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THE EQUILIBRIUM EXISTENCE DUALITY: EQUILIBRIUM WITH INDIVISIBILITIES & INCOME EFFECTS

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ABSTRACT. We show that, with indivisible goods, the existence of competitive equilibrium fundamentally depends on agents' substitution effects, not their income effects. Our Equilibrium Existence Duality allows us to transport results on the existence of competitive equilibrium from settings with transferable utility to settings with income effects. One consequence is that net substitutability—which is a strictly weaker condition than gross substitutability—is sufficient for the existence of competitive equilibrium. We also extend the “demand types” classification of valuations to settings with income effects and give necessary and sufficient conditions for a pattern of substitution effects to guarantee the existence of competitive equilibrium.

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1. INTRODUCTION

This paper shows that, when goods are indivisible and there are income effects, the existence of competitive equilibrium fundamentally depends on agents' substitution effects—i.e., the effects of compensated price changes on agents' demands. We provide general existence results that do not depend on income effects.

In contrast to the case of divisible goods, competitive equilibrium does not generally exist in settings with indivisible goods (Henry, 1970). Moreover, most previous results about when equilibrium does exist with indivisible goods assume that utility is transferable—ruling out income effects but allowing tractable characterizations of (Pareto-)efficient allocations and aggregate demand that can be exploited to analyze competitive equilibrium.¹ But understanding the role of income effects is important for economies with indivisible goods, as these goods may comprise large fractions of agents' budgets. Furthermore, in the presence of income effects, the distribution of wealth among agents affects both Pareto efficiency and aggregate demand, making it necessary to develop new methods to analyze competitive equilibrium with indivisible goods.

The cornerstone of our analysis is an application of the relationship between Marshallian and Hicksian demand. As in classical demand theory, Hicksian demand is defined by fixing a utility level and minimizing the expenditure of obtaining it. We combine Hicksian demands to construct a family of “Hicksian economies” in which prices vary, but agents' utilities—rather than their endowments—are held constant. Our key result, which we call the Equilibrium Existence Duality, states that competitive equilibria exist for all endowment allocations if and only if competitive equilibria exist in the Hicksian economies for all utility levels.

Preferences in each Hicksian economy reflect agents' substitution effects. Therefore, by the Equilibrium Existence Duality, the existence of competitive equilibrium fundamentally depends on substitution effects. Moreover, as fixing a utility level precludes income effects, agents' preferences are quasilinear in each Hicksian economy. Hence, the Equilibrium Existence Duality allows us to transport (and so generalize) *any* necessary or sufficient condition for equilibrium existence from settings with transferable utility to settings with income effects.² In particular, our most general existence result gives a necessary and sufficient condition for a pattern of agents' substitution effects to guarantee the existence of competitive equilibrium in the presence of income effects.

¹For example, methods based on integer programming (see, e.g., Koopmans and Beckmann (1957), Bikhchandani and Mamer (1997), Ma (1998), Candogan et al. (2015), and Tran and Yu (2019)) rely on characterizations of the set of Pareto-efficient allocations as the solutions to a welfare maximization problem, while methods based on convex programming (see, e.g., Murota (2003), Ikebe et al. (2015), and Candogan, Epitropou, and Vohra (2020)) and tropical geometry (Baldwin and Klempere, 2014, 2019) rely on representing aggregate demand as the demand of a representative agent.

²Outside the case of substitutes (which we describe in detail), Bikhchandani and Mamer (1997) and Ma (1998) gave necessary and sufficient conditions on profiles of valuations, and Candogan et al. (2015) gave sufficient conditions on agents' individual valuations, for the existence of competitive equilibrium in transferable utility economies.

Consider, for example, the case of substitutable goods in which each agent demands at most one unit of each good. With transferable utility, substitutability is sufficient for the existence of competitive equilibrium (Kelso and Crawford, 1982) and defines a maximal domain for existence (Gul and Stacchetti, 1999). With income effects, Fleiner et al. (2019) showed that competitive equilibrium exists under gross substitutability. The Equilibrium Existence Duality tells us that, with income effects, competitive equilibrium in fact exists under *net* substitutability and that net substitutability defines a maximal domain for existence. Moreover, we show that gross substitutability implies net substitutability; the reverse direction is not true in the presence of income effects.

An implication of our results is that it is unfortunate that Kelso and Crawford (1982), and much of the subsequent literature, used the term “gross substitutes” to refer to a condition on quasilinear preferences. Indeed, gross and net substitutability are equivalent without income effects, and our work shows that it is net substitutability, not gross substitutability, that is critical to the existence of competitive equilibrium with substitutes.³

To appreciate the distinction between gross and net substitutability, suppose that Martine owns a house and is thinking about selling her house and buying one of two different other houses: a spartan one and a luxurious one (Quinzii, 1984). If the price of her own house increases, she may wish to buy the luxurious house instead of the spartan one—exposing a gross complementarity between her existing house and the spartan one. However, Martine regards the houses as net substitutes: the complementarity emerges entirely due an income effect. Competitive equilibrium is therefore guaranteed to exist in economies with Martine if all other agents see the goods as net substitutes, despite the presence of gross complementarities.

Our most general equilibrium existence theorem characterizes the combinations of substitution effects that guarantee the existence of competitive equilibrium. It is based on Baldwin and Klemperer’s (2019) classification of valuations into “demand types.” A demand type is defined by the set of vectors that summarize the possible ways in which demand can change in response to a small generic price change. For example, the set of all substitutes valuations forms a demand type, as does the set of all complements valuations, etc.

Applying Baldwin and Klemperer’s taxonomy to changes in Hicksian demands, we see that their definition easily extends to general utility functions, capturing agents’ substitution effects. Examples of demand types in our setting with income effects, therefore, include the set of all net substitutes preferences, the set of all net complements preferences, etc. The Equilibrium Existence Duality then makes it straightforward that the Unimodularity Theorem⁴—which encompasses many standard results on the existence of

³Kelso and Crawford (1982) were aware of the equivalence between gross and net substitutability in their setting (see their Footnote 1) but used the term “gross substitutes” due to an analogy of their arguments for existence with tâtonnement from general equilibrium theory.

⁴See Theorem 4.3 of Baldwin and Klemperer (2019); an earlier version was given by Danilov et al. (2001).

competitive equilibrium as special cases⁵—is unaffected by income effects. Therefore, as with the case of substitutes, conditions on complementarities and substitutabilities that guarantee the existence of competitive equilibrium in settings with transferable utility translate to conditions on net complementarities and substitutabilities that guarantee the existence of competitive equilibrium in settings with income effects. In particular, there are patterns of net complementarities that are compatible with the existence of competitive equilibrium.

Our results may have significant implications for the design of auctions that seek competitive equilibrium outcomes, and in which bidders face financing constraints. For example, they suggest that versions of the Product-Mix Auction (Klemperer, 2008), used by the Bank of England since the Global Financial Crisis, may work well in this context.

Several other papers have considered the existence of competitive equilibrium in the presence of indivisibilities and income effects. Quinzii (1984), Gale (1984), and Svensson (1984) showed the existence of competitive equilibrium in a housing market economy in which agents have unit demand and endowments. Building on those results, Kaneko and Yamamoto (1986), van der Laan et al. (1997, 2002), and Yang (2000) analyzed settings with multiple goods, but restricted attention to separable preferences. By contrast, our results—even for the case of substitutes—allow for interactions between the demand for different goods. We also clarify the role of net substitutability for the existence of competitive equilibrium.

In a different direction, Danilov et al. (2001) proved a version of the sufficiency direction of the Unimodularity Theorem for settings with income effects. Danilov et al. (2001) also defined domains of preferences using an optimization problem that turns out to be equivalent to the expenditure minimization problem. However, they did not note the connection to the expenditure minimization problem or Hicksian demand, and, as a result, did not interpret their sufficient conditions in terms of substitution effects or establish the role of substitution effects in determining the existence of equilibrium.

We proceed as follows. Section 2 describes our setting—an exchange economy with indivisible goods and money. Section 3 develops the Equilibrium Existence Duality. Since the existing literature has focused mostly on the case in which indivisible goods are substitutes, we consider that case in Section 4. Section 5 develops demand types for settings with income effects and states our Unimodularity Theorem with Income Effects. Section 6 remarks on implications for auction design, and Section 7 is a conclusion. Appendix A proves the Equilibrium Existence Duality. Appendix B proves the connection between gross and net substitutability. Appendices C and D adapt the proofs of results from the literature to our setting.

⁵It generalizes the quasilinear case of Kelso and Crawford (1982), and results of Sun and Yang (2006), Milgrom and Strulovici (2009), Hatfield et al. (2013), and Teytelboym (2014).

2. THE SETTING

We work with a model of exchange economies with indivisibilities—adapted to allow for income effects. There is a finite set J of agents, a finite set I of indivisible goods, and a divisible numéraire that we call “money.” We allow goods to be undesirable, i.e., to be “bads.” We fix a *total endowment* $\mathbf{y}_I \in \mathbb{Z}^I$ of goods in the economy.⁶

2.1. Preferences and Marshallian Demand. Each agent $j \in J$ has a finite set $X_I^j \subseteq \mathbb{Z}^I$ of *feasible bundles* of indivisible goods and a lower bound $\underline{x}_0^j \geq -\infty$ on her consumption of money. As bundles that specify negative consumption of some goods can be feasible, our setting implicitly allows for production.⁷ The principal cases of \underline{x}_0^j are $\underline{x}_0^j = -\infty$, in which case all levels of consumption of money are feasible, and $\underline{x}_0^j = 0$, in which case the consumption of money must be positive. Hence, the set of feasible consumption bundles for agent j is $X^j = (\underline{x}_0^j, \infty) \times X_I^j$. Given a bundle $\mathbf{x} \in X^j$, we let x_0 denote the amount of money in \mathbf{x} and \mathbf{x}_I denote the bundle of goods specified by \mathbf{x} , so $\mathbf{x} = (x_0, \mathbf{x}_I)$.

The utility levels of agent j lie in the range $(\underline{u}^j, \bar{u}^j)$, where $-\infty \leq \underline{u}^j < \bar{u}^j \leq \infty$. Furthermore, each agent j has a *utility function* $U^j : X^j \rightarrow (\underline{u}^j, \bar{u}^j)$ that we assume to be continuous and strictly increasing in x_0 , and to satisfy

$$(1) \quad \lim_{x_0 \rightarrow (\underline{x}_0^j)^+} U^j(x_0, \mathbf{x}_I) = \underline{u}^j \quad \text{and} \quad \lim_{x_0 \rightarrow \infty} U^j(x_0, \mathbf{x}_I) = \bar{u}^j$$

for all $\mathbf{x}_I \in X_I^j$. Condition (1) requires that some consumption of money above the minimum level \underline{x}_0^j be essential to agent j .⁸ We let $p_0 = 1$.

Given an endowment $\mathbf{w} = (w_0, \mathbf{w}_I) \in X^j$ of a feasible consumption bundle and a price vector $\mathbf{p}_I \in \mathbb{R}^I$, agent j ’s *Marshallian demand* for goods is

$$D_M^j(\mathbf{p}_I, \mathbf{w}) = \left\{ \mathbf{x}_I^* \left| \mathbf{x}^* \in \arg \max_{\mathbf{x} \in X^j | \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{w}} U^j(\mathbf{x}) \right. \right\}.$$

As usual, Marshallian demand is given by the set of bundles of goods that maximize an agent’s utility, subject to a budget constraint, given a price vector and an endowment. An *income effect* is a change in an agent’s Marshallian demand induced by a change in her money endowment, holding prices fixed.⁹

⁶In particular, we allow for multiple units of some goods to be present in the aggregate, unlike Gul and Stacchetti (1999) and Candogan et al. (2015).

⁷Technological constraints on production (in the sense of Hatfield et al. (2013) and Fleiner et al. (2019)) can be represented by the possibility that some bundles of goods are infeasible for an agent to consume (see Example 2.15 in Baldwin and Klemperer (2014)).

⁸Henry (1970, pages 543–544), Mas-Colell (1977, Theorem 1(i)), and Demange and Gale (1985, Equation (3.1)) made similar assumptions. If consuming money is inessential but consumption of money must be nonnegative, then it is known that competitive equilibrium may not exist (Mas-Colell, 1977)—even in settings in which agents have unit demand for goods (see, e.g., Herings and Zhou (2019)). However, the existence of competitive equilibrium can be guaranteed when the agents trade lotteries over goods (Gul et al., 2020).

⁹Note that income effects also correspond to changes in an agent’s Marshallian demand induced by changes in the value of her endowment, holding prices fixed.

Our setup is flexible enough to capture a wide range of preferences with and without income effects, as the following two examples illustrate.

Example 1 (Quasilinear Utility). Given a *valuation* $V^j : X_I^j \rightarrow \mathbb{R}$, letting $\underline{x}_0^j = \underline{u}^j = -\infty$ and $\bar{u}^j = \infty$, one obtains a quasilinear utility function given by

$$U^j(x_0, \mathbf{x}_I) = x_0 + V^j(\mathbf{x}_I).$$

When agents utility functions are quasilinear, they do not experience income effects. When all agents have quasilinear utility functions, we say that utility is *transferable*.

Example 2 (Quasilogarithmic Utility). Given a function $V_Q^j : X_I^j \rightarrow (-\infty, 0)$, which we call a *quasivaluation*,¹⁰ and letting $\underline{u}^j = -\infty$, $\bar{u}^j = \infty$, and $\underline{x}_0^j = 0$, there is a *quasilogarithmic* utility function given by

$$U^j(\mathbf{x}) = \log x_0 - \log(-V_Q^j(\mathbf{x}_I)).$$

Unlike with quasilinear utility functions, agents with quasilogarithmic utility functions exhibit income effects.

2.2. Hicksian Demand, Hicksian Valuations, and the Hicksian Economies. The concept of Hicksian demand from consumer theory plays a key role in our analysis. Given a utility level $u \in (\underline{u}^j, \bar{u}^j)$ and a price vector \mathbf{p}_I , agent j 's *Hicksian demand* for goods is

$$(2) \quad D_H^j(\mathbf{p}_I; u) = \left\{ \mathbf{x}_I^* \mid \mathbf{x}^* \in \arg \min_{\mathbf{x} \in X^j \mid U^j(\mathbf{x}) \geq u} \mathbf{p} \cdot \mathbf{x} \right\}.$$

As in the standard case with divisible goods, Hicksian demand is given by the set of bundles of goods that minimize the expenditure of obtaining a utility level given a price vector. A *substitution effect* is a change in an agent's Hicksian demand induced by a change in prices, holding her utility level fixed.

As in classical demand theory, Marshallian and Hicksian demand are related by the duality between the utility maximization and expenditure minimization problems. Specifically, a bundle of goods is expenditure-minimizing if and only if it is utility-maximizing.¹¹

Fact 1 (Relationship between Marshallian and Hicksian Demand). *Let \mathbf{p}_I be a price vector.*

(a) *For all endowments \mathbf{w} , we have that $D_M^j(\mathbf{p}_I, \mathbf{w}) = D_H^j(\mathbf{p}_I; u)$, where*

$$u = \max_{\mathbf{x} \in X^j \mid \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{w}} U^j(\mathbf{x}).$$

¹⁰Here, we call V_Q^j a quasivaluation, and denote it by V_Q^j instead of V^j , to distinguish it from the valuation of an agent with quasilinear preferences.

¹¹Although Fact 1 is usually stated with divisible goods (see, e.g., Proposition 3.E.1 and Equation (3.E.4) in Mas-Colell et al. (1995)), the standard proof applies with multiple indivisible goods and money under Condition (1). For sake of completeness, we give a proof of Fact 1 in Appendix C.

(b) For all utility levels u and endowments \mathbf{w} with

$$\mathbf{p} \cdot \mathbf{w} = \min_{\mathbf{x} \in X^j | U^j(\mathbf{x}) \geq u} \mathbf{p} \cdot \mathbf{x},$$

we have that $D_H^j(\mathbf{p}_I; u) = D_M^j(\mathbf{p}_I, \mathbf{w})$.

If an agent has a quasilinear utility function, then, as she experiences no income effects, her Marshallian and Hicksian demands coincide and do not depend on endowments or utility levels. Under quasilinearity, we therefore refer to both Marshallian and Hicksian demand simply as *demand*, which we denote by $D^j(\mathbf{p}_I)$. Formally, if j has quasilinear utility with valuation V^j , defining $D^j(\mathbf{p}_I)$ as the solution to the quasilinear maximization problem

$$(3) \quad D^j(\mathbf{p}_I) = \arg \max_{\mathbf{x}_I \in X_I^j} \{V^j(\mathbf{x}_I) - \mathbf{p}_I \cdot \mathbf{x}_I\},$$

we have that $D_M^j(\mathbf{p}_I, \mathbf{w}) = D^j(\mathbf{p}_I)$ for all endowments \mathbf{w} and that $D_H^j(\mathbf{p}_I; u) = D^j(\mathbf{p}_I)$ for all utility levels u .

We next show that the interpretation of the expenditure minimization problem as a quasilinear maximization problem persists in the presence of income effects. Specifically, we can rewrite the expenditure minimization problem of Equation (2) as a quasilinear optimization problem by using the constraint to solve for x_0 as a function of \mathbf{x}_I . Formally, for a bundle $\mathbf{x}_I \in X_I^j$ of goods and a utility level $u \in (\underline{u}^j, \bar{u}^j)$, we let $S^j(\mathbf{x}_I; u) = U^j(\cdot, \mathbf{x}_I)^{-1}(u)$ denote the level of consumption of money (or savings) needed to obtain utility level u given \mathbf{x}_I .¹² By construction, we have that

$$D_H^j(\mathbf{p}_I; u) = \arg \min_{\mathbf{x}_I \in X_I^j} \{S^j(\mathbf{x}_I; u) + \mathbf{p}_I \cdot \mathbf{x}_I\}.$$

It follows that agent j 's expenditure minimization problem at utility level u can be written as a quasilinear maximization problem for the valuation $-S^j(\cdot; u)$, which we therefore call the Hicksian valuation.

Definition 1. The *Hicksian valuation* of agent j at utility level u is $V_H^j(\cdot; u) = -S^j(\cdot; u)$.

Note that $S^j(\cdot; u)$ is continuous and strictly increasing in u , and hence $V_H^j(\cdot; u)$ is continuous and strictly decreasing in u . The following lemma formally states that agent j 's Hicksian demand at utility level u is the demand correspondence of an agent with valuation $V_H^j(\cdot; u)$.

Lemma 1. For all price vectors \mathbf{p}_I and utility levels u , we have that

$$D_H^j(\mathbf{p}_I; u) = \arg \min_{\mathbf{x}_I \in X_I^j} \{S^j(\mathbf{x}_I; u) + \mathbf{p}_I \cdot \mathbf{x}_I\} = \arg \max_{\mathbf{x}_I \in X_I^j} \{V_H^j(\mathbf{x}_I; u) - \mathbf{p}_I \cdot \mathbf{x}_I\}.$$

¹²The function S^j is the *compensation function* of Demange and Gale (1985) (see also Danilov et al. (2001)).

Proof. As $U^j(\mathbf{x})$ is strictly increasing in x_0 , we have that

$$D_H^j(\mathbf{p}_I; u) = \left\{ \mathbf{x}_I^* \left| \mathbf{x}^* \in \arg \min_{\mathbf{x} \in X^j | U^j(\mathbf{x}) = u} \mathbf{p} \cdot \mathbf{x} \right. \right\}.$$

Applying the substitution $x_0 = S^j(\mathbf{x}_I; u) = -V_H^j(\mathbf{x}_I; u)$ to remove the constraint from the minimization problem yields the lemma. \square

It follows from Lemma 1 that an agent's Hicksian valuation at a utility level gives rise to a quasilinear utility function that reflects the agent's substitution effects at that utility level. Lemma 1 also yields a relationship between the family of Hicksian valuations and income effects. Indeed, by Fact 1, an agent's income effects correspond to changes in her Hicksian demand induced by changes in her utility level, holding prices fixed. By Lemma 1, these changes in Hicksian demand reflect the changes in the Hicksian valuation that are induced by the changes in utility levels. Hence, the Hicksian valuations at each utility level determine an agent's substitution effects, while the variation of the Hicksian valuations with the utility level captures her income effects.

