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Mean Oscillation Gradient Estimates for Elliptic Systems in Divergence Form with VMO Coefficients

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Dedicated to Professor Duong Minh Duc on the occasion of his 70th birthday

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Abstract

We consider gradient estimates for H^1 solutions of linear elliptic systems in divergence form $\partial_\alpha(A_{ij}^{\alpha\beta}\partial_\beta u^j) = 0$. It is known that the Dini continuity of coefficient matrix $A = (A_{ij}^{\alpha\beta})$ is essential for the differentiability of solutions. We prove the following results:

(a) If A satisfies a condition slightly weaker than Dini continuity but stronger than belonging to VMO, namely that the L^2 mean oscillation $\omega_{A,2}$ of A satisfies

$$X_{A,2} := \limsup_{r\to 0} r \int_r^2 \frac{\omega_{A,2}(t)}{t^2} \exp\left(C_* \int_t^R \frac{\omega_{A,2}(s)}{s}\,ds\right) dt < \infty,$$

where C_* is a positive constant depending only on the dimensions and the ellipticity, then $\nabla u \in BMO$.

(b) If $X_{A,2} = 0$, then $\nabla u \in VMO$.

(c) Finally, examples satisfying $X_{A,2} = 0$ are given showing that it is not possible to prove the boundedness of ∇u in statement (b), nor the continuity of ∇u when $\nabla u \in L^\infty \cap VMO$.

Keywords Oscillation estimates · Gradient estimates · Linear elliptic systems · Divergence form · VMO coefficients

Mathematics Subject Classification (2020) 35B65 · 35J15 · 35J47 · 35R05

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1 Introduction

Let $n \geq 2$, $N \geq 1$ and consider the elliptic system for $u = (u^1, \ldots, u^N)$

$$\partial_\alpha(A_{ij}^{\alpha\beta}\partial_\beta u^j) = 0 \quad \text{in } B_4, \quad i = 1, \ldots, N, \tag{1.1}$$

where B_4 is the ball in $\mathbb{R}^n$ of radius four and centered at the origin, and the coefficient matrix $A = (A_{ij}^{\alpha\beta})$ is assumed to be bounded and measurable in $\bar{B}_4$ and to satisfy, for some positive constants λ and Λ,

$$|A(x)| \leq \Lambda \quad \text{for a.e.} \quad x \in B_4, \tag{1.2}$$

$$\int_{B_2} A_{ij}^{\alpha\beta}\partial_\beta\varphi^j\partial_\alpha\varphi^i\,dx \geq \lambda\|\nabla\varphi\|_{L^2(B_4)}^2 \text{ for all } \varphi \in H_0^1(B_4). \tag{1.3}$$

It is well known that if the coefficient matrix A belongs to $C_{\rm loc}^{0,\alpha}(B_4)$ then every solution $u \in H^1(B_4)$ of (1.1) belongs to $C_{\rm loc}^{1,\alpha}(B_2)$; see, e.g., Giaquinta [13, Theorem 3.2] where the result is attributed to Campanato [7] and Morrey [24]. It was conjectured by Serrin [26] that the assumption $u \in H^1(B_4)$ can be relaxed to $u \in W^{1,1}(B_4)$. This has been settled in the affirmative by Brezis [2, 3]. (See Hager and Ross [14] for the relaxation from $u \in H^1(B_4)$ to $u \in W^{1,p}(B_4)$ for some $1 < p < 2$.) Moreover, in [2, 3], it was shown that if A satisfies the Dini condition

$$\int_0^2 \frac{\bar{\omega}_A(t)}{t}\,dt < \infty, \quad \text{where } \bar{\omega}_A(r) := \sup_{x,y\in B_2,|x-y|<r} |A(x) - A(y)|, \tag{1.4}$$

then every solution $u \in W^{1,1}(B_4)$ of (1.1) belongs to $C^1(B_2)$. For related works on the differentiability of weak solutions under suitable conditions on $\bar{\omega}_A$, see also [15, 22, 23].

Differentiability of weak solutions under weaker Dini conditions involving integral mean oscillation of A has also been studied. For $0 < r \leq 2$, let

$$\begin{aligned}
\bar{\varphi}_A(r) &:= \sup_{x\in B_2}\left\{\frac{1}{|B_r(x)|}\int_{B_r(x)}|A(y) - A(x)|^2\,dy\right\}^{1/2},\\
\omega_A(r) &:= \sup_{x\in B_2}\frac{1}{|B_r(x)|}\int_{B_r(x)}|A(y) - (A)_{B_r(x)}|\,dy,\\
(A)_{B_r(x)} &:= \frac{1}{|B_r(x)|}\int_{B_r(x)}A(y)\,dy, \quad 0 < r \leq 2.
\end{aligned}$$

In Li [21] it was shown that if

$$\int_0^2 \frac{\bar{\varphi}_A(t)}{t}\,dt < \infty, \tag{1.5}$$

then every solution $u \in H^1(B_4)$ of (1.1) belongs to $C^1(B_2)$. In Dong and Kim [12] (see also [9]), this conclusion was shown to remain valid under the weaker condition that

$$\int_0^2 \frac{\omega_A(t)}{t}\,dt < \infty. \tag{1.6}$$

(Note that the finiteness of $\int_0^2 \frac{\omega_A(t)}{t}\,dt$ or $\int_0^2 \frac{\bar{\varphi}_A(t)}{t}\,dt$ implies that A is continuous.) For related works for nonlinear elliptic equations, see, e.g., [19, 25, 28] and the references therein.

The Dini condition (1.4) and its integral variants (1.5) and (1.6) are phenomenologically sharp for the differentiablity of weak solutions of (1.1). In Jin, Maz'ya and van Schaftingen

[17], examples of continuous coefficient matrices A with moduli of continuity $\bar{\omega}_A(t) \sim \frac{1}{|\ln t|}$ as $t \to 0$ were given showing the following phenomena:

• There exists a solution $u \in W^{1,1}(B_4)$ of (1.1) such that $u \in W^{1,p}(B_4)$ for all $p \in [1,\infty)$, and $\nabla u \in BMO_{\rm loc}(B_4)$ but $\nabla u \notin L^\infty_{\rm loc}(B_2)$ and $\nabla u \notin VMO_{\rm loc}(B_2)$[1].

• There exists a solution $u \in W^{1,1}(B_4)$ of (1.1) such that $u \in W^{1,p}(B_4)$ for all $p \in [1,\infty)$ but $\nabla u \notin BMO_{\rm loc}(B_2)$.

In this paper, we consider mean oscillation estimates for ∇u when A slightly fails the Dini conditions (1.4), (1.5) and (1.6). For $1 \leq p < \infty$, let $\omega_{A,p} : (0,2] \to [0,\infty)$ denote the L^p mean oscillation of A

$$\omega_{A,p}(r) = \sup_{x \in B_2} \left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} |A(y) - (A)_{B_r(x)}|^p \, dy \right\}^{1/p}.$$

It is clear that $\omega_{A,1} = \omega_A$, $\omega_{A,2} \leq \bar{\varphi}_A$, $\omega_{A,p}$ is non-decreasing in p, and $\omega_{A,p} \leq \bar{\omega}_A$ for all $p \in [1,\infty)$.

We now state our first result.

