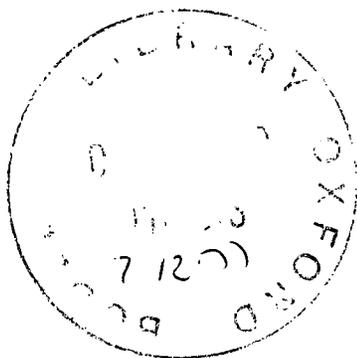


SOME PROBLEMS IN DIFFERENTIAL OPERATORS
(ESSENTIAL SELF-ADJOINTNESS)

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ABSTRACT

We consider a formally self-adjoint elliptic differential operator in \mathbb{R}^n , denoted by τ . T_0 and T are operators given by τ with specific domains. We determine conditions under which T_0 is essentially self-adjoint, introducing the topic by means of a brief historical survey of some results in this field.

In Part I, we consider an operator of order 4, and in Part II, we generalise the results obtained there to ones for an operator of order $2m$. Thus, the two parts run parallel.

In Chapter 1, we determine the domain of T_0^* , denoted by $\mathcal{D}(T_0^*)$, where T_0^* denotes the adjoint of T_0 , and introduce operators \underline{T}_0 and \underline{T} which are modifications of T_0 and T .

In Chapter 2, we use a theorem of Schechter to give conditions under which \underline{T}_0 is essentially self-adjoint.

Working with the operator \underline{T} , in Chapter 3 we show that we can approximate functions u in $\mathcal{D}(T_0^*)$ by a particular sequence of test-functions, which enables us to derive an identity involving u , Tu and the coefficient functions of the operator concerned.

In Chapter 4, we determine an upper bound for the integral of a function involving a derivative of u in $\mathcal{D}(T_0^*)$ whose order is half the order of the operator concerned, and we use the identity from the previous chapter to reformulate this upper bound.

In Chapter 5, we give conditions which are sufficient for the essential self-adjointness of T_0 . In the main theorem itself, the major step is the derivation of the integral of the function involving the

particular derivative of u in $\mathcal{D}(T_0^*)$ whose order is half the order of the operator concerned, referred to above, itself as a term of an upper bound of an integral we wish to estimate. Hence, we can employ the upper bound from Chapter 4. This "sandwiching" technique is basic to the approach we have adopted.

We conclude with a brief discussion of the operators we considered, and restate the examples of operators which we showed to be essentially self-adjoint.

CONTENTS

INTRODUCTION	i
NOTATION AND PRELIMINARY RESULTS	vii
<u>PART I:</u> The operator τ : $\tau u = \nabla^2(\nabla^2 u) - \nabla \cdot (p \nabla u) + qu$	
Chapter 1: The domain of T_0^*	1
Chapter 2: The essential self-adjointness of \underline{T}_0	4
Chapter 3: An identity for $\int \phi^4 u \tau u$	6
Chapter 4: An upper bound for $\int (\phi \nabla^2(\phi u))^2$	16
Chapter 5: The essential self-adjointness of T_0	20
<u>PART II:</u> The operator τ : $\tau u = (-1)^m \ell_{2m} u + \sum_{t=1}^{m-1} (-1)^t \ell_t \cdot (p_t \ell_t u) + qu$	
Chapter 1: The domain of T_0^*	33
Chapter 2: The essential self-adjointness of \underline{T}_0	35
Chapter 3: An identity for $\int \phi^{2m} u \tau u$	37
Chapter 4: An upper bound for $\int \phi^{m-1} \ell_m(\phi u) ^2$	62
Chapter 5: The essential self-adjointness of T_0	76
CONCLUSION	96
ADDENDUM: $q = q_1 + q_2$, $q_2 \neq 0$	98
APPENDICES	
§1 Some conditions equivalent to essential self-adjointness	ci
§2 The proof of Theorem 2.1 (Schechter [17])	ciii
§3 The proof of Lemma 3.4 (Ikebe and Kato [12])	cx
REFERENCES AND BIBLIOGRAPHY	cxii

INTRODUCTION

When considering the topic of essential self-adjointness of differential operators, one considers whether or not a symmetric operator T_0 has a unique self-adjoint extension; equivalently, whether or not $\overline{T_0} = \overline{T_0}^*$, where $\overline{T_0}$ is the closure of the operator T_0 . Therefore, the first differential operator to be considered was formally given by

$$\tau_1 u = -(p(x)u')' + q(x)u \quad (1)$$

for $x \in [0, \infty)$, p and q real-valued, and working in $L^2(0, \infty)$.

Naturally, the next step was to consider operators given by

$$\tau_2 u = \sum_{j,k=1}^n [i\partial_j + b_j(x)] a_{jk}(x) [i\partial_k + b_k(x)] u + q(x)u \quad (2)$$

$$\text{and } \tau_3 u = (r(x)u'')'' - (p(x)u')' + q(x)u \quad (3)$$

In (2), $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, the matrix $\{a_{jk}(x)\}$ is symmetric, a_{jk}, b_j and q are real-valued, $\partial_j = \partial/\partial x_j$, and $i = \sqrt{-1}$;

$T_0 u = \tau_3 u$ with $\mathcal{D}(T_0) = C_0^\infty(\mathbb{R}^n)$, $Tu = \tau_3 u$ with $\mathcal{D}(T) = L^2(\mathbb{R}^n)$.

In (3), $x \in [0, \infty)$, p, q and r are real-valued, and we work in $L^2(0, \infty)$.

Considering a $2m$ -th-order real symmetric ordinary differential operator

$$\tau_4 u = \sum_{s=0}^m (-1)^s (p_s(x)u^{(s)})^{(s)} \quad (4)$$

for $x \in [0, \infty)$ and p_s real-valued, the deficiency index, $d(\tau_4)$, of τ_4 is defined as the number of linearly independent $L^2(0, \infty)$ solutions of

$\tau_4 u = \lambda u$ ($\Im m \lambda \neq 0$). This is independent of λ and $m \leq d(\tau_4) \leq 2m$.

According to Weyl's classification, if $d(\tau_4) = 1$ then τ_4 is "limit-point" (l.p.), and if $d(\tau_4) = 2$ then τ_4 is "limit-circle". This terminology has been extended, so that if $d(\tau_4) = m$ then τ_4 is l.p. The notion of being l.p. corresponds to that of being essentially self-adjoint (see, for instance, Hellwig [10, §14.1]).

Much of the early work done on (1) and (2) was by Hartman and Wintner [8], Titchmarsh [20], Sears [18], and Kato [13], the latter being motivated by the arising in quantum mechanics of the Schrödinger equation $\nabla^2 u + \{\lambda - q(x)\}u = 0$. (In (2) with $x \in \mathbb{R}^3$, take $b_j(x) = 0$, $\{a_{jk}(x)\} = I$, the identity matrix, and set $\tau_2 u = \lambda u$.) This is the description of the motion of a particle under the influence of forces represented by $q(x)$, here taken as repulsive from the origin when $q > 0$. The notion of being essentially self-adjoint corresponds roughly to quantum mechanical completeness, i.e. if the operator is essentially self-adjoint, then the particle cannot escape to infinity in a finite time. That this correspondence is not exact is shown by Rauch and Reed [16].

The problem is to derive conditions on the coefficient functions which imply that τ_1 is l.p. (T_0 is e.s.a.), and it was Hartman [9], considering $u'' + q(x)u = 0$, who first showed that these need hold only on a sequence of intervals of $[0, \infty)$.

The research then diverged. On the one hand, work was done on (1), generalising and improving known conditions to "interval" conditions, notably by Eastham [2], and by Atkinson and Evans [1]. We give the former's result.

Theorem 0.1 Let τu be given by (1) with $p(x) \equiv 1$ and $q \in L^2_{loc}$, i.e.

$$\tau u = -u'' + q(x)u.$$

Let there be a sequence of non-overlapping intervals (a_ν, b_ν) and a sequence of real numbers w_ν such that

- (i) $(b_\nu - a_\nu)^2 w_\nu \geq K$, where $K > 0$ is a constant;
- (ii) $\sum w_\nu^{-1}$ is a divergent infinite series;
- (iii) $\int_{a_\nu}^{b_\nu} q_-(x) \geq -k(b_\nu - a_\nu)^3 w_\nu^2$, where $q_-(x) = \min(q(x), 0)$,
for some constant k .

Then τ is l.p. \square

On the other hand, the work done on (2) (see Ikebe and Kato [12], Kato [14], Simon [19]) did not involve interval conditions, but the results were very powerful. We give that of Kato.

