cup Q_x^i$. We have that $P_x^i \cap Q_y^i \neq \emptyset$ if and only if $\rho(P_x^i) \cap \rho(Q_y^i) \neq \emptyset$, and also that $P_y^i \cap Q_x^i \neq \emptyset$ if and only if $\rho(P_y^i) \cap \rho(Q_x^i) \neq \emptyset$. The first of these is because P_x^i does not intersect Q which is the only non-trivial fibre of ρ . The second is because $\rho(P_y^i)$ is a continuum and if it contains $\rho(Q)$ then it must be the whole of X/Q , so either way it intersects $\rho(Q_x^i)$ somewhere other than $\rho(Q)$. This proves that $x \sim y$ if and only if they are similarly related in X/Q i.e. if they are in the same fibre of π . $\square$

Definition 4.2.19. Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. Define subsets of X as follows.

$$M_i := \{x \in X \mid \forall Q \subseteq Q_x^i \text{ with } p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n \in Q \text{ we have } Q \cap P_x^i \neq \emptyset\}$$

$$M = \bigcap_{i=1}^n M_i$$

Note that in the definition of M_i it does not matter which $Q_x^i \in \mathcal{Q}_x^i$ is chosen. If there exists Q_x^i witnessing that $x \in M_i$ and another $\tilde{Q}_x^i \in \mathcal{Q}_x^i$, with $Q \subseteq \tilde{Q}_x^i$ containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ then consider the intersection $Q \cap P_x^i$. If this is the empty set then as $X = P_x^i \cup Q_x^i$ we have that $Q \subseteq Q_x^i$. However, this contradicts that Q_x^i witnesses $x \in M_i$. Thus it must be that $Q \cap P_x^i \neq \emptyset$ and that $\tilde{Q}_x^i$ also witnesses $x \in M_i$.

For a similar reason it does not matter which P_x^i is chosen. Again if P_x^i witnesses that $x \in M_i$ and $\tilde{P}_x^i \in \mathcal{P}_x^i, Q \subseteq Q_x^i \in \mathcal{Q}_x^i$ with $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n \in Q$ then as $P_x^i \cup Q = X$ it must either be that $\tilde{P}_x^i$ intersects Q non-trivially or $\tilde{P}_x^i$ is a proper subcontinuum of P_x^i . Since $P_x^i = \text{irr}(p_i, x)$ with both of these points lying in $\tilde{P}_x^i$ it must be that $\tilde{P}_x^i \cap Q \neq \emptyset$.

It is not necessary for $Q_x^i = \text{irr}(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ in order for $x \in M_i$, but if this is not the case then x will be p_i -sided. It may help to consider M as a subset of X lying almost in the middle, whose points are not particularly close to any one of the p_i compared to the others. It should come as no surprise given this perspective that M will be the preimage of the central point of our n -od, when we come to construct the map.

Proposition 4.2.20. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let $\sim$ be as in Definition 4.2.13 and let M be as in Definition 4.2.19. Given any $x \in M$ we have that $x_\sim = M$.*

Proof. If $y \in x_\sim$ but $y \notin M$ then $y \notin M_i$ for some $1 \leq i \leq n$. Let $P_x^i \in \mathcal{P}_x^i$, $Q_x^i \in \mathcal{Q}_x^i$ and $Q \subseteq Q_x^i$ witness this. Applying Proposition 4.2.18 gives us that $x_\sim = y_\sim = (\pi \circ \rho)^{-1}(\alpha)$ for some $\alpha < 1$. This in turn implies that there exists $\tilde{P}_x^i \subseteq (\pi \circ \rho)^{-1}([0, \alpha])$ so $\tilde{P}_x^i \cap Q = \emptyset$. From the comments in Definition 4.2.19 we can conclude that $x \notin M_i$ and $x \notin M$. We have therefore shown that if $x \in M$ then $x_\sim \subseteq M$.

Now let $x, y \in M$. For any $1 \leq i \leq n$, if $P_x^i \cap Q_y^i = \emptyset$ then $Q_y^i \subseteq Q_x^i$ which contradicts the fact that $x \in M_i$. Thus $P_x^i \cap Q_y^i \neq \emptyset$ and by symmetry $P_y^i \cap Q_x^i \neq \emptyset$. This gives us that $x \sim y$, completing the proof. $\square$

Proposition 4.2.21. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let $\sim$ be as in Definition 4.2.13 and let M be as in Definition 4.2.19. M is compact*

Proof. Since X is a compact metric space this is equivalent to the claim that M is a closed set. Let $x \in \overline{M} \setminus M$ and for some i let $Q \subseteq Q_x^i \in \mathcal{Q}_x^i$ witness that $x \notin M_i$. Let ρ, π be as in Proposition 4.2.18 with $[0, 1]$ as the image of π , $(\pi \circ \rho)(Q) = 1$. As $P_x^i \cap Q = \emptyset$ it cannot be that $1 \in (\pi \circ \rho)(P_x^i)$, so in particular $(\pi \circ \rho)(x) \neq 1$. Let $(\pi \circ \rho)(x) < \alpha < \beta < (\pi \circ \rho)(Q)$. Then $U = (\pi \circ \rho)^{-1}([0, \alpha])$ and $V = (\pi \circ \rho)^{-1}((\beta, 1])$ are connected open sets separating P_x^i and Q . Let $Y = \overline{U}$. Y is a continuum but as $x \in U$ it must be that U , and Y , intersect M , say at y . But there exists a $P_y^i \subseteq Y$ and $Y \cap Q = \emptyset$ which contradicts that $y \in M$. Thus it must be that $M = \overline{M}$ i.e. M is compact. $\square$

Having established one unusual equivalence class, the remaining classes will all be much simpler. The complement of M consists of n sets each of which can be easily handled by Proposition 4.2.18.

Lemma 4.2.22. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let C be a proper subcontinuum of X containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$. Then $X \setminus C$ is connected.*

Proof. Let $\rho : X \mapsto X/C$ be the natural quotient map to the quotient space X/C . By Lemma 3.2.21 we have that $X/C = \text{irr}(\rho(C), \rho(p_i))$. By Proposition 2.2.4 we have that $X/C \setminus \rho(C)$ is connected, and as ρ is monotone it follows that $X \setminus C$ is connected. $\square$

Lemma 4.2.23. *Let X be a continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let C and D be subcontinua of X with C and D both irreducible about $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$. Then $\overline{X \setminus C} = \overline{X \setminus D}$.*

Proof. If $C = D$ then this result is trivial. If they are not equal then as both of them are irreducible about $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ it must be that $C \not\subseteq D \not\subseteq C$. From this we have that $D \setminus C \neq \emptyset$, which implies that $D \cap (X \setminus C) \neq \emptyset$. From Lemma 4.2.22 we have that $\overline{X \setminus C}$ is a continuum which means $D \cup \overline{X \setminus C}$ is a subcontinuum of X containing $p_1, \dots, p_n$. This means $X = D \cup \overline{X \setminus C}$ and in turn implies that $X \setminus D \subseteq \overline{X \setminus C}$. Taking the closure we see that $\overline{X \setminus D} \subseteq \overline{X \setminus C}$. By symmetry we have that the same result holds with C and D swapped, so $\overline{X \setminus C} = \overline{X \setminus D}$. $\square$

Lemma 4.2.24. *Let X be an almost hereditarily decomposable, n -irreducible continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let $Y_i \subsetneq X$ be a proper subcontinuum such that $Y_i = \text{irr}(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ (Lemma 3.4.4) and let $Z_i = \overline{X \setminus Y_i}$. Then Z_i is a continuum and for all $x \in Y_i \cap Z_i$, $Z_i = \text{irr}(x, p_i)$ and $X \setminus M_i = \kappa_{Z_i}(p_i)$.*

Proof. The subset Z_i is a continuum by Lemma 4.2.22. It is well defined in that it does not depend on the choice of Y_i by Lemma 4.2.23.

Let $x \in Y_i \cap Z_i$ and let $Z' \subseteq Z$ be a subcontinuum containing x and p_i . Then $x \in Y_i \cap Z'$ so $Y_i \cup Z'$ is a continuum containing $p_1, \dots, p_n$, which means it must be X . It follows that $X \setminus Y_i \subseteq Z'$ and since Z' is compact $Z_i \subseteq Z'$. Thus $Z_i = Z'$ which gives us that $Z_i = \text{irr}(x, p_i)$.

We need to show that Z_i is almost hereditarily decomposable. Let $C \subseteq Z_i$ be indecomposable and let $V \subseteq X$ be X -open such that $V \cap Z_i \subseteq C$. Then $V \cap (X \setminus Y_i)$ is an X -open set with $V \cap (X \setminus Y_i) \subseteq V \cap Z_i \subseteq C$. Since X is almost hereditarily decomposable it must be that C has an empty X -interior, so $V \cap (X \setminus Y_i) = \emptyset$. This implies that $V \cap Z_i$ is a Z_i -open set which does not intersect the Z_i -dense subset $X \setminus Y_i$, meaning $V \cap Z_i = \emptyset$ and C has empty interior. Therefore Z_i is almost hereditarily decomposable.

Since Z_i is almost hereditarily decomposable and irreducible we can apply Theorem 2.4.11, so let $\pi_i : Z_i \mapsto \mathbb{I}$ be the universal monotone map onto the unit interval

with $\pi(p_i) = 0, \pi(Y_i \cap Z_i) = 1$. Let $x \in Y_i$. Then $Y_i \in \mathcal{Q}_x^i$ which clearly implies that $x \in M_i$. Now let $x \in \pi^{-1}(1)$. We have that $Z_i \in \mathcal{P}_x^i$. Given any $Q \subseteq Q_x^i$ containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ if $Q \cap Z_i = \emptyset$ then $Q \subseteq Y_i$ so $Q = Y_i$ which in turn implies $Q \cap Z_i \neq \emptyset$. Thus it must be that $Z_i \cap Q \neq \emptyset$. From this we get that $x \in M_i$, so $X \setminus M_i \subseteq \kappa_{Z_i}(p_i)$. For the other inclusion let $\pi_i(x) = \alpha < 1$. This gives us that there exists $P_x^i \in \mathcal{P}_x^i$ with $P_x^i \subseteq \pi_i^{-1}[0, \alpha]$. For each $Q_x^i \in \mathcal{Q}_x^i$ we therefore have that $Y_i \subseteq Q_x^i$. As $Y_i \cap P_x^i = \emptyset$ we have that $x \in X \setminus M_i$ which completes the proof. $\square$

Remark 4.2.25. Each M_i must be closed, as $\kappa_{Z_i}(p_i)$ is an open subset of Z_i and is contained in $X \setminus Y_i \subseteq Z_i$, an open subset of X . This gives an alternative proof that M is compact.

Corollary 4.2.26. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let Y_i and Z_i be as in Lemma 4.2.24. If $x \in Z_i \setminus \kappa_{Z_i}(p_i)$ then $x \in M$.*

Proof. We know that $x \in M_i$ by Lemma 4.2.24. Consider $\sigma : X \mapsto X/Y_i$ and note that $X/Y_i = \text{irr}(\sigma(p_i), \sigma(Y_i))$ by Lemma 3.2.21. For any $j \neq i$ and any $Q \subseteq Q_x^j \in \mathcal{Q}_x^j$ with $p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_n \in Q$ we have that $\sigma(Q) = X/Y_i$. This implies $\kappa_{Z_i}(p_i) \subseteq Q$ and as Q is closed, $Z_i \subseteq Q$. Thus $x \in Q$ and for each $P_x^j \in \mathcal{P}_x^j$ we have that $Q \cap P_x^j \neq \emptyset$, so $x \in M_j$. $\square$

Corollary 4.2.27. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let $\sim$ be as in Definition 4.2.13 and let Y_i, Z_i be as in Lemma 4.2.24 and let $x \in \kappa_{Z_i}(p_i)$. Then $x_\sim \subseteq \kappa_{Z_i}(p_i)$ and $x_\sim$ is a continuum. Indeed if $\pi_i : Z_i \mapsto \mathbb{I}$ is the monotone map from Theorem 2.4.11 then $x_\sim$ is a fibre of π_i .*

Proof. From Lemma 4.2.24 we have that $x \notin M_i$ from which we can deduce that $Q_x^i \neq \text{irr}(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$. Taking $\pi_i : Z_i \mapsto [0, 1]$ from Theorem 2.4.11, let $\pi_i(x) = \alpha < 1$. Then P_x^i can be chosen to lie in $\pi_i^{-1}([0, \alpha])$, so $P_x^i \cap Y_i = \emptyset$. This in turn implies $Y_i \subsetneq Q_x^i$.

By applying Proposition 4.2.18 we have that $x_\sim$ is homeomorphic to the equivalence class of the image of x in X/Y_i , which will be a continuum. Let $\rho : X \mapsto X/Y_i$. The restriction of ρ to Z_i is a map which only affects one of the E-continua of Z_i , so the equivalence classes in $\kappa_{X/Y_i}(p_i)$ will be homeomorphic to fibres of π_i . This means that $x_\sim$ is a fibre of π_i and lies in $\kappa_{Z_i}(p_i)$. $\square$

We have now found all of the equivalence classes, namely M and the equivalence classes of the Z_i lying in $\kappa_{Z_i}(p_i)$ corresponding to an application of Theorem 2.4.11 to

Z_i . We know that all of the equivalence classes are compact and with the exception of M we know they are all connected. The following lemma will be used to show that M is also connected.

Lemma 4.2.28. *Let X be an almost hereditarily decomposable, n -irreducible continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let Z_i be as defined in Lemma 4.2.24. For $i \neq j$ we have that $\kappa_{Z_i}(p_i) \cap \overline{\kappa_{Z_j}(p_j)} = \emptyset$.*

Proof. The proof is simply the manipulation of various sets. Note that $\overline{\kappa_{Z_j}(p_j)} = Z_j$.

$$\begin{aligned}
Y_i \subseteq M_i &\Rightarrow X = Y_i \cup Y_j \subseteq M_i \cup M_j \\
&\Rightarrow \kappa_{Z_j}(p_j) = Z_j \setminus M_j \subseteq M_i \\
&\Rightarrow Z_j \subseteq M_i \\
&\Rightarrow M_i \cup (X \setminus Z_j) = X \\
&\Rightarrow (X \setminus M_i) \cap Z_j = \emptyset \\
&\Rightarrow \kappa_{Z_i}(p_i) \cap \overline{\kappa_{Z_j}(p_j)} = \emptyset
\end{aligned}$$

□

Proposition 4.2.29. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$ and let M be as defined in Definition 4.2.19. M is connected.*

Proof. Let $M = A \cup B$ for disjoint closed A and B . From Lemmas 4.2.24 and 4.2.28 we have that $M = \bigcap_{i=1}^n M_i = X \setminus \bigcup_{i=1}^n \kappa_{Z_i}(p_i)$ and that each $\lambda_{Z_i}(Y_i)$ lies in M . Here $\lambda_{Z_i}(Y_i)$ is the E-continuum of Z_i containing $Y_i \cap Z_i$. As these E-continua are connected they must each lie wholly in one of A or B . Let $A' := A \cup \bigcup \{\kappa_{Z_i}(p_i) \mid \lambda_{Z_i}(Y_i) \subseteq A\}$ and define B' similarly. Now $X = A' \cup B'$ and both A' and B' are closed and disjoint, with closure again a corollary of Lemma 4.2.28. Since X is connected one of A' and B' must be empty, which implies that one of A and B must be empty. □

We are now in a position to prove the major theorem of this section.

Theorem 4.2.30. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. There exists a monotone map π from X to an n -od.*

Proof. Let $\sim$ be as in Definition 4.2.13 and let $\pi : X \mapsto X/\sim$ be the natural quotient map. Each fibre of π is either M by Proposition 4.2.20 or a fibre of some $\pi_i : Z_i \mapsto \mathbb{I}$ by Corollary 4.2.27. From Proposition 4.2.21, Corollary 4.2.27 and Proposition 4.2.29 we have that these are all continua. For any $1 \leq i \leq n$ it is clear that $\pi|_{Z_i} = \pi_i$ so $\pi(Z_i)$ is an arc. Each of these n arcs all intersect at $\pi(M)$ and nowhere else by Lemma 4.2.28, so $X/\sim$ is the union of n arcs all meeting at a point, which is an n -od. □

Corollary 4.2.31. *X is locally connected at each fibre of π .*

Proof. Any n -od is locally connected, and by Proposition 2.5.1 we have our result. $\square$

Lemma 4.2.32. *Let X be an almost hereditarily decomposable, n -irreducible continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let $\sim$ and M be as in Definitions 4.2.13 and 4.2.19. Let $x \notin M$, then $x_\sim$ has empty interior.*

Proof. Each such $x_\sim$ is also an equivalence class of some Z_i under the equivalence relation defined in Definition 2.4.1. Thus $\text{int}_{Z_i}(x_\sim) = \emptyset$ by Lemma 2.4.14 and it follows that $\text{int}_X(x_\sim) = \emptyset$. $\square$

When dealing with classical irreducibility in Chapter 2 we not only constructed a monotone map to an arc, but proved that it was universal. The same will be done here, followed by an example to show that it is only universal amongst monotone maps to n -ods and not other k -ods.

Theorem 4.2.33. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. let $\pi : X \mapsto X/\sim$ be as in Theorem 4.2.30. The map $\pi : X \mapsto X/\sim$ is universal amongst monotone maps to n -ods.*

Proof. Let $\rho : X \mapsto Y$ be another monotone map to an n -od Y . We will first show that ρ preserves equivalence classes. Let $m \in Y$ be the point of order n . Since the map is monotone $Y = \min(\rho(p_1), \dots, \rho(p_n))$ so the end points of the stalks of Y are each the image of some p_i . As n -ods are n -irreducible it must therefore be that $Y = \text{irr}(\rho(p_1), \dots, \rho(p_n))$.

Given any subcontinuum $X' \subseteq X$ contained in some $\kappa_{Z_i}(p_i)$ we have that $X \setminus X'$ has at most two components so $\rho(X \setminus X')$ has at most two as well, which means that $X' \neq \rho^{-1}(m)$. From this we have that $\rho^{-1}(m) \cap M \neq \emptyset$.

Now let $x \in X$ and suppose $\rho(x) \neq m$. Then $\rho(x)$ must lie in one of the stalks of Y , say the one ending in $\rho(p_i)$. Define a trio of subcontinua of Y as follows. Firstly, if $\rho(x) = \rho(p_i)$ then $A = \{\rho(x)\}$ and $B = Y$. Otherwise let $A = \text{irr}(\rho(x), \rho(p_i))$ and $B = \min(\rho(x), \rho(p_1), \dots, \rho(p_{i-1}), \rho(p_{i+1}), \dots, \rho(p_n))$. Either way, let C be the $(n-1)$ -od containing every stalk except the one with $\rho(x)$ in it. Now pick P_x^i, Q_x^i and Q such that $P_x^i \subseteq \rho^{-1}(A)$, $Q_x^i \subseteq \rho^{-1}(B)$ and $Q \subseteq \rho^{-1}(C)$ with $Q = \text{irr}(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$. Then $\rho(Q) \cap \rho(P_x^i) = \emptyset$ so $Q \cap P_x^i = \emptyset$. This gives us that $x \notin M$ so $\rho(M) = \{m\}$.

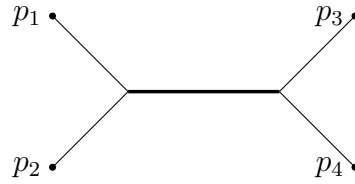
Now let $x_\sim \subseteq \kappa_{Z_i}(p_i)$ for some x and i . The image of this is a subcontinuum of Y so if it is non-degenerate it must have non empty interior. This would imply, via the irreducibility of X , that $x_\sim$ has non-empty interior which contradicts Lemma

4.2.32. This is because if $Y_1, \dots, Y_m$ are the components of $\overline{Y \setminus \rho(x_\sim)}$ then we have that $X = x_\sim \cup \rho^{-1}(Y_1) \cup \dots \cup \rho^{-1}(Y_m)$, and so $X \setminus \rho^{-1}(Y_1) \cup \dots \cup \rho^{-1}(Y_m)$ is an open subset of $x_\sim$. This subset is non-empty as its image under ρ is not the whole of Y . This implies that ρ maps $x_\sim$ to a single point, so we have shown that ρ preserves equivalence classes.

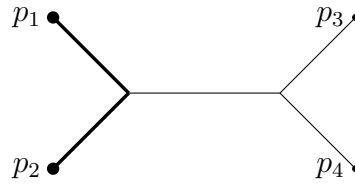
Now define $f : X/\sim \mapsto Y$ by $f(x_\sim) = \rho(x)$. As we have just seen this map is well defined and clearly $f \circ \pi = \rho$. To check that f is continuous let U be an open subset of Y and note that $f^{-1}(U) = \pi(\rho^{-1}(U))$. Since ρ preserves equivalence classes we have that $\pi^{-1}(\pi(\rho^{-1}(U))) = \rho^{-1}(U)$. This is open so it follows that $f^{-1}(U)$ is open, so f is continuous. This completes the proof. $\square$

It is worth pointing out that the map f is monotone, as $f^{-1}(y) = \pi(\rho^{-1}(y))$. Now π is continuous and ρ monotone, so it follows that f is monotone.

Example 4.2.34. Below is a continuum X with $X = \text{irr}(p_1, p_2, p_3, p_4)$. The subcontinuum M is highlighted in bold.



We can define a monotone map $\rho : X \mapsto Y$ where Y is a triad by quotienting the bold subcontinuum of the following diagram to a point.



Since the image of M under ρ is not a singleton there cannot exist a map $f : X/\sim \mapsto Y$ such that $f \circ \pi = \rho$.

We shall end this section by proving a result analogous to that of Miller, similar to Theorem 2.5.10.

Theorem 4.2.35. *Let X be an almost hereditarily decomposable continuum with $X = \text{irr}(p_1, \dots, p_n)$. Let π be as in Theorem 4.2.30. The continuum X is unicoherent if each fibre of π is unicoherent.*

Proof. Let $X = A \cup B$. If either A or B do not intersect M , say A , then A must be contained in the inverse image of one of the stalks of $X/\sim$. Let this stalk contain p_i . Let $Y \subseteq X$ be a subcontinuum such that $Y = \text{irr}(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ and let $\rho : X \mapsto X/Y$. Then $X/Y = \text{irr}(\rho(p_i), \rho(Y))$ by Lemma 3.2.21 and is almost hereditarily decomposable by Lemma 3.2.20. Furthermore, $X/Y = \rho(A) \cup \rho(B)$ and $A \cap B \cong \rho(A) \cap \rho(B)$ since $A \cap Y = \emptyset$. If $\pi' : X/Y \mapsto \mathbb{I}$ is the usual universal monotone map then each fibre of π' , with the possible exception of the one containing $\rho(Y)$, will be unicoherent as they will be homeomorphic to fibres of π . This is enough to see that $\rho(A) \cap \rho(B)$ is connected, and therefore $A \cap B$ is as well.

Now suppose A and B both intersect M . By considering the stalks of $X/\sim$ which A intersects we can see that $A \setminus M$ is the union of finitely many connected A -open sets $U_1, \dots, U_r$. For any k , $\overline{U_k}$ is irreducible with $\overline{U_k} \setminus U_k \subseteq M$ as one of the E-continua. If $A \cap M$ is disconnected, into say D and E , then each $\overline{U_k}$ must intersect one of D or E and be disjoint from the other. Relabel so that $\overline{U_1}, \dots, \overline{U_s}$ intersect D and $\overline{U_{s+1}}, \dots, \overline{U_r}$ intersect E . Now $A = (D \cup \bigcup_{i=1}^s \overline{U_i}) \cup (E \cup \bigcup_{i=s+1}^r \overline{U_i})$ which implies A is disconnected. This contradiction gives us that $A \cap M$ is connected, and the same with B .

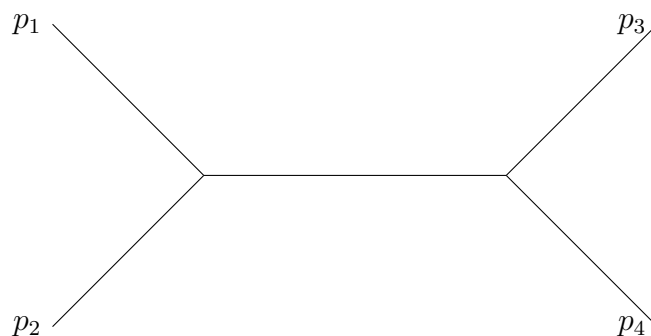
Now consider $A \cap B$, assuming both A and B intersect M . Since $X = A \cup B$ we have that $M = (A \cap M) \cup (B \cap M)$ and as M is unicoherent we have that $A \cap B \cap M$ is connected. The inverse image of each stalk of $X/\sim$, not including the central point $m = \pi(M)$, lies in either A or B (or both). Say one, X' , lay in A . Let the continuum Y be defined as $X \setminus X'$ and consider $\rho : X \mapsto X/Y$. Then $B \cap X'$ is homeomorphic to $\rho(B) \setminus \rho(Y)$ which is connected. Furthermore, $\overline{B \cap X'} \subseteq (B \cap X') \cup (A \cap B \cap M)$. From here it is clear that $A \cap B$ is connected as it consists of $A \cap B \cap M$ and n connected sets each limiting onto it. $\square$

4.3 Monotone Maps to Trees

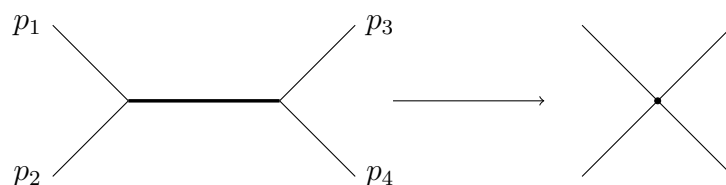
In the previous section we took an almost hereditarily decomposable, n -irreducible continuum and constructed a universal monotone map to an n -od. This can be seen as analogous to the map from Chapter 2 but there is a key difference. Not every equivalence class from Definition 4.2.16 has empty interior, in contrast to Lemma

2.4.14. The only one which might have non-empty interior is M , defined in Definition 4.2.19. The following example demonstrates this.

Example 4.3.1. Let X be the continuum shown below.



Then it is clear that X is not just almost hereditarily decomposable, but in fact hereditarily decomposable. It is also clear that $X = \text{irr}(p_1, p_2, p_3, p_4)$. In this continuum the set M is the central arc between two branch points and is the only non-trivial equivalence class. This equivalence class is shown in bold below, as well as the image of X under the map from Theorem 4.2.30.



This example raises another issue. In Chapter 2 we discussed local connectedness and the fact that the equivalence classes are the minimal subcontinua such that X is locally connected about them. In this way the map could be thought of as collapsing each non-locally connected bit of X to a point. This is clearly not the case here as X is in fact locally connected everywhere. Part of the underlying structure of X has been removed because M is such a comparatively large equivalence class.

In this section we will construct a different map, this time with the image a tree. In doing so the structure of X will be better preserved and the properties of the equivalence classes will be more in line with those of Chapter 2. To do this we will define a new equivalence relation, first on the Z_i from Lemma 4.2.24. Next these

Z_i will be quotiented away and we will see that the resulting continuum is almost hereditarily decomposable and m irreducible, for $m < n$. We will repeat this process until we have covered every part of X .

Definition 4.3.2. Let X be an almost hereditarily decomposable, n -irreducible continuum. We shall define a number of continua, maps and relations which will be used throughout this section. This definition is recursive, so at each stage assume that $\sim_k$ has finitely many non-degenerate equivalence classes, each a continuum.

1. Let $X_1 = X$, $X_{k+1} = X_k / \sim_k$. Each X_k is almost hereditarily decomposable by repeatedly applying Lemma 3.2.20.
2. Let $\rho_{k,l} : X_k \mapsto X_l$ be the composition of natural quotient maps $\rho_{i,i+1} : X_i \mapsto X_{i+1} = X_i / \sim_i$ for $k < l$
3. As $\rho_{k-1,k}$ is a monotone map, we can take points $p_1^k, \dots, p_{n_k}^k$ in X_k such that $X_k = \text{irr}(p_1^k, \dots, p_{n_k}^k)$
4. Let Y_i^k and Z_i^k be subcontinua of X_k as defined in Lemma 4.2.24.
5. Define $x \sim_k y$ if and only if $x = y$ or $x \in Z_i^k, y \in Z_j^k, Z_i^k \cap Z_j^k \neq \emptyset$.

Keep defining these until X_k is 2-irreducible or $X_k = \bigcup_{i=1}^{n_k} Z_i^k$.

We shall now prove a pair of lemmas which give us that the sequence of continua X_k defined in Definition 4.3.2 is finite.

Lemma 4.3.3. *Taking the definitions from Definition 4.3.2, let $Y_{i,j} \subseteq X_k$ be a subcontinuum with $Y_{i,j} = \text{irr}(p_i^k, p_j^k)$ as in Theorem 2.2.5. Then the only points of $\{p_1^k, \dots, p_{n_k}^k\}$ lying in $Y_{i,j}$ are p_i^k and p_j^k .*

Proof. Suppose some $p_l^k \in Y_{i,j}$ for $l \neq i, j$. Let C be a proper subcontinuum of X_k containing $p_1^k, \dots, p_{l-1}^k, p_{l+1}^k, \dots, p_{n_k}^k$ and not containing p_l^k . We will first show that p_l^k is not a point of irreducibility of $Y_{i,j}$. Let $K = \lambda_{Y_{i,j}}(p_i^k)$. If $p_l^k \in K$ and K is compact then $C \cup K$ is a continuum containing $p_1^k, \dots, p_{n_k}^k$ so is the whole of X_k . This implies $X_k \setminus C \subseteq K$ and that K has non-empty interior. However, K has empty interior in $Y_{i,j}$ so must have empty interior in X_k , leading to a contradiction. If instead K is not compact then by Lemma 3.5.6 there exists an indecomposable subcontinuum L of $Y_{i,j}$ such that $K \subseteq L$. The same argument implies that L has non-empty interior in X_k which contradicts the fact that X_k is almost hereditarily decomposable. Therefore it must be that p_l^k is not a point of irreducibility of $Y_{i,j}$.

Let $\rho : X_k \mapsto X_k/C$ be the quotient map to the natural quotient space. By Lemma 3.2.21 we have that $X_k/C = \text{irr}(\rho(p_l^k), \rho(C))$ and by Lemma 3.2.20 we have that X_k/C is almost hereditarily decomposable, so by Theorem 2.4.11 there exists a universal monotone map $\pi : X_k/C \mapsto \mathbb{I}$. Let D and E be proper subcontinua of $Y_{i,j}$ with $D = \text{irr}(p_i^k, p_l^k)$ and $E = \text{irr}(p_j^k, p_l^k)$. Then $\rho(D) = X_k/C = \rho(E)$ by the irreducibility of X_k/C . Let $\alpha \in \mathbb{I}$ not be a point of irreducibility, and let $d \in D$ be such that $\pi \circ \rho(d) = \alpha$. Take $e \in E$ similarly. Neither d nor e are points of irreducibility of D or E respectively so let $D' \subsetneq D$ be a proper subcontinuum containing p_i^k and d , and define E' similarly. Then $D \cup (\pi \circ \rho)^{-1}(\alpha) \cup E$ is a proper subcontinuum of $Y_{i,j}$ as it does not contain p_l^k . However, p_i^k and p_j^k both lie in this subcontinuum, which contradicts the irreducibility of $Y_{i,j}$. Thus it cannot be that $p_l^k \in Y_{i,j}$. $\square$

Lemma 4.3.4. *Take the definitions from Definition 4.3.2 and let $k > 1$. Each $\lambda_{X_k}(p_i^k)$ contains the image of more than one p_j^{k-1} .*

Proof. Suppose some $\lambda_{X_k}(p_i^k)$ contains the image of $p_{i'}^{k-1}$ but no others. Consider the continuum $Q = \rho_{k-1,k}^{-1}(\bigcup_{j,l \neq i} Y_{j,l})$. From Lemma 4.3.3 we have that $p_i^k \notin \bigcup_{j,l \neq i} Y_{j,l}$ so it follows that $Z_{i'}^{k-1} \cap Q = \emptyset$. For all $x \in Z_{i'}^{k-1} \setminus \kappa_{Z_{i'}^{k-1}}(p_{i'}^{k-1})$ we have $Z_{i'}^{k-1} = P_x^{i'}$ with Q witnessing that $x \notin M_{i'}$. This contradicts Lemma 4.2.24. $\square$

Corollary 4.3.5. *Take the definitions from Definition 4.3.2. For each k with X_{k+1} defined we have that $n_{k+1} \leq n_k$. This implies the sequence is finite.*

Proof. As the map $\rho_{k,k+1}$ is monotone we have that $X_{k+1} = \min(\rho(p_1^k), \dots, \rho(p_{n_k}^k))$ and by Proposition 3.2.3 we can reduce this set to one about which X_{k+1} is irreducible. By Lemma 4.3.4 we have that each point of irreducibility must be the image of at least two points of irreducibility of X_k , so not only is $n_{k+1} < n_k$, it is in fact at most half of n_k . $\square$

We will now define a new equivalence relation on X . As has been mentioned earlier this will involve the images of points in the various X_k . It is worth considering the map $\rho_{k,k+1}$ as removing the composants $\kappa_{Z_i^k}(p_i^k)$ from X_k and then shrinking the corresponding E-continua to a point.

Definition 4.3.6. For $x, y \in X$ we have $x \sim y$ if and only if one of the following holds

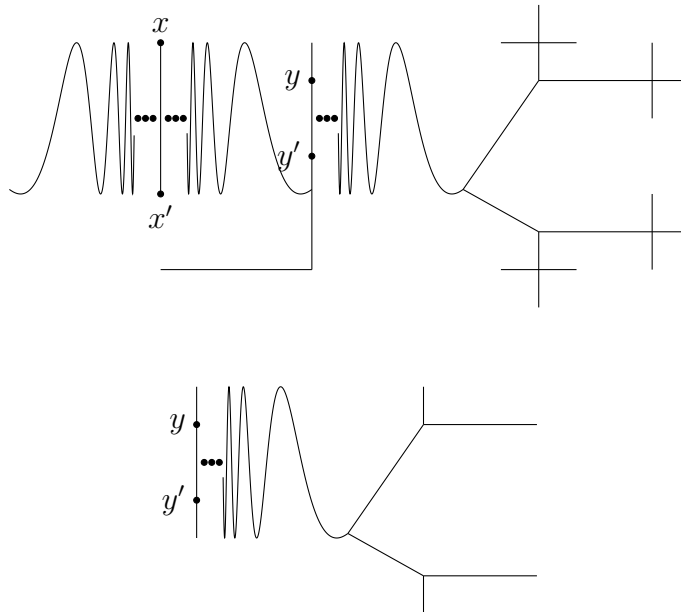
- $x = y$

- There exists k such that $\rho_{1,k}(x) \neq \rho_{1,k}(y)$ and both lie in Z_i^k for some i , neither are points of irreducibility of Z_i^k and they are related in Z_i^k using the relation for 2-irreducible spaces defined in Definition 2.4.1.
- There exists k such that $\rho_{1,k}(x) \neq \rho_{1,k}(y)$, $\rho_{1,k+1}(x), \rho_{1,k+1}(y) \in \lambda_{X_{k+1}}(p_i^{k+1})$ for some i and if $\rho_{1,k}(x)$ or $\rho_{1,k}(y)$ lie in Z_j^k for any j then they do not lie in $\kappa_{Z_j^k}(p_j^k)$.
- There exists k such that $\rho_{1,k}(x) \neq \rho_{1,k}(y)$, X_k is the union of the subcontinua Z_i^k and $\rho_{1,k}(x)$ and $\rho_{1,k}(y)$ are related in X_k under the equivalence relation defined in Definition 4.2.13.

Define the equivalence class of x under this relation to be $x_\sim$.

The second of these three criteria states that there comes a point where both x and y lie in the composant of some Z_i^k and they are related in Z_i^k . The third states that there comes a point where they lie in the E-continuum of some p_i^{k+1} and if they are the image of some Z_j^k then they were in the E-continuum of Z_j^k not containing p_j^k . The fourth criterion states that points whose images are distinct in the final continuum are related if and only if they would be related under the relation from Section 4.2.

Example 4.3.7. In the following continuum $x \sim x'$ and $y \sim y'$. The points x and x' are related due to the second criteria, and the points y and y' due to the third. The space $X = X_1$ is shown first, then the space X_2 below with the images of y and y' identified.



We will now prove that the equivalence classes are as expected, namely that they are continua about which X is locally connected.

Proposition 4.3.8. *Take the definitions from Definition 4.3.2. Let $\sim$ be as in Definition 4.3.6. The equivalence classes of $\sim$ are continua.*

Proof. Observe that each $\rho_{k,k+1}$ is a homeomorphism on $X_k \setminus \bigcup Z_i^k$. If an equivalence class $x_\sim$ arises as a result of the second of the three options then it is homeomorphic to the class of $\rho_{1,k}(x)$ in Z_i^k . This is a continuum, so $\sim$ is also a continuum.

If $x_\sim$ arises from the third of the options then it is homeomorphic to

$$A := \rho_{k,k+1}^{-1} \left(\lambda_{X_{k+1}}(p_i^{k+1}) \right) \setminus \bigcup_{j \in J} \kappa_{Z_j^k}(p_j^k)$$

where J is the subset of $\{1, \dots, n_k\}$ of indices such that the corresponding Z_j^k is mapped into $\lambda_{X_{k+1}}(p_i^{k+1})$. The set A is closed and therefore compact. Also, $\rho_{k,k+1}^{-1}(\lambda_{X_{k+1}}(p_i^{k+1}))$ is connected and if A were disconnected into $B \cup C$ then each $\kappa_{Z_j^k}(p_j^k)$ limits on either B or C but not both. This implies that one of B or C is empty, so A is connected.

If $x_\sim$ arises from the fourth option and not the third then we have that it is homeomorphic to the equivalence class of $\rho_{1,k}(x)$ under the relation defined in Definition 4.2.13, which we know is a continuum. Thus $x_\sim$ is also a continuum. $\square$

Proposition 4.3.9. *X is locally connected at each $x_\sim$.*

The proof is in two halves and the second of these gets quite complicated with a lot of not particularly clear notation. It does not provide much insight, at least in my opinion, so afterwards there will be a worked example of how to construct the connected open set.

Proof. Let U be an open set containing $x_\sim$. We will again consider separately the two ways an equivalence class can arise. If $x_\sim$ corresponds to points of reducibility of some Z_i^k then shrink U to miss all Z_j^1 and let $U_2 = \rho_{1,2}(U)$. U_2 is open and contains $\rho_{1,2}(x_\sim)$. Keep doing this until you get to $U_k \subseteq X_k$. Now $\rho_{1,k}(x_\sim) \subseteq \kappa_{Z_i^k}(p_i^k) \setminus \lambda_{X_k}(p_i^k)$ which is an open set, so shrink U_k to lie in this. Since Z_i^k is locally connected at $\rho_{1,k}(x_\sim)$ take connected open V with $\rho_{1,k}(x_\sim) \subseteq V \subseteq U_k$. V is Z_i^k -open and lies in the X_k -open subset $\kappa_{Z_i^k}(p_i^k)$, so V is X_k -open. Then $x_\sim \subseteq \rho_{1,k}^{-1}(V) \subseteq U$ and $\rho_{1,k}^{-1}(V)$ is open and connected.

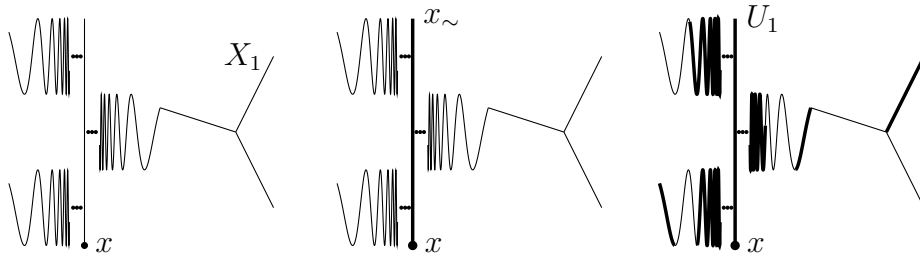
Now consider the other case, where

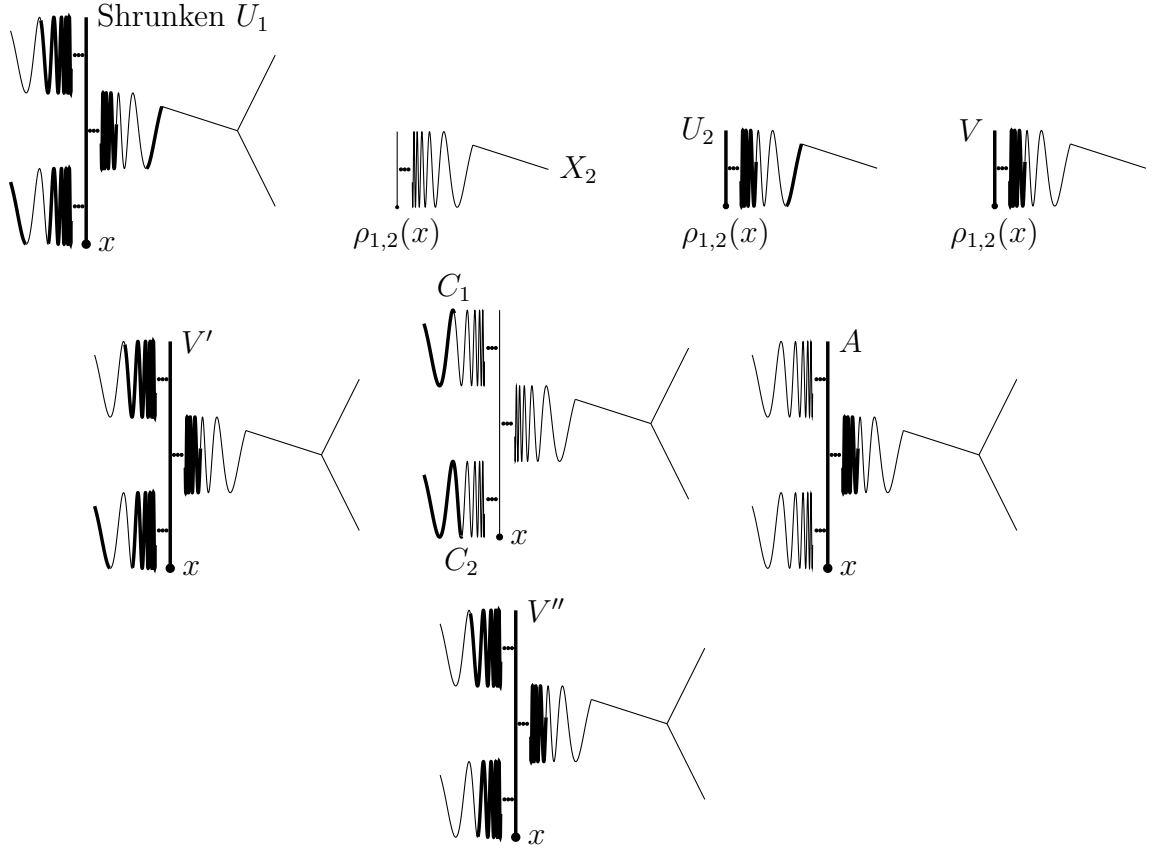
$$x_{\sim} = \rho_{1,k}^{-1} \left(\rho_{k,k+1}^{-1} \left(\lambda_{X_{k+1}}(p_i^{k+1}) \right) \right) \setminus \bigcup_{j \in J} \kappa_{Z_j^k}(p_j^k)$$

where J is the subset of $\{1, \dots, n_k\}$ of indices such that the corresponding Z_j^k is mapped into $\lambda_{X_{k+1}}(p_i^{k+1})$. Do as before, shrinking U and mapping forward until you get $U_k \subseteq X_k$ with U_k and open set containing $\rho_{1,k}(x_{\sim})$. Shrink U_k to avoid any $Z_{j'}^k$ for $j' \notin J$. Let $U_{k+1} = \rho_{k,k+1}(U_k)$. Since $\rho_{k,k+1}^{-1}(U_{k+1}) = U_k \cup \bigcup_{j \in J} \kappa_{Z_j^k}(p_j^k)$ we have that U_{k+1} is open. It contains $\lambda_{X_{k+1}}(p_i^{k+1}) = \rho_{1,k+1}(x_{\sim})$ about which X_{k+1} is locally connected so let V be a connected open set with $\lambda_{X_{k+1}}(p_i^{k+1}) \subseteq V \subseteq U_{k+1}$. Define $V' = \rho_{k,k+1}^{-1}(V) \cap U_k$ and let $Y = X_k \setminus \bigcup_{j \in J} \kappa_{Z_j^k}(p_j^k)$. Define a new equivalence relation for points of Y by $x \sim' y \Leftrightarrow x \sim_k y$ in X_k . Now $Y/\sim' \cong X_{k+1}$ so let A be the inverse image of V under this map. A is connected as the map is monotone and V is connected. For all $j \in J$, $Z_j^k \cap V'$ is Z_j^k -open and contains the E-continuum $Z_j^k \setminus \kappa_{Z_j^k}(p_j^k)$, so there is a continuum $C_j \subsetneq Z_j^k$ containing p_j^k such that $Z_j^k \setminus C_j$ is connected and lies in V' . Let $V'' = V' \setminus \bigcup_{j \in J} C_j = A \cup \bigcup_{j \in J} (Z_j^k \setminus C_j)$. The first expression of V'' gives that it is open and the second gives that it is connected. Thus $\rho_{1,k}^{-1}(V'')$ is open and connected and $x_{\sim} \subseteq \rho_{1,k}^{-1}(V'') \subseteq U$.