Theorem 1.1 *Let $A = (A^{\alpha\beta}_{ij})$ satisfy (1.2) and (1.3). There exists a constant $C_* > 0$, depending only on n, N, Λ and λ such that if*

$$X_{A,2} := \limsup_{r \to 0} r \int_r^2 \frac{\omega_{A,2}(t)}{t^2} \exp\left(C_* \int_t^2 \frac{\omega_{A,2}(s)}{s} \, ds \right) dt < \infty, \tag{1.7}$$

then every solution $u \in H^1(B_4)$ of (1.1) satisfies $\nabla u \in BMO_{\rm loc}(B_2)$. Moreover, if

$$X_{A,2} = 0, \tag{1.8}$$

then every solution $u \in H^1(B_4)$ of (1.1) satisfies $\nabla u \in VMO_{\rm loc}(B_2)$.

Note that condition (1.7) implies that $\omega_{A,2}(t) \to 0$ as $t \to 0$, i.e., $A \in VMO_{\rm loc}(B_2)$.

Remark 1.2 Let $1 < p < \infty$. Theorem 1.1 remains valid if $\omega_{A,2}$ is replaced by $\omega_{A,p}$ and the regularity assumption $u \in H^1(B_4)$ is replaced by $u \in W^{1,p}(B_4)$, where the constant C_* is now allowed to depend also on p. For $p \geq 2$, this follows from the inequality $\omega_{A,2} \leq \omega_{A,p}$ for those p. For $1 < p < 2$, see Proposition 3.2.

It is clear that if $\omega_{A,2}$ satisfies (1.5), then it satisfies (1.8) (and hence (1.7)). The following lemma gives examples which satisfy (1.8) but not necessarily (1.5).

Lemma 1.3 *If $\limsup_{t \to 0} \omega_{A,2}(t) \ln \frac{1}{t} < \frac{1}{C_*}$, then $X_{A,2} = 0$. If $\liminf_{t \to 0} \omega_{A,2}(t) \ln \frac{1}{t} > \frac{1}{C_*}$, then $X_{A,2} = \infty$.*

We note that, in case $\omega_{A,2}(t) \ln \frac{1}{t} \to 0$ as $t \to 0$, the BMO regularity of ∇u was proved by Acquistapace [1]. (See also [16].)

[1] The statement that $\nabla u \notin VMO_{\rm loc}(B_2)$ was not explicitly stated in [17], but can be seen from the proof of Proposition 1.5 therein.

By Lemma 1.3, an explicit example of $\omega_{A,2}$ satisfying (1.8) (for any constant C_*) but not (1.5) is

$$\omega_{A,2}(t) \sim \frac{1}{\ln\frac{64}{t}(\ln\ln\frac{64}{t})^\beta}, \quad \beta \in (0,1].$$

In addition, unlike (1.5) or (1.6), (1.8) does not imply that A is continuous, e.g.,

$$A_{ij}^{\alpha\beta}(x) = \left(2 + \sin\ln\ln\ln\frac{64}{|x|}\right)\delta_{ij}\delta^{\alpha\beta}.$$

(This can be checked using the fact that the function $s \mapsto \sin s$ is Lipschitz on $\mathbb{R}$ and the fact that the function $x \mapsto \ell(x) := \ln\ln\ln\frac{64}{|x|}$ has L^2 mean oscillation $\omega_{\ell,2}(t) \sim \frac{1}{\ln\frac{64}{t}\ln\ln\frac{64}{t}}$.)

When A is merely of vanishing mean oscillation, we note that

Remark 1.4 Let $A = (A_{ij}^{\alpha\beta})$ belong to $VMO(B_4)$ and satisfy (1.2) and (1.3). Then every solution $u \in W^{1,\infty}(B_4)$ of (1.1) satisfies $\nabla u \in VMO(B_2)$. See [12, equation (2.14)]. See also Section 2 for a different proof.

The obtained regularity in Theorem 1.1 and Remark 1.4 appears sharp. As in [17], counterexamples can be produced to show that, under (1.8),

- Solutions of (1.1) may not have bounded gradients (though their gradients are of vanishing mean oscillation by Theorem 1.1),
- $W^{1,\infty}$ solutions of (1.1) may not be differentiable (though their gradients are of vanishing mean oscillation by Remark 1.4).

Proposition 1.5 *There exist a coefficient matrix $A = (A_{ij}^{\alpha\beta}) \in C(\bar{B}_4)$ satisfying (1.2), (1.3) and (1.8) and a solution $u \in H^1(B_4)$ of (1.1) such that $\nabla u \in VMO(B_4)$ but $\nabla u \notin L^\infty_{\mathrm{loc}}(B_2)$.*

Proposition 1.6 *There exist a coefficient matrix $A = (A_{ij}^{\alpha\beta}) \in C(\bar{B}_4)$ satisfying (1.2), (1.3) and (1.8) and a solution $u \in H^1(B_4)$ of (1.1) such that $\nabla u \in L^\infty(B_4) \cap VMO(B_4)$ but $\nabla u \notin C(B_2)$.*

Theorem 1.1 is a consequence of the following proposition on the mean oscillation of the gradient ∇u in terms of the L^2 mean oscillation $\omega_{A,2}$ of A.

Proposition 1.7 *Let $A = (A_{ij}^{\alpha\beta})$ satisfy (1.2) and (1.3). Then there exists a constant $C_* > 0$, depending only on n, N, Λ and λ such that for every $u \in H^1(B_4)$ satisfying (1.1) and for $0 < r \leq R/4 \leq 1/2$, there hold*

$$\int_{B_r} |\nabla u|^2\,dx \leq \frac{C_* r^n}{R^n}\exp\left(2C_*\int_{2r}^R \frac{\omega_{A,2}(t)}{t}\,dt\right)\int_{B_R}|\nabla u|^2\,dx, \tag{1.9}$$

and

$$\int_{B_r}|\nabla u - (\nabla u)_r|^2\,dx \leq \frac{C_* r^{n+2}}{R^n}\int_{B_R}|\nabla u|^2\,dx \times \\ \times\left\{\int_{2r}^R \frac{\omega_{A,2}(t)}{t^2}\exp\left(C_*\int_t^R\frac{\omega_{A,2}(s)}{s}\,ds\right)dt\right\}^2, \tag{1.10}$$

where $(\nabla u)_r = \frac{1}{|B_r|}\int_{B_r} \nabla u\, dx$ *for* $0 < r \le 2$.

Moreover, if $u \in W^{1,\infty}(B_4)$*, then, for* $0 < r \le R/4 \le 1/2$,

$$\int_{B_r} |\nabla u - (\nabla u)_r|^2\, dx \le \frac{C_* r^{n+2}}{R^n}\left\{\int_{2r}^{R} \frac{\omega_{A,2}(t)}{t^2}\, dt\right\}^2 \sup_{B_R} |\nabla u|^2. \tag{1.11}$$

Remark 1.8 Let $1 < p < 2$. Under an additional assumption that $[A]_{BMO(B_4)}$ is sufficiently small, the estimates in Proposition 1.7 hold if $\omega_{A,2}$ is replaced by $\omega_{A,p}$ and the regularity assumption $u \in H^1(B_4)$ is replaced by $u \in W^{1,p}(B_4)$. We do not know if this smallness assumption can be dropped except for p close to 2. See Proposition 3.2.

