Weakly deformable poroelastic particle
in an unbounded Stokes flow
Supplementary materials

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I. ASYMPTOTIC REDUCTION OF THE NON-LINEAR PROBLEM

A. Governing equations

We again work in a frame of reference which translates with the poroelastic particle, though now assume the particle may exhibit large deformations. The dimensional governing equations for the external Stokes flow are

$$\nabla \cdot \boldsymbol{\Sigma} = \mathbf{0}, \tag{I.1}$$

$$\nabla \cdot \boldsymbol{V} = 0, \tag{I.2}$$

where $\boldsymbol{V}$ is the fluid velocity and $\boldsymbol{\Sigma}$ is the stress tensor for a Newtonian viscous fluid given by

$$\boldsymbol{\Sigma} = -P\boldsymbol{I} + \mu_f \left[\nabla \boldsymbol{V} + (\nabla \boldsymbol{V})^T\right], \tag{I.3}$$

where P is the fluid pressure and μ_f is the fluid viscosity. The superscript T denotes the transpose. The net force $\boldsymbol{F}_{\text{force}}$ and torque $\boldsymbol{T}$ exerted on the particle by the surrounding fluid are given by the integrals

$$\boldsymbol{F}_{\text{force}} = \oiint_S \boldsymbol{\Sigma} \cdot \boldsymbol{n} \, \mathrm{d}S = \mathbf{0}, \tag{I.4}$$

$$\boldsymbol{T} = \oiint_S \boldsymbol{r} \times (\boldsymbol{\Sigma} \cdot \boldsymbol{n}) \, \mathrm{d}S = \mathbf{0}, \tag{I.5}$$

where we use the subscript 'force' to distinguish from the deformation gradient tensors presented below.

For the poroelastic particle, we write all non-linear equations for the particle using an Eulerian framework, similar to the presentation in MacMinn *et al.* [1] as this facilitates the coupling to the external Newtonian fluid. To account for the rotation of the particle, we define $\boldsymbol{X}(\boldsymbol{y}, t)$ as the position of a material point in the undeformed configuration, $\boldsymbol{y}(\boldsymbol{x})$ as the position of this point in a frame which rotates with the angular velocity $\boldsymbol{\Omega}$, and $\boldsymbol{x}$ as the current position of this point. We define the deformation gradient tensors,

$$\boldsymbol{F}_0 = \left(\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{y}}\right)^{-1}, \quad \text{and} \quad \boldsymbol{F}_1 = \left(\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\right)^{-1}. \tag{I.6}$$

Using the chain rule the total deformation gradient tensor

$$\boldsymbol{F} = \left(\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}}\right)^{-1} = \boldsymbol{F}_1 \boldsymbol{F}_0. \tag{I.7}$$

To be clear, $\boldsymbol{F}_0(\boldsymbol{y}, t)$ maps from the undeformed configuration to the rigid-body rotating state, then $\boldsymbol{F}_1(\boldsymbol{x})$ captures the steady mapping from the rotating state to the current position. Since we focus on capturing the shape change of the particle we define the particle deformation as

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{y}(\boldsymbol{x}), \tag{I.8}$$

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where we have used $\boldsymbol{y}(\boldsymbol{x})$ as the reference state.

Since $\boldsymbol{F}_0$ describes a rigid-body rotation, we have

$$\det \boldsymbol{F}_0 = 1 \quad \text{and} \quad \boldsymbol{F}_0 \boldsymbol{F}_0^T = \boldsymbol{I}. \tag{I.9}$$

Using (I.9), we define the left Cauchy-Green deformation tensor

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}} = \boldsymbol{F}_1 \boldsymbol{F}_0 \boldsymbol{F}_0^T \boldsymbol{F}_1^T = \boldsymbol{F}_1 \boldsymbol{F}_1^T, \tag{I.10}$$

and the Jacobian

$$J(\boldsymbol{x}) = \det(\boldsymbol{F}) = \det(\boldsymbol{F}_1) = \frac{1}{\det(\boldsymbol{F}_1^{-1})}, \tag{I.11}$$

which measures the local volume change within the particle. Here, $\boldsymbol{F}_0$ does not appear in (I.10) and (I.11) as no strain or volume change is generated by the rigid-body rotation. The Jacobian is directly related to the porosity via

$$J(\boldsymbol{x}) = \frac{1 - \phi_{f,0}}{1 - \phi_f(\boldsymbol{x})}. \tag{I.12}$$

We assume that the internal flow is governed by Darcy's law,

$$\phi_f(\boldsymbol{v}_f - \boldsymbol{v}_s) = -\frac{\kappa(\phi_f)}{\mu_f} \nabla p_f, \tag{I.13}$$

where $\boldsymbol{v}_f$ and $\boldsymbol{v}_s$ are the internal fluid and solid velocities measured in the translating frame (undeformed configuration) and, assuming finite deformations, the permeability $\kappa = \kappa(\phi_f)$ is non-uniform. In general the permeability will change as the solid skeleton deforms, which is captured through the dependence on porosity. It will be useful to define $\kappa_0 = \kappa(\phi_{f,0})$ as the uniform permeability field associated with the undeformed porosity.