If $x_{\sim}$ arises from the fourth option but not the third then we have that its image under $\rho_{1,k}$ lies in X_k and does not contain any of the points $p_1^k, \dots, p_{n_k}^k$. From Corollary 4.2.31 we have that X_k is locally connected about $\rho_{1,k}(x_{\sim})$. As $X_K \text{ setminus } \{p_1^k, \dots, p_{n_k}^k\}$ is an open set and $\rho_{1,k}$ is monotone it follows that X is locally connected about $x_{\sim}$. $\square$

Example 4.3.10. This example will be of how to find a connected open subset in the manner described in the previous proposition. The various important subsets will be highlighted in bold.





Corollary 4.3.11. *Take the definitions from Definition 4.3.2. Let $\sim$ be as in Definition 4.3.6. Then $X/\sim$ is locally connected.*

Proof. This is a direct consequence of Proposition 4.3.9 and Proposition 2.5.1. $\square$

Lemma 4.3.12. *Let $\rho : Y \mapsto Z$ be a monotone map between compact spaces. Then for all $z \in Z$, $Z \setminus \{z\}$ is disconnected if and only if $Y \setminus \rho^{-1}(z)$ is disconnected.*

Proof. If $Y \setminus \rho^{-1}(z)$ is connected then as $Z \setminus \{z\}$ is the continuous image of this it must be connected. If $Z \setminus \{z\}$ is connected then as $Y \setminus \rho^{-1}(z) = \rho^{-1}(Z \setminus \{z\})$ and ρ is monotone it must be that $Y \setminus \rho^{-1}(z)$ is connected. $\square$

This lemma will be used a number of times, firstly to prove that $X/\sim$ is a continuum and then to show that it is a tree.

Proposition 4.3.13. *Take the definitions from Definition 4.3.2. Let $\sim$ be as in Definition 4.3.6. Then $X/\sim$ is Hausdorff.*

Proof. Let $x, y \in X$ and let $x \approx y$. We need to show that there exist disjoint open sets U and V consisting of whole equivalence classes with $x \in U$ and $y \in V$. If neither x nor y ever lie in a Z_i^k then they must be in the final X_k , and this must be an irreducible continuum in the classical sense. The equivalence classes $x_\sim$ and $y_\sim$ are homeomorphic to the corresponding classes in X_k , and it was proved in Proposition 2.4.7 that there exists a point $z \in X_k$ not related to either $\rho_{1,k}(x)$ or $\rho_{1,k}(y)$ such that removing the equivalence class of z from X_k would split this continuum into two components and that one of $\rho_{1,k}(x)$ and $\rho_{1,k}(y)$ would lie in each. Let $z' \in X$ be such that $\rho_{1,k}(z') = z$. By applying Lemma 4.3.12 we know that $X \setminus z'_\sim$ is disconnected, and that x and y lie in different components. Further, these two components consist of whole equivalence classes and are open sets.

Now suppose that one of $\rho_{1,k}(x) \in Z_i^k$ or $\rho_{1,k}(y) \in Z_i^k$ holds and that this is the smallest k for which this is true. We saw in the proof of Lemma 4.2.24 that Z_i^k is almost hereditarily decomposable so let $\sigma : Z_i^k \mapsto \mathbb{I}$ be the universal monotone map with $\sigma(p_i^k) = 0$ and let $\sigma' = \rho_{1,k} \circ \sigma$. If both $\rho_{1,k}(x)$ and $\rho_{1,k}(y)$ lie in Z_i^k then let $\alpha \in (\sigma'(x), \sigma'(y))$ or $\alpha \in (\sigma'(y), \sigma'(x))$. Then $(\sigma')^{-1}(\alpha)$ is an equivalence class of $\sim$, and removing it from X splits the space into two components as before. If only $\rho_{1,k}(x)$ lies in Z_i^k and $\sigma'(x) < 1$ then choose $\alpha \in (\sigma'(x), 1)$ and proceed as before. If only $\rho_{1,k}(y) \in Z_i^k$ then do the same thing.

We must finally consider the case where $\rho_{1,k}(x) \in Z_i^k$ and $\rho_{1,k}(y) \in Z_j^k$, both lying in the E-continuum of these subcontinua which gets mapped to 1. It cannot be that X_k is the union of some Z_l^k 's as $x \approx y$, and the E-continua of all the Z_l^k 's of the final X_k are related. Thus the points $\rho_{1,k+1}(x)$ and $\rho_{1,k+1}(y)$ are both points of irreducibility of X_{k+1} . Again as $x \approx y$ they must lie in distinct E-continua. We can then repeat what was done in the previous paragraph for when only one of the points lies in the Z_i^k . $\square$

Corollary 4.3.14. *Take the definitions from Definition 4.3.2. Let $\sim$ be as in Definition 4.3.6. Then $X/\sim$ is a continuum.*

Proof. Apply Lemma 4.1.1 and note that compactness and connectedness are preserved under continuous maps. $\square$

We will now see that this continuum is in fact a tree. To do this we will look at the cut points of $X/\sim$.

Lemma 4.3.15. *Take the definitions from Definition 4.3.2. Let $\sim$ be as in Definition 4.3.6. If $x \in X$ is not a point of irreducibility then $X/\sim \setminus \{x_\sim\}$ is disconnected.*

Proof. By Lemma 4.3.12 this is equivalent to $X \setminus x_\sim$ being disconnected. If $\rho_{1,k}(x_\sim)$ is homeomorphic to $\pi_i^{k-1}(\alpha)$ for $\alpha \neq 0, 1$ then $X_k \setminus \rho_{1,k}(x_\sim)$ is disconnected as it can be expressed as $\pi_i^{k-1}([0, \alpha)) \cup (Y_i^k \cup \pi_i^{k-1}((\alpha, 1]))$. As this is the image of $X \setminus x_\sim$ under $\rho_{1,k}$, it follows that $X \setminus x_\sim$ is also disconnected.

If $x_\sim$ instead arises from $\lambda_{X_{k+1}}(p_i^{k+1})$ then

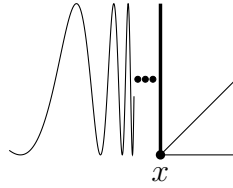
$$X_k \setminus \rho_{1,k}(x_\sim) = \rho_{k,k+1}^{-1}(X_{k+1} \setminus \lambda_{X_{k+1}}(p_i^{k+1})) \cup \bigcup_{j \in J} \kappa_{Z_j^k}(p_j^k)$$

where J is the set of indices with $\rho_{k,k+1}(Z_j^k) \in \lambda_{X_{k+1}}(p_i^{k+1})$. This is again disconnected and the image of $X \setminus x_\sim$ under $\rho_{1,k}$.

If $x_\sim$ arises as a result of the fourth possibility and not the third, then by Lemma 4.3.12 we have that $\rho_{1,k}(x_\sim)$ disconnects X_k , as its image in the corresponding n -od would be a cut point. Applying Lemma 4.3.12 again gives us that $x_\sim$ disconnects X . $\square$

It is an immediate corollary to this lemma that if x is not a point of irreducibility and $x_\sim = \{x\}$ then x is a cut point of X . We also know that no point of irreducibility can be a cut point of X . This does not characterise cut points however, as the following example shows.

Example 4.3.16. Let X be the continuum below. Then X is hereditarily decomposable and 3-irreducible. The point x is a cut point, but the bold arc is $x_\sim$ so $\{x\} \neq x_\sim$.



We are now in a position to prove the major theorem of this section.

Theorem 4.3.17. *Let X be an almost hereditarily decomposable, n -irreducible continuum. There exists a monotone surjection from X to a tree with n end points.*

Proof. Take the definitions from Definition 4.3.2. Let $\sim$ be as in Definition 4.3.6. Let $\pi : X \mapsto X/\sim$ be the natural quotient map; we will show that $X/\sim$ is a tree. The point $x_\sim \in X/\sim$ is not a cut point if and only if $x_\sim = p_{i_\sim}$ for some i , so $X/\sim$ has precisely n non-cut points. By Theorem 4.1.3 it is a tree and by Proposition 4.1.2 the end points are the $p_{i_\sim}$. $\square$

We have constructed a monotone map from an almost hereditarily decomposable, n -irreducible continuum onto a tree. As before, we will now prove that this map is universal amongst monotone maps to trees. This includes maps to trees other than $X/\sim$.

Lemma 4.3.18. *Take the definitions from Definition 4.3.2. Let $\sim$ be as in Definition 4.3.6. Every $x_\sim$ has empty interior.*

Proof. There exists k such that $\rho_{1,k}(x_\sim) \cong x_\sim$. This set has empty interior in X_k , either because it does in Z_i^j or because it lies in the closure of

$$X_{k+1} \setminus \lambda_{X_{k+1}}(p_i^{k+1}) \cup \bigcup_j \kappa_{Z_j^k}(p_j^k)$$

Since we can find an open set U containing $x_\sim$ such that $\rho_{1,k}$ is a homeomorphism on U , we have that $x_\sim$ has empty interior. $\square$

Theorem 4.3.19. *Take the definitions from Definition 4.3.2. Let $\sim$ be as in Definition 4.3.6. The map $\pi : X \mapsto X/\sim$ is universal amongst monotone maps to trees.*

Proof. Let T be a tree and let $\rho : X \mapsto T$ be a monotone map. For any $x \in X$ if $\rho(x_\sim)$ is non-degenerate then it is a subcontinuum and has non-empty interior. Let $\overline{T \setminus \rho(x_\sim)}$ have m components. We know m is finite as T is k -irreducible for some k by applying Proposition 3.2.16 and Proposition 3.2.3. Let $C_1, \dots, C_m$ be the inverse images of these components under ρ . Then $x_\sim \cup \bigcup_{i=1}^m C_i$ is a continuum and intersects each equivalence class, so must contain $p_1, \dots, p_n$. This means that $X = x_\sim \cup \bigcup_{i=1}^m C_i$. If $U \subseteq \rho(x_\sim)$ is the (non-empty) interior then $\rho^{-1}(U) \cap C_i = \emptyset$ for each i , so $\rho^{-1}(U) \subseteq x_\sim$. Thus $x_\sim$ has non-empty interior which contradicts Lemma 4.3.18. From this contradiction we can deduce that ρ preserves equivalence classes.

Define $f : X/\sim \mapsto T$ by $f(x_\sim) = \rho(x)$. We have just seen that this is well defined and clearly $f \circ \pi = \rho$. To check that f is open let U be an open subset of T and note that $f^{-1}(U) = \pi(\rho^{-1}(U))$. Since ρ preserves equivalence classes we have that $\pi^{-1}(\pi(\rho^{-1}(U))) = \rho^{-1}(U)$. Since this is open it follows that $f^{-1}(U)$ is open, so f is continuous. This completes the proof. $\square$

We will finish this section by again proving a similar result to that of Theorem 2.4 of [Mil50].

Theorem 4.3.20. *Let X be an almost hereditarily decomposable, n -irreducible continuum and let π be as in Proposition 4.3.19. X is unicoherent if each fibre of π is unicoherent.*

Proof. Let $T = T_1 \cup \dots \cup T_r$ where each T_i is an arc and the only end or branch points of T contained in T_i are the two end points of T_i . Let $X_i = \pi^{-1}(T_i)$. Let $X = A \cup B$ for proper subcontinua A and B . A similar argument as in Theorem 4.2.35 gives us that if A or B intersect the inverse image of a branch point then this intersection is connected, and thus if both of them intersect the same one, $\pi^{-1}(b)$, then $A \cap \pi^{-1}(b)$ and $B \cap \pi^{-1}(b)$ form a decomposition and therefore have connected intersection.

For $x, y \in X$ define $x \sim_i y$ if and only if $\pi(x)$ and $\pi(y)$ lie in the same component of $\overline{T \setminus T_i}$. We will consider $X/\sim_i$, which can be thought of as the inverse image of T_i with the preimages of the ends of T_i shrunk to a single point. Now $X/\sim_i$ is 2-irreducible and the union of $A/\sim_i$ and $B/\sim_i$. By applying Lemma 3.2.20 twice we have that it is almost hereditarily decomposable so by Theorem 2.4.11 it has a monotone map to an arc. This monotone map from $X/\sim_i$ will have as its fibres a pair of singletons and the fibres of π mapping to the interior of T_i , so $X/\sim_i$ is unicoherent by Theorem 2.5.10. $A \cap B \cap X_i$ contains a homeomorphic copy of their intersection in $X/\sim_i$ without the points of irreducibility (call this C). The rest of it consists of the intersection of $A \cap B$ with the inverse images of the end points of T_i , both of which must be connected and intersect the closure of C . Thus $A \cap B \cap X_i$ is connected.

Suppose $A \cap B$ was not connected and relabel the X_i s such that the subsets $A \cap B \cap (X_1, \dots, X_s)$ and $A \cap B \cap (X_{s+1}, \dots, X_r)$ are disjoint and compact. Then the images of these sets witness that $\pi(A) \cap \pi(B)$ is disconnected, but $T = \pi(A) \cup \pi(B)$. However, every tree is a dendrite so by Theorem 4.1.4 T is unicoherent. Therefore it must be that $A \cap B$ is connected, which in turn implies that X is unicoherent. $\square$

4.4 $1/n$ -homogeneous Spaces

In this section we will use the map from Theorem 4.3.17 to look at the structure of our continuum. In particular we will use it to investigate the homeomorphisms from X to itself.

Definition 4.4.1. Let X be a space and $\text{Aut}(X) = \{f : X \rightarrow X \mid f \text{ is a homeomorphism}\}$ with composition as the group operation, and the natural group action on X . X is $1/k$ -homogeneous if the action of $\text{Aut}(X)$ on X has precisely k orbits.

We will see that there is a link between the $1/k$ -homogeneity of an almost hereditarily decomposable, n -irreducible continuum X and of the tree it maps on to. Throughout this section X will be an almost hereditarily decomposable, n -irreducible continuum and $\pi : X \rightarrow T$ will be the universal monotone map onto a tree T .

Definition 4.4.2. Let $f : X \mapsto X$ be a homeomorphism. Since $\pi \circ f : X \mapsto T$ is a composition of monotone maps, it is monotone. Thus by the universal property of π and T (Theorem 4.3.19) there exists a map $g : T \mapsto T$ such that $\pi \circ f = g \circ \pi$.

This correspondence will be key, as we will see that g is also a homeomorphism. As the homeomorphisms of trees are easier to study than those of more general continua this correspondence allows us to more easily study the homeomorphisms of X .

Lemma 4.4.3. *Let $C \subseteq X$ be a subcontinuum, $\pi(C)$ non-degenerate. Then C has non-empty interior.*

Proof. $T \setminus \pi(C)$ has finitely many components. Let $A_1, \dots, A_m$ be the closures of these components. Since $\pi(C)$ is non-degenerate, $\text{int}(\pi(C)) \neq \emptyset$ so $T \neq \bigcup_{i=1}^m A_i$. This in turn implies $X \neq \bigcup_{i=1}^m \pi^{-1}(A_i)$. Now $C \cup \bigcup_{i=1}^m \pi^{-1}(A_i)$ is a continuum containing $p_1, \dots, p_n$ so it is equal to X . Thus $C \supseteq X \setminus \bigcup_{i=1}^m \pi^{-1}(A_i) \neq \emptyset$ which means C has non-empty interior. $\square$

Proposition 4.4.4. *Let $f : X \mapsto X$ be a homeomorphism and let g be as in Definition 4.4.2. The map g is a homeomorphism.*

Proof. We know that g is continuous, and it is surjective as $\pi \circ f$ is surjective. As it is a continuous map between compact T_2 spaces it preserves closed sets, so all we need to check is that it is injective. Suppose $g(x) = g(y)$ for distinct $x = \pi(x')$ and $y = \pi(y')$. Then $g \circ \pi(x') = g \circ \pi(y')$ which implies $\pi \circ f(x') = \pi \circ f(y')$. This means that $f(x')$ and $f(y')$ are in the same equivalence class of X . Define $C = f^{-1}(f(x')_{\sim})$, which is a continuum as it is homeomorphic to an equivalence class. Clearly $x', y' \in C$ so $\pi(C)$ is a non-degenerate continuum. We can apply Lemma 4.4.3 to see that C has non-empty interior. This in turn implies $f(C) = f(x')_{\sim}$ has non-empty interior which contradicts Lemma 4.3.18. Thus g is injective, and therefore a homeomorphism. $\square$

Theorem 4.4.5. *If T is $1/k$ -homogeneous and X is $1/l$ -homogeneous then $k \leq l$.*

Proof. Let $x, y \in X$ be distinct points and $f : X \mapsto X$ a homeomorphism with $f(x) = y$. Let $g : T \mapsto T$ be the corresponding homeomorphism from Definition 4.4.2 and Proposition 4.4.4. Then $g(\pi(x)) = \pi(f(x)) = \pi(y)$. This means that the orbits of $\text{Aut}(X) = \{f : X \mapsto X \mid f \text{ is a homeomorphism}\}$ lie in the inverse images under π of the orbits of $\text{Aut}(T) = \{g : T \mapsto T \mid g \text{ is a homeomorphism}\}$, so F has at least as many orbits as G . $\square$

While every homeomorphism $f : X \mapsto X$ induces a homeomorphism $g : T \mapsto T$, the reverse is unfortunately not true as the following example shows.

Example 4.4.6. Not every homeomorphism $g : T \mapsto T$ corresponds to a homeomorphism $f : X \mapsto X$. For example, let X be the $\sin^{1/x}$ continuum expressed as $\overline{\{(x, \sin(1/x)) | x \in (0, 1]\}}$, let $T = [0, 1]$ and $g(x) = 1 - x$. A homeomorphism $f : X \mapsto X$ with $g \circ \pi = \pi \circ f$ would need to map $\{0\} \times [0, 1]$ to $(1, \sin(1))$, so clearly it cannot exist. In this example T is $1/2$ -homogeneous and X is $1/4$ homogeneous. The orbits in X are

- $\{(1, \sin(1))\}$
- $\{(x, \sin(1/x)) | x \in (0, 1)\}$
- $\{(0, 0), (0, 1)\}$
- $\{0\} \times (0, 1)$

While there may not be a strict lifting, the homeomorphisms of T can still tell us a lot about the homeomorphisms of X . We know that the only homeomorphisms $f : X \mapsto X$ that can exist are the ones which induce a homeomorphism of T , which in turn informs us that the image of an equivalence class under f must also be an equivalence class, in fact a homeomorphic one. We also know that f must map points of irreducibility to other points of irreducibility, corresponding to the fact that a homeomorphism on T must map end points to other end points. The same also holds for equivalence classes which in T have order m for some fixed m , which must be mapped to other such equivalence classes.

In Example 4.4.6 we saw a continuum which was $1/4$ -homogeneous, with the corresponding tree being only $1/2$ -homogeneous. This discrepancy does not always exist, as the following example shows.

Example 4.4.7. There exist a continuum X and a corresponding tree T , with π non-trivial, such that both X and T are $1/n$ -homogeneous for the same n . Take X to be a compactification of an open interval such that the remainder is a pair of circles. The two orbits of this continuum are the original open interval and the pair of circles, so it is $1/2$ -homogeneous. The corresponding tree T is the closed unit interval, which is also $1/2$ -homogeneous.

4.5 Topological Dynamics

The definitions and notation in this section are taken from [Vri14]. As previously mentioned in Proposition 4.4.4, every homeomorphism $f : X \mapsto X$ induces one $g : T \mapsto T$ such that $\pi \circ f = g \circ \pi$. Thus if (X, f) is a dynamical system with f a homeomorphism then (T, g) is a factor of this system and g is a homeomorphism. This second system will be simpler than the first, but will share some of the same properties. In the cases of these properties it will be easier to show they exist in the system (T, g) and thus deduce they exist in (X, f) , than it would be to prove they exist in (X, f) from scratch. Where properties can hold in one system but not the other this will be shown via an example. This section will focus on the topological properties of systems and not properties related to metrics as there is no link between the metrics of X and T and the map π . The first results are an expansion on the following proposition of [Vri14]. Undefined terms will be defined below.

Proposition 4.5.1. *[Vri14, 1.5.4] Let $\phi : (X, f) \mapsto (Y, g)$ be a morphism of dynamical systems, let $x \in X$ and let $A \subseteq X$. Then;*

1. $\phi(\mathcal{O}_f(x)) = \mathcal{O}_g(\phi(x))$, hence $\phi(\overline{\mathcal{O}_f(x)}) \subseteq \overline{\mathcal{O}_g(\phi(x))}$ with equality if the orbit closure $\overline{\mathcal{O}_f(x)}$ of x is compact.
2. If the point x is periodic under f then the point $\phi(x)$ is periodic under g and the primitive period of $\phi(x)$ is a divisor of the primitive period of x . In particular, if x is invariant under f then $\phi(x)$ is invariant under g .
3. If the set A is invariant under f then the set $\phi(A)$ is invariant under g . Similarly, if the set A is completely invariant under f then so is $\phi(A)$ under g .
4. If the orbit of a point x converges under f in X to the point $z \in X$ (or has limit point z) then the orbit of the point $\phi(x)$ converges under g to $\phi(z)$ (or has limit point $\phi(z)$).
5. If the orbit of x is dense in X then the orbit of $\phi(x)$ is dense in $\phi(X)$. In particular, a factor of a transitive system is transitive and a factor of a minimal system is minimal.
6. If the set A is minimal under f and $\phi(A)$ is a closed subset of Y then the set $\phi(A)$ is minimal under g .
7. If $\phi(X)$ is dense in Y and (X, f) is topologically ergodic then (Y, g) is topologically ergodic.

It will be used several times that if X is a $\sin(\frac{1}{x})$ continuum then $\pi : X \mapsto T$ is the projection onto the 1st coordinate, and T is a unit interval. In these cases T will be instead denoted $\mathbb{I}$. For this chapter it will be assumed that X is an almost hereditarily decomposable, finitely irreducible continuum.

Definition 4.5.2. Let (S, h) be a dynamical system and let $x \in S$. Then x is called a periodic point if there exists $n \in \mathbb{N}^+$ such that $h^n(x) = x$.

Proposition 4.5.3. Let (X, f) be a dynamical system and let $x \in X$ be a periodic point. Then $\pi(x)$ is a periodic point of (T, g) .

Proof. Let $n \in \mathbb{N}^+$ be such that $f^n(x) = x$. Then $g^n(\pi(x)) = \pi(f^n(x)) = \pi(x)$. $\square$

Proposition 4.5.4. There exists a dynamical system (X, f) such that a non-periodic point $x \in X$ has periodic image $\pi(x) \in T$.

Proof. Let X be the $\sin(\frac{1}{x})$ continuum and define a homeomorphism $f : X \mapsto X$ to be $f(x, y) = (x', y|y|)$, where x' lies in the same half period of the sine curve as x , either positive or negative, and has $\sin(\frac{1}{x'}) = y|y|$. If $x = 0$ then so does x' . Now $(0, \frac{1}{2})$ is a non-periodic point of f but its image under π is not just periodic but invariant. $\square$

Definition 4.5.5. Let (S, h) be a dynamical system and let $A \subseteq S$. The set A is invariant if $h(A) \subseteq A$ and completely invariant if $h(A) = A$.

Proposition 4.5.6. Let (X, f) be a dynamical system and let $B \subseteq T$ be (completely) invariant. Then $A = \pi^{-1}(B)$ is also (completely) invariant.

Proof. Let $x \in A$. Then $\pi(x) \in B$ implies $g \circ \pi(x) \in B$ which in turn implies $\pi \circ f(x) \in B$. From this we can deduce that $f(x) \in A$. Thus A is invariant.

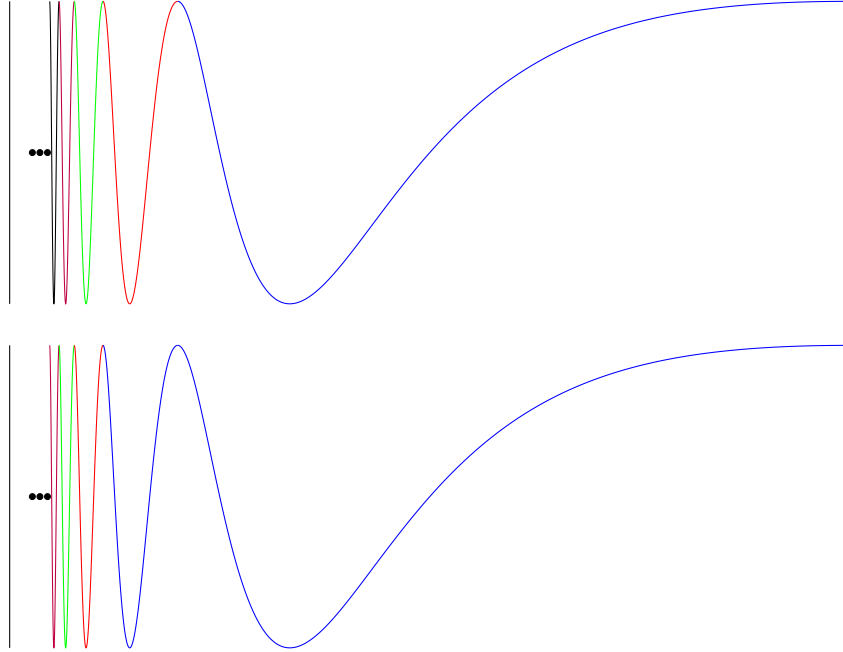
Now suppose B is completely invariant i.e. $g(B) = B$. Let $x \in A$. There exists $y \in B$ such that $g(y) = \pi(x)$. Now $g \circ \pi(f^{-1}(x)) = \pi \circ f(f^{-1}(x)) = \pi(x)$ so since g is injective $y = \pi(f^{-1}(x))$. From this we have that $f^{-1}(x) \in A$ so $f(A) = A$. $\square$

Proposition 4.5.7. There exists a dynamical system (X, f) and points $x, x' \in X$ such that the orbit of $\pi(x)$ converges to $\pi(x')$ but the orbit of x does not converge to x' .

Proof. Again take X to be the $\sin(\frac{1}{x})$ continuum. Define f by "shifting" each period one to the left and doubling the length of the rightmost section of the curve. Take as x any point not in the final loop, its orbit will converge to the point $(0, y)$ with the same 2nd-coordinate as x . Thus the orbit of $\pi(x)$ converges to $0 \in \mathbb{I}$ but there are clearly elements of $\pi^{-1}(0)$ to which x does not converge. $\square$

Remark 4.5.8. The same holds for limit points.

The next diagram illustrates the map f from Proposition 9.3, with each coloured section being mapped onto the next.



Definition 4.5.9. Let (S, h) be a dynamical system and let $x \in S$. The point x is called transitive if for each non-empty open subset $U \subseteq S$ and for each $n \in \mathbb{N}$ there exists $k > n$ such that $h^k(x) \in U$. A system is called transitive if it contains a transitive point.

Proposition 4.5.10. Let (X, f) be a dynamical system. Suppose every open subset of X contains a fibre of π . If $\pi(x)$ is transitive in (T, g) then x is transitive in (X, f) .

Proof. Let $U \subseteq X$ be an open set and let $y \in T$ be such that $\pi^{-1}(y) \subseteq U$. We will construct an open set $V \subseteq U$ consisting of whole equivalence classes.

Let $A = \{z \in T \mid \pi^{-1}(z) \subseteq U\}$. We know that $y \in A$. Suppose y is not in the interior of A . Then there exists a sequence y_n converging to y with each $y_n \notin A$. For each n let x_n belong to $\pi^{-1}(y_n) \setminus U$ witnessing this. There must be a point $x' \in X$ such that some subsequence x_{n_k} converges to x' . As π is continuous $\pi(x_{n_k})$ converges to $\pi(x')$. Since $\pi(x_{n_k}) = y_{n_k}$, this implies that $\pi(x') = y$ and that $x' \in U$. Since U is an open set and no tail of x_{n_k} lies in U , this is a contradiction. It must therefore be that y lies in the interior of A , and in particular that A has non-empty interior. Let

V be the inverse image under π of the interior of A , so V is a non-empty open subset of X consisting of whole equivalence classes and $V \subseteq U$.

Let k be a positive integer. As $\pi(x)$ is transitive there exists $n \geq k$ such that $g^n \circ \pi(x) \in V$. Now $g^n \circ \pi = \pi \circ f^n$ so $\pi^{-1}(g^n \circ \pi(x))$ lies in $\pi^{-1}(V) \subseteq U$ and contains $f^n(x)$. Thus x is transitive. $\square$

Definition 4.5.11. Let (S, h) be a dynamical system and let $A \subseteq S$. The set A is called minimal if it is non-empty, closed, invariant and whenever $B \subseteq A$ is closed and invariant either $B = \emptyset$ or $B = A$. The system (S, h) is minimal if S is minimal as a subspace of itself.

Remark 4.5.12. No system (X, f) can be minimal if X is almost hereditarily decomposable and n -irreducible. Since f is a homeomorphism the subspace $\bigcup_{i=1}^n \lambda(p_i)$ is closed and invariant.

Proposition 4.5.13. *There exists a dynamical system (X, f) and a minimal subset $B \subseteq T$ such that $A = \pi^{-1}(B)$ is not minimal.*

Proof. Use the same example as in Proposition 4.5.7. The point $0 \in \mathbb{I}$ is invariant and therefore a minimal subsystem, but its inverse image is the system $(\mathbb{I}, id)$ which is obviously not minimal. $\square$

Definition 4.5.14. Let (S, h) be a dynamical system. Then (S, h) is ergodic if for any pair of non-empty open sets $U, V \subseteq S$ there exists $n \in \mathbb{N}$ such that $h^n(U) \cap V \neq \emptyset$. The system is weakly mixing if $(S \times S, h \times h)$ is ergodic. The system is strongly mixing if for any pair of non-empty open sets $U, V \subseteq S$ the set $\{n \in \mathbb{N} | f^n(U) \cap V \neq \emptyset\}$ has finite complement.

Proposition 4.5.15. *Let (X, f) be a dynamical system. Suppose every open set $U \subseteq X$ contains a fibre of π . Then the system (X, f) is ergodic if and only if (T, g) is.*

Proof. First suppose (T, g) is ergodic. The proof will be similar to Proposition 4.5.10. Let U, V be open subsets of X and take open subsets $U', V' \subseteq T$ such that $\pi^{-1}(U') \subseteq U, \pi^{-1}(V') \subseteq V$. Take n such that $g^n(U') \cap V' \neq \emptyset$, then it is clear that $\emptyset \neq \pi^{-1}(g^n(U')) \cap \pi^{-1}(V') \subseteq f^n(U) \cap V$. Thus (X, f) is ergodic.

Now suppose (X, f) is ergodic. Let $U, V \subseteq T$ be open sets, so $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are open subsets of X . There exists n such that $f^n(\pi^{-1}(U)) \cap \pi^{-1}(V) \neq \emptyset$. Thus $\emptyset \neq \pi(f^n(\pi^{-1}(U)) \cap \pi^{-1}(V)) = g^n(U) \cap V$. Thus (T, g) is ergodic. $\square$

Corollary 4.5.16. *Suppose every open set $U \subseteq X$ contains a fibre of π . Then the system (X, f) is weakly mixing if and only if (T, g) is.*

Proof. The proof is identical to that of Proposition 4.5.15, with $\pi \times \pi$ as the map between $X \times X$ and $T \times T$ and taking basic open sets $U_1 \times U_2$ and $V_1 \times V_2$ in place of U and V . $\square$

Corollary 4.5.17. *Suppose every open set $U \subseteq X$ contains a fibre of π . Then the system (X, f) is strongly mixing if and only if (T, g) is.*

Proof. We saw in the proof of Proposition 4.5.15 that $f^n(\pi^{-1}(U)) \cap \pi^{-1}(V) = \emptyset$ if and only if $g^n(U) \cap V = \emptyset$. The proof is clear from this fact. $\square$

Chapter 5

On $\sin \frac{1}{x}$ continua

The $\sin(\frac{1}{x})$ continuum is probably the most fundamental irreducible continuum other than the unit interval. This chapter of the thesis will study a nested family of continua similar to the $\sin \frac{1}{x}$ continuum. The family is constructed recursively by adding a half open interval to the previous continuum. This interval will limit onto the previous continuum in the same way that a half open interval limits onto a closed interval in the $\sin \frac{1}{x}$ continuum. Each of these continua will be decomposable and irreducible with a pair of E-continua. Results relating to mapping properties and the E-continua will be presented first, and then the continua will be used to construct an indecomposable continuum.

5.1 Referenced Theorems

The following results from other authors will be used in this chapter.

Sam B. Nadler Jr, Continuum Theory, an Introduction

These results can be found in [Nad92].

Theorem 5.1.1 (2.4). *An inverse limit of continua is a continuum. Also, an inverse limit of non-empty compact metric spaces is a compact metric space.*

Lemma 5.1.2 (2.6). *Let $\{X_i, f_i\}_{i=1}^{\infty}$ be an inverse sequence of metric spaces, with inverse limit X . For each $i = 1, 2, \dots$ let $\pi_i : X \mapsto X_i$ be the i^{th} projection map. Let A be a compact subset of X . Then, $\{\pi_i(A), f_i|_{\pi_{i+1}(A)}\}_{i=1}^{\infty}$ is an inverse sequence with onto bonding maps and*

$$\varprojlim \{\pi_i(A), f_i|_{\pi_{i+1}(A)}\} = A = X \cap \prod_{i=1}^{\infty} \pi_i(A)$$

Lemma 5.1.3 (3.2). *If a Hausdorff space is a continuous image of a compact metric space, then it is metrizable.*

Theorem 5.1.4 (8.18). *A Hausdorff continuum X is a Peano continuum if and only if X is the continuous image of $\mathbb{I}$.*

Theorem 5.1.5 (8.23). *Any Peano continuum is arcwise connected.*

Ryszard Engelking, General Topology

This result can be found in [Eng89].

Proposition 5.1.6 (6.1.11). *Let T be a topological space and let C be a connected subspace of T . For each subspace A satisfying $C \subseteq A \subseteq \overline{C}$, the subspace A is connected.*

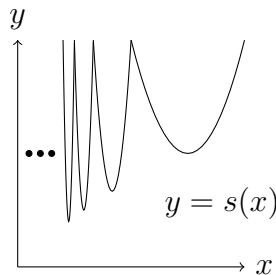
5.2 A Nested Sequence of Continua

In this section we will construct the nested sequence of continua and present a number of results on their mappings. First we need a function which can produce the necessary limiting behaviour. While the construction of these continua are motivated by the $\sin \frac{1}{x}$ continuum, that function will not work as it maps the half open interval to the closed interval. We will need to compose our map with itself, so we need a map $s : (0, 1] \mapsto (0, 1]$. In order to produce the limiting behaviour needed we require that as x approaches 0, $s(x)$ moves ever more rapidly over more of $(0, 1]$. The function I use is given below, although there are plenty of others which would also work.

Definition 5.2.1. For $n \in \omega \setminus \{0\}$ define $s_n : [\frac{1}{n+1}, \frac{1}{n}] \mapsto (0, 1]$ by

$$s_n(x) = 4n^3(n+1)\left(x - \frac{1}{n}\right)\left(x - \frac{1}{n+1}\right) + 1$$

This satisfies $s_n(\frac{1}{n}) = s_n(\frac{1}{n+1}) = 1$ and $\min(s_n) = \frac{1}{n+1}$. Define the continuous function $s : (0, 1] \mapsto (0, 1]$ by piecing together these functions. It is clear to see that s satisfies the properties mentioned above.



Definition 5.2.2. Let $A^n \subseteq \mathbb{I}^n$ be a sequence of continua defined as follows:

A^1 is the unit interval, which can be thought of as $\overline{(0, 1]}$.

If $A^n = \overline{\{(f(y), y) | y \in (0, 1]\}}$ for some $f : (0, 1] \mapsto [0, 1]^{n-1}$ then

$$A^{n+1} := \overline{\{(f(s(z)), s(z), z) | z \in (0, 1]\}}$$

Equivalently,

$$A^n = \overline{\{(s^{n-1}(x_n), \dots, s(x_n), x_n) | x_n \in (0, 1]\}}$$

It is worth noting that A^2 is homeomorphic to the $\sin \frac{1}{x}$ continuum. For $n = 1, 2$ the following results, or an analog, are well established so in the interests of clarity they will be omitted from this theorem.

Theorem 5.2.3. For all $n \geq 3$,

1. A^n is a continuum
2. If $B^n := \{(f(y), y) | y \in (0, 1]\}$, $A^n = \overline{B^n}$ and $\pi : B^n \mapsto [0, 1]^{n-1}$ is the projection onto the first $(n - 1)$ coordinates, then $\pi(B^n) = B^{n-1}$
3. $A^n = (A^{n-1} \times \{0\}) \cup B^n$
4. $\lambda(A^n) = \{A^{n-1}, \{1\}\}$

Proof. The point $(f(1), 1)$ will be denoted as 1 where there is no ambiguity.

1. B^n is the continuous image of $(0, 1]$ so it is connected, meaning A^n is as well by Proposition 5.1.6. As a closed subset of $[0, 1]^n$, A^n is compact and metric, and therefore a continuum.
2. It is clear from the definitions that π maps into B^{n-1} . Since $s : (0, 1] \mapsto (0, 1]$ is a surjection we have that π is as well.
3. Suppose the statement is true for $n - 1$ and let $(\underline{x}, y, z) \in A^n \setminus B^n$, with $\underline{x}$ representing multiple coordinates. It is clearly the case that $z < 0$ or $z > 1$ easily leads to a contradiction, so $z \in [0, 1]$. If $z \neq 0$ then $(\underline{x}, y, z) \in \overline{B^n \cap \mathbb{R}^{n-1} \times [z, 1]}$. However, this is the closure of a closed set, so $(\underline{x}, y, z) \in B^n$ giving a contradiction. Thus we have that $z = 0$. Now suppose $(\underline{x}, y) \notin A^{n-1}$. There exists an open set $U \subseteq \mathbb{R}^{n-1}$ such that $(\underline{x}, y) \in U$ and $U \cap A^{n-1} = \emptyset$. Then $U \times \mathbb{R}$ is an open set containing $(\underline{x}, y, 0)$ and not intersecting B^n , which is a contradiction. Thus we have that $A^n \subseteq B^n \cup A^{n-1} \times \{0\}$. Now consider $(f(y), y, 0)$ for some

$y \in (0, 1]$. Let $\epsilon > 0$. Note that for all $n > \frac{1}{y}$, y lies in the image of s_n . Let $m > \frac{1}{y}, \frac{1}{\epsilon}$. There exists $z \in [\frac{1}{m+1}, \frac{1}{m}]$ such that $s_m(z) = y$. Then we have that

$$d((f(y), y, 0), (f(s(z)), s(z), z)) = z \leq \frac{1}{m} < \epsilon$$

Thus we have that $B^n \cup A^{n-1} \times \{0\} \subseteq A^n$. Proof by induction on n completes the proof.

4. Let π_n denote projection onto the n^{th} coordinate. Let X be a subcontinuum of A^n containing 1 and intersecting $A^{n-1} \times \{0\}$. Then $\pi_n(X)$ is connected and contains 0 and 1, so equals $[0, 1]$. Thus $B^n \subseteq X$ and since X is closed we have $X = A^n$. Thus A^n is irreducible between 1 and any point of $A^{n-1} \times \{0\}$. Further, given any two points, at least one not from these sets, it is easy to construct a proper subcontinuum containing both: let their n^{th} coordinates be $\alpha \leq \beta$. If $\alpha = 0$ then $A^n \cap (\mathbb{R}^{n-1} \times [0, \beta])$ is such a continuum, else $A^n \cap (\mathbb{R}^{n-1} \times [\alpha, 1])$ is.