Remark 1.9 It would be interesting to see if Theorem 1.1 and Proposition 1.7 remain valid if $\omega_{A,2}$ is replaced by $\omega_{A,1}$. In view of [9, 12], it is plausible that the answer is affirmative, but this is not clear from the techniques used in the present paper.

Remark 1.10 In the above, to keep things simple, we chose to state our results for the homogeneous system (1.1) without lower order terms. They can be generalized for non-homogeneous systems or to allow for lower order terms. See, e.g., Proposition 3.1.

2 Proofs of the Main Results

Proof of Lemma 1.3 We claim: For $\delta \in (0,1)$ and $a \in (0,\infty)$, the limit

$$L_a = \limsup_{r\to 0} r\int_r^{\delta} \frac{1}{t^2}\left(\ln\frac{1}{t}\right)^{a-1} dt$$

satisfies $L_a = \infty$ if $a > 1$, $L_a = 1$ if $a = 1$ and $L_a \le (\ln\frac{1}{\delta})^{a-1}$ if $a < 1$.

When $a = 1$, the claim is clear. By integrating by parts, we have

$$\int_r^{\delta} \frac{1}{t^2}\left(\ln\frac{1}{t}\right)^{a-1} dt = -\frac{1}{t}\left(\ln\frac{1}{t}\right)^{a-1}\Bigg|_r^{\delta} - (a-1)\int_r^{\delta} \frac{1}{t^2}\left(\ln\frac{1}{t}\right)^{a-2} dt. \tag{2.1}$$

If $a < 1$, we see from (2.1) that

$$\begin{aligned} L_a &= |a-1|\limsup_{r\to 0} r\int_r^{\delta} \frac{1}{t^2}\left(\ln\frac{1}{t}\right)^{a-2} dt \le |a-1|\limsup_{r\to 0}\int_r^{\delta} \frac{1}{t}\left(\ln\frac{1}{t}\right)^{a-2} dt \\ &= \limsup_{r\to 0}\left(\ln\frac{1}{t}\right)^{a-1}\Bigg|_r^{\delta} = \left(\ln\frac{1}{\delta}\right)^{a-1}. \end{aligned}$$

To prove the claim in the case $a > 1$, we may assume without loss of generality that $a < 2$. Note that (2.1) implies

$$L_a + (a-1)L_{a-1} = \limsup_{r\to 0} r\left\{-\frac{1}{t}\left(\ln\frac{1}{t}\right)^{a-1}\Bigg|_r^{\delta}\right\} = \infty.$$

As L_{a-1} is finite (as $1 < a < 2$), we thus have that $L_a = \infty$. The claim is proved.

We now apply the claim to obtain the desired conclusions. Consider first the case that $\limsup_{t\to 0} \omega_{A,2}(t)\ln\frac{1}{t} < \frac{1}{C_*}$. Then there exist $\varepsilon \in (0, \frac{1}{C_*})$ and $\delta \in (0,1)$ so that $\omega_{A,2}(t) \le$

$\varepsilon(\ln \frac{1}{t})^{-1}$ in $(0, \delta)$. For $\hat{\delta} \in (0, \delta)$, we compute

$$\begin{aligned} X_{A,2} &= \limsup_{r \to 0} r \int_r^{\hat{\delta}} \frac{\omega_{A,2}(t)}{t^2} \exp\left(C_* \int_t^2 \frac{\omega_{A,2}(s)}{s}\,ds\right) dt \\ &\leq \varepsilon \left(\ln \frac{1}{\hat{\delta}}\right)^{-C_*\varepsilon} \exp\left(C_* \int_{\hat{\delta}}^2 \frac{\omega_{A,2}(s)}{s}\,ds\right) \limsup_{r \to 0} r \int_r^{\hat{\delta}} \frac{1}{t^2} \left(\ln \frac{1}{t}\right)^{C_*\varepsilon - 1} dt. \end{aligned}$$

As $C_*\varepsilon < 1$, we can apply the claim to obtain

$$\begin{aligned} X_{A,2} &\leq \varepsilon \left(\ln \frac{1}{\hat{\delta}}\right)^{-1} \exp\left(C_* \int_{\hat{\delta}}^2 \frac{\omega_{A,2}(s)}{s}\,ds\right) \\ &\leq \varepsilon \left(\ln \frac{1}{\hat{\delta}}\right)^{-1+C_*\varepsilon} \left(\ln \frac{1}{\delta}\right)^{-C_*\varepsilon} \exp\left(C_* \int_{\delta}^2 \frac{\omega_{A,2}(s)}{s}\,ds\right). \end{aligned}$$

Sending $\hat{\delta} \to 0$, we obtain that $X_{A,2} = 0$.

Consider next the case that $\liminf_{t \to 0} \omega_{A,2}(t) \ln \frac{1}{t} > \frac{1}{C_*}$. Then there exist $b > \frac{1}{C_*}$ and $\delta \in (0, 1)$ so that $\omega_{A,2}(t) \geq b(\ln \frac{1}{t})^{-1}$ in $(0, \delta)$. We then have

$$\begin{aligned} X_{A,2} &= \limsup_{r \to 0} r \int_r^{\delta} \frac{\omega_{A,2}(t)}{t^2} \exp\left(C_* \int_t^2 \frac{\omega_{A,2}(s)}{s}\,ds\right) dt \\ &\geq b \left(\ln \frac{1}{\delta}\right)^{-C_*b} \exp\left(C_* \int_{\delta}^2 \frac{\omega_{A,2}(s)}{s}\,ds\right) \limsup_{r \to 0} r \int_r^{\delta} \frac{1}{t^2} \left(\ln \frac{1}{t}\right)^{C_*b - 1} dt. \end{aligned}$$

As $C_*b > 1$, we deduce from the claim that $X_{A,2} = \infty$ as desired. □

Proofs of Theorem 1.1 and Remark 1.4 The results follow immediately from Proposition 1.7. □

In order to prove Proposition 1.7, we need the following estimate for harmonic replacements. (Compare [5, Lemma 3.5], [20, Lemma 3.1].)

Lemma 2.1 *Let* $A, \bar{A}$ *satisfy* (1.2) *and* (1.3) *with* $\bar{A}$ *being constant in* B_4 *and* $f = (f_i^\alpha) \in L^2(B_4)$. *Let* $R \in (0, 2)$ *and suppose* $u, h \in H^1(B_{2R})$ *satisfy*

$$\begin{aligned} \partial_\alpha(A_{ij}^{\alpha\beta} \partial_\beta u^j) &= \partial_\alpha f_i^\alpha \quad \text{in } B_{2R}, \quad i = 1, \ldots, N, \\ \partial_\alpha(\bar{A}_{ij}^{\alpha\beta} \partial_\beta h^j) &= 0 \quad \text{in } B_{2R}, \quad i = 1, \ldots, N, \\ u &= h \quad \text{on } \partial B_{2R}. \end{aligned}$$

Then there exists a constant $C > 0$ *depending only on* n, N, Λ *and* λ *such that*

$$\|\nabla(u - h)\|_{L^2(B_{3R/2})} \leq C\Big[\|f\|_{L^2(B_{2R})} + R^{-n/2}\|A - \bar{A}\|_{L^2(B_{2R})}\|\nabla u\|_{L^2(B_{2R})}\Big].$$

