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Large-order perturbation theory of linear eigenvalue problems

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We consider a class of linear eigenvalue problems depending on a small parameter ϵ in which the series expansion for the eigenvalue in powers of ϵ is divergent. We develop a new technique to determine the precise nature of this divergence. We illustrate the technique through its application to four examples: the anharmonic oscillator, a simplified model of equatorially trapped Rossby waves and two simplified models based on quasinormal modes of Reissner–Nordström–de Sitter black holes.

1. Introduction

Linear eigenvalue problems of the form

$$L_\epsilon g = \lambda g,$$

where L_ϵ is a differential operator (depending on a parameter $\epsilon > 0$), are ubiquitous in applied mathematics and theoretical physics. The eigenvalue λ might correspond to an energy level, the frequency of a normal mode of oscillation or the growth rate in a linear stability analysis. Often, ϵ is a small parameter, in which case it is common to develop the perturbation series for the eigenvalue λ in powers of ϵ,

$$\lambda \sim \sum_{n=0}^{\infty} \epsilon^n \lambda_n.$$

Sometimes, this series diverges, and it is of interest to determine the nature of this divergence. From a purely numerical point of view understanding the large-order behaviour tells us how many terms to take in an optimal approximation, how large the smallest achievable error is and whether the series can be meaningfully resummed.

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However, as has long been known, large-order growth is controlled by non-perturbative effects that are not visible in ordinary perturbation theory. For eigenvalue problems, this can mean an exponentially small imaginary part of the eigenvalue, indicating an instability not visible in the regular perturbation expansion, or, in quantum mechanics, tunnelling between classically distinct regions, or exponentially small-level splittings. Here, we provide a straightforward approach to determining the precise asymptotic behaviour of λ_n for large n.

One of the earliest examples of such a problem is the quantum anharmonic oscillator [1,2], which we revisit in §3. This problem has a long history; it was the first non-exactly solvable problem tackled by the newly written Schrödinger equation in 1926, has practical applications ranging from quantum chemistry and atomic-molecular physics to crystal lattice vibrations in solid-state theory, and serves as a simple model for quantum field theory [3]. The seminal work by Bender & Wu [1,2] established the nature of the divergence of the perturbation series for the ground state energy. This work became the prototype for similar analyses in many other quantum mechanical systems, in what is now known as large-order perturbation theory [4,5].

The main technique of Bender & Wu is to analytically continue in ϵ (typically until ϵ is negative) and then solve the resulting problem by a combination of Liouville–Green (Wentzel–Kramers–Brillouin (WKB)) and matched asymptotic approximations. This is a delicate procedure, since the goal is to identify an exponentially small component of λ beyond-all-orders of the divergent asymptotic series. Cauchy's integral formula is then used to determine the coefficients in the power series expansion of λ on the positive real ϵ axis in terms of the values of λ on either side of the negative real ϵ axis. This technique is ingenious and has proved successful, but the details can be very complicated. The present work aims to present an alternative method, which we hope will be useful.

Intriguing also is the work of Dunne & Ünsal [6,7], who use a uniform WKB approach for quantum mechanical systems with degenerate minima, focusing in particular on double-well and sine-Gordon potentials. They write the wave function in terms of a scaled parabolic cylinder function $\Psi(x) = D_\nu(u(x)/\sqrt{\epsilon})/\sqrt{u'(x)}$, where the index ν is a parameter to be determined along with $u(x)$. Although $u(x)$ and λ can be determined locally in terms of ν by a regular perturbation expansion, ν itself is determined by imposing a global boundary condition (a symmetry condition at the midpoint between wells). This shows ν is exponentially close to an integer and provides an exponential correction to λ. In [6], this exponential correction is used to determine the late-order terms in the power series, in a similar way to Bender & Wu.

In the applied mathematics literature, an early example of such a problem occurs in the work of Boyd & Natarov [8], who consider a model problem for an equatorially trapped Rossby wave in a shear flow in the ocean or atmosphere. There the main interest is in the imaginary part of the eigenvalue (corresponding to the growth rate of instability)—the divergent perturbation series is purely real, but there is an exponentially small imaginary part beyond all orders. In [9], this problem is attacked in almost the reverse direction to Bender & Wu—the divergent series is first found and used to determine the exponentially small imaginary component of the eigenvalue via optimal truncation and Stokes' phenomenon, rather than the other way round. Simplifying and extending the procedure from [9] forms the basis of the present work.

We present our procedure through its application to four examples. Each follows the same general framework, but the final part of the analysis differs slightly in each case. We hope this allow the interested reader to adapt the method to their own particular problem.

The general framework is:

(i) *Inner region*. Each problem starts with a regular perturbation expansion. Typically, the coefficients in the expansion of the eigenvalue are determined by imposing a regularity condition at the origin $x = 0$, and each term in the expansion is a polynomial in x. This region determines the eigenvalue expansion completely, but it is difficult to extract the large-order behaviour from the resulting recurrence relations.

(ii) *Outer region*. The regular expansion in (i) is not uniform and rearranges for large x, leading to a new expansion once $x = X/\epsilon$ has been rescaled. The outer problem is a singular perturbation problem, so that this expansion diverges in the usual form of factorial/power, driven by singularities away from $X = 0$. The large-order behaviour of this divergence is easy to determine by now standard methods. In addition, the late terms have an independent component driven by the divergent eigenvalue expansion.

(iii) *Boundary layer in the late terms near* $X = 0$. This is the new ingredient to the method. The key observation is that the large-order approximation of the outer expansion (ii) is also non-uniform, so that there is another inner region near $X = 0$, now not in the small-ϵ expansion but in the large-order expansion. The resolution of this inner region links the two parts of the expansion in (ii) and determines the large-order behaviour of the eigenvalue.

2. Example 1: simplified black holes

We consider the model problem

$$2(1-\epsilon x)(-\omega g + xg') + g + (xg')' = 0, \quad -\infty < x < 0, \tag{2.1}$$

with $g(0) = 1$ and $g(x) = o(e^{-x})$ as $x \to -\infty$, where $0 < \epsilon \ll 1$. This is a much-simplified version of the problem in [10] concerning quasinormal modes of Reissner–Nordström–de Sitter black holes, keeping only those ingredients necessary to illustrate the methodology; very roughly speaking g is the charged scalar field perturbation, x is the radial distance measured from the cosmological horizon, ϵ is the charge and the eigenvalue ω is the frequency of the mode.

As $x \to -\infty$, the two possible behaviours are

$$g(x) \sim e^{\epsilon x^2} \quad \text{and} \quad g(x) \sim x^{\omega},$$

while as $x \to 0$, the two possible behaviours are

$$g(x) \sim 1 \quad \text{and} \quad g(x) \sim \log x.$$

The boundary conditions at $x = 0$ and $x = -\infty$ each remove one degree of freedom, so that there is a non-zero solution only if ω takes particular values. The goal is to find the asymptotic expansion of the eigenvalues,

$$\omega \sim \sum_{n=0}^{\infty} \epsilon^n \omega_n,$$

as $\epsilon \to 0$, and, in particular, the form of the divergence of ω_n as $n \to \infty$. To simplify the presentation, we focus on the leading eigenvalue, here and in each of our examples, but of course the methodology works for any eigenvalue.

(a) Inner region

We start with

$$2(1-\epsilon x)(-\omega g + xg') + g + (xg')' = 0.$$

We expand

$$g = \sum_{n=0}^{\infty} \epsilon^n g_n \quad \text{and} \quad \omega = \sum_{n=0}^{\infty} \epsilon^n \omega_n, \tag{2.2}$$

to give at leading order

$$2(-\omega_0 g_0 + xg_0') + g_0 + (xg_0')' = 0.$$

The solution which is regular at the origin is

$$g_0 = L_{\omega_0 - 1/2}(-2x),$$

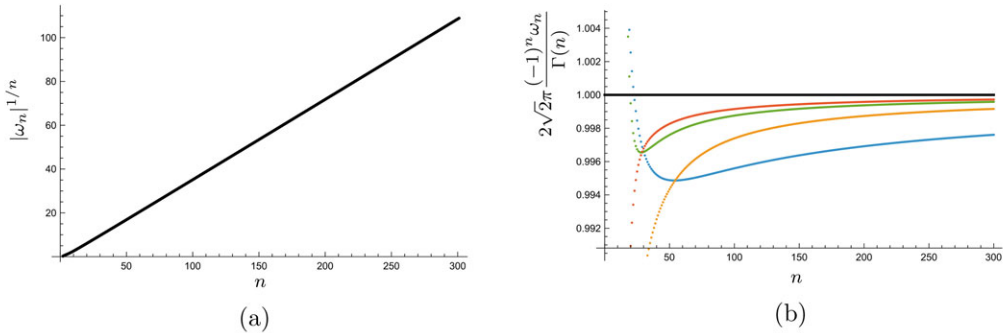

Figure 1. Divergence of the coefficients in the asymptotic expansion of ω. (a) coefficients determined numerically from equations (2.4)–(2.5). The linear growth is consistent with factorial divergence. (b) The ratio of the numerical value to the asymptotic prediction equation (2.17). Blue is the base series, while orange, green and red correspond to enhanced convergence using Richardson extrapolation on two, three and four terms, respectively. The black line is the asymptote, included to aid the eye. The convergence is slower than expected because of the presence of log terms in the higher-order corrections, unaccounted for in the extrapolation.

