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The spectrum of asymptotic Cayley trees

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Abstract
We characterize the spectrum of the transition matrix for simple random walk on graphs consisting of a finite graph with a finite number of infinite Cayley trees attached. We show that there is a continuous spectrum identical to that for a Cayley tree and, in general, a non-empty pure point spectrum. We apply our results to studying continuous time quantum walk on these graphs. If the pure point spectrum is nonempty the walk is in general confined with a nonzero probability.

Keywords: graph spectrum, Cayley tree, random walk, quantum walk

1. Introduction

The spectrum of the Laplacian and similar operators defined on graphs, both finite and infinite, has long been a topic of interest and many results are known, see for example [1]. These systems are not only of intrinsic interest but also, for example, describe the propagation of signals in discrete models of media or in networks of various kinds. In this paper we study the spectrum of the transition, or hopping, matrix on a class of infinite graphs that we call asymptotic Cayley trees. These graphs, which are defined in section 3.1, are trees with a fixed coordination number outside a finite subgraph; they can be constructed by grafting planted Cayley trees by the root to the vertices of a finite graph.

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The transition matrix is the adjacency matrix scaled by the degree of vertices and is therefore the generator of simple random walk on the graph. Furthermore, walks on an asymptotic Cayley tree can easily be decomposed into walks on the finite graph and walks on the grafted tree(s). We exploit these two facts by using elementary probabilistic methods to find an exact rational relationship between the resolvent (or Green's) functions of the graph of interest and those of the simpler constituents. These relationships enable us to characterize the spectrum of the transition matrix on an asymptotic Cayley tree. The spectrum consists of a continuous component with support on the same interval as for the regular tree, and a pure point component. For the latter we establish bounds on the multiplicity of any given eigenvalue, and both upper and lower bounds on the total number of such normalizable eigenfunctions. These bounds are directly expressed in terms of the basic properties of the finite graph and its spectrum. The total number of normalizable eigenfunctions is shown to be bounded above by the number of vertices in the finite graph. The lower bound demonstrates that large classes of graph must have normalizable eigenfunctions. In particular if the number of eigenfunctions of the finite sub-graph having eigenvalues lying outside the continuous spectrum exceeds twice the number of graft vertices, then the graph must have a non-empty pure point spectrum. In addition to these general theorems that apply to all asymptotic Cayley trees, we discuss as examples special cases where more precise statements can be made.

Our results are complementary to, and extend, those of [2] where the spectrum of the adjacency matrix on these graphs was studied by mapping the problem onto one of the adjacency matrix on a regular Cayley tree plus a finite rank perturbation. The mapping is shown always to exist but the relationship between the perturbation and the properties of the original graph is rather indirect. The perturbation problem is then analysed using the apparatus of the theory of Schrödinger operators; there is a continuous component of the spectrum with support on the same interval as the regular tree and a pure point spectrum. The dimension of the pure point component is bounded above by the rank of the perturbation, and examples are given to show that normalizable eigenfunctions are indeed present in some cases. However, neither a lower bound on the dimension, nor results for the multiplicity of individual eigenvalues are given.

As an example of the application of our results we examine the properties of continuous time quantum walk on asymptotic Cayley trees. Quantum walks on graphs have been studied extensively in recent years, the main motivation being the development of efficient quantum algorithms. For recent reviews see, e.g. [3–5]. There are two classes of quantum walk: the discrete time walk which introduces a new quantum degree of freedom usually called the 'coin'; and the continuous time quantum walk which is governed by the Schrödinger equation and does not require extra degrees of freedom. The relation between these two types of quantum walks is complicated [6, 7], and seems not to be completely understood. In this paper we are solely concerned with continuous time walks whose evolution Hamiltonian is taken to be (minus) the transition matrix.

This paper is organized as follows. In section 2 we introduce our notation for graphs and the operators defined on them, and study classical random walk on the pure Cayley tree. In section 3 we define asymptotic Cayley trees, establish a number of results for classical random walk on such graphs, and then prove our main theorems about their spectra. In section 4 we discuss some special cases, and in section 5 we use the results from the earlier sections to analyse the behaviour of continuous time quantum walks on asymptotic Cayley trees.

2. Graphs and random walks

2.1. Basic definitions

Let $G = (V, E)$ be an undirected simple (i.e. with no multiple edges and no loops) connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For $u, v \in V$ we denote by $d_G(u, v)$ the graph distance between u and v. We will consider both finite and infinite graphs but they are all locally finite, i.e. the number of neighbours σ_u of any vertex u, called the degree of u, is assumed to be finite. In fact, for the graphs to be considered the degree is uniformly bounded,

$$\sigma_u \leqslant C, \quad u \in V(G), \tag{1}$$

for some finite constant C. We consider the vector space $\ell^2(G)$ consisting of square summable complex valued functions $f: V \mapsto \mathbb{C}$ equipped with two different inner products, $\langle \cdot | \cdot \rangle$ and $(\cdot, \cdot)$, given by

$$\langle f|g\rangle = \sum_{u \in V} \bar{f}(u)\, g(u) \quad \text{and} \quad (f, g) = \sum_{u \in V} \sigma_u^{-1} \bar{f}(u)\, g(u)\,. \tag{2}$$

Due to (1), the corresponding norms $\|\cdot\|_s$ and $\|\cdot\|_r$ are equivalent,

$$\|f\|_r \leqslant \|f\|_s \leqslant C\|f\|_r, \tag{3}$$

and $\ell^2(G)$ is a Hilbert space with respect to both inner products. The functions in $\ell^2(G)$ will be referred to as ℓ^2-functions. For $u \in V$ we denote by $|u\rangle$ the function which takes the value 1 at u and is zero elsewhere, and by $\langle u|$ its covector fulfilling $\langle u|v\rangle = \delta_{u,v}$ for all $v \in V$.

Given a graph $G = (V, E)$, its adjacency matrix $A = (A_{uv})$, indexed by V, is defined by $A_{uv} = 1$ if $(u, v) \in E$ and $A_{uv} = 0$ otherwise. The matrix A defines an operator on $\ell^2(G)$ by

$$(Af)(u) = \sum_{v:(u,v) \in E} f(v)\,. \tag{4}$$

Using (1), it is easily seen that A is a bounded operator and it is self-adjoint with respect to $\langle \cdot | \cdot \rangle$. The degree matrix is defined as the diagonal matrix $D_{uv} = \delta_{uv}\sigma_u$ and it likewise defines an operator D on $\ell^2(G)$ which is bounded by (1) and clearly has a bounded inverse. Moreover, it is selfadjoint w.r.t. both inner products and we have

$$(f, g) = \langle f|D^{-1}g\rangle, \quad f, g \in \ell^2(G)\,. \tag{5}$$

The transition matrix K^G for G is defined by[4]

$$K^G = AD^{-1} \tag{6}$$

and is related to the normalized Laplacian L^G by

$$L^G = I - K^G, \tag{7}$$

[4] The transition matrix is also called the hopping matrix in the physics literature.

where I is the unit operator on $\ell^2(G)$. It is important to note that K^G, and hence also L^G, is a self-adjoint operator w.r.t. the inner product $(\cdot,\cdot)$ as a consequence of (6) and (5) and the fact that A is self-adjoint w.r.t. $\langle\cdot|\cdot\rangle$. Moreover, L^G is non-negative since

$$\left(f, L^G f\right) = \sum_{(u,v)\in E(G)} \left(D^{-1}f(u) - D^{-1}f(v)\right)^2 . \tag{8}$$

A central issue in this paper is to determine the spectrum of L^G for a class of graphs defined below. Because of the relation (7) this is equivalent to determining the spectrum of K^G, which we shall refer to as the spectrum of G, and will be our main object of study in sections 3 and 4.[5]

Given an arbitrary graph G, we next recall the relation between K^G and the simple random walk on G. By definition, a simple discrete time random walk, located at a vertex u at integer time n, moves with equal probabilty, σ_u^{-1}, to one of the neighbours of u at time $n+1$. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_{n+1})$ be a path from a vertex u to a vertex v, i.e. $\omega_j \in V$, $j = 1, \ldots, n+1$, $\omega_1 = u$, $\omega_{n+1} = v$ and $(\omega_j, \omega_{j+1}) \in E$ for $j = 1, \ldots n$. We will refer to n as the length of the path ω and denote it $|\omega|$. If the walk is located at u at time $n = 0$, then the probability that it is at v after n steps is given by

$$p_n(u,v) = \sum_{\omega: u\to v} \prod_{j=1}^{n} \sigma_{\omega_j}^{-1} , \tag{9}$$

where the sum runs over all paths of length n from u to v. If there is no such path then the probability is zero. By convention $p_0(u,u) = 1$.

One checks easily that

$$p_n(u,v) = \langle v| \left(K^G\right)^n |u\rangle . \tag{10}$$

In view of (10) one says that K^G generates simple random walk on G. The generating function for the probabilities $p_n(u,v)$ is defined as

$$Q^G_{u,v}(z) = \sum_{n=0}^{\infty} p_n(u,v) z^n , \tag{11}$$

which, using (10), is also given by

$$Q^G_{u,v}(z) = \langle v| \left(I - zK^G\right)^{-1} |u\rangle . \tag{12}$$

Note that $\lambda^{-1}Q^G_{u,v}(\lambda^{-1})$ is a matrix element of the resolvent $(\lambda - K^G)^{-1}$ of the transition matrix. If $u = v$ then we write $Q^G_{u,v} = Q^G_u$, and $Q^G_u(z)$ is the generating function for return probabilities to u.

