

A dislocation model of plasticity with particular application to fatigue crack closure



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Dougan Kelk M^cKellar
Trinity College
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To
Mom and Dad
and especially
Gramps

Abstract

D.K. McKellar
Trinity College

D.Phil.
Hilary Term, 2001

A DISLOCATION MODEL OF PLASTICITY WITH PARTICULAR APPLICATION TO FATIGUE CRACK CLOSURE

The ability to predict fatigue crack growth rates is essential in safety critical systems. The discovery of fatigue crack closure in 1970 caused a flourish of research in attempts to simulate this behaviour, which crucially affects crack growth rates. Historically, crack tip plasticity models have been based on one-dimensional rays of plasticity emanating from the crack tip, either co-linear with the crack (for the case of plane stress), or at a chosen angle in the plane of analysis (for plane strain). In this thesis, one such model for plane stress, developed to predict fatigue crack closure, has been refined. It is applied to a study of the relationship between the apparent stress intensity range (easily calculated using linear elastic fracture mechanics), and the true stress intensity range, which includes the effects of plasticity induced fatigue crack closure. Results are presented for all load cases for a finite crack in an infinite plane, and a method is demonstrated which allows the calculation of the true stress intensity range for a growing crack, based only on the apparent stress intensity range for a static crack.

Although the yield criterion is satisfied along the plastic ray, these one-dimensional plasticity models violate the yield criterion in the area immediately surrounding the plasticity ray. An area plasticity model is therefore required in order to model the plasticity more accurately. This thesis develops such a model by distributing dislocations over an area. Use of the model reveals that current methods for incremental plasticity algorithms using distributed dislocations produce an over-constrained system, due to misleading assumptions concerning the normality condition. A method is presented which allows the system an extra degree of freedom; this requires the introduction of a parameter, derived using the Prandtl-Reuss flow rule, which relates the magnitude of slip on complementary shear planes. The method is applied to two problems, confirming its validity.

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Preface

This thesis is an account of the work carried out by the author in the Department of Engineering Science and the University of Oxford, whilst part of the University Technology Centre for Solid Mechanics. The work was carried out under the supervision of Prof. D.A. Hills and Prof. A. Sackfield.

No part of this thesis has been submitted for a degree at any other university. The research described here is original, although the work of others has been drawn upon freely with due acknowledgements in the text.

Nomenclature

Symbol (and equation)	Description
a	crack half-length
α_{ij}	planes of maximum shear stress
b, b_x, b_y	Burgers vector (and components)
B	dislocation density distribution
B_x, B_y	dislocation strength per unit area
C, m	Paris law crack growth constants
δ	crack opening displacement
$\Delta K_{app} = (\sigma_{max} - \sigma_{min})\sqrt{\pi a}$	apparent stress intensity range
$\Delta K_{true} = (\sigma_{max} - \sigma_{open})\sqrt{\pi a}$	true stress intensity factor range
ΔK_{th}	threshold stress intensity range
$\epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy}$	strain components in Cartesian space
E	Young's modulus
G_{ijk}	Stress kernel function
$\Im$	imaginary part of complex number
$\kappa = \frac{3-\nu}{1+\nu}$	Kolosov's constant in plane stress
$\kappa = 3 - 4\nu$	Kolosov's constant in plane strain
$K_I = \sigma_{\infty}\sqrt{\pi a}$	mode I stress intensity factor
$\mu = \frac{E}{2(1+\nu)}$	shear modulus
n	number of fatigue cycles
ν	Poisson's ratio
r_p	plastic zone length
r_{rp}	reversed plastic zone length
$R = \frac{\sigma_{min}}{\sigma_{max}}$	R -ratio
$\Re$	real part of complex number
$\sigma_{\infty}, \sigma_{ij}^{\infty}$	far field stress, i.e. bulk load
σ_Y	yield stress in tension
$\sigma_{max}, \sigma_{min}$	maximum and minimum applied stresses
σ_{open}	stress at which crack becomes fully open
σ_{close}	stress at which crack tip first closes

Symbol (and equation)	Description
σ_{ij}^e	elastic stress for current increment
σ_{ij}^d	dislocation stress for current increment
$\sigma_{ij}^h = \Sigma(\sigma_{ij}^e + \sigma_{ij}^d)$	history stress for all previous increments
$\sigma_{ij}^t = \sigma_{ij}^h + \sigma_{ij}^e + \sigma_{ij}^d$	total stress for current increment
$\sigma_{ij}^i = \sigma_{ij}^t - \sigma_{ij}^d = \sigma_{ij}^h + \sigma_{ij}^e$	stress for this increment <i>excluding</i> the dislocation contribution
$\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$	stress components in Cartesian space
$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}$	stress components in polar co-ordinates
$\sigma_1, \sigma_2, \sigma_3$	principal stresses
σ_T	Tresca stress
σ_{vM}	von Mises equivalent stress
u_x, u_y	Cartesian displacements
U_{ij}	displacement kernel function

Glossary

Term	Description
<i>Burgers vector</i>	The vector describing the magnitude of the discontinuity due to a dislocation in a crystal lattice
<i>CMOD</i>	Crack mouth opening displacement.
<i>CTOD</i>	Crack tip opening displacement.
<i>EPFM</i>	Elastic Plastic Fracture Mechanics: Significant plastic deformation occurs ahead of the crack tip, causing the violation of the small scale yielding (SSY) assumptions. The size and shape of the plastic zone must be modelled in order to ascertain the load-bearing ability, or crack growth rate, of a flawed component. The stress distribution cannot be modelled by means of a single parameter as is the case for LEFM. See §1.2.
<i>LEFM</i>	Linear Elastic Fracture Mechanics: Small scale yielding (SSY) is assumed to be valid, and hence all micro-processes which control crack growth are characterised by the K_I -dominated field. This single parameter defines material deformation which is assumed to be essentially elastic. Also known as the “stress intensity” approach. See §1.1.
<i>Elastic limit</i>	The load above which plasticity first occurs. Also referred to as the <i>elastic-plastic</i> boundary.

<i>Proportional loading</i>	A change in the applied load causes all stress components to scale in proportion. This implies, for example, that the direction of maximum shear stress will remain constant (the 2-D Mohr's circle grows in diameter or translates along the principal stress axis).
<i>Plane strain</i>	The strains in the out-of-plane direction remain constant, and are assumed to be zero. Thus any points in the plane of analysis must remain in the same plane after deformation. An example is an infinitely long tube, subjected to internal or external pressure.
<i>Plane stress</i>	The stresses in the out-of-plane direction remain constant, and are zero. An example is a sheet of negligible thickness loaded only at its boundaries.
<i>Self-similar</i>	The spatial variation of the stress (or strain, or displacement) distribution remains the same, despite "zooming in or out", or changing the applied load.
<i>SSY</i>	Small scale yielding. All plasticity is assumed to occur within a "process zone", of arbitrary shape, which itself is enclosed in a K_I -dominated region. Beyond this K_I -dominated field, higher order terms in the series expansion of the K_I expression are required to correctly model the stress distribution. Yet further afield, the material is assumed to carry only the remote applied stress. See §1.1.4.

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Chapter 1

Introduction

The ten year period from 1942 to 1952 saw the sudden and catastrophic failure of more than 200 tanker and cargo ships built in the U.S.A. during the Second World War. Bridge collapses in 1938, 1940 and 1962 added to the concern of design engineers working with welded steel structures. The introduction of the first commercial jet-powered passenger aircraft, the de Havilland Comet, was disastrous. In 1954, two crashes in close succession resulted of the grounding of the entire fleet. It was this series of failures that spurred on the examination of structural failure, resulting in a new approach to structural design in the form of fracture mechanics¹.

Whereas conventional engineering design follows the strength-of-materials approach, fracture mechanics is based on the fundamental assumption that all materials are flawed. Cracks may initiate due to atomic-scale lattice defects (dislocations, stacking faults), granular defects (inclusions, voids, oxide particles), the manufacturing process (stress concentrations at a weld), unavoidable stress concentrations due to design (for instance keyways, splined shafts), or external damage (fretting, impact). Therefore any component which is subjected to cyclic stress in its operational life has the potential for crack growth. A crack which goes unno-

¹An excellent introduction to the origins of fracture mechanics is given by Gdoutos [2].

ticed, and so continues to grow to its critical length, could result in sudden catastrophic failure with the possibility of damage to both equipment and personnel. The difficulty for the engineer lies in the fact that this crack growth can and does occur at stresses which are *below* the calculated “safe operating” stresses based on a strength-of-materials approach.

A pertinent question is therefore: “Assuming we can detect a crack of a given size, *how long* will it take for that crack to reach a critical size where catastrophic failure is imminent?” In order to answer this, we need to understand the mechanisms governing fatigue crack growth.

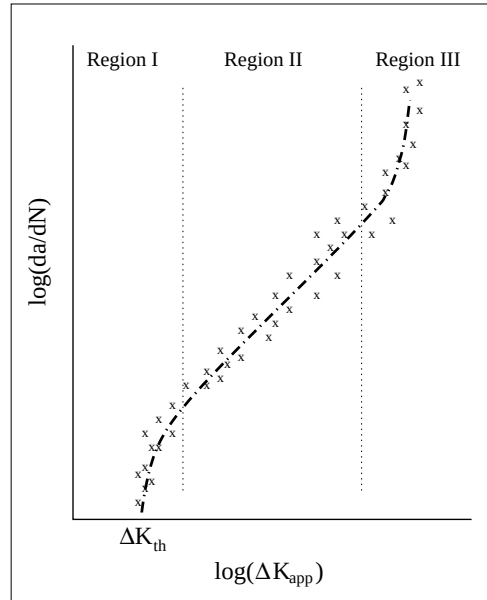


Figure 1.1: Schematic plot of fatigue crack growth data.

Experimental data revealed three characteristic modes of crack growth (see figure 1.1): Region I is associated with near-threshold behaviour where the crack grows slowly, if at all. Region II represents a region where the crack grows in a stable manner, with each applied load cycle, and this growth can be captured by a linear log-log relationship. Region III represents accelerating crack growth which is followed by catastrophic brittle failure.

Linear elastic fracture mechanics (LEFM) is founded on the assumption that the extent of the plastic zone at the crack tip is negligible in compar-

ison to the crack size and any other characteristic length dimensions in the body. Under these assumptions, the Paris law can be used to estimate crack growth rate in Region II:

$$\frac{da}{dn} = C(\Delta K_{app})^m, \quad (1.1)$$

where a = crack length,

n = number of stress cycles,

C, m = empirical constants,

ΔK_{app} = apparent stress intensity.

The constants C and m are determined experimentally, and the apparent stress intensity factor, $\Delta K_{app} = \sigma_{\infty} \sqrt{\pi a}$, is related to the applied remote stress and the crack length. This will be discussed further in section 1.1. It was hoped that this would capture the salient qualities that controlled fatigue crack growth rate. However, this single-parameter approach was not successful in describing all crack growth regimes, and this led to the development of elastic plastic fracture mechanics (EPFM), where the plastic zone *is* modelled. This is presented in section 1.2.

It was found that the Paris formulation gave good correlation for intermediate values of ΔK_{app} , but could not be fitted to crack growth rates at either low or high ΔK_{app} values. Further investigation revealed that crack growth rate is dependent on both (1) the stress intensity range and (2) the R -ratio (which is a measure of the *mean stress*). The R -ratio, or stress-ratio, is defined² as

$$R = \frac{\sigma_{min}}{\sigma_{max}}. \quad (1.2)$$

Thus attempts were made to find new growth rate laws of the form:

$$\frac{da}{dN} = f(\Delta K_{app}, R) \quad (1.3)$$

²The R -ratio is sometimes defined as $R = K_{min}/K_{max}$, iff $K_{min}, K_{max} \geq 0$. The definition in equation 1.2 shall be used in this thesis.

The Paris equation was therefore modified by a number of authors. Forman [3] suggested:

$$\frac{da}{dn} = \frac{C(\Delta K_{app})^m}{(1-R)K_c - \Delta K_{app}}, \quad (1.4)$$

where K_c = fracture toughness,
 R = R-ratio, or stress ratio.

Another possible formulation [4] is:

$$\frac{da}{dn} = C(\Delta K_{app})^m \left(\frac{1 - \left(\frac{\Delta K_{th}}{\Delta K_{app}} \right)^{n_1}}{1 - \left(\frac{K_{max}}{K_c} \right)^{n_2}} \right)^{n_3} \quad (1.5)$$

where n_1, n_2, n_3 = empirical constants.

As can be seen from equations 1.4 and 1.5, the original Paris equation requires small adjustments in order to account for the R -ratio effect.

The R -ratio dependency was little understood until 1970 when Elber [5, 6] made the revolutionary discovery of fatigue crack closure. This is when the *crack tip* is closed for part of the load cycle, even when the bulk load (also called the remote load) is consistently in the tensile regime. A new variable was introduced in order to describe the true stress intensity suffered by the crack tip, K_{true} . The discovery of crack closure suggests the reason why the crack growth rate, $\frac{da}{dn}$, is dependent on R : at high R -ratios the crack tip is more likely to be open for the entire cycle and hence $\frac{da}{dn}$ is higher; whereas at lower R -ratios the crack tip is potentially closed for part of the load cycle and so the crack may grow more slowly. The crucial point would then be to find a method which enables the accurate prediction of the applied stress or load at which the crack becomes fully open (to the crack tip). At this point, $K_{app} \neq 0$, and $K_{true} = 0$. It was then proposed that crack growth rate is controlled solely by the *true* stress intensity range:

$$\frac{da}{dN} = f(\Delta K_{true}) \quad (1.6)$$

The ability to remove the R -ratio dependency would have significant implications for the engineering fraternity. Presently, only simple load histories can be modelled, and then only by numerically intensive methods

such as finite element modelling. Crack growth rate prediction currently relies on Paris law data which is applicable only at the R -ratio at which the data was collected for that material. Any new load conditions require another series of tests to be performed in order to obtain the new Paris law constants for that material at the new load conditions. Removal of the R -ratio dependency would therefore allow Paris law data to be used more widely, in theory being applied for any load conditions. This is obviously highly desirable and has thus been the focus of much research since the discovery of crack closure. At this stage, however, nobody has presented a compelling argument which facilitates the prediction of crack growth rates at varying R -ratios *without the use of empirical correction factors*.

Much debate has circulated around the cause of crack closure, including plasticity, roughness, oxide, entrapped-fluid, and transformation-induced fatigue crack closure. Other possible causes, strongly material-dependent, include crack deflection, crack bridging by fibres, crack bridging (trapping) by particles, crack shielding by micro-cracks, and crack shielding by dislocations. A useful review of these mechanisms is presented by Fellows [7]. In this thesis, it is assumed that the dominant cause of fatigue crack closure is plasticity induced.

Modelling the plastic zone at crack tips is a complex task, and has received much attention over the last three decades. In particular, the development of dislocation theory has allowed the modelling of complicated stress distributions. An overview of dislocation theory in the context of modelling cracks and plasticity is presented in chapter 2. Based on these methods, a model employing a one-dimensional plastic zone, used to determine the load or stress at which the crack becomes fully open, is presented in chapter 3. This enables the calculation of ΔK_{true} , and a comparison of ΔK_{app} and ΔK_{true} is presented.

The limitations of one-dimensional plasticity models suggest the development of an area plasticity model. This is presented in chapters 4 and 5.

Ultimately, the goal is to apply an area plasticity model to the crack tip problem.

We continue by reviewing the foundations of fracture mechanics with a synopsis of LEFM. Reference is made to various modes of loading, and these are illustrated in figure 1.2. The Westergaard and K_I solution are

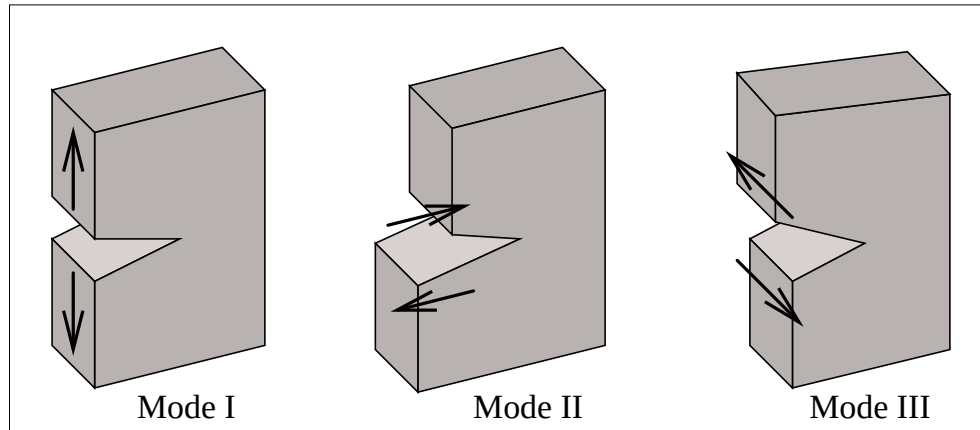


Figure 1.2: Illustration of the three loading modes

presented. A comparison is made between these solutions in the discussion on small scale yielding. We then turn to elastic plastic fracture mechanics, and briefly review the plasticity models currently being used to simulate the complex stress field around a crack.

1.1 LEFM crack tip stress analysis

Linear elastic fracture mechanics (LEFM) is based on the K_I solution, which is derived from the Westergaard [8] solution. This is discussed in some detail by Dugdale and Ruiz [9], and by Gdoutos [10], and will not be recorded in detail here. However, we will present the most important results.

1.1.1 Westergaard solution

Westergaard [8] proposed a complex analytical solution for a finite crack in an infinite plane, loaded in equi-biaxial tension. The Westergaard

equations have been modified to model a general biaxial remote stress field by Sih [11] and Eftis & Liebowitz [12]. For plane stress or plane strain:

$$\begin{aligned}\sigma_{xx} &= \Re[\phi'(z)] - y \Im[\phi''(z)] - A, \\ \sigma_{yy} &= \Re[\phi'(z)] + y \Im[\phi''(z)] + A, \\ \sigma_{xy} &= -y \Re[\phi''(z)].\end{aligned}\tag{1.7}$$

where

$$\begin{aligned}z &= x + iy, \\ A &= (1 - m) \frac{\sigma_\infty}{2}, \\ \phi(z) &= \sigma_\infty \sqrt{z^2 - a^2} - \frac{1}{2} \sigma_\infty z (1 - m), \\ \phi'(z) &= \frac{\sigma_\infty z}{\sqrt{z^2 - a^2}} - \frac{1}{2} \sigma_\infty (1 - m), \\ \phi''(z) &= \frac{\sigma_\infty}{\sqrt{z^2 - a^2}} - \frac{\sigma_\infty z^2}{\sqrt{(z^2 - a^2)^3}}\end{aligned}$$

The coordinates are shown in figure 1.3. For equi-biaxial tension (i.e. the original Westergaard solution), $m = 1$, and for uniaxial tension, $m = 0$. The Westergaard stress distributions (normalised with respect to the far field stress, σ_∞) are shown in figure 1.4 for the case of uniaxial tension. The top half of the figure is for conditions of plane strain; the bottom half for plane stress.

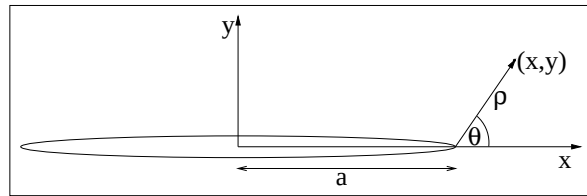


Figure 1.3: Coordinate system used for crack tip stress analysis

The displacement fields in the presence of a finite crack in an infinite plane can also be modelled using the modified Westergaard formulation in plane stress [12] or plane strain [11]. For plane stress:

$$\begin{aligned}u_x &= \frac{1 + \nu}{E} \left(\frac{1 - \nu}{1 + \nu} \Re[\phi(z)] - y \Im[\phi'(z)] - Ax \right) \\ u_y &= \frac{1 + \nu}{E} \left(\frac{2}{1 + \nu} \Im[\phi(z)] - y \Re[\phi'(z)] + Ay \right).\end{aligned}\tag{1.8}$$

For plane strain:

$$\begin{aligned} u_x &= \frac{1+\nu}{E} \left((1-2\nu) \Re[\phi(z)] - y \Im[\phi'(z)] - Ax \right) \\ u_y &= \frac{1+\nu}{E} \left(2(1-\nu) \Im[\phi(z)] - y \Re[\phi'(z)] + Ay \right). \end{aligned} \quad (1.9)$$

These equations (1.7 – 1.9) represent the complete, exact state of stress and displacement for a static elastic finite crack in an infinite plane.

The Westergaard solution is unique to a given geometry and remote loading, and the von Mises stresses are also critically dependent on the degree of transverse constraint present. Results will therefore be presented under conditions of generalised plane stress, in each case in the lower half of each figure, and plane strain, for the particular case of Poisson's ratio equal to 0.3, in the upper half of the figure.

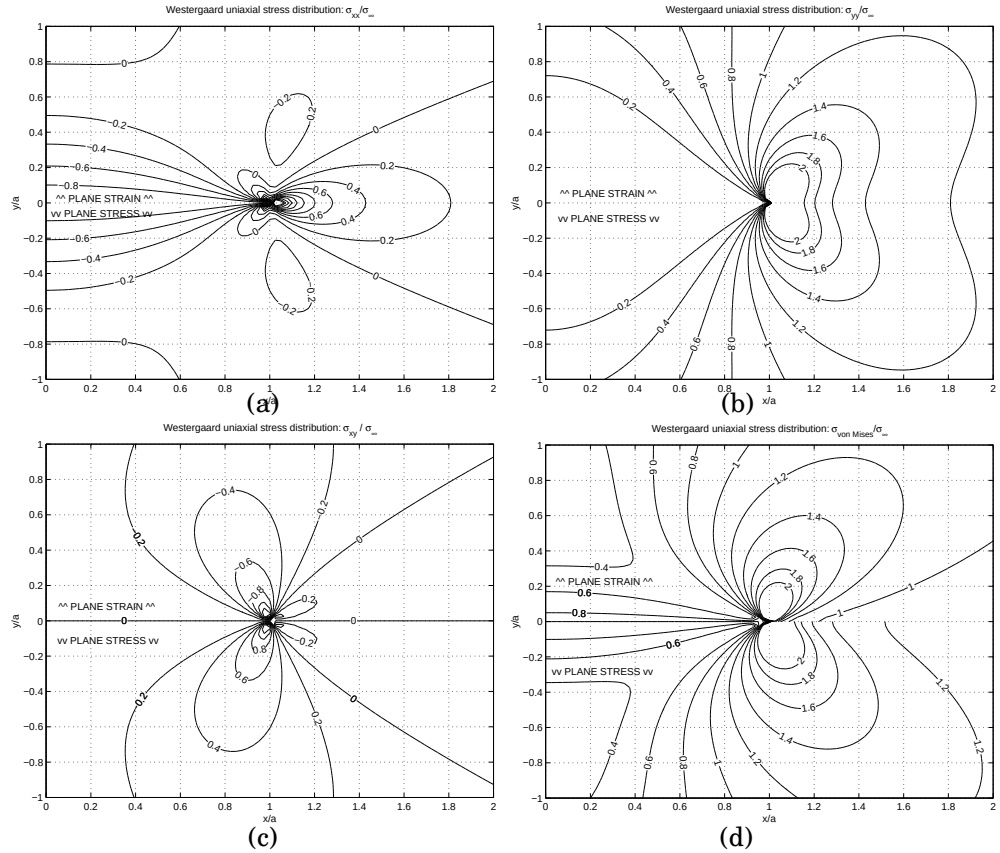


Figure 1.4: Westergaard stress distributions for a crack of length $2a$ in an infinite plane, normalised with respect to σ_∞ for plane strain (top half) and plane stress (bottom half) under uniaxial loading: (a) σ_{xx} , (b) σ_{yy} , (c) σ_{xy} , (d) von Mises stress, σ_{vM} (with $\nu = 0.3$).

1.1.2 K_I solution

The standard K_I equations (e.g. [4]) were modified by Liebowitz *et al* [13] in order to account for general biaxial loading. They showed that only the σ_{xx} distribution needs to be corrected, as follows:

$$\begin{aligned}\sigma_{xx} &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) - 2A + O(r^{\frac{1}{2}}) \dots \\ \sigma_{yy} &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) + O(r^{\frac{1}{2}}) \dots \\ \sigma_{xy} &= \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} + O(r^{\frac{1}{2}}) \dots\end{aligned}\quad (1.10)$$

where $K_I = \sigma_\infty \sqrt{\pi a}$ is the stress intensity factor, and A is as for equation 1.7. The latter is the so called T-stress which acts parallel with the crack faces, and represents the only non-vanishing stress component present, other than the stress intensity terms. Illustration of the modified K_I stress fields normalised with respect to the far field stress, σ_∞ , are shown in figure 1.5, with plane strain in the top half and plane stress in the bottom half.

Examination of the above equations reveals that the stress intensity solution has the K_I term as a scaling factor. The advantage of the stress intensity solution is that a single parameter, related to the applied remote load, can describe the entire crack tip stress distribution.

In each equation the next term in the series expansion is of order $O(r^{1/2})$, which vanishes as the crack tip is approached. Conversely, as r increases, the contribution from the higher order terms becomes significant. Thus the region in which the K_I leading term is dominant within a chosen tolerance is finite, and can be calculated. This is discussed in section 1.1.4 when considering the validity of small scale yielding assumptions.

The displacement fields in the vicinity of the crack tip (i.e. $r \ll a$) are provided by Liebowitz *et al* [13]:

$$\begin{aligned}u_x &= \frac{K_I}{\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left(\frac{\kappa - 1}{2} + \sin^2 \frac{\theta}{2} \right) - \frac{(\kappa + 1)(1 - m)\sigma_\infty}{8\mu} (r \cos \theta - a), \\ u_y &= \frac{K_I}{\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left(\frac{\kappa + 1}{2} - \cos^2 \frac{\theta}{2} \right) + \frac{(3 - \kappa)(1 - m)\sigma_\infty}{8\mu} (r \sin \theta),\end{aligned}$$

$$(1.11)$$

where $\kappa = (3 - \nu)/(1 + \nu)$ in plane stress, and $\kappa = 3 - 4\nu$ in plane strain. Equations 1.10 and 1.11 can be applied to uniaxial or biaxial loading by choosing suitable m . They represent the asymptotic solutions for $r \ll a$, and the complete solution for a semi-infinite elastic crack in an infinite plane.

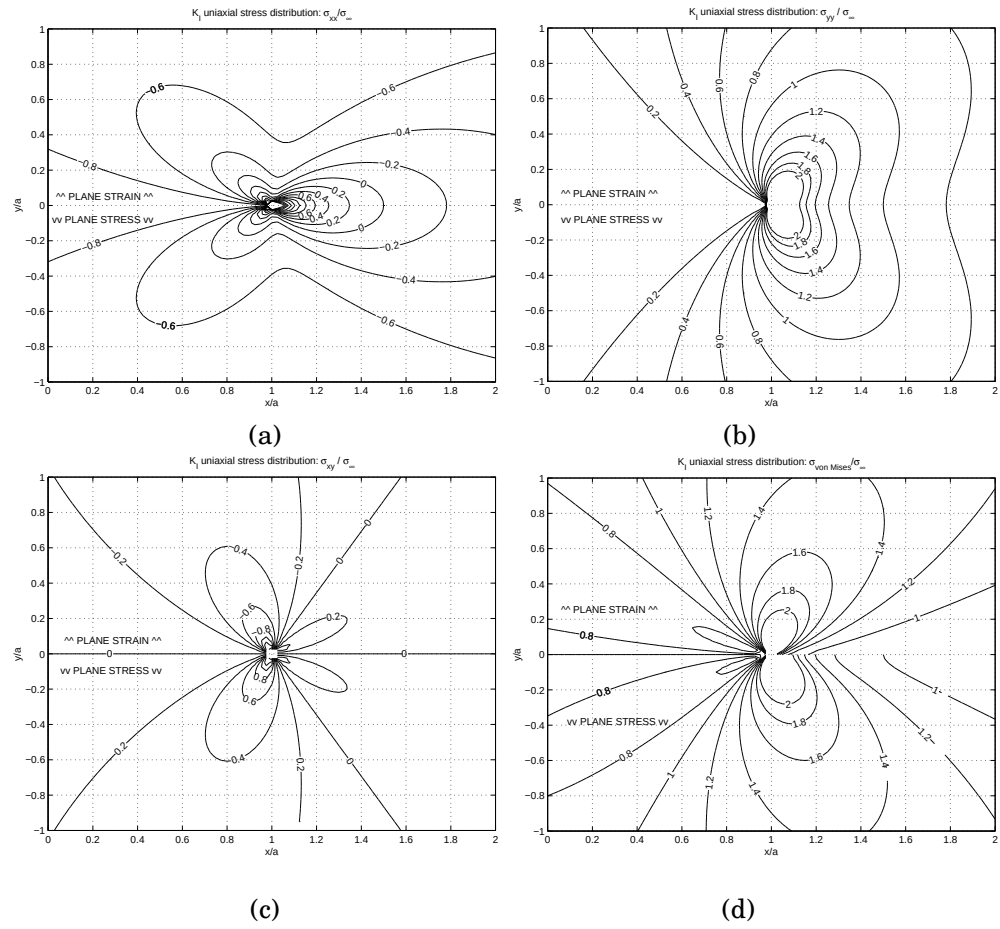


Figure 1.5: K_I stress distributions for a crack of length $2a$, normalised with respect to σ_∞ for plane strain (top half) and plane stress (bottom half) under uniaxial loading: (a) σ_{xx} , (b) σ_{yy} , (c) σ_{xy} , (d) von Mises stress, σ_{vM} (with $\nu = 0.3$).

1.1.3 Estimation of plastic zone size and shape

Examination of equations 1.7 (for the Westergaard complete solution), or 1.10 (for the K_I approximation) reveals that the stress distribution around a crack tip is singular. Recognising that engineering materials have a finite yield stress, we accept that there must be a region of plasticity at the crack tip which is not taken into consideration in either of these solutions.

Accepting that there must be some redistribution of stresses due to this plastic zone, we can nonetheless make a coarse estimate of the size and shape of the plastic zone by simply plotting the region in which the yield parameter is exceeded. The choice of yield criterion is a subtle issue and will be revisited at various stages in this thesis. Here, we simply present the expressions used to evaluate the von Mises and Tresca stresses:

$$\sigma_{vM} = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{2}}, \quad (1.12)$$

$$\sigma_T = |\sigma_1 - \sigma_2|. \quad (1.13)$$

For plane stress, $\sigma_3 = 0$, and for plane strain³, $\sigma_3 = \nu(\sigma_1 + \sigma_2)$. The yield criteria associated with these stresses predict that failure will occur when $\sigma_{vM} = \sigma_Y$, or $\sigma_T = 2k$ (where $k = \sigma_Y/2$ is the yield stress in shear). Establishing the contour $\sigma_{vM}/\sigma_Y = 1$ (or $\sigma_T/\sigma_Y = \sigma_T/2k = 1$) gives a first order approximation of the yield zone, showing the contour where the yield stress is first exceeded. Graphical results will be presented after a discussion on small scale yielding.

³Since this thesis deals only with planar analysis, the notation used here is that the third principal stress always lies in the out-of-plane direction. That is, equation 1.13 applies iff $\sigma_1 > \sigma_3 > \sigma_2$.

1.1.4 Small scale yielding

Consider figure 1.6. Small scale yielding asserts that region in which the stresses are evaluated falls outside of the plastic zone, but inside the area in which the K_I solution is accurate (say to within 10%), viz. region 1. Outside the K_I region, the higher order terms in the solution become significant and are then required for an accurate solution (region 2). Beyond this the material sustains only the far-field stress (region 3). Because

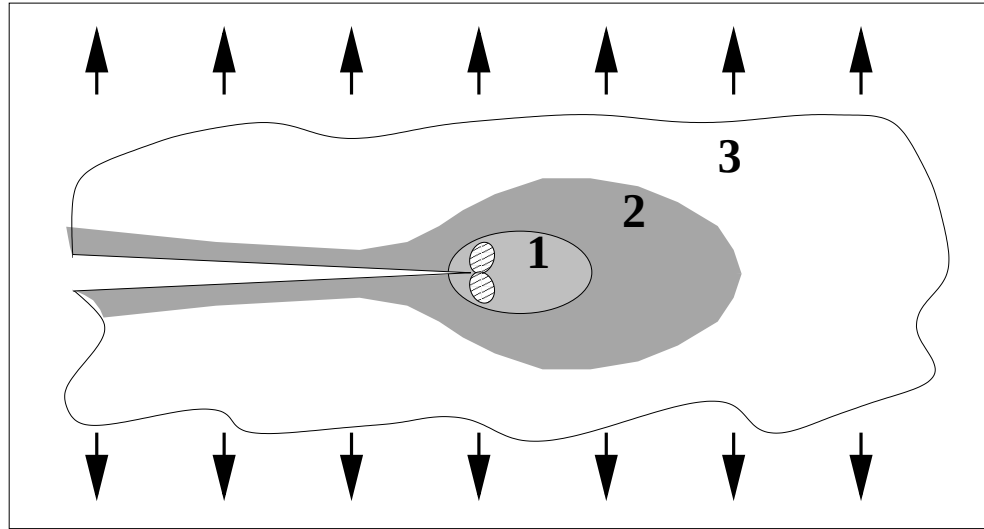


Figure 1.6: Small scale yielding assumptions apply when the plastic zone (hatched region) is contained within the K_I dominated zone (region 1). Further afield (region 2), higher order terms are required in the stress intensity formulation to describe the stress-state at the crack tip accurately. Beyond this (region 3), the material carries only the far-field stress.

of the simplicity of the K_I solution, much engineering work is conducted under “small scale yielding” assumptions. For this reason, a brief study of the error incurred in considering only the first term will now be performed. That is to say, we wish to establish some tolerances as to the accuracy of the K_I solution.

Whereas some texts (e.g. Evans [14]) normalise the error with respect to the K_I solution, in this thesis the error plots are presented in the form

$$\frac{(\sigma_{\text{full solution}} - \sigma_{K_I \text{ solution}})}{\sigma_{\text{full solution}}},$$

since the Westergaard solution is the complete solution. We then plot the error contours at $\pm 2\%$, $\pm 5\%$, and $\pm 10\%$, as shown in figure 1.7. The

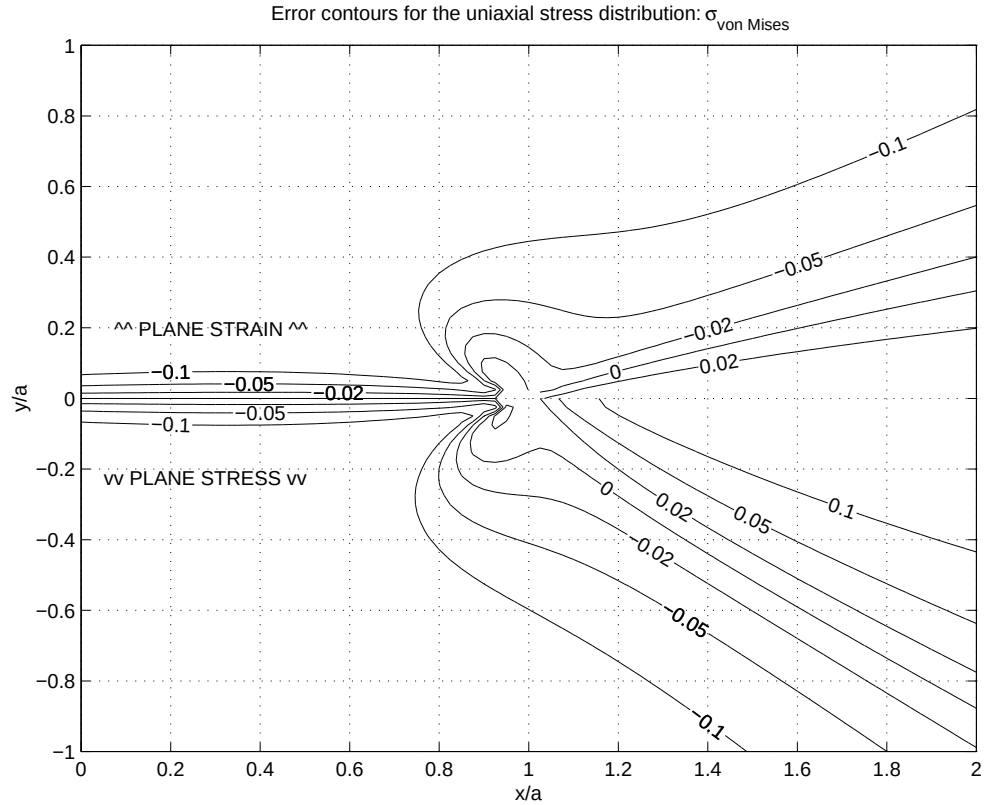


Figure 1.7: Small scale yielding error distributions of the K_I solution with respect to the exact solutions for the von Mises stress (with $\nu = 0.3$) for a crack of half-length a , for plane strain (top half) and plane stress (bottom half) under uniaxial loading in a full plane.

remarkable feature of these results is just how stringent the requirement is that crack tip quantities be encapsulated by the K_I dominant field. The fact that the stress intensity solution uses only the leading term in the series expansion, means that there is a finite region in which the stress state is accurately described by this formulation. The decision to be made by any engineer making use of this method is what level of discrepancy from the true solution is acceptable.

The amount of data available from this calculation are very great, and it is not really feasible to plot them all. In the context of assessing the relevance of a stress intensity solution, the most important physical parameter is the size of the plastic zone, conveniently judged by the value

of the von Mises yield parameter, σ_{vM} . Figure 1.5(d) shows the value of von Mises parameter normalised with respect to the remote stress for the K_I solution. Hence, to determine the extent of the plastic zone, we need to specify the ratio $\frac{\sigma_\infty}{\sigma_Y}$. For example, for $\frac{\sigma_\infty}{\sigma_Y}$ in the range 0.3–0.7, under both transverse plane stress and transverse plane strain conditions, the extent of the plastic zones as predicted by the stress intensity solution are shown in comparison with the full plane values in figure 1.8, where only the $\pm 2\%$, $\pm 5\%$, and $\pm 10\%$ error contours are plotted. This is all based on a determination of where the yield criterion is exceeded, *in the elastic solution*, i.e. without considering the redistribution of stresses which always accompanies plasticity. However, much useful information can be obtained from this figure: For example for a full plane under plane strain conditions, the size of the plastic zone as predicted by the stress intensity solution for $\frac{\sigma_\infty}{\sigma_Y} = 0.5$ falls completely within the 5% error bound.

It must be emphasised that both the Westergaard solution (equation 1.7) and the singular solution (equation 1.10) are valid only for a *stationary* elastic crack in an infinite plane. Thus the application of these equations are limited to theoretical studies and make a rather crude approximation when examining real engineering components, eloquently outlined by Lu and Chow [15]. For this reason, models have been proposed which aim to incorporate plasticity, and to predict the stress intensity factor for geometries other than an infinite plane. These models will be discussed next.

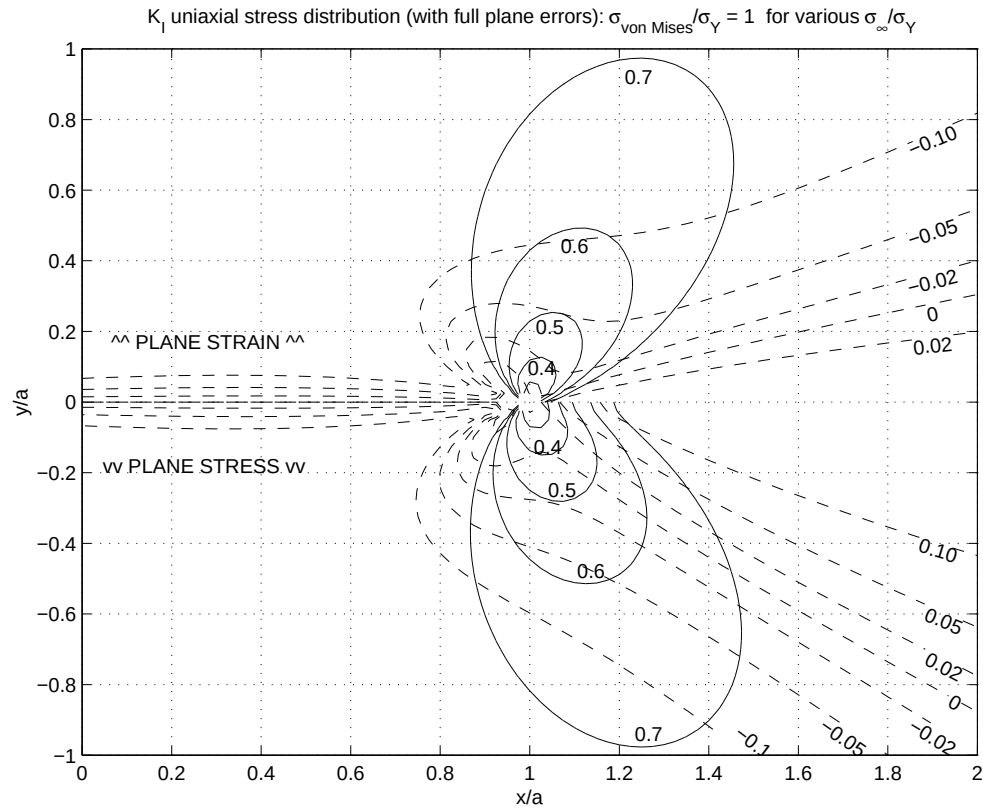


Figure 1.8: Comparison of the size and shape of plastic zones as predicted by the von Mises (with $\nu = 0.3$) stress intensity solution (for various $\frac{\sigma_\infty}{\sigma_Y}$) [solid lines] with the errors inherent in using the stress intensity solution [dashed lines] for a full plane.

1.2 EPFM crack tip stress analysis

Elastic plastic fracture mechanics (EPFM) recognises that significant plastic deformation occurs ahead of the crack tip, causing the violation of the small scale yielding (SSY) assumptions. The size and shape of the plastic zone must be modelled in order to ascertain the load-bearing ability, or crack growth rate, of a flawed component. This means that the stress distribution cannot be modelled by means of a single parameter as is the case for LEFM. Modelling plasticity in EPFM has been the focus of much research over the past three decades. We shall now briefly review this work.

1.2.1 Plasticity models currently in use

There are currently four main plasticity models being used to model plasticity at the crack tip, namely the Dugdale, Budiansky and Hutchinson, Bilby-Cottrell-Swinden (BCS), and Atkinson and Kanninen models. The range of problems these four types of solution are aimed at are shown in table 1.1, together with some of the other models that have been proposed to model plasticity. All of these models are based on a full plane analysis,

Crack type		Plane stress	Plane strain
Static	Semi-infinite	—	—
	Finite	Dugdale '60 [16]	BCS '63 [17] Vitek '76 [18] Riedel '76 [19] Atkinson '77 [20]
Growing	Semi-infinite	Budiansky '78 [21]	—
	Finite	Newman '81 [22] Nowell '98 [25]	Kanninen '80 [23, 24]

Table 1.1: Summary of plasticity models

with the exception of Newman (1981) where the standard formulations for an infinite plane have been modified to model a centre crack in a strip under tension. A brief review will now be presented, in chronological order, on the development of models to describe the complex nature of the stress distribution around a crack tip.