We define the solid velocity implicitly through the velocity gradient tensor

$$\boldsymbol{L}_s = \nabla \boldsymbol{v}_s = \dot{\boldsymbol{F}} \boldsymbol{F}^{-1}. \tag{I.14}$$

By writing the solid velocity via the velocity gradient tensor, we may separate the contributions from the rigid body rotation described by $\boldsymbol{F}_0$ and the steady deformation map $\boldsymbol{F}_1$. Substituting (I.7) into (I.14) we have

$$\boldsymbol{L}_s = (\dot{\boldsymbol{F}}_1 \boldsymbol{F}_0 + \boldsymbol{F}_1 \dot{\boldsymbol{F}}_0) \boldsymbol{F}_0^T \boldsymbol{F}_1^{-1} = \dot{\boldsymbol{F}}_1 \boldsymbol{F}_1^{-1} + \boldsymbol{F}_1 \boldsymbol{W}_0 \boldsymbol{F}_1^{-1} = \boldsymbol{F}_1 \boldsymbol{W}_0 \boldsymbol{F}_1^{-1}, \tag{I.15}$$

where the superscript $\cdot = \mathrm{d}/\mathrm{d}t$ and $\boldsymbol{W}_0 = \dot{\boldsymbol{F}}_0 \boldsymbol{F}_0^{-1} = \boldsymbol{\Omega} \times \boldsymbol{I}$ is the spin tensor which describes the angular velocity of the rotating state and $\dot{\boldsymbol{F}}_1 = 0$ due to the steady assumption [2].

Imposing conservation of mass on both the fluid and solid phases within the particle we have

$$\nabla \cdot (\phi_f \boldsymbol{v}_f) = 0, \quad \nabla \cdot ((1 - \phi_f) \boldsymbol{v}_s) = 0, \tag{I.16}$$

such that $\nabla \cdot \boldsymbol{v}_f$ and $\nabla \cdot \boldsymbol{v}_s$ are in general non-zero. Adding the equations in (I.16) we define the total internal fluid flux

$$\boldsymbol{q} = \phi_f \boldsymbol{v}_f + (1 - \phi_f) \boldsymbol{v}_s \tag{I.17}$$

which has zero divergence.

To describe how the fluid and solid phases share internal stresses we write the total stress,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - p_f \boldsymbol{I}, \tag{I.18}$$

where $\boldsymbol{\sigma}'$ is Terzaghi's effective stress which is the force per unit area carried by the solid skeleton. Details of this stress decomposition may be found in [1]. Imposing mechanical equilibrium within the particle we have

$$\nabla \cdot \boldsymbol{\sigma} = \nabla \cdot \boldsymbol{\sigma}' - \nabla p_f = \boldsymbol{0}. \tag{I.19}$$

We adopt a compressible neo-Hookean model for the particle which will reduce to linear elasticity under the assumption of small strain,

$$\boldsymbol{\sigma}' = \lambda(J - 1)\boldsymbol{I} + \frac{\mu_s}{J} (\boldsymbol{B} - \boldsymbol{I}), \tag{I.20}$$

where λ is Lamé's first parameter, μ_s is the shear modulus, and $\frac{1}{2}(\boldsymbol{B} - \boldsymbol{I})$ is the solid strain.

B. Boundary conditions

Far from the particle we expect any disturbance to the external flow caused by the particle to be small such that

$$\boldsymbol{V} - \boldsymbol{V}_\infty - \boldsymbol{V}_{tr} \to 0 \quad \text{as} \quad r \to \infty, \tag{I.21}$$

where $\boldsymbol{V}_\infty$ and $\boldsymbol{V}_{tr}$ are the external far-field flow and particle's constant translational velocity in the lab frame, respectively.

We define the deformed particle surface S as

$$\boldsymbol{r} = \boldsymbol{r}_s(\theta, \phi) = R_0 \boldsymbol{e}_r + \boldsymbol{u}(r = R_0). \tag{I.22}$$

On the deformed particle interface S we impose the continuity of normal fluid flux, continuity of total stress, continuity of fluid stresses and the Beavers & Joseph slip condition [3],

$$\boldsymbol{V} \cdot \boldsymbol{n} = \boldsymbol{q} \cdot \boldsymbol{n}, \tag{I.23}$$

$$\boldsymbol{\Sigma} \cdot \boldsymbol{n} = \boldsymbol{\sigma} \cdot \boldsymbol{n}, \tag{I.24}$$

$$\boldsymbol{n} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{n} = -p_f, \tag{I.25}$$

$$\boldsymbol{t}^i \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{n} = \frac{\gamma \mu_f}{\sqrt{\kappa}}((\boldsymbol{V} - \boldsymbol{q}) \cdot \boldsymbol{t}^i), \tag{I.26}$$

where γ is defined as the slip parameter such that $\gamma = 0$ corresponds to a surface with perfect slip and $\gamma \to \infty$ corresponds to a surface with no-slip.