Proof In the proof, C denotes a generic positive constant which depends only on n, N, Λ and λ. Using that $\bar{A}$ is constant, we have by standard elliptic estimates that

$$\|\nabla h\|_{L^\infty(B_{7R/4})} \leq C R^{-n/2}\|\nabla h\|_{L^2(B_{2R})} \leq C R^{-n/2}\|\nabla u\|_{L^2(B_{2R})}.$$

Observing that

$$\partial_\alpha (A_{ij}^{\alpha\beta} \partial_\beta (u-h)^j) = \partial_\alpha (f_i^\alpha + (\bar{A}-A)_{ij}^{\alpha\beta} \partial_\beta h^j) \quad \text{in } B_{2R}, \quad i=1,\dots,N,$$

we deduce that

$$\begin{aligned} \|\nabla(u-h)\|_{L^2(B_{3R/2})} &\le C\Big[\|f\|_{L^2(B_{7R/4})} + \|A-\bar{A}\|_{L^2(B_{7R/4})}\|\nabla h\|_{L^\infty(B_{7R/4})} \\ &\qquad + R^{-(n+2)/2}\|u-h\|_{L^1(B_{7R/4})}\Big] \\ &\le C\Big[\|f\|_{L^2(B_{2R})} + R^{-n/2}\|A-\bar{A}\|_{L^2(B_{2R})}\|\nabla u\|_{L^2(B_{2R})} \\ &\qquad + R^{-(n+2)/2}\|u-h\|_{L^1(B_{2R})}\Big]. \end{aligned} \tag{2.2}$$

To estimate $\|u-h\|_{L^1(B_{2R})}$, fix some $t>0$ and consider an auxiliary equation

$$\begin{aligned} \partial_\beta\left(\bar{A}_{ij}^{\alpha\beta}\partial_\alpha \phi^i\right) &= \frac{(u-h)^j}{\sqrt{|u-h|^2+t^2}} \text{ in } B_{2R}, \quad j=1,\dots,N, \\ \phi &= 0 \quad \text{on } \partial B_{2R}. \end{aligned}$$

Testing the above against $u-h$, we obtain

$$\int_{B_{2R}} \frac{|u-h|^2}{\sqrt{|u-h|^2+t^2}}\,dx = \int_{B_{2R}} \bar{A}_{ij}^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta(u-h)^j\,dx. \tag{2.3}$$

As $u-h$ satisfies

$$\partial_\alpha\left(\bar{A}_{ij}^{\alpha\beta}\partial_\beta(u-h)^j\right) = \partial_\alpha\left(f_i^\alpha + (\bar{A}-A)_{ij}^{\alpha\beta}\partial_\beta u^j\right) \quad \text{in } B_{2R}, \quad i=1,\dots,N,$$

we have

$$\int_{B_{2R}} \bar{A}_{ij}^{\alpha\beta}\partial_\beta(u-h)^j\partial_\alpha\phi^i\,dx = \int_{B_{2R}}\left(f_i^\alpha + (\bar{A}-A)_{ij}^{\alpha\beta}\partial_\beta u^j\right)\partial_\alpha\phi^i\,dx. \tag{2.4}$$

Inserting (2.4) into (2.3) and noting that $\|\nabla\phi\|_{L^\infty(B_{2R})} \le CR$ (as $|\partial_\beta(\bar{A}_{ij}^{\alpha\beta}\partial_\alpha\phi^i)| \le 1$), we arrive at

$$\int_{B_{2R}} \frac{|u-h|^2}{\sqrt{|u-h|^2+t^2}}\,dx \le C\Big[R^{(n+2)/2}\|f\|_{L^2(B_{2R})} + R\|A-\bar{A}\|_{L^2(B_{2R})}\|\nabla u\|_{L^2(B_{2R})}\Big].$$

Noting that the constant C is independent of t, we may send $t\to 0$ to obtain

$$\|u-h\|_{L^1(B_{2R})} \le CR^{(n+2)/2}\Big[\|f\|_{L^2(B_{2R})} + R^{-n/2}\|A-\bar{A}\|_{L^2(B_{2R})}\|\nabla u\|_{L^2(B_{2R})}\Big]. \tag{2.5}$$

The conclusion follows from (2.2) and (2.5). □

Proof of Proposition 1.7 We only need to give the proof for a fixed R, say $R=2$. Our proof is inspired by that of [21].

In the proof, C denotes a generic positive constant which depends only on n, N, Λ and λ. In particular it is independent of the parameter k which will appear below. Also, we will simply write ω instead of $\omega_{A,2}$.

Proof of (1.9): For $k\ge 0$, let $R_k = 4^{-k}$, $\bar{A}_k = (A)_{B_{2R_k}}$ and $h_k \in H^1(B_{2R_k})$ be the solution to

$$\begin{aligned} \partial_\alpha\left((\bar{A}_k)_{ij}^{\alpha\beta}\partial_\beta h_k{}^j\right) &= 0 \quad \text{in } B_{2R_k}, \quad i=1,\dots,N, \\ h_k &= u \quad \text{on } \partial B_{2R_k}. \end{aligned}$$

Let $a_k = R_k^{-n/2}\|\nabla(u-h_k)\|_{L^2(B_{R_k})}$ and $b_k = \|\nabla h_k\|_{L^\infty(B_{R_k})}$.

Note that, by triangle inequality, we have

$$\|\nabla u\|_{L^2(B_{R_k})} \leq R_k^{n/2}(a_k + b_k). \tag{2.6}$$

By elliptic estimates for h_k, we have

$$\|\nabla h_k\|_{L^2(B_{2R_k})} \leq C\|\nabla u\|_{L^2(B_{2R_k})}, \tag{2.7}$$

$$\|\nabla h_k\|_{L^\infty(B_{3R_k/2})} \leq CR_k^{-n/2}\|\nabla u\|_{L^2(B_{2R_k})}, \tag{2.8}$$

$$\|\nabla^2 h_k\|_{L^\infty(B_{3R_k/2})} + R_k\|\nabla^3 h_k\|_{L^\infty(B_{3R_k/2})} \leq CR_k^{-(n+2)/2}\|\nabla u\|_{L^2(B_{2R_k})}. \tag{2.9}$$

By Lemma 2.1,

$$\|\nabla(u-h_k)\|_{L^2(B_{3R_k/2})} \leq C\omega(2R_k)\|\nabla u\|_{L^2(B_{2R_k})}. \tag{2.10}$$

By (2.10) and (2.7),

$$R_k^{n/2}(a_k + b_k) \leq C\|\nabla u\|_{L^2(B_{2R_k})}.$$

By (2.6) and (2.10), we have

$$\begin{aligned}\|\nabla(u-h_{k+1})\|_{L^2(B_{R_{k+1}})} &\leq C\omega(2R_k)\|\nabla u\|_{L^2(B_{R_k})}\\ &\leq C\omega(2R_k)R_k^{n/2}(a_k+b_k).\end{aligned}$$

Hence

$$a_{k+1} \leq C\omega(2R_k)(a_k + b_k). \tag{2.11}$$

Next, we have by (2.10) that

$$\begin{aligned}&\|\nabla(h_{k+1}-h_k)\|_{L^2(B_{3R_{k+1}/2})}\\ &\leq \|\nabla(u-h_{k+1})\|_{L^2(B_{3R_{k+1}/2})} + \|\nabla(u-h_k)\|_{L^2(B_{3R_{k+1}/2})}\\ &\leq C\omega(2R_k)\|\nabla u\|_{L^2(B_{R_k})}\\ &\leq C\omega(2R_k)R_k^{n/2}(a_k+b_k).\end{aligned}$$