Theorem 0.2 Let τu be given by (2) with $\{a_{jk}(x)\} \equiv I$, i.e.

$$\tau u = -\sum_{j=1}^n (i\partial_j + b_j(x))^2 u + q(x)u.$$

Let the following conditions hold:

(C1) $b_j \in C^1$;

(C2) $q = q_1 + q_2$, where $q_1 \in L^2_{loc}$ with $q(x) \geq -q^*(|x|)$, $q^*(r)$ being monotonic non-decreasing in $r > 0$ and $q^*(r) = o(r^2)$ as $r \rightarrow \infty$;

(C3) $q_2 \in L^2_{loc}$ with $\int_{|x| \leq r} |q_2(x)|^2 dx \leq (Kr^s)^2$ ($1 \leq r < \infty$) for some K, s , and $\int_{|y| \leq r} |q_2(x-y)| |y|^{2-n} dy \rightarrow 0$ as $r \rightarrow 0$ uniformly for $x \in \mathbb{R}^n$,

where $|y|^{2-n}$ should be replaced by $1 - \log|y|$ if $n=2$,

and by 1 if $n=1$.

If $n \geq 5$, (C3) may be replaced by $q_2 \in L^{\frac{1}{2}n}$.

Then T_0 is e.s.a. \square

Quite recently, Eastham, Evans and McLeod [4] introduced interval conditions for (2), making considerable use of Kato's theorem above. Their conditions are imposed in a suitably defined sequence $\{A_\nu\}$ of "annuli", and the statement of their result makes use of functions $\theta \in C^{1,+}_0(A_\nu)$. (With Ω_ν the "inner" boundary of A_ν and $h_\nu(x) = \text{dist.}(x, \Omega_\nu)$, $\theta \in C^{1,+}_0(A_\nu)$ if it is a function only of h_ν , is non-negative, continuous and piecewise C^1 , and $\text{supp } \theta \subset A_\nu$. Also, d_ν is the "width" of A_ν .)

Theorem 0.3 Let τu be given by (2) with

(i) $a_{jk} \in C^{1+\alpha}$, for some $\alpha > 0$;

(ii) $b_j \in C^1$;

(iii) $q = q_1 + q_2$, where $q_1 \in L^2_{loc}$ and q_1 is locally bounded below, and q_2 is as in (C3) of Thm.0.2. In particular, $q \in L^2_{loc}$.

(iv) τ is elliptic, i.e. $\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \geq K(x)|\xi|^2$ for any real vector $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, where $K(x)$ is a strictly positive function of x .

Suppose that in A_ν

- (a) $q_1(x) \geq -Q(h_\nu)$ with $Q \geq 1$;
- (b) $\lambda^+(x) \leq p(h_\nu)$ where λ^+ is the greatest eigen-value of $\{a_{jk}(x)\}$;
- (c) $\lambda^-(x) \geq K > 0$ where λ^- is the least eigen-value of $\{a_{jk}(x)\}$, this condition being unnecessary if $q_2 = 0$ in A_ν .

Then given any $\theta \in C_0^{1,+}(A_\nu)$, any $\varepsilon > 0$ and any u and $v \in \mathcal{D}(T_0^*)$ for which $\int_{\Omega_\nu} (u \overline{T_0^* v} - \overline{v} T_0^* u) dx \rightarrow L > 0$ as $\nu \rightarrow \infty$, we have, for ν sufficiently large, that

$$\int_0^{\nu} \theta(h_\nu) p^{-\frac{1}{2}}(h_\nu) dh_\nu \leq K(L - \varepsilon)^{-1} \int_{A_\nu} \{ |u|^2 + |v|^2 + |Tu|^2 + |Tv|^2 \} \times \{ 1 + Qh^2 + ph'^2 \} dx. \quad \square$$

Several corollaries can then be deduced, generalising many earlier results.

Meanwhile, Evans [5] proved the results below for (3). The former generalises the results of himself and Atkinson [1] for a 2nd.-order operator to a 4th.-order operator, and those of Everitt [7] and Eastham [3] to interval conditions, and the latter generalises a result of Everitt [6] to interval conditions.

Theorem 0.4 Let $Tu = r_3u$ with r positive, r' locally absolutely continuous, p locally absolutely continuous, q locally Lebesgue integrable, and $\mathcal{D}(T)$ given by $\{u: u \in L^2(0, \infty), r_3u \in L^2(0, \infty), u''$ locally absolutely continuous on $[0, \infty)\}$.

For suitably defined sequences of intervals $\{I_\nu\}$ and $\{J_\nu\}$ with $J_\nu \subset I_\nu$, suppose that there exists a non-negative real function σ which is twice continuously differentiable in I_ν

and satisfies, in I_ν ,

- (i) $|\sigma| \leq K_0, |\sigma'| \leq K_0$;
- (ii) $|r'\sigma| \leq K_1 r$;
- (iii) $-p\sigma^2 \leq K_2 r$;
- (iv) $-q\sigma^4 \leq K_3$;
- (v) $r(1 + |\sigma\sigma''|)^2 \leq K_4, p\sigma^2 \leq K_4$;
- (vi) $\sum \int_{I_\nu} \sigma^3 dx = \infty$.

Then T is l.p. \square

Theorem 0.5 Let T_ν be as Thm.0.4.

Let $I_\nu = [a_\nu, b_\nu]$, $\nu = 1, 2, \dots$, be a sequence of mutually disjoint intervals such that $a_\nu \rightarrow \infty$ as $\nu \rightarrow \infty$ and $b_\nu < K(b_\nu - a_\nu)$. Suppose that in these intervals

- (i) $0 < r(x) \leq K_1 x^4$;
- (ii) $|xr'(x)| \leq K_2 r(x)$;
- (iii) $x^2 p(x) \geq -K_3 r(x)$ and either $p(x) \leq K_3 x^2$ or $p(x) \leq K_3 x^2 |q|^{1/2}$
- (iv) $q \geq -K_4$.

Then T is l.p. \square

Consequently, in view of these last three theorems, the aim of Part I of this thesis is to establish generalisations of Theorem 0.3 to a 4th.-order partial differential operator, the corollary of the first of which can be read as a generalisation of Theorem 0.4 to dimension greater than 1, and the corollary of the second as a similar generalisation of Theorem 0.5.

However, the proof in [4] of Theorem 0.3 relies crucially on Kato's Lemma A [14], the generalisation of the statement of which does not seem to be valid for 4th.- (or higher) order operators. To overcome this problem, we have had to assume that the leading coefficient of the operator under

consideration (i.e. that of $\nabla^2(\nabla^2 u)$) is a constant, which, for convenience, we take as 1. We also take q_2 from Theorems 0.2 and 0.3 as zero, and the coefficient matrix as $p(x)I$, i.e. we shall consider

$$\tau u = \nabla^2(\nabla^2 u) - \nabla \cdot (p(x)\nabla u) + q(x)u$$

for $x \in \mathbb{R}^n$, p and q real-valued, $p \in C^1$ and $q \in L^2_{loc}$.

The aim of Part II of the thesis is to generalise the theorems we shall prove in Part I to ones for a partial differential operator of order $2m$.

Whilst drafting the thesis, I received from Dr.W.D.Evans a preprint ["On the Deficiency Indices of Powers of Real $2n$ th.-Order Symmetric Differential Expressions"; J.London Math.Soc.(2) 13 (1976) 543-556] and a typescript ["Interval Limit-Point Criteria for Differential Expressions and their Powers"; to appear] of some work done by himself and A.Zettl. Although they deal with powers T^k of the $2m$ th.-order ordinary differential operator given by (4), the results have a bearing on the operator under consideration in Part II here, when $\kappa=1$. They obtain a slight extension of Hinton's result [11] with interval conditions, and the conditions imposed correspond closely with those derived in Part II here (see Corollary II.5.4.1).

NOTATION AND PRELIMINARY RESULTS

Definitions and the proofs of statements here are to be found in Mizohata [15] and/or Schechter [17].

Here and throughout we use symbols such as L^2 and C^t , denoting square integrable functions and t -times continuously differentiable functions respectively, the properties to hold in all of \mathbb{R}^n unless stated otherwise.

For $x = (x_1, x_2, \dots, x_n)$, ∂_j is defined as $\partial/\partial x_j$. For $\alpha \in \mathbb{R}^n$, we define $D^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n}$. In this context of α being a multi-index, $|\alpha|$ is the order of the operator D^α , i.e. $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$. Similarly, for $\xi \in \mathbb{R}^n$, $\xi x = \xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n$ and $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \dots \xi_n^{\alpha_n}$.

