BANACH SPACES WITH FEW OPERATORS AND MULTIPLIER RESULTS

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Abstract

The construction of a non-separable reflexive Banach space on which every operator is the sum of a scalar multiple of the identity operator and an operator of separable range is presented.

Using a result of Rao, a sufficient condition is given for Banach spaces with smooth norms to be decomposable.

It is shown that operators on Banach spaces of co-dimension one in their biduals are the sum of a scalar multiple of the identity operator and a weakly compact operator. The Banach spaces of bounded operators $L(l^1, l^p)$ $(1<p<\infty)$ and $L(l^p, l^r)$, $1 < p \leq r \leq p^1 < \infty$, where $1/p + 1/p^1 = 1$, are shown to be primary.

The spaces of bounded diagonal operators and compact diagonal operators on a semi-normalized Schauder basis β, the multiplier algebras $L_d(X, \beta)$ and $K_d(X, \beta)$, are introduced and studied. New examples of these multiplier algebras are presented and a theorem of Sersouri is extended. A necessary and sufficient condition is given for c_o to embed in $K_d(X, \beta)$. A sufficient condition is given on a semi-normalized Schauder basis β of a reflexive hereditarily indecomposable Banach space Y to ensure that $K_d(Y, \beta)$ has the RNP. It is shown that the algebra $L_d(X, \beta)$ is semisimple and that on the algebra $K_d(X, \beta)$ derivations are automatically continuous.

By representing diagonal operators as stochastic processes a general method of constructing multiplier algebras is given. A non trivial multiplier invariance for the normalized Haar basis of $L^1[0,1]$ is proved.

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Introduction

This thesis is concerned with the problem of whether there exists an infinite dimensional Banach space X on which every operator $T : X \to X$ is the sum of a scalar multiple of the identity and a compact operator.

There are two reasons for asking this question. The first is that on such a Banach space every operator would have pleasant spectral properties. The second reason, and the more important one from the point of view of this thesis, is that there is a sense in which such a Banach space would have "few" operators.

It follows from the axiom of choice, by means of the Hahn-Banach theorem, that any infinite dimensional Banach space is well endowed with linear functionals. As a consequence it is always possible to construct finite rank operators on an infinite dimensional Banach space.

If one considers the closure of the space of finite rank operators on a Banach space X in the operator norm topology, the resulting space is determined entirely by the linear and topological properties of the Banach space X. For instance, if the Banach space X has a Schauder basis, then the closure of the space of finite rank operators in the operator norm topology coincides with the space of compact operators on X. In this case the Banach space X has few operators if every operator on X is o f the form $\lambda I + K$, where λ is a scalar, I is the identity operator and K is a compact operator.

Recently Barnes [1] has shown that any Banach algebra of bounded operators on a Banach space X which contains the finite rank operators must contain the ideal of nuclear operators with continuous inclusion. Therefore a more extreme few operators question is whether there exists an infinite dimensional Banach space on which every operator is the sum of a scalar multiple of the identity and a nuclear operator. By Barnes' result such a Banach space would have as few operators as the axioms of set theory al low a Banach space to possess.

Other few operators questions may be asked. Is there a non separable Banach space X on which every operator is the sum of a scalar multiple of the identity and an operator of separable range? Is there a non reflexive Banach space on which every operator is the sum of a scalar multiple of the identity and a weakly compact operator? Is there a Banach space on which every operator is the sum of a scalar multiple of the identity and a strictly singular operator?

Thus it is natural to search for a Banach space X where every operator is a small perturbation of a scalar multiple of the identity, where the notion of smallness will depend on the properties of the Banach space X.

A more general approach to this class of problems is to consider how the linear and topological properties of Banach spaces X and Y determine the linear and topological properties of the space of bounded operators between X and Y, $L(X,Y)$. The few operators questions which arise in this setting include the following. Is there a pair a Banach spaces X and Y such that every bounded, linear operator from X to Y is compact and every bounded, linear operator from Y to X is compact? Is th ere a pair of Banach spaces X and Y such that every operator from X to Y is nuclear and every bounded operator from Y to X is nuclear?

A very important class of operators in the space of bounded operators on a Banach space X is the idempotents or projections, i.e. the operators $P : X \to X$ such that $P^2 = P$. In [2] it was shown that Hilbert spaces are characterized among the Banach spaces by having the property that every closed subspace is the range of a bounded projection. The range of a projection on a Banach space X is closed and is said to be a complemented subspace of X. Therefore Hilbert spaces are very rich in proj ections. A Banach space X would have as few projections as possible, and is said to be indecomposable, if there is no projection $P : X \to X$ such that rank $(P) =$ rank $(I - P) = \infty$.

It is also true that if Banach spaces X and Y are abundant in projections, then it is possible to study the closed subspaces of X, Y or even $L(X, Y)$ which are the ranges of projections on these spaces. When the Banach spaces X and Y are rich in projections it is the case that $L(X, Y)$ is "large " in some sense. For instance many complemented subspaces of $L(X, Y)$ may in fact be isomorphic to $L(X, Y), X^*$ or Y.

It is a result of Mazur that every infinite dimensional Banach space contains a subspace with a Schauder basis. ([4], p.7). Therefore, for such a subspace, the space of diagonal operators may be investigated.

For a Banach space X with a Schauder basis $\beta = (e_n)_{n=1}^{\infty}$, the space of diagonal operators on $\beta, L_d(X, \beta)$, consists of operators $T : X \to X$ where $Te_n = \lambda_n e_n$ for all n for some scalars λ_n. The richness of the structure of the space $L_d(X, \beta)$ depends on the properties of the Schauder basis β and the few operators questions asked for $L(X)$ can easily be modified to this setting.

Banach spaces which are rich in projections include those which have unconditional Schauder bases. (A Banach space X has unconditional Schauder basis $\beta = (e_n)_{n=1}^{\infty}$ if $L_d(X, \beta)$ can be canonically identified with the Banach space l^{∞}). In these spaces, any sequence consisting of zeros or ones in $L_d(X, \beta)$ corresponds to a projection on X. For a Banach space X with an unconditional basis it is natural to ask whether every operator on X is of the form $D + K$, where D is a diagonal operator on β and K is a compact operator.

An important theorem which underlies several results in this thesis is due to Ramsey [3]. The version we use in Chapter 4 can be stated as follows.

Think of the integers as ordinals so that each integer n is the set consisting of its predecessors and $n \in m$ means $n < m$. By $[n]^{(r)}$ we mean the set of all subsets of n of size r. By $[n]$ we shall mean $[n]^{(1)}$, i.e. simply n itself. A k-colouring of a set S is a map γ of S into a set $\{c_1, c_2, \ldots, c_k\}$ of k colours. If γ is a k-colouring of $L^{(r)}$ then $M \subseteq L$ is said to be monochromatic of γ is constant on $M^{(r)}$. We now state a version of Ramsey's theorem.

Given natural numbers k, r and m there is a natural number n such that for any k-

colouring of $[n]^{(r)}$ there is a monochromatic m-subset of $[n]$.

The results of Chapter 1 depend on a very powerful anti-Ramsey theorem for colourings of graphs on the first uncountable ordinal w_1. In Chapter 3 results due to Gowers are presented which rely heavily on topological Ramsey theory.

Another important result used extensively in Chapter 4 is Pelczyński's decomposition theorem. If a Banach space X is complemented in a Banach space Y and Y is complemented in X and both X and Y are isomorphic to their Cartesian squares, $X \cong X \oplus X$ and $Y \cong Y \oplus Y$, then X is isomorphic to Y. This result enables us to study the richness of the structure of certain spaces of bounded operators.

Chapter 2 of this thesis is a brief survey of few operator results in finite dimensional Banach spaces and contains no original material. Similarly Chapter 3 represents a survey of very important recent results on hereditarily indecomposable Banach spaces and with the exception of Theorem 3.35 contains no new results. The remainder of the thesis represents the author's own contribution to the subject, except for Chapter 4, Theorem 4 which is due to J. Bourgain and F. Delbaen. Original results are italicised and known results are not. The results discussed in the thesis represent recent progress on problems which were open for many years.

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Chapter 1

A non-separable Reflexive Banach space on which there are few operators

0. Introduction

In [1], using an axiom of set theory in addition to ZFC, the $\diamondsuit_{\aleph_1}$ condition, Shelah constructed a non separable space on which every bounded linear operator is the sum of a scalar multiple of the identity operator and an operator of separable range. In [2], Shelah and Steprans obtained this result without the use of $\diamondsuit_{\aleph_1}$ or any other additional set theoretic axioms.

Lindenstrauss identified the following large class of Banach spaces which are rich in projections [4], [5].

Definition 1 A Banach space has the *separable complementation property* if every separable subspace is contained in a separable and complemented subspace.

Reflexive spaces [4], [5] and more generally weakly compactly generated spaces [6] are known to have the separable complementation property. (Recall that a Banach space is weakly compactly generated if it is the closed linear span of a weakly compact subset.)

The aim of this chapter is to observe that it is possible to modify the construction of Shelah and Steprans [2] to give a non separable weakly compactly generated Banach space on which every bounded linear operator is the sum of a scalar multiple of the identity operator and an operator of separable range and then to construct a reflexive Banach space with this few operators property.

Since these spaces have the separable complementation property there is a sense in which they have as few operators as their linear and topological properties will allow.

1. Notation

As in [2], the integers will be thought of as ordinals so that $n \in m$ means $n < m$. By $[X]^\lambda$ and $[X]^{<\lambda}$ is meant the set of all subsets of X of size λ and of size less than λ respectively. By X^Y we mean the set of functions from Y to X. In particular $\mathbb{R}^{(\omega_1)}$ is the real vector space of all real "ω_1-tuples".

If $X \subseteq \omega_1$, $\mathbb{I}_X$ denotes the characteristic function of X.

2. The Construction of Shelah and Steprans.

Given $\mathcal{H} \subseteq [\omega_1]^{<\omega}$ and $\mathcal{G} \subseteq [\omega_1]^{<\omega}$, define $\| \cdot \|_{\mathcal{H},p}$ on $\mathbb{R}^{(\omega_1)}$ by

$$\| x \|_{\mathcal{H},p} = \sup\{(\sum_{\alpha \in H} |x(\alpha)|^p)^{\frac{1}{p}} : H \in \mathcal{H}\}$$

Given $\mathcal{K} \subseteq [[\omega_1]^2]^{<\omega}$, define $\| \cdot \|_{\mathcal{K},p}$ on $\mathbb{R}^{(\omega_1)}$ by

$$\| x \|_{\mathcal{K},p} = \sup\{\left(\sum_{\{\alpha,\beta\} \in K} |x(\alpha) - x(\beta)|^p\right)^{\frac{1}{p}} : K \in \mathcal{K}\}$$

If we define $\| \cdot \|_{\mathcal{H},\mathcal{K},p}$ on $\mathbb{R}^{(\omega_1)}$ by

$$\| x \|_{\mathcal{H},\mathcal{K},p} = \max[\| x \|_\infty, \| x \|_{\mathcal{H},p}, \| x \|_{\mathcal{K},p}]$$

and take $X_p(\mathcal{H},\mathcal{K})$ to be the completion of the space $\mathbb{R}^{(\omega_1)}$, then $X_p(\mathcal{H},\mathcal{K})$ is a Banach space under the norm $\| \cdot \|_{\mathcal{H},\mathcal{K},p}$.

Shelah and Steprans considered the case $p = 1$ and chose $\mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ to satisfy

(A) Each element K of $\mathcal{K}$ can be written as $K = \{\{\alpha_{2j}, \alpha_{2j+1}\} : j \in n\}$ where $\alpha_0 < \alpha_1 < \ldots < \alpha_{2n-1}$

(B) If $G \in \mathcal{G}, H \in \mathcal{H}$ and $K \in \mathcal{K}$ then

(a) $\#(G \cap H) \leq 1$

(b) $\#(G \cap \cup K) \leq 2$.

(c) $\#(H \cap \cup K) \leq 2$.

(C) If $K, K^1 \in \mathcal{K}$ then $\#\{\alpha \in w_1 : \exists \beta \neq \beta^1$ with $\{\alpha,\beta\} \in K$ and $\{\alpha,\beta^1\} \in K^1\} \leq 5$.

(D) If $(\{\alpha_\xi, \beta_\xi\})_{\xi \in w_1}$ is a disjoint family in $[w_1]^2$ then for every $n \in \omega$ there exists

(a) an injection $\varphi : n \to \omega$, such that $\{\alpha_{\varphi(m)} : m \in n\} \in \mathcal{G}$ and $\{\beta_{\varphi(m)} : m \in n\} \in \mathcal{H}$.

(b) and injection $\psi : n \to \omega_1$ such that

$$\{\{\alpha_{\psi(m)}, \beta_{\psi(m)}\} : m \in n\} \in \mathcal{K}$$

The proof of the existence of $\mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ depends on an extension of a powerful anti-Ramsey result of Todorcevic [9].

Lemma 1

Let I be a set of cardinality $\aleph_1$. There exists a function $F : [\omega_1]^2 \to I$ such that for every disjoint family $(\{\alpha^0(\zeta), \alpha^1(\zeta)\})_{\zeta\in\omega}$, in $[\omega_1]^2$, for every $m \in \omega$ and every pair of functions $f_0, f_1 : [m]^2 \to I$ there exists an injection $\varphi : m \to \omega_1$ such that $F(\{\alpha^u(\phi(j)), \alpha^u(\phi(k))\}) = f_u(\{j,k\})$ for $u \in \{0,1\}$ and each $\{j,k\} \in [m]^2$.

To define $\mathcal{G}, \mathcal{H}$ and $\mathcal{K}$, Shelah and Steprans consider a function

$$F = (F_0, F_1) : [\omega_1]^2 \to \{0,1,2\} \times [\omega_1]^{<\aleph_0}$$

as in Lemma 1.

Then

$$\mathcal{G} = \{A \in [\omega_1]^2 : F_0(B) = 0,\ \forall B \in [A]^2\}$$

$$\mathcal{H} = \{A \in [\omega_1]^2 : F_0(B) = 1, \quad \forall B \in [A]^2\}$$

and K in $[[\omega_1]^2]^{<\aleph_0}$ lies in $\mathcal{K}$ if and only if

$K = \{\{\alpha^0_m, \alpha^1_m\} : m \in n\}$ for some $n \in \omega$, where

$$\alpha^u_j < \alpha^v_k \quad \forall j \in n \in K,\ u \in v \in 2$$

$$F_0(\{\alpha^0_j, \alpha^0_k\}) = F_0(\{\alpha^1_j, \alpha^1_k\}) = 2,\ \forall j \in k \in n.$$

$$\begin{aligned} F(\{\alpha^0_j, \alpha^0_k\}) &= F_1(\{\alpha^1_j, \alpha^1_k\}) \\ &= \cup\{\{\alpha^0_i, \alpha^1_i\} : i \in j\},\ \forall j \in k \in n. \end{aligned}$$

To prove (D) for $\mathcal{K}$, the next lemma, which we shall use later, is employed.

Lemma 2 Let $(\{\alpha^0_\zeta, \alpha^1_\zeta\})_{\zeta\in\omega}$, be a family of mutually disjoint sets in $[\omega_1]^2$. Let $F : [\omega_1]^2 \to \{0,1,2\} \times [\omega_1]^{<\aleph_0}$ be the function given by Lemma 1. Let $m \in \{0,1,2\}$ be given. There exists an injection $\psi_m : \omega \to \omega_1$ such that

$$F_0(\{\alpha^u(\psi_m(j)),\ \alpha^u(\psi_m(k))\}) = m$$

and

$$F_1(\{\alpha^u(\psi_m(j)), \alpha^u(\psi_m(k))\})$$

$$= \cup\{\{\alpha^0(\psi_m(i)), \alpha^1(\psi_m(i))\} : i \in j\},$$

whenever $u \in \{0,1\}$ and $j \in k \in \omega$.

Proof We shall define ψ_m recursively for $m \in \{0,1,2\}$. At stage n suppose that $\psi_{m|n}$ is defined and that there exists uncountable Λ_n such that

$$\begin{aligned} & F_0(\{\alpha^u(\psi_m(j)), \alpha^u(\psi_m(k))\}) \\ = \;& F_0(\{\alpha^u(\psi_m(j)), \alpha^u_\zeta\}) = m, \text{ and} F_1(\{\alpha^u(\psi_m(j)), \alpha^u(\psi_m(k))\}) \\ = \;& F_1(\{\alpha^u(\psi_m(j)), \alpha^u_\zeta\}) = \cup\{\{\alpha^0(\psi_m(i)), \alpha^1(\psi_m(i))\} : i \in j\} \end{aligned}$$

whenever $u \in \{0,1\}, \zeta \in \Lambda_n$ and $j \in k \in n$.

CLAIM There exists $\zeta \in \omega_1$, such that the set $\{\eta \in \Lambda_n : F(\{\alpha^u_\zeta, \alpha^u_\eta\}) = (m, \cup\{\alpha^0(\psi_m(j)), \alpha^1(\psi_m(j))$ $j \in n\}) \, \forall u \in \{0,1\}$ is uncountable.

If the CLAIM is false, we can find uncountable $\Gamma \subseteq \Lambda_n$ such that, for all $\{\zeta, \eta\} \in [\Gamma]^2$ there exists $u \in \{0,1\}$ with $F(\{\alpha^0_\zeta, \alpha^1_\zeta\}) \neq (m, \cup\{\{\alpha^0(\psi_m(j)), \alpha^1(\psi_n(j))\} : j \in n\} = \delta$, say.

Consider the family of disjoint sets $(\{\alpha^0_\zeta, \alpha^1_\zeta\})_{\zeta \in \omega_1}$. Define $f_0, f_1 : [\{0,1\}]^2 \to I$ by $f_0\{\{0,1\}) = f_1(\{0,1\} = \delta$.

By Lemma 1 there exist an injection $\mathcal{X} : \{0,1\} \to \Gamma$ such that $F(\{\alpha^u(\mathcal{X}(j)), \alpha^u(\mathcal{X}(k)\}) = \delta$, $\forall u \in \{0,1\}$ and each $\{j,k\} \in [\{0,1\}]^2$. This is a contradiction, so the CLAIM is true.

Now define $\psi_m(n)$ to be any $\zeta \in \omega$, predicted by the CLAIM.

The proof of Lemma 2 is now complete.□

Let X_1 denote the space $X_1(\mathcal{H}, \mathcal{K})$ which is considered in [2] and X_2 denote the space $X_2(\mathcal{H}, \mathcal{K})$.

3. Properties of X_1 and X_2.

As in [2] the properties of $\mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ imply the next result.

Lemma 3 *Let* $\|\cdot\|$ *denote the norm of* X_1 *or* X_2. *If* y *is a vector supported on a set* $G \in \mathcal{G}$ *and* z *is a vector supported on a set* $\cup K$, *where* $K \in \mathcal{K}$, *then*

$$\| y \| \leq 2 \| y \|_\infty \text{ and } \| z \| \leq 10 \| z \|_\infty .$$

Proof From property (B) we have that

$$\#(G \cap H) \leq 1 \text{ and } (G \cap (\cup K^1)) \leq 2$$

for any $H \in \mathcal{H}$ and $K^1 \in \mathcal{K}$.

So,

$$\begin{aligned}
\| y \|_{\mathcal{H},2} \leq \| y \|_{\mathcal{H},1} &= \sup\{(\textstyle\sum_{\alpha\in H} |y(\alpha)|) : H \in \mathcal{H}\} \\
&\leq \sup\{\| y \|_\infty \#(G \cap H) : H \in \mathcal{H}\} \\
&\leq \| y \|_\infty \\
\| y \|_{\mathcal{K},2} \leq \| y \|_{\mathcal{K},1} &\leq \sup\{\textstyle\sum_{\{\alpha,\beta\}\in K^1} |y(\alpha) - y(\beta)| : K^1 \in \mathcal{K}\} \\
&\leq \sup\{\textstyle\sum_{\{\alpha,\beta\}\in \mathcal{K}^1} [|y(\alpha)| + |y(\beta)|] : \mathcal{K}^1 \in \mathcal{K}\} \\
&\leq \sup\{\textstyle\sum \| y \|_\infty \#(G \cap (\cup K^1)) : K^1 \in \mathcal{K}\} \\
&\leq 2 \| y \|_\infty .
\end{aligned}$$

So, using the definition of the norm $\| \cdot \|$, we have

$$\| y \|_\infty \leq \| y \| \leq 2 \| y \|_\infty .$$

For the vector z,

$$\begin{aligned}
\| z \|_{\mathcal{H},2} \leq \| z \|_{\mathcal{H},1} &= \sup\{\textstyle\sum_{\alpha\in H} |z(\alpha)| : H \in \mathcal{H}\} \\
&\leq \sup\{\| z \|_\infty \#(H \cap (\cup K)) : H \in \mathcal{H}\} \\
&\leq 2 \| z \|_\infty \\
\| z \|_{\mathcal{K},2} \leq \| z \|_{\mathcal{K},1} &= \sup\{(\textstyle\sum_{\{\alpha,\beta\}\in K^1} |z(\alpha - z(\beta)| : K^1 \in \mathcal{K}\} \\
&\leq \sup\{\textstyle\sum_{\{\alpha,\beta\}\in K^1} (|z(\alpha)| + |z(\beta)|) : K^1 \in \mathcal{K}\}
\end{aligned}$$

By property (C), if $K, K^1 \in \mathcal{K}$ then

$$\#(\{\alpha \in \omega_1 : \exists \beta \neq \beta_1 \text{ with } \{\alpha,\beta\} \in K \text{ and } \{\alpha,\beta^1\} \in K^1\}) \leq 5,$$

and so we may write

$$\| z \|_{\mathcal{K},2} \leq \sup\{2 \| z \|_\infty \#(\{\alpha \in \omega_1 : \exists \beta \neq \beta^1 \text{ with } \{\alpha,\beta\} \in K \text{ and}$$

$$\{\alpha,\beta^1\} \in K^1\}) : K^1 \in \mathcal{K}^1\} \leq 10 \| z \|_\infty .$$

Thus,

$$\| z \|_\infty \leq \| z \| \leq 10 \| z \|_\infty$$

and we are done.□

Theorem 4 *Let X be X_1 or X_2. Every bounded linear operator on $X, T : X \to X$, can be expressed as $T = \lambda I + S$ where λ is a scalar and S has separable range.*

Proof of Theorem 4

In the case X_2, the proof is as follows. The proof for X_1 may be found in [2].

Let $T : X_2 \to X_2$ be a bounded linear operator.

Let $e_\alpha = \mathbb{I}_{\{\alpha\}}$ for each $\alpha \in \omega_1$.

CLAIM 1 For any $\gamma \in \omega_1$ there exists $\zeta \in \omega_1$ such that $(Te_\alpha)(\gamma) = 0$ for all $\alpha \geq \zeta$.

If CLAIM 1 were false, there exists $\gamma \in \omega_1$ such that for all $\zeta \in \omega_1$ there exists $\alpha(\zeta) \geq \zeta$ such that $|(Te_{\alpha(\zeta)})(\gamma)| > 0$. Taking Γ to be $\{\alpha(\zeta) : \zeta \in \omega_1\}$, because $\alpha(\zeta) \geq \zeta\, \forall \zeta \in \omega_1$, Γ has cardinality $\aleph_1$.

Also, Γ is the countable union $\Gamma = \cup\{\{\alpha(\zeta) : \zeta \in \omega_1, |(Te_{\alpha(\zeta)})(\gamma)| \geq r\} : r > 0, r \in \mathbb{Q}\}$, and so for some $r > 0$, the set $\Delta = \{\alpha(\zeta) : \zeta \in \omega_1,\ |(Te_{\alpha(\zeta)})(\gamma)| \geq r\}$ is uncountable.

Fix $n \in \mathbb{N}$. We may as well write $\Delta = \{\alpha(\zeta) : \zeta \in \omega_1\}$. Using lemma 2, we can find an injection $\phi : \omega \to \Delta$ such that every finite subset of $\{\alpha(\varphi(m)) : m \in \omega\}$ lies in $\mathcal{G}$.

Let $G = \{\alpha(\phi(m)) : m \in n\} \in \mathcal{G}$ and consider $y \in \mathbb{R}^{(\omega_1)}$ defined by

$y(\alpha) = \text{sign}((\text{Te}_\alpha)(\gamma))$ if $\alpha \in G$ and $y(\alpha) = 0$ if $\alpha \notin G$.

Then,

$$\begin{aligned} \| T \| \| y \|_{X_2} &\geq \| Ty \|_{X_2} \\ &\geq \| Ty \|_\infty \\ &\geq |(Ty)(\gamma)| \\ &= \textstyle\sum_{\alpha \in G} |(\text{Te}_\alpha(\gamma)| \\ &\geq rn \end{aligned}$$

So we have $\forall n \in \mathbb{N}, \| T \| \sqrt{n} \geq \| T \| \| y \|_{X_2} \geq rn$, i.e. $\| T \| \geq r\sqrt{n}$.

For sufficiently large n this is a contradiction, so CLAIM 1 is true.

With a view to a contradiction, suppose now that T is not of the form $D + S$ where D is diagonal on $(e_\alpha)_{\alpha \in \omega_1}$ and S is of separable range.

CLAIM 2 There exists a family of disjoint two element subsets $\{\alpha(\zeta), \beta(\zeta)\}_{\zeta \in \omega_1}$ of ω_1 such that

$$|(\text{Te}_{\alpha(\zeta)})(\beta(\zeta))| > 0\ \forall \zeta \in \omega_1,$$

Suppose CLAIM 2 is false. Also suppose that for each $\zeta \in \omega_1$ there exists $\eta = \eta(\zeta) \geq \zeta$ such that

$|(\text{Te}_\eta)(\gamma)| > 0$ for some $\gamma = \gamma(\zeta) \geq \zeta$ with $\gamma \neq \eta$.

The condition $\gamma(\zeta) \neq \eta(\zeta)$ allows us to extract an uncountable subfamily from $(\{\gamma(\zeta), \eta(\zeta)\})_{\zeta\in\omega_1}$ which satisfies CLAIM 2. So, if CLAIM 2 is false, there exists $\zeta \in \omega_1$, such that for all $\eta \geq \zeta, (\mathrm{Te}_\eta)(\beta) = 0 \, \forall \, \beta > \zeta, \, \beta \neq \eta$.

For each $\eta \geq \zeta$ there exists a scalar r_η and some g_η lying in the closed linear span of $\{e_\alpha : \alpha \in \zeta\}$ such that

$$Te_\eta = r_\eta \cdot e_\eta + g_\eta.$$

Let us define a linear mapping $D : X \to X$ by

$$(Dx)(\alpha) = \begin{cases} r_\alpha x(\alpha), & \text{if } \alpha \geq \zeta \\ 0, & \text{if } \alpha < \zeta \end{cases}$$

If also $\{\Pi_\zeta\}_{\zeta\in\omega_1}$ is the sequence of basis projections associated to $(e_\alpha)_{\alpha\in\omega_1}$, we have

$$T = D + \Pi_\zeta T + T\Pi_\zeta - \Pi_\zeta T \Pi_\zeta.$$

Since Π_ζ is of separable range, $T = D + S$ where D is diagonal and S is of separable range. We have assumed that T does not have this form so CLAIM 2 holds.

Using CLAIM 1 and the fact that every element x has countable support we may assume in CLAIM 2 that $(\mathrm{Te}_{\alpha(\zeta)})(\beta(\eta)) = 0$ whenever $\zeta \neq \eta$. Also $\{\{\alpha(\zeta), \beta(\zeta)\} : \zeta \in \omega_1\} = \cup\{\{\alpha(\zeta), \beta(\zeta)\} : |\,\mathrm{Te}_{\alpha(\zeta)}(\beta(\zeta))| \geq r, \, \zeta \in \omega_1, r > 0, r \in \mathbb{Q}\}$, and so we may as well suppose that there exists $r > 0$ such that

$$|(\mathrm{Te}_{\alpha(\zeta)})(\beta(\zeta))| \geq r, \, \forall \, \zeta \in \omega_1.$$

Using 2D, given $n \in \mathbb{N}$, we find an injection $\varphi : n \to \omega$, such that

$G = \{\alpha(\varphi(m)) : m \in n\} \in \mathcal{G}$ and $H = \{\beta(\varphi(m)) : m \in n\} \in \mathcal{H}$.

Then

$$\begin{aligned} 2 \parallel T \parallel & \geq \parallel T\mathbb{I}_G \parallel_{X_2} \\ & \geq (\textstyle\sum_{\beta\in H} |(\mathbb{I}_G)(\beta)|^2)^{\frac{1}{2}} \\ & \geq \sqrt{n} r. \end{aligned}$$

This is a contradiction for n sufficiently large.

It follows that every operator on X_2 has form $S + D$, where S is of separable range and D is a diagonal operator on $\{e_\alpha\}_{\alpha\in\omega_1}$.

Consider the diagonal operator D. Let us suppose

$(Dx)(\alpha) = \lambda(\alpha)x(\alpha), \, \forall \alpha \in \omega_1$, for scalars $\{\lambda(\alpha)\}_{\alpha\in\omega_1}$.

CLAIM 3 There exists $\zeta \in \omega_1$ such that for all $\eta^1 > \eta \geq \zeta$, $\lambda(\eta) = \lambda(\eta^1)$.

CLAIM 3 asserts that the family $\{\lambda(\alpha)\}_{\alpha\in\omega_1}$ is eventually constant. If it were false, for every $\zeta \in \omega_1$ there exist $\eta^1 > \eta \geq \zeta$ such that $|\lambda(\eta) - \lambda(\eta^1)| > 0$. As before, we can find $r > 0$ and mutually disjoint two element subsets $(\{\alpha(\zeta),\ \beta(\zeta)\})_{\zeta\in\omega_1}$ in $[\omega_1]^2$ such that

$$|\lambda(\alpha(\zeta)) - \lambda(\beta(\zeta))| \geq r \,\forall \zeta \in \omega_1.$$

Let $n \in \mathbb{N}$ be given. By $2(D)$, there exists an injection $\psi : n \to \omega_1$ such that

$$K = \{\{\alpha(\psi(m)),\ \beta(\psi(m))\} : m \in n\}$$

lies in $\mathcal{K}$.

$$\begin{aligned} \| \mathcal{D}\mathbb{I}_{\cup(K)} \|_{X_2} &\geq (\textstyle\sum_{m\in n} |(\mathcal{D}\mathbb{I}_{\cup(K)})(\alpha(\psi(m))) - (\mathcal{D}\mathbb{I}_{\cup(K)})(\beta(\psi(m)))|^2)^{\frac{1}{2}} \\ &= (\textstyle\sum_{m\in n} |\lambda(\alpha(\psi(m))) - \lambda(\beta(\psi(m)))|^2)^{\frac{1}{2}} \\ &\geq \sqrt{n}r \end{aligned}$$

So,

$$5 \| D \| > r\sqrt{n}.$$

Again this is a contradiction for sufficiently large n and the proof of Theorem 4 is complete.□

Some further properties of X_1 and X_2 are inherent in the construction.

Proposition 5 *If X is X_1 or X_2, then X contains isomorphic copies of c_o. In particular X is non reflexive. Also X_1 contains isometric copies of ℓ^1.*

Proof Consider an uncountable family $(\{\alpha^0_\zeta, \alpha^1_\zeta\})_{\zeta\in\omega_1}$ in $[\omega_1]^2$. By Lemma 2, we can find an injection $\psi : \omega \to \omega_1$ such that for all $\{j, k\} \in [\omega]^2$,

$$F_o(\{\alpha^o(\psi(j)), \alpha^o(\psi(k))\}) = 0$$

Thus, the set $C_1 = \{(\alpha^o(\psi(j)) : j \in \omega\}$ is such that every finite subset lies in $\mathcal{G}$.

If $(e_\alpha)_{\alpha\in\omega_1}$ denotes the basis $\{\mathbb{I}_{\{\alpha\}}\}_{\alpha\in\omega_1}$, then it is easily checked that $\{e_{\alpha^o(\psi(j))} : j \in \omega\}$ is a copy of the standard c_o basis in X.

If $H \in \mathcal{H}$ is given, then

$$\sum_{\alpha\in H} |\sum_{k=0}^{m-1} c_k e_{\alpha(\psi(k))}| \leq \sum_{k=0}^{m-1} |c_k| \sum_{\alpha\in H} e_{\alpha(\psi(k))}(\alpha)$$

Since, $G = \{\alpha(\psi(k)) : k \in m\} \in \mathcal{G}$ and $\#(G \cap H) \leq 1$, the expression

$$\sum_{\alpha\in H} e_{\alpha(\psi(k))}(\alpha)$$

takes a non zero value for at most one $k \in m$, and this value is 1.

So,

$$\sum_{\alpha\in H}|\sum_{k=0}^{m-1} c_k e_{\alpha(\psi(k))}(\alpha)| \leq \sup\{|c_k| : k\in m\}$$

It follows that for $1\leq p<\infty$,

$$\|\sum_{k=1}^{m-1} c_k e_{\alpha(\psi(k))}\|_{\mathcal{H};p} \leq \sup\{|c_k| : k\in m\}$$

Consider now, $K=\{\{\gamma_0,\gamma_1\},\{\gamma_2,\gamma_3\},\cdots,\{\gamma_{2p-2},\gamma_{2p-1}\}\}\in\mathcal{K}$.

Then

$$\begin{aligned}
&\left(\sum_{j=0}^{p-1}|\sum_{k=0}^{m-1} c_k(e_{\alpha(\psi(k))}(\gamma_{2j})-e_{\alpha(\psi(k))}(\gamma_{2j+1}))|^2\right)^{1/2}\\
&\leq \sum_{k=0}^{m-1}|c_k|\sum_{j=0}^{p-1}(e_{\alpha(\psi(k))}(\gamma_{2j})+e_{\alpha(\psi(k))}(\gamma_{2j+1}))\\
&\leq \sum_{k=0}^{m-1}|c_k|\#(\{\alpha(\psi(k))\}\cap(\cup K)).
\end{aligned}$$

Since $G=\{\alpha(\psi(k)) : k\in m\}\in\mathcal{G}$, $\#(G\cap(\cup K))\leq 2$, so the expression $\#(\{\alpha(\psi(k))\}\cap(\cup K))$ takes the non zero value 1 at most twice, so that

$$\|\sum_{k=0}^{m-1} c_k e_{\alpha(\psi(k))}\|_{\mathcal{K},p} \leq 2\sup\{|c_k| : k\in m\}$$

Thus in X_1 or X_2 we have

$$\begin{aligned}
\sup\{|c_k| : k\in m\} &\leq \|\sum_{k=0}^{m-1} c_k e_{\alpha(\psi(k))}\|\\
&\leq 2\sup\{|c_k| : k\in m\}
\end{aligned}$$

We can also find, by Lemma 2, an injection $\phi:\omega\to\omega_1$ such that for all $\{j,k\}\in[\omega]^2$,

$$f_0(\{\alpha^0(\varphi(j)),\alpha^0(\varphi(k))\})=1.$$

Thus, the set $C_2=\{\alpha^0(\varphi(j)) : j\in\omega\}$ is such that every finite subset lies in $\mathcal{H}$. Then, for any $n\in\mathbb{N}$,

$$\begin{aligned}
&\|\sum_{k=1}^{m-1} c_k e_{\alpha^0(\phi(k))}\|_{\mathcal{H},p}\\
&\geq \sum_{\alpha\in H}|\sum_{k=1}^{m-1} c_k e_{\alpha^0(\varphi(k))}(\alpha)|
\end{aligned}$$

where $H=\{\alpha^0(\varphi(k)) : k\in m\}\in\mathcal{H}$.

$$\begin{aligned}
&=\sum_{j=0}^{m-1}|\sum_{k=1}^{m-1} c_k e_{\alpha^0(\phi(k))}(\alpha^0(\phi(j)))|\\
&=\sum_{j=0}^{m-1}|c_j|.
\end{aligned}$$

It follows that, $\|\sum_{j=0}^{m-1} c_j e_{\alpha^0(\phi(j))}\|_{X_1}=\sum_{j=0}^{m-1}|c_j|$ and $\{e_{\alpha^0(\phi(j))} : j\in\omega\}$ is an isometric copy of the standard ℓ^1 basis and we are done.□

Proposition 6 *The Banach space X_1 is not weakly compactly generated.*

Proof A Banach space X is weakly compactly generated if and only if there exists a reflexive space R and a bounded linear injection $T : R \to X$ such that $T(R)$ is norm dense in X [10].

Consider $(e_\alpha)_{\alpha \in \omega_1}$ in X_1. If X_1 is weakly compactly generated there exists $r_\alpha \in R$, R reflexive, such that

$$\| T(r_\alpha) - e_\alpha \| < \frac{1}{2}$$

where $T : R \to X_1$ is a bounded linear injection.

Since ω_1 is uncountable and $\omega_1 = \cup\{\{\alpha \in \omega_1 : \| r_\alpha \| \leq n\} : n \in \mathbb{N}\}$ we can find $M < \infty$ and an uncountable set Γ in ω_1 such that $\| r_\alpha \| \leq M \; \forall \alpha \in \Gamma$.