Note that $h_{k+1} - h_k$ satisfies

$$\partial_\alpha\left((\bar{A}_k)_{ij}^{\alpha\beta}\partial_\beta(h_{k+1}-h_k)^j\right) = \partial_\alpha\left((\bar{A}_k - \bar{A}_{k+1})_{ij}^{\alpha\beta}\partial_\beta h_{k+1}{}^j\right) \text{ in } B_{2R_{k+1}},\ i=1,\ldots,N,$$

As $\bar{A}_k$ is constant and

$$|\bar{A}_k - \bar{A}_{k+1}| \leq C\omega(2R_k),$$

we have by standard estimates for elliptic equations with constant coefficients and (2.8) and (2.9) (applied to h_{k+1}) that

$$\|\nabla(h_{k+1}-h_k)\|_{L^\infty(B_{R_{k+1}})} \leq C\omega(2R_k)(a_k+b_k), \tag{2.12}$$

$$R_{k+1}\|\nabla^2(h_{k+1}-h_k)\|_{L^\infty(B_{R_{k+1}})} \leq C\omega(2R_k)(a_k+b_k). \tag{2.13}$$

By (2.12),

$$b_{k+1} \leq b_k + C\omega(2R_k)(a_k+b_k). \tag{2.14}$$

By (2.11) and (2.14), we have

$$a_{k+1} + b_{k+1} \leq (1 + C\omega(2R_k))(a_k+b_k).$$

We deduce that

$$
\begin{aligned}
a_k + b_k &\le \prod_{j=0}^{k}(1 + C\omega(2R_j))(a_0 + b_0) \le C \exp\left(C\sum_{j=0}^{k}\omega(2R_j)\right)\|\nabla u\|_{L^2(B_2)} \\
&\le C \exp\left(C\Omega(2R_k)\right)\|\nabla u\|_{L^2(B_2)},
\end{aligned} \tag{2.15}
$$

where

$$
\Omega(t) := \int_t^2 \frac{\omega(s)}{s}\,ds,
$$

and where we have used the fact that $\omega(t) \le C\omega(s)$ whenever $0 < t \le s \le 4t$. We have thus shown that

$$
\int_{B_{R_k}} |\nabla u|^2\,dx \le CR_k^n \exp\big(C\Omega(2R_k)\big) \int_{B_2} |\nabla u|^2\,dx \text{ for } k \ge 0.
$$

Estimate (1.9) is readily seen.

Proof of (1.10): We write

$$
h_{R_k} = \sum_{j=0}^{k} w_j, \text{ where } w_0 = h_{R_0} \text{ and } w_j = h_{R_j} - h_{R_{j-1}} \text{ for } j \ge 1.
$$

Using the estimate $\|\nabla^2 h_{R_0}\|_{L^\infty(B_1)} \le C\|\nabla u\|_{L^2(B_2)}$ together with (2.13) and (2.15), we have

$$
\begin{aligned}
|\nabla h_{R_k}(x) - \nabla h_{R_k}(0)| &\le C|x|\sum_{j=0}^{k}\frac{\omega(2R_j)}{R_j}\exp\big(C\Omega(2R_j)\big)\|\nabla u\|_{L^2(B_2)} \\
&\le C|x|\int_{2R_k}^{2}\frac{\omega(t)}{t^2}\exp\big(C\Omega(t)\big)\,dt\|\nabla u\|_{L^2(B_2)},
\end{aligned} \tag{2.16}
$$

where we have again used the fact that $\omega(t) \le C\omega(s)$ whenever $0 < t \le s \le 4t$. This implies

$$
\begin{aligned}
&\|\nabla h_{R_k} - \nabla h_{R_k}(0)\|_{L^2(B_{R_k})} \\
&\le CR_k^{(n+2)/2}\int_{2R_k}^{2}\frac{\omega(t)}{t^2}\exp\big(C\Omega(t)\big)\,dt\|\nabla u\|_{L^2(B_2)}.
\end{aligned} \tag{2.17}
$$

Combining (2.17) with (2.10) and (2.15), we get

$$
\begin{aligned}
\|\nabla u - (\nabla u)_{R_k}\|_{L^2(B_{R_k})} &\le \|\nabla u - \nabla h_{R_k}(0)\|_{L^2(B_{R_k})} \\
&\le \|\nabla(u - h_{R_k})\|_{L^2(B_{R_k})} + \|\nabla h_{R_k} - \nabla h_{R_k}(0)\|_{L^2(B_{R_k})} \\
&\le CR_k^{(n+2)/2}\int_{2R_k}^{2}\frac{\omega(t)}{t^2}\exp\big(C\Omega(t)\big)\,dt\|\nabla u\|_{L^2(B_2)} \\
&\quad + CR_k^{n/2}\omega(2R_k)\exp\big(C\Omega(2R_k)\big)\|\nabla u\|_{L^2(B_2)}.
\end{aligned} \tag{2.18}
$$

As $\omega(2R_k) \le C\omega(t)$ whenever $2R_k \le t \le 4R_k$, we have

$$
\int_{2R_k}^{4R_k}\frac{\omega(t)}{t^2}\exp\big(C\Omega(t)\big)\,dt \ge \frac{\omega(2R_k)}{CR_k}\exp\big(C\Omega(2R_k)\big).
$$

Using this in (2.18), we deduce that for $k \ge 1$ that

$$
\|\nabla u - (\nabla u)_{R_k}\|_{L^2(B_{R_k})} \le CR_k^{(n+2)/2}\int_{2R_k}^{2}\frac{\omega(t)}{t^2}\exp\big(C\Omega(t)\big)\,dt\|\nabla u\|_{L^2(B_2)}.
$$

Estimate (1.10) follows.

Proof of (1.11): We adjust the proof of (1.10) exploiting the fact that $\nabla u \in L^\infty(B_2)$. First, using the fact that $a_k + b_k \le C\|\nabla u\|_{L^\infty(B_2)}$ in (2.13) we get instead of (2.16) the stronger estimate

$$|\nabla h_{R_k}(x) - \nabla h_{R_k}(0)| \le C|x| \int_{2R_k}^{2} \frac{\omega(t)}{t^2}\,dt \|\nabla u\|_{L^\infty(B_2)},$$

and so

$$\|\nabla h_{R_k} - \nabla h_{R_k}(0)\|_{L^2(B_{R_k})} \le C R_k^{(n+2)/2} \int_{2R_k}^{2} \frac{\omega(t)}{t^2}\,dt \|\nabla u\|_{L^\infty(B_2)}. \tag{2.19}$$

Combining (2.19) with (2.10), we get for $k \ge 1$ that

$$\begin{aligned}
\|\nabla u - (\nabla u)_{R_k}\|_{L^2(B_{R_k})} &\le \|\nabla u - \nabla h_{R_k}(0)\|_{L^2(B_{R_k})}\\
&\le \|\nabla(u - h_{R_k})\|_{L^2(B_{R_k})} + \|\nabla h_{R_k} - \nabla h_{R_k}(0)\|_{L^2(B_{R_k})}\\
&\le C R_k^{(n+2)/2} \int_{2R_k}^{2} \frac{\omega(t)}{t^2}\,dt \|\nabla u\|_{L^\infty(B_2)}\\
&\quad + C R_k^{n/2} \omega(2R_k) \|\nabla u\|_{L^\infty(B_2)}\\
&\le C R_k^{(n+2)/2} \int_{2R_k}^{2} \frac{\omega(t)}{t^2}\,dt \|\nabla u\|_{L^\infty(B_2)}.
\end{aligned}$$

Estimate (1.11) follows. □

Remark 2.2 If the Dini condition (1.4) or (1.5) holds, it can be seen from (2.12) that $\{\nabla h_k(0)\}$ converges to some $P \in \mathbb{R}^{N\times n}$, from which it follows that

$$\lim_{r\to 0} r^{-n/2} \|\nabla u - P\|_{L^2(B_r)} = 0,$$

yielding the continuity of ∇u at the origin. We have thus recovered the results on the continuous differentiability of H^1 solutions of Brezis [2, 3] and Li [21].