The support of a function ϕ is given by $\text{supp } \phi = \overline{\Omega}$ where $\Omega = \{x: \phi(x) \neq 0\}$.

\mathcal{D} will denote the set of infinitely differentiable functions with compact support. Elsewhere in the literature, \mathcal{D} appears as C_0^∞ . A function $\phi \in \mathcal{D}$ is called a test-function. If we have a sequence $\{\phi_\nu\}$, $\phi_\nu \in \mathcal{D}$, then $\phi_\nu \rightarrow 0$ means there exists a compact set Ω such that $\text{supp } \phi_\nu \subset \Omega$ and, for arbitrary α , $D^\alpha \phi_\nu(x) \rightarrow 0$ uniformly.

\mathcal{D}' will denote the set of continuous linear functionals on \mathcal{D} , i.e. if $A \in \mathcal{D}'$ then for any complex λ and μ , and any ϕ and $\psi \in \mathcal{D}$ we have

$$A(\lambda\phi + \mu\psi) = \lambda A(\phi) + \mu A(\psi), \text{ and}$$

$$\text{if } \{\phi_\nu\} \text{ is such that } \phi_\nu \in \mathcal{D} \text{ and } \phi_\nu \rightarrow 0, \text{ then } A(\phi_\nu) \rightarrow 0.$$

A is called a distribution, and instead of $A(\phi)$, we write $\langle A, \phi \rangle$.

Unless stated otherwise, all integrations are over \mathbb{R}^n and with respect to x . The basic space with which we shall work is L^2 . It is a Hilbert space and accordingly we denote the inner product of two functions

f and g in L^2 by $(f, g)_0$. (The use of the suffix will become clear and it will be omitted when no confusion would arise.) As usual, we define

$$(f, g)_0 = \int f \bar{g} \quad \text{and} \quad \|f\|_0^2 = (f, f)_0.$$

We define $\langle f, g \rangle_0 = (f, \bar{g})_0 = \int f g$.

When $f \in L^2$ and we say $f \in \mathcal{D}'$ we are equating f with the functional on test-functions ϕ given by $\langle f, \phi \rangle_0$, i.e. $\langle f, \phi \rangle =_{\text{def.}} \langle f, \phi \rangle_0$.

Also $D^\alpha f \in \mathcal{D}'$, it being the functional given by

$$\langle D^\alpha f, \phi \rangle =_{\text{def.}} (-1)^{|\alpha|} \langle f, D^\alpha \phi \rangle_0.$$

When f is in fact $|\alpha|$ -times differentiable, we have $\langle D^\alpha f, \phi \rangle = \langle D^\alpha f, \phi \rangle_0$.

Let S denote the set of infinitely differentiable functions f such that $|x|^\kappa |D^\alpha f(x)|$ is bounded for each $\kappa \geq 0$ and α . $\mathcal{D} \subset S \subset L^2$.

For $f \in S$, the Fourier transform Ff of f is defined by

$$Ff(\xi) = (2\pi)^{-\frac{1}{2}n} \int e^{-i\xi x} f(x) dx \quad (i = \sqrt{-1}).$$

If $f \in S$ and $Ff = g(\xi)$, then $g \in S$ and we define the inverse Fourier transform of g by

$$F^{-1}g(x) = (2\pi)^{-\frac{1}{2}n} \int e^{ix\xi} g(\xi) d\xi.$$

We shall make use of the following properties of Fourier transforms:

$$(f, g)_0 = (Ff, Fg)_0;$$

$$\|Ff\|_0 = \|f\|_0 \quad \text{and} \quad \|g\|_0 = \|F^{-1}g\|_0;$$

$$F[D^\alpha f] = (i\xi)^\alpha F[f];$$

$$F(f * g) = Ff Fg \quad \text{where } f * g \text{ is the convolution of } f \text{ and } g.$$

For $f \in S$ we define, for any real s ,

$$\|f\|_s = \|(1 + |\xi|)^s Ff\|_0.$$

We define H_0^s as the completion of \mathcal{D} with respect to the s -norm given above. (Note that $H_0^0 = L^2$.) H_0^s is a Hilbert space with inner product

$$(f, g)_s = ((1 + |\xi|)^s Ff, (1 + |\xi|)^s Fg)_0.$$

For f and $g \in \mathcal{D}$

$$\langle f, g \rangle_0 = (f, \bar{g})_0 = (Ff, F\bar{g})_0 = ((1 + |\xi|)^s Ff, (1 + |\xi|)^{-s} F\bar{g})_0.$$

Therefore $|\langle f, g \rangle_0| \leq \|f\|_s \|g\|_{-s}$.

Thus we can define $\langle f, g \rangle$ for $f \in H_0^s$ and $g \in H_0^{-s}$, and for fixed $f \in H_0^s$ $\langle f, \cdot \rangle$ is a bounded linear functional on H_0^{-s} .

Various properties of the H_0^s spaces of which we make use are:

H_0^s is dual to H_0^{-s} ;

if $s > t$ then $\mathcal{D} \subset H_0^s \subset H_0^t \subset \mathcal{D}'$ densely;

F and F^{-1} have natural extensions to bounded linear operators on H_0^s .

The original definition of the s -norm given above was, for s a non-negative integer, $f \in L^2$, $D^\alpha f$ taken as a distribution:

$$\|f\|_s^2 = \sum_{|\alpha| \leq s} \|D^\alpha f\|_0^2,$$

giving a norm equivalent to the one we use. Thus a function f in H_0^s can be viewed as, in some sense, s -times differentiable.

The spaces $H_{0,loc}^s$ are treated similar to L_{loc}^2 , and the norms in these spaces will be denoted by $\|\cdot\|_{s,\Omega}$, for a compact set Ω .

For any operator A , $\mathcal{D}(A)$ will denote its domain.

B_R will denote the n -dimensional ball of radius R .

ε (ε' , ε'' , η) will be an arbitrarily small positive quantity, and $K(\varepsilon)$ a positive constant depending on ε . K (K' , K'') will be an absolute positive constant, not necessarily the same in subsequent occurrences. If we wish to keep track of a particular constant we shall use k , c , etc.

Ω (Ω' , Ω'') will be a compact subset of \mathbb{R}^n , and for any Ω

$$\text{dist}(x, \Omega) = \inf_{y \in \Omega} |x - y|.$$

We make frequent use of the inequalities:

$$|ab| \leq \frac{1}{2}\varepsilon|a|^2 + \frac{1}{2}\varepsilon^{-1}|b|^2, \text{ and}$$

$$(\sum a_j b_j)^2 \leq (\sum a_j^2)(\sum b_j^2),$$

or a combination of these:

$$|\sum a_j b_j| \leq \frac{1}{2}\varepsilon(\sum a_j^2) + \frac{1}{2}\varepsilon^{-1}(\sum b_j^2), \text{ where } a_j \text{ and } b_j \text{ are real.}$$

We also use a generalisation of the first of these:

$$|a|^s |b|^{t-s} \leq \varepsilon |a|^t + K(\varepsilon) |b|^t, \text{ for } 0 \leq s < t, t \geq 2.$$

[This can be shown by taking $N=t$ in Proposition II.3.0.1, and the function $a(s)$ there as $|a|^s |b|^{t-s}$ for $0 \leq s < t$.]

We shall frequently use the technique of integration by parts, and in all cases, one of the functions will be sufficiently differentiable and have compact support, so that the boundary terms will be zero.

PART I

CHAPTER 1

We shall consider a 4-th. order elliptic differential operator formally given by:

$$\tau u = \sum_{j,k=1}^n \partial_j^2 (\partial_k^2 u) - \sum_{j=1}^n \partial_j (p(x) \partial_j u) + q(x)u$$

for $x \in \mathbb{R}^n$. The coefficient functions $p(x)$ and $q(x)$ are real-valued, $\partial_j = \partial/\partial x_j$ where $x = (x_1, \dots, x_n)$. With $\nabla = (\partial_1, \dots, \partial_n)$ in Cartesian co-ordinates we have

$$\tau u = \nabla^2 (\nabla^2 u) - \nabla \cdot (p \nabla u) + qu.$$

τ is formally self-adjoint and we wish to establish conditions under which τ does or does not determine a unique self-adjoint operator in L^2 .