where $L_n(z)$ is the Laguerre function. To avoid exponential growth as $x \to -\infty$ we need the Laguerre function to be a polynomial, i.e. we need $\omega_0 - 1/2$ to be a non-negative integer. Choosing the first of these, $n = 0$, gives the solution $g_0 = 1$, $\omega_0 = 1/2$. At the next order,

$$2xg_1' + (xg_1')' - 2\omega_1 = -x.$$

The solution which is regular at $x = 0$ and does not grow exponentially at minus infinity is

$$g_1 = -\frac{x}{2} \quad \text{and} \quad \omega_1 = -\frac{1}{4}.$$

In general,

$$2xg_n' + (xg_n')' - 2\omega_n = -2x(\omega_{n-1}g_0 + \cdots + \omega_0 g_{n-1} - xg_{n-1}') + 2(\omega_{n-1}g_1 + \cdots \omega_1 g_{n-1}), \tag{2.3}$$

and the solution is of the form

$$g_n = \sum_{i=1}^{n} a_{ni}x^i,$$

with

$$2ja_{n,j} + (j+1)^2 a_{n,j+1} = -2\sum_{k=j-1}^{n-1} \omega_{n-1-k}a_{k,j-1} + 2(j-1)a_{n-1,j-1} + 2\sum_{k=j}^{n-1} \omega_{n-k}a_{k,j}, \quad \text{for } j = 1, \ldots, n, \tag{2.4}$$

and

$$a_{n1} - 2\omega_n = 0. \tag{2.5}$$

We can iterate to find ω_n numerically. Figure 1a shows $|\omega_n|^{1/n}$ as a function of n; the linear growth in n is consistent with factorial growth in ω_n at large n. In principle, we could extract the asymptotic behaviour as $n \to \infty$ from equations (2.4)–(2.5), but this is not so straightforward. The method we now highlight determines ω_n for large n without the need to analyse equations (2.4)–(2.5).

(b) Outer region

The expansion equation (2.2) is not uniform in x—it rearranges when x is large. In this section, we develop the corresponding expansion valid for large x.

To this end, we set $\epsilon x = X$ to give

$$2(1-X)(-\omega g + Xg') + g + \epsilon(Xg')' = 0.$$

Now, expanding

$$g = \sum_{n=0}^{\infty} \epsilon^n g_n \tag{2.6}$$

gives, at leading order,

$$2(1-X)(-\omega_0 g_0 + Xg_0') + g_0 = 0,$$

so that

$$g_0 = B(1-X)^{1/2} X^{\omega_0 - 1/2},$$

for some constant B. For there to be no singularity at $X=0$, we require $\omega_0 = 1/2$, in agreement with §2a. To match with the inner expansion as $X \to 0$ we require $g_0 \to 1$ so that $B=1$. In general, equating coefficients of ϵ^n,

$$X(g_n + 2(1-X)g_n') = -(Xg_{n-1}')' + 2(1-X)(\omega_1 g_{n-1} + \cdots + \omega_n g_0). \tag{2.7}$$

We need to determine the late terms in the expansion, that is, the behaviour of g_n as $n \to \infty$. There are two sources of divergence in g_n: the usual factorial/power divergence driven by differentiating g_{n-1} and a factorial/constant divergence driven by ω_n. For the first, we follow the usual procedure [11] by supposing that

$$g_n \sim \frac{G\Gamma(n+\gamma)}{\chi^{n+\gamma}} \tag{2.8}$$

as $n \to \infty$, where G and χ are functions of x and γ is constant. Then, equating coefficients of powers of n gives, at leading order,

$$\chi' = 2(1-X).$$

Since this divergence is driven by the singularity in g_0 at $X=1$, we have $\chi(1)=0$, so that

$$\chi = -(1-X)^2.$$

At the next order, we find

$$(2-5X)G + 2X(1-X)G' = 0,$$

giving

$$G = \frac{\Lambda}{X(1-X)^{3/2}},$$

for some constant Λ. Thus, (absorbing $(-1)^{-\gamma}$ into Λ) this part of g_n satisfies

$$g_n \sim \frac{\Lambda(-1)^n \Gamma(n+\gamma)}{X(1-X)^{3/2}(1-X)^{2n+2\gamma}}.$$

As $X \to 1$,

$$g_n \sim \frac{\Lambda(-1)^n \Gamma(n+\gamma)}{(1-X)^{3/2}(1-X)^{2n+2\gamma}}.$$

Comparing powers of $1-X$ with g_0 gives

$$-\frac{3}{2} - 2\gamma = \frac{1}{2} \Rightarrow \gamma = -1,$$

so that

$$g_n \sim \frac{\Lambda(-1)^n \Gamma(n-1)}{X(1-X)^{2n-1/2}}. \tag{2.9}$$

The other part of g_n, driven by the divergence of ω_n, is given by $g_n \sim Q\omega_n$ where

$$X(Q + 2(1-X)Q') = 2(1-X)g_0 = 2(1-X)^{3/2},$$

so that

$$Q = (1 - X)^{1/2}(\log X + C).$$

The presence of $\log X$ here means we need to modify slightly the ansatz $g_n \sim Q\omega_n$ (essentially we need C to include a term proportional to $\log n$). If we set instead $g_n = (Q_0 \log n + Q_1)\omega_n$, then,

$$X(Q_0 - 2(1-X)Q_0') = 0$$

and

$$X(Q_1 - 2(1-X)Q_1') = 2(1-X)^{3/2},$$

so that

$$Q_0 \log n + Q_1 = (1-X)^{1/2}(C_0 \log n + C_1 + \log X).$$

Putting the two parts of g_n together gives

$$g_n \sim \frac{\Lambda(-1)^n \Gamma(n-1)}{X(1-X)^{2n-1/2}} + (1-X)^{1/2}(C_0 \log n + C_1 + \log X)\omega_n. \tag{2.10}$$

To determine Λ, we need to match with an inner region in the vicinity of the singularity at $X = 1$.

(c) Inner region near $X = 1$

Motivated by both $g_0(X) = \sqrt{1-X}$ and by equation (2.9), we set $X = 1 - \epsilon^{1/2}\hat{x}$, $g = \epsilon^{1/4}\hat{g}$ to give

$$(1 - 2\hat{x}\epsilon^{1/2}\omega)\hat{g} + (2\hat{x}(-1 + \epsilon^{1/2}\hat{x}) - \epsilon^{1/2})\hat{g}' + (1 - \epsilon^{1/2}\hat{x})\hat{g}'' = 0. \tag{2.11}$$

In terms of the inner variable,

$$g_0 = \epsilon^{1/4}\hat{x}^{1/2} \tag{2.12}$$

and

$$\epsilon^n g_n \sim \frac{\epsilon^{1/4}\Lambda(-1)^n \Gamma(n-1)}{\hat{x}^{2n-1/2}}. \tag{2.13}$$

At leading order in equation (2.11),

$$\hat{g}_0 - 2\hat{x}\hat{g}_0' + \hat{g}_0'' = 0.$$

Writing

$$\hat{g}_0 = \sum_{n=0}^{\infty} c_n x^{1/2-2n},$$

we find

$$c_n = -\frac{(2n-5/2)(2n-3/2)c_{n-1}}{4n}, \quad c_0 = 1,$$

where the latter equation comes from matching with equation (2.12). Thus,

$$c_n = -(-1)^n \frac{(3/4)_{n-1}(5/4)_{n-1}}{16(2)_{n-1}},$$

where $(x)_n = \Gamma(x+n)/\Gamma(x)$ is Pochammer's symbol. Matching with equation (2.13) gives

$$\Lambda = \lim_{n\to\infty} \frac{(-1)^n c_n}{\Gamma(n-1)} = -\frac{1}{16\Gamma(3/4)\Gamma(5/4)} = -\frac{1}{4\sqrt{2}\pi}.$$

(d) Boundary layer in the late terms near $X = 0$

So far everything we have done has followed the standard approach to finding the late terms of the expansion, as described in [11], for example. In this section we make one crucial observation, which extends this standard approach, and allows us link the two parts of the expansion in equation (2.10) and determine ω_n.

This observation is that the large-n asymptotic approximation for g_n in the outer region is non-uniform, and rearranges when X is small. We can see this directly from the asymptotic behaviour equation (2.9), which is singular at $X = 0$, while we know that g_n is in fact regular at $X = 0$.

Thus, there is another inner region near the origin, now not in the small-ϵ expansion of g, but in the large-n expansion of g_n. To examine this inner region, we rescale X by setting $X = \xi/n$. Then, the equation for g_n, equation (2.7), becomes

$$\frac{\xi}{n} g_n + 2\xi \left(1 - \frac{\xi}{n}\right) g_n' = -n(\xi g_{n-1}')' + 2\left(1 - \frac{\xi}{n}\right)(\omega_1 g_{n-1} + \cdots + \omega_n g_0), \tag{2.14}$$

where $'$ is now $\mathrm{d}/\mathrm{d}\xi$. Writing $X = \xi/n$ in equation (2.10), the inner limit of the outer is

$$\begin{aligned} g_n &\sim \frac{\Lambda(-1)^n n\Gamma(n-1)}{\xi(1-\xi/n)^{2n-1/2}} + (1-\xi/n)^{1/2}\left(C_0 \log n + C_1 + \log(\xi/n)\right)\omega_n \\ &\sim \Lambda(-1)^n \Gamma(n)\frac{\mathrm{e}^{2\xi}}{\xi} + ((C_0 - 1)\log n + C_1 + \log \xi)\omega_n. \end{aligned} \tag{2.15}$$

This motivates writing

$$\omega_n \sim \Omega(-1)^n \Gamma(n) \quad \text{and} \quad g_n \sim H(\xi)\Omega(-1)^n\Gamma(n),$$

which, on substituting into equation (2.14), gives, at leading order,

$$-(\xi H')' + 2\xi H' = 2.$$

Thus,

$$H = \alpha_1 + \alpha_2 \operatorname{Ei}(2\xi) + \log \xi, \tag{2.16}$$

where

$$\operatorname{Ei}(z) = \int_{-\infty}^{z} \frac{\mathrm{e}^t}{t}\,\mathrm{d}t$$

is the exponential integral. Now, g_n should be regular as $\xi \to 0$. Since $\operatorname{Ei}(2\xi) \sim \log \xi$ as $\xi \to 0$, we need

$$\alpha_2 = -1,$$

to remove the logarithmic singularity at $\xi = 0$. Since $\operatorname{Ei}(2\xi)$ exhibits Stokes' phenomenon for large ξ, there will be a switch in the behaviour of the late terms depending on the argument of ξ—this is what is known as the higher-order Stokes phenomenon, a Stokes phenomenon not in the asymptotic expansion of g as a function of ϵ, but in the late-term approximation of g_n [12,13]. There is a higher-order Stokes line when ξ crosses the positive real axis, across which the constant contribution to the large-ξ approximation of H (i.e. in the outer limit of the inner expansion) changes. Note that there is no Stokes' phenomenon associated with the particular solution $\log \xi$, so that the coefficient of $\mathrm{e}^{2\xi}/\xi$ is fixed. This is not be the case in our other examples.