Therefore $\kappa(1) = A^n \setminus (A^{n-1} \times \{0\})$ and $\kappa((f(y), y, 0)) = A^n \setminus \{1\}$

□

Proposition 5.2.4. *Any continuous non-degenerate Hausdorff image of A^n is a decomposable continuum.*

Proof. We know that the Hausdorff image of a continuum is a continuum by Lemma 5.1.3. We shall prove this proposition by induction on n . First, let X be a continuous image of A^1 . From Theorem 5.1.4 we have that X is a Peano continuum. If X were indecomposable it could not be locally connected at any of its points, as any non-dense open set U would intersect every composant, so U being connected would imply $\overline{U}$ is a proper subcontinuum intersecting two distinct composants. Now suppose any image of A^{n-1} is decomposable. Let $f : A^n \mapsto X$ be a continuous surjection. Suppose there exists $y \in (0, 1)$ such that

$$f(A^n \cap ([0, 1]^{n-1} \times [0, y])) \neq X$$

Either the image of the other half of A^n , which is homeomorphic to an arc, completes the decomposition, or is equal to the whole of X , making X a Peano continuum. If no such y exists however, then for all $x \in X$ and for all $y \in (0, 1)$ we have that $f^{-1}(x) \cap \pi_n^{-1}([0, y)) \neq \emptyset$. Thus we can find a sequence y_m such that for each $m \in \omega$, $f(y_m) = x$ and $\pi_n(y_m) < \frac{1}{m}$. As A^n is a compact metric space this sequence has a subsequence which converges, say to y' . Then $y' \in A^{n-1} \times \{0\}$ and $f(y') = x$. Thus $X = f(A^{n-1} \times \{0\})$, so by the inductive hypothesis X is decomposable. □

The next three results are lemmas which will be used in the proof of Theorem 5.2.8. Lemmas 5.2.6 and 5.2.7 are very similar and are used to handle two different cases in the proof of the theorem.

Lemma 5.2.5. *Any continuous Hausdorff image of the half open interval $(0, 1]$ is arcwise connected.*

Proof. Let $f : (0, 1] \mapsto X$ be a continuous surjection, and let $x, y \in X$. Take points $x', y' \in (0, 1]$ such that $f(x') = x, f(y') = y$. We can suppose without loss of generality that $x' < y'$. Then $x, y \in Y := f([x', 1])$. From Theorem 5.1.4 we have that Y is a Peano continuum which means it is arcwise connected by Theorem 5.1.5. Thus there exists an embedding $i : \mathbb{I} \hookrightarrow Y$ such that $i(0) = x, i(1) = y$. Then when considered as a map into X it is clear that i is continuous, injective and an isomorphism onto its image, so it is an embedding into X . Thus there is an arc in X with end points x, y , which proves the lemma. $\square$

This lemma will clearly generalise to any space which can be expressed as the nested union of arcwise connected spaces.

Lemma 5.2.6. *Any continuous Hausdorff image of A^n can be expressed as the union of $m \leq n$ arcwise connected subspaces $P_1, \dots, P_m$ such that for all k , $\overline{P_k} = \bigcup_{i \leq k} P_i$ and $P_k \not\subseteq \overline{P_{k-1}}$.*

Proof. The proof will be done by induction. The base case $n = 1$ is equivalent to the claim that all Peano continua are arcwise connected, which we know from Theorem 5.1.5.

Suppose the inductive hypothesis is true for A^n and let $g : A^{n+1} \mapsto X$ be a continuous surjection. Let $Y := A^{n+1} \setminus A^n$. Then g restricts to a continuous surjection $g|_{A^n} : A^n \mapsto g(A^n)$, and we can apply the hypothesis to this to get $P_1, \dots, P_m$ for $m \leq n$. Let $P_{m+1} := g(Y)$. The subspace Y is homeomorphic to $(0, 1]$ so we know from Lemma 5.2.5 that P_{m+1} is arcwise connected. We also know this is true for P_k for $k \leq m$. Since $g(A^n)$ is compact it is also closed, which means the closure condition is met for all the P_k except possibly P_{m+1} . However,

$$\overline{P_{m+1}} = \overline{g(Y)} \supseteq g(\overline{Y}) = g(A^{n+1}) = X = \bigcup_{i=1}^{m+1} P_i$$

This gives everything except the condition that $P_{m+1} \not\subseteq \overline{P_m}$. If this is not the case then simply omit P_{m+1} from the list, and $P_1, \dots, P_m$ will satisfy the conditions of the lemma. $\square$

Lemma 5.2.7. *Any continuous Hausdorff image of A^n can be expressed as the union of $m < n$ arcwise connected subspaces $P_1, \dots, P_m$ such that for each pair of integers $k \neq l$, $\overline{P_k} \supseteq \bigcup_{i \leq k} P_i$ and $P_k \cap P_l = \emptyset$.*

Proof. This proof will be similar to the previous lemma, with the base case identical. Suppose the result holds for A^n and let $g : A^{n+1} \mapsto X$ be a continuous surjection, with arcwise connected $Q_1, \dots, Q_{m'}$ satisfying the conditions of the inductive hypothesis for the restriction $g|_{A^n} : A^n \mapsto g(A^n)$. Let $Q_{m'+1} = g(A^{n+1} \setminus A^n)$ and define

$$M = \{k = 1, \dots, m' + 1 \mid Q_k \cap Q_{m'+1} \neq \emptyset\}$$

Now let $P_1 = Q_\alpha$ for minimum $\alpha \notin M$, $P_2 = Q_\beta$ for minimum $\beta \notin M \cup \{\alpha\}$ and so on, with the final subspace being $P_m = \bigcup_{k \in M} Q_k$.

If two arcwise connected spaces have a non-trivial intersection then their union is also arcwise connected, so P_m is arcwise connected, and each of the other P_k are arcwise connected by construction. For $k, l < m$ distinct with $P_k = Q_\alpha, P_l = Q_\beta$ we have from the inductive hypothesis that $P_k \cap P_l = Q_\alpha \cap Q_\beta = \emptyset$. Further, we have that $\bigcup_{i < k} P_i \subseteq \bigcup_{\gamma < \alpha} Q_\gamma \subseteq \overline{Q_\alpha} = \overline{P_k}$. If $k \neq m$ then by construction $P_k \cap P_m = \emptyset$. Finally,

$$\overline{P_m} \supseteq \overline{g(A^{n+1} \setminus A^n)} \supseteq g(\overline{A^{n+1} \setminus A^n}) = g(A^{n+1}) = \bigcup_{i < m} P_i$$

□

The next theorem concerns irreducible images of A^n . The proof begins by showing that one of the elements of $\lambda(X)$ must be a singleton. It then considers two cases, whether or not the second one is a singleton. In each case we conclude that X is the image of A^k for a smaller k or that the other element of $\lambda(X)$ is the image of A^{n-1} .

Theorem 5.2.8. *Let $g : A^n \mapsto X$ be a surjection for $n \geq 2$ and X an irreducible continuum. Either X is the continuous image of A^k for some $k < n$ or $\lambda(X)$ consists of a singleton and a continuous image of A^{n-1} .*

Proof. Suppose that X is not the continuous image of A^k for any $k < n$. Define $Y = g(A^{n-1} \times \{0\})$. Let $\lambda(X) = \{a, b\}$. If $a \cap Y \neq \emptyset \neq b \cap Y$ then since Y is a subcontinuum, $Y = X$ which is a contradiction. Without loss of generality let $b \cap Y = \emptyset$. If $a \cap Y = \emptyset$ then both a and b lie in the image of $A^n \setminus A^{n-1}$, which is homeomorphic to a half open interval. By taking an arc with end points in a and b , then taking the image of this arc under g we see that X is the image of an arc, so is the image of A^1 . Thus we can assume $a \cap Y \neq \emptyset$.

Suppose b is not a singleton. Pick two points $x, y \in b$. Both $g^{-1}(x)$ and $g^{-1}(y)$ are compact sets, so their projections onto the last coordinate are compact. Let x', y' be the minima of these sets respectively. Without loss of generality let $x' < y'$. Then $g(A^n \cap (\mathbb{R}^{n-1} \times [0, x']))$ is a proper subcontinuum of X since it does not contain y , but it intersects both a and b . This is a contradiction, which gives that b is a singleton. From now on b will represent the unique element of X such that $\kappa(b) = X \setminus a$. If $b' := \min \pi_n(g^{-1}(b))$ then $g(A^n \cap (\mathbb{R}^{n-1} \times [0, b'])) = X$. Since $b' \neq 0$ we can assume without loss of generality that $b' = 1$ and $b = g(f(1), 1)$. We will now consider two cases, namely whether or not a is a singleton.

Case 1: Suppose a is not a singleton, and let $x \in a, x = g(y, z)$ for $y \in [0, 1]^{n-1}$ and $z \in [0, 1]$. Then as $X = \text{irr}(x, b)$ we have that $X = g(A^n \cap [z, 1] \times \mathbb{R}^{n-1})$ and since X is not the image of the unit interval (A^1) we have that $z = 0$ and therefore $a \subseteq Y$.

From Lemma 5.2.6 we can express Y as the union of arcwise connected sets $P_1, \dots, P_m$ for $m \leq n - 1$. Suppose for some k , $a \cap P_k \neq \emptyset$ and $P_k \not\subseteq a$. Let $x \in a \cap P_k, y \in P_k \setminus a$ and let Z be a proper subcontinuum of X containing b and y . As P_k is arcwise connected there exists $i : \mathbb{I} \hookrightarrow P_k$ such that $i(0) = x, i(1) = y$. Then $Z \cup i(\mathbb{I})$ is a continuum containing x and b which is therefore X . Let $x' \in a \setminus \{x\}$, then as Z is a proper subcontinuum of X containing b it must be that $x' \notin Z$. We therefore know that $x' \in i(\mathbb{I})$ and thus can find α with $x' = i(\alpha)$. Then since i is injective we have that $Z \cup i([\alpha, 1])$ is a proper subcontinuum, as it does not contain $x = i(0)$. It intersects a at x' and contains b , which gives a contradiction. Therefore we can conclude that if $a \cap P_k \neq \emptyset$ then $P_k \subseteq a$.

If $P_k \cap a = \emptyset$ then there exists a proper subcontinuum X' of X which contains b and intersects P_k . Further, $X' \cup \overline{P_k}$ is a continuum. Consider two situations. Situation one is if $X' \cup \overline{P_k} = X$. For any $l > k$ we have that

$$P_l \not\subseteq \overline{P_k} \Rightarrow P_l \cap X' \neq \emptyset \Rightarrow P_l \not\subseteq a \Rightarrow P_l \cap a = \emptyset$$

Since this is true of all $l > k$ we have that $a \subseteq P_1 \cup \dots \cup P_{k-1}$.

Situation two is if $X' \cup \overline{P_k} \neq X$. In this case as for all $l < k, P_l \subseteq \overline{P_k}$ we have that $P_l \cap a = \emptyset$. This implies that $a \subseteq P_{k+1} \cup \dots \cup P_m$.

Suppose $\{k \in \{1, \dots, m\} | a \cap P_k = \emptyset\} \neq \emptyset$. Let α be its minimal element and β be its maximal element. Let X_α and X_β be proper subcontinua of X containing b and intersecting P_α, P_β respectively. As α is minimal we know that $X_\alpha \cup \overline{P_\alpha} = X$ since it is not the case that for all $l < \alpha, P_l \cap a = \emptyset$. Thus $a \subseteq P_1 \cup \dots \cup P_{\alpha-1}$. Similarly

we have that $X_\beta \cup \overline{P_\beta} \neq X$ and therefore $a \subseteq P_{\beta+1} \cup \cdots \cup P_m$. This gives our final contradiction, from which we see that

$$a = P_1 \cup \cdots \cup P_m = Y = g(A^{n-1} \times \{0\})$$

Case 2: We will now consider a to be a singleton. As with b , a will now be a point of X instead of a subset. Using Lemma 5.2.7 we can express $X = \bigcup_{i=1}^m P_i$ for arcwise connected P_i , with $P_i \cap P_j = \emptyset$ for $i \neq j$ and $\overline{P_k} \supseteq \bigcup_{i \leq k} P_i$. We know from the construction of the P_k that $b \in P_m$. If $a \in P_m$ then since P_m is arcwise connected there exists $i : \mathbb{I} \hookrightarrow P_m$ such that $i(0) = a, i(1) = b$. Since the image of $\mathbb{I}$ under i is a subcontinuum of X we have that $X = i(\mathbb{I})$, so unless $n = 1$ this contradicts the initial assumption that X is not the continuous image of A^k for $k < n$.

Suppose $a \in P_\alpha$ for $\alpha \neq m$. Suppose further that P_α is not a singleton. Then there exists $X' \subsetneq X$ a proper subcontinuum with $b \in X', X' \cap P_\alpha \neq \emptyset$. Let $x \in X' \cap P_\alpha$, let $i : \mathbb{I} \hookrightarrow P_\alpha, i(0) = a, i(1) = x$. Then since $X' \cup i(\mathbb{I})$ is a continuum containing a and b we have that $X = X' \cup i(\mathbb{I})$. Since $P_{\alpha+1} \cap P_\alpha = \emptyset$ we have that $P_{\alpha+1} \subseteq X'$, implying that $\overline{P_{\alpha+1}}$ is a subset of X' and therefore P_α is a subset of X' as well, which is a contradiction.

Therefore $P_\alpha = \{a\}$. From its construction in Lemma 5.2.7 we know that P_α contains the image of some $A^k \setminus A^{k-1}$, and therefore since singletons are closed and $A^k \setminus A^{k-1}$ is a dense subset of A^k we have that $g(A^k) = \{a\}$. We will now show that X is the image of A^{n-1} .

Let $\pi(x_1, \dots, x_n) = (0, x_2, \dots, x_n)$ be the projection onto the last $n-1$ coordinates. On $\{\underline{x} | x_2 \neq 0\}$, π is a bijection with inverse $(0, x_2, \dots, x_n) \mapsto (s(x_2), x_2, \dots, x_n)$. Thus we can define a continuous function $h : A^{n-1} \mapsto X$ as follows: $h|_{x_2 \neq 0} = g \circ \pi^{-1}$ and $h|_{x_2=0} = a$. Then h is a surjection and $g = h \circ \pi$. If U is an open subset of X then $h^{-1}(U) = \pi(g^{-1}(U))$. We have that g is continuous and π is a projection, so $h^{-1}(U)$ is open, making h continuous and X the continuous image of A^{n-1} . This final contradiction completes the proof of the theorem. $\square$

Definition 5.2.9. A continuum X is arc-like if for all $\epsilon > 0$ there exists a map $f_\epsilon : X \mapsto \mathbb{I}$ such that for any pair of points $x, y \in X$ satisfying $f_\epsilon(x) = f_\epsilon(y)$, the distance between x and y in X is less than ϵ . Such a map is called an ϵ -map

Proposition 5.2.10. *Each A^n is arc-like.*

Proof. This shall be proved by induction on n . Indeed, we will show the very slightly stronger claim that each is witnessed by a map that sends the point $(\dots, s(1), 1)$ to

1. Since A^1 is the unit interval, the base case is trivial. Suppose A^n is arc-like, and for all $\epsilon > 0$ this is witnessed by $f_\epsilon : A^n \mapsto \mathbb{I}$. Fix $\epsilon > 0$. Pick $x_\epsilon < \frac{\epsilon}{2}$ such that $s(x_\epsilon) = 1$, and let $\pi : A^{n+1} \mapsto A^n$ be the projection onto the first n coordinates. We will construct a map $g_\epsilon : A^{n+1} \mapsto \mathbb{I}$. Let $g_\epsilon|_{A^n \times [0, x_\epsilon]} = \frac{1}{2}f_{\frac{\epsilon}{2}} \circ \pi$, and let g_ϵ map the rest of A^{n+1} linearly onto $[\frac{1}{2}, 1]$. This map is continuous as $\frac{1}{2}f_{\frac{\epsilon}{2}} \circ \pi(\dots, 1, x_\epsilon) = \frac{1}{2}$. Let $x, x' \in g_\epsilon^{-1}(y)$ for some y . If $y > \frac{1}{2}$ then $x = x'$. Otherwise,

$$\begin{aligned} d(x, x') &\leq |x_\epsilon| + d(\pi(x), \pi(x')) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

Thus g_ϵ is an ϵ -map from A^{n+1} to $\mathbb{I}$ for arbitrary ϵ , which proves that A^n is arc-like for all n . $\square$

5.3 An Indecomposable Continuum

We have discussed the spaces A^n as a nested sequence of continua, each defined recursively containing the previous one. It is natural then to consider the union of all of them in $\mathbb{I}^\omega$, or rather its closure to get a continuum.

Definition 5.3.1. Consider each A^n as a subspace of $\mathbb{I}^\omega$ with the metric $d(x, y) = \sum 2^{-n}|x_n - y_n|$. Each A^n is therefore seen as $A^n \times \{0\}^{\omega \setminus n}$. Let $A^\infty := \bigcup_{n \in \omega} A^n$ and $A^\omega := \overline{A^\infty}$.

A^∞ is connected as it is the nested union of connected spaces. This means that A^ω is also connected, and is therefore a continuum.

Lemma 5.3.2. Let $L = \{x \in (0, 1]^\omega \mid \text{For all } n, s(x_{n+1}) = x_n\}$. Then $A^\omega = A^\infty \cup L$.

Proof. Let $x \in A^\omega \setminus A^\infty$ and suppose $x \notin L$. Either there is an $n \in \omega$ such that $s(x_{n+1}) \neq x_n$ or there is a coordinate of x which is a zero, or both. Either way there exists $n \in \omega$ such that $(x_1, \dots, x_n) \notin A^n$. Let σ_n be the projection onto the first n coordinates. Then for any $k \in \omega$, $\sigma_n(A^k) \subseteq A^n$, which implies that $\sigma_n(A^\infty) \subseteq A^n$. Thus $\sigma_n^{-1}(\mathbb{I}^n \setminus A^n)$ is an open subset of $\mathbb{I}^\omega$ containing x but not intersecting A^∞ , which is a contradiction. We therefore have that $A^\omega \subseteq A^\infty \cup L$.

Now let $x \in L$ and let U be a basic open subset of $\mathbb{I}^\omega$ which contains x . Without loss of generality say that U restricts the first n coordinates. Then $(x_1, \dots, x_n, 0, \dots)$ is an element of $A^\infty \cap U$, so $A^\infty \cap U$ is non-empty and $x \in A^\omega$. Thus $A^\infty \cup L \subseteq A^\omega$. $\square$

We have that A^ω is the union of A^∞ and an inverse limit on $(0, 1]$. It could also be seen as a generalised inverse limit on $[0, 1]$ with bonding map given by the graph of A^2 .

Proposition 5.3.3. *A^ω is arc-like.*

Proof. Fix $\epsilon > 0$. Let $N \in \omega$ such that $\sum_{n \geq N} 2^{-n} < \frac{\epsilon}{2}$. Let $f_{\frac{\epsilon}{2}} : A^N \mapsto \mathbb{I}$ be an $\frac{\epsilon}{2}$ -map, whose existence is guaranteed by Proposition 5.2.10, and let $\pi : A^\omega \mapsto A^N$ be the projection map and $g_\epsilon : A^\omega \mapsto \mathbb{I}$ be the composition of $f_{\frac{\epsilon}{2}}$ and π . If $x, x' \in g_\epsilon^{-1}(y)$ for some $y \in \mathbb{I}$ then

$$d(x, x') \leq \sum_{n \geq N} 2^{-n} + d_N(\pi(x), \pi(x')) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

Thus g_ϵ is an ϵ -map for arbitrary ϵ , which proves that A^ω is arc-like. $\square$

Most of the rest of this section will focus on the composants of A^ω . We will see that A^ω is an indecomposable continuum and that A^∞ is a component of it. It is unusual for an indecomposable continuum to have such an easily defined component, for example if you consider the pseudoarc there is no comparable expression for the components.

Lemma 5.3.4. *Let $C \subseteq A^\omega$ be a subcontinuum and suppose $\{k \in \omega \mid C \cap A^k \not\subseteq A^{k-1}\}$ is unbounded. Then $C = A^\omega$.*

Proof. Let $m \in \{k \in \omega \mid C \cap A^k \not\subseteq A^{k-1}\}$ and let $\pi_m : A^\omega \mapsto \mathbb{I}$ be the projection onto the m^{th} coordinate. Since C is a continuum, $\pi_m(C) = [0, \alpha]$ for some $\alpha > 0$. Let $x \in A_{m-1}$ and take a sequence $(x_k)_{k \in \omega} \in A_m$ with $\pi_m(x_k) \in (0, \alpha]$ and $x_k \rightarrow x$. Then for each $k \in \omega$ there exist $y_k \in C$ such that $\pi_m(y_k) = \pi_m(x_k)$. Since each of the first $(m-1)$ coordinates of a point of A_ω are defined by the m^{th} when it is non-zero we have that the projection of y_k onto A_m is x_k . There is a convergent subsequence of (y_k) which converges to a point y . By the continuity of projections, the projection of y onto A_m is equal to x , which also implies $\pi_m(y) = 0$. Since y has a coordinate of value 0 it must lie in A^∞ and every subsequent coordinate must also be 0, so $y = x$. Since C is closed we have that $x \in C$. This means that $A_{m-1} \subseteq C$. As this is true of every m in $\{k \in \omega \mid C \cap A^k \not\subseteq A^{k-1}\}$ we have that $A^\infty \subseteq C$. Since C is closed we have that $A^\omega \subseteq C$ and therefore $C = A^\omega$. $\square$

Theorem 5.3.5. *A^ω is indecomposable.*

Proof. Let $A^\omega = C \cup D$ for two subcontinua C, D . Then as $A^\infty \subseteq C \cup D$, one of C or D satisfies the conditions of Lemma 5.3.4. Thus one of C and D is equal to A^ω , which completes the proof. $\square$

Proposition 5.3.6. A^∞ is a composant of A^ω

Proof. Let $x, y \in A^\infty$. By the definition of A^∞ there are $n, m \in \omega$ such that $x \in A^n, y \in A^m$. Without loss of generality let $n \leq m$. Then $x, y \in A^m$ which is a proper subcontinuum of A^ω . This proves that A^∞ is continuumwise connected, so is contained in some composant of A^ω .

Let C be a subcontinuum of A^ω which intersects both A^∞ and L . We will show that $\{n \in \omega \mid C \cap A^n \not\subseteq A^{n-1}\}$ is unbounded, and apply Lemma 5.3.4. We can assume this set is non-empty as $C \cap A^\infty \neq \emptyset$ so $C \cap A^m \neq \emptyset$ for some m . Thus if $\{n \in \omega \mid C \cap A^n \not\subseteq A^{n-1}\} = \emptyset$ then replace C with $C \cup A^m$ and note that the first continuum is equal to A^ω if and only if the second one is.

Let $C \cap A^n \not\subseteq A^{n-1}$. Let $x = (x_1, \dots, x_{n+1}, 0, \dots) \in A^{n+1} \setminus A^n$. Since C is connected and intersects L we know that for any $k \in \omega$ the projection onto the k^{th} coordinate satisfies $\pi_k(C) = [0, \alpha_k]$ for some $\alpha_k > 0$. Given any point $p \in (0, 1], s^{-1}(p)$ contains 0 in its closure. Using all of these properties we can define a sequence $(z_k) \in C \cap L$ such that for all n , $\pi_1(z_k) = x_1, \dots, \pi_{n+1}(z_k) = x_{n+1}$ and $\pi_{n+2}(z_k), \dots, \pi_{k+n+1}(z_k) < 2^{-k}$. Then we have that

$$d(x, z_k) < \sum_{i=1}^k 2^{-k} \cdot 2^{-(n+i+1)} + \sum_{i=n+k+2}^{\infty} 2^{-i}$$

The right hand side tends to 0 as $k \rightarrow \infty$, so $x \in C$. Thus $\{n \in \omega \mid C \cap A^n \not\subseteq A^{n-1}\}$ is unbounded, which means $C = A^\omega$. This proves that any proper subcontinuum which intersects A^∞ is contained in A^∞ . Therefore A^∞ is a composant of A^ω . $\square$

We will now investigate the remaining composants of A^ω , namely those lying in L . Central to this is the observation that L is an inverse limit in which every coordinate space is $(0, 1]$ and every bonding map is s . We will look into what a continuum contained in L could look like, and extrapolate what the composants are from this.

Recall Lemma 5.1.2. If C is a compact subset of L then it is defined by its projections. For the rest of this section if C is a compact set, or a subcontinuum, then let $C_n := \pi_n(C)$.

Lemma 5.3.7. Let C be a compact subset of L , or indeed any inverse limit of metric spaces. Then C is a continuum if and only if each C_n is a continuum.

Proof. Since projections are continuous, if C is a continuum then so are all of the C_n .

Any inverse limit in which all of the coordinate spaces are continua is itself a continuum by Theorem 5.1.1, so if each C_n is a continuum then so is C . $\square$

It is a straightforward algebraic manipulation to see that if $x > \frac{1}{n}$ then

$$s_n^{-1}(x) = \frac{2n+1 \pm \sqrt{\frac{(n+1)x-1}{n}}}{2n(n+1)}$$

Given any continuum $[x, y] \subseteq (0, 1]$ we therefore have that if m_x, m_y are respectively the minimum integers such that $\frac{1}{m_x} \leq x, \frac{1}{m_y} \leq y$ then

$$\begin{aligned} s^{-1}([x, y]) &= \bigcup_{n \in \omega} s_n^{-1}([x, y]) \\ &= \bigcup_{n \geq m_y} s_n^{-1}([x, y]) \\ &= \bigcup_{m_y \leq n \leq m_x} s_n^{-1}([x, y]) \cup \bigcup_{n > m_x} s_n^{-1}([x, y]) \\ &= \bigcup_{m_y \leq n \leq m_x} \left[\frac{2n+1 - \sqrt{\frac{(n+1)y-1}{n}}}{2n(n+1)}, \frac{2n+1 + \sqrt{\frac{(n+1)y-1}{n}}}{2n(n+1)} \right] \\ &\quad \cup \bigcup_{n > m_x} \left[\frac{2n+1 - \sqrt{\frac{(n+1)y-1}{n}}}{2n(n+1)}, \frac{2n+1 - \sqrt{\frac{(n+1)x-1}{n}}}{2n(n+1)} \right] \\ &\quad \cup \bigcup_{n > m_x} \left[\frac{2n+1 + \sqrt{\frac{(n+1)x-1}{n}}}{2n(n+1)}, \frac{2n+1 + \sqrt{\frac{(n+1)y-1}{n}}}{2n(n+1)} \right] \end{aligned}$$

While the specific subsets do not look particularly useful, the important point here is that the inverse image of a copy of the interval $[x, y]$ in $(0, 1]$ consists of a countable, infinite union of disjoint copies of $\mathbb{I}$, all but finitely many of which surjects under s onto $[x, y]$. Thus constructing a continuum C in L is equivalent to choosing an $[x, y]$ to act as C_1 , then choosing one of the countably infinite components of $s^{-1}(C_1)$ which surjects onto C_1 to be C_2 , then one of the components of $s^{-1}(C_2)$ to be C_3 , and so on. In some cases it would be possible to take as C_{n+1} only a subcontinuum of a component of C_n . However, since we are investigating the composants of A^ω it makes sense to look at subcontinua which are as large as possible. For this same reason, we

will also only consider continua of the form $[x, 1]$ for C_1 , and we can further simplify by restricting x to be the reciprocal of some natural number. Thus two elements x and y of L lie in the same composant of A^ω if and only if there exists $n \in \omega$ such that $x_1, y_1 \in D_1 := [\frac{1}{n}, 1]$ and for all $m \in \omega$, x_m and y_m belong to the same component of $s^{-1}(D_{m-1})$, which then becomes denoted D_m .

Chapter 6

Compactifications

In this chapter we will look at irreducibility and compactifications. In particular we will focus on compactifications whose remainder consists entirely of points of irreducibility. Studying the properties of spaces with such compactifications will provide an insight into the structure of irreducible continua.

6.1 Referenced Theorems

The following results from other authors will be used in this chapter.

Sam B. Nadler Jr, Continuum Theory, an Introduction

These results can be found in [Nad92].

Lemma 6.1.1 (3.2). *If a Hausdorff space is a continuous image of a compact metric space, then it is metrizable.*

Theorem 6.1.2 (5.4). *Let X be a continuum and let U be a nonempty, proper, open subset of X . If K is a component of $\overline{U}$, then $K \cap \partial U \neq \emptyset$. Equivalently, $K \cap (X \setminus U) \neq \emptyset$.*

Proposition 6.1.3 (6.3). *Let T be a connected topological space and let C be a connected subset of T such that $T \setminus C$ is disconnected, $T \setminus C = A \cup B$. Then $A \cup C$ and $B \cup C$ are connected. Hence, if T and C are continua, $A \cup C$ and $B \cup C$ are continua.*

Theorem 6.1.4 (11.17). *If X is a nondegenerate indecomposable continuum then the composants of X are mutually disjoint.*

D.E. Bennett and J.B. Fugate, Continua and their non-separating Subcontinua

This result can be found in [BF77].

Theorem 6.1.5 (1.30). *The E -continua of X are exactly those end continua at which X is locally connected.*

Ryszard Engelking, General Topology

These results can be found in [Eng89].

Theorem 6.1.6 (3.5.8). *For every Tychonoff space X the following conditions are equivalent.*

1. *The space X is locally compact.*
2. *For every compactification cX of the space X the remainder $cX \setminus X$ is closed in cX .*
3. *There exists a compactification cX of the space X such that the remainder $cX \setminus X$ is closed in cX .*

Corollary 6.1.7 (6.1.11). *Let T be a topological space and let C be a connected subspace of T . For each subspace A satisfying $C \subseteq A \subseteq \overline{C}$, the subspace A is connected.*

K. D. Magill, n -point Compactifications

This result can be found in [Mag65].

Theorem 6.1.8 (2.1). *The following statements concerning a space X are equivalent:*

1. *X has an n -point compactification.*
2. *X is locally compact and contains an n -star; a compact subset K whose complement is the union of n mutually disjoint, open subsets $\{G_i\}_{i=1}^n$ such that $K \cup G_i$ is not compact for each i .*

Max Pitz, Topological reconstruction and compactification theory

These results can be found in [Pit15].

Theorem 6.1.9 (3.1.4). *If a space X has a maximal N -point compactification νX and γX is any other finite compactification then $\gamma X \leq \nu X$.*

Corollary 6.1.10 (3.1.5). *If X has a maximal N -point compactification then all N -point compactifications of X are equivalent.*

6.2 Irreducibility and Remainders

Suppose that Y is an irreducible, decomposable continuum with a full set of E-continua. The subset consisting of the union of all the E-continua has an empty interior, so if Y' is the complement of this set then we know that Y' is dense in Y . This holds regardless of whether we are considering classical irreducibility or finite irreducibility. Another way of saying this is that Y is a compactification of Y' . In this chapter we will look at the properties of such a space Y' . We will also investigate which properties a space X must have in order to have a compactification which is an irreducible continuum with a full set of E-continua. This will conclude with a set of necessary and sufficient conditions arising as the intersection of these two approaches.

Definition 6.2.1. A space X is open ended if for all closed and connected proper subsets $A, B \subsetneq X$ such that $A \cup B = X$, neither A nor B is compact.

Lemma 6.2.2. *Let X be a connected space with metric compactification γX . Then γX is a continuum.*

Proof. X is connected, so its closure in a space is connected by Corollary 6.1.7, and it is dense in γX . As it is a compactification γX is clearly compact, and we have stated that it is metric. $\square$

Definition 6.2.3. A space X is said to be densely irreducible if it is not compact, is continuumwise connected, locally compact and open ended.

The following theorem shows that we can safely narrow our focus to densely irreducible spaces, as any space with an irreducible continuum as a compactification, whose remainder is the set of E-continua, must have these properties.

Theorem 6.2.4. *Let Y be a decomposable, n -irreducible continuum. Suppose that $Y = \text{irr}(p_1, \dots, p_n)$ and each $\lambda(p_i)$ compact. Let Y' be the set of non-irreducible points of Y . Then Y' is non-compact, locally compact, continuum-wise connected and open ended.*

Proof. It is immediate that $Y' = Y \setminus (\lambda(p_1) \cup \dots \cup \lambda(p_n))$. This clearly shows that Y' is an open subset of Y . Since Y is connected, Y' cannot be closed and is therefore not compact. Now Y' is a dense subset of Y so Y is a compactification of Y' which also gives us that Y' is locally compact by Theorem 6.1.6.

For proof that Y' is continuumwise connected see Proposition 3.4.9.

Let A, B be closed connected subsets of Y' such that $Y' = A \cup B$. Suppose that A is compact. Then $\overline{A}^Y = A$ which implies $A \cup \overline{B}^Y = Y$. No point p_i can lie in A so $p_1, \dots, p_n \in \overline{B}^Y$ implying $\overline{B}^Y = Y$ and $B = \overline{B}^Y \cap Y' = Y'$. Thus Y' is open ended. $\square$

Note that if X is densely irreducible then being continuumwise connected implies that it is also connected. We will first see some of the properties of a compactification of such a space, in particular looking at the points of irreducibility of γX .

Proposition 6.2.5. *If X is a densely irreducible space and γX a metric compactification of it then γX is decomposable.*

Proof. X is a proper continuumwise connected subset of γX . If γX were indecomposable then the composants would form a partition (Theorem 6.1.4). Since X is locally compact, Theorem 6.1.6 gives us that it is an open subset of γX . This means that it would have to intersect each of the composants of γX , which would imply there is a proper subcontinuum of X intersecting two distinct composants of γX . This is a contradiction as indecomposable continua are irreducible between any pair of points from distinct composants. $\square$

Proposition 6.2.6. *Let X be a densely irreducible space and suppose there exist two proper closed connected subsets A, B of X such that $X = A \cup B$. Then these can be extended to a decomposition of γX for any metric compactification γX of X .*

Proof. Let $A' = \overline{A}^{\gamma X}$, and $B' = \overline{B}^{\gamma X}$. Since A and B are connected, A' and B' will be as well by Corollary 6.1.7, which makes them continua. $X \neq A = \overline{A}^X = A' \cap X$ which means that A' , and by symmetry B' , are proper subcontinua of γX . Now $A' \cup B' = \overline{A \cup B}^{\gamma X} = \overline{X}^{\gamma X} = \gamma X$, so A' and B' form a decomposition of γX . $\square$

We will now look at the points of irreducibility of γX , and show that all of them must lie in the remainder $\gamma X \setminus X$.

Lemma 6.2.7. *Let X be a densely irreducible space, γX a metric compactification of X and suppose $\gamma X = \text{irr}(p_1, \dots, p_n)$. Let $\lambda(p_i) \subseteq \gamma X \setminus X$. Then $\lambda(p_i)$ is an n -E-continuum, and is a component of $\gamma X \setminus X$.*

Proof. X is locally compact, so by Theorem 6.1.6 we have that $\gamma X \setminus X$ is a closed and compact subset of γX . Let K be the component of $\gamma X \setminus X$ containing $\lambda(p_i)$. Since components are closed (Corollary 6.1.7) we have that K is closed in $\gamma X \setminus X$ and is therefore compact and a continuum. As K is a subset of $\gamma X \setminus X$ and X is a dense subset of γX we know that K has empty interior so by applying Proposition 3.2.8 we can therefore deduce that $\lambda(p_i) = K$, which proves that $\lambda(p_i)$ is a component of $\gamma X \setminus X$ and is compact. $\square$

Theorem 6.2.8. *Let X be a densely irreducible space and γX a metric compactification of X . If γX is n -irreducible, then all of its points of irreducibility lie in $\gamma X \setminus X$.*

Proof. Let $\gamma X = \text{irr}(p_1, \dots, p_n)$. Since X is continuumwise connected it cannot be the case that $p_1, \dots, p_n \in X$, or indeed that each $\lambda(p_i)$ intersects X . Suppose $p_1, \dots, p_k \in X$ and for each $i > k$ we have $\lambda(p_i) \cap X = \emptyset$. From Lemma 6.2.7 we have that for each $i > k$, $\lambda(p_i)$ is an E-continuum of γX so in particular γX is locally connected at $\lambda(p_i)$ by Theorem 6.1.5. Suppose $x \in \gamma X \setminus (X \cup \lambda(p_{k+1}) \cup \dots \cup \lambda(p_n))$. Then as γX is normal we can construct disjoint open sets $U_{k+1}, \dots, U_n, V$ such that $\lambda(p_i) \subseteq U_i$ and $x \in V$. By local connectedness we can assume each U_i is connected. Each U_i intersects X so we can take finitely many subcontinua $K_1, \dots, K_m$ of X such that $\bigcup_{i=1}^m K_i \cup \bigcup_{j=1}^n U_j$ is connected. Let C be the closure of this union. Then take subcontinua $C_1, \dots, C_k \subseteq X$ each intersecting C and with $p_i \in C_i$. Let $K = C \cup C_1 \cup \dots \cup C_k$. This is a subcontinuum of γX containing $p_1, \dots, p_n$ but not containing x , which is a contradiction. Thus we have shown that $\gamma X \setminus X = \lambda(p_{k+1}) \cup \dots \cup \lambda(p_n)$.

We will now consider a pair of proper subcontinua A and B such that $p_1 \in A$ and $p_2, \dots, p_n \in B$. We know that B exists as $\gamma X = \text{irr}(p_1, \dots, p_n)$. For A , take any subcontinuum of X such that $p_1 \in A$ and $A \cap B \neq \emptyset$. Now $\gamma X = A \cup B$ and this is a proper decomposition.

Define $B_0 = B$ and $B_i = B_{i-1} \setminus \lambda(p_{k+i})$. These are all non-empty and intersect X as B cannot lie entirely in $\gamma X \setminus X$. We will show by induction that each B_i is connected.

The base case $i = 0$ is trivial, so suppose B_{i-1} is connected and let $B_i = U \cup V$ for disjoint B_i -open subsets U and V . Since $B_i = B_{i-1} \setminus \lambda(p_{k+i})$, and both B_{i-1} and $\lambda(p_{k+i})$ are connected, we have that $U \cup \lambda(p_{k+i})$ and $V \cup \lambda(p_{k+i})$ are connected by Proposition 6.1.3. Let $C \subsetneq \gamma X$ be a proper subcontinuum, $p_1, \dots, p_{k+i-1}, p_{k+i+1}, \dots, p_n \in C$. One of U and V must intersect X so assume it is U . Take $K \subseteq X$ a subcontinuum intersecting both C and U . Then $\overline{C \cup K \cup U \cup \lambda(p_{k+i})}$ is a continuum, and is equal to $C \cup K \cup \overline{U} \cup \lambda(p_{k+i})$. Since this contains $p_1, \dots, p_n$ it must equal γX , and as $V \cap \overline{U} = \emptyset$ it must be that $V \subseteq C \cup K$. The same argument then gives us that $U \subseteq C \cup K$. Consequently $(C \cup K) \cup \lambda(p_{k+i}) = \gamma X$, which is a contradiction as these are disjoint closed sets and γX is connected. Thus B_i must be connected.

The above induction shows that $B_{n-k} = B \cap X$ is connected. From this we have that $X = (X \cap A) \cup (X \cap B) = A \cup (X \cap B)$. Now A is a continuum and $X \cap B$ is a connected X -closed set, so this contradicts the fact that X is open ended. Thus it must be that $p_1, \dots, p_n \notin X$. $\square$

Proposition 6.2.9. *Let X be a densely irreducible space and let γX be a metric compactification of X . If γX is n -irreducible then $\gamma X \setminus X$ has precisely n components.*

Proof. We know from Theorem 6.2.8 that all points of irreducibility lie in $\gamma X \setminus X$. Each $\lambda(p_i)$ is a component of $\gamma X \setminus X$ so there are at least n components. Now suppose there are more than n components. This means that $\gamma X \setminus X \neq \lambda(p_1) \cup \dots \cup \lambda(p_n)$, so take a point x witnessing this. Take open sets $U_1, \dots, U_n, V$ containing $\lambda(p_i)$ and x respectively. Since γX is normal we can assume these open sets are pairwise disjoint. From Theorem 6.1.5 we know that γX is locally connected about each $\lambda(p_i)$ so we can assume each U_i is connected. Each U_i intersects X so take subcontinua C_i of X such that $C_i \cap U_i \neq \emptyset \neq C_i \cap U_n$ for each i between 1 and $n-1$. Then $C = C_1 \cup \dots \cup C_{n-1} \cup \overline{U_1} \cup \dots \cup \overline{U_n}$ is a continuum. As $x \in V$ it does not lie in $\overline{U_i}$ for any $1 \leq i \leq n$ and as it is not in X it cannot be in any of the subcontinua C_i . Thus C is a proper subcontinuum of γX which contains $p_1, \dots, p_n$. This is a contradiction, so no such x can exist. $\square$

It is worth noting that the converse is not true, and there exist metric, decomposable compactifications of spaces, with exactly n components in their remainder which are not n -irreducible. For example, take the two point compactification of a triod without its end points.

6.3 Finite Compactifications

We have shown that it is necessary for a space to be continuumwise connected, non-compact, locally compact and open ended in order to have an irreducible decomposable compactification whose points of irreducibility are precisely the elements of the remainder. We have also proved a number of results in the opposite direction, that if a space with these properties has an irreducible compactification then the remainder will be precisely the union of the E-continua of the compactification. We will now consider finite compactifications. First it will be shown that this question can be rephrased in terms of only finite compactifications, which are much simpler than general compactifications. A link between maximal finite compactifications and irreducible finite compactifications will be examined and we will find a new pair of properties that our underlying space must possess. This section will conclude with the proof of the chapter's major theorem, a list of necessary and sufficient conditions for a space to have in order that it has an irreducible metric compactification whose remainder is the set of irreducible points.

Proposition 6.3.1. *If an n -point compactification of a densely irreducible space X is irreducible, then it is maximal.*

Proof. If a finite compactification αX is not maximal then by the proof of Theorem 6.1.9 it is the continuous image of a larger finite compactification δX with this map the identity on X . As δX has more than n components in its remainder we can see by applying Proposition 6.2.9 that it is not n -irreducible, or indeed minimal about a set of n points. Let $q_1, \dots, q_n$ be points in the remainder of δX which are mapped to the points $p_1, \dots, p_n$ in the remainder of αX . There exists a proper subcontinuum $C \subsetneq \delta X$ containing $q_1, \dots, q_n$. Since $C \subsetneq \delta X$ it must be that $X \not\subseteq C$ so the image of C is a proper subcontinuum of αX containing all points of the remainder, which means αX is not n -irreducible. $\square$

Theorem 6.3.2. *If a densely irreducible space X has a metric irreducible compactification with n E-continua then it has a metric irreducible n -point compactification. Conversely if X has an irreducible metric n -point compactification then every metric compactification whose remainder has n components is n -irreducible.*

Proof. Let γX be a metric, n -irreducible compactification of X and let $\gamma X = \text{irr}(p_1, \dots, p_n)$ with each $\lambda(p_i)$ compact. Now take a monotone map $q : \gamma X \mapsto \alpha X$ which maps each E-continuum to a point and is a homeomorphism on X . Since γX is normal and there

are only finitely many non-degenerate fibres of q it follows that the quotient space αX is Hausdorff, so by Lemma 6.1.1 it is a continuum. X is dense in γX so it is dense in αX , making αX a compactification of X , and therefore decomposable by Proposition 6.2.5. Since q is monotone we have $\alpha X = \min(q(p_1), \dots, q(p_n))$ by Proposition 3.2.16. For any $1 \leq i \leq n$ there exists $C_i \subsetneq \gamma X$ containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ and not intersecting $\lambda(p_i)$. This C_i can not contain the whole of X , so $q(C_i)$ is a proper subcontinuum of αX . It is clear that $q(C_i)$ contains $q(p_1), \dots, q(p_{i-1}), q(p_{i+1}), \dots, q(p_n)$. Thus $\alpha X = \text{irr}(p_1, \dots, p_n)$.