C. Non-dimensionalisation

We non-dimensionalise lengths with the particle radius R_0, velocities with the velocity scale V_0 which is determined by the magnitude of the external far-field flow in (I.21), and the fluid pressure and stress with the viscous pressure scaling $\mu_f V_0 / R_0$. Additionally, the permeability is scaled on κ_0. The stress condition (I.24) states that the total stress on the particle must balance with the viscous stress exerted by the external flow at the interface. Hence, the total stress, elastic stress, and pore pressure are also scaled on $\mu_f V_0 / R_0$. We define the dimensionless parameter

$$\epsilon = \frac{\mu_f V_0}{\mu_s R_0}, \tag{I.27}$$

which is the ratio of typical viscous stresses in the external flow to the elastic stiffness of the solid skeleton. From the constitutive relation (I.20) we see that ϵ characterises the magnitude of the solid strain. It follows from the strain tensor that the solid displacement is non-dimensionalised on ϵR_0.

Applying the above non-dimensionalisations, the governing equations for the external Stokes flow are given by

$$\nabla \cdot \boldsymbol{\Sigma} = \boldsymbol{0}, \tag{I.28}$$

$$\nabla \cdot \boldsymbol{V} = 0, \tag{I.29}$$

$$\boldsymbol{\Sigma} = -P\boldsymbol{I} + \nabla \boldsymbol{V} + (\nabla \boldsymbol{V})^T, \tag{I.30}$$

$$\boldsymbol{F}_{\text{force}} = \boldsymbol{T} = \boldsymbol{0}, \tag{I.31}$$

where $\boldsymbol{F}_{\text{force}}$ and $\boldsymbol{T}$ are calculated by non-dimensionalising (I.4) and (I.5).

The non-linear governing equations in the particle are

$$\phi_f(\boldsymbol{v}_f - \boldsymbol{v}_s) = -\tilde{\kappa}_0 \kappa(\phi_f) \nabla p_f, \tag{I.32}$$

where $\boldsymbol{v}_s$ is given through

$$\boldsymbol{L}_s = \nabla \boldsymbol{v}_s = \boldsymbol{F}_1 \boldsymbol{W}_0 \boldsymbol{F}_1^{-1} \tag{I.33}$$

and $\tilde{\kappa}_0 = \kappa_0/R_0^2$ is the Darcy number, which represents the particle permeability in reference to its cross-sectional area,

$$\nabla \cdot (\phi_f \boldsymbol{v}_f) = 0, \quad \nabla \cdot ((1-\phi_f)\boldsymbol{v}_s) = 0, \tag{I.34}$$

$$\nabla \cdot \boldsymbol{\sigma}' - \nabla p_f = \boldsymbol{0}, \tag{I.35}$$

$$\boldsymbol{\sigma}' = \frac{1}{\epsilon}\frac{2\nu}{1-2\nu}(J-1)\boldsymbol{I} + \frac{1}{\epsilon}(\boldsymbol{B}-\boldsymbol{I}), \tag{I.36}$$

where Poisson's ratio, $\nu = \lambda/(2(\lambda+\mu_s))$, measures the magnitude of transverse strain induced in the particle when deformed and

$$\boldsymbol{F}_1 = (\boldsymbol{I} - \epsilon\nabla\boldsymbol{u})^{-1}. \tag{I.37}$$

The far-field condition for the external Stokes flow is,

$$\boldsymbol{V} - \boldsymbol{V}_\infty - \boldsymbol{V}_{tr} \to 0 \quad \text{as} \quad r \to \infty. \tag{I.38}$$

The deformed particle surface S is implicitly determined by the equations

$$\boldsymbol{r}_s = \boldsymbol{e}_r + \epsilon\boldsymbol{u}(r=1). \tag{I.39}$$

We then have the dimensionless interfacial conditions

$$\boldsymbol{V} \cdot \boldsymbol{n} = \boldsymbol{q} \cdot \boldsymbol{n}, \tag{I.40}$$

$$\boldsymbol{\Sigma} \cdot \boldsymbol{n} = \boldsymbol{\sigma} \cdot \boldsymbol{n}, \tag{I.41}$$

$$\boldsymbol{n} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{n} = -p_f, \tag{I.42}$$

$$\boldsymbol{t}^i \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{n} = \frac{\tilde{\gamma}}{\sqrt{\kappa(\phi_f)}}\left(\boldsymbol{V} - \phi_f \boldsymbol{v}_f\right) \cdot \boldsymbol{t}^i, \tag{I.43}$$

where $\tilde{\gamma} = \gamma/\sqrt{\tilde{\kappa}_0}$.