We define operators T_0 and T by:

$$T_0 u = \tau u, \mathcal{D}(T_0) = \mathcal{D}; \quad Tu = \tau u, \mathcal{D}(T) = L^2.$$

We shall introduce conditions on p and q as and when we need them, conditions $\mathcal{E}(i)$ - $\mathcal{E}(iii)$ being assumed throughout Part I.

$$\mathcal{E}(i) \quad p \in C^1; \quad \mathcal{E}(ii) \quad q \in L^2_{loc}.$$

Lemma 1.1 T_0 maps \mathcal{D} into L^2 and is symmetric.

Proof: Trivial. \square

Lemma 1.2 T maps L^2 into \mathcal{D}' .

Proof: $u \in L^2$ and $q \in L^2_{loc}$ imply that $qu \in L^1_{loc} \subset \mathcal{D}'$.

$\partial_j (p(x) \partial_j u)$ is the functional (on test-functions ϕ) $\int u \partial_j (p(x) \partial_j \phi)$ and is continuous because $p \in C^1$.

$\partial_j^2 (\partial_k^2 u)$ is the continuous functional $\int u \partial_k^2 (\partial_j^2 \phi)$. \square

Lemma 1.3 T_0^* (the adjoint of T_0 as an operator in L^2) has as its domain $\{u: u \in L^2, Tu \in L^2\}$ and then $T_0^*u = Tu$.

Proof: Let $u \in L^2$, $Tu \in L^2$ and $v \in \mathcal{D}$.

Then $(v, Tu) = (T_0 v, u)$.

Therefore, by definition of adjoint, $u \in \mathcal{D}(T_0^*)$ and $T_0^*u = Tu$.

But T_0^* and T are formally equivalent, so if $u \in L^2$ and $T_0^*u \in L^2$, then $Tu \in L^2$ and $T_0^*u = Tu$. \square

Using the conclusion of Lemma 1.3, the following conditions are seen to be equivalent:

(A) T_0 is essentially self-adjoint.

(B) If $u \in L^2$ and $Tu \in L^2$ then there exists a sequence $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, such that $u_\nu \rightarrow u$ and $Tu_\nu \rightarrow Tu$ in L^2 .

(C) $(T_0 + \zeta)\mathcal{D}$ is dense in L^2 for every complex ζ such that $\text{Im } \zeta \neq 0$.

(D) $T + \zeta$ is 1-1 from L^2 to \mathcal{D}' for ζ as in (C).

(C) is often taken as the definition of essential self-adjointness rather than "an operator A is essentially self-adjoint iff the closure of A is self-adjoint". The proofs of the other equivalences are to be found in Appendix 1.

We define a function $q(x)$ following a further condition on $q(x)$.

(iii) $q(x) \geq -q^*(|x|)$ where $q^*(r)$ is a real-valued monotonic non-decreasing function of $r \geq 0$ and is locally bounded.

We now define $q(x)$ by:

$$q(x) = q(x) \text{ if } x \in B_R;$$

$$q(x) = q(x) + q^*(|x|) - q^*(R) \text{ if } x \notin B_R.$$

Therefore we have $q(x) \geq -q^*(R)$ for all $x \in \mathbb{R}^n$.

We define a formal operator \underline{T} by:

$$\underline{T}u = \nabla^2(\nabla^2 u) - \nabla \cdot (p \nabla u) + qu,$$

and obtain \underline{T}_0 and \underline{T} similarly to above.

Lemmas 1.1, 1.2, 1.3 and the equivalence of $\mathcal{E}(A)$ - $\mathcal{G}(D)$ hold with τ replaced by \underline{T} .

The general idea of the preliminaries is to work with \underline{T} . We impose conditions to obtain \underline{T}_0 essentially self-adjoint and proceed to prove that $\mathcal{E}(B)$ (for \underline{T}) holds with the added restriction: if $\text{supp } u \not\subseteq B_R$, then $\text{supp } u_\nu \subset B_R$. This enables us to demonstrate the existence of a sequence $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, such that $u_\nu \rightarrow \phi u$ and $Tu_\nu \rightarrow T(\phi u)$ in L^2 for any $\phi \in \mathcal{D}$ with $\text{supp } \phi \not\subseteq B_R$, whenever $u \in \mathcal{D}(T^*)$. (See Lemma 3.8.)

CHAPTER 2

In this chapter, we shall state conditions on p and q which render \underline{T}_0 essentially self-adjoint.

The following theorem is due to Schechter and it makes use of the following notation:

For $\alpha > 0$, define

$$\begin{aligned} \omega_\alpha(x) &= \omega_\alpha^{(n)}(x) = |x|^{\alpha-n} && \text{when } 0 < \alpha < n \\ &= 1 - \log|x| && \text{when } \alpha = n \\ &= 1 && \text{when } \alpha > n \end{aligned}$$

$$N_{\alpha, \delta, y}(f) = \int_{|x| < \delta} |f(x-y)|^2 \omega_\alpha(x) dx$$

$$N_{\alpha, \delta}(f) = \sup_y N_{\alpha, \delta, y}(f) \quad N_{\alpha, y}(f) = N_{\alpha, 1, y}(f)$$

$$N_\alpha(f) = N_{\alpha, 1}(f) \quad (= \sup_y N_{\alpha, 1, y}(f))$$

$$N_\alpha = \{f: f \in L^2_{loc}, N_\alpha(f) < \infty\}.$$

The proof of the theorem is rather long, so we just give a very brief sketch of the lines along which the proof runs. Fuller details are given in Appendix 2, and we refer the reader to Schechter [17] for a complete proof.

Theorem 2.1 Consider the operator given by

$$L(x, D) = P(D) + \sum_{j=1}^r q_j(x) Q_j(D) \quad \text{where}$$

(i) $P(D)$ is a symmetric elliptic operator of order μ with real coefficients;

(ii) $Q_j(D)$ is an operator of order $\mu_j < \mu$ with real coefficients;

(iii) $q_j(x)$ is a real-valued function and $q_j \in N_{2(\mu-\mu_j)}$;

also, if $2(\mu-\mu_j) \leq n$, then $N_{2(\mu-\mu_j), \delta}(q_j) \rightarrow 0$ as $\delta \rightarrow 0$;

(iv) $\sum q_j(x)Q_j(D)$ is symmetric.

Then $L(x,D)$ on \mathcal{D} is essentially self-adjoint in L^2 .

Proof: Use is made of the functional analytic lemma:

If \bar{A} is self-adjoint and \bar{B} is symmetric with $\mathcal{D}(\bar{A}) \subset \mathcal{D}(\bar{B})$ and

$$\|\bar{B}u\| \leq a \|\bar{A}u\| + b \|u\| \quad \text{for all } u \in \mathcal{D}(\bar{A}) \text{ with } a < 1,$$

then $\overline{\bar{A} + \bar{B}}$ is self-adjoint,

and so the proof is in three stages:

(I) \bar{P} (the closure of $P(D)$ in L^2) is self-adjoint;

(II) $\mathcal{D}(\bar{P}) = H_0^\mu \subset \mathcal{D}(q_j \bar{Q}_j)$;

(III) $\sum \|q_j \bar{Q}_j u\| \leq a \|\bar{P}u\| + b \|u\|$ for all $u \in H_0^\mu$ with $a < 1$. \square

$$\begin{aligned} \text{With } q_0(x) &= q(x), & Q_0(D) &= I \text{ (the identity);} \\ q_{1,j}(x) &= -\partial_j p(x), & Q_{1,j}(D) &= \partial_j \quad \text{for } j=1, \dots, n; \\ q_2(x) &= -p(x) & Q_2(D) &= \nabla^2; \\ & & \text{and } P(D) &= \nabla^2(\nabla^2). \end{aligned}$$

we state condition $\mathcal{E}(\text{iv})$.

$\mathcal{E}(\text{iv})$ $q_\kappa \in N_{2(4-\kappa)}$ and if $2(4-\kappa) \leq n$, then $N_{2(4-\kappa), \delta}(q_\kappa) \rightarrow 0$
as $\delta \rightarrow 0$, for $\kappa = 0, 1, 2$.

Corollary 2.1.1 Let p and q satisfy $\mathcal{E}(\text{iv})$.

Then \underline{T}_0 is essentially self-adjoint. \square

$\mathcal{E}(\text{iv})$ will be assumed to hold in any lemma requiring \underline{T}_0 to be essentially self-adjoint.