To illustrate how an agent's family of Hicksian valuations reflects her income effects, we consider the cases of quasilinear and quasilogarithmic utility.

Example 3 (Example 1 continued). With quasilinear utility, the Hicksian valuation at utility level u is $V_H^j(\mathbf{x}_I; u) = V^j(\mathbf{x}_I) - u$. Changes in u do not affect the relative values of bundles under $V_H^j(\cdot; u)$, so changes in the utility level do not affect Hicksian demand. Indeed, there are no income effects. By construction, a utility function $U^j(\mathbf{x})$ is quasilinear in x_0 if and only if $S^j(\mathbf{x}_I; u)$ is quasilinear in u —or, equivalently, $V_H^j(\mathbf{x}_I; u)$ is quasilinear in $-u$.

In general, it follows from Fact 1 and Lemma 1 that agent j 's preferences exhibit income effects if and only if $S^j(\mathbf{x}_I; u)$ —or, equivalently, $V_H^j(\mathbf{x}_I; u)$ —is not additively separable between \mathbf{x}_I and u .

Example 4 (Example 2 continued). With quasilogarithmic utility, the Hicksian valuation at utility level u is $V_H^j(\mathbf{x}_I; u) = e^u V_Q^j(\mathbf{x}_I)$. In this case, each Hicksian valuation is a positive linear transformation of V_Q^j . Income effects are reflected by the fact that $V_H^j(\mathbf{x}_I; u)$ is not additively separable between \mathbf{x}_I and u .

We use Lemma 1 to convert preferences with income effects into families of valuations. It turns out that each continuously decreasing family of valuations is the family of Hicksian valuations of a utility function, so a utility function can be represented equivalently by a family of Hicksian valuations.

Fact 2 (Duality for Preferences). *Let $F : X_I^j \times (\underline{u}^j, \bar{u}^j) \rightarrow (-\infty, -\underline{x}_0^j)$ be a function. There exists a utility function $U^j : X^j \rightarrow (\underline{u}^j, \bar{u}^j)$ whose Hicksian valuation at each utility level u is $F(\cdot, u)$ if and only if for each $\mathbf{x}_I \in X_I^j$, the function $F(\mathbf{x}_I, \cdot)$ is continuous,*

strictly decreasing, and satisfies^{13,14}

$$(4) \quad \lim_{u \rightarrow (\underline{u}^j)^+} F(\mathbf{x}_I, u) = -\underline{x}_0^j \quad \text{and} \quad \lim_{u \rightarrow (\bar{u}^j)^-} F(\mathbf{x}_I, u) = -\infty.$$

Finally, we combine the families of Hicksian valuations to form a family of Hicksian economies, in each of which utility is transferable and agents choose consumption bundles to minimize the expenditure of obtaining given utility levels.

Definition 2. The *Hicksian economy* for a profile of utility levels $(u^j)_{j \in J}$ is the transferable utility economy in which agent j 's valuation is $V_H^j(\cdot; u^j)$.

The family of Hicksian economies consists of the “duals” of the original economy in which income effects have been removed and price effects are given by substitution effects. Like the construction of Hicksian valuations, the construction of the Hicksian economies allows us to convert economies with income effects to families of economies with transferable utility and is a key step of our analysis.

3. THE EQUILIBRIUM EXISTENCE DUALITY

We now turn to the analysis of competitive equilibrium in exchange economies. An *endowment allocation* consists of an endowment $\mathbf{w}^j \in X^j$ for each agent j such that $\sum_{j \in J} \mathbf{w}_I^j = \mathbf{y}_I$, where \mathbf{y}_I is the total endowment. Given an endowment allocation, a competitive equilibrium specifies a price vector such that markets for goods clear when agents maximize utility. By Walras's Law, it follows that the market for money clears as well.

Definition 3. Given an endowment allocation $(\mathbf{w}^j)_{j \in J}$, a *competitive equilibrium* consists of a price vector \mathbf{p}_I and a bundle $\mathbf{x}_I^j \in D_M^j(\mathbf{p}_I, \mathbf{w}^j)$ for each agent such that $\sum_{j \in J} \mathbf{x}_I^j = \mathbf{y}_I$.

In transferable utility economies, a competitive equilibrium consists of a price vector \mathbf{p}_I and a bundle $\mathbf{x}_I^j \in D^j(\mathbf{p}_I)$ for each agent such that $\sum_{j \in J} \mathbf{x}_I^j = \mathbf{y}_I$. In this case, the endowment allocation does not affect competitive equilibrium because endowments do not affect (Marshallian) demand. We therefore omit the endowment allocation when considering competitive equilibrium in transferable utility economies in which an endowment allocation exists—i.e., $\mathbf{y}_I \in \sum_{j \in J} X_I^j$. On the other hand, the total endowment \mathbf{y}_I affects competitive equilibrium even when utility is transferable.

Recall that utility is transferable in the Hicksian economies. Furthermore, by Lemma 1, a competitive equilibrium in the Hicksian economy for a profile $(u^j)_{j \in J}$ of utility levels

¹³A version of Fact 2 for the function S^j in a setting in which utility is increasing in goods is proved in Lemma 1 in Danilov et al. (2001). For sake of completeness, we give a proof of Fact 2 in Appendix C. Fact 2 is also similar in spirit to the duality between utility functions and expenditure functions (see, e.g., Propositions 3.E.2 and 3.H.1 in Mas-Colell et al. (1995)). However, the arguments of the expenditure function (at each utility level) are prices, while the arguments of the Hicksian valuation (at each utility level) are quantities.

¹⁴Condition (4) is analogous to Condition (1) and ensures that the corresponding utility function is defined everywhere on X^j . Note that Condition (4) is essentially automatic in the context of Danilov et al. (2001) and therefore does not appear explicitly in their result (Lemma 1 in Danilov et al. (2001)).

consists of a price vector \mathbf{p}_I and a bundle $\mathbf{x}_I^j \in D_H^j(\mathbf{p}_I; u^j)$ for each agent such that $\sum_{j \in J} \mathbf{x}_I^j = \mathbf{y}_I$. Thus, agents act as if they minimize expenditure in competitive equilibrium in the Hicksian economies.¹⁵

Building on Fact 1 and Lemma 1, our Equilibrium Existence Duality connects the equilibrium existence problems in the original economy (which can feature income effects) and the Hicksian economy (in which utility is transferable). Specifically, we show that competitive equilibrium always exists in the original economy if and only if it always exists in the Hicksian economies. Here, we hold agents' preferences and the total endowment (of goods) fixed but allow the endowment allocation to vary.

Theorem 1 (Equilibrium Existence Duality). *Suppose that the total endowment and the sets of feasible bundles are such that an endowment allocation exists. Competitive equilibria exist for all endowment allocations if and only if competitive equilibria exist in the Hicksian economies for all profiles of utility levels.*

By Lemma 1, agents' substitution effects determine their preferences in each Hicksian economy. Therefore, Theorem 1 tells us that *any* condition that ensures the existence of competitive equilibria can be written as a condition on substitution effects alone. That is, substitution effects fundamentally determine whether competitive equilibrium exists.

Both directions of Theorem 1 also have novel implications for the analysis of competitive equilibrium in economies with indivisibilities. As demands in the Hicksian economies are given by Hicksian demand in the original economy (Lemma 1), the “if” direction of Theorem 1 implies that every condition on demand D^j that guarantees the existence of competitive equilibrium in settings with transferable utility translates into a condition on *Hicksian* demand D_H^j that guarantees the existence of competitive equilibrium in settings with income effects. In Sections 4 and 5, we use the “if” direction of Theorem 1 to obtain new domains for the existence of competitive equilibrium with income effects from previous results on the existence of competitive equilibrium in settings with transferable utility (Kelso and Crawford, 1982; Baldwin and Klemperer, 2019). Conversely, the “only if” direction of Theorem 1 shows that if a condition on demand defines a maximal domain for the existence of competitive equilibrium in settings with transferable utility, then the translated condition on Hicksian demand defines a maximal domain for the existence of competitive equilibrium in settings with income effects. In Sections 4 and 5, we also use this implication to derive new maximal domain results for settings with income effects.

¹⁵As a result, competitive equilibria in the Hicksian economies coincide with *quasiequilibria with transfers* from the modern treatment of the Second Fundamental Theorem of Welfare Economics (see, e.g., Definition 16.D.1 in Mas-Colell et al. (1995)). As the set of feasible levels of money consumption is open, agents always can always reduce their money consumption slightly from a feasible bundle to obtain a strictly cheaper feasible bundle. Hence, quasiequilibria with transfers coincide with equilibria with transfers in the original economy (see, e.g., Proposition 16.D.2 in Mas-Colell et al. (1995) for the case of divisible goods). If the endowments of money were fixed in the Hicksian economies, this concept would coincide with the concept of *compensated equilibrium* of Arrow and Hahn (1971) and the concept of *quasiequilibrium* introduced by Debreu (1962).

To prove the “only if” direction of Theorem 1, we exploit a version of the Second Fundamental Theorem of Welfare Economics for settings with indivisibilities. To understand connection to the existence problem for the Hicksian economies, note that the existence of competitive equilibrium in the Hicksian economies is equivalent to the conclusion of the Second Welfare Theorem—i.e., that each Pareto-efficient allocation can be supported in an equilibrium with endowment transfers—as the following lemma shows.¹⁶

Lemma 2. *Suppose that the total endowment and the sets of feasible bundles are such that an endowment allocation exists. Competitive equilibria exist in the Hicksian economies for all profiles of utility levels if and only if, for each Pareto-efficient allocation $(\mathbf{x}^j)_{j \in J}$ with $\sum_{j \in J} \mathbf{x}^j = \mathbf{y}_I$, there exists a price vector \mathbf{p}_I such that $\mathbf{x}^j \in D_M^j(\mathbf{p}_I, \mathbf{x}^j)$ for all agents j .*

We prove Lemma 2 in Appendix A. Intuitively, as utility is transferable in the Hicksian economies, variation in utility levels between Hicksian economies plays that same role as endowment transfers in the Second Welfare Theorem. It is well-known that the conclusion of the Second Welfare Theorem holds whenever competitive equilibria exist for all endowment allocations (Maskin and Roberts, 2008).¹⁷ It follows that competitive equilibrium always exists in the Hicksian economies whenever it always exists in the original economy, which is the “only if” direction of Theorem 1.

We use a different argument to prove the “if” direction. Our strategy is to show that there exists a profile of utility levels and a competitive equilibrium in the corresponding Hicksian economy in which all agents’ expenditures equal their budgets in the original economy. To do so, we apply a topological fixed-point argument that is similar in spirit to standard proofs of the existence of competitive equilibrium. Specifically, we consider an auctioneer who, for a given profile of candidate equilibrium utility levels, evaluates agents’ expenditures over all competitive equilibria in the Hicksian economy and adjusts candidate equilibrium utility levels upwards (resp. downwards) for agents who under- (resp. over-) spend their budgets.¹⁸ The existence of competitive equilibrium in the Hicksian economies ensures that the process is nonempty-valued, and the transferability of utility in the Hicksian economies ensures that the process is convex-valued. Kakutani’s Fixed Point Theorem implies the existence of a fixed-point utility profile. By construction, there exists a competitive equilibrium in the corresponding Hicksian economy at which

¹⁶Recall that an allocation $(\mathbf{x}^j)_{j \in J} \in \prod_{j \in J} X^j$ is *Pareto-efficient* if there does not exist an allocation $(\hat{\mathbf{x}}^j)_{j \in J} \in \prod_{j \in J} X^j$ such that

$$\sum_{j \in J} \hat{\mathbf{x}}^j = \sum_{j \in J} \mathbf{x}^j,$$

and $U^j(\hat{\mathbf{x}}^j) \geq U^j(\mathbf{x}^j)$ for all agents j with strict inequality for some agent.

¹⁷While Maskin and Roberts (2008) assumed that goods are divisible, their arguments apply even in the presence of indivisibilities—as we show in Appendix A.

¹⁸This approach is similar in spirit to Negishi’s (1960) proof of the existence of competitive equilibrium with divisible goods. Negishi (1960) instead applied an adjustment process to the inverses of agents’ marginal utilities of money. However, Negishi’s (1960) approach does not generally yield a convex-valued adjustment process in the presence of indivisibilities.

agents' expenditures equal the values of their endowments. By Lemma 1, agents must be maximizing utility given their endowments at this equilibrium, and hence once obtains a competitive equilibrium in the original economy. The details of the argument are in Appendix A.

3.1. Examples. We next illustrate the power of Theorem 1 using the two examples.

Our first example is a “housing market” in which agents have unit-demand preferences, may be endowed with a house, and can experience arbitrary income effects. We can use Theorem 1 to reduce the existence problem to the assignment game of Koopmans and Beckmann (1957)—reproving a result originally due to Quinzii (1984).

Example 5 (A Housing Market—Quinzii, 1984; Gale, 1984; Svensson, 1984). For each agent j , let $X_I^j \subseteq \{\mathbf{0}\} \cup \{\mathbf{e}^i \mid i \in I\}$ be nonempty. In this case, in Hicksian economy, utility is transferable and agents have unit demand for the goods. As the endowment allocation does not affect competitive equilibrium when utility is transferable, the results of Koopmans and Beckmann (1957) imply that competitive equilibria exist in the Hicksian economies for all profiles of utility levels (provided that an endowment allocation exists). Hence, Theorem 1 implies that competitive equilibria exist for all endowment allocations—even in the presence of income effects.

In the second example, we revisit the quasilogarithmic utility functions from Example 2. We provide sufficient conditions on agents' quasivaluations for competitive equilibrium to exist. These conditions are related to, but not in general implied by, the conditions developed in Sections 4 and 5.

Example 6 (Existence of Competitive Equilibrium with Quasilogarithmic Preferences). For each agent j , let $V_Q^j : X_I^j \rightarrow (-\infty, 0)$ be a quasivaluation. Let agent j 's utility function be quasilogarithmic for the quasivaluation V_Q^j , as in Example 2. In this case, agent j 's Hicksian valuation at each utility level is a positive linear transformation of V_Q^j (Example 4). Hence, by Theorem 1, competitive equilibria exist for all endowment allocations as long as competitive equilibrium exists when utility is transferable and each agent j 's valuation is an (agent-dependent) positive linear transformation of V_Q^j —e.g., if the quasivaluations V_Q^j are all strong substitutes valuations (Milgrom and Strulovici, 2009), or all valuations of a unimodular demand type (Baldwin and Klemperer, 2019). Additionally, in the case in which one unit of each good is available in total (i.e., $y_i = 1$ for all goods i), Candogan et al. (2015) showed that competitive equilibrium exists when utility is transferable and all agents have sign-consistent tree valuations. Hence, if one unit of each good is available in total, then Theorem 1 implies that competitive equilibria exist with quasilogarithmic utility for all endowment allocations if all agents' quasivaluations are sign-consistent tree valuations.

In the remainder of the paper, we use Theorem 1 to develop novel conditions on preferences that ensure the existence of competitive equilibrium.

4. THE CASE OF SUBSTITUTES

In this section, we apply the Equilibrium Existence Duality (Theorem 1) to prove a new result regarding the existence of competitive equilibrium with substitutable indivisible goods and income effects: we show that a form of *net* substitutability is sufficient for, and in fact defines a maximal domain for, the existence of competitive equilibrium. We begin by reviewing previous results on the existence of competitive equilibrium under (gross) substitutability. We then derive our existence theorem for net substitutability and relate it to the previous results.

In this section, we focus on the case in which each agent demands at most one unit of each good. Formally, we say that an agent j *demands at most one unit of each good* if $X_I^j \subseteq \{0, 1\}^I$. We extend to the case in which agents can demand multiple units of some goods in Section 5.3.

4.1. Gross Substitutability and the Existence of Competitive Equilibrium. We recall a notion of gross substitutability for preferences over indivisible goods from Fleiner et al. (2019), which extends the gross substitutability condition from classical demand theory. It requires that *uncompensated* increases in the price of a good weakly raise demand for all other goods. With quasilinear utility, the modifier “gross” can be dropped—as in classical demand theory (see also Footnote 1 in Kelso and Crawford (1982)).

Definition 4 (Gross Substitutability). Suppose that agent j demands at most one unit of each good.

- (a) A utility function U^j is a *gross substitutes utility function at endowment* $\mathbf{w}_I \in X_I^j$ of goods if for all money endowments $w_0 > \underline{x}_0^j$, price vectors \mathbf{p}_I , and $\lambda > 0$, whenever $D_M^j(\mathbf{p}_I, \mathbf{w}) = \{\mathbf{x}_I\}$ and $D_M^j(\mathbf{p}_I + \lambda \mathbf{e}^i, \mathbf{w}) = \{\mathbf{x}'_I\}$, we have that $x'_k \geq x_k$ for all goods $k \neq i$.¹⁹
- (b) A *substitutes valuation* is a valuation for which the corresponding quasilinear utility function is a gross substitutes utility function.²⁰

Technically, Definition 4 imposes a substitutability condition on the locus of prices at which Marshallian demand is single-valued—following Ausubel and Milgrom (2002), Hatfield et al. (2013), Baldwin and Klemperer (2019), and Fleiner et al. (2019).²¹

It is well-known that when utility is transferable, competitive equilibrium exists under substitutability.

¹⁹Our definition of gross substitutability holds the endowment of goods fixed, but, unlike Fleiner et al. (2019), imposes a condition at every feasible endowment of money. Imposing the “full substitutability in demand language” condition from Assumption D.1 in Supplemental Appendix D of Fleiner et al. (2019) at every money endowment is equivalent to our gross substitutability condition.

²⁰Note that substitutability is independent of the endowment of goods as endowments do not affect the demands of agents with quasilinear utility functions. Our definition of substitutability coincides with Kelso and Crawford’s (1982) definition (Danilov, Koshevoy, and Lang, 2003).

²¹By contrast, Kelso and Crawford (1982) imposed a gross substitutability condition at all price vectors. Imposing Kelso and Crawford’s (1982) condition at every money endowment leads to a strictly stronger condition than Definition 4(a) in the presence of income effects (Schlegel, 2020).

Fact 3. *Suppose that utility is transferable and that an endowment allocation exists. If each agent demands at most one unit of each good and has a substitutes valuation, then competitive equilibrium exists.*²²

Moreover, the class of substitutes valuations forms a maximal domain for the existence of competitive equilibrium in transferable utility economies. Specifically, if an agent has a non-substitutes valuation, then competitive equilibrium may not exist when the other agents have substitutes valuations. Technically, we require that one unit of each good be present among agents' endowments (i.e., that $y_i = 1$ for all goods i) as complementarities between goods that are not present are irrelevant for the existence of competitive equilibrium.

Fact 4. *Suppose that $y_i = 1$ for all goods i . If $|J| \geq 2$, agent j demands at most one unit of each good, and V^j is not a substitutes valuation, then there exist sets $X_I^k \subseteq \{0, 1\}^I$ of feasible bundles and substitutes valuations $V^k : X_I^k \rightarrow \mathbb{R}$ for agents $k \neq j$, for which there exists an endowment allocation but no competitive equilibrium.*²³

While Fact 4 shows that there is no domain strictly containing the domain of substitutes valuations for which the existence of competitive equilibrium can be guaranteed in transferable utility economies, it does not rule out the existence of other domains for which the existence of competitive equilibrium can be guaranteed. For example, Sun and Yang (2006), Candogan et al. (2015), and Baldwin and Klemperer (2019) gave examples of domains other than substitutability for which the existence of competitive equilibrium is guaranteed.