1.2.2 The Dugdale model

Dugdale [16] was the first to develop a crack model incorporating plasticity. The formulation considered a stationary finite crack of length $2a$ in an elastic-perfectly plastic material. The crack is in an infinite plane, subject to an applied remote uniform uniaxial⁴ stress. Plane stress is assumed (i.e. the plate is thin compared with the crack size), which implies that

⁴Many authors report that the Dugdale formulation employs biaxial loading. Examination of the original paper shows that this is not the case. This confusion may have arisen through derivations incorporating the Westergaard solution, which was originally formulated under equi-biaxial remote load.

the planes of maximum shear stress lie at $\pm 45^\circ$ to the plane normal in the out-of-plane direction. Dugdale's strip yield model assumes that there is an infinitesimally thin strip of yielded material co-linear with the crack, ahead of the crack tip, at the yield stress, of length r_p (see figure 1.9).

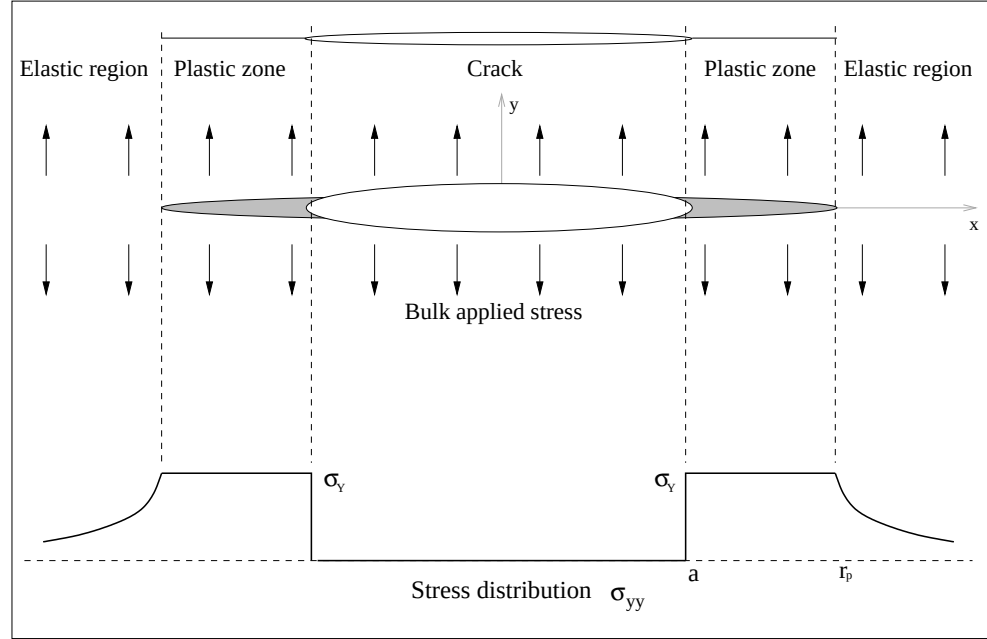


Figure 1.9: The Dugdale plane stress strip-yield model

The derivation is based on the superposition of stress functions in order to account for the redistribution of stresses due to the presence of the plastic zone. These stress functions are based on Muskhelishvili's work [26], and Dugdale's original derivation presents the predicted length of the plastic ray as

$$\frac{r_p}{a} = 2 \sin^2 \left(\frac{\pi}{4} \frac{\sigma_{yy}^\infty}{\sigma_Y} \right). \quad (1.14)$$

A slightly different derivation is presented in Ewalds and Wanhill [4], where the sine term is expanded as a series. When SSY occurs, the series may be truncated to one term with the result

$$\frac{r_p}{a} = \frac{\pi}{8} \left(\frac{K_I}{\sigma_Y} \right)^2. \quad (1.15)$$

The errors in taking only the first term of this expansion are small, and in fact equation 1.15 *over-predicts* the length of the plastic ray (equation

1.14) by a small amount⁵.

The advantage of the Dugdale model is that it gives a simple closed-form result for the plastic zone size in plane stress. However, since the model is for a stationary crack, it cannot be used for the prediction of crack closure.

1.2.3 The BCS model

The Bilby, Cottrell and Swinden (“BCS”, 1963) model [17] was pioneering in the application of dislocation theory⁶ to the study of crack problems. It was developed independently of the Dugdale (1960) model, and in their paper they point out that the results of the Dugdale model were drawn to their attention after their work had been completed.

A finite crack is assumed to reside in an elastic isotropic infinite plane, subject to uniform remote anti-plane applied shear. The case of anti-plane strain is therefore modelled (Mode III deformation), with displacements occurring in the out-of-plane (z) direction. Screw dislocations (see section 2.1) are sited along the length of the crack, and in a plastic zone region ahead of the crack tip in a Dugdale-type yield zone. The objective of the study (as in the Dugdale paper) was to determine the length of the plastic zone as a function of applied remote stress. A second case was also analysed: that of edge dislocations with an applied remote shear stress, then representing plane strain (Mode II deformation). Although Dugdale’s model analysed Mode I deformation in plane stress, these two models are comparable in that they both assume the plastic zone to be co-linear with the crack with all the plasticity confined to a thin strip.

The solution of the Cauchy integral equation in the BCS formulation, using Muskhelishvili [26] potentials, yielded the expression of the length of the plastic zone as

$$\frac{a}{r_p + a} = \cos \left(\frac{\pi \tau^\infty}{2 k} \right) \quad (1.16)$$

⁵At $\sigma_{yy}^\infty/\sigma_Y = 0.3, 0.5, 0.7$ the errors are 1.9, 5.3, 10.7% respectively.

⁶Dislocation theory will be discussed in chapter 2. A detailed understanding of this theory is not required for the present discussion.

where τ^∞ is the remote applied shear stress and k the yield stress in shear. This is in agreement with the length of the Dugdale plastic zone, which can be re-written as

$$\frac{a}{r_p + a} = \cos \left(\frac{\pi \sigma_\infty}{2 \sigma_Y} \right) \quad (1.17)$$

where σ_∞ is the remote applied tensile stress and σ_Y the yield stress in tension.

The BCS model introduced the possibilities of modelling cracks and plastic zones using dislocation theory. Another model for plane strain, but considering Mode I deformation, will be discussed next.

1.2.4 The Atkinson and Kanninen model

The Atkinson model [20] is based on earlier work where dislocation theory (see chapter 2) was used to model relaxation at a crack tip [27]. It uses inclined strip yield “superdislocations” to model crack growth in plane strain. The superdislocations are aligned along the planes of maximum shear stress, which are assumed to occur at an angle of 70.53° to the plane of the crack. The superdislocations are orientated such that their Burgers vectors are co-incident with the direction of the assumed maximum shear stress trajectory. The Atkinson model was developed to model fatigue crack growth by Kanninen *et al* [23][24]. Superdislocations left “embedded” in the material behind the crack tip represent the size of the previous plastic zones, and in this manner the plastic wake is modelled. However, computations for complicated load histories are lengthy.

Although this model has limitations concerning the prediction of plastic zone size, it was shown that the superdislocation method permits accurate prediction of the crack tip opening displacement. The advantage of the dislocation method is that it can be applied to finite geometries by using the appropriate kernels. A further advantage is that the superdislocation approach requires less computational effort than the distributed dislocation approach. The model’s main shortfall lies in the fact that the

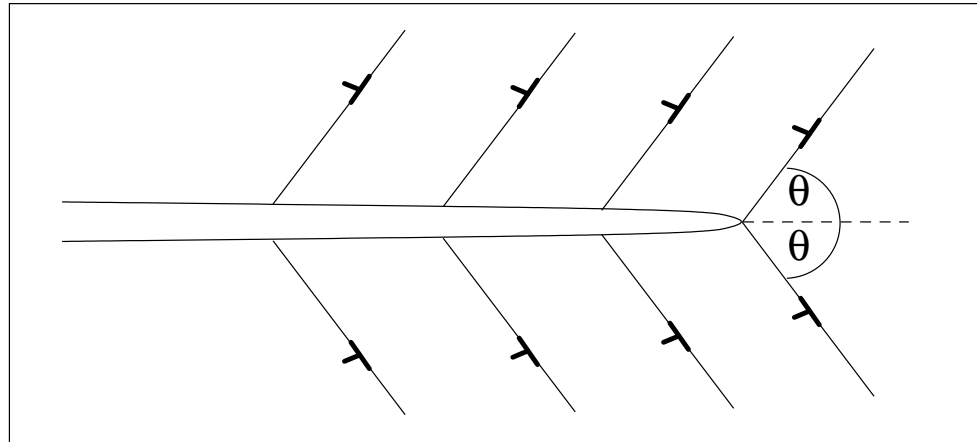


Figure 1.10: The Kanninen plane strain inclined slip-yield model. The bold symbols represent “superdislocations” which account for all the plasticity along that ray.

plastic zone is modelled by 1-D plasticity rays. Furthermore, the normality condition is violated since the dislocation is not free to rotate such that the direction of the Burgers vector is co-incident with the instantaneous direction of maximum shear stress. Implementation of the flow rule will be discussed in detail in section 4.2.3.

1.2.5 The Budiansky and Hutchinson model

The Budiansky and Hutchinson model [21] was preceded by Rice [28], Bilby [29], and Weertman [30]. Although these models gave closed-form results for steady-state crack growth, none accounted for fatigue crack closure.

The Budiansky and Hutchinson model was the first to attempt to account for fatigue crack closure using the Dugdale model for plasticity. It applies to thin sheets (i.e. plane stress), under uniform tension, with a *semi-infinite crack*, i.e. the plastic wake evolves to form a layer of material adjacent to the crack faces of *constant* thickness. It assumes small scale yielding to be valid (see section 1.1.4). The model highlights the presence of a cyclic plastic zone for a growing crack, which is one-quarter of the length of the plastic zone for a stationary crack owing to the effect of the material yielding in compression on the unloading part of the

cycle. Although it does shed some light on the mechanisms involved in fatigue crack closure, it is not applicable to finite cracks in finite geometries. The model uses function-analytic methods rather than dislocation theory, hence the implementation is outside the scope of this thesis.

1.2.6 The Newman model

The Newman model [22] uses the Dugdale model as its basis, extending it to account for crack growth. It employs bar elements in the crack which carry a compressive load when in contact, and similar elements in the plastic zone, which can sustain both tensile and compressive loads. In both sets the elements are permitted to yield when the stress exceeds a certain flow stress which is defined as the average of the yield stress and the ultimate tensile stress.

The main downfall of this model is that it uses five empirical constants derived from experimental data on one aluminium alloy (2219-T851) in order to facilitate the simulation of crack growth. The model was adapted to account for plane strain, by the introduction of a sixth empirical constant α which characterises the degree of through-thickness constraint. In this manner plane stress, plane strain, or something between these extremes could be modelled by altering α . This was then used to model random spectrum loading (e.g. stress loading for an aircraft wing in flight) and was found to be in good agreement with experimental tests.

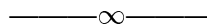
1.3 Summary

The standard strength-of-materials approach to engineering design was found wanting in the explanation for failures of structures well within their designed operational stress range. Fracture mechanics emerged as a discipline in the post-war era in an attempt to quantify fracture behaviour.

Linear elastic fracture mechanics assumes that the shape of the plastic zone is arbitrary, and that it falls within a “process zone”. The assumptions of small scale yielding then require this process zone to fall within the K_I -dominated zone such that the mechanisms controlling crack behaviour are all described by this single parameter approach. This technique can be used when the size of the plastic zone is negligible with respect to the crack length, or any other characteristic length dimension of the body in which the crack is located, for example a semi-infinite crack in an infinite plane.

The assumptions of small scale yielding are often invalid, and this led to the development of elastic plastic fracture mechanics. Here it is recognised that in order to model crack behaviour, the plastic zone itself cannot be ignored and must be modelled. Five such models have been reviewed briefly, addressing the cases of both plane stress and plane strain, for static and growing cracks.

The latest developments in the field will be discussed in detail in chapter 3 when we present the Nowell [25] model. In order to facilitate that analysis, an overview of dislocation theory is required. This will be examined in the next chapter.



Chapter 2

An overview of the dislocation technique

In this chapter, the fundamental principles in the use of the dislocation technique for modelling both fatigue cracks and plasticity will be explained. Our aim is to build a composite picture by looking first at those small blocks which contribute to the larger model. Once an understanding of the basics is obtained, the method of modelling static elastic mode I cracks is described. Finally, a brief description of the modelling of plasticity is given which introduces the possibility of modelling both static plastic and growing plastic cracks.

2.1 Fundamentals of dislocation theory

The term “dislocation” is used to describe the discontinuity in an otherwise orderly structured lattice, an example of which would be an extra half-plane of atoms being wedged in to the parent material. Scholars in materials science will be familiar with the concept of dislocations in a material, allowing shear deformation to occur on slip bands at a granular scale. The motion of these dislocations is the mechanism by which plas-

ticity occurs in metals. A good introduction to this topic is given by Hull [31]. A concise history of the development of dislocation theory is documented by Hirth [32]. Hirth notes that the initiation of the dislocation theory of slip occurred in 1934 with the presentation of papers by Orowan [33], Polanyi [34] and Taylor [35]. It was not until 1963 that dislocation theory was applied to the solution of crack problems, when Bilby *et al* [17] proposed a model to explain plastic yield from a notch.

2.1.1 Dislocation types

There are two types of dislocations: screw and edge. In order to facilitate their distinction, the concept of a Burgers vector is introduced: this is the vector that describes the displacement discontinuity in an otherwise orderly lattice. The direction of the Burgers vector relative to the dislocation line (see figure 2.1) denotes the type of dislocation, which shall be discussed next. The following explanation is based on figures 2.1–2.3.

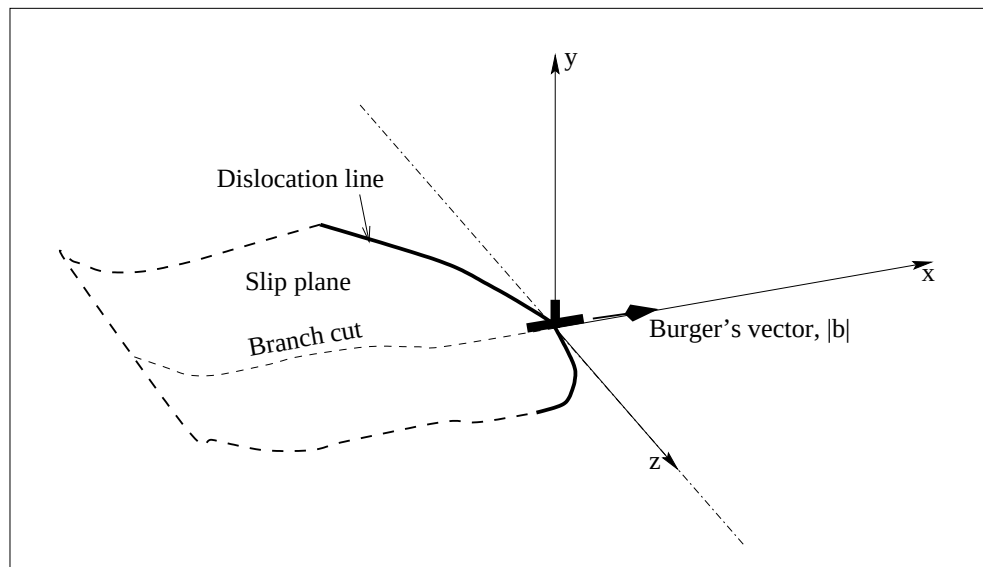


Figure 2.1: Terminology used with edge dislocations.

Screw dislocations

The Burgers vector lies parallel to the dislocation line. Screw dislocations arise due to out-of-plane shear stresses (Mode III deformation – see figure 1.2), and are associated with out-of-plane material flow, as is the case in anti-plane strain. Since out-of-plane shear is not encountered in any of the problems described in this text, screw dislocations shall not be discussed further.

Edge dislocations

An edge dislocation has a Burgers vector lying in the plane normal to the dislocation line. The orientation of the Burgers vector with respect to the slip plane then defines one of two categories: climb edge dislocations and glide edge dislocations.

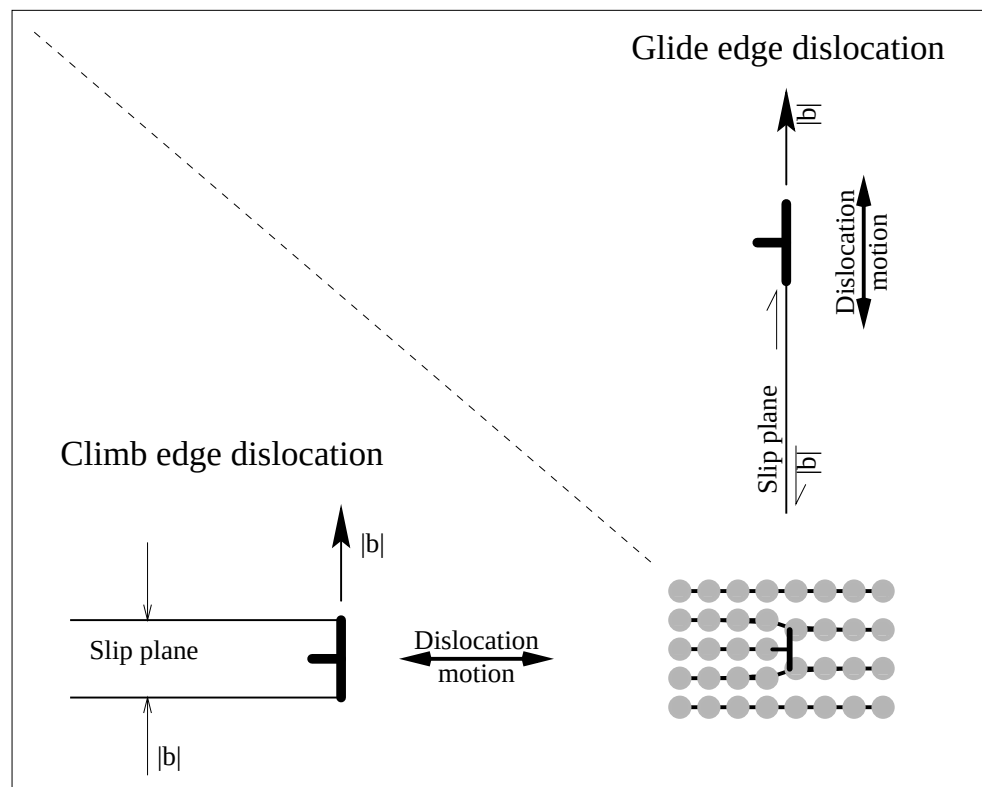


Figure 2.2: Schematic diagram of the two types of edge dislocation

Climb edge dislocations have a Burgers vector lying *perpendicular to the slip plane* (and perpendicular to the dislocation line). The effect of a climb edge dislocation can be easily envisaged if one considers the analogy to a pack of playing cards: forcing an extra card half-way into the pack (see figure 2.3(a)) would represent a climb edge dislocation. Since the adjacent cards are forced apart to accommodate the extra card, climb edge dislocations are used either to model extra material being added to the plane of analysis (as in a Dugdale-type plastic zone); or to model crack openings (by considering the *effect* on adjacent cards of inserting a card without actually doing so). The magnitude of the Burgers vector corresponds to the opening created behind the dislocation, analogous to the number of extra cards being forced into the pack.

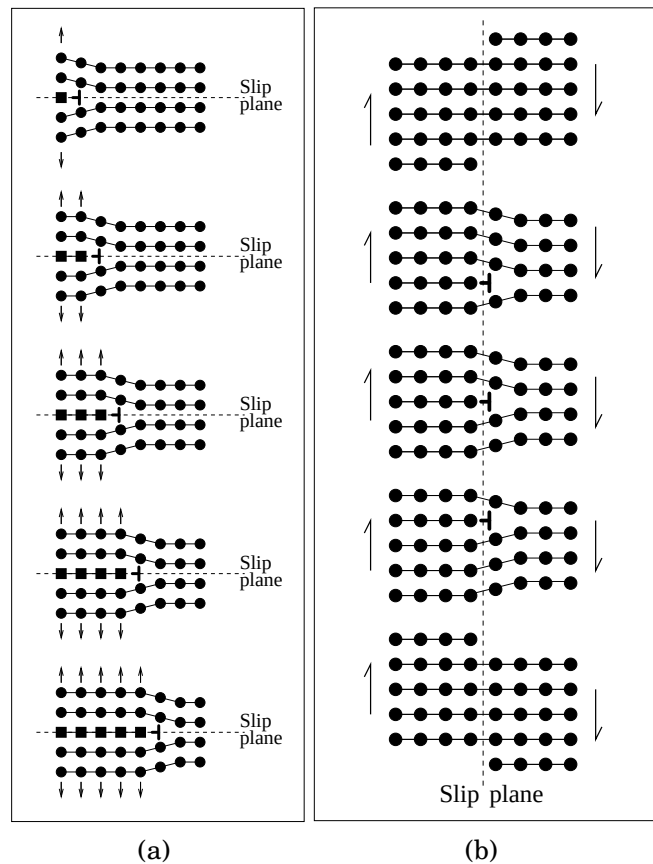


Figure 2.3: An edge dislocation moving (a) by climb and (b) by glide through a 2-D lattice. The square symbols in (a) represent extra material arriving into the plane of analysis.

Glide edge dislocations have a Burgers vector lying *parallel to the slip plane* (and perpendicular to the dislocation line). Using the pack of cards analogy, we would now require the help of a magician, who would mysteriously make the extra card (which is half inserted into the pack) move up and down through the pack without actually removing it at any time. Returning to a crystal lattice analysis (see figure 2.3(b)), we can see that this type of dislocation movement arises from shear stresses in the plane of analysis. Glide edge dislocations are used to model shear flow and hence are used in the modelling of plastic zones in plane strain.

It should be borne in mind that in practice, dislocations seldom move solely in one mode, but rather as a combination of for example climb and glide (movement in the same plane), or a combination of edge and screw [31]. This can be envisaged if one returns to figure 1.2. We can now see that Mode-I loading has similarities with a climb edge dislocation; Mode-II with a glide edge dislocation; and Mode-III with a screw dislocation. A combination of say climb-edge and screw dislocations would thus represent an opening-twisting defect in the lattice, much like removing a cork from a bottle. Since the choice of branch cut (and orientation of axes) is simply one of mathematical convention, these combined dislocations can always be resolved into their component dislocation types by simple vector analysis.

2.1.2 A single edge dislocation in an infinite plane

The stress distribution, σ_{ij}^d , around a single dislocation in an infinite plane is given [36, 37] as:

$$\begin{Bmatrix} \sigma_{xx}^d \\ \sigma_{yy}^d \\ \sigma_{xy}^d \end{Bmatrix} = \frac{2\mu}{\pi(\kappa + 1)} \begin{bmatrix} G_{xx} & G_{yx} \\ G_{xy} & G_{yy} \\ G_{xy} & G_{yx} \end{bmatrix} \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} \quad (2.1)$$

where the dislocation influence functions G_{ijk} are listed in Appendix A and describe the spatial variation of the stresses due to Burgers vectors b_x and b_y ; μ is the shear modulus and κ is Kolosov's constant.

For illustrative purposes, let us consider the case $b_x = 0$ and $b_y = 1$ and choose to examine the σ_{yy} component of stress and u_y component of displacement:

$$\sigma_{yy}^d(x, y) = \frac{2\mu}{\pi(\kappa + 1)} \left(\frac{1}{x - x_d} \right) b_y \quad (2.2)$$

The displacements are also of relatively simple form:

$$u_y^d = \frac{1}{2\pi(\kappa + 1)} \left((\kappa + 1)\theta - \frac{2xy}{r^2} \right) b_y \quad (2.3)$$

with $\theta \in [-\pi, \pi]$.

The stress and displacement fields from these equations, along with the other components, are plotted in figure 2.4.

There are many similarities between dislocations in the materials science context and those which are the subject of this study. It is important to note, however, that the dislocations which we shall be discussing in this text are *not* the same as physical dislocations as used in the materials science context, in that here they do not denote the presence of a lattice defect. In the materials science context, dislocations *move* through a lattice, piling up at grain boundaries and then pushing through to the next grain when sufficient strain energy is accumulated. For the purposes of this study, this movement is an unnecessary complexity. We choose instead to identify convenient sites at which to “activate” dislocations. By this we mean that there are no physical dislocations present: we are concerned only with the stress distribution due to a dislocation. We are thus simply using the concept of a dislocation in order to have as a tool at our disposal a means of modelling stress fields. For example a model of a crack in an infinite plane requires that an open crack have faces which are traction free – dislocations may be “activated” along the line of the crack in order to cancel the far field stress. This technique will be discussed in more detail in the next section, and for now it suffices to say that for the remainder of the thesis, a “dislocation” will refer not to a physical dislocation but rather to a mathematical dislocation at a fixed site.

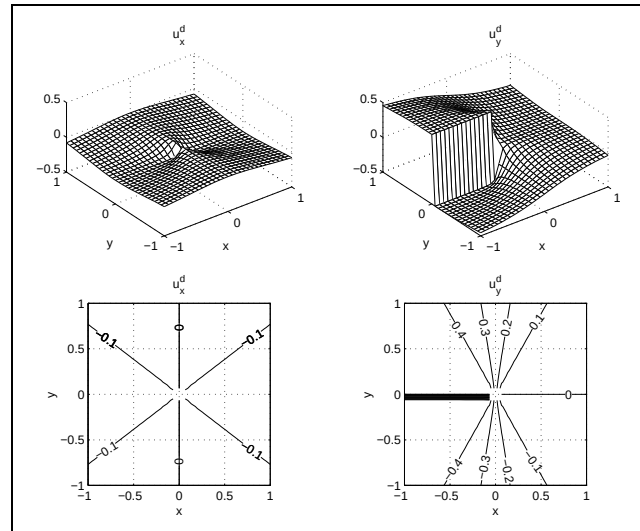
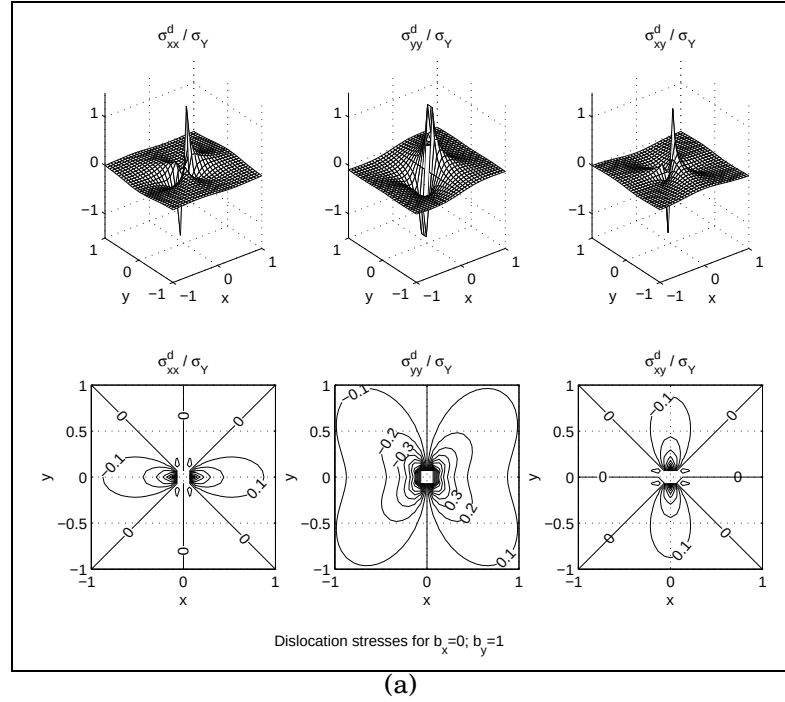


Figure 2.4: (a) Stress and (b) displacement field distributions for a single climb edge dislocation ($b_x=0, b_y=1$) located at the origin of an infinite plane.

2.2 Modelling cracks using edge dislocations

Figure 2.5 shows, in a very simple way, how the effect of a number of dislocations can be summed in order to obtain a nett stress distribution. Placing a number of climb dislocations (with Burgers vector components $b_x=0$; $b_y=1$) behind one another enables one to “pack out” a crack by summing of the Burgers vectors. Returning to the pack of cards analogy, this

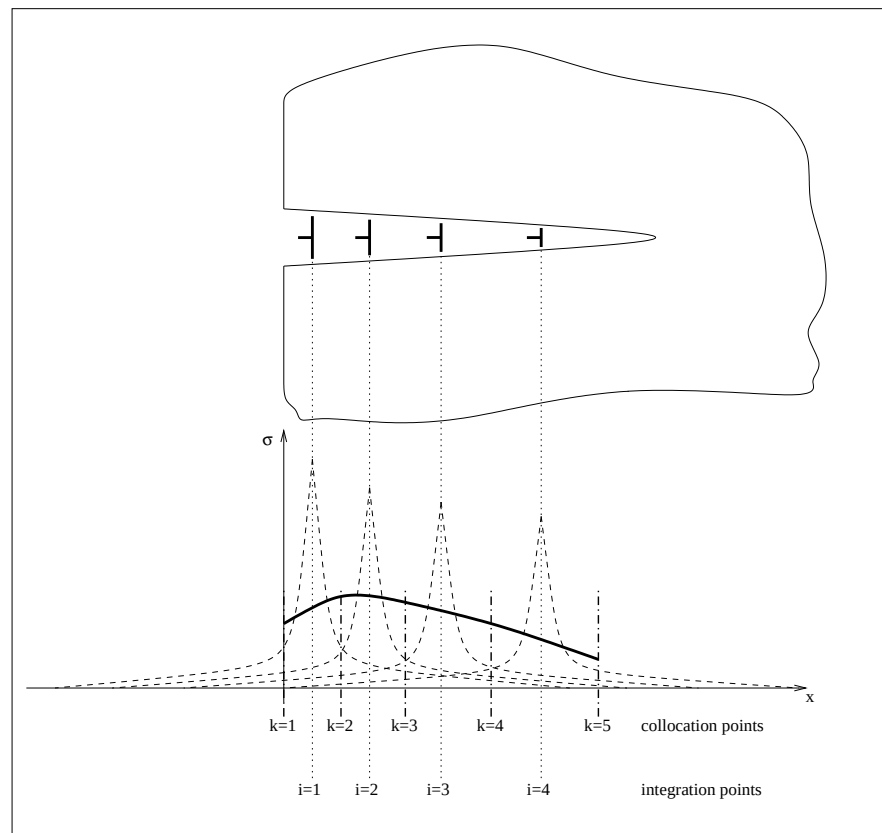


Figure 2.5: The effect of a number of dislocations is summed to obtain the nett stress.

would be akin to sliding a number of cards into the deck, some cards further than others, building up the layers until the desired crack shape is obtained. Figure 2.6 shows how a finite crack in an infinite plane may be modelled – here symmetry is employed to reduce the problem size by including terms in the kernel functions to account for the mirror-image dislocations in the left half plane. The pack of cards analogy is a little misleading, in that what actually happens is that we solve the problem

by cancellation of stresses. We will show that having solved for the magnitude of the Burgers vectors, the crack shape can readily be determined.

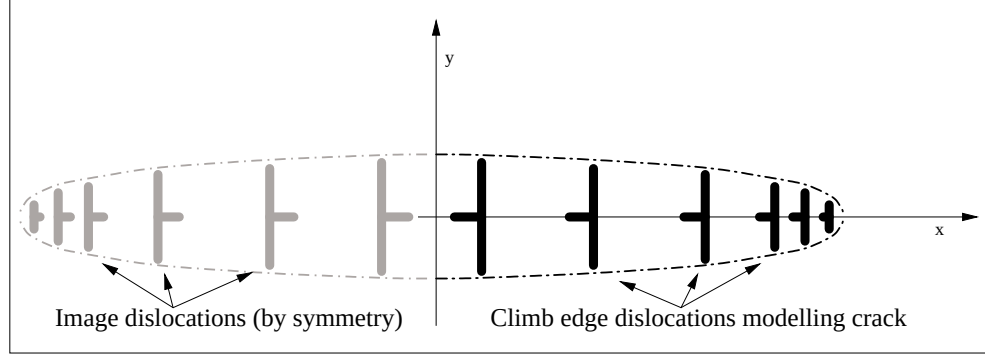


Figure 2.6: A crack is modelled by “packing” climb edge dislocations together.

Instead of using discrete dislocations, a dislocation density distribution can be employed to give a smooth function corresponding to an infinite number of dislocation sites along the line of the crack. This distribution is represented as follows:

$$B_y(x_d) = \frac{db_y(x_d)}{dx_d} \quad (2.4)$$

The stress at any point in the plane containing the crack can then be expressed by:

$$\begin{aligned} \sigma_{ij}(x, y) &= \sigma_{ij}^{\infty}(x, y) + \sigma_{ij}^d(x, y) \\ &= \sigma_{ij}^{\infty}(x, y) + \frac{2\mu}{\pi(\kappa + 1)} \int_0^a G_{yij}(x, y, x_d) B_y(x_d) dx_d, \quad i, j \in [x, y] \end{aligned} \quad (2.5)$$

where (x, y) = collocation point (point of inspection),

x_d = integration point (position of dislocation),

μ = shear modulus,

κ = Kolosov's constant,

G_{yij} = stress kernel (influence) function.

The displacement distributions can also be calculated using the appropriate kernels:

$$u_i = \frac{1}{2\pi(\kappa + 1)} \int_0^a U_{yi}(x, y, x_d) B_y(x_d) dx_d, \quad i \in [x, y] \quad (2.6)$$

kernels (e.g. half plane, strip) which cannot be analytically inverted, we present the numerical method which in principle is applicable to other geometries [37]. We use the normalised intervals:

$$\begin{aligned} p_i &= \frac{2x_d}{a} - 1 & -1 \leq p_i \leq 1, \\ r_k &= \frac{2x}{a} - 1 & -1 \leq r_k \leq 1. \end{aligned} \quad (2.8)$$

For the crack faces to be traction free, $N(x) = 0$ for $0 \leq x \leq a$. Thus equation 2.7 can be expressed as:

$$\frac{2\mu}{\pi(\kappa + 1)} \int_{-1}^1 B_y(p_i) G_{yy}(r_k, p_i) dp = -\sigma_{yy}^\infty(r_k), \quad (2.9)$$

where $-1 \leq r_k \leq 1$.

The dislocation density distribution function, $B_y(p)$, is now re-written as the product of a weight function W_i , which encapsulates the end point behaviour required, and a function $\phi(p)$ to be determined [39]. For an edge crack in a half-plane, the required behaviour is bounded at the free surface ($p = -1$) and singular at the crack tip ($p = 1$). The appropriate weight function is then $\sqrt{1+p}/\sqrt{1-p}$ so that:

$$B_y(p) = \phi_y(p) \frac{\sqrt{1+p}}{\sqrt{1-p}} \quad (2.10)$$

Now equation 2.9 can be discretised using Gauss-Jacobi quadrature in the general form [37]

$$\sum_{i=1}^n W_i G_{fgh}(r_k, p_i) \phi(p_i) = F(r_k) \quad k = 1, \dots, n \quad (2.11)$$

giving:

$$\sum_{i=1}^n \frac{2(1+p_i)}{2n+1} G_{yy}(r_k, p_i) \phi_y(p_i) = -\frac{\kappa+1}{2\mu} \sigma_{yy}^\infty(r_k) \quad (2.12)$$

$$\begin{aligned} \text{where } r_k &= \cos\left(\frac{2k}{2n+1}\pi\right) & k = 1, \dots, n \\ p_i &= \cos\left(\frac{2i-1}{2n+1}\pi\right) & i = 1, \dots, n. \end{aligned}$$

Equation 2.12 is a system of n simultaneous linear equations which can be solved for $\phi(p_i)$.

We can then apply the discretised format of equation 2.5 in order to calculate the stress field for a crack in a half plane:

$$\begin{aligned} \sigma_{gh}(x(p_i), y) &= \sigma_{gh}^{\infty}(x(p_i), y) + \\ &\quad \frac{2\mu}{\pi(\kappa + 1)} \sum_{i=1}^n \frac{2(1 + p_i)}{2n + 1} G_{y gh}(x(p_i), y, x_d(r_k)) \phi_y(p_i). \end{aligned} \quad (2.13)$$

It should be noted from the above equation that the stresses can be evaluated only at the collocation points given by equations 2.8 and 2.12, because the Cauchy integral has been solved using a numerical quadrature. It is possible to evaluate the stresses at other points, but this requires an interpolation formula.

Having determined $\phi(p)$ (equation 2.12) we evaluate B_y from equation 2.10 and can now calculate the shape of the crack using:

$$\delta(x) = \int_x^a B_y(x_d) dx_d. \quad (2.14)$$

2.2.2 Determination of the stress intensity factor

Hills *et al* [37] present the determination of the stress intensity factor, but not in a single exposition. It would also appear that a typographical error has included a factor of $\frac{1}{\pi}$ in their equation 2.81, and so the derivation of the stress intensity factor for a crack in a half plane is presented next. Assuming small scale yielding, we start by considering the crack tip elastic displacement field:

$$u_y(r, \theta) = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \sin\left(\frac{\theta}{2}\right) (\kappa - \cos \theta) \quad (2.15)$$

The Burgers vector magnitude must be defined by the discontinuity in the displacement field along the line of the crack:

$$\begin{aligned} b(r) &= u_y(r, +\pi) - u_y(r, -\pi) \\ &= \frac{\kappa + 1}{\mu} K_I \sqrt{\frac{r}{2\pi}} \end{aligned} \quad (2.16)$$

Differentiating with respect to r :

$$\frac{db(r)}{dr} = \frac{\kappa + 1}{2\mu} \frac{K_I}{\sqrt{2\pi r}}, \quad (2.17)$$

and solving for K_I :

$$K_I = \lim_{r \rightarrow 0} \left[\frac{2\mu}{\kappa + 1} \sqrt{2\pi} \sqrt{r} \frac{db(r)}{dr} \right] \quad (2.18)$$

Now, confining our attention to the r dependence:

$$\begin{aligned} \lim_{r \rightarrow 0} \left[\sqrt{r} \frac{db(r)}{dr} \right] &= \lim_{p \rightarrow 1} [\sqrt{a - x_d} B_y(p)] \\ &= \lim_{p \rightarrow 1} \left[\sqrt{\frac{a}{2}(1-p)} \phi(p) \frac{\sqrt{1+p}}{\sqrt{1-p}} \right] \\ &= \lim_{p \rightarrow 1} \left[\sqrt{\frac{a}{2}(1+p)} \phi(p) \right] \\ &= \sqrt{a} \phi(1) \end{aligned} \quad (2.19)$$

Therefore, the corrected version of Hills' equation 2.81 reads

$$K_I = \sqrt{\pi a} \frac{2\mu}{\kappa + 1} \sqrt{2} \phi(1). \quad (2.20)$$

In order to obtain the stress intensity factor, the value of $\phi(p)$ at the crack tip ($p = 1$) needs to be calculated. This is achieved using Krenk's [40] interpolation formula:

$$\phi_y(1) = \frac{2}{2n+1} \sum_{i=1}^n \cot \left(\frac{2i-1}{2n+1} \frac{\pi}{2} \right) \sin \left(\frac{2i-1}{2n+1} n\pi \right) \phi_y(p_i). \quad (2.21)$$

The stress intensity factor is then given by:

$$K_I = \sqrt{\pi a} \frac{2\mu}{\kappa + 1} \sqrt{2} \phi_y(1). \quad (2.22)$$

When considering a crack in an infinite plane, we need to take into consideration the fact that the end point behaviour (singular–singular) will be different from that expected in a half plane (bounded–singular). A different numerical quadrature scheme is then required: Gauss-Chebyshev. With such a scheme the formulae used to calculate the stress intensity factor require modifications.

Alternatively, it is possible to take advantage of the symmetry of a crack in an infinite plane about the y -axis and to model only the right-hand half of the crack. This enables us to use the same discretisation as that for a crack in a half plane (and indeed, strip). All that is required is to modify

the kernel used for a crack in a full plane by adding a term to place a *symmetry-based* dislocation of equal and opposite strength at the position $-x_d$ for every dislocation x_d . Similarly, a crack in a half-plane or a strip can be modelled by simply including the appropriate influence function.

The kernel functions for both stress and displacement calculations are listed in appendix A.

2.3 Modelling plasticity using edge dislocations

We now present a means of emulating a Dugdale-type plastic zone where all plasticity is confined to a line collinear with the crack tip. The method employed in modelling of the plasticity zone is very similar to that used when modelling cracks (see section 2.2), with only the boundary conditions being slightly different. Whereas for cracks we require the normal tractions along the crack faces to be zero, in the plastic zone we require the stresses along the line of plasticity to satisfy the yield criterion being used.

In plane stress, the plane of maximum shear stress lies at $\pm 45^\circ$ to the plane of analysis (i.e. the x - y plane). The intersection of these two planes forms a line collinear with the crack ahead of the crack tip. For this reason, a strip yield model allowing for material flow in the out-of-plane direction is viable. Climb dislocations are therefore used to simulate the extra material arriving in the plastic zone. In plane strain, however, the plane of maximum shear stress lies *in* the plane of the analysis, and can be shown to have an angle of approximately $\pm 70.5^\circ$ to the x - z plane [20]. Here, glide dislocations are necessary to model the shear flow along these plasticity rays.

2.3.1 Classical metal plasticity models

Classical metal plasticity models incorporate three key components:

- The *yield surface* defines the stress boundary within which a material behaves elastically.
- The associated *flow rule* defines the plastic deformation when the stress characterising the yield parameter moves beyond the confines of the yield surface.
- The *hardening law* defines the manner in which the yield surface and flow rule change as the plastic deformation occurs.

Most engineers have an understanding of a yield surface and the necessity for the stress at any point in a body to lie within or on this surface, however it may be defined (e.g. von Mises, Tresca, Hill). A more subtle issue is the flow rule (see, for example, Mendelson [41]) such as the Prandtl-Reuss equations for plastic flow in the elastoplastic¹ range. These state that in any direction, the ratio of the plastic strain increment to the current deviatoric stress is instantaneously constant:

$$\frac{d\varepsilon_{xx}^p}{\sigma'_{xx}} = \frac{d\varepsilon_{yy}^p}{\sigma'_{yy}} = \frac{d\varepsilon_{zz}^p}{\sigma'_{zz}} = \frac{d\gamma_{yz}^p}{\tau_{yz}} = \frac{d\gamma_{zx}^p}{\tau_{zx}} = \frac{d\gamma_{xy}^p}{\tau_{xy}} = d\lambda \quad (2.23)$$

or

$$d\varepsilon_{ij}^p = \sigma'_{ij} d\lambda \quad (2.24)$$

where $\sigma'_{ij} = (\sigma_{ij} - \frac{1}{3}I_1)$ denotes the deviatoric stress and $d\lambda$ is an instantaneous non-negative constant of proportionality (which may vary throughout a straining process). $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$ is the first invariant of stress.

It is crucial to note that: (1) a small increment in plastic strain depends only on the current total deviatoric stresses (not on the stress increment

¹The Levy-Mises equations relating the increments of total strain to the stress deviations, proposed by Lévy in 1871 and von Mises in 1913, are only valid for large scale plasticity (i.e. they ignore elastic strains). The elastoplastic range is that where elastic strains are of the same order as the plastic strains.