Proof of Proposition 1.5 We take $N = 1$ and drop the indices i, j in the expression of A (so that $A = (A^{\alpha\beta})$). Following [17, Lemma 2.1], we make the ansatz that

$$\begin{aligned}
A^{\alpha\beta}(x) &= \delta^{\alpha\beta} + a(|x|)\left(\delta^{\alpha\beta} - \frac{x^\alpha x^\beta}{|x|^2}\right),\\
u(x) &= x^1 v(|x|).
\end{aligned}$$

Then

$$\partial_\alpha(A^{\alpha\beta}\partial_\beta u) = x^1\left(v''(|x|) + \frac{n+1}{|x|} v'(|x|) - \frac{n-1}{|x|^2} a(|x|) v(|x|)\right).$$

Selecting now

$$\begin{aligned}
a(r) &= -\frac{1 + n \ln \frac{64}{r}}{(n-1)(\ln \frac{64}{r})^2 \ln\ln \frac{64}{r}},\\
v(r) &= \ln\ln \frac{64}{r},
\end{aligned}$$

we see that A is continuous in $\bar{B}_4$, satisfies (1.2) and (1.3) and u is an H^1 solution of (1.1). The matrix A admits a modulus of continuity $\bar{\omega}_A(t) \sim \frac{1}{\ln \frac{64}{t} \ln\ln \frac{64}{t}}$ as $t \to 0$ and so (1.8) holds. It is readily seen that $u \in W^{1,p}(B_4)$ for all $p \in [1,\infty)$, $\nabla u \in VMO(B_4)$ but $\nabla u \notin L^\infty_{\text{loc}}(B_2)$. □

Proof of Proposition 1.6 Instead of the choice in the proof of Proposition 1.5, we now choose

$$a(r) = -\frac{\sin\ln\ln\ln\frac{64}{r} + \cos\ln\ln\ln\frac{64}{r}(1+\ln\frac{64}{r} + n\ln\frac{64}{r}\ln\ln\frac{64}{r})}{(n-1)(\ln\frac{64}{r})^2(\ln\ln\frac{64}{r})^2(2+\sin\ln\ln\frac{64}{r})},$$
$$v(r) = 2+\sin\ln\ln\ln\frac{64}{r}.$$

It is readily checked that A is continuous in $\bar{B}_4$, satisfies (1.2), (1.3) and (1.8) and u is an H^1 solution of (1.1), $\nabla u \in L^\infty(B_4)\cap VMO(B_4)$ but $\nabla u \notin C(B_2)$. □

3 Some Extensions

As announced in the introduction, Proposition 1.7 can be adapted for non-homogeneous systems with or without lower order terms. To illustrate, consider for example the system

$$\partial_\alpha(A_{ij}^{\alpha\beta}\partial_\beta u^j) = \partial_\alpha f_i^\alpha \quad \text{in } B_4, \quad i=1,\ldots,N. \tag{3.1}$$

Let $\omega_{f,2} : (0,2] \to [0,\infty)$ denote the L^2 mean oscillation of f

$$\omega_{f,2}(r) = \sup_{x\in B_2}\left\{\frac{1}{|B_r(x)|}\int_{B_r(x)}|f(y)-(f)_{B_r(x)}|^2\,dy\right\}^{1/2}.$$

We have

Proposition 3.1 *Let $A = (A_{ij}^{\alpha\beta})$ satisfy (1.2) and (1.3). Then there exists a constant $C_* > 0$, depending only on n, N, Λ and λ such that for every $u \in H^1(B_4)$ satisfying (3.1) with $f \in L^2(B_4)$ and for $0 < r \le R/4 \le 1/2$, there hold*

$$\int_{B_r}|\nabla u|^2\,dx \le \frac{C_* r^n}{R^n}\exp\left(2C_*\int_{2r}^R\frac{\omega_{A,2}(t)}{t}\,dt\right)\int_{B_R}|\nabla u|^2\,dx + \frac{C_* r^n}{R^n}\left\{\int_{2r}^2\exp\left(C_*\int_{2r}^t\frac{\omega_{A,2}(s)}{s}\,ds\right)\frac{\omega_{f,2}(t)}{t}\,dt\right\}^2 \tag{3.2}$$

and

$$\int_{B_r}|\nabla u-(\nabla u)_r|^2\,dx \le \frac{C_* r^{n+2}}{R^n}\left\{\int_{2r}^2\frac{\omega_{A,2}(t)}{t^2}\left[\exp\left(C_*\int_t^2\frac{\omega_{A,2}(s)}{s}\,ds\right)\|\nabla u\|_{L^2(B_2)} + \int_t^2\exp\left(C_*\int_t^\tau\frac{\omega_{A,2}(s)}{s}\,ds\right)\frac{\omega_{f,2}(\tau)}{\tau}\,d\tau\right]dt + \int_{2r}^2\frac{\omega_{f,2}(t)}{t^2}dt\right\}^2. \tag{3.3}$$

Moreover, if $u \in W^{1,\infty}(B_4)$, then, for $0 < r \le R/4 \le 1/2$,

$$\int_{B_r}|\nabla u-(\nabla u)_r|^2\,dx \le \frac{C_* r^{n+2}}{R^n}\left\{\int_{2r}^R\frac{\omega_{A,2}(t)}{t^2}\,dt\|\nabla u\|_{L^\infty(B_R)} + \int_{2r}^R\frac{\omega_{f,2}(t)}{t^2}\,dt\right\}^2. \tag{3.4}$$

Proof of Proposition 3.1 The proof is a simple adjustment of that of Proposition 1.7. We will only point out the main changes.