For the second statement let γX again be a metric compactification with n components $K_1, \dots, K_n$. Define an equivalence relation whose only non-degenerate equivalence classes are the components $K_1, \dots, K_n$, and let ρ be the resulting quotient map. Call αX the image of ρ . As before it is clearly Hausdorff so it must be a continuum. The map $\rho|_X$ is a homeomorphism so αX is a compactification, as X is densely embedded in it. It is an n -point compactification with remainder $\{\rho(K_1), \dots, \rho(K_n)\}$. Since X has an irreducible, hence maximal by Proposition 6.3.1, n -point compactification and since, by Corollary 6.1.10, maximal finite compactifications are unique up to homeomorphism it must be that αX is homeomorphic to an irreducible n -point compactification, and is therefore irreducible itself.

Let $C \subseteq \gamma X$ be a subcontinuum containing a point p_i from each of the K_i 's. Then $\rho(C)$ is a subcontinuum of αX containing $\alpha X \setminus X$, so $\rho(C) = \alpha X$. Thus $\rho(X) \subseteq \rho(C)$ and consequently it must be that $X \subseteq C$. Since X is dense and C is closed we have $C = \gamma X$ and $\gamma X = \min(p_1, \dots, p_n)$. Now let $C_i \subseteq \alpha X$ be a proper subcontinuum containing $\rho(K_1), \dots, \rho(K_{i-1}), \rho(K_{i+1}), \dots, \rho(K_n)$. As it is a proper subcontinuum we have that $\rho(X) \not\subseteq C_i$ which implies $\rho^{-1}(C_i)$ is a proper subcontinuum of γX containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$. Thus $\gamma X = \text{irr}(p_1, \dots, p_n)$. $\square$

Lemma 6.3.3. *Let αZ be a maximal finite compactification of a locally compact space Z with remainder $\{p_1, \dots, p_n\}$. Let U be a connected open set which contains a point p_i and suppose for each $j \neq i$ we have $p_j \notin \bar{U}$. Then $U \setminus \{p_i\}$ is connected.*

Proof. Suppose otherwise, and let $U \setminus \{p_i\} = U_1 \cup U_2$ for disjoint open sets U_1, U_2 . Let V_j be disjoint open sets each containing the corresponding p_j and with only V_i intersecting $\bar{U}$ non-trivially. Let $V'_j = V_j \cap Z = V_j \setminus \{p_j\}$ for $j \neq i$ and let $U'_1 = U_1 \cap V_i, U'_2 = U_2 \cap V_i$. Since U is connected both U_1 and U_2 must contain p_i in their closure by Proposition 6.1.3, so $U'_1 \neq \emptyset \neq U'_2$. Define

$$K = Z \setminus (U'_1 \cup U'_2 \cup V'_1 \cup \dots \cup V'_{i-1} \cup V'_{i+1} \cup \dots \cup V'_n)$$

Since K is the complement in αZ of an open set containing $p_1, \dots, p_n$ we have that K is compact. If $K \cup V'_j$ is compact then $\alpha Z \setminus (K \cup V')$ is open, so $\{p_j\} = V_j \cap \alpha Z \setminus (K \cup V'_j)$ is also open, which is not the case.

Now suppose $K \cup U'_i$ is compact. Then it is a closed set of αZ , and so we have that $U \cap (K \cup U'_i) = U'_i$ is a closed subset of U . As it is also an open subset of U this contradicts that U is connected. From this we have that $\{U'_1, U'_2, V'_1, \dots, V'_{i-1}, V'_{i+1}, \dots, V'_n\}$ forms an $(n+1)$ -star of Z , which by Theorem 6.1.8 implies that Z has an $(n+1)$ point compactification. This contradicts the maximality of αZ so $U \setminus \{p_i\}$ must be connected. $\square$

Remark 6.3.4. It is clear that can also be applied to a compactification γZ whose remainder has n components. There is a natural quotient map from γZ to αZ and this map clearly allows us to apply Lemma 6.3.3 to γZ .

The pair of propositions following the next provide us with two new necessary conditions for our underlying space X .

Lemma 6.3.5. *Let X be a space and let $U_1, \dots, U_n$ be a maximal n -star of X with $K = X \setminus \bigcup U_i$. Let $\alpha X = X \cup \{p_1, \dots, p_n\}$ be a finite compactification with $p_i \in \overline{U_i}^{\alpha X}$. Then $U_i \cup \{p_i\}$ is an open subset of αX for each i .*

Proof. We know that U_i is open in αX as it is an open subset of X , and X being locally compact means that it is an open subset of each of its compactifications by Theorem 6.1.6. Thus if $U_i \cup \{p_i\}$ is not an open subset then it must be that $p_i \in \overline{\alpha X \setminus (U_i \cup \{p_i\})}$. It follows that $p_i \in \overline{\bigcup_{j \neq i} U_j} = \bigcup_{j \neq i} \overline{U_j}$ since X is dense and K is compact. Let V be an αX -open set containing p_i with $K \cap \overline{V} = \emptyset$ and for all $j \neq i, p_j \notin \overline{V}$. Such a V exists as αX is regular and $K \cup \{p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n\}$ is a closed subset.

We will show that $U_1 \setminus \overline{V}, \dots, U_{i-1} \setminus \overline{V}, U_{i+1} \setminus \overline{V}, \dots, U_n \setminus \overline{V}, V \cap U_i, V \setminus (U_i \cup \{p_i\})$ is an $(n+1)$ -star of X . It is clear that these sets are all disjoint and all lie in X . The first n of them are clearly open and $V \setminus (U_i \cup \{p_i\}) = (V \cap \bigcup_{j \neq i} \overline{U_j}) \setminus \{p_i\} = V \cap \bigcup_{j \neq i} U_j$, which is also open.

The union of these sets is $\bigcup_{j \neq i} (U_j \setminus \overline{V}) \cup V \setminus \{p_i\}$. This set contains $\bigcup_{j \neq i} U_j$ so its X -complement L lies in $K \cup U_i$. Thus $\overline{L}^{\alpha X} \subseteq K \cup U_i \cup \{p_i\}$. But as $p_i \in V$ and $V \cap L = \emptyset$ we have that $\overline{L}^{\alpha X} \subseteq K \cup U_i \subseteq X$. Since L is a closed subset of X it follows that $L = \overline{L}^{\alpha X}$ and as such L is compact.

This means that we have indeed constructed an $(n+1)$ star, which contradicts the maximality of $U_1, \dots, U_n$. This contradiction in turn implies that $U_i \cup \{p_i\}$ must be an open subset of αX . $\square$

Proposition 6.3.6. *Let Y be a decomposable, n -irreducible continuum and suppose $Y = \text{irr}(p_1, \dots, p_n)$ with each $\lambda(p_i)$ compact. Let Y' be the set of non-irreducible points of Y . Let $U_1, \dots, U_n$ be open subsets of Y' which form an n -star. Then there exist connected open subsets $V_i \subseteq U_i$ which also form an n -star of Y' .*

Proof. Let $Y = \text{irr}(p_1, \dots, p_n)$ and let $\lambda(p_i) \subseteq \overline{U_i}$. By applying the same argument as in Lemma 6.3.5 we have that $U_i \cup \lambda(p_i)$ is an open subsets of Y . From Theorem 3.4.7 we know that Y is locally connected at its E-continua. Thus there are connected open sets V_i with $\lambda(p_i) \subseteq V_i \subseteq U_i$. Let $W_i = V_i \cap Y' = V_i \setminus \lambda(p_i)$. By Lemma 6.3.3 we know that the W_i are connected, and it is clear from their construction that they form an n -star. $\square$

Proposition 6.3.7. *Let Y be a decomposable, n -irreducible continuum and suppose $Y = \text{irr}(p_1, \dots, p_n)$ with each $\lambda(p_i)$ compact. Let Y' be the set of non-irreducible points of Y . Let C be a closed, connected, non-compact subset of Y' . Then $Y' \setminus C$ has at most $n - 1$ components.*

Proof. If $C = Y'$ then this proposition is clearly true. Otherwise there is some $p_i \in \overline{C}^Y$ meaning $\overline{C}$ is n -terminal by Corollary 3.3.11. This implies that $Y \setminus \overline{C}$ has at most $n - 1$ components by Proposition 3.4.1, one for each p_j not in $\overline{C}$. By applying Lemma 6.3.3 we see that $Y' \setminus C$ also has at most $n - 1$ components. $\square$

We are now in a position to prove the major result of this chapter.

Theorem 6.3.8. *Let X be a topological space satisfying the following criteria*

1. X is locally compact
2. X is continuumwise connected
3. X is not compact
4. X is open ended
5. X has a maximal n -star for $n \geq 2$
6. For any closed, connected, non-compact subset $C \subseteq X$, $X \setminus C$ has at most $n - 1$ components.
7. For any open sets $U_1, \dots, U_n \subseteq X$ which form an n -star, there exist connected open subsets $V_1, \dots, V_n$ with $V_i \subseteq U_i$ which also form an n -star.

Then any metric compactification of X whose remainder has precisely n components will be an n -irreducible, decomposable continuum. Further, the remainder of this compactification will consist of precisely its points of irreducibility.

Conversely, if X is a space with an n -irreducible metric compactification then X must have all of the properties listed.

Proof. To prove the final statement, see Theorem 6.2.4, Proposition 6.3.1, Theorem 6.3.2 and Propositions 6.3.6 and 6.3.7.

We only need to show that an n -point compactification αX of X is irreducible about the points $p_1, \dots, p_n$ of the remainder. This requires showing that no proper subcontinuum of αX contains $p_1, \dots, p_n$ and that a proper subcontinuum contains $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ for each i . Then we can apply Theorem 6.3.2.

Let $C \subseteq \alpha X$ be a proper subcontinuum containing $p_1, \dots, p_n$ and let $x \in \alpha X \setminus C$. Let U_i be disjoint open subsets of αX each containing their respective p_i with $x \notin \overline{U_i}$ for each i . Then $U_1 \setminus \{p_1\}, \dots, U_n \setminus \{p_n\}$ is an n -star of X , so there exists connected $V_i \subseteq U_i \setminus \{p_i\}$ which form an n -star by Condition 7. By applying Theorem 6.1.2 we have that each component of $C \cap X$ must have some p_i in its closure, which in turn implies that each component must intersect some V_i . Let $C' = (C \cap X) \cup \overline{V_1}^X \cup \dots \cup \overline{V_n}^X$. Each $\overline{V_i}^X$ is connected and each component of $C \cap X$ intersects one of them, so C' must be connected. If it were not, say $C' = A \cup B$ then we have that $C \subseteq \overline{A} \cup \overline{B}$, with these sets disjoint and both intersecting C non-trivially.

By Condition 6 we have that $X \setminus C'$ has at most $n - 1$ components $K_1, \dots, K_m$. Now $\overline{K_j}^{\alpha X}$ cannot contain any of the p_i 's. This is because $p_i \in V_i \cup \{p_i\} \subseteq \alpha X \setminus K_j$ and by Lemma 6.3.5 this is an open set. Thus $\overline{K_j}^{\alpha X}$ lies in X . Since there are finitely many of them and by Condition 2 X is continuumwise connected, there exists a subcontinuum $K \subseteq X$ with $K_j \subseteq K$ for each j . Now $X = K \cup (C' \cap X)$ and this is the union of two connected closed sets, one of which is compact. This contradicts Condition 4, so there cannot be a proper subcontinuum of αX containing $p_1, \dots, p_n$. Hence, $\alpha X = \min(p_1, \dots, p_n)$.

Now we will show that for each p_i there exists a proper subcontinuum of αX containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$. As before let $U_1, \dots, U_n$ be disjoint open sets containing their respective p_i 's and let $V_i \subseteq U_i \setminus \{p_i\}$ be connected open sets forming an n -star. Then $V'_i = V_i \cup \{p_i\}$ is a connected open set for each i . By Condition 2 there exist subcontinua $C_{j,k} \subseteq X$ intersecting V'_j and V'_k non-trivially. Let $C = \bigcup_{j,k \neq i} C_{j,k} \cup \bigcup_{j \neq i} \overline{V'_j}$. Then C is a continuum containing $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$. It does not contain p_i as each U_j is disjoint from U_i and each $C_{j,k}$ lies in X . Thus $\alpha X = \text{irr}(p_1, \dots, p_n)$. $\square$

6.4 Redundancies

We have shown that if X is a metric space then X has a compactification which is an irreducible continuum whose remainder consists of each of its E-continua if and only if

1. X is locally compact
2. X is continuumwise connected
3. X is not compact
4. X is open ended
5. X has a maximal n -star and the corresponding n -point compactification is metric
6. If $C \subseteq X$ is closed, connected and non-compact then $X \setminus C$ has at most $n - 1$ components.
7. For any maximal star $\{U_1, \dots, U_n\}$ in X there exists a star $\{V_1, \dots, V_n\}$ such that each V_i is connected and $V_i \subseteq U_i$.

These properties have been expressed such that each is a property of a space that does not require that space to have any of the others in order to make sense. The list is long, and several of the properties seem to be closely related. Thus we will try and reduce the list by showing that some properties can be deduced from a combination of the others. We will also look at examples of spaces which do not have the right kind of compactifications and see which properties they lack, to get an idea of what must necessarily be on our list. For example if a space had six of the seven and did not have the right kind of compactification then the seventh property is independent of the first six.

A space can only have a finite compactification if it is locally compact, which means it is impossible to discuss Property 5 fully if Property 1 does not hold. As such we will first show that Property 1 is independent of the other six, as far as these six make sense, so that for the rest of this section we can focus solely on locally compact spaces.

Example 6.4.1. Recall the indecomposable continuum A^ω defined in Chapter 5 with composant A^∞ . Let X be the space attained by attaching a half open interval $[0, 1)$ to A^∞ by identifying the point $0 \in [0, 1)$ with the point $x = (0, 0, \dots) \in A^\infty$. By applying Theorem 6.1.6 we can see that X is not locally connected as $Y := [0, 1] \cup A^\omega$

is a compactification of it in which X is not an open set. Both $[0, 1)$ and A^∞ are continuumwise connected so X is also continuumwise connected. It is clear that X is not compact as for example $[0, 1)$ is a closed subset which is not compact.

Let $X = A \cup B$ for A and B closed connected subsets and suppose A is compact. Then $Y = A \cup \overline{B}^Y$ so every component of A^ω other than A^∞ must lie in $\overline{B}^Y$ and as such, the whole of A^ω must lie in $\overline{B}^Y$. Since A is compact we also have that $1 \in \overline{B}^Y$ and because B is connected it must be that $[0, 1] \subseteq \overline{B}^Y$. Thus we have shown that $Y = \overline{B}^Y$, which implies that $X = \overline{B}^Y \cap X = \overline{B}^X = B$. Thus X is open ended.

The space X has a two star consisting of $(0.8, 1)$ and $A^\infty \cup [0, 0.2)$. To show that this is maximal we must show that A^∞ does not have a two star as if X had a three star it would induce a two star in A^∞ . For a contradiction let U, V form a two star of A^∞ with $K = A^\infty \setminus (U \cup V)$ compact. Let $\pi_n : A^\omega \mapsto \mathbb{I}$ be the projection onto the n^{th} coordinate. We will first see that if $\pi_n(K \cap A_n) \subseteq [0, \alpha_n]$ then α_n must tend to 0 as n tends to infinity. If it did not then there would exist $\epsilon > 0$ such that for arbitrarily large n , there exists $x_n \in A_n \cap K$ such that the distance between x and A_{n-1} is greater than ϵ , implying that x_n does not lie in the ϵ -ball about A_{n-1} , $B(\epsilon, A_{n-1})$. Since K is compact the sequence x_n converges to some point $x \in K$. Now $x \in A_m$ for some m , which means $x \in B(\epsilon, A_m)$ which for all $n > m$ is a subset of $B(\epsilon, A_n)$. No tail of the sequence $\{x_k\}_{k \in \omega}$ lies in this open set, which contradicts the fact that the sequence converges to x .

Now suppose both U and V are non-empty, with $x \in U$ and $y \in V$. Take basic open sets $x \in U_1 \times \cdots \times U_n \times \mathbb{I}^{\omega \setminus n} \subseteq U$ and $y \in V_1 \times \cdots \times V_m \times \mathbb{I}^{\omega \setminus m} \subseteq V$. Without loss of generality we can assume $n = m$ and $s(U_i) = U_{i-1}$, and the same for V_i 's. As U_n and V_n are both open subsets of $\mathbb{I}$ we have that there exist x' and y' non-zero lying in U_n and V_n respectively. Take $k \in \omega$ such that $\frac{1}{k} < x', y'$ and an arbitrarily large N such that $\pi_N(A_N \cap K)$ does not contain $\frac{1}{k}$. Then the component of 1 in $A_N \setminus K$ contains $[\frac{1}{k}, 1]$ in its projection and lies in $U \cup V$. As it is connected we can say without loss of generality that it lies in U . The set of N^{th} coordinates of this component includes $[\frac{1}{k}, 1]$ so the set of $(N-1)^{\text{th}}$ coordinates do as well. Continuing inductively we have that the set of n^{th} coordinates of U also contains $[\frac{1}{k}, 1]$, so there exists a point $u \in U$ with n^{th} coordinate $y' \in V_n$. The 1st to $(n-1)^{\text{th}}$ coordinates of u must lie in V_1 to V_{n-1} respectively as $s(V_i) = V_{i-1}$, so $u \in V$. Since U and V are disjoint this is a contradiction. Thus no such 2-star can exist, so no 3-star of X can exist and X has a maximal 2-star.

Let $C \subseteq X$ be closed, connected and not compact. We need to show that $X \setminus C$ is connected. Suppose $X \setminus C = U \cup V$ for disjoint open U and V . It clearly cannot

be the case if U and V are non-empty that $A^\infty \subseteq C$ or that $A^\infty \cap C = \emptyset$. Suppose $C \cap A^\infty$ were not compact. Then $\overline{C \cap A^\infty}^Y \not\subseteq A^\infty$ and as $C \cap A^\infty$ must be connected, this continuum must be the whole of A^ω . We have already established that $A^\infty \not\subseteq C$ so this is a contradiction. Now it must be that $[0, 1) \subseteq C$ in order for C to not be compact. This means that $U \cup V \subseteq A^\infty$. $Y = \overline{U} \cup \overline{V} \cup \overline{C}$, and as $C \cap A^\infty$ is compact we have that $A^\omega \setminus A^\infty \subseteq \overline{U} \cup \overline{V}$. This set is dense in A^ω so $A^\omega \subseteq \overline{U} \cup \overline{V}$. Now $C \cup U$ and $C \cup V$ are both connected by Proposition 6.1.3, and it is clear that $C \cup U \setminus (0, 1)$ and $C \cup V \setminus (0, 1)$ will also be connected. The closures of these two sets in A^ω form a proper decomposition of A^ω , which is indecomposable (Theorem 5.3.5). This contradiction gives us that $X \setminus C$ must be connected.

Let U_1 and U_2 form a 2-star in X , with $K = X \setminus (U_1 \cup U_2)$. We will construct connected open subsets V_1 and V_2 which form a 2-star. One of $U_1 \cap A^\infty$ and $U_2 \cap A^\infty$ must have compact closure in X as seen above, so without loss of generality suppose this is $U_1 \cap A^\infty$. Then $1 \in \overline{U_1}^Y$ so there exists some $\epsilon > 0$ with $(1 - \epsilon, 1) \subseteq U_1$. Set $V_1 = (1 - \epsilon, 1)$. Let $V_2 = U_2 \setminus [0, 1)$ and express $V_2 = W \cup W'$ for disjoint open sets W and W' . One of W or W' must have compact closure or they would form a 2-star in A^∞ . As $A^\omega = \overline{A^\infty \cap (K \cup U_2)}^Y = \overline{(A^\infty \cap K) \cup V_2}^Y$ and K is compact we have that $A^\omega = \overline{V_2}^Y \cup (K \cap A^\infty)$. It follows from this that if, say, W does not have compact closure in X then $\overline{W}^Y = A^\omega$. This implies $W' \subseteq \overline{W}^X$ which is a contradiction. Thus V_2 must be connected.

Thus we have shown that X satisfies every property except for the first, so Property 1 is independent of the other six.

The next example will show that continuumwise connectedness is independent of the other properties.

Example 6.4.2. Let $X = (0, 1) \cup (2, 3)$ as subsets of the real line and let $\alpha X = [0, 1] \cup [2, 3]$ be the closure. It is clear that X satisfies every condition except for continuumwise connectedness. Indeed X is not even connected.

Remark 6.4.3. If X is compact then there is no point discussing its compactifications and many of the other conditions cannot be reasonably discussed. As such we will not consider whether the compactness of X is dependent on the other properties.

We will now show that being open ended is a consequence of the other conditions. First a lemma.

Lemma 6.4.4. *Let X be a locally compact space with a metric, maximal finite compactification $\alpha X = X \cup \{p_1, \dots, p_n\}$ and suppose αX is locally connected at each point p_i . Suppose X satisfies Condition 6. Let $U_1, \dots, U_n$ be an n -star. Then there exists an n star $V_1, \dots, V_n$ with each V_i connected, $V_i \subseteq U_i$ and $X \setminus \overline{V_i}$ connected.*

Proof. Let αX be a finite compactification with p_i in the closure of U_i . We will show that such a V_1 can be constructed for U_1 , and by symmetry the rest of the proof follows immediately. As αX is metric it is first countable, so there exists a local basis of p_1 , denoted $\{W_1, W_2, \dots\}$, of connected open sets with $W_{k+1} \subseteq W_k$ for all k . We can insist that $W_k \subseteq U_1 \cup \{p_1\}$ for all k so to prove the lemma we need to show that one of the W_k satisfies the conditions required for our V_1 . If $\alpha X \setminus \overline{W_k}$ is disconnected then so is $X \setminus \overline{W_k} \cap X^X$, so we shall assume for contradiction that for each k , $\alpha X \setminus \overline{W_k}$ is disconnected.

We have from Lemma 6.3.3 that $W_k \setminus \{p_1\} = W_k \cap X$ is connected. Thus $\overline{W_k} \cap X^X$ is a closed, connected, non-compact set so its complement has at most $n - 1$ components. It follows that $\alpha X \setminus \overline{W_k}$ also has at most $n - 1$ components as the points p_i cannot be components. Thus we can express $\alpha X \setminus \overline{W_k} = C_1^k \cup \dots \cup C_{m_k}^k$ where each C_i^k is a component.

Define $D_i^k = C_i^{k+1} \cap \overline{W_k}$. These will each be non-empty as if one were then the corresponding C_i^{k+1} would be a component of αX which is impossible. Since $\overline{W_k} \setminus \overline{W_{k+1}} \subseteq \alpha X \setminus \overline{W_{k+1}}$ we have that $\overline{W_k} \setminus \overline{W_{k+1}} = D_1^k \cup \dots \cup D_{m_{k+1}}^k$.

We will now show that there can only be finitely many C_i^k which do not contain any C_j^{k-1} . Suppose there were infinitely many, each with a different value of k . Call these $E_1, E_2, \dots$ with corresponding $k_1, k_2, \dots$. By removing the intervening sets from our local basis we can assume that $k_{i+1} = k_i + 1$. Then

$$E := (\alpha X \setminus \overline{W_{k_1}}) \cup E_1 \cup \dots \cup E_n = C_1^{k_1} \cup \dots \cup C_{m_{k_1}}^{k_1} \cup E_1 \cup \dots \cup E_n$$

The right hand side is a disjoint union. When E is considered as a subset of $\alpha X \setminus \overline{W_{k_n}}$ it is clear that E_n is a clopen subset of E . Next considering $E \setminus E_n = E \cap \alpha X \setminus \overline{W_{k_{n-1}}}$ it is clear that E_{n-1} is a clopen subset of this clopen subset of E . Thus E_{n-1} is also a clopen subset of E . Repeat this to see that each E_i is a clopen subset of E . It then follows that $C_1^{k_1} \cup \dots \cup C_{m_{k_1}}^{k_1}$ is a clopen subset of E and the $C_i^{k_1}$ are the components

of it. Thus we have that E has more than n components. Now note that

$$\begin{aligned}
E &= (\alpha X \setminus \overline{W_{k_1}}) \cup E_1 \cup \cdots \cup E_n \\
&= \alpha X \setminus (\overline{W_{k_1}} \setminus (E_1 \cup \cdots \cup E_n)) \\
&= \alpha X \setminus ((\overline{W_{k_1+1}} \cup \bigcup_{D_i^{k_1} \neq E_1} D_i^{k_1}) \cup E_2 \cdots \cup E_n) \\
&= \alpha X \setminus ((\overline{W_{k_2}} \cup \bigcup_{D_i^{k_1} \neq E_1} D_i^{k_1}) \cup E_2 \cdots \cup E_n)
\end{aligned}$$

Now we have that $\overline{W_{k_2}} \cup \bigcup_{D_i^{k_1} \neq E_1} D_i^{k_1}$ is a continuum by Proposition 6.1.3. We can repeat this for each subsequent k_i to see that $E = \alpha X \setminus K$ where K is a continuum containing p_1 . This contradicts the fact that E has more than n components.

Thus we have concluded that there are only finitely many sets C_i^k which do not contain an earlier C_j^l , and in fact there cannot be n such sets. Therefore we can discard from our local basis any W_k with such a C_i^k and still be left with a local basis. This means that if $k < l$ then $m_k \geq m_l$. Since this means m_k is a descending sequence of natural numbers there exists some K such that for all $k > K$ we have $m_k = m_K$. By discarding every W_l with $l < K$ we now have that for every set W_k in our basis, the set $\alpha X \setminus \overline{W_k}$ has the same number of components, say m . A minor relabelling means we can express these as $C_1^k, \dots, C_m^k$ and can say that if $k < l$ then $C_i^k \subseteq C_i^l$. Since αX is metric we have that $\{p_1\} = \bigcap_{k=1}^{\infty} W_k = \bigcap_{k=1}^{\infty} \overline{W_k}$. We now have that

$$\begin{aligned}
\alpha X \setminus \{p_1\} &= \alpha X \setminus \bigcap_{k=1}^{\infty} \overline{W_k} \\
&= \bigcup_{k=1}^{\infty} \alpha X \setminus \overline{W_k} \\
&= \bigcup_{k=1}^{\infty} C_1^k \cup \cdots \cup C_m^k \\
&= \bigcup_{i=1}^m \bigcup_{k=1}^{\infty} C_i^k
\end{aligned}$$

From the definition it is clear that each C_i^k is an open set, being an open subset of $\alpha X \setminus \overline{W_k}$. This implies each $\bigcup_{k=1}^{\infty} C_i^k$ is also open, so we have shown that $\alpha X \setminus \{p_1\}$ is disconnected. On the other hand $X \subseteq \alpha X \setminus \{p_1\} \subseteq \overline{X}$ so it cannot be disconnected without contradicting Corollary 6.1.7. We have arrived at our final contradiction, so we can conclude that the lemma is true and that there exists a connected open set $V_1 \subseteq U_1$ with $X \setminus V_1$ connected. $\square$

Lemma 6.4.5. *Let C be a closed, non-compact subset of a locally compact space Z . Then C is locally compact.*

Proof. Take any compactification γZ of Z , say the Stone-Ćech compactification. In order to show that C is locally compact consider $\overline{C}^{\gamma Z}$. This is a compactification of C . In this compactification $C = Z \cap \overline{C}^{\gamma Z}$, and as Z is an open subset of γZ we have that C is an open subset of $\overline{C}^{\gamma Z}$. This gives us that C is locally compact by Theorem 6.1.6. $\square$

We are now in a position to prove that the fourth condition, being open ended, is redundant.

Proposition 6.4.6. *Let X be a space with each condition except for being open ended. Then X is open ended.*

Proof. Suppose $X = A \cup B$ where both A and B are closed and connected, and where A is compact. Take an n -star $U_1, \dots, U_n$ such that $\overline{U_i} \cap A = \emptyset$. Condition 7 implies that αX is locally connected about each p_i so we can apply Lemma 6.4.4. We can therefore assume each U_i is connected and $X \setminus \overline{U_i}$ is also connected. We will show that we can construct A' and B' closed and connected, A' compact, such that $X = A' \cup B'$ and $B' \setminus \bigcup_{i=1}^{n-1} \overline{U_i}$ is connected.

Let B_1 be the closure of a component of $B \setminus \overline{U_1}$ not lying in $\bigcup_{i \neq 1} \overline{U_i}$. This intersects $\overline{U_1}$. For every other component K let $K' = K \setminus \bigcup_{i \neq 1} U_i$. As $X \setminus \overline{U_1}$ is connected it must be that $A \cap K'$ is non-empty. Let A_1 be the closure of the union of A with every such K' . Note that this is compact as it is a subset of $X \setminus \bigcup U_i$.

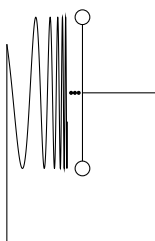
Now suppose for $i = 1 \dots k$ we have defined A_i to be a continuum and B_i to be a continuum intersecting $\overline{U_i}$ and B_{i-1} . Let B_{k+1} be the closure of a component of $B \setminus \overline{U_{k+1}}$ which contains a point of $B_k \setminus \bigcup_{i \neq k} \overline{U_i}$. This intersects $\overline{U_{k+1}}$. Define A_{k+1} in the same way as before.

We can now define $A' = A_1 \cup \dots \cup A_n$ and $B' = (B_1 \cup \overline{U_1}) \cup \dots \cup (B_n \cup \overline{U_n})$. These are both closed sets and A' is compact. They are the union of intersecting connected sets so are both connected. $X = A' \cup B'$ by construction. $B' \setminus \bigcup_{i=1}^{n-1} \overline{U_i} = B_1 \cup \dots \cup B_n \cup \overline{U_n}$ which is connected as the union of intersecting connected sets. Thus we have our construction.

Now consider $X \setminus (B' \setminus \bigcup_{i=1}^{n-1} \overline{U_i})$. This is the complement of a connected, closed, non-compact set so can have at most $n - 1$ components. However, it is equal to the disjoint union $(A \setminus B') \cup (\overline{U_1} \setminus B') \cup \dots \cup (\overline{U_{n-1}} \setminus B')$. Each of these is a non-empty set and they are all closed as subsets of $X \setminus (B' \setminus \bigcup_{i=1}^{n-1} \overline{U_i})$. This contradicts Condition 6, which gives us our conclusion that X must be open ended. $\square$

We have now seen that local compactness and continuumwise connectedness are both independent of the other properties, that compactness cannot be meaningfully discussed and that open endedness is redundant. The following two examples show that Properties 6 and 7 are independent of the rest. Neither space is open ended but as we have shown this to be redundant this is not a problem.

Example 6.4.7. This space is the Warsaw Circle with two points removed.



The space is clearly not compact and is locally compact with a maximal 2-point compactification. It is continuumwise connected and in fact arcwise connected as any two points can be joined by following around the circle and avoiding the $\sin(\frac{1}{x})$ part of the space. The complement of any connected closed non-compact subspace is connected. The intersection of a subcontinuum of the two point compactification with X is also connected. The only condition which fails is Condition 7 as any open sets U_1 and U_2 forming a 2-star will contain infinitely many components from the $\sin(\frac{1}{x})$ curve.

The next example shows that Condition 6 is independent of the others.

Example 6.4.8. The space X is shown on the left and on the right, highlighted in bold, is a closed, connected, non-compact subset with disconnected complement.



It is clear that this space satisfies every condition except for the 6th.

We can now conclude that a space X has an irreducible compactification with the remainder as the set of irreducible points if and only if X meets each of the following criteria, and that each condition is independent of the others.

1. X is locally compact

2. X is continuumwise connected
3. X is not compact
4. X has a maximal n -star and the corresponding n -point compactification is metric
5. If $C \subseteq X$ is closed, connected and non-compact then $X \setminus C$ has at most $n - 1$ components.
6. For any maximal star $\{U_1, \dots, U_n\}$ in X there exists a star $\{V_1, \dots, V_n\}$ such that each V_i is connected and $V_i \subseteq U_i$.

Chapter 7

Infinite Irreducibility

This will be similar to the start of Chapter 3, generalising classical results about irreducibility to the infinite case.

7.1 Referenced Theorems

The following results from other authors will be used in this chapter.

Sam B. Nadler Jr, Continuum Theory, an Introduction

These results can be found in [Nad92].

Lemma 7.1.1 (3.2). *If a Hausdorff space is a continuous image of a compact metric space, then it is metrizable.*

Theorem 7.1.2 (5.4). *Let X be a continuum and let U be a nonempty, proper, open subset of X . If K is a component of $\overline{U}$, then $K \cap \partial U \neq \emptyset$. Equivalently, $K \cap (X \setminus U) \neq \emptyset$.*

Proposition 7.1.3 (6.3). *Let T be a connected topological space and let C be a connected subset of T such that $T \setminus C$ is disconnected and can be expressed as $A \cup B$. Then $A \cup C$ and $B \cup C$ are connected. Hence, if T and C are continua, $A \cup C$ and $B \cup C$ are continua.*

Proposition 7.1.4 (11.14). *If X is any nondegenerate continuum, then the component $\kappa(p)$ of any point $p \in X$ is a union of countably many proper subcontinua of X , each of which contain p .*

Theorem 7.1.5 (11.17). *If X is a nondegenerate indecomposable continuum, then the composants of X are mutually disjoint.*

D.E. Bennett and J.B. Fugate, Continua and their non-separating Subcontinua

These results can be found in [BF77].

Theorem 7.1.6 (1.3). *Each terminal subcontinuum of a continuum X is non-separating in X .*

Theorem 7.1.7 (1.22). *Suppose that X is a continuum and $p \in X$. The following are equivalent:*

- $X \setminus \kappa(p)$ is closed;
- $X \setminus \kappa(p)$ is a continuum;
- $X \setminus \kappa(p)$ is continuum-wise connected;
- If K is a subcontinuum of X such that $K \cap (X \setminus \kappa(p)) \neq \emptyset$ and $K \cap \kappa(p) \neq \emptyset$, then K is decomposable.

Theorem 7.1.8 (1.30). *The E -continua of X are exactly those end continua at which X is locally connected.*

Ryszard Engelking, General Topology

These results can be found in [Eng89].

Lemma 7.1.9 (Kuratowski-Zorn Lemma, page 8). *In a partially ordered set in which every chain has an upper bound, every element has a maximal element above it.*

Theorem 7.1.10 (2.1.7). *A space X is hereditarily normal if and only if for every pair of separated subsets A, B of X there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.*

Theorem 7.1.11 (3.1.1). *A Hausdorff space is compact if and only if every family of closed subsets of X which has the finite intersection property has non-empty intersection.*

Proposition 7.1.12 (4.3.6). *A topological space is metrizable by a totally bounded metric if and only if it is a regular second countable space.*

Theorem 7.1.13 (4.3.27). *Every metric on a compact space is totally bounded.*

Theorem 7.1.14 (6.1.19). *The intersection of a decreasing sequence of continua is a continuum.*

Theorem 7.1.15 (6.1.23). *In a compact Hausdorff space X the component of a point $x \in X$ coincides with the quasicomponent of the point x .*

7.2 Defining infinite irreducibility

The following is a further generalisation of the idea of irreducibility to the infinite case. The definitions will be similar to the finite case but there will be several differences in the results. Before giving the final definition we will first explore some of the issues which can arise from generalising from the classical case to the infinite case. We will denote this more basic definition as $X = \text{IRR}(A)$, so $X = \text{IRR}(A)$ if no proper subcontinuum of X contains A . The same property will be denoted $X = \text{min}(A)$ once a more appropriate definition of irreducibility is introduced.

When working with finite irreducibility we had that whenever a subcontinuum $X = \text{min}(p_1, \dots, p_n)$ then there exists some subset $\{p'_1, \dots, p'_m\} \subseteq \{p_1, \dots, p_n\}$ with $X = \text{irr}(p'_1, \dots, p'_m)$ (Proposition 3.2.3). The next few examples will be of continua with infinite subsets about which no proper subcontinuum exists. Each of them will demonstrate that this is not a meaningful definition of infinite irreducibility, either because the subset shows us nothing about the structure of X or because it cannot be reduced to a set with the second property, that for each point $a \in A$ there is a proper subcontinuum containing $A \setminus \{a\}$.

Example 7.2.1. Every continuum is separable so given a continuum X let D be a countable dense subset. An intuitive definition of irreducibility, that a continuum is irreducible about a set if no proper subcontinuum contains it, would give us that $X = \text{IRR}(D)$. All subcontinua are closed so any subcontinuum containing D must contain $\overline{D} = X$.

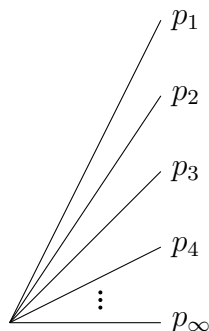
The preceding example shows that the simple definition is not good enough to tell us anything about the structure of our continuum. Connectedness is completely irrelevant as the existence of D is guaranteed by X being compact and metric, and there are no proper closed sets let alone subcontinua containing D . There can also clearly be no hope for minimality or unique cardinality.

The next example, using the same definition, demonstrates another problem.

Example 7.2.2. Let $X = [0, 1]$ and let $A = \{2^{-n} | n \in \omega\}$. Then A can be thought of as 1 and a sequence converging to 0. Any subcontinuum containing A must contain $\overline{A}$ so contains 0 and 1. Since $X = \text{irr}(0, 1)$ we have that $X = \text{IRR}(A)$.

With the exception of 1, every point in A would be called redundant. There are no proper subcontinua containing all but one of these points. Yet removing all these redundant points gives a set A' with $X \neq \text{IRR}(A')$. It is clear that the points themselves are unimportant and all that matters is that they converge to 0, which is the real point of irreducibility. It would seem sensible to insist on A being a closed set, but the next example will show that this can also lead to problems.

Example 7.2.3. Let X be the harmonic fan below.



If $A = \{p_1, p_2, \dots\}$ and $A' = A \cup \{p_\infty\}$ then we have that $\overline{A} = A'$. Now $X = \text{IRR}(A')$ but also $X = \text{IRR}(A)$. This means that the point p_∞ is redundant because again it is the limit of the sequence $\langle p_n \rangle$. However unlike Example 1.2 none of the points p_i are redundant.

Example 1.3 has shown that simply requiring the set A be closed is not enough to guarantee that the set can be minimal. The following proposition solidifies this.

Proposition 7.2.4. *Let X be a continuum. If $X = \text{IRR}(A)$ for an infinite closed set A then A is not minimal, meaning there exists some proper subset $B \subsetneq A$ such that $X = \text{IRR}(B)$.*

Proof. Since A is a closed subset of X it must be compact. Infinite compact sets cannot be discrete so there must be a non-isolated point $a \in A$ such that $A = \overline{A \setminus \{a\}}$. Now if $Y \subseteq X$ is a subcontinuum with $A \setminus \{a\} \subseteq Y$ then since Y is closed $a \in Y$. It follows from the irreducibility of X that $Y = X$. Thus we have that $X = \text{IRR}(A \setminus \{a\})$ and that A was not minimal. $\square$

We therefore have that closed sets will not be the way to go. The following final example will show that even a proper closed set may not have a minimal (not closed) subset.

Example 7.2.5. Let X be the union of a simple closed curve and an arc whose only point of intersection is one of the end points of the arc, and call this point x . Let $A \subset X$ be the simple closed curve and the other end point of the arc, which is clearly closed and not equal to X . Any subcontinuum of X containing A must be equal to X . Take a subset B of A such that no proper subcontinuum of X contains B . Then $\overline{B}$ must contain the simple closed curve as if it did not then consider a continuum Y consisting of $\overline{B}$, the arc, and a segment of the simple closed curve joining x to $\overline{B}$. Then the segment can be chosen such that $Y \subsetneq X$ and $B \subset Y$. Thus we have that B contains a dense subset of the simple closed curve. Since X is T_1 it is clear that no dense set is a minimal dense set. This means that B is not a minimal irreducible set by our intuitive definition of irreducibility.

At this point it seems hopeless to try and turn any intuitively irreducible set into one without redundancies, so it seems the only option is to insist on the lack of redundancies from the beginning, in the definition of irreducibility.

Definition 7.2.6. Let A be an infinite subset of a continuum X . We say that X is ∞ -irreducible about A if and only if no proper subcontinuum of X contains A , but for all $a \in A$ there exists a proper subcontinuum $C_a \subsetneq X$ such that $A \setminus \{a\} \subseteq C_a$. This is written $X = \text{irr}(A)$. We say that X is ∞ -irreducible if there exists some infinite $A \subseteq X$ such that $X = \text{irr}(A)$.

If there exists $A \subseteq X$ such that no proper subcontinuum of X contains all of A we write $X = \text{min}(A)$ and say that X is minimal about A . There may or may not be proper subcontinua of X containing each set $A \setminus \{a\}$.

This definition immediately yields the following proposition which gives us that a continuum cannot be ∞ -irreducible about subsets of different sizes.

Proposition 7.2.7. *Let X be a continuum with $X = \text{irr}(A)$. Then A is countable and discrete.*

Proof. Let $a \in A$ and let $C \subsetneq X$ be a proper subcontinuum containing $A \setminus \{a\}$. Then $a \in X \setminus C$ which is an open set. This means that $A \cap (X \setminus C) = \{a\}$ is an open subset of the subspace A . Since this is true of every point $a \in A$ it follows that A is a discrete subspace. X is a compact metric space so it is second countable by Proposition 7.1.12

and Theorem 7.1.13, which means every subspace of X is also second countable. The only second countable discrete spaces are the countable ones, so A is countable. $\square$

We will now define composants of X for the irreducible case in the same manner as before.

Definition 7.2.8. Let X be a continuum and let $A' \subseteq X$ be an infinite subset. Then define

$$\kappa(A') = \{x \in X \mid \exists C \text{ a proper subcontinuum of } X \text{ s.t. } A' \cup \{x\} \subseteq C\}$$

This set is called the ∞ -composant of A' .