For $n > 0$, let $p_n^{(1)}(u)$ denote the probability that a walk starting at u is back at u after n steps and that this is the first return to u. The associated generating function is

$$P^G_u(z) = \sum_{n=2}^{\infty} p_n^{(1)}(u) z^n , \tag{13}$$

[5] Note that sometimes a different definition is used; namely, that the spectrum of A is called the spectrum of G.

called the first return generating function. We have $P_u^G(1) \leqslant 1$ since the probabilities $p_n^{(1)}(u)$ refer to mutually exclusive events. A walk returning to the starting point u can visit u arbitrarily many times before ending at u. It follows that

$$Q_u^G(z) = \frac{1}{1 - P_u^G(z)}. \tag{14}$$

2.2. The Cayley tree and its spectrum

A regular Cayley tree of degree q is an infinite tree graph T_q where all the vertices have the same degree q. (This graph is sometimes referred to as the Bethe lattice.) A planted Cayley tree of degree q is an infinite tree graph T'_q where all the vertices have the same degree q except one of them, called the root, which has degree 1.

The first return and all-returns generating functions for T_q are independent of the starting vertex, and we denote them by P and Q, respectively. Moreover, the first return generating function for T_q obviously equals the first return generating function for T'_q at the root. In order to calculate the latter we note that in the first step the walk moves from the root u to its neighbour v with probability 1. Clearly, the walk must return to v before taking the final step to u which has probability q^{-1}, and it can return to v arbitrarily often before returning to u. Hence, the first return generating function satisfies the equation

$$\begin{aligned} P(z) &= z \sum_{n=0}^{\infty} \left(\frac{q-1}{q} P(z) \right)^n \frac{z}{q} \\ &= \frac{z^2}{q - (q-1) P(z)}. \end{aligned} \tag{15}$$

The factor of z is associated with the first step and the factor z/q comes from the last step. The solution of (15) satisfying the initial condition $P(0) = 0$ is

$$P(z) = \frac{q - \sqrt{q^2 - a^2 z^2}}{2(q-1)}, \tag{16}$$

where

$$a = 2\sqrt{q-1}. \tag{17}$$

By (14) we then find that the return generating function for T_q is

$$Q(z) = \frac{2 - q + \sqrt{q^2 - a^2 z^2}}{2(1 - z^2)}. \tag{18}$$

We can calculate $Q_{u,v}(z)$ for arbitrary vertices u and v in a similar way. First, note that by (9) and (11) we have

$$Q_{u,v}(z) = \sum_{\omega : u \to v} \left(\frac{z}{q} \right)^{|\omega|}, \tag{19}$$

where the sum is over all paths ω from u to v. Let $(u, u_1, \ldots, u_{n-1}, v)$ be the shortest path from u to v. Any path ω from u to v can be uniquely decomposed into a sequence of $n+1$ paths

$\omega^1, \omega^2, \ldots, \omega^{n+1}$ where ω^1 is any path from u and back to u; the first step in ω^2 is from u to u_1, ω^2 avoids u and returns to u_1; the first step in ω^3 is from u_1 to u_2, ω^3 avoids u_1 and returns to u_2 and so on for the subsequent vertices, the first step in ω^{n+1} being from u_{n-1} to v and otherwise avoiding u_{n-1} before returning to v. Clearly

$$\sum_{j=1}^{n+1} |\omega^j| = |\omega| \,. \tag{20}$$

The sum over ω^1 yields a factor $Q(z)$. Summing over ω^2 gives a factor of

$$\frac{z/q}{1-\frac{q-1}{q}P(z)}\,, \tag{21}$$

where z/q is associated with the initial step in ω_2. The same factor is obtained by summing over each of the subsequent ω^j's. Using (15) we find that

$$Q_{u,v}(z) = Q(z)\left(z^{-1}P(z)\right)^{d(u,v)} \,. \tag{22}$$

Note that the same formula applies for a planted Cayley tree T_q' if u is the root, and v any vertex.

It is important to note that, as a consequence of (16) and (18), the functions $Q(\lambda^{-1})$ and $\lambda P(\lambda^{-1})$ are both analytic in the complex λ-plane with a cut on the real axis along the interval $[-\frac{a}{q}, \frac{a}{q}]$, and they are real-valued on the real axis outside the cut. The limit $\lim_{\epsilon\downarrow 0} Q_u((\lambda+i\epsilon)^{-1})$ exists for all $\lambda \in \mathbb{R}$ and in view of (12) this entails (see, e.g. [8] section XIII.6) that $|u\rangle$ belongs to the absolutely continuous subspace of $\ell^2(T_q)$ and has spectral density ρ_u concentrated on $[-\frac{a}{q}, \frac{a}{q}]$ given by

$$\rho_u(\lambda) = \lim_{\epsilon\downarrow 0} \frac{1}{\pi\lambda} \mathrm{Im}\, Q_u\left((\lambda+i\epsilon)^{-1}\right) = \frac{1}{2\pi}\frac{\sqrt{a^2-q^2\lambda^2}}{1-\lambda^2}\,, \quad \lambda \in \left[-\tfrac{a}{q}, \tfrac{a}{q}\right]. \tag{23}$$

Since the vectors $|u\rangle$ span all of $\ell^2(T_q)$, it follows that T_q has absolutely continuous spectrum equal to $[-\frac{a}{q}, \frac{a}{q}]$, which was first proven in [9]. Furthermore,

$$\langle u|f(K)|v\rangle = \int_{-\frac{a}{q}}^{\frac{a}{q}} f(\lambda)\,\rho_{u,v}(\lambda)\,d\lambda \tag{24}$$

for any continuous function f on $[-\frac{a}{q}, \frac{a}{q}]$, where

$$\rho_{u,v}(\lambda) = \lim_{\epsilon\downarrow 0}\frac{1}{2\pi i\lambda}\left(Q_{u,v}\left((\lambda-i\epsilon)^{-1}\right) - Q_{u,v}\left((\lambda+i\epsilon)^{-1}\right)\right). \tag{25}$$

Clearly, $\rho_{u,v}$ depends only on the distance ℓ between u and v in T_q and hence we use the notation $\rho^{(\ell)}$ for this function. Recalling (22) we get from (25) that

$$\rho^{(\ell)}(\lambda) = -\frac{1}{\rho_u(\lambda)}\lim_{\epsilon\downarrow 0}\mathrm{Im}\left((\lambda+i\epsilon)^{\ell-1} Q\left((\lambda+i\epsilon)^{-1}\right) P\left((\lambda+i\epsilon)^{-1}\right)^{\ell}\right). \tag{26}$$

Setting $\lambda = aq^{-1}\cos\theta$, $\theta \in [0,\pi]$, and defining $\tan\beta = \frac{q}{q-2}\tan\theta$, gives

$$\begin{aligned}\rho^{(\ell)}(\lambda(\theta)) &= \frac{1}{(q-1)^{\frac{1}{2}\ell}}\frac{\sin(\ell\theta+\beta)}{\sin\beta} \\ &= \frac{1}{q(q-1)^{\frac{1}{2}\ell}}\left(2C_\ell(\cos\theta) + (q-2)\,U_\ell(\cos\theta)\right),\end{aligned} \tag{27}$$

where C_ℓ and U_ℓ are Chebyshev polynomials of the first and second kind. (We use C_ℓ for the polynomial of the first kind to avoid confusion with our notation for trees.)

3. Asymptotic Cayley trees

3.1. Definition

Given a graph G and a subgraph F we denote by $G\setminus F$ the subgraph of G spanned by vertices of G that are not in F (i.e. the edges of $G\setminus F$ are those whose end vertices are both outside F). If T'_q is a planted Cayley tree of degree q with root r, we set $S_q = T'_q\setminus r$; thus S_q is a tree, commonly known as the semi-infinite Cayley tree of order q, with all vertices of degree q, except one which is of degree $q-1$ and is defined as the root of S_q.

Let B be a graph and let T'_q be a planted tree disjoint from B. A graph defined by identifying the root r of T'_q with some vertex v_0 of B is then said to be obtained by grafting T'_q onto B at v_0, which we call the graft vertex.

Given a vertex u_0 of G, let $B_R(u_0)$ denote the subgraph spanned by the vertices at graph distance at most R from u_0, i.e. the (closed) ball in G of radius R around u_0. We define an *asymptotic Cayley tree* of degree q to be a connected graph G with the property that there exists a vertex u_0 and an integer $R \geqslant 0$ such that $G\setminus B_R(u_0)$ is a finite, non-empty union of disjoint trees isomorphic to S_q, whose roots are at graph distance $R+1$ from u_0 in G. By the triangle inequality it follows that if G satisfies the stated condition for some vertex u_0 and $R \geqslant 0$, then it also satisfies the condition with u_0 replaced by any vertex u_1 and with R replaced by any integer larger than or equal to $R + d_G(u_0,u_1)$.

While the definition just given is formally simple, we shall make frequent use of the following more constructive characterisation of an asymptotic Cayley tree. First, note that if G is obtained by successively grafting planted Cayley trees of degree q onto some finite graph B, then G is an asymptotic Cayley tree of degree q. Indeed, let u_0 be some vertex in B and choose

$$R \geqslant \max\{d_B(v,v') \mid v,v' \in V(B)\}. \tag{28}$$

Then G, u_0, R satisfy the defining condition of an asymptotic Cayley tree above. Conversely, if G, u_0, R satisfy this condition, then G is obtained by grafting $q-1$ planted Cayley trees onto $B_{R+1}(u_0)$ at each vertex at distance $R+1$ from u_0. This proves that asymptotic Cayley trees of degree q are precisely those graphs that can be obtained by grafting a finite (positive) number of planted Cayley trees onto some finite graph B.

We call the minimal such finite graph B the *core* of G and denote it by $G^{(0)}$. In order to see that $G^{(0)}$ is unique we define a subtree of G to be Cayley-maximal if it is a maximal subtree of G that is isomorphic to T'_q and such that all its vertices except possibly the root have degree q in G. It is then easily verified that two Cayley-maximal subtrees of G are either identical or share at most their roots. It follows that removing all such subtrees, save their roots, is a well defined process and yields a finite subgraph B that is obviously the smallest one with the desired property.