By a similar argument to the Proof of Proposition 5, we can find countable $\Gamma_1 = \{e_{\alpha_k} : k \in \omega\}$ such that $\{e_{\alpha_k}\}$ is a copy of the standard ℓ^1 basis in X_1. So,

$$\begin{aligned}
&\| T \| \| \textstyle\sum_{k=1}^n c_k r_{\alpha_k} \| \geq \| \sum_{k=1}^n c_k T(r_{\alpha_k}) \| \\
&= \| \textstyle\sum_{k=1}^n c_k e_{\alpha_k} - \sum_{k=1}^n c_k (e_{\alpha_k} - T(r_{\alpha_k})) \| \\
&\geq \| \textstyle\sum_{k=1}^n c_k e_{\alpha_k} \| - \| \sum_{k=1}^n c_k (T(r_{\alpha_k}) - e_{\alpha_k}) \| \\
&\geq \textstyle\sum_{k=1}^n |c_k| - \sum_{k=1}^n |c_k| \, \| T r_{\alpha_k} - e_{\alpha_k} \| \\
&\geq \tfrac{1}{2} \textstyle\sum_{k=1}^n |c_k|.
\end{aligned}$$

So,

$$\frac{1}{2} \sum_{k=1}^{n} |c_k| \leq \| T \| \, \| \sum_{k=1}^{n} c_k r_{\alpha_k} \| \leq M \| T \| \sum_{k=1}^{n} |c_k|$$

Thus, $(r_{\alpha_k})_{k=1}^{\infty}$ is a copy of the standard ℓ^1 basis in R. This is impossible since R is reflexive and so X_1 is not weakly compactly generated.□.

Proposition 7 *If X is a Banach space and is either of X_1 or X_2, then X is not isomorphic to a dual space.*

Proof

If $X \cong Y^*$ for some Y, since by Proposition 5 c_0 embeds in Y^*, by a result of Bessaga and Pelczyński, Y contains a complemented copy of ℓ^1 and so X contains a complemented copy of ℓ^∞; (cf.[12], Proposition 2. e.8). So, there is a subspace Z of X such that

$$X \cong \ell^\infty \oplus Z.$$

Since $\ell^\infty \cong \ell^\infty \oplus \ell^\infty$, we have $X \cong \ell^\infty \oplus \ell^\infty \oplus Z \cong \ell^\infty \oplus X$.

So X is the direct sum of two non separable summands. This is impossible by Theorem 4. So X is not isomorphic to a dual space.□.

In fact the Banach space X_2 is weakly compactly generated.

Proposition 8 *The space X_2 is weakly compactly generated. It therefore has the separable complementation property.*

Proof Consider the mapping $T : \ell_2(\omega_1) \to X_2$ given by

$$(Tx)(\alpha) = x(\alpha).$$

Then,

$$\| T(x) \|_{X_2} \leq 2 \| x \|_{\ell_2(\omega_1)},$$

T is an injection and $T(\ell^2(\omega_1))$ is dense in X_2. Therefore X_2 is weakly compactly generated.$\square$.

Therefore X_2 is a Banach space satisfying the conclusion of Theorem 4 but which also has the separable complementation property.

4. A Reflexive Example.

We shall construct our reflexive example using an interpolation method introduced in [10].

The Banach space X_2 with its norm $\| \cdot \|$ contains $\ell_2(\omega_1)$ as a dense subspace. Define a Banach space Y as follows: let $| \cdot |_n$ be the Minkowski functional of the set 2^n Ball $(\ell_2(\omega_1)) + 2^{-n}(\text{Ball}(X_2))$ for $n \geq 0$. Let Y be given by

$$Y = \{y \in \ell_2(\omega) + X_2 : \| y \|_Y = (\sum_{n=1}^{\infty} |y|_n^2)^{1/2} < \infty\}.$$

It is shown in [10] that Y is a reflexive Banach space when endowed with norm $\| \cdot \|_Y$ and that $\ell_2(\omega_1) \subseteq Y \subseteq X_2$ with continuous inclusions.

Notice that $|\omega|_n < r$ if and only if $\omega = x + z$, with $x \in X_2$ and $z \in \ell_2(\omega_1)$ with $\| x \| < 2^{-n}r$ and $\| z \|_2 < 2^n r$

The crucial observation is lemma 4. First observe that by property (D) we can construct sets in $\mathcal{G}$ of size n for any $n \in \omega$.

Lemma 1 *Let $G_n = \{\alpha_k : 1 \leq k \leq 2^0 + 2^4 + \cdots + 2^{4n}\}$ be an element of $\mathcal{G}$ where $n \geq 0$.*

If y is a vector of the form

$$y = e_{\alpha_1} + \sum_{p=1}^{n} \frac{1}{2^{2p}} \sum_{k=2^0+2^4+\cdots+2^{4(p-1)}+1}^{2^0+2^4+\cdots+2^{4p}} e_{\alpha_k}$$

(i.e. y has the form

$$= (1, \overbrace{\frac{1}{4}, \frac{1}{4}, \cdots, \frac{1}{4}}^{16 \text{ terms}}, \overbrace{\frac{1}{16}, \frac{1}{16}}^{256 \text{ terms}}, \cdots, \underbrace{\frac{1}{2^{2n}}, \cdots, \frac{1}{2^{2n}}}_{2^{4n} \text{ terms}})$$

and is supported on $G_n \in \mathcal{G}$), then $\| y \|_Y \leq 2$.

Proof We can first of all split y as $x + z$ where

$$x = \sum_{p=1}^{n} \frac{1}{2^{2p}} \sum_{k=2^0+2^4+\cdots+2^{4(p-1)}+1}^{2^0+2^4+\cdots+2^{4p}} e_{\alpha_k}$$

and

$$z = e_{\alpha_1}.$$

This shows that $|y|_0 \leq 1$.

This same splitting shows, since $\| x \| \leq 2 \| x \|_\infty = \frac{1}{2}$ and $\| z \|_2 = 1 < 2$ that $|y|_2 < 1$.

For $j \geq 2$, consider

$$\begin{aligned} y &= x_j + z_j \text{ where} \\ x_j &= \textstyle\sum_{p=j}^{n} \frac{1}{2^{2p}} \sum_{k=2^0+2^4+\cdots+2^{4(p-1)}+1}^{2^0+2^4+\cdots+2^{4p}} e_{\alpha_k} \\ z_j &= e_{\alpha_1} + \textstyle\sum_{p=1}^{j-1} \frac{1}{2^{2p}} \sum_{k=2^0+2^4+\cdots+2^{4(p-1)}+1}^{2^0+2^4+\cdots+2^{4p}} e_{\alpha_k}. \end{aligned}$$

Since $\| x_j \| \leq 2 \| x_j \|_\infty$ (by Lemma 3) $\leq 2^{-2j+1}$ and $\| z_j \|_2 = (1 + \sum_{p=1}^{j-1} \frac{2^{4p}}{2^{4p}})^{1/2} = \sqrt{j}$ we have that

$$|y|_j \leq 2^{-j} \max(2, \sqrt{j})$$

and so

$$\| y \|_Y \leq (1 + 1 + \frac{1}{4} + \frac{1}{16} + \sum_{j=4}^{n} \frac{j}{2^{2j}})^{1/2} < 2,$$

as required.□.

We require one further lemma which is similar to lemma 4. By property (D) we can construct a set $K \in \mathcal{K}$ of size n for $n \in \omega$.

Lemma 2 *Let*

$$K_n = \{\{\alpha_k, \beta_k\} : 1 \leq k \leq \frac{1}{2}(2^4 + 2^8 + \cdots + 2^{4n})\}$$

be an element of $\mathcal{K}$ *where* $n \geq 0$. *If* ω *is a vector of the form*

$$\omega = \sum_{p=1}^{n} \frac{1}{2^{2p}} \sum_{k=\frac{1}{2}(2^4+\cdots+2^{4(p-1)})+1}^{\frac{1}{2}(2^4+\cdots+2^{4p})} (e_{\alpha_k} + e_{\beta_k})$$

(i.e. w has form

$$= (1, \overbrace{\frac{1}{4}\alpha_1, \frac{1}{4}\beta_1, \frac{1}{4}\alpha_2, \frac{1}{4}, \beta_2, \cdots, \frac{1}{4}\alpha, \frac{1}{4}\beta}^{16\ terms}, \cdots, \underbrace{\frac{1}{2^{2n}}\alpha, \frac{1}{2^{2n}}\beta, \cdots, \frac{1}{2^{2n}}\alpha, \frac{1}{2^{2n}}\beta}_{2^{4n}\ terms})$$

and is supported on $\cup K_n, K_n \in \mathcal{K}$) *then*

$$\| w \|_Y \leq 12$$

Proof We can split w as $w = x_0 + z_0$ where $z_0 = 0$. It is clear that $|w|_0 \leq 1$.

Also, by Lemma 3, $\| x_0 \| \leq 10 \| x_0 \|_\infty = \frac{10}{4}$ and $\| z_0 \|_2 = 0$, which shows that $|w|_1 < \frac{5}{2}$.

If we split $w = x_j + z_j$ where

$$z_j = \sum_{p=1}^{j} \frac{1}{2^{2p}} \sum_{k=\frac{1}{2}(2^4+\cdots+2^{4(p-1)})+1}^{\frac{1}{2}(2^4+\cdots+2^{4p})} (e_{\alpha_k} + e_{\beta_k})$$

$$x_j = \sum_{p=j+1}^{n} \frac{1}{2^{2p}} \sum_{k=\frac{1}{2}(2^4+\cdots+2^{4(p-1)})+1}^{\frac{1}{2}(2^4+\cdots+2^{4p})} (e_{\alpha_k} + e_{\beta_k})$$

Then,

$$\| x_j \| \leq 10 \| x_j \|_\infty \leq \frac{10}{2^{2j+2}} = \frac{1}{2^{j+1}}[\frac{5}{2^j}]$$

and $\| z_j \|_2 = \sqrt{j}$.

Then, for $j \geq 1 |w|_{j+1} \leq \frac{1}{2^{j+1}} \max[10, \sqrt{j}]$ so that

$$\| w \|_Y \leq (1 + \frac{25}{4} + \frac{100}{4} + \frac{100}{16} + \frac{100}{64} + \cdots + \sum_{j=101}^{\infty} \frac{j}{2^{2j}})^{\frac{1}{2}} < 12.\square$$

We can now state the main result of this chapter.

Theorem 10

Every bounded linear operator on $Y, T : Y \to Y$, *can be expressed as* $T = \lambda I + S$, *where* λ *is a scalar and* S *has separable range.*

Proof

Let $T : Y \to Y$ be a bounded linear operator.

Let $e_\alpha = \Pi_{\{\alpha\}}$ for each $\alpha \in w_1$

CLAIM 1 For any $\gamma \in \omega_1$ there exists $\zeta \in \omega_1$ such that $(Te_\alpha)(\gamma) = 0$ for all $\alpha \geq \zeta$.

If CLAIM 1 were false, there exists $\gamma \in \omega_1$ such that for all $\zeta \in \omega_1$ there exists $\alpha(\zeta) \geq \zeta$ such that $|(Te_{\alpha(\zeta)})(\gamma)| > 0$. Taking Γ to be $\{\alpha(\zeta) : \zeta \in \omega_1\}$, because $\alpha(\zeta) \geq \zeta \; \forall \zeta \in \omega$, Γ has cardinality $\aleph_1$.

Also, Γ is the countable union $\Gamma = \cup\{\{\alpha(\zeta) : \zeta \in \omega_1, |(Te_{\alpha(\zeta)})(\gamma)| \geq r\} : r > 0, r \in \mathbb{Q}\}$, and so for some $r > 0$, the set $\Delta = \{\alpha(\zeta) : \zeta \in \omega_1, \; |(Te_{\alpha(\zeta)})(\gamma)| \geq r\}$ is uncountable.

Fix $n \in \mathbb{N}$. We may as well write $\Delta = \{\alpha(\zeta) : \zeta \in \omega_1\}$. Using lemma 2, we can find an injection $\phi : \omega \to \Delta$ such that every finite subset of $\{\alpha(\varphi(m)) : m \in \omega\}$ lies in $\mathcal{G}$.

Let $G = \{\alpha(\phi(m)) : m \in n\} \in \mathcal{G}$ and consider $y \in \mathbb{R}^{(\omega_1)}$ defined by

$$y = \operatorname{sign}(\mathrm{Te}_{\alpha(\varphi(1))}(\gamma))e_{\alpha(\varphi(1))} + \sum_{p=1}^{k} \frac{1}{2^{2p}} \sum_{r=2^0+2^4+\cdots 2^{4(p-1)}+1}^{2^0+2^4+\cdots+2^{4p}} \operatorname{sign}(\mathrm{Te}_{\alpha(\varphi(r))}(\gamma))e_{\alpha(\varphi(r))}$$

where n is chosen to be equal to $2^0 + 2^4 + 2^8 + 2^{16} + \cdots + 2^{4k}$ for some $k \in \mathbb{N}$.

By Lemma 8, $\| y \|_Y \leq 2$ and so

$$\begin{aligned} 2 \| T \| \geq \| Ty \|_Y &\geq \| Ty \|_{X_2} \\ &\geq \| Ty \|_\infty \\ &\geq |(Ty)(\gamma)| \\ &= |(\mathrm{Te}_{\alpha(\varphi(1))}(\gamma))| + \sum_{p=1}^{k} \frac{1}{2^{2p}} \sum_{r=2^0+2^4+\cdots+2^{4(p-1)}+1}^{2^0+2^4+\cdots+2^{4p}} |\mathrm{Te}_{\alpha(\varphi(\gamma))}(\gamma)| \\ &\geq r + \sum_{p=1}^{k} \frac{2^{4p}r}{2^{2p}} \\ &> 2^{2k}r. \end{aligned}$$

For sufficiently large k, this is a contradiction, so CLAIM 1 is true.

With a view to a contradiction, suppose now that T is not of the form $D + S$ where D is diagonal on $(e_\alpha)_{\alpha\in\omega_1}$ and S is of separable range.

CLAIM 2 There exists a family of disjoint two element subsets $\{\alpha(\zeta), \beta(\zeta)\}_{\zeta\in\omega_1}$ of ω_1 such that

$$|(\mathrm{Te}_{\alpha(\zeta)})(\beta(\zeta))| > 0 \, \forall \zeta \in \omega_1.$$

Suppose CLAIM 2 is false. Also suppose that for each $\zeta \in \omega_1$ there exists $\eta = \eta(\zeta) \geq \zeta$ such that $|(\mathrm{Te}_\eta)(\gamma)| > 0$ for some $\gamma = \gamma(\zeta) \geq \zeta$ with $\gamma \neq \eta$.

The condition $\gamma \neq \eta$ allows us to extract an uncountable subfamily from $(\{\gamma(\zeta), \eta(\zeta)\})_{\zeta\in\omega_1}$ which satisfies CLAIM 2. So, if CLAIM 2 is false, there exists $\zeta \in \omega_1$, such that for all $\eta \geq \zeta, (\mathrm{Te}_\eta)(\beta) = 0 \, \forall \, \beta > \zeta, \, \beta \neq \eta$.

For each $\eta \geq \zeta$ there exists a scalar r_η and some g_η lying in the closed linear span of $\{e_\alpha : \alpha \in \zeta\}$ such that

$$Te_\eta = r_\eta \cdot e_\eta + g_\eta.$$

Let us define a linear mapping $D : X \to X$ by

$$(Dx)(\alpha) = \begin{cases} r_\alpha x(\alpha), & \text{if } \alpha \geq \zeta \\ 0, & \text{if } \alpha < \zeta \end{cases}$$

If also $\{\Pi_\zeta\}_{\zeta\in\omega_1}$ is the sequence of basis projections associated to $(e_\alpha)_{\alpha\in\omega_1}$, we have

$$T = D + \Pi_\zeta T + T\Pi_\zeta - \Pi_\zeta T \Pi_\zeta.$$

Since Π_ζ is of separable range, $T = D + S$ where D is diagonal and S is of separable range. We have assumed that T does not have this form so CLAIM 2 holds.

Using CLAIM 1 and the fact that every element x has countable support we may assume in CLAIM 2 that $(\mathrm{Te}_{\alpha(\zeta)})(\beta(\eta)) = 0$ whenever $\zeta \neq \eta$. Also $\{\{\alpha(\zeta),\beta(\zeta)\} : \zeta \in \omega_1\} = \cup\{\{\alpha(\zeta),\beta(\zeta)\} : |\,\mathrm{Te}_{\alpha(\zeta)}(\beta(\zeta))| \geq r,\ \zeta \in \omega_1, r > 0, r \in \mathbb{Q}\}$, and so we may as well suppose that there exists $r > 0$ such that

$$|(\mathrm{Te}_{\alpha(\zeta)})(\beta(\zeta))| \geq r,\ \forall\, \zeta \in \omega_1.$$

Using 2D, given $n \in \mathbb{N}$, we find an injection $\varphi : n \to \omega$, such that

$G = \{\alpha(\varphi(m)) : m \in n\} \in \mathcal{G}$ and $H = \{\beta(\varphi(m)) : m \in n\} \in \mathcal{H}$.

Let y be defined by

$$y = e_{\alpha(\varphi(1))} + \sum_{p=1}^{k} \frac{1}{2^{2p}} \sum_{r=2^0+2^4+\cdots+2^{4(p-1)}+1}^{2^0+2^4+\cdots+2^{4p}} e_{\alpha(\varphi(r))}.$$

where n is chosen to be $2^0 + 2^4 + \cdots + 2^{4k}$ for some $k \in \mathbb{N}$.

By Lemma 4, $\| y \|_Y \leq 2$ and so

$$2 \| T \| \geq \| Ty \|_Y \geq \| Ty \|_{X_2} \geq \left(\textstyle\sum_{\beta\in H} |(Ty)(\beta)|^2\right)^{1/2}$$

$$= \left(|(\mathrm{Te}_{\alpha(\varphi(1))})(\beta(\varphi(1)))|^2 + \textstyle\sum_{p=1}^{k} \frac{1}{2^{4k}} \sum_{s=2^0+\cdots+2^{4(p-1)}+1}^{2^0+2^4+\cdots+2^{4p}} |(\mathrm{Te}_{\alpha(\varphi(s))})(\beta(\varphi(s))|^2\right)^{\frac{1}{2}}$$

$$\geq \left(r^2 + \textstyle\sum_{p=1}^{k} r^2 \frac{2^{4k}}{2^{4k}}\right)^{\frac{1}{2}}$$

$$= r\sqrt{k+1}$$

This is a contradiction for k sufficiently large.

It follows that every operator on Y has form $S + D$, where S is of separable range and D is diagonal on $\{e_\alpha\}_{\alpha\in\omega_1}$.

Consider the diagonal operator D.

Let us suppose

$(Dx)(\alpha) = \lambda(\alpha)x(\alpha)$, $\forall \alpha \in \omega_1$, for scalars $\{\lambda(\alpha)\}_{\alpha\in\omega_1}$.

CLAIM 3 There exists $\zeta \in \omega_1$ such that for all $\eta^1 > \eta \geq \zeta$, $\lambda(\eta) = \lambda(\eta^1)$.

CLAIM 3 asserts that the family $\{\lambda(\alpha)\}_{\alpha \in \omega_1}$ is eventually constant. If it were false, for every $\zeta \in \omega_1$ there exist $\eta^1 > \eta \geq \zeta$ such that $|\lambda(\eta) - \lambda(\eta^1)| > 0$. As before, we can find $r > 0$ and mutually disjoint two element subsets $(\{\alpha(\zeta), \beta(\zeta)\})_{\zeta \in \omega_1}$ in $[\omega_1]^2$ such that

$$|\lambda(\alpha(\zeta)) - \lambda(\beta(\zeta))| \geq r \, \forall \zeta \in \omega_1.$$

Let $n \in \mathbb{N}$ be given. By $2(D)$, there exists an injection $\psi : n \to \omega_1$ such that

$$K = \{\{\alpha(\psi(m)), \beta(\psi(m))\} : m \in n\}$$

lies in $\mathcal{K}$.

Let ω be an element of $\mathbb{R}^{(\omega_1)}$ of the form

$$\omega = \sum_{p=1}^{k} \frac{1}{2^{2p}} \sum_{s=\frac{1}{2}(2^4+\cdots+2^{4(p-1)})+1}^{\frac{1}{2}(2^4+\cdots+2^{4p})} e_{\alpha(\psi(s))}$$

where $n = \frac{1}{2}(2^4 + 2^8 + \cdots + 2^{4k})$

for some $k \in \mathbb{N}$.

It follows from Lemma 9 that $\| \omega \|_Y \leq 12$.

$$\begin{aligned}
& 12 \| D \| \geq \| D\omega \|_Y \geq \| D\omega \|_{X_2} \\
\geq \; & (\textstyle\sum_{m \in n} |(D\omega)(\alpha(\psi(m))) - ((D\omega)(\beta(\psi(m))))|^2)^{1/2} \\
= \; & (\textstyle\sum_{p=1}^{k} \frac{1}{2^{4p}} \sum_{s=\frac{1}{2}(2^4+\cdots 2^{4(p-1)})+1}^{\frac{1}{2}(2^4+\cdots+2^{4p})} |\lambda(\alpha(\psi(s))) - \lambda(\beta(\psi(s)))|^2|^{\frac{1}{2}} \\
\leq \; & (\textstyle\sum_{p=1}^{k} \frac{1}{2^{4p}} 2^{4p-1} r^2)^{\frac{1}{2}} \\
\geq \; & \frac{\sqrt{k}r}{\sqrt{2}}.
\end{aligned}$$

This is a contradiction for sufficiently large k and the proof of Theorem 10 is complete.□.

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Chapter 2
Finite Dimensional Spaces with Few Operators

The Minkowski compactum $\mathcal{M}_n$ is the set of all equivalence classes of isometric n-dimensional Banach spaces. It is a compact metric space when endowed with the metric $\log d(\cdot,\cdot)$ induced by the Banach-Mazur distance $d(\cdot,\cdot)$ where

$$d(X,Y) = \inf\{\| T \| . \| T^{-1} \| : T : X \to Y \text{ is an isomorphism } \}.$$

The problem of computing the diameter of the compactum arises naturally. Let us define

$$d(\mathcal{M}_n) = \sup\{d(X,Y) : \quad X, Y \in \mathcal{M}_n\}.$$

By a result of F. John [1], it is well known that

$$d(X, \ell_n^2) \leq \sqrt{n}, \ \forall X \in \mathcal{M}_n$$

and so $d(\mathcal{M}_n) \leq n$.

However an exact estimate for $d(\mathcal{M}_n)$ was not obtained for over thirty years. In 1980 Gluskin [2] proved the following result.

Theorem 2.1

There exists an absolute constant $c > 0$ such that

$$d(\mathcal{M}_n) \geq c\, n, \ \forall\, n \in \mathbb{N}.$$

Gluskin's proof depends on a construction of elements of $\mathcal{M}_n$ which are quotients of ℓ_N^1 spaces and uses measure theory in a crucial fashion in the selection of random Banach spaces of finite dimension.

Another problem which was known to the Polish school of analysts in the 1930's in Lwów was the finite dimensional basis problem: is there an absolute constant K such that every finite dimensional space has a basis with basis constant at most K? (The basis constant of a Banach space X is given by $b(X) = \inf\{b((e_i)_{i=1}^{\infty}) : (e_i)_{i=1}^{\infty}$ is a Schauder basis of $X\}$, where $b((e_i)_{i=1}^{\infty})$ is the basis constant of the Schauder basis $(e_i)_{i=1}^{\infty}$.)

Motivated by the methods of his previous results Gluskin [3] answered this question negatively. In fact he proved:

Theorem 2.2

For any n there exists an n-dimensional space X such that the operator norm of any projection P, of rank at most $\frac{n}{2}$, on X is such that

$$\| P \| \geq c \frac{\text{rank } P}{\sqrt{n \log n}}$$

where $c > 0$ is an absolute constant. For such a space the basis constant exceeds $c\sqrt{\frac{n}{\log n}}$.

At around the same time and independently, Szarek [4] using similar methods constructed n-dimensional Banach spaces whose basis constants exceeded $c^1\sqrt{n}$, where $c^1 > 0$ is some absolute constant.

By $(e_1, e_2, \cdots, e_m)$ we mean the standard unit basis of $\mathbb{R}^m$.

Identify a normed space B by its unit ball and write $\|\cdot\|_B$ for the norm generated by an absolutely convex body $B \subseteq \mathbb{R}^m$.

For a Hilbert space H and its subspace F denote by P_F the orthogonal projection of H onto F. If $\dim H = m$ and $|\cdot|$ is the Hilbert norm, then by a Gaussian variable with distribution $N(0, 1, H)$ we mean an H-valued random variable g with density

$$\ell g(x) = (\frac{m}{2\pi})^{\frac{m}{2}} \exp(\frac{-m|x|^2}{2})$$

against Lebesgue measure.

The Gaussian variable g can also be expressed as

$$\frac{1}{\sqrt{m}} \sum_{j=1}^{m} \gamma_j \, h_j,$$

where γ_j are independent real Gaussian variables with $N(0, 1)$ distribution on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and (h_j) is an orthonormal basis of H.

It is easily established that

1. $\mathbb{E}(|g|^2) = 1$

2. $\mathbb{P}[\omega \in \Omega : \frac{1}{2} \leq |g(\omega)| \leq 2] \geq 1 - e^{-cm}$, c an absolute constant

3. If E is another Hilbert space, $V : H \to E$ an isometry onto and E_0 a k-dimensional subspace of E, then

$$\sqrt{\frac{m}{k}} \, P_{E_0} \, Vg$$

is a Gaussian variable with distribution $N(0, 1, E_0)$.

Let n be given $g_1, g_2, \ldots, g_m$ be Gaussian variables with distribution $N(0, 1, \ell_n^2)$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For each $\omega \in \Omega$ denote $B_n(\omega)$ the absolutely convex hull of the set

$$\{e_1, e_2, \cdots, e_n, g_1, g_2, \cdots, g_m\}.$$

Identify $B_n(\omega)$ with the normed space $(\mathbb{R}^n, \|\cdot\|_{B_{n(\omega)}})$ whose unit ball is $B_n(\omega)$.

It may be observed that each $B_n(\omega)$ is a quotient of ℓ_1^N for some N.

It was shown in [5] that with high probability the spaces $B_n(\omega)$ posesses certain properties including a few operators property: operators of sufficiently small operator norm are in some sense a small perturbation of a scalar multiple of the identity.

We require first a definition

Definition 2.3

A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the mixing condition $(M_{k,\alpha})$ if there exists a subspace $H \subseteq \mathbb{R}^n, \dim H \geq k$ such that $\| P_{H^\perp}Tx \|_2 \geq \alpha \| x \|_2$ for $x \in H$.

A normed space X satisfies the Grothendieck theorem with constant C if and only if $\pi_1(T) \leq C \| T \|$ for all $T \in L(X,Y)$ for any normed space Y, where $\pi_1(\cdot)$ is the one-absolutely summing norm. (An operator $T : X \to Y$ is one-absolutely summing if there is a constant C such that for all finite subsets (x_i) in X $\sum ||Tx_i|| \leq C \sup\{\sum |x^*(x_i)| : x^* \in X^*, ||x^*|| \leq 1\}$.).

If H is a Hilbert space and $T \in L(H)$ is compact denote by $(s_j(T))_{j=1}^{\dim H}$ the sequence of s-numbers of T; i.e. the eigenvalues of $(T^*T)^{1/2}$, counted with multiplicity and arranged in non increasing order.

The quasi-norm $\| \cdot \|_{C_0}$ is defined as

$$\| T \|_{C_0} = \sum_{j=1}^{\dim H} \min(s_j(T), 1).$$

Let E be an n dimensional normed space. The volume ratio of E is defined to be

$$\mathrm{vr}(E) = \inf \left\{ \left(\frac{\mathrm{vol}(B_E)}{\mathrm{vol}(D)}\right)^{\frac{1}{n}} : D \subseteq B_E \text{ is an ellipsoid} \right\},$$

where B_E is the unit ball of E.

The important properties of the spaces $\{B_n(\omega) : \omega \in \Omega\}$ were established by Szarek [5].

Theorem 2.4

Given $\delta \in (0,1)$ for n sufficiently large the set of $\omega \in \Omega$ which satisfy

(i) $B_n(\omega)$ is isometric to a quotient of ℓ_1^N with $N \leq 2n$,

(ii) $\| \cdot \|_2 \leq \| \cdot \|_{B_N(\omega)} \leq \| \cdot \|_1$,

(iii) $vr(B_n(\omega)) \leq C$, where C is an absolute constant.

(iv) $B_n(\omega)$ satisfies the Grothendieck theorem with constant C, C an absolute constant.

(v) if $\| T \|_{L(B_n(\omega))} \leq c(\delta)\sqrt{n}$, then $\| T - \lambda I \|_{C_0} \leq \delta n$ for some $\lambda \in \mathbb{R}$,

has probability at least $1 - \epsilon^n$, for some $0 < \epsilon < 1$.

Condition (v) is the required few operators property.

It states that if an operator on $B_n(\omega)$ has sufficiently small operator norm, then it is a perturbation of a scalar multiple of the identity by an operator of small quasi-norm $\|\cdot\|_{C_0}$. Moreover the probability of this event for the spaces $B_n(\omega)$ is asymptotically one as n tends to infinity.

The existence of n-dimensional spaces satisfying results similar to Theorem 2.4 led to a string of counter examples in both finite dimensional and infinite dimensional Banach space theory.

The space constructed in Theorem 2.4 solves the finite dimensional basis problem ([3], [4]).

Definition 2.5 Given a finite dimensional Banach space E, the asymmetry constant, $S(E)$, is the infimum of all numbers ϱ with the following property: there is a group G of invertible operators on E such that $\sup\{\| g \|: g \in G\} \leq \varrho$ and for $u : E \to E$ the condition $ug = gu$ for $g \in G$ implies $u = cI_E$ for some scalar c.

In [9] using random spaces Mankiewicz showed that for each n there exists an n-dimensional Banach space E such that $s(E) \geq c\sqrt{n}$ where $c > 0$ is an absolute constant. In [10] Mankiewicz gave further results along these lines and investigated the subspace mixing condition for operators.

Szarek proved in [5] that given n there exists a $2n$-dimensional real normed space X such that whenever Y is an n-dimensional complex normed space and $Y_{\mathbb{R}}$ is Y treated as a real space, then the Banach-Manzur distance $d(X, Y_{\mathbb{R}}) \geq c\sqrt{n}$ where c is a constant.

Szarek also shows in [5] that given n there exists an n dimensional space Y over $\mathbb{C}$ such that Y admits a second complex multiplication (denote the resulting space $\bar{Y}$) where $d(Y, \bar{Y}) \geq cn$, c a numerical constant.

Bourgain [11] then constructed an infinite dimensional real Banach space with two complex structures which are not complex isomorphic.

In [12], Szarek constructed an infinite dimensional super-reflexive real Banach space which does not admit complex structure. Moreover this space is not isomorphic to the Cartesian square of any Banach space.

Mankiewicz [13] showed that there exists a separable superreflexive (real or complex) Banach space X such that the Banach algebra $L(X)$ admits a homomorphism onto the Banach algebra $C(\beta\mathbb{N})$. This provides a complex example of a Banach space not isomorphic to any Cartesian square.

Further results on random spaces and random matrices are contained in [14] and [15]. However the most spectacular result obtained using randomly consructed finite dimensional was proved by Szarek in [16]: there exists a Banach space with the bounded approximation property that fails to have a Schauder basis.

More recently [17] Mankiewicz and Szarek have investigated limitations of the random approach adopted above. Amongst other results they show that a "generic" d-dimensional quotient of ℓ_1^n contains a C-complemented subspace C-isomorphic to ℓ_p^k, $k \geq c\sqrt{d}$, either for $p = 1$ or $p = 2$.

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Chapter 3
The Distortion Problem and Hereditarily Indecomposable Banach Spaces

In this chapter we shall first of all review the construction of Banach spaces, by Gowers and Maurey, which have few operators. Their results turn out to be related to a long standing open problem in the geometry of Banach spaces whose solution has recently been obtained. [1], [2].

Definition 3.1

For $\lambda > 1$ an infinite dimensional Banach space X is *λ-distortable* if there exists an equivalent norm $|\cdot|$ on X such that for all infinite dimensional subspaces Y of X,

$$\sup\{\frac{|y|}{|z|} : y, z \in S(Y)\} > \lambda.$$

[N.B. for a Banach space X, we denote

$$S(X) = \{x \in X : \| x \| = 1\} \text{ and } \text{ball}(X) = \{x \in X : \| x \| \leq 1\}.]$$

An infinite dimensional Banach space X is said to be *distortable* if X is λ-distortable for some $\lambda > 1$.

The "distortion problem" for an infinite dimensional Banach space X is whether X is *distortable* or not.

An infinite dimensional Banach space X is *arbitrarily distortable* if X is λ-distortable for all $\lambda > 1$.

R.C. James [3] proved that the Banach space c_0 and ℓ^1 are not distortable. Milman [4] showed that if an infinite dimensional Banach space X is not distortable then X contains a subspace isomorphic to c_0 or ℓ^p, $1 \leq p < \infty$. In fact Milman studied a somewhat wider class of stability problems.

Definition 3.2

A function $f : S(X) \to \mathbb{R}$ is *oscillation stable* on X if for all infinite dimensional subspaces Y of X and for all $\epsilon > 0$ there exists an infinite dimensional subspace Z of Y with

$$\sup\{|f(y) - f(z)| : y, z \in S(Z)\} < \epsilon.$$

A function $f : S(X) \to \mathbb{R}$ is finitely oscillation stable if for all infinite dimensional subspaces Y of X and for all $\epsilon > 0$ there exists a subspace Z of Y of arbitrarily large finite dimension with

$$\sup\{|f(y) - f(z)| : y, z \in S(Z)\} < \epsilon.$$

It may be shown that X does not contain a distortable infinite dimensional subspace Y if and only if for every equivalent norm $\|\cdot\|$ on X, $\|\cdot\|: S(X) \to \mathbb{R}$ is oscillation stable.

Theorem 3.3 Let X be a Banach space. Every Lipschitz (or even uniformly continuous) function $f : S(X) \to \mathbb{R}$ is finitely oscillation stable.

Theorem 3.4 There exist Lipschitz functions on unit spheres of Banach spaces which are not oscillation stable.

Tsirelson [5] was the first to construct an infinite dimensional Banach space which failed to contain c_0 or ℓ^p for some $1 \leq p < \infty$. Figiel and Johnson [6] described the dual T of the space displayed by Tsirelson and showed that it also had this property. In [7] an explicit distortable norm is displayed for T. Theorem 3.4 follows from the work of Tsirelson [5] and Milman [4].

In [8] Gowers proved a result for the space c_0.

Theorem 3.5

Every uniformly continuous function $f : S(c_0) \to \mathbb{R}$ is oscillation stable.

The proof depends heavily on combinatorial arguments.

Schlumprecht [9] was the first to construct a space S which was arbitrarily distortable. The space S is a space of sequences x for which $\| x \|_S$ is finite, where $\| x \|_S$ is defined by the expression

$$\| x \|_S = \max[\| x \|_{c_0}, \sup_{\ell \geq 2 \, E_1 < E_2 < \cdots < E_\ell} \frac{1}{\varphi(\ell)} \sum_{i=1}^{\ell} \| E_\ell x \|]$$

where $\varphi(\ell) = \log_2(1+\ell)$; $E_1 < E_2 < \cdots < E_\ell$ means that E_i is a finite subset of $\mathbb{N}$ which $\max E_i < \min E_{i+1}, 1 \leq i \leq \ell$ and if $x = \sum_{i=1}^{\infty} a_i e_i$ where $\{e_i\}$ are the standard sequences, and $E \subseteq \mathbb{N}$ then $Ex = \sum_{i \in E} a_i e_i$.

Theorem 3.6 The Banach space S with norm given by $\| \cdot \|_S$ above is arbitrarily distortable.

The results stated so far show that the distortion problem for arbitrary Banach spaces essentially reduces to the distortion problem for the space ℓ^p $(1 < p < \infty)$.

Odell and Schlumprecht solved this problem in [1].

Theorem 3.7

The spaces ℓ^p $(1 < p < \infty)$ are arbitrarily distortable.

Odell and Schlumprecht used the space S to prove Theorem 3.7. We now give a sketch of their argument.

It may be shown that an infinite dimensional uniformly convex space X is distortable if there exist asymptotic separated sets, $C, D \subseteq S(X)$; the sets C, D are separated if

$$d(C, D) = \inf\{\| x - y \| : x \in C, y \in D\} > 0$$

and a set C is asymptotic if $d(C, S(Y)) = 0$ for all infinite dimensional subspaces Y of X. A distorted norm is given on such X by the closed convex hull of $C \cup (-C) \cup \delta\, \text{ball}(X)$ for some $\delta > 0$. The Mazur map $M_p : S(\ell^1) \to S(\ell^p)$ is given by

$$M_p((x_i)_{i=1}^{\infty}) = (\text{sign}(x_i)|x_i|^{\frac{1}{p}})_{i=1}^{\infty}.$$

The mapping M_p is a uniformly continuous bijection with uniformly continuous inverse. It follows that M_p preserves the separated sets property. A further argument shows that M_p preserves asymptotic sets. By consideration of the Mazur map M_p, it is shown that $\ell^p(1 < p < \infty)$ is distortable if and only if $S(\ell^1)$ contains a pair of separated asymptotic sets.