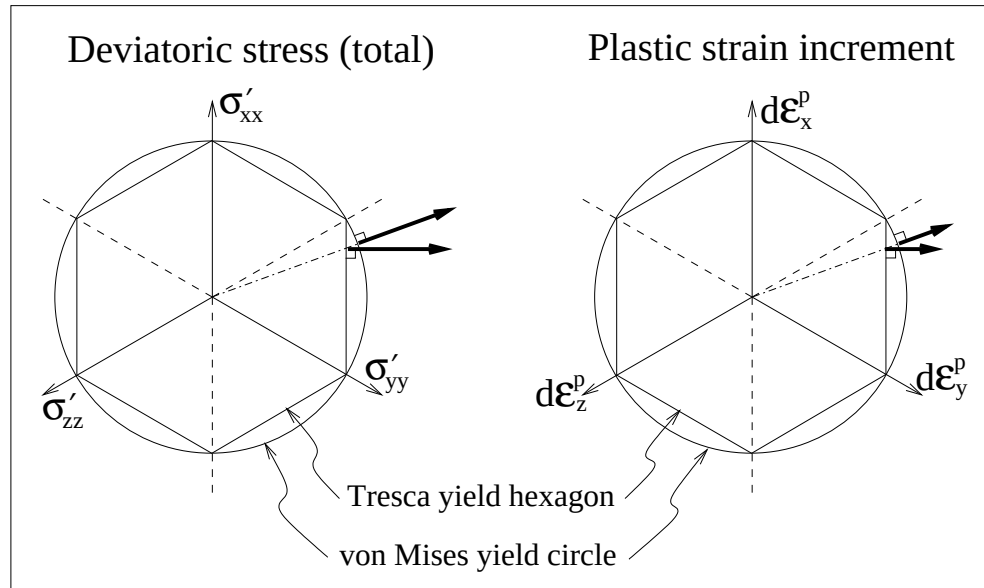


Figure 2.8: Schematic diagram of the Prandtl-Reuss flow rule in the π -plane. The instantaneous deviatoric stress direction must coincide with the direction of the instantaneous plastic strain increment.

required to bring it about); (2) the principal axes of stress and strain increment coincide. A schematic diagram showing the parallel vectors is given in figure 2.8.

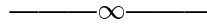
On the matter of hardening laws: throughout this text, we shall be considering materials defined as *elastic perfectly plastic*. This means that the yield stress does not change with plastic strain. No further reference will be made to hardening laws.

The ability to model the plastic zone ahead of the crack tip allows us to develop the next degree of complexity: growing cracks. Growing cracks are very much more difficult to model in that there are many subtleties which need to be addressed, the most pressing of which is crack closure. This can all be done using models which rely on plasticity being confined to a line, or ray, although there are limitations to this simplification, which will be discussed at a later stage.

2.4 Summary

We have discussed the basic concepts in dislocation theory, starting with a single dislocation in an infinite plane, and then extending these ideas to incorporate a means of modelling static and growing cracks with plasticity in an infinite plane. By simply changing the influence functions being used, the geometry of the problem can be changed to model a crack in an infinite plane, half-plane or strip.

In the next chapter we shall discuss a plane stress 1-D ray-plasticity model which employs dislocation theory to model static plastic and growing plastic cracks.



Chapter 3

One-dimensional models of plasticity

An efficient means of modelling the plastic wake will be demonstrated. This model has been refined and extended, in order to obtain data for cracks under various load conditions. A study of the comparison of true and apparent stress intensity factors due to closure in a plane stress model will be presented. Some remarks are then made concerning the crack initiation problem. Having demonstrated the relative simplicity of the one-dimensional model, we examine their limitations in modelling plasticity problems.

3.1 The Nowell model for closure in plane stress

Nowell's plane stress model for the calculation of crack closure [25] is based on the Dugdale and Newman plasticity models (see section 1.2.1). The method employs dislocation theory in the form of boundary elements to model both the crack shape and the plasticity ahead of the crack tip. See section 2.1 for a description of the fundamental dislocation theory used in this model. A displacement discontinuity boundary element con-

sists of two dislocations “facing” one another, as shown in figure 3.1.

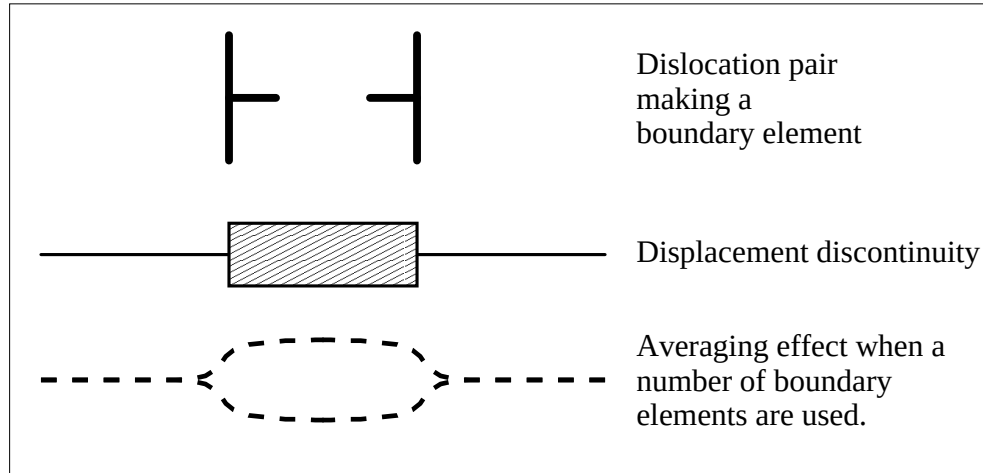


Figure 3.1: Schematic diagram of a boundary element.

The original model was implemented in FORTRAN code. This has been re-worked in MATLAB and C, using the C N.A.G. routines to speed up the numerically intensive procedure of solving the quadratic programming problem. Re-writing of this code required a complete dismantling of the algorithm in order to gain a thorough understanding.

A brief explanation of the workings of the model in terms of schematic diagrams will now be made. Full details of the formulation can be found in the original paper [25].

The key points which need to be made are as follows (see figure 3.2):

1. Three sets of dislocations are employed in the model. The first represents the crack itself. The second represents the Dugdale-type [16] plastic zone ahead of the crack tip. The third is present only for a growing crack, and represents the plastic wake which arises from the crack growing through its own plastic zone, manifesting itself as extra material “pasted” onto the crack flanks.
2. A solution of the static plastic crack in an isotropic infinite plane under transverse plane stress and subject to uniaxial tension could be achieved by analytical inversion. However, anticipating complex-

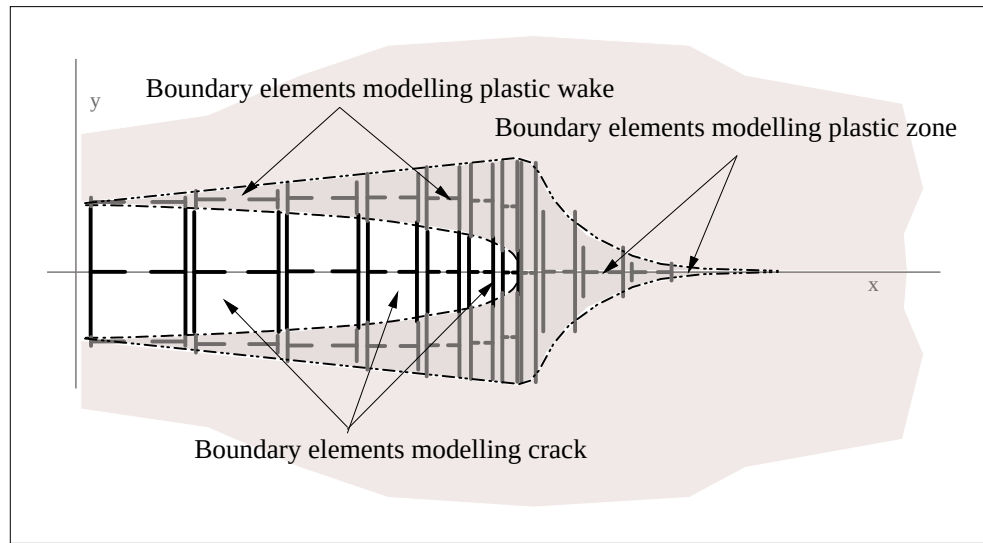


Figure 3.2: A growing plastic crack is modelled using three sets of boundary elements.

ities introduced when modelling growing cracks, even for the simplest load histories, a numerical quadratic programming technique, reported in [25, 42] is employed. The additional features may, for example, include remote closure (i.e. the crack tip is “propped open” by material remote from the crack tip itself as a result of an overload cycle); or sudden changes in the size of the plastic zone as a result of block overloads/underloads, which may cause a sudden change in the residual plastic displacement adjacent to the crack tip.

3. The model incorporates both tensile and compressive yield. Thus, if parts of the crack faces are pushed together very hard, the possibility of compressive yield is not ignored.
4. No attempt is made to infer directly crack growth rate from the characteristics of the crack tip process zone. Crack growth rate information is assumed to be available from experimental data such as the Paris equation 1.1. The key point is that we assume that it is the true stress intensity range which controls crack growth rate.

3.1.1 Static plastic crack

The method is based on a static crack as described in sections 2.2 and 2.3. Instead of a continuous distribution of dislocations, we employ a set of finite dipoles, each formed from a dislocation pair. The plastic zone may also be represented in this way. In this region, the boundary condition to be satisfied is simply that the normal traction must equal the yield stress, as the through thickness stress is zero, and hence, if Tresca's criterion is adopted, the maximum stress difference is between σ_{yy} and σ_{zz} . The crack is subjected to a maximum load and the magnitude of the plastic zone is estimated by considering those elements which carry a stress in excess of the yield criterion. Boundary elements, or extended dislocation dipoles, are activated within this plastic zone, allowing the material to extend in the direction of the applied load, in order to return the stress state to the yield surface. This procedure results in "load shedding" away from the plastic zone, causing the calculated plastic zone to grow slightly. Additional elements are placed in the extended plasticity zone, and the process is repeated until convergence is achieved. To reduce the amount of manual intervention required a quadratic programming technique may be used, by minimising object functions defining the boundary conditions to be satisfied.

To illustrate this, we shall consider the boundary conditions along the crack flanks only. In this region, one of two conditions prevails: (a) the crack is open and the crack flanks are traction free; or (b) the crack is closed and the crack flanks carry a compressive stress. These conditions may be expressed as

$$\begin{aligned} \text{Closed:} \quad & \sigma_{yy} \leq 0, \quad b_y = 0 \\ \text{Open:} \quad & \sigma_{yy} = 0, \quad b_y \geq 0. \end{aligned} \tag{3.1}$$

The object function

$$F_1 = -\sigma_{yy}b_y \tag{3.2}$$

therefore has the desired properties of being always positive, and zero

when the boundary conditions are satisfied. The object function is minimised in the quadratic programming routine, subject to the constraints

$$\begin{aligned}\sigma_{yy} &\leq 0 \\ b_y &\geq 0 \\ \sigma_{yy} &= f(b_y).\end{aligned}\tag{3.3}$$

For a static plastic crack, a similar object function (F_2) is required which defines the yield behaviour anticipated in the plastic zone. Further complexity is introduced when considering a growing plastic crack (introducing a third object function, F_3), which will be discussed shortly. For now it suffices to say that it is the novel definition of these object functions which makes this method efficient. Full details of the technique are presented in the original paper [25].

Although it is clear that the full field displacements will not match those of a more sophisticated area-plasticity model (because of the localisation of strains in the plastic zone), this relatively simple model can be used to predict both the crack tip opening displacement, and the stress distribution along the line of the crack with considerable accuracy. The major source of error is that plasticity which in reality exists over an area has been collapsed onto a line, and this is discussed in section 3.4.

One issue of critical importance which is not raised in the original paper [25] concerns the plastic flow rule. Since the plastic zone is confined to a line on the x -axis ahead of the crack tip, it is clear that the direction of maximum shear stress at any point along the plastic zone is perpendicular to the x -axis. Thus climb edge dislocations deployed in the plastic zone with $b_x = 0$ and $b_y \neq 0$ will *always* satisfy the normality condition since there are no changes in the direction of maximum shear stress at any point in time.

The effect of a crack in a half-plane, in a finite width strip, or other geometries, can be obtained by simply modifying the kernel functions, $G_{yij}(x, y, x_d)$, in order to account for the different remote boundary conditions.

3.1.2 Growing plastic crack

For the more complex case of a growing plastic crack, some means of modelling the fact that the crack is growing through plastically deformed material (i.e. its own plastic zone) is necessary. The region of cyclic plasticity immediately ahead of the crack tip and approximately one-quarter of the length of the full plastic zone (Dugdale, [16]) will at the minimum load of any cycle be suffering a compressive stress. Thus, when the crack grows through this compressive region, the crack flanks will be pressed closed for a portion of the load cycle. This is modelled by “pasting” additional material of an appropriate thickness onto the crack flanks as the crack tip grows through its own plastic zone. The extra material can then support a compressive load, and can yield in compression, but cannot sustain a tensile load (separation occurs); nor for obvious reasons can it yield in tension.

In order to track the load history accurately, an incremental method is required when modelling a growing plastic crack. This is implemented as follows: a crack is assumed to exist in an isotropic material, with both a plastic wake along the crack flanks and a plastic zone ahead of the crack tip, the dimensions of which are not known beforehand. The crack is subjected to a maximum load, and the magnitude of the plastic zone calculated as described in the preceding section for a static crack. To simulate cyclic loading the load is now reduced to its minimum value. Reverse (compressive) yielding is predicted, and its extent is found by returning the material to the yield surface at all points, in a similar manner to that described previously. This completes the simulation of the first loading cycle. The crack front is advanced by an arbitrarily small amount, which is achieved by moving the crack tip forward by a small increment (typically 1-2% of the crack length), and moving the mesh such that the focused region is retained at the crack tip. Some of the material from the plastic region which carries a residual compressive stress will now lie adjacent to the crack flanks, and in order to model the effect of this extra material, a

third set of dislocation dipoles is deployed along the crack flanks to model the developing plastic wake. The loading history is repeated, so that the crack grows progressively through its own plastic zone. For a finite crack in an infinite isotropic plane subjected to a periodically varying remote load, a steady state solution will evolve due to self-similarity. It can be seen that this method is computationally expensive.

For a periodic driving force, and a body devoid of other geometric features, it is sensible to exploit the argument of self-similarity in order to obtain the steady state solution in a single cycle [25]. For a semi-infinite crack (so that there is no intrinsic length dimension in the problem), a plastic wake of constant thickness arises, as demonstrated by Budiansky & Hutchinson [21]. For a finite crack, the size of the plastic wake is proportional to the crack length [22]. All that remains is to find the magnitude of the residual crack tip stretch. This is done by iteratively applying a single load cycle to the crack until the residual stretch converges to the solution. In practice using the Newton-Raphson method, five to eight iterations are required in order to obtain a value for the residual stretch which is consistent to within 0.02%.

Any deviation from a uniform load cycle (such as an overload or underload), or any geometry which introduces further length scales (for example remote bending, half plane or strip, inclusion or hole), requires the incremental method to be used to track the load history.

Refinements made to the model will be discussed in the next section.

3.2 Refinement of the Nowell model

The following refinements were made in the MATLAB implementation of the Nowell model:

- For a growing crack, the algorithm was automated (using the Newton-Raphson method) to enable convergence on an initial solution for the residual stretch.
- The algorithm was embedded within a state-sensitive framework which enabled the detection of opening and closing loads without user intervention.
- A database was created to allow more complex load histories such as block overloads and underloads to be analysed.
- The model was extended to handle half plane and strip geometries.
- Remote applied tension was supplemented with the ability to apply remote bending.
- Code was developed to plot crack shape and stress distribution at the minimum, maximum, opening and closing loads automatically.

These refinements are now discussed in more detail.

To model a growing crack, the original FORTRAN code required the user to type in estimates of the magnitude of residual stretch. The program would then compute the stress distributions and report a new residual stretch. The user would then consider whether the value returned fell within a given tolerance of the initial guess; success would mean that the steady-state solution for a growing crack had been determined and that detection of the opening and closing loads could be attempted. In practice this process is tedious, but more seriously, failure to determine this value with great accuracy jeopardises the remaining analysis.

Incorporation of the algorithm into a state-sensitive framework allowed the method to proceed independently of user input. Tolerances are specified at the outset and the program then computes a solution within these limits. Thus, results for varying loads can be compared with the confidence that each case has been solved to the same degree of accuracy.

Initially, the main objective was to allow the theoretical results from the Nowell model to be compared to the experimental results on crack closure being produced by Fellows [43, 42, 7, 44] by means of moiré interferometry. One of the aspects of Fellows' research was the influence of overloads and underloads (see figure 3.3) on crack closure. By developing a system to drive the model from a database, comparisons could be made for any load history.

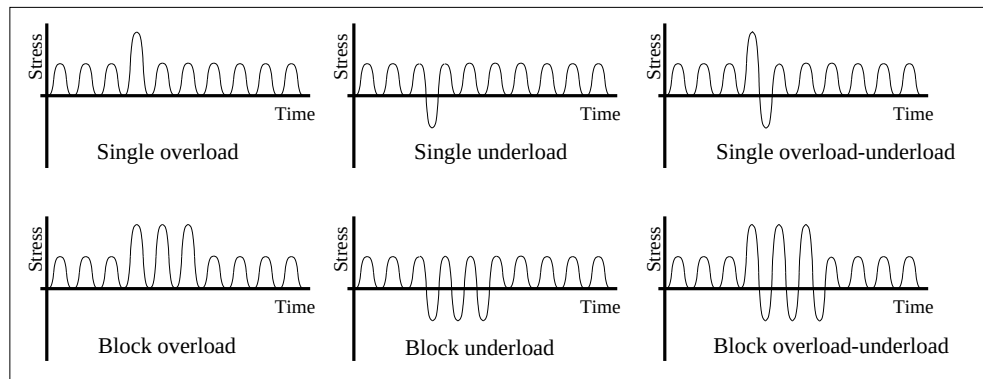


Figure 3.3: The refined Nowell model can readily compute response to applications of variable amplitude loading.

The argument of self-similarity in the establishment of a linear plastic wake (see preceding section) is valid only for a model involving one length dimension, *viz.* a steady-state growing crack under remote uniform tension in a full or half plane. The introduction of any further length parameters, for example due to remote bending (distance to neutral axis), a crack in a strip (strip depth or thickness), or any model involving a non-steady state solution, precludes the use of a linear plastic wake. In this case the crack must be “grown” from a small size in order to model the load history correctly. The implementation of the half plane and strip kernels with remote applied bending allowed more accurate comparisons to

be made to the experimental results, where the cracks were being grown in a beam subjected to four-point bending.

The flow diagram outlining the refined Nowell model is shown in figure 3.4. A more detailed version of this flow chart can be found in appendix C. The associated MATLAB and C code are on the enclosed CD-ROM. Example output from the program will be discussed next.

The objective of this study is to undertake a comparison of the true and apparent stress intensity ranges for a crack in an infinite plane under plane stress for all load conditions. Under these conditions, the assumption of a plastic wake growing proportionally to crack length (i.e. employing the self-similarity argument) is valid and has been used to generate data. Both static plastic and growing plastic cracks will be studied. Prior to presenting this analysis, we choose one load case to demonstrate the output obtained for each type of crack.

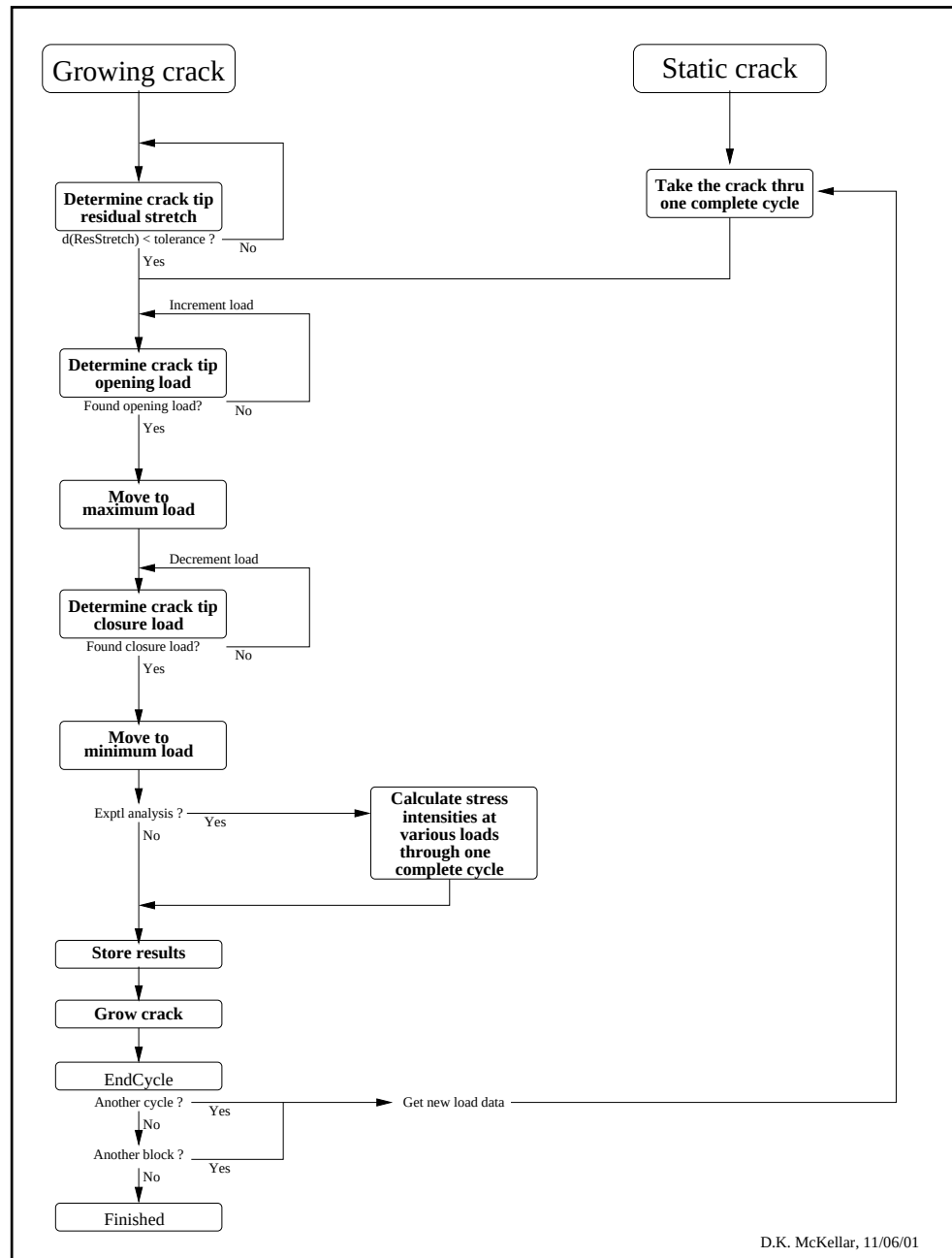


Figure 3.4: Algorithm implemented for Nowell model.

3.2.1 Results: static plastic crack example

The amount of data available from this analysis are substantial. A method of presenting all this data on one diagram will now be outlined. Consider figure 3.5, which shows three data sets: (a) the crack opening displacement is marked by ovals; (b) the plastic displacements are marked by crosses; and (c) the stress distribution is marked by squares.

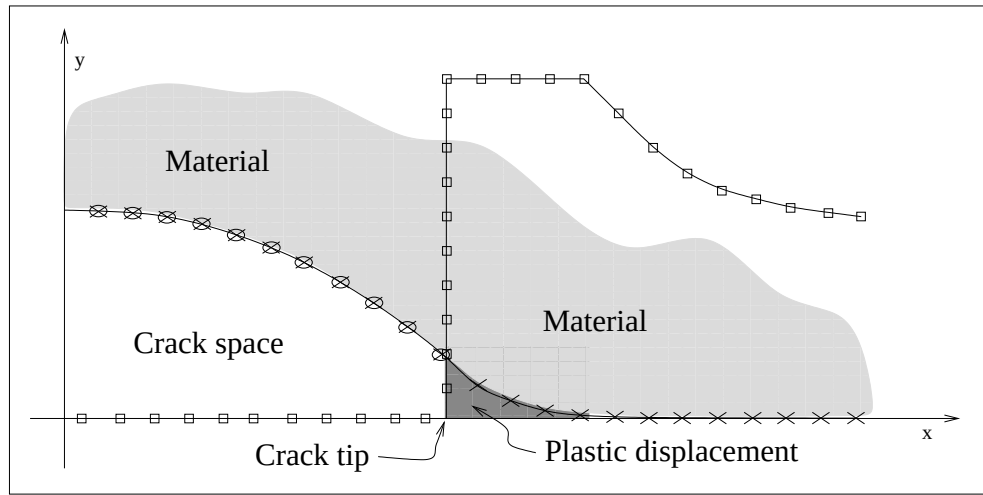


Figure 3.5: Schematic illustration for a static crack. The crack opening displacement is shown by the ovals; the plastic displacement is marked with crosses; and the stress distribution is represented by squares.

Figure 3.6 shows an example set of the results for a static plastic crack in an infinite plane. The crack lies along $|\frac{x}{a}| \leq 1$; the crack tips lie at $|\frac{x}{a}| = 1$, and the plastic zones are in the regions $|\frac{x}{a}| > 1$. Note the focusing of boundary elements towards the crack tip to obtain a more accurate solution. Here, $R = 0$ and $\frac{\sigma_{max}}{\sigma_Y} = 0.4$. The choice of load case is arbitrary, since all loads in the range $\frac{\sigma_\infty}{\sigma_Y} \in [-0.8, 0.8]$ have been analysed¹, and thus we select a case which is representative of typical engineering design range. Consider first the output at maximum load, figure 3.6(a). Note the crack itself, $0 \leq x/a \leq 1$. The crack opening displacement, d , has been normalised with respect to d_0 , which is the crack mouth opening displacement under steady state conditions for this applied stress cycle ($\sigma_{min} = 0.00 \sigma_Y$, $\sigma_{max} = 0.40 \sigma_Y$). This gives the actual crack shape, based

¹The range of remote loads applied will be discussed in more detail in section 3.3.

on

$$\delta(0) = \frac{2d_0\mu}{\pi a(1 + \kappa)\sigma_Y} \quad (3.4)$$

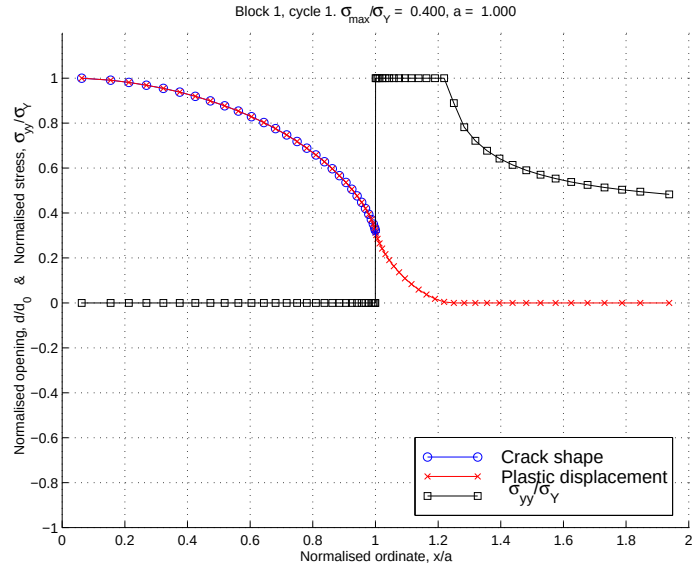
where $\delta(x)$ is the crack shape, $\delta(0)$ is the crack mouth opening displacement, μ is the shear modulus, and κ is Kolosov's constant. The values of d_0 are recorded for every case analysed and written to a text file which documents the progress of the algorithm².

The plastic displacement along the crack flanks is zero, (as expected, since we are modelling a static plastic crack), thus in figure 3.6(a), the circles and crosses trace the same line. It can be seen that the crack flanks are traction free. The crack tip ($x/a = 1$) has been stretched open. A plastic zone has developed ahead of the crack tip, approximately $0.21a$ in length. The stresses within the plastic zone are on the yield surface, and beyond the plastic zone ($x/a > \sim 1.21$) the stress distribution is a combination of the elastic stress and the redistributed dislocation stresses due to the plasticity.

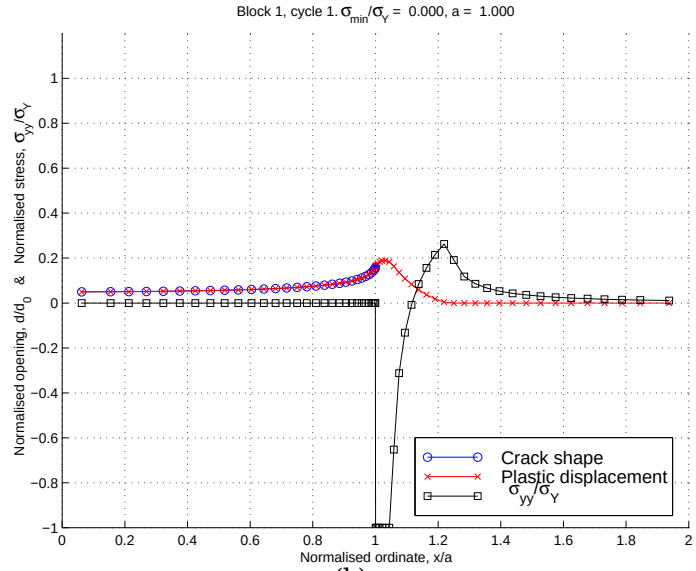
The applied load is now removed completely, since $\sigma_{min} = 0.00 \sigma_Y$. At this minimum load (figure 3.6(b)), the crack is still completely open, and a state of compressive yield is attained immediately ahead of the crack tip, propping it open. Some portion of the original plastic zone continues to carry a residual tension, and beyond the plastic zone the elastic stress distribution (plus a small component of redistributed stress) can again be seen.

It should be noted that, for a given load case, the length of the plastic zone (r_p) is the same for all load cycles. The length of the reversed plastic zone, r_{rp} , does *not* affect the length of the full plastic zone. This is because the compressive plastic zone r_{rp} falls within the full plastic zone r_p , and thus the first stage of loading annihilates the pre-existing residual stresses from the previous load cycle. Therefore the first cycle, and all subsequent steady state cycles, are identical *for a static plastic crack*.

²These results can be found in the files ProgSummary.txt.gz on the attached CD-ROM.



(a)



(b)

Figure 3.6: Example results for a **static crack** in a full plane for $R = 0$, $\frac{\sigma_{\max}}{\sigma_Y} = 0.4$. The crack tip lies at $\frac{x}{a} = 1$, and the plastic zone is in the region $\frac{x}{a} > 1$. (a) Maximum load $\frac{\sigma_{\max}}{\sigma_Y} = 0.400$. (b) Minimum load $\frac{\sigma_{\min}}{\sigma_Y} = 0.000$.

3.2.2 Results: growing plastic crack example

A growing crack is represented in much the same manner as that of a static crack, with the exception that now there is a plastic wake which is formed when the crack grows through its own plastic zone, illustrated schematically in figure 3.7. As discussed previously, this causes closure to

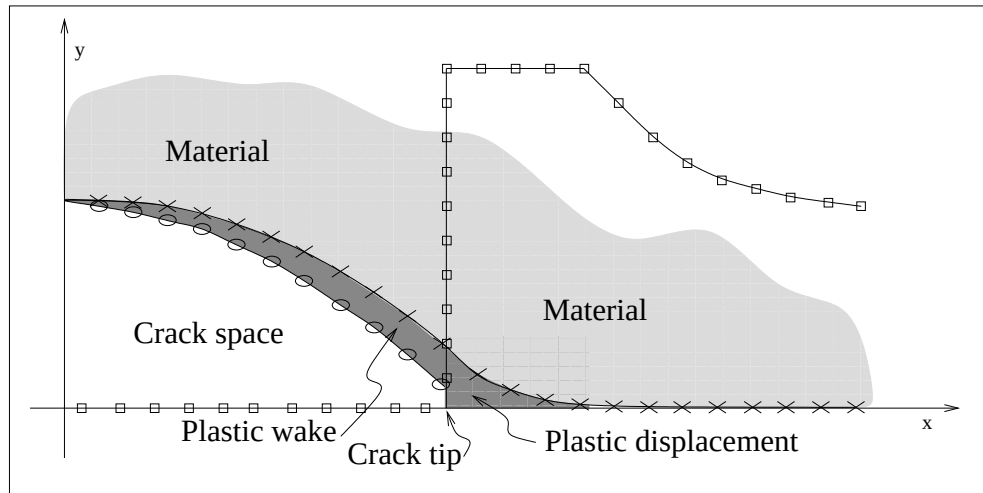


Figure 3.7: Schematic illustration for a growing crack. The crack opening displacement is shown by the ovals; the plastic displacement is marked with crosses; and the stress distribution is represented by squares. Note the plastic wake which is “pasted” to the crack flanks.

occur. The points at which the opening, maximum, closing, and minimum loads might occur are labelled (a) through (d) as shown in figure 3.8. Each of these points corresponds to each of the diagrams in figure 3.9, respectively, which presents an example of the results obtained for one of the

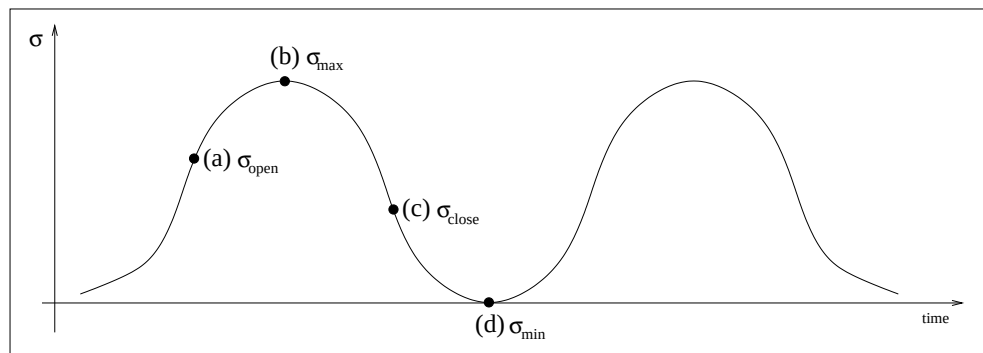


Figure 3.8: Diagram representing the opening, maximum, closing, and minimum loads.

load cases run. Note that, unlike the static crack, we now have a plastic wake which is shown clearly in the figures. Here, $R = 0$, $\frac{\sigma_{max}}{\sigma_Y} = 0.4$ (as for the example of the static crack). Figure 3.9(a) provides stress and displacement data for the load at which the crack tip becomes fully open. Moving through the opening load, the stress at the crack tip switches (instantaneously) from compressive yield to tensile yield. The criterion used to identify the load at which opening occurs is that the stress suffered by the collocation point in the crack, closest to the crack tip, is no longer compressive. Calculation of the crack tip stress intensity at minimum load (by considering the stress distribution within the plastic zone) produces a “negative” stress intensity. The crack should become open if a load increment corresponding to this negative stress intensity is added to the minimum load, and thus the opening load can usually be determined in one step. The stress distribution is very complex ahead of the crack tip.

Figure 3.9(b) shows the crack at maximum load. Note the region of tensile yield which corresponds to the plastic zone, and the fact that the crack tip itself is slightly open (there are both plastic and elastic components contributing to the CTOD).

The applied load is then reduced until the closure load is reached, as shown in figure 3.9(c). Since the closure load will always be lower than the opening load, the first step in finding the closure load is to subtract the load increment corresponding to the difference between the maximum and opening loads. The remote load is then further reduced, whilst monitoring the CTOD and the stress at the crack tip. The Newton-Raphson method is used to decrement the remote load based on the history of the CTOD. Again, the criterion used to identify the closure load is that which first provides a negligible compressive stress at the collocation point in the crack nearest to the crack tip. The closure load is a difficult point in the load cycle to identify, since, as this plot shows, there can be remote closure (and resulting compressive stress) along the crack flanks before the crack tip itself actually closes. The closure point has to be approached

gradually – if it is inadvertently passed, the entire load cycle analysis must be repeated because plasticity history is path dependent. As for the opening load, the stress distribution is very complex.

Finally, figure 3.9(d) shows the crack at minimum load. Here, the effect of the closure can easily be seen in that approximately half of the crack is open, with the other half under compressive stress due to the plastic wake.

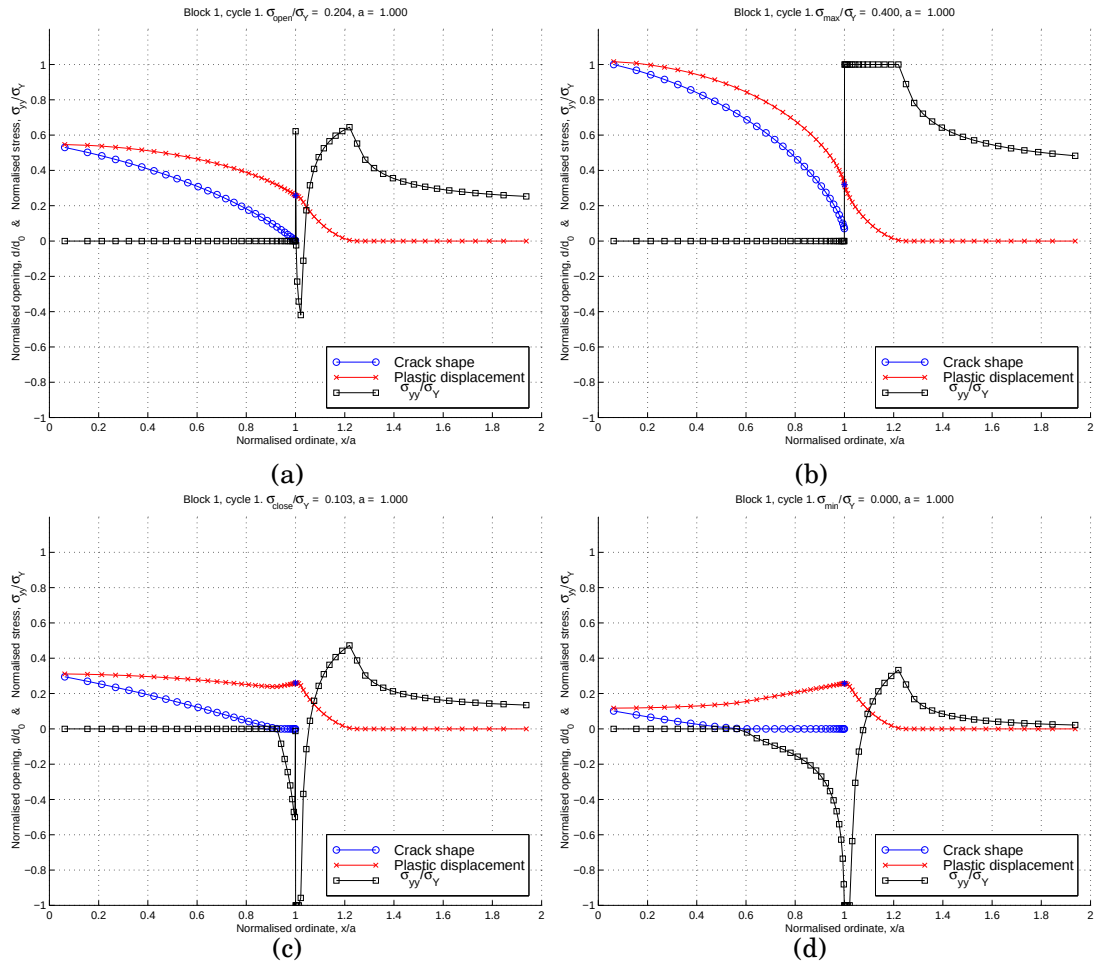


Figure 3.9: Example results for a **growing crack** in a full plane for $R = 0$, $\frac{\sigma_{max}}{\sigma_Y} = 0.4$. The crack tip lies at $\frac{x}{a} = 1$, and the plastic zone is in the region $\frac{x}{a} > 1$. (a) Opening load $\frac{\sigma_{open}}{\sigma_Y} = 0.204$. (b) Maximum load $\frac{\sigma_{max}}{\sigma_Y} = 0.400$. (c) Closing load $\frac{\sigma_{close}}{\sigma_Y} = 0.103$. (d) Minimum load $\frac{\sigma_{min}}{\sigma_Y} = 0.000$.

3.2.3 Results: single overload example

As has been mentioned previously, the refinements of the Nowell model were originally motivated by the need to obtain theoretical data which could be compared to the experimental work being conducted by Fellows. One of the last experiments she conducted involved a single overload, and it was desired to have a theoretical comparison³. A steady state crack was grown at $\frac{\sigma_{max}}{\sigma_Y} = 0.507$. A single overload of $\frac{\sigma_{max}}{\sigma_Y} = 0.845$ was applied, whereupon the maximum applied stress was returned to the previous steady state value and the crack was grown through the enlarged plastic zone.

Figure 3.10 shows the simulation at the time of the applied overload. At the maximum applied overload (figure 3.10(a)), the CTOD is very large, and the length of the plastic zone is approximately $0.8 a$. It can also be seen that the CMOD is nearly twice that for a steady state growing crack.

The minimum load is shown in figure 3.10(b). Note that, despite the fact that the applied remote load is zero, the crack is propped open. This is due to the large plastic deformation suffered at the maximum overload.

Figure 3.11 shows the simulation for the same crack, which has grown to a length of 1.5 times that at which the overload was applied. Since the crack has grown through its own plastic wake, closure is now occurring. The effect of the overload can clearly be seen in both the extra material which has been “pasted” to the crack flanks, and in the complex stress distribution for the crack at minimum load.

³The results from this comparison are documented in Fellows’ D.Phil thesis [44].

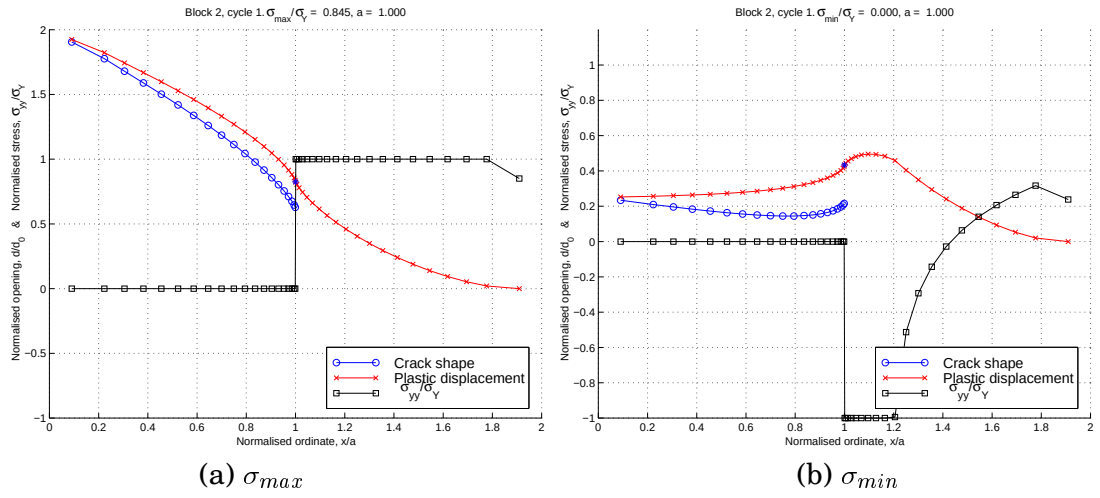


Figure 3.10: Specimen 51 single overload test: predicted crack shapes during the overload.