Given an asymptotic Cayley tree G, we define $V_0^G \subseteq V(G^{(0)})$ to be the set of vertices at which planted Cayley trees are grafted to obtain G. Any sequence of the form

$$\mathcal{G} = \left\{ G^{(0)}, G^{(1)}, G^{(2)}, \ldots, G^{(n)} = G \right\}, \tag{29}$$

where $G^{(j+1)}$ is obtained from $G^{(j)}$ by grafting one or more trees isomorphic to T_q' onto $G^{(j)}$ at a vertex $v_j \in V_0^G$, will be called a G-sequence in the following.

We note that, as the reader may also easily verify, the definition given here of an asymptotic Cayley tree is equivalent to the one given in definition 2.1 of [2] of a graph isomorphic to a regular tree at infinity.

3.2. Generating functions

We now establish the relationship between the resolvents of the transition matrices on two graphs that are related by the grafting of planted trees at a single vertex.

Lemma 1. *Let B be a (finite or infinite) graph and denote by G the graph that is obtained by grafting p copies of T_q' at a vertex v_0, of degree σ_{v_0}, in B, which is then identified with $v_0 \in V(G)$. Then the following statements relating $Q_{u,v}^G$ and $Q_{u,v}^B$ hold:*

(i) For $u = v = v_0$,

$$Q_{v_0}^G(z) = \frac{(\sigma_{v_0} + p)\, Q_{v_0}^B(z)\, Q(z)}{\sigma_{v_0} Q(z) + p\, Q_{v_0}^B(z)}, \tag{30}$$

where $Q_{v_0}^B(z)$ is given by (14).

(ii) For $u \in V(B)$, $v \in V(B) \setminus \{v_0\}$,

$$Q_{u,v}^G(z) = Q_{u,v}^B(z) - \frac{p\, Q_{u,v_0}^B(z)\, Q_{v_0,v}^B(z)}{\sigma_{v_0} Q(z) + p\, Q_{v_0}^B(z)}. \tag{31}$$

(iii) For $u \in V(B)$ and $v \in G \setminus B$,

$$Q_{u,v}^G(z) = \frac{q\, Q(z)\, Q_{u,v_0}^B(z)}{\sigma_{v_0} Q(z) + p\, Q_{v_0}^B(z)} \left(z^{-1} P(z)\right)^{d_G(v_0,v)}. \tag{32}$$

(iv) For $u, v \in G \setminus B$,

$$Q_{u,v}^G(z) = Q(z) \left(1 - \alpha_p(z) \left(z^{-1} P(z)\right)^{d_G(v_0,u) + d_G(v_0,v) - d_G(u,v)}\right) \left(z^{-1} P(z)\right)^{d_G(u,v)}, \tag{33}$$

where

$$\alpha_p(z) = 1 - \frac{q\, Q_{v_0}^B(z)}{\sigma_{v_0} Q(z) + p Q_{v_0}^B(z)}. \tag{34}$$

Remark 1. Note that, since

$$\sigma_u Q_{u,v}^G(z) = \sigma_v Q_{v,u}^G(z) \tag{35}$$

holds for any graph G, the lemma is sufficient to determine $Q_{u,v}^G(z), \forall u, v \in V(G)$ given $Q_{u,v}^B(z), \forall u, v \in V(B)$.

Proof. First note that walks starting from and returning to v_0 consist of repeated excursions into either B or a tree which gives

$$Q^G_{v_0}(z) = \frac{1}{1 - p\frac{P(z)}{\sigma_{v_0}+p} - \frac{\sigma_{v_0} P^B_{v_0}(z)}{\sigma_{v_0}+p}}, \tag{36}$$

where $P^B_{v_0}(z)$ is given by (13). Statement (*i*) follows by (14).

To prove statement (*ii*) we first need the identity

$$\begin{aligned} Q^G_{v_0,v}(z) &= Q^G_{v_0}(z)\,\frac{\sigma_{v_0}}{\sigma_{v_0}+p}\,\frac{Q^B_{v_0,v}(z)}{Q^B_{v_0}(z)} \\ &= \frac{\sigma_{v_0} Q^B_{v_0,v}(z)\,Q(z)}{\sigma_{v_0} Q(z) + p\,Q^B_{v_0}(z)}, \qquad v \in V(B)\setminus\{v_0\}, \end{aligned} \tag{37}$$

which is easily obtained by decomposing a walk contributing to the left-hand side into a (possibly trivial) walk from v_0 and back to v_0 and a walk from v_0 to u that does not return to v_0, and then using (30). The statement for $u = v_0$ follows by rearranging (37). For $u \neq v_0$ separate the contributing walks into those that do, or do not, visit v_0 gives

$$\begin{aligned} Q^G_{u,v}(z) &= \frac{Q^B_{u,v_0}(z)\,Q^G_{v_0,v}(z)}{Q^B_{v_0}(z)} + \left(Q^B_{u,v}(z) - \frac{Q^B_{u,v_0}(z)\,Q^B_{v_0,v}(z)}{Q^B_{v_0}(z)}\right) \\ &= Q^B_{u,v}(z) - \frac{p\,Q^B_{u,v_0}(z)\,Q^B_{v_0,v}(z)}{\sigma_{v_0} Q(z) + p\,Q^B_{v_0}(z)}, \end{aligned} \tag{38}$$

where we have used (37) in the second step. This completes the proof of statement (*ii*).

Next consider the case when $u \in V(B)$ and $v \in G\setminus B$. Decompose the contributing walks into a component that starts at u and ends at v_0, followed by a walk from v_0 to v without revisiting v_0. Applying the same reasoning that led to (22), but noting that $v_0 \in G$ has degree $\sigma_{v_0} + p$ rather than q, we obtain

$$Q^G_{u,v}(z) = Q^G_{u,v_0}(z)\,\frac{q}{\sigma_{v_0}+p}\left(z^{-1}P(z)\right)^{d_G(v_0,v)}. \tag{39}$$

Using (30) if $u = v_0$, respectively (35) and (37) if $u \neq v_0$, we obtain (32) which proves statement (*iii*).

Denote by $\widetilde{G}$ the subgraph of G obtained by removing all of B except the vertex v_0, i.e. $\widetilde{G} = G\setminus\{B\setminus\{v_0\}\}$. Then, for $u,v \in G\setminus B$, separate the contributing walks into those that do, or do not, visit v_0 to obtain

$$Q^G_{u,v}(z) = \frac{Q^{\widetilde{G}}_{u,v_0}(z)\,Q^G_{v_0,v}(z)}{Q^{\widetilde{G}}_{v_0}(z)} + \left(Q^{\widetilde{G}}_{u,v}(z) - \frac{Q^{\widetilde{G}}_{u,v_0}(z)\,Q^{\widetilde{G}}_{v_0,v}(z)}{Q^{\widetilde{G}}_{v_0}(z)}\right). \tag{40}$$

To evaluate the first term note that, by (35) and then (22),

$$\frac{Q^{\widetilde{G}}_{u,v_0}(z)}{Q^{\widetilde{G}}_{v_0}(z)} = \frac{p}{q}\,\frac{Q^{\widetilde{G}}_{v_0,u}(z)}{Q^{\widetilde{G}}_{v_0}(z)} = \frac{p}{q}\,\frac{\frac{q}{p}Q_{v_0,u}(z)}{Q(z)} = \left(z^{-1}P(z)\right)^{d_G(v_0,u)}. \tag{41}$$

Combining this with (32) gives

$$\frac{q\,Q(z)\,Q^B_{v_0}(z)}{\sigma_{v_0}Q(z)+p\,Q^B_{v_0}(z)}\left(z^{-1}P(z)\right)^{d_G(v_0,v)+d_G(v_0,u)}. \tag{42}$$

The second term in (40) occurs only when u,v are in the same planted tree T'_q. Since the contributing walks to the term do not reach v_0, we can compute it by replacing $\widetilde{G}$ with the regular Cayley tree obtained by identifying the root of T'_q with the root of the semi-infinite Cayley tree S_q. Using (22) we then obtain

$$Q(z)\left(\left(z^{-1}P(z)\right)^{d_G(u,v)}-\left(z^{-1}P(z)\right)^{d_G(v_0,v)+d_G(v_0,u)}\right). \tag{43}$$

Combining the two contributions proves statement (*iv*).

□

3.3. The spectrum of asymptotic Cayley trees

Here we study the spectrum of an asymptotic Cayley tree G. Our strategy is to deploy lemma 1 on the sequence $\mathcal{G}$ (29). By grafting trees to one vertex at a time, we can follow the evolution of the spectrum from one member of the sequence to its successor.

Theorem 1. *The spectrum of an asymptotic Cayley tree G of degree q consists of an absolutely continuous part $[-\frac{a}{q},\frac{a}{q}]$ and a finite set of eigenvalues S^G_{pp} (the pure point spectrum).*

Proof. Let $\mathcal{G}$ be any G-sequence as in (29) and assume for some index i and arbitrary vertices u,v in $G^{(i)}$ that $Q^{G^{(i)}}_{u,v}$ has the form

$$Q^{G^{(i)}}_{u,v}(z)=A^{G^{(i)}}_{u,v}(z)+\tilde{A}^{G^{(i)}}_{u,v}(z)\sqrt{q^2-a^2z^2}, \tag{44}$$

where $A^{G^{(i)}}_{u,v}$ and $\tilde{A}^{G^{(i)}}_{u,v}$ are rational functions of z. Since $Q(z)$ and $z^{-1}P(z)$ are of this form it follows from lemma 1 applied to $B=G^{(i)}$, that the same holds for $Q^{G^{(i+1)}}_{u,v}(z)$. Recalling that $G^{(0)}$ is a finite graph, we have that $Q^{G^{(0)}}_{u,v}$ is a rational function and it follows by repeating the argument that (44) also holds for $G^{(n)}=G$.