To complete the analysis and determine Ω, we need to match equation (2.16) with equation (2.15). As $\xi \to \infty$,

$$\operatorname{Ei}(2\xi) \sim \frac{\mathrm{e}^{2\xi}}{2\xi}.$$

Matching with equation (2.15) gives $C_0 = 1$ and $\Omega = -2\Lambda$ so that

$$\omega_n \sim -2\Lambda(-1)^n\Gamma(n) = \frac{(-1)^n\Gamma(n)}{2\sqrt{2\pi}}, \tag{2.17}$$

as $n \to \infty$.

In figure 1b, this result is compared with ω_n found by numerically iterating equations (2.4)–(2.5). The agreement is found to be good, though the convergence is slightly slower than expected because of the presence of log terms in the higher-order corrections.

3. Example 2: anharmonic oscillator

Having introduced the procedure with a simple model problem, we now consider the classical problem of the anharmonic oscillator [1,2]. Much of the analysis follows the same framework, though the details of the boundary layer in the late-term approximation analogous to §2d are a little different.

Consider[1]

$$\left(-\frac{d^2}{dx^2}+\frac{x^2}{4}+\frac{\epsilon x^4}{4}\right)\Psi=\lambda\Psi,$$

with

$$\Psi\to 0 \quad \text{as} \quad x\to\pm\infty.$$

(a) Inner region

We first factor out the decay at infinity by writing $\Psi=e^{-x^2/4}g$ to give

$$-g''+xg'+\frac{g}{2}+\frac{\epsilon x^4 g}{4}-\lambda g=0. \tag{3.1}$$

Now expand

$$g=\sum_{n=0}^{\infty}\epsilon^n g_n \quad \text{and} \quad \lambda=\sum_{n=0}^{\infty}\epsilon^n\lambda_n, \tag{3.2}$$

to give

$$-g_0''+xg_0'+\frac{g_0}{2}-\lambda_0 g_0=0 \tag{3.3}$$

and

$$-g_n''+xg_n'+\frac{g_n}{2}-\lambda_0 g_n=-\frac{x^4 g_{n-1}}{4}+\sum_{k=1}^{n}\lambda_k g_{n-k}, \quad n\geq 1. \tag{3.4}$$

For the first eigenvalue, the leading-order solution is $g_0=1$, $\lambda_0=1/2$, and, in general

$$g_n=\sum_{k=1}^{2n}a_{n,k}x^{2k}, \tag{3.5}$$

with

$$2ka_{n,k}=(2k+2)(2k+1)a_{n,k+1}-\frac{1}{4}a_{n-1,k-2}+\sum_{i=1}^{n}\lambda_i a_{n-i,k}, \quad k=2n,\ldots,1 \tag{3.6}$$

and

$$-2a_{n,1}=\lambda_n, \tag{3.7}$$

with the convention that $a_{n,k}=0$ for $k>2n$ and $k<1$. Equations (3.6)–(3.7) are equivalent to eqn (6.3) in [2]. It is argued in [2] that the leading-order late-term behaviour of equations (3.6)–(3.7) is the same as that of the linearized equation (i.e. with the final sum omitted). With further approximation, and quite a bit of analysis, Bender & Wu manage to extract the leading-order behaviour of λ_n. Here, we show how this may be obtained by following the systematic procedure outlined in §2.

[1]Note the typo in eqn (1.1) of [2] in which the minus sign is missing.

(b) Outer region

As before, the expansion equation (3.2) is not uniform in x—it rearranges when x is large. In this section, we develop the corresponding expansion valid for large x. We subtract off the leading-order eigenvalue by writing

$$\lambda = \frac{1}{2} + \epsilon\bar{\lambda}. \tag{3.8}$$

We rescale into the far field by setting $\epsilon^{1/2}x = X$ to give

$$-\epsilon^2 g'' + \epsilon X g' + \frac{X^4 g}{4} - \epsilon^2\bar{\lambda} g = 0. \tag{3.9}$$

The solution this time is of the Liouville–Green (WKB) form,

$$g = \mathrm{e}^{\phi/\epsilon} A \quad \text{and} \quad A \sim \sum_{n=0}^{\infty} \epsilon^n A_n. \tag{3.10}$$

Substituting equation (3.10) into equation (3.9) and equating coefficients of powers of ϵ gives, following a routine calculation,

$$\phi = \frac{1}{6} + \frac{X^2}{4} - \frac{(1+X^2)^{3/2}}{6} \quad \text{and} \quad A_0 = \frac{\sqrt{2}}{(1+X^2)^{1/4}\sqrt{1+\sqrt{1+X^2}}}, \tag{3.11}$$

where the normalization $\sqrt{2}$ comes from matching with the inner region. In general, the equation for A_n is

$$X(1+X^2)^{1/2}A_n' + \left((1+X^2)^{1/2} - \frac{1}{2} - \frac{1}{2(1+X^2)^{1/2}}\right)A_n - A_{n-1}'' - \bar{\lambda}_0 A_{n-1} - \bar{\lambda}_1 A_{n-2} - \cdots - \bar{\lambda}_{n-1}A_0 = 0.$$

As before, there are two sources of divergence in A_n: the usual factorial/power from repeated differentiation of the singularity in A_0, and a factorial/constant divergence driven by $\bar{\lambda}_n$. For the first, we use the usual factorial/power ansatz following the procedure in [11] to find

$$A_n \sim \frac{\Lambda}{(1+X^2)^{1/4}\sqrt{1-\sqrt{1+X^2}}}\,\frac{(-1)^n 3^n \Gamma(n)}{(1+X^2)^{3n/2}}. \tag{3.12}$$

The other part of A_n satisfies

$$X(1+X^2)^{1/2}A_n' + \left((1+X^2)^{1/2} - \frac{1}{2} - \frac{1}{2(1+X^2)^{1/2}}\right)A_n \sim \bar{\lambda}_{n-1}A_0,$$

giving

$$A_n \sim \frac{\bar{\lambda}_{n-1}}{(1+X^2)^{1/4}\sqrt{1+\sqrt{1+X^2}}}\left(C_0 \log n + C_1 - \tanh^{-1}\sqrt{1+X^2}\right),$$

where, as in §2b, the presence of a logarithm in X necessitates the inclusion of a logarithm in n. Together,

$$A_n \sim \frac{\Lambda}{(1+X^2)^{1/4}\sqrt{1-\sqrt{1+X^2}}}\,\frac{(-1)^n 3^n \Gamma(n)}{(1+X^2)^{3n/2}} + \frac{\bar{\lambda}_{n-1}}{(1+X^2)^{1/4}\sqrt{1+\sqrt{1+X^2}}}\left(C_0 \log n + C_1 - \tanh^{-1}\sqrt{1+X^2}\right). \tag{3.13}$$

To determine Λ, we need to match with an inner region in the vicinity of either $X = \mathrm{i}$ or $X = -\mathrm{i}$. This problem is slightly unusual in that there are two singularities in the leading-order solution, but they each produce a late-term behaviour with the same singulant χ, so that there is only one factorial/power divergence in the late terms.

(c) Inner region near $X = \mathrm{i}$

We set $X = \mathrm{i} - \mathrm{i}\epsilon^{2/3}\hat{x}/2$, $A = \epsilon^{-1/6}\,\hat{A}$ to give, at leading order,

$$-2\hat{x}^{1/2}\hat{A}' - \frac{\hat{A}}{2\hat{x}^{1/2}} + 4\hat{A}'' = 0.$$

To match with equation (3.11) requires

$$\hat{A} \sim \frac{\sqrt{2}}{\hat{x}^{1/4}} \quad \text{as } \hat{x} \to \infty. \tag{3.14}$$

Writing

$$\hat{A} = \sqrt{2}\sum_{n=0}^{\infty} \frac{g_n}{\hat{x}^{1/4+3n/2}}, \tag{3.15}$$

gives

$$g_n = -\frac{(6n-1)(6n-5)g_{n-1}}{12n}, \quad g_0 = 1,$$

where the latter condition comes from equation (3.14). Thus,

$$g_n = \frac{5(-1)^n 3^{n-2}(7/6)_{n-1}(11/6)_{n-1}}{4(2)_{n-1}}. \tag{3.16}$$

The inner limit of equation (3.12) is

$$\epsilon^n A_n \sim \frac{\Lambda}{\epsilon^{1/6}\hat{x}^{1/4}} \frac{(-1)^n 3^n \Gamma(n)}{\hat{x}^{3n/2}}. \tag{3.17}$$

Matching equation (3.17) with equation (3.15) gives

$$\Lambda = \sqrt{2}\lim_{n\to\infty} \frac{g_n}{(-1)^n 3^n \Gamma(n)} = \frac{5\sqrt{2}}{36\Gamma(7/6)\Gamma(11/6)} = \frac{1}{\pi\sqrt{2}}.$$