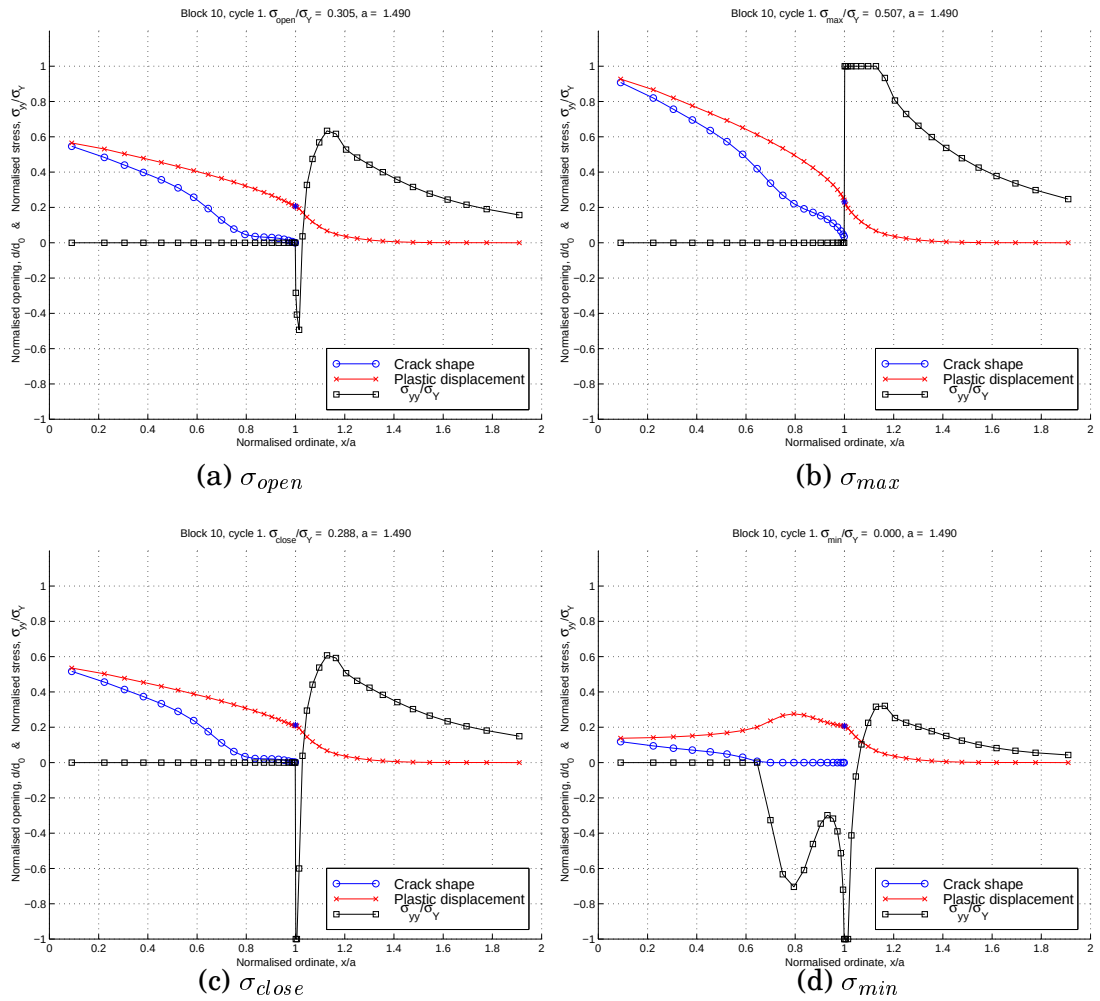


Figure 3.11: Specimen 51 single overload test: predicted crack shapes after the overload at a crack length 1.5 times that at which the overload occurred.

3.3 Comparison of ΔK_{true} and ΔK_{app} in plane stress

The example output shown in sections 3.2.1 and 3.2.2 are for a static and growing crack with $\frac{\sigma_{min}}{\sigma_Y} = 0.00$ and $\frac{\sigma_{max}}{\sigma_Y} = 0.40$. This analysis was repeated for both a static and a growing crack for a wide range of values⁴ with $\frac{\sigma_{min}}{\sigma_Y}$ varying from -0.80 to $+0.80$ and $\frac{\sigma_{max}}{\sigma_Y}$ from 0.00 to $+0.80$. The objective is to plot values of ΔK_{true} and ΔK_{app} enabling a study of those load conditions where closure can be expected to occur (closure regions). The combination of stress values produces 445 cases for the static crack and 445 cases for a growing crack. We shall discuss the results for static and growing cracks separately.

A brief explanation of the layout of figures 3.12 to 3.17 is required. The axes denote the maximum and minimum applied stresses, normalised with respect to the yield stress in tension, σ_Y . Reference is made to ΔK_{app} , where this is the apparent stress intensity as predicted by the elasticity solution $\Delta K_{app} = (K_{max} - K_{min})$ and $K = \sigma\sqrt{\pi a}$ with $K \geq 0$. This results in an artificial discontinuity at $\sigma_{min} = 0$. The true stress intensity range ΔK_{true} incorporates the effects of an active plastic zone ahead of the crack tip $\Delta K_{true} = (\sigma_{max} - \sigma_{open})\sqrt{\pi a}$, where σ_{open} is the stress at which the crack tip starts to open (i.e. starts to suffer a stress intensity). Note that this may occur when the applied remote stress is negative. In summary, ΔK_{app} is based on the elasticity solution, whereas ΔK_{true} incorporates plasticity effects.

The region above the north-east/south-west line is undefined as $\sigma_{min} > \sigma_{max}$. Centred at the origin and moving clockwise from the line $y = x$, the R -ratio⁵ varies from plus one ($y = x, x > 0$) to minus one ($y = -x, x > 0$). The line $x = 0, y < 0$ defines an asymptote where the R -ratio is undefined. Approaching this asymptote clockwise from the line $y = -x$, the R -ratio changes rapidly as it approaches minus infinity, and so it is very difficult

⁴The values used were $\frac{\sigma_{min}}{\sigma_Y} = [-0.80 : 0.10 : -0.10, -0.09 : 0.01 : 0.09, 0.10 : 0.10 : 0.80]$ and $\frac{\sigma_{max}}{\sigma_Y} = [0.00 : 0.01 : 0.09, 0.10 : 0.10 : 0.80]$ with $\sigma_{min} < \sigma_{max}$. Using MATLAB notation—[J:D:K] is the same as [J, J+D, ..., J+m*D] where $m = \text{fix}((K-J)/D)$; and [A,B] is the horizontal concatenation of matrices A and B.

⁵The R -ratio is defined as $R = \sigma_{min}/\sigma_{max}$

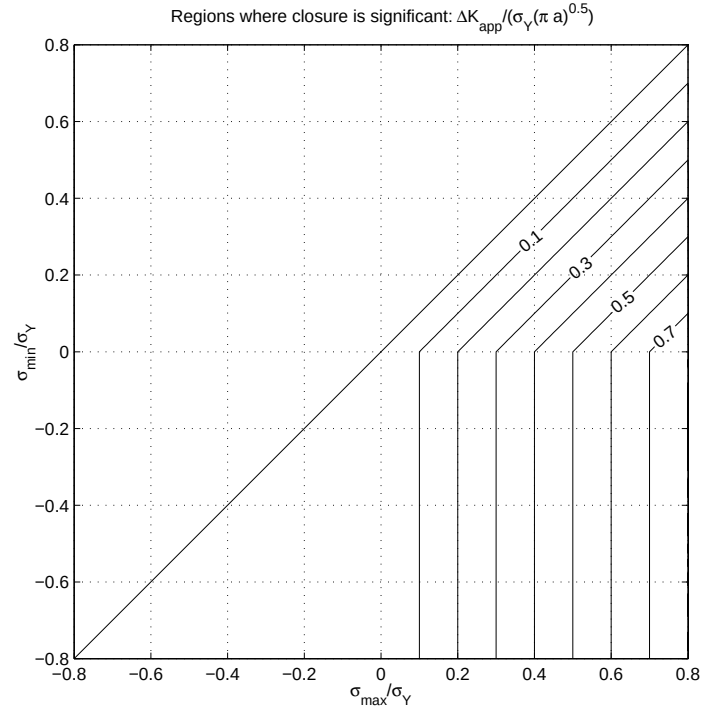
to obtain clean data in this region as a result. In the region from the line $x = 0, y < 0$ to $y = x, x < 0$, the R -ratio is positive, but the crack is always closed since $\sigma_{max} < 0$. Thus the layout of these diagrams enables one to display results for all possible loads or R -ratios.

Static plastic crack

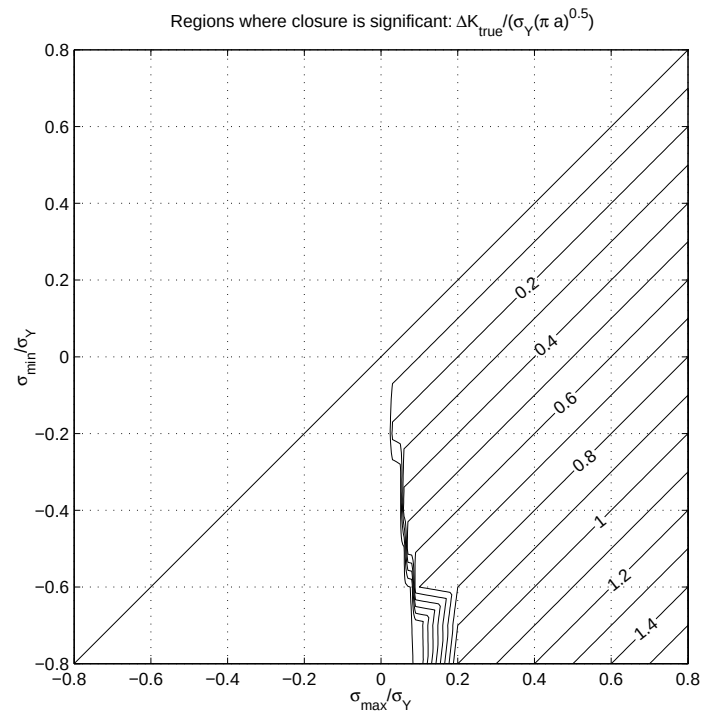
The apparent and true stress intensity ranges for a static plastic crack are plotted in figure 3.12(a) and 3.12(b), respectively. Note the discontinuity in the values of ΔK_{app} along the positive x -axis, due to the definition of ΔK_{app} .

The data presented in figure 3.12 are subjected to simple manipulation, and these results are displayed in figure 3.13. In the first instance the difference between these quantities is presented (figure 3.13(a)). The first point of interest is that, for a static crack, the true stress intensity range can be *greater than* the apparent stress intensity range, which at first sight may be counter-intuitive. We then present the discrepancy between the true and apparent stress intensity ranges (with respect to the apparent stress intensity range) in figure 3.13(b). The second interesting result is that this relative discrepancy is a function of the R -ratio only. For $R = -1$, there is a 100% error, and this gets dramatically worse as the R -ratio continues to decrease.

These results are summarised schematically in figure 3.14. The top-right quadrant is the most well understood. Here, $\Delta K_{true} = \Delta K_{app} \geq 0$ as no closure occurs in either the elastic problem, or that with crack tip plasticity. The bottom-left quadrant is also easy to predict: In this case $\sigma_{max} < 0$ so the crack is always closed and $\Delta K_{true} = \Delta K_{app} = 0$. The bottom-right quadrant merits comment. In the elastic problem, whenever the applied stress is negative the crack tip closes, and hence the crack tip stress intensity vanishes. When plasticity is present the crack is held open by the (compressive) residual stress state ahead of the tip, and hence closure of the crack faces does not occur. The *range* of crack tip stress intensity experienced is greater than that suggested by elastic analysis.



(a)



(b)

Figure 3.12: (a) Apparent and (b) true stress intensity ranges for **static cracks** in an infinite plane.

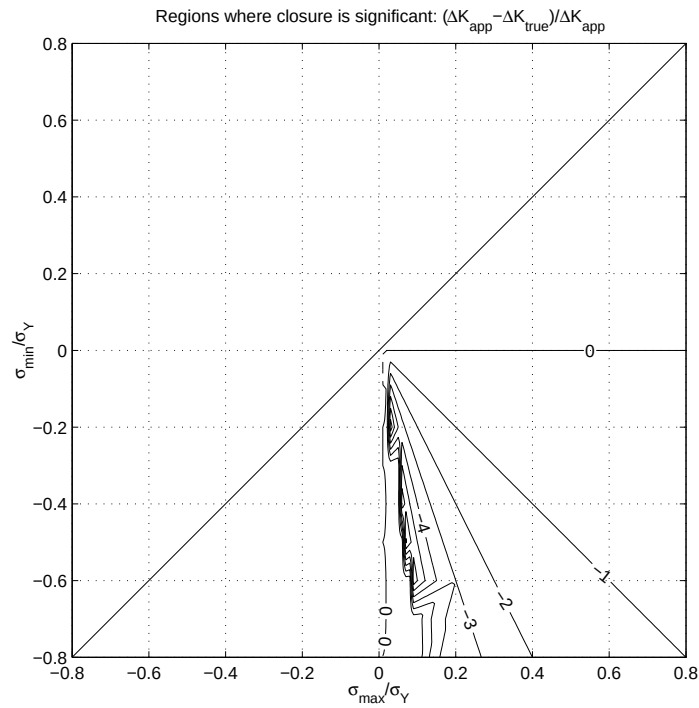
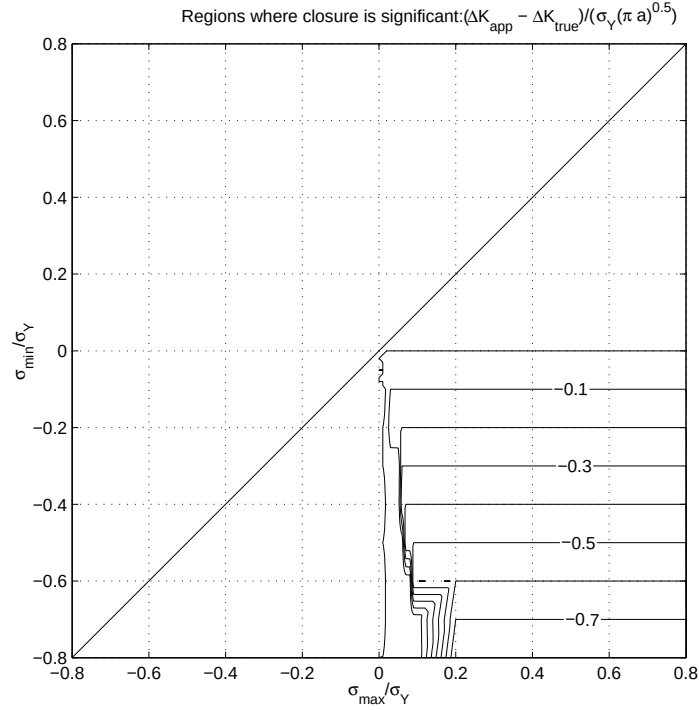


Figure 3.13: Regions where closure is significant for **static cracks** in an infinite plane. (a) Difference between ΔK_{app} and ΔK_{true} . (b) Relative discrepancy between ΔK_{app} and ΔK_{true} with respect to ΔK_{app} .

Closure regions for a static crack

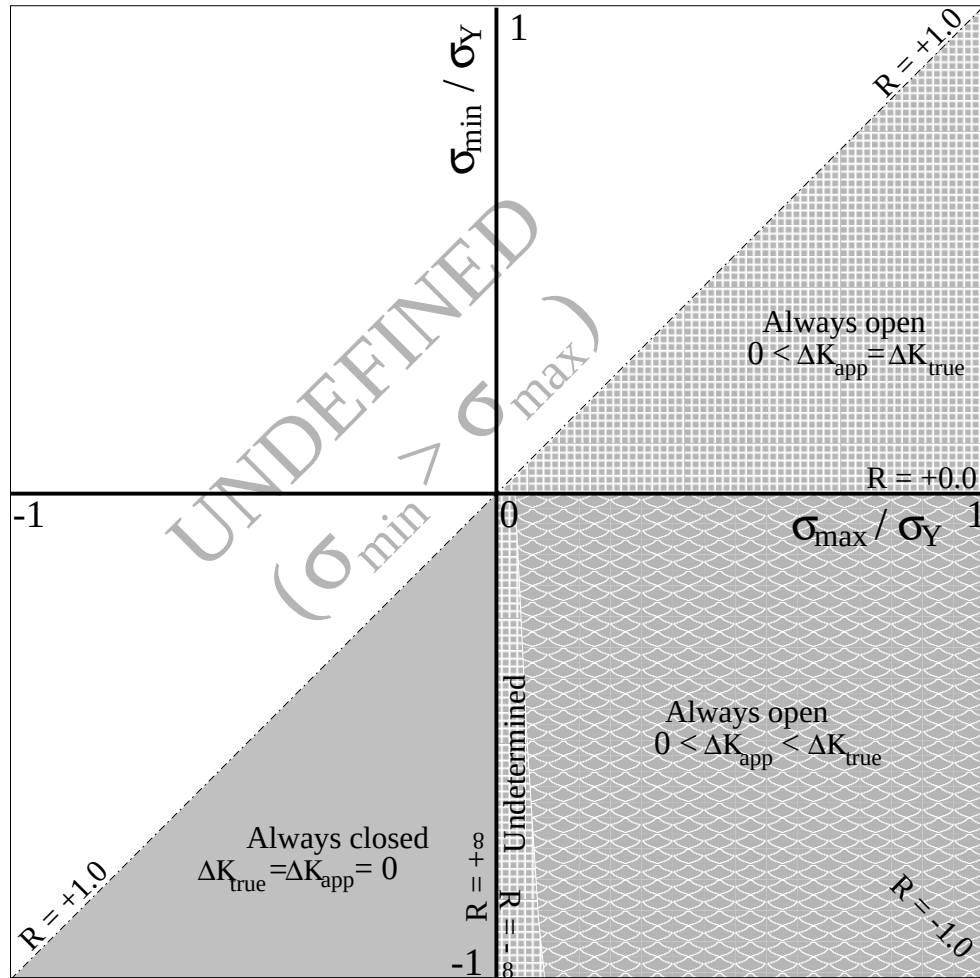


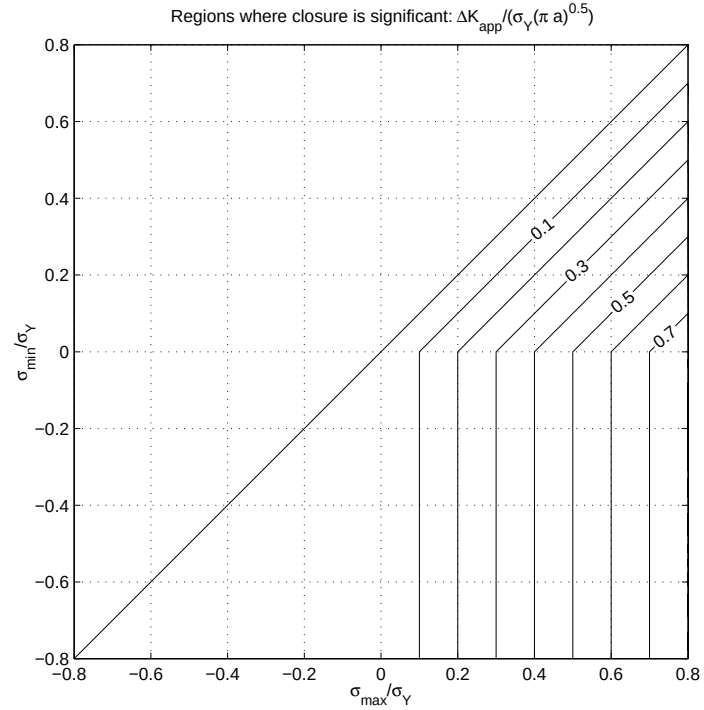
Figure 3.14: Schematic summary of closure regions applicable to **static plastic** cracks. Centred at the origin and moving clockwise from the line $y = x$, the R -ratio varies from $+1$ to -1 ($y = -x, x > 0$). The line $x = 0, y < 0$ defines an asymptote where the R -ratio is undefined. In the bottom-left quadrant, the R -ratio is positive, but the crack is always closed since $\sigma_{max} < 0$.

Growing plastic crack

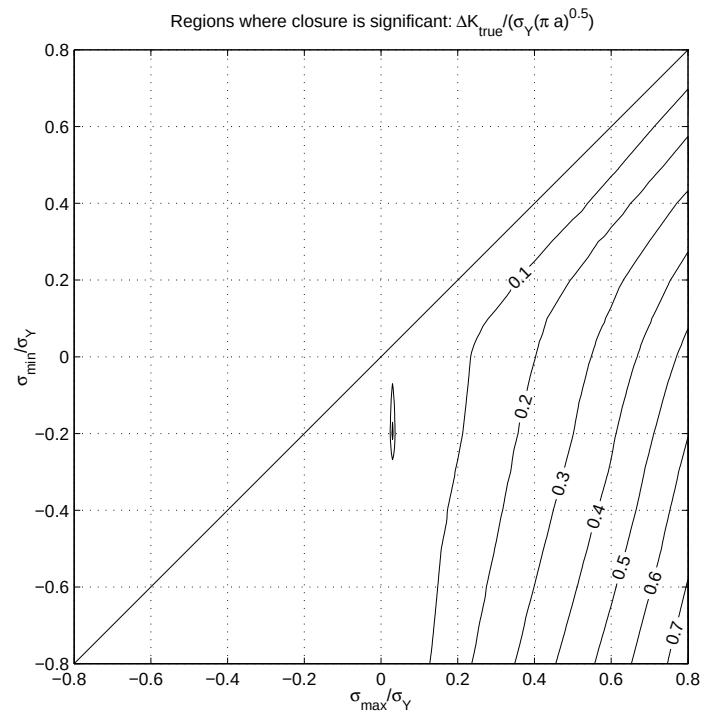
We now turn our attention to the more complex problem involving a crack growing through its own plastic zone. The results are shown in figure 3.15. The values for ΔK_{app} are, of course, identical to those for a static crack, as an elastic analysis cannot differentiate between the two cases. This important point will be exploited in section 3.3.1. However, the plastic wake significantly alters the loads at which the crack becomes fully open, as was demonstrated in section 3.2.2. The plot of ΔK_{true} (figure 3.15(b)) reflects this. Note that the true stress intensity range is continuous across the x -axis.

Looking at the plot showing the difference between ΔK_{app} and ΔK_{true} (figure 3.16(a)), we can see that maximum difference for a growing crack (≈ 0.2) is much less than that for a static crack (up to 0.8). A study of the discrepancy between the true and apparent stress intensity ranges $\frac{\Delta K_{app} - \Delta K_{true}}{\Delta K_{app}}$ (figure 3.16(b)) reveals that the true stress intensity may differ from the apparent stress intensity by as much as 60%, and (more significantly) that this maximum deviance from the apparent stress intensity factor range occurs for mid-range values of σ_{max} where $\sigma_{min} = 0$, which is where the majority of engineering design takes place.

These results are summarised in figure 3.17. This diagram is significantly different from that applicable to a static plastic crack (figure 3.14). Generally, closure occurs for most R -ratios, with the exception of very high R -ratios (i.e. $0.85 < R < 1.0$) and very low R -ratios (i.e. $-\infty < R < -10$). The reason for no closure occurring for very high R -ratios is that the range of applied remote load is not sufficient to induce reversed plasticity. The region corresponding to very negative R -ratios is interesting – the results indicate that the crack remains open. However, it is very difficult to get accurate data in this region. It is thought that the very negative R -ratio introduces numerical discretisation dependencies due to the fact that such a small portion of the load cycle exists in the tensile regime.

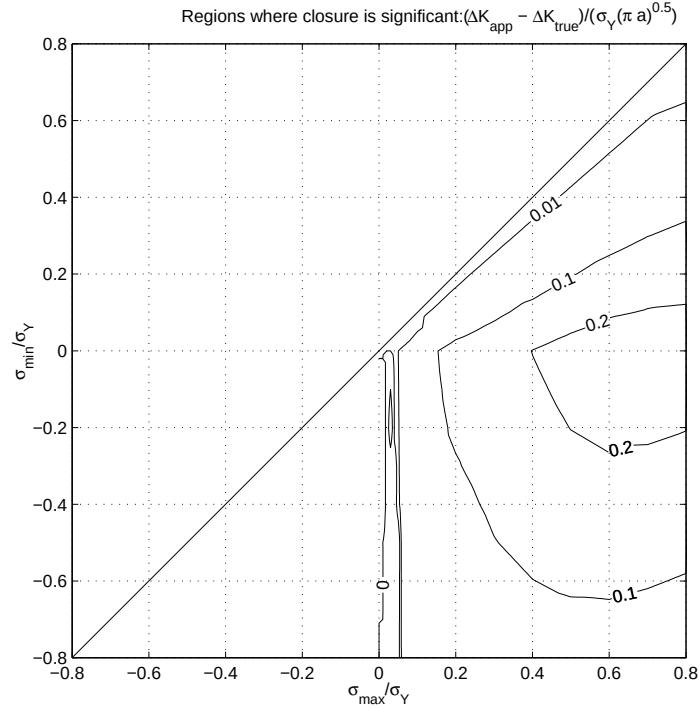


(a)

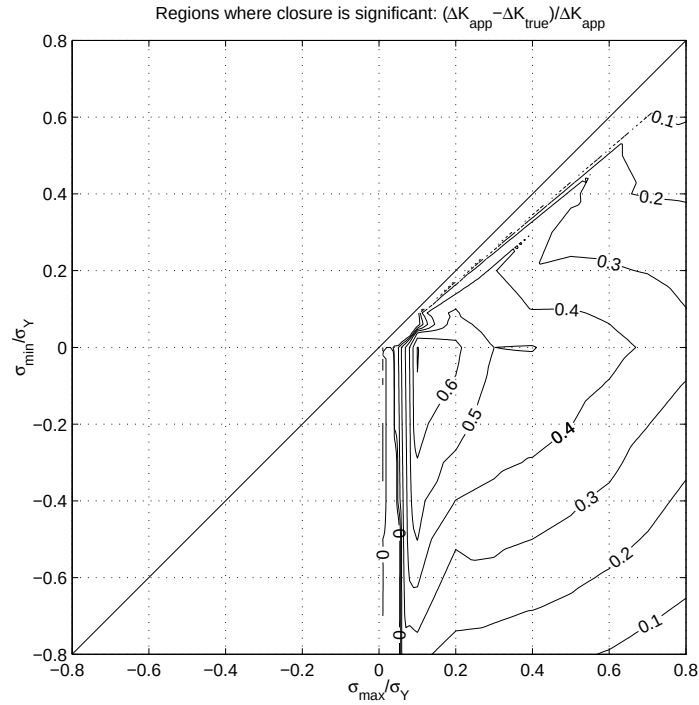


(b)

Figure 3.15: (a) Apparent and (b) true stress intensity ranges for *growing cracks* in an infinite plane.



(a)



(b)

Figure 3.16: Regions where closure is significant for **growing cracks** in an infinite plane. (a) Difference between ΔK_{app} and ΔK_{true} . (b) Relative discrepancy between ΔK_{app} and ΔK_{true} with respect to ΔK_{app} .

Closure regions for a growing crack

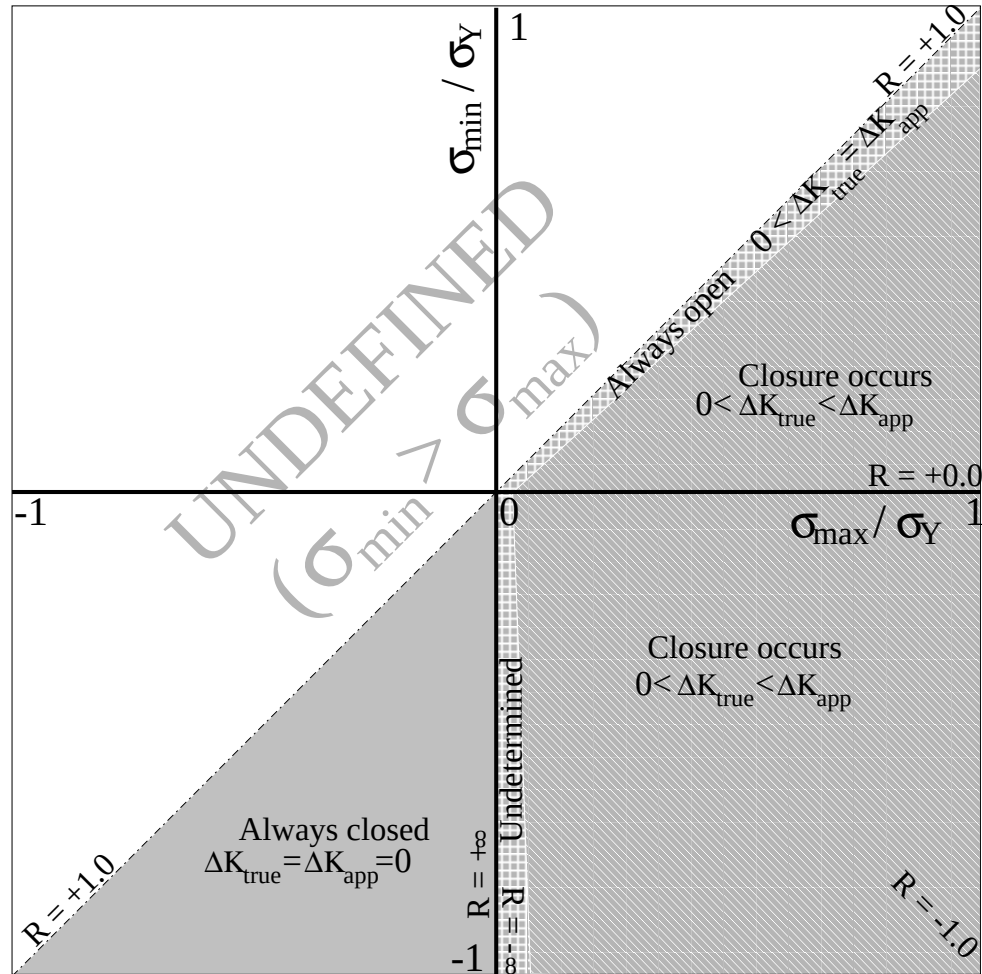


Figure 3.17: Schematic summary of closure regions applicable to **growing plastic** cracks in an infinite plane. Centred at the origin and moving clockwise from the line $y = x$, the R -ratio varies from $+1$ to -1 ($y = -x, x > 0$). The line $x = 0, y < 0$ defines an asymptote where the R -ratio is undefined. In the bottom-left quadrant, the R -ratio is positive, but the crack is always closed since $\sigma_{max} < 0$.

3.3.1 Predicting threshold behaviour

The results presented above have an interesting application to crack initiation. Consider a latent flaw in an otherwise perfect body, suffering cyclic loading. We can now appreciate an effect not widely discussed in the literature, for loading where $R < 0$. This is that, while a crack is in the initiation phase (and therefore essentially static) it may experience a true stress intensity range *larger* than that implied by an elastic analysis, but, as soon as the crack begins to propagate, the plastic wake affects the true stress intensity range which will drop significantly, to well below that implied by an elastic analysis. If the true stress intensity range is now below the apparent threshold value, crack arrest will occur. A comparison of the schematic summary plots for static (figure 3.14) and growing (figure 3.17) cracks will help to highlight these scenarios. A plot of $\Delta K_{true}^{STATIC} - \Delta K_{true}^{GROWING}$ (figure 3.18) demonstrates the drop in true stress intensity suffered by the crack tip as it moves from the static to growing plastic crack regime. This effect shows a small dependence on the nominal maximum stress, but a strong dependence on the nominal minimum stress.

A method for predicting crack growth or arrest can now be outlined. There are two methods described – the second is far more elegant.

Method 1

1. Using LEFM, calculate ΔK_{app}^{STATIC} (or use figure 3.12(a))
2. Use figure 3.13(b) to calculate ΔK_{true}^{STATIC}
3. Use figure 3.18 to calculate $\Delta K_{true}^{GROWING}$
4. Crack grows iff $\Delta K_{true}^{GROWING} > \Delta K_{threshold(app)}$

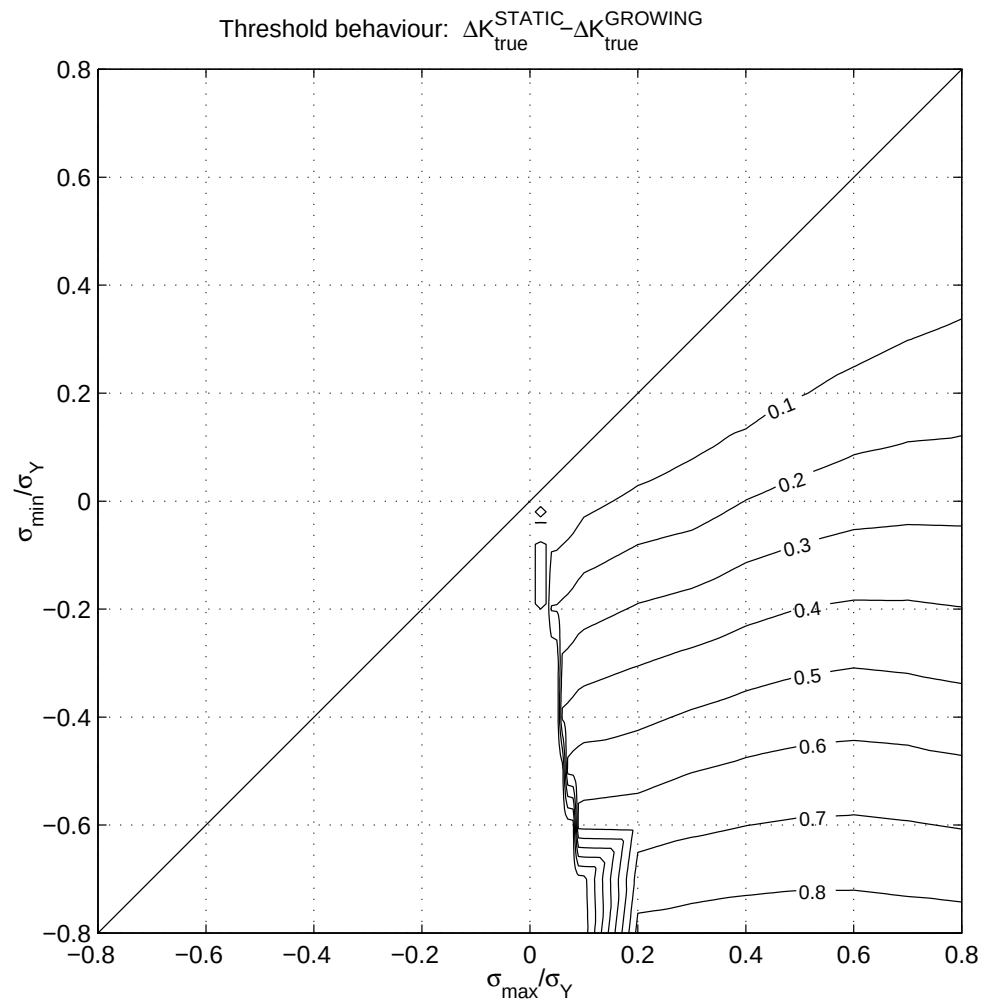


Figure 3.18: Comparison of true stress intensity for a static and growing crack.

The second method exploits the fact that $\Delta K_{app}^{STATIC} = \Delta K_{app}^{GROWING}$ since neither of these considers the effect of a plastic wake:

Method 2

1. Using LEFM, calculate ΔK_{app}^{STATIC} (or use figure 3.12(a))
2. $\Delta K_{app}^{GROWING} = \Delta K_{app}^{STATIC}$
3. Use figure 3.16(b) (or 3.19) to calculate $\Delta K_{true}^{GROWING}$
4. Crack grows iff $\Delta K_{true}^{GROWING} > \Delta K_{threshold(app)}$

This method is more elegant in that only one plot is required to predict the true stress intensity range. For this reason 3.16(b) is repeated in an enlarged format in its own right (figure 3.19).

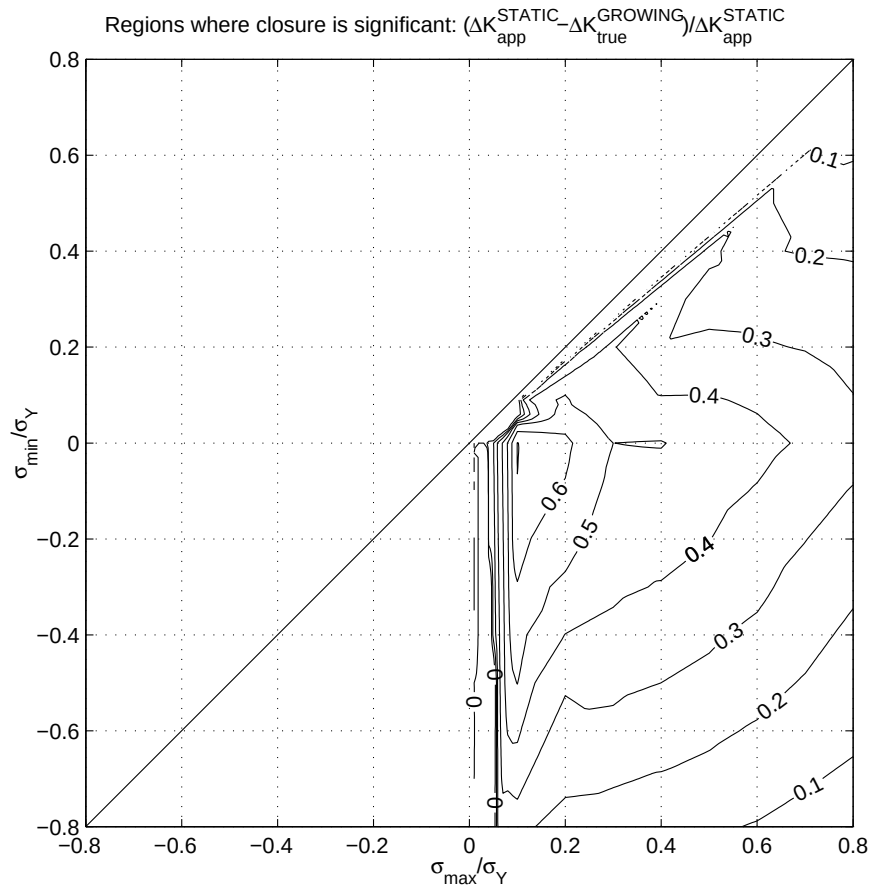


Figure 3.19: Plot used to establish threshold behaviour: since $\Delta K_{app}^{GROWING} = \Delta K_{app}^{STATIC}$, the true stress intensity for a growing crack $\Delta K_{true}^{GROWING}$ can be predicted. The crack then grows iff $\Delta K_{true}^{GROWING} > \Delta K_{threshold(app)}$

3.4 Limitations of 1-D plasticity models

One-dimensional ray-plasticity models are useful to establish crack shape, crack tip opening displacement, and to estimate plastic zone lengths. However, these models are inherently flawed in that the plasticity is collapsed onto a single line, causing strains in the plastic zone to be infinite. The three primary drawbacks associated with ray-plasticity models are: (1) there is a finite region around the plastic zone where the yield criterion is violated; (2) current plane strain models violate the flow rule; (3) calculation of strains within the plastic zone is not possible.

Limitations of plane stress models

Figure 3.20 shows the regions where the BCS model (see section 1.2.3) exceeds the yield criterion for various remote loads $\frac{\sigma_{yy}^{\infty}}{\sigma_Y}$ in the range 0.3 to 0.7. Since the direction of maximum shear stress remains constant for the

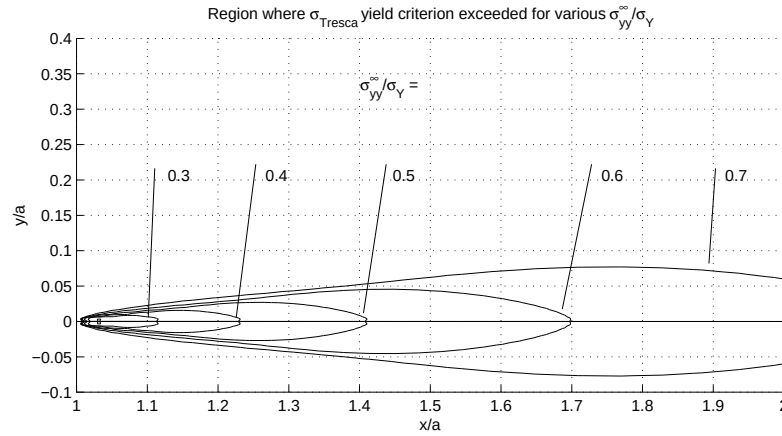


Figure 3.20: Region over which the BCS model violates the Tresca yield criterion for various remote applied loads.

plane stress formulation, and the Burgers vectors are originally deployed coincident with this direction, the flow rule is inadvertently satisfied.

Limitations of plane strain models

Figure 3.21 shows the region where an inclined slip yield model (for example Atkinson & Kanninen – see section 1.2.4) violates the Tresca yield criterion for $\frac{\sigma_\infty}{\sigma_Y} = 0.5$ and the angle of the inclined plasticity ray is 50° to 80° . In this case the directions of the Burgers vectors are fixed coinci-

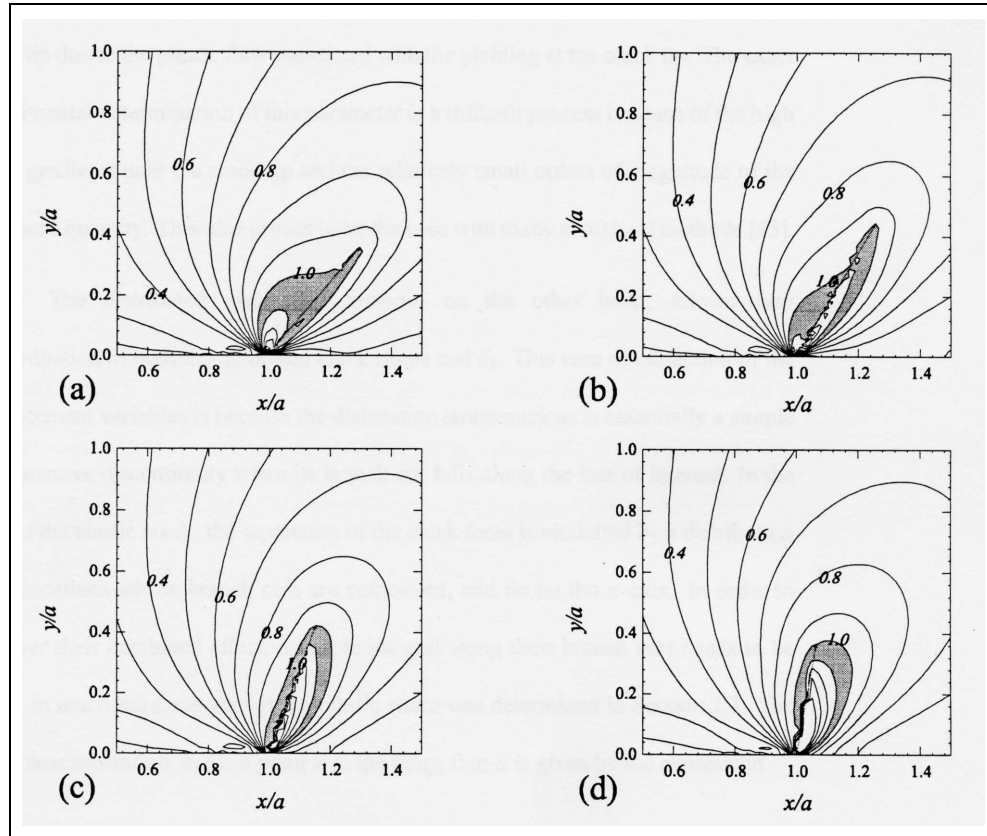


Figure 3.21: Tresca stress contours for the inclined slip yield model with Burgers vectors fixed in the direction of the plasticity ray. $\frac{\sigma_\infty}{\sigma_Y} = 0.50$ and the angle of the ray is (a) 50° , (b) 60° , (c) 70° , (d) 80° . The shaded area shows the region in which the yield criterion is exceeded. Source: Blomerus [1] pg 64.

dent to the direction of the plastic ray. The direction of maximum shear stress does not necessarily lie in this orientation and indeed does change as stress re-distribution occurs. For this reason, inclined slip yield models violate the normality condition. Blomerus [1] improved on these models by allowing the dislocations to rotate so as to satisfy the normality condition. However, since the plasticity is still confined to a line, the reduction

of the region in which the yield criterion is violated is small (see figure 3.22).