Likewise, it follows from the identities of lemma 1 that if $Q^{G^{(i)}}_{u,v}$ is finite outside a finite set $F_i\subset\mathbb{C}$, independent of u,v, then $Q^{G^{(i+1)}}_{u,v}(z)$ is finite if z is not in F_i and $pQ^{G^{(i)}}_{v_0}(z)+\sigma_{v_0}Q(z)\neq 0$, where v_0 is the graft vertex in $G^{(i)}$ and p is the number of trees grafted at v_0. In view of (44), the equation

$$pQ^{G^{(i)}}_{v_0}(z)+\sigma_{v_0}Q(z)=0 \tag{45}$$

has finitely many solutions. Thus $Q^{G^{(i+1)}}_{u,v}(z)$ is finite outside a finite set F_{i+1}. By repeating this argument we conclude that the poles of the functions $A^G_{u,v}$ and $\tilde{A}^G_{u,v}$ are contained in a finite set F independent of u,v. In particular, it follows that the limit $\lim_{\epsilon\downarrow 0}(\lambda+i\epsilon)^{-1}Q^G_u((\lambda+i\epsilon)^{-1})$ exists outside F for all vertices u, which rules out the presence of a singular continuous spectrum (see, e.g. the Proposition in section XIII.6 of [8]). The remaining singularities of $\lambda^{-1}Q^G_{v_0}(\lambda^{-1})$ are then poles contained in F. Thus we have that

$$\lambda^{-1}Q^G_u\left(\lambda^{-1}\right)=B^G_u(\lambda)+\tilde{B}^G_u(\lambda)\sqrt{q^2\lambda^2-a^2}, \tag{46}$$

where B_u^G and $\tilde{B}_u^G$ are rational functions of λ that are real-valued on the real axis (by the self-adjointness of K^G) and $\tilde{B}_u^G$ has no poles in $[-\frac{a}{q}, \frac{a}{q}]$, which therefore equals the continuous spectrum with the spectral density of the state $|u\rangle$ given by

$$\rho_u(\lambda) = -\frac{1}{\pi}\tilde{B}_u^G(\lambda)\sqrt{a^2 - q^2\lambda^2}\,. \tag{47}$$

This completes the proof. □

The characterization of the spectrum given by theorem 1 and its proof implies, by the spectral theorem, that for $\lambda \notin S_{pp}^G$ we have

$$\lambda^{-1}Q_{u,v}^G\left(\lambda^{-1}\right) = \sum_{\mu\in S_{pp}^G}\frac{\langle u|e_\mu|v\rangle}{\lambda-\mu} - \frac{1}{\pi}\int_{-\frac{a}{q}}^{\frac{a}{q}}\frac{\tilde{B}_{u,v}^G(\mu)\sqrt{a^2-q^2\mu^2}}{\lambda-\mu}\,d\mu\,, \tag{48}$$

where e_μ denotes the spectral projection onto the eigenspace of K^G corresponding to μ and $\tilde{B}_{u,v}^G(\mu)$ is a rational function of μ with no poles in $[-\frac{a}{q}, \frac{a}{q}]$. For an arbitrary, say continuous, function f on the spectrum of G the following generalisation of (24) holds:

$$\langle u|f\left(K^G\right)|v\rangle = \sum_{\mu\in S_{pp}^G} f(\mu)\langle u|e_\mu|v\rangle - \frac{1}{\pi}\int_{-\frac{a}{q}}^{\frac{a}{q}} f(\mu)\tilde{B}_{u,v}^G(\mu)\sqrt{a^2-q^2\mu^2}\,d\mu\,. \tag{49}$$

We next establish some basic estimates for the multiplicity of the individual eigenvalues of K^G. Denote by $\hat{S}_0^G$ the set of eigenvalues of $K^{G^{(0)}}$ that have at least one non-trivial eigenfunction vanishing on the set V_0^G of graft vertices in $G^{(0)}$. Moreover, denote the subspace of such eigenfunctions with eigenvalue λ by $E_0^G(\lambda)$, and its dimension by $d_0^G(\lambda)$.

Theorem 2. *Let G be an asymptotic Cayley tree of degree q. All eigenvalues $\lambda \in S_{pp}^G$ have finite multiplicity and fulfill:*

(i) If $\lambda \in [-\frac{a}{q}, \frac{a}{q}]$, then $\lambda \in \hat{S}_0^G$ and the multiplicity of λ equals $d_0^G(\lambda)$;
(ii) If $\lambda \notin [-\frac{a}{q}, \frac{a}{q}]$, then the multiplicity of λ is at most $\sharp V_0^G + d_0^G(\lambda)$, where $\sharp V$ denotes the number of vertices in V.

For the proof of this result we need the following lemma.

Lemma 2. *Let G be an asymptotic Cayley tree with a subtree T'_q grafted at vertex v_0, and assume $\phi \in \ell^2(G)$ is an eigenfunction of K^G with eigenvalue λ. Then the following statements hold:*

(i) If $\lambda \in [-\frac{a}{q}, \frac{a}{q}]$, then ϕ vanishes on T'_q;
(ii) If $\lambda \notin [-\frac{a}{q}, \frac{a}{q}]$, then ϕ is spherical in T'_q, i.e. its value at $v \in T'_q$ depends only on the height $d(v_0, v)$, and is given by

$$\phi(v) = A\,\xi^{-d(v_0,v)}\,, \quad v \in T'_q\,, \tag{50}$$

where $\xi^{-1} = \lambda P(\lambda^{-1})$ and $A = \phi(v_0)$.

Proof. Denoting by v_1 the unique neighbour of v_0 in T_q', we define $\psi(v) = \phi(v)$, if $v \notin T_q'$ or if $v = v_0$ or if $v = v_1$, and

$$\psi(v) = \frac{1}{(q-1)^{h-1}} \sum_{w \in T_q' : d(v_0, w) = h} \phi(w) \tag{51}$$

for vertices v at height $h > 1$ in T_q'. Then ψ is a function of $h = d(v_0, v)$ alone for $v \in T_q'$, and it is easily seen that it fulfils

$$K^G \psi(v) = \lambda \psi(v), \quad v \in G. \tag{52}$$

Moreover, ψ belongs to $\ell^2(G)$ as a consequence of the following estimate, where ϕ_0 denotes the restriction of ϕ to $V(G) \setminus V(T_q')$:

$$\begin{aligned} \|\psi\|_s^2 &= \|\phi_0\|_s^2 + |\psi(v_0)|^2 + \sum_{h=1}^{\infty} (q-1)^{-h+1} \Big| \sum_{w \in T_q' : d(v_0, w) = h} \phi(w) \Big|^2 \\ &\leqslant \|\phi_0\|_s^2 + |\phi(v_0)|^2 + \sum_{h=1}^{\infty} \sum_{w \in T_q' : d(v_0, w) = h} |\phi(w)|^2 = \|\phi\|_s^2, \end{aligned} \tag{53}$$

where we have used the Cauchy-Schwarz inequality. Hence, ψ is an eigenfunction of K^G with eigenvalue λ.

Writing $\psi(v) = \psi(h)$ for $v \in T_q'$ at height h, the eigenvalue condition (52) gives

$$(q-1)\psi(h+1) + \psi(h-1) = \lambda q \psi(h), \quad h \geqslant 1. \tag{54}$$

The general solution to this equation is

$$\psi(h) = A\xi_+^{-h} + B\xi_-^{-h}, \quad h \geqslant 0, \tag{55}$$

where A and B are constants and $\xi_\pm$ are the solutions to the quadratic equation

$$\xi^2 - \lambda q \xi + q - 1 = 0. \tag{56}$$

For $|\lambda| < a/q$ we find

$$\xi_\pm = \sqrt{(q-1)} e^{\pm i\theta}, \tag{57}$$

where θ is real, so ψ cannot be in $\ell^2(G)$ unless it vanishes on T_q'. In particular, $\phi(v_0) = \psi(v_0) = 0$ and $\phi(v_1) = \psi(v_1) = 0$. It follows that if we consider q copies of T_q' with the restriction of ϕ defined on each of them, then by identifying their roots we obtain an eigenfunction on the Cayley tree T_q with eigenvalue λ. But since the pure point spectrum of T_q is empty, it follows that ϕ vanishes on T_q', thus proving statement (*i*) in this case. A similar argument applies if $\lambda = \pm \frac{a}{q}$, in which case the general solution to (54) is $\psi(h) = (A + Bh)(q-1)^{-\frac{h}{2}}$. This completes the proof of statement (*i*).

If $|\lambda| > \frac{a}{q}$, then $\xi_+ \neq \xi_-$ are real and $\xi_+ \xi_- = q - 1$. Hence, exactly one of the roots, say ξ_+, fulfills $|\xi_+| > \sqrt{q-1}$ and the only solutions in $\ell^2(T_q')$ are of the form $A\xi_+^{-h}, h \geqslant 0$. In particular, we have

$$\phi(v_0) = \psi(v_0) = \xi_+ \psi(v_1) = \xi_+ \phi(v_1). \tag{58}$$

Applying this argument to any subtree of T_q' spanned by the descendants of any vertex $w_1 \neq v_0$ in T_q' and its predecessor w_0 it follows that $\phi(w_0) = \xi_+\phi(w_1)$. This proves that the restriction of ϕ to T_q' is of the form $A\,\xi_+^{-d(v_0,v)}$ for $v \in T_q'$. Finally, it is straighforward to check that $\xi_+^{-1} = \lambda P(\lambda^{-1})$ by direct substitution in (56) and applying (15). This proves statement (*ii*). □

Proof of theorem 2. Let $\phi \in \ell^2(G)$ be an eigenfunction of K^G with eigenvalue $\lambda \in [-\frac{a}{q}, \frac{a}{q}]$. By lemma 2(*i*) the function ϕ vanishes on all grafted trees T_q', from which it follows that the restriction $\phi^{(0)}$ of ϕ to $G^{(0)}$ belongs to $E_0^G(\lambda)$. On the other hand, extending any $\phi^{(0)} \in E_0^G(\lambda)$ to $V(G)$ by assigning the value 0 at vertices outside $G^{(0)}$ we obtain an eigenfunction of K^G with eigenvalue λ. This proves statement (*i*).