Proof of (3.2): Define h_k, a_k, b_k as in the proof of Proposition 1.7. Proceeding as before but paying attention to the application of Lemma 2.1 which now involves an inhomogeneous right hand side, we consecutively obtain the following estimates:

$$\begin{aligned}
&\|\nabla u\|_{L^2(B_{R_k})} \leq R_k^{n/2}(a_k + b_k),\\
&\|\nabla(u - h_k)\|_{L^2(B_{3R_k/2})} \leq C\omega(2R_k)\|\nabla u\|_{L^2(B_{2R_k})} + C\omega_{f,2}(2R_k)R_k^{n/2},\\
&R_k^{n/2}(a_k + b_k) \leq C\|\nabla u\|_{L^2(B_{2R_k})} + C\omega_{f,2}(2R_k)R_k^{n/2},\\
&a_{k+1} \leq C\omega(2R_k)(a_k + b_k) + C\omega_{f,2}(2R_k),\\
&R_{k+1}\|\nabla^2(h_{k+1} - h_k)\|_{L^\infty(B_{R_{k+1}})} \leq C\omega(2R_k)(a_k + b_k) + C\omega_{f,2}(2R_k),\\
&b_{k+1} \leq b_k + C\omega(2R_k)(a_k + b_k) + C\omega_{f,2}(2R_k),\\
&a_{k+1} + b_{k+1} \leq (1 + C\omega(2R_k))(a_k + b_k) + C\omega_{f,2}(2R_k).
\end{aligned}$$

Hence, instead of (2.15), we now get

$$\begin{aligned}
a_k + b_k \leq\ & C \exp\big(C\Omega(2R_k)\big)\|\nabla u\|_{L^2(B_2)}\\
&+ C\int_{2R_k}^2 \exp\big(C[\Omega(2R_k) - \Omega(t)]\big)\frac{\omega_{f,2}(t)}{t}\,dt,
\end{aligned}$$

where we recall the notation $\Omega(t) = \int_t^2 \frac{\omega(s)}{s}\,ds$. Estimate (3.2) follows.

Proof of (3.3): Instead of (2.16) and (2.17), we now have

$$\begin{aligned}
&|\nabla h_{R_k}(x) - \nabla h_{R_k}(0)|\\
&\leq C|x|\sum_{j=0}^k \frac{\omega_{f,2}(2R_j)}{R_j} + C|x|\sum_{j=0}^k \frac{\omega(2R_j)}{R_j}\Big[\exp\big(C\Omega(2R_j)\big)\|\nabla u\|_{L^2(B_2)}\\
&\quad + \int_{2R_j}^2 \big(C[\Omega(2R_j) - \Omega(t)]\big)\frac{\omega_{f,2}(t)}{t}\,dt\Big]\\
&\leq C|x|\int_{2R_k}^2 \frac{\omega_{f,2}(t)}{t^2}dt + C|x|\int_{2R_k}^2 \frac{\omega(t)}{t^2}\Big[\exp\big(C\Omega(t)\big)\|\nabla u\|_{L^2(B_2)}\\
&\quad + \int_t^2 \exp\big(C[\Omega(t) - \Omega(\tau)]\big)\frac{\omega_{f,2}(\tau)}{\tau}\,d\tau\Big]dt,
\end{aligned}$$

and

$$\begin{aligned}
&\|\nabla h_{R_k} - \nabla h_{R_k}(0)\|_{L^2(B_{R_k})}\\
&\leq CR_k^{(n+2)/2}\int_{2R_k}^2 \frac{\omega_{f,2}(t)}{t^2}dt + CR_k^{(n+2)/2}\int_{2R_k}^2 \frac{\omega(t)}{t^2}\Big[\exp\big(C\Omega(t)\big)\|\nabla u\|_{L^2(B_2)}\\
&\quad + \int_t^2 \exp\big(C[\Omega(t) - \Omega(\tau)]\big)\frac{\omega_{f,2}(\tau)}{\tau}\,d\tau\Big]dt.
\end{aligned}$$

It then follows that

$$\begin{aligned}
&\|\nabla u-(\nabla u)_{R_k}\|_{L^2(B_{R_k})}\\
\le\ &\|\nabla u-\nabla h_{R_k}(0)\|_{L^2(B_{R_k})}\\
\le\ &\|\nabla (u-h_{R_k})\|_{L^2(B_{R_k})}+\|\nabla h_{R_k}-\nabla h_{R_k}(0)\|_{L^2(B_{R_k})}\\
\le\ &CR_k^{(n+2)/2}\int_{2R_k}^2\frac{\omega_{f,2}(t)}{t^2}dt+CR_k^{(n+2)/2}\int_{2R_k}^2\frac{\omega(t)}{t^2}\Big[\exp\big(C\Omega(t)\big)\|\nabla u\|_{L^2(B_2)}\\
&+\int_t^2\exp\big(C[\Omega(t)-\Omega(\tau)]\big)\frac{\omega_{f,2}(\tau)}{\tau}\,d\tau\Big]dt\\
&+CR_k^{n/2}\omega_{f,2}(2R_k)\\
&+CR_k^{n/2}\omega(2R_k)\exp\big(C\Omega(2R_k)\big)\|\nabla u\|_{L^2(B_2)}\\
&+CR_k^{n/2}\omega(2R_k)\int_{2R_k}^2\exp\big(C[\Omega(2R_k)-\Omega(t)]\big)\frac{\omega_{f,2}(t)}{t}\,dt.
\end{aligned}$$

Noting that, on the right hand side, the third term can be absorbed into the first term, and the fourth and fifth terms can be absorbed into the second term, we arrive at estimate (3.3).

Proof of (3.4): Using $\nabla u\in L^\infty(B_2)$, we obtain this time that

$$|\nabla h_{R_k}(x)-\nabla h_{R_k}(0)|\ \le C|x|\left\{\int_{2R_k}^2\frac{\omega(t)}{t^2}\,dt\|\nabla u\|_{L^\infty(B_2)}+\int_{2R_k}^2\frac{\omega_{f,2}(t)}{t^2}\,dt\right\},$$

and so

$$\begin{aligned}
&\|\nabla h_{R_k}-\nabla h_{R_k}(0)\|_{L^2(B_{R_k})}\\
\le\ &CR_k^{(n+2)/2}\left\{\int_{2R_k}^2\frac{\omega(t)}{t^2}\,dt\|\nabla u\|_{L^\infty(B_2)}+\int_{2R_k}^2\frac{\omega_{f,2}(t)}{t^2}\,dt\right\}.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\|\nabla u-(\nabla u)_{R_k}\|_{L^2(B_{R_k})}\ &\le\ \|\nabla u-\nabla h_{R_k}(0)\|_{L^2(B_{R_k})}\\
&\le\ \|\nabla (u-h_{R_k})\|_{L^2(B_{R_k})}+\|\nabla h_{R_k}-\nabla h_{R_k}(0)\|_{L^2(B_{R_k})}\\
&\le\ CR_k^{(n+2)/2}\left\{\int_{2R_k}^2\frac{\omega(t)}{t^2}\,dt\|\nabla u\|_{L^\infty(B_2)}+\int_{2R_k}^2\frac{\omega_{f,2}(t)}{t^2}\,dt\right\}\\
&\quad+CR_k^{n/2}\omega(2R_k)\|\nabla u\|_{L^\infty(B_2)}+CR_k^{n/2}\omega_{f,2}(2R_k)\\
&\le\ CR_k^{(n+2)/2}\left\{\int_{2R_k}^2\frac{\omega(t)}{t^2}\,dt\|\nabla u\|_{L^\infty(B_2)}+\int_{2R_k}^2\frac{\omega_{f,2}(t)}{t^2}\,dt\right\}.
\end{aligned}$$

Estimate (3.4) is readily seen. □

Finally, we briefly touch on the validity of Theorem 1.1 when $\omega_{A,2}$ is replaced by $\omega_{A,p}$ for $1<p<2$. For this, we only need the following L^p version of Proposition 1.7.