It is usually clear from context whether we are discussing a finitely irreducible or an ∞ -irreducible continuum. If it is clear then the space will just be called irreducible and the ∞ -composants will just be called composants.

Proposition 7.2.9. *Let X be a continuum. For any subset $A' \subseteq X$ the set $\kappa(A')$ is either empty or connected and dense.*

Proof. Suppose that $\kappa(A')$ is not empty. It is clear that $\kappa(A')$ can be thought of as the union of all proper subcontinua of X which contain A' , so it is the union of connected intersecting sets, which makes it connected. If it were not dense then it must be closed, else the closure would be a strictly larger, proper subcontinuum containing A' . But even if it were closed, take a point $x \in X \setminus \kappa(A')$ and an open set U such that $\kappa(A') \subseteq U$ and $x \notin \overline{U}$. Let K be the component of $\overline{U}$ containing $\kappa(A')$. By Theorem 7.1.2 we have that $K \setminus U \neq \emptyset$ so $K \not\subseteq \kappa(A')$. However, $x \notin K$ so K is a proper subcontinuum of X containing A' but not contained in the composant of A' . This contradiction indicated that $\kappa(A')$ is dense. $\square$

Definition 7.2.10. Let X be a continuum and suppose $X = \text{irr}(A)$. Then for each $a \in A$ the set $\lambda_A(a) = X \setminus \kappa(A \setminus \{a\})$. If it is not ambiguous then this will just be denoted $\lambda(a)$. Note that for each $a \in A$ we have that $a \in \lambda(a)$.

Proposition 7.2.11. *Let X be a continuum with $X = \text{irr}(A)$. For each $a \in A$ the set $\lambda(a)$ is connected*

Proof. Let $\lambda(a) = M \cup N$ for disjoint subsets M, N clopen in $\lambda(a)$. Without loss of generality let $a \in M$. By Theorem 7.1.10 there exist disjoint X -open sets U and V containing M and N respectively. Let P be the component of a in U . By Theorem 7.1.2, $\overline{P} \setminus U \neq \emptyset$ and it is clear that $\overline{P} \cap N = \emptyset$ so $\overline{P} \cap \kappa(A \setminus \{a\}) \neq \emptyset$. From this

we have that there exists a subcontinuum $Y \subsetneq X$ containing $A \setminus \{a\}$ and intersecting $\overline{P}$. Thus $Y \cup \overline{P}$ is a subcontinuum of X , containing A and not intersecting N . Since $X = \text{irr}(A)$ it must be that $X = Y \cup \overline{P}$, so $N = \emptyset$. From this we have that $\lambda(a)$ is connected. $\square$

Proposition 7.2.12. *Let X be a continuum, $C \subseteq X$ a subcontinuum and suppose $X = \text{irr}(A)$. Suppose $C \cap \lambda(a) \neq \emptyset$ and $C \not\subseteq \lambda(a)$. Then $\lambda(a) \subseteq C$, and in fact lies in the interior of C .*

Proof. Let $x \in C \setminus \lambda(a)$. There exists a proper subcontinuum $D \subsetneq X$ containing $A \setminus \{a\}$ and x . As it is a proper subcontinuum, $D \cap \lambda(a) = \emptyset$ and $C \cup D$ is a continuum containing $A \setminus \{a\}$ and intersecting $\lambda(a)$. This implies that $C \cup D = X$, and therefore $\lambda(a) \subseteq X \setminus D \subseteq \text{int}(C) \subseteq C$. $\square$

Corollary 7.2.13. *Let X be a continuum with $X = \text{irr}(A)$ and let $a, b \in A$ be distinct points. Then $\lambda(a) \cap \lambda(b) = \emptyset$.*

Proof. Suppose for contradiction that $\lambda(a) \cap \lambda(b) \neq \emptyset$. Let C be a proper subcontinuum of X containing $A \setminus \{a\}$. Then C intersects $\lambda(b)$ at b so contains it, which means it intersects $\lambda(a)$ and therefore contains that. This is the desired contradiction. $\square$

Remark 7.2.14. It is clear to see from the proof of this Corollary that for distinct a, b we have $\lambda(a) \cap \overline{\lambda(b)} = \emptyset$, since C would contain the closure of $\lambda(b)$.

Corollary 7.2.15. *Let X be a continuum. If $X = \text{irr}(A)$ and for all $a \in A$ we pick $b_a \in \lambda(a)$ then $X = \text{irr}(\{b_a | a \in A\})$.*

Proof. Any continuum containing each b_a will contain each $\lambda(a)$ so will contain each a . Thus no proper subcontinuum of X contains $\{b_a | a \in A\}$. Given $C_a \subsetneq X$ containing $A \setminus \{a\}$ we have that for any $a' \neq a$, $\lambda(a') \subseteq C_a$ and $C_a \cap \lambda(a) = \emptyset$. Therefore $\{b_{a'} | a' \in A \setminus \{a\}\} \subseteq C_a$ and $b_a \notin C_a$. This proves that $X = \text{irr}(\{b_a | a \in A\})$. $\square$

Proposition 7.2.16. *Let X and Y be continua, let X be infinitely irreducible with $X = \text{irr}(A)$ and let $\rho : X \mapsto Y$ be a monotone surjection. Then $Y = \text{min}(\rho(A))$.*

Proof. The proof is identical to Proposition 3.2.16. $\square$

Lemma 7.2.17. *Let X be an ∞ -irreducible continuum with $X = \text{irr}(A)$. Suppose there exists disjoint proper subcontinua $C, D \subseteq X$ with $A \subseteq C \cup D$. Let $\rho : X \mapsto X/C, D$ be the quotient map to the quotient space $X/C, D$, so the only two non-degenerate fibres of ρ are C and D . Then $X/C, D$ is a continuum with $X/C, D = \text{irr}(\rho(C), \rho(D))$. If $D = \{a\}$ for some point a in A then $\rho(\lambda_X(a)) = \lambda_{X/C}(\rho(a))$.*

Proof. The proof is identical to Lemma 3.2.21. $\square$

Theorem 7.2.18. *Let X be a continuum with $X = \text{irr}(A)$ and $X = \text{irr}(B)$. There exists a bijection $f : A \mapsto B$ such that for all $a \in A$, $\lambda_A(a) = \lambda_B(f(a))$.*

Proof. Take a point $a \in A$, we will define its image $f(a)$. We have by Proposition 7.2.11 and Remark 7.2.14 that $\overline{\lambda_A(a)}$ is a proper subcontinuum of X so it cannot contain the whole of B . There is therefore a point $b \in B \cap \kappa(A \setminus \{a\})$ so take a proper subcontinuum $C \subseteq X$ with $b \in C$, $A \setminus \{a\} \subseteq C$ and consider the monotone map $\rho : X \mapsto X/C$. We have from Lemma 7.2.17 that $X/C = \text{irr}(\rho(C), \rho(a))$.

Note that the composant of $\rho(C)$ is the image under ρ of $\kappa(A \setminus \{a\})$. By Proposition 7.1.4 we have that $\kappa(\rho(C)) = \bigcup_{i=1}^{\infty} K_i$ for a nested sequence of continua K_i each containing $\rho(C)$, and that every proper subcontinuum of X/C containing $\rho(C)$ must lie in some K_i . For each K_i , $\rho^{-1}(K_i)$ is a proper subcontinuum of X so there exists some $b_i \in B$ with $b_i \notin \rho^{-1}(K_i)$. We will show that one of these b_i must lie in $\lambda_A(a)$.

If there are only finitely many distinct b_i and all of them lie in $\kappa(A \setminus \{a\})$ then there is a proper subcontinuum of X containing all of them as well as $A \setminus \{a\}$. The union of this with C will contain the whole of B but not a , which is a contradiction. We can now suppose there are infinitely many distinct b_i , none in $\lambda_A(a)$, and show this leads to a contradiction. Let $D \subseteq X$ be a proper subcontinuum of X containing $B \setminus \{b_1\}$. Then D intersects C as C contains some point of B which is clearly not b_1 . Thus $\rho(D)$ is a subcontinuum of X/C containing $\rho(C)$ and is a proper subcontinuum as it does not contain $\rho(b_1)$, so it must lie in some K_i . However, $\rho(D)$ contains the image of every b_j except for b_1 and the way the sequence was defined implies that the whole of it does not lie in any K_i . Thus we have a contradiction and it must be that some b_i lies in $\lambda_A(a)$. Define $f(a)$ to be this point.

We shall show now that the map f is well defined and a bijection. To show it is well defined we must show that only one point of B can lie in each $\lambda_A(a)$. This is clear however from Proposition 7.2.12, as if two points $b_1, b_2 \in B$ both lie in $\lambda_A(a)$ then it would be impossible for a continuum to contain one and not the other. Since $X = \text{irr}(B)$ we have that f is well defined. As the collection of $\lambda_A(a)$ are disjoint by Corollary 7.2.13 we have that the map f is injective, as no b can lie in two of them. No proper subcontinuum of X can contain $\{f(a) | a \in A\} \subseteq B$, which implies that $\{f(a) | a \in A\} = B$ and the map f is surjective.

All that is left to prove is that $\lambda_A(a) = \lambda_B(f(a))$. Since f is a bijection we only need to show that $\lambda_B(f(a)) \subseteq \lambda_A(a)$, then apply the same argument to f^{-1} . Let $x \in \lambda_B(f(a))$ and let $K \subseteq X$ be a subcontinuum containing x and $A \setminus \{a\}$. Then

for each $a' \in A \setminus \{a\}$ we have that $K \cap \lambda_A(a') \neq \emptyset$ which by Proposition 7.2.12 gives us that $\lambda_A(a')$, and in particular $f(a')$, must lie in K . Thus $B \setminus f(a) \subseteq K$ and $K \cap \lambda_B(f(a)) \neq \emptyset$ as it contains x . This gives us that $K = X$, and consequently $x \in \lambda_A(a)$. Thus $\lambda_B(f(a)) \subseteq \lambda_A(a)$, completing the proof. $\square$

We earlier defined a composant $\kappa(A \setminus \{a\})$ as the union of all proper subcontinua containing $A \setminus \{a\}$. It follows that no proper subcontinuum of X contains $A \setminus \{a\}$ as well as a point $x \in \lambda_A(a)$. While this does not directly prove that $X = \text{irr}(\{x\} \cup A \setminus \{a\})$, we are now in a position to show that this is the case.

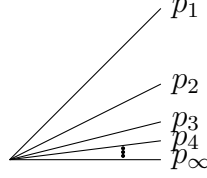
Proposition 7.2.19. *Let X be an ∞ -irreducible continuum with $X = \text{irr}(A)$ and let $a \in A$. Then $\lambda(a) = \{x \in X \mid X = \text{irr}(\{x\} \cup A \setminus \{a\})\}$*

Proof. It is clear that $\lambda(a) \supseteq \{x \in X \mid X = \text{irr}(\{x\} \cup A \setminus \{a\})\}$ so we only need to prove the other inclusion. Take a point $x \in \lambda(a)$ and let $A' = \{x\} \cup A \setminus \{a\}$. If $C \subseteq X$ is a subcontinuum containing A' then it intersects $\lambda(a)$ but is not contained in it so by Proposition 7.2.12 we have that $\lambda(a) \subseteq C$. Therefore $A \subseteq C$ and as $X = \text{irr}(A)$ we have that $C = X$. Now let $a' \in A \setminus \{a\}$ and let $D \subsetneq X$ be a proper subcontinuum containing $A \setminus \{a'\}$. Then $D \cap \lambda(a) \neq \emptyset$ and $D \not\subseteq \lambda(a)$ so $x \in \lambda(a) \subseteq D$ by Proposition 7.2.12. We therefore have a proper subcontinuum containing $A' \setminus \{a'\}$. All that is left to show is that there is a proper subcontinuum containing $A' \setminus \{x\}$, but we know a proper subcontinuum exists containing $A \setminus \{a\}$ and by again applying Proposition 7.2.12 it is clear that x does not lie in such a continuum. This completes the proof. $\square$

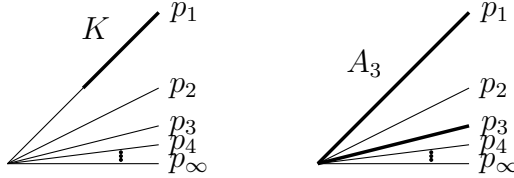
We have seen that if a continuum is ∞ -irreducible then there exists a unique set of infinitely many disjoint subspaces such that $X = \text{irr}(A)$ if and only if A contains precisely one point from each subset, namely the sets $\lambda(a)$.

At this point in the study of finite irreducibility, terminal and end continua were introduced. These concepts cannot be meaningfully generalised to ∞ -irreducibility. This is because these definitions are tied up with decompositions of a continuum and for the finite case there was a meaningful difference between a decomposition $A_1, \dots, A_n$ and some subset $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n$, namely the number of subcontinua. This is no longer the case for ∞ -irreducibility, as removing one object from an infinite set does not change its size. Some of the results can however be interpreted in a different light to avoid talking about terminal or end continua, and these results will be presented here. First however, will be an example of an infinitely irreducible space which illustrates the problems trying to view terminal continua in an infinite setting.

Example 7.2.20. Let X be the harmonic fan below.



Now $X = \text{irr}(\{p_i | i \in \omega\})$, with p_∞ not a point of irreducibility as no proper subcontinuum of X contains every other p_i . Let K be an arc containing p_1 and half of the corresponding arm of X . For $2 \leq n \leq \infty$ let A_n be the arc in X with p_1 and p_n as end points.



The subcontinuum K seems an ideal candidate to be a terminal continuum as it contains a point of irreducibility and is confined to the vicinity of this point, not reaching the branch point of X or containing any other irreducible points. Note however that $K \cap A_n \neq \emptyset$ for each n , that $X = \bigcup_{n=2}^{\infty} A_n$ and that X is not the union of any subcollection of these arcs. It would therefore seem like this is a valid decomposition of X , with each subcontinuum intersecting K . However, the union of K with all but one of these arcs, say A_n , will not be the whole of X as it will not include p_n . It seems that despite being an ideal candidate, K has failed to fulfil the natural condition to be called an infinitely terminal continuum.

Proposition 7.2.21. *Let X be an ∞ -irreducible continuum with $X = \text{irr}(A)$. If K is a non-cutting subcontinuum with non-empty interior then $A \cap K \neq \emptyset$.*

Proof. We have that $U := X \setminus K$ is a continuumwise connected set. Take any point $x \in U$, then for each $a \in A \cap U$ there exists a continuum $C_a \subseteq U$ with $x, a \in C_a$. Let $C = \overline{\bigcup_{a \in A} C_a}$. Since each C_a contains x we have that $\bigcup_{a \in A} C_a$ is connected, which implies that C is a continuum. Clearly $A \cap U \subseteq C$. However, $C \subseteq \overline{U}$ which is not the whole of X as K has non-empty interior. Thus C is a proper subcontinuum of X , which means $A \not\subseteq C$. There must therefore be some point $a \in A \cap K$. $\square$

Note that the requirement that K has non-empty interior is essential. Also note that the converse is not true, and that there exist subcontinua intersecting A which are cutting. Examples of each are shown below as subcontinua of the fan in Example 7.2.20.



Proposition 7.2.22. *Let X be a continuum with $X = \text{irr}(A)$. Then $X \setminus \bigcup_{a \in A} \lambda(a)$ is dense.*

Proof. Let $L = \bigcup_{a \in A} \lambda(a)$ and let $U \subseteq L$ be an open set. We know that each composant is dense from Proposition 7.2.9 so each $\lambda(a)$ has empty interior. We also know that $X = \text{irr}(B)$ if and only if there is precisely one point of B in each $\lambda(a)$ from Theorem 7.2.18. To prove that L has empty interior we will show that for each $a' \in A$, $\lambda(a')$ is an open subset of L and consequently $U \cap \lambda(a)$ is an open subset of X . If $\lambda(a')$ were not open in L then it intersects $\overline{\bigcup_{a \neq a'} \lambda(a)}^L$ at say x . Let $C \subseteq X$ be a proper subcontinuum containing $A \setminus \{a'\}$. Then for each $a \neq a'$ we have that $\lambda(a) \cap C \neq \emptyset$ so by Proposition 7.2.12 we have that $\lambda(a) \subseteq C$. This in turn implies that $\overline{\bigcup_{a \neq a'} \lambda(a)} \subseteq C$ as C is closed. Thus $x \in C$ and $C \cap \lambda(a') \neq \emptyset$ so again $\lambda(a') \subseteq C$, which contradicts that $X = \text{irr}(A)$ and that C is a proper subcontinuum of X . This contradiction gives us that each $\lambda(a')$ is indeed an open subset of L and $U \cap \lambda(a')$ is an open subset of X . Since $\kappa(A \setminus \{a'\})$ is dense we have that $U \cap \lambda(a') = \emptyset$. This is true of every $a' \in A$ so $U \cap L = \emptyset$, meaning that L does indeed have an empty interior and its complement is dense. $\square$

7.3 Irreducible Subcontinua and Local Connectedness

This section will focus on minimal or irreducible subcontinua of a continuum. There will also be a number of results related to open sets containing $\lambda(a)$ and their complements. We will reintroduce the definition of an E-continuum, but without definitions for terminal and end continua we will be unable to look at adaptations of results from [BF77] as we did for n -irreducibility in Chapter 3.

Definition 7.3.1. Let X be a continuum and let $A \subseteq X$ be an infinite subset with $X = \text{irr}(A)$. Let $a \in A$. If $\lambda(a)$ is compact then it is an ∞ -E-continuum. Where it is unambiguous $\lambda(a)$ will just be referred to as an E-continuum.

Proposition 7.3.2. Let X be an ∞ -irreducible continuum and let $A \subseteq X$ be such that $X = \text{irr}(A)$. Let $a \in A$. Then $\lambda(a)$ is compact if and only if for each subcontinuum K of X with $K \cap \lambda(a) \neq \emptyset$ and $K \not\subseteq \lambda(a)$, K is decomposable.

Proof. First suppose $\lambda(a)$ is compact and let K be a counterexample i.e. an indecomposable subcontinuum of X with $K \cap \lambda(a) \neq \emptyset$ and $K \not\subseteq \lambda(a)$. The composants of K are continuumwise connected, disjoint and dense by Theorem 7.1.5. Suppose one of the composants $\kappa_K(x)$ intersects $\kappa(A \setminus \{a\})$. Take a proper subcontinuum C of X containing $A \setminus \{a\}$ and intersecting $\kappa_K(x)$. For each point $y \in \kappa_K(x)$ there is a subcontinuum D such that $y \in D$ and $C \cap D \neq \emptyset$. Now D has empty interior in K so must have empty interior in X , meaning $X \neq C \cup D$ and $y \in \kappa(A \setminus \{a\})$. From this we can conclude that if $\kappa(A \setminus \{a\})$ intersects a composant of K then it must contain that composant.

From Proposition 7.2.12 we have that $\lambda(a) \subseteq K$ and as $K \not\subseteq \lambda(a)$ we know that this is a strict inclusion. As $\lambda(a)$ is a proper subcontinuum of K it is contained in a composant of K , and since it is compact it cannot be the whole of this composant, so the composant containing $\lambda(a)$ intersects, and is therefore contained in, $\kappa(A \setminus \{a\})$. This is a contradiction, so no such K can exist.

For the reverse direction suppose the condition holds, but $\lambda(a)$ is not compact. Let $K = \overline{\lambda(a)}$. We know that K is decomposable so let $K = C \cup D$. Clearly it is not the case that $\lambda(a) \subseteq C$ or $\lambda(a) \subseteq D$ as $\lambda(a)$ is dense in K . By our assumption $\lambda(a)$ cannot contain both C and D so without loss of generality let $C \not\subseteq \lambda(a)$. Since $C \cap \kappa(A \setminus \{a\}) \neq \emptyset$ there exists a proper subcontinuum $E \subsetneq X$ containing $A \setminus \{a\}$ and intersecting C . Then $C \cup E$ is a proper subcontinuum of X , as it does not contain D , with $A \setminus \{a\} \subseteq C \cup E$ and some point of $\lambda(a)$ in $C \cup E$. This is a contradiction, so we have that $\lambda(a)$ must be compact. $\square$

Lemma 7.3.3. Let X be a continuum and let $B \subseteq X$. Then there exists a subcontinuum $Y \subseteq X$ such that $Y = \min(B)$.

Proof. If $X = \min B$ then we are done. Otherwise let $\mathcal{C}(B)$ be the set of subcontinua of X containing B , partially ordered by reverse inclusion. As X contains B we have that $X \in \mathcal{C}(B)$, and in particular that $\mathcal{C}(B)$ is not empty. Given a chain of continua $K_i \in \mathcal{C}(B)$ let $K = \bigcap K_i$. Then by Theorem 7.1.14 we have that K is a

continuum which will contain B . Thus every chain in $\mathcal{C}(B)$ lies under a maximal element. Applying Lemma 7.1.9 gives us that there is a maximal element M of $\mathcal{C}(B)$. Now M is a subcontinuum of X and as it is a minimal element of $\mathcal{C}(B)$ it must be that $M = \min(B)$. $\square$

Remark 7.3.4. We cannot say that $Y = \text{irr}(B)$ since Y may not have proper subcontinua containing each $B \setminus \{b\}$. As an example, take any convergent sequence along with its limit point to be B . Then no proper subcontinuum of Y can contain the sequence without the limit point.

Lemma 7.3.5. *Let X be a continuum. If $X = \text{irr}(A)$, each $\lambda(a)$ is an E -continuum, $B \subseteq A$ is not a singleton and $Y = \min(B)$ then $Y = \text{irr}(B)$.*

Proof. We need to show that for any $b \in B$ there exists a proper subcontinuum of Y containing $B \setminus \{b\}$. Let $Z \subsetneq X$ be a proper subcontinuum containing $A \setminus \{b\}$. Let $U_Z = X \setminus Z$. This open set is connected, as if $U_Z = V \cup W$ with $b \in V$ then $Z \cup V$ is a continuum containing A by Proposition 7.1.3, so $W = \emptyset$. By irreducibility we have that $X = Y \cup Z$ so $U_Z \subseteq Y$.

Claim. Each component of $Y \cap Z$ has non-trivial intersection with $\overline{U_Z}$.

Let K be a component of $Y \cap Z$. By Theorem 7.1.15 K is a nested intersection of $(Y \cap Z)$ -clopen sets, $K = \bigcap Q_\alpha$. If $K \cap \overline{U_Z} = \emptyset$ then $\bigcap (Q_\alpha \cap \overline{U_Z}) = \emptyset$. Each $(Y \cap Z)$ -clopen set is X -closed, so each $Q_\alpha \cap \overline{U_Z}$ is also closed. X is compact so if the nested intersection of closed sets is empty we can infer that there is some $Q_\alpha \cap \overline{U_Z}$ which is also empty by Theorem 7.1.11. But then as $Y = (Y \cap Z) \cup U_Z$,

$$\overline{Y \setminus Q_\alpha} = \overline{(Y \cap Z) \setminus Q_\alpha \cup U_Z \setminus Q_\alpha} = (Y \cap Z) \setminus Q_\alpha \cup \overline{U_Z}.$$

Thus we have that $\overline{Y \setminus Q_\alpha} \cap Q_\alpha = \emptyset$ which means Q_α is Y -open. Q_α is compact so it is also Y -closed, meaning Q_α is a proper non-trivial clopen subset of Y . Y is connected so this contradiction gives us that K must intersect $\overline{U_Z}$ non-trivially, proving our claim.

Let $\rho : X \mapsto X/Z$. Lemma 7.2.17 gives us that $X/Z = \text{irr}(\rho(b), \rho(Z))$ and that $\rho(\lambda(b))$ is an E -continuum of X/Z .

Claim. $\overline{U_Z}$ is irreducible between b and any point of $Z \cap \overline{U_Z}$

Let $L \subseteq \overline{U_Z}$ contain b and intersect Z . Then $\rho(L) = X/Z$ which in turn implies that $\rho^{-1}(X/Z \setminus \rho(Z)) \subseteq L$. Since $U_Z = \rho^{-1}(X/Z \setminus \rho(Z))$ this implies $\overline{U_Z} \subseteq L$. Thus $L = \overline{U_Z}$, which proves our claim.

Claim. $\overline{U}_Z$ is decomposable.

As was mentioned above, $\rho(\lambda(b))$ is an E-continuum of X/Z and thus X/Z is decomposable by Theorem 7.1.7. Let $X/Z = C \cup D$ for proper subcontinua C, D , and without loss of generality let $\rho(Z) \in C$. We will show that $\rho(Z)$ is not a cut point of C . If $C \setminus \rho(Z) = V \cup W$ for disjoint clopen V, W then one of them, say V , must intersect D . By Proposition 7.1.3 $V \cup \rho(Z)$ is a continuum and thus $D \cup V \cup \rho(Z)$ is a continuum. By irreducibility of X/Z we have that $D \cup V \cup \rho(Z)$ is the whole of X/Z , which in turn implies that $W \subseteq D$ and applying the same argument we get that $V \subseteq D$, so $X/Z \setminus D = \rho(Z)$. This is a contradiction as a continuum has no isolated points. Thus $C \setminus \rho(Z)$ must be connected, and $\overline{\rho^{-1}(C \setminus \rho(Z))}$ must be connected as well since ρ is monotone. We know that $\overline{U}_Z = \overline{\rho^{-1}(C \setminus \rho(Z))} \cup \rho^{-1}(D)$ and both of these sets are proper subcontinua. This is a decomposition so $\overline{U}_Z$ is decomposable.

So far we have that each component of $Y \cap Z$ intersects $\overline{U}_Z$ and these intersections lie in the complement of a composant κ of $\overline{U}_Z$. This is a composant in the classic sense, of a continuum irreducible about a pair of points. Thus $(Y \cap Z) \cup \overline{U}_Z \setminus \kappa$ is a continuum contained in Y and containing $B \setminus \{b\}$ but not b . This completes the proof. $\square$

For the following corollary it will be useful to recall Proposition 7.2.7. In particular, we could apply to corollary to an enumeration $a_1, a_2, \dots$ of A with each $A_k = A \setminus \{a_1, \dots, a_k\}$.

Corollary 7.3.6. *Let X be a continuum with $X = \text{irr}(A)$ and suppose each $\lambda(a)$ is an E-continuum. Let $A_1 \supsetneq A_2 \supsetneq \dots$ be a strictly descending family of subsets of A . Then there exists a strictly descending chain $C_1 \supsetneq C_2 \supsetneq \dots$ of subcontinua of X with $C_n = \text{irr}(A_n)$.*

Proof. We define the sequence by induction. Let C_1 be as in Lemma 7.3.3, with $C_1 = \min(A_1)$. Then suppose C_{n-1} has been defined and apply Lemma 7.3.3 again to find $C_n \subseteq C_{n-1}$ with $C_n = \min(A_n)$. By Lemma 7.3.5 we have that for each n , $C_n = \text{irr}(A_n)$. $\square$

We will now look at open sets containing $\lambda(a)$, focusing on local connectedness and the connectedness of complements.

Theorem 7.3.7. *Let X be a continuum with $X = \text{irr}(A)$ and let U be an open subset of X such that for some $a \in A$, $\lambda(a) \subseteq U$ and $\lambda(a)$ is an E-continuum. Then there exists a connected open set V such that $\lambda(a) \subseteq V \subseteq U$ and $X \setminus \overline{V}$ is connected.*

Proof. Let $C_a \subsetneq X$ be a subcontinuum with $A \setminus \{a\} \subseteq C_a$. Let $U_a = X \setminus C_a$. Define an equivalence relation on X such that C_a is the only non-trivial equivalence class, and let $\rho : X \mapsto Y := X/C_a$ be the resulting quotient map. Clearly ρ is continuous and monotone, and Y is a continuum by Lemma 7.1.1. Y is irreducible, in the classic sense, between $\rho(a)$ and $\rho(C_a)$ by Proposition 7.2.17 with $\rho(\lambda(a))$ an E-continuum. The set $\rho(U \cap U_a)$ is an open subset of Y containing $\rho(\lambda(a))$. Thus Y is locally connected at $\rho(\lambda(a))$ by Theorem 7.1.8 meaning there exists a connected open subset V' of Y such that $\rho(\lambda(a)) \subseteq V' \subseteq \rho(U \cap U_a)$ and $Y \neq \overline{V'}$. If $V = \rho^{-1}(V')$ then

$$\lambda(a) \subseteq V \subseteq U \cap U_a \subseteq U$$

and V is a connected open set. $X \setminus \overline{V} = \rho^{-1}(Y \setminus \overline{V'})$ and since $\overline{V'}$ is a terminal subcontinuum of Y its complement is connected by Theorem 7.1.6. Consequently, $X \setminus \overline{V}$ is connected. $\square$

Lemma 7.3.8. *Let X be a continuum with $X = \text{irr}(A)$ and suppose for each $a \in A$ we have that $\lambda(a)$ is an E-continuum. Take an ordering $a_1, a_2, \dots$ of A . For each a_n let U_n be an open set containing $\lambda(a_n)$. Then there exist V_n connected and open, $\lambda(a_n) \subseteq V_n \subseteq U_n$ such that for all n we have $X \setminus \bigcup_{i=1}^n \overline{V_i}$ connected.*

Proof. We will order A as $a_1, a_2, \dots$ and construct the V_n recursively. As we saw in the proof of Proposition 7.2.22, $\lambda(a) \cap \overline{\bigcup_{a' \neq a} \lambda(a')} = \emptyset$ so we have that each U_a can be shrunk to ensure they are all disjoint. Let $C_1, C_2, \dots$ be a nested sequence of continua as in Corollary 7.3.6, with $C_{n-1} = \text{irr}(a_n, a_{n+1}, \dots)$. Suppose $V_1, \dots, V_{n-1}$ have been defined and in such a way that $C_{n-1} \subseteq X \setminus \bigcup_{i=1}^{n-1} \overline{V_i}$.

We have from Proposition 7.2.12 that $\lambda(a_n) \subseteq \text{int}(C_{n-1})$. Take U'_n to be the open set $U_n \cap \text{int}(C_{n-1}) \setminus C_n$ and by Theorem 7.3.7 take V_n to be open and connected with $\lambda(a_n) \subseteq V_n \subseteq \overline{V_n} \subseteq U'_n$. Then

$$X \setminus \bigcup_{i=1}^n \overline{V_i} = (C_{n-1} \setminus \overline{V_n}) \cup ((X \setminus \bigcup_{i=1}^{n-1} \overline{V_i}) \setminus C_{n-1})$$

Every component of $((X \setminus \bigcup_{i=1}^{n-1} \overline{V_i}) \setminus C_{n-1})$ limits on to C_{n-1} and as $\overline{V_n} \subseteq \text{int}(C_{n-1})$ it follows that $X \setminus \bigcup_{i=1}^n \overline{V_i}$ is connected. $\square$

Theorem 7.3.9. *Let X be a continuum with $X = \text{irr}(A)$ and suppose for each $a \in A$ we have $\lambda(a)$ is an E-continuum. Then $K = X \setminus \bigcup_{a \in A} \lambda(a)$ is continuumwise connected.*

Proof. Let $x, y \in K$. Let $a_1, a_2, \dots$ be an ordering of A and take U_n be open sets such that $\lambda(a_n) \subseteq U_n$ and $x, y \notin \overline{U_n}$. For each n take V_n as in Lemma 7.3.8 and define $C = \bigcap_{n=1}^{\infty} \overline{X \setminus \bigcup_{i=1}^n \overline{V_i}}$. Each $X \setminus \bigcup_{i=1}^n \overline{V_i}$ is connected, so their closures are as well. This means C is the nested intersection of continua, making C a continuum (Theorem 7.1.14). Since $x, y \notin \overline{U_n}$ we know that they are not in any $\overline{V_n}$, so must lie in C . Finally, $C \subseteq K$ as each $\lambda(a_n)$ lies in the open set V_n and

$$V_n \cap X \setminus \bigcup_{i=1}^n \overline{V_i} = \emptyset \quad \text{so} \quad V_n \cap \overline{X \setminus \bigcup_{i \leq n} \overline{V_i}} = \emptyset \quad \text{so} \quad V_n \cap C = \emptyset$$

□

Proposition 7.3.10. *Suppose X is a continuum with $X = \text{irr}(A)$ and each $\lambda(a)$ is compact. Then given any $x \in X \setminus \bigcup_{a \in A} \lambda(a)$ and any $a \in A$ there is a subcontinuum $Y \subseteq X$ such that $x \in Y$ and $Y \cap A = \{a\}$.*

Proof. Take an open set U containing $\lambda(a)$ such that the closure of U does not intersect $\lambda(a')$ for any $a' \in A \setminus \{a\}$. Take $V \subseteq U$ a connected open subset containing $\lambda(a)$. Since $X \setminus \bigcup_{i=1}^n \lambda(a)$ is dense by Proposition 7.2.22 there exists a point $y \in V \cap X \setminus \bigcup_{a' \in A} \lambda(a')$ and by Theorem 7.3.9 a subcontinuum $X' \subseteq X \setminus \bigcup_{a' \in A} \lambda(a')$ containing x and y . Then $Y = X' \cup \overline{V}$ is the desired continuum. □

Chapter 8

Infinite Irreducibility and Monotone Maps

This chapter will proceed in a similar way to Chapter 4. We will start with an almost hereditarily decomposable continuum X which is irreducible about some infinite set of points A . By defining irreducible subcontinua for each non-irreducible point $x \in X$ an equivalence relation will be defined on X and we will see that the corresponding quotient space is a fan, by which we mean a dendrite with exactly one branch point and countably many end points. By adapting the equivalence relation a second quotient map will be constructed with a dendrite as its image.

8.1 Referenced Theorems

The following results from other authors will be used in this chapter.

Sam B. Nadler Jr, Continuum Theory, an Introduction

These results can be found in [Nad92].

Lemma 8.1.1 (3.2). *If a Hausdorff space is a continuous image of a compact metric space, then it is metrizable.*

Proposition 8.1.2 (4.14). *Let X be a compact metric space. If $(A_i)_{i \in \omega}$ is a sequence of nonempty subsets of X then there exists a subsequence A_{i_j} such that $\lim(A_{i_j})$ exists and is compact.*

Proposition 8.1.3 (4.18). *Let X be a compact metric space. Then every sequence of subcontinua of X has a subsequence converging to a subcontinuum of X and thus every convergent subsequence of subcontinua of X has a subcontinuum of X as its limit.*

Theorem 8.1.4 (5.4). *Let X be a continuum and let U be a nonempty, proper, open subset of X . If K is a component of $\overline{U}$, then $K \cap \partial U \neq \emptyset$. Equivalently, $K \cap (X \setminus U) \neq \emptyset$.*

Theorem 8.1.5 (10.2). *A continuum X is a dendrite if and only if any two points of X are separated in X by a third point of X .*

Proposition 8.1.6 (10.6). *Every subcontinuum of a dendrite is a dendrite.*

Theorem 8.1.7 (10.10). *A continuum X is a dendrite if and only if the intersection of any two connected subsets of X is connected. In particular, dendrites are unicoherent.*

Theorem 8.1.8 (11.8). *Let X be a non-degenerate continuum such that X is irreducible between p and q . If A and B are subcontinua of X such that $p \in A$ and $q \in B$ then $X \setminus (A \cup B)$ is connected.*

Proposition 8.1.9 (11.20). *A nondegenerate continuum X is indecomposable if and only if there are three points of X such that X is irreducible between each two of these three points.*

D.E. Bennett and J.B. Fugate, Continua and their non-separating Subcontinua

These results can be found in [BF77].

Theorem 8.1.10 (1.22). *Suppose that X is a continuum and $p \in X$. The following are equivalent:*

- $X \setminus \kappa(p)$ is closed;
- $X \setminus \kappa(p)$ is a continuum;
- $X \setminus \kappa(p)$ is continuum-wise connected;
- If K is a subcontinuum of X such that $K \cap (X \setminus \kappa(p)) \neq \emptyset$ and $K \cap \kappa(p) \neq \emptyset$, then K is decomposable.

Proposition 8.1.11 (1.23). *Suppose that M is a continuum such that each indecomposable subcontinuum has empty interior. Then for each $p \in M$, $M \setminus \kappa(p)$ is a continuum.*

Ryszard Engelking, General Topology

This result can be found in [Eng89].

Theorem 8.1.12 (6.1.9). *Let $\{C_s\}_{s \in S}$ be a family of connected subspaces of a topological space X . If there exists an $s_0 \in S$ such that the set C_{s_0} is not separated from any of the sets C_s then the union $\bigcup_{s \in S} C_s$ is connected.*

8.2 Monotone Maps Onto A Fan

We will begin by proving the following proposition, which parallels Proposition 8.1.11.

Proposition 8.2.1. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum. Let $A \subseteq X$ with $X = \text{irr}(A)$ and let $a \in A$. Then $\lambda(a)$ is compact.*

Proof. The proof is a straightforward application of Proposition 7.2.12 and Proposition 7.3.2 □

Definition 8.2.2. Let X be an almost hereditarily decomposable continuum and suppose $X = \text{irr}(A)$. Let $x \in X$ not be a point of irreducibility and let $B \subsetneq A$. Define two sets of subcontinua as follows.

$$\begin{aligned}\mathcal{P}_x^B &= \{C \subseteq X \mid C \text{ is a subcontinuum, } C = \min(B \cup \{x\})\} \\ \mathcal{Q}_x^B &= \{C \subseteq X \mid C \text{ is a subcontinuum, } C = \min(A \setminus B \cup \{x\})\}\end{aligned}$$

From Lemma 7.3.3 we know that both of these sets are non-empty. If $B = \{a\}$ then these sets will be denoted $\mathcal{P}_x^a$ and $\mathcal{Q}_x^a$. A typical element of each will be denoted $P_x^B \in \mathcal{P}_x^B$ and $Q_x^B \in \mathcal{Q}_x^B$. Note that $\mathcal{Q}_x^B = \mathcal{P}_x^{A \setminus B}$.

Proposition 8.2.3. *Let X be an almost hereditarily decomposable continuum and suppose $X = \text{irr}(A)$. Let $x \in X$ not be a point of irreducibility and let $B \subsetneq A$. Let $P_x^B \in \mathcal{P}_x^B$. Either $P_x^B = \text{irr}(B \cup \{x\})$ or $P_x^B = \text{irr}(B)$.*

Proof. If there does not exist $Y \subsetneq P_x^B$ containing B then P_x^B is a minimal subcontinuum of X containing B , which by Lemma 7.3.5 means $P_x^B = \text{irr}(B)$. Suppose otherwise, that there does exist $Y \subsetneq P_x^B$ containing B .

All that is left is to show that for each $b \in B$ there exists $Y \subsetneq P_x^B$ containing $\{x\} \cup B \setminus \{b\}$. This will be identical to the proof of Lemma 7.3.5 but with Z chosen to contain $\{x\} \cup A \setminus \{b\}$. Such a continuum Z exists as $x \notin \lambda(b)$. □

Based on the definition of $\mathcal{Q}_x^B$ it clearly follows from Proposition 8.2.3 that the same result holds for $Q_x^B \in \mathcal{Q}_x^B$ and $A \setminus B$. We will now prove a number of results similar to the p -sidedness results from Section 4.2.

Proposition 8.2.4. *Let X be an almost hereditarily decomposable space and suppose $X = \text{irr}(A)$. Let $x \in X$ not be a point of irreducibility and $B \subsetneq A$. Let $P_x^B \in \mathcal{P}_x^B$ and let $Q_x^B \in \mathcal{Q}_x^B$. Then one of the following holds.*

- $\forall y \in P_x^B \cap Q_x^B, P_x^B = \min(y, B)$
- $\forall y \in P_x^B \cap Q_x^B, Q_x^B = \min(y, A \setminus B)$

Proof. Suppose neither are the case and let subcontinua $Y \subsetneq P_x^B$ and $Z \subsetneq Q_x^B$ witness this i.e. $Y \cap Q_x^B \neq \emptyset \neq Z \cap P_x^B$ with $B \subseteq Y$ and $A \setminus B \subseteq Z$. Since they are proper subcontinua it must be that $x \notin Y \cup Z$ so $X \neq Y \cup Z$, which in turn implies $Y \cap Z = \emptyset$. Let $X/(Y, Z)$ be the quotient space whose only non trivial equivalence classes are Y and Z , and let $\rho : X \mapsto X/(Y, Z)$ be the natural quotient map. Applying Lemma 7.2.17 gives us that $X/(Y, Z) = \text{irr}(\rho(Y), \rho(Z))$ so by Theorem 8.1.8 we have that $X/(Y, Z) \setminus (\rho(Y) \cup \rho(Z))$ is connected. Since ρ is monotone we also have that $X \setminus (Y \cup Z)$ is connected. Let $C = \overline{X \setminus (Y \cup Z)}$. C is a continuum. As $Y \cup Q_x^B = X = Z \cup P_x^B$ we have that $X \setminus Y \subseteq Q_x^B$ and $X \setminus Z \subseteq P_x^B$, which in turn implies that $C \subseteq P_x^B \cap Q_x^B$. Since $X = Y \cup Z \cup C$ and Y and Z are disjoint it must be that $C \cap Y \neq \emptyset \neq C \cap Z$. Let y and z lie in these intersections respectively.

Suppose $C \neq \text{irr}(x, y)$ and let $C' \subsetneq C$ witness this. Then $Y \cup C'$ is a subcontinuum of P_x^B containing $B \cup \{x\}$ so $Y \cup C' = P_x^B$. Since $C \subseteq P_x^B$ we have that $C \setminus C' \subseteq Y$. This set is C -open so there exists X -open U such that $C \cap U = C \setminus C'$. As C is the closure of $X \setminus (Y \cup Z)$ it must be that $U \cap X \setminus (Y \cup Z) \neq \emptyset$. This is a subset of $U \cap C \subseteq Y$ which is a contradiction. Thus $C = \text{irr}(x, y)$. An identical argument gives that $C = \text{irr}(x, z)$.

Now suppose $C \neq \text{irr}(y, z)$ witnessed by C' . As $C = \text{irr}(x, y)$ it must be that $x \notin C'$. Consequently, $x \notin C' \cup Y \cup Z$ which is a continuum containing A . This contradiction gives us that $C = \text{irr}(y, z)$. Since C is irreducible between each pair of x, y, z it must be indecomposable by Proposition 8.1.9, but X is almost hereditarily decomposable and the open set $X \setminus (Y \cup Z)$ lies in C . This final contradiction completes the proof. $\square$

Definition 8.2.5. Let X be an almost hereditarily decomposable space such that $X = \text{irr}(A)$. A point x which is not a point of irreducibility is B -sided if the first option from Proposition 8.2.4 holds. It is possible for a point to be both B -sided and $A \setminus B$ -sided.