(d) Boundary layer in the late terms near $X = 0$

Here, we come to the key step. The large n asymptotic series for A_n in the outer region is non-uniform and rearranges when X is small—there is another inner region near the origin. We emphasize again that this is a boundary layer not in the small-ϵ expansion of A, but in the large n-expansion of A_n. Again, this non-uniformity is evident because the asymptotic formula equation (3.13) is singular at $X = 0$, while A_n should be regular there. This time the appropriate scaling for the inner region is $X = \xi/n^{1/2}$ (so that XA_n' balances with A_{n-1}''), giving

$$\xi A_n' + \frac{3\xi^2}{4n}A_n - nA_{n-1}'' + \cdots - \bar{\lambda}_0 A_{n-1} - \bar{\lambda}_1 A_{n-2} - \cdots - \bar{\lambda}_{n-1}A_0 = 0. \tag{3.18}$$

As $X \to 0$ in equation (3.13),

$$\epsilon^n A_n \sim \frac{\sqrt{2}\Lambda}{-\mathrm{i}\sqrt{X^2}} \frac{(-1)^n 3^n \Gamma(n)}{(1+X^2)^{3n/2}} + \bar{\lambda}_{n-1}\left(C_0 \log n + C_1 \pm \frac{\mathrm{i}\pi}{2} - \log 2 + \log X\right). \tag{3.19}$$

Note that there are two choices of branch to be made here—one for $\sqrt{X^2}$ arising from $\sqrt{1-\sqrt{1+X^2}}$ and one for the constant $\pm\mathrm{i}\pi/2$ arising from $\tanh^{-1}\sqrt{1+X^2}$. In particular note that when matching to find Λ, we took $\sqrt{1-\sqrt{1+X^2}}$ to be real and positive when X approached $\pm\mathrm{i}$, which means we need $-\mathrm{i}\sqrt{X^2}$ to be real and positive when X is on the imaginary axis; in turn this means we need $\sqrt{X^2} = X$ when X is positive imaginary, and $\sqrt{X^2} = -X$ when X is negative imaginary. We return to this choice and the position of the branch cuts shortly, when we match with the inner solution.

With $X=\xi/n^{1/2}$, equation (3.19) is

$$\epsilon^n A_n \sim \mathrm{i}\sqrt{2}\Lambda n^{1/2}(-1)^n 3^n \Gamma(n)\frac{\mathrm{e}^{-3\xi^2/2}}{\sqrt{\xi^2}} + \bar{\lambda}_{n-1}\left(\left(C_0-\frac{1}{2}\right)\log n + C_1 \pm \frac{\mathrm{i}\pi}{2} - \log 2 + \log\xi\right). \tag{3.20}$$

This motivates setting $A_n \sim H\Omega(-1)^n 3^n \Gamma(n+1/2)$, $\bar{\lambda}_{n-1} \sim \Omega(-1)^n 3^n\Gamma(n+1/2)$. Using this ansatz in equation (3.18) gives

$$3\xi H' + H'' - 3 = 0,$$

so that

$$H = \alpha_1 + \alpha_2\int_0^\xi \mathrm{e}^{-3u^2/2}\,\mathrm{d}u + 3\int_0^\xi \mathrm{e}^{-3t^2/2}\int_0^t \mathrm{e}^{3u^2/2}\,\mathrm{d}u\,\mathrm{d}t. \tag{3.21}$$

Both the particular integral and the complementary function exhibit Stokes' phenomenon for large ξ, so that there will be a switch in the behaviour of the late terms depending on the argument of ξ, corresponding to the higher-order Stokes phenomenon. There is a higher-order Stokes line owing to the particular integral when ξ crosses the real axis, across which the coefficient of $\mathrm{e}^{-\xi^2}/\xi$ in the far field (i.e. in the outer limit of the inner expansion) changes. The branch cut associated with $\sqrt{\xi^2}$ in equation (3.20) must be chosen to line up with this higher-order Stokes line. In addition, there is a higher-order Stokes line on the imaginary axis, across which the constant in the far field changes. The branch cut associated with $\pm\mathrm{i}\pi/2$ must be chosen to align with this higher-order Stokes line. As $\xi\to\infty$ in the first quadrant,

$$H \sim \alpha_1 + \frac{\alpha_2\sqrt{\pi}}{\sqrt{6}} + \left(-\frac{\alpha_2}{3} - \frac{\mathrm{i}\sqrt{\pi}}{\sqrt{6}}\right)\frac{\mathrm{e}^{-3\xi^2/2}}{\xi} + \cdots + \log\xi + \frac{1}{2}(\gamma_E + \log 6) + \cdots. \tag{3.22}$$

where γ_E is the Euler gamma. As $\xi\to\infty$ in the third quadrant,

$$H \sim \alpha_1 - \frac{\alpha_2\sqrt{\pi}}{\sqrt{6}} + \left(-\frac{\alpha_2}{3} + \frac{\mathrm{i}\sqrt{\pi}}{\sqrt{6}}\right)\frac{\mathrm{e}^{-3\xi^2/2}}{\xi} + \cdots + \log\xi + +\frac{1}{2}(\gamma_E + \log 6 - \mathrm{i}\pi) + \cdots. \tag{3.23}$$

In the model problem of §2, the key coefficient α_2 was determined by imposing that H was regular at $\xi=0$. In this case, equation (3.21) is regular at the origin for all α_1, α_2, and it is matching with the outer solution which determines α_2.

Matching equation (3.22) with equation (3.20) as $\xi\to\infty$ in the first quadrant gives

$$\left(-\frac{\alpha_2}{3} - \frac{\mathrm{i}\sqrt{\pi}}{\sqrt{6}}\right)\Omega = \mathrm{i}\sqrt{2}\Lambda \quad \text{and} \quad C_0 = 1/2, \quad C_1 + \frac{\mathrm{i}\pi}{2} - \log 2 = \alpha_1 + \frac{\alpha_2\sqrt{\pi}}{\sqrt{6}} + \frac{\gamma_E}{2} + \log 6.$$

Matching equation (3.23) with equation (3.20) as $\xi\to\infty$ in the third quadrant gives

$$\left(-\frac{\alpha_2}{3} + \frac{\mathrm{i}\sqrt{\pi}}{\sqrt{6}}\right)\Omega = -\mathrm{i}\sqrt{2}\Lambda, \quad C_0 = \frac{1}{2}, \quad C_1 - \frac{\mathrm{i}\pi}{2} - \log 2 = \alpha_1 - \frac{\alpha_2\sqrt{\pi}}{\sqrt{6}} + \frac{\gamma_E}{2} + \log 6 - \mathrm{i}\pi.$$

Thus, $\alpha_2 = 0$ and

$$\Omega = -\frac{2\sqrt{3}}{\sqrt{\pi}}\Lambda = -\frac{\sqrt{6}}{\pi^{3/2}}.$$

This gives, finally,

$$\bar{\lambda}_{n-1} = \lambda_n \sim \frac{(-1)^{n+1}\sqrt{6}}{\pi^{3/2}}3^n\Gamma(n+1/2), \tag{3.24}$$

in agreement with [2]. In figure 2, this result is compared with λ_n found by numerically iterating equations (3.6)–(3.7); the agreement is excellent.

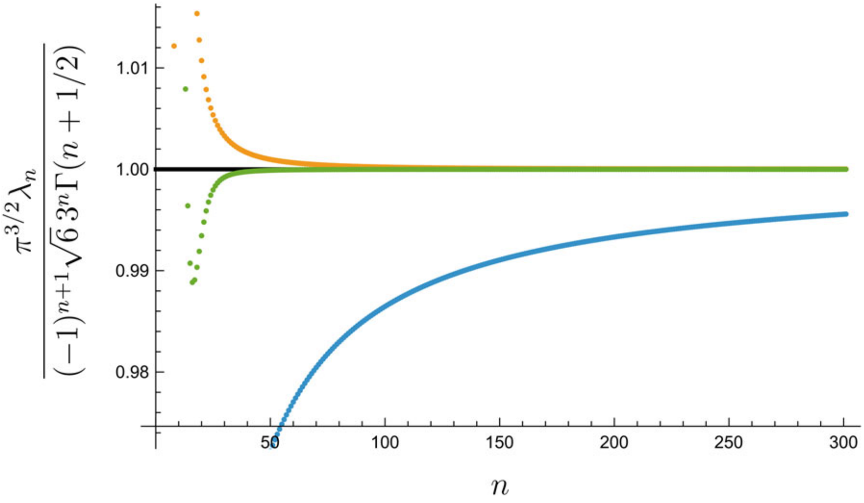

Figure 2. The ratio of the numerical value found by iterating equations (3.6)–(3.7) to the asymptotic prediction equation (3.24). Blue is the base series, while orange and green correspond to enhanced convergence using Richardson extrapolation on two and three terms, respectively. The black line is the asymptote, included to aid the eye.

4. Example 3: simplified Rossby waves

Our third example is a simplified version of the model problem for an equatorially trapped Rossby wave considered in [9]. Consider

$$\frac{d^2\psi}{dx^2} - 2x\frac{d\psi}{dx} + \frac{\epsilon^2\psi}{1+\epsilon x} = \lambda\psi, \quad e^{-x^2/2}\psi \to 0 \quad \text{as } x \to \pm\infty,$$

with $\psi(0) = 1$. Essentially, the problem considered in [9] has ϵ^2 in the third term replaced with ϵ. The switch to ϵ^2 makes the inner region below more complicated, but significantly simplifies all the other regions of the analysis. This weakens the strength of the pole in Boyd & Natarov's Hermite-with-pole equation [8], while keeping the same structure.