Let $\lambda \in S_{pp}^G \backslash [-\frac{a}{q}, \frac{a}{q}]$ and denote the graft vertices in $G^{(0)}$ by $v_1, \ldots, v_N$, where $N = \sharp V_0^G$. Moreover, let $\phi_i, 1 \leqslant i \leqslant N$, denote an eigenfunction of K^G with eigenvalue λ such that

$$\phi_i(v_i) \neq 0 \quad \text{and} \quad \phi_i(v_j) = 0 \quad \text{for } j < i, \tag{59}$$

if such a function exists. Otherwise, set $\phi_i = 0$. By lemma 2(*ii*) the restriction of ϕ_i to the trees T_q' grafted at v_i is uniquely determined up to multiplication by a constant and it vanishes on the trees grafted at any vertex $v_j, j < i$. It follows that for any eigenfunction ϕ of K^G with eigenvalue λ there exist coefficients $\mu_1, \ldots, \mu_N$ such that $\phi - \mu_1\phi_1 - \cdots - \mu_N\phi_N$ vanishes on all grafted trees T_q'. By the same arguments as above it follows that the dimension of the space of such functions equals $d_0^G(\lambda)$. Clearly this proves statement (*ii*). □

Our final result on the spectrum of G provides bounds on the total dimension of eigenspaces of G in terms of the same quantity for the core of G.

Theorem 3. *Let G be an asymptotic Cayley tree and let $d^G(\lambda)$ denote the multiplicity of the eigenvalue $\lambda \in S_{pp}^G$, i.e. the dimension of the eigenspace of K^G corresponding to λ. Then the following relations hold:*

$$\sum_{\mu \in S_{pp}^{G^{(0)}} \backslash \left[-\frac{a}{q}, \frac{a}{q}\right]} d^{G^{(0)}}(\mu) - 2\sharp V_0^G \leqslant \sum_{\mu \in S_{pp}^{G} \backslash \left[-\frac{a}{q}, \frac{a}{q}\right]} d^{G}(\mu) \leqslant \sum_{\mu \in S_{pp}^{G^{(0)}} \backslash \left[-\frac{a}{q}, \frac{a}{q}\right]} d^{G^{(0)}}(\mu)\,. \tag{60}$$

Before proving this result it is convenient to establish the following lemma.

Lemma 3. *Let G be an asymptotic Cayley tree obtained by grafting $p \geqslant 1$ trees isomorphic to T_q' at a vertex v_0 of a graph B (which is either finite or an asymptotic Cayley tree), and assume $w_k, k = 1\ldots m$, and $z_j, j = 1\ldots n$, satisfying $w_m < w_{m-1} < \ldots < w_1 < 0 < z_1 < z_2 < \ldots < z_n$ are the poles of $Q_{v_0}^B$ in $(-\frac{q}{a}, \frac{q}{a})$. Then α_p, given by (34), has exactly one pole $z_i^* \in (z_i, z_{i+1})$ for each $i = 1, \ldots, n-1$, and $w_j^* \in (w_{j+1}, w_j)$ for each $j = 1, \ldots, m-1$. Moreover, α_p has no pole in (w_1, z_1) and at most one pole in each of the intervals $(-\frac{q}{a}, w_m)$ and $(z_n, \frac{q}{a})$.*

Proof. By (48) the poles of $Q_{v_0}^B(z)$ are of the form λ^{-1}, where $\lambda \in S_{pp}^B \setminus \{0\}$ has an eigenfunction ϕ_λ that overlaps $|v_0\rangle$, i.e. $\phi_\lambda(v_0) \neq 0$. By theorem 2(*i*) this implies that $\lambda \notin [-\frac{a}{q}, \frac{a}{q}]$ and so the eigenvalues in question are z_i^{-1}, $i = 1, \ldots, n$, and $w_j^{-1}, j = 1, \ldots, m$, and (48) yields

$$zQ_{v_0}^B(z) = \int_{-a/q}^{a/q} \frac{z\rho_{v_0}^B(\lambda)}{1 - z\lambda}\, d\lambda + \sum_{i=1}^{n} \frac{za_i}{1 - zz_i^{-1}} + \sum_{j=1}^{m} \frac{zb_j}{1 - zw_j^{-1}}\,, \tag{61}$$

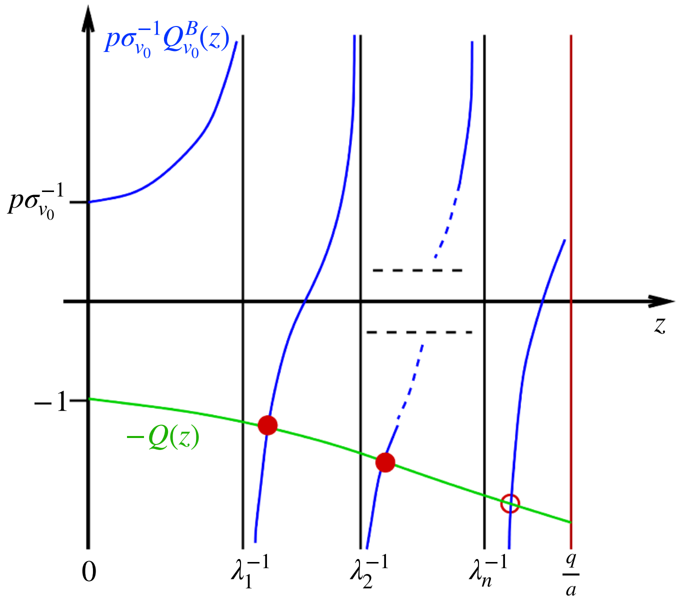

Figure 1. The solutions to the pole condition (63). Blue (dark grey) lines represent the branches of $p\sigma_{v_0}^{-1}Q_{v_0}^B(z)$, and the green (light grey) line is $-Q(z)$. Solid points indicate solutions z^* that must exist; the solution indicated by the open point exists only if the condition (64) is satisfied.

where the coefficients a_i and b_j are positive numbers given by

$$a_i = \langle v_0|e_i|v_0\rangle \quad \text{and} \quad b_j = \langle v_0|f_j|v_0\rangle\,. \tag{62}$$

Here e_i and f_j denote the spectral projections of K^B in $\ell^2(B)$ corresponding to z_i^{-1} and w_j^{-1}, respectively. It follows by differentiation that $zQ_{v_0}^B(z)$ is a strictly increasing function on each of the open subintervals of $(-\frac{q}{a},\frac{q}{a})$ arising from the subdivision $-\frac{q}{a} < w_m < w_{m-1} < \cdots < w_1 < 0 < z_1 < z_2 < \cdots < z_n < \frac{q}{a}$.

Since $Q(z) \neq 0$ for $z \in (-\frac{q}{a},\frac{q}{a})$ we see from (34) that α_p has poles at values of z for which

$$p\,Q_{v_0}^B(z) = -\sigma_{v_0}Q(z)\,, \tag{63}$$

and is otherwise analytic on $(-\frac{q}{a},\frac{q}{a})$. The function $-zQ(z)$ (18) takes the value 0 at $z=0$ and is decreasing for $0<z<q/a$, and the function $zQ_{v_0}^B(z)$ vanishes at $z=0$ and is increasing on the interval $(0,z_1)$, so in this interval there is no solution to (63). In the intervals (z_i,z_{i+1}) the function $zQ_{v_0}^B(z)$ increases monotonically from $-\infty$ at $z=z_i$ to $+\infty$ at z_{i+1}, so in each of these intervals there is a unique solution, z_i^*, to (63), see figure 1. In the interval $(z_n,q/a)$ there is a solution z_n^* if and only if

$$\sigma_{v_0}^{-1}p\,Q_{v_0}^B\left(\frac{q}{a}\right) > -Q\left(\frac{q}{a}\right) = -\frac{2q-2}{q-2}\,. \tag{64}$$

This proves the statement concerning poles in $(0,\frac{a}{q})$. Similar arguments apply for the poles in $(-\frac{a}{q},0)$, and clearly there is no pole at $z=0$. Hence, the proof is complete. □

Proof of theorem 3. Let $\mathcal{G}$ be a G-sequence as in (29) with $n = \sharp V_0^G$, i.e. such that all trees T_q' to be grafted at any vertex in V_0^G are grafted in one step. Using the notation of lemma 3 with $B = G^{(i)}$ and with G denoting $G^{(i+1)}$, it then suffices to prove that

$$\sum_{\mu \in S_{pp}^B \setminus \left[-\frac{a}{q}, \frac{a}{q}\right]} d^B(\mu) - 2 \leqslant \sum_{\lambda \in S_{pp}^G \setminus \left[-\frac{a}{q}, \frac{a}{q}\right]} d^G(\lambda) \leqslant \sum_{\mu \in S_{pp}^B \setminus \left[-\frac{a}{q}, \frac{a}{q}\right]} d^B(\mu) . \tag{65}$$

Consider first $Q_u^G(z)$, where $u \notin B$. Recalling again that $Q(z)$ and $z^{-1}P(z)$ are analytic in $(-\frac{q}{a}, \frac{q}{a})$, it follows from (33) that the poles of $Q_u^G(z)$ are precisely those of $\alpha_p(z)$ as given by lemma 3 and hence their inverse values, $(z_i^*)^{-1}$ and $(w_i^*)^{-1}$, belong to S_{pp}^G.