Proposition 8.2.6. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$ and let $B \subseteq A$. Let $x \in X$ not be a point of irreducibility. If x is B -sided, witnessed by $P_x^B \in \mathcal{P}_x^B$ with $P_x^B = \text{irr}(B \cup \{x\})$, then the set $\mathcal{P}_x^B$ only has one element.*

Proof. Suppose P_x^B and $\tilde{P}_x^B$ are distinct elements of $\mathcal{P}_x^B$ with P_x^B satisfying the first condition of Proposition 8.2.4 and $P_x^B = \text{irr}(B \cup \{x\})$ and note that this implies that for all $y \in P_x^B \cap Q_x^B$, $P_x^B = \text{irr}(y, B)$. Since both subcontinua are minimal about $B \cup \{x\}$, it must be that $P_x^B \not\subseteq \tilde{P}_x^B \not\subseteq P_x^B$. As $P_x^B \cup Q_x^B = X$ we have that $\tilde{P}_x^B \setminus P_x^B \subseteq Q_x^B$. Similarly, $P_x^B \setminus \tilde{P}_x^B \subseteq Q_x^B$, so $(P_x^B \cap \tilde{P}_x^B) \cup Q_x^B = X$. As P_x^B is irreducible between B and each point of $P_x^B \cap Q_x^B$ we have that $P_x^B \setminus \tilde{P}_x^B \subseteq \lambda_{P_x^B}(B)$, and therefore $\kappa_{P_x^B}(B) \subseteq P_x^B \cap \tilde{P}_x^B$. Since this is a dense subset of P_x^B and is contained in a closed subset $P_x^B \cap \tilde{P}_x^B$ of $\tilde{P}_x^B$, we have that $P_x^B \subseteq \tilde{P}_x^B$. This contradiction gives us the result we need. $\square$

Proposition 8.2.7. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. Let $B \subsetneq A$ and let x not be a point of irreducibility. Let P_x^B be in $\mathcal{P}_x^B$ and let $y \in P_x^B$. There exists P_y^B in $\mathcal{P}_y^B$ such that $P_y^B \subseteq P_x^B$. If x is B -sided and $P_x^B = \text{irr}(B \cup \{x\})$ then $P_y^B \subseteq P_x^B$ holds for each P_y^B in $\mathcal{P}_y^B$.*

Proof. For the first part of this proposition we can apply Lemma 7.3.3 to P_x^B to find the P_y^B we want.

Now suppose that x is B -sided and that $P_x^B = \text{irr}(B \cup \{x\})$. We have that $X = P_x^B \cup Q_x^B$, so suppose $P_y^B \cap Q_x^B \neq \emptyset$. Then $X = P_y^B \cup Q_x^B$ so

$$\kappa_{P_x^B}(B) \subseteq P_x^B \setminus Q_x^B = X \setminus Q_x^B = P_y^B \setminus Q_x^B \subseteq P_y^B$$

As $\kappa_{P_x^B}(B)$ is a dense subset of P_x^B and P_y^B is compact it follows that $P_x^B \subseteq P_y^B$. Then as $B \cup \{y\} \subseteq P_x^B$ we have that $P_x^B = P_y^B$. This completes the proof. $\square$

We will now define our equivalence relation on X .

Definition 8.2.8. Let X be an almost hereditarily decomposable continuum and let A be an infinite set of points of X with $X = \text{irr}(A)$. Let $x, y \in X$. Let $x \sim y$ if and only if for some $a \in A$ we have $x, y \in \lambda(a)$, or if for all $a \in A$ we have $P_x^a \cup Q_y^a = X = P_y^a \cup Q_x^a$.

Note that as $P_x^a = \min(\{x, a\})$ it is clear that in fact $P_x^a = \text{irr}(x, a)$. Also note that $P_x^a \cup Q_y^a = X$ is equivalent to $P_x^a \cap Q_y^a \neq \emptyset$.

Proposition 8.2.9. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$ and let $\sim$ be as defined in Definition 8.2.8. This relation does not depend on which elements of $\mathcal{P}_x^a, \mathcal{Q}_x^a, \mathcal{P}_y^a$ and $\mathcal{Q}_y^a$ are chosen.*

Proof. Suppose P_x^a and $\tilde{P}_x^a$ are two distinct elements of $\mathcal{P}_x^a$. As both are irreducible and they are not equal to each other, we have that $P_x^a \not\subseteq \tilde{P}_x^a \not\subseteq P_x^a$. Let $y \in X$ and $Q_y^a \in \mathcal{Q}_y^a$ be such that $P_x^a \cup Q_y^a = X$. Then $\emptyset \neq \tilde{P}_x^a \setminus P_x^a \subseteq Q_y^a$ which means $\tilde{P}_x^a \cap Q_y^a \neq \emptyset$ and $\tilde{P}_x^a \cup Q_y^a = X$. An identical argument gives the same result for $Q_x^a, \tilde{Q}_x^a$ and from this we can conclude that if for all $a \in A$ the subcontinua P_x^a and Q_x^a give $x \sim y$ then so do any other choices for these subcontinua. $\square$

Having shown that this relation is well defined we will now prove that it is an equivalence relation.

Theorem 8.2.10. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$ and let $\sim$ be as defined in Definition 8.2.8. The relation $\sim$ is an equivalence relation.*

Proof. It is clear to see that $\sim$ is reflexive and symmetric. Suppose $x \sim y$ and $y \sim z$ for three distinct points $x, y, z \in X$. If any one of them lies in $\lambda(a)$ for some $a \in A$ then all three must do so and $x \sim z$. Suppose none of them lie in an E-continuum of X .

Let $a \in A$ and take $P_x^a \in \mathcal{P}_x^a$ and $Q_z^a \in \mathcal{Q}_z^a$. We only need to show that $P_x^a \cup Q_z^a = X$ as x, z and a can be interchanged freely and from Proposition 8.2.9 the choice of P_x^a and Q_z^a is irrelevant. If $z \in P_x^a$ or $x \in Q_z^a$ then this is the case so let us consider when $z \notin P_x^a$ and $x \notin Q_z^a$. Since $x \sim y$ and $y \sim z$ we know that for all $P_y^a \in \mathcal{P}_y^a$ and $Q_y^a \in \mathcal{Q}_y^a$ we have that $P_y^a \cup Q_z^a = X = P_x^a \cup Q_y^a$. Thus without loss of generality we have that $x \in P_y^a$ and $z \in Q_y^a$.

If y is $\{a\}$ -sided then from Proposition 8.2.7 $P_x^a \subseteq P_y^a$. Since $X = P_x^a \cup Q_y^a$ we have that

$$\emptyset \neq P_x^a \cap Q_y^a \subseteq P_y^a \cap Q_y^a \subseteq \lambda_{P_y^a}(y)$$

Therefore $P_x^a = P_y^a$ which gives us that $P_x^a \cup Q_z^a = X$.

If y is Q_y^a -sided and $Q_y^a = \text{irr}(\{y\} \cup A \setminus \{a\})$ then use the same argument as for y being $\{a\}$ -sided. If $Q_y^a = \text{irr}(A \setminus \{a\})$ then as $z \in Q_y^a$ we have that Q_y^a is an element of $\mathcal{Q}_z^a$. Proposition 8.2.9 and the fact that $P_x^a \cup Q_y^a = X$ give us that $P_x^a \cup Q_z^a = X$. $\square$

Definition 8.2.11. Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$ and let $\sim$ be as defined in Definition 8.2.8. Let $x \in X$. We denote by $x_\sim$ the equivalence class of x under $\sim$.

In order to study the equivalence classes we will split X into various subsets and consider the equivalence classes of points in each of those subsets. It will turn out that each subset is the union of whole equivalence classes and that the interactions between the subsets will create the fan structure we are hoping to map to.

Lemma 8.2.12. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$ and let $x \in X$ not be a point of irreducibility. Suppose there exists Q_x^a in $\mathcal{Q}_x^a$ with $Q_x^a = \text{irr}(\{x\} \cup A \setminus \{a\})$. Suppose this is witnessed by Q , so $A \setminus \{a\} \subseteq Q \subsetneq Q_x^a$, and that $Q \cap P_x^a = \emptyset$. Let $\rho : X \mapsto X/Q$ be the natural quotient map. Then X/Q is almost hereditarily decomposable and $X/Q = \text{irr}(\rho(a), \rho(Q))$.*

Proof. From Lemma 3.2.20 we know that X/Q is almost hereditarily decomposable and from Lemma 7.2.17 we know that $X/Q = \text{irr}(\rho(a), \rho(Q))$. $\square$

Proposition 8.2.13. *Let X, x, Q_x^a, Q and ρ be as in Lemma 8.2.12 and $\sim, x_\sim$ be as in Definitions 8.2.8 and 8.2.11. By applying Theorem 2.4.11 let $\pi : X/Q \mapsto \mathbb{I}$ be a monotone quotient map onto an arc. Then $x_\sim$ is homeomorphic under ρ to the fibre of π containing $\rho(x)$.*

Proof. First suppose y is in Q . Then there exists an element Q_y^a of $\mathcal{Q}_y^a$ with $Q_y^a \subseteq Q$ and as $Q \cap P_x^a = \emptyset$ we have that $P_x^a \cap Q_y^a = \emptyset$. This implies that $x \approx y$, that $x_\sim \subseteq X \setminus Q$ and that $\rho|_{x_\sim}$ is a homeomorphism.

Now take $y \notin Q$. Since $X/Q = \rho(Q_x^b) = \rho(Q_y^b)$ for any $b \in A \setminus \{a\}$ it must be that $x \in Q_y^b, y \in Q_x^b$. Thus we only have to check whether $X = P_x^a \cup Q_y^a = P_y^a \cup Q_x^a$. We have that $P_x^a \cap Q_y^a \neq \emptyset$ if and only if $\rho(P_x^a) \cap \rho(Q_y^a) \neq \emptyset$, and also that $P_y^a \cap Q_x^a \neq \emptyset$ if and only if $\rho(P_y^a) \cap \rho(Q_x^a) \neq \emptyset$. The first of these is because P_x^a does not intersect Q , which is the only non-trivial fibre of ρ . The second is because $\rho(P_y^a)$ is a continuum and if it contains $\rho(Q)$ then it must be the whole of X/Q , so either way it intersects $\rho(Q_x^a)$ somewhere other than $\rho(Q)$. This proves that $x \sim y$ if and only if they are similarly related in X/Q i.e. if they are in the same fibre of π . $\square$

Definition 8.2.14. Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. Define subsets of X as follows.

$$M_a := \{x \in X \mid \forall Q \subseteq Q_x^a \text{ with } A \setminus \{a\} \subseteq Q \text{ we have } Q \cap P_x^a \neq \emptyset\}$$

$$M = \bigcap_{a \in A} M_a$$

Note that in the definition of M_a it does not matter which $Q_x^a \in \mathcal{Q}_x^a$ is chosen. If there exists Q_x^a witnessing that $x \in M_a$ and $\tilde{Q}_x^a \in \mathcal{Q}_x^a$, with $Q \subseteq \tilde{Q}_x^a$ containing $A \setminus \{a\}$

then consider the intersection $Q \cap P_x^a$. If this is the empty set then as $X = P_x^a \cup Q_x^a$ we have that $Q \subseteq Q_x^a$. However, this contradicts that Q_x^a witnesses $x \in M_a$. Thus it must be that $Q \cap P_x^a \neq \emptyset$ and that $\tilde{Q}_x^a$ also witnesses $x \in M_a$.

For a similar reason it does not matter which P_x^a is chosen. Again if P_x^a witnesses that $x \in M_a$ and $\tilde{P}_x^a \in \mathcal{P}_x^a$, $Q \subseteq Q_x^a \in \mathcal{Q}_x^a$ with $A \setminus \{a\} \subseteq Q$ then as $P_x^a \cup Q = X$ it must either be that $\tilde{P}_x^a$ intersects Q non-trivially or $\tilde{P}_x^a$ is a proper subcontinuum of P_x^a . Since $P_x^a = \text{irr}(a, x)$ with both of these points lying in $\tilde{P}_x^a$ it must be that $\tilde{P}_x^a \cap Q \neq \emptyset$.

It is not necessary for $Q_x^a = \text{irr}(A \setminus \{a\})$ in order that $x \in M_a$, but if this is not the case then x will be $\{a\}$ -sided. As it was in Section 4.2, M is the set of points that are not particularly close to any of the points of irreducibility but instead lie in the middle of X . Indeed the set M is again an equivalence class and will be the preimage of the branch point of the fan when we construct our map.

Proposition 8.2.15. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. Let $\sim$ be as in Definition 8.2.8 and let M be as in Definition 8.2.14. Given any $x \in M$ we have that $x_\sim = M$.*

Proof. Let $x \in M$. If $y \in x_\sim$ but $y \notin M$ then $y \notin M_a$ for some $a \in A$. Let $P_y^a \in \mathcal{P}_y^a$, $Q_y^a \in \mathcal{Q}_y^a$ and $Q \subseteq Q_y^a$ witness this. Applying Proposition 8.2.13 to y gives us that $y_\sim = (\pi \circ \rho)^{-1}(\alpha)$ for some $\alpha < 1$. This in turn implies that there exists an element $P_x^a \in \mathcal{P}_x^a$ with $P_x^a \subseteq (\pi \circ \rho)^{-1}([0, \alpha])$ so $P_x^a \cap Q = \emptyset$. Thus $x \notin M_a$ and $x \notin M$. This contradiction shows that if $x \in M$ then $x_\sim \subseteq M$.

Now let $y \in M$, let $a \in A$ and let P_x^a and Q_y^a be elements of $\mathcal{P}_x^a$ and $\mathcal{Q}_y^a$ respectively. If $P_x^a \cap Q_y^a = \emptyset$ then $Q_y^a \subseteq Q_x^a$ which contradicts the fact that $x \in M$. Thus $P_x^a \cap Q_y^a \neq \emptyset$ and by symmetry $P_y^a \cap Q_x^a \neq \emptyset$. This gives us that $y \in x_\sim$, completing the proof. $\square$

Proposition 8.2.16. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. Let $\sim$ be as in Definition 8.2.8 and let M be as in Definition 8.2.14. M is compact*

Proof. Let $x \in \overline{M} \setminus M$ and for some a let $Q \subseteq Q_x^a \in \mathcal{Q}_x^a$ witness that $x \notin M_a$. Let ρ, π be as in Proposition 8.2.13 with $[0, 1]$ as the image of π , $(\pi \circ \rho)(Q) = 1$. As $P_x^a \cap Q = \emptyset$ it cannot be that $1 \in (\pi \circ \rho)(P_x^a)$, so in particular $(\pi \circ \rho)(x) \neq 1$. Let $(\pi \circ \rho)(x) < \alpha < \beta < (\pi \circ \rho)(Q)$. Then $U = (\pi \circ \rho)^{-1}([0, \alpha])$ and $V = (\pi \circ \rho)^{-1}((\beta, 1])$ are connected open sets separating P_x^a and Q . Let $Y = \overline{U}$. Y is a continuum but as $x \in U$ and $x \in \overline{M}$ it must be that U , and Y , intersect M non-trivially. Let $y \in Y \cap M$. There exists an element P_y^a of $\mathcal{P}_y^a$ with $P_y^a \subseteq Y$. Now $Y \cap Q = \emptyset$ which contradicts that $y \in M$. Thus it must be that $M = \overline{M}$ i.e. M is compact. $\square$

Having established one unusual equivalence class, the remaining classes will all be much simpler. The complement of M consists of countably many sets each of which can be easily handled by Proposition 8.2.13. In order to do this we will introduce two new types of subcontinua of X and explore their relation with M and the sets M_a .

Lemma 8.2.17. *Let X be an ∞ -irreducible continuum and $A \subseteq X$ a subset with $X = \text{irr}(A)$. Let $a \in A$ and let C be a proper subcontinuum of X containing $A \setminus \{a\}$. Then $X \setminus C$ is connected.*

Proof. Let $\rho : X \mapsto X/C$ be the natural quotient map to the quotient space X/C . By Lemma 7.2.17 we have that $X/C = \text{irr}(\rho(C), \rho(a))$. By Proposition 2.2.4 we have that $X/C \setminus \rho(C)$ is connected, and as ρ is monotone it follows that $X \setminus C$ is connected. $\square$

Lemma 8.2.18. *Let X be an ∞ -irreducible continuum and $A \subseteq X$ a subset with $X = \text{irr}(A)$. Let $a \in A$ and let C and D be subcontinua of X with C and D both irreducible about $A \setminus \{a\}$. Then $\overline{X \setminus C} = \overline{X \setminus D}$.*

Proof. If $C = D$ then this result is trivial. If they are not equal then as both of them are irreducible about $A \setminus \{a\}$ it must be that $C \not\subseteq D$ and $D \not\subseteq C$. From this we have that $D \setminus C \neq \emptyset$, which implies that $D \cap (X \setminus C) \neq \emptyset$. From Lemma 8.2.17 we have that $\overline{X \setminus C}$ is a continuum which means $D \cup \overline{X \setminus C}$ is a subcontinuum of X containing A . This means $X = D \cup \overline{X \setminus C}$ and in turn implies that $X \setminus D \subseteq \overline{X \setminus C}$. Taking the closure we see that $\overline{X \setminus D} \subseteq \overline{X \setminus C}$. By symmetry we have that the same result holds with C and D swapped, so $\overline{X \setminus C} = \overline{X \setminus D}$. $\square$

Lemma 8.2.19. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. For each $a \in A$ let $Y_a \subsetneq X$ be a proper subcontinuum such that $Y_a = \text{irr}(A \setminus \{a\})$ and let $Z_a = \overline{X \setminus Y_a}$. Then for all $x \in Y_a \cap Z_a$, $Z_a = \text{irr}(x, a)$ and $X \setminus M_a = \kappa_{Z_a}(a)$.*

Proof. Let $x \in Y_a \cap Z_a$ and let $Z' \subseteq Z_a$ be a subcontinuum containing x and a . Then $x \in Y_a \cap Z'$ so $Y_a \cup Z'$ is a continuum containing A , which means it must be X . It follows that $X \setminus Y_a \subseteq Z'$ and since Z' is compact $Z_a \subseteq Z'$. Thus $Z_a = Z'$ which gives us that $Z_a = \text{irr}(x, a)$.

We need to show that Z_a is almost hereditarily decomposable. Let $C \subseteq Z_a$ be indecomposable and let $V \subseteq X$ be X -open such that $V \cap Z_a \subseteq C$. Then $V \cap (X \setminus Y_a)$ is an X -open set with $V \cap (X \setminus Y_a) \subseteq V \cap Z_a \subseteq C$. Since X is almost hereditarily decomposable it must be that C has an empty X -interior, so $V \cap (X \setminus Y_a) = \emptyset$. This implies that $V \cap Z_a$ is a Z_a -open set which does not intersect the Z_a -dense subset

$X \setminus Y_a$, meaning $V \cap Z_a = \emptyset$ and C has empty interior. Therefore Z_a is almost hereditarily decomposable.

Since Z_a is almost hereditarily decomposable and irreducible we can apply Theorem 2.4.11, so let $\pi_a : Z_a \mapsto \mathbb{I}$ be the universal monotone map onto the unit interval with $\pi(a) = 0, \pi(Y_a \cap Z_a) = 1$. Let $x \in Y_a$. Then Y_a is an element of $\mathcal{Q}_x^a$ which clearly implies that $x \in M_a$.

Now let $x \in \pi^{-1}(1)$. We have that $Z_a \in \mathcal{P}_x^a$. Given any $Q \subseteq Q_x^a \in \mathcal{Q}_x^a$ containing $A \setminus \{a\}$ we have that if $Q \cap Z_a = \emptyset$ then $Q \subseteq Y_a$ so $Q = Y_a$ which in turn implies $Q \cap Z_a \neq \emptyset$. From this contradiction we infer that $x \in M_a$, so $X \setminus M_a \subseteq \kappa_{Z_a}(p_a)$. For the other inclusion let $\pi_a(x) = \alpha < 1$. This gives us that there exists $P_x^a \in \mathcal{P}_x^a$ with $P_x^a \subseteq \pi_a^{-1}[0, \alpha]$ so $Y_a \subseteq Q_x^a$ and $Y_a \cap P_x^a = \emptyset$. This gives us that $x \in X \setminus M_a$ which completes the proof. $\square$

Note that in the previous lemma the choice of Y_a was unimportant, as the subcontinuum Z_a was the true focus and Lemma 8.2.18 showed that Z_a is well defined.

Remark 8.2.20. Each M_a must be closed, as $\kappa_{Z_a}(p_a)$ is an open subset of Z_a and is contained in $X \setminus Y_a \subseteq Z_a$, an open subset of X . This gives an alternative proof that M is compact.

Corollary 8.2.21. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. Let Y_a and Z_a be as in Lemma 8.2.19 for some $a \in A$. If $x \in Z_a \setminus \kappa_{Z_a}(p_a)$ then $x \in M$.*

Proof. We know that $x \in M_a$ by Lemma 8.2.19. Consider $\sigma : X \mapsto X/Y_a$ and note that $X/Y_a = \text{irr}(\sigma(a), \sigma(Y_a))$ by Lemma 3.2.21. For any $b \in A \setminus \{a\}$ and any $Q \subseteq Q_x^b \in \mathcal{Q}_x^b$ with $A \setminus \{b\} \subseteq Q$ we have that $\sigma(Q) = X/Y_a$. This implies $\kappa_{Z_a}(a) \subseteq Q$ and as Q is closed, $Z_a \subseteq Q$. Thus $x \in Q$ and $Q \cap P_x^b \neq \emptyset$. Therefore $x \in M_b$, and as b was arbitrary it follows that $x \in M$. $\square$

Corollary 8.2.22. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. Let $\sim$ be as in Definition 8.2.8, let Y_a, Z_a be as in Lemma 8.2.19 and let $x \in \kappa_{Z_a}(a)$. Then $x_\sim \subseteq \kappa_{Z_a}(a)$ and $x_\sim$ is a continuum. Indeed if $\pi_a : Z_a \mapsto \mathbb{I}$ is the monotone map from Theorem 2.4.11 then $x_\sim$ is a fibre of π_a .*

Proof. From Lemma 8.2.19 we have that $x \notin M_a$ which implies that $Q_x^a \neq \text{irr}(A \setminus \{a\})$. Taking $\pi_a : Z_a \mapsto [0, 1]$ from Theorem 2.4.11 with $\pi_a(a) = 0$, and let $\pi_a(x) = \alpha < 1$. Then there exists P_x^a in $\mathcal{P}_x^a$ lying in $\pi_a^{-1}([0, \alpha])$, so $P_x^a \cap Y_a = \emptyset$. This in turn implies $Y_a \subsetneq Q_x^a$.

By applying Proposition 8.2.13 we have that $x_\sim$ is homeomorphic to the equivalence class of the image of x in X/Y_a , which will be a continuum. Let $\rho : X \mapsto X/Y_a$. The restriction of ρ to Z_a is a map which only affects one of the E-continua of Z_a , so the equivalence classes in $\kappa_{X/Y_a}(a)$ will be homeomorphic to fibres of π_a . This means that $x_\sim$ is a fibre of π_a and lies in $\kappa_{Z_a}(a)$. $\square$

We have now found all of the equivalence classes, namely M and the equivalence classes of the Z_a lying in $\kappa_{Z_a}(a)$ corresponding to an application of Theorem 2.4.11 to Z_a . We know that all of the equivalence classes are compact and with the exception of M we know they are all connected. The following lemma will be used to show that M is also connected.

Lemma 8.2.23. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. Let Z_a be as defined in Lemma 8.2.19. For $a \neq b$ we have that $\kappa_{Z_a}(a) \cap \overline{\kappa_{Z_b}(b)} = \emptyset$.*

Proof. The proof is simply the manipulation of various sets. Note that $\overline{\kappa_{Z_b}(b)} = Z_b$.

$$\begin{aligned} Y_a \subseteq M_a &\Rightarrow X = Y_a \cup Y_b \subseteq M_a \cup M_b \\ &\Rightarrow \kappa_{Z_b}(b) = Z_b \setminus M_b \subseteq M_a \\ &\Rightarrow Z_b \subseteq M_a \\ &\Rightarrow M_a \cup (X \setminus Z_b) = X \\ &\Rightarrow (X \setminus M_a) \cap Z_b = \emptyset \\ &\Rightarrow \kappa_{Z_a}(a) \cap \overline{\kappa_{Z_b}(b)} = \emptyset \end{aligned}$$

$\square$

Proposition 8.2.24. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. Let M be as in Definition 8.2.14. Then M is connected.*

Proof. Suppose that $M = K \cup L$ for disjoint closed K and L . Each Z_a has an E-continuum as its intersection with M so each intersection must lie in K or in L . Define $K' = K \cup \bigcup \{\kappa_{Z_a}(a) \mid Z_a \cap K \neq \emptyset\}$ and L' similarly. Now $X = K' \cup L'$ and the two sets are disjoint.

Suppose K' were not closed. There exists a sequence $(a_n) \subset A$ such that for all n we have $Z_{a_n} \subseteq K'$ and that each Z_{a_n} contains a point x_n forming a sequence whose limit lies in L' . By Proposition 8.1.2 the sequence (Z_{a_n}) has a convergent subsequence (Y_k) and by Proposition 8.1.3 the limit of this subsequence is a continuum. Call it Z . Since each Z_{a_n} intersects K so does Z . There is a subsequence of (x_n) corresponding to (Y_k) so the limit of this subsequence lies in Z , meaning $Z \cap L' \neq \emptyset$. Moreover $L' \setminus L$ is an open set as the union of open $\kappa_{Z_a}(a)$ and $L' \setminus L$ does not intersect any of

the Z_{a_n} so $Z \cap L \neq \emptyset$. This also implies that $Z \subseteq M$. Thus we have that K and L disconnect Z which is a contradiction.

Thus it must be that K' is closed and by symmetry so is L' . As X is connected one of K' and L' must be empty, implying one of K and L are as well. Therefore M is connected. $\square$

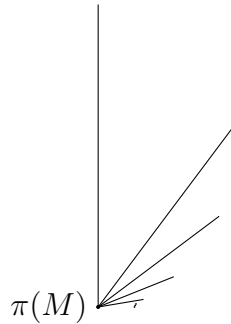
For the following theorem recall that we are using the word fan to refer to dendrites with a single branch point and countably infinitely many end points.

Theorem 8.2.25. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum. There exists a monotone map $\pi : X \mapsto F$ where F is a fan.*

Proof. Let π be the quotient map $X \mapsto X/\sim$, with $\sim$ as defined in Definition 8.2.8. Then $F := X/\sim$ is the union of $|A|$ arcs each intersecting at a single point $\pi(M)$ and nowhere else. Thus F is a fan if and only if it is a dendrite. We will use Theorem 8.1.5 to show this. Let $x_\sim$ and $y_\sim$ be distinct points of F . They cannot both be $\pi(M)$ so without loss of generality $x_\sim$ lies in one of the arcs, call it K . If both $x_\sim$ and $y_\sim$ lie in K then suppose without loss of generality that $x_\sim$ lies closer to the image of the E-continuum of X . Take a point $z_\sim$ in K which is not an end point and separates $x_\sim$ from $y_\sim$ if $y_\sim$ lies in K , or otherwise separates $x_\sim$ from the end point of K which is not the image of an E-continuum.

Let U be the component of $K \setminus \{z_\sim\}$ containing $x_\sim$. Then U is an open subset of K and $\pi^{-1}(U)$ is an open subset of $X \setminus Y_a$ for some $a \in A$. As $X \setminus Y_a$ is an X -open subset, we have that $\pi^{-1}(U)$ is also X -open, and therefore U is F -open. This implies that U is an $(F \setminus \{z_\sim\})$ -clopen subset which contains $x_\sim$ but not $y_\sim$, so $x_\sim$ and $y_\sim$ are separated in F by $z_\sim$. Thus by Theorem 8.1.5 F is a dendrite, and therefore a fan. $\square$

The following diagram depicts an example of F .



We will end this section by proving another analogue to the result of Miller.

Theorem 8.2.26. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum and let π be as in Theorem 8.2.25. X is unicoherent if each fibre of π is unicoherent.*

Proof. Let $X = K \cup L$ for a pair of proper subcontinua K and L and let m be the branch point of the fan F . Suppose one of these continua, say K , does not intersect $M := \pi^{-1}(m)$. Then $\pi(K)$ must lie in one of the stalks of F . Let F' be the subcontinuum of F consisting of every other stalk and let $Y = \pi^{-1}(F')$. Consider the monotone map $\rho : X \mapsto X/Y$. By Lemma 7.2.17 X/Y is 2-irreducible and is almost hereditarily decomposable by Lemma 3.2.20. Let $\pi' : X/Y \mapsto \mathbb{I}$ be the universal monotone map corresponding to an application of Theorem 2.4.11; its fibres are the same as for π , with the exception of the E-continuum containing $\rho(Y)$. As K does not intersect Y this is enough to apply Theorem 2.5.10 to give us that $\rho(K) \cap \rho(L)$ is connected. This is homeomorphic to $K \cap L$ as seen by ρ , so $K \cap L$ is connected.

Now suppose both K and L intersect $\pi^{-1}(m)$. We will first show that $K \cap M$ is connected. Write $K \cap M = D \cup E$ for disjoint compact D and E . By considering the stalks of F we can see that $K \setminus M$ is the union of open, connected sets U_α , $\alpha \in I$ for $I \subseteq \omega$. For any β , $\overline{U_\beta}$ is irreducible with $\overline{U_\beta} \setminus U_\beta \subseteq M$ as one of its E-continua. Each $\overline{U_\beta}$ must intersect one of D or E and be disjoint from the other. Define $D' = D \cup \bigcup \{\overline{U_\beta} \mid \overline{U_\beta} \cap D \neq \emptyset\}$ and define E' similarly. D' is closed as its closure would consist of itself and the limit of the $\{\overline{U_\beta} \mid \overline{U_\beta} \cap D \neq \emptyset\}$, which would be a continuum in $K \cap M$ intersecting D so would lie in D . Similarly E' is closed and as $K = D' \cup E'$ it must be that one of them, and one of D and E , is empty. Thus $K \cap M$ is connected, as is $L \cap M$.

Now consider $K \cap L$, assuming both K and L intersect M . Since $M = (K \cap M) \cup (L \cap M)$ and M is unicoherent we have that $K \cap L \cap M$ is connected. The inverse image of each stalk of F , not including the point m , lies in either K or L (or both, and can intersect both). Say one, X' , lay in K . Let the continuum Y be defined as $X \setminus X'$ and consider $\rho : X \mapsto X/Y$. Then $L \cap X'$ is homeomorphic to $\rho(L) \setminus \rho(Y)$ which is connected. Furthermore, $\overline{L \cap X'} \subseteq (L \cap X') \cup (K \cap L \cap M)$. From here it is clear that $K \cap L$ is connected as it consists of $K \cap L \cap M$ and a collection of connected sets each limiting onto it. $\square$

8.3 Monotone Maps Onto Dendrites

In Section 4.2 one of the problems with mapping onto an n -od was that a lot of the structure would be lost, specifically the equivalence class M being much larger than the rest and all of its internal structure being reduced to a point. This lead us to investigate mapping onto trees instead to preserve this structure. The problem is even more pronounced for ∞ -irreducible continua. We know from Proposition 7.2.7 that if a continuum X is irreducible about an infinite set A then A is countable. The fan defined in Theorem 8.2.25 is characterised by the number of end points it has, which will be one for each point $a \in A$. This means that the maps constructed in Theorem 8.2.25 all have the same image, regardless of which ∞ -irreducible continuum it is applied to. This is clearly a huge loss of information, so we will again construct a different equivalence relation whose corresponding quotient map preserves more of the structure of our initial space. In Section 4.3 we used trees so here we will use dendrites.

Definition 8.3.1. A continuum C is a dendrite if it is a Peano continuum containing no simple closed curves.

Definition 8.3.2. Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. For two points $x, y \in X$ we have that $x \sim y$ if and only if $x, y \in \lambda(a)$ for some $a \in A$ or for all non-empty $B \subsetneq A$ and for some $P_x^B \in \mathcal{P}_x^B, Q_x^B \in \mathcal{Q}_x^B, P_y^B \in \mathcal{P}_y^B$ and $Q_y^B \in \mathcal{Q}_y^B$ we have $P_x^B \cup Q_y^B = X = P_y^B \cup Q_x^B$.

The difference between this and Definition 8.2.8 is that every subset of A will be considered, not just the singletons. In the following example, the central line will be characterised under this equivalence relation when you take B to be the points of irreducibility on the left. Under Definition 8.2.8 this line could not be so characterised and would be contained in M .

Example 8.3.3. The continuum below is irreducible about the set of end points of the two fans.



We will start similarly to how we did for maps to fans, with a number of key results about the relation $\sim$. Some of the proofs will be very similar to their counterparts from the previous section.

Proposition 8.3.4. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. Let $\sim$ be as defined in Definition 8.3.2. This relation does not depend on which elements of $\mathcal{P}_x^B, \mathcal{Q}_x^B, \mathcal{P}_y^B$ or $\mathcal{Q}_y^B$ are chosen.*

Proof. Suppose P_x^B and $\tilde{P}_x^B$ are two different subcontinua of X , each elements of $\mathcal{P}_x^B$. As both are minimal about $B \cup \{x\}$ and they are not equal to each other, we have that $P_x^B \not\subseteq \tilde{P}_x^B \not\subseteq P_x^B$. Let $y \in X$ and $Q_y^B \in \mathcal{Q}_y^B$ be such that $P_x^B \cup Q_y^B = X$. Then $\emptyset \neq \tilde{P}_x^B \setminus P_x^B \subseteq Q_y^B$ which means $\tilde{P}_x^B \cap Q_y^B \neq \emptyset$ and $\tilde{P}_x^B \cup Q_y^B = X$. An identical argument gives the same result for $Q_x^B, \tilde{Q}_x^B$ and from this we can conclude that the choices of P_x^B and P_y^B do not determine whether or not $x \sim y$. $\square$

Theorem 8.3.5. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. The relation $\sim$, as defined in Definition 8.3.2, is an equivalence relation.*

Proof. It is clear to see that $\sim$ is reflexive and symmetric. Suppose $x \sim y$ and $y \sim z$ for three distinct points $x, y, z \in X$. If any one of them lies in $\lambda(a)$ for some $a \in A$ then all three must do and $x \sim z$. Suppose this is not the case.

Let $B \subseteq A$ be a proper non-empty subset and let $P_x^B \in \mathcal{P}_x^B$ and $Q_z^B \in \mathcal{Q}_z^B$. We need to show that $P_x^B \cup Q_z^B = X$ as the symmetry of the theorem means we could then swap x and z to have our result. If $z \in P_x^B$ or $x \in Q_z^B$ for any elements of $\mathcal{P}_x^B$ and $\mathcal{Q}_z^B$ then this is the case for that pair so let us further assume that neither of these occur. Since $x \sim y$ and $y \sim z$ we know that $P_y^B \cup Q_x^B = X = P_z^B \cup Q_y^B$ so we have that $z \in P_y^B$ and $x \in Q_y^B$.

First suppose that y is B -sided. If $P_y^B = \text{irr}(B)$ then $P_y^B \in \mathcal{P}_x^B$ which by applying Proposition 8.2.9 gives us that $P_x^B \cup Q_z^B = X$. If $P_y^B \neq \text{irr}(B)$ then by Proposition 8.2.3 we have that $P_y^B = \text{irr}(B \cup \{y\})$. Since $X = P_x^B \cup Q_y^B$ we have that

$$\emptyset \neq P_x^B \cap Q_y^B \subseteq P_y^B \cap Q_y^B \subseteq \lambda_{P_y^B}(B)$$

Therefore $P_x^B = P_y^B$ which gives us that $P_x^B \cup Q_z^B = X$.

If y is not B -sided it must be $A \setminus B$ -sided and if $Q_y^B = \text{irr}(\{y\} \cup A \setminus B)$ then use the same argument as for y being B -sided. If $Q_y^B = \text{irr}(A \setminus B)$ then as $z \in Q_y^B$ we again have that Q_y^B is an element of $\mathcal{Q}_z^B$. Proposition 8.3.4 and the fact that $P_x^B \cup Q_y^B = X$ give us that $P_x^B \cup Q_z^B = X$. $\square$

Definition 8.3.6. The equivalence class of a point x will be denoted by $x_\sim$.

The following lemma and other results like it were not necessary for finite irreducibility because it was immediate that the union of finitely many closed sets is closed. We will now have to deal with infinite collections of closed sets more frequently because our continuum X is not finitely irreducible but ∞ -irreducible.

Lemma 8.3.7. *Let K_n be a sequence of sets converging to a set K . Let L be a closed set. Then $K_n \cup L$ converges to $K \cup L$.*

Proof. First take a point x in $\lim(K_n \cup L)$. If x lies in L then it clearly lies in $K \cup L$. If it does not lie in L then take any open set $U \ni x$. $U \setminus L$ is also open and contains x so must intersect each element of a tail of $(K_n \cup L)$. As $U \setminus L$ cannot intersect L it must intersect each element of a tail of the sequence (K_n) , implying $x \in K$.

Now let $x \in K \cup L$. If $x \in L$ then every open set containing it will intersect L and therefore a tail of $(K_n \cup L)$. Otherwise $x \in K$ and every open set around x intersects a tail of (K_n) and therefore a tail of $(K_n \cup L)$. Thus $x \in \lim K_n \cup L$. $\square$

Proposition 8.3.8. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. Let $\sim$ be as defined in Definition 8.3.2. For any $x \in X$, $x_\sim$ is a closed set.*

Proof. If $x \in \lambda(a)$ for any $a \in A$ then $x_\sim = \lambda(a)$, which is compact by Proposition 8.2.1. Otherwise, let $y \in \overline{x_\sim}$ and let x_k be a sequence of points in $x_\sim$ converging to y . We will show that $y \in x_\sim$. Let $B \subsetneq A$. We only need to show that $P_y^B \cup Q_{x'}^B = X$ for some $x' \in x_\sim$ and some $P_y^B \in \mathcal{P}_y^B$ and $Q_{x'}^B \in \mathcal{Q}_{x'}^B$, and the symmetry of the problem along with the fact that $\sim$ is transitive will give us our proof. If for any $x' \in x_\sim$ and $Q_{x'}^B \in \mathcal{Q}_{x'}^B$ we have $y \in Q_{x'}^B$ then $P_y^B \cup Q_{x'}^B = X$ so we shall consider the case where for all $x' \in x_\sim$ and all $Q_{x'}^B \in \mathcal{Q}_{x'}^B$, $y \notin Q_{x'}^B$.

We will first inductively define a subsequence (y_n) of (x_k) and corresponding $P_{y_n}^B$. Define $y_1 = x_1$ and suppose $y_1, \dots, y_n$ have been defined with $y_n = x_m$. Since $y \in X \setminus Q_{y_n}^B \subseteq P_{y_n}^B$ and as $X \setminus Q_{y_n}^B$ is an open set there must be a tail of (x_k) lying in $P_{y_n}^B$. Pick one of these points with index greater than m to be y_{n+1} . There exists an element $P_{y_{n+1}}^B \in \mathcal{P}_{y_{n+1}}^B$ with $P_{y_{n+1}}^B \subseteq P_{y_n}^B$. Thus we have a new sequence (y_n) in $x_\sim$ converging to y and a nested sequence of continua $(P_{y_n}^B)$.

By Proposition 8.1.2 there exists a convergent subsequence of $(P_{y_n}^B)$ and by Proposition 8.1.3 it converges to a continuum. Without loss of generality we can assume that the sequence itself is convergent and call the limit P . For any $x' \in x_\sim$ and any

n we have that $x' \sim y_n$ and therefore that $P_{y_n}^B \cup Q_{x'}^B = X$. By applying Lemma 8.3.7 we have that $P \cup Q_{x'}^B = X$.

If $P \in \mathcal{P}_y^B$ then clearly we are done. If this is not the case then as $y \in P$ this means we can assume $P \neq \min(B)$ and by Lemmas 7.3.3 and 7.3.5 we can take a proper subcontinuum $C \subsetneq P$ such that $C = \text{irr}(B)$. Let $\rho : P_{y_1}^B \mapsto P_{y_1}^B/C$ be the natural quotient map and call the image D . Now $D = \text{irr}(\rho(C), \rho(y_1))$ by Lemma 7.2.17 and is almost hereditarily decomposable by Lemma 3.2.20, so let $\pi : D \mapsto [0, 1]$ be the universal monotone map attained by applying Theorem 2.4.11. Define α_n so that $\pi \circ \rho(P_{y_n}^B) = [0, \alpha_n]$. Let $\alpha = \pi \circ \rho(y)$, so (α_n) converges to α .

It is clear that $\pi \circ \rho(P) = \pi \circ \rho(P_y^B) = [0, \alpha]$ and from this we can deduce that $\text{int}(\rho(P) \setminus \rho(P_y^B)) = \emptyset$. As $C \neq P$ we have that $\pi \circ \rho(P) \neq 0$ so $\rho(P) \setminus \rho(P_y^B)$ is homeomorphic to $P \setminus P_y^B$ i.e. $P \setminus P_y^B$ also has empty interior. Now for any $x' \in x_\sim$

$$X \setminus (P_y^B \cup Q_{x'}^B) = (P \cup Q_{x'}^B) \setminus (P_y^B \cup Q_{x'}^B) \subseteq P \setminus P_y^B$$

As the left hand side is an open set and the right hand side has empty interior it must be that $X \setminus (P_y^B \cup Q_{x'}^B) = \emptyset$ or rather $X = P_y^B \cup Q_{x'}^B$. This completes our proof. $\square$

Corollary 8.3.9. *The set $x_\sim$ is compact.*

Our intention is to construct a monotone map out of the equivalence relation $\sim$. In order to do this we must obviously prove that each equivalence class is connected, which we will do with the help of the following lemma.

Lemma 8.3.10. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. Let $\sim$ be as defined in Definition 8.3.2. Let $x, y \in C$ where C is a subcontinuum of X with empty interior. Then $x \sim y$.*

Proof. For any $B \subseteq A$ and any $P_x^B \in \mathcal{P}_x^B$ and $Q_y^B \in \mathcal{Q}_y^B$ we have that $P_x^B \cup C \cup Q_y^B = X$ as it is a continuum containing the whole of A . Thus $X \setminus (P_x^B \cup Q_y^B) \subseteq C$. This is an open set and C has empty interior, so $X = P_x^B \cup Q_y^B$. The same is true if x and y are swapped around and as the choice of B was arbitrary this implies that $x \sim y$. $\square$

Proposition 8.3.11. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. Let $\sim$ be as defined in Definition 8.3.2. Each equivalence class $x_\sim$ is connected.*

Proof. Let $x, y \in X$ such that $x \sim y$. If either is a point of irreducibility of X then $x_\sim$ is an E-continuum so is connected. Otherwise by applying Theorem 2.2.5 consider a subcontinuum C of X such that $C = \text{irr}(x, y)$. We will show that $C \subseteq x_\sim$ to prove

that $x_\sim$ is continuumwise connected, and therefore connected. To do this, take a point $z \in C$ and a proper non-empty subset $B \subsetneq A$. We will show that for some $P_x^B \in \mathcal{P}_x^B$ and $Q_z^B \in \mathcal{Q}_z^B$ we have $P_x^B \cup Q_z^B = X$, by the symmetry of the problem the same argument will indicate that $P_z^B \cup Q_x^B = X$ and therefore $z \in x_\sim$.