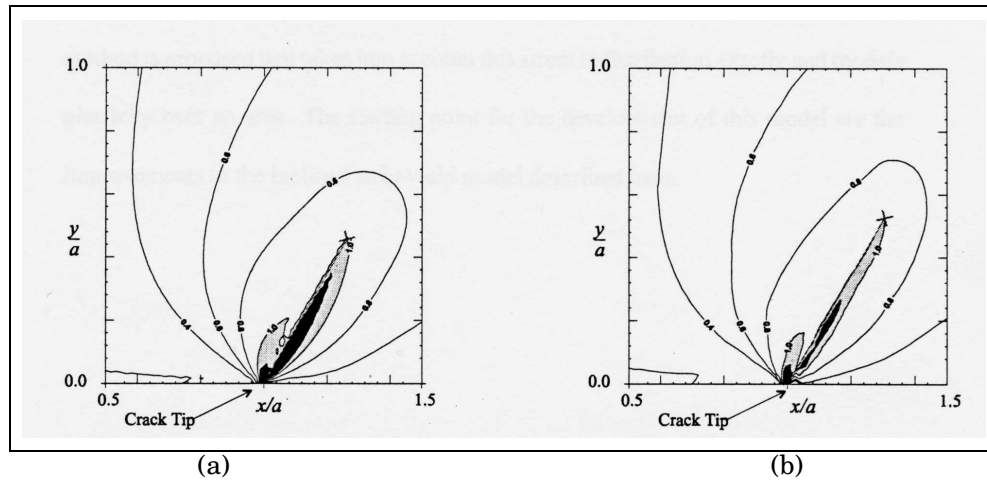


Figure 3.22: Tresca stress contours for the inclined slip yield model showing (a) original results for $\frac{\sigma_\infty}{\sigma_Y} = 0.50$ and ray angle of 60° , and (b) allowing the direction of the Burgers vectors to rotate into the direction of maximum shear stress. The shaded region in which the yield criterion is exceeded is noticeably reduced. Source: Blomerus [1] pg 85.

3.5 Summary

In this chapter we have described the Nowell model, and the refinements that were made to the implementation of the model. Some limitations of the model have been highlighted: all plasticity is collapsed onto a single line ahead of the crack tip creating a region around the plastic zone where the yield condition is violated; and the assumption of a linear plastic wake is only applicable to a crack under remote tension in a full or half plane.

Despite these drawbacks, the model is attractive in its relative simplicity and efficiency. The analysis of 890 load cases has resulted in the compilation of comprehensive ΔK_{true} values, determining regions of closure for both static and growing cracks. These diagrams have been used to illustrate a method by which basic LEFM theory may be used to predict the more sophisticated *true* stress intensity range which more accurately characterises a growing crack.

It should be noted that inaccuracies which apply to the Nowell model due to the strip-yield plastic zone apply equally to all models approximating the plastic zone as a one-dimensional line of infinite strain. Furthermore, the ray-plasticity models for plane strain also violate the Prandtl-Reuss flow rule, since the orientation of the Burgers vector is fixed and thus unable to match the evolving direction of maximum shear stress.

An area plasticity model will be far more computationally expensive than a simple strip-yield model. However, we must concede that in order to quantify the effects of small scale yielding, and to determine the penalty associated with a simple ray plasticity model, we need a model for distributing plasticity over an area. The development of such a model will be discussed in the next chapter.

Chapter 4

Development of a 2-D area-plasticity model

As can be seen from Chapter 3, the line plasticity model has a severe drawback in that the plasticity is confined to a line, preventing calculation of the strains within the plastic zone (which are undefined). Furthermore, plasticity is, in reality, distributed over a finite area. For these reasons, an area plasticity model has been developed, based on a triangular element of piecewise-constant distributed dislocations. This chapter describes the development and verification of this model.

4.1 Distributing dislocations over a triangle

A triangular element is chosen since this can be used as the fundamental unit of any area-type formulation. Here we shall develop the equations relating the stress at any collocation point in a plane to a triangular element of piecewise constant distributed dislocations, and then explore the practical issues concerned with the implementation of this model. Finally, we will verify that the derivation is correct by comparing this new work to the known results for a point dislocation.

Ideally, we would like to model the continuous stress distribution throughout a three dimensional body. However, given the complexities involved in this task, the first logical step is to develop a two dimensional planar analysis. Chapter 3 describes a one dimensional plasticity model which corresponds to the case of plane stress. We look now to the case of plane strain. To model the continuously varying stresses in the plane of analysis, a continuously varying distribution of dislocations would be required. We choose to restrict the shape function of the elements to the simplest form – piecewise constant – and seek the solution for a triangular element of this type in an isotropic infinite plane. For ease of readability, any future reference to an “element”, “triangular element”, or “element of dislocation” should be understood as “a triangular element of piecewise constant distributed dislocation in an isotropic infinite plane”, unless specified otherwise.

4.1.1 Derivation of stress equations

We begin by recalling the equations defining the stress distribution around a single (discrete) dislocation (see section 2.1.2):

$$\begin{Bmatrix} \sigma_{xx}^d \\ \sigma_{yy}^d \\ \sigma_{xy}^d \end{Bmatrix} = \begin{bmatrix} G_{xx} & G_{yx} \\ G_{xy} & G_{yy} \end{bmatrix} \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} \quad (4.1)$$

where the G_{ijk} are the standard full plane kernels for an edge dislocation positioned at the origin (presented in appendix A) of the form

$$G_{xxx} = -\frac{y}{r^4} (3x^2 + y^2) . \quad (4.2)$$

We shall now present the influence functions for a triangle of piecewise constant distributed dislocations [45]. The bulk of the analytical derivation is presented in appendix B, from which the key equations are extracted.

The stresses at any point in an infinite plane due to a piecewise constant distribution of dislocations distributed over a domain S (equation B.3) are:

$$\begin{aligned}
\sigma_{xx} + \sigma_{yy} &= 4\Re [\Phi(z)] \\
\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} &= 2 \left[\Omega(z) - \frac{B_x - iB_y}{B_x + iB_y} \Phi(z) \right]
\end{aligned} \tag{4.3}$$

where $B_i = \frac{\partial b_i}{\partial S}$ is the Burgers vector strength per unit area and

$$\begin{aligned}
\Phi(z) &= -\frac{\mu(B_x + iB_y)}{\pi i(\kappa + 1)} G_1 \\
\Omega(z) &= +\frac{\mu(B_x + iB_y)}{\pi i(\kappa + 1)} G_2
\end{aligned} \tag{4.4}$$

with

$$\begin{aligned}
G_1 &= \iint_S e^{-i\theta} dr d\theta \\
G_2 &= \iint_S e^{-3i\theta} dr d\theta
\end{aligned} \tag{4.5}$$

The stress components can be obtained from equation 4.3:

$$\begin{aligned}
\sigma_{xx} &= \Re \left[2\Phi(z) - \Omega + \frac{B_x - iB_y}{B_x + iB_y} \Phi(z) \right] \\
\sigma_{yy} &= \Re \left[2\Phi(z) + \Omega - \frac{B_x - iB_y}{B_x + iB_y} \Phi(z) \right] \\
\sigma_{xy} &= \Im \left[+\Omega - \frac{B_x - iB_y}{B_x + iB_y} \Phi(z) \right]
\end{aligned} \tag{4.6}$$

Substituting Φ and Ω into equations 4.6, and taking σ_{xx} as an example:

$$\begin{aligned}
\sigma_{xx} &= \frac{\mu}{\pi(\kappa + 1)} \Re \left[\left(\frac{-2(B_x + iB_y) - (B_x - iB_y)}{i} \right) G_1 - \left(\frac{B_x + iB_y}{i} \right) G_2 \right] \\
&= \frac{\mu}{\pi(\kappa + 1)} \Re \left[\left(\frac{-3B_x - iB_y}{i} \right) G_1 - \left(\frac{B_x + iB_y}{i} \right) G_2 \right] \\
&= \frac{\mu}{\pi(\kappa + 1)} \left[\Re [+3iG_1 + iG_2], \Re [-G_1 - G_2] \right] \left\{ \begin{matrix} B_x \\ B_y \end{matrix} \right\}
\end{aligned} \tag{4.7}$$

This procedure is repeated to find the remaining four influence functions, all six of which are now listed:

$$\begin{aligned}
G_{xxx} &= \Re [3iG_1 + iG_2] \\
G_{xyy} &= \Re [iG_1 - iG_2] \\
G_{xxy} &= \Im [-iG_1 - iG_2] \\
G_{yxx} &= \Re [-G_1 - G_2] \\
G_{yyy} &= \Re [-3iG_1 + G_2] \\
G_{xyx} &= \Im [-G_1 + G_2]
\end{aligned} \tag{4.8}$$

where G_1 and G_2 are determined from equation 4.5 and the stresses are then

$$\begin{Bmatrix} \sigma_{xx}^d \\ \sigma_{yy}^d \\ \sigma_{xy}^d \end{Bmatrix} = \begin{bmatrix} G_{xx} & G_{yx} \\ G_{xy} & G_{yy} \\ G_{xy} & G_{yx} \end{bmatrix} \begin{Bmatrix} B_x \\ B_y \end{Bmatrix} \quad (4.9)$$

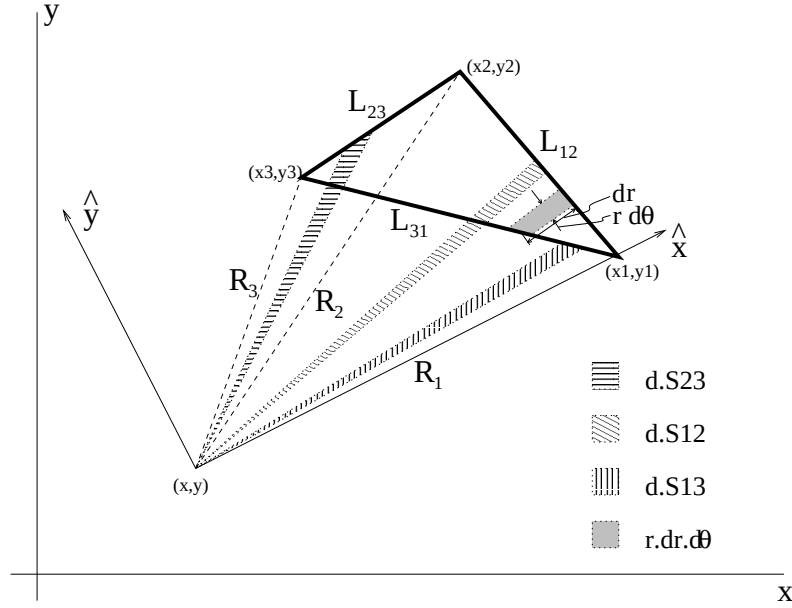


Figure 4.1: Local coordinate set to evaluate the influence functions for a triangular element of piecewise constant dislocation.

We turn our attention, now, to the evaluation of these influence functions for the case when S is a general triangle such as that shown in figure 4.1. We locate the origin of the local coordinate set at the desired evaluation point, and ensure that the x -axis of the local coordinate set passes through the vertex labelled (x_1, y_1) . Recalling equations 4.5, we see that the small element $r dr d\theta$ must sweep through a discontinuity in the triangle as the limit of the integration changes, in this case from L_{12} to L_{23} . Note that the convention is to sweep counter-clockwise. In order to evaluate the nett effect of the triangular dislocation, we need to evaluate three integrals corresponding to the three sides of the triangle L_{12} , L_{23} , and L_{31} . For example, in figure 4.1, to evaluate the influence function at the evaluation point (x, y) , the following method is required: (1) calculate the contribution of $d.S23$ which is effectively triangle R_2, L_{23}, R_3 ; (2) add to this the contribution of $d.S12$ (R_1, L_{12}, R_2); and then (3) subtract

the contribution of $d.S13 (R_1, L_{31}, R_3)$. This will give the influence of the intended triangle (L_{12}, L_{23}, L_{31}).

It will be seen that the selection of these regions is a function of the position of the collocation point relative to the triangular element. It is clear, then, that some method of determining the boundaries of the regions around the triangular element is required.

4.1.2 Identifying regions around an element

Each of the two integrals G_1, G_2 (equations 4.5) needs to be evaluated for each region around the triangle (see figure 4.2). Each section is labelled based on the triangle's vertices: for example region 3 is always outside the triangle adjacent to the side from vertex 1 to vertex 2. The regions were

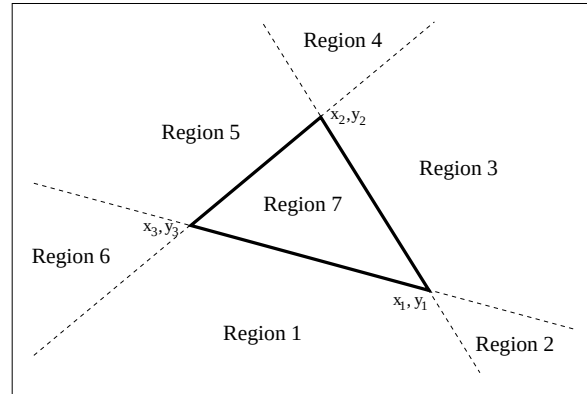


Figure 4.2: Regions associated with each triangle of dislocation.

identified by considering the cross products of the vectors describing the three sides of each triangle, and then taking logical combinations in order to ascertain the boundaries of each region.

The results are shown in figure 4.3. The code performing this function (`Triangle.m`) can be found in appendix C.

In the preceding section the method of determining the influence function at the evaluation point by considering the constituent regions was discussed. Figure 4.4 shows the three possible combinations which en-

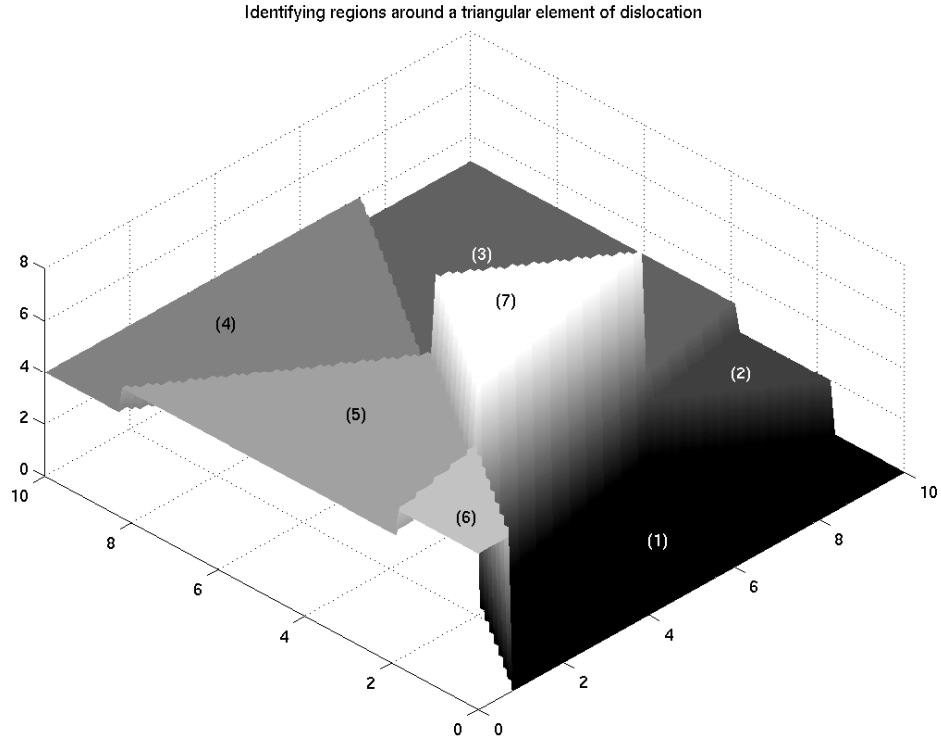


Figure 4.3: Example showing regions identified for a triangle of dislocations.

able the stress state to be evaluated at any point in the plane.

Since we are now able to identify precisely in which region the evaluation point lies, we can examine the full expression defining the integrals G_1, G_2 (equations 4.5). The analytical derivation is detailed in appendix B, and only equations B.6 and B.7 are repeated here:

$$\begin{aligned}
 G_1 &= c \cos \gamma e^{-i\gamma} \left[\ln(\sin \alpha) - i\alpha \right]_{\theta_1}^{\theta_2} \\
 G_2 &= c \cos \gamma e^{-3i\gamma} \left[\ln(\sin \alpha) + \cos(2\alpha) - i(\sin(2\alpha) + \alpha) \right]_{\theta_1}^{\theta_2} \quad (4.10)
 \end{aligned}$$

where

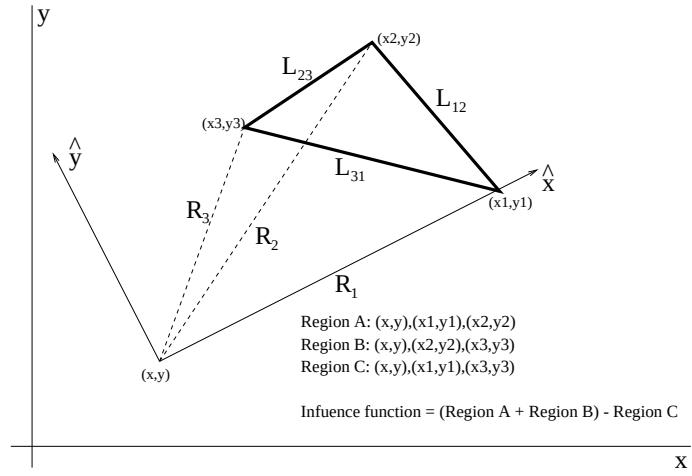
c = intersection of line of triangle with local y -axis

γ = $\arctan m$

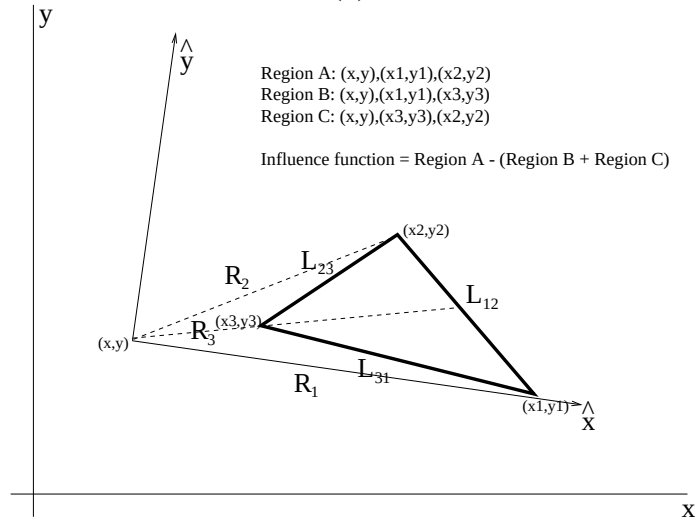
m = gradient of line of triangle

α = $\theta - \gamma$

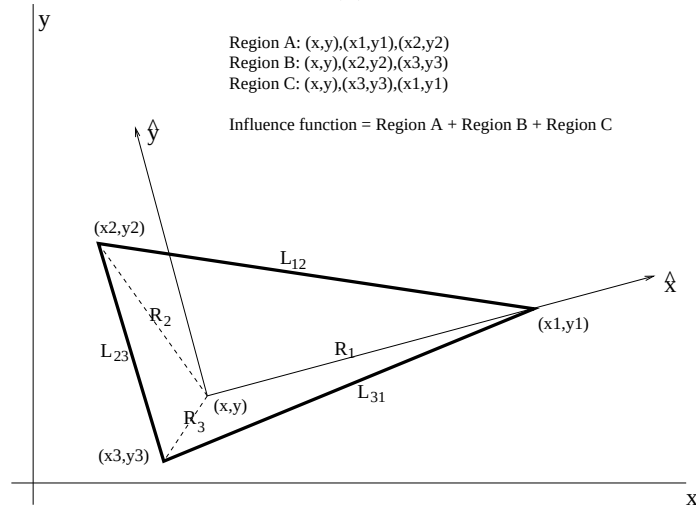
θ = variable of integration



(a)



(b)



(c)

Figure 4.4: Determining the correct combinations of influence functions: (a) Region 1. (b) Region 6. (c) Region 7.

The importance of establishing the region in which the evaluation point lies can now be appreciated in that the limits to be used for the evaluation of these integrals change for each region.

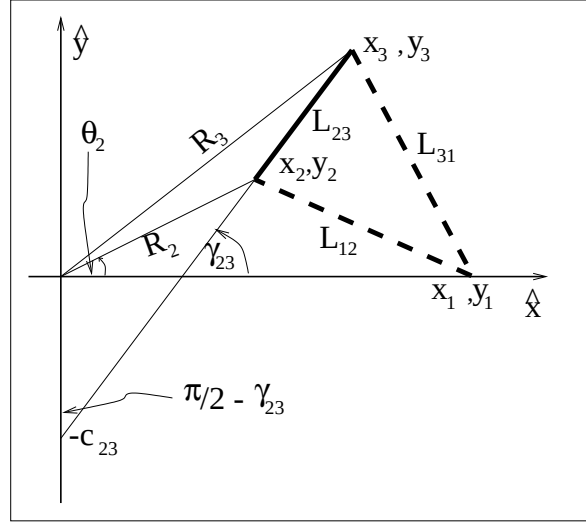


Figure 4.5: Schematic showing nomenclature used in equation 4.10

Two problems were encountered in the implementation of equation 4.10:

- The value of the constant c was numerically unstable (approaches $\pm\infty$) as the collocation point approached any position such that the side of the triangle under consideration became parallel to the y -axis.
- The value of α has range $[0, 2\pi]$, but $\ln(\sin \alpha)$ is defined only for $\alpha = (0, \pi)$.

Considering figure 4.5, the simple manipulation

$$\begin{aligned} \frac{R_2}{\sin(\frac{\pi}{2} - \gamma_{23})} &= \frac{c_{23}}{\sin(\gamma_{23} - \theta_2)} \\ \therefore \frac{R_2}{\cos \gamma_{23}} &= \frac{c_{23}}{\sin(-(\theta_2 - \gamma_{23}))} \\ \therefore c_{23} \cos \gamma_{23} &= -R_2 \sin \alpha_{23} \end{aligned} \quad (4.11)$$

(where $\alpha_{23} = \theta_2 - \gamma_{23}$) allows us to overcome the first of these two problems. The second point requires us to place the restriction $0 < \alpha < \pi$. The latter point is only significant when we consider Region 7, that is the internal

region of the triangle. We restrict α by calculating the contribution from three sub-triangles, each time rotating the local axes so that the local x -axis goes through the appropriate vertex in order to ensure that $0 < \alpha < \pi$.

The expressions to evaluate the integrals for each region are then:

$$\begin{aligned}
G_1.R_1 &= (-s_{21} + s_{23} + s_{31}) \\
G_1.R_2 &= (-s_{21} - s_{32} + s_{31}) \\
G_1.R_3 &= (+s_{12} - s_{32} + s_{31}) \\
G_1.R_4 &= (+s_{12} - s_{32} - s_{13}) \\
G_1.R_5 &= (+s_{12} + s_{23} - s_{13}) \\
G_1.R_6 &= (-s_{21} + s_{23} - s_{13}) \\
G_1.R_7 &= (+s_{12} + s_{23} + s_{13})
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
G_2.R_1 &= (-q_{21} + q_{23} + q_{31}) \\
G_2.R_2 &= (-q_{21} - q_{32} + q_{31}) \\
G_2.R_3 &= (+q_{12} - q_{32} + q_{31}) \\
G_2.R_4 &= (+q_{12} - q_{32} - q_{13}) \\
G_2.R_5 &= (+q_{12} + q_{23} - q_{13}) \\
G_2.R_6 &= (-q_{21} + q_{23} - q_{13}) \\
G_2.R_7 &= (+q_{12} + q_{23} + q_{13})
\end{aligned} \tag{4.13}$$

where, for example,

$$s_{12} = -R_1 e^{-i\gamma_{12}} \sin(\alpha_{12}) \left[\ln |\sin(\alpha_{12})| - i(\alpha_{12}) \right]_{\theta_1}^{\theta_2} \tag{4.14}$$

$$\begin{aligned}
q_{12} &= -R_1 e^{-3i\gamma_{12}} \sin(\alpha_{12}) \times \\
&\quad \left[\ln |\sin(\alpha_{12})| + \cos(2\alpha_{12}) - i(\sin(2\alpha_{12}) + \alpha_{12}) \right]_{\theta_1}^{\theta_2}
\end{aligned} \tag{4.15}$$

Note that $s_{12} \neq s_{21}$, since the limits will be different. The implementation of this method can be found in `Tri_IFn.m` in appendix C.

Of course, a viable alternative would be to evaluate these functions numerically, and it may well be that such an implementation is computationally more efficient than this analytical method. The motivation behind using an analytical method was to provide a reliable benchmark against which future models could be compared with confidence.

4.1.3 Verification of the distributed dislocation formulation

Before proceeding with the solution of problems involving more than one element, it is important that we verify that the formulation derived in sections 4.1.1 and 4.1.2 is correct.

We have derived the expression for the distribution of piecewise constant dislocations over a triangle in an isotropic infinite plane. Therefore, by the argument of self-similarity, reducing the element to an arbitrarily small size should produce an identical stress distribution to that of a discrete edge dislocation in an infinite plane for regions external to the triangle (given that $b_i = B_i \Delta S$). Figures 4.6 and 4.7 show the results of this comparison for all six influence functions. In each figure, the top row recalls the influence function for a discrete dislocation, which is singular at the integration point. The second row presents the influence function for a triangular element, and we note that this is finite and continuous. The area over which the dislocations have been distributed is marked by the bold triangle. The bottom row displays the difference between these two quantities¹. This difference is negligible when the evaluation point is remote from the triangle, and increases as the triangle is approached. Beyond a distance of approximately twice the characteristic element length², the difference is negligible. This will be discussed again at a later stage. For now, it suffices to say that the bottom row of these two figures prove that the external regions of the triangle have been correctly derived and

¹Plotting the normalised error is not feasible, as both the discrete and distributed dislocation functions have contours which are zero, which would produce “infinite” normalised errors.

²We use the longest side of the triangle as the characteristic length in order to avoid difficulties with triangles that are, for example, very long and thin.

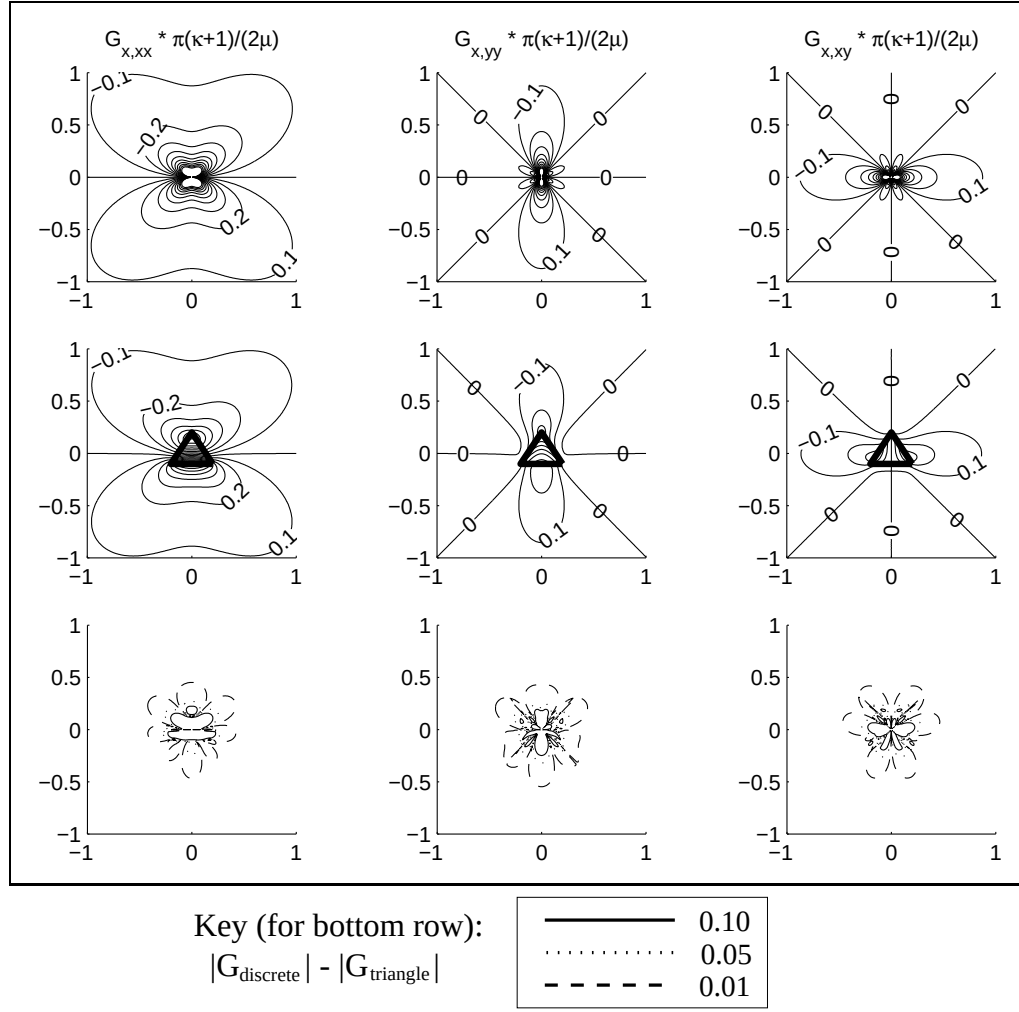


Figure 4.6: Verification of the G_{xij} external influence distributions. The top row plots the distributions for a discrete dislocation; the second row for a triangle of piecewise constant distributed dislocation; the bottom row shows the difference between these distributions (to which the key applies).

implemented.

Verification of the formulation for the interior of the triangle is more difficult, since there is no source in the literature available for comparison. Thus the only means of testing the derivation is to take advantage of our knowledge that the external formulation is correct. To this end, a novel “puzzle problem” was used: we take a triangle of dislocations of arbitrary finite size (call it ‘A’) and divide it into four sub-triangles (call them ‘B’ to ‘E’). We know that the distributions of triangles ‘B’ to ‘E’ over the domain

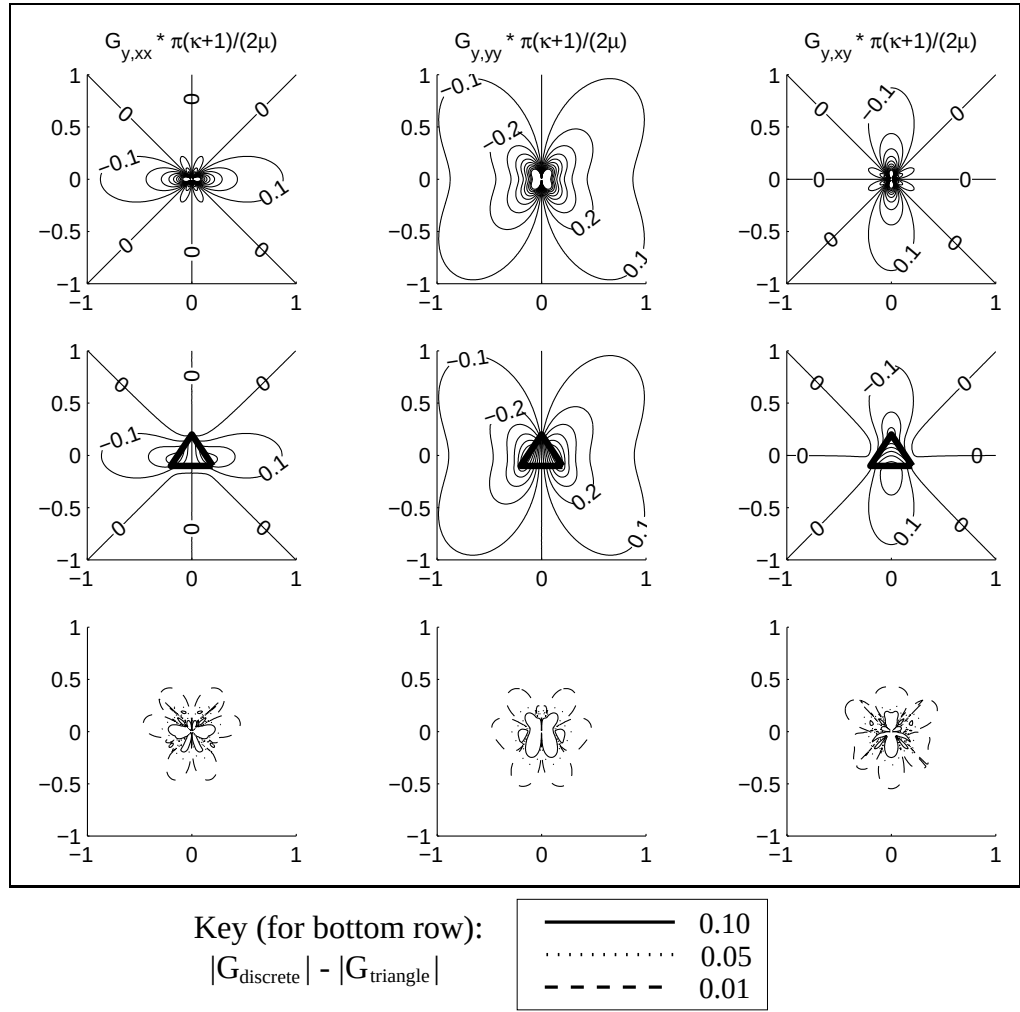


Figure 4.7: Verification of the $G_{y\ ij}$ external influence distributions. The top row plots the distributions for a discrete dislocation; the second row for a triangle of piecewise constant distributed dislocation; the bottom row shows the difference between these distributions (to which the key applies).

of 'A' include contributions from the *external* regions of 'B' to 'E' which we have shown to be correct by comparison to well-known discrete dislocation distributions. Thus we conclude that if the internal distribution due to 'A' matches the distribution over 'A' from 'B' to 'E', the internal distribution of 'A' to 'E' must be correct.

Figure 4.8 shows that the output for triangle 'A' is identical to that of the sum of the outputs of triangles 'B' to 'E' for $G_{y\ yy}$. Similar tests were executed for the other five influence functions with the same positive result.

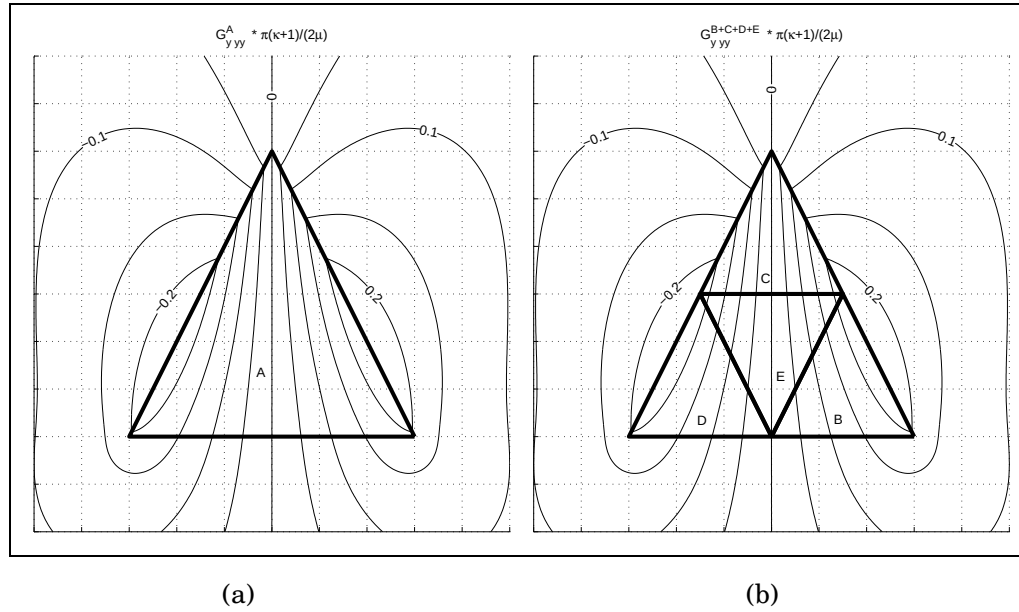


Figure 4.8: Verification of internal influence function by means of the “puzzle problem”: (a) the influence distribution within the internal region of the large triangle ‘A’ corresponds (b) to that of the sum of four smaller triangles ‘B’ to ‘E’ which incorporate regions external to themselves.

We have therefore proved that the internal influence function distribution of the triangle of dislocation is correct both in derivation and implementation.

The task now is to develop a model which incorporates plasticity distributed over an area.

4.2 Area plasticity model algorithm

With the knowledge that the derivation of the triangular element is correct, we now proceed by activating a single element in an infinite plane under uniform remote tension.

4.2.1 Algorithm structure – single element

The incremental plasticity algorithm is complex. It is important that the method allows elements exceeding the yield parameter to (a) become active, (b) return the stress state to the yield surface, and (c) to do this in such a way as to obey the normality condition (flow rule).

It may have been noted that the last of the above three conditions is expressed as a verification procedure performed *after* a solution has been obtained, as opposed to a *requirement* which is enforced as part of the solution algorithm. This subtle point will be discussed in detail in section 4.2.3. In either case, it is imperative that the flow rule is utilised in some form in order to maintain the integrity of the solution.

Some comments on proportional and non-proportional loading are now required. Proportional loading refers to the case when all stress components scale in proportion to one another as the load is increased. Most elastic problems in engineering are proportional, whereas most plasticity problems are non-proportional. Table 4.1 highlights a few exceptional examples to this general statement. The most important feature of proportional loading is that the direction of maximum shear stress remains constant for any chosen load increment.

	Elastic	Plastic
Proportional	(Common)	Spherical cavity subject to internal pressure. 1-D problems such as isotropic homogeneous uniaxial tension test, and beam in bending
Non-proportional	Round punch in isotropic homogeneous half plane.	(Common)

Table 4.1: Examples of unusual load cases

For the purposes of this discussion, we now introduce the nomenclature used in the program `APM.m` (see appendix C). We shall be using an incremental plasticity model (see section 2.3.1) and so require a notation which distinguishes total from incremental stress contributions. Table 4.2 reviews the notation.

Symbol	Description
σ^e	Elastic stress for current increment.
σ^d	Dislocation stress for current increment.
$\sigma^h = \Sigma(\sigma^e + \sigma^d)$	History stress for all previous increments.
$\sigma^t = \sigma^h + \sigma^e + \sigma^d$	Total stress for current increment.
$\sigma^i = \sigma^t - \sigma^d = \sigma^h + \sigma^e$	Stress for this increment <i>excluding</i> the dislocation contribution.

Table 4.2: Incremental plasticity nomenclature

An infinite plane containing one potentially active triangular element is established by assigning all surrounding elements a purely elastic response (infinite yield stress). The plane is subjected to remote uniform tension in the y -direction. For as long as $\sigma_{yy}^\infty < \sigma_Y$, the plane behaves in a linear-elastic fashion. The remote tension is then increased by a fixed amount above the yield stress, say 5%. Although it would appear that all of the elements should now yield, only one is allowed to do so. Thus we expect all points remote from the element under consideration to continue to carry a stress of $1.05 \sigma_Y$ elastically. The stress decreases as we approach the element, and the collocation point within the element lies precisely on the yield surface.

For the Tresca yield criterion:

$$\sigma_{\hat{x}\hat{y}}^t = k \quad (4.16)$$

where the superscript t denotes the total stress, the local $\hat{x}\hat{y}$ coordinate set has been rotated to coincide with the direction of maximum shear stress, and $k = \frac{\sigma_Y}{2}$ is the yield stress in shear. We assume that the direction of the Burgers vector coincides with the direction of maximum shear stress (i.e. that all slip occurs on this $\hat{x}\hat{y}$ slip plane). We isolate the dislocation stress by writing

$$\sigma_{\hat{x}\hat{y}}^d = k - \sigma_{\hat{x}\hat{y}}^i. \quad (4.17)$$

It is very important to note at this point that the above step can *only* be performed if the Tresca yield criterion is used, since the yield criterion is a linear combination of the principal stresses. Conversely, the von Mises yield criterion does not permit the above decomposition. The dislocation stress is now decomposed into the product of influence function and Burgers vector

$$\begin{bmatrix} G \end{bmatrix} \begin{Bmatrix} b \end{Bmatrix} = k - \sigma_{\hat{x}\hat{y}}^i \quad (4.18)$$

which allows us to solve for the magnitude of the Burgers vectors which will return the stress to the yield surface. We have not explicitly specified that this solution satisfies the normality condition. To verify that this is the case, the dislocation stress due to the active element is computed, and the new total stress (which now includes the effect of stress redistribution due to the active element) is used to calculate a new direction of maximum shear stress. If the direction of the Burgers vector is within a given tolerance of the new direction of maximum shear stress, then both the yield criterion and flow rule have been satisfied. If not, the solution process is repeated until the correct Burgers vector is obtained.

As we are activating only one element in a case of proportional loading, we expect to achieve a solution in one step, irrespective of the size of the load increment. This matter of key importance will be discussed further in section 4.2.2 when we consider the nature of slip systems. The question of iterations within each load increment will also be discussed in section

4.2.3 where we comment on the implementation of the flow rule. The algorithm for the area plasticity model (APM), where multiple elements become active, will be discussed in section 4.2.4.

Algorithm efficiency – discrete *vs* element influence functions

Reconsider figures 4.6 and 4.7. These figures display the influence function due to a discrete dislocation in the top row; the influence of a triangular element in the middle row; and the difference in the bottom row. We now concern ourselves with the bottom row. It is observed that should the distance from the collocation point to the nearest vertex be greater than approximately twice the length of the longest side of the triangle, the difference between the discrete and element influence functions becomes negligible. We can then substitute the computationally expensive triangular influence function with that for the discrete dislocation.

Algorithm stability

Having covered the method employed for a single element subjected to remote uniform tension, we now focus on an issue raised indirectly in equation 4.16, namely that at every collocation point, the local coordinate set is rotated so as to coincide with the direction of maximum shear stress at that point. This is tantamount to saying that the local coordinate set is rotated so as to orientate the $\hat{x}$ -axis in the direction of the anticipated slip plane. We now wish to examine this issue in further detail.

A review of the literature reveals only one paper on the topic of modelling plasticity using distributed dislocations [46]. In this pioneering work, Blomerus *et al* make no explicit reference to slip planes, but do discuss slip and “flow field” directions. No further explanation is made concerning the nature of the relationship between the slip and flow field directions, as it was assumed that these were one and the same. Furthermore, it was thought that due to the complementarity of shears, the flow

field corresponded to the orientation of either (but only one) slip plane. For the purposes of this discussion, this is now termed the “single slip plane” model.

Blomerus [1] used distributed dislocations over what he called “pseudo-squares” consisting of radial lines and circular arcs. The problem considered was that of a hole in an infinite plane under plane strain remote uniform tension. Symmetry conditions were exploited, modelling only one quarter of the problem and activating symmetric pseudo-square elements of piecewise constant dislocations as appropriate. Comparison of the size and shape of the plastic zone *only* were made to a similar analysis executed using the finite element (FEM) technique (Abaqus v5.5), and it was reported that good agreement was obtained for a fairly coarse mesh³. It was subsequently found that refinement of the mesh resulted in an unstable divergent algorithm, with the sign of B_i oscillating. This problem was attributed to the “... ill-posed nature of the simply discretised numerical solution of the integral equation...”⁴, and was overcome by employing a singular value decomposition (SVD) technique in order to obtain convergence⁵.