D. Asymptotic reduction

We now assume $\epsilon \ll 1$, such that the particle strain and deformation are small, and expand the variables, $\boldsymbol{V}$, P, $\boldsymbol{F}_{\text{force}}$, $\boldsymbol{T}$, $\boldsymbol{\Sigma}$, ϕ_f, $\boldsymbol{v}_f$, $\boldsymbol{v}_s$, κ, p_f, $\boldsymbol{u}$ and $\boldsymbol{\sigma}'$ in powers of ϵ, e.g.

$$\boldsymbol{V} = \boldsymbol{V}^{(0)} + \epsilon\boldsymbol{V}^{(1)} + \cdots, \tag{I.44}$$

where the superscript (i) denotes the associated power of ϵ. We similarly expand the position of the deformed particle surface as $\boldsymbol{r}_s = \boldsymbol{r}_s^{(0)} + \epsilon\boldsymbol{r}_s^{(1)} + \cdots$. We proceed by assuming $\phi_{f,0}$, $\tilde{\kappa}_0$, $\tilde{\gamma}$ and ν remain fixed as $\epsilon \to 0$.

The leading-order incompressible Stokes equations are simply

$$\nabla p_f^{(0)} = \nabla^2\boldsymbol{v}^{(0)}, \tag{I.45}$$

$$\nabla \cdot \boldsymbol{v}^{(0)} = 0. \tag{I.46}$$

In the limit of small ϵ we have the expansions

$$\boldsymbol{F}_1 = \boldsymbol{I} + \epsilon\nabla\boldsymbol{u} + O\left(\epsilon^2\right), \tag{I.47}$$

$$\det(\boldsymbol{F}_1) = 1 + \epsilon\nabla\cdot\boldsymbol{u} + O\left(\epsilon^2\right). \tag{I.48}$$

The porosity is given by expanding (I.11) and (I.12) and using (I.48) to write

$$\phi_f = \phi_{f,0} + \epsilon(1-\phi_{f,0})\nabla\cdot\boldsymbol{u}^{(0)} + \cdot\,. \tag{I.49}$$

Expanding (I.32), the leading-order flow in the particle is governed by Darcy's law (I.32) with constant porosity, $\phi_f^{(0)} = \phi_{f,0}$ and associated permeability $\tilde{\kappa}_0$,

$$\phi_{f,0}(\boldsymbol{v}_f^{(0)} - \boldsymbol{v}_s^{(0)}) = -\tilde{\kappa}_0 \nabla p_f^{(0)}. \tag{I.50}$$

We also expand the local continuity conditions (I.34) to give

$$\nabla \cdot \boldsymbol{v}_f^{(0)} = 0. \quad \nabla \cdot \boldsymbol{v}_s^{(0)} = 0. \tag{I.51}$$

Substituting (I.47) into (I.33) the velocity gradient tensor is constant to leading order, $\boldsymbol{L}_s^{(0)} = \boldsymbol{W}_0^{(0)}$, such that the solid velocity becomes the rigid-body rotation

$$\boldsymbol{v}_s^{(0)} = \boldsymbol{W}_0^{(0)} \boldsymbol{x} = r[\boldsymbol{\Omega} \times \boldsymbol{e}_r], \tag{I.52}$$

where $\boldsymbol{\Omega}$ remains to be determined. We note that in (I.52), $\boldsymbol{v}_s$ is identical to $\boldsymbol{v}_\Omega$ as given in II.1 in the main text.

The leading-order mechanical equilibrium equation for the solid skeleton is given by expanding (I.35),

$$\nabla \cdot \boldsymbol{\sigma}^{(0)} = \nabla \cdot \boldsymbol{\sigma}'^{(0)} - \nabla p_f^{(0)} = \boldsymbol{0}, \tag{I.53}$$

where expanding (I.36) and using (I.47) we reduce to a linearly elastic stress-strain relationship to leading order,

$$\boldsymbol{\sigma}'^{(0)} = \frac{2\nu}{1-2\nu}(\nabla \cdot \boldsymbol{u}^{(0)})\boldsymbol{I} + 2\boldsymbol{\varepsilon}, \tag{I.54}$$

where $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \boldsymbol{u}^{(0)} + \left(\nabla \boldsymbol{u}^{(0)}\right)^T\right)$ is the linear elastic strain.

The leading-order far-field condition for the external Stokes flow is,

$$\boldsymbol{V} - \boldsymbol{V}_\infty - \boldsymbol{V}_{tr} \to 0 \quad \text{as} \quad r \to \infty. \tag{I.55}$$