After these preliminaries, the first step in Odell and Schlumprecht's proof was the next result.

Theorem 3.8

If X is an infinite dimensional Banach space with an unconditional basis then $S(X)$ and $S(\ell^1)$ are uniformly homeomorphic if and only if X does not contain ℓ_n^{∞}'s uniformly.

The difficult direction in proving Theorem 3.8 is the "if" direction.

If X has a 1-unconditional normalized basis $(e_i)_{i=1}^{\infty}$ then we define $|x| = \sum_{i=1}^{\infty} |a_i| e_i$ where $x = \sum_{i=1}^{\infty} a_i e_i$. Then $S(X)^+ = \{x \in S(X) : x = |x|\}$ and $X^+ = \{x \in X : x = |x|\}$. If h is a finitely supported element in $S(\ell^1)^+$ and $y = \sum y_i e_i \in X^+$ we define an entropy map $E(h, y) = \sum h_i \log(y_i)$ where by convention $0 \log 0 = 0$.

It turns out that there is a unique $x \in S(X)^+$ having the same support as h and maximizing $E(h, \cdot)$ on $S(X)^+$. Define $F_X(h)$ to be that x. It further transpires that if X is uniformly convex and uniformly smooth then the map F_X extends to a uniform homeomorphism between $S(\ell^1)$ and $S(X)$.

One obstacle remains: the map F_X does not trivially preserve asymptotic sets. Odell and Schlumprecht overcame this as follows.

In the arbitrarily distortable space S there are sets $A_k \subseteq S(S)$ and $A_k^* \subseteq S(S^*)$ with $\epsilon_k \downarrow 0$ such that

(i) A_k is asymptotic in S $\forall k$

(ii) $|x_k^*(x_\ell)| < \epsilon_{\min(k,\ell)}$ if $k \neq \ell$ and $x_k^* \in A_k^*$ and $x_l \in A_\ell$

(iii) For all k and $x \in A_k$ there is $x^* \in A_k^*$ with $x^*(x) > 1 - \epsilon_k$.

Define a sequence of sets $B_k \subseteq S(\ell^1)$ by setting

$$B_k = \{\frac{x_k^* \circ x_k}{|x_k^*|(|x_k|)} : x_k \in A_k, x_k^* \in A_k^* \text{ and } |x_k^*|(|x_k|) > 1 - \epsilon_k\}$$

where $\sum a_i e_i^* \circ \sum b_i e_i$ is the element of ℓ^1 given by $(a_i b_i)_{i=1}^{\infty}$.

By considering the map F_{S^*}, Odell and Schlumprecht show that each B_k is asymptotic in ℓ^1. Therefore the sets $C_k = M_p(B_k) \subseteq S(\ell^p)$ $(1 < p < \infty)$ are also asymptotic.

The set C_k is unconditional means that $|x| \in C_k$ if and only if $x \in C_k$. The set C_k is spreading means that if $x = (x_i)_{i=1}^{\infty} \in C_k$ then

$$(0, 0, \cdots, 0, x_1, 0, \cdots, 0, x_2, \cdots 0, x_3, \cdots) \in C_k,$$

no matter where the 0's are placed.

An infinite dimensional Banach space X is sequentially arbitrarily distortable if there exists a sequence of equivalent norms $|\cdot|_k \leq \|\cdot\|$ and $\epsilon_k \downarrow 0$ so that for all $i_0 \in \mathbb{N}$ and all infinite dimensional subspaces $Y \subseteq X$ there exists $y \in Y$, $|y|_{i_0} = 1$ and $|y|_k < \epsilon_{\min(i_0,k)}$ if $k \neq 0$.

To be precise Odell and Schlumprecht prove.

Theorem 3.9 The sets $C_k \subseteq S(\ell^1)$ are unconditional, asymptotic and spreading. Also for some sequence $\epsilon_k \downarrow 0$

$$\langle |x_k|, |x_\ell| \rangle < \epsilon_{\min(k,\ell)} \text{ if } k \neq \ell \text{ and } x_k \in C_k \text{ and } x_\ell \in C_\ell.$$

To show that ℓ^2 is sequentially arbitrarily distortable and so arbitrarily distortable, we set

$$|x|_k = \sup\{\langle y, x \rangle : y \in C_k \cup \epsilon_k \operatorname{ball}(\ell^2)\}.$$

The proof for $p \neq 2$ is very similar.

In [10] Maurey and Rosenthal constructed an infinite dimensional Banach space with a Schauder basis which is weakly null but has no unconditional subsequences. By combining ideas from [10] and Schlumprecht's paper [9], Gowers and Maurey independently constructed an infinite dimensional reflexive Banach space X with a Schauder basis which contains no unconditional basic sequences, [11], [12]. We give a brief outline of their approach and of some of the properties of this space.

In [11] three definitions of the norm of X are given. We give two which reveal clearly that the space X is a Tsirelson type contruction.

Let c_{00} be the finitely supported sequences. If $E \subseteq \mathbb{N}$ then, $E : c_{00} \to c_{00}$ is given by

$$E(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i \in E} a_i e_i$$

The range of a vector $x, \operatorname{ran}(x)$ is the smallest interval containing its support. For finite subsets of $\mathbb{N}$ E and F, we write $E < F$ provided $\max(E) < \min(F)$. We write $x < y$ to mean $\operatorname{ran}(x) < \operatorname{ran}(y)$. If $x_1 < x_2 < x_3 < \cdots < x_n$, we shall say that $x_1, x_2, \cdots, x_n$ are successive. Let $Q = \{(x_n)_{n=1}^{\infty} : |\operatorname{supp}((x_n)_{n=1}^{\infty})| < \infty, x_n \in \mathbb{Q} \quad \forall n \in \mathbb{N}, |x_n| \leq 1, \forall n \in \mathbb{N}\}$.

Let $J = \{j_1, j_2, j_3, \cdots\}$ be such that $j_1 < j_2 < j_3 < \ldots$ and $\log \log n \geq 4m^2$ whenever $m, n \in J, m < n$.

Let $f : [0,\infty) \to [0,\infty)$ be given by

$$f(t) = \log_2(1+t),\ \forall t \geq 0.$$

We assume that $f(j_1) > 256$.

Define $K \subseteq J$ and $L \subseteq \mathbb{N}$ by

$$\begin{aligned} K &= \{j_1, j_3, j_5, \cdots\} \\ L &= \{j_2, j_4, j_6, \cdots\} \end{aligned}$$

Consider an injection σ from the collection of finite sequences of successive elements of Q to L such that, if $z_1, z_2, \cdots, z_s$ is such a sequence, $S = \sigma(z_1, z_2, \cdots, z_s)$ and $z = \sum_{i=1}^{s} z_i$, then $\frac{1}{20} f(S^{\frac{1}{40}})^{\frac{1}{2}} \geq |\operatorname{supp}(z)|$.

Let $Y = (c_{00}, \|\cdot\|)$ be a normed space. For $m \in \mathbb{N}$.

We define $A_m^*(Y) = \{f \in Y^* : g = \frac{1}{f(m)} \sum_{i=1}^{m} f_i$ such that $f_1 < f_2 < \cdots < f_m$ and $\| f_i \| \leq 1,\ \forall i\}$.

For $k \in \mathbb{N}$, let Γ_k^Y be the set of sequences $g_1, g_2, \cdots g_k$ such that $g_i \in Q$ for each $i, g_1 \in A_{j_{2k}}^*(Y)$ and $g_{i+1} \in A_{\sigma(g_1, g_2, \cdots, g_i)}^*(Y)$ for each $1 \leq i \leq k-1$. These are called special sequences.

For $k \in \mathbb{N}$, let $B_k^*(Y)$ be the set of functionals of the form $\frac{1}{f(k)^{\frac{1}{2}}} \sum_{j=1}^{k} g_j$ such that $(g_1, \cdots, g_k) \in \Gamma_k^Y$. These are called special functionals. Let $X_0 = (c_{00}, \|\cdot\|_0)$ where $\| x \|_0 = \| x \|_{\ell^\infty}$. For $N \geq 0$ set

$$\| x \|_{x_{N+1}} = \sup\{\frac{1}{f(n)} \sum_{i=1}^{n} \| E_i x \|_{X_N} : n \in \mathbb{N}, E_1 < E_2 < \cdots < E_n\}$$

$$\vee \sup\{|g(Ex)| : k \in K, g \in B_k^*(X_N), E \subseteq \mathbb{N}\}$$

where in the supremum the sets E_i are intervals of integers.

The sets $B_k^*(X_N)$ increase with N so $\|\cdot\|_{X_N} \leq \|\cdot\|_{X_{N+1}}$. Also, $\|\cdot\|_{X_N} \leq \|\cdot\|_{\ell^1}$; so we may define $\|\cdot\|$ by,

$$\| x \| = \lim_{n\to\infty} \| x \|_{X_N}.$$

The norm on X may be defined implicitly.

$$\| x \| = \| x \|_{c_0} \vee \sup\{\frac{1}{f(n)} \sum_{i=1}^{n} \| E_i x \| : 2 \leq n \in \mathbb{N}, E_1 < E_2 < E_3 < \cdots < E_n\}$$

$$\vee \sup\{|g(Ex)| : k \in K, g \in B_k^*(X), E \subseteq \mathbb{N}\}.$$

To prove that X has no unconditional basic sequences, several technical lemmas are proved about a crucial class of sequences in X.

Definition 3.10

Given $x \in X$ we say that x is an ℓ^1_{n+}-average with constant C if $\| x \| = 1$ and $x = \sum_{i=1}^n x_i$ for some sequence $x_1 < x_2, \cdots < x_n$ of non zero elements of X such that $\| x_i \| \leq \frac{C}{n} \forall i$.

A vector x is an ℓ^1_{n+} vector with constant C if it is a positive multiple of an ℓ^1_{n+}-average.

A sequence $x_1 < x_2 < \cdots < x_n$ is a rapidly increasing sequence of ℓ^1_+-averages or *R.I.S.*, for f of length N with constant $1+\epsilon$ if x_k is an $\ell^1_{n_k+}$-average with constant $1+\epsilon$ for each k, $n_1 \geq 2(1+\epsilon)M_f(\frac{N}{\epsilon^1})f^1(1)$, and

$$\frac{\epsilon^1}{2} f(n_k)^{\frac{1}{2}} \geq |\operatorname{supp}(x_{k-1})|$$

for $k = 2, \cdots, N$. Where $\epsilon^1 = \min(\epsilon, 1)$ and $M_f : \mathbb{R} \to \mathbb{R}$ is given by

$$M_f(t) = f^{-1}(36t^2),\ \forall t \in \mathbb{R}.$$

In fact Gowers and Maurey prove a stronger property than that so far stated.

Definition 3.11

A Banach space X is hereditarily indecomposable (*H.I*) if every closed subspace Y of X is indecomposable.

Using an inductive argument, for two infinite dimensional subspaces Y and Z of X such that $Y \cap Z = 0$, Gowers and Maurey construct, for every $\delta > 0$ vectors $y \in Y$ and $z \in Z$ such that $\delta \| y + z \| > \| y - z \|$. From this it follows that the projection from $Y + Z$ to Y has operator norm at least $(1-\delta)\frac{\delta^{-1}}{2}$, and so X is hereditarily indecomposable.

The vectors y and z are constructed by inductively defining a pair of sequences $x_1, \cdots, x_k$ and $x_1^*, x_2^*, \cdots, x_k^*$ where $x_1, x_2, \cdots, x_k$ is a *R.I.S.* of length k and $x_1^*, x_2^*, \cdots, x_k^*$ is a special sequence of length k. Additional properties of these sequences ensure that

$$\| \sum_{i=1}^k x_i \| \geq f(k)^{\frac{-1}{2}}(\frac{k}{2} - 1)$$

$\| \sum_{i=1}^k (-1)^{i-1} x_i \| \leq (1+2\epsilon)kf(k)^{-1}$ and when i is odd $x_i \in Y$, when i is even $x_i \in Z$.

Taking $y \in Y$ to be the sum of the odd numbered x_i's and $z \in Z$ to be the sum of the even numbered x_i's, we have

$$\| y + z \| \geq \frac{1}{3} f(k)^{\frac{1}{2}} \| y - z \| \geq \frac{1}{\delta} \| y - z \|$$

and the proof is complete.

Gowers and Maurey then go on to study operators on hereditarily indecomposable spaces of $\mathbb{C}$. Their theorem is as follows.

Theorem 3.12

If X is a complex hereditarily indecomposable Banach space then every bounded linear operator T on X can be written $T = \lambda I + S$, where $\lambda \in \mathbb{C}$ and S is a strictly singular operator on X.

The spectrum of T is finite or consists of λ and a sequence (λ_n) of eigenvalues with finite multiplicity converging to λ. $\Box$.

We recall a definition.

Definition 3.13 An operator $S : X \to X$ on an infinite dimensional Banach space X is strictly singular if the restriction of S to any infinite dimensional subspace of X is not an isomorphism.

Gowers and Maurey also prove Theorem 3.12 for the real case of the example X they constructed.

Proof of Theorem 3.12

The proof of Theorem 3.12 proceeds by several definitions and lemmas.

Definition 3.14

For a complex Banach space X and an operator T mapping X into itself, $\lambda \in \mathbb{C}$ is infinitely singular for T if there exists for every $\epsilon > 0$ an infinite dimensional subspace Y_ϵ of X such that the restriction of $T - \lambda I$ to Y_ϵ is of operator norm at most ϵ.

A complex number λ is not infinitely singular for T if and only if $T - \lambda I$ is an isomorphism on some finite co-dimensional subspace of X. This property will remain unaffected by a sufficiently small perturbation so if we define

$$F_T = \{\lambda \in \mathbb{C} : \lambda \text{ not infinitely singular for } T\},$$

then F_T is an open subset of $\mathbb{C}$. Also if $\lambda \in F_T$, then $\ker(T - \lambda I)$ is finite dimensional.

Lemma 3.15

If $\lambda \in F_T$ and if (x_n) is a bounded sequence such that $(T - \lambda I)x_n$ is norm convergent, then (x_n) has a norm convergent subsequence and the image of $T - \lambda I$ of any closed subspace of X is closed.

Proof Consider $S = T - \lambda I$ and let Y be some finite co-dimensional subspace on which S is an isomorphism. Let Z be a subspace of X such that $X = Y \oplus Z$ and suppose $x_n = y_n + z_n$ where $y_n \in Y, z_n \in Z$.

Since Z is finite dimensional and x_n is bounded, $Sx_n = Sy_n + Sz_n$, we can pass to a subsequence such that Sz_n converges; consequently the corresponding subsequence Sy_n is norm convergent. Since S is an isomorphism on Y, y_n converges. Passing to a further subsequence we may assume z_n converges, so (x_n) converges as required.

If F is a closed subspace of $X, F = F \cap Y + G$, where G is finite dimensional. So, $S(F) = S(F \cap Y) + S(G)$. Since S is an isomorphism on Y, and $F \cap Y$ is closed in $Y, S(F \cap Y)$ is closed so $S(F)$ is also closed.□.

Lemma 3.16 If $\lambda \in \partial sp(T) \cap F_T$, then λ is an eigenvalue of T with finite multiplicity.

Proof The complex number $\lambda \in \partial\,\text{sp}(T)$ is an approximate eigenvalue of T. That is to say there exist $(x_n), \| x_n \| = 1$ with $(T - \lambda I)x_n \to 0$ as $n \to \infty$. By Lemma 3.15, (x_n) has a convergent subsequence and so the limit of this sequence is an eigenvector with eigenvalue λ. Also the eigenspace of $\lambda, \ker(T - \lambda I)$ is finite dimensional since $\lambda \in F_T$ by a previous observation, so λ has finite multiplicity.□.

Lemma 3.17

If $\lambda \in \partial\,\text{sp}(T) \cap F_T$ then λ is an isolated point of $\text{sp}(T)$.

Proof The set F_T is open, so it suffices to show that λ is an isolated point of $\partial\,\text{sp}(T) \cap F_T$. Suppose not, then $\exists \lambda_n \in \partial\,\text{sp}(T) \cap F_T$ such that $\lambda_n \to \lambda$ with $\lambda_n \neq \lambda\, \forall n$. The complex numbers λ_n lie in F_T, so each λ_n is eigenvalue, by Lemma 3.16. If x_n is such that $\| x_n \| = 1$ and $Tx_n = \lambda_n x_n$ then since $(T - \lambda I)x_n = (\lambda_n - \lambda)x_n$ converges to 0 as $n \to \infty$, we may assume by Lemma 3.15 that (x_n) is norm convergent to some x with $\| x \| = 1$ such that $Tx = \lambda x$.

Let Y be the closed subspace of X generated by the sequence (x_n). Let U be the restriction of $T - \lambda I$ to Y.

Then $U(Y) \subseteq Y$ and UY is dense in Y. Also, $(T - \lambda I)Y = UY$ and $\lambda \in F_T$, so by Lemma 3.15, UY is closed and so $UY = Y$. However, $x \in Y$ so $Y_0 \cap \ker U$ is not $\{0\}$ and $\ker U$ is finite dimensional. Thus Y can be written as a direct sum $Y_0 + Y_1$ and $UY_1 = Y$, so given small enough $\epsilon > 0$, $(U - \epsilon I)Y_1 = Y$. When $\epsilon \neq 0, (U - \epsilon I)Y = Y_0$ and so $\ker(U - \epsilon I) \neq \{0\}$, for small enough $\epsilon > 0$. This contradicts the fact that $\lambda \in \partial\,\text{sp}(T)$.□.

Lemma 3.18

Let S be a bounded linear operator from X to itself. If $\text{sp}(S) = \{0\}$, provided X is infinite dimensional, then 0 is infinitely singular for S.

Proof If $0 \in F_S$ and X is infinite dimensional, then S is an isomorphism on some infinite dimensional subspace Z of X of finite co-dimension in X. If necessary replacing S with kS for some $k \in \mathbb{C}$, we may assume $\| Sz \| \geq \| z \|$ for every $z \in Z$.

Define subspaces $(Z_n)_{n=0}^{\infty}$ by $Z_0 = Z, Z_1 = Z \cap SZ, \cdots Z_{k+1} = Z \cap SZ_k$. Then each Z_n is an infinite dimensional subspace of X. If $z \in Z_k$, $\| z \| > 0$ then $z = S^k z_0$ for some $z_0 \in Z$ and $0 < \| z_0 \| \leq \| S^k z_0 \|$, so $\| S^k \| \geq 1\, \forall k$. So the spectral radius of S exceeds 1. This contradicts the fact that $\text{sp}(S) = \{0\}$.

Lemma 3.19

If X is infinite dimensional then $F_T \neq \mathbb{C}$.

Proof If $F_T = \mathbb{C}$, by Lemma 3.17 every point in $\partial \operatorname{sp}(T)$ is isolated and so the spectrum of T, $\operatorname{sp}(T)$ is finite. Let us suppose $\operatorname{sp}(T) = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$. Let us consider

$$P(z) = \Pi_{i=1}^{m}(z - \lambda_i), \quad \forall z \in \mathbb{C}.$$

For $\lambda \neq 0$, we have

$$P(z) - \lambda = \Pi_{i=1}^{m}(z - \mu_i) \text{ where } \mu_i \notin \operatorname{sp}(T)\, \forall\, 1 \leq i \leq m.$$

It follows that $(P - \lambda)(T) = p(T) - \lambda I$ is invertible if $\lambda \neq 0$. So, $Sp(P(T)) = \{0\}$. However X is infinite dimensional, so by Lemma 3.18, 0 is infinitely singular for $P(T)$. So there exists (x_n), $\| x_n \| = 1$ such that $(P - \lambda)(T)x_n \to 0$ as $n \to \infty$.

If we write, $P(T) = (T - \lambda I)P_1(T)$ and repeatedly apply Lemma 3.15, we prove that some sequence of (x_n) is convergent. This is impossible so $F_T \neq \mathbb{C}$.□.

To complete the proof of Theorem 3.12, suppose X is hereditarily indecomposable. Since $F_T \neq \mathbb{C}$, there exists at least one $\lambda_0 \in \mathbb{C}$ such that λ_0 is infinitely singular for T.

Since λ_0 is infinitely singular for T, for every $\epsilon > 0$ there is an infinite dimensional subspace Y of X such that $\| T|_Y - \lambda_0 I|_Y \| < \epsilon$.

If $T - \lambda_0 I$ is an isomorphism on some subspace Z, say $\| (T - \lambda_0 I)z \| \geq \delta \| z \|$ for all $z \in Z$. Let $\epsilon = \frac{\delta}{M}$ and consider Y as in the paragraph above. For $y \in Y$ and $z \in Z$, we have

$$\begin{aligned} (\| T \| + |\lambda_0|) \| y + z \| &\geq \| (T - \lambda_0)(y - z) \| \\ &\geq \delta \| z \| - \epsilon \| y \| \\ &\geq \delta \| z \| - \tfrac{\delta}{M}[\| z \| + \| y + z \|]. \end{aligned}$$

That is,

$$(1 - \frac{\delta}{M}) \| z \| \leq (\| T \| + |\lambda_0| + \frac{\delta}{M}) \| y + z \| .$$

This contradicts the fact that X is hereditarily indecomposable. So $(T - \lambda_0 I)$ is not an isomorphism on any infinite dimensional subspace of X, so $T - \lambda_0 I$ is strictly singular.

By Lemma 3.17 the spectrum of T is finite or consists of a sequence of eigenvalues converging to λ_0.

If T has a finite spectrum, then if $\lambda \in \partial \operatorname{sp}(T) \cap F_T$, λ is isolated in $\operatorname{sp}(T)$ by Lemma 3.17. Consider the spectral projection Q associated with λ. Then $Y = QX$ is finite dimensional, since λ_0 is not infinitely singular for T and by the spectral mapping theorem. For a finite spectrum, one value $\lambda \in \operatorname{sp}(T)$ must be infinitely singular.

This completes the proof of Theorem 3.12.□.

A corollary of Theorem 3.12 may also be stated.

Corollary 3.20

A complex hereditarily indecomposable Banach space X is not isomorphic to any proper subspace.

A long standing question asked by Banach is whether every Banach space X is isomorphic to the direct sum $X \oplus \mathbb{R}$ (in the real case) or $X \oplus \mathbb{C}$ in the complex case. This is the hyperplane problem for Banach spaces. Corollary 3.20 answers the hyperplane conjecture negatively. Gowers [13] gave a second construction of a space which solved this problem.

Theorem 3.21

There exists an infinite dimensional Banach space with an unconditional basis which is not isomorphic to any of its proper subspaces.

The definition of the norm of the space of Theorem 3.21 is very similar to that of Theorem 3.12.

In the description of the norm of Gowers and Maurey's hereditarily indecomposable Banach space, let the set J be such that if $m, n \in J$ then $\log\log\log n \geq 2m$. Define a norm on c_{00} by the implicit equation

$$\| x \| = \| x \|_{c_0} \vee \sup\{\tfrac{1}{f(n)} \textstyle\sum_{k=1}^{n} \| E_k x \| : 2 \leq n \in \mathbb{N},$$
$$E_1 < E_2 < \cdots < E_n\} \vee \sup\{|x^*(x)| : k \in K, x^* \in B_k^*(X)\}.$$

where the supremum for the sets E_k now can be over arbitrary sets $E_1 < E_2 < \cdots < E_n$ and not just for intervals E_k. This fact ensures that the resulting Banach space has an unconditional basis.

In the proof of Theorem 3.21 Gowers again uses the notion of *R.I.S.* sequence and also a result of Casazza.

Theorem 3.22 Let X be a Banach space with a basis with the property that whenever (y_n) and (z_n) are two sequences in X such that $y_n < z_n < y_{n+1}$ (with respect to the basis), they are not equivalent. Then X is not isomorphic to any proper subspace.

In [14], Gowers and Maurey also establish a few operators result for the Banach space of Theorem 3.21.

Theorem 3.23

There exists an infinite dimensional Banach space X with unconditional basis $(e_n)_{n=1}^{\infty}$ such that every bounded linear operator T from X to itself may be written $T = S + D$, where S is a strictly singular operator and D is a diagonal operator on $(e_n)_{n=1}^{\infty}$.

The ideas outlined above have also enabled Gowers to construct further examples of Banach spaces with surprising properties.

Definition 3.23

A Banach space X satisfies the Schroeder-Bernstein conjecture if given a Banach space Y, if Y is isomorphic to a complemented subspace of X and X is isomorphic to a complemented subspace of Y, then X is isomorphic to Y.

Gowers [15] next constructed a Banach space X which contains no c_0, ℓ^1 or reflexive subspaces. In [16] he solved the Schroeder-Bernstein problem for Banach spaces by showing that this space X is isomorphic to its cube but not its square: $X \cong X \oplus X \oplus X$ but $X \ncong X \oplus X$.

So far the results discussed have consisted mainly of counter-examples to long standing questions in Banach space theory. However Gowers [17] finally established a positive result.

Theorem 3.24

Let X be an infinite dimensional Banach space. Then X contains an infinite dimensional subspace with an unconditional basis or an infinite dimensional hereditarily indecomposable subspace.

The statement of Theorem 3.24 allows another long standing problem in Banach space theory to be settled.

Definition 3.25

An infinite dimensional Banach space X is homogeneous if it is isomorphic to all of its closed infinite dimensional subspaces.

Schlumprecht had shown in [9] that any homogeneous Banach space is decomposable. In [37] the next theorem was proved.

Theorem 3.26

If X is an infinite dimensional Banach space all of whose subspaces have an unconditional basis, then X is hereditarily ℓ^2.

If X is homogeneous, it is decomposable and so by Theorem 3.24 X must contain an unconditional basic sequence, and being homogeneous X must have an unconditional basis. Since X is homogeneous, by Theorem 3.26, X must be isomorphic to ℓ^2. For a historical review of the homogeneous space problem see [18].

In [19] Gowers commenced a study of the combinatorial ideas underlying [17].

Definition 3.27

A subset A of a topological space is analytic if it is the continuous image of a complete separable metrisable topological space.

If A is a subset of $\mathbb{N}^\omega$ in the product topology and either A or its complement contains all infinite subsets for some $X \in \mathbb{N}^\omega$, then A is said to be a Ramsey set.

This property holds for open, analytic or Borel subsets of $\mathbb{N}^\omega$ in the product topology (cf. [20]]).

Let X be a Banach space with basis $(e_n)_{n=1}^{\infty}$. A generalized block basis is a sequence of pairs (x_n, λ_n) where $x_1 < x_2 < \cdots$ are vectors of norm one and $\lambda_n \in [0,1]\, \forall n$.

Let $\sum(X)$ be the set of all infinite generalized block bases of the basis $(e_n)_{n=1}^{\infty}$.

Let $\sigma \subseteq \sum(X)$ be arbitrary and consider a two player game between players A and B. At the nth stage, A chooses a block subspace $X_n \subseteq X$ infinite dimensional and B chooses a pair $x_n = (x_n^1, \lambda_n)$ with $x_n^1 \in X_n$. Player A aims to contruct a sequence $(x_1, x_2, \cdots) \in \sigma$ and B aims to prevent this.

A strategy for A is a function ϕ which, given any finite block basis $x_1, x_2, \cdots, x_n$ and any subspace $Y \subseteq X$, gives $x = \varphi(x_1, \cdots x_n; Y) \in Y$.

The strategy φ is a winning strategy for A if, given a sequence $X_1, X_2, \cdots$ of subspaces of X, the sequence $(x_1, x_2, \cdots)$ defined by $x_1 = \varphi(\phi; X_1), x_{n+1} = \varphi(x_1, \ldots x_n; X_{n+1}\}$ is in σ.

Let $\Delta = (\delta_1, \delta_2, \cdots)$ be a sequence of positive scalars and $\sigma \subseteq \sum(X)$, then σ_Δ is the set of block bases $(x_1, x_2, \cdots) \in \sum(X)$ such that there exists $(y_1, y_2, \cdots) \in \sigma$ with $\| y_n - x_n \| \leq \delta_n \; \forall n$.

Denote by $\sigma_{-\Delta}$ the set $((\sigma^c))^C_\Delta$.

Definition 3.28

A set $\sigma \subseteq \sum(X)$, for a Banach space X with basis $(e_n)_{n=1}^\infty$, is weakly Ramsey if, for every $\Delta > 0$, either X has an infinite dimensional block subspace Y such that every sequence $(y_1, y_2, \cdots) \in \sum(X)$ consisting of vectors in Y is an element of $(\sigma^C)_\Delta$, or X has a subspace Y such that A has a winning strategy for σ_Δ, provided all of B's moves are subspaces of Y.

Two topologies can be defined on $\sum(X)$. First take the norm topology on X, then the product topology: we call this N. Second take the discrete topology on X, then the product topology: we call this D.

Gowers theorem can now be stated.

Theorem 3.29 Let X be a Banach space with a basis $(e_n)_{n=1}^\infty$. Every N-analytic subset of $\sum(X)$ is weakly Ramsey.

Using Theorem 3.29, Gowers gave a new proof of Theorem 3.24 which we now sketch.

Proof of Theorem 3.24.

Without loss assume X has a bimonotone basis. If X contains no unconditional block basic sequence, then every block subspace contains a block basis $x_1, x_2, \cdots$ such that $\| x_i \| \geq 10^{-i} \; \forall i$ and given $k \; \exists \, n < m$ such that

$$\| \sum_{i=n}^{m} x_i \| > k \, \| \sum_{i=n}^{m} (-1)^i x_i \| .$$

The set of such sequences is a G_δ set for N and so is N analytic. By Theorem 3.29, player A has a winning strategy in a subspace W for producing sequences like this with k replaced by $\frac{k}{2}$. By considering strategies for player B of form $Y, Z, Y, Z, Y, Z, \cdots$ with Y, Z arbitrary subspaces of W, it may be shown that the sequence produced by A gives for all $\epsilon > 0$ elements $y \in Y$ and $z \in Z$ with $\epsilon \parallel y+z \parallel > \parallel y-z \parallel$ and so W is hereditarily indecomposable.□.

Gowers then goes on to to give a classification of Banach spaces.

Definition 3.30

An infinite dimensional Banach X space is minimal if every infinite dimensional subspace of X has a further subspace isomorphic to X.

Two infinite dimensional Banach spaces X and Y are totally incomparable if no infinite dimensional subspace of X is isomorphic to a subspace of Y.

An infinite dimensional Banach space X is quasi-minimal if it does not contain a pair of totally incomparable infinite dimensional subspaces.

Let $Y \leq Z$ mean Y embeds isomorphically in Z. Write $Y \sim Z$ if $Y \leq Z$ and $Z \leq Y$. Then given a Banach space $X, \sim$ defines an equivalence relation on the infinite dimensional subspaces of X. If X is quasi-minimal there is a collection ζ of infinite dimensional subspaces of X such that the set ζ is partially ordered by $\leq$, if $Y, Z \in \zeta$ then there exists $W \in \zeta$ such that $Y \geq W$ and $Z \geq W$ and every infinite dimensional subspace of X has a subspace isomorphic to a subspace in ζ. Moreover the existence of such a collection ζ implies that X is quasi-minimal. (Take ζ to consist of a representative member of each equivalence class under $\sim$).

As a result, if a quasi-minimal space X has a minimal subspace Y, then Y embeds in every infinite dimensional subspace of X. Therefore, X has a minimal subspace if and only if ζ has cardinality one for every choice of ζ.

If ζ has cardinality greater than one, we say that X is strictly quasi-minimal.

It may be shown that if $|\zeta| > 1$, then ζ is uncountable.

Also an infinite dimensional hereditarily indecomposable space is quasi-minimal.

We now state Gowers final result.

Theorem 3.31

Let X be an infinite dimensional Banach space. Then X has an infinite dimensional subspace Y with one of the following properties, which are mutually exclusive and all possible:

(i) The space Y is hereditarily indecompsable and therefore every operator from a subspace Z of Y into Y is a strictly singular perturbation of a scalar multiple of the inclusion map.

(ii) The space Y has an unconditional basis and every isomorphism between block subspaces W and Z of Y is a strictly singular perturbation of some invertible diagonal operator on Y.

(iii) The space Y has an unconditional basis and is strictly quasi-minimal.

(iv) The space Y has an unconditional basis and is minimal.

It should be mentioned that Schlumprecht showed that the space S of [9] is complementably minimal: every finite dimensional subspace of S contains a subspace isomorphic to S and complemented in S. [21]. It was also shown that Tsirelson's space T contains no minimal subspace [22].

Odell and Schlumprecht also solved another open problem in the circle of ideas of this chapter [23].

Brunel and Succheston proved a result which allowed the notion of spreading model to be defined. [24]

Proposition 3.32

Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence in a separable Banach space E. There exists a subsequence $(x_n^1)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that, for all $x \in E$, for all $k \geq 1$ and scalars $(a_1, \cdots, a_k) = a$,

$$L(x,a) = \lim_{k\to\infty} \| x + a_1 x_{n_1}^1 + a_2 x_{n_2}^1 + \cdots + a_k x_{n_k}^1 \|$$

exists whenever $n_1 < n_2 < \cdots < n_k$ tends to infinity.

Let $(e_i)_{i=1}^{\infty}$ be the canonical basis of $\mathbb{R}^{(\mathbb{N})}$. Let us suppose we can extract a bounded subsequence by Propostion 3.32, we set

$$\| x + \sum_i a_i e_i \| = L(x,a)\; \forall x \in E, \forall\, a \in \mathbb{R}^{(\mathbb{N})}.$$

This defines a semi-norm for $E \times \mathbb{R}^{(\mathbb{N})}$.

It may easily be shown that this seminorm defines a norm on $E \times \mathbb{R}^{(\mathbb{N})}$ if and only if the sequence (x_n^1) does not converge in E.

Definition 3.33

A sequence (x_n^1) such as in Proposition 3.32 for which the seminorm above is a norm is called a spreading sequence. If $(x_n)_{n=1}^{\infty}$ is a spreading sequence in a separable Banach space E the completion of $E \times \mathbb{R}^{(\mathbb{N})}$ for the norm described above is the extension of E defined by the sequence $(x_n)_{n=1}^{\infty}$. Denote it by $\mathcal{F}$.

The closed subspace of $\mathcal{F}$ generated by the sequence $(e_i)_{i=1}^{\infty}$ is the spreading model defined by the sequence $(x_n)_{n=1}^{\infty}$.

It had been widely conjectured that every infinite dimensional Banach space X had a spreading model which contained c_0 or $\ell^p (1 \leq p < \infty)$. However the following result is true. [23]

Theorem 3.34

There is an infinite dimensional Banach space X which has no spreading model which contains c_0 or ℓ^p $(1 \leq p < \infty)$.

The space is a modification of Schlumprecht's space S [9] along the lines of Gowers' space in [15].

For an extensive study of spreading models we refer to [24].

Further results on hereditarily indecomposable Banach spaces have been found by Ferenczi [31]. In particular he has constructed a uniformly convex hereditarily indecomposable space [32] and studied the K-theory of the algebra of bounded operators of two of the spaces constructed by Gowers and Maurey [31]. Properties of C_0-groups and C_0-semigroups of operators on hereditarily indecomposable Banach spaces were studied in [35]. Further results on hereditarily indecomposable Banach spaces can be found in [34], [33]. In [33] hereditarily indecomposable Banach spaces were shown to be precisely those spaces X where every bounded operator $T : Y \to X$ mapping a subspace Y into X has form $T = \lambda i_Y + S$, where λ is a scalar, $i_Y : Y \to X$ is the inclusion mapping and S is a strictly singular operator. This result clarifies the few operators property enjoyed by hereditarily indecomposable Banach spaces. It has also been announced that the final version of [14] contains an example of an indecomposable Banach space that is not hereditarily indecomposable. In [38] it is shown that a hereditarily indecomposable Banach space is arbitrarily distortable.

A Positive Result

To complete this chapter we give our own result which yields a condition for a Banach space to be decomposable.

Theorem 3.35

Let X be a Banach space such that X and X^* have Fréchet differentiable norms. If there is a basic sequence $(x_n)_{n=1}^{\infty}$ in X such that $X_m = [x_1, x_2, \cdots, x_m]$ is complemented by a projection $P_m : X \to X_m$ with $\| I - P_m \| = 1$, $\forall m \in \mathbb{N}$, then the space $Y = [x_n]_{n=1}^{\infty}$ is complemented in X.

If Y is of infinite co-dimension in X, then the Banach space X is decomposable.

Proof of Theorem 3.35

In [27], Rao proved the equivalence of the following properties:

(i) X is strictly convex, reflexive and if a sequence $(x_n) \subseteq X$ weakly converges to $x \in X$ and $\| x_n \| \to \| x \|$ as $n \to \infty$, then $\| x - x_n \| \to 0$ as $n \to \infty$.

(ii) X^* has Fréchet differentiable norm.

We may therefore suppose that X is strictly convex and reflexive.