Some comments are now made concerning this work by Blomerus. First, comparison of the size and shape of the plastic zone (i.e. the elastic-plastic boundary) is a necessary but insufficient proof of agreement with FEM: for example, the addition or subtraction of a hydrostatic stress would alter the stress distribution but have no effect on the size or shape of the plastic zone. The agreement of the entire stress distribution would constitute a unique solution. Secondly, exploiting symmetry (without first proving the validity of the solution ignoring symmetry) has potential for errors in the method by which symmetry is implemented, since no datum is obtained to facilitate comparison. Lastly, the numerical instability was attributed to the discretisation of the numerical solution. It is not clear that this is

³Ref: [1], pg 132–134

⁴Ref: [1], pg 133/4

⁵Ref: [1], pg 134–135

necessarily the case. This will be discussed now in further detail.

During the course of the author's work, similar problems were experienced with convergence, and these were always attributed to the nature of the numerical solution. Convergence could only be achieved if SVD was employed, and it was noted that at least one of the singular values returned in the SVD analysis was always zero. This corresponds to a rank-deficient matrix, and prompted the examination of the exact assumptions associated with the slip-plane or "flow direction" concept. We now begin this analysis by elaborating on the underlying assumptions.

4.2.2 Slip planes and flow direction

Consider figure 4.9. In any planar analysis, there are two directions of maximum shear stress, due to the complementarity of shear stresses, and these are orientated at directions of $\pm \frac{\pi}{4}$ to the direction of maximum principal stress. These slip planes are termed α_{12} and $\alpha_{21} = \alpha_{12} - \frac{\pi}{2}$ respectively. We have a Burgers vector $|\mathbf{b}|$ which can be decomposed into $\mathbf{b}_{12}$ and $\mathbf{b}_{21}$ on the corresponding planes, representing the magnitude of slip on that plane.

Single slip plane

As has been discussed previously, it has always been assumed that due to the complementarity of shear stresses, choosing either (but only one) of these slip planes would suffice in the algorithm to return the stress state to the yield surface. Once a solution had been obtained, the normality condition was checked to see that it coincided with the direction of maximum shear stress as described earlier in this section. This method was employed for a single element in an infinite plane under remote uniform tension in the y -direction, and the results are shown in figure 4.10. It can be seen that within the assumptions presented, both solutions are correct. The stress at the evaluation point within each triangle has been

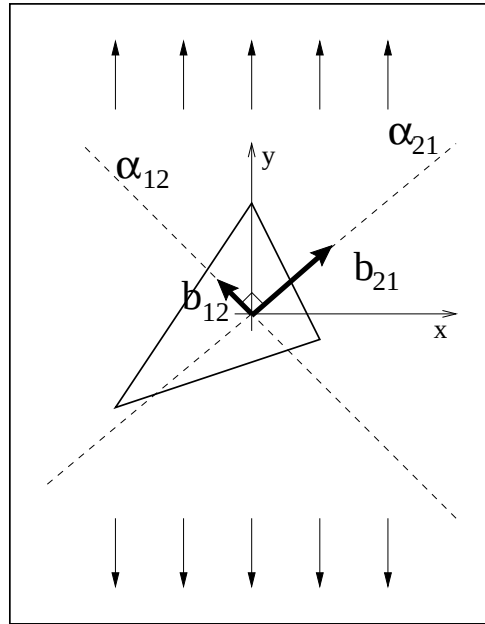


Figure 4.9: Schematic diagram illustrating slip planes.

returned to the yield surface, and the directions of the Burgers vectors (arrowed) do coincide with the expected directions of maximum shear stress. However,

- the stress distributions themselves are not symmetric about the y -axis as one would expect them to be⁶;
- several iterations are required in order to converge on a solution;
- the direction of maximum shear stress *changes* from the expected 45° , and is dependent on the increment size;
- a solution can only be obtained if the perturbation is within approximately 7% of the yield stress.

These characteristics do not correspond to a case of proportional loading. It is clear that further investigation is necessary⁷.

⁶Consider the minimisation of work done in deforming a material. For the case of a material which work-softens, we expect all slip to occur on one plane. For work-hardening materials, we expect both slip planes to be active.

⁷It should be noted that had this formulation been executed using one element in quadrant 1 and employing symmetry to account for those in quadrants 2, 3 and 4, the lack of symmetry of the stress field would not have been exposed since each element would have a complementary element which “balances” the stress distribution.

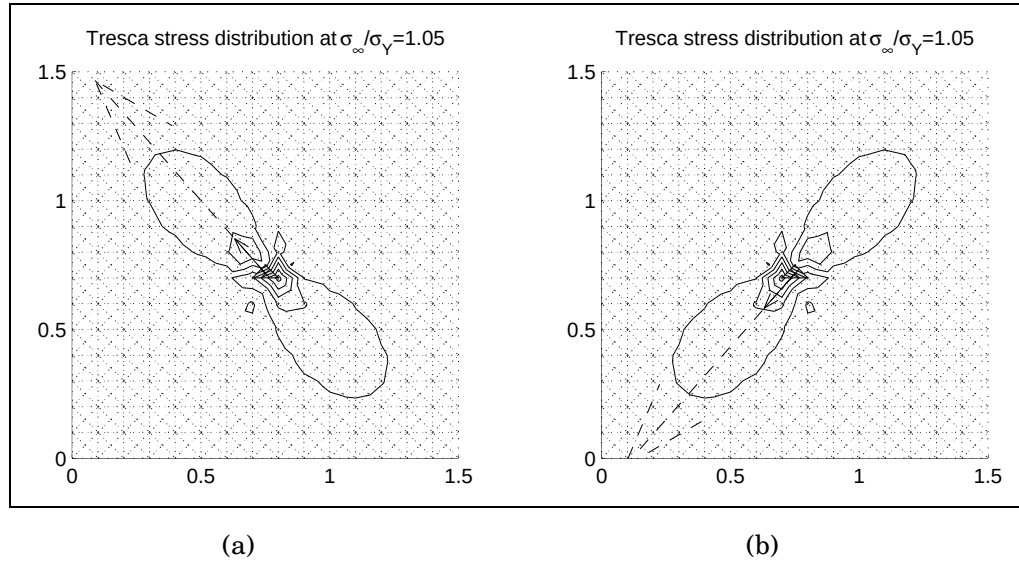


Figure 4.10: Implementation of the single slip plane method for a single element in an infinite plane under remote uniform tension: Contours of constant Tresca stress at intervals of $\sigma_T/\sigma_Y = 0.01$ for a remote applied load of $\sigma_\infty/\sigma_Y = 1.05$ with (a) α_{12} active (b) α_{21} active.

The problem is the assumption that the direction of material flow corresponds to one of the directions of maximum shear stress. A simple thought experiment proves this assumption untenable: take a uniaxial tensile test specimen, and load it such that plastic flow occurs. Although at a microscopic scale the slip is occurring on planes oriented at $\pm \frac{\pi}{4}$ to the direction of maximum principal stress, both of these planes are active and the nett result is that the macroscopic “flow direction” is on planes coincident with the principal stress directions. This is illustrated in figure 4.11.

We shall now develop a model to incorporate a dual slip system formulation into the plasticity model.

Dual slip planes

A “dual slip system” is one in which both slip planes are considered active when returning the stress to the yield surface. The merits of a dual slip system are first that it is intuitively correct at a macroscopic scale, and secondly that it is a more general case. By this we mean that if we allow unequal slip, this gives the system the highest degree of freedom possible.

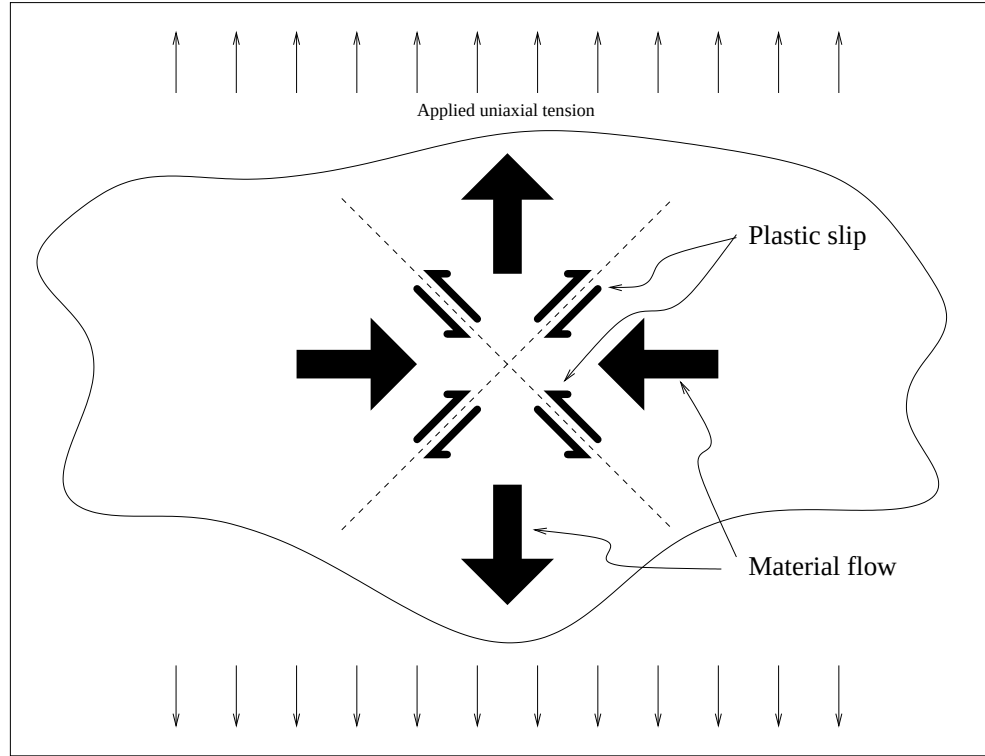


Figure 4.11: Illustration of the difference between the direction of slip (a microscopic concept), and the direction of material flow (a macroscopic concept).

The single slip system can be considered a special case of the more general solution. We turn our attention to equation 4.18 and rewrite it in detail for a dual slip system, assuming the most general case of unequal slip:

$$\begin{bmatrix} G_{12,12} & G_{21,12} \\ G_{12,21} & G_{21,21} \end{bmatrix} \begin{Bmatrix} b_{12} \\ b_{21} \end{Bmatrix} = \begin{Bmatrix} \tilde{\sigma}_{12} \\ \tilde{\sigma}_{21} \end{Bmatrix} \quad (4.19)$$

where the notation $G_{b,s}$ reads the “influence function for Burgers vector b on slip plane s ”, and the term $\tilde{\sigma}_s$ represents the right hand side of equation 4.18 evaluated on slip plane s . We note that we now have an expression which accounts for the cross-terms of the influence function, for example the influence of Burgers vector b_{21} on slip plane α_{12} is $G_{21,12}$.

By the complementarity of shear stresses, the second row of equation 4.19 is always ± 1 times the first row, hence the second row is superfluous information and is disregarded⁸. We introduce a variable $\beta = b_{21}/b_{12}$

⁸This accounts for the rank deficient matrix which causes the SVD analysis to return a zero singular value.

which represents the ratio of the Burgers vectors:

$$\begin{bmatrix} G_{12,12} & G_{21,12} \end{bmatrix} \begin{Bmatrix} b_{12} \\ \beta b_{12} \end{Bmatrix} = \begin{Bmatrix} \tilde{\sigma}_{12} \end{Bmatrix} \quad (4.20)$$

Note that β does *not* characterise the ratio of slip on the slip planes, since there may be a contribution due to the shape and orientation of the triangle of distributed dislocation. If, for example, a triangle has an axis of symmetry that coincides with the axis of symmetry of the applied stress field, the influence functions will be equal in magnitude on each slip plane, and in this special case the β parameter will represent the ratio of *slip* on these planes (since the ratio of slip will equal the ratio of Burgers vectors). For a more general case, the ratio of slip could be calculated by evaluating the displacements on each slip plane (having determined the Burgers vectors). This would require the displacement kernels for a triangle of dislocation which are yet to be determined.

It now appears that we have one equation in two unknowns β , b_{12} . The case of a single slip system is obtained by specifying β , in which case the normality condition is not *required* for the algorithm. For the general case, the β parameter requires definition based on the associated flow rule.

4.2.3 Enforcing the normality condition

We have previously commented on the fact that although the *verification* of the flow rule does satisfy the requirements of a classical plasticity model, a stronger assertion would be to derive an expression which *employs* the flow rule in the plasticity algorithm.

Derivation of the β parameter using the Prandtl-Reuss flow rule

The Burgers vectors can be decomposed into their xy components (see figure 4.9) as

$$\begin{aligned}
 b_x &= b_{12} \cos \alpha_{12} + b_{21} \cos \alpha_{21} \\
 &= b_{12} (\cos \alpha_{12} - \beta \sin \alpha_{12}), \\
 \text{and } b_y &= b_{12} \sin \alpha_{12} + b_{21} \sin \alpha_{21} \\
 &= b_{12} (\sin \alpha_{12} + \beta \cos \alpha_{12}).
 \end{aligned} \tag{4.21}$$

The stresses can then be written as

$$\begin{aligned}
 \sigma_{xx}^d &= G_{xxx} b_x + G_{yxx} b_y \\
 &= b_{12} (G_{xxx} (\cos \alpha_{12} - \beta \sin \alpha_{12}) + G_{yxx} (\sin \alpha_{12} + \beta \cos \alpha_{12})) \\
 \sigma_{yy}^d &= b_{12} (G_{xyy} (\cos \alpha_{12} - \beta \sin \alpha_{12}) + G_{yyy} (\sin \alpha_{12} + \beta \cos \alpha_{12})) \\
 \sigma_{xy}^d &= b_{12} (G_{xxy} (\cos \alpha_{12} - \beta \sin \alpha_{12}) + G_{yyx} (\sin \alpha_{12} + \beta \cos \alpha_{12}))
 \end{aligned} \tag{4.22}$$

If we make the substitutions

$$\begin{aligned}
 a_1 &= G_{xxx} \cos \alpha_{12} & c_1 &= G_{xyy} \cos \alpha_{12} \\
 a_2 &= G_{xxx} \sin \alpha_{12} & c_2 &= G_{xyy} \sin \alpha_{12} \\
 a_3 &= G_{yxx} \sin \alpha_{12} & c_3 &= G_{yyy} \sin \alpha_{12} \\
 a_4 &= G_{yxx} \cos \alpha_{12} & c_4 &= G_{yyy} \cos \alpha_{12}
 \end{aligned} \tag{4.23}$$

then the stresses are

$$\begin{aligned}
 \sigma_{xx}^d &= b_{12} (a_1 - \beta a_2 + a_3 + \beta a_4) \\
 \sigma_{yy}^d &= b_{12} (c_1 - \beta c_2 + c_3 + \beta c_4)
 \end{aligned} \tag{4.24}$$

We now recall the Prandtl-Reuss equation (2.23)

$$\frac{d\varepsilon_{xx}^p}{\sigma'_{xx}} = \frac{d\varepsilon_{yy}^p}{\sigma'_{yy}} = \frac{d\varepsilon_{zz}^p}{\sigma'_{zz}} = \frac{d\gamma_{yz}^p}{\tau_{yz}} = \frac{d\gamma_{zx}^p}{\tau_{zx}} = \frac{d\gamma_{xy}^p}{\tau_{xy}} = d\lambda \tag{4.25}$$

and, considering only the first two terms:

$$\therefore \frac{d\varepsilon_{xx}^p}{d\varepsilon_{yy}^p} = \frac{\sigma'_{xx}}{\sigma'_{yy}} \tag{4.26}$$

Since we are using an incremental plasticity model, the plastic strain increment is represented entirely by the dislocation strain of the current step. Using Hooke's law for plane strain we write the plastic strain increments as

$$\begin{aligned} d\varepsilon_{xx}^p &= \frac{1-\nu^2}{E} \left(\sigma_{xx}^d - \frac{\nu}{1-\nu} \sigma_{yy}^d \right) \\ d\varepsilon_{yy}^p &= \frac{1-\nu^2}{E} \left(\sigma_{yy}^d - \frac{\nu}{1-\nu} \sigma_{xx}^d \right) \end{aligned} \quad (4.27)$$

Thus the ratio of the plastic strains is

$$\frac{d\varepsilon_{xx}^p}{d\varepsilon_{yy}^p} = \frac{\left(\sigma_{xx}^d - \frac{\nu}{1-\nu} \sigma_{yy}^d \right)}{\left(\sigma_{yy}^d - \frac{\nu}{1-\nu} \sigma_{xx}^d \right)} \quad (4.28)$$

and substituting in equations 4.24 we have

$$\begin{aligned} \frac{d\varepsilon_{xx}^p}{d\varepsilon_{yy}^p} &= \frac{(a_1 - \beta a_2 + a_3 + \beta a_4) - \frac{\nu}{1-\nu} (c_1 - \beta c_2 + c_3 + \beta c_4)}{(c_1 - \beta c_2 + c_3 + \beta c_4) - \frac{\nu}{1-\nu} (a_1 - \beta a_2 + a_3 + \beta a_4)} \\ &= \frac{\beta \left(-a_2 + a_4 + \frac{\nu}{1-\nu} (c_2 - c_4) \right) + \left(a_1 + a_3 - \frac{\nu}{1-\nu} (c_1 + c_3) \right)}{\beta \left(-c_2 + c_4 + \frac{\nu}{1-\nu} (a_2 - a_4) \right) + \left(c_1 + c_3 - \frac{\nu}{1-\nu} (a_1 + a_3) \right)} \end{aligned} \quad (4.29)$$

We now consider the ratio of the deviatoric stresses:

$$\frac{\sigma'_{xx}}{\sigma'_{yy}} = \frac{2\sigma_{xx}^i - \sigma_{yy}^i - \sigma_{zz}^i}{2\sigma_{yy}^i - \sigma_{xx}^i - \sigma_{zz}^i} = \mathcal{D} \quad (4.30)$$

where $\sigma_{zz}^i = \nu(\sigma_{xx}^i + \sigma_{yy}^i)$ for plane strain, and we are using the increment stress σ^i which excludes the contribution from the dislocations.

Finally, using equations 4.29 and 4.30, equation 4.26 can be written as

$$\beta = \frac{\mathcal{D} \left(c_1 + c_3 - \frac{\nu}{1-\nu} (a_1 + a_3) \right) - \left(a_1 + a_3 - \frac{\nu}{1-\nu} (c_1 + c_3) \right)}{\left(-a_2 + a_4 + \frac{\nu}{1-\nu} (c_2 - c_4) \right) - \mathcal{D} \left(-c_2 + c_4 + \frac{\nu}{1-\nu} (a_2 - a_4) \right)} \quad (4.31)$$

which is the definition of the β parameter for a dual slip system in plane strain.

This dual slip system was implemented for a single element in an infinite plane subject to remote uniform tension. An example of the output is shown in figure 4.12. Because the axis of symmetry of the element co-

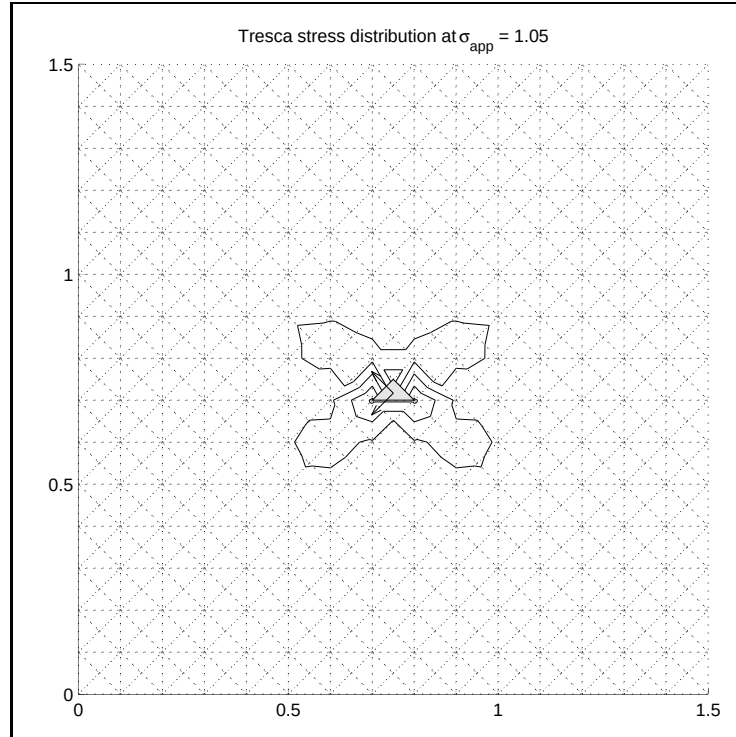


Figure 4.12: Implementation of dual slip planes for a single element in an infinite plane under remote uniform tension.

incides with the axis of symmetry of the stress field, dual equal slip is predicted with $\beta = \pm 1$. We note that the yield criterion has been satisfied, and that the direction of either of the slip planes corresponds with the appropriate direction of maximum shear stress. Furthermore, unlike the case for a single slip system, we see that the stress distribution is now symmetrical about the y -axis as we expect it to be; the solution is obtained without iteration; the direction of maximum shear stress remains $\pm 45^\circ$; and a solution is possible irrespective of the choice of load increment. These observations now agree with those which we would expect for this case of proportional loading.

We are now in a position to elaborate on the difference between the “flow direction” and the slip plane. The slip planes are those planes on which

the shear stress is a maximum and at a microscopic scale we expect plastic flow to occur on these planes. The flow direction, however, is a macroscopic concept and coincides with the planes of principal stress. The flow direction cannot be determined by the vector summation of the Burgers vectors on each of the slip planes alone: the use of piecewise constant dislocation distribution requires the introduction of an element shape and/or orientation factor⁹.

Comments on the implementation of the Prandtl-Reuss flow rule

Equation 4.31 was implemented in the code `Solve_DUAL_SLIP_PLANES.m` which can be found in appendix C. We shall now discuss some of the subtleties involved in this implementation.

There are two yield criteria which could be used: Tresca and von Mises. The Tresca yield criterion is simpler to implement than the von Mises because it results in a linear system of equations which can be easily solved. However, the flow rule which is explicitly associated with the Tresca yield criterion is multi-valued under some conditions, causing difficulties in the implementation of the Tresca flow rule¹⁰. The Prandtl-Reuss flow rule is applicable to the von Mises criterion. We are therefore caught in a difficult situation: in order to use the incremental method of plasticity, we would prefer to use the Tresca yield criterion, but the associated flow rule introduces complications. Implementation of the von Mises yield criterion would be very complex since equation 4.16 could not be written as a linear system.

Consider figure 4.13. Point 1 indicates the location of the system stress outside of the yield surface. We utilise the Tresca yield criterion to determine the *magnitude* of the vector required to return to the Tresca yield surface. The *direction* of this vector is determined by the associated flow rule. The flow rule associated with the Tresca yield criterion would return

⁹A continuous variation of the dislocation distribution within an element would result in the summation of the Burgers vectors corresponding to the flow direction.

¹⁰See, for example, Mendelson [41] §7.4

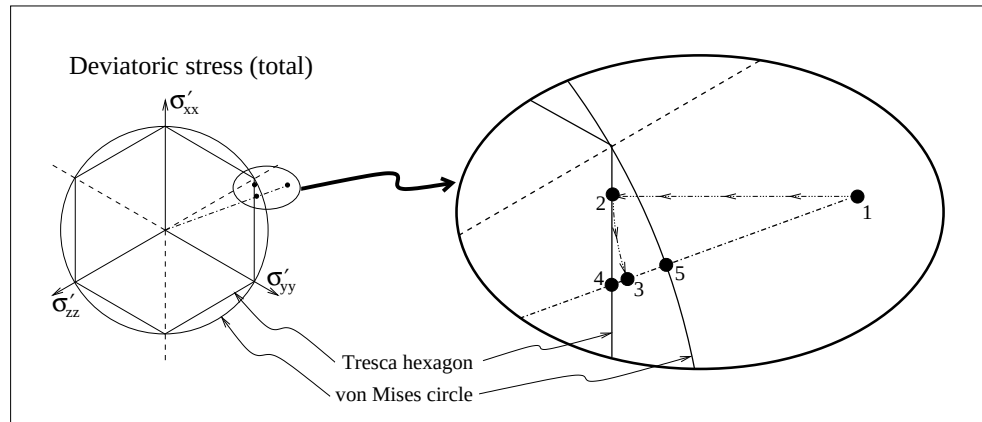


Figure 4.13: Schematic diagram illustrating implementation of the Prandtl-Reuss flow rule with the Tresca yield criterion. Points 1-5 are discussed in the text.

the stress to the yield surface at point 2, since this is the path normal to the Tresca yield surface. However, such a flow rule is not attractive (for the reasons outlined previously), and hence we use the Prandtl-Reuss flow rule. This gives the vector a direction on a path normal to the von Mises circle, and so the stress is moved to point 3 instead of point 2. Point 3, however, lies slightly outside the Tresca yield surface and hence a few iterations are required to converge on a solution, represented by point 4. Were it practical for the von Mises yield criterion to be used in conjunction with the Prandtl-Reuss flow rule, the solution for a single element would proceed directly from point 1 to point 5. The modelling of a plastic zone consisting of a number of active elements would result only in the slight oscillation on the line 1-5 due to redistribution of stresses.

Despite the fact that using the Prandtl-Reuss flow rule in conjunction with the Tresca yield criterion requires a few iterations to converge on a solution, it is still preferable to use this method rather than simply checking the normality condition in hindsight, since the former is a more rigorous application of the classical metal plasticity model.

4.2.4 Algorithm structure – multiple elements

The algorithm implemented was similar to that used for a single element in an infinite plane, with the exception that the number of elements which are active becomes variable. See figure 4.14 for the program flow diagram.

A criterion is required which identifies when an element becomes active (i.e. yields) as the elastic-plastic boundary expands. There are three scenarios from which to choose: an element is active if (a) one vertex exceeds the yield surface, (b) the centroid exceeds the yield surface, or (c) all three vertices lie outside the yield surface. Considering the growth of a plastic zone as the load is incremented, the first of these options will produce the largest estimate of plasticity; conversely, the last would provide a conservative estimate of the smallest region of plasticity. The method implemented corresponds to choice (b), in that it is a compromise between these limits.

The implementation of the incremental theory of plasticity also bears mentioning. This assumes that over the load step attempted, the direction of maximum shear stress remains constant. In the case of proportional loading, the direction of maximum shear stress remains constant and so there is no upper bound on the size of the load step (this corresponds to deformation plasticity models). For non-proportional loading, the direction of maximum shear stress changes, requiring many smaller steps to be taken in order to track the plasticity history accurately.

It is clear that there are two factors affecting the number of elements which become active: (a) the size of load increment chosen, and (b) the coarseness of the mesh discretisation. Obviously it would be foolhardy to choose a very small increment with a very coarse mesh; likewise, it would be pointless to select a large increment with a very fine mesh. Therefore there is an optimum value for the chosen load increment, which is related to the coarseness of the mesh.

Assume, then, that we are part way through a problem and have a plas-

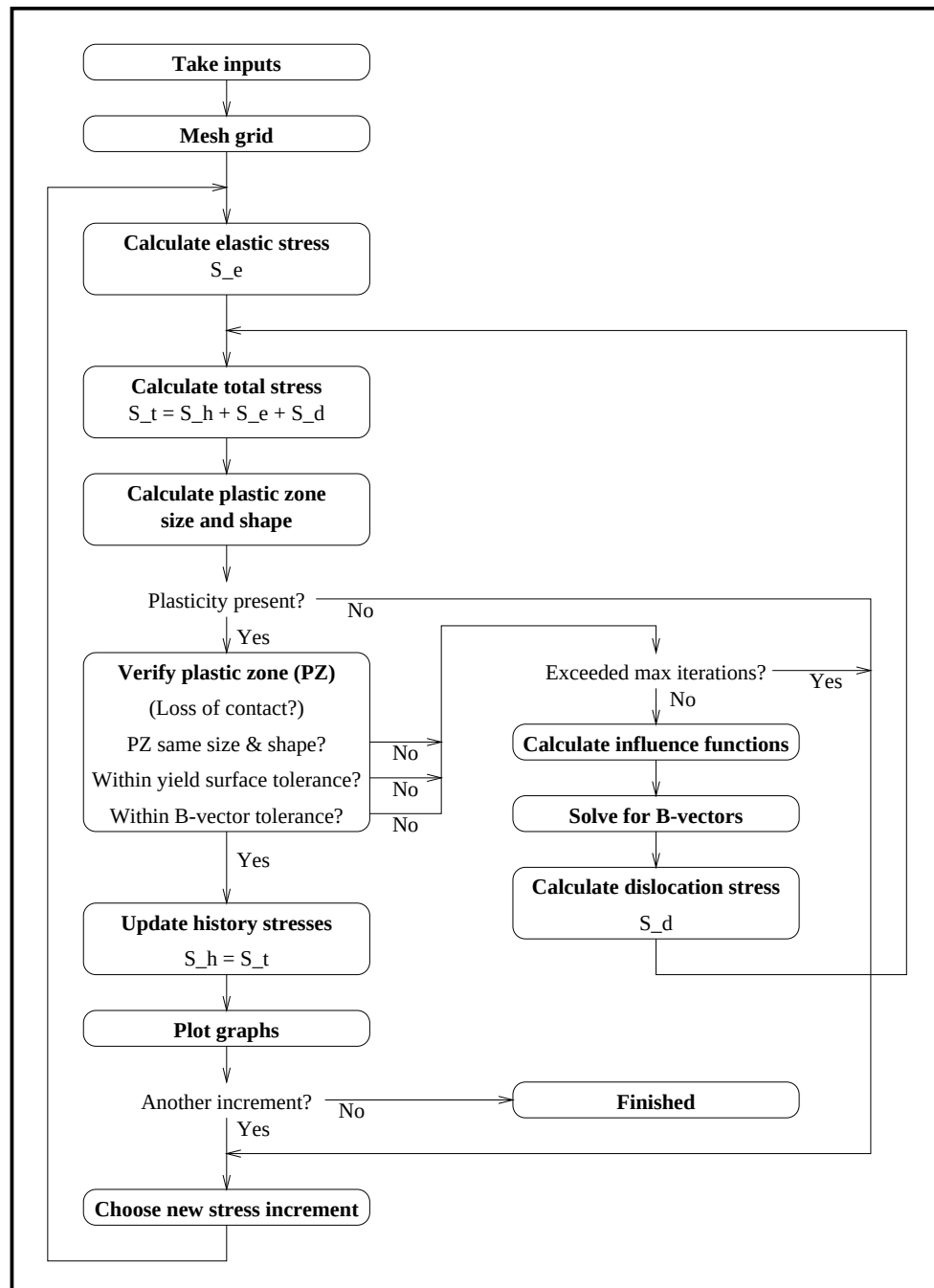


Figure 4.14: Algorithm implemented for the area plasticity model.

tic zone of finite size. We now increment the applied load (or remote stress) as a perturbation to the result previously obtained. Should the chosen load increment be sufficiently small, it is feasible that the plastic zone does not extend far enough to activate new elements. In this case, were this increment to proceed, the strength of the elements already active increase in order to return the stress to the yield surface. This will cause a small load-shedding effect, which in turn might activate a small number of new elements. However, more often than not this does not happen and the plastic zone remains the same size with only the stress contours outside of the plastic zone moving slightly due to the stress redistribution.

We therefore have two choices: either (a) select arbitrarily small load increments, calculating the strength of the active elements even if the plastic zone hasn't grown (and gain a thoroughly accurate description of the plasticity history); or (b) determine the elastic perturbation required to activate new elements. Of course, method (b) is computationally more efficient since the intensive process of solving for the strengths of the dislocation elements (at arbitrarily small increments which do not necessarily cause more elements to become active) is omitted. Experimental observation has shown that the errors associated with employing method (b) over option (a) are negligible. This is attributed to the fact that these errors are small relative to those arising from other assumptions in the model (yield surface tolerance, normality rule tolerance, solution accuracy due to mesh size). A further complication is that if the increment is too large and the number of active elements increases significantly, the algorithm may not converge. This is caused by the violation of the assumption that the direction of maximum shear stress remains constant over a chosen increment.

We therefore have a lower increment size determined by the refinement of the mesh, and an upper increment size determined by the degree of non-proportionality associated with the redistribution of stress due to the active plasticity elements.

4.3 Summary

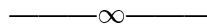
In this chapter we have presented the influence functions for a triangular element of piecewise constant dislocation in an infinite plane under plane strain conditions. The evaluation of these influence functions requires careful implementation, dependent on the region in which the collocation point lies relative to the triangular element. It has been proven that both the derivation and implementation of the element influence function is correct.

The case of a single element in an infinite plane subjected to remote uniform tension was then discussed. This case study drew out the necessity for a dual slip system, that is, for slip to occur on both planes. The β parameter was introduced to describe the ratio of the Burgers vectors. An explanation of the difference between the microscopic concept of slip planes, and the macroscopic nature of a “flow field” was presented.

Previous algorithms for plasticity have relied on a verification procedure to honour the flow rule. Here, the β parameter was derived from the Prandtl-Reuss flow rule, enabling the *enforcement* of the normality condition in the solution of plasticity problems.

The algorithm for a multiple element incremental plasticity model was presented, and comment was made on the subtle issue concerning choice of increment size.

Having established an algorithm that enforces both the yield criterion and the normality condition, we now look to apply this method to two problems involving area plasticity.



Chapter 5

Application of the 2-D area-plasticity model

Two problems to demonstrate the use of the kernel for a triangle of dislocation in an infinite plane are discussed. We consider first the solution of the Kelvin problem, in an attempt to model this incorporating plasticity. We then analyse the effect of a peg journalled in a hole in an elastically similar infinite plane, subject to a lateral force.

5.1 The Kelvin problem

The Kelvin elastic solution for a line force in an infinite plane [47]¹[48] is:

$$\begin{aligned}\sigma_{rr} &= -\frac{(3 + \nu)F \cos \theta}{4\pi r} \\ \sigma_{\theta\theta} &= \frac{(1 - \nu)F \cos \theta}{4\pi r} \\ \sigma_{r\theta} &= \frac{(1 - \nu)F \sin \theta}{4\pi r}\end{aligned}\tag{5.1}$$

The Tresca stress distribution for the Kelvin elasticity solution is shown in figure 5.1. Contours within the elastic-plastic boundary have not been

¹Note that in Barber [47], equation 13.12 for σ_{rr} is missing a negative sign; cf. Timoshenko [48] has the correct equations which are reproduced here.

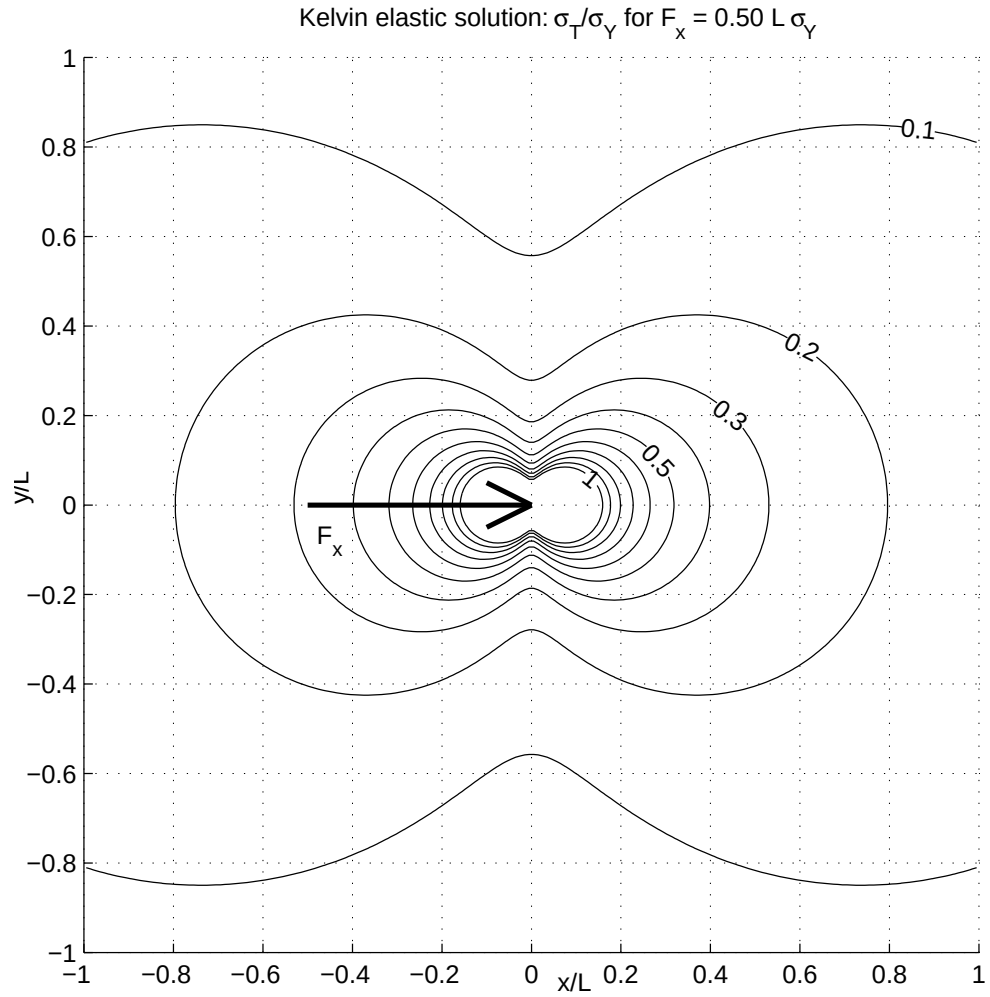


Figure 5.1: Tresca stress contours for the Kelvin solution, normalised with respect to the yield stress, for an applied force of $F_x = 0.50 L \sigma_Y$.

plotted, since they become infinitely dense as the origin is approached, and it is informative to depict the elastic-plastic boundary as predicted by the purely elastic formulation. As can be seen from equations 5.1, the Kelvin solution is singular at the origin and is inherently self-similar as there are no length dimensions present. For this reason we introduce a fictitious length scale, L , which is used to normalise both the force F and the axis-set.

The Kelvin problem was selected as there are no boundaries present, and hence the infinite plane kernel for the area plasticity model, developed in chapter 4, is valid. Paradoxically, this lack of length dimension is

precisely the cause of many complexities which will be explained shortly.

Conflicting requirements then arise: on the one hand, we want to activate a large number of elements to remove any mesh dependency; on the other, we have to start with the smallest plastic zone and “grow” it in order to honour the incremental theory of plasticity allowing for the redistribution of stresses.

The plasticity solution to the Kelvin problem was first attempted using the area plasticity model (APM), as derived in chapter 4. This was compared with a finite element method (FEM) implementation. Difficulties in these methods resulted in the use of a plasticity exclusion region, where the solution was again attempted using both the APM and the FEM. These are now discussed in further detail.

Modelling the Kelvin problem using the area plasticity model

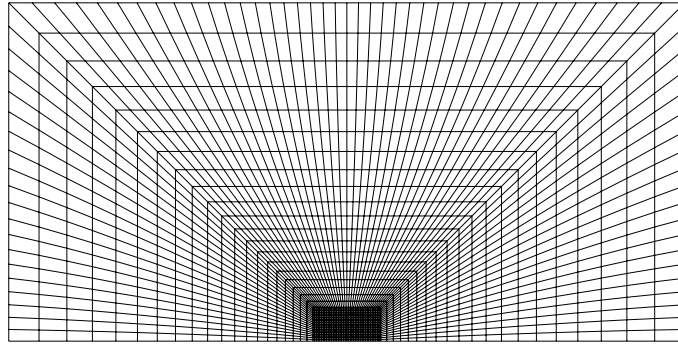
Attempts to model the Kelvin problem using the area plasticity model were unsuccessful. It was found that, irrespective of the mesh refinement or size of load increment applied, the size of the plastic zone grew in an unbounded manner. The reason for this behaviour will be discussed at a later stage. In order to gain a further understanding of the processes involved, a finite element model was implemented.

Modelling the Kelvin problem using the finite element method

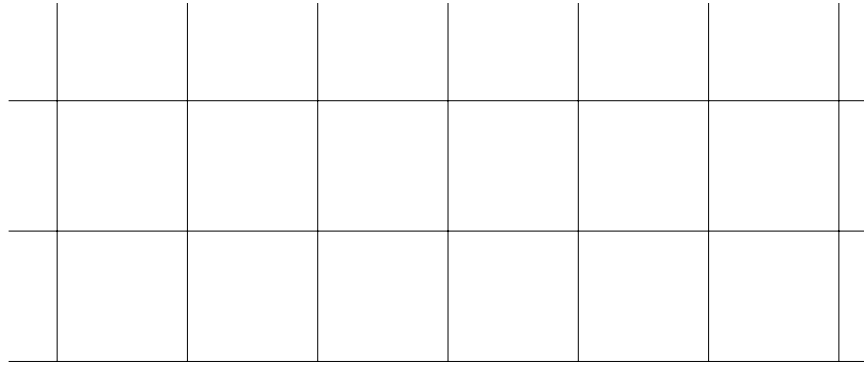
The mesh chosen is displayed in figure 5.2(a). Only the top half plane is analysed due to symmetry². The outer nodes of the meshed region are subjected to the “ENCASTRE” boundary condition (i.e. all displacement and rotational degrees of freedom removed), with the exception of the symmetry axis at the bottom of the mesh, which is subjected to the “YSYMM” constraint. The entire mesh lies in $x \in [-100, 100]$, $y \in [0, 100]$,

²Since the FE technique does not involve the modelling of slip on certain planes, symmetry about the x -axis can be exploited with confidence.

and is focused towards the origin, where the region $x \in [-1, 1]$, $y \in [0, 1]$ has square elements with sides of length $c = 0.05$. A force, F_{app} , is applied in the positive x direction at the origin.



(a) Entire mesh: $x \in [-100, 100]$, $y \in [0, 100]$.



(b) Mesh at same zoom scale as figure 5.3

Figure 5.2: The mesh used for the Kelvin problem in ABAQUS. (a) The entire mesh lies in $x \in [-100, 100]$, $y \in [0, 100]$. The mesh is focused towards $(0, 0)$ where the line force F_{app} is applied to a single node. (b) The mesh on the same scale as that shown in figure 5.3. In this region, each square element has characteristic length of $c = 0.05$.

Modelling the Kelvin problem using a finite element package is fraught with difficulties. Obtaining a response for an elastic perfectly plastic material with a point force applied to one of the nodes produces instabilities once the displacement of the node exceeds the characteristic element length in that region. Also, it should be emphasised that the ABAQUS package does not have the ability to implement the Tresca yield criterion, and employs instead the von Mises criterion. Figure 5.3 shows the output from the ABAQUS model.