Expanding the kinematic condition (I.39) we find

$$\boldsymbol{r}_s = \boldsymbol{e}_r + \epsilon \boldsymbol{u}^{(0)}(r=1), \tag{I.56}$$

such that $\boldsymbol{r}_s^{(0)} = \boldsymbol{e}_r$, implying interfacial conditions are applied on the undeformed particle surface with $\boldsymbol{n} = \boldsymbol{e}_r$, $\boldsymbol{t}^1 = \boldsymbol{e}_\theta$, and $\boldsymbol{t}^2 = \boldsymbol{e}_\phi$. Expanding the boundary conditions (I.40) – (I.43) to leading order, and noting that $\boldsymbol{v}_s$ in (I.52) has no radial component, we have

$$\boldsymbol{V} \cdot \boldsymbol{e}_r = \phi_{f,0} \boldsymbol{v}_f \cdot \boldsymbol{e}_r, \tag{I.57}$$

$$\boldsymbol{\Sigma} \cdot \boldsymbol{e}_r = \boldsymbol{\sigma} \cdot \boldsymbol{e}_r, \tag{I.58}$$

$$\boldsymbol{e}_r \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{e}_r = -p_f, \tag{I.59}$$

$$\boldsymbol{e}_\theta \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{e}_r = \frac{\gamma}{\sqrt{\tilde{\kappa}_0}}((\boldsymbol{V} - \boldsymbol{q}) \cdot \boldsymbol{e}_\theta), \tag{I.60}$$

$$\boldsymbol{e}_\phi \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{e}_r = \frac{\gamma}{\sqrt{\tilde{\kappa}_0}}((\boldsymbol{V} - \boldsymbol{q}) \cdot \boldsymbol{e}_\phi). \tag{I.61}$$

Expanding the force and torque balances (I.4) and (I.5) to leading order then gives

$$\boldsymbol{F}_{\text{force}}^{(0)} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \boldsymbol{\Sigma}^{(0)}|_{r=1} \cdot \boldsymbol{e}_r \ \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = \boldsymbol{0}, \tag{I.62}$$

$$\boldsymbol{T}^{(0)} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \boldsymbol{r} \times (\boldsymbol{\Sigma}^{(0)}|_{r=1} \cdot \boldsymbol{e}_r) \ \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = \boldsymbol{0}. \tag{I.63}$$

Dropping the superscripts, (I.45), (I.46) and (I.49) to (I.63) which form the leading-order problem to calculate the fluid flow and resulting deformation are equivalently presented in the main text as II.3 to II.25. We note that this system of equations could equivalently be derived from a linearly poroelastic description.

II. REFORMULATION OF THE BOUNDARY/INTERFACIAL CONDITIONS

To exploit the orthogonality of the solid spherical harmonics we reformulate the boundary conditions for both the fluid and solid problems in terms of radial components and radial derivatives. This approach is similar to that developed by Happel and Brenner [4], and later extended by Murata [5] to consider an elastic particle in a general Stokes flow.

A. The fluid problem

We use the interfacial conditions II.22, II.24, and II.25 in the main text to calculate the coefficients in the solid spherical harmonic functions $\Phi_{-n-1}, \Psi_{-n-1}, P_{-n-1}$ and p_n. The normal flux condition II.22 (main text) and continuity of fluid stress II.24 (main text) are already in the desired form, since $\boldsymbol{v}_s$ has no radial component,

$$V_r|_{r=1} = \phi_{f,0} v_{f,r}|_{r=1} = -\tilde{\kappa}_0 \frac{\partial p_f}{\partial r}|_{r=1}, \tag{II.1}$$

$$\Sigma_{rr}|_{r=1} = -p_f|_{r=1}. \tag{II.2}$$

Separating II.25 in the main text into the θ and ϕ components, the Beavers & Joseph conditions are

$$\boldsymbol{e}_\theta \cdot \Sigma \cdot \boldsymbol{e}_r = \tilde{\gamma}(V + \tilde{\kappa}_0 \nabla p_f - v_s) \cdot \boldsymbol{e}_\theta, \tag{II.3}$$

$$\boldsymbol{e}_\phi \cdot \Sigma \cdot \boldsymbol{e}_r = \tilde{\gamma}(V + \tilde{\kappa}_0 \nabla p_f - v_s) \cdot \boldsymbol{e}_\phi, \tag{II.4}$$

applied on the undeformed particle surface ($r = 1$). We reformulate these interfacial conditions using the divergence and curl operators. Beginning with divergence, we write

$$r\nabla \cdot (\boldsymbol{\Sigma} \cdot \boldsymbol{e}_r(r=1)) - 2\Sigma_{rr}|_{r=1} = \tilde{\gamma}\left(r\nabla \cdot (\boldsymbol{V}(r=1)) - 2V_r|_{r=1} + \tilde{\kappa}_0 r \nabla^2 (p_f(r=1)) - 2\tilde{\kappa}_0 \frac{\partial p_f}{\partial r}|_{r=1}\right), \tag{II.5}$$

and use the following relations,

$$r\nabla \cdot (\boldsymbol{\Sigma} \cdot \boldsymbol{e}_r(r=1)) = [r\nabla \cdot \boldsymbol{\Sigma}]_{r=1} - [3P]_{r=1} - [\Sigma_{rr}]_{r=1} - \left[r\frac{\partial \Sigma_{rr}}{\partial r}\right]_{r=1}, \tag{II.6}$$

$$r\nabla \cdot (\boldsymbol{V}(r=1)) = [r\nabla \cdot (\boldsymbol{V})]_{r=1} - \left[r\frac{\partial V_r}{\partial r}\right]_{r=1}, \tag{II.7}$$