Generalizing Fact 3 to settings with income effects, Fleiner et al. (2019) showed that competitive equilibrium exists for an endowment allocation $(\mathbf{w}^j)_{j \in J}$ if each agent j 's utility function is a gross substitutes utility function at her endowment \mathbf{w}_I^j of goods.²⁴ However, Fleiner et al. (2019) did not offer a maximal domain result for gross substitutability. In the next section, we show that gross substitutability does not actually drive existence of competitive equilibrium with substitutable indivisible goods.

²²Fact 3 is a version of Theorem 1 in Hatfield et al. (2013) for exchange economies and follows from Proposition 4.6 in Baldwin and Klemperer (2019). See Kelso and Crawford (1982) and Gul and Stacchetti (1999) for earlier versions that assume that valuations are monotone.

²³Fact 4 is a version of Theorem 2 in Gul and Stacchetti (1999) and Theorem 4 in Yang (2017) that applies when X_I^k can be strictly contained in $\{0, 1\}^I$, as well as a version of Theorem 7 in Hatfield et al. (2013) for exchange economies. For sake of completeness, we give a proof of Fact 4 in Appendix D. The proof shows that the statement would hold if $|J| \geq |I|$ and agents $k \neq i$ were restricted to unit-demand valuations—as in Theorem 2 in Gul and Stacchetti (1999).

²⁴Fleiner et al. (2019) worked with a matching model and considered equilibrium with personalized pricing, but their arguments also apply in exchange economies without personalized pricing. However, Fleiner et al. (2019) only required that each agent sees goods as gross substitutes for a fixed endowment of goods and money. Our notion of gross substitutability considers a fixed endowment of goods but a variable endowment of money, and therefore the existence result of Fleiner et al. (2019) is not strictly a special case of Theorem 2. Moreover, Fleiner et al. (2019) also allowed for frictions such as transaction taxes and commissions in their existence result.

4.2. Net Substitutability and the Existence of Competitive Equilibrium. In light of Theorem 1 and Fact 3, competitive equilibrium exists if agents' Hicksian demands satisfy an appropriate substitutability condition—i.e., if preferences satisfy a *net* analogue of substitutability.

We build on Definition 4 to define a concept of net substitutability for settings with indivisibilities. Net substitutability is a version of the net substitutability condition from classical consumer theory. It requires that *compensated* increases in the price of a good (i.e., price increases that are offset by compensating transfers) weakly raise demand for all other goods.

Definition 5 (Net Substitutability). Suppose that agent j demands at most one unit of each good. A utility function U^j is a *net substitutes utility function* if for all utility levels u , price vectors \mathbf{p}_I , and $\lambda > 0$, whenever $D_H^j(\mathbf{p}_I; u) = \{\mathbf{x}_I\}$ and $D_H^j(\mathbf{p}_I + \lambda \mathbf{e}^i; u) = \{\mathbf{x}'_I\}$, we have that $x'_k \geq x_k$ for all goods $k \neq i$.

For quasilinear utility functions, net substitutability coincides with (gross) substitutability. More generally, net substitutability can be expressed as a condition on Hicksian valuations.

Remark 1. By Lemma 1, if an agent demands at most one unit of each good, then she has a net substitutes utility function if and only if her Hicksian valuations at all utility levels are substitutes valuations.

We can apply Fact 2 and Remark 1 to construct large classes of net substitutes preferences with income effects from families of substitutes valuations. There are several rich families of substitutes valuations, including endowed assignment valuations (Hatfield and Milgrom, 2005) and matroid-based valuations (Ostrovsky and Paes Leme, 2015). This leads to a large class of quasilogarithmic net substitutes utility functions.

Example 7 (Example 2 continued). A quasilogarithmic utility function U^j is a net substitutes utility function if and only if the quasivaluation V_Q^j is a substitutes valuation.²⁵

More generally, in light of Fact 2 and Remark 1, each family of substitutes valuations leads to a class of net substitutes utility functions with income effects consisting of the utility functions whose Hicksian valuations all belong to the family. These classes are defined by conditions on substitution effects and do not restrict income effects. By contrast, gross substitutability places substantial restrictions on the form of income effects.²⁶

To understand the difference between gross and net substitutability, we compare the conditions in a setting in which agents have unit demand for goods.

Example 8 (Example 5 continued). Consider an agent, Martine, who owns a house i_1 and is considering selling it to purchase (at most) one of houses i_2 and i_3 . If Martine

²⁵Indeed, recall that Example 4 tells us that agent j 's Hicksian valuation at each utility level is a positive linear transformation of V_Q^j . The conclusion follows by Remark 1.

²⁶See Remark E.1 in the Supplemental Material of Fleiner et al. (2019).

experiences income effects, then her choice between i_2 and i_3 generally depends on the price she is able to procure for her house i_1 . For example, if i_3 is a more luxurious house than i_2 , then Martine may only demand i_3 if the value of her endowment is sufficiently large—i.e., if the price of her house i_1 is sufficiently high. As a result, when Martine is endowed with i_1 , she does not generally have gross substitutes preferences: increases in the price of i_1 can lower Martine’s demand for i_2 . That is, Martine can regard i_2 as a gross complement for i_1 . In contrast, Martine has net substitutes preferences—no compensated increase in the price of i_1 could make Martine stop demanding i_2 —a condition that holds generally in the housing market economy.²⁷ Note also that, unlike net substitutability, gross substitutability generally depends on endowments: if Martine were not endowed a house, she would have gross substitutes preferences (Kaneko, 1982, 1983; Demange and Gale, 1985).

While Example 8 shows that net substitutability does not imply gross substitutability, it turns out that gross substitutability implies net substitutability.

Proposition 1. *If agent j demands at most one unit of each good and there exists an endowment \mathbf{w}_I of goods at which U^j is a gross substitutes utility function, then U^j is net substitutes utility function.*

Proposition 1 and Example 8 show that gross substitutability (at any one endowment of goods) implies net substitutability but places additional restrictions on income effects. Nevertheless, the restrictions on substitution effects alone, entailed by net substitutability, are sufficient for the existence of competitive equilibrium.

Theorem 2. *If all agents demand at most one unit of each good and have net substitutes utility functions, then competitive equilibria exist for all endowment allocations.*

Theorem 2 is an immediate consequence of the Equilibrium Existence Duality and the existence of competitive equilibria in transferable utility economies under substitutability.

Proof. Remark 1 implies that the agents’ Hicksian valuations at all utility levels are substitutes valuations. Hence, Fact 3 implies that competitive equilibria exist in the Hicksian economies for all profiles of utility levels if an endowment allocation exists. The theorem follows by the “if” direction of Theorem 1. \square

As gross substitutability implies net substitutability (Proposition 1), the existence of competitive equilibrium under gross substitutability is a special case of Theorem 2. But Theorem 2 is more general: as Example 8 shows, net substitutability allows for forms of gross complementarities between goods, in addition to gross substitutability. The following example illustrates how the distinction between gross substitutability and

²⁷Danilov et al. (2001, Example 2) also showed the connection between Quinzii’s (1984) housing market economy and a substitutability condition, but formulated their discussion in terms of the shape of the convex hull at domains at which demand is multi-valued instead of net substitutability. Their discussion is equivalent to ours by Corollary 5 in Danilov, Koshevoy, and Lang (2003) and Remark 1.

net substitutability relates to the existence of competitive equilibrium when agents can demand multiple goods.

Example 9 (Gross Substitutability versus Net Substitutability and the Existence of Competitive Equilibrium). There are two goods and the total endowment is $\mathbf{y}_I = (1, 1)$. There are two agents, which we call j and k , and j 's feasible set of consumption bundles of goods is $X_I^j = \{0, 1\}^2$.

We consider the price vectors $\mathbf{p}_I = (2, 2)$ and $\mathbf{p}'_I = (4, 2)$ and consider two examples in which agent j 's Marshallian demand changes from $(1, 1)$ to $(0, 0)$ as prices change from \mathbf{p}_I to \mathbf{p}'_I —a gross complementarity. But the consequences for the existence of competitive equilibrium are different across the two cases. In Case (a), the gross complementarity reflects a net complementarity for j , and competitive equilibrium may not exist if k sees goods as net substitutes. In Case (b), the gross complementarity reflects only an income effect for j , as in Example 8, so competitive equilibrium is guaranteed to exist if k sees goods as net substitutes.

(a) Suppose that j has a quasilinear utility function with valuation given by

$$V^j(\mathbf{x}_I) = \begin{cases} 0 & \text{if } \mathbf{x}_I = (0, 0), (0, 1), (1, 0) \\ 5 & \text{if } \mathbf{x}_I = (1, 1). \end{cases}$$

Here, V^j is not a substitutes valuation because $D^j(\mathbf{p}_I) = \{(1, 1)\}$ while $D^j(\mathbf{p}'_I) = \{(0, 0)\}$: i.e., increasing the price of the first good can lower j 's demand for the second good. If $X_I^k = \{(0, 0), (0, 1), (1, 0)\}$ and agent k has a quasilinear utility function with a substitutes valuation given by

$$(5) \quad V^k(\mathbf{x}_I) = \begin{cases} 0 & \text{if } \mathbf{x}_I = (0, 0) \\ 4 & \text{if } \mathbf{x}_I = (1, 0) \\ 3 & \text{if } \mathbf{x}_I = (0, 1), \end{cases}$$

then no competitive equilibrium exists.²⁸

(b) Suppose instead that U^j is quasilogarithmic (as defined in Example 2) with quasi-valuation given by

$$V_Q^j(\mathbf{x}_I) = \begin{cases} -11 & \text{if } \mathbf{x}_I = (0, 0) \\ -7 & \text{if } \mathbf{x}_I = (0, 1) \\ -4 & \text{if } \mathbf{x}_I = (1, 0) \\ -1 & \text{if } \mathbf{x}_I = (1, 1). \end{cases}$$

²⁸The existence of a feasible set of bundles of goods and a substitutes valuation for k for which no competitive equilibrium exists follows from Fact 4. To check that V^k is an example of such a valuation, suppose, for sake of deriving a contradiction, that $(\mathbf{x}_I^j, \mathbf{x}_I^k)$ is the allocation of goods in a competitive equilibrium. The First Welfare Theorem implies that $\mathbf{x}_I^j = (1, 1)$ and that $\mathbf{x}_I^k = (0, 0)$. But for agent j to demand $(1, 1)$, the equilibrium prices would have to sum to at most 5, while for agent k to demand $(0, 0)$, the equilibrium prices would both have to be at least 3—a contradiction. Hence, we can conclude that no competitive equilibrium exists.

At the endowment $\mathbf{w}_I^j = (0, 1)$ of goods, U^j is not a gross substitutes utility function as, letting $w_0^j = 3$, we have that $D_M^j(\mathbf{p}_I, \mathbf{w}^j) = \{(1, 1)\}$ while $D_M^j(\mathbf{p}'_I, \mathbf{w}^j) = \{(0, 0)\}$.²⁹ That is, increasing the price of the first good can lower j 's Marshallian demand for the second good. By contrast, as V_Q^j is a substitutes valuation, Example 7 implies that U^j is a net substitutes utility function: the gross complementarity is entirely due to an income effect. For example, at the utility level

$$u = \max_{\mathbf{x} \in X^j | \mathbf{p}' \cdot \mathbf{x} \leq \mathbf{p}' \cdot \mathbf{w}^j} U^j(\mathbf{x}) = \log \frac{5}{11},$$

we have that $D_H^j(\mathbf{p}_I; u) = \{(1, 0)\}$ and that $D_H^j(\mathbf{p}'_I; u) = \{(0, 0)\}$,³⁰ so the decrease in the Marshallian demand for the second good as prices change from \mathbf{p}_I to \mathbf{p}'_I at the endowment \mathbf{w}^j reflects an income effect. By Theorem 2, competitive equilibrium exists whenever k has a net substitutes utility function. For example, if k has a quasilinear utility function with a substitutes valuation given by Equation (5), then for the endowment allocation defined by $\mathbf{w}_I^j = (0, 1)$, $\mathbf{w}_I^k = (1, 0)$, and $w_0^j = w_0^k = 3$, the price vector $(3, 2)$ and the allocation of goods defined by $\mathbf{x}_I^j = (1, 0)$ and $\mathbf{x}_I^k = (0, 1)$ comprise a competitive equilibrium.³¹

In Case (b), agent j has net substitutes preferences—leading to the guaranteed existence of competitive equilibrium when agent k has net substitutes preferences. By contrast, in Case (a), agent j does not have net substitutes preferences—and competitive equilibrium may not exist when k has net substitutes preferences.

In general, net substitutability forms a maximal domain for the existence of competitive equilibrium. Specifically, if an agent does not have net substitutes preferences, then competitive equilibrium may not exist when the other agents have substitutes quasilinear preferences.

Proposition 2. *Suppose that $y_i = 1$ for all goods i . If $|J| \geq 2$, agent j demands at most one unit of each good, and U^j is not a net substitutes utility function, then there exist*

²⁹To show this, note that $w_0^j - \mathbf{p}'_I \cdot ((1, 1) - \mathbf{w}_I^j) = -1$, so it would violate j 's budget constraint to demand $(1, 1)$ at the price vector \mathbf{p}'_I . For the other bundles, note that

\mathbf{x}_I	$(0, 0)$	$(0, 1)$	$(1, 0)$	$(1, 1)$
$U^j(w_0^j - \mathbf{p}_I \cdot (\mathbf{x}_I - \mathbf{w}_I^j), \mathbf{x}_I)$	$\log \frac{5}{11}$	$\log \frac{3}{7}$	$\log \frac{3}{4}$	$\log 1$
$U^j(w_0^j - \mathbf{p}'_I \cdot (\mathbf{x}_I - \mathbf{w}_I^j), \mathbf{x}_I)$	$\log \frac{5}{11}$	$\log \frac{3}{7}$	$\log \frac{1}{4}$	undef.

so $D_M^j(\mathbf{p}_I, \mathbf{w}^j) = \{(1, 1)\}$ and $D_M^j(\mathbf{p}'_I, \mathbf{w}^j) = \{(0, 0)\}$.

³⁰The expressions for $D_H^j(\mathbf{p}_I; u)$ and $D_H^j(\mathbf{p}'_I; u)$ hold because agent j 's Hicksian valuation at utility level u is $\frac{5}{11}$ times the quasivaluation V_Q^j (by Example 4).

³¹To show this, let $\hat{\mathbf{p}}_I = (3, 2)$. It is clear that $(0, 1) \in D^k(\hat{\mathbf{p}}_I)$. It remains to show that $(1, 0) \in D_M^j(\hat{\mathbf{p}}_I, \mathbf{w}^j)$. Note that $w_0^j - \hat{\mathbf{p}}_I \cdot ((1, 1) - \mathbf{w}_I^j) = 0$, so it would violate j 's budget constraint to demand $(1, 1)$ at the price vector $\hat{\mathbf{p}}_I$. For the other bundles, note that

$$U^j(w_0^j - \hat{\mathbf{p}}_I \cdot (\mathbf{x}_I - \mathbf{w}_I^j), \mathbf{x}_I) = \begin{cases} \log \frac{5}{11} & \text{if } \mathbf{x}_I = (0, 0) \\ \log \frac{3}{7} & \text{if } \mathbf{x}_I = (0, 1) \\ \log \frac{1}{2} & \text{if } \mathbf{x}_I = (1, 0), \end{cases}$$

so $D_M^j(\hat{\mathbf{p}}_I, \mathbf{w}^j) = \{(1, 0)\}$.

sets $X_I^k \subseteq \{0, 1\}^I$ of feasible bundles and substitutes valuations $V^k : X_I^k \rightarrow \mathbb{R}$ for agents $k \neq j$, and an endowment allocation for which no competitive equilibrium exists.

Proposition 2 is an immediate consequence of the Equilibrium Existence Duality and the fact that substitutability defines a maximal domain for the existence of competitive equilibrium with transferable utility.

Proof. By Remark 1, there exists a utility level u at which agent j 's Hicksian valuation $V_H^j(\cdot; u)$ is not a substitutes valuation. Fact 4 implies that there exist feasible sets $X_I^k \subseteq \{0, 1\}^I$ and substitutes valuations V^k for agents $k \neq j$, for which an endowment allocation exists but no competitive equilibrium would exist with transferable utility if agent j 's valuation were $V_H^j(\cdot; u)$. With those sets X_I^k of feasible bundles and valuations V^k for agents $k \neq j$, the “only if” direction of Theorem 1 implies that there exists an endowment allocation for which no competitive equilibrium exists. \square

Proposition 2 entails that any domain of preferences that contains all substitutes quasilinear preferences and guarantees the existence of competitive equilibrium must lie within the domain of net substitutes preferences. Therefore, Proposition 2 and Theorem 2 suggest that net substitutability is the most general way to incorporate income effects into a substitutability condition to ensure the existence of competitive equilibrium.

By contrast, the relationship between the nonexistence of competitive equilibrium and failures of gross substitutability depends on why gross substitutability fails. Gross substitutability can fail due to substitution effects that reflect net complementarities, as in Example 9(a), or due to income effects, as in Example 9(b). If the failure of gross substitutability reflects a net complementarity, then Proposition 2 tells us that competitive equilibrium may not exist if the other agents have substitutes quasilinear preferences, as in Example 9(a). On the other hand, the failure of gross substitutability is only due to income effects, then Theorem 2 tells us that competitive equilibrium exists if the other agents have net substitutes preferences (e.g., substitutes quasilinear preferences), as in Example 9(b).

5. DEMAND TYPES AND THE UNIMODULARITY THEOREM

In this section, we characterize exactly what conditions on patterns of substitution effects guarantee the existence of competitive equilibrium. Specifically, we consider Baldwin and Klemperer's (2019) classification of valuations into “demand types” based on sets of vectors that summarize the possible ways in which demand can change in response to a small generic price change. We first review the definition of demand types from Baldwin and Klemperer (2019). We then extend the concept of demand types to settings with income effects, and develop a version of the Baldwin and Klemperer's (2019) Unimodularity Theorem that allows for income effects and characterizes which demand types guarantee the existence of competitive equilibrium (see also Danilov et al. (2001)). A

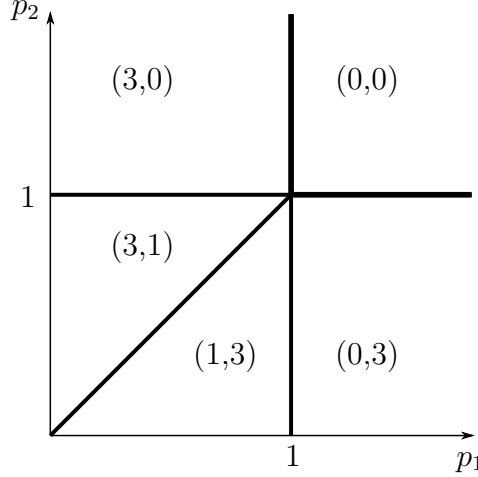


FIGURE 1. Depiction of Agent j 's Demand in Example 10. The labels indicate demand in the regions of price vectors at which demand is single-valued.

special case of the Unimodularity Theorem with Income Effects extends Theorem 2 to settings in which agents can demand multiple units of some goods.

5.1. Demand Types and the Unimodularity Theorem with Transferable Utility. We first review the concept of demand types for quasilinear settings, as developed by Baldwin and Klemperer (2019).

An integer vector is *primitive* if the greatest common divisor of its components is 1. By focusing on the directions of demand changes, we can restrict to primitive demand change vectors. A *demand type vector set* is a set $\mathcal{D} \subseteq \mathbb{Z}^I$ of primitive integer vectors such that if $\mathbf{d} \in \mathcal{D}$ then $-\mathbf{d} \in \mathcal{D}$.

Definition 6 (Demand Types for Valuations). Let V^j be a valuation.