Clearly if for any $P_x^B \in \mathcal{P}_x^B$ we have $z \in P_x^B$ then our argument is complete so suppose this is never the case. Then as $x \sim y$ we have that $X = P_x^B \cup Q_y^B$ for any choices of P_x^B and Q_y^B , so $z \in Q_y^B$. Again if $Q_y^B \in \mathcal{Q}_z^B$ then we are done so assume this is also not the case.

If C is indecomposable then it must have empty interior so by Lemma 8.3.10 we would have that $x \sim z$. If $z \in \lambda_C(y)$ and $\lambda_C(y)$ is compact then again by Lemma 8.3.10 $y \sim z$ and we are done, and if $\lambda_C(y)$ is not compact then by Theorem 8.1.10 it lies in an indecomposable continuum, so once more Lemma 8.3.10 would give $z \in x_\sim$. We can therefore assume $C \neq \text{irr}(x, z)$ and by the same arguments can assume $C \neq \text{irr}(y, z)$.

Apply Theorem 2.2.5 twice to get proper subcontinua $D, E \subseteq C$ with $D = \text{irr}(x, z)$ and $E = \text{irr}(y, z)$. Let $P_x^B \in \mathcal{P}_x^B$ and $Q_z^B \in \mathcal{Q}_z^B$ with $Q_z^B \subseteq Q_y^B$ for some $Q_y^B \in \mathcal{Q}_y^B$. Let $N = P_x^B \cup Q_z^B$, which we are assuming is a proper subset of X . Let $\rho : X \mapsto X/N$ be the natural quotient map with two non-degenerate fibres, P_x^B and Q_z^B . Note that ρ is a monotone map. From Lemma 7.2.17 we know that $X/N = \text{irr}(\rho(P_x^B), \rho(Q_z^B))$ and by applying Lemma 3.2.20 twice we have that X/N is almost hereditarily decomposable. Thus we have from Theorem 2.4.11 that there exists monotone $\pi : X/N \mapsto [0, 1]$. Let $K = \overline{(\pi \circ \rho)^{-1}((0, 1))}$. Then K is a continuum and if $p \in K \cap P_x^B$ and $q \in K \cap Q_z^B$ then $K = \text{irr}(p, q)$. This is because if $K' \subseteq K$ contained p and q then $\pi \circ \rho(K')$ would be the whole of $[0, 1]$ so $(\pi \circ \rho)^{-1}((0, 1)) \subseteq K'$.

We can see from the irreducibility of X about A that $X = P_x^B \cup D \cup Q_z^B$. As Q_z^B is a proper subcontinuum of Q_y^B we can deduce from this that $y \in P_x^B$. We also have that $\rho(D) = \rho(E) = X/N$ by irreducibility, and from this it is clear that $K \subseteq D$ and $K \subseteq E$. Let L_x be the component of x in $D \cap P_x^B$ and let L_y be the component of y in $E \cap P_x^B$. The intersection $L_x \cap K$ must be non-empty or L_x would be a component of D , which cannot be the case. This intersection is in $K \cap P_x^B$ which is itself a subset of $\lambda_K(p)$. The same is true of L_y . Consider the continuum $L_x \cup \overline{\lambda_K(p)} \cup L_y$. This is a proper subcontinuum of C as it does not contain z , but it contains both x and y which contradicts that $C = \text{irr}(x, y)$. This contradiction indicates that one of the earlier cases must occur, each of which implied that $z \in x_\sim$. Thus $C \subseteq x_\sim$, so $x_\sim$ is continuumwise connected and therefore connected. $\square$

Theorem 8.3.12. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. Let $\sim$ be as defined in Definition 8.3.2. Let $\pi : X \mapsto X/\sim$. Then $X/\sim$ is a dendrite.*

Proof. We must prove that $X/\sim$ is Hausdorff to show that it is a continuum, by Lemma 8.1.1, then prove that it is a dendrite via Theorem 8.1.5. Finding a cut point will satisfy both of these requirements as the resulting disjoint open sets will be the unions of equivalence classes. Take $x_\sim$ and $y_\sim$ distinct. If either of these are E-continua of X , say $x_\sim$, then let Y be a proper subcontinuum of X irreducible about $A \setminus \lambda(x)$, by Lemmas 7.3.3 and 7.3.5. From Lemma 8.2.19 we can see that the continuum $\overline{X \setminus Y}$ is irreducible between x and any point of Y it contains and, with the exception of the E-continua $\overline{X \setminus Y} \setminus \kappa(x)$, it has the same equivalence classes for the map from Theorem 2.4.11. As the image of this map is an arc it is clear that one of these classes separates $x_\sim$ from either Y or $y_\sim$ in $\overline{X \setminus Y}$ and therefore separates $x_\sim$ from $y_\sim$ in X .

We will now suppose that neither $x_\sim$ nor $y_\sim$ are E-continua of X . There must exist a proper non-trivial subset $B \subsetneq A$ and subcontinua $P_x^B \in \mathcal{P}_x^B$ and $Q_y^B \in \mathcal{Q}_y^B$ such that $P_x^B \cap Q_y^B = \emptyset$. Let ρ be a quotient map from X to a continuum Y with the only non-degenerate fibres being P_x^B and Q_y^B . Note that if the images under ρ of two sets intersect in Y at a point other than $\rho(P_x^B)$ or $\rho(Q_y^B)$ then the two sets intersect in X . By Lemma 3.2.20 Y is almost hereditarily decomposable and by Lemma 7.2.17 we have that $Y = \text{irr}(\rho(P_x^B), \rho(Q_y^B))$. Thus from Theorem 2.4.11 we have that there is a monotone map $\pi_Y : Y \mapsto \mathbb{I}$.

We will show that with the exception of the E-continua, the fibres of π_Y are the images under ρ of equivalence classes of $\sim$. Let c and d lie in the same fibre of π_Y , so if C_x is a subcontinuum irreducible between $\rho(P_x^B)$ and c , and D_y is a subcontinuum irreducible between $\rho(Q_y^B)$ and d then $C_x \cap D_y \neq \emptyset$, and the same for C_y and D_x . Let $B' \subsetneq A$ and let P and Q be sets as in Lemma 7.3.3 for $B' \cup \{\rho^{-1}(c)\}$ and $B' \cup \{\rho^{-1}(d)\}$ respectively. Then if $B' \neq B$ or $A \setminus B$ it is clear that one of $\rho(P)$ or $\rho(Q)$ is the whole of Y , so contains c and d and thus $P \cap Q \neq \emptyset$. If $B' = B$ then we can find $C_x \subseteq \rho(P)$ and $D_y \subseteq \rho(Q)$ so again $P \cap Q \neq \emptyset$. An identical argument can be applied if $B' = A \setminus B$, but with $D_x \subseteq \rho(P)$ and $C_y \subseteq \rho(Q)$. Thus if $c = \rho(c')$ and $d = \rho(d')$ then $c' \sim d'$.

Now suppose c' and d' are two points of X whose images $c = \rho(c')$ and $d = \rho(d')$ do not lie in an E-continuum of Y , and $c' \sim d'$. Let C_x, C_y, D_x and D_y be as in the last paragraph. There exists a choice of $P_{c'}^B \in \mathcal{P}_{c'}^B$ which is a subcontinuum of $\rho^{-1}(C_x)$ and $Q_{d'}^B \subseteq \rho^{-1}(D_y)$. As $c' \sim d'$ it must be that these intersect. As c and d do not lie in an E-continuum of Y we have that C_x and D_y both contain one E-continuum and do not intersect the other, so $\rho^{-1}(C_x) \cap \rho^{-1}(D_y)$ does not lie in $P_x^B \cup Q_y^B$. It follows

from this that $C_x \cap D_y \neq \emptyset$ and similarly $C_y \cap D_x \neq \emptyset$. Thus c and d lie in the same fibre of π_Y , and the fibres of π_Y other than the E-continua are the homeomorphic images of equivalence classes of $\sim$.

Every point of an arc other than the end points is a cut point, so by applying Lemma 4.3.12 every fibre of π_Y except the E-continua is separating in Y . This implies that its preimage under ρ is separating in X by the same lemma. This preimage is an equivalence class $z_\sim$. By Proposition 8.3.11 we have that each equivalence class is connected so if $X \setminus z_\sim = U \cup V$ then both U and V are the unions of whole equivalence classes and without loss of generality $x_\sim \subseteq U$ and $y_\sim \subseteq V$. From this we have that $X/\sim$ is Hausdorff so by Lemma 8.1.1 it is a continuum. We have shown that $\pi(z_\sim)$ disconnects $\pi(x_\sim)$ and $\pi(y_\sim)$ in $X/\sim$ so by Theorem 8.1.5 we have that $X/\sim$ is a dendrite. $\square$

Corollary 8.3.13. *The map π is monotone and X is locally connected about each $x_\sim$.*

Proof. That π is monotone is immediate from Proposition 8.3.11 and that X is locally connected about the fibres follows from Proposition 2.5.1. $\square$

Here we reach a divergence from Section 4.3 on maps to trees. While in that instance each equivalence class has an empty interior, the same is not true for ∞ -irreducible continua, in part because an infinite collection of components in the complement of an equivalence class can intersect that class in enough places that no proper subcontinua can avoid them. The next example shows such an equivalence class.

Example 8.3.14. Let Y be the continuum constructed in Example 2.5.8, and consider $\pi_Y : Y \mapsto \mathbb{I}$, the universal monotone map onto the unit interval. For each point $k/2^n$ there is a vertical line in Y mapped to $k/2^n$. Define X by attaching a line of length $1/2$ to an end point of the vertical line of Y mapped to $1/2$, lines of length $1/4$ to end points of vertical lines mapping to $1/4$ and $3/4$, and so on. Choose the points of intersection to all be in the image of $[0, 1] \times \{0\}$ so they all lie "at the bottom" of Y . The continuum X is hereditarily decomposable as Y is hereditarily decomposable and adding these arcs to Y will not change this. This implies that X is also almost hereditarily decomposable.

We will now show that X is irreducible about the set of end points of the arcs we have attached to Y and that Y is an equivalence class of the resulting equivalence relation, as defined in Definition 8.3.2. Let $\rho : X \mapsto Y$ be the projection of each

attached line onto the point it is attached to. Let $X' \subseteq X$ contain the set A of end points of these attached lines. Then $\pi_Y \circ \rho(X')$ clearly contains both end points of $\mathbb{I}$, so $\rho(X')$ contains a pair of points p and q such that $Y = \text{irr}(p, q)$. This in turn gives us that $\rho(X') = Y$ and it clearly follows that $X' = X$. It is immediate that for each $a \in A$ there is a proper subcontinuum $C_a \subsetneq X$ with $A \setminus \{a\} \subseteq C_a$; simply take Y and every attached line except for the one ending in a .

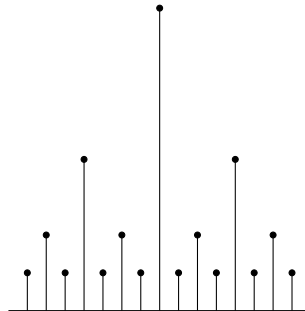
Since each attached line lies "at the bottom" of Y we have that $\overline{X \setminus Y}$ can only hit the parts of Y with pre-image in $[0, 1] \times \{0\}$. Therefore Y has non-empty interior in X . Let y_1, y_2 be distinct points of Y and let B be a proper non-empty subset of A . Then P_x^B contains B so $(\pi_Y \circ \rho)(P_x^B)$ must contain $\pi_Y \circ \rho(B)$ and $\overline{\pi_Y \circ \rho(B)}$. Similarly $(\pi_Y \circ \rho)(Q_y^B)$ must contain $\overline{\pi_Y \circ \rho(A \setminus B)}$. Thus $(\pi_Y \circ \rho)(P_x^B \cup Q_y^B)$ contains $\overline{\pi_Y \circ \rho(A)}$ and as $\pi_Y \circ \rho(A)$ is a dense subset of $\mathbb{I}$ we have that $\mathbb{I} = \pi_Y \circ \rho(P_x^B) \cup \pi_Y \circ \rho(Q_y^B)$. Since these are compact sets and $\mathbb{I}$ is connected there must exist $\alpha \in \pi_Y \circ \rho(P_x^B) \cap \pi_Y \circ \rho(Q_y^B)$. Thus both $\rho(P_x^B)$ and $\rho(Q_y^B)$ intersect $\pi_Y^{-1}(\alpha)$ and one of them must contain the whole fibre. This implies one of P_x^B or Q_y^B contains the whole fibre, and that these two continua intersect non-trivially. As this is true for every set $B \subseteq A$ it gives us that $x \sim y$. From this we know that Y lies in an equivalence class of $\sim$ thus proving that there is an equivalence class with non-empty interior.

We will now see another inconsistency between finitely and ∞ -irreducible continua. The map from a finitely irreducible continua to a tree was universal amongst monotone maps to trees, as we saw in Theorem 4.3.19. The same does not hold for ∞ -irreducible continua being mapped onto dendrites, as the following example shows. This example is a similar continuum to the one in Example 8.3.14.

Example 8.3.15. Let X be the continuum in $\mathbb{R}^2$ consisting of a horizontal line $[0, 1] \times \{0\}$ and vertical lines $\{k \cdot 2^{-n}\} \times [0, 2^{-n}]$ for each integer n and each positive odd $k < n$. Define a set A as

$$A = \{(k \cdot 2^{-n}, 2^{-n}) | n \in \omega, k < n \text{ positive, odd}\}$$

Thus A is the set of end points of the vertical lines.



Let f be the projection onto the first coordinate. If $Y \subseteq X$ is a subcontinuum containing A then Y must clearly contain each vertical line, and must contain $f(A)$. This is a dense subset of $[0, 1] \times \{0\}$ so $Y = X$. Thus $X = \text{irr}(A)$, and it is clear to see that in fact $X = \text{irr}(A)$.

Since X is hereditarily decomposable it is almost hereditarily decomposable. It is also locally connected. Given a point $x \in X$ and an open set $U \ni x$, if the second coordinate of x is non-zero then U can be shrunk to miss the horizontal line and every vertical line except the one containing x . It is then clear that U can be further shrunk to an open set about x which is connected. If x lies on the horizontal line then take $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. This set will be connected as if a vertical line intersects $B(x, \epsilon)$ then said intersection will include $(k \cdot 2^{-n}, 0)$.

Consider the relation $\sim$ on X . For each point x not on the horizontal line we have that $x_\sim = \{x\}$, witnessed by taking as B a singleton, specifically the end point of the vertical line containing x . Let $x = (\alpha, 0)$ and $y = (\beta, 0)$ be points of X and let $B \subsetneq A$ be a proper non-trivial subset. Then P_x^B contains B so must contain $f(B)$ and $\overline{f(B)}$. Similarly Q_y^B must contain $\overline{f(A \setminus B)}$. Thus $P_x^B \cup Q_y^B$ contains $\overline{f(A)}$ and as $f(A)$ is a dense subset of $[0, 1] \times \{0\}$ we have that the horizontal line lies in $P_x^B \cup Q_y^B$. It is clear that every vertical line must lie in $P_x^B \cup Q_y^B$ so $P_x^B \cup Q_y^B = X$. Thus $x \sim y$, and the horizontal line is the only non-degenerate equivalence class of $\sim$.

As has already been proved, X is locally connected. It does not contain any simple closed curves so X is a dendrite. If $\pi : X \mapsto X/\sim$ were universal amongst monotone maps to dendrites then there would exist a map $g : X/\sim \mapsto X$ such that $g \circ \pi$ is the identity map on X . This is because the identity is a monotone map from X to a dendrite, namely itself. Such a map g cannot exist however, as π is not injective. Therefore $\pi : X \mapsto X/\sim$ is not universal.

We end this section as we have done previously, with a result on unicoherence. Unlike the previous results this one does hold for ∞ -irreducible continua, although it will be harder to prove than in previous cases. This is because, unlike for trees, dendrites can have points which separate them into infinitely many components. Where previously we could partition such a set and know it was closed this is no longer possible. Instead, to prove it is closed we will have to consider the limits of the components, and show that such a limit will lead to contradictions if it does not stay in the right half of the partition.

Lemma 8.3.16. *Let X be an almost hereditarily decomposable, ∞ -irreducible continuum with $X = \text{irr}(A)$. Let $\sim$ be as defined in Definition 8.3.2. Let $\pi : X \mapsto X/\sim$.*

Let $Y \subseteq X$ be a subcontinuum. For each $x \in Y$ we have that $x_\sim \subseteq Y$ or $x_\sim$ is an end point of $\pi(Y)$ or a branch point of $X/\sim$.

Proof. Let $x \in Y$ be such that $x_\sim$ is not an end point of $\pi(Y)$ or a branch point of $X/\sim$. This implies that $x_\sim$ does not lie in $\overline{X/\sim \setminus \pi(Y)}$. Let $\{D_\alpha\}$ be the collection of components of $\overline{X/\sim \setminus \pi(Y)}$ and let $X_\alpha = \pi^{-1}(D_\alpha)$. By Theorem 8.1.4 each D_α intersects $\pi(Y)$ so it follows that each X_α intersects Y . Thus $Y \cup \pi^{-1}(\overline{X/\sim \setminus \pi(Y)})$ is a continuum which intersects every equivalence class, including all of the E-continua. This means that it must be the whole of X . As $x_\sim$ is not a branch or end point we have that $x_\sim \notin \overline{X/\sim \setminus \pi(Y)}$ so $x_\sim \cap X_\alpha = \emptyset$ for each α . Thus $x_\sim \subseteq Y$. This completes the proof. $\square$

Theorem 8.3.17. *If each equivalence class $x_\sim$ is unicoherent then so is X .*

Proof. Let K and L be proper subcontinua of X with $X = K \cup L$. Note that $X/\sim = \pi(K) \cup \pi(L)$ and that dendrites are unicoherent by Theorem 8.1.7 so

$$\begin{aligned} \pi(K) \cap \pi(L) \text{ is connected} &\Rightarrow \pi^{-1}(\pi(K) \cap \pi(L)) \text{ is connected} \\ &\Rightarrow \pi^{-1}(\pi(K)) \cap \pi^{-1}(\pi(L)) \text{ is connected} \end{aligned}$$

Now $\pi^{-1}(\pi(K)) = K \cup \{x_\sim | x_\sim \text{ is a branch or end point of } \pi(K)\}$, and a similar statement is true for L . We will show that $K \cap x_\sim$ is connected for each x , which will imply that $(K \cap x_\sim) \cup (L \cap x_\sim)$ is a decomposition and therefore $K \cap L \cap x_\sim$ is connected.

If $K \cap x_\sim = \emptyset$ or $x_\sim \subseteq K$ then clearly $K \cap x_\sim$ is connected. If neither of these hold then $x_\sim \in X/\sim$ is either an end point of $\pi(K)$ or a branch point of $X/\sim$ by Lemma 8.3.16. If it is an end point of $\pi(K)$ then consider $K' = \overline{\pi^{-1}(\pi(K) \setminus \{x_\sim\})}$. It is clear from the map π that $K' \cap x_\sim$ is an E-continuum so is connected. Suppose $K \cap x_\sim = M \cup N$ for disjoint compact M and N . Then $K' \cap x_\sim$ must lie entirely in one or the other so without loss of generality say $K' \cap x_\sim \subseteq M$. Then $(K' \cup M) \cup N$ is the union of K along with each equivalence class K intersects other than $x_\sim$. This union must be connected by Theorem 8.1.12 and the two sets $(K' \cup M)$ and N are compact and disjoint. It must therefore be that $N = \emptyset$, which in turn implies $K \cap x_\sim$ is connected.

Now suppose $x_\sim$ is a branch point of $X/\sim$ and not an end point of $\pi(K)$. Let $D_1, D_2, \dots$ be the components of $\pi(K) \setminus \{x_\sim\}$ and let $K_i = \overline{\pi^{-1}(D_i)}$. Then as before, $x_\sim$ is an end point of $\pi(K_i)$ so $K_i \cap x_\sim$ is connected. Suppose $K \cap x_\sim = M \cup N$ for disjoint compact M, N . Each K_i can only intersect one of M or N , so we can define

$K_M = M \cup \bigcup \{K_i \mid K_i \cap M \neq \emptyset\}$ and K_N similarly. Now $K = K_M \cup K_N$ with these two sets disjoint, so we will show that both are closed.

Let $y \in \overline{K_M} \setminus K_M$. There exists a sequence $(y_n) \subseteq \bigcup \{K_i \mid K_i \cap M \neq \emptyset\}$ which converges to y . As each K_i is compact we can assume without loss of generality that each y_n lies in a different K_i , call it K_{i_n} . By Proposition 8.1.3 we have that there is a convergent subsequence of (K_{i_n}) with limit a subcontinuum C . As each K_{i_n} intersects M so too does C , and $y \in C$. Suppose there exists $z \in C \setminus x_\sim$. By Proposition 8.1.6 we have that $\pi(K)$ is a dendrite, so is locally connected about $z_\sim$ but for every open subset U containing $z_\sim$ and every open subset V of $U \setminus \{x_\sim\}$ containing $z_\sim$ we have that V intersects infinitely many D_i so cannot be connected. This contradiction gives us that $C \subseteq x_\sim$, so C is a continuum contained in $M \cup N$. We know $C \cap M \neq \emptyset$ and as $y \notin M \subseteq K_M$ we have that $C \cap N \neq \emptyset$. This final contradiction proves that both K_M and K_N are closed subsets of K so one must be empty, implying one of M and N is empty. Consequently we have that $K \cap x_\sim$ is connected for each $x \in X$, and a parallel result holds for L .

If $K \cap L = U \cup V$ for disjoint $(K \cap L)$ -clopen subsets U and V then each set $K \cap L \cap x_\sim$ must lie entirely in one of U or V . Define $U' = \pi^{-1}(\pi(U))$ and V' similarly. Both U and V are compact, so U' and V' are as well. Then $\pi^{-1}(\pi(K)) \cap \pi^{-1}(\pi(L)) = U' \cap V'$ and these two sets are disjoint and closed. By connectedness one of them must be empty, implying one of U and V must be empty and that $K \cap L$ is connected. $\square$

Chapter 9

Maximal and Minimal Subcontinua

In previous chapters we have constructed equivalence relations whose equivalence classes are all subcontinua of our continuum X , and used these to create monotone maps. These maps turned out to have a number of illuminating properties through which we could examine the structure of X . It therefore seems natural to ask why these subcontinua were so fundamental to the structure of X and whether anything similar exists in other continua. This chapter will look at the properties of these subcontinua and examine them in the context of non-irreducible continua.

9.1 Referenced Theorems

The following results from other authors will be used in this chapter.

Sam B. Nadler Jr, Continuum Theory, an Introduction

These results can be found in [Nad92].

Lemma 9.1.1 (3.2). *If a Hausdorff space is a continuous image of a compact metric space, then it is metrizable.*

Proposition 9.1.2 (4.18). *Let X be a compact metric space. Then every sequence of subcontinua of X has a subsequence converging to a subcontinuum of X , and thus every convergent sequence of subcontinua of X has a subcontinuum of X as its limit.*

Theorem 9.1.3 (5.4). *Let X be a continuum and let U be a nonempty, proper, open subset of X . If K is a component of $\overline{U}$, then $K \cap \partial U \neq \emptyset$. Equivalently, $K \cap (X \setminus U) \neq \emptyset$.*

Proposition 9.1.4 (6.3). *Let T be a connected topological space and let C be a connected subset of T such that $T \setminus C$ is disconnected, $T \setminus C = A \cup B$. Then $A \cup C$ and $B \cup C$ are connected. Hence, if T and C are continua, $A \cup C$ and $B \cup C$ are continua.*

Theorem 9.1.5 (8.26). *Any connected open subset of a Peano continuum is arcwise connected.*

Theorem 9.1.6 (11.17). *If X is non-degenerate and indecomposable then the composants of X are mutually disjoint.*

Ryszard Engelking, General Topology

These results can be found in [Eng89].

Lemma 9.1.7 (Kuratowski-Zorn Lemma, page 8). *In a partially ordered set in which every chain has an upper bound, every element has a maximal element above it.*

Theorem 9.1.8 (2.1.7). *A space X is hereditarily normal if and only if for every pair of separated subsets A, B of X there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.*

Proposition 9.1.9 (3.1.5). *Let U be an open subset of a topological space X . If a family $\{F_s\}_{s \in S}$ of closed subsets of X contains at least one compact set - in particular, if X is compact - and if $\bigcap_{s \in S} F_s \subseteq U$ then there exists a finite set $\{s_1, s_2, \dots, s_k\} \subseteq S$ such that $\bigcap_{i=1}^k F_{s_i} \subseteq U$.*

Theorem 9.1.10 (3.9.3). *In a Čech-complete space the countable union of nowhere dense sets has empty interior.*

Theorem 9.1.11 (4.3.6). *A topological space is metrizable by a totally bounded metric if and only if it is a regular second countable space.*

Theorem 9.1.12 (4.3.27). *Every metric on a compact space is totally bounded.*

Theorem 9.1.13 (6.1.9). *Let $\{C_s\}_{s \in S}$ be a family of connected subspaces of a topological space X . If there exists an $s_0 \in S$ such that the set C_{s_0} is not separated from any of the sets C_s then the union $\bigcup_{s \in S} C_s$ is connected.*

Theorem 9.1.14 (6.1.19). *The intersection of a decreasing sequence of continua is a continuum.*

Theorem 9.1.15 (6.1.23). *In a compact Hausdorff space X the component of a point $x \in X$ coincides with the quasicomponent of the point x .*

Theorem 9.1.16 (6.3.3). *A space X is locally connected if and only if the components of all open subspaces of X are open.*

Theorem 9.1.17 (7.1.12). *For every non-empty locally compact paracompact space X the condition $\dim(X) = 0$ is equivalent to X being totally disconnected.*

9.2 Continua which are Full of Maximal Nowhere Dense Subcontinua

Proposition 9.2.1. *For the equivalence relations defined in Definitions 2.4.1 and 4.3.6, each equivalence class has empty interior and is maximal amongst subcontinua with this property. Thus if C is a continuum and $x_\sim \subsetneq C$ for some $x \in X$ then C does not have empty interior.*

Proof. The proof is basically identical to the proof that the map π is universal. An example of the proof for a map $\pi : X \mapsto T$ with T a tree is given, the proof for a map to an arc is identical.

From the construction of the equivalence classes $x_\sim$ it is clear that they have empty interior (Lemma 4.3.18). Let $\pi(x) = t$ and $x_\sim = \pi^{-1}(t) \subsetneq C$. Then $\pi(C)$ is a non-degenerate subcontinuum of T so has non empty interior. There are finitely many components of $T \setminus \pi(C)$, let their closures be denoted $T_1, \dots, T_k$. Since $\pi(C)$ has non-empty interior $T \neq \bigcup_{i=1}^k T_i$. If $X_i = \pi^{-1}(T_i)$ then it follows that $X \neq \bigcup_{i=1}^k X_i$. Each X_i is a continuum as π is monotone. By irreducibility we have that $X = C \cup \bigcup_{i=1}^k X_i$ and so $X \setminus \bigcup_{i=1}^k X_i \subseteq C$. This is an open set so C has a non-empty interior. $\square$

Each of these subcontinua are maximal amongst nowhere dense subcontinua of X and X can be expressed as the disjoint union of them all. In this section we will study continua who can also be expressed as such a union to see what this implies about the structure of X .

Definition 9.2.2. A continuum X is said to have a full set of maximal, nowhere dense subcontinua if for each point $x \in X$ there is a subcontinuum A_x containing x which is maximal amongst continua with empty interiors. Such a continuum will be said to be an FMND continuum.

For this section if X is an FMND continuum, unless specified otherwise, the maximal empty interior subcontinuum containing a point x will be denoted A_x .

Lemma 9.2.3. *Let T be a topological space, let C be a closed subset of T and let D be a subset of T . Then $C \cup D$ has non-empty interior if and only if at least one of C or D has non-empty interior.*

Proof. It is clear that if one of C or D has non-empty interior then so does $C \cup D$. For the other direction let U be an open set with $U \subseteq C \cup D$. Then

$$U = (U \cap C) \cup (U \setminus C)$$

If $U \setminus C$ is non-empty then this is an open set contained in D , so D has non-empty interior. If $U \setminus C$ is empty but $U \cap C$ is not, then $U \cap C = U \subseteq C$ so C has non-empty interior. If both $U \cap C$ and $U \setminus C$ are empty for all choices of U then the only open subset of $C \cup D$ is the empty set, so $C \cup D$ has empty interior. $\square$

Proposition 9.2.4. *Let X be an FMND continuum and let $x \in X$. The subcontinuum A_x is unique.*

Proof. If A_x and B_x were two such distinct examples of maximal nowhere dense subcontinua containing x then neither one can contain the other and $A_x \cup B_x$ is a continuum properly containing A_x and B_x . It follows from the maximality of A_x and B_x that $A_x \cup B_x$ has non-empty interior, which by Lemma 9.2.3 gives us that one of A_x or B_x has non-empty interior. This contradicts their definition, so it cannot be that there are two such distinct subcontinua. $\square$

Proposition 9.2.5. *Let X be an FMND continuum. The collection of subcontinua $\{A_x | x \in X\}$ forms a partition of X into subcontinua.*

Proof. Let $y \in A_x$. Then A_x is a maximal nowhere dense subcontinuum of X containing y , so by applying Proposition 9.2.4 we have that $A_x = A_y$. This in turn gives us that no two maximal nowhere dense subcontinua can intersect, or each would act as A_y for any point y in the intersection, contradicting Proposition 9.2.4. Each x lies in some A_x so the set $\{A_x | x \in X\}$ is indeed a partition. $\square$

For any of the equivalence relations defined in previous chapters we have that the continuum X is locally connected about their equivalence classes. This was important as it lead to the quotient space being locally connected and therefore easily understood. Fortunately, the maximal nowhere dense subcontinua also have this property. In the following proposition the continuum does not need to be FMND, as the result holds for any maximal nowhere dense subcontinua regardless of whether there are some points of X which do not lie in one.

Proposition 9.2.6. *Let X be any metric continuum and let C be a subcontinuum, $\text{int}(C) = \emptyset$ and for every subcontinuum $D \subseteq X$ properly containing C , D has non-empty interior. Then X is locally connected at C .*

Proof. Suppose it is not, witnessed by an open set U . Consider the components of $U \setminus C$. There must be infinitely many of them, and indeed countably many of them U_i limiting on to C . Let $K_i = \overline{U_i}$. By Proposition 9.1.2 there exists a convergent subsequence of $\{K_i\}$ so without loss of generality let $\{K_i\}$ converge to a continuum K . It is clear that K intersects C non-trivially. By applying Theorem 9.1.3 we have that for each i , $K_i \setminus U$ is non-empty so take a point y_i in this set. The sequence $\{y_i\}$ has a convergent subsequence which converges to a point y so y must lie in K . Since no point y_i lies in U it cannot be that $y \in U$, and therefore $y \notin C$. This in turn implies that $\text{int}(K) \neq \emptyset$.

Since $\text{int}(K)$ is an open set in the limit K there must exist $N \in \mathbb{N}$ such that for all $n \geq N$ we have $K_n \cap \text{int}(K) \neq \emptyset$. Furthermore, as $K_n = \overline{U_n}$, we have that $U_n \cap \text{int}(K) \neq \emptyset$. Let $U' = \bigcup_{n \geq N} U_n$ and express U' as $A \cup B$ for disjoint U' -clopen sets A and B . Each U_n is connected so each must lie wholly in A or wholly in B , which means one of A or B must contain a subsequence of $\{U_i\}$. Suppose this is A . By applying Theorem 9.1.8 we can take disjoint X -open sets V and W with $A \subseteq V$ and $B \subseteq W$. Since for each $n \geq N$ we have that $U_n \cap \text{int}(K) \neq \emptyset$ it follows that if B is non-empty then there must be a point of K in B . This point lies in W so there exists $M \in \mathbb{N}$ such that for all $m \geq M$, $W \cap K_m \neq \emptyset$. This contradicts the fact that $W \cap A = \emptyset$, so it must be that B is empty. From this we can conclude that U' is connected. However, each U_i is a component of $U \setminus C$ and $U' \subseteq U \setminus C$, so it cannot be that U' is connected. This final contradiction proves that no such U can exist and that X is locally connected about C . $\square$

Remark 9.2.7. If C is defined as in Proposition 9.2.6 then any subcontinuum D which intersects C but is not contained in it has non-empty interior. This is because $C \cup D$ is a continuum, so will have non-empty interior. By applying Lemma 9.2.3 we have that D also has non-empty interior.

Remark 9.2.8. This gives a proof that a continuum is locally connected about its E-continua which does not use global properties of the space.

The continua considered in Chapters 2, 4 and 8 were all almost hereditarily decomposable. As is shown below, this is a necessary condition of being FMND, but not a sufficient one.

Proposition 9.2.9. *Let X be an FMND continuum. X is almost hereditarily decomposable.*

Proof. Let $C \subseteq X$ be an indecomposable subcontinuum and let $x \in C$. Every proper subcontinuum of C has empty interior in C and therefore in X . If B is any proper subcontinuum of C containing x then $A_x \cup B$ has empty interior by Lemma 9.2.3, so $A_x \cup B = A_x$. Thus the composant of x in C must lie in A_x and as this is dense in C and A_x is closed, $C \subseteq A_x$. This gives $\text{int}(C) \subseteq \text{int}(A_x) = \emptyset$. $\square$

Remark 9.2.10. The converse is not true. A unit square is almost hereditarily decomposable (proof below) but does not have maximal empty-interior subcontinua. We can see this as for any $A \subseteq X$ with empty interior, pick any point in $X \setminus A$ and add an arc from A to that point. This new continuum also has empty interior by Lemma 9.2.3. For a non-locally connected example, take the product of a $\sin \frac{1}{x}$ continuum with an arc.

To prove that the unit square, and indeed any Peano continuum, is almost hereditarily decomposable we will make use of the following well known result.

Lemma 9.2.11. *Let X be a continuum. X is decomposable if and only if it has a proper subcontinuum with non-empty interior.*

Proof. First suppose that X is decomposable, with $X = A \cup B$ an example of a decomposition. Then as both A and B are proper subcontinua of X we have that $X \setminus A$ is a non-empty open subset. This open subset lies in B , so B is a proper subcontinuum with non-empty interior.

Now suppose X is indecomposable. Each proper subcontinuum of X lies in one of the composants of X , which are disjoint by Theorem 9.1.6. As each composant is dense they must all have empty interiors, which implies each proper subcontinuum has empty interior. $\square$

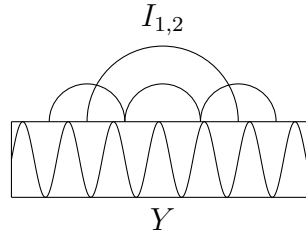
Proposition 9.2.12. *Let Y be a locally connected continuum. Then Y is almost hereditarily decomposable.*

Proof. Take a proper subcontinuum $X \subseteq Y$ with non-empty interior U . Let $x \in X$ be a point of that interior. Then there exists a connected open subset V such that $x \in V \subseteq \overline{V} \subseteq U$. This implies that $\overline{V}$ is a proper subcontinuum of X , and as it contains V it has non-empty interior. By Lemma 9.2.11 we have that X is decomposable, and in turn that Y is almost hereditarily decomposable. $\square$

As we did previously, we will use this partition (Proposition 9.2.5) of our continuum to define a quotient map onto a new space. Various properties of this space will be examined and used to deduce results about the underlying structure of our original continuum.

Definition 9.2.13. Let X be an FMND continuum and for each $x \in X$ let A_x be the maximal nowhere dense subcontinuum of X containing x . We shall denote by L the quotient space and $\pi : X \mapsto L$ the quotient map corresponding to the partition $\{A_x | x \in X\}$.

Example 9.2.14. Let X be an FMND continuum. The quotient space L defined in Definition 9.2.13 is not necessarily a continuum. We shall construct an example similar to Example 8.3.14 to demonstrate this. Let Y be the continuum defined in Example 2.5.8. Consider the monotone map $\pi_Y : Y \mapsto \mathbb{I}$ and let $y_{k,n}$ be the point of $\pi_Y^{-1}(k \cdot 2^{-n})$ which would have second coordinate 1 in K , so each point $y_{k,n}$ lies "at the top" of Y . For each $n \neq 1$ and each odd $k < 2^n$ attach an arc to Y with end points $y_{k,n}$ and $y_{k+2,n}$ such that the maximum distance between the arc and Y is 2^{-n} . Call this arc $I_{k,n}$.



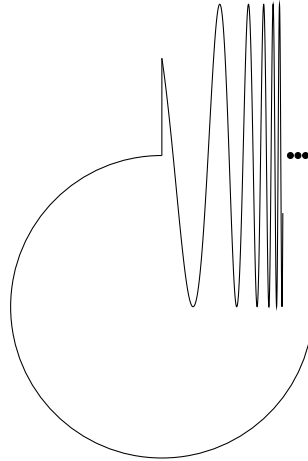
Attach the arcs in such a way that for distinct n and m and for all k and l , the arcs $I_{k,n}$ and $I_{l,m}$ are disjoint. Finally, for each arc $I_{k,n}$ add a half open interval to the continuum limiting on to $I_{k,n}$ so that the half open interval and $I_{k,n}$ form a $\sin(\frac{1}{x})$ continuum and the whole of the half open interval lies within a distance of 2^{-n} of $I_{k,n}$. Call the whole space X .

Note that Y is an FMND continuum. If a point lies in Y and its fibre of π_Y does not contain any points $y_{k,n}$ then this fibre is a maximal nowhere dense subcontinuum of X . For a point $y_{k,n}$, the subcontinuum consisting of each fibre $\pi_Y^{-1}(l \cdot 2^{-n})$ and the arcs joining them together forms a maximal nowhere dense subcontinuum of X containing $y_{k,n}$. Each point in the half open intervals forming $\sin(\frac{1}{x})$ continua is a maximal nowhere dense subcontinuum, so we have that X is an FMND continuum.

Let x and y be two points in Y and let A_x and A_y be their corresponding maximal nowhere dense subcontinua. Let U be an open subset of X with $A_x \subseteq U$ and for each $z \in X$ if $A_z \cap U \neq \emptyset$ then $A_z \subseteq U$. There exists some $N \in \mathbb{N}$ such that for all $n \geq N$ there exists odd $k < n$ such that $\pi_Y^{-1}(k \cdot 2^{-n}) \cap U \neq \emptyset$. This implies that $A_{y_{k,n}} \subseteq U$. The set $\bigcup_{n \geq N} \bigcup_k \pi_y^{-1}(k \cdot 2^{-n})$ is dense in Y and lies in U , which implies that A_y is in the closure of U . From this we can conclude that the quotient space L defined in Definition 9.2.13 is not Hausdorff, as every open subset of L containing A_x will contain A_y in its closure. Hence L is not a continuum.

Although the quotient space L is not necessarily a continuum the cases where it is a continuum will give us a useful insight into the structure of X . We will next present a number of results concerning FMND continua for which the quotient space L is a continuum.

Example 9.2.15. The following space is an example of an FMND continuum and the corresponding map π . Highlighted in bold is the only non-degenerate maximal nowhere dense subcontinuum. The map π takes this subcontinuum to a point and maps X onto a simple closed curve.



The continuum X is not in any way irreducible so this map could not be produced by any of the theorems from Chapters 2, 4 and 8.

Theorem 9.2.16. *Let X be an FMND continuum and L be as in Definition 9.2.13 with L a continuum. Then L is a Peano continuum.*

Proof. From Proposition 9.2.6 we have that X is locally connected about each fibre of π . By applying Proposition 2.5.1 we in turn have that L is locally connected about each of its points, so is a locally connected continuum. $\square$

The following lemma will show that an FMND continuum is not just locally connected about each maximal nowhere dense subcontinuum, but that the corresponding connected open set can be chosen to be a union of A_x 's and to be continuumwise connected.

Lemma 9.2.17. *Let X be a continuum and let $\mathcal{C}$ be a partition of X into subcontinua with X locally connected about each $C \in \mathcal{C}$ and the corresponding quotient space a continuum. Let U be an open subset of X . Suppose there exists $C \in \mathcal{C}$ such that $C \subseteq U$. Then there exists an open subset V with $C \subseteq V \subseteq U$ such that V is continuumwise connected.*

Proof. Define $Y = X/\mathcal{C}$, $\pi : X \mapsto Y$ as the quotient space and quotient map respectively. Define $U' := \bigcup\{C \in \mathcal{C} \mid C \subseteq U\}$. We know that Y is locally connected as X is locally connected about each fibre of π (Proposition 2.5.1). We will first show that U' is open. Since $U' = \pi^{-1}(\pi(U'))$ we know that U' is open if and only if $\pi(U')$ is open. If this were not the case there would exist a sequence of points $y_n \in Y$ such that for all n , $y_n \notin \pi(U')$ but that $y_n \rightarrow y$ for some $y \in \pi(U')$. Let $C_n \in \mathcal{C}$ be the inverse image of y_n under π . Since each y_n is not in $\pi(U')$ we know that each $C_n \not\subseteq U$, so take $x_n \in C_n \setminus U$. Then $\{x_n\}$ is a sequence in a compact space X , so it has a subsequence which converges to some x . Since π is continuous it must be that $y_n \rightarrow \pi(x)$ so $\pi(x) = y$. But $y \in \pi(U')$ so $x \in \pi^{-1}(y) \subseteq U$. This gives us a contradiction as x lies in the open set U , but the convergent subsequence of x_n does not ever intersect this open set. Thus it must be that $\pi(U')$ is open, and by extension U' is open.

