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A preconditioner for the 3D Oseen equations

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We describe a preconditioner for the linearised incompressible Navier-Stokes equations (the Oseen equations) which requires as components only a preconditioner/solver for each of a discrete Laplacian and a discrete advection-diffusion operator. With this preconditioner, convergence of an iterative method such as GMRES is independent of the mesh size and depends only mildly on the viscosity parameter (the inverse Reynolds number). Thus when the component preconditioner/solvers are effective on their respective sub-problems (as one expects with an appropriate multigrid cycle for instance) a fast Oseen solver results.

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1 Introduction

Much work on fast (and perhaps parallel) solution algorithms for scalar elliptic partial differential equation problems has been reported in the last 25 years and there are very many effective and efficient ways to solve for example the discrete Laplacian ([13], [12]). For iterative approaches it is also the case that approximate solutions can be obtained very quickly: for example a single multigrid cycle will typically reduce the error in an initial guess by an order of magnitude. For non-self-adjoint problems such as advection-diffusion there are also good approximate solvers based on multigrid ([8],[9],[17],[7]) or on other iterations, but this problem is certainly more challenging, particularly when advection dominates and this is an active research area.

In this short paper we will not consider such scalar partial differential equation problems, but will assume that good approximate solvers for both the Laplacian and the advection-diffusion operator are available. (In the numerical results that we present, we will employ a multigrid V-cycle.) Our goal is rather to describe a new preconditioned iterative solution algorithm for the Oseen problem (the linearised incompressible Navier-Stokes problem) which uses approximate Laplace and advection-diffusion solves.

We consider only steady-state problems here: for also time-dependent problems see [11]. Multigrid approaches for the Oseen problem are available for certain discretisations ([16]).

2 The Oseen problem

For a flow domain Ω the Oseen problem is to find the velocity $\mathbf{u}$ and pressure p satisfying

$$\begin{aligned} \mathbf{w}\cdot\nabla\mathbf{u} - \nu\nabla^2\mathbf{u} - \nabla p &= 0 \qquad \text{in } \Omega && (2.1)\\ \nabla\cdot\mathbf{u} &= 0 \qquad \text{in } \Omega. && (2.2) \end{aligned}$$

together with appropriate boundary conditions. It arises as the simplest linearisation of the incompressible Navier-Stokes equations

$$\begin{aligned} \mathbf{u}\cdot\nabla\mathbf{u} - \nu\nabla^2\mathbf{u} - \nabla p &= 0 \qquad \text{in } \Omega && (2.3)\\ \nabla\cdot\mathbf{u} &= 0 \qquad \text{in } \Omega. && (2.4) \end{aligned}$$

describing the steady flow of an incompressible viscous fluid with viscosity ν. A Picard iteration for (2.3,2.4) for example gives rise to an Oseen problem of the form (2.1,2.2) at each iteration where the known velocity field or 'wind' $\mathbf{w}$ is the velocity field from the previous Picard iteration. The usual choice $\mathbf{w} = \mathbf{0}$ starts the process with a linear Stokes problem. Convergence of this Picard iteration is established in [4].

Discretisation of (2.1,2.2) for example by mixed finite elements which are stable in the sense of Babuvska-Brezzi ([1]) or by the MAC finite difference scheme yields a linear system of equations of the form

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \qquad (2.5)$$

where F is a discrete advection-diffusion operator applying to each of the velocity components, B^T is a discrete gradient and B is a discrete negative divergence. The solution of this system of equations is a significant task particularly for 3-dimensional flow problems: iterative methods usually present the only feasible approach and preconditioning is required to ensure that an appropriate linear iteration will converge in a reasonable number of steps. The purpose of this paper is to describe such a preconditioner for (2.5) which requires the approximate solution of only two subproblems:

- a discrete Laplace problem
- a discrete advection-diffusion problem.