- (a) A bundle \mathbf{x}_I is *uniquely demanded by agent j* if there exists a price vector \mathbf{p}_I such that $D^j(\mathbf{p}_I) = \{\mathbf{x}_I\}$.
- (b) A pair $\{\mathbf{x}_I, \mathbf{x}'_I\}$ of uniquely demanded bundles are *adjacently demanded by agent j* if there exists a price vector \mathbf{p}_I such that $D^j(\mathbf{p}_I)$ contains \mathbf{x}_I and \mathbf{x}'_I but no other bundle that is uniquely demanded by agent j .
- (c) If \mathcal{D} is a demand type vector set, then V^j is *of demand type \mathcal{D}* if for all pairs $\{\mathbf{x}_I, \mathbf{x}'_I\}$ that are adjacently demanded by agent j , the difference $\mathbf{x}'_I - \mathbf{x}_I$ is a multiple of an element of \mathcal{D} .³²

For intuition, suppose that a small price change causes a change in demand. Then, generically, demand changes between adjacently demanded bundles. Thus, the demand type vectors represent the possible directions of changes in demand in response to small generic price changes (see Proposition 3.3 in Baldwin and Klemperer (2019) for a formal statement). To illustrate Definition 6, we consider an example.

³²Definition 6(c) coincides with Definition 3.1 in Baldwin and Klemperer (2019) by Proposition 2.20 in Baldwin and Klemperer (2019).

Example 10. Suppose that there are two goods and let

$$X_I^j = \{0, 1, 2, 3\}^2 \setminus \{(2, 3), (3, 2), (3, 3)\}.$$

Consider the valuation defined by $V^j(\mathbf{x}_I) = x_1 + x_2$. As Figure 1 illustrates, the uniquely demanded bundles are $(0, 0)$, $(0, 3)$, $(1, 3)$, $(3, 0)$, and $(3, 1)$.

When $1 = p_1 < p_2$, agent j 's demand is $D^j(\mathbf{p}_I) = \{(0, 0), (1, 0), (2, 0), (3, 0)\}$. Hence, as the bundles $(1, 0)$ and $(2, 0)$ are not uniquely demanded, the bundles $(0, 0)$ and $(3, 0)$ are adjacently demanded. As a result, for V^j to be of demand type \mathcal{D} , the set \mathcal{D} must contain the vector $(1, 0)$, which is the primitive integer vector proportional to the demand change $(3, 0) - (0, 0) = (3, 0)$. Similarly, the bundles $(0, 0)$ and $(0, 3)$ are adjacently demanded, and any demand type vector set \mathcal{D} such that V^j is of demand type \mathcal{D} must contain the vector $(0, 1)$.

When $p_1 < p_2 = 1$, demand is $D^j(\mathbf{p}_I) = \{(3, 0), (3, 1)\}$. Hence, the bundles $(3, 0)$ and $(3, 1)$ are adjacently demanded. Similarly, the bundles $(0, 3)$ and $(1, 3)$ are adjacently demanded. These facts respectively imply, again, that $(0, 1)$ and $(1, 0)$ are in any demand type vector set \mathcal{D} such that V^j is of demand type \mathcal{D} .

Last, when $p_1 = p_2 < 1$, agent j 's demand is $D^j(\mathbf{p}_I) = \{(1, 3), (2, 2), (3, 1)\}$. Hence, as the bundle $(2, 2)$ is not uniquely demanded, the bundles $(1, 3)$ and $(3, 1)$ are adjacently demanded. As a result, for V^j to be of demand type \mathcal{D} , the set \mathcal{D} must contain the vector $(1, -1)$, which is the primitive integer vector proportional to the demand change $(3, 1) - (1, 3) = (2, -2)$.

By contrast, the bundles $(0, 0)$ and $(3, 1)$ are not adjacently demanded: the only price vector at which agent j demands them both is $\mathbf{p}_I = (1, 1)$, but $D^j(1, 1)$ also contains the uniquely demanded bundles $(0, 3)$, $(1, 3)$, and $(3, 0)$. Similarly, the bundles $(0, 0)$ and $(1, 3)$ are not adjacently demanded. Hence,

$$\mathcal{D} = \pm \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

is the minimal demand type vector set \mathcal{D} such that V^j is of demand type \mathcal{D} .

Consider any valuation of the same demand type \mathcal{D} as in Example 10, and a change in price from \mathbf{p}_I to $\mathbf{p}'_I = \mathbf{p}_I + \lambda \mathbf{e}^1$ for some $\lambda > 0$. For generic choices of \mathbf{p}_I and λ , the demand at any price on the straight line from \mathbf{p}_I to \mathbf{p}'_I either is unique, or demonstrates the adjacency of two bundles uniquely demanded at prices on this line. The change in demand between such bundles must therefore be a multiple of an element of \mathcal{D} (by Definition 6). Moreover, since only the price of good 1 is changing and that price is increasing, the law of demand entails that demand for good 1 must strictly decrease upon any change in demand.³³ Thus, the change in demand between the two consecutive uniquely demanded bundles must be a positive multiple of either $(-1, 0)$ or $(-1, 1)$.

³³As there are no income effects here, the compensated law of demand (see, e.g., Proposition 3.E.4 in Mas-Colell et al. (1995)) reduces to the law of demand.

Therefore, demand for good 2 must (weakly) increase, reflecting substitutability between the goods. This two-good example is a special case of an important class of demand types.

Example 11 (The Strong Substitutes Demand Type). The *strong substitutes demand type vector set* consists of all vectors in \mathbb{Z}^I with at most one +1 component, at most one −1 component, and no other nonzero components. As illustrated in Example 10, this demand type vector set captures one-to-one substitution between goods through demand type vectors with one component of 1 and one component of −1. Furthermore, if an agent k demands at most one unit of each good, then V^k is a substitutes valuation if and only if it is of the strong substitutes demand type (see Theorems 2.1 and 2.4 in Fujishige and Yang (2003)).

In settings in which agents can demand multiple units of each good, a form of concavity is needed to ensure the existence of competitive equilibrium. A valuation is concave if, under that valuation, each bundle of goods that is a convex combination of feasible bundles of goods is demanded at some price vector. For the formal definition, we let $\text{Conv}(T)$ denote the *convex hull* of a set $T \subseteq \mathbb{R}^I$.

Definition 7 (Concavity). A valuation V^j is *concave* if for each bundle $\mathbf{x}_I \in \text{Conv}(X_I^j) \cap \mathbb{Z}^n$, there exists a price vector \mathbf{p}_I such that $\mathbf{x}_I \in D^j(\mathbf{p}_I)$.

In Section 4.1, we discussed that substitutability guarantees the existence of competitive equilibrium in transferable utility economies when agents demand at most one unit of each good. Generalizing that result, Baldwin and Klemperer (2019) identified a necessary and sufficient condition for the concave valuations of a demand type to form a domain for the guaranteed existence of competitive equilibrium.

Definition 8 (Unimodularity). A set of vectors in \mathbb{Z}^I is *unimodular* if every linearly independent subset can be extended to be a basis for \mathbb{R}^I , of integer vectors, such that any square matrix whose columns are these vectors has determinant ± 1 .

For example, the demand type vector set in Example 10 is unimodular, while the demand type vector set

$$(6) \quad \pm \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

is not unimodular, because

$$\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2.$$

The demand type vector set in (6) represents that the two goods can be substitutable or complementary for agents—a possibility that can cause competitive equilibrium to fail to exist, as in Example 9(a). Baldwin and Klemperer (2019) showed that the unimodularity of a demand type vector set is precisely the condition for the corresponding demand type to guarantee the existence of competitive equilibrium.

Fact 5 (Unimodularity Theorem with Transferable Utility). *Let \mathcal{D} be a demand type vector set. Competitive equilibria exist for all finite sets J of agents with concave valuations of demand type \mathcal{D} and for all total endowments for which endowment allocations exist if and only if \mathcal{D} is unimodular.*³⁴

Danilov et al. (2001) used conditions on the ranges of agents' demand correspondences to describe classes of concave valuations, which correspond to the concave valuations of Baldwin and Klemperer's (2019) unimodular demand types;³⁵ they formulated a version of the “if” direction of Fact 5 with those conditions.³⁶

As Poincaré (1900) showed, the strong substitutes demand type vector set is unimodular. Therefore, in light of Example 11, the existence of competitive equilibrium in transferable utility economies in which agents demand at most one unit of each good and have substitutes valuations (Fact 3) is a special case of Fact 5. Moreover, Fact 5 is strictly more general: as Baldwin and Klemperer (2019) showed, there are unimodular demand type vector sets for which the existence of competitive equilibrium cannot be deduced from the corresponding result for strong substitutes by applying a change of basis to the space of bundles of goods.³⁷ To illustrate the additional generality, we discuss an example of such a demand type.³⁸

Example 12. There are five goods. Consider the demand type vector set

$$\mathcal{D} = \pm \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Intuitively, this demand type vector set allows for independent changes in the demand for each good (through the first five vectors), as well as for substitution from a good to the bundle consisting of its two neighbors if the goods are arranged in a circle (through the last five vectors). This demand type vector set is unimodular, and cannot be obtained from the strong substitutes demand type vector set by a change of basis of the space of integer bundles of goods (see, e.g., Section 19.4 of Schrijver (1998)).

³⁴The “if” direction of Fact 5 is a case of the “if” direction of Theorem 4.3 in Baldwin and Klemperer (2019). The “only if” direction of Fact 5, which we prove in Appendix D, is a mild strengthening of the “only if” direction of Theorem 4.3 in Baldwin and Klemperer (2019) that applies in exchange economies.

³⁵To understand the correspondence, let \mathcal{D} be a unimodular demand type vector set. In the terminology of Danilov et al. (2001), a valuation V^j is $\mathcal{D}(\mathcal{P}t(\mathcal{D}, \mathbb{Z}))$ -concave if, for each price vector \mathbf{p}_I , we have that $D^j(\mathbf{p}_I) = \text{Conv}(D^j(\mathbf{p}_I)) \cap \mathbb{Z}^I$ and each edge of $\text{Conv}(D^j(\mathbf{p}_I))$ is parallel to an element of \mathcal{D} (see Definition 4 and pages 264–265 in Danilov et al. (2001)). It follows from Lemma 2.11 and Proposition 2.16 in Baldwin and Klemperer (2019) that a valuation is $\mathcal{D}(\mathcal{P}t(\mathcal{D}, \mathbb{Z}))$ -concave if and only if it is concave and of demand type \mathcal{D} .

³⁶See Definition 4, Theorem 3, and pages 264–265 in Danilov et al. (2001).

³⁷By contrast, the existence results of Sun and Yang (2006) and Teytelboym (2014) can be deduced from Fact 3 applying an appropriate change of basis. Those results are also special cases of Fact 5.

³⁸Section 6.1 in Baldwin and Klemperer (2019) provides another example that includes only complements valuations.

Moreover, the demand types defined by maximal, unimodular demand type vector sets turn out to define maximal domains for the existence of competitive equilibrium in settings with transferable utility. Here, we say that a unimodular demand type vector set is *maximal* if it is not strictly contained in another unimodular demand type vector set.

Fact 6. *Let \mathcal{D} be a maximal unimodular demand type vector set. If $|J| \geq 2$ and V^j is non-concave or not of demand type \mathcal{D} , then there exist sets X_I^k of feasible bundles and concave valuations $V^k : X_I^k \rightarrow \mathbb{R}$ of demand type \mathcal{D} for agents $k \neq j$, as well as a total endowment, for which there exists an endowment allocation but no competitive equilibrium.*³⁹

While Fact 5 shows that there exist valuations in each non-unimodular demand type for which competitive equilibrium does not exist, Fact 6 shows that for *every* valuation outside a maximal unimodular demand type, there exist concave valuations within the demand type that lead to non-existence. Hence, the necessity direction of Fact 5, together with Fact 6, provide complementary perspectives on the way in which competitive equilibrium can fail to exist outside the context of unimodular demand types.

5.2. Demand Types and the Unimodularity Theorem with Income Effects. We now use Fact 2 to extend the demand types framework to settings with income effects.

Definition 9 (Demand Types with Income Effects). An agent's preferences are *of demand type \mathcal{D}* if her Hicksian valuations at all utility levels are of demand type \mathcal{D} .

Lemma 1 leads to an economic interpretation of Definition 9: a utility function is of demand type \mathcal{D} if \mathcal{D} summarizes the possible ways in which Hicksian demand can change in response to a small generic price change. In particular, Definition 9 extends the concept of demand types to settings with income effects by placing conditions on substitution effects. Indeed, Definition 9 considers only the properties of Hicksian valuations at each utility level (which, by Lemma 1, reflect substitution effects), and not how an agent's Hicksian valuations vary with her utility level (which, by Fact 1 and Lemma 1, reflects income effects).

Danilov et al. (2001) translated their conditions on the ranges of agents' demand correspondences from quasilinear settings to settings with income effects by using Fact 2 in an analogous manner (see Assumption 3' in Danilov et al. (2001)). However, the economic interpretation in terms of substitution effects that Lemma 1 leads to was not clear from Danilov et al.'s (2001) formulation.

As with the case of transferable utility, a concavity condition is needed to ensure the existence of competitive equilibrium. With income effects, the relevant condition is a

³⁹Fact 6 is related to Proposition 6.10 in Baldwin and Klemperer (2014), which connects failures of unimodularity to the non-existence of competitive equilibrium in specific economies. We supply a proof of Fact 6 in Appendix D.

version of the quasiconcavity condition from classical demand theory for settings with indivisible goods. We define quasiconcavity based on concavity and duality.⁴⁰

Definition 10 (Quasiconcavity). An agent's utility function is *quasiconcave* if her Hicksian valuations at all utility levels are concave.

As with the case of transferable utility, unimodularity is a necessary and sufficient condition for the existence of competitive equilibrium to be guaranteed for all quasiconcave preferences of a demand type when income effects are present.

Theorem 3 (Unimodularity Theorem with Income Effects). *Let \mathcal{D} be a demand type vector set. Competitive equilibria exist for all finite sets J of agents with quasiconcave utility functions of demand type \mathcal{D} , for all total endowments, and for all endowment allocations if and only if \mathcal{D} is unimodular.*

The “only if” direction of Theorem 3 is a special case of the Unimodularity Theorem with Transferable Utility (Fact 5). The “if” direction of Theorem 3 is an immediate consequence of the Equilibrium Existence Duality and Fact 5.

Proof of the “if” direction of Theorem 3. Consider a finite set J of agents with quasiconcave preferences of demand type \mathcal{D} and a total endowment for which an endowment allocation exists. By definition, the agents' Hicksian valuations at all utility levels are concave and of demand type \mathcal{D} . Hence, competitive equilibria exist in the Hicksian economies for all profiles of utility levels by the “if” direction of Fact 5. By the “if” direction of Theorem 1, competitive equilibria must therefore exist in the original economy for all endowment allocations. \square

Danilov et al. (2001) proved a version of the “if” direction of Theorem 3 under the assumptions that utility functions are monotone in goods, that consumption of goods is nonnegative, and that the total endowment is strictly positive (see Theorems 2 and 4 in Danilov et al. (2001)).⁴¹ Note that they formulated their result in terms of Fact 2 and a condition on the ranges of demand correspondences (see their Assumption 3') instead of in terms of unimodular demand types.

Danilov et al.'s (2001) approach was to show the existence of competitive equilibrium in a convexified economy and that, under unimodularity, competitive equilibria in the convexified economy give rise to competitive equilibria in the original economy. In contrast, our approach of using the Equilibrium Existence Duality illuminates the role of substitution effects in ensuring the existence of competitive equilibrium. Moreover, it yields a maximal domain result for unimodular demand types with income effects.

⁴⁰It is equivalent to define quasiconcavity in terms of the convexity of the upper contour sets, but Definition 10 is more immediately applicable for us.

⁴¹Danilov et al.'s (2001) existence result is not formally a special case of ours because they allowed for unbounded sets X_I^j of feasible bundles of goods.

Proposition 3. *Let \mathcal{D} be a maximal unimodular demand type vector set. If $|J| \geq 2$ and U^j is not quasiconcave or not of demand type \mathcal{D} , then there exist sets X_I^k of feasible bundles and concave valuations $V^k : X_I^k \rightarrow \mathbb{R}$ of demand type \mathcal{D} for agents $k \neq j$, as well as a total endowment and an endowment allocation, for which no competitive equilibrium exists.*

Proposition 3 is an immediate consequence of the Equilibrium Existence Duality and the maximal domain result for unimodular demand types under the transferability of utility.

Proof. By definition, there exists a utility level u at which agent j 's Hicksian valuation $V_H^j(\cdot; u)$ is non-concave or not of demand type \mathcal{D} . In either case, Fact 6 implies that there exist sets X_I^k of feasible bundles and concave valuations $V^k : X_I^k \rightarrow \mathbb{R}$ of demand type \mathcal{D} for agents $k \neq j$, and a total endowment for which an endowment allocation exists but no competitive equilibrium would exist with transferable utility if agent j 's valuation were $V_H^j(\cdot; u)$. With those sets X_I^k of feasible bundles and valuations V^k for agents $k \neq j$ and that total endowment, the “only if” direction of Theorem 1 implies that there exists an endowment allocation for which no competitive equilibrium exists. \square

Intuitively, Proposition 3 and Theorem 3 suggest that Definition 9 is the most general way to incorporate income effects into unimodular demand types from the quasilinear setting and ensure the existence of competitive equilibrium. Indeed, Proposition 3 entails that any domain of preferences that contains all concave quasilinear preferences of a maximal, unimodular demand type and guarantees the existence of competitive equilibrium must lie within the corresponding demand type constructed in Definition 9.

5.3. The Strong Substitutes Demand Type and Net Substitutability with Multiple Units. We now use the case of Theorem 3 for the strong substitutes demand type to extend Theorem 2 to settings in which agents can demand multiple units of some goods. In such settings, if utility is transferable, the substitutability condition needed to ensure the existence of competitive equilibrium is *strong substitutability*—the condition requiring that agents see units of goods as substitutes (Milgrom and Strulovici, 2009). As Shioura and Tamura (2015) and Baldwin and Klemperer (2019) showed, there is a close relationship between strong (net) substitutability and the strong substitutes demand type.⁴²

Definition 11 (Strong Substitutability). (a) A valuation is a *strong substitutes valuation* if it corresponds to a substitutes valuation when each unit of each good is regarded as a separate good.

⁴²Requiring that different goods, rather than different units of goods, be substitutes leads to a condition called *ordinary substitutability*. However, ordinary substitutability does not ensure the existence of competitive equilibrium when agents can demand multiple units of some goods (Danilov, Koshevoy, and Lang, 2003; Milgrom and Strulovici, 2009; Baldwin and Klemperer, 2019). Ordinary substitutability in turn corresponds to an “ordinary substitutes” demand type (see Definitions 3.4 and 3.5 and Proposition 3.6 in Baldwin and Klemperer (2019)).

- (b) A utility function is a *strong net substitutes utility function* if it corresponds to a net substitutes utility function when each unit of each good is regarded as a separate good.

Fact 7. *A valuation (resp. utility function) is a strong (net) substitutes valuation (resp. utility function) if and only if it is concave (resp. quasiconcave) and of the strong substitutes demand type.*^{43,44}

As the strong substitutes demand type vector set is unimodular (Poincaré, 1900), the existence of competitive equilibrium under strong net substitutability is therefore a special case of the Unimodularity Theorem with Income Effects.

Corollary 1. *If all agents have strong net substitutes utility functions, then competitive equilibria exist for all endowment allocations.*

Corollary 1 can also be proven directly using the Equilibrium Existence Duality and the existence of competitive equilibrium under strong substitutability in transferable utility economies (Milgrom and Strulovici, 2009; Ikebe et al., 2015). Theorem 2 is the special case of Corollary 1 for settings in which agents demand at most one unit of each good. As there are unimodular demand type vector sets unrelated to the strong substitutes demand type vector set (such as the one in Example 12), Theorem 3 is strictly more general than Corollary 1 (and hence Theorem 2). In particular, Theorem 3 also illustrates that certain patterns of net complementarities can also be compatible with the existence of competitive equilibrium.

As the strong substitutes demand type vector set is maximal as a unimodular demand type vector set (see, e.g., Example 9 in Danilov and Koshevoy (2004)), Proposition 3 yields a maximal domain result for strong net substitutability.