Now take V to be the component of C in U' . Since the map π is monotone, the components of $\pi(U')$ are the images of the components of U' , and the components of U' are the inverse images of the components of $\pi(U')$. As $\pi(U')$ is an open subset of Y , and as Y is locally connected, the components of $\pi(U')$ are also open (Theorem 9.1.16). This gives us that V is an open set with open image in Y . Take two points $p, q \in V$. Then by applying Theorem 9.1.5 we have that there is an arc $\mathbb{I} \subseteq \pi(V)$ containing $\pi(p)$ and $\pi(q)$. Let $K = \pi^{-1}(\mathbb{I})$. Since π is monotone K is a continuum, and we know $p, q \in K \subseteq V$. Thus V is continuumwise connected. It is also open and from the construction it is clear that $C \subseteq V \subseteq U$. $\square$

Corollary 9.2.18. *The statement of the above lemma holds for an FMND continuum X with the collection of maximal, empty interior subcontinua as the partition if said partition is Hausdorff.*

Proposition 9.2.19. *Let X be an FMND continuum and suppose L is a continuum. Let $U \subseteq X$ be a connected open set with the property that for each $x \in U$, we have that $A_x \subseteq U$. Then U is continuumwise connected.*

Proof. We know that $\pi^{-1}(\pi(U)) = U$ so $\pi(U)$ is an open subset of L . By applying Theorem 9.1.5 to this open, connected subset we have that $\pi(U)$ is arcwise connected. Then inverse image under π of any such arc will be a continuum contained in U , so U is continuumwise connected. $\square$

Corollary 9.2.20. *Let X be an FMND continuum and suppose L is a continuum. Let x be a point of X . Then each component of $X \setminus A_x$ is open and continuumwise connected.*

Proof. Each component of $X \setminus A_x$ is connected and for any y in such a component A_y must also lie in that component. Each component is therefore the inverse image of a component of $L \setminus \pi(x)$, and as L is locally connected these components are open by Theorem 9.1.16. Now apply Proposition 9.2.19. $\square$

We shall next prove that, as in Chapters 2, 4 and 8, the continuum L is one dimensional. In fact there is a stronger result, as seen in the following lemma, but for our purposes we are more interested in its application to FMND continua.

Lemma 9.2.21. *Let $f : Y \mapsto Z$ be a continuous surjection between compact metric spaces such that for every non-degenerate subcontinuum $Z' \subseteq Z$ we have that $f^{-1}(Z')$ has non-empty interior. Then $\dim(Z) \leq 1$.*

Proof. Let $z \in Z$ and let $U \subseteq Z$ be an open set containing z . Let d be a metric on Z and $B(x, r)$ the open ball of radius r about x under this metric. Let $\epsilon > 0$ be such that $B(z, \epsilon) \subseteq U$. Suppose for contradiction that for each $\delta \in (0, \epsilon)$ we have that $\partial B(z, \delta)$ is not zero dimensional. Since $\partial B(z, \delta)$ is a compact metric space we can apply Theorem 9.1.17 to see that it cannot be totally disconnected, so there must be a non-degenerate subcontinuum $C_\delta \subseteq \partial B(z, \delta)$.

Now for all $z' \in C_\delta$ we have that $d(z, z') = \delta$, so for $\delta \neq \delta'$ it follows that $C_\delta \cap C_{\delta'} = \emptyset$. Consequently $\{f^{-1}(C_\delta) | \delta \in (0, \epsilon)\}$ is a disjoint collection of subsets of Y . For each δ we have that $f^{-1}(C_\delta)$ has non-empty interior. These interiors form an uncountable set of pairwise disjoint open subsets of Y . However, Y is compact and metric so it is second countable by Theorem 9.1.11 and Corollary 9.1.12. This means that no such collection of open sets can exist, as each must contain a basic open set.

This contradiction gives us that there exists some δ such that $\partial B(z, \delta)$ is zero dimensional, which in turn implies that $\dim(z) \leq 1$. As z was an arbitrary point of Z it follows that $\dim(Z) \leq 1$. $\square$

Theorem 9.2.22. *If X is an FMND continuum and L is as in Definition 9.2.13 with L a continuum then $\dim(L) = 1$.*

Proof. To prove that X satisfies the conditions of Lemma 9.2.21 we only need to show that the inverse image of a non-degenerate subcontinuum of L has non-empty interior. If $L' \subseteq L$ is such a continuum and $X' = \pi^{-1}(L')$ then it is clear that X' does not lie in any of the fibres of π , which are precisely the maximal nowhere dense subcontinua of X . Thus X' has non-empty interior, so by Lemma 9.2.21, $\dim(L) \leq 1$. Since L is a non-degenerate continuum it cannot be zero dimensional, so $\dim(L) = 1$. $\square$

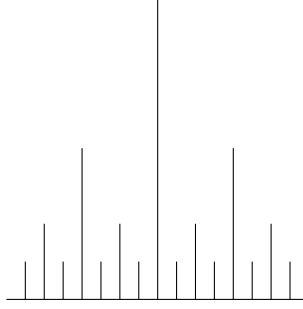
Proposition 9.2.23. *If X is FMND and L is as in Definition 9.2.13 with L a continuum then L is hereditarily decomposable.*

Proof. Suppose for contradiction that an indecomposable subcontinuum $C \subseteq L$ exists. There are uncountably many composants of C and by Theorem 9.1.6 they are pairwise disjoint. They are all dense, so none of them are singletons. Thus there are uncountably many disjoint, non-degenerate subcontinua of C . We can repeat the same argument as in Lemma 9.2.21, that the inverse images of these continua give rise to an uncountable collection of disjoint open subsets and that this is a contradiction in a second countable space. $\square$

It would seem reasonable to suppose that L has no non-degenerate nowhere dense subcontinua. The map $\pi : X \mapsto L$ after all has taken each such subcontinuum of X and shrunk it to a point. This is not the case however, as the following example shows.

Example 9.2.24. Let Y be the continuum constructed in Example 2.5.8, and consider $\pi_Y : Y \mapsto \mathbb{I}$, the universal monotone map onto the unit interval from Theorem 2.4.11. For each point $k/2^n$ there is a vertical line in Y mapped to $k/2^n$. Define X by attaching a line of length $1/2$ to an end point of the vertical line of Y mapped to $1/2$, lines of length $1/4$ to end points of vertical lines mapping to $1/4$ and $3/4$, and so on. Choose the end points to either all be at the "bottom" of "Y" or all be at the "top". Then X is FMND and the non-trivial maximal nowhere dense subcontinua are precisely those of Y . If $\pi_x : X \mapsto L$ is the map induced from X being FMND

then L is the continuum below. Clearly the horizontal line, which is the image of Y , has empty interior in L .



These leads to the question of whether or not L is FMND. While there is not yet a proof or counterexample for this, we do have the following proposition.

Proposition 9.2.25. *Let X be an FMND continuum and suppose that L is also an FMND continuum. Let $\pi_L : L \mapsto L'$ be monotone map corresponding to Definition 9.2.13 with L' also a continuum. Then there are no non-degenerate subcontinua of L' with empty interiors.*

Proof. Let $C \subseteq L'$ be a non-degenerate subcontinuum. C is an uncountable subset of L' so $\pi_L^{-1}(C)$ is the union of uncountably many maximal nowhere dense subcontinua. We have shown in the proof of Lemma 9.2.21 that there are at most countably many non-degenerate maximal nowhere dense subcontinua of L . Let U be the interior of $\pi_L^{-1}(C)$, we know U is non-empty as C is not a singleton. Now by Theorem 9.1.10 we have that the union of the non-degenerate fibres of π_L in C has empty interior, so U must contain a point l which is a fibre of π_L . By Lemma 9.2.17 there exists a non-empty open subset $V \subseteq U$ with V consisting of whole fibres of π_L . Thus $\pi_L^{-1}(\pi_L(V)) = V$ so $\pi_L(V)$ is an open subset of L' . Further $\pi_L(V) \subseteq C$, so C has non-empty interior. $\square$

It would be useful to know the circumstances under which a continuum is FMND. We have already seen that an FMND continuum must be almost hereditarily decomposable but that this is not enough to guarantee being FMND (Proposition 9.2.9 and Remark 9.2.10). The following example shows that being hereditarily decomposable is not enough to ensure that a continuum is FMND.

Example 9.2.26. Let X be the Cantor fan, which is the cone over the Cantor set C . Any subcontinuum of X will either be contained in an arc, so will be an arc itself, or will be homeomorphic to a cone over some subset of C . Either way such a subcontinuum will be decomposable, so X is hereditarily decomposable. Let $x \in X$ be the base point of the cone, and let $\{x_c | c \in C\}$ be the set of end points of arcs in X . As C has no isolated points each arc from x to x_c is nowhere dense in X . If X were FMND then each of these arcs would have to lie in A_x . However, their union is the whole of X , and X clearly does not have an empty interior as a subset of itself. Thus no such A_x exists and X is not FMND.

It is worth noting that the continuum X in Example 9.2.26 is one dimensional. From Theorem 9.2.22 we know there is a link between FMND continua and one dimensional continua, but as we saw in the example X is hereditarily decomposable and one dimensional but is not FMND.

We will now look at a few results concerning FMND continua arising as subcontinua or images of FMND continua.

Proposition 9.2.27. *Let X be an FMND continuum and $U \subseteq X$ an open connected subset of X . Then $\overline{U}$ is an FMND continuum.*

Proof. For each $x \in \overline{U}$ define

$$B_x = \bigcup \{C \subseteq \overline{U} | x \in C \text{ a subcontinuum, } \text{int}_{\overline{U}}(C) = \emptyset\}$$

If we can show that $\overline{B_x}^{\overline{U}}$ has empty interior then this will prove that $\overline{U}$ is FMND. Note that $A_x = \bigcup \{C \subseteq X | x \in C \text{ a subcontinuum, } \text{int}_X(C) = \emptyset\} = \overline{A_x}$. If C is a subcontinuum which is part of the union defining B_x then $\text{int}_X(C) = \emptyset$, else the intersection of this interior with $\overline{U}$ would give $\text{int}_{\overline{U}}(C) \neq \emptyset$. Thus $B_x \subseteq A_x$, and indeed $\overline{B_x}^{\overline{U}} \subseteq A_x$. Let $V = \text{int}_{\overline{U}}(\overline{B_x}^{\overline{U}})$ and suppose V is not empty. Then $V \cap U \neq \emptyset$ as V is open in $\overline{U}$, and $V \cap U$ is an X -open set contained in $\overline{B_x}^{\overline{U}}$ and therefore in A_x . This cannot be the case, so it must be that $V = \emptyset$, which completes the proof. $\square$

Corollary 9.2.28. *Let B_x be the maximal nowhere dense subcontinuum containing x in $\overline{U}$. Then B_x is the component of $A_x \cap \overline{U}$ containing x .*

Proof. Let C be the component of x in $A_x \cap \overline{U}$. We have shown in the proof of Proposition 9.2.27 that $B_x \subseteq A_x$ and since B_x is connected it must lie in C . Since $C \subseteq A_x$ it must have empty interior in X , and we saw in the proof of Proposition 9.2.27 that this implies C has empty interior in $\overline{U}$. Thus $C \subseteq B_x$ as $x \in C$. Hence $C = B_x$. $\square$

Proposition 9.2.29. *Let X be an FMND continuum with L Hausdorff and let Y be a subcontinuum of X such that there are at most countably many x with $A_x \cap Y \neq \emptyset$ and $A_x \not\subseteq Y$. Then X/Y is an FMND continuum.*

Proof. Let $\rho : X \mapsto X/Y$ be the usual monotone map. We will prove the proposition by considering $B_y = \bigcup \{C \subseteq X/Y \mid y \in C, \text{int}(C) = \emptyset\}$. If $y = \rho(x)$ then first consider the case $A_x \cap Y = \emptyset$ and note that $\rho|_{X \setminus Y}$ is a homeomorphism. Given any subcontinuum $C \subseteq A_x$ with empty interior in X it is clear that $\rho(C)$ has empty interior in X/Y , so $\rho(A_x) \subseteq B_y$. Conversely, let D be a subcontinuum of X/Y with empty interior containing y . If $\rho^{-1}(D)$ does not have an empty interior then it must contain Y , so $\rho(Y) \in D$. Take disjoint open subsets U and V of D containing $\rho(A_x)$ and $\rho(Y)$ respectively and consider the component K of $\bar{U}$ containing $\rho(A_x)$. By Theorem 9.1.3 we have that $K \neq \rho(A_x)$. Now K lies in D so it has an empty interior in X/Y , but $\rho^{-1}(K)$ properly contains A_x so does not have empty interior in X . This is a contradiction as ρ is a homeomorphism on $X \setminus Y$, which contains $\rho^{-1}(K)$. Thus we have that $\rho^{-1}(D)$ has empty interior so lies in A_x . We have therefore shown that $B_y = \rho(A_x)$, implying that each y satisfying $A_{\rho^{-1}(y)} \cap Y = \emptyset$ lies in a maximal nowhere dense subcontinuum of X/Y .

Now consider $B_{\rho(Y)}$. Suppose for some $x \in X$, $A_x \cap Y \neq \emptyset$. Take any subcontinuum $C \subseteq A_x$ containing x . Then

$$\begin{aligned} \text{int}_X(C) = \emptyset & \quad \text{therefore} \quad \text{int}_X(C \setminus Y) = \emptyset \\ & \quad \text{therefore} \quad \text{int}_{X/Y}(\rho(C) \setminus \{\rho(Y)\}) = \emptyset \\ & \quad \text{therefore} \quad \text{int}_{X/Y}(\rho(C)) = \emptyset \end{aligned}$$

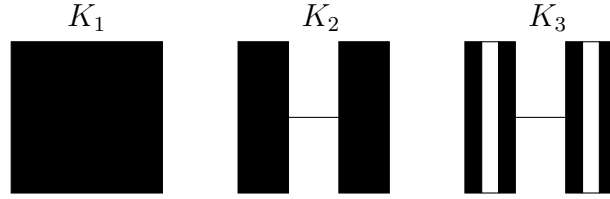
This gives us that $\rho(C) \subseteq B_{\rho(Y)}$ so $\rho(A_x) \subseteq B_{\rho(Y)}$. As the B_y 's partition X/Y it is clear that $B_{\rho(Y)} = \bigcup_{A_x \cap Y \neq \emptyset} \rho(A_x) = \rho(\bigcup_{A_x \cap Y \neq \emptyset} (A_x))$. This is equal to $\rho(\pi^{-1}(\pi(Y)))$ so it must be compact as L is Hausdorff. To show it has empty interior, let $U \subseteq B_{\rho(Y)}$ be an open subset of X/Y . For each maximal nowhere dense subcontinuum of X which intersects Y but is not contained in Y take a point x_n , so we have a countable, possibly finite, set of points x_n with $A_{x_n} \cap Y \neq \emptyset$ and $A_{x_n} \not\subseteq Y$. The set $U \setminus \rho(Y)$ is an open subset of X/Y . If U is non-empty then so is $U \setminus \rho(Y)$ as $\rho(Y)$ is a singleton, so $U \not\subseteq \rho(Y)$. Then $\rho^{-1}(U \setminus \rho(Y))$ is an open set contained in $\bigcup_{n=1}^{\infty} A_{x_n}$, which by the Theorem 9.1.10 has empty interior. Thus $U = \emptyset$.

We have shown that each B_y is compact and nowhere dense, so X/Y is FMND. $\square$

It is not the case that given any FMND continuum X and any monotone map $\rho : X \mapsto Y$, that Y is an FMND continuum as the following example shows. In

this example the map is similar to in Proposition 9.2.29, but the subcontinuum in question intersects uncountably many sets A_x without containing them.

Example 9.2.30. Consider the continuum X constructed in a similar way to Example 2.5.8, but with two copies one on top of the other. Take as the continuum K_i as below and let K be the intersection of all of them. Let X be the image of K where each horizontal line is reduced to a point.



Let X_1 correspond to the upper copy of the continuum from Example 2.5.8 and let X_2 correspond to the lower one. X is FMND as both X_1 and X_2 are and a maximal subcontinuum of one can only intersect the corresponding subcontinuum in the other, so this union forms the maximal nowhere dense subcontinuum in X . Let $\rho : X \mapsto Y$ be a monotone map whose only non-degenerate fibre is X_1 . We will show that Y is not FMND. Let $y \in Y$ be the point with $y = \rho(X_1)$ and consider what A_y would need to include. For each end point of the Cantor set the corresponding vertical line in X_2 intersects X_1 , so the image of this line under ρ contains y . Thus each such vertical line must lie in A_y as each has empty interior. These lines form a dense subset of X_2 , so form a dense subset of Y . As A_y is closed it would follow that $A_y = Y$. This clearly contradicts that A_y ought to have empty interior, so no such A_y can exist and Y is not FMND.

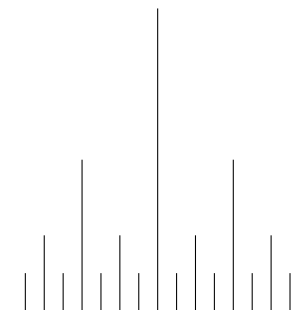
The equivalence relations defined in Chapters 2 and 4 lead to monotone maps which were universal to the corresponding set of continua (Theorems 2.4.15 and 4.3.19). While it is possible for the map π defined in Definition 9.2.13 to be universal, and indeed has been in a number of example so far, this is not always the case.

Proposition 9.2.31. *Let X be an FMND continuum. Suppose there exists $x \in X$ such that A_x can be expressed as the disjoint union of proper subcontinua, with X locally connected at each such subcontinuum. Let $\mathcal{C}$ be this collection of subcontinua and suppose the partition $\{A_y | y \notin A_x\} \cup \mathcal{C}$ is Hausdorff. Then $\pi_X : X \mapsto L_X$ is not universal amongst monotone maps to locally connected continua.*

Proof. As the partition $\{A_y | y \notin A_x\} \cup \mathcal{C}$ of X is Hausdorff it defines a quotient map $\rho : X \mapsto Y$ to a continuum by Lemma 9.1.1. This map is monotone and X is locally connected about each fibre of ρ , so Proposition 2.5.1 gives us that Y is locally connected. However, this map cannot factor through $\pi_X : X \mapsto L_X$ because $A_x \in L_X$ would need to be mapped to more than one point, namely each $\rho(C)$ for $C \in \mathcal{C}$. Thus $\pi_X : X \mapsto L_X$ is not universal. $\square$

An example of such a continuum is below.

Example 9.2.32. Let X be the below continuum and let A be the horizontal line.



A is the only non-degenerate maximal nowhere dense subcontinuum of X . However, X is locally connected so each singleton is a subcontinuum of X about which X is locally connected. Thus A can be expressed as the union of these singletons, satisfying the criteria of Proposition 9.2.31. In this case the map witnessing that $\pi : X \mapsto L$ is not universal is the identity map on X .

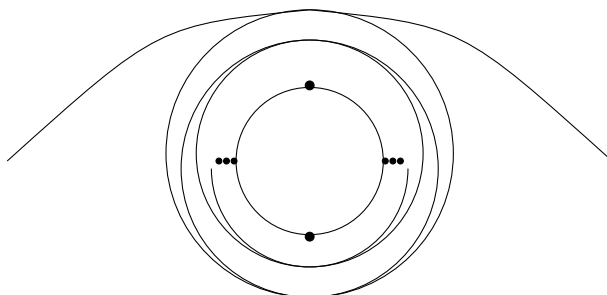
9.3 Minimal Locally Connected

Another property of the equivalence classes of Definition 2.4.1 and Definition 4.3.6 is that they are minimal subcontinua about which X is locally connected, and they are unique in this regard. This is also frequently the case for Definition 8.3.2, although as Example 8.3.14 shows there are some counterexamples. We shall investigate such minimal subcontinua, as well as their interactions with the subcontinua of the previous section.

Theorem 9.3.1. *Let X be a continuum. Given any point $x \in X$ there exists a subcontinuum C such that $x \in C$, X is locally connected at C and for each subcontinuum $D \subseteq C$ with $x \in D$, we have that X is not locally connected about D .*

Proof. Let x be a point of X and let $\mathcal{C}(x)$ be the set of subcontinua C of X containing x such that X is locally connected about C , and let $\mathcal{C}(x)$ be ordered by reverse inclusion. Since X is locally connected about itself, we have that $X \in \mathcal{C}(x)$ and that $\mathcal{C}(x)$ is non-empty. Let γ be a limit ordinal and for each $\alpha < \gamma$ let $C_\alpha \in \mathcal{C}(x)$, so that these continua form a chain in $\mathcal{C}(x)$. Let $C_\gamma = \bigcap_{\alpha \in \gamma} C_\alpha$. This will be a continuum by Theorem 9.1.14 and if $C_\gamma \subseteq U$ for some open set U then by Proposition 9.1.9 there exists some α such that $C_\alpha \subseteq U$. By local connectedness of X at C_α there exists connected open V such that $C_\gamma \subseteq C_\alpha \subseteq V \subseteq U$. Thus X is locally connected at C_γ and $C_\gamma \in \mathcal{C}(x)$. This gives us that every chain in $\mathcal{C}(x)$ has an upper bound. We can therefore apply Lemma 9.1.7 to see that $\mathcal{C}(x)$ contains a maximal element, which will therefore satisfy the conditions of this theorem. $\square$

Example 9.3.2. The subcontinuum defined in Theorem 9.3.1 is not necessarily unique. Let A and B be two copies of the $\sin \frac{1}{x}$ continuum. Pair off the local maxima and minima of the sin curve as well as the end points of the non-degenerate E-continua and take a quotient of this relation to be the space X . Then X is locally connected at the images of the two E-continua, both of which are minimal with this property. Thus either of the points in their intersection prove this proposition.



Remark 9.3.3. The space above is hereditarily decomposable and the minimal continua have empty interior, so it does not seem likely that adding extra conditions to X or C will guarantee uniqueness. It also proves that the maximal subcontinua with empty interior defined in Section 9.2 are not necessarily minimal subcontinua at which X is locally connected.

Remark 9.3.4. It is reasonable to ask whether for each maximal nowhere dense subcontinuum A_x there exists some $y \in A_x$ such that A_x is the union of each minimal subcontinuum of X containing y about which X is locally connected. However, if the continuum in Example 9.3.2 were constructed with three $\sin(\frac{1}{x})$ continua not two then it would witness that this is not the case.

As mentioned above, these minimal subcontinua are of interest because of the role they play in irreducible continua. In an irreducible, almost hereditarily decomposable continuum X each point x has a unique minimal subcontinuum containing X about which X is locally connected, as we saw in Proposition 2.5.6. We shall consider other continua with this property.

Lemma 9.3.5. *Let X be a continuum, $x \in X$ and $K \subseteq X$ a subcontinuum. Suppose X is locally connected about K and $x \in K$. Then there exists a subcontinuum C with $x \in C \subseteq K$ such that C satisfies the properties of Theorem 9.3.1.*

Proof. The proof is almost identical to that of Theorem 9.3.1. We simply redefine $\mathcal{C}(x)$ to be the collection of subcontinua of K containing x about which X is locally connected, and note that $K \in \mathcal{C}(x)$ so $\mathcal{C}(x)$ is not empty. The rest of the proof proceeds identically. $\square$

Corollary 9.3.6. *Let X be an FMND continuum, and let $x \in X$. Let A_x be the maximal nowhere dense subcontinuum of X containing x . Then there exists a subcontinuum C of X with $x \in C \subseteq A_x$ with C satisfying the properties of Theorem 9.3.1*

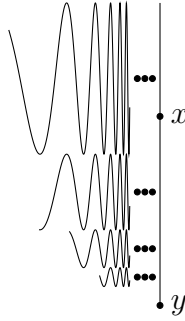
Proof. This is a straightforward application of Proposition 9.2.6 and Lemma 9.3.5. $\square$

Proposition 9.3.7. *Let X be a continuum and suppose for each $x \in X$ there is a unique subcontinuum M_x satisfying the conditions of Theorem 9.3.1. For all $x \in X$ and all $y \in M_x$, $M_y \subseteq M_x$.*

Proof. By applying Lemma 9.3.5 we have that there exists a subcontinuum C of M_x satisfying the properties of the theorem and containing y . Since M_y is unique it must be that $M_y = C$ and consequently $M_y \subseteq M_x$. $\square$

It will often be the case that if $y \in M_x$ then M_y is not just a subset of M_x but these two continua are in fact equal. The next example shows that this is not always true.

Example 9.3.8. Let X be the continuum shown, with points x and y labelled.



Let $S_1, S_2, \dots$ be the $\sin \frac{1}{x}$ continua which make up X , with S_1 the top one containing x . For the point x , the continuum M_x consists of the whole of the vertical line on the right. This is because M_x must contain the non-degenerate E-continuum of S_1 as this contains x . This intersects the corresponding E-continuum of S_2 , so M_x must contain that, and the same is true of each of the S_n . It must therefore contain y , as M_x is closed.

Now let U be an open set containing y . There exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $S_n \subseteq U$. Let z be the top point of the non-degenerate E-continuum of S_N , so $\{z\} = S_{N-1} \cap S_N$. Take a subset $V = (\bigcup_{n \geq N} S_n) \setminus \{z\}$. It is clear that $V = X \setminus \bigcup_{k=1}^{N-1} S_k$, so V is open. It is also clear that $y \in V \subseteq U$. Finally as Proposition 2.2.4 implies z is not a cut point of S_N we have that V is connected by applying Theorem 9.1.13 to the sets $T_n = S_n \cup \dots \cup S_{N+1} \cup S_N \setminus \{z\}$.

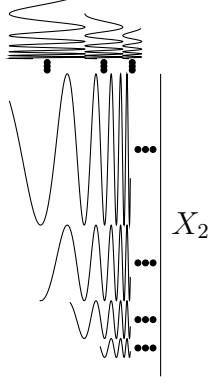
From this we have shown that X is locally connected at y , so the only subcontinuum M_y is $\{y\}$. This is despite the fact that $y \in M_x$.

For an example of a continuum with $M_x \cap M_z \neq \emptyset$ but with neither contained in the other, consider two copies X_1 and X_2 of the continuum X above and take as Y the union of these two continua, intersecting at the points y_1 and y_2 corresponding to y . If $x_i \in X_i$ corresponds to x then we have that $M_{x_1} \cap M_{x_2} = \{y_1\} = \{y_2\}$.

The following two examples show that there can be infinite ascending chains $M_{x_1} \subsetneq M_{x_2} \subseteq \dots$ and infinite descending chains $M_{x_1} \supsetneq M_{x_2} \supsetneq \dots$ of minimal subcontinua about which X is locally connected.

Example 9.3.9. In this example we will construct a continuum X with an infinite, strictly ascending chain of subcontinua $M_{x_1} \subsetneq M_{x_2} \subsetneq \dots$ satisfying the conditions of Theorem 9.3.1. The continuum X will consist of a nested union of continua, each similar to the continuum in Example 9.3.8. The highest $\sin \frac{1}{x}$ continuum in one will act as the vertical line in the next, with infinitely many $\sin \frac{1}{x}$ continua limiting onto it. Let X_1 be the continuum from Example 9.3.8 and let S_k^1 be the various $\sin \frac{1}{x}$

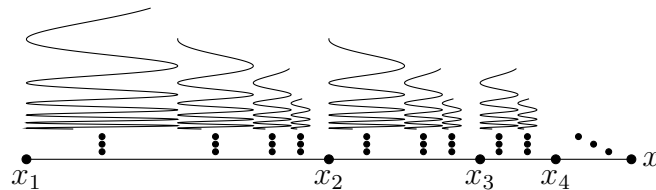
continua in X_1 . Let $X_2 \subseteq \mathbb{I}^2 \times [0, 1/2]$ be X_1 with infinitely many $\sin \frac{1}{x}$ continua limiting onto S_1^1 , each referred to as S_k^2 . Keep defining X_n in this fashion with $X_n \subseteq \mathbb{I}^2 \times [0, 1/2] \times \cdots \times [0, 1/2^{n-1}]$. Let the vertical line in X_1 be denoted L .



Now let $x \in S_1^n \setminus X_{n-1}$ and consider M_x . There is some m such that $x \in S_m^{n+1}$, so the non-degenerate E-continuum of S_m^{n+1} must lie in M_x . This will intersect S_{m-1}^{n+1} and S_{m+1}^{n+1} , so their non-degenerate E-continua must also lie in M_x . Repeating this we have that the whole of $S_1^n \subseteq M_x$. This means that S_1^{n-1} intersects M_x , so the same argument gives us that $S_1^{n-1} \subseteq M_x$. Continuing this process gives us that $S_1^1 \cup \cdots \cup S_1^n \subseteq M_x$ and M_x also contains L . Let this subcontinuum be denoted $K := S_1^1 \cup \cdots \cup S_1^n \cup L$.

Let U be an open subset of X with $K \subseteq U$. There exists a value of $M > 1$ such that for each $m \geq M$ we have that $S_m^{n+2} \subseteq U$. Let $U' = (U \setminus (S_1^{n+2} \cup \cdots \cup S_{m-1}^{n+2})) \cap X_{n+2}$. Then U' is an open subset of X as $U \setminus (S_1^{n+2} \cup \cdots \cup S_{m-1}^{n+2})$ lies in the interior of X_{n+2} . The component of U' containing K will be the union over $k \leq n+1$ of the components of $U' \cap S_l^k$ containing the non-degenerate E-continuum of S_l^k , so this component will be an open set. Thus X is locally connected about K which means $K = M_x$. We have shown that for each point $x \in S_1^n$ we have that $M_x = S_1^1 \cup \cdots \cup S_1^n \cup L$. Thus if we take a sequence of points x_n with each lying in S_1^n then $M_{x_1} \subsetneq M_{x_2} \subsetneq \cdots$.

Example 9.3.10. We will now construct a continuum X by taking the union of infinitely many copies of the continuum from Example 9.3.8 arranged in a line.



Let L be the horizontal line containing the points $x_1, x_2, \dots$ and x and let $[x_n, x_m]$ be the subarc of L between x_n and x_m . Let y be a point of $[x_{n-1}, x_n]$ and consider M_y . This subcontinuum will have to contain the E-continuum of the $\sin \frac{1}{x}$ continuum containing y , which will intersect the adjacent E-continua and therefore contain these. Repeating this gives us that M_y contains the whole of $[x_{n-1}, x_n]$. The same argument can then be applied to $[x_n, x_{n+1}]$, and by a simple induction we have that M_y contains $[x_{n-1}, x]$. The continuum X is locally connected about this subset of L , so it is equal to M_y . From this we can conclude that $M_{x_1} \supsetneq M_{x_2} \supsetneq M_{x_3} \supsetneq \dots$ is a strictly descending chain of subcontinua.

As we did with FMND continua, we can use the minimal subcontinua from Theorem 9.3.1 to define monotone maps onto locally connected continua. This will not be as simple as for FMND continua however, as the minimal subcontinua about which X is locally connected do not form a partition of X , are not always unique and can intersect each other both with and without being contained in each other.

Proposition 9.3.11. *Let X be a continuum such that for each point x in X there is a unique subcontinuum M_x satisfying the conditions of Theorem 9.3.1. Let L be a locally connected continuum and let $\rho : X \mapsto L$ be a monotone map. Then for all x the image of M_x under ρ is a singleton.*

Proof. Let x be a point of X and let $y = \rho(x)$. We need to show that $\rho(M_x) = y$. However, by Proposition 2.5.1 we have that X is locally connected about $\rho^{-1}(y)$ and by applying Lemma 9.3.5 we have that there exists a subcontinuum K of X with $x \in K \subseteq \rho^{-1}(y)$ such that X is locally connected about K and K is a minimal subcontinuum with this property. Since M_x is unique we have that $K = M_x$ and therefore $M_x \subseteq \rho^{-1}(y)$. Thus $\rho(M_x) = y$, which completes our proof. $\square$

Proposition 9.3.12. *Let X be a continuum and suppose for each $x \in X$ there is a unique subcontinuum M_x satisfying the conditions of Theorem 9.3.1. Suppose further that there exists a set $P \subseteq X$ such that $\mathcal{P} = \{M_x | x \in P\}$ is a partition of X with the corresponding quotient space Hausdorff. This partition defines a monotone map from X onto a locally connected continuum, and this map is universal amongst monotone maps to locally connected continua.*

Proof. Define a quotient map $\sigma : X \mapsto L$ where the fibres of σ are the continua M_x with $x \in P$. This is monotone and X is locally connected at each M_x so the image L is locally connected by Proposition 2.5.1. We will show that this map is universal.

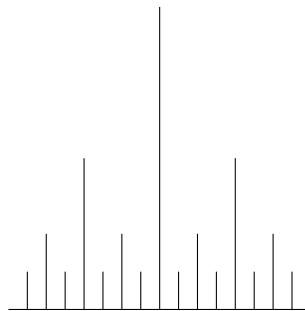
Let $\rho : X \mapsto Y$ be a monotone map onto a locally connected continuum Y . By applying Proposition 9.3.11 we have that $\rho(M_x) = y$ so we can define a map $f : L \mapsto Y$ by $f(l) = \rho(\sigma^{-1}(l))$. It is clear from the construction that $f \circ \sigma = \rho$. To show that f is continuous take an open set $U \subseteq Y$. Then $f^{-1}(U) = \sigma(\rho^{-1}(U))$. Call $V = \rho^{-1}(U)$, then V is an open set consisting of whole fibres of σ . Thus $\sigma^{-1}(\sigma(V)) = V$ which means that $\sigma(V)$ is open under the quotient topology. Thus f is a continuous function which proves that $\sigma : X \mapsto L$ is universal. $\square$

Corollary 9.3.13. *Let X be an FMND continuum and suppose for each $x \in X$ there exists a unique minimal subcontinuum M_x containing x such that X is locally connected at M_x , and suppose $A_x = M_x$. Then the map defined in Definition 9.2.13 is universal amongst monotone maps to locally connected continua provided that L itself is a continuum.*

Proof. We have from Proposition 9.2.5 that the subcontinua A_x form a partition of X . Thus the whole set X will fulfil the requirements of P in Proposition 9.3.12 so the corresponding map $\pi : X \mapsto L$ is universal. $\square$

Remark 9.3.14. The map in Corollary 9.3.13 will not necessarily be universal if each M_x is merely a subcontinuum of A_x , instead of being equal to it as the following example shows.

Example 9.3.15. Define $X \subseteq [0, 1]^2$ as $[0, 1] \times \{0\} \cup \bigcup_{n \in \omega} \bigcup_{k \text{ odd}} \{k/2^n\} \times [0, 1/2^n]$.



This continuum is locally connected, so for each point x there exists a unique $M_x = \{x\}$. However, the line $[0, 1] \times \{0\}$ is a subcontinuum. It has empty interior as the vertical lines clearly limit on to every point $(k/2^n, 0)$, which forms a dense subset of $[0, 1] \times \{0\}$, meaning the vertical lines contain the whole of the horizontal line in their closure. Thus for any x on the horizontal line $M_x = \{x\}$ but $A_x = [0, 1] \times \{0\}$.

The monotone map π as described in Corollary 9.3.13 will have the horizontal line as its only non-degenerate fibre. The image of this map will be homeomorphic to the fan



The identity map $id : X \mapsto X$ cannot factor through π , so π is not universal.

This chapter started by considering the equivalence classes of the equivalence relations defined in Chapters 2, 4 and 8 and asking which properties of these subcontinua led to Theorems 2.4.11, 4.3.17 and 8.3.12. We introduced two properties, the first having a full set of maximal nowhere dense subcontinua and the second having unique minimal subcontinua about which X is locally connected. If a continuum is FMND then we immediately know a great deal about its structure as a result of Theorem 9.2.22, but we have nothing to say about universality. In contrast the second property leads to a number of universality properties but very little can be said about structure. It would certainly not be possible to place limits on the dimension of a monotone image, as for each $n \in \mathbb{N}$ we have that $[0, 1]^n$ is a continuum with unique minimal subcontinua, namely the singletons. We will end this chapter with a theorem and two corollaries which tie together the FMND property, universal maps and dimensions.

Theorem 9.3.16. *Let X be a continuum and let $\pi_X : X \mapsto L_X$ be a monotone surjection onto a locally connected continuum. Suppose π_X is universal amongst such maps. Let $\rho : X \mapsto Y$ be a monotone surjection onto a continuum Y . Then Y has a universal monotone map onto a locally connected continuum $\pi_Y : Y \mapsto L_Y$.*

Proof. Define $\mathcal{F} = \{\rho(\pi_X^{-1}(l)) \mid l \in L_X\}$ to be the images of the fibres of π_X under ρ . Using $\mathcal{F}$ define an equivalence relation $\sim$ on Y by taking the transitive and sequential closures of the sets in $\mathcal{F}$. This means that if there exist $y_n \in Y$ with $y \sim y_n$ for all n , $y_n \rightarrow y_\infty$ then $y \sim y_\infty$.

We will first show that each equivalence class is a continuum. Let $y \in Y$ and let $y_\sim$ denote the equivalence class under $\sim$. Let $A_1 = \pi_X^{-1}(\pi_X(\rho^{-1}(y)))$ and let $B_1 = \rho(A_1)$. For all ordinals α define $A_{\alpha+1} = \pi_X^{-1}(\pi_X(\rho^{-1}(B_\alpha)))$ and let $B_{\alpha+1} = \rho(A_{\alpha+1})$. For

a limit ordinal γ define $B_\gamma = \overline{\bigcup_{\alpha < \gamma} B_\alpha}$. A_γ does not need defining in this case. Each B_α is a continuum and is a subset of $y_\sim$ as $y_\sim$ is sequentially and transitively closed. Each subcontinuum of Y is also a closed subset of Y so there are at most $2^{|Y|}$ subcontinua. Thus the transfinite sequence B_α must terminate at some point. Such a termination can only occur when $B_\alpha = y_\sim$ for some α , which proves that $y_\sim$ is a continuum.

We now define $L_Y = Y/\sim$ and π_Y to be the quotient map. L_Y is locally connected as Y is locally connected about each equivalence class of $\sim$ (Proposition 2.5.1). We know this because their inverse image under ρ is connected and consists of unions of fibres of π_X , so X is locally connected about them. Let $f : L_X \mapsto L_Y$ be induced from $\pi_Y \circ \rho$ by universality of π_X .

Let M be a locally connected continuum and let $\sigma : Y \mapsto M$ be monotone. Let $g : L_X \mapsto M$ be induced from $\sigma \circ \rho$ by universality of π_X . Consider $\{\sigma^{-1}(m) | m \in M\}$. This is a partition of Y into continua. Furthermore, we have that $g \circ \pi_X = \sigma \circ \rho$, so each $\rho(\pi_X^{-1}(l))$ must lie in some $\sigma^{-1}(m)$. This means that the partition $\{\sigma^{-1}(m) | m \in M\}$ is refined by $\{y_\sim | y \in Y\}$ i.e. for each $y \in Y$ there exists $m \in M$ such that $y_\sim \subseteq \sigma^{-1}(m)$. This is another way of saying that σ preserves the equivalence classes of $\sim$. Now recall that $f \circ \pi_X = \pi_Y \circ \rho$. Given any $l' \in L_Y$ we have that

$$g(f^{-1}(l')) = \bigcup_{x \in \rho^{-1}(y), y \in \pi_Y^{-1}(l')} g \circ \pi_X(x) = \bigcup_{x \in \rho^{-1}(y), y \in \pi_Y^{-1}(l')} \sigma \circ \rho(x) = \bigcup_{y \in \pi_Y^{-1}(l')} \sigma(y)$$

We have shown above that for all $y \in \pi_Y^{-1}(l')$ the image $\sigma(y)$ will be the same, so $g(f^{-1}(l'))$ is a singleton.

Because of this we can define $h : L_Y \mapsto M$ by $h(l') = g(f^{-1}(l'))$. From the construction we have that $h \circ \pi_Y = \sigma$, so all that is left is to show that h is continuous. Let $U \subseteq M$ be an open set. Then $h^{-1}(U) = \pi_Y(\sigma^{-1}(U))$. Let $V = \sigma^{-1}(U)$, we know V is open. It only contains whole equivalence classes of $\sim$ so we know that $\pi_Y^{-1}(\pi_Y(V)) = V$. This gives us that $h^{-1}(U) = \pi_Y(V)$ is open under the quotient topology, and thus h is continuous. $\square$

Corollary 9.3.17. *Let X, Y be as in Theorem 9.3.16 with monotone maps π_X, π_Y onto L_X and L_Y . Then L_Y is a monotone image of L_X .*

Proof. It is clear from the construction in the previous theorem that $f : L \mapsto L'$ is monotone. $\square$

Corollary 9.3.18. *Let X, Y be as in Theorem 9.3.16 with monotone maps π_X, π_Y onto L_X and L_Y . If X is FMND with π_X the map from Definition 9.2.13 then $\dim(L_Y) = 1$ or L_Y is a singleton.*

Proof. The map $f \circ \pi_X : X \mapsto L'$ satisfies the conditions of Lemma 9.2.21. Since L' is a continuum it can only be zero dimensional if L' is a singleton, otherwise $\dim(L') = 1$. $\square$

Chapter 10

Open Questions

Finite Irreducibility

Question. In Sections 3.4 and 3.5 we saw a number of theorems which required each subset $\lambda(p_i)$ to be compact. Counterexamples with non-compact sets $\lambda(p_i)$ were often similar to the continuum in Example 3.4.6. This continuum contains points p_1, p_2, p_3 and p_4 such that $X = irr(p_1, p_2, p_3, p_4)$, which implies that for each $1 \leq i \leq 4$ there exists a proper subcontinuum of X containing each point except p_i . However, every subcontinuum $Y \subseteq X$ containing p_1 and p_3 must also contain one of p_2 or p_4 . Suppose the definition of n -irreducibility were changed to require that for each proper subset $P \subsetneq \{p_1, \dots, p_n\}$ there exists a subcontinuum Y of X such that $Y \cap \{p_1, \dots, p_n\} = P$. Would the compactness assumption of these theorems still be required under this new definition?

On $\sin \frac{1}{x}$ continua

Question. Are the composants of A_ω all homeomorphic? Are the composants other than A^∞ homeomorphic to each other? Is the component of $(1, 1, \dots)$ unique?

Compactifications

Question. Can a similar list of conditions be found such that any space X satisfying each condition will have a compactification γX which is ∞ -irreducible and whose remainder consists of the irreducible points of X ? In the introduction of Chapter 7 we have seen that X will not be locally compact as if $\gamma X = irr(A)$ then A is not a closed set.

Maximal and Minimal Subcontinua

Question. Are there properties of continua, other than irreducibility, such that any continuum with these properties must also be FMND? Similarly, are there properties of a continuum X which guarantee that each point $x \in X$ lies in a unique minimal subcontinuum of X about which X is locally connected?

Question. If X is a FMND continuum and $\pi : X \mapsto L$ is the corresponding quotient map, can conditions be placed upon X to guarantee that L is Hausdorff, and therefore a continuum? One obvious example would be if the partition $\{A_x | x \in X\}$ is upper semicontinuous, but are there any which relate more directly to the structure or properties of X ?

Question. If X is a FMND continuum and $\pi : X \mapsto L$ is the corresponding quotient map, if L is Hausdorff is L an FMND continuum?

Question. Suppose X is an FMND continuum such that for each point $x \in X$ there is a unique minimal subcontinuum $x \in M_x \subseteq X$ with X locally connected about M_x . Under what circumstances do we have that $A_x = M_x$?

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