$$r\nabla^2 p_f(r=1) = \left[r\nabla^2 p_f\right]_{r=1} - \left[r\frac{\partial^2 p_f}{\partial r^2}\right]_{r=1}, \tag{II.8}$$

to write, noting the divergence of $\boldsymbol{V}$, $\boldsymbol{\Sigma}$ and $\boldsymbol{v_s}$ all equal zero,

$$\left[r\frac{\partial \Sigma_{rr}}{\partial r} + 3(P + \Sigma_{rr})\right]_{r=1} = \tilde{\gamma}\left[r\frac{\partial V_r}{\partial r} + 2V_r + \tilde{\kappa}_0 r \frac{\partial^2 p_f}{\partial r^2} + 2\tilde{\kappa}_0 \frac{\partial p_f}{\partial r}\right]_{r=1}. \tag{II.9}$$

To be clear, in the left-hand side of each of the relations (II.6) – (II.8) we evaluate the expression on $r = 1$ before applying any derivatives. On the right-hand side we perform any derivatives before evaluating on $r = 1$.

We then write the curl relation as

$$[\boldsymbol{r} \cdot \nabla \times (\boldsymbol{\Sigma} \cdot \boldsymbol{e}_r)]_{r=1} = \tilde{\gamma}\,[\boldsymbol{r} \cdot \nabla \times \boldsymbol{V} - \boldsymbol{r} \cdot \nabla \times \boldsymbol{v}_s]_{r=1}, \tag{II.10}$$

since there are no r-derivatives present and the curl of ∇p_f is trivially zero.

The reformulated interfacial conditions (II.1), (II.2), (II.9) and (II.10) allow us to apply the solid spherical harmonic identities in B 1 in the main text to solve the fluid problem separately for each degree n. For example, applying Equation B.8 to the general solutions for the Stokes velocity and Darcy pressure, III.5 and III.9 in the main text, we find the expressions

$$V_r = \sum_{n=1}^{\infty}\left[\frac{n}{r}\Phi_n^* + \frac{n}{2(2n+3)} r P_n^*\right] + \sum_{n=1}^{\infty}\left[-\frac{n+1}{r}\Phi_{-n-1} + \frac{n+1}{2(2n-1)} r P_{-n-1}\right], \tag{II.11}$$

$$r\frac{\partial V_r}{\partial r} = \sum_{n=1}^{\infty}\left[\frac{n(n-1)}{r}\Phi_n^* + \frac{n(n+1)}{2(2n+3)} r P_n^*\right] + \sum_{n=1}^{\infty}\left[\frac{(n+1)(n+2)}{r}\Phi_{-n-1} - \frac{n(n+1)}{2(2n-1)} r P_{-n-1}\right], \tag{II.12}$$

$$\frac{\partial p_f}{\partial r} = \sum_{n=1}^{\infty} \frac{n}{r} p_n. \tag{II.13}$$

$$r\frac{\partial^2 p_f}{\partial r^2} = \sum_{n=1}^{\infty} \frac{n(n-1)}{r} p_n. \tag{II.14}$$

Using the constitutive relation II.5 (main text) we then obtain

$$\Sigma_{rr} = \sum_{n=1}^{\infty}\left[\frac{2n(n-1)}{r^2}\Phi_n^* + \frac{n^2-n-3}{2n+3}P_n^*\right] + \sum_{n=1}^{\infty}\left[\frac{2(n+1)(n+2)}{r^2}\Phi_{-n-1} - \frac{n^2+3n-1}{2n-1}P_{-n-1}\right], \tag{II.15}$$

$$r\frac{\partial \Sigma_{rr}}{\partial r} = \sum_{n=1}^{\infty}\left[\frac{2n(n-1)(n-2)}{r^2}\Phi_n^* + \frac{(n^2-n-3)n}{2n+3}P_n^*\right] + \sum_{n=1}^{\infty}\left[-\frac{2(n+1)(n+2)(n+3)}{r^2}\Phi_{-n-1} + \frac{(n^2+3n-1)(n+1)}{2n-1}P_{-n-1}\right], \tag{II.16}$$

It may also be shown using B.8 and B.10 that

$$\boldsymbol{r}\cdot\nabla\times\boldsymbol{V} = \sum_{n=1}^{\infty} n(n+1)\Psi_n^* + \sum_{n=1}^{\infty} n(n+1)\Psi_{-n-1}, \tag{II.17}$$

$$\boldsymbol{r}\cdot\nabla\times(\boldsymbol{\Sigma}\cdot\boldsymbol{e}_r) = \sum_{n=1}^{\infty}\frac{n(n+1)(n-1)}{r}\Psi_n^* - \sum_{n=1}^{\infty}\frac{n(n+1)(n+2)}{r}\Psi_{-n-1}, \tag{II.18}$$

$$\boldsymbol{r}\cdot\nabla\times\boldsymbol{v}_s = 2r(\Omega^x\sin\theta\cos\phi + \Omega^y\sin\theta\sin\phi + \Omega^z\cos\theta). \tag{II.19}$$