3 The preconditioner

The starting point is the realisation that convergence of an appropriate Krylov subspace iterative method such as GMRES [10] will be extremely rapid for a linear system written in the form $\mathcal{K}x = b$ when $\mathcal{K}$ has only a few distinct eigenvalues. More precisely, termination will occur in the same number of steps as the dimension of the Krylov subspace

$$\mathrm{span}\{r, \mathcal{K}r, \mathcal{K}^2 r, \mathcal{K}^3 r, \ldots\} \tag{3.1}$$

where the vector r is the residual of the linear system for a given initial guess (and so in general must be considered arbitrary). Thus if we (right) precondition (2.5) by the block triangular matrix

$$\begin{pmatrix} F & B^T \\ 0 & -S \end{pmatrix} \tag{3.2}$$

with $S = BF^{-1}B^T$ being the Schur complement, then the preconditioned system

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} F & B^T \\ 0 & -S \end{pmatrix}^{-1} \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} I & 0 \\ BF^{-1} & I \end{pmatrix} \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{3.3}$$

would require only 2 iterations for termination of GMRES since

$$\left(\begin{pmatrix} I & 0 \\ BF^{-1} & I \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which ensures that the Krylov subspace (3.1) with

$$\mathcal{K} = \begin{pmatrix} I & 0 \\ BF^{-1} & I \end{pmatrix}$$

as in (3.3) is of dimension 2 for any r. The required solution is recovered from

$$\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} F & B^T \\ 0 & -S \end{pmatrix}^{-1} \begin{pmatrix} v \\ q \end{pmatrix}.$$

Conveniently one can factor

$$\begin{pmatrix} F & B^T \\ 0 & -S \end{pmatrix}^{-1} = \begin{pmatrix} F^{-1} & F^{-1}B^TS^{-1} \\ 0 & -S^{-1} \end{pmatrix} = \begin{pmatrix} F^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & -B^T \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -S^{-1} \end{pmatrix} \tag{3.4}$$

from which it is explicitly seen that application of the preconditioner requires the action of F^{-1} and S^{-1} as well as a simple multiplication by B^T.

The above and related observations are described in [6] where choice of signs of the blocks in the preconditioner is also considered.

The outcome is that *if* the action of F^{-1} and of S^{-1} were readily available then a suitable Krylov subspace iterative methods like GMRES used with the preconditioner (3.2) implemented via the factorisation (3.4) would terminate after 2 iterations. Correspondingly, if good approximations to the action of F^{-1} and of S^{-1} are available then one can expect convergence in a small number of iterations - the eigenvalues of $\mathcal{K}$ which are all at 1 in the exact case will 'spread out' when approximations are used, but will remain clustered for good approximations.

The key point here is that if we have an approximate solver for the advection-diffusion problem as we have assumed, then the action of the inverse advection-diffusion matrix F^{-1} can be approximated. It is less obvious that the action of S^{-1} can be approximated with a Laplace solve and simpler operations, but this is precisely what we show in the next section. If such an approximate solver for S was not available then there would seem little point in pursuing this approach.

4 Schur complement approximation

Considering the underlying differential operators which comprise the Schur complement, S, we have

$$-S = -BF^{-1}B^T \sim \nabla \cdot (\mathbf{w} \cdot \nabla - \nu\nabla^2)^{-1}\nabla$$

Since the differential (as opposed to the discrete) operators commute, it follows that

$$-S \sim \nabla^2(\mathbf{w} \cdot \nabla - \nu\nabla^2)^{-1}$$

so that

$$-S^{-1} \sim (\mathbf{w} \cdot \nabla - \nu\nabla^2)(\nabla^2)^{-1}.$$

Thus this motivational argument (which certainly lacks rigour) points to the action of $-S^{-1}$ as the product of an advection-diffusion operator and an inverse Laplacian. A more detailed argument based on the fundamental solution tensor for the Oseen operator given in [3] indicates more clearly the scaling, order and dimension of these operators. For the approximation of $-S^{-1}$ we are lead to construct matrices F_p and A_p which are of the dimension of the discrete pressure space and are discretisations on that space of respectively the advection-diffusion operator $\mathbf{w} \cdot \nabla - \nu\nabla^2$ and the Laplacian, ∇^2. In order to ensure the correct scaling we also employ the mass matrix, M_p, for the pressure space - this also ensures that the Schur complement preconditioner reduces in the case of the Stokes problem ($\mathbf{w} = 0$) to M_p which is optimal in that case (see [15]).