CHAPTER 3

Our aim in this chapter is to derive the identity

$$\int \phi^4 u \Delta u - \int q \phi^4 u^2 = \int \nabla^2(\phi^4 u) \nabla^2 u + \int \nabla(\phi^4 u) \cdot p \nabla u$$

for $u \in \mathcal{D}(T_0^*)$ and $\phi \in \mathcal{D}$, with u and ϕ real-valued. We first give a formal identity which will be used often:

$$\begin{aligned} \tau(\phi u) &= \phi \tau u + u[\nabla^2(\nabla^2 \phi) - \nabla \cdot (p \nabla \phi)] - 2p \nabla \phi \cdot \nabla u \\ &\quad + 2 \sum_{j,k} (\partial_j \partial_k^2 \phi) (\partial_j u) + 2 \sum_{j,k} (\partial_k^2 \partial_j \phi) (\partial_j u) \\ &\quad + 2 \sum_{j,k} (\partial_k^2 \phi) (\partial_j^2 u) + 4 \sum_{j,k} (\partial_j \partial_k \phi) (\partial_j \partial_k u) \\ &\quad + 2 \sum_{j,k} (\partial_j \phi) (\partial_j \partial_k^2 u) + 2 \sum_{j,k} (\partial_j \phi) (\partial_k^2 \partial_j u) \end{aligned}$$

This is obtained by simply writing out $\tau(\phi u)$ in full and calculating the required derivatives of the product ϕu . When ϕ is a test-function and u is a distribution, we get the alternative formulation:

$$\begin{aligned} \tau(\phi u) &= \phi \tau u + u[\nabla^2(\nabla^2 \phi) - \nabla \cdot (p \nabla \phi)] - 2p \nabla \phi \cdot \nabla u + 4 \nabla(\nabla^2 \phi) \cdot \nabla u \\ &\quad + 2 \nabla^2 \phi \nabla^2 u + 4 \sum_j (\partial_j \nabla \phi) \cdot (\partial_j \nabla u) + 4 \nabla \phi \cdot \nabla(\nabla^2 u) \end{aligned}$$

The identities remain valid with τ replaced by $\underline{\tau}$.

Not all of the following lemmas use the assumption that \underline{T}_0 is e.s.a. (essentially self-adjoint), but we shall make it clear in the proofs when we are making this assumption.

Throughout this chapter, we shall be concerned with $\mathcal{D}(T_0^*)$, i.e. $\{u: u \in L^2, Tu \in L^2\}$. With $u \in \mathcal{D}(T_0^*)$, consider that

$$\underline{T}u = Tu + (q - \underline{q})u.$$

Now $Tu \in L^2$ and $(q - \underline{q})$ is locally bounded ($q - \underline{q} = 0$ if $|x| \leq R$, and otherwise $q - \underline{q} = q^*(|x|) - q^*(R)$) and so $(q - \underline{q})u \in L^2_{loc}$. So $u \in \mathcal{D}(T_0^*)$ implies $\underline{T}u \in L^2_{loc}$.

Lemma 3.1 If $u \in L^2$ and $\underline{T}u \in L^2_{loc}$, then $\nabla^2 u \in L^2_{loc}$.

Proof: Let $\phi \in \mathcal{D}$ with $\text{supp } \phi \subset \Omega''$ where Ω'' is some compact set, and consider $\int (\nabla^2 + i)\phi \nabla^2 u$, where $i = \sqrt{-1}$.

$$\begin{aligned}
\int(\nabla^2 + i)\phi\nabla^2 u &= \int i\phi\nabla^2 u + \int\nabla^2\phi\nabla^2 u = \int iu\nabla^2\phi + \int\phi\nabla^2(\nabla^2 u) \\
&= \int iu\nabla^2\phi + \int\phi\underline{\Delta}u + \int\phi\nabla\cdot(p\nabla u) - \int\phi\underline{q}u \\
&= \int iu\nabla^2\phi + \int\phi\underline{\Delta}u + \int u\nabla\cdot(p\nabla\phi) - \int\phi\underline{q}u
\end{aligned}$$

$$\begin{aligned}
\text{so } \left| \int(\nabla^2 + i)\phi\nabla^2 u \right| &\leq \int |u| |\nabla^2\phi| + \int |\phi| |\underline{\Delta}u| + \int |u| [|\nabla p| |\nabla\phi| + |p| |\nabla^2\phi|] + \int |\phi\underline{q}| |u| \\
&\leq \|u\| \|\nabla^2\phi\| + \|\phi\| \|\underline{\Delta}u\|_{\Omega''} + K(\Omega'') \|u\| [\|\nabla\phi\| + \|\nabla^2\phi\|] + \\
&\quad + K(\Omega'')M \|u\|, \quad \text{where } M = \sup |\phi|,
\end{aligned}$$

because $|\nabla p|$ and $|p|$ are bounded on Ω'' , and $\|\phi\underline{q}\| \leq M \|\underline{q}\|_{\Omega''} = K(\Omega'')M$.

$$\text{Now } \|\phi\| = \|F\phi\| \leq \|(\xi^2 + i)F\phi\| = \|F((\nabla^2 + i)\phi)\| = \|(\nabla^2 + i)\phi\|$$

$$\text{and } \|\nabla^2\phi\| = \|F(\nabla^2\phi)\| = \|\xi^2 F\phi\| \leq \|(\xi^2 + i)F\phi\| = \|(\nabla^2 + i)\phi\|.$$

$$\text{Also } \|\nabla\phi\| \leq \|\nabla^2\phi\|^{\frac{1}{2}} \|\phi\|^{\frac{1}{2}} \leq \frac{1}{2}(\|\nabla^2\phi\| + \|\phi\|) \leq \|(\nabla^2 + i)\phi\|.$$

Now $|\phi(y')| - |\phi(y)| \leq |\phi(y') - \phi(y)| = \left| \int_y^{y'} \nabla\phi \right| \leq K'(\Omega'') \|\nabla\phi\|$, and so

$|\phi(y)| \geq |\phi(y')| - K'(\Omega'') \|\nabla\phi\|$; as this is valid for any y' , we have

$|\phi(y)| \geq M - K'(\Omega'') \|\nabla\phi\|$, and so $\|\phi\| \geq K''(\Omega'') [M - K'(\Omega'') \|\nabla\phi\|]$;

hence $M \leq K(\Omega'') [\|\phi\| + \|\nabla\phi\|]$, and so we have $M \leq K(\Omega'') \|(\nabla^2 + i)\phi\|$.

$$\text{Therefore } \left| \int(\nabla^2 + i)\phi\nabla^2 u \right| \leq K(u, \Omega'') \|(\nabla^2 + i)\phi\|.$$

But ∇^2 as an operator from \mathcal{D} to L^2 is e.s.a. Therefore $(\nabla^2 + i)\mathcal{D}$ is dense in L^2 . So consider any $v \in L^2$ with $\text{supp } v \subset \Omega$, say. Then there exists a sequence $\{\phi_\nu\}$, $\phi_\nu \in \mathcal{D}$, such that $(\nabla^2 + i)\phi_\nu \rightarrow v$ in L^2 . We must show that the functions in this sequence have common support.

For some $d > 0$, let $\Omega' = \{x: \text{dist}(x, \Omega) \leq d\}$, and $\Omega'' = \{x: \text{dist}(x, \Omega') \leq d\}$.

Let $\psi(x) \in \mathcal{D}$ be a real-valued function such that $0 \leq \psi \leq 1$, $\psi(x) = 1$ if $x \in \Omega'$, $\psi(x) = 0$ if $x \notin \Omega''$, $|\partial_j \psi| \leq K$ and $|\partial_j \partial_k \psi| \leq K$ for $1 \leq j, k \leq n$. Then $\psi\phi_\nu \in \mathcal{D}$.