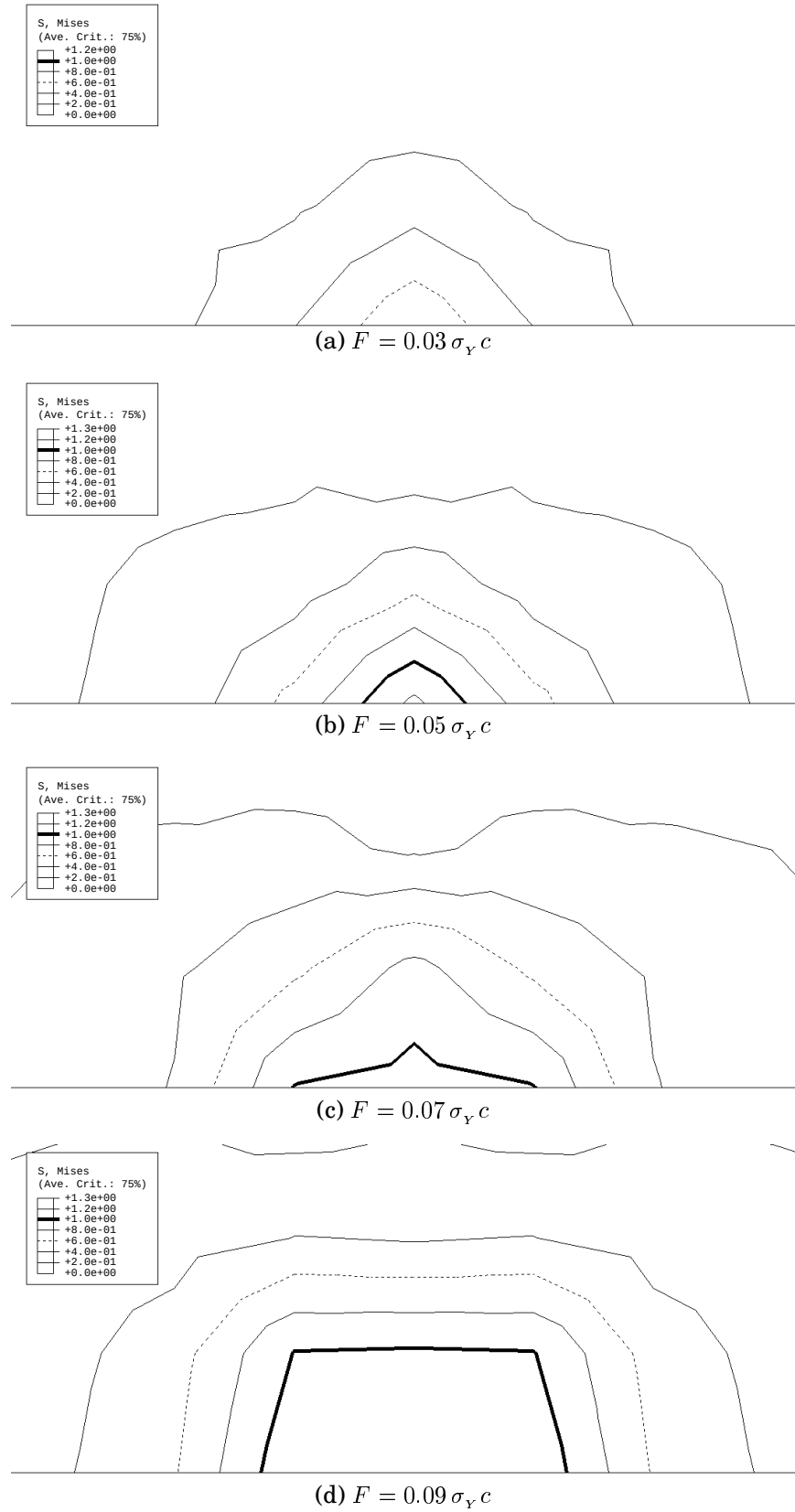


Figure 5.3: The Kelvin problem solved by finite element methods using ABAQUS. Only the top half plane is analysed due to symmetry. A point force F_{app} is applied in the positive x direction. Von Mises stress contours are plotted at intervals of $\frac{\sigma_{vM}}{\sigma_Y} = 0.2$. The contour $\frac{\sigma_{vM}}{\sigma_Y} = 0.6$ is dotted and the elastic-plastic boundary is the bold line, first visible in (b) where $F = 0.05 \sigma_Y c$. (c is the characteristic element length – see text.)

Von Mises contours are plotted³ at intervals of $\frac{\sigma_{vM}}{\sigma_Y} = 0.2$, with $\frac{\sigma_{vM}}{\sigma_Y} = 0.6$ as a dashed line and $\frac{\sigma_{vM}}{\sigma_Y} = 1.0$ (i.e. the elastic-plastic boundary) as a bold line. The analysis became unstable at $F_{app} = 0.11 \sigma_Y c$ (where c is the characteristic element size) and aborted.

It is appreciated that, due to self-similarity, all four plots in figure 5.3 should look qualitatively similar. However, it can be seen that this is certainly not the case, and that there is a mesh dependency. Indeed, for a given applied force, the size and shape of the plastic zone is related only to the mesh size, since this is the only length dimension in the problem. Thus, some other means of implementing the model was required.

Modelling the Kelvin problem using a plasticity exclusion region with the area plasticity model

The conundrum of the only length dimension being introduced by the mesh may be overcome by artificially introducing another length dimension. This is implemented by creating a circular region of finite radius $R_0 = 1$, at the centre of which the point force is applied. Within this region, the material is assigned an infinite yield stress, i.e. it is purely elastic. The surrounding material remains elastic perfectly plastic. The Kelvin problem incorporating plasticity can then be solved, since as the size of the plastic zone becomes large compared to the region of infinite yield stress, the plastic zone will evolve into a constant shape for any load increment.

Many attempts were made to obtain a solution to the Kelvin problem incorporating plasticity using the area plasticity model. Two sets of results are now presented. Figure 5.4 contains the output using load increments of $F_{app} = 0.50 \sigma_Y R_0$. It was found that the plastic zone grew at each increment, and that the analysis became unstable after $F_{app} = 6.00 \sigma_Y R_0$.

³If Tresca contours were plotted, it would not be a true representation of the Tresca stress since the von Mises yield criterion has been used. The Tresca stress is simply calculated by ABAQUS from the difference in principal stresses.

Figures 5.5 and 5.6 show the results from an analysis which used much smaller increments of $F_{app} = 0.01 \sigma_Y R_0$ with calculation of the dislocation stress only being executed when the elastic perturbation was sufficiently large to activate new elements. This technique was discussed in detail in section 4.2.4.

The mesh used in both cases⁴ was composed of radial lines at fifteen degrees and circumferential lines forming an aspect ratio of 1.0, surrounding an exclusion zone of radius 1.0. The arrows in the figures produced by the APM code represent the Burgers vectors in each of the active elements. The magnitude of these arrows represents the element strength per unit area, and the direction represents the planes on which the shear stresses are a maximum. It should be emphasised that in each case, these are the dislocation distributions required to return the stress to the yield surface *for the current load increment*.

The results presented in these figures are incomplete in the sense that the output has not progressed to the point where the size of the plasticity exclusion zone is small with respect to the size of the plastic zone. Despite a number of attempts with various mesh designs, this algorithm does not allow such a solution to be obtained. As the plasticity begins to enclose the exclusion region, it grows very suddenly. This is due to the characteristic shape of the elastic plastic boundary of the elastic solution (bean-shape) shown in figure 5.1, relative to the shape of the exclusion region (circular). The sudden growth of the plastic zone as it encloses the exclusion region causes failure to converge at that load increment. The difference in the size and shape of the plastic zones presented in these two figures will be discussed after the results from the finite element method have been presented.

⁴Mesh generation proceeds as follows: (1) Angular increment, aspect ratio, and the size of the element nearest the origin are chosen. (2) Triangles are generated in the first “row” which surrounds the origin. (3) The remainder of the plane is divided into quadrilaterals based on angular increment and aspect ratio. (4) Each quadrilateral is then split into four triangles. Thus the specification of angular increment, aspect ratio and size of first element uniquely identifies a given mesh.

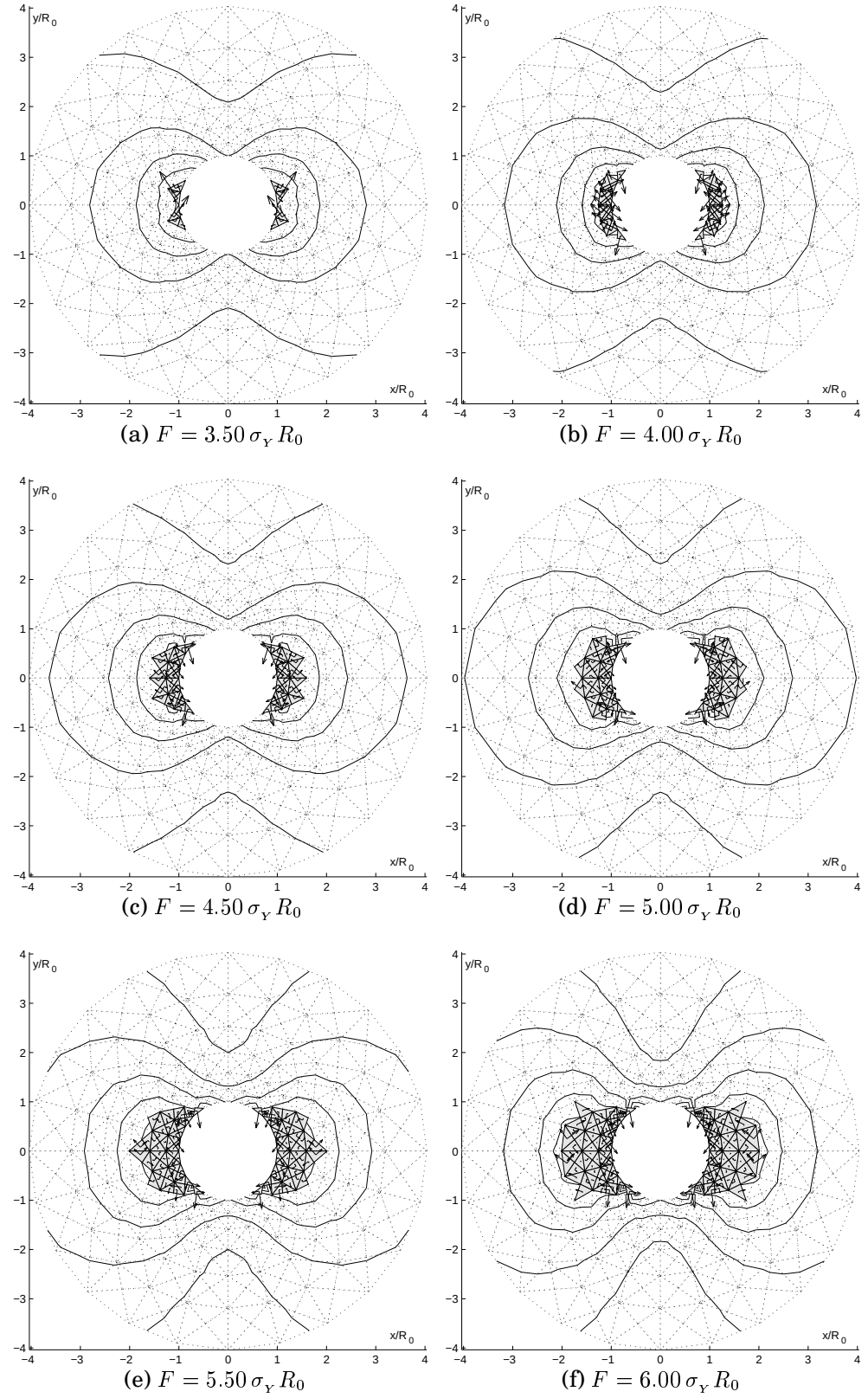


Figure 5.4: The Kelvin problem solved using the area plasticity model with a circular plasticity exclusion region $R_0 = 1$. Tresca stress contours are plotted at intervals of $0.2 \sigma_Y$. The load was applied in increments of $F = 0.50 \sigma_Y R_0$. Results are shown for (a) $F = 3.50 \sigma_Y R_0$, (b) $F = 4.00 \sigma_Y R_0$, (c) $F = 4.50 \sigma_Y R_0$, (d) $F = 5.00 \sigma_Y R_0$, (e) $F = 5.50 \sigma_Y R_0$, (f) $F = 6.00 \sigma_Y R_0$.

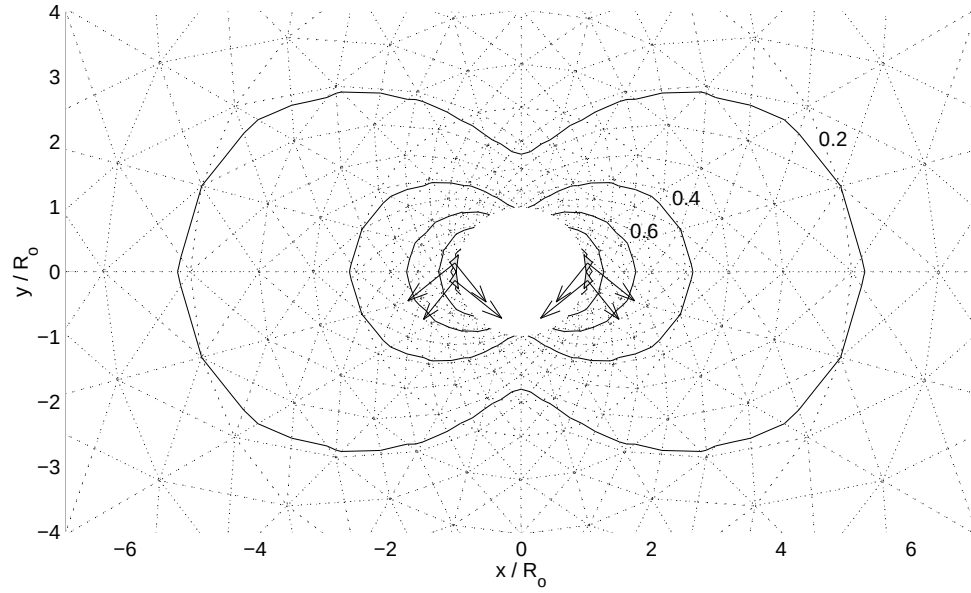
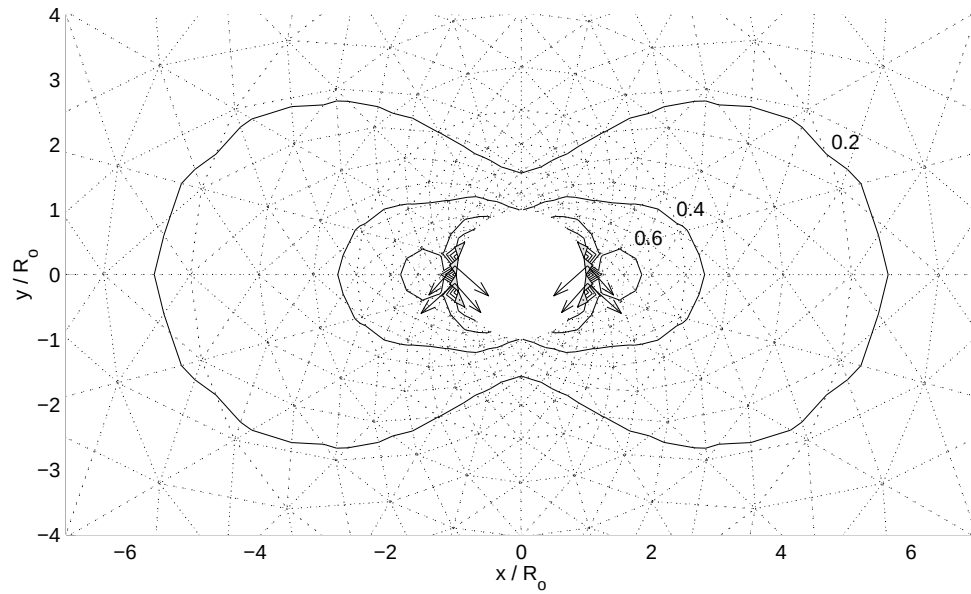
(a) $F = 3.30 \sigma_Y R_0$ (b) $F = 3.50 \sigma_Y R_0$

Figure 5.5: The Kelvin problem solved using the area plasticity model with a circular plasticity exclusion region $R_0 = 1$. Tresca stress contours are plotted at intervals of $0.2 \sigma_Y$. The load was applied in increments of $F = 0.01 \sigma_Y R_0$. Results are shown for each increment where the perturbation caused the plastic zone to grow: (a) $F = 3.30 \sigma_Y R_0$, (b) $F = 3.50 \sigma_Y R_0$.

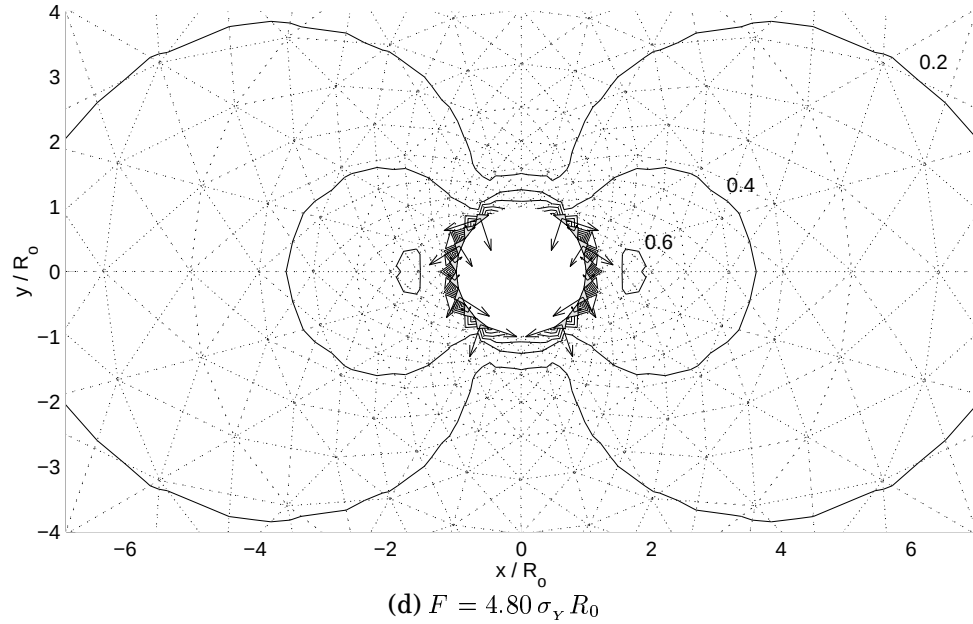
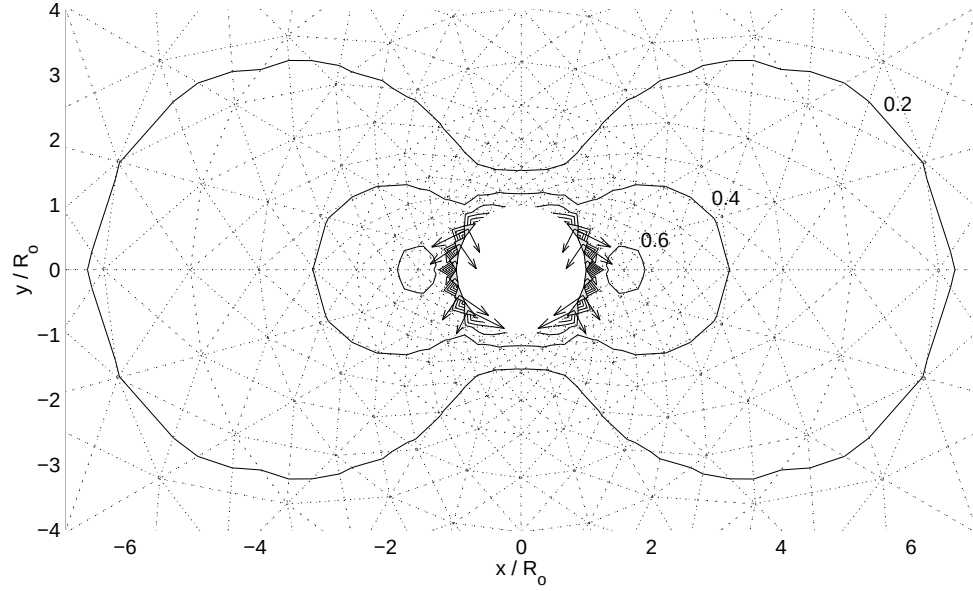
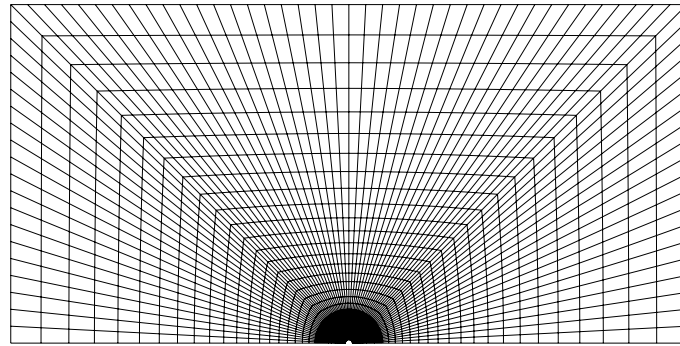


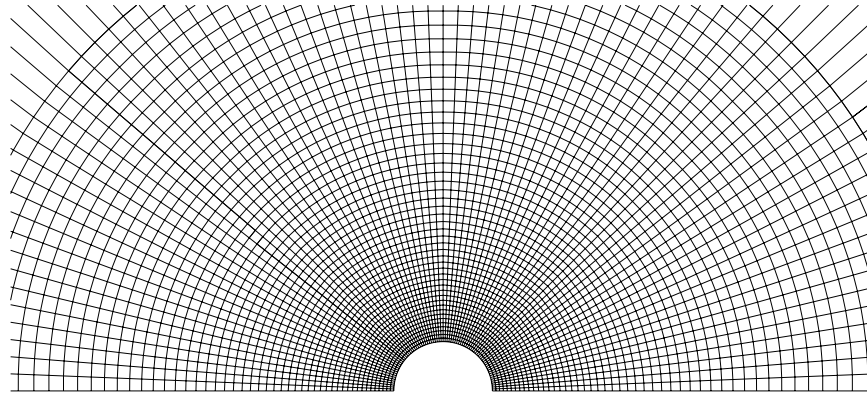
Figure 5.6: The Kelvin problem solved using the area plasticity model with a circular plasticity exclusion region $R_0 = 1$. Tresca stress contours are plotted at intervals of $0.2 \sigma_Y$. The load was applied in increments of $F = 0.01 \sigma_Y R_0$. Results are shown for each increment where the perturbation caused the plastic zone to grow: (c) $F = 4.20 \sigma_Y R_0$, (d) $F = 4.80 \sigma_Y R_0$.

Modelling the Kelvin problem using a plasticity exclusion region with the finite element method

The mesh chosen is similar to that used previously for the Kelvin problem, but now has a circular plasticity exclusion region of unitary radius, as displayed in figure 5.7(a). An infinite plane was modelled by meshing an area



(a) Entire mesh: $x \in [-100, 100]$, $y \in [0, 100]$.



(b) Mesh at same zoom scale as figure 5.8.

Figure 5.7: The mesh used for the Kelvin problem in ABAQUS. (a) The entire mesh lies in $x \in [-100, 100]$, $y \in [0, 100]$. The mesh is focused towards $(0, 0)$ where the line force F_{app} is applied to a single node. (b) The mesh on the same scale as that shown in figure 5.8.

two orders of magnitude larger than the radius of the circular plasticity exclusion region. Symmetry about the x -axis was exploited. Increments of $F_{app} = 1.0 \sigma_Y R_0$ were applied. The results from the FEM analysis are shown in figure 5.8, which are in agreement with those from the APM analysis (see figure 5.4), given the differences in yield criteria employed by the respective models (i.e. APM uses Tresca; FEM uses von Mises).

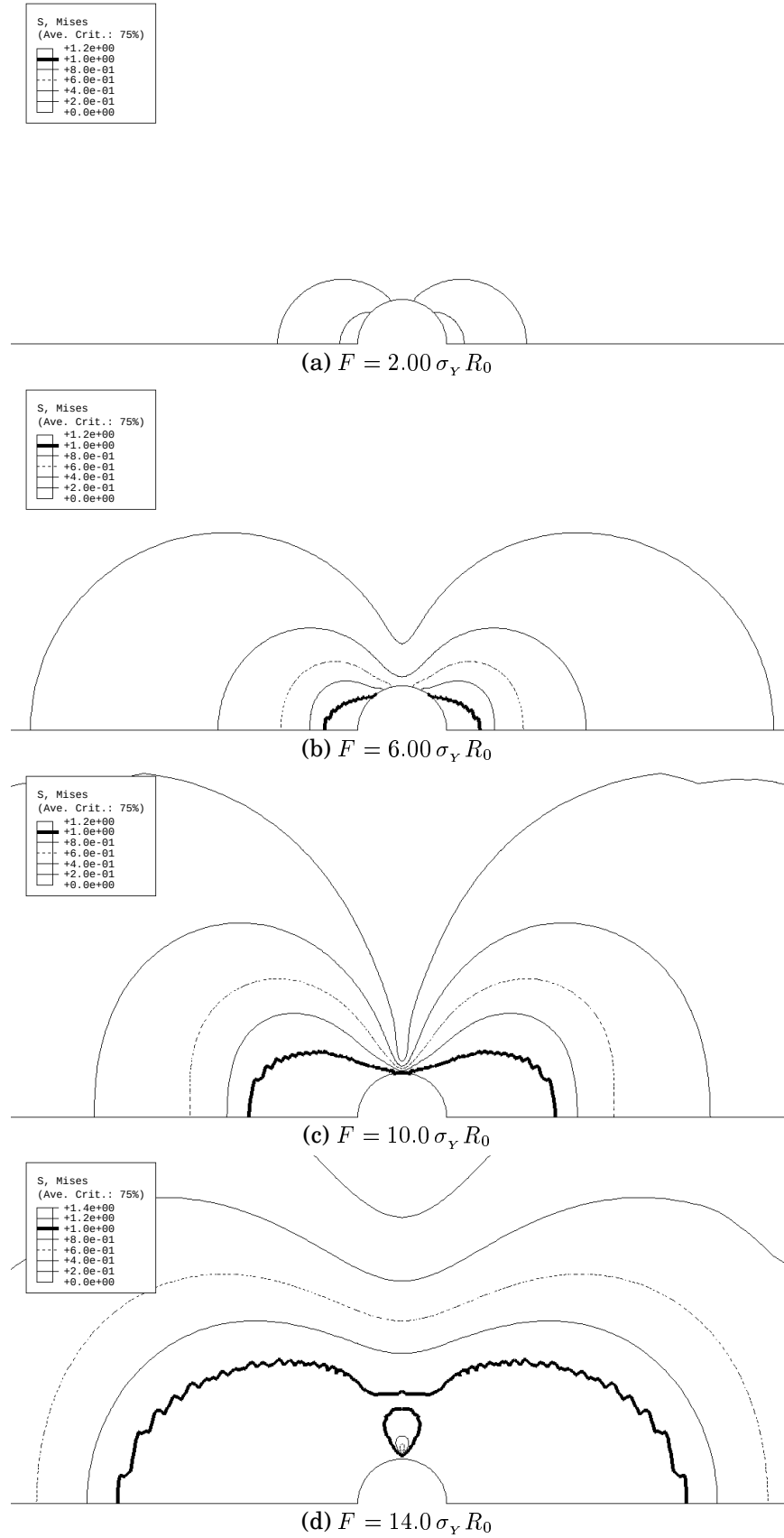


Figure 5.8: The Kelvin problem using a plasticity exclusion region $R_0 = 1$, solved by finite element methods using ABAQUS. Only the top half plane is analysed due to symmetry. A point force is applied in the positive x direction in increments of $F_{app} = 1.0 \sigma_Y R_0$. Tresca stress contours are plotted at intervals of 0.2, and the elastic-plastic boundary is the bold line.

Discussion on the implementation of a plasticity exclusion region

The first analysis (see figure 5.4) used load increments of $F_{app} = 0.50 \sigma_Y R_0$, and the last increment which converged was $F_{app} = 6.00 \sigma_Y R_0$. The second analysis (figures 5.5 and 5.6) used increments of $F_{app} = 0.01 \sigma_Y R_0$ with the last stable increment at $F_{app} = 4.80 \sigma_Y R_0$. These differences reveal an interesting characteristic of this problem, *viz.* that for a given mesh, the size and shape of the plastic zone is a function of the size of the load increment attempted. This question of increment size was discussed in section 4.2.4, where it was stated that there is an optimum range of load increments for a given mesh. Increment sizes chosen within this range will converge. However, within this range, different results are obtainable, as illustrated by figures 5.4 to 5.6. Two questions must then be answered. (a) Why is there a difference? (b) Which is correct? Using larger increments predicts a larger area over which the yield stress is exceeded, causing more elements to become active, each of which has a moderate strength. Selecting smaller increments causes fewer elements to become active, but these elements are then of a higher Burgers vector strength (per unit area). Both cases do return the stress state to the yield condition, and satisfy the normality rule. As to which is correct, the incremental theory of plasticity implies that the size of the plastic zone can be over-estimated but cannot be under-estimated, i.e. the smaller plastic zone (figures 5.5 and 5.6) is closer to the true solution.

Consider now the matter of mesh dependence. A finer mesh will allow, say, the first four elements to become active for a smaller load increment than a coarser mesh would. This is because the elements become active once the stress at the collocation point (in this case the centroid of the triangle) exceeds the yield condition, and for a finer mesh, the collocation point lies closer to the origin than it would for a larger element. In short, the force required to create some initial plasticity in the region immediately surrounding the hole is prescribed by the ratio of the element size to the size of that hole.

The behaviour observed in figures 5.5 and 5.6 is for a ring of elements immediately bordering the exclusion region to become active. Furthermore, we have seen that the shape of the plasticity exclusion region (circular) relative to the bean-shaped elastic plastic boundary in the elastic Kelvin solution (figure 5.1) is crucial, since there are no length dimensions in this problem. We therefore conclude that, were the plasticity exclusion region itself bean-shaped⁵, a load increment of any size would converge. The size of the plastic zone could then be grown arbitrarily large, until the plasticity exclusion region was small relative to the size of the entire plastic zone. We therefore conclude that the Kelvin solution could be solved incorporating plasticity, and that this simply involves a plastic zone of arbitrary size (due to self-similarity) which has the same shape as the elastic plastic boundary as shown in figure 5.1.

We turn now to a problem which is well defined (in terms of length dimensions), and also of practical use. This is that of a peg, journalled in a hole in an infinite plane, subjected to a lateral force.

⁵Implementation of such a mesh is non-trivial: first, the bean-shaped region would have to be identified, and then a meshing routine would be required to march around the bean shaped contours, discretising the region in such a way as to create bean-shaped rings of triangles around the original contour. This would be required to remove the mesh dependency.

5.2 A journalled peg subject to a lateral force

The results from the Kelvin problem with plasticity were encouraging. However, further evidence that the area plasticity model converges on the correct solution was desired. A problem where the kernel for a triangular element of dislocations in an infinite plane may be applied, yet circumventing the difficulties associated with modelling a point force, is that of a peg of finite radius in an infinite plane. Upon applying a force to the peg (in much the same manner as the Kelvin problem) we anticipate the development of the plastic zone on the compressive side of the peg. In order to use the infinite plane kernel for the triangular elements, it is necessary to assert that the peg and plane are elastically similar. Furthermore, no separation between the peg and plane must be allowed to occur⁶. For this reason, the radial stresses σ_{rr} are monitored, and the model becomes invalid when $\sigma_{rr} \geq 0$. This separation is expected to occur on the side of the peg opposite the plasticity, and is counteracted by applying a shrink-fit (i.e. press-fit) pre-stress, of the form:

$$\begin{aligned}\sigma_{rr} &= -P_0 \left(\frac{R_0}{R} \right)^2 \\ \sigma_{\theta\theta} &= P_0 \left(\frac{R_0}{R} \right)^2 \\ \sigma_{r\theta} &= 0\end{aligned}\tag{5.2}$$

where P_0 is the applied pressure, R_0 is the radius of the peg ($R_0 = 1.0$), and R is the distance from the origin. We then apply a force to the peg in the positive x direction using the standard Kelvin expressions (equations 5.1). Obviously, the case where separation is least likely to occur is when the highest shrink-fit pre-stress is applied, and the upper limit to avoid plasticity due to the shrink-fit is incipient yield, which for the Tresca criterion corresponds to a value of $P_0 = 0.5 \sigma_Y$.

⁶Separation would necessitate the use of (a) a kernel other than that for an infinite plane, and (b) knowledge of the compliance of the material being modelled, i.e. the ratio σ_Y / E would affect the stress distribution

Results from the area plasticity model

Figure 5.9 shows the results from the area plasticity model with a mesh of radial lines at 15 degrees with an aspect ratio of 0.4. The analysis reported loss of contact at an applied load of $F_{app} = 0.39 \sigma_Y R_0$. Results are presented at applied loads of $F_{app}/(\sigma_Y R_0) = 0.05, 0.06, 0.09, 0.15, 0.33$, all of which are valid, since no separation has occurred. Figure 5.10 shows the results from a similar analysis but with a slightly more refined mesh: radial lines at 9 degrees with an aspect ratio of 0.4. In this case separation over a large region ($\theta = 90^\circ$) is reported at $F_{app} = 0.68 \sigma_Y R_0$. Results were obtained at loads of $F_{app}/(\sigma_Y R_0) = 0.05, 0.06, 0.08, 0.12, 0.20, 0.37, 0.68$. The reason for the much larger estimate for the separation load was that the penultimate analysis was executed at $F_{app} = 0.37 \sigma_Y R_0$ which is where separation is imminent (as we can deduce from the previous analysis). Figure 5.10(f) ($F_{app} = 0.68 \sigma_Y R_0$) is therefore invalid in that separation has occurred. Shortly after this the analysis becomes unstable as the number of active elements grows too rapidly for convergence to be achieved.

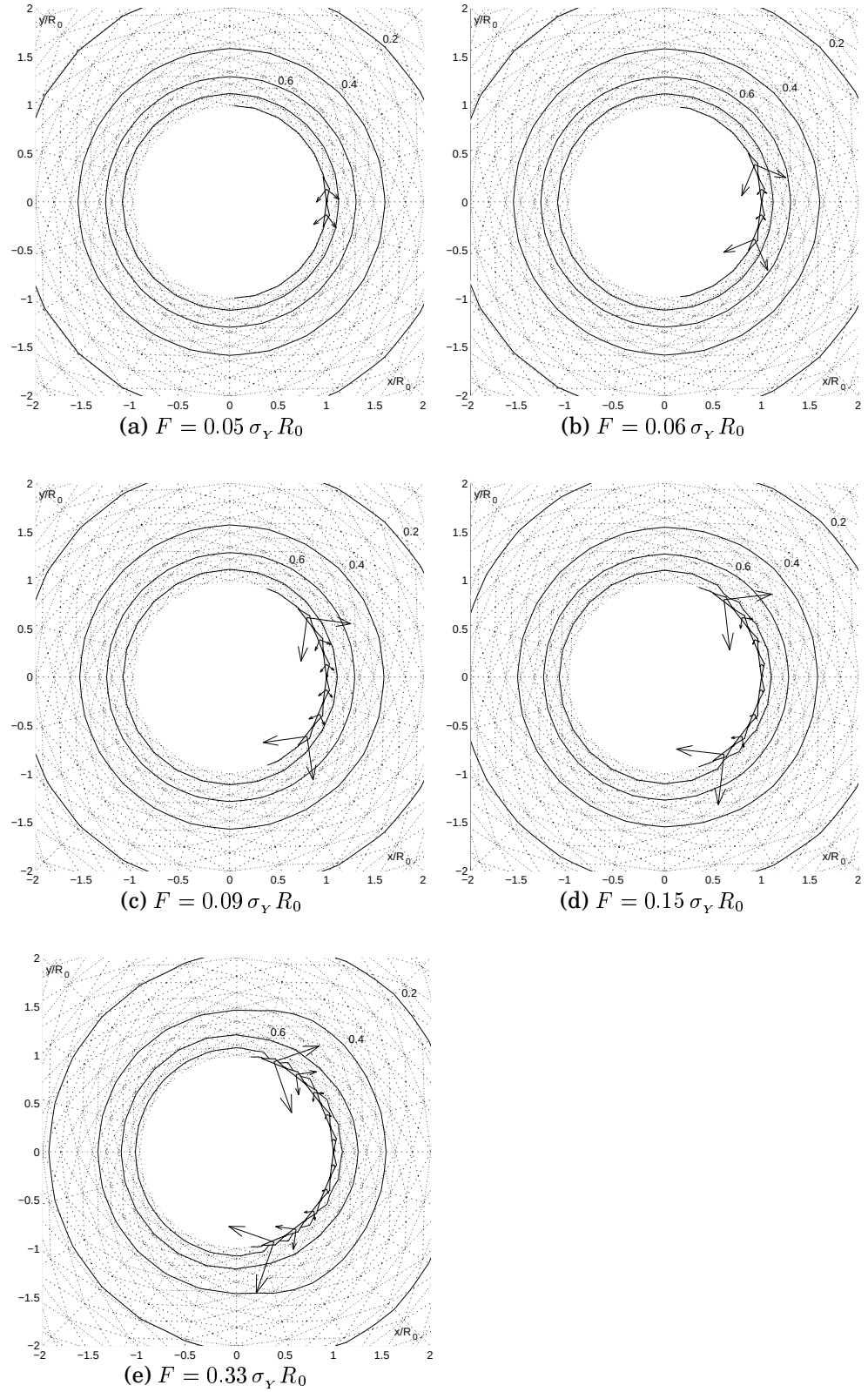


Figure 5.9: A shrink-fit peg ($R_0 = 1$) in an infinite plane using the area plasticity model with mesh of 15 degree lines and aspect ratio 0.4 and $F/\sigma_Y R_0 =$ (a) 0.05, (b) 0.06, (c) 0.09, (d) 0.15, (e) 0.33.

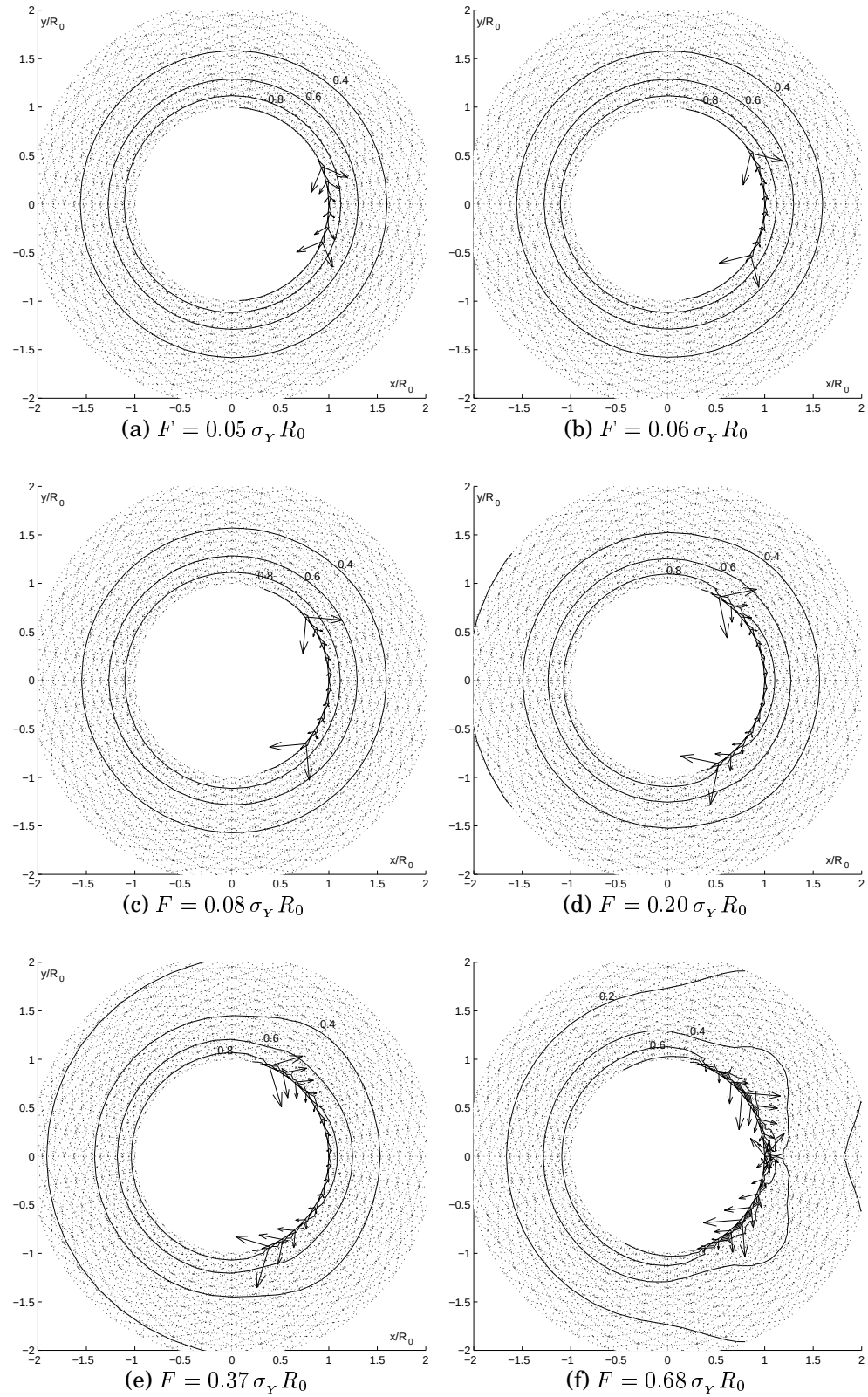


Figure 5.10: A shrink-fit peg ($R_0 = 1$) in an infinite plane using the area plasticity model with mesh of 9 degree lines and aspect ratio 0.4 and $F/\sigma_Y R_0 =$ (a) 0.05, (b) 0.06, (c) 0.08, (d) 0.20, (e) 0.37, (f) 0.68. The last figure, although informative, is invalid in that separation is reported prior to this load and hence the kernel for an infinite plane is incorrect.

Results from a finite element model

A finite element model of a peg in an infinite plane was implemented for comparison. The refined region of the mesh is displayed in figure 5.11 (the elements representing the peg itself have been removed for clarity). The peg was assigned purely elastic material properties, whilst the plane

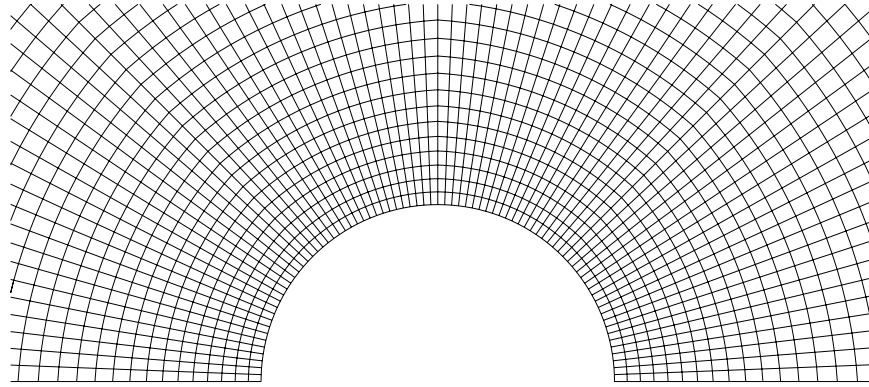


Figure 5.11: The mesh used for the journalled peg problem in ABAQUS. The entire mesh lies in $x \in [-100, 100]$, $y \in [0, 100]$, as shown in figure 5.7(a). Here, the mesh is on the same scale as that shown in figure 5.12.

was elastic perfectly plastic with the elastic parameters identical to those of the peg. A press fit stress was applied to the interior of the peg-plane contact surface acting radially outwards, of magnitude $P_0 = 0.50 \sigma_Y$. A force⁷, F_{app} , was then applied to ten nodes within the peg, in the positive x -direction. The force was applied in equal increments of $F_{app} = 0.01 \sigma_Y R_0$ up to the maximum value of $0.70 \sigma_Y R_0$.

As mentioned previously, ABAQUS uses the von Mises yield criterion, and for this reason some discrepancies in the comparison to the area plasticity model are anticipated. Comparison of figures 5.9 and 5.10 to figure 5.12 show that both the estimates of the growth of the size and shape of the plastic zone, and the stress contours outside of the plastic zone, agree within limits of the assumptions made. It should be noted that the FEM results predicted separation to occur between loads of $F_{app} = 0.30 \sigma_Y R_0$ and $F_{app} = 0.35 \sigma_Y R_0$, which is similar to that reported by the area plasticity model of $F_{app} = 0.39 \sigma_Y R_0$.