(a) Inner region

We see that the expansion in powers of ϵ proceeds as

$$\psi = \sum_{n=0}^{\infty} \epsilon^n \psi_n \quad \text{and} \quad \lambda = \sum_{n=0}^{\infty} \epsilon^{2n} \lambda_n. \tag{4.1}$$

At leading order

$$\psi_0'' - 2x\psi_0' = \lambda_0 \psi_0.$$

In order for $e^{-x^2/2}\psi(y)$ to decay as $x \to \pm\infty$, we need ψ_0 to be a Hermite polynomial. The leading eigenvalue, therefore, has $\psi_0 = 1$, $\lambda_0 = 0$. In general,

$$\psi_n'' - 2x\psi_n' + \sum_{k=2}^{n} (-x)^{k-2}\psi_{n-k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \lambda_k \psi_{n-2k}, \quad n \geq 1.$$

In particular, we find $\psi_1 = 0$, while

$$\psi_2'' - 2x\psi_2' + 1 = \lambda_1,$$

so that $\psi_2 = 0$, $\lambda_1 = 1$. Separating even and odd indices,

$$\psi''_{2n+1} - 2x\psi'_{2n+1} + \sum_{k=1}^{n}(-x)^{2k-2}\psi_{2(n-k)+1} + \sum_{k=1}^{n}(-x)^{2k-1}\psi_{2(n-k)} = \sum_{k=0}^{n}\lambda_k\psi_{2(n-k)+1}$$

and

$$\psi''_{2n} - 2x\psi'_{2n} + \sum_{k=1}^{n}(-x)^{2k-2}\psi_{2(n-k)} + \sum_{k=1}^{n-1}(-x)^{2k-1}\psi_{2(n-k)-1} = \sum_{k=0}^{n}\lambda_k\psi_{2(n-k)}.$$

The solutions are of the form

$$\psi_{2n+1} = \sum_{k=1}^{n} a_{2n+1,k}x^{2k-1} \quad \text{and} \quad \psi_{2n} = \sum_{k=0}^{n-1} a_{2n,k}x^{2k}, \tag{4.2}$$

with

$$2(2k-1)a_{2n+1,k} = 2k(2k+1)a_{2n+1,k+1} + \sum_{m=1}^{n} a_{2(n-m)+1,k-m+1} - \sum_{m=1}^{n} a_{2(n-m),k-m} - \sum_{m=0}^{n}\lambda_m a_{2(n-m)+1,k} \tag{4.3}$$

and

$$4ka_{2n,k} = (2k+2)(2k+1)a_{2n,k+1} + \sum_{m=1}^{n} a_{2(n-m),k-m+1} - \sum_{m=1}^{n-1} a_{2(n-m)-1,k-m+1} - \sum_{m=0}^{n}\lambda_m a_{2(n-m),k}, \tag{4.4}$$

with $a_{n,0} = 0$ for $n > 0$ and the convention that $a_{2n+1,k} = 0$ and $a_{2n,k} = 0$ if $k > n$ or $k < 1$. For each n, equations (4.3)–(4.4) may be solved iteratively stepping down from $k = n$. The solvability condition determining λ_n comes from setting $k = 0$ in equation (4.4), giving $\lambda_n = 2a_{2n,1}$. We now follow the procedure of §2 to determine the asymptotic behaviour of λ_n as $n \to \infty$.

(b) Outer region

As usual, the expansion equation (4.1) is not uniform in x and rearranges when x is large. We set $\epsilon x = X$ to give

$$\epsilon^2\frac{d^2\psi}{dX^2} - 2X\frac{d\psi}{dX} + \frac{\epsilon^2\psi}{1+X} = \lambda\psi.$$

The outer expansion now proceeds straightforwardly in powers of ϵ^2 as

$$\psi = \sum_{n=0}^{\infty}\epsilon^{2n}\psi_n \quad \text{and} \quad \lambda = \sum_{n=0}^{\infty}\epsilon^{2n}\lambda_n. \tag{4.5}$$

At leading order

$$-2X\frac{d\psi_0}{dX} = \lambda_0\psi_0,$$

with solution

$$\psi_0 = B_0X^{-\lambda_0/2}.$$

For there to be no singularity at $X = 0$ requires $\lambda_0/2$ to be a non-positive integer, in agreement with the inner analysis in §4a. The leading eigenvalue therefore has $\lambda_0 = 0$, $\psi_0 = 1$. At the next

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order,

$$-2X\frac{\mathrm{d}\psi_1}{\mathrm{d}X}+\frac{1}{1+X}=\lambda_1,$$

with solution

$$\psi_1=B_1+\frac{(1-\lambda_1)}{2}\log X-\frac{1}{2}\log(1+X).$$

For there to be no singularity at $X=0$ we require $\lambda_1=1$, in agreement with §4a. The boundary condition $\psi_1(0)=0$ (more properly a matching condition with the inner region) gives $B_1=0$. In general,

$$\frac{\mathrm{d}^2\psi_{n-1}}{\mathrm{d}X^2}-2X\frac{\mathrm{d}\psi_n}{\mathrm{d}X}+\frac{\psi_{n-1}}{1+X}=\sum_{k=1}^{n}\lambda_k\psi_{n-k}. \tag{4.6}$$

As usual, there are two types of divergence: a factorial/power from repeated differentiation of the singularity $\log(1+X)$ in ψ_1 and a factorial/constant from λ_n. For the first, we use the usual ansatz to give

$$\psi_n\sim\frac{\Lambda\Gamma(n-1)}{X(1-X^2)^{n-1}}. \tag{4.7}$$

Here, we see a curious feature of this example—the late term behaviour ψ_n was driven by a singularity at $X=-1$, but the singulant vanishes also at $X=+1$. Whereas in the anharmonic oscillator problem of §3, both singularities $X=\pm\mathrm{i}$ were present in the early terms, here only $X=-1$ is present in the early terms. The resolution of this apparent paradox, as described in [9], is a higher-order Stokes line, which turns off the contribution equation (4.7) in a region enclosing $X=1$. We return to this point later when matching with an inner region near $X=0$.

The other part of ψ_n satisfies $\psi_n\sim Q\lambda_n$ where

$$-2XQ'=1,$$

giving

$$Q=C-\frac{1}{2}\log X.$$

As usual, the presence of a logarithm means that we need to modify the large n ansatz to essentially allow C to depend on $\log n$. Putting both parts of ψ_n together, we have

$$\psi_n\sim\frac{\Lambda\Gamma(n-1)}{X(1-X^2)^{n-1}}+\left(-\frac{1}{2}\log X+C_0\log n+C_1\right)\lambda_n. \tag{4.8}$$

The next step is to determine the constant Λ, by matching with an inner region near the singularity at $X=-1$.

(c) Inner region near $X=-1$

We set $X=-1+\epsilon^2\hat{x}$. Then, the inner limit of the outer expansion satisfies

$$\psi_0+\epsilon^2\psi_1\sim1-\epsilon^2\log\epsilon-\frac{\epsilon^2}{2}\log\hat{x} \tag{4.9}$$

and

$$\epsilon^{2n}\psi_n\sim-\frac{\epsilon^2\Lambda\Gamma(n-1)}{(2\hat{x})^{n-1}}. \tag{4.10}$$

Equation (4.9) motivates writing $\psi=1-\epsilon^2\log\epsilon+\epsilon^2\hat{\psi}$ to give the inner equation as

$$\frac{\mathrm{d}^2\hat{\psi}}{\mathrm{d}\hat{x}^2}-2(-1+\epsilon^2\hat{x})\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}\hat{x}}+\frac{(1-\epsilon^2\log\epsilon+\epsilon^2\hat{\psi})}{\hat{x}}=\lambda(1-\epsilon^2\log\epsilon+\epsilon^2\hat{\psi}).$$

At leading order

$$\frac{\mathrm{d}^2\hat{\psi}_0}{\mathrm{d}\hat{x}^2}+2\frac{\mathrm{d}\hat{\psi}_0}{\mathrm{d}\hat{x}}+\frac{1}{\hat{x}}=0.$$

Thus,

$$\hat{\psi}_0 = \beta_1 + \beta_2 \mathrm{e}^{-2\hat{x}} - \frac{1}{2}\log\hat{x} + \frac{1}{2}\mathrm{e}^{-2\hat{x}}\,\mathrm{Ei}(2\hat{x}) = \beta_1 + \beta_2 \mathrm{e}^{-2\hat{x}} - \frac{1}{2}\log\hat{x} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{(2\hat{x})^{n+1}}.$$

Matching with equation (4.10) gives

$$\Lambda = -\frac{1}{2}.$$

(d) Boundary layer in the late terms near $X = 0$

As usual, the large n asymptotic series for ψ_n in the outer region rearranges when X is small. This is again clear from the fact that the asymptotic approximation for ψ_n, equation (4.8), is singular at $X = 0$, while ψ_n is not. There is a boundary layer near the origin in the large n expansion of ψ_n. The appropriate scaling of this inner region is $X = \xi/n^{1/2}$, so that $\mathrm{d}^2\psi_{n-1}/\mathrm{d}X^2$ balances $X\mathrm{d}\psi_n/\mathrm{d}X$, so that equation (4.6) becomes

$$n\frac{\mathrm{d}^2\psi_{n-1}}{\mathrm{d}\xi^2} - 2\xi\frac{\mathrm{d}\psi_n}{\mathrm{d}\xi} + \frac{\psi_{n-1}}{1+\xi/n^{1/2}} = \sum_{k=1}^{n}\lambda_k\psi_{n-k}. \tag{4.11}$$