For any non empty closed convex subset C of X, define $\pi(x, C) = \{y \in C : \| x - y \| = \inf\{\| x - z \| : z \in C\}\}$

When X is strictly convex, the set valued map $\pi(x, C)$ takes values the empty set or singleton sets, so $x \mapsto \pi(x, C)$ defines an indempotent mapping, provided $\pi(x, C)$ is non empty for all $x \in C$. We prove a simple lemma on this mapping.

Lemma 3.36

If X is a strictly convex Banach space and C is a finite dimensional subspace complemented by a projection $P : X \to C$ with $\| I - P \| = 1$, then $Px = \pi(x, C)\, \forall x \in X$. In particular $\pi(., C)$ is a bounded, linear mapping.

Proof Since C is finite dimensional, a well known result from the theory of Hilbert spaces gives that $\pi(x, C)$ is non empty for all $x \in C$. Therefore $x \mapsto \pi(x, C)$ defines an idempotent mapping on X, since X is strictly convex. For $x \in X$,

$$\| (I - P)(x - \pi(x, C)) \| = \| x - Px \|,$$

since

$$(I - P)(\pi(x, C)) = 0 \text{ for } \pi(x, C) \in C.$$

Therefore

$$\begin{aligned} \|(I - P)(x - \pi(x, C))\| &\leq \|I - P\| \cdot \|x - \pi(x, C)\| \\ &= \| x - \pi(x, C) \| \\ &\leq \| x - Px \|, \end{aligned}$$

since $Px \in C$, by the definition of $\pi(x, C)$.

Thus, given $x \in X$

$$\| x - Px \| = \| x - \pi(x, C) \| .$$

Since X is strictly convex, $\pi(x, C)$ is uniquely defined so that,

$$Px = \pi(x, C), \forall x \in X,$$

as required.□.

For a sequence of non empty, closed convex sets $\{C_n\}$ in X, Mosco [28] made the following definitions $s - \liminf C_n = \{x \in X : \exists\, x_n \in C_n$ for almost all n and $x_n \to x$ in norm as $n \to \infty\}$

$\omega - \limsup C_n = \{x \in X : \exists$ as subsequence $\{C_{n^1}\}$ of $\{C_n\}$ and $x_{n^1} \in C_{n^1}\, \forall n^1$ and $x_{n^1} \to x$ in the weak topology as $n^1 \to \infty\}$

If $s - \liminf C_n = \omega - \liminf C_n$ the common value is denoted $\lim\limits_n C_n$.

In [29] the next result is proved. It supersedes an earlier version from [27].

Proposition 3.37

If X is a Banach space such that both X and X^* have Fréchet differentiable norm, for any sequence of non empty closed convex sets $\{C_n\}$ in X, the following are equivalent:

(i) $\lim\limits_n C_n$ exists and is non empty.

(ii) there exists non empty, closed convex C such that $d(x, C_n)$ tends to $d(x, C)$ as $n \to \infty$.

(iii) $\pi(x, C_n)$ is a norm convergent sequence for every $x \in X$.

If we consider $C_n = [x_1, x_2, \cdots, x_n]$ and $C = [(x_n)_{n=1}^{\infty}]$, since $(x_n)_{n=1}^{\infty}$ is a basic sequence property (ii) of Proposition 3.37 holds; so using Lemma 3.36 we can define $P : X \to C$ by

$$Px = \lim_{n \to \infty} \pi(x, C_n) = \lim_{n \to \infty} P_n x.$$

The mapping P is clearly a linear indempotent onto C. By the Banach-Steinhaus theorem, P is continuous and we are done.□

Deville has shown that if the norm of a Banach space X is C^{∞}, then X contains an isomorphic copy of ℓ^p and so is not hereditarily indecomposable. Deville has also shown that a space X such that both X and X^* have C^{∞} bump functions must be a Hilbert space. [36].

This prompts the following question.

Question 3.36 How smooth need the norm of a Banach space X be before the space X is decomposable and are smoothness conditions sufficient on their own to imply that a space is decomposable?

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Chapter 4
Spaces of Bounded Operators.

1. Two Few Operator Problems.

It is well known to Banach space theorists that for certain pairs of Banach spaces X and Y, the space of bounded linear operators from X to Y, $L(X,Y)$, coincides with the space of compact operators from X to Y, $K(X,Y)$. (e.g. Take $X = \ell^p, Y = \ell^q$, $1 \leq q < p < \infty$). The following question has been raised.

Question 1 Is there a pair of Banach spaces X and Y such that $L(X,Y) = K(X,Y)$ and $L(Y,X) = K(Y,X)$.

The solution of this problem has been known for some time now, but has never been published. The short proof below was communicated by F. Delbaen. It is due to J. Bourgain and F. Delbaen.

Lemma 2 Let X be any Banach space.

An operator $T : \ell^2 \to X$ is compact if and only if $\| Te_n \| \to 0$ as $n \to \infty$ for any orthonormal sequence (e_n) in ℓ^2.

Proof cf [1], p.46.

A Banach space is somewhat reflexive if every infinite dimensional subspace contains an infinite dimensional reflexive subspace.

Let $1 < \lambda < \infty$ and $1 \leq p \leq \infty$. A Banach space is called a $\mathcal{L}_{p,\lambda}$ -space if given a finite dimensional subspace B of X there is a finite dimensional subspace of E of X containing B and an invertible linear map $T : E \to \ell_p^{\dim E}$ such that $\| T \| \| T^{-1} \| \leq \lambda$.

A Banach space is said to be a $\mathcal{L}^p$ space if it is a $\mathcal{L}_{p,\lambda}$ -space for some $\lambda > 1$.

Lemma 3 ([2], Lemma 5.9) Let X be an example of a somewhat reflexive $\mathcal{L}^\infty$ space with X^* isomorphic to ℓ^1 as constructed by Bourgain and Delbaen in [2].

Let (x_n) be a weakly null sequence in X with $\| x_n \| \geq \delta$. We can extract a subsequence (x_{nk}) such that

$$\| \sum_{k=1}^{N} x_{n_k} \| > CN^{1/p},$$

where the constant C and exponent $p > 1$ depend on δ and the parameters used in the construction of the $\mathcal{L}^\infty$ space X. The constant C and exponent p are both independent of the arbitrary integer N and of the weakly null sequence $(x_n)_{n=1}^{\infty}$.

Proof This is [2], lemma 5.9. We may assume $1 < p < 2$ by varying the parameters in the construction of X.□.

Theorem 4

There exists a $\mathcal{L}^\infty$-space X which is somewhat reflexive and such that $X^* \cong \ell^1$, so that

$$L(X,\ell^2) = K(X,\ell^2) \text{ and } L(\ell^2,X) = K(\ell^2,X).$$

Proof Take X to be a $\mathcal{L}^\infty$-space of Bourgain-Delbaen constructed to satisfy Lemma 3.

Let $T : X \to \ell^2$ be a bounded linear operator. Since $T^* : \ell^2 \to X^*$, where $X^* \cong \ell^1$, maps ℓ^2, a reflexive space, into an ℓ^1 copy, the adjoint T^* is compact and so T is compact.

So $L(X,\ell^2) = K(X,\ell^2)$.

Let $T : \ell^2 \to X$ be a bounded linear operator. If T is not compact, by Lemma 2, there is an orthonormal sequence (z_n) in ℓ^2 such that $y_n = Tz_n$ is not norm null. The sequence (y_n) must be weakly null, so by considering a subsequence (y_{n_k}) we have $\| y_{n_k} \| \geq \delta > 0$ for some δ. By Lemma 3 there exists $C^1 < \infty$ and $1 < p < 2$ such that

$$C^1 \| T \| N^{\frac{1}{2}} \geq \| \sum_{k=1}^{N} y_{n_k} \| = \| \sum_{k=1}^{N} T z_{n_k} \| > CN^{\frac{1}{p}}.$$

So there is a constant D such that for all N,

$$N^{\frac{1}{2}} \geq DN^{\frac{1}{p}}.$$

Since $1 < p < 2$ this is a contradiction and so T must be compact.

Hence $L(\ell^2,X) = K(\ell^2,X)$.□

A second question has also been considered by Banach space theorists.

Question 2

Is there a Banach space X for which some proper closed ideal of operators contained in $L(X)$, the Banach algebra of bounded linear operators on X, is complemented in $L(X)$?

In particular is the ideal of compact operators on $X, K(X)$, ever complemented in $L(X)$?

In [3] it was shown that for $X = J$, the James space, $L(X)$ admits a non trivial multiplicative linear functional and so $L(X)$ has an ideal of co-dimension one. In fact such an ideal of operators can be explicitly identified.

Lemma 5 (Gantmacher, cf [4], p.21).

An operator $T : X \to Y$ between Banach spaces X and Y is weakly compact if and only if $T^{**}(X^{**}) \subseteq Y$.

Theorem 6 *If X is a Banach space of co-dimension one in its bidual, then every operator on X is of the form $\lambda I + W$, where λ is a scalar, I is the identity operator on X and W is a weakly compact operator.*

Proof We have that $X^{**} = X \oplus Y$, where Y is a one dimensional space. Consider the projection $P : X^{**} \to Y$ onto Y along X. If $T : X \to X$ is an arbitrary bounded linear operator, since $P \circ T^{**}/_Y$ maps one dimensional space Y onto itself there exists a scalar λ such that

$$P \circ T^{**}|_Y = \lambda I_Y$$

The mapping $P \circ (T - \lambda I_X)^{**}$ maps X^{**} onto Y and in fact

$$P \circ (T - \lambda I_X)^{**} = 0.$$

It follows that $(T-\lambda I_X)^{**}(X^{**})$ is contained in $(I-P)(X^{**}) = X$ i.e. $(T-\lambda I_X)^{**}(X^{**}) \subseteq X$ and so by Lemma 5, $T - \lambda I_X$ is weakly compact. □.

Thus, for the James space J, the ideal of weakly compact operators $W(J)$ is of co-dimension one in $L(J)$.

G. Emmanuele has studied the complementation of the space of weakly compact operators between Banach spaces X and Y, $W(X,Y)$, in the space of bounded operators $L(X,Y)$ in [5].

2. The Banach Space $L(\ell^1, \ell^p), 1 < p < \infty$, is primary.

The space of bounded linear operators between ℓ^1 and a Banach space X can be explicitly identified.

Proposition 7 *Let X be any Banach space, then the space $L(\ell^1, X)$ is isometrically isomorphic to $(X \oplus X \oplus X \oplus \cdots)_{\ell^\infty}$.*

Proof Let $(e_n)_{n=1}^\infty$ denote the standard basis of ℓ^1. Define a mapping from $L(\ell^1, X)$ to $(X \oplus X \oplus \cdots)_{\ell^\infty}$ by

$$T \to (\mathrm{Te}_n)_{n=1}^\infty$$

Then, the mapping is linear and

$$\sup_n \| \mathrm{Te}_n \|_X \leq \| T \|$$

so the mapping is bounded. Also, if $\mathrm{Te}_n = 0$, for all $n \in \mathbb{N}$, then $T = 0$, so the mapping is injective. If $x = \sum_{n=1}^\infty x_n e_n \in \ell^1$, then

$$\| T(x) \| = \| \sum_{n=1}^\infty x_n \mathrm{Te}_n \| \leq \| x \|_{\ell^1} \sup_n \| \mathrm{Te}_n \|_X$$

So that

$$\| T \| = \sup_n \| \mathrm{Te}_n \|_X .$$

If $(y_n)_{n=1}^\infty \in (X \oplus X \oplus \cdots)_{\ell^\infty}$ define $S : \ell^1 \to X$ by $Se_n = y_n$. Then S is bounded with

$$\| S \| = \sup_n \| y_n \|_X$$

and under the mapping the image of S is $(y_n)_{n=1}^\infty \in (X \oplus X \oplus \cdots)_{\ell^\infty}$, so the mapping is an isometric isomorphism.□.

We shall now study the projections on the space of bounded operators $L(\ell^1, \ell^p)$ for $1 < p < \infty$.

A Banach space X is said to be primary if, when X is isomorphic to $Y \oplus Z$, then X is isomorphic to Y or X is isomorphic to Z. In [6] Blower proved that $L(\ell^2)$ is primary. We shall modify Blower's arguments to show that $L(\ell^1, \ell^p), 1 < p < \infty$, is primary.

2.1 A Finite Dimensional Decomposition

We first express $L(\ell^1, \ell^p)$ as a direct sum of finite dimensional subspaces.

Proposition 8

Let (n_k) be a sequence in $\mathbb{N}$ with $\sup_k n_k = \infty$. Then,

$$L(\ell^1, \ell^p) \cong (\oplus_{k=1}^{\infty} \ell^p_{n_k})_{\ell^\infty}.$$

Proof By Proposition 7 we have that

$$L(\ell^1, \ell^p) \cong (\ell^p \oplus \ell^p \oplus \cdots)_{\ell^\infty}.$$

It is obvious that $(\oplus_{n=1}^{\infty} \ell^p_n)_{\ell^\infty}$ is complemented in $L(\ell^1, \ell^p)$.

CLAIM The space $(\oplus_{n=1}^{\infty} \ell^p_n)_{\ell^\infty}$ contains a complemented copy of ℓ^p.

To prove the CLAIM, let P_n denote the basis projections of the standard basis $(e_n)_{n=1}^{\infty}$ in ℓ^p. Let $\widetilde{S}$ be subspace of $(\oplus_{n=1}^{\infty} \ell^p_n)_{\ell^\infty}$ given by

$$\widetilde{S} = \{(x_n)_{n=1}^{\infty} : P_n x_{n+1} = x_n, \forall\, n \in \mathbb{N}\}.$$

If $(y_n)_{n=1}^{\infty} = y \in \widetilde{S}, P_n y_{n+1} = y_n$ and so considering the y_n as elements of ℓ^p,

$$|y_n| \leq |y_{n+1}| \text{ on } \mathbb{N},$$

$$\sup_n \| y_n \|^p_{\ell^p_n} < \infty.$$

So since every montone increasing sequence of real numbers bounded above converges, there exists unique $y \in \ell^p$ such that $P_n y = y_n$ for all n.

If we define $\psi : \ell^p \to \widetilde{S}$ by

$$\psi(y) = (P_n y)_{n=1}^{\infty},$$

then ψ is bounded linear and onto $\widetilde{S}$.

Let $\varphi : (\oplus_{n=1}^{\infty} \ell^p_n)_{\ell^\infty} \to \ell^p$ be given by

$$\langle \varphi((x_n)_{n=1}^{\infty}), \eta \rangle = LIM \langle x_n, \eta \rangle,$$

for all $\eta \in \ell^{p^1}, \frac{1}{p} + \frac{1}{p^1} = 1$, where LIM denotes a Banach limit. Then φ is bounded and linear and $\psi \circ \varphi$ is a projection of $(\oplus_{n=1}^{\infty} \ell^p_n)_{\ell^\infty}$ onto $\psi(\ell^p) = \widetilde{S}$ and $\varphi \circ \psi$ is the identity mapping onto ℓ^p. So ℓ^p is a complemented subspace of $(\oplus_{n=1}^{\infty} \ell^p_n)_{\ell^\infty}$.

Let $L = (\oplus_{n=1}^{\infty} \ell_n^p)_{\ell^\infty}$. If we consider a partition of $\mathbb{N}$ into infinite sets $(N_k)_{k=1}^{\infty}$, then $L = (\oplus_{k=1}^{\infty}(\oplus_{m\in N_k} \ell_m^p)_{\ell^\infty})_{\ell^\infty}$, and so we can find a subspace W of L such that

$$L \cong (L \oplus L \oplus L \oplus \cdots)_{\ell^\infty} \oplus W,$$

since $(\oplus_{m\in N_k} \ell_m^p)_{\ell^\infty}$ contains a complemented copy of $(\oplus_{n=1}^{\infty} \ell_n^p)_{\ell^\infty} = L$ for any infinite set N_k.

Therefore,

$$\begin{aligned} L \oplus L &\cong L \oplus ((L \oplus L \oplus \cdots)_{\ell^\infty} \oplus W) \\ &\cong (L \oplus L \oplus L \oplus \cdots)_{\ell^\infty} \oplus W \\ &\cong L. \end{aligned}$$

So that

$$\begin{aligned} (L \oplus L \oplus L \oplus \cdots)_{\ell^\infty} &\cong (L \oplus L \oplus \cdots)_{\ell^\infty} \oplus L \\ &\cong (L \oplus L \oplus \cdots)_{\ell^\infty} \oplus ((L \oplus L \oplus \cdots)_{\ell^\infty} \oplus W) \\ &\cong ((L \oplus L) \oplus (L \oplus L) \oplus \cdots)_{\ell^\infty}) \oplus W \\ &\cong (L \oplus L \oplus L \oplus \cdots)_{\ell^\infty} \oplus W \\ &\cong L. \end{aligned}$$

If $L^\circ = (\ell^p \oplus \ell^p \oplus \ell^p \oplus \cdots)_{\ell^\infty}$, for any partitition of $\mathbb{N}$ into infinite sets N_k,

$$\begin{aligned} L^\circ &= (\oplus_{k=1}^{\infty}(\oplus_{m\in N_k} \ell^p)_{\ell^\infty})_{\ell^\infty} \\ &\cong (L^\circ \oplus L^\circ \oplus L^\circ \oplus \cdots)_{\ell^\infty} \end{aligned}$$

since $(\oplus_{m\in N_k} \ell^p)_{\ell^\infty} \cong L^\circ$.

By Pelczyński's decomposition method we have that

$$L^\circ = L(\ell^1, \ell^p) \cong (\oplus_{k=1}^{\infty} \ell_k^p)_{\ell^\infty} = L$$

To finish the proof observe that given n_k with $\sup_k n_k = \infty$, $L^{\circ\circ} = (\oplus_{k=1}^{\infty} \ell_{n_k}^p)_{\ell^\infty}$ is complemented in L and L is complemented in L^{00}. An argument similar to that used for L, shows that

$$L^{\circ\circ} \cong (L^{\circ\circ} \oplus L^{\circ\circ} \oplus \cdots)_{\ell^\infty}.$$

Applying Pelczyński's decomposition theorem gives $L \cong L^{\circ\circ}$, which is the required result.□.

2.2 A Finite Dimensional Result.

Let M_n be the set of $n \times n$ matrices endowed with the $\ell_{n^2}^p$ norm. By Proposition 8, we have that for a sequence (n_k) in $\mathbb{N}$ with $\sup_k n_k = \infty$,

$$L(\ell^1, \ell^p) \cong (\oplus_{k=1}^{\infty} M_{n_k})_{\ell^\infty}.$$

Definition 9

Let $\sigma = \{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ and $\tau = \{\tau_1, \tau_2, \cdots, \tau_n\}$ be two subsets of $\{1, 2, 3, \cdots, N\}$ of size n. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices in M_n. Define $J_{\sigma,\tau} : M_n \to M_N$ by

$$(J_{\sigma,\tau}(A))_{k\ell} = \begin{cases} a_{ij}, & \text{if } k = \sigma_i, \ell = \tau_j \\ 0, & \text{otherwise} \end{cases}$$

Let $K_{\sigma,\tau} : M_N \to M_n$ be defined by

$$(K_{\sigma,\tau}(B))_{ij} = b_{\sigma(i),\tau(j)}$$

Observe that $K_{\sigma,\tau}J_{\sigma,\tau} = I_n$, the identity map on M_n and $J_{\sigma,\tau}K_{\sigma,\tau} = P_{\sigma,\tau}$ is a projection onto $J_{\sigma,\tau}(M_n)$. Let us call $P_{\sigma,\tau}$ a block projection of order n.

Proposition 10 Given $n, \epsilon > 0$ and $K < \infty$ there exists N_0 such that if $N \geq N_0$ and $T \in L(M_N, M_N)$ with $\| T \| \leq K$, then there exist disjoint subsets σ, τ of $\{1, 2, \cdots, N\}$ of size n and a constant c such that

$$\| K_{\sigma,\tau}TJ_{\sigma,\tau} - cI_n \| < \epsilon.$$

Therefore, one of $K_{\sigma,\tau}TJ_{\sigma,\tau}$ or $K_{\sigma,\tau}(I_N - T)J_{\sigma,\tau}$ is invertible.

Proof Let $\epsilon > 0$ be given. Divide $D = \{z \in \mathbb{C} : |z| \leq K\}$ into finitely many disjoint subsets V_k of diameter $\frac{\epsilon}{4n^2}$. Define a colouring on pairs of $\{1, 2, \cdots, N\}$ by

$$\{i, j\} \mapsto k \text{ if } \langle \mathrm{T}e_{ij}, e_{ij} \rangle \in V_k$$

where $i < j$.

[$e_{ij} = e_i \otimes e_j$ is the usual matrix unit and we consider the $\ell^p - \ell^{p^1}$, where $\frac{1}{p} + \frac{1}{p^1} = 1$, duality].

By Ramsey's theorem [7], we can find an index $\mathcal{K}$ and a large monochromatic subset σ_1 of $\{1, 2, \cdots, N\}$ whose colour is $\mathcal{K}$. Let c be any non zero point in $V_{\mathcal{K}}$.

Let $\delta > 0$ be given to be specified later. Consider the 2-colouring of the quadruples of σ_1 given by $\{i, j, k, \ell\}$ is bad if $i < k < j < \ell$ and $|\langle \mathrm{T}e_{ij}, e_{k\ell} \rangle| \geq \delta$.

By Ramsey's theorem there is a large monochromatic subset σ_2 of σ_1. If σ_2 is a bad monochromatic subset of σ_1, let $k < j < \ell$ be the three largest elements of σ_2 and i range from the smallest element of σ_2 to the largest element σ_2 smaller than k. For suitable complex numbers ϵ_i with $|\epsilon_i| = 1$,

$$\begin{aligned} \| T(\textstyle\sum_i \epsilon_i e_{ij}) \| \geq \textstyle\sum_i \epsilon_i \langle \mathrm{T}e_{ij}, e_{k\ell} \rangle & \\ & \geq \delta(|\sigma_2| - 3) \end{aligned}$$

However, $\| \sum_i \epsilon_i e_{ij} \| = (|\sigma_2| - 3)^{\frac{1}{p}}$ and hence $K \geq \delta(|\sigma_2| - 3)^{1-\frac{1}{p}}$.

Since $1 < p < \infty$, by a suitable choice of N we ensure that σ_2 is good. Continuing in this way we obtain a large subset σ_3 of σ_2 such that

$$|\langle \mathrm{T}e_{ij}, e_{k\ell} \rangle| \leq \delta$$

whenever i, j, k and ℓ are distinct elements of σ_3 with $\max\{i,k\} < \min\{j,\ell\}$.

For the case $i = k < j < \ell$, consider the two colouring of triples of σ_3 given by

$\{i, j, \ell\}$ is bad if $i < j < \ell$ and $|\langle \mathrm{Te}_{ij}, e_{i\ell}\rangle| \geq \delta$.

Ramsey's theorem gives a large monochromatic set. If ϱ is a bad monochromatic subset, fix i as small as we can, ℓ as large as we can and let j range through the bad subset so that $i < j < \ell$. For suitable complex numbers ϵ_j with $|\epsilon_j| = 1$ we have

$$\| \sum_j \epsilon_j e_{ij} \| = (|\varrho| - 2)^{\frac{1}{p}}$$

while

$$\begin{aligned} \| \textstyle\sum_j T\epsilon_j e_{ij} \| &\geq \textstyle\sum_j \epsilon_j \langle \mathrm{Te}_{ij}, e_{i\ell}\rangle \\ &\geq \delta(|\varrho| - 2). \end{aligned}$$

This ensures that we have a large good monochromatic subset ϱ_4 of σ_3. The cases $i = k < \ell < j$ and $i < k < j = \ell$ are dealt with similarly.

In conclusion, there is a large subset σ_5 of σ_4 such that when (i, j) and (k, ℓ) are distinct pairs in σ_5 with $\max\{i, j\} < \min\{k, \ell\}$ then

$$|\langle \mathrm{Te}_{ij}, e_{k\ell}\rangle| \leq \delta.$$

We split up σ_5 by taking the first n elements to form σ and the last n to form τ. The construction of σ and τ gives

$$|\langle \mathrm{Te}_{ij}, e_{ij}\rangle - c| \leq \tfrac{\epsilon}{4n^2}.$$

We can write

$$\begin{aligned} & \| (K_{\sigma,\tau} T J_{\sigma,\tau} - cI_n) \textstyle\sum_{(i,j)} a_{ij} e_{ij} \| \\ \leq\ & \| \textstyle\sum_{(i,j)} a_{ij} (\langle \mathrm{Te}_{\sigma_i,\tau_j}, e_{\sigma_i,\tau_j}\rangle - c) e_{ij} \| \\ +\ & \| \textstyle\sum_{(i,j)} a_{ij} \sum_{(k,\ell):(k,\ell)\neq(i,j)} \langle \mathrm{Te}_{\sigma_i,\tau_j}, e_{\sigma_k,\tau_\ell}\rangle e_{k\ell} \| \\ \leq\ & n^2 \max |a_{ij}| \tfrac{\epsilon}{4n^2} + n^4 \delta \max |a_{ij}| \\ \leq\ & \tfrac{\epsilon}{2} \max |a_{ij}|, \end{aligned}$$

where $\delta = \frac{\epsilon}{4n^4}$ ensures the above holds. So we have

$$\| K_{\sigma,\tau} T J_{\sigma,\tau} - cI_n \| < \epsilon.$$

The final statement follows by considering Neumann series to obtain inverses of the operators, noticing that c was chosen to be non zero.□.

2.3. A Local Result.

We shall call operators $T \in L(\mathcal{M})$, where $\mathcal{M} = (\oplus_{n=1}^{\infty} M_n)_{\ell^\infty}$, diagonal operators if T has form $T = \oplus_{k=1}^{\infty} T_k$ where $T_n \in L(M_n, M_n)$. The analysis reduces to a study of these diagonal operators.

We shall say that an operator $S : X \to Y$ between Banach spaces X and Y *factors through* an operator $R : X_1 \to Y_1$ between Banach spaces X_1 and Y_1, if there exist operators $U : X \to X_1$ and $V : Y_1 \to Y$ such that

$$S = VRU$$

If an operator $S_1 : X \to Y$ factors through $R_1 : X_1 \to Y_1$ and an operator $S_2 : X \to Y$ factors through $R_2 : X_1 \to Y_1$, we shall say that the factorizations coincide provided there exist operators $U : X \to X_1$ and $V : Y_1 \to Y$ such that

$$\begin{aligned} S_1 &= VR_1U \\ and S_2 &= VR_2U \end{aligned}$$

Lemma 11

Given $\epsilon > 0$ and a bounded, linear operator $T : \mathcal{M} \to \mathcal{M}$ there is a bounded, linear operator T^1 on $\mathcal{M}$ such that

(1) T^1 may be factored through T and $I - T^1$ may be factored through $I - T$ and the factorizations coincide.

(2) T^1 is almost diagonal in the sense that $\| T^1 - D \| < \epsilon$ where D is diagonal.

To prove lemma 11 we also need to prove a further lemma.

Lemma 12

Given $n \in \mathbb{N}, \epsilon > 0$ there is an $N^1(n, \epsilon)$ such that if $N \geq N^1(n, \epsilon)$ and E is an n-dimensional subspace of M_N there is a subspace F of M_N and a block projection q of order n of M_N onto F such that

$$\| qx \| \leq \epsilon \| x \| \text{ for } x \in E.$$

Proof Let $\epsilon > 0$ and $n \in \mathbb{N}$ be given. Let E be an n-dimensional subspace of M_{nm}, where m will be chosen later to be sufficiently large. By a volume argument we can find an $\frac{\epsilon}{2}$ net $x_1, x_2, \cdots, x_L$ of the unit sphere of E where L depends only on ϵ and n. Consider the natural projections Q_r onto the m^2 $n \times n$ blocks of M_{nm}. We have

$$L = \sum_{k=1}^{L} \| x_k \|^p = \sum_{k=1}^{L} \sum_{r=1}^{m^2} \| Q_r x_k \|^p$$

by a blocking property of the ℓ^p norm on M_{nm}

If we let m be sufficiently large that $L < m^2 (\frac{\epsilon}{2})^p$, we have

$$m(\frac{\epsilon}{2})^p \geq \frac{1}{m} \sum_{k=1}^{L} \sum_{r=1}^{m^2} \| Q_r x_k \|^p$$

So for some $r, 1 \leq r \leq m^2$,

$$(\frac{\epsilon}{2})^p \geq \sum_{k=1}^{L} \| Q_r x_k \|^p$$

We can choose our q of the lemma to be this Q_r, since (x_k) is an $\frac{\epsilon}{2}$ net for the unit sphere of E.□.

Definition 13

Let I be any subset of $\mathbb{N}$. We introduce projections on $\mathcal{M}$.

$$(P_I(x))_j = \begin{cases} x_j, & j \in I \\ 0, & \text{otherwise} \end{cases}$$

for $x = (x_j) \in \mathcal{M}$.

$$\begin{aligned} P_k(x_1, x_2, \cdots) &= (x_1, x_2, \cdots, x_k, 0, 0, \cdots) \\ R_k(x_1, x_2, \cdots) &= (0, 0, \cdots, 0, x_{k+1}, x_{k+2} \cdots) \\ p_k(x_1, x_2, \cdots) &= (0, 0, \cdots, x_k, 0, 0, \cdots). \end{aligned}$$

Proof of Lemma 11

We need to select T^1 such that

$$\| p_n T^1 P_{n-1} \| < \frac{\epsilon}{2^n}, \| p_n T^1 R_n \| < \frac{\epsilon}{2^n}$$

Given an infinite subset I of $\mathbb{N}, m \in I$ and $\epsilon > 0$, we can select a proper infinite subset J of I such that $m \neq J$ and $\| p_m T P_J \| < \epsilon$.

This is possible because of the fact that for each n, p_n is finite rank. If $\epsilon > 0$ and F is a finite dimensional space with $S \in L(\mathcal{M}, F)$ there is an infinite subset of $\mathbb{N}$ such that $\| SP_I \| < \epsilon$. It suffices to consider the case $F = \mathbb{C}$ so that $S \in \mathcal{M}^*$. We can partition $\mathbb{N}$ into infinitely many disjoint subsets φ and to each subset φ there corresponds $S_\varphi = SP_\varphi \in \mathcal{M}^*$. Since $\mathcal{M}$ is formed from an ℓ^∞ direct sum,

$$\| \sum_\varphi \pm S_\varphi \| \leq 2,$$

for any choice of signs and so $\| S_\varphi \| < \epsilon$ for some choice of φ.

Define recursively an increasing sequence of integers m_k, a decreasing sequence (I_k) of infinite subsets of $\mathbb{N}$ and block projections q_k on M_{m_k} such that

(i) q_k is of order k

(ii) if $x \in p_{m_k} T(\oplus_{s=1}^{k-1} q_s(M_{m_s}))$ then $\| q_k(x) \| \leq \frac{\epsilon}{2^k} \| x \|$

(iii) $m_k \in I_k - I_{k+1}$, and

(iv) $\| p_{m_k} T P_{I_{k+1}} \| \leq \frac{\epsilon}{2^k}$

Lemma 12 can be used to satisfy conditions (i) and (ii), while the remark of the above paragraph gives (iii) and (iv).

Introduce the mapping $r_k : M_k \to q_k(M_{m_k})$ as the natural isometry and let

$$T^1 = (\oplus r_k^{-1} q_k) T (\oplus r_k).$$

Then $\| T^1 \| \leq \| T \|$ and $I = (\oplus r_k^{-1} q_k)(\oplus r_k)$. Hence T^1 factors through T and $I - T^1$ factors through $I - T$. Moreover the factorizations coincide. Also since $\| p_n T^1 P_{n-1} \| < \frac{\epsilon}{2^n}$ and $\| p_n T^1 R_n \| \leq \frac{\epsilon}{2^n}$ we see that if $D = \oplus p_n T^1 p_n$ then

$$\| T^1 - D \| < \epsilon$$

where D is diagonal and the proof of Lemma 12 is complete. □.

2.4. The Main Result.

We can now prove our main result.

Theorem 14 *The Banach space* $L(\ell^1, \ell^p) \quad (1 < p < \infty)$ *is primary.*

Proof Let us recall that $\mathcal{M}$ is isomorphic to $L(\ell^1, \ell^p)$. Let $P : \mathcal{M} \to \mathcal{M}$ be a projection. Using the notation used above let $P = T$.

By Proposition 10 we can find an increasing sequence of integers n_j and bounded maps U_{n_j}, V_{n_j} where $\| U_{n_j} \| \leq 4$ and $\| V_{n_j} \| \leq 4$ such that Diagram 1 (see Appendix) commutes, where $Q_{n_j} = p_{n_j} T p_{n_j}$ for all j or $Q_{n_j} = p_{n_j}(I - T^1)p_{n_j}$ for all j

We conclude that there are bounded mappings U^1, V^1 such that Diagram 2 (See Appendix) commutes where $D^1 = D$ or $D^1 = I - D$.

A perturbation argument combined with lemma 12 gives bounded maps U, V such that Diagram 3 (see Appendix) commutes, where $T^{11} = T$ or $T^{11} = I - T$.

Hence, $(\oplus_j M_{n_j})_{\ell^\infty}$ is isomorphic to the subspace $T^{11}U(\oplus_j M_{n_j})_{\ell^\infty}$ of $T^{11}(\mathcal{M})$ complemented by the projection $T^{11}UV$. However $(\oplus_j M_{n_j})_{\ell^\infty}$ is isomorphic to $\mathcal{M}$.

We have that there exist spaces W and Z such that

$$\begin{aligned} T^{11}(\mathcal{M}) &\cong \mathcal{M} \oplus W \\ \mathcal{M} &\cong T^{11}(\mathcal{M}) \oplus Z \end{aligned}$$

So, since $\mathcal{M} \cong (\mathcal{M} \oplus \mathcal{M} \oplus \cdots)_{\ell^\infty}$

$$\begin{aligned} T^{11}(\mathcal{M}) &\cong W \oplus (\mathcal{M} \oplus \mathcal{M} \oplus \cdots)_{\ell^\infty} \\ &\cong W \oplus ((T^{11}(\mathcal{M}) \oplus Z) \oplus (T^{11}(\mathcal{M}) \oplus Z) \oplus \cdots)_{\ell^\infty} \\ &\cong (T^{11}(\mathcal{M}) \oplus T^{11}(\mathcal{M}) \oplus \cdots)_{\ell^\infty} \oplus Z^1, \end{aligned}$$

for some space Z^1. So that

$$\begin{aligned} T^{11}(\mathcal{M}) \oplus T^{11}(\mathcal{M}) &\cong T^{11}(\mathcal{M}) \oplus (T^{11}(\mathcal{M}) \oplus \cdots)_{\ell^\infty} \oplus Z^1 \\ &\cong (T^{11}(\mathcal{M}) \oplus T^{11}(\mathcal{M}) \oplus \cdots)_{\ell^\infty} \oplus Z^1 \\ &\cong T^{11}(\mathcal{M}) \end{aligned}$$

and

$$\begin{aligned} & (T^{11}(\mathcal{M}) \oplus T^{11}(\mathcal{M}) \oplus \cdots)_{\ell^\infty} \cong T^{11}(\mathcal{M}) \oplus (T^{11}(\mathcal{M}) \oplus \cdots)_{\ell^\infty} \\ \cong\ & Z^1 \oplus (T^{11}(\mathcal{M}) \oplus T^{11}(\mathcal{M}) \oplus \cdots)_{\ell^\infty} \oplus (T^{11}(\mathcal{M}) \oplus \cdots)_{\ell^\infty} \\ \cong\ & Z^1 \oplus ((T^{11}(\mathcal{M}) \oplus (T^{11}(\mathcal{M})) \oplus (T^{11}(\mathcal{M}) \oplus T^{11}(\mathcal{M})) \oplus \cdots)_{\ell^\infty} \\ \cong\ & Z^1 \oplus (T^{11}(\mathcal{M}) \oplus T^{11}(\mathcal{M}) \oplus \cdots)_{\ell^\infty} \\ \cong\ & T^{11}(\mathcal{M}). \end{aligned}$$

So that by Pelczyński's decomposition method, $T^{11}(\mathcal{M})$ is isomorphic to $\mathcal{M}$ and we are done. $\square$.

3. The Banach Space $L(\ell^p, \ell^r)$, $1 < p \leq r \leq p^1 < \infty$ where $\frac{1}{p} + \frac{1}{p^1} = 1$ is primary.

3.1 A Finite dimensional Decomposition.

Let W_n be the space of $n \times n$ matrices endowed with the operator norm of $L(\ell^p_n, \ell^r_n)$.

Proposition 15

Let (n_k) be a sequence in $\mathbb{N}$ with $\sup_k n_k = \infty$. Then,

$$L(\ell^p, \ell^r) \cong (\oplus_{k=1}^{\infty} W_{n_k})_{\ell^\infty}$$

for $1 < p \leq r < \infty$.

Proof

Let $(e_n)_{n=1}^{\infty}$ be the standard basis of ℓ^p (or ℓ^r). Let $(P_n^p)_{n=1}^{\infty}$ be the basis projections in $\ell^p, 1 < p < \infty$.