We consider $\{(\nabla^2 + i)(\psi\phi_\nu)\}$. Let $\phi \in \{\phi_\nu\}$ such that $\|(\nabla^2 + i)\phi - v\| \leq \eta$.

$$\text{Now } (\nabla^2 + i)(\psi\phi) = \psi(\nabla^2 + i)\phi + 2\nabla\psi \cdot \nabla\phi + \phi\nabla^2\psi,$$

$$\begin{aligned}
\text{and so } \|(\nabla^2 + i)(\psi\phi) - v\| &\leq \|(\nabla^2 + i)\phi - v\| + \|(\nabla^2 + i)(\psi\phi - \phi)\| \\
&\leq \eta + \|(\psi-1)(\nabla^2 + i)\phi\| + \|\phi\nabla^2\psi\| + 2\|\nabla\psi \cdot \nabla\phi\|.
\end{aligned}$$

$$\begin{aligned}
\text{But } \|(\psi-1)(\nabla^2 + i)\phi\| &\leq \|(\psi-1)(\nabla^2 + i)\phi\|_{\Omega'}, \quad \text{because } (\psi-1) = 0 \text{ in } \Omega' \\
&\leq \|(\nabla^2 + i)\phi\|_{\Omega'}, \quad \text{because } 0 \leq (\psi-1) \leq 1 \\
&\leq \|(\nabla^2 + i)\phi - v\|_{\Omega'}, \quad \text{because } v = 0 \text{ outside } \Omega \subset \Omega' \\
&\leq \eta.
\end{aligned}$$

If this taking of limits is not justifiable, we may proceed as follows:

Consider $\int (\nabla^2 + \zeta)\phi \cdot (\nabla^2 - \zeta)u$, where $\Im \zeta \neq 0 \neq \Im \zeta^2$ and $|\zeta| \leq K$, to obtain, much as before,

$$\left| \int (\nabla^2 + \zeta)\phi \cdot (\nabla^2 - \zeta)u \right| \leq K(u, \Omega'') \left\| (\nabla^2 + \zeta)\phi \right\|.$$

From this it follows, as before, that $(\nabla^2 - \zeta)u$ is a continuous linear functional on a dense subspace of the dual space of L^2_{loc} .

According to the Hahn-Banach Theorem, this can be extended uniquely to $f \in L^2_{loc}$ such that, for all $\phi \in \mathcal{D}$ with $\text{supp } \phi \subset \Omega''$,

$$\int (\nabla^2 + \zeta)\phi \cdot (\nabla^2 - \zeta)u = \int (\nabla^2 + \zeta)\phi \cdot f \quad (\dagger)$$

Now $(\nabla^2 - \zeta)$ on L^2 is invertible because ∇^2 is symmetric and $\Im \zeta \neq 0$, and if g is such that $(\nabla^2 - \zeta)g = f$, then $g \in L^2_{loc}$. But we have $\nabla^2 g = f + \zeta g$, and so $\nabla^2 g \in L^2_{loc}$.

So from (\dagger) we have $\int (\nabla^2 + \zeta)\phi \cdot (\nabla^2 - \zeta)(u - g) = 0$,

and therefore we obtain $\int (\nabla^2(\nabla^2 \cdot) - \zeta^2)\phi \cdot (u - g) = 0 \quad (\ddagger)$

As $\nabla^2(\nabla^2 \cdot)$ is e.s.a. and $\Im \zeta^2 \neq 0$, $(\nabla^2(\nabla^2 \cdot) - \zeta^2)\mathcal{D}$ is dense in L^2 .

By (\ddagger) , this implies that $u - g = 0$, and so $\nabla^2 u \in L^2_{loc}$. \square

$$\begin{aligned}
\text{and } \|\phi \nabla^2 \psi\| &\leq \|\phi \nabla^2 \psi\|_{\Omega''/\Omega}, \text{ because } \nabla^2 \psi = 0 \text{ outside } \Omega''/\Omega, \\
&\leq K \|\phi\|_{\Omega''/\Omega}, \\
&\leq K \|(\nabla^2 + i)\phi\|_{\Omega''/\Omega}, \text{ as above} \\
&\leq K\eta \text{ as above}
\end{aligned}$$

$$\begin{aligned}
\text{and } \|\partial_j \psi |\nabla \phi|\|^2 &\leq K \|\partial_j \nabla \psi |\phi|\|^2 + \|\partial_j \psi \phi\|^2 + \|\partial_j \psi \nabla^2 \phi\|^2 \leq K\eta^2 \text{ as above} \\
\text{so } \|\nabla \psi \cdot \nabla \phi\| &\leq K\eta.
\end{aligned}$$

So, by choice of η , $\|(\nabla^2 + i)(\psi\phi) - v\| < \varepsilon$, and we have that $(\nabla^2 + i)(\psi\phi) \rightarrow v$ in L^2 with $\text{supp } \psi\phi \subset \Omega''$.

Therefore $|\int v \nabla^2 u| \leq K(u, \Omega) \|v\|$, where the infimum of the K 's for which this is valid for v 's in L^2 having the same support, Ω , is $\|\nabla^2 u\|_{\Omega}$. Therefore $\nabla^2 u \in L^2_{\text{loc}}$. \square See facing page.

Corollary 3.1.1 If $u \in L^2$ and $\underline{T}u \in L^2_{\text{loc}}$, then $|\nabla u| \in L^2_{\text{loc}}$. \square

Lemma 3.2 If $u \in L^2$ and $\underline{T}u \in L^2_{\text{loc}}$, then $\underline{T}(\phi u) \in H_0^{-1}$ for any $\phi \in \mathcal{D}$.

Proof: Let $\phi \in \mathcal{D}$ with $\text{supp } \phi \subset \Omega$. We shall show that each term in the expansion of $\underline{T}(\phi u)$ is in H_0^{-1} .

$\underline{T}u \in L^2_{\text{loc}}$ and so $\phi \underline{T}u \in L^2$.

$u[\nabla^2(\nabla^2 \phi) - \nabla \cdot (p \nabla \phi)] \in L^2$ because $|p|$ and $|\nabla p|$ are bounded on Ω and $u \in L^2$.

$p \nabla \phi \cdot \nabla u \in L^2$ because $|p|$ is bounded on Ω and, by Corol. 3.1.1, $|\nabla u| \in L^2_{\text{loc}}$.

$\nabla(\nabla^2 \phi) \cdot \nabla u \in L^2$ because, by Corol. 3.1.1, $|\nabla u| \in L^2_{\text{loc}}$.

$\nabla^2 \phi \nabla^2 u \in L^2$ because, by Lemma 3.1, $\nabla^2 u \in L^2_{\text{loc}}$.

$$\begin{aligned}
\text{Let } \psi \in H_0^1: \left| \int \psi \langle \psi, (\partial_j \nabla \phi) \cdot (\partial_j \nabla u) \rangle \right| &= \left| \int \psi \int \partial_j \psi (\partial_j \nabla \phi) \cdot (\partial_j \nabla u) \right| \\
&= \left| \int \int \partial_j \psi (\partial_j \nabla \phi) \cdot (\nabla u) \right| \\
&\leq c \|\psi\|_1 \|\nabla u\|_{\Omega}
\end{aligned}$$

$$\text{where } c^2 = \int \|\partial_j \nabla \phi\|_{L^\infty}^2 + \|\nabla(\nabla^2 \phi)\|_{L^\infty}^2.$$

By Corol. 3.1.1, $\|\nabla u\|_{\Omega}$ is finite and therefore $\int (\partial_j \nabla \phi) \cdot (\partial_j \nabla u) \in H_0^{-1}$.

Similarly, $|\langle \psi, (\nabla \phi) \cdot (\nabla(\nabla^2 u)) \rangle| \leq c \|\psi\|_1 \|\nabla^2 u\|_{\Omega}$

$$\text{where here } c^2 = \|\nabla \phi\|_{L^\infty}^2 + \|\nabla^2 \phi\|_{L^\infty}^2.$$

By Lemma 3.1, $\|\nabla^2 u\|_\Omega$ is finite and therefore $(\nabla\phi) \cdot (\nabla(\nabla^2 u)) \in H_0^{-1}$.

Therefore, as $L^2 \subset H_0^{-1}$, we have $\underline{T}(\phi u) \in H_0^{-1}$. \square

In order to prove the next lemma, we shall use Gårding's inequality:

If L is a uniformly strongly elliptic operator in any bounded domain Ω , with leading coefficients uniformly continuous and other coefficients

bounded, then there exist constants $c > 0$ and k such that for all

$u \in H_0^s(\Omega)$ we have $\Re(Lu, u)_{0,\Omega} \geq c\|u\|_{s,\Omega}^2 - k\|u\|_{0,\Omega}^2$,

where $2s$ is the order of the operator L .

Lemma 3.3 If $u \in L^2$ and $\underline{T}u \in H_0^{-2}$, then $u \in H_0^2$, and for any $\varepsilon > 0$

$$(c - \varepsilon)\|u\|_2^2 \leq (k + q^*(R))\|u\|^2 + K(\varepsilon)\|\underline{T}u\|_{-2}^2. \quad (1)$$

Proof: $\underline{T}_0 - q$ satisfies the conditions of Gårding's inequality

with $s = 2$.

Assume that $u \in \mathcal{D} \subset H_0^2$.