The inner limit of the outer expansion equation (4.8) is, for $X < 0$,

$$\begin{aligned}\psi_n &\sim -\frac{\Gamma(n-1)n^{1/2}}{2\xi(1-\xi^2/n)^{n-1}} + \left(-\frac{1}{2}\log(\xi) + C + \frac{1}{4}\log n\right)\lambda_n \\ &\sim -\frac{\Gamma(n-1/2)}{2}\frac{\mathrm{e}^{\xi^2}}{\xi} + \left(-\frac{1}{2}\log(\xi) + (C_0 + 1/4)\log n + C_1\right)\lambda_n.\end{aligned} \tag{4.12}$$

This motivates setting $\lambda_n = \Omega\Gamma(n-1/2)$, $\psi_n \sim H\Omega\Gamma(n-1/2)$ in equation (4.11), to give

$$H'' - 2\xi H' = 1,$$

where $' \equiv \mathrm{d}/\mathrm{d}\xi$, so that

$$H = \alpha_1 + \alpha_2\int_0^{\xi}\mathrm{e}^{t^2}\,\mathrm{d}t + \int_0^{\xi}\mathrm{e}^{t^2}\int_0^{t}\mathrm{e}^{-p^2}\,\mathrm{d}p\,\mathrm{d}t.$$

Both the particular integral and the complementary function exhibit Stokes' phenomenon for large ξ, corresponding to the higher-order Stokes phenomenon in ψ. There is a higher-order Stokes line owing to the particular integral when ξ crosses the imaginary axis, across which the coefficient of e^{ξ^2}/ξ in the far field changes. In addition, there is a higher-order Stokes line on the real axis, across which the constant in the far field changes. For real ξ, as $\xi \to -\infty$

$$H \sim \alpha_1 + \mathrm{i}\frac{\alpha_2\sqrt{\pi}}{2} + \left(\frac{\alpha_2}{2} - \frac{\sqrt{\pi}}{4}\right)\frac{\mathrm{e}^{\xi^2}}{\xi} + \cdots - \frac{1}{4}(\gamma_E + \log(-4\xi^2) + \cdots),$$

while as $\xi \to +\infty$

$$H \sim \alpha_1 - \mathrm{i}\frac{\alpha_2\sqrt{\pi}}{2} + \left(\frac{\alpha_2}{2} + \frac{\sqrt{\pi}}{4}\right)\frac{\mathrm{e}^{\xi^2}}{\xi} + \cdots - \frac{1}{4}(\gamma_E + \log(-4\xi^2) + \cdots),$$

where γ_E is the Euler gamma.

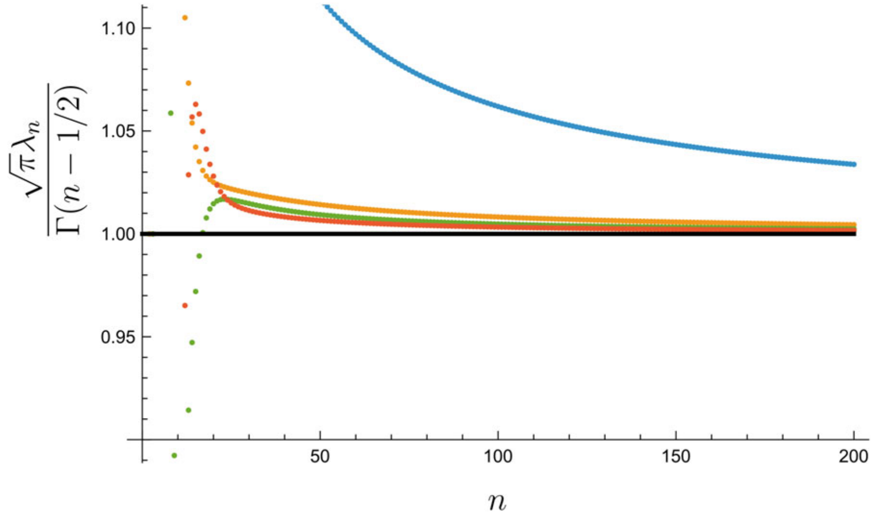

Figure 3. The ratio of the numerical value found by iterating equations (4.3)–(4.4) to the asymptotic prediction equation (4.13). Blue is the base series, while orange, green and red correspond to enhanced convergence using Richardson extrapolation on two, three and four terms, respectively. The black line is the asymptote, included to aid the eye. The convergence is slower than expected because of the presence of logarithmic terms in the higher-order corrections, unaccounted for in the extrapolation.

(e) Matching with the outer

The outer limit of the inner is

$$\psi_n \sim \Omega\left(\frac{\alpha_2}{2} - \frac{\sqrt{\pi}}{4}\right)\frac{e^{\xi^2}}{\xi}\Gamma(n-1/2) + \cdots - \frac{\lambda_n}{2}(\log\xi + \cdots) \quad \text{as } \xi \to -\infty$$

and

$$\psi_n \sim \Omega\left(\frac{\alpha_2}{2} + \frac{\sqrt{\pi}}{4}\right)\frac{e^{\xi^2}}{\xi}\Gamma(n-1/2) + \cdots - \frac{\lambda_n}{2}(\log\xi + \cdots) \quad \text{as } \xi \to +\infty.$$

Matching with equation (4.12) as $\xi \to -\infty$ gives

$$\Omega\left(\frac{\alpha_2}{2} - \frac{\sqrt{\pi}}{4}\right) = -\frac{1}{2}.$$

For $X > 0$, as per the discussion following equation (4.7), there can be no exponential term in the outer, because there must be no singularity at $X = 1$. Thus, matching as $\xi \to \infty$ gives

$$\Omega\left(\frac{\alpha_2}{2} + \frac{\sqrt{\pi}}{4}\right) = 0.$$

Together

$$\alpha_2 = -\frac{\sqrt{\pi}}{2} \quad \text{and} \quad \Omega = \frac{1}{\sqrt{\pi}},$$

so that

$$\lambda_n \sim \frac{\Gamma(n-1/2)}{\sqrt{\pi}}. \tag{4.13}$$

In figure 3, this result is compared with λ_n found by numerically iterating equations (4.3)–(4.4); the agreement is good, though the convergence is slower than expected because of the presence of logarithmic terms in the higher-order corrections.

5. Example 4: divergence driven by two singularities

Our final example is chosen to illustrate that the divergence of the eigenvalue can be driven by more than one singularity in the outer solution, leading to more exotic behaviour. This is exactly what happens in the model in [10] concerning quasinormal modes of Reissner–Nordström–de Sitter black holes. The form of this divergence is more difficult to pick up with other methods, and the interaction between two singularities makes it difficult to guess the form of the divergence from numerical calculations of the leading terms in the series.

Consider, as a model problem,

$$\frac{b^2+(c+\epsilon x)^2}{b^2+c(c+\epsilon x)}(-\omega g+xg')+g+(xg')'=0, \quad -\infty<x<0, \tag{5.1}$$

with

$$g(0)=1, \quad g(x)=o(\mathrm{e}^{-x}) \quad \text{as } x\to\infty.$$

The relationship with equation (2.1) is clear—the coefficient of the first term has been modified to generate an outer solution with two singularities.

(a) Inner region

We expand

$$g=\sum_{n=0}^{\infty}\epsilon^n g_n \quad \text{and} \quad \omega=\sum_{n=0}^{\infty}\epsilon^n\omega_n,$$

to give at leading order

$$-\omega_0 g_0+xg_0'+g_0+(xg_0')'=0,$$

with solution $g_0=1$, $\omega_0=1$. At the next order,

$$-\frac{cx}{b^2+c^2}+(-\omega_1+xg_1')+(xg_1')'=0,$$

with solution

$$g_1=-\frac{cx}{b^2+c^2} \quad \text{and} \quad \omega_1=\frac{c}{b^2+c^2}.$$

In general,

$$g_n=\sum_{i=1}^{n}a_{ni}x^i,$$

with

$$\begin{aligned} ia_{ni}=&\sum_{k=1}^{n}\omega_k a_{n-k,i}-(i+1)^2a_{n,i+1}\\ &-\frac{c}{(b^2+c^2)}\left(-2\sum_{k=0}^{n-1}\omega_k a_{n-k-1,i-1}+2(i-1)a_{n-1,i-1}+a_{n-1,i-1}+i^2a_{n-1,i}\right)\\ &-\frac{1}{(b^2+c^2)}\left(-\sum_{k=0}^{n-2}\omega_k a_{n-2-k,i-2}+(i-2)a_{n-2,i-2}\right), \end{aligned} \tag{5.2}$$

and $\omega_n=a_{n,1}$. As usual, we can iterate equation (5.2) numerically. Figure 4 shows $|\omega_n|^{1/n}$ as a function of n; the linear growth in n is consistent with factorial growth in ω_n at large n. Note that this growth is not nearly as smooth as that in figure 1a, with some ripples present. Similar ripples can be seen in fig. 2 of [10]. These ripples are a direct result of the interaction of the contributions from the two singularities in the outer problem.

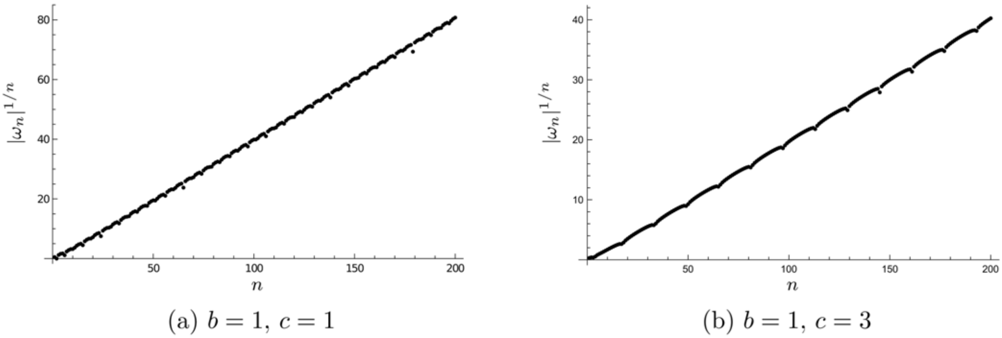

Figure 4. Divergence of the coefficients in the asymptotic expansion of ω, determined numerically from equation (5.2). The linear growth is consistent with factorial divergence.