Define a mapping $\varphi : (\oplus_{k=1}^{\infty} W_k)_{c_0} \to K(\ell^p, \ell^r)$ by

$$\langle \varphi((T_n)_{n=1}^{\infty}), x \rangle = \sum_{k=1}^{\infty} (P_{m_k}^r - P_{m_{k-1}}^r) T_k (P_{m_k}^p - P_{m_{k-1}}^p) x,$$

for $x \in \ell^p$, where m_k is chosen so that $P_{m_0} = 0, m_1 = 1$ and $m_k = m_{k-1} + k$.

Then, given $(T_n)_{n=1}^{\infty} \in (\oplus_{n=1}^{\infty} W_n)_{c_0}$ and $x \in \ell^p$

$$\begin{aligned}
&\| \textstyle\sum_{k=m+1}^{n} (P_{m_k}^r - P_{m_k}^r) T_k (P_{m_k}^p - P_{m_{k-1}}^p) x \|_{\ell^r} \\
&= (\textstyle\sum_{k=m+1}^{n} \| (P_{m_k}^r - P_{m_{k-1}}^r) T (P_{m_k}^p - P_{m_{k-1}}^p) x \|_{\ell^r}^r)^{\frac{1}{r}} \\
&\leq \sup_{m+1 \leq k \leq n} \| (P_{m_k}^r - P_{m_{k-1}}^r) T_k (P_{m_k}^p - P_{m_{k-1}}^p) \| (\textstyle\sum_{k=m+1}^{n} \| (P_{m_k}^p - P_{m_{k-1}}^p) x \|_{\ell^p}^r)^{\frac{1}{r}} \\
&\leq \sup_{m+1 \leq k \leq n} \| T_k \| (\textstyle\sum_{k=m+1}^{n} \| (P_{m_k}^p - P_{m_{k-1}}^p) x \|_{\ell^p}^p)^{\frac{1}{p}},
\end{aligned}$$

since $1 < p \leq r < \infty$.

$$\leq \sup_{m+1 \leq k \leq n} \| T_k \| \| x \|.$$

Therefore, the operator $\varphi((T_n)_{n=1}^{\infty})$ is a well defined element of $K(\ell^p, \ell^r)$ for $(T_n)_{n=1}^{\infty} \in (\oplus_{k=1}^{\infty} W_k)_{c_0}$.

Let $Q : K(\ell^p, \ell^r) \to K(\ell^p, \ell^r)$ be given by

$$Q(T)(x) = \sum_{k=1}^{\infty} (P_{m_k}^r - P_{m_{k-1}}^r) T (P_{m_k}^p - P_{m_{k-1}}^p) x$$

for $x \in \ell^p$.

Then,

$$\begin{aligned}
&(\sum_{k=1+m}^{n} \| (P^r_{m_k} - P^r_{m_{k-1}})T(P^p_{m_k} - P^p_{m_{k-1}})x \|^r_{\ell^r})^{\frac{1}{r}} \\
\leq\ & \sup_{m+1\leq k\leq n} \| (P^r_{m_k} - P^r_{m_{k-1}})T(P^p_{m_k} - P^p_{m_{k-1}}) \| (\sum_{k=m+1}^{n} \| (P^p_{m_k} - P^p_{m_{k-1}})x \|^r_{\ell^p})^{\frac{1}{r}} \\
\leq\ & \sup_{m+1\leq k\leq n} \| (P^r_{m_k} - P^r_{m_{k-1}})T(P^p_{m_k} - P^p_{m_{k-1}}) \| (\sum_{k=m+1}^{n} \| (P^p_{m_k} - P^p_{m_{k-1}})x \|^p_{\ell^p})^{\frac{1}{p}}
\end{aligned}$$

since $1 < p \leq r < \infty$,

$$\leq \sup_{m+1\leq k\leq n} \| (P^r_{m_k} - P^r_{m_{k-1}})T(P^p_{m_k} - P^p_{m_{k-1}}) \| \| x \|_{\ell^p}$$

$\to 0$ as $m, n \to \infty$ since $T \in K(\ell^p, \ell^r)$ and $(e_n)_{n=1}^{\infty}$ is a basis of ℓ^r.

So $Q(T)$ is well defined and it is easily seen that $Q^2 = Q$ and $\| Q(T) \| \leq \| T \|$. Also, $Q(K(\ell^p, \ell^r))$ is contained in $\varphi((\oplus_{k=1}^{\infty} W_k)_{c_0})$ and if $T \in \varphi((\oplus_{n=1}^{\infty} W_n)_{c_0})$, then $Q(T) = T$. So $K(\ell^p, \ell^r)$ contains a complemented copy of $(\oplus_{k=1}^{\infty} W_k)_{c_0}$. If we consider double duals, we see that $L(\ell^p, \ell^r)$ contains a complemented copy of $(\oplus_{k=1}^{\infty} W_k)_{\ell^\infty}$.

Let $\mathcal{W} = (\oplus_{k=1}^{\infty} W_k)_{\ell^\infty}$. Let $\mathcal{X}_n$ be the natural mapping used to identify W_n with $P^r_n L(\ell^p, \ell^r) P^p_n$ and $\mathcal{X}_{n+1}(w_n \oplus 0) = \mathcal{X}_n(w_n)$ for all $w_n \in W_n$. Let $\psi : \mathcal{W} \to L(\ell^p, \ell^r)$ be given by

$$\langle \psi((w_n)_{n=1}^{\infty})\xi, \eta \rangle = LIM \langle \mathcal{X}_n(w_n)\xi, \eta \rangle$$

for $\xi \in \ell^p, \eta \in \ell^{r^1}$ where $\frac{1}{r} + \frac{1}{r^1} = 1$, where LIM denotes a Banach limit.

If $\varphi : L(\ell^p, \ell^r) \to \mathcal{W}$ is given by

$$\varphi(T) = (P^r_n T P^p_n)_{n=1}^{\infty}$$

then φ and ψ are bounded, linear mappings. In fact $\psi \circ \varphi$ is the identity on $L(\ell^p, \ell^r)$ and $\varphi \circ \psi$ is a projection of $\mathcal{W}$ onto $\varphi(L(\ell^p, \ell^r))$.

Thus $L(\ell^p, \ell^r)$ is isomorphic to a complemented subspace of $\mathcal{W}$.

If we consider a partition of $\mathbb{N}$ into infinite sets N_k, we see that

$$\mathcal{W} \cong (\oplus_{k=1}^{\infty} (\oplus_{m \in N_k} W_m)_{\ell^\infty})_{\ell^\infty}$$

So there exists a subspace Z of $\mathcal{W}$ such that

$$\mathcal{W} \cong (\mathcal{W} \oplus \mathcal{W} \oplus \cdots)_{\ell^\infty} \oplus Z.$$

Therefore,

$$\begin{aligned}
\mathcal{W} \oplus \mathcal{W} &\cong W \oplus (\mathcal{W} \oplus \mathcal{W} \oplus \cdots)_{\ell^\infty} \oplus Z \\
&\cong (\mathcal{W} \oplus \mathcal{W} \oplus \mathcal{W} \oplus \cdots)_{\ell^\infty} \oplus Z \\
&\cong \mathcal{W}
\end{aligned}$$

So that

$$\begin{aligned}(\mathcal{W}\oplus\mathcal{W}\oplus\mathcal{W}\oplus\cdots)_{\ell^\infty} &\cong (\mathcal{W}\oplus\mathcal{W}\oplus\cdots)_{\ell^\infty}\oplus\mathcal{W}\\ &\cong (\mathcal{W}\oplus\mathcal{W}\oplus\cdots)_{\ell^\infty}\oplus(\mathcal{W}\oplus\mathcal{W}\oplus\cdots)_{\ell^\infty}\oplus Z\\ &\cong ((\mathcal{W}\oplus\mathcal{W})\oplus(\mathcal{W}\oplus\mathcal{W})\oplus\cdots)_{\ell^\infty}\oplus Z\\ &\cong (\mathcal{W}\oplus\mathcal{W}\oplus\mathcal{W}\oplus\cdots)_{\ell^\infty}\oplus Z\\ &\cong \mathcal{W}.\end{aligned}$$

If $L = L(\ell^p,\ell^r)$, let $(M_k)_{k=1}^\infty$ be a partition of $\mathbb{N}$ into infinite sets with $\min M_k \geq k$. If $P^p_{M_k}$ is the projection in ℓ^p given by

$$P^p_{M_k}(\sum_{i=1}^{\infty} x_i e_i) = \sum_{i\in M_k} x_i e_i$$

then $T \mapsto P^r_{M_k} T P^p_{M_k}$ maps $K(\ell^p,\ell^r)$ onto a copy of $K(\ell^p,\ell^r)$. By essentially repeating the same argument as given at the start of this proof, we can see that $(K(\ell^p,\ell^r)\oplus K(\ell^p,\ell^r)\oplus\cdots)_{c_0}$ is a complemented subspace of $K(\ell^p,\ell^r)$. By considering double duals, we can find a subspace Z^1 of L such that

$$L \cong (L\oplus L\oplus L\oplus\cdots)_{\ell^\infty}\oplus Z^1.$$

Repeating an argument used for $\mathcal{W}$ gives,

$$L \cong (L\oplus L\oplus L\oplus\cdots)_{\ell^\infty}.$$

By Pelczyński's decomposition method

$$L = L(\ell^p,\ell^r) \cong \mathcal{W} = (\oplus_{k=1}^\infty W_k)_{\ell^\infty}.$$

If $\sup_k n_k = \infty$, then $\mathcal{W}_{(n_k)} = (\oplus_{k=1}^\infty W_{n_k})_{\ell^\infty}$ is complemented in $\mathcal{W}$ and $\mathcal{W}$ is complemented in $\mathcal{W}_{(n_k)}$. Repeating an argument used for $\mathcal{W}$ for $\mathcal{W}_{(n_k)}$ shows that

$$\mathcal{W}_{(n_k)} \cong (\mathcal{W}_{(n_k)}\oplus\mathcal{W}_{(n_k)}\oplus\cdots)_{\ell^\infty}$$

and so by Pelczyński's decomposition method, $\mathcal{W}_{(n_k)} \cong \mathcal{W}$, and we are done.

3.2. The Main Result

The proof that $L(\ell^p,\ell^r)$ is primary can now proceed almost verbatim as that for $L(\ell^1,\ell^p)$.

There are two points where the proofs differ.

Proof of Proposition 10

The proof proceeds as above, except now when considering sets σ_2, the estimate

$$\| \sum_i \epsilon_i e_{ij} \| = (|\sigma_2|-3)^{\frac{1}{p^1}},$$

where $\frac{1}{p}+\frac{1}{p^1}=1$, guarantees the existence of a large subset σ_3 of σ_2 such that

$$|\langle \mathrm{T}e_{ij}, e_{k\ell}\rangle| \leq \delta.$$

whenever i, j, k and ℓ are distinct elements of σ_3 with $\max\{i,k\} < \min\{i,\ell\}$.

Similarly when considering the set ϱ, the estimate

$$\| \sum_j \epsilon_j e_{ij} \| = (|\varrho| - 2)^{\frac{1}{r}}$$

ensures a large good monochromatic subset ϱ_4 of ϱ_3.

The rest of the proof now proceeds as before.□.

To prove Lemma 11 we also need Lemma 12. Here we need to take a little more care than before.

Proposition 16

Let $1 < p \leq r < \infty$. Let $A = (a_{ij})_{i,j=1}^{\infty}$ be a bounded linear mapping from ℓ^p to ℓ^r. Then

$$\| A \| \leq (\sum_i (\sum_j |a_{ij}|^{p^1})^{\frac{r}{p^1}})^{\frac{1}{r}}$$

and also

$$\| A \| \leq (\sum_i \sum_j |a_{ij}|^t)^{\frac{1}{t}},$$

for $t = \min(r, p^1)$ and $\frac{1}{p} + \frac{1}{p^1} = 1$.

If $A : \ell_n^p \to \ell_n^r$ is a bounded linear mapping and $r \leq p^1$, then

$$(\sum_i \sum_j |a_{ij}|^r)^{\frac{1}{r}} \leq n^{\frac{1}{r}} \| A \| .$$

Proof If $x = (x_j) \in \ell^p$, by Minkowski's inequality

$$\begin{aligned} \| Ax \|_{\ell^r} &= (\textstyle\sum_i (|\sum_j a_{ij} x_j|^r)^{\frac{1}{r}} \\ &\leq (\textstyle\sum_i (\sum_j |a_{ij}|^{p^1})^{\frac{r}{p^1}})^{\frac{1}{r}} (\sum_j |x_j|^p)^{\frac{1}{p}} \end{aligned}$$

So, $\| A \| \leq (\sum_i (\sum_j |a_{ij}|^{p^1})^{\frac{r}{p^1}})^{\frac{1}{r}}$.

If $r \geq p^1$, $\| \cdot \|_{\ell^{p^1}} \geq \| \cdot \|_{\ell^r}$, so,

$$\begin{aligned} & (\textstyle\sum_i (\sum_j |a_{ij}|^{p^1})^{\frac{r}{p^1}})^{\frac{1}{r}} \\ \geq\ & (\textstyle\sum_i (\sum_j |a_{ij}|^{p^1})^{\frac{p^1}{p^1}})^{\frac{1}{p^1}} \\ =\ & (\textstyle\sum_i \sum_j |a_{ij}|^{p^1})^{\frac{1}{p^1}} \end{aligned}$$

If $p^1 \geq r$, $\| \cdot \|_{\ell^{p^1}} \leq \| \cdot \|_{\ell^r}$, so

$$\begin{aligned} & (\textstyle\sum_i (\sum_j |a_{ij}|^{p^1})^{\frac{r}{p^1}})^{\frac{1}{r}} \\ \leq\ & (\textstyle\sum_i (\sum_j |a_{ij}|^r)^{\frac{r}{r}})^{\frac{1}{r}} = (\sum_i \sum_j |a_{ij}|^r)^{\frac{1}{r}} \end{aligned} \quad \square$$

If $r \leq p^1$ then

$$\sum_i \sum_j |a_{ij}|^r = \sum_j \| Ae_j \|_{\ell_n^r}^r \leq \| A \|^r \ n,$$

and we are done.

Modified proof of Lemma 12

Let $\epsilon > 0$ and $n \in \mathbb{N}$ be given.

Let E be an n-dimensional subspace of W_{nm}, where m will be chosen later to be sufficiently large. By a volume argument we can find an $\frac{\epsilon}{2}$ net $x_1, x_2, \cdots, x_L$ of the unit sphere of E where L depends only on ϵ and n.

Consider the natural projections Q_s onto the $m^2 \quad n \times n$ blocks of W_{nm}. We have

$$L = \sum_{k=1}^{L} \| x_k \|^t \geq \frac{1}{(mn)} \sum_{k=1}^{L} (\ell(x_k))^t$$

where $\ell(x_k)$ is the $\| \cdot \|_t$ norm on the space of matrices W_{nm}.

$$\geq \frac{1}{(mn)} \sum_{k=1}^{L} \sum_{s=1}^{m^2} \ell(Q_s x_k)^t$$

by the blocking property of the $\| \cdot \|_t$ norm.

$$\geq \frac{1}{(mn)} \sum_{k=1}^{L} \sum_{s=1}^{m^2} \| Q_s x_k \|^t$$

Choose m sufficiently large that $Ln < (\frac{\epsilon}{2})^t m$, where $t > 1$.

Then there is some $s, 1 \leq s \leq m^2$ such that

$$\sum_{k=1}^{L} \| Q_s x_k \|^t < (\frac{\epsilon}{2})^t.$$

Since x_k is an $\frac{\epsilon}{2}$-net for the unit sphere of E, we may choose this Q_s to be the required q of the lemma.□.

The rest of the proof of Lemma 11 proceeds as before verbatim, only replacing $\mathcal{M}$ with $\mathcal{W}$ and M_n with W_n. The remainder of the proof of the main result is also unchanged.

We have proved.

Theorem 17

Let $1 < p \leq r \leq p^1 < \infty$ where $\frac{1}{p} + \frac{1}{p^1} = 1$. The Banach space $L(\ell^p, \ell^r)$ is primary.

Closing Remarks

The Ramsey theory technique used here was introduced by Blower [6]. The method of the rest of the proof was first used by Bourgain [8] to prove that H^∞ is primary. Subsequently Müller used it to prove that BMO is primary [9].

The decomposition of Banach spaces as ℓ^∞ sums of finite dimensional subspaces may also have relevance for another problem. Pfitzner has shown that C^*-algebras have Pełczyński's property (V) and so $L(\ell^2)$ is a Grothendieck space.

If $\infty > p \geq 2, \ell^{p^1}$ embeds isometrically in $L^1[0,1]$ [cf. [10], p.167], and so ℓ_p is a quotient of ℓ^∞. Thus $L(\ell^1, \ell^p) \cong (\ell^p \oplus \ell^p \oplus \cdots)_{\ell^\infty}$ is a quotient of ℓ^∞. Since ℓ^∞ has (V) and (V) is inherited by quotients, $L(\ell^1, \ell^p)$ has (V) and so is a Grothendieck space for $p \geq 2$.

Question 18

For which $1 \leq p \leq r < \infty$ is the Banach space $L(\ell^p, \ell^r)$ a Grothendieck space?

The primarity of tensor products of l^p spaces has been studied in [11].

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Chapter 5
Spaces of Diagonal Operators

1. Introduction

Let $\beta = (e_n)_{n=1}^{\infty}$ be a Schauder basis of a Banach space X. We say that β is semi-normalized if

$$0 < \inf_n \| e_n \| \leq \sup_n \| e_n \| < \infty.$$

Let X be a Banach space having a semi-normalized Schauder basis $(e_n)_{n=1}^{\infty}$, which we denote by β. Denote $K_d(X,\beta)$ and $L_d(X,\beta)$ the following spaces of operators:

$$K_d(X,\beta) = \{T \in K(X) : \exists (\lambda_n)_{n=1}^{\infty} \lambda_n \text{ scalars, } \quad \text{with} \quad Te_n = \lambda_n e_n \quad \forall n\}$$
$$L_d(X,\beta) = \{T \in L(X) : \exists (\lambda_n)_{n=1}^{\infty} \lambda_n \text{ scalars, } \quad \text{with} \quad Te_n = \lambda_n e_n \quad \forall n\};$$

i.e. $L_d(X,\beta)$ is the space of diagonal operators on β and $K_d(X,\beta)$ is the space of compact diagonal operators on β.

These spaces have been studied in [1] and [2]. They are closed subspaces of $K(X)$, the space of compact operators on X, and $L(X)$, the space of bounded operators on X, respectively.

The purpose of this chapter is to continue the study of these spaces initiated in [1] and [2]. In particular we shall be interested in similarities between the properties of $K(X)$ and $L(X)$ and the properties of $K_d(X,\beta)$ and $L_d(X,\beta)$. Our motivation is the following question.

Question 1 Let X be a Banach space with a seminormalized Schauder basis $\beta = (e_n)_{n=1}^{\infty}$. Is there a semi-normalized Schauder basis β^1 of X such that there is a diagonal operator in $L_d(X,\beta^1)$ not of the form $\lambda I + K$, where K is a compact diagonal operators and λ is a scalar?

We recall some notation from [1].

If X is a Banach space with a basis $\beta = (e_n)_{n=1}^{\infty}$, let $\hat{X} = \bar{sp}[e_n^* : n \geq 1]$: i.e. the space generated in X^* by the sequence $(e_n^*)_{n=1}^{\infty}$ biorthogonal to $(e_n)_{n=1}^{\infty}$. Also denote by $\tilde{X}$, the space

$$\tilde{X} = \{(\alpha_n)_{n=1}^{\infty} : \sup_n \| \sum_{k=1}^{n} \alpha_k e_k \| < \infty\}.$$

The space $\tilde{X}$ is a Banach space with the norm

$$\| (\alpha_n)_{n=1}^{\infty} \| = \sup_n \| \sum_{k=1}^{n} \alpha_k e_k \|_X$$

and X embeds naturally in $\tilde{X}$.

For a semi-normalized basis $\beta = (e_n)_{n=1}^{\infty}$ of a Banach space X, we define operators $\bar{e_n} : X \to X$ by

$$\bar{e_n}(\sum_{k=1}^{\infty} x_k e_k) = x_n e_n.$$

Let ω denote the vector space of all scalar sequences. By a sequence space we mean a linear subspace of ω.

Let S be a sequence space. The γ-dual of a sequence space S is the sequence space S^{γ} given by

$$S^{\gamma} = \{(\beta_n) \text{ scalars } : \sup_n |\sum_{i=1}^{n} \beta_i \alpha_i| < \infty \text{ for all } (\alpha_n) \in S\}.$$

A sequence space S is γ-perfect if $S^{\gamma\gamma} = S$.

A sequence space S is a BK space if it is a Banach space and the co-ordinate functionals are continuous on S; ie. $x_n = (\alpha_j^{(n)}), x = (\alpha_j) \in S$, $\lim_{n\to\infty} x_n = x$ imply $\lim_{n\to\infty} \alpha_j^{(n)} = \alpha_j$.

Define a multiplication operation on ω by

$$(\lambda_k).(\mu_k) = (\lambda_k \mu_k).$$

Under this multiplication and its other operations, ω is an algebra. A BK-algebra is BK-space which is a sub-algebra of ω.

2. Some Previous Results.

In this section we recall some results from [1] and [2] which we shall use throughout the sequel.

Proposition 2 [1] Let $\beta = (e_n)_{n=1}^{\infty}$ be a semi-normalized Schauder basis of a Banach space X. Then $(\bar{e_n})_{n=1}^{\infty}$ is a Schauder basis of $K_d(X, \beta)$.

Also, the basic sequences $(\bar{e_n})_{n=1}^{\infty}, (\bar{e_n^*})_{n=1}^{\infty}$ and $(\bar{\bar{e}}_n)_{n=1}^{\infty}$ are equivalent.

Proposition 3 [1] Let $\beta = (e_n)_{n=1}^{\infty}$ be a semi-normalized Schauder basis of a Banach space X. The space $L_d(X, \beta)$ can be identified with $\tilde{K_d}\,(X, \beta)$ under a natural isomorphism.

Let $e_n (n \in \mathbb{N})$ denote the sequences $(\delta_{pn})_{p=1}^{\infty}$ and e the sequence $(1, 1, 1, 1, \dots)$ in ω.

Proposition 4 [2].

A sequence space S is the sequence space associated to the space of diagonal operators on a semi-normalized Schauder basis β of a Banach space X, $L_d(X,\beta)$, if and only if S is a γ-perfect BK-algebra containing each $e_n (n \in \mathbb{N})$ and e.

It was shown in [29] that a sequence space S is associated to a Schauder basis of a Banach space if and only if S contains all unit vectors e_n and there exists a γ-perfect BK-space T such that the closed span of (e_n) in T is S. In this case $T = S^{\gamma\gamma}$.

Thus to construct spaces of diagonal operators it is necessary to find a BK-algebra where $(e_n)_{n=1}^{\infty}$ is a Schauder basis in S and the sequence $(1, 1, 1, \cdots)$ lies in S.

Henceforth we shall refer to spaces satisfying Proposition 4 as multiplier algebras.

Proposition 5 [2]

If X is a Banach space with semi-normalized Schauder basis β and S is the sequence space associated with $L_d(X,\beta)$ and S_0 the sequence space associated with $K_d(X,\beta)$, then

$$\begin{array}{ccccc} bv & \subseteq & S & \subseteq & \ell^{\infty} \\ bv_0 & \subseteq & S_0 & \subseteq & c_0, \end{array}$$

with continuous inclusions.

We make one further remark.

Proposition 6 [1]

Let X be a Banach space with semi-normalized Schauder basis β.

The Banach space $L_d(X,\beta)$ is isomorphic to a dual space.

Proposition 6 was an observation of Sersouri in [1]. An early observation of Figa-Talamanca was that the space of multipliers on the trigonometric basis of $L^p(\mathbb{T})$, $\mathcal{M}_p, 1 < p < \infty$, is isometric to a dual space [25].

3. Examples

The first question raised in [1] is the construction of some examples of spaces of diagonal operators. The examples given in [1] are ℓ^{∞}, bv and J^{**} where J is the James space.

Example 7

Consider the summing basis in $c_o, (z_n)_{n=1}^{\infty} = \beta_1$ where

$$z_n = \sum_{i=1}^{n} e_i,$$

where $(e_i)_{i=1}^{\infty}$ is the standard basis of c_o.

If $\Lambda : c_o \to c_o$ lies in $L_d(c_o, \beta_1)$, $\Lambda z_n = \lambda_n z_n \quad \forall n$, then $\| \Lambda \|$ can be explicitly computed.

Proposition 8 If $T : \ell^1 \to \ell^1$ is a bounded linear operator then

$$\| T \| = \sup_n \| \mathrm{Te}_n \|_{\ell^1} .$$

Also, if $T : c_o \to c_o$ is a bounded linear operator then

$$\| T \| = \sup_n \| T^* e_n \|_{\ell^1} = \sup_n \sum_{m=1}^{\infty} | < e_n, \mathrm{Te}_m > |$$

Proof cf. [3], p.175.

Using the relations

$$\begin{aligned} e_1 &= z_1 \\ e_n &= z_n - z_{n-1}, \quad n \geq 2. \end{aligned}$$

We can compute

$$\| \Lambda \| = |\lambda_1| + \sum_{j=1}^{\infty} |\lambda_j - \lambda_{j+1}|.$$

Thus $L_d(c_o, \beta_1)$ can be identified with bv.□.

Example 9 Consider a sequence of real numbers (t_n) such that

$$0 < t_n < 1, \quad \lim_{n \to \infty} t_n = 1, \quad \Pi_{n=1}^{\infty} t_n = 0.$$

Define (x_n) in c_o by

$$x_n = \sum_{i=n}^{\infty} (\Pi_{j=n}^{i} t_j) e_i, n \in \mathbb{N}.$$

Lindenstrauss (cf. [4], p.634) observed that $(x_n)_{n=1}^{\infty} = \beta_2$ is a monotone conditional basis of c_o.

If $\Lambda : c_o \to c_o$ is a diagonal operator β_2 with $\Lambda x_n = \lambda_n x_n \; \forall n$, using

$$e_n = \frac{1}{t_n} x_n - x_{n+1}, \; \forall n \in \mathbb{N}$$

and Proposition 8, we find

$$\| \Lambda \| = \sup_{m \geq 1} [\sum_{j=1}^{m-1} |\lambda_j - \lambda_{j+1}| (\Pi_{k=j+1}^{m} t_k) + |\lambda_m|].$$

The space of diagonal operators $L_d(c_o, \beta_2)$ is identified with a space of sequences of weighted bounded variation.

Example 10 In [5], Holub and Retherford consider the following shrinking basis, β_3, for c_o:

$$\begin{aligned} x_1 &= e_1 \\ x_n &= (-1)^{n-1} [\tfrac{1}{n-1} e_1 + \textstyle\sum_{j=2}^{n} (-1)^{j-1} \tfrac{j-1}{n-1} e_j] \quad n \geq 2. \end{aligned}$$

The co-efficient functionals in ℓ^1, $(f_n)_{n=1}^{\infty}$ are given by

$$\begin{aligned} f_1 &= e_1 + e_2 \\ f_n &= e_n + \tfrac{n-1}{n} e_{n+1},\ n \geq 2. \end{aligned}$$

If $\Lambda : \ell^1 \to \ell^1$ is given by $\Lambda f_n = \lambda_n f_n, \forall n \in \mathbb{N}$, we can compute $\| \Lambda \|$ using Proposition 2.2 as

$$\| \Lambda \| = \max[|\lambda_1| + \sum_{j=1}^{\infty} \frac{1}{j} |\lambda_j - \lambda_{j+1}|, \sup_{n \geq 2} [|\lambda_n| + (n-1) \sum_{j=n}^{\infty} \frac{1}{j} |\lambda_j - \lambda_{j+1}|]$$

By results in [1], $(\bar{x}_n)_{n=1}^{\infty}$ and $(\bar{f}_n)_{n=1}^{\infty}$ are equivalent Schauder bases and moreover are shrinking. Once more, the space of diagonal operators $L_d(c_o, \beta_3)$ is a space of sequences of weighted bounded variation.

Example 11 (cf. [5]).

Let $(f_n)_{n=1}^{\infty}$ be the basic sequence in ℓ^1 given by

$$f_n = e_n - \frac{1}{2}(e_{2n} + e_{2n+1}), n \in \mathbb{Z}^+$$

Define integers $\gamma_j(n)$ by

$$\gamma_0(n) = n$$

and $\gamma_{j+1}(n) = \left[\frac{\gamma_j(n)-1}{2}\right]$.

Define x_n in c_0 by

$$x_n = \sum_{j=0}^{n} \frac{1}{2^j} e_{\gamma_j(n)}$$

where $e_i = 0$ for $i \leq 0$.

Then $(x_n)_{n=1}^{\infty}$ is a basis for c_0 with co-efficient functionals $(f_n)_{n=1}^{\infty}$. We shall compute the multipliers on $(x_n)_{n=1}^{\infty}$.

Consider $T : c_0 \to c_0$ of the form $Tx_n = \lambda_n x_n\ \forall n \in \mathbb{N}$. It can easily be shown that

$$\begin{aligned} \mathrm{T}e_1 &= \lambda_1 e_1 \\ \mathrm{T}e_2 &= \lambda_2 e_2 \\ \mathrm{T}e_3 &= \tfrac{1}{2}(\lambda_3 - \lambda_1) e_1 + \lambda_3 e_3 \\ \mathrm{T}e_4 &= \tfrac{1}{2}(\lambda_4 - \lambda_1) e_1 + \lambda_4 e_4 \end{aligned}$$

and generally

$$\mathrm{T}e_n = \lambda_n e_{\gamma_0(n)} + \textstyle\sum_{j=1}^{n} \frac{1}{2^j} (\lambda_{\gamma_{j-1}(n)} - \lambda_{\gamma_j(n)}) e_{\gamma_j(n)}$$

Using Proposition 8 we find

$$\| T \| = \sup_{n\in\mathbb{N}}[|\lambda_n| + \sum_{j=1}^{\infty}\frac{1}{2^j}\sum_{k=2^j n+2^j-1}^{2^j n+2^{j+1}-2}|\lambda_k - \lambda_{[k/2]}|]$$

Example 12

The convergence field of the strong convergence method is the Banach space $[cs]$ defined by

$$[cs] = \{(x_k)_{k=1}^{\infty} : \sum_{2^j \le k < 2^{j+1}} |x_k| = o(1)\ (j \to \infty) \text{ and } \sum_{k=1}^{\infty} x_k \text{ exists } \}$$

The norm on $[cs]$ is defined by

$$\| x \|_{[cs]} = \sup_n |\sum_{k=1}^{n} x_k| + \sup_j \sum_{2^j \le k < 2^{j+1}} |x_k|.$$

It is easily established that the standard basis $(e_n)_{n=1}^{\infty}$ is a Schauder basis of $[cs]$. In [26] it was shown that the multiplier algebra of this basis is the space ℓv, where

$$\ell v = \{x = (x_k)_{k=1}^{\infty} : \sum_j \max_{2^j \le k < 2^{j+1}} |x_k - \alpha_j(x)| + \sum_j |\alpha_j(x) - \alpha_{j+1}(x)| < \infty\}$$

is endowed with the norm

$$\| x \|_{\ell v} = \sum_j \max_{2^j \le k < 2^{j+1}} |x_k - \alpha_j(x)| + \sum_j |\alpha_j(x) - \alpha_{j+1}(x)| + \sup_j |\alpha_j(x)|, \alpha_j(x) = \frac{1}{2^j}\sum_{2^j \le k < 2^{j+1}} x_k$$

It has been shown that $(x_k)_{k=1}^{\infty}$ lies in ℓv if and only if

$$\sum_{k=1}^{\infty} |x_{\lambda_k} - x_{\lambda_{k+1}}| < \infty$$

for every lacunary sequence of integers (λ_k). [Recall a sequence of integers is lacunary if there exists $q > 1$ such that $\lambda_{k+1} > q\lambda_k$ for all k].

We can also show that the space ℓv is isomorphic as a Banach space to none of the multiplier algebras so far exhibited.

Let $Z = (\oplus_{n=0}^{\infty} \ell_{2^n}^{\infty})_{\ell^1}$ be endowed with the norm

$$\| (z_k)_{k=1}^{\infty} \| = \sum_{n=0}^{\infty} \max_{2^n \le k < 2^{n+1}} |z_k|$$

Let $d^j = \sum_{2^j \le k < 2^{j+1}} e_k$ in ℓv for $j \ge 0$.

Define $P : \ell v_0 \to \ell v_0$ by $P((x_k)_{k=1}^{\infty}) = \sum_{j=0}^{\infty} \alpha_j(x) d^j$.

Then,

$$\begin{aligned} \| P(x) \|_{\ell v_0} &= \sup_j |\alpha_j(x)| + \sum_{j=0}^{\infty} |\alpha_j(x) - \alpha_{j+1}(x)| \\ &\leq \| x \|_{\ell v} . \end{aligned}$$

Also, $P^2(x) = P(x) \quad \forall x \in \ell v$.

Observe that

$$\| \sum_{j=1}^{m} c_j d^j \|_{\ell v} = \sup_{0 \leq j \leq m} |c_j| + \sum_{j=0}^{m-1} |c_j - c_{j+1}|.$$

Therefore, in $\ell v_0, (d^j)_{j=1}^{\infty}$ spans a copy of bv_0. If $(x_j)_{j=1}^{\infty} \in bv_0$, then

$$P(\sum_{j=0}^{\infty} x_j d^j) = \sum_{j=1}^{\infty} x_j d^j$$

so the projection P is onto $[d^j]_{j=0}^{\infty}$ in ℓv_0. Also,

$$(I - P)(x) = \sum_{j=0}^{\infty} \sum_{2^j} (x_k - \alpha_j(x)) e_k$$

Let

$$W = \{x \in Z : \alpha_j(x) = 0, \quad \forall j \in \mathbb{Z}^+\}.$$

Then,

$$(I - P)(\ell v_0) \subseteq W.$$

If $\omega \in W$, then $(I - P)(\omega) = \omega$, so $(I - P)(W) = W$.

For $x \in W$,

$$\| x \|_{\ell v_0} = \sum_{j=0}^{\infty} \max_{2^j} |x_k| = \| x \|_Z .$$

So W is complemented in ℓv_0 and

$$\ell v_0 \cong bv_0 \oplus W.$$

Let $Q : Z \to Z$ be given by

$$Q(x) = \sum_{j=0}^{\infty} \sum_{2^j} (x_k - \alpha_j(x)) e_k$$

$$\begin{aligned} \| Q(x) \|_Z &\leq \sum_{j=0}^{\infty} \max_{2^j} |x_k - \alpha_j(x)| \\ &\leq 2 \sum_{j=0}^{\infty} \max_{2^j} |x_k| = 2 \| x \|_Z \end{aligned}$$

Also $Q^2(x) = Q(x)$.

If $\omega \in W$, $Q(\omega) = \omega$ and $Q(Z) \subseteq W$, so W is closed in Z. Observe that for $x \in Z$

$$(I - Q)(x) = \sum_{j=0}^{\infty} \alpha_j(x) d^j.$$

In Z,

$$\| \sum_{j=0}^{\infty} c_j d^j \|_Z = \sum_{j=0}^{\infty} |c_j|,$$

so $(d^j)_{j=0}^{\infty}$ spans a copy of ℓ^1.

If $(c_j)_{j=1}^{\infty} \in \ell^1$, then $x = \sum_{j=0}^{\infty} c_j d^j \in [d^j]_{j=0}^{\infty}$ and

$$(I - Q)(x) = x,$$

so $[d^j]_{j=0}^{\infty}$ is complemented in Z.

Therefore

$$Z = (\oplus_{n=0}^{\infty} \ell_{2^n}^{\infty})_{\ell^1} \cong \ell^1 \oplus W.$$

Since bv_0 is isomorphic to ℓ^1, we have

$$Z \cong \ell^1 \oplus W \cong bv_0 \oplus W \cong \ell v_0.$$

That is $\ell v_0 \cong (\oplus_{n=0}^{\infty} \ell_{2^n}^{\infty})_{\ell^1}$.

Notice that since ℓv_0 contains ℓ^1, the multiplier basis $(\bar{e}_n)_{n=1}^{\infty}$ is not shrinking and since ℓv_0 contains ℓ_n^{∞} uniformly ℓv_0 is not of finite cotype.

Example 13

For $q > 1$, let

$$\Lambda_q = \{(\lambda_k)_{k=1}^{\infty}; \lambda_k \in \mathbb{N}, \lambda_{k+1} > q\lambda_k\}$$

For $q^1 > q, \Lambda_{q^1} \subseteq \Lambda_q$. The set of all lacunary sequence is $\cup_{q>1}\Lambda_q = \Lambda$ say.

If $\| \cdot \|_M$ is the norm of a multiplier algebra M, define $\| \cdot \|_{M_{\Lambda_q}}$ by

$$\| (x_k)_{k=1}^{\infty} \|_{M_{\Lambda_q}} = \sup_{(\lambda_k) \in \Lambda_q} \| (x_{\lambda_k}) \|_M$$

for $q > 1$.