The action of $-S^{-1}$ is thus replaced by

$$M_p^{-1} F_p A_p^{-1} \tag{4.1}$$

which defines the Schur complement preconditioner. Inversion of the discrete (pressure space) Laplacian A_p in practice is replaced by an approximate Laplace solve.

Thus if $\hat{F}^{-1}$ represents the approximate inversion of the advection-diffusion operator F and $\hat{A}_p^{-1}$ the approximate inversion of the pressure Laplacian, then the inverse of the Oseen preconditioner (cf. 3.4) is

$$\begin{pmatrix} \hat{F}^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & -B^T \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \hat{M}_p^{-1} F_p \hat{A}_p^{-1} \end{pmatrix}. \tag{4.2}$$

The mass matrix M_p is readily approximated by its diagonal, so that $\hat{M}_p^{-1}$ represents 2 or 3 diagonally preconditioned conjugate gradient iterations (see [14]).

The factored form (4.2) readily shows how the preconditioner is implemented with one approximate Laplace solve (on the pressure space) and one approximate solve of an advection-diffusion problem on the velocity space.

Note that boundary conditions for F and thus $\hat{F}$ will be prescribed (that is, $\hat{F}$ is taken to approximate the same boundary value operator as F). Pressure boundary conditions are however not specified for the Oseen problem, and we aim to preserve this feature in the construction of the Schur complement. Thus, we employ homogeneous Neumann boundary conditions in the construction of F_p and $\hat{A}_p$. More details can be found in [5].

A final comment concerns the use of low order approximation spaces for pressure which do not straightforwardly allow (conforming) approximations of second order differential operators: an example illustrates how one might proceed. For a piecewise constant (P0) approximation the centroids of the triangular elements may be connected to form a Delaunay triangulation on which a (conforming) Laplacian (and advection-diffusion operator) may be constructed in the usual way employing a piecewise linear (P1) approximation.

5 A finite element discretization

We consider the Navier-Stokes equations in three space dimensions. We linearise the equation by applying a simple Picard iteration, as mentioned earlier. Thus, at each iteration we require the solution of the Oseen system of equations (2.1) and (2.2).

We discretise in space using the $Q2Q1$ finite element discretisation described in the following. Note that, it can be seen in [3] and [11], that the solver presented in section 3 performs well for many different stable and stabilised, finite element and finite difference schemes.

The domain $\Omega = (0,1)^3$ is split into finitely many non-overlapping brick elements $\tau \in \mathcal{T}_h$, such that for each element $\tau \in \mathcal{T}_h$, there exists a trilinear mapping from τ to the unit cube. We define

$$h = \max_{\tau \in \mathcal{T}_h} diam(\tau).$$

The approximation space for each velocity component, denoted by V^h, is given by the set of all continuous functions that are tri-quadratic on each element τ, denote the basis for this space by $\{\phi_h^i\}_{i=1}^{N_{\mathbf{v}}}$. The approximation space for pressure, P^h, is given by the set of all continuous functions that are tri-linear on each element τ, denote the basis for this space by $\{\psi_h^i\}_{i=1}^{N_p}$. The $Q2Q1$ finite element is LBB-stable, see [1]. We will denote the space $Q2Q1$ by $X^h = \mathbf{V}^h \times P^h$, where $\mathbf{V}^h = V^h \times V^h \times V^h$. Note, in this case P^h is a subspace of V^h. We denote the total number of degrees of freedom for a given mesh, by $N := 3N_{\mathbf{v}} + N_p$.