(b) Outer region

We set $\epsilon x = X$ to give

$$\frac{b^2+(c+X)^2}{b^2+c(c+X)}(-\omega g + Xg') + g + \epsilon (Xg')' = 0.$$

Expanding

$$g = \sum_{n=0}^{\infty} \epsilon^n g_n \quad \text{and} \quad \omega = \sum_{n=0}^{\infty} \epsilon^n \omega_n, \tag{5.3}$$

and using $\omega_0 = 1$ gives

$$\frac{b^2+(c+X)^2}{b^2+c(c+X)}(-g_0 + Xg_0') + g_0 = 0,$$

with solution

$$g_0 = \frac{\sqrt{b^2+(c+X)^2}}{\sqrt{b^2+c^2}}, \tag{5.4}$$

where we have used the fact that $g_0 \to 1$ as $X \to 0$. We see that g_0 has singularities when $b^2 + (c + X)^2 = 0$, i.e. $X = -c \pm ib$ (of course, the coefficient of the first term in equation (5.1) was chosen to make this the case). In general,

$$\frac{b^2+(c+X)^2}{b^2+c(c+X)}(-g_n + Xg_n') + g_n = -(Xg_{n-1}')' + \frac{b^2+(c+X)^2}{b^2+c(c+X)}(\omega_1 g_{n-1} + \cdots + \omega_n g_0).$$

As usual there are two types of divergence: a factorial/power from the differentiation and a factorial/constant from the ω_n. For the first, we use the usual ansatz,

$$g_n = \frac{G\Gamma(n+\gamma)}{\chi^{n+\gamma}}. \tag{5.5}$$

At leading order in n this gives

$$-\frac{b^2+(c+X)^2}{b^2+c(c+X)} = -\chi'$$

so that

$$\chi = \frac{(c+X)(cX+c^2-2b^2)}{2c^2} + \frac{b^2(b^2+c^2)\log(b^2+c^2+cX)}{c^3} + \text{const.}$$

This time there are two possible late-term divergences, one corresponding to each of the two singularities of the leading-order solution,

$$\chi_1 = \frac{2ib^3 + c(c+X)^2 - b^2(c+2X)}{2c^2} + \frac{b^2(b^2+c^2)}{c^3}\log\left(\frac{b^2+c(c+X)}{b^2+ibc}\right)$$

and

$$\chi_2 = \frac{-2\mathrm{i}b^3 + c(c+X)^2 - b^2(c+2X)}{2c^2} + \frac{b^2(b^2+c^2)}{c^3}\log\left(\frac{b^2+c(c+X)}{b^2-\mathrm{i}bc}\right).$$

At the next order,

$$\frac{b^2+(c+X)^2}{b^2+c(c+X)}(-G+XG') + G = G\chi' + X(2G'\chi' + G\chi'').$$

Thus,

$$\frac{(b^4+c(c+X)^2(c+3X)+b^2(2c^2+5cX+4X^2))}{(b^2+c(c+X))^2}G + \frac{X(b^2+(c+X)^2)}{b^2+c(c+X)}G' = 0,$$

giving

$$G = \frac{\Lambda(b^2+c(c+X))}{X(b^2+(c+X)^2)^{3/2}}.$$

Thus, this part of g_n satisfies

$$g_n \sim \frac{\Lambda_1(b^2+c(c+X))}{X(b^2+(c+X)^2)^{3/2}}\frac{\Gamma(n+\gamma_1)}{\chi_1^{n+\gamma_1}} + \frac{\Lambda_2(b^2+c(c+X))}{X(b^2+(c+X)^2)^{3/2}}\frac{\Gamma(n+\gamma_2)}{\chi_2^{n+\gamma_2}}.$$

To determine γ_1 and γ_2 we match the order of the singularity as $X \to -c \pm \mathrm{i}b$ with the early terms. As $X \to -c + \mathrm{i}b$,

$$\chi_1 \sim \frac{(X+c-\mathrm{i}b)^2}{c-\mathrm{i}b},$$

and

$$g_n \sim -\frac{\mathrm{i}b}{2\sqrt{2}(\mathrm{i}b)^{3/2}(X+c-\mathrm{i}b)^{3/2}}\frac{(c-\mathrm{i}b)^{n+\gamma_1}}{(X+c-\mathrm{i}b)^{2n+2\gamma_1}}\Lambda_1\Gamma(n+\gamma_1).$$

Comparing powers of $X+c-\mathrm{i}b$ with g_0 gives

$$-\frac{3}{2} - 2\gamma_1 = \frac{1}{2} \Rightarrow \gamma_1 = -1.$$

A similar comparison as $X \to -c - \mathrm{i}b$ gives $\gamma_2 = -1$ also, so that

$$g_n \sim \frac{\Lambda_1(b^2+c(c+X))}{X(b^2+(c+X)^2)^{3/2}}\frac{\Gamma(n-1)}{\chi_1^{n-1}} + \frac{\Lambda_2(b^2+c(c+X))}{X(b^2+(c+X)^2)^{3/2}}\frac{\Gamma(n-1)}{\chi_2^{n-1}}.$$

The other part of g_n satisfies $g_n = (Q_0\log n + Q_1)\omega_n$,

where
$$\frac{b^2+(c+X)^2}{b^2+c(c+X)}(-Q_0+XQ_0') + Q_0 = 0$$

and

$$\frac{b^2+(c+X)^2}{b^2+c(c+X)}(-Q_1+XQ_1') + Q_1 = \frac{b^2+(c+X)^2}{b^2+c(c+X)}g_0,$$

giving

$$Q_0\log n + Q_1 = \frac{\sqrt{b^2+(c+X)^2}}{b^2+c^2}(\log X + C_0\log n + C_1).$$

Together,

$$g_n \sim \frac{\Lambda_1(b^2+c(c+X))}{X(b^2+(c+X)^2)^{3/2}}\frac{\Gamma(n-1)}{\chi_1^{n-1}} + \frac{\Lambda_2(b^2+c(c+X))}{X(b^2+(c+X)^2)^{3/2}}\frac{\Gamma(n-1)}{\chi_2^{n-1}} + \frac{\sqrt{b^2+(c+X)^2}}{b^2+c^2}(\log X + C_0\log n + C_1)\omega_n. \tag{5.6}$$

The next step is to determine Λ_1 and Λ_2 through matching with inner regions near $X = -c \pm \mathrm{i}b$.

(c) Inner region near $X = -c + ib$

To determine Λ_1, we look near $X = -c + ib$. We set $X = -c + ib + \epsilon^{1/2}(c - ib)^{1/2}\hat{x}$, $g = \epsilon^{1/4}\hat{g}$ to give

$$\frac{2i\epsilon^{1/2}(c-ib)^{1/2}\hat{x}}{b+ic}\left(-\omega\hat{g} + \frac{1}{(c-ib)^{1/2}\epsilon^{1/2}}(-c+ib+\epsilon^{1/2}(c-ib)^{1/2}\hat{x})\hat{g}'\right)$$
$$+\hat{g} + \epsilon\left(\frac{1}{(c-ib)^{1/2}\epsilon^{1/2}}\hat{g}' + (-c+ib+(c-ib)^{1/2}\epsilon^{1/2}\hat{x})\frac{1}{(c-ib)\epsilon}\hat{g}''\right) = 0.$$

At leading order

$$-2\hat{x}\hat{g}_0' + \hat{g}_0 - \hat{g}_0'' = 0.$$

Writing

$$\hat{g}_0 = \frac{\sqrt{2}(ib)^{1/2}(c-ib)^{1/4}}{\sqrt{b^2+c^2}}\sum_{n=0}^{\infty} c_n x^{1/2-2n}, \tag{5.7}$$

gives

$$c_n = \frac{(2n-5/2)(2n-3/2)c_{n-1}}{4n}, \quad c_0 = 1,$$

where the latter condition comes from matching with equation (5.4). Thus,

$$c_n = -\frac{(3/4)_{n-1}(5/4)_{n-1}}{16(2)_{n-1}}.$$

The inner limit of the outer expansion is

$$\epsilon^n g_n \sim -\frac{ib\epsilon^n}{2\sqrt{2}(ib)^{3/2}(X+c-ib)^{3/2}}\frac{(c-ib)^{n-1}}{(X+c-ib)^{2n-2}}\Lambda_1\Gamma(n-1)$$
$$\sim -\frac{\epsilon^{1/4}}{2\sqrt{2}(ib)^{1/2}}\frac{\Lambda_1\Gamma(n-1)\hat{x}^{1/2-2n}}{(c-ib)^{3/4}}. \tag{5.8}$$

Matching equation (5.7) with equation (5.8) gives

$$\Lambda_1 = -\frac{\sqrt{2}(ib)^{1/2}(c-ib)^{1/4}}{\sqrt{b^2+c^2}}2\sqrt{2}(ib)^{1/2}(c-ib)^{3/4}\lim_{n\to\infty}\frac{c_n}{\Gamma(n-1)}$$
$$= \frac{(ib)(c-ib)^{1/2}}{(c+ib)^{1/2}}\frac{4}{16\Gamma(3/4)\Gamma(5/4)} = \frac{(ib)(c-ib)^{1/2}}{\sqrt{2}\pi(c+ib)^{1/2}}.$$

A similar calculation near $X = -c - ib$ shows

$$\Lambda_2 = \frac{(-ib)(c+ib)^{1/2}}{\sqrt{2}\pi(c-ib)^{1/2}} = \bar{\Lambda}_1,$$

where an overbar denotes complex conjugation.