⁷This can be interpreted as a bearing pressure $\sigma_{app} = \frac{F_{app}}{R_o t}$ where R_o is the radius of the hole and t is the thickness of the plane, both of which are chosen to be unity.

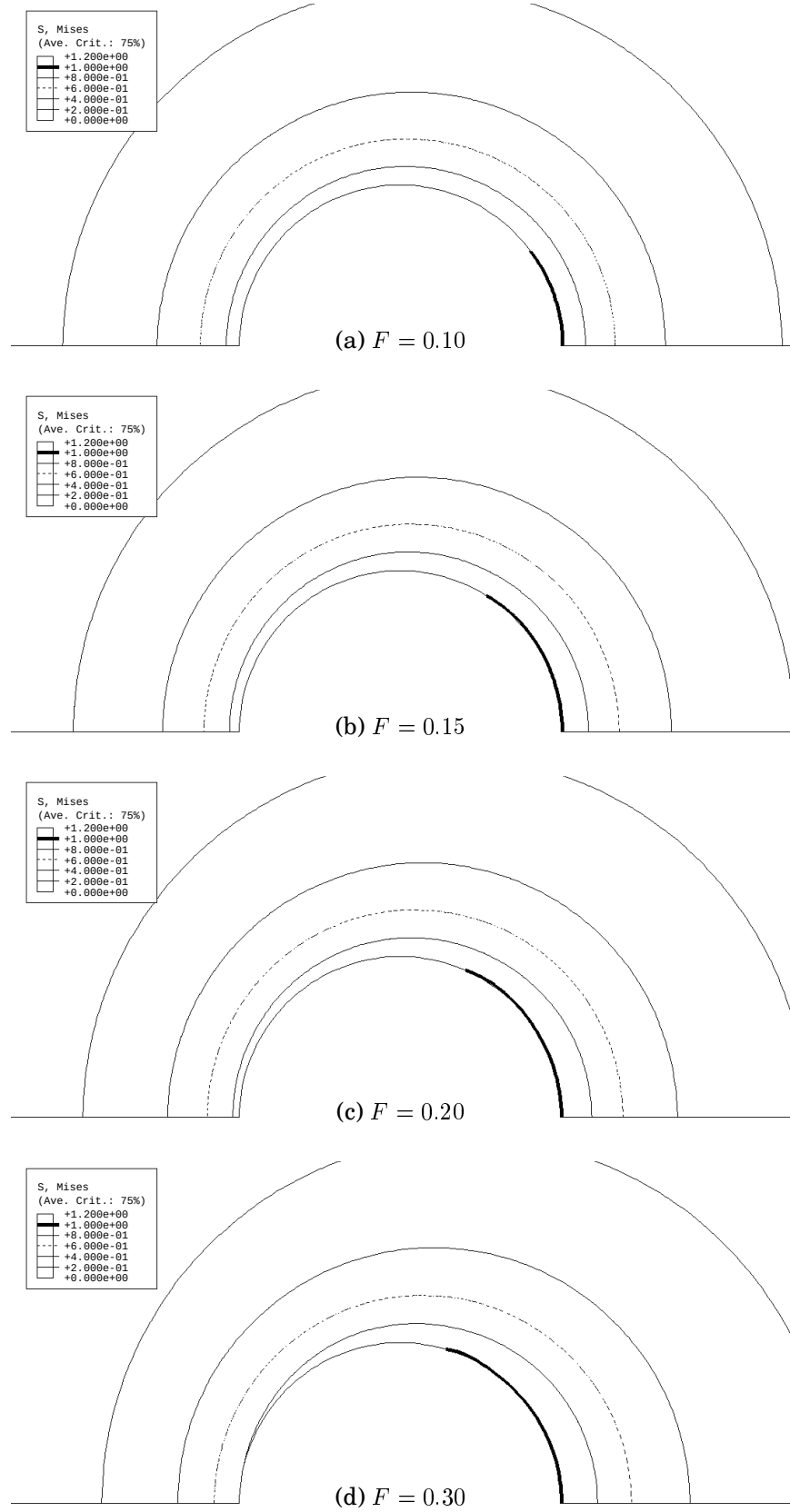


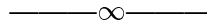
Figure 5.12: The peg problem solved by finite element methods using ABAQUS. Only the top half plane is analysed due to symmetry. A force F_{app} is applied in the positive x direction. Von Mises stress contours are plotted at intervals of 0.2. The contour $\sigma_{vM} = 0.6 \sigma_Y$ is dotted and the elastic-plastic boundary is the bold line.

5.3 Summary

We have presented two problems for which the influence function for a triangle in an infinite plane is valid.

The first attempts to solve the Kelvin problem for a point force in an infinite plane with plasticity were unsuccessful. A finite element model demonstrated that the lack of a length dimension in the problem causes the result to be mesh dependent. An attempt was made to circumvent this by introducing a plasticity exclusion region, where the material is similar to that in the plane in all respects except that it has an infinite yield stress. Results showing the development of the plastic zone at various increments revealed that the solution could only be achieved up to a certain point, this being determined by the shape of the plasticity exclusion region. It was concluded that the Kelvin solution incorporating plasticity is qualitatively identical to the elastic solution, with the exception that the stresses do nowhere exceed the yield criterion.

The more practical problem of a shrink-fit peg in an infinite plane subjected to a lateral force was then discussed. Solutions were presented which confirm that the area plasticity model produces similar output to that of a finite element model.



Chapter 6

Conclusions

The introductory chapters briefly explained the origins of fracture mechanics, and provided an overview of dislocation theory applicable to the modelling of cracks and crack-tip plasticity. The Nowell model (see chapter 3) employs dislocation theory with a one-dimensional plastic zone relevant under plane stress conditions. A comparison of the apparent and true stress intensity ranges, ΔK_{app} and ΔK_{true} , revealed some interesting features concerning both static and growing cracks. Nevertheless, ray-plasticity models have limitations as they model plasticity over a line rather than an area. Plastic flow over an area in plane strain was addressed in chapter 4, where a method employing triangular elements was developed. Simple tests showed that the assumptions concerning the activation of a single slip plane results in an over-constrained system. Therefore, a dual slip plane model was developed. The Prandtl-Reuss flow rule was used to derive an expression for a new parameter, β , which describes the ratio of the Burgers vector magnitudes on the appropriate slip planes. In chapter 5, this new method was applied to two problems to demonstrate the viability of the algorithm. A comparison with the finite element method validated the new model.

6.1 What has been achieved?

1-D ray-plasticity models

Research based on the one-dimensional plane stress ray-plasticity model due to Nowell has resulted in the following:

1. Nowell's model for plane stress has been extended to incorporate a strip under remote bending or tension. This enabled comparison with experimental work being performed by a colleague using moiré interferometry (see §3.2).
2. A comparison of the apparent and true stress intensity ranges, ΔK_{app} and ΔK_{true} , was performed for the case of a finite crack in a full plane subject to a wide range of remote tension loads. These results were presented for both static and growing cracks (§3.3). The motivation for this study was twofold. First, we wish to determine the load at which a growing crack first suffers a stress intensity (i.e. the point in the load cycle at which the crack becomes fully open) for all load combinations. Secondly, these data are then used to calculate the *true* stress intensity range suffered by the crack tip for all load combinations. It was found that:
 - (a) Under certain load conditions, a static crack was found to have $\Delta K_{true} > \Delta K_{app}$, i.e. the crack is subjected to a *more* severe stress intensity range than that predicted by LEFM analysis. This is counter-intuitive. Thus, under certain conditions, a static crack has a higher propensity to grow than is predicted by LEFM theory (see figure 3.14).
 - (b) For a growing crack, $\Delta K_{true} \leq \Delta K_{app}$. The true stress intensity range can be as little as 40% of the apparent stress intensity range, and this occurs within a stress range that is commonly used in engineering design. This result means that LEFM-based design codes may be unnecessarily conservative (see figure 3.17).

These two points imply that under certain conditions a short crack (where the effects of plasticity induced closure are small relative to other causes of closure – *cf.* chapter 1) will suffer a large true stress intensity range which will rapidly diminish as the crack grows through its own plastic zone (where the effects of plasticity induced closure *do* then become dominant), possibly leading to crack arrest. This behaviour has been noted experimentally and recorded on typical Paris law-type $\log(da/dn)$ vs $\log(\Delta K)$ plots [49].

3. A method to calculate $\Delta K_{true}^{GROWING}$, based only on ΔK_{app}^{STATIC} , has been demonstrated for a finite crack in an infinite plane under uniaxial tension, using figure 3.19. Since the apparent stress intensity range for a static crack is easily calculated using LEFM, engineers now have access to the sophisticated concept of the true stress intensity range for a growing crack without needing to perform complex calculations (see §3.3.1).

Discussion

The following observations have been made during the course of the work on one-dimensional models:

The Dugdale-type plastic zone is often used as a basis for plane stress models. A significant caveat is that as the direction of maximum shear stress remains constant (perpendicular to the length of the plastic ray), the associated flow rule is intrinsically obeyed and need not be enforced in the solution procedure.

The ray-plasticity model appears to emulate a crack in plane stress with a fair degree of success. The main limitation is that it cannot be used to determine the strain distributions within the plastic zone. However, plane stress conditions are not readily encountered in practical engineering conditions — a plane strain model would be far more useful.

Using rays to model plasticity in plane strain is more difficult. Early

models assumed that the direction of maximum shear stress was coincident with the direction of the plasticity ray, and that it remained so despite the redistribution of stresses. This is a violation of the associated flow rule, or normality condition.

The fact that all plasticity is confined to a line means that the yield condition is satisfied only along the plastic ray. However, the yield condition is violated in the area immediately surrounding the ray. Both the plane stress and plane strain implementations of the ray-plasticity model suffer from this flaw.

An improvement to the plane strain models can be made, whereby the direction of the Burgers vector is freed, and allowed to rotate so as to satisfy the normality condition. This reduces the size of the region in which the yield criterion is violated, but does not remove it completely.

Despite the elegance and relative simplicity of the one-dimensional ray-plasticity models, in either plane stress or plane strain, the violation of the yield parameter in the region surrounding the plasticity ray required that a method of distributing dislocations over an area be developed.

2-D area-plasticity models

The research on area plasticity models has achieved the following:

1. An area plasticity model (APM) has been developed for an infinite plane (see chapter 4). This method can readily be extended to incorporate other geometries. The stage is now set for research into the nature of the crack tip plastic zone, which controls the crack growth rate.
2. The influence functions have been solved analytically for a triangular element of piecewise constant distributed dislocation in an infinite plane, in order to provide a benchmark against which future work (which might involve numerical integration) may be measured (see §4.1.1).
3. The region in which the evaluation point lies relative to the triangular element was established. This is required in the implementation of the influence function to ensure that the correct limits are taken in the evaluation of the appropriate integrals (see §4.1.2).
4. It was shown that the dislocation influence functions had been derived and implemented correctly, by comparing the influence functions *external* to the triangle to that of the well-known discrete dislocation in an infinite plane. The verification of the *internal* distribution relied only upon the the correct external distribution (§4.1.3).
5. It was demonstrated that plasticity models using only a single slip plane result in an *over-constrained* problem. The use of a single slip plane is based on the misleading assumption that the direction of slip (a microscopic concept) coincides with the direction of material flow (a macroscopic concept). Choosing a single slip plane does, in fact, arbitrarily fix the direction of material flow (hence the over-constrained problem) and results in a violation of the associated flow rule. This occurs despite a “verification” that the Burgers vector

coincides with the direction of maximum shear stress. See §4.2.2, and figure 4.10.

6. A dual slip plane method has been developed, which allows the problem the necessary degrees of freedom. This required the introduction of a new parameter, β , which describes the ratio of the Burgers vector magnitude on one slip plane with respect to the other. The Prandtl-Reuss flow rule was used to derive an expression for β . Therefore, despite the fact that the Prandtl-Reuss flow rule is not associated with the Tresca criterion, we now have an algorithm which correctly implements the normality condition. See §4.2.3, and figure 4.12.
7. The APM was applied to two problems, and the results were compared to output from the finite element method. The agreement of these results means that this model can be developed as an alternative to the computationally intensive finite element method (see chapter 5).

Progress has been made towards the understanding of incremental plasticity models distributing plasticity over an area. The primary step forward has been the implementation of the Prandtl-Reuss flow rule, in accordance with classical plasticity models. It is no longer necessary simply to expect the plastic strain increments to occur in the correct direction — we can now *enforce* this as part of the solution procedure.

6.2 Further work

ΔK_{app} vs ΔK_{true} in alternate geometries

The technique presented in chapter 3 for the plane stress one-dimensional model was applicable to a finite crack in an infinite plane. However, since the kernels for a half plane and strip have already been implemented, everything is in place to obtain data for these more useful geometries, subjected to either remote tension or bending. Although this work could be easily executed, the time taken to run the model for 900 remote loads under each configuration will be significant. Collation of the data would also be time consuming.

Implementation of APM algorithm

During the course of the research, the method of implementing the area plasticity algorithm was radically overhauled and consciously replaced with a computationally *less* efficient implementation. This was done to facilitate the examination of stress and influence distributions across each individual element. Now, with the knowledge that a reliable method has been obtained, this implementation can be greatly improved upon, returning to a *very* efficient means of solving plasticity problems.

Initially, the collocation points (centroids of the triangles) and integration points (vertices of the triangles) were stored in one data structure, and a separate data structure was used as a look-up table in order to associate the correct points in the plane with a given triangular element. This method is efficient because the stress distribution is calculated once at each necessary point. For the purposes of plotting results or organising data, the look-up table is used to produce element-specific information.

Prior to discovering that the assumptions concerning slip systems were incorrect (see §4.2), much time was spent trying to debug the code for this area plasticity model. It was decided that the look-up method was a po-

tential source of error, and for this reason, the method was abandoned and a new element-oriented approach was adopted. This meant that the information for each element was stored independently, allowing the monitoring of stresses within each element with greater confidence. The drawback of this method is that it introduces a large degree of redundancy. For instance, consider a point in the plane which is the vertex of a triangle, as shown in figure 6.1. We see that point A is a vertex not only to trian-

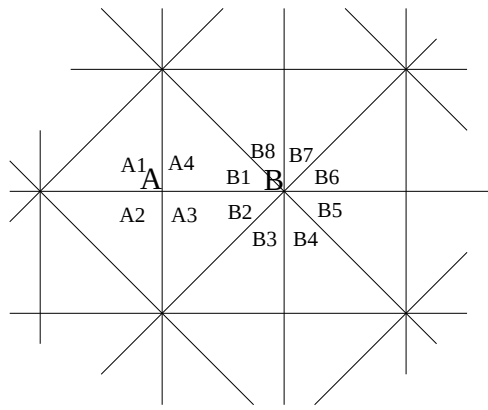


Figure 6.1: The use of an element-orientated implementation introduces a large degree of redundancy.

gle A1, but also to triangles A2, A3, and A4. This means that the elastic, dislocation, increment, and total stresses are being calculated four times when once would suffice. The situation at point B is even worse, since this point is a common vertex to eight triangles. Thus the decision to convert to an element-orientated implementation was not taken lightly, knowing that it would be a very inefficient implementation of what is in practice a highly efficient algorithm. MATLAB in particular is excellent at performing matrix-based calculations, which the former method enabled; turning to a element-oriented method required the adoption of a loop structure which should be avoided at all costs in efficient code.

Introduction of higher-order elements

The piecewise constant distribution of dislocation over an element could be replaced with a more complex formulation, such as linear, or perhaps quadratic, variation. This would result in a more accurate description of the plasticity within a given element. In order to achieve this, the derivation presented in appendix B would have to be re-worked, and then new expressions analogous to those in equations 4.8 would need to be derived. Mathematically, this would be a challenging exercise. The cost of pursuing such a course of action also needs to be offset against the fact that it *may* not be worthwhile, since a higher order element can easily be modelled by simply using a finer mesh in conjunction with lower order elements. It should also be borne in mind that elements of piecewise constant distributed dislocation, having strength $B_i = \partial b_i / \partial i$ per unit area, are in fact equivalent to *first order* displacement elements such as those employed in the finite element method.

Extension of the APM to other geometries

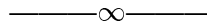
The immense flexibility of the dislocation method in modelling plasticity is due to the fact that the kernel used to determine the influence function in an infinite plane can be replaced with that of a half plane, or strip; or the effect of an inclusion. Such extensions would be preparatory work for an investigation of the behaviour of the crack tip plastic zone in more realistic geometries. The derivation of the influence functions for an element of piecewise constant distributed dislocations in a *half plane* is currently in progress. The use of numerical integration for these more complex geometries may be judicious. For instance, the current work could have been simplified by evaluating equations 4.5 numerically.

Comparison of APM with ray-plasticity models

Modelling plasticity using rays is far less computationally demanding than modelling plasticity over an area. The development of a model which incorporates area plasticity in the neighbourhood of a crack tip would permit an informative study into the trade-offs associated with the simpler line-plasticity models. The more accurate analysis of the extent of the plastic zone following overloads and underloads could also be investigated.

Quantification of small scale yielding

The possibilities listed above require the APM to be applied to modelling area plasticity in the neighbourhood of a crack, perhaps using a combination of the APM technique with that developed by Nowell. The ramifications of such a model would be immense: for the first time, an exact solution to the crack tip plasticity zone could be obtained. This would enable the discrepancies associated with small scale yielding assumptions to be determined unequivocally, for any geometry – a question which has remained unanswered since the introduction of LEFM in the post-war era.



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Appendix A

Kernel functions for discrete dislocations

Kernel equations for a full plane

The kernel equations for a dislocation in an infinite plane are:

$$\begin{aligned} G_{xxx} &= -\frac{y}{r^4}(3x^2 + y^2) \\ G_{xyy} &= +\frac{y}{r^4}(x^2 - y^2) \\ G_{xxy} &= +\frac{x}{r^4}(x^2 - y^2) \\ G_{yxx} &= +\frac{x}{r^4}(x^2 - y^2) \\ G_{yyy} &= +\frac{x}{r^4}(x^2 + 3y^2) \\ G_{yyx} &= +\frac{y}{r^4}(x^2 - y^2) \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} x &= (x - x_d) \\ y &= (y - y_d) \\ r^2 &= x^2 + y^2 \end{aligned}$$

with the dislocation sited at (x_d, y_d) (the integration point) and the point of inspection at (x, y) (the collocation point).

The kernel equations for an edge dislocation with Burgers vector $(0, 1)$ located at $(x_d, 0)$ with an image dislocation (for symmetry) at $(-x_d, 0)$ in a full plane are as follows:

$$\begin{aligned}
G_{yxx} &= \frac{-2(x-x_d)}{r_1^2} + \frac{4(x-x_d)^3}{r_1^4} + \frac{2(x+x_d)}{r_2^2} - \frac{4(x+x_d)^3}{r_2^4}, \\
G_{yyy} &= \frac{6(x-x_d)}{r_1^2} - \frac{4(x-x_d)^3}{r_1^4} - \frac{6(x+x_d)}{r_2^2} + \frac{4(x+x_d)^3}{r_2^4}, \\
G_{yxy} &= y \left(\frac{-2}{r_1^2} + \frac{4(x-x_d)^2}{r_1^4} + \frac{2}{r_2^2} - \frac{4(x+x_d)^2}{r_2^4} \right), \\
U_{yx} &= \left(\frac{\kappa-1}{2} \log[r_1] - \frac{(x-x_d)^2}{r_1^2} \right) - \\
&\quad \left(\frac{\kappa-1}{2} \log[r_2] - \frac{(x+x_d)^2}{r_2^2} \right), \\
U_{yy} &= \left(\frac{\kappa+1}{2} \arctan \left[\frac{x-x_d}{y} \right] - \frac{(x-x_d)y}{r_1^2} \right) - \\
&\quad \left(\frac{\kappa+1}{2} \arctan \left[\frac{x+x_d}{y} \right] - \frac{(x+x_d)y}{r_2^2} \right), \\
\frac{du_y}{dx} &= \frac{2(x-x_d)^2 y}{((x-x_d)^2 + y^2)^2} - \frac{y}{(x-x_d)^2 + y^2} - \frac{(1+\kappa)y}{2((x-x_d)^2 + y^2)} - \\
&\quad \frac{2(x+x_d)^2 y}{(x+x_d)^2 + y^2} + \frac{y}{(x+x_d)^2 + y^2} + \frac{(1+\kappa)y}{2((x+x_d)^2 + y^2)},
\end{aligned}$$

(A.2)

where $r_1 = \sqrt{(x-x_d)^2 + y^2}$,
 $r_2 = \sqrt{(x+x_d)^2 + y^2}$.

Kernel equations for a half plane

The kernel equations for an edge dislocation with Burgers vector $(0, 1)$ located at $(x_d, 0)$ with an image dislocation (for symmetry) at $(-x_d, 0)$ in a half plane are as follows:

$$\begin{aligned}
 G_{yxx} &= \frac{-2(x-x_d)}{r_1^2} + \frac{4(x-x_d)^3}{r_1^4} + \frac{2(x+x_d)}{r_2^2} - \frac{4(x+x_d)^3}{r_2^4} - \\
 &\quad \frac{4x_d}{r_2^2} - \frac{16x_d(x+x_d)^2}{r_2^4} + \frac{32x_d(x+x_d)^4}{r_2^6} + \frac{24x_d^2(x+x_d)}{r_2^4} - \frac{32x_d^2(x+x_d)^3}{r_2^6}, \\
 G_{yyy} &= \frac{6(x-x_d)}{r_1^2} - \frac{4(x-x_d)^3}{r_1^4} - \frac{6(x+x_d)}{r_2^2} + \frac{4(x+x_d)^3}{r_2^4} - \\
 &\quad \frac{4x_d}{r_2^2} + \frac{32x_d(x+x_d)^2}{r_2^4} - \frac{32x_d(x+x_d)^4}{r_2^6} - \frac{24x_d^2(x+x_d)}{r_2^4} + \frac{32x_d^2(x+x_d)^3}{r_2^6}, \\
 G_{yxy} &= y \left(\frac{-2}{r_1^2} + \frac{4(x-x_d)^2}{r_1^4} + \frac{2}{r_2^2} - \frac{4(x+x_d)^2}{r_2^4} - \frac{8x_d(x+x_d)}{r_2^4} \right) + \\
 &\quad y \left(\frac{32x_d(x+x_d)^3}{r_2^6} + \frac{8x_d^2}{r_2^4} - \frac{32x_d^2(x+x_d)^2}{r_2^6} \right), \\
 U_{yx} &= \left(\frac{\kappa-1}{2} \log[r_1] - \frac{(x-x_d)^2}{r_1^2} \right) - \\
 &\quad \left(\frac{\kappa-1}{2} \log[r_2] - \frac{4x_d(x+x_d)^2x}{r_2^4} - \frac{2x_d^2 + (\kappa-1)(x+x_d)x_d - (x+x_d)^2}{r_2^2} \right), \\
 U_{yy} &= \left(\frac{\kappa+1}{2} \arctan \left[\frac{x-x_d}{y} \right] - \frac{(x-x_d)y}{r_1^2} \right) - \\
 &\quad \left(\frac{\kappa+1}{2} \arctan \left[\frac{x+x_d}{y} \right] - \frac{4x_d(x+x_d)xy}{r_2^4} + \frac{y(\kappa x_d + x)}{r_2^2} \right), \\
 \frac{du_y}{dx} &= \frac{y}{2} \left(\frac{4(-x+x_d)^2}{((x_d-x)^2+y^2)^2} - \frac{3+\kappa}{(x_d-x)^2+y^2} \right) + \\
 &\quad \frac{y}{2} \left(\frac{32x_dx(x_d+x)^2}{((x_d+x)^2+y^2)^3} - \frac{4(x_d^2(2+\kappa) + x_d(5+\kappa)x + x^2)}{((x_d+x)^2+y^2)^2} + \frac{3+\kappa}{(x_d+x)^2+y^2} \right),
 \end{aligned} \tag{A.3}$$

where $r_1 = \sqrt{(x-x_d)^2 + y^2}$,
 $r_2 = \sqrt{(x+x_d)^2 + y^2}$.

Kernel equation for a strip

The G_{yyy} kernel for an edge dislocation with Burgers vector $(0, 1)$ located at $(x_d, 0)$ with an image dislocation (for symmetry) at $(-x_d, 0)$ in a half plane are as follows [50]¹ (h is the depth of the beam):

$$G_{yyy} = \frac{1}{x_d - x} + g(x, x_d) + g(x - h, x_d - h) + k_{fl}(x, x_d) \quad (\text{A.4})$$

where

$$g(x, x_d) = -\frac{1}{x + x_d} + \frac{6x}{(x + x_d)^2} - \frac{4x^2}{(x + x_d)^3} \quad (\text{A.5})$$

and

$$k_{fl}(x, x_d) = \int_0^\infty \frac{1}{D(\alpha)} \left[R_1(x, x_d, \alpha) + R_2(x, x_d, \alpha) - R_1(h - x, h - x_d, \alpha) - R_2(h - x, h - x_d, \alpha) \right] d\alpha \quad (\text{A.6})$$

$$R_1(x, x_d, \alpha) = \left[\left(4\alpha^2 h^2 - e^{-2\alpha h} \right) \left(2 - 3\alpha x_d - \alpha x + 2\alpha^2 x x_d \right) + 2 - 3\alpha x_d - \alpha x + 2\alpha^2 x x_d - 4\alpha^2 h^2 + 2\alpha^2 x_d h + 2\alpha^2 x h - 4\alpha h \right] e^{-\alpha(x+x_d)} \quad (\text{A.7})$$

$$R_2(x, x_d, \alpha) = \left[\left(1 - e^{-2\alpha h} \right) (\alpha x - \alpha x_d - 2) - 2\alpha h (\alpha x - 2 + 2\alpha^2 x x_d - 2\alpha^2 x h - 3\alpha x_d + 3\alpha h) \right] e^{\alpha(x_d-x)} \quad (\text{A.8})$$

$$D(\alpha) = e^{2\alpha h} + e^{-2\alpha h} - 4\alpha^2 h^2 - 2 \quad (\text{A.9})$$

¹Note that there is a typographical error in [50] equation 21, which has been corrected here.

Appendix B

Kernel functions for a triangle of dislocation

Stress due to a discrete dislocation in an infinite plane

The stresses due to a discrete edge dislocation in an infinite plane can be written as [45]:

$$\begin{aligned}\sigma_{xx}^1 + \sigma_{yy}^1 &= 2 \left[\phi(z - z_d) + \overline{\phi(z - z_d)} \right] = 4\Re [\phi(z - z_d)] \\ \sigma_{yy}^1 - \sigma_{xx}^1 + 2i\sigma_{xy}^1 &= 2 \left[(\bar{z} - \bar{z}_d)\phi'(z - z_d) + \psi(z - z_d) \right]\end{aligned}\quad (\text{B.1})$$

where $\phi(z - z_d)$ and $\psi(z - z_d)$ are the Muskhelishvili [26] potentials:

$$\begin{aligned}\phi(z - z_d) &= \frac{\mu}{\pi i(\kappa + 1)} \frac{b_x + ib_y}{z - z_d} \\ \phi'(z - z_d) &= \frac{\partial \phi}{\partial z} = -\frac{\mu}{\pi i(\kappa + 1)} \frac{b_x + ib_y}{(z - z_d)^2} \\ \psi(z - z_d) &= -\frac{\mu}{\pi i(\kappa + 1)} \frac{b_x - ib_y}{z - z_d} = \bar{\phi}(z - z_d) = -\frac{b_x - ib_y}{b_x + ib_y} \phi(z - z_d)\end{aligned}$$

with $z_d = x_d + iy_d$ is the integration point,

$z = x + iy$ is the collocation point.

Distributing dislocations over a region S

We now replace $b_x + ib_y$ with $B_x + iB_y$ as strengths of dislocations per unit area ($B = \frac{\partial b}{\partial S}$). Then the stress field due to this distribution is:

$$\sigma_{ij}(x, y) = \iint_S \sigma_{ij}^1 dS$$

Thus B.1 gives

$$\begin{aligned} \sigma_{xx} + \sigma_{yy} &= 4\Re \iint_S \phi(z - z_d) dS \\ \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} &= 2 \iint_S \left[(\bar{z} - \bar{z}_d) \phi'(z - z_d) - \frac{B_x - iB_y}{B_x + iB_y} \phi(z - z_d) \right] dS \end{aligned}$$

Now put

$$\begin{aligned} \Phi(z) &= \iint_S \phi(z - z_d) dS \\ \Omega(z) &= \iint_S (\bar{z} - \bar{z}_d) \phi'(z - z_d) dS \end{aligned} \quad (\text{B.2})$$

Therefore

$$\begin{aligned} \sigma_{xx} + \sigma_{yy} &= 4\Re [\Phi(z)] \\ \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} &= 2 \left[\Omega(z) - \frac{B_x - iB_y}{B_x + iB_y} \Phi(z) \right] \end{aligned} \quad (\text{B.3})$$

We thus need to evaluate two complex integrals (B.2) in order to determine the full stress field.

Evaluating the integrals when S is chosen to be a triangle

We introduce polar coordinates with z as the origin: $(z_d - z) = re^{i\theta}$. Then $dS = r dr d\theta$.

$$\begin{aligned} \Phi(z) &= \iint_S \phi(z - z_d) dS \\ &= \frac{\mu(B_x + iB_y)}{\pi i(\kappa + 1)} \iint_S \frac{1}{z - z_d} dS \\ &= -\frac{\mu(B_x + iB_y)}{\pi i(\kappa + 1)} \iint_S \frac{1}{re^{i\theta}} r dr d\theta \\ &= -\frac{\mu(B_x + iB_y)}{\pi i(\kappa + 1)} \iint_S e^{-i\theta} dr d\theta \\ &= -\frac{\mu(B_x + iB_y)}{\pi i(\kappa + 1)} \int re^{-i\theta} d\theta \\ &= -\frac{\mu(B_x + iB_y)}{\pi i(\kappa + 1)} I_1 \end{aligned} \quad (\text{B.4})$$

where $I_1 = \int r e^{-i\theta} d\theta$.

Turning our attention to Ω :

$$\begin{aligned}
 \Omega(z) &= -\frac{\mu(B_x + iB_y)}{\pi i(\kappa + 1)} \iint_S \frac{\bar{z} - \bar{z}_d}{(z - z_d)^2} dS \\
 &= \frac{\mu(B_x + iB_y)}{\pi i(\kappa + 1)} \iint_S \frac{r e^{-i\theta}}{r^2 e^{2i\theta}} r dr d\theta \\
 &= \frac{\mu(B_x + iB_y)}{\pi i(\kappa + 1)} \iint_S e^{-3i\theta} dr d\theta \\
 &= \frac{\mu(B_x + iB_y)}{\pi i(\kappa + 1)} \int r e^{-3i\theta} d\theta \\
 &= \frac{\mu(B_x + iB_y)}{\pi i(\kappa + 1)} I_2
 \end{aligned} \tag{B.5}$$

where $I_2 = \int r e^{-3i\theta} d\theta$.

Now the distance from z to one of the lines $y_t - y = m(x_t - x) + c$ making up the triangle is given by

$$\begin{aligned}
 r^2 &= (x_t - x)^2 + (y_t - y)^2 \\
 &= (x_t - x)^2 + (m(x_t - x) + c)^2 \\
 &= (1 + m^2)(x_t - x)^2 + 2mc(x_t - x) + c^2 \\
 &= (1 + m^2)(r^2 \cos^2 \theta + 2mcr \cos \theta + c^2)
 \end{aligned}$$

Thus r is given by the positive solution of the quadratic equation:

$$(m^2 \cos^2 \theta - \sin^2 \theta)r^2 + 2mcr \cos \theta + c^2 = 0$$

$$\therefore r = \frac{c}{\sin \theta - m \cos \theta}$$

We will use $m = \tan \gamma = \frac{\sin \gamma}{\cos \gamma}$:

$$\begin{aligned}
 I_1 &= \int r e^{-i\theta} d\theta \\
 &= \int \frac{c e^{-i\theta}}{\sin \theta - m \cos \theta} d\theta \\
 &= \int \frac{c \cos \gamma e^{-i\theta}}{\sin \theta \cos \gamma - \cos \theta \sin \gamma} d\theta \\
 &= c \cos \gamma \int \frac{e^{-i\theta}}{\sin(\theta - \gamma)} d\theta \\
 &= c \cos \gamma e^{-i\gamma} \int \frac{e^{-i(\theta - \gamma)}}{\sin(\theta - \gamma)} d\theta \\
 &= c \cos \gamma e^{-i\gamma} \int \frac{e^{-i\alpha}}{\sin \alpha} d\alpha
 \end{aligned}$$

where $\alpha = \theta - \gamma$. Now using Euler

$$\frac{e^{-i\alpha}}{\sin \alpha} = \frac{\cos \alpha - i \sin \alpha}{\sin \alpha} = \cot \alpha - i$$

$$\begin{aligned}
 \therefore \quad & c \cos \gamma e^{-i\gamma} \int (\cot \alpha - i) d\alpha \\
 & = c \cos \gamma e^{-i\gamma} [\ln(\sin \alpha) - i\alpha] \\
 \therefore \quad I_1 & = c \cos \gamma e^{-i\gamma} \left[\ln(\sin(\theta - \gamma)) - i(\theta - \gamma) \right]_{\theta_1}^{\theta_2} \quad (\text{B.6})
 \end{aligned}$$

The method is repeated for integral I_2 , giving:

$$I_2 = c \cos \gamma e^{-3i\gamma} \int \frac{e^{-3i\alpha}}{\sin \alpha} d\alpha$$

But

$$\begin{aligned}
 \frac{e^{-3i\alpha}}{\sin \alpha} & = \frac{\cos 3\alpha - i \sin 3\alpha}{\sin \alpha} \\
 & = \frac{\cos \alpha \cos 2\alpha}{\sin \alpha} - \sin 2\alpha - i(2 \cos 2\alpha - 1)
 \end{aligned}$$

Therefore

$$\begin{aligned}
 I_2 & = c \cos \gamma e^{-3i\gamma} \times \\
 & \quad \left[\ln(\sin(\theta - \gamma)) + \cos(2(\theta - \gamma)) - i(\sin(2(\theta - \gamma)) + (\theta - \gamma)) \right]_{\theta_1}^{\theta_2} \quad (\text{B.7})
 \end{aligned}$$

Appendix C

Source code

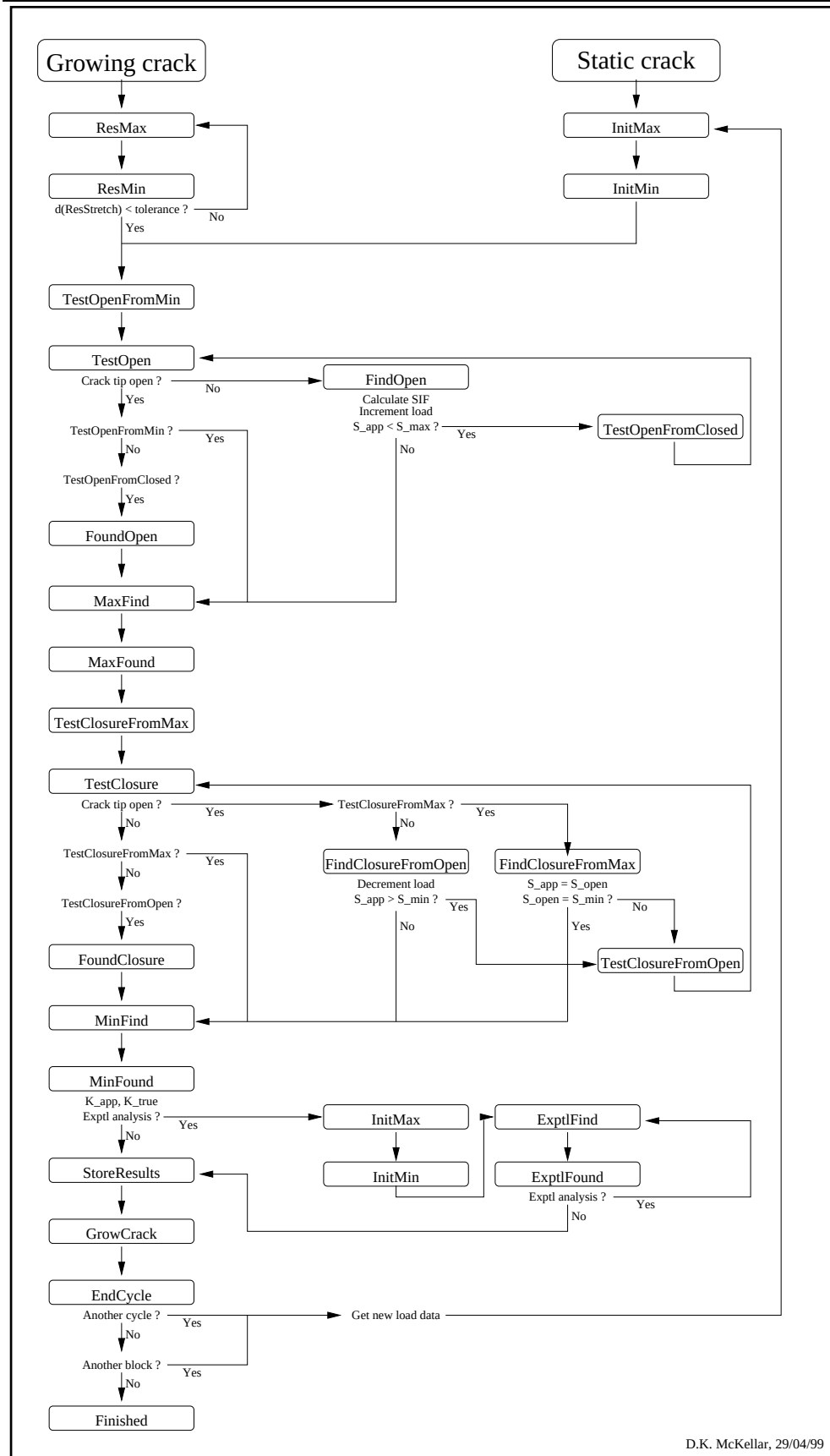
The CD-ROM¹ attached on the inside back cover contains source code written during the course of this research. The bulk of the programs are written in MATLAB v5.3; some are in C. All APM files run on a Linux platform; the QP files run on a UNIX platform. The files for the area plasticity model (APM), Nowell's model (QuadPro), some results, and a PostScript copy of this thesis are listed in table C.1.

The flow diagrams for the Nowell model (figure 3.4) and the APM (figure 4.14) are repeated from the text for convenience.

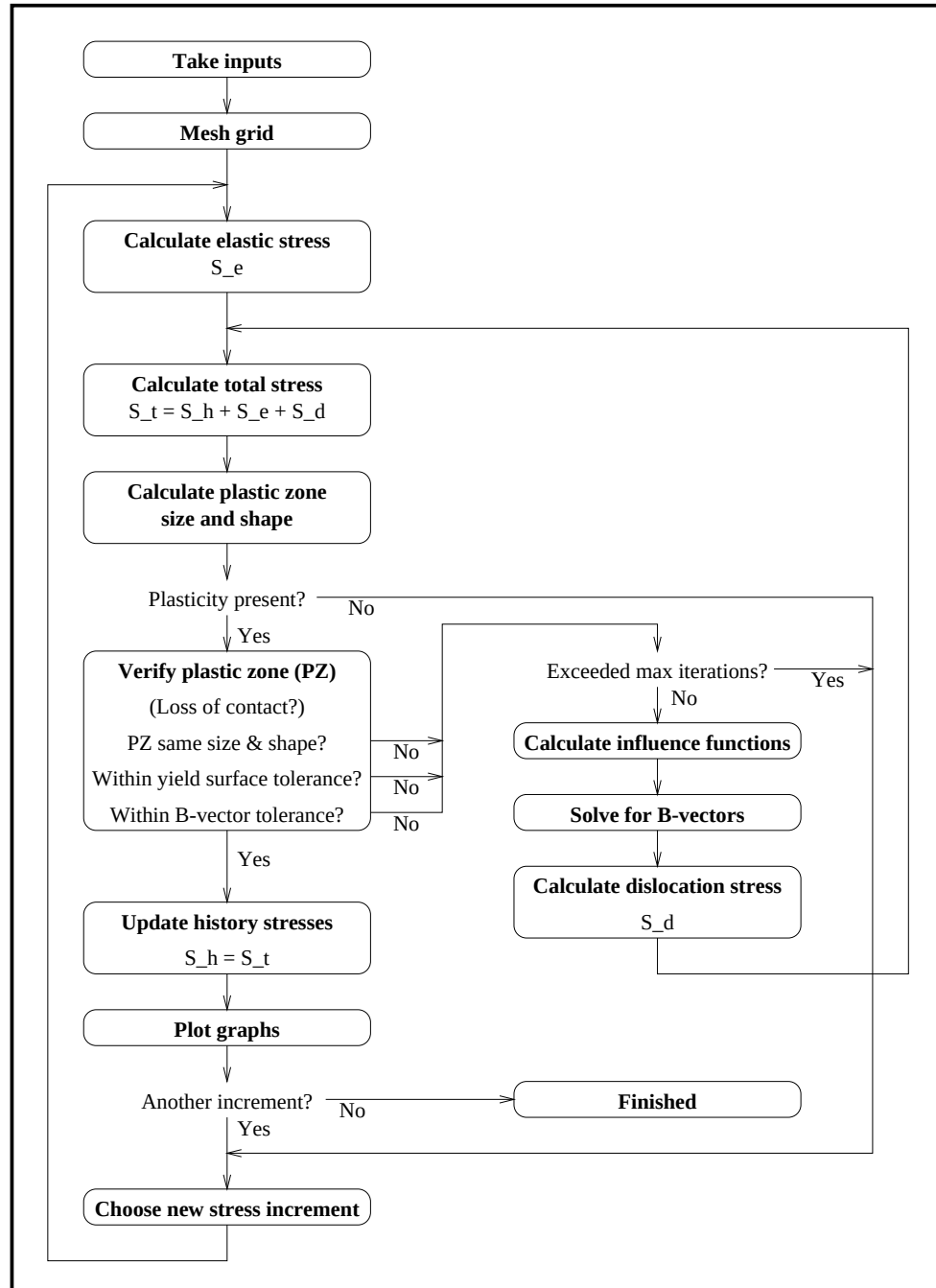
¹The CD-ROM was burnt using Windows98.

Directory	Filename
APM	APM.m Combine.m Dislocation_DUAL_SLIP_PLANES.m Dislocation_SINGLE_SLIP_PLANE.m Display_plots.m Elastic_Hole.m Elastic_Kelvin.m Elastic_Shrink_Fit_Peg.m Elastic_Tension.m Exclusion.m Influence.m Initialise.m Inputs.m Mesh.m PZone.m Plot.m Principal_Stresses.m Rotate_Axes.m Solve_DUAL_SLIP_PLANES.m Solve_SINGLE_SLIP_PLANE.m Tresca_vonMises.m Tri_IFn.m Triangle.m Update.m Verify.m tricont.m
QP	QPLoadData.m QPPlot.m QPSolver.m QP_Analysis.m QP_CTOD_v_dSapp.m QP_S_v.t.m QP_bat.m QP_dB0_v.t.m QuadPro.m SIF.m
QP_Results	Static_Data.tar.gz Growing_Data.tar.gz
Thesis	DKM_PhD.ps DKM_PhD_images.ps.tar.gz

Table C.1: Table of selected files on attached CD-ROM.



Detailed flow chart of algorithm implemented for Nowell model (see figure 3.4 for less detailed version).



Algorithm implemented for the area plasticity model (APM).