Then $\Re((\underline{T}_0 - q)u, u) \geq c\|u\|_2^2 - k\|u\|^2$

$$\Re(\underline{T}_0 u, u) \geq c\|u\|_2^2 - k\|u\|^2 + (qu, u)$$

$$\geq c\|u\|_2^2 - (k + q^*(R))\|u\|^2 \quad (2)$$

$$\Re((\underline{T}_0 + \zeta)u, u) \geq c\|u\|_2^2 + (\Re\zeta - k - q^*(R))\|u\|^2$$

where ζ is any complex number with $\Im\zeta \neq 0$.

Taking $\Re\zeta$ sufficiently large:

$$\Re((\underline{T}_0 + \zeta)u, u) \geq c\|u\|_2^2.$$

$u \in \mathcal{D}$ and therefore $\underline{T}_0 u \in L^2$ and $\zeta u \in L^2$. $L^2 \subset H_0^{-2}$ and therefore $(\underline{T}_0 + \zeta)u \in H_0^{-2}$.

Therefore $c\|u\|_2^2 \leq \|(\underline{T}_0 + \zeta)u\|_{-2}\|u\|_2$

and so $c\|u\|_2 \leq \|(\underline{T}_0 + \zeta)u\|_{-2}$ (3)

\underline{T}_0 is e.s.a. and so $(\underline{T}_0 + \zeta)\mathcal{D}$ is dense in L^2 and hence in H_0^{-2} too.

Therefore if $v \in H_0^{-2}$ there exists a sequence $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, such that

$(\underline{T}_0 + \zeta)u_\nu \rightarrow v$ in H_0^{-2} . So, by (3), $\{u_\nu\}$ converges in H_0^2 , say $u_\nu \rightarrow u'$ in H_0^2 ,

and $(\underline{T}_0 + \zeta)u' = v$. So $(\underline{T}_0 + \zeta)$ maps a subset of H_0^2 onto H_0^{-2} .

Now if $u \in L^2$ and $\underline{T}u \in H_0^{-2}$, then $(\underline{T} + \zeta)u \in H_0^{-2}$. Therefore there exists $u'' \in H_0^2$ such that $(\underline{T} + \zeta)u'' = (\underline{T} + \zeta)u$. But as \underline{T}_0 is e.s.a., $\underline{T} + \zeta$ is 1-1 from L^2 to \mathcal{D} . As $H_0^{-2} \subset \mathcal{D}$ and $u'' \in L^2$, we have $u = u''$, and so $u \in H_0^2$.

So, from (3), if $u \in L^2$ and $\underline{T}u \in H_0^{-2}$, then we have

$$c \|u\|_2 \leq \|(\underline{T} + \zeta)u\|_{-2} \quad (4)$$

Now, for $u \in \mathcal{D}$, we have from (2)

$$\begin{aligned} c \|u\|_2^2 &\leq \operatorname{Re}(\underline{T}_0 u, u) + (k + q^*(R)) \|u\|^2 \\ &\leq (k + q^*(R)) \|u\|^2 + \|\underline{T}_0 u\|_{-2} \|u\|_2 \\ &\leq (k + q^*(R)) \|u\|^2 + \varepsilon \|u\|_2^2 + K(\varepsilon) \|\underline{T}_0 u\|_{-2}^2 \end{aligned}$$

and so (1) holds for $u \in \mathcal{D}$.

When $u \in L^2$ and $\underline{T}u \in H_0^{-2}$, $(\underline{T} + \zeta)u \in H_0^{-2}$ and there exists $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, such that $(\underline{T} + \zeta)u_\nu \rightarrow (\underline{T} + \zeta)u$ in H_0^{-2} . By (4), $u_\nu \rightarrow u$ in H_0^2 and so $\zeta u_\nu \rightarrow \zeta u$ in H_0^2 and therefore also in H_0^{-2} . So $\underline{T}u_\nu \rightarrow \underline{T}u$ in H_0^{-2} . Also $u_\nu \rightarrow u$ in L^2 . Therefore (1) holds for $u \in L^2$ when $\underline{T}u \in H_0^{-2}$. \square

We now need another condition on q :

$\mathcal{E}(v) N_{4-\alpha}(\underline{q})$ is locally bounded, for some $\alpha > 0$ with $n < 4 - \alpha < 0$.

The condition and the following lemma in which it is used are essentially due to Ikebe and Kato, the condition being reformulated using the notation established in the previous chapter. Their proof is to be found in [12] and is given in Appendix 3.

Lemma 3.4 If $u \in H_{0,loc}^2$ and \underline{q} satisfies $\mathcal{E}(v)$, then $\underline{q}u \in L_{loc}^2$, and

$$\|\underline{q}u\|_{\Omega} \leq K \|u\|_{2,\Omega'}$$

for any compact set Ω , where Ω' is Ω extended in width by $d > 0$, and K depends on Ω , d and n .

Proof: See Appendix 3. \square

$\mathcal{E}(v)$ will be assumed to hold in any lemma depending on Lemma 3.4.

Lemma 3.5 If $u \in L^2$ and $\underline{T}(\phi u) \in H_0^{-1}$, then $\phi u \in H_0^3$, for any $\phi \in \mathcal{D}$.

Proof: $H_0^{-1} \subset H_0^{-2}$, so $\underline{T}(\phi u) \in H_0^{-2}$. Also, $\phi u \in L^2$.

Therefore, by Lemma 3.3 applied to ϕu instead of u , $\phi u \in H_0^2$, i.e. $u \in H_{0,loc}^2$.

Therefore, by Lemma 3.4, $\underline{q}u \in L_{loc}^2$, and so $\underline{q}\phi u \in L^2$.

Also, $\nabla \cdot (p\nabla(\phi u)) \in L^2$. But $L^2 \subset H_0^{-1}$, and therefore

$$\nabla^2(\nabla^2(\phi u)) = \underline{T}(\phi u) + \nabla \cdot (p\nabla(\phi u)) - \underline{q}\phi u \in H_0^{-1}.$$

Now, for any $\psi \in \mathcal{D}$ and any s

$$\|\psi\|_{s+\mu} \leq K(\|P(D)\psi\|_s + \|\psi\|_s)$$

where $P(D)$ is an elliptic operator of order μ with real coefficients (see Lemma A2.2). Putting $P(D) = \nabla^2(\nabla^2 \cdot)$, $\mu = 4$ and $s = -1$, we obtain

$$\|\psi\|_3 \leq K(\|\nabla^2(\nabla^2\psi)\|_{-1} + \|\psi\|_{-1}) \quad \text{for any } \psi \in \mathcal{D}.$$

Since \mathcal{D} is dense in H_0^3 , and $\nabla^2(\nabla^2(\phi u))$ and $\phi u \in H_0^{-1}$, this implies that $\phi u \in H_0^3$. \square

Lemma 3.6 If $u \in L^2$ and $\underline{T}u \in L_{loc}^2$, then $\underline{T}(\phi u) \in L^2$ for any $\phi \in \mathcal{D}$.

Proof: Let $\phi \in \mathcal{D}$. By Lemma 3.2, $\underline{T}(\phi u) \in H_0^{-1}$. Then by Lemma 3.5, $\phi u \in H_0^3$, and so every term in the expansion of $\underline{T}(\phi u)$ is in L^2 .

Therefore $\underline{T}(\phi u) \in L^2$. \square

Lemma 3.7 If $u \in L^2$ and $\underline{T}u \in L^2$ with $\text{supp } u \subsetneq B_R$, then there exists

a sequence $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, with $\text{supp } u_\nu \subset B_R$, such that $u_\nu \rightarrow u$ and $\underline{T}u_\nu \rightarrow \underline{T}u$ in L^2 .

Proof: \underline{T}_0 is e.s.a. and so there exists $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, such that $u_\nu \rightarrow u$ and $\underline{T}u_\nu \rightarrow \underline{T}u$ in L^2 .

Let $\phi \in \mathcal{D}$ with $\text{supp } \phi \subset B_R$ and $\phi = 1$ on $\text{supp } u$.

Then $\phi u_\nu \in \mathcal{D}$ and $\text{supp } \phi u_\nu \subset B_R$.

We have $\phi u_\nu \rightarrow \phi u = u$ in L^2 and also $\phi \underline{T}u_\nu \rightarrow \phi \underline{T}u = \underline{T}u$ in L^2 since $\text{supp } \underline{T}u \subset \text{supp } u$.

We shall show that the rest of the terms in the expansion of $\underline{T}(\phi u_\nu)$ converge to 0 in L^2 , in which case $\{\phi u_\nu\}$ will be the required sequence.