Substituting the explicit representations (II.11) – (II.19) into the reformulated boundary conditions (II.1), (II.2), (II.9) and (II.10) and grouping terms together we obtain the system of equations

$$n\Phi_n^* + \frac{n}{2(2n+3)}P_n^* - (n+1)\Phi_{-n-1} + \frac{n+1}{2(2n-1)}P_{-n-1} = -n\tilde{\kappa}_0 p_n, \tag{II.20}$$

$$2n(n-1)\Phi_n^* + \frac{n^2-n-3}{2n+3}P_n^* + 2(n+1)(n+2)\Phi_{-n-1} + \frac{n^2+3n-1}{2n-1}P_{-n-1} = -p_n, \tag{II.21}$$

$$\begin{aligned}&[2n(n-1)(n-2) - n(n+1)\tilde{\gamma}]\,\Phi_n^* + \left[\frac{n(n^2-n+3)+9}{2n+3} - \frac{n(n+3)\tilde{\gamma}}{2(2n+3)}\right]P_n^*\\ &\quad - \left[(n+1)\left(2(n+2)(n+3) + n^2\tilde{\gamma}\right)\right]\Phi_{-n-1} + \left[4 + \frac{n^2(n+4)}{2n-1} - \frac{(n-2)(n+1)\tilde{\gamma}}{2(2n-1)}\right]P_{-n-1}\\ &\quad = (n(n+1)+3)p_n,\end{aligned} \tag{II.22}$$

$$\begin{aligned}&(n-1)n(n+1)\Psi_n^* - n(n+1)(n+2)\Psi_{-n-1}\\ &\quad = n(n+1)\tilde{\gamma}(\Psi_n^* + \Psi_{-n-1}) - 2(\Omega^z\cos\theta + \Omega^x\cos\phi\sin\theta + \Omega^y\sin\theta\sin\phi).\end{aligned} \tag{II.23}$$

By prescribing Φ_n^*, Φ_n^* and P_n^* through the far-field condition II.19 (main text), we may exploit the linearity of the solid spherical harmonics to solve the system of equations (II.20) – (II.23) simultaneously for each degree n. This results in the solutions III.14 to III.17 in the main text and gives the unknown scalar harmonics, $\Phi_{-n-1}, \Psi_{-n-1}, P_{-n-1}$ and p_n which determine the exterior and interior flow profiles through the general solutions III.5, III.6, and III.9 (main text).

B. The solid problem

We now reformulate the boundary conditions for the solid problem,

$$\sigma'_{rr}|_{r=1} = 0, \quad \sigma'_{r\theta}|_{r=1} = \Sigma_{r\theta}|_{r=1}, \quad \sigma'_{r\phi}|_{r=1} = \Sigma_{r\phi}|_{r=1}. \tag{II.24}$$

The first of these boundary conditions is already suitable and so we reformulate only the θ and ϕ components. The analysis is similar to that carried out in the previous section, though since the particle is compressible we have $\nabla \cdot \boldsymbol{u} \neq 0$. We begin by writing,

$$r\nabla \cdot (\boldsymbol{\sigma}' \cdot \boldsymbol{e}_r(r=1)) - 2\sigma'_{rr}|_{r=1} = r\nabla \cdot (\boldsymbol{\Sigma} \cdot \boldsymbol{e}_r(r=1)) - 2\Sigma_{rr}|_{r=1}, \tag{II.25}$$

where we may again use (II.6) to reformulate the right-hand side to the desired form. For the left-hand side we have,

$$\begin{aligned} r\nabla \cdot (\boldsymbol{\sigma}' \cdot \boldsymbol{e}_r(r=1)) &= r\left[\nabla \cdot (\boldsymbol{\sigma}' \cdot \boldsymbol{e}_r)\right]_{r=1} - \left[r\frac{\partial \sigma'_{rr}}{\partial r}\right]_{r=1} \\ &= \left[\nabla \cdot (\boldsymbol{\sigma}' \cdot \boldsymbol{r})\right]_{r=1} - \left[r\frac{\partial \sigma'_{rr}}{\partial r}\right]_{r=1} \\ &= \left[\boldsymbol{r} \cdot \nabla \cdot \boldsymbol{\sigma}' + \mathrm{tr}(\boldsymbol{\sigma}') - r\frac{\partial \sigma'_{rr}}{\partial r} - \sigma'_{rr}\right]_{r=1} \\ &= \left[r\frac{\partial p_f}{\partial r} + (3\lambda + 2\mu_s)(\nabla \cdot \boldsymbol{u}) - r\frac{\partial \sigma'_{rr}}{\partial r} - \sigma'_{rr}\right]_{r=1}. \end{aligned} \tag{II.26}$$

Substituting (II.6) and (II.26) into (II.25), and using $\sigma'_{rr} = 0$ and $\Sigma_{rr} = -p_f$ on $r = 1$ we obtain III.29 in the main text.