Then if

$$M_{\Lambda_q} = \{(x_k)_{k=1}^{\infty} : \| (x_k) \|_{M_{\Lambda_q}} < \infty\},$$

$M_{\Lambda_q}(q > 1)$ is a multiplier algebra and $bv \subseteq M_{\Lambda_q} \subseteq \ell v$.

If

$$\| (x_k)_{k=1}^{\infty} \|_{M_\Lambda} = \sup_{(\lambda_k) \in \Lambda} \| (x_{\lambda_k}) \|_M$$

and

$$M_\Lambda = \{(x_k)_{k=1}^{\infty} : \| (x_k) \|_{M_\Lambda} < \infty\},$$

M_Λ is a multiplier algebra and $bv \subseteq M_\Lambda \subseteq \ell v$.

Example 14

It is well known that under certain orderings the matrix units $(e_i \otimes e_j)_{i,j=1}^{\infty}$ form Schauder bases of $K(\ell^p)$ for $1 < p < \infty$. Using Grothendieck's inequality, the spaces of diagonal operators on these bases are observed in [24], Chapter 5, to be isomorphic to $\Gamma_p(\ell^1, \ell^\infty)$, the space of all operators from ℓ^1 to ℓ^∞ which factor through L^p.

We now consider method of producing new examples of spaces of diagonal operators from existing spaces of diagonal operators.

Theorem 15

Let M_0^1 and M_0^2 be the sequence spaces associated to the spaces of compact diagonal operators $K_d(Y_1, \beta_1)$ and $K_d(Y_2, \beta_2)$ on semi-normalized Schauder bases β_1 and β_2. That is

$$\begin{aligned} M_0^1 &= K_d(Y_1, \beta_1) \\ M_0^2 &= K_d(Y_2, \beta_2) \end{aligned}$$

Let $Z_\theta = (M_0^1, M_0^2)_{\theta;1}$ be the interpolation space obtained by the real interpolation method.

Then there exists a Banach space Y_3 and a semi-normalized Schauder basis β_3 such that

$$\begin{aligned} Z_\theta &= K_d(Y_3, \beta_3) \\ \text{So,} \quad \tilde{Z}_\theta &= L_d(Y_3, \beta_3). \end{aligned}$$

Proof As a consequence of the main result of [2], and the fact that the bases are semi-normalized we have

$$\begin{aligned} & bv_o \subseteq M_0^1 \subseteq c_o \\ \text{and} \quad & bv_o \subseteq M_0^2 \subseteq c_0, \end{aligned}$$

with continuous inclusions.

Therefore, M_0^1 and M_0^2 form an interpolation couple of Banach algebras with compatible multiplication. By a result of Löfstrom, ([6],p.79, Exercise 12), $(M_0^1, M_0^2)_{\theta;1}$ is a Banach algebra under pointwise multiplication of sequences, ie. a BK-algebra.

Each e_n lies in Z_θ since

$$\| e_n \|_{Z_\theta} \leq \max[\| e_n \|_{M_0^1}, \| e_n \|_{M_0^2}]$$

for all n.

Since the span of e_n is dense in both bv_0 and c_0 it is dense in Z_θ. In $L(Z_\theta)$,

$$\begin{aligned} & \| \bar{e_1} + \cdots + \bar{e_n} : Z_\theta \to Z_\theta \| \\ & \leq \max[\| \bar{e_1} + \cdots + \bar{e_n} : Y_1 \to Y_1 \|, \| \bar{e_1} + \cdots + \bar{e_n} : Y_2 \to Y_2 \|] \end{aligned}$$

by a simple interpolation property

$$\leq K < \infty,$$

for some uniform constant K since β_1 and β_2 are Schauder bases of Y_1 and Y_2.

Thus $(e_n)_{n=1}^{\infty}$ is a Schauder basis of Z_θ and so by the main result of [7], $\tilde{Z}_\theta$ is a γ-perfect BK algebra containing each e_n. Also $\tilde{Z}_\theta$ contains the sequence

$(1, 1, 1, 1, \dots)$, since

$$\begin{aligned} & \sup_n \| e_1 + e_2 + \cdots + e_n \|_{Z_\theta} \\ \leq\ & \sup_n \max[\| e_1 + \cdots + e_n \|_{M_0^1}, \| e_1 + \cdots + e_n \|_{M_0^2}] \\ \leq\ & \max[\sup_n \| e_1 + \cdots + e_n \|_{M_0^1}, \sup_n \| e_1 + \cdots + e_n \|_{M_0^2}] \\ \leq\ & C < \infty \end{aligned}$$

for some universal constant C since M_0^1 and M_0^2 are spaces of compact diagonal operators.

So $\tilde{Z}_\theta = L_d(Y_3, \beta_3)$ for some Banach space Y_3 and semi-normalized Schauder basis β_3 by the main result of [2], and $Z_\theta = K_d(Y_3, \beta_3)$.□.

By an exactly similar proof and using a result of Caldéron ([6], p.104) we can also consider the complex method of interpolation.

Theorem 16

Let M_1^0 and M_2^0 be the sequence spaces associated to the spaces of compact diagonal operators $K_d(Y_1, \beta_1)$ and $K_d(Y_2, \beta_2)$ on semi-normalized Schauder bases β_1 and β_2.

Let $Y_\theta = (M_1^0, M_2^0)_{[\theta]}$. Then there is a Banach space Y_3 and semi-normalized Schauder basis β_3 such that

$$\begin{aligned} \tilde{Y}_\theta &= L_d(Y_3, \beta_3) \text{ and} \\ Y_\theta &= K_d(Y_3, \beta_3). \end{aligned}$$

Example 17

The interpolation spaces $Z_\theta = (bv_0, c_0)_{\theta;1}$ are spaces of compact diagonal operators for some semi-normalized Schauder basis. Also by [8], Z_θ is of cotype $\max[\frac{1}{1-\theta}, 2]$ for $\theta \neq \frac{1}{2}$ and cotype $2 + \varepsilon$, $\forall \varepsilon > 0$ for $\theta = \frac{1}{2}$ and by [38] ℓ^1 embeds in Z_θ, so the multiplier basis is not shrinking.

A result of Montgomery-Smith [39] shows that cotype is not in general preserved by the complex interpolation method.

It would be interesting to know if the next result also holds for the spaces $K(X)$ and $L(X)$ instead of $K_d(X, \beta)$ and $L_d(X, \beta)$.

Theorem 18

Let X be a Banach space with a semi-normalized Schauder basis $(e_n)_{n=1}^{\infty} = \beta$.

The space $K_d(X,\beta)$ has type p if and only if $L_d(X,\beta)$ has type p.

Also $K_d(X,\beta)$ has cotype q if and only if $L_d(X,\beta)$ has cotype q.

Proof We give the proof for type. The argument for cotype is analogous.

Let us suppose that $K_d(X,\beta)$ has type p with constant C. Let $(x_i)_{i=1}^{n}$ lie in $L_d(X,\beta)$ if $x_i = (c_1^i, c_2^i, c_3^i, \dots)$, then

$$\| x_i \|_{L_d} = \sup_m \| \sum_{j=1}^{m} c_j^i \bar{e}_j \| .$$

So that,

$$\left(\int_0^1 \| \sum_{i=1}^{n} r_i(t) x_i^m \|^2 \, dt \right)^{\frac{1}{2}} \leq C \left(\sum_{i=1}^{n} \| x_i^m \|^p \right)^{\frac{1}{p}}$$

for $x_i^j = \sum_{j=1}^{m} c_m^i \bar{e}_j \in K_d$.

By renorming X if necessary we may assume that $(e_n)_{n=1}^{\infty} = \beta$ is monotone, so that $(\bar{e_n})_{n=1}^{\infty}$ is monotone. Thus,

$$\| x_i \|_{L_d} = \lim_{m \to \infty} \| x_i^m \| .$$

Taking limits of both sides of the above inequality and applying the dominated convergence theorem, we have

$$\left(\int_0^1 \| \sum_{i=1}^{n} r_i(t) x_i \|_{L_d}^2 \right)^{\frac{1}{2}} \leq C \left(\sum_{i=1}^{n} \| x_i \|^p \right)^{\frac{1}{p}}$$

and so $L_d(X,\beta)$ has type p.□.

The following two questions are raised in Theorem 18.

Question 19 Is there a Banach space X such that $L(X)$ is not finitely representable in $K(X)$?

Question 20 Is there a Banach space X and a semi-normalized Schauder basis β of X such that $L_d(X,\beta)$ is not finitely representable in $K_d(X,\beta)$?

If X is reflexive with the compact approximation property $K(X)^{**}$ is isomorphic to $L(X)$ and $L(X)$ is always finitely representable in $K(X)$ by local reflexivity. Also if X is reflexive with a semi-normalized Schauder basis β, $L_d(X,\beta) \cong K_d(X,\beta)^{**}$ and so by local reflexivity $L_d(X,\beta)$ is finitely representable in $K_d(X,\beta)$. (cf. [9], p.33)

It is not known if there is an infinite dimensional Banach space X such that either $K(X)$ or $L(X)$ have finite cotype or non trivial type. This is relevant to the results of Section 5.

Theorem 21

There is a continuum of mutually non isomorphic multiplier algebras.

Proof

The spaces $\tilde{Z}_\theta$ of Example 17 are multiplier algebras each having the same cotype as Z_θ for $0 < \theta < 1$, i.e. $\max[\frac{1}{1-\theta}, 2], \theta \neq \frac{1}{2}$ and $2+\epsilon$, $\forall_{\epsilon>0}$ for $\theta = \frac{1}{2}$. Thus the $\tilde{Z}_\theta$ are mutually non isomorphic.□.

We shall return to the construction of families of multiplier algebras in Chapter 6.

4. An Analogue of Sersouri's Theorem

The main result of [1] is that if $(e_n)_{n=1}^\infty$ is shrinking or boundedly complete, then $(\bar{e_n})_{n=1}^\infty$ is shrinking. Similar results can be obtained by considering wider classes of Schauder bases.

Definition 22

A Schauder basis $(e_i)_{i=1}^\infty$ is of type wc_0 if e_i converges weakly to zero. (This notion has also been called semi-shrinking).

A Schauder basis $(e_i)_{i=1}^\infty$ of Banach space X is of type $(wc_0)^*$ if e_i^* converges weakly to zero in X^*. [This notion has been called semi-boundedly complete].

A Schauder basis $(e_i)_{i=1}^\infty$ is of type (swc_0) if some subsequence (e_{n_i}) is of type wc_0.

A Schauder basis $(e_i)_{i=1}^\infty$ is of type $(swc_0)^*$ if some subsequence (e_{n_i}) is of type $(wc_0)^*$.

Theorem 23

Let $(e_i)_{i=1}^\infty$ be a seminormalized basis of a Banach space X. If $(e_i)_{i=1}^\infty$ is of type wc_0 or of type $(wc_0)^*$ then $(\bar{e_n})_{n=1}^\infty$ is of type wc_0.

Proof Consider the injective tensor product $X \overset{v}{\otimes} X^*$.

For each element v of $(X \overset{v}{\otimes} X^*)^*$ there is a measure μ on the product $S \times T$ of the weak-$*$ compact spaces $S = \text{ball}\, X^*$, $T = \text{ball}\, X^{**}$ such that

$$v(x \otimes x^*) = \int_{S\times T} \langle y^*, x\rangle \langle y^{**}, x^*\rangle d\mu(y^*, y^{**})$$

Since e_i is (wc_0) or $(wc_0)^*$, the sequence $\langle y^*, e_i\rangle\langle y^{**}, e_i^*\rangle$ tends to zero for all $(y^*, y^{**}) \in S \times T$. Hence, $v(e_i \otimes e_i^*) \to 0$ as $i \to \infty$ by Lebesgue's dominated convergence theorem.

Thus, $(\bar{e}_i)_{i=1}^\infty$ is of type (wc_0).□.

Example 24 In [1], Sersouri remarked he could not find an example of a semi-normalized Schauder basis, $(e_n)_{n=1}^\infty$ such that $(\bar{e_n})_{n=1}^\infty$ is shrinking but $(e_n)_{n=1}^\infty$ is neither shrinking nor boundedly complete.

Let $(e_n)_{n=1}^\infty$ be a semi-normalized unconditional basis of a Banach space X which contains both c_0 and ℓ^1. Then $(\bar{e_n})_{n=1}^\infty$ is equivalent to the standard basis of c_0 and so is shrinking.

However $(e_n)_{n=1}^{\infty}$ can neither be shrinking nor boundedly complete (cf. [9]), p.21-22, Theorems 1c.9 and 1c.10).

Indeed if we take $X = (c_0 \oplus \ell^1)_{\ell^\infty}$ and let

$$\begin{aligned} x_{2n-1} &= (e_n, 0), \quad n \in \mathbb{N} \\ x_{2n} &= (0, e_n), \quad n \in \mathbb{N} \end{aligned}$$

then $(x_n)_{n=1}^{\infty}$ is unconditional and $(\bar{x_n})_{n=1}^{\infty}$ is equivalent to the standard c_0 basis so is of type (wc_0). However $(x_n)_{n=1}^{\infty}$ is neither of type wc_0 or of type $(wc_0)^*$.

However we can obtain a converse in the next result.

Theorem 25

Let $(e_i)_{i=1}^{\infty}$ be a semi-normalized Schauder basis of a Banach space X. Then $(\bar{e}_i)_{i=1}^{\infty}$ is of type (swc_0) if and only if $(e_i)_{i=1}^{\infty}$ is either of type (swc_0) or of type $(swc_0)^*$

Proof

If $e_{n_i} \to 0$ weakly, then by Theorem 23, $\bar{e_{n_i}} \to 0$ weakly. So $(\bar{e_n})_{n=1}^{\infty}$ is of type (swc_0).

If $e^*_{n_i} \to 0$ weakly, then by Theorem 23, $(\bar{e_{n_i}})_{i=1}$ is of type (wc_0) and so $(\bar{e_n})_{n=1}^{\infty}$ is of type (swc_0).

If $(e_i \otimes e_i^*)_{i=1}^{\infty} = (\bar{e}_i)_{i=1}^{\infty}$ is of type (swc_0), then $e_{n_i} \otimes e^*_{n_i} \to 0$ weakly in $X \overset{v}{\otimes} X^*$. If $(e^*_{n_i})_{i=1}^{\infty}$ is not weakly null, there exists $x^{**} \in X^{**}$ and a subsequence $(e^*_{m_k})$ of $(e^*_{n_k})$ such that

$$|x^{**}(e^*_{m_k})| \geq 1 \quad \forall k \in \mathbb{N}.$$

Since $(e_{n_i} \otimes e^*_{n_i}) \to 0$ weakly for every $x^* \in X^*$,

$$\begin{aligned} & |(x^* \otimes x^{**})\,(e_{m_k} \otimes e^*_{m_k})| \\ = \; & |x^*(e_{m_k}) x^{**}(e^*_{m_k})| \\ \geq \; & |x^*(e_{m_k})| \end{aligned}$$

so e_{m_k} is of type (wc_0). That is $(e_i)_{i=1}^{\infty}$ is of type (swc_0).

Similarly if $(e_{n_i})_{i=1}^{\infty}$ is not weakly null, then $(e_i^*)_{i=1}^{\infty}$ is of type $(swc_0)^*$.□.

Corollary 26

If $(x_n)_{n=1}^{\infty}$ is a semi-normalized unconditional basic sequence in a Banach space X, then $(x_n)_{n=1}^{\infty}$ has a (wc_0) (ie. semi-shrinking) subsequence or a $(wc_0)^*$ (ie. semi-boundedly complete) subsequence.

Proof The multiplier basis $(\bar{x_n})_{n=1}^{\infty}$ is equivalent to the standard basis of c_0 so is (wc_0) and hence (swc_0). By Theorem 25, the basic sequence $(x_n)_{n=1}^{\infty}$ is (swc_0) or $(swc_0)^*$.□.

Gowers [29] has constructed a Banach space with no shrinking or boundedly complete basic sequences. Corollary 26 prompts a natural question.

Question 27. Is there a Banach space X which has no semi-normalized (wc_0) (ie. semi-shrinking) or $(wc_0)^*$ (ie. semi-boundedly complete) basic sequences?

Clearly a Banach space answering Question 27 would have no semi-normalized unconditional basic sequences.

5. The Containment of c_0 in $K_d(X,\beta)$.

In [1] Sersouri observes that the space of compact diagonal operators is uncomplemented in the space of diagonal operators if it contains a copy of c_0. This result also holds for the space of compact operators $K(X)$ on a Banach space X and the space of bounded operators on $X, L(X)$. ([10], [11]).

Using ideas originating in work of Feder [12], [13], Emmanuele gave a necessary and sufficient condition for the containment of c_0 in $K(X)$ [10]. Emmanuele proved that c_0 embeds in the space of compact operators on a Banach space X if and only if there is a non compact operator $T : X \to X$ and compact operators T_n, such that $T(x) = \sum_{n=1}^{\infty} T_n(x)$, for all $x \in X$; where the expression $\sum_{n=1}^{\infty} T_n(x)$ converges unconditionally to $T(x)$ for each $x \in X$.

Definition 28

An operator $T \in L_d(X,\beta)$, where X is a Banach space and β is a semi-normalized Schauder basis of X, has *an unconditional, compact multiplier expansion with respect to β* if and only if

$$T(x) = \sum_{n=1}^{\infty} T_n(x), \quad \forall x \in X$$

where each $T_n \in K_d(X,\beta)$ and $\sum_{n=1}^{\infty} T_n(x)$ converges unconditionally to $T(x)$ for each $x \in X$.

In analogy with Emmanuele's result [10] we have the following.

Theorem 29

Let X be a Banach space with a seminormalized Schauder basis β.

The space c_0 is isomorphic to a subspace of $K_d(X,\beta)$ if and only if there exists a non compact diagonal operator in $L_d(X,\beta)$ which has an unconditional compact multiplier expansion with respect to β.

Proof Let β be $(e_n)_{n=1}^{\infty}$ and suppose c_0 embeds isomorphically in $K_d = K_d(X,\beta)$.

The bases $\beta^1 = (\bar{e}_n)_{n=1}^{\infty}$ and $(\bar{\bar{e}}_n)_{n=1}^{\infty}$ are equivalent, by a result from [1], so we may consider instead the space $K_d^1 = K_d(K_d,\beta^1)$.

Let (k_n) be a copy of the standard basis of c_0 in K_d^1 and let $(k_n^*)_{n=1}^{\infty}$ be its co-efficient functionals.

By Proposition 1.a.12 of [9], since $(k_n)_{n=1}^{\infty}$ is a weakly null basic sequence, we may assume that k_n is a block basic sequence of $(\bar{e}_n)_{n=1}^{\infty}$, say

$$k_n = \sum_{j=p_n+1}^{p_{n+1}} a_j \bar{e}_j \; \forall n \in \mathbb{N}$$

where $p_1 < p_2 < p_2 < \dots$.

Define $T_n \in L(K_d)$ by

$$T_n = \sum_{j=p_n}^{p_{n+1}} (\bar{e}_j)^* \otimes a_j \bar{e}_j.$$

Each T_n is a compact multiplier in K_d given by

$$T_n = \sum_{j=p_n+1}^{p_{n+1}} a_j \;\bar{\bar{e}}_j$$

For $\zeta = (\zeta_i) \in \ell^\infty$ and $x \in K_d$,

$$\begin{aligned}
& \| \textstyle\sum_{n=M+1}^{N} \zeta_n T_n(x) \| \\
= & \| \textstyle\sum_{n=M+1}^{N} \zeta_n \sum_{j=p_n+1}^{p_{n+1}} (\bar{e}_j)^*(x) a_j \bar{e}_j \| \\
= & \| (\textstyle\sum_{n=M+1}^{N} \zeta_n \sum_{j=p_n+1}^{p_{n+1}} a_j \bar{e}_j) \cdot (x) \| \\
= & \| (\textstyle\sum_{n=M+1}^{N} \zeta_n \sum_{j=p_n+1}^{p_{n+1}} a_j \bar{e}_j) \cdot (\sum_{n=p_M+1}^{p_{N+1}} (\bar{e}_j)^*(x) \bar{e}_j) \| \\
= & \| (\textstyle\sum_{n=M+1}^{N} \zeta_n k_n) \cdot (\sum_{j=p_M+1}^{p_{N+1}} (\bar{e}_j)^*(x) \bar{e}_j) \| \\
\leq & \| \textstyle\sum_{n=M+1}^{N} \zeta_n k_n \| \cdot \| \sum_{j=p_M+1}^{p_{N+1}} (\bar{e}_j)^*(x) \bar{e}_j \| \\
\leq & \; C \sup_{M+1 \leq n \leq N} |\zeta_n| \; \| \textstyle\sum_{j=p_M+1}^{p_{N+1}} (\bar{e}_j)^*(x) \bar{e}_j \| .
\end{aligned}$$

for some constant C, since $(k_n)_{n=1}^{\infty}$ is equivalent to a standard basis of c_0.

This final expression tends to zero as $M, N \to \infty$ since $x \in K_d$ and $(\bar{e}_n)_{n=1}^{\infty}$ is a basis of K_d.

So for each $x \in K_d$ the series $\sum_{n=1}^{\infty} T_n(x)$ converges unconditionally and defines an element $T(x)$ of X.

The mapping T in $L(K_d)$ is linear and bounded since there exists constant C such that

$$\begin{aligned}
\| \sum_{n=1}^{M} T_n(x) \| &= \| \sum_{n=1}^{M} \sum_{j=p_n+1}^{p_{n+1}} (\bar{e}_j)^*(x) a_j \bar{e}_j \| \\
&= \| (\sum_{n=1}^{M} \sum_{j=p_n+1}^{p_{n+1}} a_j \bar{e}_j) \cdot (x) \| \\
&\leq \| \sum_{n=1}^{M} k_n \| \| x \| \\
&\leq C \| x \|,
\end{aligned}$$

since $(k_n)_{n=1}^{\infty}$ is a standard c_0 basis.

The operator T lies in $L_d(K_d, \beta^1)$. Let $(\hat{k}_n)^*$ denote the Hahn-Banach extension of (k_n^*) to the whole of K_d.

Suppose T is compact. Then T^* is compact and $(T^*((\hat{k}_n)^*))_{n=1}^{\infty}$ is sequentially compact. However for $n > m$

$$\begin{aligned}
&\langle T^*((\hat{k}_n)^*) - T^*((\hat{k}_m)^*), \sum_{r=p_{m+1}+1}^{p_{n+1}} \bar{e}_r \rangle \\
= \ &\langle (\hat{k}_n)^* - (\hat{k}_m)^*, \sum_{r=p_{m+1}+1}^{p_{n+1}} T\bar{e}_r \rangle \\
= \ &\langle (\hat{k}_n)^* - (\hat{k}_m)^*, \sum_{t=1}^{\infty} \sum_{r=p_{m+1}+1}^{p_{n+1}} \sum_{j=p_t+1}^{p_{t+1}} a_j \bar{\bar{e}}_j (\bar{e}_r) \rangle \\
= \ &\langle (\hat{k}_n)^* - (\hat{k}_m)^*, \sum_{r=p_{m+1}+1}^{p_{n+1}} \sum_{j=p_r+1}^{p_{r+1}} a_j e_j \rangle \\
= \ &\langle (\hat{k}_n)^* - (\hat{k}_m)^*, \sum_{r=m+1}^{n} k_r \rangle \\
= \ &\langle k_n^* - k_m^*, \sum_{r=m+1}^{n} k_r \rangle \\
= \ &\langle k_n^*, \sum_{r=m+1}^{n} k_r \rangle \\
= \ &\langle k_n^*, k_n \rangle \\
= \ &1
\end{aligned}$$

Therefore,

$$\inf_{n>m} \| T^*((\hat{k}_n)^*) - T^*((\hat{k}_m)^*) \| > 0$$

and so T is not compact.

The operators T_n in K_d^1 correspond to diagonal operators S_n in K_d given by

$$S_n = \sum_{j=p_n+1}^{p_{n+1}} a_j \bar{e}_j = k_n.$$

If we define $S \in L_d(X,\beta)\backslash K_d(X,\beta)$ by

$$S(y) = \sum_{n=1}^{\infty} S_n(y), \quad \forall y \in X$$

then S is well defined and $\sum_{n=1}^{\infty} S_n(y)$ converges unconditionally to $S(y)$ by similar arguments as used for T and (T_n).

This completes one half of the proof.

Let us now suppose that there exists $T \in L_d(X,\beta)\backslash K_d(X,\beta)$ such

$$T(x) = \sum_{n=1}^{\infty} T_n(x), \quad \forall x \in X$$

where each $T_n \in K_d(X,\beta)$ and $\sum_{n=1}^{\infty} T_n(x)$ converges unconditionally to $T(x)$ for each $x \in X$.

Let $S_n(x) = \sum_{m=1}^{n} T_n(x)$, then (S_n) is not a Cauchy sequence in $L(X)$, else T would be compact.

Thus, there exists $\eta > 0$, sequences $(m_k), (n_k)\ldots$ in $\mathbb{N}$ with $m_k < n_k < m_{k+1}$ such that

$$\| S_{m_k} - S_{n_k} \| = \| \sum_{p=m_k+1}^{n_k} T_p \| > \eta \quad \forall k \in \mathbb{N}$$

Let $\mathcal{F}$ be the field of finite subsets of $\mathbb{N}$ and their complements. Define $G : \mathcal{F} \to K_d(X,\beta)$ by

$$G(\Delta) = \begin{cases} \sum_{k\in\Delta}(S_{m_k} - S_{n_k}) \text{ if } \Delta \text{ finite} \\ \sum_{k\in\mathbb{N}\backslash\Delta}(S_{n_k} - S_{m_k}) \text{ if } \Delta \text{ infinite} \end{cases}$$

Since $T = \sum_n T_n$ is an unconditional compact expansion, the set $\{G(\Delta) : \Delta \in \mathcal{F}\}$ is a bounded subset of $K_d(X,\beta)$.

If $\Delta_1, \Delta_2 \in \mathcal{F}$ and $\Delta_1 \cap \Delta_2 = \emptyset$ then

$$G(\Delta_1 \cap \Delta_2) = G(\Delta_1) + G(\Delta_2),$$

since if $\Delta_1 \cap \Delta_2 = \emptyset$ either Δ_1 or Δ_2 is finite. In the first case, $G(\Delta_1 \cup \Delta_2) = G(\Delta_1) + G(\Delta_2)$ trivially. In the second case

$$\begin{aligned} G(\Delta_1) + G(\Delta_2) &= \textstyle\sum_{k\in\Delta_1}(S_{m_k} - S_{n_k}) + \sum_{k\in\mathbb{N}\backslash\Delta_2}(S_{n_k} - S_{m_k}). \\ &= \textstyle\sum_{k\in(\mathbb{N}\backslash\Delta_2)\backslash\Delta_1})(S_{n_k} - S_{m_k}) + \sum_{k\in\mathbb{N}\backslash(\Delta\cup\Delta_2)}(S_{n_k} - S_{m_k}) \\ &= G(\Delta_1 \cup \Delta_2) \end{aligned}$$

Therefore G is a bounded, finitely additive vector measure which is not strongly additive since $\| S_{n_k} - S_{m_k} \| \geq \eta > 0 \quad \forall k \in \mathbb{N}$. By ([14], p.20, Theorem 2), the space c_0 embeds isomorphically in $K_d(X, \beta)$, and the theorem is proved.□

The characterization of when c_0 embeds in $K_d(X, \beta)$ allows further results to be proved.

We shall say that a semi-normalized Schauder basis β of a Banach space X has Property M_0 if $L_d(X, \beta) \cap c_0 = K_d(X, \beta)$. Property M_0 is enjoyed by unconditional bases. The examples given earlier in this chapter show that the standard bases of bv_0, J and lv_0 enjoy Property M_0 as does the summing basis of c_0.

Theorem 30

Let $\beta = (e_n)_{n=1}^{\infty}$ be a weakly null (ie. (wc_0)) semi-normalized Schauder basis of a Banach space X. Suppose β has Property M_0. If c_0 embeds isomorphically in $K_d(X, \beta)$, then $\beta = (e_n)_{n=1}^{\infty}$ has an unconditional subsequence.

Proof If $c_0 \hookrightarrow K_d(X, \beta)$, by Theorem 29 there exists a non compact operator $T \in L_d(X, \beta)$. Let us suppose

$$\mathrm{T}e_n = \lambda_n e_n, \quad \forall n \in \mathbb{N},$$

where λ_n are scalars. Moreover, T has an unconditional compact multiplier expansion

$$(**) \quad T(x) = \sum_{n=1}^{\infty} T_n(x), \quad \forall x \in X$$

where each $T_n \in K_d(X, \beta)$ and the convergence in $(**)$ is unconditional for each $x \in X$.

Also, the proof of Theorem 29 shows that we may assume T_n has the form

$$T_n = \sum_{j=p_n+1}^{p_{n+1}} a_j \bar{e}_j$$

for $p_1 < p_2 < p_3 < \cdots$.

Since T is non compact, and since β has Property M_0, there is a subsequence $(m_k)_{k=1}^{\infty}$ such that $\inf_k |\lambda_{m_k}| > 0$. With out loss we may assume that at most one m_k lies in each interval
$\{p_k, p_k + 1, p_k + 2, \cdots, p_{k+1} - 1\}$; by taking another subsequence if necessary.

Let $X_1 = [e_{m_k}]_{k=1}^{\infty}$. The operator $T|_{X_1}$ maps X_1 into X_1 and is non compact. We may also write

$$T(x_1) = \sum_{k=1}^{\infty} e_{m_k}^*(T_{r_k}(x_1))e_{m_k}$$

for some subsequence (r_k) of $\mathbb{N}$, where the convergence is unconditional for each $x_1 \in X_1$.

An operator $S : Z \to Y$ is lucid if there exist sequences (z_n^*) in Z^* and (y_n) in Y such that

$$S(z) = \sum_{n=1}^{\infty} z_n^*(z) y_n, \quad \forall z \in Z$$

where the series converges unconditionally.

In [30] it was shown that lucid operators are precisely those which factor through a Banach space with an unconditional basis.

Thus, the operator $T|_{X_1}$ is lucid, so there is a Banach space U with an unconditional basis $(u_n)_{n=1}^{\infty}$ and a factorization as in Figure 1 (See Appendix).

Since $T|_{X_1}$ is non compact and $T|_{X_1} = S_2S_1$, S_1 is non compact. So for some subsequence $(n_k)_{k=1}^{\infty}$ of $(m_k)_{k=1}^{\infty}$, $\inf_k \| S_1e_{n_k} \| > 0$. Since $(S_1e_{n_k})_{k=1}^{\infty}$ is weakly null, $(S_1e_{n_k})_{k=1}^{\infty}$ can be taken to be a block basis of the unconditional basis $(u_n)_{n=1}^{\infty}$, by ([9] p.7, Proposition 1.a.12). Therefore $(S_1e_{n_k})_{k=1}^{\infty}$ is an unconditional basic sequence. Now,

If $m = \inf_k |\lambda_{n_k}|$ and $A_1 = \inf_n \| e_n \|, B_1 = \sup_n \| e_n \|$,

$$|\lambda_{n_k}| \leq \frac{\| \mathrm{T}e_{n_k} \|}{\| e_{n_k} \|} \leq \frac{\| T \| B_1}{A_1} < \infty.$$

Thus the sequence $\{\lambda_{n_k}\}_{k=1}^{\infty}$ lies in the compact set

$$\{z \in \mathbb{C} : 0 < m \leq |z| \leq \frac{\| T \| B_1}{A_1} < \infty\}.$$

By the Bolzano-Weierstrass theorem, $\{\lambda_{n_k}\}_{k=1}^{\infty}$ has a convergent subsequence. Using the Cauchy condition we can choose a further subsequence $\{\lambda_{q_k}\}_{k=1}^{\infty}$ such that

$$|\lambda_{q_k} - \lambda_{q_{k+1}}| < \frac{1}{k^2}, \quad \forall k \in \mathbb{N}.$$

Then,

$$\sum_{k=1}^{\infty} |\frac{1}{\lambda_{q_k}} - \frac{1}{\lambda_{q_{k+1}}}| \leq \frac{\pi^2}{6m^2} < \infty.$$

Thus the sequence $\{\frac{1}{\lambda_{q_k}}\}$ lies in bv and so is a bounded multiplier on the basis $\{e_{q_k}\}_{k=1}^{\infty}$.

Let $X_2 = [e_{q_k}]_{k=1}^{\infty}$. Let $S_3 : X_2 \to X_2$ be given by

$$S_3(\sum_{k=1}^{\infty} c_k e_{q_k}) = \sum_{k=1}^{\infty} \frac{1}{\lambda_{q_k}} c_k e_{q_k}.$$

For signs $\theta_k = \pm 1$, scalars $(c_k)_{k=1}^{\infty}$

$$\begin{aligned} & \| \textstyle\sum_{k=1}^{\infty} c_k\theta_k e_{q_k} \| \\ = & \| S_3(\textstyle\sum_{k=1}^{\infty} c_k\theta_k\lambda_{q_k} e_{q_k}) \| \\ \leq & \| S_3 \| \| \textstyle\sum_{k=1}^{\infty} c_k\theta_k\lambda_{q_k} e_{q_k} \| \\ = & \| S_3 \| \| T(\textstyle\sum_{k=1}^{\infty} c_k\theta_k e_{q_k}) \| \\ = & \| S_3 \| \| S_2S_1(\textstyle\sum_{k=1}^{\infty} c_k\theta_k e_{q_k}) \| \\ = & \| S_3 \| \| S_2 \| \| \textstyle\sum_{k=1}^{\infty} c_k\theta_k S_1 e_{q_k} \| \\ \leq & K \| S_3 \| \| S_2 \| \| \textstyle\sum_{k=1}^{\infty} c_k S_1 e_{q_k} \|, \end{aligned}$$

where K is the unconditional constant of $(S_1 e_{n_k})_{k=1}^{\infty}$

$$\leq K \parallel S_3 \parallel \parallel S_2 \parallel \parallel S_1 \parallel \parallel \sum_{k=1}^{\infty} c_k e_{q_k} \parallel .$$

Therefore, $(e_{q_k})_{k=1}^{\infty}$ is an unconditional basic sequence and the required result is proven.□.

Example 31

Consider the uniformly convex Banach space E with normalized monotone basis $\beta = (e_n)_{n=1}^{\infty}$ so that no subsequence of β is an unconditional basic sequence, constructed by Maurey and Rosenthal ([31], p.231). Then c_0 does not embed isomorphically in $K_d(E,\beta)$ provided β has Property M_0.

In fact the methods of the proof of Theorem 30 yield another result.

Theorem 32

Let X be a Banach space with a semi-normalized Schauder basis $\beta = (e_n)_{n=1}^{\infty}$ which has Property M_0.

If $S \in L_d(X,\beta)$ is strictly singular then S is compact.

Proof Let $S : X \to X$ be given by

$$Se_n = \lambda_n e_n, \quad \forall n \in \mathbb{N}$$

where λ_n is a scalar.

If S is not compact since β has Property M_0, there exists a subsequence $\{m_k\}$ so that

$$0 < m = \inf_k |\lambda_{m_k}|.$$

Then $\{\lambda_{m_k}\}_{k=1}^{\infty}$ lies in some compact set and so by Bolzano Weierstrass, some subsequence of $\{\lambda_{m_k}\}$ converges. By applying the Cauchy condition, there is a further subsequence (n_k) such that

$$|\lambda_{n_k} - \lambda_{n_{k+1}}| < \frac{1}{k^2}.$$

So that,

$$\sum_{k=1}^{\infty} |\frac{1}{\lambda_{n_k}} - \frac{1}{\lambda_{n_{k+1}}}| < \frac{\pi^2}{6m^2} < \infty.$$

Therefore the sequence $\{\frac{1}{\lambda_{n_k}}\}_{k=1}^{\infty}$ is of bounded variation and so is a bounded multiplier of the basic sequence $\{e_{n_k}\}_{k=1}^{\infty}$. Let $X_1 = [e_{n_k}]_{k=1}^{\infty}$. If S_1 is the multiplier on $[e_{n_k}]_{k=1}^{\infty}$ given by $(\frac{1}{\lambda_{n_k}})_{k=1}^{\infty}$, then for scalars (c_k).

$$\begin{aligned} \parallel \textstyle\sum_{k=1}^{\infty} c_k e_{n_k} \parallel &= \parallel S_1(\textstyle\sum_{k=1}^{\infty} c_k \lambda_{n_k} e_{n_k}) \parallel \\ &\leq \parallel S_1 \parallel \parallel S(\textstyle\sum_{k=1}^{\infty} c_k e_{n_k}) \parallel \\ &\leq \parallel S_1 \parallel \parallel S \parallel \parallel \textstyle\sum_{k=1}^{\infty} c_k e_{n_k} \parallel . \end{aligned}$$

So that, $S|_{X_1}$ is an isomorphism on X_1 and hence S is not strictly singular.□.