Corollary 2. *If $|J| \geq 2$ and U^j is not a strong net substitutes utility function, then there exist strong substitutes valuations V^k for agents $k \neq j$, as well as a total endowment and an endowment allocation, for which no competitive equilibrium exists.*

6. AUCTION DESIGN

Our work has several implications for auction design. First, our perspective of analyzing preferences by using the expenditure-minimization problem may yield new approaches for extending auction bidding languages to allow for income effects.

Second, our equilibrium existence results suggest that some auctions with competitive equilibrium pricing may work well for indivisible goods even in the presence of financing constraints. One set of examples are Product-Mix Auctions, such as the one implemented

⁴³The quasilinear case of this fact is part of Theorem 4.1(i) in Shioura and Tamura (2015) (see also Proposition 3.10 in Baldwin and Klemperer (2019)). The general case follows from the quasilinear case by Lemma 1 and Remark 1.

⁴⁴In particular, if agent j demands at most one unit of each good, then U^j is a net substitutes utility function if and only if it is of the strong substitutes demand type.

by the Bank of England⁴⁵—these implement competitive equilibrium allocations assuming that the submitted sealed bids represent bidders’ actual preferences, since truth-telling is a reasonable approximation in these auctions when there are sufficiently many bidders.

However, while we have shown that gross complementarities do not lead to the nonexistence of competitive equilibrium, they do create problems for dynamic auctions. When agents see goods as gross substitutes, iteratively increasing the prices of over-demanded goods leads to a competitive equilibrium (Kelso and Crawford, 1982; Fleiner et al., 2019). In contrast, when there are gross complementarities between goods, increases in the price of an over-demanded good can lead to other goods being under-demanded due to an income effect. So, even though competitive equilibrium always exists when agents see goods as (strong) net substitutes, it may not be possible to find a competitive equilibrium using a monotone, dynamic auction. In particular, simple “activity rules” that require bidders to bid on a smaller total number of units of goods as prices increase may result in inefficient outcomes. So, the Product-Mix Auction approach of finding competitive equilibrium based on a single round of sealed bids seems especially useful in the presence of income effects.

7. CONCLUSION

The Equilibrium Existence Duality is a useful tool for analyzing economies with indivisible goods. It is based on the relationship between Marshallian and Hicksian demands, and on an interpretation of Hicksian demand in terms of a quasilinear maximization problem. The Equilibrium Existence Duality shows that competitive equilibrium exists (for all endowment allocations) if and only if competitive equilibrium exists in each of a family of Hicksian economies. An application is that it is net substitutability, not gross substitutability, that is relevant to the existence of equilibrium. And extending the demand types classification of valuations (Baldwin and Klemperer, 2019) allows us to state a Unimodularity Theorem with Income Effects that gives conditions on the patterns of substitution effects that guarantee the existence of competitive equilibrium. In short, with income effects, just as without them, existence does not depend on agents seeing goods as substitutes; rather, substitution effects are fundamental to the existence of competitive equilibrium.

Our results point to a number of potential directions for future work. First, it would be interesting to investigate applications of the Equilibrium Existence Duality to other results on the existence of equilibrium with transferable utility—such as those of Bikhchandani and Mamer (1997), Ma (1998), and Candogan et al. (2015). Second, our results could be used to further develop auction designs that find competitive equilibrium outcomes given the submitted bids, such as Klemperer’s (2008) Product-Mix Auction. More

⁴⁵See Klemperer (2008, 2010, 2018) and Baldwin and Klemperer (in preparation). Iceland planned a Product-Mix Auction for bidders with budget constraints (Klemperer, 2018), but that auction was for a setting with divisible goods.

broadly, our approach may lead to new results about the properties of economies with indivisibilities and income effects.

APPENDIX A. PROOF OF THEOREM 1 AND LEMMA 2

We prove the following result, which combines Theorem 1 and Lemma 2.

Theorem A.1. *Suppose that the total endowment and the sets of feasible bundles are such that an endowment allocation exists. The following are equivalent.*

- (I) *Competitive equilibria exist for all endowment allocations.*
- (II) *For each Pareto-efficient allocation $(\mathbf{x}^j)_{j \in J}$ with $\sum_{j \in J} \mathbf{x}_I^j = \mathbf{y}_I$, there exists a price vector \mathbf{p}_I such that $\mathbf{x}^j \in D_M^j(\mathbf{p}_I, \mathbf{x}^j)$ for all agents j .*
- (III) *Competitive equilibria exist in the Hicksian economies for all profiles of utility levels.*

The remainder of this appendix is devoted to the proof of Theorem A.1.

A.1. Proof of the (I) \implies (II) Implication in Theorem A.1. The proof of this implication is essentially identical to the proof of Theorem 3 in Maskin and Roberts (2008). Consider a Pareto-efficient allocation $(\mathbf{x}^j)_{j \in J}$ with $\sum_{j \in J} \mathbf{x}^j = \mathbf{y}_I$.

Let agent j 's endowment be $\mathbf{w}^j = \mathbf{x}^j$. By Statement (I) in the theorem, there exists a competitive equilibrium, say consisting of the price vector \mathbf{p}_I and the allocation $(\hat{\mathbf{x}}_I^j)_{j \in J}$ of goods. By the definition of competitive equilibrium, we have that $\hat{\mathbf{x}}_I^j \in D_M^j(\mathbf{p}_I, \mathbf{x}_I^j)$ for all agents j . In particular, letting $\hat{x}_0^j = x_0^j - \mathbf{p}_I \cdot (\hat{\mathbf{x}}_I^j - \mathbf{x}_I^j)$ for each agent j , we have that $\sum_{j \in J} \hat{\mathbf{x}}^j = \sum_{j \in J} \mathbf{x}^j$ and that $U^j(\hat{\mathbf{x}}^j) \geq U^j(\mathbf{x}^j)$ for all agents j . As the allocation $(\mathbf{x}^j)_{j \in J}$ is Pareto-efficient, we must have that $U^j(\hat{\mathbf{x}}^j) = U^j(\mathbf{x}^j)$ for all agents j . It follows that $\mathbf{x}_I^j \in D_M^j(\mathbf{p}_I, \mathbf{x}^j)$ for all agents j —as desired.

A.2. Proof of the (II) \implies (III) Implication in Theorem A.1. Let $(u^j)_{j \in J}$ be a profile of utility levels. Consider any allocation $(\mathbf{x}_I^j)_{j \in J} \in \times_{j \in J} X_I^j$ of goods with $\sum_{j \in J} \mathbf{x}_I^j = \mathbf{y}_I$ that minimizes

$$\sum_{j \in J} S^j(\mathbf{x}_I^j; u^j)$$

over all allocations $(\hat{\mathbf{x}}_I^j)_{j \in J} \in \times_{j \in J} X_I^j$ of goods with $\sum_{j \in J} \hat{\mathbf{x}}_I^j = \mathbf{y}_I$. Such an allocation exists because each set X_I^j is finite and an endowment allocation exists. For each agent j , let $x_0^j = S^j(\mathbf{x}_I^j; u^j)$ —so $U^j(\mathbf{x}^j) = u^j$.

Claim A.2. *The allocation $(\mathbf{x}^j)_{j \in J}$ is Pareto-efficient.*

Proof. Consider any allocation $(\hat{\mathbf{x}}^j)_{j \in J} \in \times_{j \in J} X^j$ with $\sum_{j \in J} \hat{\mathbf{x}}_I^j = \mathbf{y}_I$, and $U^j(\hat{\mathbf{x}}^j) \geq U^j(\mathbf{x}^j) = u^j$ for all agents j with strict inequality for some $j = j_1$. As $S^j(\hat{\mathbf{x}}_I^j; \cdot)$ is strictly increasing for each agent j , we must have that

$$\hat{x}_0^j = S^j(\hat{\mathbf{x}}_I^j; U^j(\hat{\mathbf{x}}^j)) \geq S^j(\hat{\mathbf{x}}_I^j; u^j)$$

for all agents j with strict inequality for $j = j_1$. Hence, we must have that

$$\sum_{j \in J} \hat{x}_0^j > \sum_{j \in J} S^j(\hat{\mathbf{x}}_I^j; u^j) \geq \sum_{j \in J} S^j(\mathbf{x}_I^j; u^j) = \sum_{j \in J} x_0^j,$$

where the second inequality follows from the definition of $(\mathbf{x}_I^j)_{j \in J}$, so the allocation $(\mathbf{x}^j)_{j \in J}$ cannot be Pareto-dominated. \square

By Claim A.2 and Statement (II) in the theorem, there exists a price vector \mathbf{p}_I such that $\mathbf{x}_I^j \in D_M^j(\mathbf{p}_I, \mathbf{x}^j)$ for all agents j . Fact 1 implies that $\mathbf{x}_I^j \in D_H^j(\mathbf{p}_I; u^j)$ for all agents j . By Lemma 1, it follows that the price vector \mathbf{p}_I and the allocation $(\mathbf{x}_I^j)_{j \in J}$ of goods comprise a competitive equilibrium in the Hicksian economy for the profile $(u^j)_{j \in J}$ of utility levels.

A.3. Proof of the (III) \implies (I) Implication in Theorem A.1. Let $(\mathbf{w}^j)_{j \in J}$ be an endowment allocation. For each agent j , we define a utility level $u_{\min}^j = U^j(\mathbf{w}^j)$ and let

$$K^j = w_0^j - \min_{\mathbf{x}_I \in X_I^j} S^j(\mathbf{x}_I; u_{\min}^j),$$

which is non-negative by construction. Furthermore, let $K = 1 + \sum_{j \in J} K^j$ and let

$$u_{\max}^j = \max_{\mathbf{x}_I \in X_I^j} U^j(w_0^j + K, \mathbf{x}_I).$$

Given a profile $\mathbf{u} = (u^j)_{j \in J}$ of utility levels, let

$$T(\mathbf{u}) = \left\{ \left(\begin{array}{c} S^j(\mathbf{x}_I^j; u^j) - w_0^j \\ + \mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j) \end{array} \right)_{j \in J} \left| \begin{array}{l} (\mathbf{p}_I, (\mathbf{x}_I^j)_{j \in J}) \text{ is a competitive} \\ \text{equilibrium in the Hicksian economy} \\ \text{for the profile } (u^j)_{j \in J} \text{ of utility levels} \end{array} \right. \right\}$$

denote the set of profiles of net expenditures over all competitive equilibria in the Hicksian economy for the profile $(u^j)_{j \in J}$ of utility levels. As discussed in Section 3, the strategy of the proof is to solve for a profile $\mathbf{u} = (u^j)_{j \in J}$ of utility levels such that $\mathbf{0} \in T(\mathbf{u})$.

We first show that the correspondence $T : \times_{j \in J} [u_{\min}^j, u_{\max}^j] \rightrightarrows \mathbb{R}^J$ is upper hemicontinuous and has compact, convex values. We then apply a topological fixed point argument to show that there exists a profile $\mathbf{u} = (u^j)_{j \in J} \in \times_{j \in J} [u_{\min}^j, u_{\max}^j]$ of utility levels such that $\mathbf{0} \in T(\mathbf{u})$. We conclude the proof by constructing a competitive equilibrium for the endowment allocation $(\mathbf{w}^j)_{j \in J}$ in the original economy from a competitive equilibrium in the Hicksian economy for the profile $(u^j)_{j \in J}$ of utility levels.

Proof of the Regularity Conditions for T . We begin by proving that the correspondence $T : \times_{j \in J} [u_{\min}^j, u_{\max}^j] \rightrightarrows \mathbb{R}^J$ is upper hemicontinuous and has compact, convex values. We actually give explicit bounds for the range of T . Let

$$\overline{M} = \max_{j \in J} \{S^j(\mathbf{w}_I^j; u_{\max}^j) - w_0^j\}$$

and let

$$\underline{M} = \sum_{j \in J} \left(\min_{\mathbf{x}_I \in X_I^j} \{S^j(\mathbf{x}_I; u_{\min}^j)\} - w_0^j \right) - (|J| - 1)\overline{M}.$$

Claim A.3. *The correspondence $T : \times_{j \in J} [u_{\min}^j, u_{\max}^j] \rightrightarrows \mathbb{R}^J$ is upper hemicontinuous and has compact, convex values and range contained in $[\underline{M}, \overline{M}]^J$.*

The proof of Claim A.3 uses the following technical description of T .

Claim A.4. *Let $\mathbf{u} = (u^j)_{j \in J} \in \times_{j \in J} [u_{\min}^j, u_{\max}^j]$ be a profile of utility levels and let $(\mathbf{x}_I^j)_{j \in J} \in \times_{j \in J} X_I^j$ be an allocation of goods with $\sum_{j \in J} \mathbf{x}_I^j = \mathbf{y}_I$. If $(\mathbf{x}_I^j)_{j \in J}$ minimizes*

$$\sum_{j \in J} S^j(\hat{\mathbf{x}}_I^j; u^j)$$

over all allocations $(\hat{\mathbf{x}}_I^j)_{j \in J} \in \times_{j \in J} X_I^j$ of goods with $\sum_{j \in J} \hat{\mathbf{x}}_I^j = \mathbf{y}_I$, then we have that

$$T(\mathbf{u}) = \left\{ (S^j(\mathbf{x}_I^j; u^j) - w_0^j + \mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j))_{j \in J} \mid \mathbf{p}_I \in \mathcal{P} \right\},$$

where

$$\mathcal{P} = \{ \mathbf{p}_I \mid S^j(\mathbf{x}_I^j; u^j) + \mathbf{p}_I \cdot \mathbf{x}_I^j \leq S^j(\mathbf{x}'_I; u^j) + \mathbf{p}_I \cdot \mathbf{x}'_I \text{ for all } j \in J \text{ and } \mathbf{x}'_I \in X_I^j \}.$$

Proof. By construction, we have that

$$\mathcal{P} = \left\{ \mathbf{p}_I \mid \begin{array}{l} (\mathbf{p}_I, (\mathbf{x}^j)_{j \in J}) \text{ is a competitive equilibrium in the} \\ \text{Hicksian economy for the profile } (u^j)_{j \in J} \text{ of utility levels} \end{array} \right\}.$$

A standard lemma regarding competitive equilibria in transferable utility economies shows that in the Hicksian economy for the profile $(u^j)_{j \in J}$ of utility levels, if $(\mathbf{p}_I, (\hat{\mathbf{x}}^j)_{j \in J})$ is a competitive equilibrium, then so is $(\mathbf{p}_I, (\mathbf{x}_I^j)_{j \in J})$.⁴⁶ In this case, we have that

$$S^j(\mathbf{x}_I^j; u^j) + \mathbf{p}_I \cdot \mathbf{x}_I^j = S^j(\hat{\mathbf{x}}_I^j; u^j) + \mathbf{p}_I \cdot \hat{\mathbf{x}}_I^j,$$

and hence that

$$S^j(\mathbf{x}_I^j; u^j) - w_0^j + \mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j) = S^j(\hat{\mathbf{x}}_I^j; u^j) - w_0^j + \mathbf{p}_I \cdot (\hat{\mathbf{x}}_I^j - \mathbf{w}_I^j),$$

for all agents j . The claim follows. \square

Proof of Claim A.3. It suffices to show that T has convex values, range contained in $[\underline{M}, \overline{M}]^J$, and a closed graph.

We first show that $T(\mathbf{u})$ is convex for all $\mathbf{u} \in \times_{j \in J} [u_{\min}^j, u_{\max}^j]$. We use the notation of Claim A.4 to prove this assertion. Note that \mathcal{P} is the set of solutions to a set of linear inequalities, and is hence convex. Claim A.4 implies that $T(\mathbf{u})$ is the set of values of a linear function on \mathcal{P} —so it follows that $T(\mathbf{u})$ is convex as well.

⁴⁶The lemma is due to Shapley (1964, page 3); see also Bikhchandani and Mamer (1997) and Hatfield et al. (2013). Jagadeesan et al. (2020, Lemma 1) proved the lemma in a setting with multiple units that allows for non-monotone valuations.

We next show that $T(\mathbf{u}) \subseteq [\underline{M}, \overline{M}]^J$ holds for all $\mathbf{u} \in \times_{j \in J} [u_{\min}^j, u_{\max}^j]$. We again use the notation of Claim A.4. Let $\mathbf{u} \in \times_{j \in J} [u_{\min}^j, u_{\max}^j]$ and $\mathbf{t} \in T(\mathbf{u})$ be arbitrary. By Claim A.4, there exists $\mathbf{p}_I \in \mathcal{P}$ such that

$$t^j = S^j(\mathbf{x}_I^j; u^j) - w_0^j + \mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j)$$

for all agents j . Note that for all agents j , we must have that

$$t^j \leq S^j(\mathbf{w}_I^j; u^j) - w_0^j \leq S^j(\mathbf{w}_I^j; u_{\max}^j) - w_0^j \leq \overline{M},$$

where the first inequality holds due to the definition of \mathcal{P} , the second inequality holds because $S^j(\mathbf{w}_I^j; \cdot)$ is strictly increasing, and the third inequality holds due to the definition of \overline{M} . Furthermore, as $\sum_{j \in J} \mathbf{x}_I^j = \mathbf{y}_I = \sum_{j \in J} \mathbf{w}_I^j$, we have that

$$\sum_{j \in J} t^j = \sum_{j \in J} (S^j(\mathbf{x}_I^j; u^j) - w_0^j).$$

It follows that

$$\begin{aligned} t^j &= \sum_{k \in J} (S^k(\mathbf{x}_I^k; u^k) - w_0^k) - \sum_{k \in J \setminus \{j\}} t^k \\ &\geq \sum_{k \in J} (S^k(\mathbf{x}_I^k; u_{\min}^k) - w_0^k) - \sum_{k \in J \setminus \{j\}} t^k \\ &\geq \sum_{k \in J} (S^k(\mathbf{x}_I^k; u_{\min}^k) - w_0^k) - (|J| - 1)\overline{M} \\ &\geq \underline{M} \end{aligned}$$

for all agents j , where the first inequality holds because $S^k(\mathbf{x}_I^k; \cdot)$ is increasing for each agent k , the second inequality holds because $t^k \leq \overline{M}$ for all agents k , and the third inequality holds due to the definition of \underline{M} .

Last, we show that T has a closed graph. Our argument uses the following version of Farkas's Lemma.

Fact A.5 (Page 200 of Rockafellar, 1970⁴⁷). *Let L_1, L_2 be disjoint, finite sets and, for each $\ell \in L_1 \cup L_2$, let $\mathbf{v}_I^\ell \in \mathbb{R}^I$ be a vector and let α_ℓ be a scalar. There exist scalars λ_ℓ for $\ell \in L_1 \cup L_2$ with $\lambda_\ell \geq 0$ for $\ell \in L_2$ such that*

$$\sum_{\ell \in L_1 \cup L_2} \lambda_\ell \mathbf{v}_I^\ell = \mathbf{0} \quad \text{and} \quad \sum_{\ell \in L_1 \cup L_2} \lambda_\ell \alpha_\ell < 0$$

if and only if there does not exist a vector $\mathbf{p}_I \in \mathbb{R}^I$ such $\mathbf{v}_I^\ell \cdot \mathbf{p}_I \leq \alpha_\ell$ for all $\ell \in L_1 \cup L_2$ with equality for all $\ell \in L_1$.

Consider a sequence $\mathbf{u}_{(1)}, \mathbf{u}_{(2)}, \dots \in \times_{j \in J} [u_{\min}^j, u_{\max}^j]$ of profiles of utility levels. For each m , let $\mathbf{t}_{(m)} \in T(\mathbf{u}_{(m)})$. Suppose that $\mathbf{u}_{(m)} \rightarrow \mathbf{u}$ and $\mathbf{t}_{(m)} \rightarrow \mathbf{t}$ as $m \rightarrow \infty$. We need to show that $\mathbf{t} \in T(\mathbf{u})$.