Using the corresponding variational form for (2.1) and (2.2), we find that discrete problem at the $n+1$ Picard iteration is given by finding $[\mathbf{u}_h^{n+1}, p_h^{n+1}] \in X^h$, such that $\mathbf{u}_h^{n+1}$ satisfies any given Dirichlet boundary conditions and

$$\begin{aligned} \nu\left(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h\right)_{\Omega^h} + \left(\mathbf{u}_h^n \cdot \nabla \mathbf{u}_h^{n+1}, \mathbf{v}_h\right)_{\Omega^h} + \left(p_n^{n+1}, \nabla \cdot \mathbf{v}_h^{n+1}\right)_{\Omega^h} &= 0 \\ \left(\nabla \cdot \mathbf{u}_h^{n+1}, q_h\right)_{\Omega^h} &= 0, \end{aligned} \tag{5.1}$$

for all $[\mathbf{v}_h, q_h] \in X^h$ such that $\mathbf{v}_h = \mathbf{0}$ on any Dirichlet boundary and the inner product $(\cdot,\cdot)_{\Omega^h}$ uses quadrature and is exact for polynomials of up to degree seven in each variable.

When convection dominates, that is

$$\nu < h \left\| \mathbf{u}^{n+1} \right\| L^\infty(\Omega^h),$$

the approximation possibly requires stabilisation. We choose to add streamline-diffusion, see [2]. This leads to the extra term

$$\left(\delta(\nu, \mathbf{u}_h^n, h)\mathbf{u}_h^n \cdot \nabla \mathbf{u}_h^{n+1}, \mathbf{u}_h^n \cdot \nabla \mathbf{v}_h\right)_{\Omega^h},$$

where $\delta(\nu, \mathbf{u}_h^n, h)$ is a given function. Note, this term improves the multigrid solver for the velocity term, see [8].

At the $n+1$ iteration we define define the velocity Laplacian, velocity convection and divergence matrices associated with the space X^h to be $M_{\mathbf{v}}^h$, $A_{\mathbf{v}}^h$, $N_{\mathbf{v}}^h$ and $B = [B_x^h, B_y^h, B_z^h]$, respectively. The $(i,j)-$th entry of each of these matrices is

$$A_{\mathbf{v}}^h(i,j) = \left(\nabla \phi_h^i, \nabla \phi_h^j\right)_{\Omega^h},$$

$$N_{\mathbf{v}}^h(\mathbf{u}_h^n)(i,j) = \left(\mathbf{u}_h^n \cdot \nabla \phi_h^i, \phi_h^j\right)_{\Omega^h} + h\left(\delta(\nu, \mathbf{u}_h^n, h)\mathbf{u}_h^n \cdot \nabla \phi_h^i, \mathbf{u}_h^n \cdot \nabla \phi_h^j\right)_{\Omega^h},$$

$$B_x^h(i,j) = \left(\frac{\partial \phi_h^i}{\partial x}, \psi_h^j\right)_{\Omega^h}, B_y^h(i,j) = \left(\frac{\partial \phi_h^i}{\partial y}, \psi_h^j\right)_{\Omega^h}$$

and

$$B_z^h(i,j) = \left(\frac{\partial \phi_h^i}{\partial z}, \psi_h^j\right)_{\Omega^h}.$$

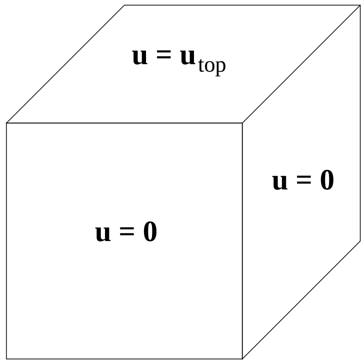

Figure 1: *Boundary conditions for the lid driven cavity problem*

Table 1: Maximum preconditioned *FpGMRES* iterations on uniform meshes for 3D lid driven cavity with $\mathbf{u}_{top} = (1, 0, 0)$.