Corollary 33

If X is a hereditarily indecomposable Banach space and $\beta = (e_n)_{n=1}^{\infty}$ is a semi-normalized Schauder basis of X which has Property M_0, then every element of $L_d(X, \beta)$ is of the form $\lambda I + K$, where λ is a scalar and K is an element of $K_d(X, \beta)$.

Proof Every operator on a hereditarily indecomposable Banach space is of the form $\lambda I + S$, where S is a strictly singular operator. If $T \in L_d(X, \beta)$, then $T = \lambda I + S$, where S is a strictly singular element of $L_d(X, \beta)$. By Theorem 32, S is an element of $K_d(X, \beta)$.□.

Example 34

If X is a reflexive hereditarily indecomposable Banach space with semi-normalized Schauder basis β which has Property M_0, then $L_d(X, \beta) = K_d(X, \beta) \oplus \mathrm{sp}(I)$.

Therefore $L_d(X, \beta)$ is a separable Banach space which is isomorphic to a dual space, and so both $L_d(X, \beta)$ and $K_d(X, \beta)$, have the RNP.

Therefore to answer the main question of [1] it suffices to find a semi-normalized basis of a reflexive hereditarily indecomposable Banach space which has Property M_0.

If we consider spaces of compact operators several results are also known. If we consider the special $\mathcal{L}^{\infty}$ space Y constructed in [36] with Y^* isomorphic to ℓ^1, $K(Y)$ is isomorphic to $\ell^1 \overset{v}{\otimes} Y = K(c_0, Y)$. By a result of Diestel and Morrison [33], since Y has RNP, $K(c_0, Y)$ and so $K(Y)$ has RNP and does not contain c_0. ([35]).

A bounded subset A of a Banach space E is limited if for every weak $*$ null sequence (x_n^*) in the dual space $E^*, x_n^*(x) \to 0$ uniformly for x in A. If every limited subset of E is relatively norm compact then E has the Gelfand-Phillips property.

In [32], Emmanuele shows that the $\mathcal{L}^{\infty}$ space X constructed in [36] such that X has the Schur property is such that $L(X)$ has the Gelfand-Phillips property. Therefore, ℓ^{∞} does not embed in $L(X)$ and c_0 does not embed in $K(X)$. It should be noted that if a Banach space is separable, it has the Gelfand-Phillips property. Further spaces of compact operators which do not contain c_0 are given in [34].

The following questions arise naturally.

Question 35 Is there a reflexive Banach space X such that c_0 does not embed in $K(X)$ or such that $K(X)$ has the RNP?

Question 36

Is there a Banach space X such that $L(X)$ is separable?

6. Banach Algebra Properties of $K_d(X, \beta)$ and $L_d(X, \beta)$.

Our first theorem concerns $L_d(X, \beta)$.

Theorem 37

Let X be a Banach space with semi-normalized Schauder basis $\beta = (e_n)_{n=1}^{\infty}$. Then $L_d(X,\beta)$ is a semisimple Banach algebra and the character space of $K_d(X,\beta)$ is homeomorphic to $\mathbb{N}$.

Proof Let $\varphi : K_d(X,\beta) \to \mathbb{C}$ be a multiplicative functional. If $x^2 = x \in K_d(X,\beta)$, then $\varphi(x^2) = \varphi(x)^2 = \varphi(x)$ so $\varphi(x) \in \{0,1\}$.

Thus for $m \neq n$,

$$\begin{aligned} \varphi(\bar{e_m} + \bar{e_n}) &= 0 \text{ or } 1 \\ \text{and} \quad \varphi(\bar{e_m}) &= 0 \text{ or } 1 \end{aligned}$$

For $a \in K_d(X,\beta)$ we must have

$$\varphi(a) = \phi(\sum_{m=1}^{\infty} a_m \bar{e_m}) = a_m = (\bar{e_m})^*(a)$$

for some $m \in \mathbb{N}$.

The spectrum of $K_d(X,\beta)$ is $\{(\bar{e_m})^* : m \in \mathbb{N}\}$.

Consider $A = \text{sp}(I) \oplus K_d(X,\beta)$ and φ on A, if $\varphi = 0$ on $K_d(X,\beta)$ and $\varphi \neq 0$ on A then

$$K_d(X,\beta) \subset \ker\varphi \subsetneq A,$$

so $K_d(X,\beta) = \ker\varphi$. So the spectrum of A is $\{(\bar{e_m})^* : m \in \mathbb{N} \cup \{\infty\}\}$ where

$$(\bar{e_\infty})^*(a) = \lim_{m\to\infty} (\bar{e_m})^*(a)$$

for all $a \in A$.

Since A has an identity its spectrum is compact. Also $\varphi \mapsto \varphi(\bar{e_m})$ is weak-$*$ continuous on $K_d(X,\beta)^*$ or A^*, and since $\{0,1\}$ is discrete the spectrum of $K_d(X,\beta)$ is discrete and continuously embedded in the spectrum of A. The one point compactification is minimal so the spectrum of A is homeomorphic to $\mathbb{N}\cup\{\infty\}$. If one considers the mappings defined on $L_d(X,\beta)$ by $S \mapsto \langle e_n^*, Se_n\rangle$, then these are clearly characters. Also if $\langle e_n^*, Se_n\rangle = 0$ for all n, then $S = 0$. Therefore the characters of $L_d(X,\beta)$ separate the points of $L_d(X,\beta)$ and by ([22], p.83) $L_d(X,\beta)$ is semisimple. □

The proof of Theorem 37 shows that if X is a Banach Spaces with a semi-normalized basis β and every multiplier on β is of the form $\lambda I + K$, where λ is a scalar and K is a compact multiplier, then the spectrum of $L_d(X,\beta)$ is homeomorphic to $\mathbb{N} \cup \{\infty\}$, the one point compactification of $\mathbb{N}$. If β is unconditional, $L_d(X,\beta)$ is l^∞ and the spectrum of $L_d(X,\beta)$ is homeomorphic to $\beta\mathbb{N}$, the Stone-*Čech* compactification of $\mathbb{N}$.

Question 37(a)

Which topological spaces are homeomorphic to the spectra of multiplier algebras $L_d(X, \beta)$, where β is a semi-normalized Schauder basis of a Banach space X?

Let A be a Banach algebra. A Banach space $\mathbb{X}$ is a A-bimodule if $\mathbb{X}$ is a bimodule with respect to the bilinear maps $(a, x) \mapsto a.x$ from $A \times \mathbb{X}$ into $\mathbb{X}$ and $(a, x) \mapsto x.a$ from $A \times \mathbb{X}$ into $\mathbb{X}$. A derivation from A into $\mathbb{X}$ is a linear map $D : A \mapsto \mathbb{X}$ such that

$$D(a.b) = a.Db + Da.b \text{ for } a, b \in A.$$

The continuity of homomorphisms and derivations on Banach algebras of operators has been extensively studied (cf. [16], [17], [18], [19], [20], [21]).

It is well known (cf [16]) that if all homomorphisms on a Banach algebra A are continuous then all derivations from A are continuous. In general the converse of this statement is false. ([23]).

We prove an automatic continuity result for the derivations from the algebra of compact multipliers on a semi-normalized Schauder basis.

Theorem 38

Let X be a Banach space with semi-normalized Schauder basis $\beta = (e_n)_{n=1}^{\infty}$. The derivations from $K_d(X, \beta)$ are continuous.

Proof Let $\mathbb{X}$ be a Banach $K_d(X, \beta)$ bimodule and $D : K_d(X, \beta) \to \mathbb{X}$ be a derivation.

Let $\mathcal{L}(D) = \{a \in K_d(X, \beta) : b \mapsto D(ab) \text{ is continuous}\}$

For $b \in K_d(X, \beta)$,

$$D(b\, \bar{e_n}) = D((\bar{e_n})^*(b)\bar{e_n}) = (\bar{e_n})^*(b)(D\bar{e_n}).$$

Since $(\bar{e_n})^*$ is continuous for all $n, \bar{e_n} \in \mathcal{L}(D)$ for all n. The smallest ideal which contains each $\bar{e_n}$ in $K_d(X, \beta)$ is $K_d(X, \beta)$ itself. Since $\mathcal{L}(D)$ is a closed ideal of $K_d(X, \beta)$ for any derivation $D, K_d(X, \beta) = \mathcal{L}(D)$.

Since $K_d(X, \beta)$ has a bounded approximate identity $(\bar{e_1} + \bar{e_2} + \cdots + \bar{e_n})_{n=1}^{\infty}$, given $T_n \to 0$ as $n \to \infty$ in $K_d(X, \beta)$, by Cohen's factorization theorem ([22], 11.12), there exists $R \in K_d(X, \beta)$ and S_n in $K_d(X, \beta)$ with $T_n = RS_n$ and $S_n \to 0$ as $n \to \infty$. Since $R \in \mathcal{L}(D)$, the mapping $S \mapsto D(RS)$ is continuous and so $D(T_n) = D(RS_n) \to 0$ as $n \to \infty$. It follows that the derivation D is continuous.□.

Dales and Woodin ([23]) have shown under the continuum hypothesis (CH) there exist discontinuous homomorphisms on c_0. Thus, under this hypothesis, homomorphisms need not be continuous on the algebra of compact multipliers of a semi-normalized Schauder basis.

A natural question is:

Question 39 If X is a Banach space with a semi-normalized Schauder basis β, are derivations from $L_d(X, \beta)$ automatically continuous?

It should be noted that a derivation of a semisimple commutative Banach algebra into itself is continuous and hence zero (cf. [22], p.95). In the section above we have considered derivations into general bimodules.

In [18] a James type space E was considered and it was shown that there are discontinuous derivations on $L(E)$, the algebra of bounded operators on E. By Theorem 38, derivations are continuous on the James space J.

If $D : A \to \mathbb{X}$ is given by

$$D(a) = a.x - x.a, \; \forall \, a \in A$$

for some $x \in \mathbb{X}$, then D is called an inner derivation.

A Banach algebra A is amenable if, for every Banach A-bimodule $\mathbb{X}$, every continuous derivation $D : A \to \mathbb{X}^*$ is inner.

Question 40 If X is a Banach space with semi-normalized Schauder basis β, under what conditions is either $K_d(X, \beta)$ or $L_d(X, \beta)$ amenable?

In [37] it was shown that under symmetrized approximation property $\mathbb{A}$ on X, the algebra $K(X)$ of compact operators on X is amenable.

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Chapter 6

A Construction of Families of Multiplier Algebras and a Multiplier Invariance for the Haar Basis.

Introduction

In Chapter 5 the properties of multiplier algebras of diagonal operators on semi-normalized Schauder bases in Banach space were investigated. Although some examples were given, the extent of known examples is still small. In the first part of this chapter a general method is given for the construction of multiplier algebras.

In the second part of this chapter invariances of the form $D.E = E$ are considered for sequence spaces D and E. In particular a non trivial multiplier invariance is proved for the normalized Haar basis of the Banach space $L^1[0,1]$.

1. Lindenstrauss Bases

Lindenstrauss [4] considered an explicit quotient mapping of ℓ^1 onto $L^1[0,1]$. Let $Q : \ell^1 \to L^1[0,1]$ be given by

$$Qe_{2^n+k-1} = 2^n \mathbb{I}_{[\frac{k}{2^n}, \frac{k+1}{2^n})},$$

for $0 \le k < 2^n, n \in \mathbb{Z}^+$.

The kernel of Q, $\ker Q$ is the subspace D of ℓ^1 which is spanned by the basic sequence $(e_n - \frac{1}{2}(e_{2n+1} + e_{2n+2}))_{n=0}^{\infty}$.

It is shown that D has no unconditional basis and is not isomorphic to a dual space. Let

$$f_n = e_n - \frac{1}{2}(e_{2n+1} + e_{2n+2}), n \in \mathbb{Z}^+.$$

The basis $(f_n)_{n=0}^{\infty}$ in ℓ^1 has the following properties [5]:

(A) $[f_n]_{n=0}^{\infty}$ has no unconditional basis.

(B) $[f_n]_{n=0}^{\infty}$ is not isomorphic to a dual space.

(C) $([f_n]_{n=0}^{\infty})^*$ is isomorphic to ℓ^{∞}

(D) $([f_n]_{n=0}^{\infty})$ is complemented in no dual space.

If $(x_i)_{i=0}^{\infty}$ is a Schauder basis for a Banach space X with co-efficient functionals $(f_n)_{n=0}^{\infty}$ which satisfy conditions (A), (B), (C) and (D) then $(x_i)_{i=1}^{\infty}$ is a Lindenstrauss basis for X.

Holub and Retherford [5] showed that c_0 has a Lindenstrauss basis.

This is the basis $(x_n)_{n=1}^{\infty}$ for c_0 considered in Example 11 of Chapter 5. The co-efficient functionals of $(x_n)_{n=1}^{\infty}$ are $(f_n)_{n=0}^{\infty}$ in ℓ^1.

In Chapter 5, the operator norm of a multiplier $\Lambda : c_0 \to c_0$ given by

$$\Lambda x_n = \lambda_n x_n, \quad \forall n \in \mathbb{Z}^+$$

was shown to be

$$\| \Lambda \| = \sup_{n=\mathbb{N}} [|\lambda_n| + \sum_{j=1}^{\infty} \frac{1}{2^j} \sum_{k=2^j n+2^j-1}^{2^j n+2^{j+1}-2} |\lambda_k - \lambda_{[k/2]}|]$$

An immediate consequence of the evaluation of $\| \Lambda \|$ for a multiplier Λ on $(x_n)_{n=1}^{\infty}$ and of the equivalence of the multiplier bases $(\bar{x_n})_{n=1}^{\infty}$ and $(\bar{f}_n)_{n=0}^{\infty}$ is the following Proposition.

Proposition 1

Let $(\lambda_k)_{k=1}^{\infty}$ and $(\mu_k)_{k=1}^{\infty}$ be sequences of scalars. Then

$$\sup_{n\in\mathbb{N}}[|\lambda_n\mu_n + \sum_{j=1}^{\infty} \tfrac{1}{2^j} \sum_{k=2^j n+2^j-1}^{2^j n+2^{j+1}-2} \lambda_k\mu_k|]$$

$$\leq \sup_n[|\lambda_n| + \sum_{j=1}^{\infty} \tfrac{1}{2^j} \sum_{k=2^j n+2^j-1}^{2^j n+2^{j+1}-2} |\lambda_k - \lambda_{[k/2]}|] \cdot \sup_n[|\mu_n + \sum_{j=1}^{\infty} \tfrac{1}{2^j} \sum_{k=2^j n+2^j-1}^{2^j n+2^{j+1}-2} \mu_k|]$$

and there exists a constant K such that

$$|\lambda_1|\,|\mu_1| + |\lambda_2||\mu_2| + \sum_{n=3}^{\infty} |\lambda_n\mu_n - \tfrac{1}{2}\lambda_{[\frac{n-1}{2}]}\mu_{[\frac{n-1}{2}]}|$$

$$\leq K \sup_n[|\lambda_n| + \sum_{j=1}^{\infty} \tfrac{1}{2^j} \sum_{k=2^j n+2^j-1}^{2^j n+2^{j+1}-2} |\lambda_k - \lambda_{[\frac{k}{2}]}|][|\mu_1| + |\mu_2| + \sum_{n=1}^{\infty} |\mu_n - \tfrac{1}{2}\mu_{[\frac{n-1}{2}]}|]$$

Proof These inequalities are restatements of

$$\| \sum_{n=1}^{\infty} \lambda_n\mu_n x_n \|_{c_0} \leq \| \sum_{n=1}^{\infty} \lambda_n \bar{x_n} \| \, \| \sum_{n=1}^{\infty} \mu_n x_n \|_{c_0}$$

and

$$\| \sum_{n=1}^{\infty} \lambda_n\mu_n f_n \|_{\ell^1} \leq \| \sum_{n=1}^{\infty} \lambda_n \bar{f}_n \|_{[\bar{f}_n]} \| \sum_{n=1}^{\infty} \mu_n f_n \|_{\ell^1} \ \square$$

2. A Stochastic Approach

Let $(\mathcal{A}_n)_{n=0}^{\infty}$ be the dyadic σ-algebras on $[0,1]$: that is

$$\begin{aligned} \mathcal{A}_0 &= \{\phi, [0,1)\} \\ \mathcal{A}_1 &= \{\phi, [0,\tfrac{1}{2}), [\tfrac{1}{2},1), [0,1)\} \end{aligned}$$

and in general $\mathcal{A}_n$ has atoms $I_{n,k} = [\frac{k}{2^n}, \frac{k+1}{2^n}), 0 \leq k < 2^n$. Consider the Lebesgue probability measure $\mathbb{P}$ on $[0,1)$.

Consider the dyadic tree consisting of nodes $\mathcal{T} = \{(n,k) : n \geq 0, 0 \leq k < 2^n\}$ and consider a function $\lambda : \mathcal{T} \to \mathbb{F}, \mathbb{F}$ is the field of scalars. Define random variables $(X_n)_{n=1}^{\infty}$ by setting

$$X_n(w) = \lambda(n,k), \ \forall \omega \in [\frac{k}{2^n}, \frac{k+1}{2^n})\ n \geq 0, 0 \leq k < 2^n.$$

If the dyadic tree $\mathcal{T}$ has nodes enumerated as in Tree 1 (see Appendix) then the process $(X_n)_{n=0}^{\infty}$ is adapted to the filtration $(\mathcal{A}_n)_{n=0}^{\infty}$. Let $\mathcal{T}_{(n,k)}$ denote the subtree of $\mathcal{T}$ commencing at node (n,k)

The multiplier norm of the Lindenstrauss basis of c_0 described in section 1 above, and in Chapter 5, Example 11, can be expressed as

$$\sup_n \| \ \mathbb{E}[\ |X_n| + \sum_{j=n}^{\infty} |X_j - X_{j+1}|\ |\mathcal{A}_n] \ \|_\infty$$

where $\lambda_{2^n+k} = \lambda(n,k)$ for $n \geq 0, 0 \leq k < 2^n$ ie. the enumeration of $(\lambda_n)_{n=0}^{\infty}$ is as in Tree 2 (see Appendix).

We set out the calculation of this expression below.

We can write

$$X_n = \sum_{0 \leq k < 2^n} \lambda(n,k) \mathbb{I}_{I_{(n,k)}}$$

Therefore,

$$\begin{aligned} &|X_n| + \textstyle\sum_{j=n}^{\infty} |X_j - X_{j+1}| \\ &= \textstyle\sum_{0 \leq k < 2^n} |\lambda(n,k)| \mathbb{I}_{I_{(n,k)}} + \sum_{j=n}^{\infty} \sum_{0 \leq k < 2^{n+1}} |\lambda(j, [\frac{k}{2}]) - \lambda(j+1,k)| \mathbb{I}_{I(j+1,k)} \end{aligned}$$

For f defined on $[0,1]$,

$$\mathbb{E}[f|\mathcal{A}_n] = \sum_{0 \leq k < 2^n} \left(\frac{1}{\mathbb{P}(I_{(n,k)})} \right) \int_{I_{n,k}} f d\mathbb{P} \mathbb{I}_{I_{(n,k)}}$$

So,

$$\begin{aligned} &\mathbb{E}[|X_n| + \textstyle\sum_{j=n}^{\infty} |X_j - X_{j+1}| |\mathcal{A}_n] \\ = \ &\textstyle\sum_{0 \leq k < 2^n} [|\lambda(n,k)| + \sum_{j=n}^{\infty} \frac{1}{2^{j+1-n}} \displaystyle\sum_{\substack{0 \leq p < 2^{j+1} \\ \{p:(j+1,p) \in \mathcal{T}_{(n,k)}\}}} |\lambda(j, [\frac{p}{2}]) - \lambda(j+1,p)|] \mathbb{I}_{I(n,k)} \end{aligned}$$

Since $\mathbb{P}(I_{(n,k)}) = \frac{1}{2^n}\ \forall (n,k) \in \mathcal{T}$.

Thus:

$$\begin{aligned} & \sup_n \mathbb{E}[|X_n| + \textstyle\sum_{j=n}^{\infty} |X_j - X_{j+1}||\mathcal{A}_n] \|_\infty \\ = & \sup_{(n,k)\in\mathcal{T}}[|\lambda(n,k)| + \textstyle\sum_{j=n}^{\infty} \frac{1}{2^{j+1-n}} \displaystyle\sum_{\substack{0\le p<2^{j+1} \\ \{p:(j+1,p)\in\mathcal{T}_{(n,k)}\}}} |\lambda(j,[\frac{p}{2}]) - \lambda(j+1,p)|] \end{aligned}$$

This is the same expression as calculated in Chapter 5, Example 11

The above observation can be generalized.

Theorem 2

Let $(\mathcal{A}_n)_{n=0}^{\infty}$ be a filtration of $[0,1)$ with each $\mathcal{A}_n$ consisting of a finite number m_n of atoms

$$\text{e.g. } \mathcal{A}_n = \{I_1^n, I_2^n, \cdots, I_{m_n}^n\}.$$

Given a scalar function on the tree

$$\mathcal{I}_{(\mathcal{A}_n)} = \{(n,k) : n \in \mathbb{Z}^+,\ 1 \le k \le m_n\},$$

$\lambda : \mathcal{I}_{(\mathcal{A}_n)} \to \mathbb{F}$, *define a process $(X_n)_{n=0}^{\infty}$ by*

$$X_n(\omega) = \lambda(n,k),$$

for $\omega \in I_k^n, n \ge 0, 1 \le k \le m_n$. Let $\mathcal{I}_{(\mathcal{A}_n),(n,k)}$ be the subtree of $\mathcal{I}_{(\mathcal{A}_n)}$ commencing at node (n,k).

The following expressions all define norms of multiplier algebras where

$$\lambda_k = \lambda(0,k) \quad 1 \le k \le m_0$$

$$\lambda_{m_n+k} = \lambda(n+1,k), \quad 1 \le k \le m_{n+1} \text{ for } n \ge 0$$

$$\begin{array}{lll} \sup_n & \| \mathbb{E}[|X_n| + \sum_{j=n}^{\infty} |X_j - X_{j+1}||\mathcal{A}_n] \|_\infty & (A) \\ \sup\{ & \| \mathbb{E}[(|X_{p_0}|^2 + \sum_{j=1}^{\infty} |X_{p_j} - X_{p_{j+1}}|^2)^{\frac{1}{2}}|\mathcal{A}_{p_0}] \|_\infty\colon p_0 < p_1 < p_2 < \cdots \} & (B) \\ \sup_n & \| \sum_{j=0}^{n-1} |X_j - X_{j+1}| + |X_n| \|_\infty & (C) \\ \sup_n\{ & \| [\sum_{j=0}^{n-1} |X_{p_j} - X_{p_{j+1}}|^2 + |X_{p_n}|^2]^{\frac{1}{2}} \|_\infty : p_0 < p_1 \cdots < p_n, n \ge 0\} & (D) \end{array}$$

Proof To show the result holds, if e_n is the sequence $(\delta_{pn})_{n=1}^{\infty}$ in ω, we need to show that $(e_n)_{n=1}^{\infty}$ is a semi-normalized Schauder basis under each of expression $(A),(B),(C),(D)$; this proves that the resulting sequence space is γ-perfect [3]. We then need to show that each of the expressions defines a BK-algebra in ω and that each e_n and $e = (1,1,1,\cdots)$ lies in this BK-algebra.

(A) We can write

$$X_n = \sum_{1\le k\le m_n} \lambda(n,k)\mathbb{I}_{I(n,k)}$$

Therefore

$$|X_n| + \sum_{j=n}^{\infty} |X_j - X_{j+1}|$$
$$= \sum_{1\leq k\leq m_n} |\lambda(n,k)| \mathbb{I}_{I_{(n,k)}} + \sum_{j=n}^{\infty} \sum_{1\leq k\leq m_{n+1}} |\lambda(j,k^1) - \lambda(j+1,k)| \mathbb{I}_{I_{(j+1,k)}}$$

where $(j,,k^1)$ is the predecessor in the tree $\mathcal{T}_{(\mathcal{A}_n)}$ of $(j+1,k)$.

For f defined on $[0,1)$

$$\mathbb{E}[f|\mathcal{A}_n] = \sum_{1\leq k\leq m_n} \left(\frac{1}{\mathbb{P}(I_{(n,k)})}\right) \int_{I_{(n,k)}} f d\mathbb{P} \mathbb{I}_{I_{(n,k)}}$$

So,

$$\mathbb{E}[|X_n| + \sum_{j=n}^{\infty} |X_j - X_{j+1}| |\mathcal{A}_n]$$
$$= \sum_{1\leq k\leq m_n} [|\lambda(n,k)| + \sum_{j=n}^{\infty} \sum_{\substack{1\leq p\leq m_{n+1} \\ \{p:(j+1,p)\in\mathcal{I}_{(\mathcal{A}_n),(n,k)}\}}} \frac{\mathbb{P}(I_{(j+1,p)})}{\mathbb{P}(I_{(j,p^1)})} |\lambda(j,p^1) - \lambda(j+1,p)|] \mathbb{I}_{I_{(n,k)}}$$

Thus,

$$\sup_n \| \mathbb{E}[|X_n| + \sum_{j=n}^{\infty} |X_j - X_{j+1}| |\mathcal{A}_n] \|_{\infty}$$
$$= \sup_{(n,k)\in\mathcal{I}_{(\mathcal{A}_n)}} [|\lambda(n,k)| + \sum_{j=n}^{\infty} \sum_{\substack{1\leq p\leq m_{n+1} \\ \{p:(j+1,p)\in\mathcal{I}_{(\mathcal{A}_n),(n,k)}\}}} \frac{\mathbb{P}(I_{(j+1,p)})}{\mathbb{P}(I_{(j,p^1)})} |\lambda(j,p^1) - \lambda(j+1,p)|]$$

It is easily seen that each e_n $(n \in \mathbb{N})$ and $e = (1,1,1,\dots)$ has finite expression (A). Moreover the expression clearly defines the norm of a BK-space.

If $(\lambda_n)_{n=1}^{\infty}$ is as enumerated in the statement,

$$\| (\lambda_k)_{k=1}^{\infty} \| = \sup_n \| \mathbb{E}[|X_n| + \sum_{j=n}^{\infty} |X_j - X_{j+1}| |\mathcal{A}_n] \|_{\infty},$$

then it is also very clear that,

$$\| \sum_{k=1}^{n} \lambda_k e_k \| \leq \| \sum_{k=1}^{n+1} \lambda_k e_k \|,$$

So that $(e_k)_{k=1}^{\infty}$ is a Schauder basis in the space and the condition, $\sup_n \| \sum_{k=1}^{n} \lambda_k e_k \| < \infty$ defines the sequences in the space. If $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$ lie in the space, for $(n,k) \in \mathcal{I}_{(\mathcal{A}_n)}$

$$\mu(n,k)|\lambda(n,k)| + \sum_{j=n}^{\infty} \sum_{\substack{1\leq p\leq m_{n+1} \\ \{p:(j+1,p)\in\mathcal{I}_{(\mathcal{A}_n),(n,k)}\}}} \frac{\mathbb{P}(I_{(j+1,p)})}{\mathbb{P}(I_{(j,p^1)})} |\lambda(j,p^1)\mu(j,p^1) - \lambda(j+1,p)\mu(j+1,p)|$$
$$\leq \| (\mu_n) \|_{\infty} [|\lambda(n,k)| + \sum_{j=n}^{\infty} \sum_{\substack{1\leq p\leq m_{n+1} \\ \{p:(j+1,p)\in\mathcal{I}_{(\mathcal{A}_n),(n,k)}\}}} \frac{\mathbb{P}(I_{(j+1,p)})}{\mathbb{P}(I_{(j,p^1)})} |\lambda(j,p^1) - \lambda(j+1,p)|]$$
$$+ \| (\lambda_n) \|_{\infty} [|\mu(n,k)| + \sum_{j=n}^{\infty} \sum_{\substack{1\leq p\leq m_{n+1} \\ \{p:(j+1,p)\in\mathcal{I}_{(\mathcal{A}_n),(n,k)}\}}} \frac{\mathbb{P}(I_{(j+1,p)})}{\mathbb{P}(I_{(j,p^1)})} |\mu(j,p^1) - \mu(j+1,p)|]$$

Since, the supremum norm is dominated by the expression (A),

$$\sup_{(n,k)\in\mathcal{I}_{(A_n)}}[|\lambda(n,k)||\mu(n,k)|+\sum_{j=n}^{\infty}\sum_{\substack{1\le p\le m_{n+1}\\ \{p:(j+1,p)\in\mathcal{I}_{(A_n),(n,k)}\}}}\frac{\mathbb{P}(I_{(j+1,p)})}{\mathbb{P}(I_{(j,p^1)})}|\lambda(j,p^1)\mu(j,p^1)-\lambda(j+1,p)\mu(j+1,p)|]$$

$$\le 2\sup_{(n,k)\in\mathcal{I}_{(A_n)}}[|\lambda(n,k)|+\sum_{j=n}^{\infty}\sum_{\substack{1\le p\le m_{n+1}\\ \{p:(j+1,p)\in\mathcal{I}_{(A_n),(n,k)}\}}}\frac{\mathbb{P}(I_{(j+1,p)})}{\mathbb{P}(I_{(j,p^1)})}|\lambda(j,p^1)-\lambda(j+1,p)|]$$

$$\times\sup_{(n,k)\in\mathcal{I}_{(A_n)}}[|\mu(n,k)|+\sum_{j=n}^{\infty}\sum_{\substack{1\le p\le m_{n+1}\\ \{p:(j+1,p)\in\mathcal{I}_{(A_n),(n,k)}\}}}\frac{\mathbb{P}(I_{(j+1,p)})}{\mathbb{P}(I_{(j,p^1)})}|\mu(j,p^1)-\mu(j+1,p)|]$$

Thus the expression (A) defines a multiplier algebra.

The proof for (B) is similar.

We consider the expression (D). Clearly each e_n and $e=(1,1,1,\cdots)$ lies in the space defined by (D).

If $(X_n)_{n=0}^{\infty}$ is the process defined by $(\lambda_n)_{n=1}^{\infty}$ and $(Y_n)_{n=0}^{\infty}$ is the process defined by $(\mu_n)_{n=1}^{\infty}$, then

$$\begin{aligned}&(\textstyle\sum_{j=0}^{n-1}|X_{p_j}Y_{p_j}-X_{p_{j+1}}Y_{p_{j+1}}|^2+|X_{p_n}Y_{p_n}|^2)^{\frac{1}{2}}\\ &\le\sqrt{2}(\textstyle\sum_{j=0}^{n-1}|X_{p_j}-X_{p_{j+1}}|^2+|X_{p_n}|^2)\sup_n\|Y_n\|_\infty\\ &+\sqrt{2}(\textstyle\sum_{j=0}^{n-1}|Y_{p_j}-Y_{p_{j+1}}|^2+|Y_{p_n}|^2)^{\frac{1}{2}}\sup_n\|X_n\|_\infty\\ &\le 2\sqrt{2}\,\|\,[\textstyle\sum_{j=0}^{n-1}|X_{p_j}-X_{p_{j+1}}|^2+|X_{p_n}|^2]^{\frac{1}{2}}\,\|_\infty\|\,[\textstyle\sum_{j=0}^{n-1}|Y_{p_j}-Y_{p_{j+1}}|^2+|Y_{p_n}|^2]^{\frac{1}{2}}\,\|_\infty\,.\end{aligned}$$

Thus the expression defined by (D) is a BK-algebra. It is clear that

$$\|\sum_{k=1}^{n}\lambda_k e_k\|\le\|\sum_{k=1}^{n+1}\lambda_k e_k\|$$

and that the condition $\sup_n\|\sum_{k=1}^{n}\lambda_k e_k\|<\infty$ is the same as expression (D) being finite, so (D) defines a multiplier algebra. The proof for (C) is similar.□.

Example 3 By varying the filtration clearly many examples can be constructed.

Let $(\mathcal{A}_n)$ be the filtration given by

$\mathcal{A}_n$ has atoms $\{[0,\frac{1}{n!}),[\frac{1}{n!},\frac{2}{n!}),\dots,[\frac{n!-1}{n!},1)\}$.

Let $(Z_n)_{n=0}^{\infty}$ be defined by

$$Z_n(\omega)=\lambda_{n!+k}\mathbb{I}_{I n,k}(\omega)$$

where $I_{n,k} = [\frac{k}{n!}, \frac{k+1}{n!}), n \geq 0$ and $0 \leq k < n!$

Then

$$\sup_n \| \ \mathbb{E}[|Z_n| + \sum_{j=n}^{\infty} |Z_j - Z_{j+1}||\mathcal{A}_n] \ \|_\infty$$

is the multiplier norm

$$\sup_m [|\lambda_m| + \sum_{j=1}^{\infty} \frac{m!}{(m+j)!} \sum_{k=p_{m,j}}^{q_{m,j}} |\lambda_k - \lambda_k^1|]$$

where we refer to Tree 3 (See Appendix) for the underlying tree. The points $p_{m,j}$ and $q_{m,j}$ are the first and last nodes of the jth level of the subtree commencing at node m and λ_k^1 is the predecessor in the tree of λ_k.□.

Example 4 Consider the basic sequence in ℓ^1 given by

$$f_n^r = e_n - (1-r)e_{2n+1} - re_{2n+2},$$

for $0 < r < 1$, $n \geq 0$.

Let $\mathbb{P}$ be the Lebesgue measure on $[0,1)$ and consider the filtration $(\mathcal{A}_n)_{n=0}^{\infty}$ of $[0,1)$, where

$\mathcal{A}_0$ has atoms $\{[0,1)\}$

$\mathcal{A}_1$ has atoms $\{[0,r),[r,1)\}$

$\mathcal{A}_2$ has atoms $\{[0,r^2),[r^2,r),[r,2r-r^2),[2r-r^2,1)\}$ etc.

Thus $\mathcal{A}_n$ has atoms $\{I_{n,k} : 0 \leq k < 2^n\}$, where $I_{n+1,2k}$ and $I_{n+1,2k+1}$ cover $I_{n,k}$ disjointly in the ratio $r : r-1$, as shown in Tree 4 (See Appendix).

If $Q^r : \ell^1 \to L^1[0,1]$ is defined by

$$Q^r e_{2^n+k-1} = (\mathbb{P}(I_{n,k}))^{-1}\mathbb{I}_{I_{n,k}}$$

$$\text{then since} \quad L^1[0,1] = C\ell(\cup_{n=0}^{\infty} L^1([0,1),\mathcal{A}_n,\mathbb{P})),$$

Q^r is a quotient mapping of ℓ^1 onto $L^1[0,1)$, since the functions $(\mathbb{P}(I_{n,k}))^{-1}\mathbb{I}_{I_{n,k}}$ are the extreme points of the unit ball of $L^1([0,1),\mathcal{A}_n,\mathbb{P})$, and the unit ball of $L^1([0,1),\mathcal{A}_n,\mathbb{P})$ is the closed convex hull of its extreme points. Also the conditions of the first lemma stated in [4] are satisfied, so $\ker Q^r$ is not complemented in any dual Banach space.

However, $\ker Q^r$ is spanned by the basic sequence $[f_n^r]_{n=1}^{\infty}$. If we consider $\gamma_j(k)$ as defined in Chapter 5, Example 11, then we can study the sequence $(x_n^r)_{n=0}^{\infty}$ in c_0 given by

$$x_n^r = \sum_{j=0}^{n} r^{p(j)}(1-r)^{q(j)} e_{\gamma_j(n)}.$$

where $p(n)$ and $q(n)$ are as determined by Tree 5 (See Appendix).

Then $(x_n^r)_{n=0}^{\infty}$ has co-efficient functionals $(f_n^r)_{n=0}^{\infty}$. The arguments of [5] may be adapted to show that $(x_n^r)_{n=0}^{\infty}$ is a basis of c_0. Then the multipliers $(\lambda_n)_{n=0}^{\infty}$ of $(x_n^r)_{n=0}^{\infty}$ may be identified using the technique of Chapter 5, Example 11.

Thus if

$$\Lambda x_n^r = \lambda_n x_n^r,\ n \geq 0$$

then

$$\| \Lambda \| = \sup_n [|\lambda_n| + \sum_{j=1}^{\infty} \sum_{k=2^j n+2^j-1}^{2^j n+2^{j+1}-2} r^{s(n,k)}(1-r)^{t(n,k)} |\lambda_k - \lambda_{[\frac{k}{2}]}|],$$

where $s(n,k)$ and $t(n,k)$ are as determined in Tree 6 (See Appendix) and we can show that this can be written

$$\sup_n \| \mathbb{E}[|X_n| + \sum_{j=n}^{\infty} |X_j - X_{j+1}| | \mathcal{A}_n] \|_\infty .$$

using the same method as Chapter 6, Section 2.

Since $(f_n^r)_{n=0}^{\infty}$ does not span a space complemented in a dual space and $\ell^1 / \ker Q^r$ is isomorphic to $L^1[0,1]$, the arguments outlined in [4] show that $(x_n^r)_{n=0}^{\infty}$ $(0 < r < 1)$ is a Lindenstrauss basis of c_0.

In [5] it was asked: "how many Lindenstrauss bases with mutually non isomorphic co-efficient spaces c_0 does admit?"