⁴⁷Theorem 22.1 in Rockafellar (1970) states the case of Fact A.5 in which $L_1 = \emptyset$. The version of Fact A.5 for $L_1 \neq \emptyset$ is left as an exercise on page 200 of Rockafellar (1970).

As each set X_I^j is finite and an endowment allocation exists, by passing to a subsequence we can assume that there exists an allocation $(\mathbf{x}_I^j)_{j \in J} \in \times_{j \in J} X_I^j$ of goods with $\sum_{j \in J} \mathbf{x}_I^j = \mathbf{y}_I$ that, for each m , minimizes

$$\sum_{j \in J} S^j(\hat{\mathbf{x}}_I^j; u_{(m)}^j)$$

over all allocations $(\hat{\mathbf{x}}_I^j)_{j \in J} \in \times_{j \in J} X_I^j$ of goods with $\sum_{j \in J} \hat{\mathbf{x}}_I^j = \mathbf{y}_I$. By the continuity of $S^j(\hat{\mathbf{x}}_I^j; u)$ in u for each agent j , the allocation $(\mathbf{x}_I^j)_{j \in J}$ minimizes

$$\sum_{j \in J} S^j(\hat{\mathbf{x}}_I^j; u^j)$$

over all allocations $(\hat{\mathbf{x}}_I^j)_{j \in J} \in \times_{j \in J} X_I^j$ of goods with $\sum_{j \in J} \hat{\mathbf{x}}_I^j = \mathbf{y}_I$.

Suppose for sake of deriving a contradiction that $\mathbf{t} \notin T(\mathbf{u})$. Let $L_1 = J$ and let $L_2 = \bigcup_{j \in J} \{j\} \times X_I^j$. Define vectors $\mathbf{v}_I^\ell \in \mathbb{R}^I$ for $\ell \in L_1 \cup L_2$ by

$$\mathbf{v}_I^\ell = \begin{cases} \mathbf{x}_I^j - \mathbf{w}_I^j & \text{for } \ell = j \in L_1 \\ \mathbf{x}_I^j - \mathbf{x}_I' & \text{for } \ell = (j, \mathbf{x}_I') \in L_2 \end{cases}$$

and scalars α_ℓ for $\ell \in L_1 \cup L_2$ by

$$\alpha_\ell = \begin{cases} S^j(\mathbf{x}_I^j; u^j) - w_0^j - t^j & \text{for } \ell = j \in L_1 \\ S^j(\mathbf{x}_I'; u^j) - S^j(\mathbf{x}_I^j; u^j) & \text{for } \ell = (j, \mathbf{x}_I') \in L_2. \end{cases}$$

By Claim A.4, there does not exist a price vector \mathbf{p}_I such that $\mathbf{v}_I^\ell \cdot \mathbf{p}_I \leq \alpha_\ell$ for all $\ell \in L_1 \cup L_2$ with equality for all $\ell \in L_1$. The “if” direction of Fact A.5 therefore guarantees that there exist scalars λ_ℓ for $\ell \in L_1 \cup L_2$ with $\lambda_\ell \geq 0$ for all $\ell \in L_2$ such that

$$\sum_{\ell \in L_1 \cup L_2} \lambda_\ell \mathbf{v}_I^\ell = \mathbf{0} \quad \text{and} \quad \sum_{\ell \in L_1 \cup L_2} \lambda_\ell \alpha_\ell < 0.$$

By the definition of the scalars α_ℓ , we have that

$$\sum_{j \in J} \lambda_j (S^j(\mathbf{x}_I^j; u^j) - w_0^j - t^j) + \sum_{j \in J} \sum_{\mathbf{x}_I' \in X_I^j} \lambda_{j, \mathbf{x}_I'} (S^j(\mathbf{x}_I'; u^j) - S^j(\mathbf{x}_I^j; u^j)) < 0.$$

Due the continuity of $S^j(\hat{\mathbf{x}}_I^j; u)$ in u for each agent j and because $\mathbf{u}_{(m)} \rightarrow \mathbf{u}$ and $\mathbf{t}_{(m)} \rightarrow \mathbf{t}$ as $m \rightarrow \infty$, there must exist m such that

$$\sum_{j \in J} \lambda_j (S^j(\mathbf{x}_I^j; u_{(m)}^j) - w_0^j - t_{(m)}^j) + \sum_{j \in J} \sum_{\mathbf{x}_I' \in X_I^j} \lambda_{j, \mathbf{x}_I'} (S^j(\mathbf{x}_I'; u_{(m)}^j) - S^j(\mathbf{x}_I^j; u_{(m)}^j)) < 0.$$

Defining scalars α'_ℓ for $\ell \in L_1 \cup L_2$ by

$$\alpha'_\ell = \begin{cases} S^j(\mathbf{x}_I^j; u_{(m)}^j) - w_0^j - t_{(m)}^j & \text{for } \ell = j \in L_1 \\ S^j(\mathbf{x}_I'; u_{(m)}^j) - S^j(\mathbf{x}_I^j; u_{(m)}^j) & \text{for } \ell = (j, \mathbf{x}_I') \in L_2, \end{cases}$$

we have that

$$\sum_{\ell \in L_1 \cup L_2} \lambda_\ell \mathbf{v}_I^\ell = \mathbf{0} \quad \text{and that} \quad \sum_{\ell \in L_1 \cup L_2} \lambda_\ell \alpha'_\ell < 0.$$

The “only if” implication of Fact A.5 therefore guarantees that there does not exist a price vector \mathbf{p}_I such that $\mathbf{v}_I^\ell \cdot \mathbf{p}_I \leq \alpha'_\ell$ for all $\ell \in L_1 \cup L_2$ with equality for all $\ell \in L_1$. By Claim A.4, it follows that $\mathbf{t}_{(m)} \notin T(\mathbf{u}_{(m)})$ —a contradiction. Hence, we can conclude that $\mathbf{t} \in T(\mathbf{u})$ —as desired. \square

Completion of the Proof of the (I) \implies (II) Implication in Theorem A.1. We first solve for a profile $\mathbf{u} = (u^j)_{j \in J}$ of utility levels such that $\mathbf{0} \in T(\mathbf{u})$.

Claim A.6. *Under Statement (III) in Theorem A.1, there exists a profile $\mathbf{u} = (u^j)_{j \in J}$ of utility levels such that $\mathbf{0} \in T(\mathbf{u})$.*

To prove Claim A.6, we apply a topological fixed point argument.

Proof. Consider the compact, convex set

$$Z = [\underline{M}, \overline{M}]^J \times \prod_{j \in J} [u_{\min}^j, u_{\max}^j].$$

As $T(\mathbf{u}) \subseteq [\underline{M}, \overline{M}]^J$ for all $\mathbf{u} \in \prod_{j \in J} [u_{\min}^j, u_{\max}^j]$, we can define a correspondence $\Phi : Z \rightrightarrows Z$ by

$$\Phi(\mathbf{t}, \mathbf{u}) = T(\mathbf{u}) \times \arg \min_{\hat{\mathbf{u}} \in \prod_{j \in J} [u_{\min}^j, u_{\max}^j]} \left\{ \sum_{j \in J} t^j \hat{u}^j \right\}.$$

Claim A.3 guarantees that $T : \prod_{j \in J} [u_{\min}^j, u_{\max}^j] \rightrightarrows \mathbb{R}^J$ is upper hemicontinuous and has compact, convex values. Statement (III) in Theorem A.1 ensures that the correspondence T has non-empty values. Because $\prod_{j \in J} [u_{\min}^j, u_{\max}^j]$ is compact and convex, it follows that the correspondence Φ is upper hemicontinuous and has non-empty, compact, convex values as well. Hence, Kakutani’s Fixed Point Theorem guarantees that Φ has a fixed point (\mathbf{t}, \mathbf{u}) .

By construction, we have that $\mathbf{t} \in T(\mathbf{u})$ and that

$$(A.1) \quad u^j \in \arg \min_{\hat{u}^j \in [u_{\min}^j, u_{\max}^j]} t^j \hat{u}^j$$

for all agents j . It suffices to prove that $\mathbf{t} = \mathbf{0}$.

Let $(\mathbf{p}_I, (\mathbf{x}^j)_{j \in J})$ be a competitive equilibrium in the Hicksian economy for the profile $(u^j)_{j \in J}$ of utility levels with

$$(A.2) \quad S^j(\mathbf{x}_I^j; u^j) - w_0^j + \mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j) = t^j$$

for all agents j . As $u^j \geq u_{\min}^j$ and $S^j(\mathbf{x}_I^j; \cdot)$ is increasing for each agent j , it follows from Equation (A.2) and the definition of K^j that

$$(A.3) \quad \begin{aligned} t^j &= S^j(\mathbf{x}_I^j; u^j) - w_0^j + \mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j) \\ &\geq S^j(\mathbf{x}_I^j; u_{\min}^j) - w_0^j + \mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j) \\ &\geq \mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j) - K^j \end{aligned}$$

for all agents j .

Next, we claim that $t^j \leq 0$ for all agents j . If $t^j > 0$, then Equation (A.1) would imply that $u^j = u_{\min}^j$. But as $\mathbf{t} \in T(\mathbf{u})$, it would follow that

$$t^j \leq S^j(\mathbf{w}_I^j; u_{\min}^j) - w_0^j + \mathbf{p}_I \cdot (\mathbf{w}_I^j - \mathbf{w}_I^j) = S^j(\mathbf{w}_I^j; u_{\min}^j) - w_0^j = 0,$$

where the last equality holds due to the definitions of S^j and u_{\min}^j , so we must have that $t^j \leq 0$ for all agents j .

As $(\mathbf{x}_I^j)_{j \in J}$ is the allocation of goods in a competitive equilibrium, we have that $\sum_{j \in J} \mathbf{x}_I^j = \mathbf{y}_I = \sum_{j \in J} \mathbf{w}_I^j$ and hence that

$$\sum_{j \in J} \mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j) = 0 \geq \sum_{j \in J} t^j,$$

where the inequality holds because $t^j \leq 0$ for all agents j . It follows that for all agents j , we have that

$$t^j - \mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j) \leq \sum_{k \in J \setminus \{j\}} (\mathbf{p}_I \cdot (\mathbf{x}_I^k - \mathbf{w}_I^k) - t^k) \leq \sum_{k \in J \setminus \{j\}} K^k \leq \sum_{k \in J} K^k < K,$$

where the second inequality follows from Equation (A.3), the third inequality holds because $K^j \geq 0$, and the fourth inequality holds due to the definition of K . Hence, by Equation (A.2), we have that

$$S^j(\mathbf{x}_I^j; u^j) = w_0^j + t^j - \mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j) < w_0^j + K$$

for all agents j . Since utility is strictly increasing in the consumption of money, it follows that

$$u^j = U^j(S^j(\mathbf{x}_I^j; u^j), \mathbf{x}_I^j) < U^j(w_0^j + K, \mathbf{x}_I^j) \leq u_{\max}^j,$$

where the equality holds due to the definition of S^j and the second inequality holds due to the definition of u_{\max}^j . Equation (A.1) then implies that $t^j \geq 0$ for all agents j , so we must have that $t^j = 0$ for all agents j . \square

By Claim A.6, there exists a profile $\mathbf{u} = (u^j)_{j \in J}$ of utility levels and a competitive equilibrium $(\mathbf{p}_I, (\mathbf{x}_I^j)_{j \in J})$ in the corresponding Hicksian economy with

$$(A.4) \quad w_0^j = S^j(\mathbf{x}_I^j; u^j) + \mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j)$$

for all agents j . Lemma 1 implies that $\mathbf{x}_I^j \in D_H^j(\mathbf{p}_I; u^j)$ for all agents j , and we have that $U^j(w_0^j - \mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j), \mathbf{x}_I^j) = u^j$ for all agents j by Equation (A.4) and the definition of S^j . It follows from Fact 1 that $\mathbf{x}_I^j \in D_M^j(\mathbf{p}_I, \mathbf{w}^j)$ for all agents j , so the price vector \mathbf{p}_I and the allocation $(\mathbf{x}_I^j)_{j \in J}$ of goods comprise a competitive equilibrium in the original economy for the endowment allocation $(\mathbf{w}^j)_{j \in J}$.

APPENDIX B. PROOF OF PROPOSITION 1

We actually prove a stronger statement.

Claim B.1. *Suppose that agent j demands at most one unit of each good and let $\mathbf{w}_I \in X_I^j$. A utility function U^j is a net substitutes utility function if for all money endowments $w_0 > \underline{x}_0^j$, price vectors \mathbf{p}_I , and $0 < \mu < \lambda$, whenever*

- (i) $D_M^j(\mathbf{p}_I, \mathbf{w}) = \{\mathbf{x}_I\}$,
- (ii) $D_M^j(\mathbf{p}_I + \lambda \mathbf{e}^i, \mathbf{w}) = \{\mathbf{x}'_I\}$,
- (iii) $\{\mathbf{x}_I, \mathbf{x}'_I\} \subseteq D_M^j(\mathbf{p}_I + \mu \mathbf{e}^i, \mathbf{w})$, and
- (iv) $x'_i < x_i$,

we have that $x'_k \geq x_k$ for all goods $k \neq i$.

To complete the proof of the proposition from Claim B.1, we work in the setting of Claim B.1. Note that, for the endowment \mathbf{w}_I of goods, U^j is a gross substitutes utility function when $x'_k \geq x_k$ holds for all goods $k \neq i$ under Conditions (i) and (ii). This property clearly implies that $x'_k \geq x_k$ holds for all goods $k \neq i$ under Conditions (i), (ii), (iii), and (iv), and hence that U^j is net substitutes utility function by Claim B.1. The proposition therefore follows from Claim B.1.

It remains to prove Claim B.1. In the argument, we use the following characterization of substitutes valuations.

Fact B.2 (Theorems 2.1 and 2.4 in Fujishige and Yang, 2003; Theorems 3.9 and 4.10(iii) in Shioura and Tamura, 2015). *Suppose that agent j demands at most one unit of each good. A valuation V^j is a substitutes valuation if and only if for all price vectors \mathbf{p}_I with $|D^j(\mathbf{p}_I)| = 2$, writing $D^j(\mathbf{p}_I) = \{\mathbf{x}_I, \mathbf{x}'_I\}$, the difference $\mathbf{x}'_I - \mathbf{x}_I$ is a vector with at most one positive component and at most one negative component.*

Proof of Claim B.1. We prove the contrapositive. Suppose that U^j is not a net substitutes utility function. We show that there exists a money endowment w_0 , a price vector \mathbf{p}_I , price increments $0 < \mu < \lambda$, and goods $i \neq k$ such that Conditions (i), (ii), (iii), and (iv) from the statement hold but $x'_k < x_k$.

By Remark 1, there exists a utility level u such that $V_H^j(\cdot; u)$ is not a substitutes valuation. Hence, by Lemma 1 and the “if” direction of Fact B.2 for $V^j = V_H^j(\cdot; u)$, there exists a price vector $\hat{\mathbf{p}}_I$ such that $|D_H^j(\hat{\mathbf{p}}_I; u)| = 2$, and writing $D_H^j(\hat{\mathbf{p}}_I; u) = \{\mathbf{x}_I, \mathbf{x}'_I\}$, the difference $\mathbf{x}'_I - \mathbf{x}_I$ has at least two positive components or at least two negative components. Without loss of generality, we can assume that the difference $\mathbf{x}'_I - \mathbf{x}_I$ has at least two negative components. Suppose that $x'_i < x_i$ (so Condition (iv) holds) and that $x'_k < x_k$, where $i, k \in I$ are distinct goods.

Define a money endowment w_0 by

$$w_0 = S^j(\mathbf{x}_I; u) + \hat{\mathbf{p}}_I \cdot (\mathbf{x}_I - \mathbf{w}_I) = S^j(\mathbf{x}'_I; u) + \hat{\mathbf{p}}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I);$$

Fact 1 implies that $D_M^j(\hat{\mathbf{p}}_I, \mathbf{w}) = \{\mathbf{x}_I, \mathbf{x}'_I\}$. Let μ be such that

$$D_M^j(\hat{\mathbf{p}}_I - \mu \mathbf{e}^i, \mathbf{w}), D_M^j(\hat{\mathbf{p}}_I + \mu \mathbf{e}^i, \mathbf{w}) \subseteq \{\mathbf{x}_I, \mathbf{x}'_I\};$$

such a μ exists due to the upper hemicontinuity of D_M^j . Let $\mathbf{p}_I = \hat{\mathbf{p}}_I - \mu \mathbf{e}^i$, let $\lambda = 2\mu$, and let $\mathbf{p}'_I = \mathbf{p}_I + \lambda \mathbf{e}^i = \hat{\mathbf{p}}_I + \mu \mathbf{e}^i$.

By construction, we have that $\{\mathbf{x}_I, \mathbf{x}'_I\} \subseteq D_M^j(\mathbf{p}_I + \mu \mathbf{e}^i, \mathbf{w}) = D_M^j(\hat{\mathbf{p}}_I, \mathbf{w})$, so Condition (iii) holds. It remains to show that $D_M^j(\mathbf{p}_I, \mathbf{w}) = \{\mathbf{x}_I\}$ and that $D_M^j(\mathbf{p}'_I, \mathbf{w}) = \{\mathbf{x}'_I\}$. As j demands at most one unit of each good, we must have that $x_i = 1$ and that $x'_i = 0$. We divide into cases based on the value of w_i to show that

$$(B.1) \quad \begin{aligned} U^j(w_0 - \mathbf{p}_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I) &> U^j(w_0 - \mathbf{p}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I) \\ U^j(w_0 - \mathbf{p}'_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I) &> U^j(w_0 - \mathbf{p}'_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I). \end{aligned}$$

Case 1: $w_i = 0$. In this case, we have that

$$\begin{aligned} U^j(w_0 - \mathbf{p}_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I) &> U^j(w_0 - \hat{\mathbf{p}}_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I) \\ &= U^j(w_0 - \hat{\mathbf{p}}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I) \\ &= U^j(w_0 - \mathbf{p}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I), \end{aligned}$$

where the inequality holds because $p_i < \hat{p}_i$ and $x_i > w_i$, the first equality holds because $\{\mathbf{x}_I, \mathbf{x}'_I\} \subseteq D_M^j(\hat{\mathbf{p}}_I, \mathbf{w}_I)$, and the second equality holds because $x'_i = w_i$. Similarly, we have that

$$\begin{aligned} U^j(w_0 - \mathbf{p}'_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I) &< U^j(w_0 - \hat{\mathbf{p}}_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I) \\ &= U^j(w_0 - \hat{\mathbf{p}}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I) \\ &= U^j(w_0 - \mathbf{p}'_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I), \end{aligned}$$

where the inequality holds because $p'_i > \hat{p}_i$ and $x_i > w_i$, the first equality holds because $\{\mathbf{x}_I, \mathbf{x}'_I\} \subseteq D_M^j(\hat{\mathbf{p}}_I, \mathbf{w}_I)$, and the second equality holds because $x'_i = w_i$.

Case 2: $w_i = 1$. In this case, we have that

$$\begin{aligned} U^j(w_0 - \mathbf{p}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I) &< U^j(w_0 - \hat{\mathbf{p}}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I) \\ &= U^j(w_0 - \hat{\mathbf{p}}_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I) \\ &= U^j(w_0 - \mathbf{p}_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I) \end{aligned}$$

where the inequality holds because $p_i < \hat{p}_i$ and $x'_i < w_i$, the first equality holds because $\{\mathbf{x}_I, \mathbf{x}'_I\} \subseteq D_M^j(\hat{\mathbf{p}}_I, \mathbf{w}_I)$, and the second equality holds because $x_i = w_i$. Similarly, we have that

$$\begin{aligned} U^j(w_0 - \mathbf{p}'_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I) &> U^j(w_0 - \hat{\mathbf{p}}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I) \\ &= U^j(w_0 - \hat{\mathbf{p}}_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I) \\ &= U^j(w_0 - \mathbf{p}'_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I), \end{aligned}$$

where the inequality holds because $p'_i > \hat{p}_i$ and $x'_i < w_i$, the first equality holds because $\{\mathbf{x}_I, \mathbf{x}'_I\} \subseteq D_M^j(\hat{\mathbf{p}}_I, \mathbf{w}_I)$, and the second equality holds because $x_i = w_i$.