N =	6934	15468	49072	112724
$\nu = 1/20$	29	28	29	28
$\nu = 1/40$	33	33	32	32
$\nu = 1/80$	47	46	43	42
$\nu = 1/160$	70	70	68	69

We also define the pressure mass, pressure Laplacian and pressure convection associated with the space X^h to be M_p^h, A_p^h, N_p^h, respectively. The $(i,j)-$th entry of each of these matrices is

$$M_p^h(i,j) = \left(\psi_h^i, \psi_h^j\right)_{\Omega^h}, A_p^h(i,j) = \left(\nabla\psi_h^i, \nabla\psi_h^j\right)_{\Omega^h},$$

$$\begin{aligned} N_p^h(R_V^P\mathbf{u}_h^n)(i,j) &= \left(R_V^P\mathbf{u}_h^n \cdot \nabla\psi_h^i, \psi_h^j\right)_{\Omega^h} + \\ &\quad h\left(\delta(\nu, R_V^P\mathbf{u}_h^n, h) R_V^P\mathbf{u}_h^n \cdot \nabla\psi_h^i, R_V^P\mathbf{u}_h^n \cdot \nabla\psi_h^j\right)_{\Omega^h}, \end{aligned}$$

where R_V^P is the restriction of the space V^h to the subspace P^h. Note, for discretisations such that P^h is not a subspace of V^h a suitable projection must be found, in most cases this is straight forward.

6 Numerical computations

We apply the above procedure to a three dimensional lid driven cavity problem on a unit cube, see Fig. 1. Two cases of horizontal flow applied across the top are considered, $\mathbf{u}_{top} = (1, 0, 0)$ and $\mathbf{u}_{top} = (1/\sqrt{3}, \sqrt{2}/\sqrt{3}, 0)$. We perform Picard iterations until the change in velocity between two iterates is below 10^{-5}.

The solver at each Picard iteration, denoted *FpGMRES*, consists of preconditioned GMRES with preconditioner given by (4.2). Two multigrid V-cycles are used for the

Table 2: Maximum preconditioned *FpGMRES* iterations on uniform meshes for 3D lid driven cavity with $\mathbf{u}_{top} = (1/\sqrt{3}, \sqrt{2}/\sqrt{3}, 0)$.

N =	6934	15468	49072	112724
$\nu = 1/20$	31	30	29	28
$\nu = 1/40$	39	37	36	35
$\nu = 1/80$	49	51	48	45
$\nu = 1/160$	58	64	64	61

Table 3: Maximum preconditioned *GMRES* iterations on stretched meshes for 3D lid driven cavity with $\mathbf{u}_{top} = (1/\sqrt{3}, \sqrt{2}/\sqrt{3}, 0)$, $N = 49072$.

$\alpha =$	1	2	4	8
$\nu = 1/50$	40	37	38	45
$\nu = 1/100$	52	49	47	49
$\nu = 1/200$	67	63	59	61

approximations $\hat{F}^{-1}$ and $\hat{A}_p^{-1}$, with point Gauss-Seidel smoothing, and three diagonally scaled conjugate gradient iterations are applied for $\hat{M}_p^{-1}$. We set a tolerance of 10^{-6} on *FpGMRES* and a maximum number of 100 iterations are allowed, this is about half the number of entries on a row of the coefficient matrix.

Tables 1 and 2 show the maximum number of *FpGMRES* iterations over all Picard iterations on a given mesh. Note that, if the number of iterations remains fixed with mesh refinement, then total work at each level of refinement grows at a rate proportional to the number of degrees of freedom. This can quite clearly seen to be the case. We can also see that the quality of the preconditioner has dependence on the viscosity that is proportional to $\sqrt{\nu}$, see [3].

Finally, we consider the use of anisotropic meshes, these are of great importance in fluid flow approximations since they allow an efficient means of capturing boundary layers. For a given mesh denote

$$\alpha = \max_{\tau \in \mathcal{T}_(} \frac{\text{Maximum edge length of } \tau}{\text{Minimum edge length of } \tau},$$

this measures the *stretching* of the mesh. A typical stretched mesh can be seen in Fig. 3.

In Table 3 we see that anisotropic mesh refinement does not effect the quality of the solver. On the contrary, we see that the number of iterations decreases initially with α, this would seem to be closely linked to the size of the boundary layers present in this problem.