Proposition 3.2 *Let* $A=(A_{ij}^{\alpha\beta})$ *satisfy* (1.2) *and* (1.3)*. Let* $1<p<2$*. Then there exist constants* $\gamma>0$ *and* $C_*>0$ *depending only on* n*,* N*,* p*,* Λ *and* λ *such that, provided*

$[A]_{BMO}(B_4) < \gamma$, there hold for every $u \in W^{1,p}(B_4)$ satisfying (1.1) and for $0 < r \le R/4 \le 1/2$ that

$$\int_{B_r} |\nabla u|^p\,dx \le \frac{C_* r^n}{R^n} \exp\left(2C_* \int_{2r}^{R} \frac{\omega_{A,p}(t)}{t}\,dt\right) \int_{B_R} |\nabla u|^p\,dx,$$

and

$$\int_{B_r} |\nabla u - (\nabla u)_r|^p\,dx \le \frac{C_* r^{n+2}}{R^n} \int_{B_R} |\nabla u|^p\,dx \times \times \left\{\int_{2r}^{R} \frac{\omega_{A,p}(t)}{t^2} \exp\left(C_* \int_t^R \frac{\omega_{A,p}(s)}{s}\,ds\right) dt\right\}^2,$$

where $(\nabla u)_r = \frac{1}{|B_r|}\int_{B_r} \nabla u\,dx$ for $0 < r \le 2$.

Moreover, if $u \in W^{1,\infty}(B_4)$, then, for $0 < r \le R/4 \le 1/2$,

$$\int_{B_r} |\nabla u - (\nabla u)_r|^p\,dx \le \frac{C_* r^{n+2}}{R^n} \left\{\int_{2r}^{R} \frac{\omega_{A,p}(t)}{t^2}\,dt\right\}^2 \sup_{B_R} |\nabla u|^p.$$

The proof of Proposition 3.2 is the same as that of Proposition 1.7, but now using the following harmonic replacement estimate:

Lemma 3.3 *Let $1 < p < 2$. Let $A, \bar{A}$ satisfy (1.2) and (1.3) with $\bar{A}$ being constant in B_4 and $f = (f_i^\alpha) \in L^{p'}(B_4)$ with $p' = \frac{p}{p-1}$. Let $R \in (0, 1)$ and suppose $u, h \in W^{1,p}(B_{4R})$ satisfy*

$$\begin{aligned} \partial_\alpha(A_{ij}^{\alpha\beta} \partial_\beta u^j) &= \partial_\alpha f_i^\alpha \quad \text{in } B_{3R}, \quad i = 1, \ldots, N, \\ \partial_\alpha(\bar{A}_{ij}^{\alpha\beta} \partial_\beta h^j) &= 0 \quad \text{in } B_{2R}, \quad i = 1, \ldots, N, \\ u &= h \quad \text{on } \partial B_{2R}. \end{aligned}$$

Then there exist constants $\gamma > 0$ and $C > 0$ depending only on n, N, p, Λ and λ such that, provided $[A]_{BMO(B_{4R})} \le \gamma$,

$$\|\nabla(u-h)\|_{L^p(B_{3R/2})} \le C\Big[R^{n(1/p-1/p')}\|f\|_{L^{p'}(B_{3R})} + R^{-n/p}\|A - \bar{A}\|_{L^p(B_{3R})}\|\nabla u\|_{L^p(B_{3R})}\Big].$$

Proof We amend the proof of Lemma 2.1 using L^p theories for elliptic systems whose leading coefficients have small BMO semi-norm.[2] In the proof, C denotes a generic positive constant which depends only on n, N, p, Λ and λ.

It is known that (see, e.g., Dong and Kim [10, 11])[3], provided $[A]_{BMO(B_{4R})} \le \gamma$ for some small enough γ depending only on n, N, p, Λ and λ, one has

$$\|\nabla u\|_{L^{p'}(B_{2R})} \le C\Big[\|f\|_{L^{p'}(B_{3R})} + R^{n(1/p'-1/p)}\|\nabla u\|_{L^p(B_{3R})}\Big]. \tag{3.5}$$

Using that $\bar{A}$ is constant, we have by standard elliptic estimates that

$$\|\nabla h\|_{L^\infty(B_{7R/4})} \le CR^{-n/p}\|\nabla h\|_{L^p(B_{2R})} \le CR^{-n/p}\|\nabla u\|_{L^p(B_{2R})}.$$

[2] When p is close to 2 such smallness assumption is not needed, see, e.g., [6, 27].

[3] For further references, see [4, 5, 8, 18, 27].

Using

$$\partial_\alpha \left(A_{ij}^{\alpha\beta} \partial_\beta (u-h)^j \right) = \partial_\alpha \left(f_i^\alpha + (\bar{A}-A)_{ij}^{\alpha\beta} \partial_\beta h^j \right) \quad \text{in } B_{2R}, \quad i=1,\dots,N,$$

and once again the fact that $[A]_{BMO(B_{4R})} \le \gamma$, we have

$$\begin{aligned} &\|\nabla(u-h)\|_{L^p(B_{3R/2})} \\ &\le C\Big[\|f\|_{L^p(B_{7R/4})} + \|A-\bar{A}\|_{L^p(B_{7R/4})} \|\nabla h\|_{L^\infty(B_{7R/4})} \\ &\quad + R^{-(n+p')/p'} \|u-h\|_{L^1(B_{7R/4})} \Big] \\ &\le C\Big[R^{n(1/p-1/p')} \|f\|_{L^{p'}(B_{2R})} + R^{-n/p} \|A-\bar{A}\|_{L^p(B_{2R})} \|\nabla u\|_{L^p(B_{2R})} \\ &\quad + R^{-(n+p')/p'} \|u-h\|_{L^1(B_{2R})} \Big]. \end{aligned} \tag{3.6}$$

To estimate $\|u-h\|_{L^1(B_{2R})}$, recall from the proof of Lemma 2.1 the chain of identities

$$\begin{aligned} \int_{B_{2R}} \frac{|u-h|^2}{\sqrt{|u-h|^2+t^2}}\, dx &= \int_{B_{2R}} \bar{A}_{ij}^{\alpha\beta} \partial_\alpha \phi^i \partial_\beta (u-h)^j \, dx \\ &= \int_{B_{2R}} \left(f_i^\alpha + (\bar{A}-A)_{ij}^{\alpha\beta} \partial_\beta u^j \right) \partial_\alpha \phi^i \, dx, \end{aligned}$$

which imply

$$\int_{B_{2R}} \frac{|u-h|^2}{\sqrt{|u-h|^2+t^2}}\, dx \le C\Big[R^{(n+p)/p} \|f\|_{L^{p'}(B_{2R})} + R \|A-\bar{A}\|_{L^p(B_{2R})} \|\nabla u\|_{L^{p'}(B_{2R})} \Big].$$

Noting that the constant C is independent of t, we may send $t \to 0$ to obtain

$$\|u-h\|_{L^1(B_{2R})} \le C R^{(n+p)/p} \Big[\|f\|_{L^{p'}(B_{2R})} + R^{-n/p} \|A-\bar{A}\|_{L^p(B_{2R})} \|\nabla u\|_{L^{p'}(B_{2R})} \Big]. \tag{3.7}$$

The conclusion follows from (3.5), (3.6) and (3.7). □

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