In order to compare eigenvalue multiplicities for B and G we consider three possible cases for an eigenvalue λ of K^G outside $[-\frac{a}{q}, \frac{a}{q}]$:

1) λ^{-1} is different from all z_i, z_i^*, w_j, w_j^*. By lemma 1(*i*), λ^{-1} is not a pole of $Q_{v_0}^G$. Hence all corresponding eigenfunctions of K^G vanish at v_0 and are extensions of eigenfunctions of K^B with eigenvalue λ. Since also $Q_{v_0}^B$ does not have a pole at λ^{-1} it follows that

$$d^G(\lambda) = d^B(\lambda) . \tag{66}$$

2) $\lambda = z_i^{-1}$. In this case, K^B has an eigenfunction ϕ_λ with eigenvalue λ that overlaps $|v_0\rangle$. Hence, the eigenspace of K^B has a basis consisting of ϕ_λ and $d^B(\lambda) - 1$ eigenfunctions that vanish at v_0. The latter can be extended trivially to eigenfunctions of K^G with eigenvalue λ. On the other hand, K^G can have no such eigenfunction overlapping $|v_0\rangle$, since otherwise $Q_{v_0}^G$ would have a pole at z_i, which is not the case by lemma 1(*i*). We conclude that

$$d^G(\lambda) = d^B(\lambda) - 1 . \tag{67}$$

Clearly, the same statement holds if $\lambda = w_j^{-1}$.

3) $\lambda = (z_i^*)^{-1}$. Since $Q_{v_0}^G$ has a pole at z_i^* by lemma 1(*iv*) we see that K^G has an eigenfunction ψ_λ with eigenvalue λ that overlaps $|v_0\rangle$ and hence the corresponding eigenspace of K^G has a basis consisting of ψ_λ and $d^G(\lambda) - 1$ eigenfunctions that vanish at v_0 (and on the trees grafted at v_0 by lemma 2). Restricting the latter to B we obtain $d^G(\lambda) - 1$ linearly independent eigenfunctions of K^B with eigenvalue λ. On the other hand, K^B does not have any such eigenfunctions that overlap $|v_0\rangle$ since Q^B is regular at z_i^*. Hence,

$$d^G(\lambda) = d^B(\lambda) + 1 \tag{68}$$

in this case. The same conclusion holds for $\lambda = (w_j^*)^{-1}$.

Note that if $\lambda \notin [-\frac{a}{q}, \frac{a}{q}]$ is an eigenvalue of K^B that is not among the z_i and w_j then $Q_u(z)$ has a pole at λ^{-1} for some $u \in V(B)$ and lemma 1(*ii*) then implies that λ is an eigenvalue of K^G. According to 1) above the contribution of such eigenvalues to the sum of all eigenvalue multiplicities is the same for K^B and K^G. For the remaining eigenvalues we get, using 2) and 3) above,

$$\sum_{i=1}^{n} d^G\left(z_i^{-1}\right) + \sum_{j=1}^{m} d^G\left(w_i^{-1}\right) = \sum_{i=1}^{n}\left(d^B\left(z_i^{-1}\right) - 1\right) + \sum_{j=1}^{m}\left(d^B\left(w_j^{-1}\right) - 1\right)$$
$$= \sum_{i=1}^{n} d^B\left(z_i^{-1}\right) + \sum_{j=1}^{m} d^B\left(w_j^{-1}\right) - (n+m) \tag{69}$$

and

$$\sum_i d^G\left(\left(z_i^*\right)^{-1}\right) + \sum_j d^G\left(\left(w_i^*\right)^{-1}\right) = \sum_i \left(d^B\left(\left(z_i^*\right)^{-1}\right) + 1\right) + \sum_j \left(d^B\left(\left(w_j^*\right)^{-1}\right) + 1\right)$$
$$= \sum_i d^B\left(\left(z_i^*\right)^{-1}\right) + \sum_j d^B\left(\left(w_j^*\right)^{-1}\right) + (k+l)\,, \tag{70}$$

where $k = n-1$ or $k = n$ and $l = m-1$ or $l = m$ by lemma 3. Finally, collecting the contributions to the sums of all eigenvalue multiplicities for K^G and K^B so displayed, corresponding to eigenvalues outside $[-\frac{a}{q}, \frac{a}{q}]$, the claimed inequalities follow. □

Remark 2. Our results have a number of implications:

1. Combining theorem 2(*ii*) and the upper bound of theorem 3 we also have

$$\sum_{\mu\in S_{pp}^G} d^G(\mu) \leqslant \sum_{\mu\in S_{pp}^{G^{(0)}}} d^{G^{(0)}}(\mu) = \sharp V\left(G^{(0)}\right)\,, \tag{71}$$

which is the analogue of the bound given in [2], Theorem 4.2, for the adjacency matrix.
2. The lower bound of theorem 3 shows that the presence of ℓ^2 states is ubiquitous. Specifically, if the number of eigenfunctions of $K^{G^{(0)}}$ with eigenvalues lying outside the continuous spectrum exceeds twice the number of graft vertices, then G necessarily has a non-empty pure point spectrum.
3. From the proof of lemma 3 we see that, as trees are added to the core graph to generate the sequence $\mathcal{G}$, discrete eigenvalues either remain unchanged or move towards the continuum.

4. Examples

In this section we consider simple examples of asymptotic Cayley trees where the pure point spectrum can be calculated rather explicitly. First we look at the case when the core is a complete graph and we graft the same number of planted Cayley trees at each vertex of the core. Then we study the case when the core is a regular graph (all vertices of the same degree) and we graft planted Cayley trees on the core vertices so that the resulting asymptotic Cayley tree is also regular.

4.1. The core is a complete graph

Let C be the complete graph on n vertices which has spectrum $\{1, -1/(n-1)\}$. The positive eigenvalue is simple, and the eigenfunction is constant on C. The negative eigenvalue has multiplicity $n-1$. The eigenfunctions can be chosen to take the value $-(n-1)$ on one vertex, which can be any vertex on C, and 1 on all other vertices. Of the n such functions, any $n-1$ are linearly independent. Now graft p copies of T_q' to each vertex of C. The resulting graph G is characterized by the 3 integers p, q, n. The permutation symmetry of the vertices of C ensures that the eigenfunctions of K^G take the same form on C as those of K^C.

In the planted Cayley trees the eigenfunction ψ of K^G with eigenvalue λ is given by (55) and (56). For the eigenfunction that is constant on C, the eigenvalue equation for a vertex on C is

$$p\xi^{-1} + n - 1 = (p + n - 1)\,\lambda. \tag{72}$$

Combining (56) and (72) shows that $\xi = 1$ or

$$\xi^{-1} = \frac{p+n-1}{q\,(n-1) - (p+n-1)}. \tag{73}$$

We see that ψ is in $\ell^2(G)$ if

$$1 + \sqrt{q-1} < \frac{q\,(n-1)}{p+n-1}, \tag{74}$$

which is always the case for large enough q if n and p are fixed. From (56) the corresponding value of λ then lies outside the continuous spectrum.

In the case of the degenerate eigenfunctions, (72) is replaced by

$$p\xi^{-1} - 1 = (p + n - 1)\,\lambda, \tag{75}$$

and we obtain

$$\xi^{-1} = \xi_{\pm}^{-1} = \frac{-q \pm \sqrt{q^2 - 4q\,(n-1)\,(p+n-1) + 4\,(p+n-1)^2}}{2\,(q\,(n-1) - (p+n-1))}. \tag{76}$$

If n and p are fixed, and q is large enough, then $(q-1)\xi_+^{-2} < 1$ and ψ is in $\ell^2(G)$. In particular, if

$$q > 4\left(n - 1 + \frac{p^2 - 1}{n-1}\right)(n + p - 1) \tag{77}$$

then, uniformly in $n \geqslant 2$ and $p \geqslant 1$, the non-constant eigenfunctions of K^C correspond to ℓ^2-eigenfunctions of K^G. This is in contrast to the eigenstate that is constant on the core which persists for all $n \geqslant 4$ and all $q \geqslant 3$.

4.2. q-Regular asymptotic trees

Theorem 1 applies uniformly to all asymptotic Cayley trees but theorems 2 and 3 are strongest for those whose number of graft vertices is small relative to the total number of vertices in $G^{(0)}$. By contrast we can make more detailed statements about S^G_{pp} for asymptotic Cayley trees consisting of equal numbers of planted Cayley trees grafted to every vertex of a core that is a regular graph.

Theorem 4. *Let $G^{(0)}$ be a $(q-p)$-regular graph, where $q > p + 2$, and let G be the q-regular asymptotic tree obtained by grafting $p \geqslant 1$ copies of T'_q at each vertex $v \in G^{(0)}$. Then the following hold:*

(i) if $p > q-1-\sqrt{q-1}$ then S^G_{pp} is empty;
(ii) if $p \leqslant q-1-\sqrt{q-1}$ there is at least one member of S^G_{pp}.