As $\mathbf{w}_I \in X_I^j \subseteq \{0, 1\}^I$, the cases exhaust all possibilities. Hence, we have proven that Equation (B.1) must hold. As $D_M^j(\mathbf{p}_I, \mathbf{w}), D_M^j(\mathbf{p}'_I, \mathbf{w}) \subseteq \{\mathbf{x}_I, \mathbf{x}'_I\}$, we must have that

$D_M^j(\mathbf{p}_I, \mathbf{w}) = \{\mathbf{x}_I\}$ and that $D_M^j(\mathbf{p}'_I, \mathbf{w}) = \{\mathbf{x}'_I\}$ —so Conditions (i) and (ii) hold, as desired. \square

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APPENDIX C. PROOFS OF FACTS 1 AND 2

C.1. **Proof of Fact 1.** We begin by proving two technical claims.

Claim C.1. *Let $\mathbf{w} \in X^j$ be an endowment and let u be a utility level. If*

$$(C.1) \quad u = \max_{\mathbf{x} \in X^j | \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{w}} U^j(\mathbf{x}),$$

then we have that

$$\mathbf{p} \cdot \mathbf{w} = \min_{\mathbf{x} \in X^j | U^j(\mathbf{x}) \geq u} \mathbf{p} \cdot \mathbf{x}$$

and that $D_M^j(\mathbf{p}_I, \mathbf{w}) \subseteq D_H^j(\mathbf{p}_I; u)$.

Proof. Letting $\mathbf{x}'_I \in D_M^j(\mathbf{p}_I, \mathbf{w})$ be arbitrary and $x'_0 = w_0 - \mathbf{p}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I)$, we have that $U^j(\mathbf{x}') = u$ and that $\mathbf{p} \cdot \mathbf{x}' \leq \mathbf{p} \cdot \mathbf{w}$ by construction. It follows that

$$\mathbf{p} \cdot \mathbf{w} \geq \min_{\mathbf{x} \in X^j | U^j(\mathbf{x}) \geq u} \mathbf{p} \cdot \mathbf{x}.$$

Suppose for the sake of deriving a contradiction that there exists $\mathbf{x}'' \in X^j$ with $\mathbf{p} \cdot \mathbf{x}'' < \mathbf{p} \cdot \mathbf{w}$ and $U^j(\mathbf{x}'') \geq u$. Then, we have that $x''_0 < w_0 + \mathbf{p}_I \cdot (\mathbf{x}''_I - \mathbf{w}_I)$; write $x''_0 = w_0 + \mathbf{p}_I \cdot (\mathbf{x}''_I - \mathbf{w}_I)$, so $x'''_0 > x''_0$. Since U^j is strictly increasing in consumption of money, it follows that $U^j(x'''_0, \mathbf{x}''_I) > u$ —contradicting Equation (C.1) as $x'''_0 + \mathbf{p}_I \cdot \mathbf{x}''_I = \mathbf{p} \cdot \mathbf{w}$. Hence, we can conclude that

$$\mathbf{p} \cdot \mathbf{w} = \min_{\mathbf{x} \in X^j | U^j(\mathbf{x}) \geq u} \mathbf{p} \cdot \mathbf{x}.$$

Since $U^j(\mathbf{x}') = u$ and $\mathbf{p} \cdot \mathbf{x}' = \mathbf{p} \cdot \mathbf{w}$, it follows that $\mathbf{x}'_I \in D_H^j(\mathbf{p}_I; u)$. Since $\mathbf{x}'_I \in D_M^j(\mathbf{p}_I, \mathbf{w})$ was arbitrary, we can conclude that $D_M^j(\mathbf{p}_I, \mathbf{w}) \subseteq D_H^j(\mathbf{p}_I; u)$. \square

Claim C.2. *Let $\mathbf{w} \in X^j$ be an endowment and let u be a utility level. If*

$$(C.2) \quad \mathbf{p} \cdot \mathbf{w} = \min_{\mathbf{x} \in X^j | U^j(\mathbf{x}) \geq u} \mathbf{p} \cdot \mathbf{x},$$

then we have that

$$u = \max_{\mathbf{x} \in X^j | \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{w}} U^j(\mathbf{x})$$

and that $D_H^j(\mathbf{p}_I; u) \subseteq D_M^j(\mathbf{p}_I, \mathbf{w})$.

Proof. Let $\mathbf{x}'_I \in D_H^j(\mathbf{p}_I; u)$ be arbitrary and $x'_0 = S^j(\mathbf{x}'_I; u)$. We have that $U^j(\mathbf{x}') \geq u$ and that $\mathbf{p} \cdot \mathbf{x}' = \mathbf{p} \cdot \mathbf{w}$ by construction. It follows that

$$u \leq \max_{\mathbf{x} \in X^j | \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{w}} U^j(\mathbf{x}).$$

We next show that

$$u = \max_{\mathbf{x} \in X^j | \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{w}} U^j(\mathbf{x}).$$

Suppose for sake of deriving a contradiction that there exists $\mathbf{x}'' \in X^j$ with $\mathbf{p} \cdot \mathbf{x}'' \leq \mathbf{p} \cdot \mathbf{w}$ and $U^j(\mathbf{x}'') > u$. By definition of S^j , we know that $x''_0 > S^j(\mathbf{x}''_I; u)$. Letting

$x_0''' = S^j(\mathbf{x}_I''; u)$, we have that $x_0''' + \mathbf{p}_I \cdot \mathbf{x}_I'' < \mathbf{p} \cdot \mathbf{w}$, which contradicts Equation (C.2) as $U^j(x_0''', \mathbf{x}_I'') = u$. Hence, we can conclude that

$$u = \max_{\mathbf{x} \in X^j | \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{w}} U^j(\mathbf{x}).$$

Since $U^j(\mathbf{x}_I') = u$ and $\mathbf{p} \cdot \mathbf{x}_I' = \mathbf{p} \cdot \mathbf{w}$, it follows that $\mathbf{x}_I' \in D_M^j(\mathbf{p}_I, \mathbf{w})$. Since $\mathbf{x}_I' \in D_H^j(\mathbf{p}_I; u)$ was arbitrary, we can conclude that $D_H^j(\mathbf{p}_I; u) \subseteq D_M^j(\mathbf{p}_I, \mathbf{w})$. \square

Let $\mathbf{w} \in X^j$ be an endowment and let u be a utility level. By Claims C.1 and C.2, Conditions (C.1) and (C.2), are equivalent, and under these equivalent conditions, we have that $D_M^j(\mathbf{p}_I, \mathbf{w}) \subseteq D_H^j(\mathbf{p}_I; u)$ and that $D_H^j(\mathbf{p}_I; u) \subseteq D_M^j(\mathbf{p}_I, \mathbf{w})$. Hence, we must have that $D_M^j(\mathbf{p}_I, \mathbf{w}) = D_H^j(\mathbf{p}_I; u)$ under the equivalent Conditions (C.1) and (C.2)—as desired.

C.2. Proof of Fact 2. We prove the “if” and “only if” directions separately.

Proof of the “If” Direction. We define a utility function U^j implicitly by

$$U^j(\mathbf{x}) = F(\mathbf{x}_I, \cdot)^{-1}(-x_0),$$

which is well-defined, continuous, and strictly increasing in x_0 by the Inverse Function Theorem because $F(\mathbf{x}_I, \cdot)$ is continuous, strictly decreasing, and satisfies Condition (4). Condition (1) holds because F is defined over the entirety of $X_I^j \times (\underline{u}^j, \bar{u}^j)$.

Proof of the “Only If” Direction. We define $F : X_I^j \times (\underline{u}^j, \bar{u}^j) \rightarrow (-\infty, -\underline{x}_0^j)$ implicitly by

$$F(\mathbf{x}_I, u) = -U^j(\cdot, \mathbf{x}_I)^{-1}(x_0),$$

which is well-defined, continuous, and strictly decreasing in u by the Inverse Function Theorem because $U^j(\cdot, \mathbf{x}_I)$ is continuous, strictly increasing, and satisfies Condition (1). Condition (4) holds because U^j is defined over the entirety of X^j .

APPENDIX D. PROOFS OF THE MAXIMAL DOMAIN AND NECESSITY RESULTS FOR SETTINGS WITH TRANSFERABLE UTILITY

In this appendix, we supply proofs of Facts 4 and 6, as well as the “only if” direction of Fact 5. Utility is transferable throughout this appendix.

We use the concept of a pseudo-equilibrium price vector.

Definition D.1 (Milgrom and Strulovici, 2009). Suppose that utility is transferable. A *pseudo-equilibrium price vector* is a price vector \mathbf{p}_I such that

$$\mathbf{y}_I \in \text{Conv} \left(\sum_{j \in J} D^j(\mathbf{p}_I) \right).$$

There is a connection between pseudo-equilibrium price vectors, competitive equilibria, and the existence problem.

Fact D.2 (Theorem 18 in Milgrom and Strulovici, 2009; Lemma 2.19 in Baldwin and Klemperer, 2019). *If utility is transferable and the total endowment is such that a competitive equilibrium exists, then, for each pseudo-equilibrium price vector \mathbf{p}_I , there exists an allocation $(\mathbf{x}_I^j)_{j \in J}$ such that \mathbf{p}_I and $(\mathbf{x}_I^j)_{j \in J}$ comprise a competitive equilibrium.*

The nonexistence of competitive equilibria may therefore be demonstrated by using the contrapositive of Fact D.2.

Our arguments use valuations that are *linear on their domain*. That is, let $\mathbf{t}_I \in \mathbb{R}^I$, let $X_I^j \subseteq \mathbb{Z}^n$ be finite, and let $V^j = V^{j, \mathbf{t}_I}: X_I^j \rightarrow \mathbb{R}$ be given by $V^{j, \mathbf{t}_I}(\mathbf{x}_I) = \mathbf{t}_I \cdot \mathbf{x}_I$ for all $\mathbf{x}_I \in X_I^j$. Recalling Equation (3) for demand sets in the quasilinear case, we observe that, for each $\mathbf{s}_I \in \mathbb{R}^I$, we have that

$$(D.1) \quad D^j(\mathbf{t}_I - \mathbf{s}_I) = \arg \max_{\mathbf{x}_I \in X_I^j} (\mathbf{t}_I \cdot \mathbf{x}_I - (\mathbf{t}_I - \mathbf{s}_I) \cdot \mathbf{x}_I) = \arg \max_{\mathbf{x}_I \in X_I^j} \mathbf{s}_I \cdot \mathbf{x}_I.$$

Lemma D.3. *If $\text{Conv}(X_I^j) \cap \mathbb{Z}^I = X_I^j$, then V^{j, \mathbf{t}_I} is concave for all $\mathbf{t}_I \in \mathbb{R}^I$.*

Proof. Observe by Equation (D.1) that $D^j(\mathbf{t}_I) = \arg \max_{\mathbf{x}_I \in X_I^j} \mathbf{0} \cdot \mathbf{x}_I = X_I^j$. So, if $\mathbf{x}_I \in \text{Conv}(X_I^j) \cap \mathbb{Z}^I = X_I^j$ then $\mathbf{x}_I \in D^j(\mathbf{t}_I)$. By Definition 7, we know V^{j, \mathbf{t}_I} is concave. \square

We will also make use of an alternative characterization of concavity.

Fact D.4 (Lemma 2.11 in Baldwin and Klemperer, 2019). *A valuation V^j is concave if and only if $\text{Conv}(D^j(\mathbf{p}_I)) \cap \mathbb{Z}^I = D^j(\mathbf{p}_I)$ for all price vectors \mathbf{p}_I .*

D.1. Additional Facts regarding Unimodularity and Demand Types. The following results are especially useful in the proof of the “only if” direction of Fact 5, and the proof of Fact 6.

We seek to construct pseudo-equilibrium price vectors (the total endowment is in the convex hull of aggregate demand) that are not competitive equilibrium price vectors (the total endowment is not demanded on aggregate). Failure of unimodularity allows such constructions because of the following property.

Fact D.5 (See, e.g., Fact 4.9 in Baldwin and Klemperer (2019)). *A demand type vector set \mathcal{D} is unimodular if and only if there is no linearly independent subset $\{\mathbf{d}^1, \dots, \mathbf{d}^r\}$ of \mathcal{D} such that there exists $\mathbf{z} = \sum_{\ell=1}^r \alpha_\ell \mathbf{d}^\ell \in \mathbb{Z}^I$ with $\alpha_\ell \in (0, 1)$ for $\ell = 1, \dots, r$.*

To see the connection to Fact D.2 and to existence of competitive equilibrium, suppose that $\{\mathbf{d}^1, \dots, \mathbf{d}^r\}$ and \mathbf{z} are as in Fact D.5. If $\mathbf{y}_I = \mathbf{z}$ and if $D_M^j(\mathbf{p}_I, \mathbf{w}^j) = \{\mathbf{0}, \mathbf{d}^j\}$ for $j = 1, \dots, r$, then \mathbf{p}_I is a pseudo-equilibrium price vector but there is no competitive equilibrium at \mathbf{p}_I .

Baldwin and Klemperer (2019) generalized Fact B.2 to the general case of transferable utility.

Fact D.6 (Proposition 2.20 in Baldwin and Klemperer, 2019). *Let V^j be a valuation of demand type \mathcal{D} . For any price \mathbf{p}_I' , if $\text{Conv}(D^j(\mathbf{p}_I'))$ has an edge E , then the difference*

between the extreme points of E is proportional to a demand type vector, and there exists a price \mathbf{p}_I such that $\text{Conv}(D^j(\mathbf{p}_I)) = E$.

Moreover if \mathbf{d} is in the minimal demand type vector set \mathcal{D} , such that V^j is of demand type \mathcal{D} , then there exists a price vector \mathbf{p}_I such that $\text{Conv}(D^j(\mathbf{p}_I))$ is a line segment, the difference between whose endpoints is proportional to \mathbf{d} .

We also demonstrate now the following useful corollary of Fact D.6.

Corollary D.7. *Let $V^j = V^{j, \mathbf{t}_I}$ for some $\mathbf{t}_I \in \mathbb{R}^I$, and let \mathcal{D} be the minimal demand type vector set such that V^j is of demand type \mathcal{D} . Then \mathcal{D} consists of the primitive integer vectors in the directions of the edges of the polytope $\text{Conv}(X_I^j)$.*

Proof. Observe that $D^j(\mathbf{t}_I) = X_I^j$ and so, by Fact D.6, each edge of $\text{Conv}(X_I^j)$ is proportional to a vector in \mathcal{D} . Conversely, if $\mathbf{d} \in \mathcal{D}$ then, by Fact D.6, there exists a price \mathbf{p}_I such that $\text{Conv}(D^j(\mathbf{p}_I))$ is a line segment, the difference between whose endpoints is proportional to \mathbf{d} . But writing $\mathbf{s}_I = \mathbf{t}_I - \mathbf{p}_I$, we see from Equation (D.1) that $D^j(\mathbf{p}_I) = \arg \max_{\mathbf{x}_I \in X_I^j} \mathbf{s}_I \cdot \mathbf{x}_I$ which tells us (cf. e.g. Grünbaum 1967, Section 2.4) that E is an edge of $\text{Conv}(X_I^j)$. \square

Our proofs of Facts 4 and 6, and the “only if” direction of Fact 5, now follow the same structure. Within each argument, we address a demand type which is not unimodular. Observe by Fact D.5 that when unimodularity fails for a set of vectors \mathcal{D} , then there exist polytopes, with integer vertices and whose edge directions are in \mathcal{D} , that contain a non-vertex integer vector, \mathbf{z} . We use Corollary D.7 construct valuations of the appropriate demand type such that, at some price \mathbf{p}_I , the convex hull of the aggregate demand set is a polytope with these properties; and such that there exists a feasible endowment allocation is the total endowment is the non-vertex integer vector \mathbf{z} . Thus \mathbf{p}_I is a pseudo-equilibrium price. Moreover, we design our individual valuations so that \mathbf{p}_I is not a competitive equilibrium. The contrapositive of Fact D.2 can then be applied to show the non-existence of competitive equilibrium.

D.2. Proof of Fact 4. By Fact B.2, there exists a price vector \mathbf{p}_I such that $D^j(\mathbf{p}_I) = \{\mathbf{x}'_I, \mathbf{x}'_I + \mathbf{g}\}$, where \mathbf{g} has at least two positive components or at least two negative components. Identify I with $\{1, \dots, |I|\}$ and without loss of generality assume that $g_1, g_2 < 0$. Because agent j demands at most one unit of each good, we know that $\mathbf{x}'_I, \mathbf{x}'_I + \mathbf{g} \in \{0, 1\}^{|I|}$ and so $\mathbf{g} \in \{-1, 0, 1\}^{|I|}$. We conclude both that $g_1 = g_2 = -1$ and that $x'_1 = x'_2 = 1$.

Let $k \in J \setminus \{j\}$ be arbitrary. For agents $j' \in J \setminus \{j, k\}$, let $X_I^{j'} = \{\mathbf{0}\}$, let $V^{j'}$ be arbitrary, and let $\mathbf{w}_I^{j'} = \mathbf{0}$.

Let $X_I^k = \{\mathbf{x}_I \in \{0, 1\}^{|I|} \mid x_1 + x_2 \leq 1\}$ and let $\mathbf{t}_I = \mathbf{p}_I - \mathbf{e}^1 - \mathbf{e}^2$. Let $V^k = V^{k, \mathbf{t}_I}$, which is a substitutes valuation by Example 11 and Corollary D.2, because each edge of $\text{Conv}(X_I^k)$ is proportional to either $\mathbf{e}^1 - \mathbf{e}^2$ or to \mathbf{e}^ℓ for some $\ell \in I$; or, alternatively, by

Theorem 4 in Hatfield et al. (2019) because, like Shapley's (1962) assignment valuations, it is the supremal convolution of $|I| - 1$ unit-demand valuations.

Set $\mathbf{w}_I^j = \mathbf{x}_I' \in X_I^j$ and set $\mathbf{w}_I^k = \mathbf{y}_I - \mathbf{x}_I'$. Since $\mathbf{x}_I' \in \{0, 1\}^{|I|}$ it follows that $\mathbf{w}_I^k \in \{0, 1\}^{|I|}$, and moreover since $x_1' = x_2' = 1$ we know $w_1^k = w_2^k = 0$; thus $\mathbf{w}_I^k \in X_I^k$. Now $(\mathbf{w}_I^{j'})_{j' \in J}$ is clearly an endowment allocation.

By Equation (D.1) we know that

$$D^k(\mathbf{p}_I) = \arg \max_{\mathbf{x}_I \in X_I^k} (\mathbf{e}^1 + \mathbf{e}^2) \cdot \mathbf{x}_I = \{\mathbf{x}_I \in \{0, 1\}^{|I|} | x_1 + x_2 = 1\}.$$

Observing that $\mathbf{e}^2 \in D^k(\mathbf{p}_I)$ and considering the vectors from \mathbf{e}^2 to other elements of the demand set, we can write $D^k(\mathbf{p}_I)$ as

$$D^k(\mathbf{p}_I) = \mathbf{e}^2 + \left\{ \alpha_2(\mathbf{e}^1 - \mathbf{e}^2) + \sum_{\ell=3}^{|I|} \alpha_\ell \mathbf{e}^\ell \mid \alpha_\ell \in \{0, 1\} \text{ for } 2 \leq \ell \leq |I| \right\}.$$

Combining this with agent j , and recalling other agents' demand sets are identically zero, we conclude that

$$\sum_{j' \in J} D^{j'}(\mathbf{p}_I) = \mathbf{x}_I' + \mathbf{e}^2 + \left\{ \alpha_1 \mathbf{g} + \alpha_2(\mathbf{e}^1 - \mathbf{e}^2) + \sum_{\ell=3}^{|I|} \alpha_\ell \mathbf{e}^\ell \mid \alpha_\ell \in \{0, 1\} \text{ for } 1 \leq \ell \leq |I| \right\}.$$

The convex hull of this set can be expressed very similarly, but the weights α_ℓ are allowed to lie in $[0, 1]$.