The Papkovich-Neuber representation of the solid deformation III.22 (main text) is written most conveniently using a Cartesian coordinate system since we have the forms for $\boldsymbol{\psi}_p$ and $\boldsymbol{\psi}_c$ given in III.26 and III.27 in the main text, respectively. To calculate, for example, the stress component σ'_{rr} using the Papkovich-Neuber representation we write

$$\sigma'_{rr} = \frac{2\nu}{1-2\nu}(\nabla \cdot \boldsymbol{u}) + 2\frac{\partial u_r}{\partial r}, \tag{II.27}$$

where $u_r = \boldsymbol{u} \cdot \boldsymbol{r}/r$ and $\boldsymbol{r} = (x, y, z)$ in a Cartesian basis. Substituting III.22 (main text) into (II.27) we obtain

$$\begin{aligned} \sigma'_{rr} = \sum_{n=0}^{\infty} \Bigg[&\frac{1}{2r^2}(3-4\nu-n)n[\boldsymbol{r} \cdot \boldsymbol{\psi}_c]_n + \nu[\nabla \cdot \boldsymbol{\psi}_c]_n \\ &+ \frac{1}{2r^2}(3-4\nu-(n+2))(n+2)[\boldsymbol{r} \cdot \boldsymbol{\psi}_p]_n + \nu[\nabla \cdot \boldsymbol{\psi}_p]_n - \frac{1}{r^2}(n+1)(n+2)\phi_n \Bigg]. \end{aligned} \tag{II.28}$$

Similarly, we write the remaining required terms as

$$(3\lambda + 2\mu_s)\nabla \cdot \boldsymbol{u} = (1+\nu)\nabla \cdot \boldsymbol{\psi} = (1+\nu)\sum_{n=0}^{\infty}\left([\nabla \cdot \boldsymbol{\psi}_c]_n + [\nabla \cdot \boldsymbol{\psi}_p]_n\right), \tag{II.29}$$

$$\begin{aligned} \frac{\partial \sigma'_{rr}}{\partial r} = \sum_{n=0}^{\infty} \Bigg[&\frac{1}{2r^2}(3-4\nu-n)n(n-1)[\boldsymbol{r} \cdot \boldsymbol{\psi}_c]_n + \nu(n-1)[\nabla \cdot \boldsymbol{\psi}_c]_n \\ &+ \frac{1}{2r^2}(3-4\nu-(n+2))(n+2)(n+1)[\boldsymbol{r} \cdot \boldsymbol{\psi}_p]_n + \nu(n+1)[\nabla \cdot \boldsymbol{\psi}_p]_n \\ &- \frac{1}{r^2}(n+1)(n+2)n\phi_n \Bigg], \end{aligned} \tag{II.30}$$

where using III.25 and III.26 in the main text we have

$$\boldsymbol{r} \cdot \boldsymbol{\psi}_p = \frac{r^2}{1-\nu}\sum_{n=0}^{\infty}\frac{np_n}{2(2n+3)}, \quad \nabla \cdot \boldsymbol{\psi}_p = \frac{1}{1-\nu}\sum_{n=0}^{\infty}\frac{np_n}{2n+3}. \tag{II.31}$$

For the curl reformulation of (II.24) given by III.30 in the main text, the right-hand side is known via (II.18) and for the left-hand side we have

$$\boldsymbol{r} \cdot \nabla \times (\boldsymbol{\sigma}' \cdot \boldsymbol{e}_r) = \sum_{n=0}^{\infty}\frac{1}{r}(1-\nu)\left[(n-1)\left(y\frac{\partial \psi_n^x}{\partial z} - z\frac{\partial \psi_n^x}{\partial y} + z\frac{\partial \psi_n^y}{\partial x} - x\frac{\partial \psi_n^y}{\partial z} + x\frac{\partial \psi_n^z}{\partial y} - y\frac{\partial \psi_n^z}{\partial x}\right)\right], \tag{II.32}$$

where we have used the fact that the curl of a gradient is zero. The expressions (II.28), (II.30) and (II.32) are evaluated on the undeformed particle surface ($r = 1$) and substituted into the reformulated boundary conditions III.28 to III.30 (main text), along with the relevant expressions for the fluid stress, in order to obtain the system of equations III.31 to III.33 in the main text.

[1] C. W. MacMinn, E. R. Dufresne, and J. S. Wettlaufer, Large deformations of a soft porous material, Physical Review Applied **5**, 044020 (2016).

[2] S. N. Atluri and A. Cazzani, Rotations in computational solid mechanics, Archives of Computational Methods in Engineering **2**, 49 (1995).

[3] G. S. Beavers and D. Joseph, Boundary conditions at a naturally permeable wall, Journal of Fluid Mechanics **30**, 197–207 (1967).

[4] J. Happel and H. Brenner, *Low Reynolds number hydrodynamics*, 5 No. 1 (Kluwer Academic Publishers, 1991).

[5] T. Murata, Deformation of an elastic particle suspended in an arbitrary flow field, Journal of the Physical Society of Japan **50**, 1009 (1981).