We have shown.

Proposition 5

There is a continuum $\mathfrak{c}$ of Lindenstrauss bases of c_0 whose co-efficient functionals form basic sequences in ℓ^1 which span subspaces isomorphic to D.

This follows from [20], p.108, Theorem 2.f.8.

3. The Haar Basis and Multiplier Invariances

Consider the space ω of all scalar sequences. By an FK space we mean a subspace of ω with a complete metrizable locally convex topology with continuous coordinate functionals $f_n : (x_k)_{k=1}^{\infty} \mapsto x_n$. An FK space whose topology is defined by a norm is called a BK-space.

Let e_k be the sequence with 1 as the kth co-ordinate and 0 elsewhere and φ the subspace of ω which consists of all finitely supported sequences.

For sequences $x = (x_k)_{k=1}^{\infty}$ and $y = (y_k)_{k=1}^{\infty}$ we define a multiplication in ω by

$$x \circ y = (x_k y_k)_{k=1}^{\infty}$$

For subsets A and B of ω we define

$$A \circ B = \{x \circ y : x \in A, y \in B\}.$$

Conditions have been obtained on an FK space E which ensure that multiplier invariances of the form

$$E = D \circ E$$

hold for specific FK spaces D. (e.g. [7], [8], [9], [10], [11], [12], [13]).

We recall for convenience some details from Chapter 5, Example 12.

The convergence field of the strong convergence method ([14]) is the space $[cs]$ defined by

$$[cs] = \{(x_k)_{k=1}^{\infty} : \sum_{2^j \leq k < 2^{j+1}} |x_k| = o(1) \text{ as } j \to \infty \text{ and } \sum_{k=1}^{\infty} x_k \text{ exists } \}$$

The space $[cs]$ is complete under the norm

$$\| (x_k)_{k=1}^{\infty} \|_{[cs]} = \sup_n |\sum_{k=1}^{n} x_k| + \sup_j \sum_{2^j \leq k < 2^{j+1}} |x_k|$$

and $(e_n)_{n=1}^{\infty}$ is a Schauder basis for $[cs]$.

It was shown in [15] that the multiplier algebra of $(e_n)_{n=1}^{\infty}$ in $[cs]$ is the space ℓv defined by

$$\ell v = \{(x_k)_{k=1}^{\infty} : \sum_j \max_{2^j \leq k < 2^{j+1}} |x_k - \alpha_j(x)| + \sum_j |\alpha_j(x) - \alpha_{j+1}(x)| < \infty\}$$

where $\alpha_j(x) = \frac{1}{2^j} \sum_{2^j \leq k < 2^{j+1}} x_k$, the space ℓv is a BK space under the norm

$$\| x \|_{\ell v} = \sum_j \max_{2^j \leq k < 2^{j+1}} |x_k - \alpha_j(x)| + \sum_j |\alpha_j(x) - \alpha_{j+1}(x)| + \sup_j |\alpha_j(x)|$$

The space ℓv_0 is defined as $\ell v \cap c_0$.

A sequence of integers is lacunary if there exists $q > 1$ such that

$$\lambda_{k+1} > q \lambda_k$$

The space ℓv is the space of sequences $(x_k)_{k=1}^{\infty}$ for which

$$\sum_{k=1}^{\infty} |x_{\lambda_k} - x_{\lambda_{k+1}}| < \infty$$

for all lacunary sequences of integers $(\lambda_k)_{k=1}^{\infty}$. It may also be shown that ℓv is isomorphic as a Banach space to $(\oplus_{n=0}^{\infty} \ell_{2^n}^{\infty})_{\ell^1}$.

Definition 3.1

Let $s^n = \sum_{k=1}^n e_k$. A sequence x in an FK space containing φ, E, has property AB if $\{s^n \circ x\}_{n=1}^\infty$ form a bounded subset of E.

A sequence x has property AK if it has AB and $s^n \cdot x$ converge to x in E.

If each element of a space has AB, we say E has AB. Similarly for property AK.

Let $\mathcal{H} = \{h \in \omega : h_k = 0 \text{ or } h_k = 1 \text{ for all } k\}$.

Let $d^n = \sum_{2^n \leq k < 2^{n+1}} e_k (n \geq 0)$.

A sequence x has property $[AB]$ in E if and only if x has property AB in E and $\{\mathcal{H} \circ d^j \circ x\}_{j=1}^\infty$ is a bounded subset of E.

A sequence x has $[AK]$ in E if and only if it has both $[AB]$ and AK in E.

Similarly E has $[AB]$ or $[AK]$ if each element of E has the property.

The next result is from [13].

It has never been published. The proof given differs from that of [13] in that use is made of the identification of ℓv as a multiplier algebra.

Theorem 6 Let E be an FK-space containing φ. The space E has property $[AB]$ if and only if $E = \ell v \circ E$.

The space E has property $[AK]$ if and only if $E = \ell v_0 \circ E$.

Proof First observe that the sets $\{\| s^n \|_{\ell v} : n \in \mathbb{N}\}$ and $\{\| h \circ d^m \|_{\ell v} : h \in \mathcal{H}, m \geq 0\}$ are bounded in the BK-space ℓv. In fact if $2^m \leq n < 2^{m+1}$

$$\| s^n \|_{\ell v} = \max[|1 - \alpha_m|, |\alpha_m|] + |1 - \alpha_m| + |\alpha_m| + 1 \leq 4$$

If $m \geq 0$ and $h \in \mathcal{H}$,

$$\| h \circ d^m \|_{\ell v} = \max_{2^m \leq k < 2^{m+1}} |h_k - \alpha_m| + 2|\alpha_m| + |\alpha_m| \leq 5.$$

Suppose now that $E = \ell v \circ E$. Given $y \in \ell v$ and $x \in E$ define $T_x(y) = x \circ y$. By the closed graph theorem each mapping T_x is continuous and maps ℓv into E. The sets $\{s^n \circ x\}_{n=1}^\infty = \{T_x(s^n)\}_{n=1}^\infty$ and $\{T_x(\mathcal{H} \circ d^j) : h \in \mathcal{H}, j \geq 0\} = \{(\mathcal{H} \circ d^j x)\}_{j=0}^\infty$ are bounded subsets of E for each $x \in E$.

Thus E has $[AB]$.

Let E be an FK space containing φ which has property $[AB]$.

Since $e = (1, 1, 1, 1, \dots) \in \ell v$, $E = e.E \subseteq \ell v \circ E$.

Let $y \in \ell v_0$ and $x \in E$. Given a continuous seminorm p on E, consider

$$p(s^{2^{n-1}} \circ y \circ x - s^{2^{m-1}} \circ y \circ x).$$

We have

$$
\begin{aligned}
& s^{2^n-1} y \circ x - s^{2^m-1} y \circ x \\
= \;& \sum_{j=m}^{n-1} \sum_{2^j \le k < 2^{j+1}} y_k x_k e_k \\
= \;& \sum_{j=m}^{n-1} \sum_{2^j \le k < 2^{j+1}} (y_k - \alpha_j(y)) x_k e_k + \sum_{j=m}^{n-1} \alpha_j(y) \sum_{2^j \le k < 2^{j+1}} x_k e_k \\
= \;& \sum_{j=m}^{n-1} \sum_{2^j \le k < 2^{j+1}} (y_k - \alpha_j(y)) x_k e_k \\
+ \;& \sum_{j=m-1}^{n-1} (\alpha_j(y) - \alpha_{j+1}(y)) s^{2^{j+1-1}} \cdot x + \alpha_n(y) s^{2^{n-1}-1} \cdot x - \alpha_{m-1}(y) s^{2^{m-1}-1} \cdot x
\end{aligned}
$$

Define a seminorm $p_{|E|}$ on E by

$$
p_{|E|}(x) = \sup_{h \in \mathcal{H}} p(h \circ x) \ \text{ for } x \in E.
$$

A simple argument shows that for each $x, y \in \varphi$,

$$
p_{|E|}(x \circ y) \le 4 \parallel x \parallel_\infty p_{|E|}(y).
$$

Thus,

$$
\begin{aligned}
& p(s^{2^n-1} \circ y \circ x - s^{2^m-1} \circ y \circ x) \\
& \le 4 \sum_{j=m}^{n-1} \max_{2^j \le k < 2^{j+1}} |y_k - \alpha_j(y)| p_{|E|}(d^j x) \\
& + \sup_j p(s^j x) \{ \sum_{j=m-1}^{n-1} |\alpha_j(y) - \alpha_j(y)| + |\alpha_n(y)| + |\alpha_{m-1}(y)| \} \\
& \le 4 \sup_j p_{|E|}(d^j \circ x) \sum_{j=m}^{n-1} \max_{2^j \le k < 2^{j+1}} |y_k - \alpha_j(y)| \\
& + \sup_j p(s^j \circ x) \{ \sum_{j=m-1}^{n-1} |\alpha_j(y) - \alpha_{j+1}(y)| + |\alpha_n(y)| + |\alpha_{m-1}(y)| \}
\end{aligned}
$$

The sequence $(e_n)_{n=1}^{\infty}$ is a Schauder basis of the algebra of compact multipliers ℓv_0 and $y \in \ell v_0$, so

$$
\sum_{j=m}^{n-1} \max_{2^j \le k < 2^{j+1}} |y_k - \alpha_j(y)| + \sum_{j=m-1}^{n-1} |\alpha_j(y) - \alpha_{j+1}(y)| + |\alpha_n(y)| + |\alpha_{m-1}(y)|
$$

$\to 0$ as $m, n \to \infty$.

Since E has $[AB]$, $\sup_j p_{|E|}(d^j x) < \infty$ and $\sup_j p(s^j x) < \infty$, so $\{s^{2^n-1} \circ y \circ x\}_{n=1}^{\infty}$ is a Cauchy sequence in E. Since E is complete and an FK-space, $y \circ x$ lies in E. Therefore, $\ell v_0 \circ E \subseteq E$. However, $\ell v = \ell v_0 \oplus sp(e)$, so $E \subseteq \ell v \circ E \subseteq E + e \circ E = E$. That is $\ell v \circ E = E$ as required.

If E is an FK-space containing φ such that $E = \ell v_0 \circ E$. We define a map $T_x(y) = x \circ y$ for each $x \in E$ and $y \in \ell v_0$, the argument outlined before shows that E has $[AB]$.

Let $z \in E$. Since $E = \ell v_0 \circ E$, $z = x \circ y$ where $x \in \ell v_0$ and $y \in E$. Since the mapping T_y is continuous from ℓv_0 to E, for any continuous seminorm p on E,

$$\begin{aligned} p(s^n \circ z - z) &= p(s^n x \circ y - x \circ y) \\ &= p(T_y(s^n x - x)) \end{aligned}$$

$\to 0$ as $n \to \infty$, since $x \in \ell v_0$ and $(e_n)_{n=1}^{\infty}$ is a Schauder basis of ℓv_0, so $s^n x - x \to 0$ as $n \to \infty$.

So E has AK and $[AB]$, that is E has $[AK]$.

If E is an FK-space containing φ has $[AK]$, then E has $[AB]$ and we have shown that $\ell v_0 \circ E \subseteq E$. Since E has $[AK]$ it has AK, and by a result of Garling [7], $E = bv_0 \circ E$. However ℓv_0 is an algebra of compact multipliers on a Schauder basis so $bv_0 \subset \ell v_0$ and hence, $E = bv_0 \circ E \subset \ell v_0 \circ E \subseteq E$; i.e. $E = \ell v_0 \circ E$ as required.□.

Consider now the Banach space $L^1[0,1]$. The normalized Haar functions are defined

$$\begin{aligned} h_1(t) &= 1, \quad \forall t \in [0,1) \\ h_2(t) &= \mathbb{I}_{[0,\frac{1}{2})}(t) - \mathbb{I}_{[\frac{1}{2},1)}(t), && \forall t \in [0,1) \\ h_{2^n+k}(t) &= 2^n \mathbb{I}(t)_{[\frac{2k-2}{2^{n+1}},\frac{2k-1}{2^{n+1}})} - 2^n \mathbb{I}(t)_{[\frac{2k-1}{2^{n+1}},\frac{2k}{2^{n+1}})}, && \forall t \in [0,1). \end{aligned}$$

It is well known that the Haar functions define a monotone Schauder basis for $L^1[0,1]$. (cf. [21], p.6).

Theorem 7

Let E be the BK-space associated with the normalized Haar basis of $L^1[0,1]$. Then E has property $[AK]$.

Therefore

$$\begin{aligned} E &= \ell v_0 \circ E \\ \text{and} \quad E &= \ell v \circ E. \end{aligned}$$

That is the spaces ℓv_0 and ℓv comprise of multipliers of the normalized Haar basis of $L^1[0,1]$.

Proof Since the Haar basis is a Schauder basis of $L^1[0,1]$ its co-ordinate space certainly has property AK.

CLAIM Given $f = \sum_{m=1}^{\infty} c_m h_m \in L^1[0,1]$, the set

$$\{\sum_{k=0}^{2^n-1} \pm c_{2^n+k} h_{2^n+k} : n \geq 0\}$$

is bounded in the $L^1[0,1]$ norm

The claim is true because the functions $h_{2^n}, h_{2^n+1}, \cdots, h_{2^{n+1}-1}$ are disjointly supported, and so

$$\begin{aligned}
&\| \sum_{k=0}^{2^n-1} \pm c_{2^n+k} h_{2^n+k} \|_{L^1} \\
&=\| \sum_{k=0}^{2^n-1} c_{2^n+k} h_{2^n+k} \|_{L^1} \\
&=\| s_{2^{n+1}-1} \circ f - s_{2^n-1} \circ f \|_{L^1} \\
&\leq 2 \| f \|_{L^1} .
\end{aligned}$$

Thus E has AK and $[AB]$, so that E has $[AK]$ as required.□.

The multiplier invariances $E = \ell v_0 \circ E$ and $E = \ell v \circ E$ follow from Theorem 6. □.

It would be interesting to find a better multiplier result for the normalized Haar basis on $L^1[0,1]$. In the space $L^p[0,1], 1 < p < \infty$, the Haar system is also a monotone Schauder basis. It is a deep result due to Paley [16] that in this case the Haar system is an unconditional basis of $L^p[0,1]$, $1 < p < \infty$, and so its multiplier algebra is ℓ^∞. Burkholder [17], [18], [19] has used martingale methods to discover the unconditional constant of the Haar basis in $L^p[0,1]$ and connections with certain boundary value problems. It would be interesting if such techniques could be applied to the space $L^1[0,1]$.

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Appendix

Chapter 4

Diagram 1

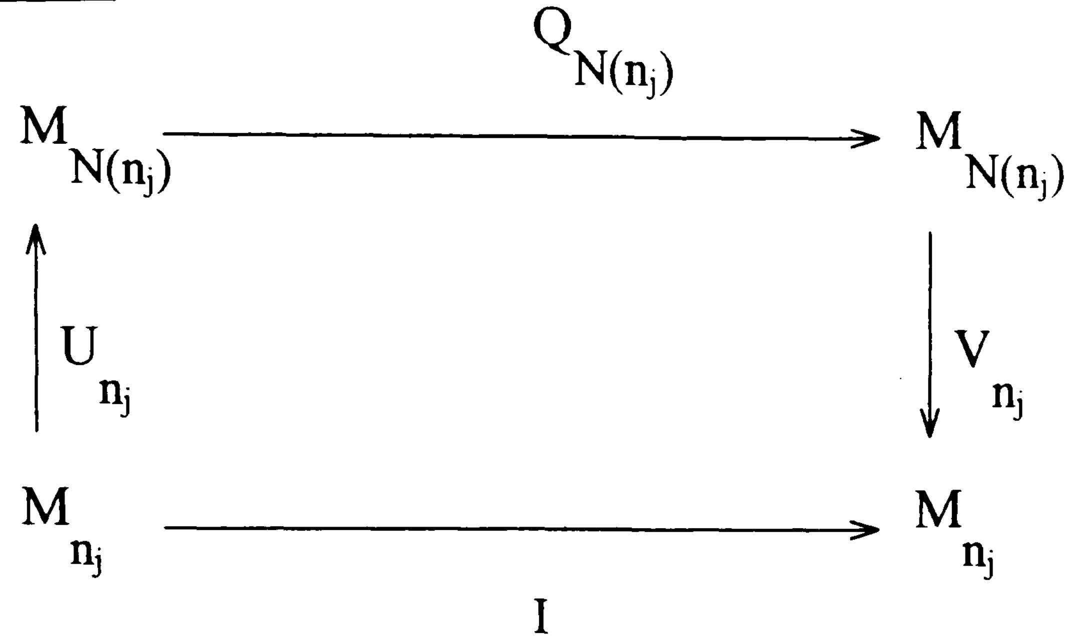
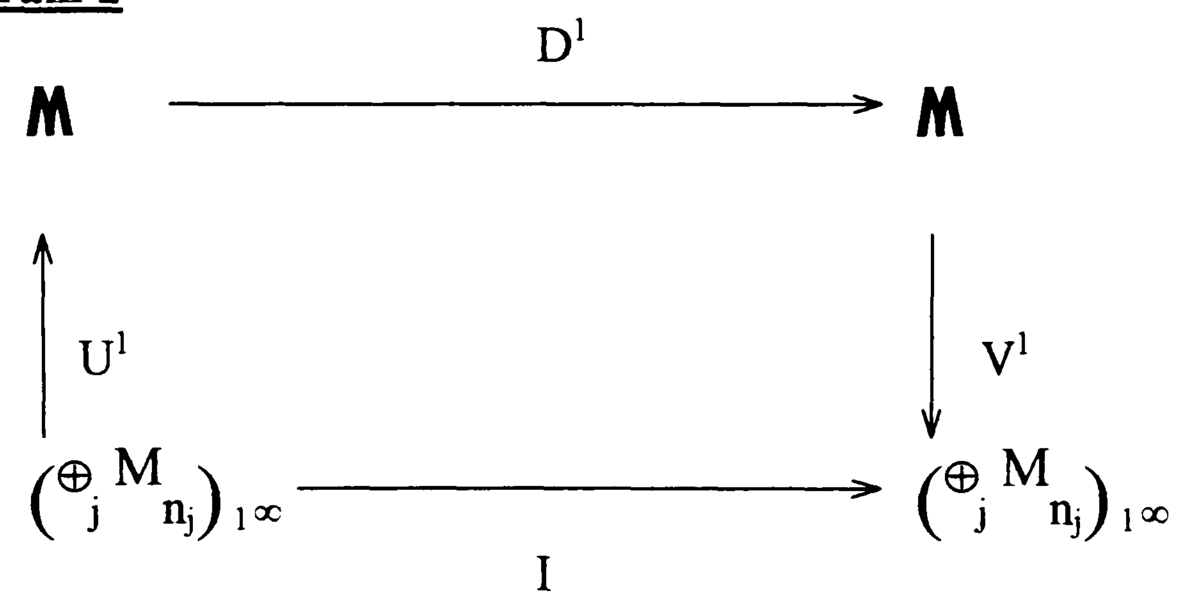
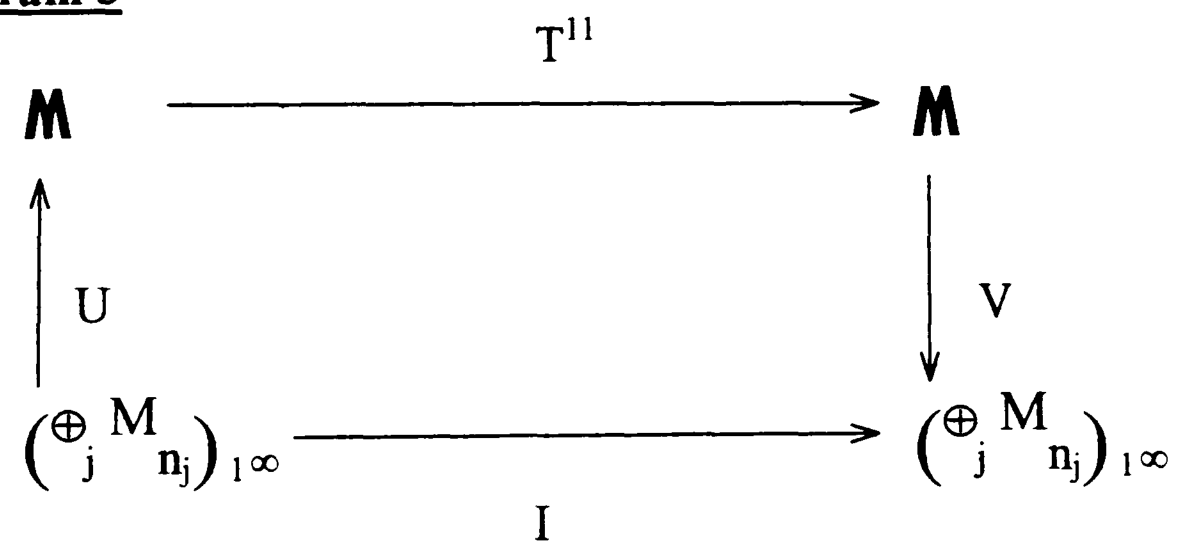


Diagram 2



$D^1 = D$ or $D^1 = I - D$

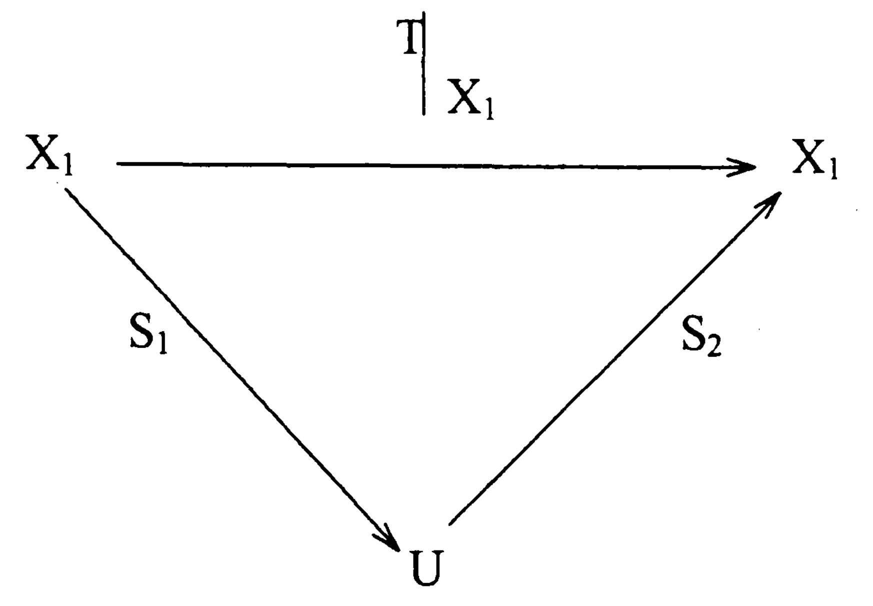
Diagram 3



$T^{11} = T$ or $T^{11} = I - T$

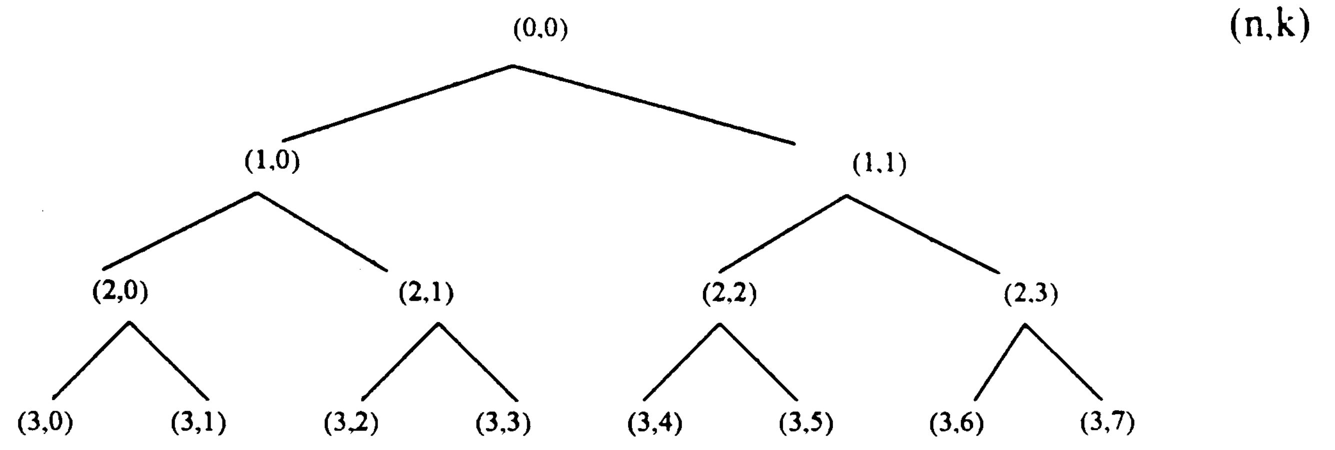
Chapter 5

Figure 1

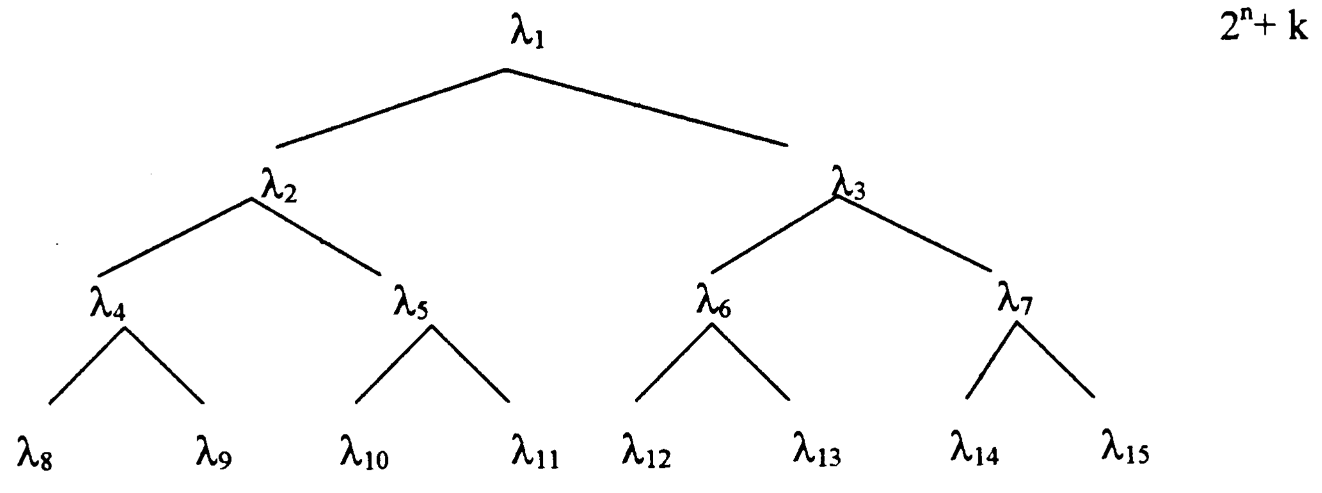


Chapter 6

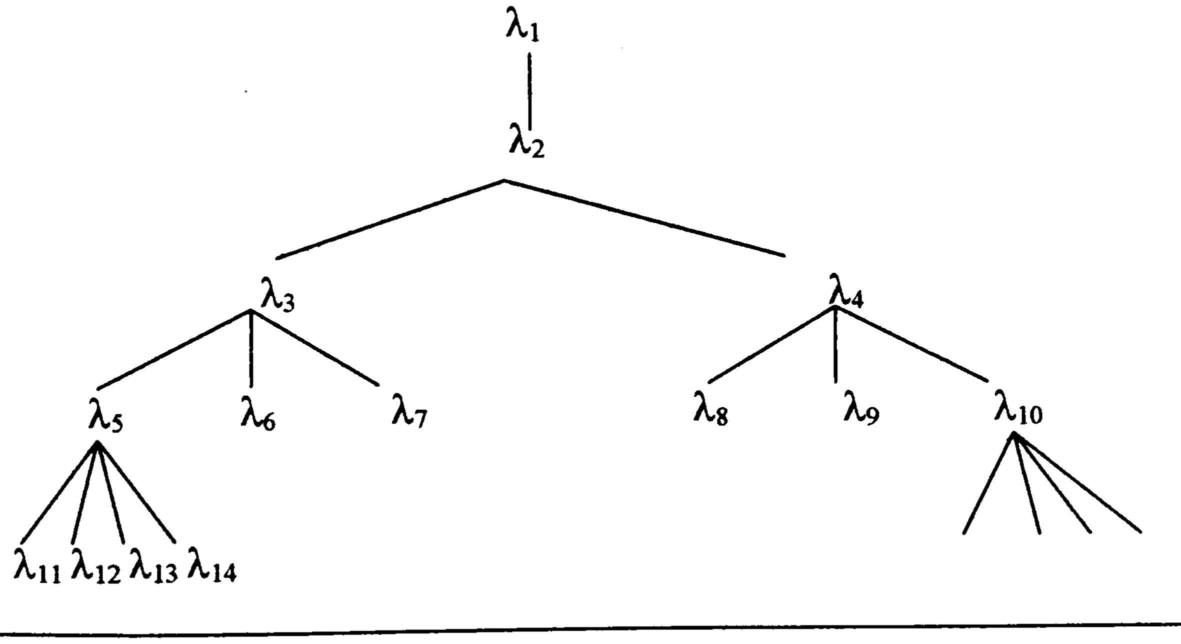
Tree 1



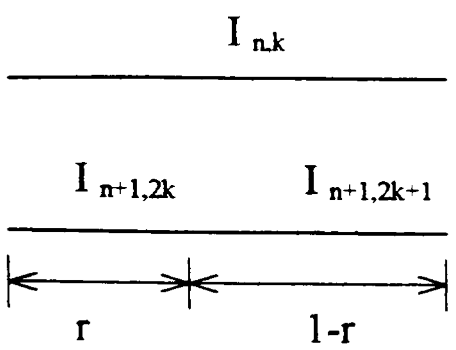
Tree 2



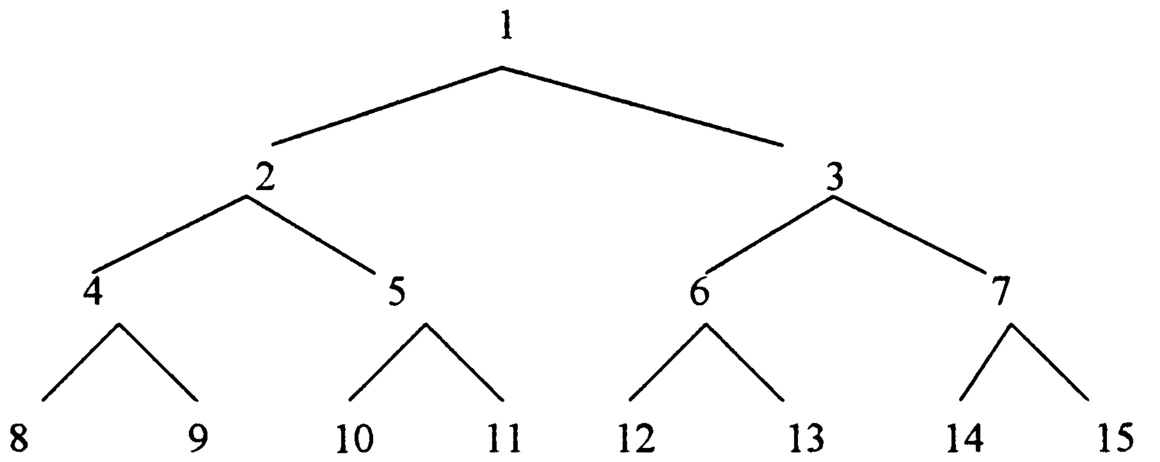
Tree 3

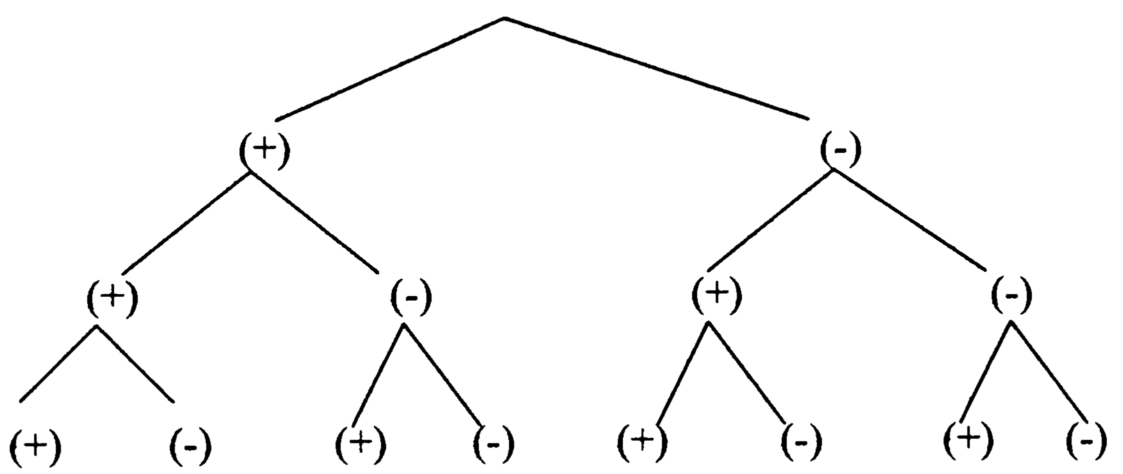


Tree 4



Tree 5



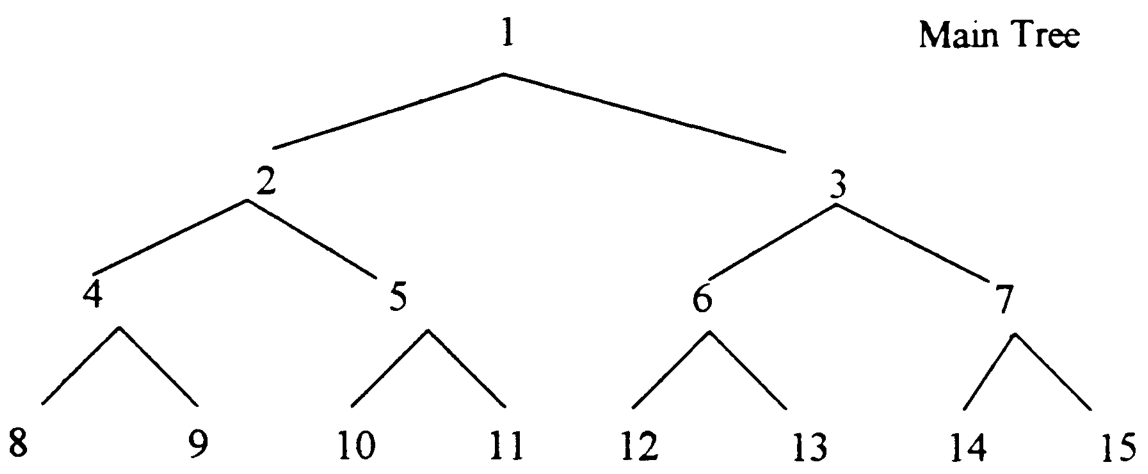


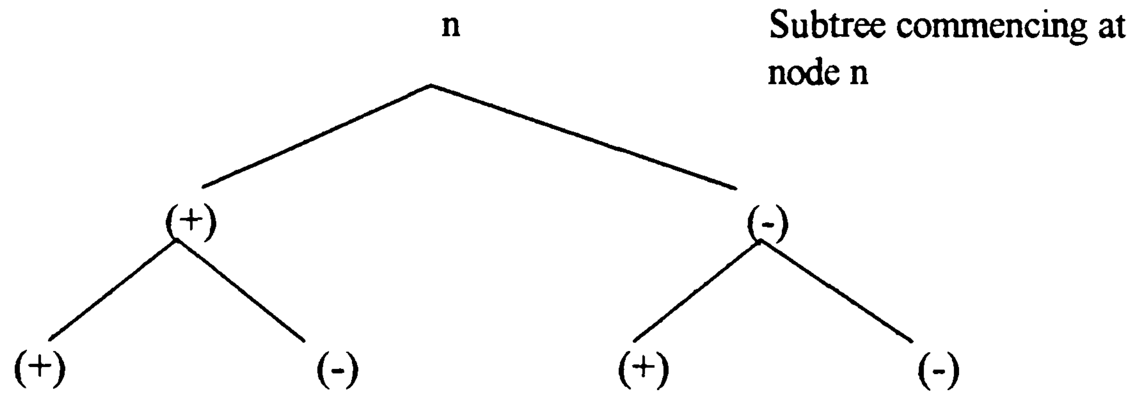
If $m = 2^n+k$ where $n \geq 0$, $0 \leq k < 2^n$ is a node of the tree.

Then p(m) is the number of plus signs on the branch joining node 1 and node m (including node m), as shown in Tree 5.

Also q(m) = n-p(m).

Tree 6





s(n,k) is the number of plus signs on the branch joining node n and node k (including node k but excluding node n) as shown in the subtree above.

t(n,k) is the number of minus signs on the branch joining node n and node k (including node k but excluding node n) as shown in the subtree above.