Proof. For a walk in G the step from a vertex $u \in G^{(0)}$ to a neighbouring vertex $v \in G^{(0)}$ can be decomposed into excursions from u into the attached trees followed by a final step along the edge (u,v). As all vertices in G have the same degree, it follows that, for $u \in G^{(0)}$,

$$Q^G_u(z) = Q^{G^{(0)}}_u(f(z)), \tag{78}$$

where

$$f(z) = \frac{z}{1+\frac{p(1-P(z))}{q-p}}. \tag{79}$$

By the spectral theorem, the poles of $Q^{G^{(0)}}_v(z)$ are at λ^{-1}, where $\lambda \in S^{G^{(0)}} \setminus \{0\}$ and those of $Q^G_v(z)$ are at λ'^{-1}, where $\lambda' \in S^G_{pp} \setminus \{0\}$. It follows from (78) that if $\lambda \in S^{G^{(0)}} \setminus \{0\}$, and a real solution $\lambda'^{-1} \in [-1,1]$ exists to $\lambda^{-1} = f(\lambda'^{-1})$, then $\lambda' \in S^G_{pp}$. From (79) we obtain

$$\lambda = \frac{\lambda' q(2q-2-p) + \operatorname{sgn}(\lambda')\, p\sqrt{q^2\lambda'^2 - a^2}}{2(q-1)(q-p)}, \tag{80}$$

and see immediately that there are real solution pairs λ, λ' only if $|\lambda'| > \frac{a}{q}$ and $|\lambda| > \frac{2q-2-p}{(q-p)\sqrt{q-1}}$. If $p > q-1-\sqrt{q-1}$ the second condition becomes $|\lambda| > 1$ which is not possible so statement (i) is proved. On the other hand, if $p \leqslant q-1-\sqrt{q-1}$, straightforward algebra shows that there are solution pairs to (80) in which $s \in [0, \log\frac{q-p-1}{\sqrt{q-1}}]$ is computed from

$$|\lambda| = 2\frac{\sqrt{q-p-1}}{q-p}\cosh\left(s + \tanh^{-1}\frac{p}{2q-p-2}\right), \tag{81}$$

and then λ' is given by

$$\lambda' = \operatorname{sgn}(\lambda)\,\frac{a}{q}\cosh(s). \tag{82}$$

In particular $S^{G^{(0)}}$ always contains $\lambda = 1$, and we obtain in that case

$$\lambda' = 1 - \frac{p(q-p-2)}{q(q-p-1)}, \tag{83}$$

which proves statement (ii). Note further that for every eigenvalue $\lambda \in S^{G^{(0)}}$ satisfying these conditions there is at least one eigenstate of K^G with corresponding eigenvalue λ' given by using (81) and (82). □

We note that if $G^{(0)}$ is a complete graph on n vertices then G is q-regular if $n+p-1=q$. In that case theorem 4(ii) gives the inequality (74).

Theorem 4 is easily generalised to the graphs for which the degree of vertices in $G^{0)}$ differs from $q-p$ but we will not give details here. On a regular graph K^G is simply proportional to the adjacency matrix.

Our result is therefore related to those of [10] which considers the class $\mathcal{C}_q$ of q-regular graphs such that the density of closed loops inside a ball of radius R goes to zero as R goes to infinity. Clearly, q-regular asymptotic Cayley trees are a special case of this construction. In [10] it is shown that the continuous spectrum of a graph $G \in \mathcal{C}_q$ is the same as that of T_q and the limiting spectral density is (23); this result can be viewed as a coarse-grained version of theorem 1.

5. Quantum walks

5.1. Definitions

As an example of the application of our results we consider a class of continuous time quantum walks. Let H be a self-adjoint operator on $\ell^2(G)$ with respect to the inner product $(\cdot,\cdot)$. A quantum walk on G is then defined by the unitary time development operator

$$U(t) = e^{-itH}. \tag{84}$$

If the state of the walk at time $t = 0$ is given by the normalized vector $|\psi_0\rangle$, then its state at time t is $|\psi(t)\rangle = U(t)|\psi_0\rangle$. In particular, if the walk is at the vertex u at time 0, then the probability amplitude that it is at a vertex v at time t is given by

$$A_t(u,v) = \sqrt{\frac{\sigma_u}{\sigma_v}}\langle v|U(t)|u\rangle, \tag{85}$$

and the probability of finding the walk at v at time t, given that it was at u at time $t = 0$ is

$$P_t(u,v) = |A_t(u,v)|^2. \tag{86}$$

For classical random walks we define the spectral dimension of a graph G to be d_s if, and only if, $p_n(u,u) \sim n^{-d_s/2}$ as $n \to \infty$. It can be shown that d_s, if it exists, does not depend on u. If $p_n(u,u)$ falls off faster than any power of n then one says that the spectral dimension is infinite. From (11) it follows that

(i) If $Q_u(z)$ is analytic at $z = 1$ then $p_n(u,u)$ decays at least exponentially with n so $d_s = \infty$. Using (18) we see that this is the case for a regular Cayley tree.
(ii) If $Q_u(z)$ has a branch point of the form $(1-z)^\beta$, then, by a Tauberian theorem [11], $d_s = 2\beta + 2$. This is the case on, for example, the graph $\mathbb{Z}$ (equivalently the regular Cayley tree with $q = 2$) which has $d_s = 1$, as can be seen by setting $q = 2$ in (18). In [12] the generating function and its corresponding spectral dimension are calculated for a number of simple graphs.

Analogously the quantum spectral dimension d_{qs} of a graph G is defined by

$$P_t(u,u) \sim t^{-d_{qs}/2}, \quad t \to \infty. \tag{87}$$

For example, in the case of $\mathbb{Z}$, and with $H = -K^G$, it is easy to see that $d_{qs} = 2$. It can be shown quite generally that $d_{qs} \leqslant 2d_s$ [13]. We will see below that this upper bound on d_{qs} is not necessarily saturated; indeed on asymptotic Cayley trees $d_{qs} = 6$.

5.2. Quantum walk on an asymptotic Cayley tree

Our results on the spectrum of asymptotic Cayley trees can be used to understand the properties of the quantum walks described above on such graphs.

Theorem 5. *Let G be an asymptotic Cayley tree, and $H = -K^G$. Quantum walk on G has the following properties for large elapsed time t.*

(i) If $S^G_{pp} = \emptyset$, then the probability of staying inside a fixed ball around the starting vertex u decays at a rate t^{-3} for large t and hence $d_{qs} = 6$. (Note that this includes the case of the pure Cayley tree T_q.)

(ii) If $S^G_{pp} \subset [-\frac{a}{q}, \frac{a}{q}]$, then a walker starting at $u \notin G^{(0)} \setminus V^G_0$ stays within a fixed ball around u with a probability decaying asymptotically as t^{-3} for t large. On the other hand, a walker starting inside $G^{(0)} \setminus V^G_0$ is trapped in this set with a probability $P_t(u)$ at time t of the form

$$P_t(u) = \sum_{\lambda \in S^G_{pp}} \langle u | e_\lambda | u \rangle + \tilde{C}^G_t(u)\, t^{-\frac{3}{2}} + O\left(t^{-\frac{5}{2}}\right), \tag{88}$$

where $\tilde{C}^G_t$ is a bounded oscillating function of t (and we note that the t-independent term on the right-hand side is positive unless all eigenfunctions of K^G vanish at u).

(iii) If S^G_{pp} contains at least one eigenvalue $\lambda \notin [-\frac{a}{q}, \frac{a}{q}] \cup S^{G^{(0)}}$, then there is at least one tree T'_q grafted to $G^{(0)}$ at some vertex v_0 such that a walker starting at $u \in T'_q$ will be trapped in a ball of fixed radius $R > 0$ around u with a probability that declines exponentially with $d^G(v_0, u)$ in the limit $t \to \infty$.

To prove these results we need the following lemma.

Lemma 4. *Let G be an asymptotic Cayley tree, and let $H = -K^G$. Then the behaviour at large t of the probability amplitude $A^G_t(u,v)$ for finite $d_G(u,v)$ is given by*

$$A^G_t(u,v) = \sum_{\lambda \in S^G_{pp}} \sqrt{\frac{\sigma_u}{\sigma_v}} \langle v | e_\lambda | u \rangle\, e^{i\lambda t} + C^G_t(u,v)\, t^{-3/2} + O\left(t^{-5/2}\right), \tag{89}$$

where C^G_t is a bounded oscillating function of t and e_λ is the spectral projection of K^G corresponding to λ.

Proof. By (49) the probability amplitude $A^G_t(u,v)$ is given by

$$A^G_t(u,v) = \sqrt{\frac{\sigma_u}{\sigma_v}} \sum_{\lambda \in S^G_{pp}} \langle v | e_\lambda | u \rangle\, e^{it\lambda} - \sqrt{\frac{\sigma_u}{\sigma_v}} \frac{1}{\pi} \int_{-\frac{a}{q}}^{\frac{a}{q}} e^{it\lambda} \tilde{B}^G_{u,v}(\lambda) \sqrt{a^2 - q^2\lambda^2}\, d\lambda, \tag{90}$$

where $\tilde{B}^G_{u,v}(\lambda)$ is a rational function with no poles in $[-\frac{a}{q}, \frac{a}{q}]$. Setting $\lambda = \frac{a}{q}\cos\theta$ the integral term in (90) becomes

$$A^G_{u,v}(t)_{cont} = -\sqrt{\frac{\sigma_u}{\sigma_v}} \frac{a^2}{q\pi} \int_0^\pi e^{i\frac{a}{q}t\cos\theta} \tilde{B}^G_{u,v}\left(\tfrac{a}{q}\cos\theta\right) \sin^2\theta\, d\theta. \tag{91}$$

By partial integration this gives

$$A^G_{u,v}(t)_{cont} = -\sqrt{\frac{\sigma_u}{\sigma_v}}\frac{a}{i\pi t}\int_0^\pi e^{i\frac{a}{q}t\cos\theta}\left(\tilde{B}^G_{u,v}\left(\tfrac{a}{q}\cos\theta\right)\cos\theta - \frac{a}{q}\frac{d\tilde{B}^G_{u,v}}{d\lambda}\left(\tfrac{a}{q}\cos\theta\right)\sin^2\theta\right)d\theta\,. \tag{92}$$

Performing a further integration by parts of the contribution from the second term in parentheses and applying a standard stationary phase approximation we get

$$A^G_t(u,v)_{cont} = -\sqrt{\frac{aq\sigma_u}{2\pi t^3\sigma_v}}\left(e^{i\left(\frac{at}{q}+\pi 4\right)}\tilde{B}^G_{u,v}\left(\tfrac{a}{q}\right) - e^{-i\left(\frac{at}{q}+\pi 4\right)}\tilde{B}^G_{u,v}\left(-\tfrac{a}{q}\right)\right) + O\left(t^{-5/2}\right)\,. \tag{93}$$

Combining (90) and (93) proves the lemma. □

Proof of theorem 5. The probability that a walker is to be found within a distance R of the starting vertex $u\in G$ after time t has elapsed is

$$P_t(R,u) = \sum_{v:\,d_G(v,u)\leqslant R}|A_t(u,v)|^2\,. \tag{94}$$

First, let $S^G_{pp}=\emptyset$. Then, by lemma 4,

$$P_t(R,u) = \frac{1}{t^3}\sum_{v:\,d_G(v,u)\leqslant R}\left|C^G_t(u,v)\right|^2 + O\left(t^{-4}\right)\,, \tag{95}$$

but $C^G_t(u,v)$ is a bounded oscillating function of t so statement (i) follows.