(d) Boundary layer in the late terms near $X = 0$

As in all our examples, the large n asymptotic series for g_n in the outer region rearranges when X is small, so that there is an inner region near the origin. As in §2 the appropriate rescaling is

$X = \xi/n$, under which the equation for g_n becomes

$$\frac{b^2+(c+\xi/n)^2}{b^2+c(c+\xi/n)}(-g_n+\xi g_n')+g_n=-n(\xi g_{n-1}')'+\frac{b^2+(c+\xi/n)^2}{b^2+c(c+\xi/n)}(\omega_1 g_{n-1}+\cdots+\omega_n g_0). \tag{5.9}$$

Writing χ_1 and χ_2 in terms of ξ gives

$$\chi_1 \sim \frac{2\mathrm{i}b^3-b^2c+c^3}{2c^2}+\frac{b^2(b^2+c^2)}{c^3}\log\left(1-\frac{\mathrm{i}c}{b}\right)+\frac{\xi}{n}+\cdots=\chi_0+\frac{\xi}{n}$$

and

$$\chi_2 \sim \frac{-2\mathrm{i}b^3-b^2c+c^3}{2c^2}+\frac{b^2(b^2+c^2)}{c^3}\log\left(1+\frac{\mathrm{i}c}{b}\right)+\frac{\xi}{n}+\cdots=\bar{\chi}_0+\frac{\xi}{n},$$

say. Thus, the inner limit of the outer solution is

$$\begin{aligned} g_n &\sim \frac{\Lambda_1}{\xi(b^2+c^2)^{1/2}}\frac{\Gamma(n)}{(\chi_0+\xi/n)^{n-1}}+\frac{\bar{\Lambda}_1}{\xi(b^2+c^2)^{1/2}}\frac{\Gamma(n)}{(\bar{\chi}_0+\xi/n)^{n-1}} \\ &\quad+(\log(\xi/n)+C_0\log n+C_1)\omega_n \\ &\sim \frac{\Lambda_1}{\xi(b^2+c^2)^{1/2}}\frac{\Gamma(n)}{\chi_0^{n-1}}\mathrm{e}^{-\xi/\chi_0}+\frac{\bar{\Lambda}_1}{\xi(b^2+c^2)^{1/2}}\frac{\Gamma(n)}{\bar{\chi}_0{}^{n-1}}\mathrm{e}^{-\xi/\bar{\chi}_0} \\ &\quad+(\log\xi+(C_0-1)\log n+C_1)\omega_n. \end{aligned} \tag{5.10}$$

From our analyses in §§2–4, we have seen that the boundary-layer approximation to g_n comprises a particular integral driven by ω_n and a complementary function matching with the remaining factorial/power divergence of the outer expansion. For the current problem, we write the particular integral as $g_n = H\omega_n$, giving

$$(\xi H')'+\xi H'=1,$$

so that

$$H=\log\xi$$

as in §2. Matching this particular solution with equation (5.10) gives $C_0=1$, $C_1=0$. The homogeneous solution may be written as

$$g_n=G(\xi)\frac{\Gamma(n)}{\chi_0^n}+\bar{G}(\xi)\frac{\Gamma(n)}{\bar{\chi}_0^n},$$

where

$$\chi_0(\xi G')'+\xi G'=0,$$

giving

$$G=\alpha_1+\alpha_2\,\mathrm{Ei}\left(-\frac{\xi}{\chi_0}\right).$$

Thus, together, we have

$$g_n\sim\left(\alpha_1+\alpha_2\,\mathrm{Ei}\left(-\frac{\xi}{\chi_0}\right)\right)\frac{\Gamma(n)}{\chi_0^n}+\left(\bar{\alpha}_1+\bar{\alpha}_2\,\mathrm{Ei}\left(-\frac{\xi}{\bar{\chi}_0}\right)\right)\frac{\Gamma(n)}{\bar{\chi}_0^n}+\omega_n\log\xi.$$

Now, as in §2, g_n should be regular as $\xi\to 0$. Thus, since $\mathrm{Ei}(\xi)\sim\log\xi$ as $\xi\to 0$, we need

$$\omega_n\sim-\alpha_2\frac{\Gamma(n)}{\chi_0^n}-\bar{\alpha}_2\frac{\Gamma(n)}{\bar{\chi}_0^n}.$$

To complete the analysis we need to match with equation (5.10) to determine α_2. As $\xi\to\infty$

$$\mathrm{Ei}\left(-\frac{\xi}{\chi_0}\right)\sim-\frac{\chi_0\mathrm{e}^{-\xi/\chi_0}}{\xi}.$$

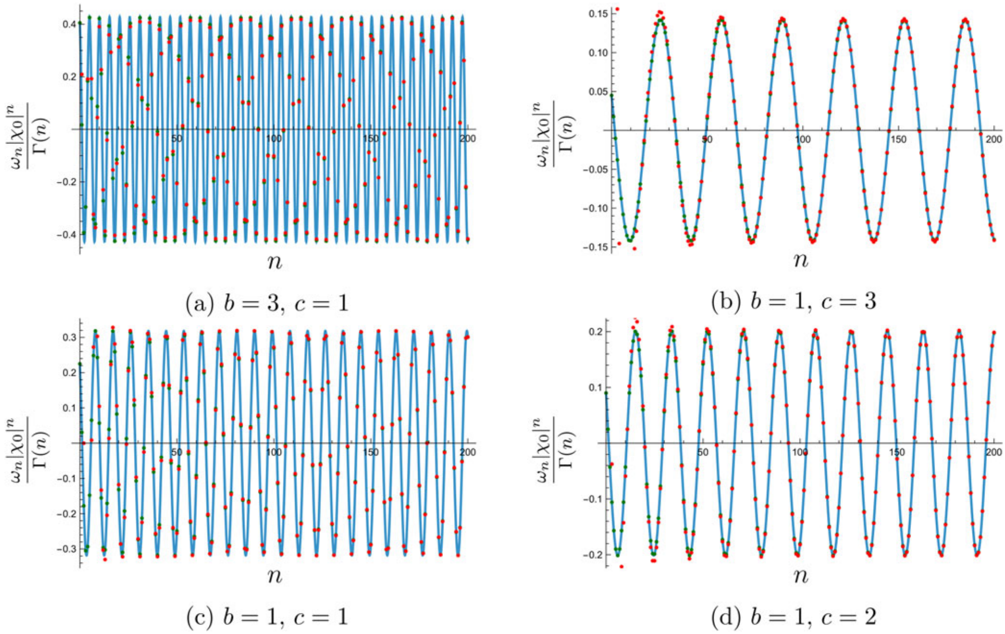

Figure 5. A comparison of the asymptotic approximation equation (5.11) with ω_n found by numerically iterating equation (5.2), for various values of b and c. We normalize by $\Gamma(n)/|\chi_0|^n$ to remove the exponential growth. The solid curve shows equation (5.11) as a continuous function of n, which is a sinusoidal oscillation of period $\arg(\chi_0)/2\pi$. The green dots show equation (5.11) evaluated at integer n. The red dots are the numerical values.

Thus, the outer limit of the inner is

$$g_n \sim \left(\alpha_1 - \alpha_2 \frac{\chi_0 e^{-\xi/\chi_0}}{\xi}\right) \frac{\Gamma(n)}{\chi_0^{n-1}} + \left(\bar{\alpha}_1 - \bar{\alpha}_2 \frac{\bar{\chi}_0 e^{-\xi/\bar{\chi}_0}}{\xi}\right) \frac{\Gamma(n)}{\bar{\chi}_0^{n-1}} + \omega_n \log \xi.$$

Matching with equation (5.10) gives

$$\alpha_2 = -\frac{\Lambda_1}{(b^2+c^2)^{1/2}} = -\frac{ib}{\sqrt{2}\pi(c+ib)}.$$

Thus,

$$\omega_n \sim \frac{ib}{\sqrt{2}\pi(c+ib)} \frac{\Gamma(n)}{\chi_0^n} - \frac{ib}{\sqrt{2}\pi(c-ib)} \frac{\Gamma(n)}{\bar{\chi}_0^n}. \tag{5.11}$$

In figure 5, this result is compared with ω_n found by numerically iterating equation (5.2) for various values of b and c. The sinusoidal oscillation predicted by equation (5.11) is clear in figure 5b,d, when the period of the oscillation is long enough that there are many integers per cycle, but when the period is short ω_n seems to jump around between different longwave oscillations because of aliasing.

6. Conclusion

Through four examples, we have demonstrated a systematic procedure for calculating the precise asymptotic behaviour of the late terms of the asymptotic expansion of the eigenvalue in a variety of linear eigenvalue problems. The framework in each of our examples is the same.

After a regular perturbation expansion, the eigenfunction at each order is a polynomial, leading to a set of recurrence relations for the coefficients of these polynomials and the coefficients of the eigenvalue expansion. While these relations are easy to iterate numerically to get the leading terms of the eigenvalue expansion, it is hard to extract the late term behaviour from them.

This regular perturbation expansion is non-uniform, and rearranges when x is large. Rescaling to an outer variable the corresponding outer solution can be found, again as an asymptotic power series. This series has a standard factorial/power divergence driven by singularities in the leading-order approximation, and an additional divergence driven by the divergent eigenvalue expansion. In contrast to the original expansion, the late terms in this outer asymptotic expansion are easy to find, using the usual factorial/power ansatz, but the divergence of the eigenvalue is still undetermined.

However, the late term approximation of this outer expansion also non-uniform, now not as $\epsilon \to 0$ but as $n \to \infty$. By introducing a local variable in the equation for the late terms of the outer expansion, a new inner expansion is generated in which the two parts of the divergence become coupled, and the eigenvalue is determined.

We hope our framework provides a template by which similar problems of interest may be solved.

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