Since $\mathbf{x}_I', \mathbf{x}_I' + \mathbf{g} \in \{0, 1\}^{|I|}$, we have that if $g_i = 1$ (resp. $g_i = -1$), then $x_i' = 0$ (resp. $x_i' = 1$). Taking

$$\alpha_\ell = \begin{cases} \frac{|g_\ell|}{2} & \text{if } g_\ell \neq 0 \\ 1 - x_\ell' & \text{if } g_\ell = 0 \end{cases}$$

for $1 \leq \ell \leq |I|$, we have that

$$\begin{aligned} x_i' + \alpha_1 g_i + \alpha_i &= 1 - \frac{1}{2} + \frac{1}{2} = 1 && \text{for all } i \in I \text{ with } g_i = -1 \\ x_i' + \alpha_1 g_i + \alpha_i &= x_i' + 0 + (1 - x_i') = 1 && \text{for all } i \in I \text{ with } g_i = 0 \\ x_i' + \alpha_1 g_i + \alpha_i &= 0 + \frac{1}{2} + \frac{1}{2} = 1 && \text{for all } i \in I \text{ with } g_i = 1. \end{aligned}$$

As $x_1' = x_2' = 1$ and $g_1 = g_2 = -1$, it follows that

$$\mathbf{x}_I' + \mathbf{e}^2 + \alpha_1 \mathbf{g} + \alpha_2(\mathbf{e}^1 - \mathbf{e}^2) + \sum_{\ell=3}^{|I|} \alpha_\ell \mathbf{e}^\ell = \mathbf{y}_I.$$

As $\alpha_\ell \in [0, 1]$ for all $1 \leq \ell \leq |I|$, we therefore have that $\mathbf{y}_I \in \text{Conv} \left(\sum_{j' \in J} D^{j'}(\mathbf{p}_I) \right)$, so \mathbf{p}_I is a pseudo-equilibrium price vector. But as $\alpha_1 \in (0, 1)$ and the vectors $\mathbf{g}, \mathbf{e}^1 - \mathbf{e}^2, \mathbf{e}^3, \dots, \mathbf{e}^{|I|}$ are linearly independent, we have that $\mathbf{y}_I \notin \sum_{j' \in J} D^{j'}(\mathbf{p}_I)$, so there is

no competitive equilibrium at \mathbf{p}_I .⁴⁸ Therefore, by the contrapositive of Fact D.2, no competitive equilibrium can exist.

D.3. Proof of the “Only If” Direction of Fact 5. Let \mathcal{D} be a demand type vector set that is not unimodular. We need to show that there exists a finite set J of agents with concave valuations of demand type \mathcal{D} , as well as a total endowment, for which there exists an endowment allocation but no competitive equilibrium. We will use $J = \{j, k\}$.

Let $L = \{\mathbf{d}^1, \dots, \mathbf{d}^n\} \subseteq \mathcal{D}$ be a minimal non-unimodular subset. By construction, L is linearly independent, and $\{\mathbf{d}^1, \dots, \mathbf{d}^{n-1}\}$ is unimodular. Let

$$\mathcal{P} = \left\{ \sum_{\ell=1}^n \alpha_\ell \mathbf{d}^\ell \mid 0 \leq \alpha_\ell \leq 1 \text{ for } \ell = 1, \dots, n \right\}$$

denote the parallelepiped spanned by L . By Fact D.5, there exists $\mathbf{z} = \sum_{\ell=1}^n \beta_\ell \mathbf{d}^\ell \in \mathcal{P} \cap \mathbb{Z}^I$ with $\beta_\ell \in (0, 1)$ for all $\ell = 1, \dots, n$.

Let $X_I^j = \mathcal{P} \cap \mathbb{Z}^n$ and let $V^j = V^{j, \mathbf{0}}$ be the linear valuation which is identically zero on its domain. Recall Equation (D.1): we know $D^j(\mathbf{0}) = X_I^j$. Observe that $\mathbf{z} \in X_I^j$. Clearly $\text{Conv}(X_I^j) \cap \mathbb{Z}^I = X_I^j$ and so V^j is concave by Lemma D.3.

Let \mathbf{s}_I satisfy $\mathbf{s}_I \cdot \mathbf{d}^\ell = 0$ for $\ell = 1, \dots, n-1$ and $\mathbf{s}_I \cdot \mathbf{d}^n > 0$. (Such an \mathbf{s}_I exists as L is linearly independent.) Then, for $\mathbf{x}_I = \sum_{\ell=1}^n \alpha_\ell \mathbf{d}^\ell \in X_I^j$, we have

$$\mathbf{s}_I \cdot \mathbf{x}_I = \sum_{\ell=1}^n \alpha_\ell \mathbf{s}_I \cdot \mathbf{d}^\ell = \alpha_n \mathbf{s}_I \cdot \mathbf{d}^n.$$

We assumed that $\mathbf{s}_I \cdot \mathbf{d}^n > 0$, so $\mathbf{s}_I \cdot \mathbf{x}_I$ is minimized when $\alpha_n = 0$; equivalently $-\mathbf{s}_I \cdot \mathbf{x}_I$ is maximized when $\alpha_n = 0$. So, by Equation (D.1), we know that

$$D^j(\mathbf{s}_I) = \arg \max_{\mathbf{x}_I \in X_I^j} -\mathbf{s}_I \cdot \mathbf{x}_I = \left\{ \sum_{\ell=1}^{n-1} \alpha_\ell \mathbf{d}^\ell \mid 0 \leq \alpha_\ell \leq 1 \text{ for } \ell = 1, \dots, n-1 \right\} \cap \mathbb{Z}^I.$$

Now set $X_I^k = \{\mathbf{0}, \mathbf{d}^n\}$ and let $V^k = V^{k, \mathbf{s}_I}$. By Equation (D.1) again, we know that $D^k(\mathbf{s}_I) = X_I^k$. As $\mathbf{d}^n \in \mathcal{D}$, which is a demand type vector set, we know that \mathbf{d}^n is a primitive integer vector, from which it follows that $\text{Conv}(X_I^k) \cap \mathbb{Z}^I = X_I^k$. Thus, by Lemma D.3, we know that V^k is concave.

Observe that

$$D^j(\mathbf{s}_I) + D^k(\mathbf{s}_I) = \left\{ \sum_{\ell=1}^n \alpha_\ell \mathbf{d}^\ell \mid 0 \leq \alpha_\ell \leq 1 \text{ for } \ell = 1, \dots, n-1 \text{ and } \alpha_n \in \{0, 1\} \right\} \cap \mathbb{Z}^I.$$

So $\text{Conv}(D^j(\mathbf{s}_I) + D^k(\mathbf{s}_I)) = \mathcal{P}$.

Let the total endowment \mathbf{y}_I be \mathbf{z} . Set $\mathbf{w}_I^j = \mathbf{z} \in X_I^j$, and set $\mathbf{w}_I^k = \mathbf{0} \in X_I^k$. This is clearly an endowment allocation. Since $\mathbf{y}_I \in \mathcal{P} = \text{Conv}(D^j(\mathbf{s}_I) + D^k(\mathbf{s}_I))$, we see \mathbf{s}_I is a pseudo-equilibrium price vector. But, since L is linearly independent and since

⁴⁸The existence of an integer vector that is in $\text{Conv}\left(\sum_{j' \in J} D^{j'}(\mathbf{p}_I)\right)$ but not $\sum_{j' \in J} D^{j'}(\mathbf{p}_I)$ follows from Fact D.5 as the vectors $\mathbf{g}, \mathbf{e}^1 - \mathbf{e}^2, \mathbf{e}^3, \dots, \mathbf{e}^{|I|}$ do not comprise a unimodular set.

$0 < \beta_n < 1$, we know $\mathbf{y}_I \notin D^j(\mathbf{s}_I) + D^k(\mathbf{s}_I)$, so there is no competitive equilibrium at \mathbf{s}_I . It follows, by the contrapositive of Fact D.2, that no competitive equilibrium can exist.

D.4. Proof of Fact 6. We will use the following claim.

Claim D.8. *If \mathcal{D} is a maximal unimodular demand type vector set then \mathcal{D} spans \mathbb{R}^I .*

Proof. Let $L \subseteq \mathcal{D}$ be a maximal, linearly independent set. As \mathcal{D} is unimodular, there exists a set T of integer vectors such that $L \cap T = \emptyset$ and $L \cup T$ is a basis of \mathbb{R}^I with determinant ± 1 . We claim that $\mathcal{D}_0 = \mathcal{D} \cup T \cup -T$ is unimodular. To see why, let $L' \subseteq \mathcal{D} \cup T \cup -T$ be a maximal linearly independent set. As \mathcal{D}_0 spans \mathbb{R}^I by construction, L' must span \mathbb{R}^I . Due to the maximality of L , we must have that $|L' \cap (T \cup -T)| = |T|$. It follows that $L' \cap \mathcal{D}$ is a basis for the span of \mathcal{D} . As \mathcal{D} is unimodular, $L' \cap \mathcal{D}$ must be the image of L under a unimodular change of basis of the span of \mathcal{D} . It follows that L' is a basis for \mathbb{R}^I with determinant ± 1 —so \mathcal{D}_0 is unimodular. Due to the maximality of \mathcal{D} , we must have that $T = \emptyset$, and hence \mathcal{D} must span \mathbb{R}^I . As \mathcal{D} is unimodular, it follows that \mathcal{D} must integrally span \mathbb{Z}^I . \square

We next divide into cases based on whether V^j is non-concave and of demand type \mathcal{D} , or not of demand type \mathcal{D} , to construct concave valuations V^k of demand type \mathcal{D} for agents $k \neq j$ and a total endowment for which no competitive equilibrium exists.

Case 1: V^j is not concave but is of demand type \mathcal{D} . By Fact D.4, there exists a price vector \mathbf{p}_I such that $D^j(\mathbf{p}_I) \neq \text{Conv}(D^j(\mathbf{p}_I)) \cap \mathbb{Z}^I$. Let $\mathbf{x}'_I \in D^j(\mathbf{p}_I)$ be an extreme point of $\text{Conv}(D^j(\mathbf{p}_I))$, so there exists $\mathbf{s}_I \in \mathbb{R}^n$ satisfying

$$(D.2) \quad \{\mathbf{x}'_I\} = \arg \max_{\mathbf{y}_I \in D^j(\mathbf{p}_I)} \mathbf{s}_I \cdot \mathbf{y}_I.$$

Let $\mathbf{x}''_I \in (\text{Conv}(D^j(\mathbf{p}_I)) \cap \mathbb{Z}^I) \setminus D^j(\mathbf{p}_I)$ be arbitrary.

Let $k \in J \setminus \{j\}$ be arbitrary. Let $X_I^k = (\text{Conv}(D^j(\mathbf{p}_I)) \cap \mathbb{Z}^I) + \{-\mathbf{x}'_I\}$. Since V^j is of demand type \mathcal{D} , it follows by Fact D.6 that every edge of $\text{Conv}(D^j(\mathbf{p}_I))$ is a multiple of a vector in \mathcal{D} , and so the same is true of $\text{Conv}(X_I^k)$. Moreover, by definition of X_I^k it is clear that $\text{Conv}(X_I^k) \cap \mathbb{Z}^I = X_I^k$.

Fix $\mathbf{t}_I = \mathbf{p}_I + \mathbf{s}_I$ and let $V^k = V^{\mathbf{t}_I, k}$, which is concave by Lemma D.3 and of demand type \mathcal{D} by Corollary D.7. By Equation (D.1) we know $D^k(\mathbf{p}_I) = \arg \max_{\mathbf{x}_I \in X_I^k} \mathbf{s}_I \cdot \mathbf{x}_I$, and so by Equation (D.2) and the definition of X_I^k , it follows that $D^k(\mathbf{p}_I) = \{\mathbf{x}_I - \mathbf{x}'_I\} = \{\mathbf{0}\}$.

Let the total endowment \mathbf{y}_I be \mathbf{x}''_I , let $\mathbf{w}_I^j = \mathbf{x}'_I \in X_I^j$, and let $\mathbf{w}_I^k = \mathbf{x}''_I - \mathbf{x}'_I \in X_I^k$. For agents $j' \in J \setminus \{j, k\}$, let $X_I^{j'} = \{\mathbf{0}\}$, let $V^{j'}$ be arbitrary, and let $\mathbf{w}_I^{j'} = \mathbf{0}$. Thus $(\mathbf{w}_I^{j'})_{j' \in J}$ is an endowment allocation. Moreover,

$$\sum_{j' \in J} D^{j'}(\mathbf{p}_I) = D^j(\mathbf{p}_I).$$

Thus $\mathbf{y}_I = \mathbf{x}_I'' \in \text{Conv}\left(\sum_{j' \in J} D^{j'}(\mathbf{p}_I)\right)$ and so \mathbf{p}_I is a pseudo-equilibrium price vector. But $\mathbf{y}_I = \mathbf{x}_I'' \notin \sum_{j' \in J} D^{j'}(\mathbf{p}_I) = D^j(\mathbf{p}_I)$ by definition of \mathbf{x}_I'' , and so there is no competitive equilibrium at \mathbf{p}_I . Therefore, by the contrapositive of Fact D.2, no competitive equilibrium can exist.

Case 2: V^j is not of demand type \mathcal{D} . By Fact D.6 there exists a primitive integer vector $\mathbf{g} \notin \mathcal{D}$ and a price vector $\mathbf{p}_I \in \mathbb{R}^n$ such that $D^j(\mathbf{p}_I) \subseteq \{\mathbf{x}_I' + \alpha \mathbf{g} \mid \alpha = 0, \dots, r\}$ where $r \geq 1$ and $\mathbf{x}_I', \mathbf{x}_I' + r\mathbf{g} \in D^j(\mathbf{p}_I)$.

As \mathcal{D} is not strictly contained in any unimodular demand type vector set, and as $\mathbf{g} \notin \mathcal{D}$, the set $\mathcal{D} \cup \{\mathbf{g}\}$ is not unimodular. Let $\{\mathbf{d}^1, \dots, \mathbf{d}^m, \mathbf{g}\}$ be a minimal non-unimodular subset of $\mathcal{D} \cup \{\mathbf{g}\}$. Thus the set $\{\mathbf{d}^1, \dots, \mathbf{d}^m, \mathbf{g}\}$ is linearly independent and, by Fact D.5, there exists

$$(D.3) \quad \mathbf{z} = \beta_0 \mathbf{g} + \sum_{\ell=1}^m \beta_\ell \mathbf{d}^\ell \in \cap \mathbb{Z}^I \text{ with } 0 < \beta_\ell < 1 \text{ for } \ell = 0, \dots, m.$$

By Claim D.8, we know that \mathcal{D} spans \mathbb{R}^I . Since \mathcal{D} is also unimodular, by Fact D.5 there exist $\mathbf{d}^{m+1}, \dots, \mathbf{d}^n \in \mathcal{D}$ for some $n \geq m$ such that $\mathbf{d}^1, \dots, \mathbf{d}^n$ are linearly independent and

$$\mathbf{z} = \sum_{\ell=1}^n \gamma_\ell \mathbf{d}^\ell \text{ with } \gamma_\ell \in \mathbb{Z} \text{ for all } \ell = 1, \dots, n.$$

Moreover, by replacing $\mathbf{d}^{m+1}, \dots, \mathbf{d}^n$ with their negations if necessary, we can assume that $\gamma_{m+1}, \dots, \gamma_n \geq 0$.

Let $k \in J \setminus \{j\}$ be arbitrary. Let $X_I^k = Y_I^k + Z_I^k$, where

$$Y_I^k = \left\{ \sum_{\ell=1}^m \alpha_\ell \mathbf{d}^\ell \mid -|\gamma_\ell| \leq \alpha_\ell \leq |\gamma_\ell| + 1 \text{ for } \ell = 1, \dots, m \right\} \cap \mathbb{Z}^I$$

$$Z_I^k = \left\{ \sum_{\ell=m+1}^n \alpha_\ell \mathbf{d}^\ell \mid 0 \leq \alpha_\ell \leq \gamma_\ell \text{ for } \ell = m+1, \dots, n \right\} \cap \mathbb{Z}^I.$$

Observe that $\mathbf{z} \in X_I^k$. Moreover, $\text{Conv}(X_I^k) \cap \mathbb{Z}^I = X_I^k$, as we may see by writing $X_I^k = \{\sum_{\ell=1}^n \alpha_\ell \mathbf{d}^\ell \mid c_\ell \leq \alpha_\ell \leq d_\ell \text{ for } \ell = 1, \dots, n\} \cap \mathbb{Z}^I$ for suitably chosen c_ℓ and d_ℓ .

Choose \mathbf{s}_I such that $\mathbf{s}_I \cdot \mathbf{d}^\ell = 0$ for $\ell = 1, \dots, m$ and $\mathbf{s}_I \cdot \mathbf{d}^\ell < 0$ for $\ell = m+1, \dots, n$. (Such an \mathbf{s}_I exists because $\mathbf{d}^1, \dots, \mathbf{d}^n$ are linearly independent.) Set $\mathbf{t}_I = \mathbf{p}_I + \mathbf{s}_I$ and set $V^k = V^{k, \mathbf{t}_I}$. Then V^k is concave by Lemma D.3. By Equation (D.1) and the definition of X_I^k , we deduce that

$$D^k(\mathbf{p}_I) = \arg \max_{\mathbf{x}_I \in X_I^k} \mathbf{s}_I \cdot \mathbf{x}_I = Y_I^k.$$

Moreover, the edges of X_I^k are parallel to $\mathbf{d}^1, \dots, \mathbf{d}^n$ and so by Corollary D.7, the valuation V^k is of demand type \mathcal{D} .

For agents $j' \in J \setminus \{j, k\}$, let $X_I^{j'} = \{\mathbf{0}\}$, let $V^{j'}$ be arbitrary, and let $\mathbf{w}_I^{j'} = \mathbf{0}$. Let the total endowment \mathbf{y}_I be $\mathbf{x}'_I + \mathbf{z}$. Set $\mathbf{w}_I^j = \mathbf{x}'_I \in X_I^j$ and $\mathbf{w}_I^k = \mathbf{z} \in X_I^k$, so $(\mathbf{w}_I^{j'})_{j' \in J}$ is an endowment allocation.

Now see that

$$(D.4) \quad \sum_{j' \in J} D^j(\mathbf{p}_I) \subseteq \{\mathbf{x}'_I + \alpha \mathbf{g} \mid \alpha = 0, \dots, r\} + Y_I^k$$

while, since $\mathbf{x}'_I + r\mathbf{g} \in D^j(\mathbf{p}_I)$, we have the equality

$$\text{Conv} \left(\sum_{j' \in J} D^j(\mathbf{p}_I) \right) = \{\mathbf{x}'_I + \alpha \mathbf{g} \mid 0 \leq \alpha \leq r\} + \text{Conv}(Y_I^k)$$

Recalling Equation (D.3), we conclude that $\mathbf{y}_I = \mathbf{x}'_I + \mathbf{z} \in \text{Conv} \left(\sum_{j' \in J} D^j(\mathbf{p}_I) \right)$, so \mathbf{p}_I is a pseudo-equilibrium price vector. But, since $0 < \beta_0 < 1$ in Equation (D.3) and since the set $\{\mathbf{d}^1, \dots, \mathbf{d}^m, \mathbf{g}\}$ is linearly independent, we conclude from Equation (D.4) that $\mathbf{y}_I = \mathbf{x}'_I + \mathbf{z} \notin \sum_{j' \in J} D^j(\mathbf{p}_I)$, so there is no competitive equilibrium at \mathbf{p}_I .

Therefore, by the contrapositive of Fact D.2, no competitive equilibrium can exist.

As the cases exhaust all possibilities, we have proven the fact.