Now let $S^G_{pp}\subset[-\frac{a}{q},\frac{a}{q}]$. If $\lambda\in S^G_{pp}$ and $u\notin V(G^{(0)})\setminus V^G_0$, then by lemma 2$(i)$ $\langle v|e_\lambda|u\rangle$ vanishes for all $v\in V(G)$ so (95) holds by lemma 4. On the other hand, if $u\in V(G^{(0)})\setminus V^G_0$ then $\langle v|e_\lambda|u\rangle=0$ for $v\notin V(G^{(0)})\setminus V^G_0$ and the probability that the walker stays inside $G^{(0)}$ at time t equals

$$\begin{aligned}P_t(u) = &\sum_{v\in V(G)\setminus V^G_0}\left|\sum_{\lambda\in S^G_{pp}}\sqrt{\frac{\sigma_u}{\sigma_v}}\langle v|e_\lambda|u\rangle e^{i\lambda t}\right|^2\\ &+2t^{-3/2}\mathrm{Re}\sum_{\mu\in S^G_{pp}}\sum_{v\in G^{(0)}\setminus V^G_0}\sqrt{\frac{\sigma_u}{\sigma_v}}\langle v|e_\lambda|u\rangle C^G_t(u,v)^*e^{i\lambda t} + O\left(t^{-5/2}\right)\,.\end{aligned} \tag{96}$$

The first term evaluates to the constant $\sum_{\lambda\in S^G_{pp}}\langle u|e_\lambda|u\rangle$, and statement (ii) follows.

Finally, if S^G_{pp} contains at least one eigenvalue $\lambda\notin[-\frac{a}{q},\frac{a}{q}]\cup S^{G^{(0)}}$ then it follows from (the proof of) lemma 2(ii) that there is at least one vertex $v_0\in V^G_0$ such that $\langle v_0|e_\lambda|v_0\rangle>0$ and if u,v belong to one of the p trees T'_q grafted at v_0 we have

$$\langle v|e_\lambda|u\rangle = \frac{\sigma_{v_0}+p}{q}\langle v_0|e_\lambda|v_0\rangle\left(\lambda P(\lambda^{-1})\right)^{d^G(v_0,u)+d^G(v_0,v)} \tag{97}$$

(where, as previously, σ_{v_0} denotes the degree of v_0 in $G^{(0)}$). Since this identity holds for all eigenvalues $\lambda \notin [-\frac{a}{q}, \frac{a}{q}]$ and $\langle v|e_\lambda|u\rangle = 0$ if $\lambda \in [-\frac{a}{q}, \frac{a}{q}]$, we obtain from (89), for any vertex $u \in T_q'$ with $d^G(v_0, u) > R$, that

$$P_t(R,u) = \sum_{v: d^G(u,v) \leqslant R} \Big| \sum_{\lambda \in S^G_{pp} \setminus \left[-\frac{a}{q}, \frac{a}{q}\right]} \tfrac{\sigma_{v_0}+p}{q} \langle v_0|e_\lambda|v_0\rangle \, e^{i\lambda t} \left(\lambda P(\lambda^{-1})\right)^{d^G(v_0,u)+d^G(v_0,v)} \Big|^2 + O(t^{-3/2}). \tag{98}$$

Here, the first term on the right-hand side is a non-vanishing bounded oscillating function of t and, recalling that $|\lambda P(\lambda^{-1})| < (q-1)^{-\frac{1}{2}}$, if $|\lambda| > \frac{a}{q}$, it decays for fixed R exponentially with $d^G(v_0, u)$. This proves statement (*iii*).

□

In the case of T_2 (i.e. the infinite chain) $S^G_{pp} = \emptyset$; using (22) and setting $q = 2$, it is easily seen that (90) reduces to the well known formula [14]

$$A_t^{T_2}(u,v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{it\cos\theta + i\ell\theta} \, d\theta = i^\ell J_\ell(t), \tag{99}$$

where J_ℓ is the ℓ-th Bessel function.

A similar analysis can be used to show that a walker who escapes into a grafted tree always behaves ballistically at large times. Here we outline the calculation to demonstrate this. Let G be an asymptotic Cayley tree obtained by grafting $p \geqslant 1$ trees T_q' at a vertex v_0 of a graph B (which hence is either finite or an asymptotic Cayley tree), and let $u \in B$. Let $\nu > 0$ be a real number and consider times t chosen so that νt is an integer. The probability that the walk is at any vertex $v_{\nu t}$ in the tree(s) a distance νt from v_0 is then given at large times by

$$P_t(v_{\nu t}, u) = (q-1)^{\nu t - 1} \left|A_t^G(u, v_{\nu t})\right|^2. \tag{100}$$

We have from (90), (32) and (50) that

$$\begin{aligned} A_t^G(u, v_{\nu t}) = & \sqrt{\frac{\sigma_u}{q}} \sum_{\lambda \in S^G_{pp} \setminus \left[-\frac{a}{q}, \frac{a}{q}\right]} \langle v_0|e_\lambda|u\rangle \left(\lambda P\left(\lambda^{-1}\right)\right)^{\nu t} \\ & - \sqrt{\frac{\sigma_u}{q}} \frac{a^2}{q\pi} \int_0^\pi e^{i\left(\frac{a}{q} t\cos\theta + \nu t\theta\right)} B_{u,v_0}\left(\tfrac{a}{q}\cos\theta\right) \sin^2\theta \, d\theta, \end{aligned} \tag{101}$$

where

$$B_{u,v_0}(\lambda) = \lambda^{-1} \frac{q\,Q\left(\lambda^{-1}\right) Q^B_{u,v_0}\left(\lambda^{-1}\right)}{\sigma_{v_0} Q\left(\lambda^{-1}\right) + p\,Q^B_{v_0}\left(\lambda^{-1}\right)}. \tag{102}$$

It follows from theorem 1 that, since $\lambda P(\lambda^{-1})$ has no poles and no zeroes, $B^G_{u,v_0}(\lambda)$ is analytic for $\lambda \notin [-\frac{a}{q}, \frac{a}{q}] \cup S^G_{pp}$. For large t, the integral can be evaluated by the method of steepest descent. If $\nu = \frac{a}{q}\sin\theta_0$, $\theta_0 \in (0, \frac{\pi}{2})$ we obtain

$$A_t^G(u, v_{\nu t})_{cont} = i\nu \sqrt{\frac{\sigma_u}{2\pi q t \lambda_+}} \left(\frac{-1}{q-1}\right)^{\frac{1}{2}\nu t} \left(B_{u,v_0}(\lambda_+)\, e^{i\phi_t} + B_{u,v_0}(-\lambda_+)\, e^{-i\phi_t}\right), \tag{103}$$

where $\lambda_+ = \frac{a}{q}\cos\theta_0$ and

$$\phi_t = t\left(\lambda_+ + \nu\left(\theta - \frac{\pi}{2}\right)\right) - \frac{\pi}{4}. \tag{104}$$

The contribution from S^G_{pp} decays exponentially with t, so $P_t(v_{\nu t}, u)$ is an oscillating function whose magnitude decays like t^{-1}. For $\nu = \frac{a}{q}\cosh\theta_0$, $\theta_0 \in (0,\infty)$, we obtain

$$A^G_t(u, v_{\nu t})_{cont} = i\nu\sqrt{\frac{\sigma_u}{2\pi\, qt|\lambda|}}\left(\frac{-1}{q-1}\right)^{\frac{1}{2}\nu t} B_{u,v_0}(\lambda)\, e^{-\phi_t}, \tag{105}$$

where $\lambda = -i\frac{a}{q}\sinh\theta_0$ and

$$\phi_t = t(\nu\theta_0 - |\lambda|), \tag{106}$$

so all contributions to $P_t(v_{\nu t}, u)$ decay exponentially with t. Hence the walker's motion is ballistic at large times, and there is an Airy type front moving with velocity $\nu_c = \frac{a}{q}$. This behaviour is qualitatively very similar to the $q = 2$ case [14], and to that of continuous time quantum walk on the infinite toothed comb [13].

In summary, the motion on a pure Cayley tree is ballistic and, as $t \to \infty$, the walker moves towards infinity at a constant rate. The probability for return to the starting point after time t decays like t^{-3}. This is in contrast with the exponential decay of the classical case; on the highly branched structure of a Cayley tree, quantum walkers remain in the vicinity of their starting point *longer* than do classical walkers. In the case of an asymptotic Cayley tree a new feature occurs if $S^G_{pp} \neq \emptyset$: the square summable eigenfunctions prevent the walk from moving to infinity with probability one so the walker is at least partially trapped in the vicinity of the core. Whether such eigenfunctions exist depends on the structure of the core graph but we have shown that in general they do, and given some explicit examples. The part of the wave function which is not trapped behaves qualitatively as on the pure Cayley tree; so a walker who escapes from the vicinity of the core moves towards infinity at a constant rate. The core graph therefore behaves like a trap and the resulting properties of quantum walk are qualitatively similar to those observed on graphs with a trap potential on a single site [15].

Data availability statement

No new data were created or analysed in this study.

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