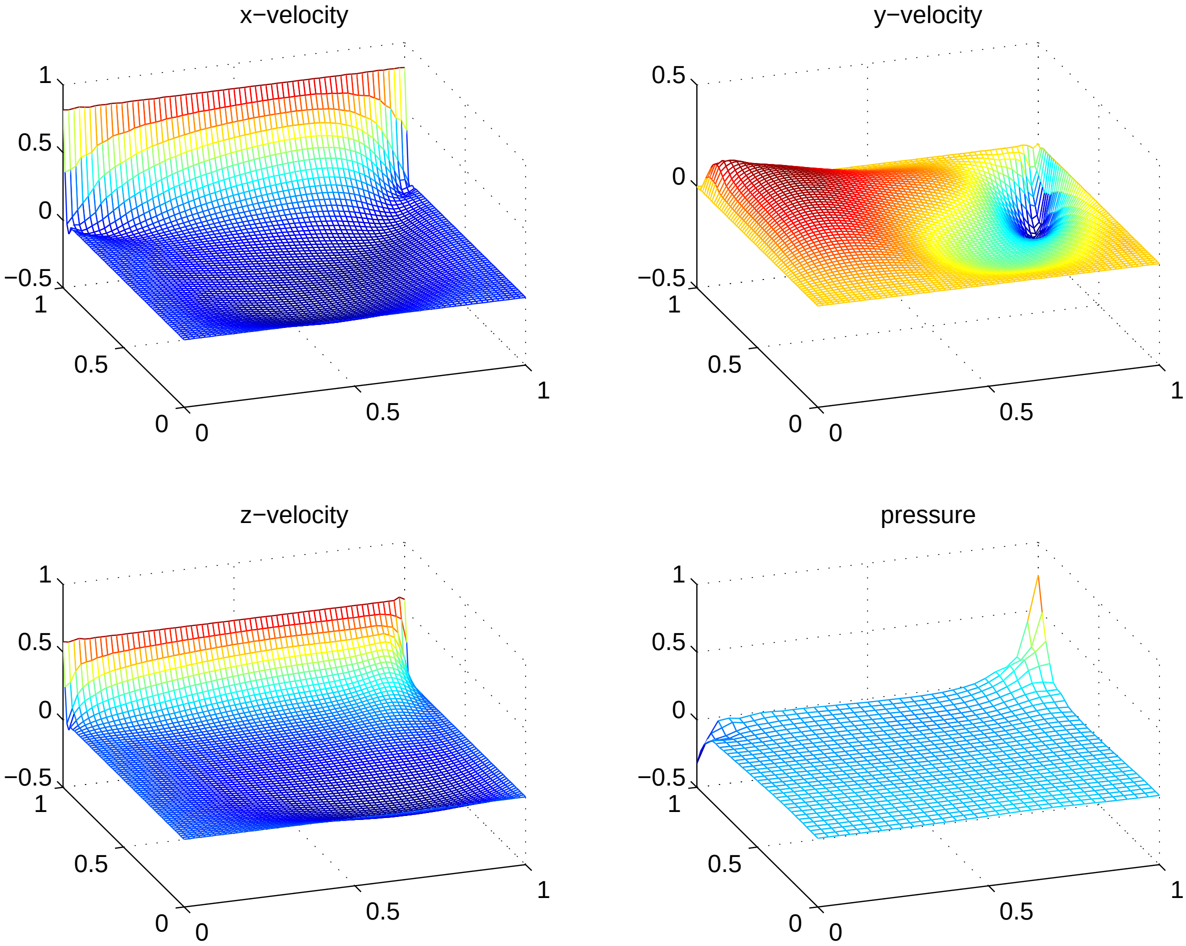

Figure 2: *Cross section view of velocities and pressure at* $z = 0.5$ *on a uniform mesh with* 112724 *degrees of freedom,* $\nu = 1/160$, $\mathbf{u}_{top} = (1/\sqrt{3}, \sqrt{2}/\sqrt{3}, 0)$.

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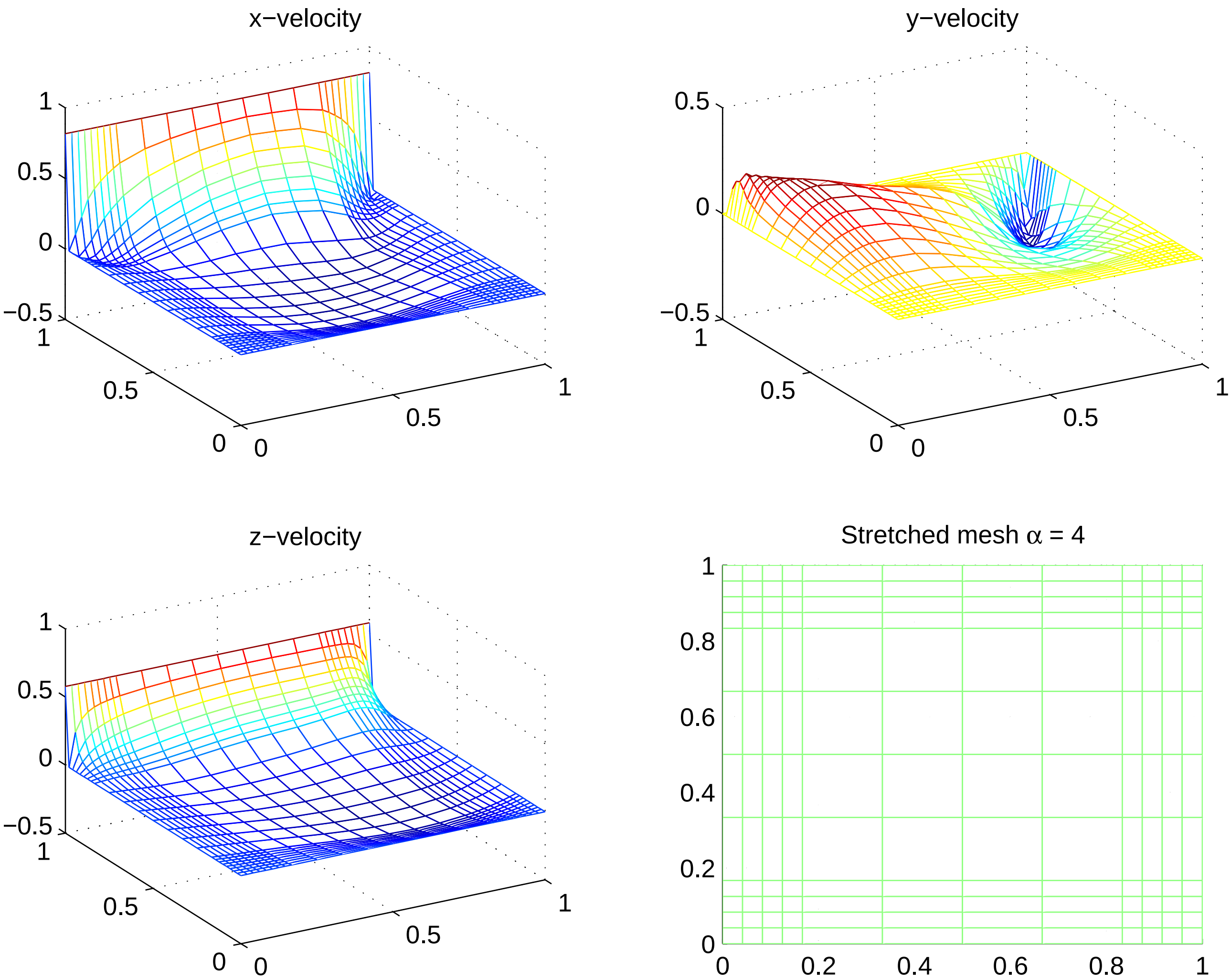

Figure 3: *Cross section view of at* $z = 0.5$ *on a stretched mesh with* 49072 *degrees of freedom,* $\nu = 1/100$, $\mathbf{u}_{top} = (1/\sqrt{3}, \sqrt{2}/\sqrt{3}, 0)$.

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