Now $\underline{T}u_\nu \rightarrow \underline{T}u$ in L^2 implies that $\underline{T}u_\nu \rightarrow \underline{T}u$ in H_0^{-2} , so we apply Lemma 3.3(1) to $u_\nu - u$ to obtain $u_\nu \rightarrow u$ in H_0^2 . So as well as having $u_\nu \rightarrow u$ in L^2 , we have

$$\partial_j u_\nu \rightarrow \partial_j u \text{ in } L^2 \text{ and } \partial_j \partial_k u_\nu \rightarrow \partial_j \partial_k u \text{ in } L^2 \text{ for } 1 \leq j, k \leq n.$$

$$\text{So } u_\nu [\nabla^2(\nabla^2 \phi) - \nabla \cdot (p \nabla \phi)] \rightarrow u [\nabla^2(\nabla^2 \phi) - \nabla \cdot (p \nabla \phi)] = 0 \text{ in } L^2$$

since $|\nabla^2(\nabla^2 \phi) - \nabla \cdot (p \nabla \phi)| \leq K$ on $\text{supp } \phi$ and $= 0$ on $\text{supp } u$,

$$\text{and } p \nabla \phi \cdot \nabla u_\nu \rightarrow p \nabla \phi \cdot \nabla u = 0 \text{ in } L^2$$

since $|p \nabla \phi| \leq K$ on $\text{supp } \phi$ and $= 0$ on $\text{supp } u$.

Similarly, $\nabla(\nabla^2 \phi) \cdot \nabla u_\nu \rightarrow \nabla(\nabla^2 \phi) \cdot \nabla u = 0$ in L^2 ,

$$\nabla^2 \phi \nabla^2 u_\nu \rightarrow \nabla^2 \phi \nabla^2 u = 0 \text{ in } L^2,$$

$$\text{and } \sum_j (\partial_j \nabla \phi) \cdot (\partial_j \nabla u_\nu) \rightarrow \sum_j (\partial_j \nabla \phi) \cdot (\partial_j \nabla u) = 0 \text{ in } L^2.$$

The remaining term to be considered is $\nabla \phi \cdot \nabla(\nabla^2 u_\nu)$, involving third-order derivatives of u_ν . We treat this term as $\sum_j (\partial_j \phi) (\partial_j (\nabla^2 u_\nu))$.

Assume $\text{supp } u \subset B_r$ with $r < R$ and let $d = (R - r)/3$. We may assume that ϕ is real-valued with $0 \leq \phi \leq 1$, $\phi(x) = 1$ if $|x| < r + d$, $\phi(x) = 0$ if $|x| > R - d$, $|\partial_j \phi| \leq K$ and $|\partial_j \partial_k \phi| \leq K$ for $1 \leq j, k \leq n$.

Now for any $\psi \in \mathcal{D}$

$$\begin{aligned} \int (\partial_j \phi)^2 |\partial_j (\nabla^2 \psi)|^2 &= \int (\partial_j \phi)^2 \partial_j (\nabla^2 \psi) \partial_j (\nabla^2 \bar{\psi}) \\ &= -\int \partial_j [(\partial_j \phi)^2 \partial_j (\nabla^2 \psi)] \nabla^2 \bar{\psi} \\ &= -2 \int (\partial_j \phi) (\partial_j^2 \phi) \partial_j (\nabla^2 \psi) \nabla^2 \bar{\psi} - \int (\partial_j \phi)^2 \partial_j^2 (\nabla^2 \psi) \nabla^2 \bar{\psi} \\ &\leq 2 \int |\partial_j \phi| |\partial_j^2 \phi| |\partial_j (\nabla^2 \psi)| |\nabla^2 \psi| + \\ &\quad + \int |\partial_j \phi|^2 |\partial_j^2 (\nabla^2 \psi)| |\nabla^2 \psi| \\ &\leq \frac{1}{2} \int (\partial_j \phi)^2 |\partial_j (\nabla^2 \psi)|^2 + K \int (\partial_j^2 \phi)^2 |\nabla^2 \psi|^2 + \\ &\quad + \frac{1}{2} \epsilon \int (\partial_j \phi)^2 |\partial_j^2 (\nabla^2 \psi)|^2 + K(\epsilon) \int (\partial_j \phi)^2 |\nabla^2 \psi|^2. \end{aligned}$$

$$\text{Therefore } \int (\partial_j \phi)^2 |\partial_j (\nabla^2 \psi)|^2 \leq K \int (\partial_j^2 \phi)^2 |\nabla^2 \psi|^2 + \epsilon \int (\partial_j \phi)^2 |\partial_j^2 (\nabla^2 \psi)|^2 + K(\epsilon) \int (\partial_j \phi)^2 |\nabla^2 \psi|^2.$$

$$\text{Therefore } \left\| (\partial_j \phi) (\partial_j (\nabla^2 u_\nu)) \right\|^2 \leq K \left\| (\partial_j^2 \phi) (\nabla^2 u_\nu) \right\|^2 + \epsilon \left\| (\partial_j \phi) (\partial_j^2 (\nabla^2 u_\nu)) \right\|^2 + K(\epsilon) \left\| (\partial_j \phi) (\nabla^2 u_\nu) \right\|^2 \quad (\dagger)$$

Let $\chi(x) = 1$ if $r \leq |x| \leq R$ and $\chi(x) = 0$ otherwise. Note that $\chi u = 0$, and

that $\chi = 0$ implies that $\partial_j \phi = 0$ and $\partial_j^2 \phi = 0$ for $1 \leq j \leq n$.

$$\text{So } \|(\partial_j \phi)(\nabla^2 u_\nu)\| = \|\chi(\partial_j \phi)(\nabla^2 u_\nu)\| \leq K \|\chi \nabla^2 u_\nu\| = K \|\nabla^2(\chi u_\nu)\|$$

with similar inequalities for the other terms on the right-hand side of (†).

$$\text{Therefore } \|(\partial_j \phi)(\partial_j(\nabla^2 u_\nu))\|^2 \leq K\varepsilon \|\partial_j^2(\nabla^2(\chi u_\nu))\|^2 + K(\varepsilon) \|\nabla^2(\chi u_\nu)\|^2 \quad (\ddagger)$$

Considering the first term on the right-hand side of (‡):

$$\begin{aligned} \|\partial_j^2(\nabla^2(\chi u_\nu))\| &\leq \|\nabla^2(\nabla^2(\chi u_\nu))\| \\ &= \|\underline{T}(\chi u_\nu) + \nabla \cdot (p \nabla(\chi u_\nu)) - \underline{q} \chi u_\nu\| \\ &\leq \|\underline{T}(\chi u_\nu)\| + \|\nabla \cdot (p \nabla(\chi u_\nu))\| + \|\underline{q} \chi u_\nu\| \end{aligned}$$

because $\underline{T}(\chi u_\nu) \in L^2$, $\nabla \cdot (p \nabla(\chi u_\nu)) \in L^2$, and, by Lemma 3.4, $\underline{q} \chi u_\nu \in L^2$.

Now $\|\underline{T}(\chi u_\nu)\| = \|\underline{T}(\chi(u_\nu - u))\| = \|\chi \underline{T}(u_\nu - u)\| \leq K$, independently of ν ,

and $\|\nabla \cdot (p \nabla(\chi u_\nu))\| = \|\nabla \cdot (p \nabla(\chi(u_\nu - u)))\| = \|\chi \nabla \cdot (p \nabla(u_\nu - u))\|$

$\leq K$, independently of ν ,

and $\|\underline{q} \chi u_\nu\| \leq K \|\chi u_\nu\|_2$ by Lemma 3.4

$$= K \|\chi(u_\nu - u)\|_2 \leq K, \text{ independently of } \nu.$$

Therefore $\|\partial_j^2(\nabla^2(\chi u_\nu))\| \leq K$, independently of ν ,

so choose ε such that $K\varepsilon \|\partial_j^2(\nabla^2(\chi u_\nu))\|^2 < \frac{1}{2}\eta$ for all ν .

With this ε , consider the second term on the right-hand side of (‡):

$$\|\nabla^2(\chi u_\nu)\| = \|\nabla^2(\chi(u_\nu - u))\| = \|\chi(\nabla^2(u_\nu - u))\| \leq \|u_\nu - u\|_2.$$

Now $u_\nu \rightarrow u$ in H_0^2 , so choose N such that for all $\nu > N$, $K(\varepsilon) \|\nabla^2(\chi u_\nu)\|^2 < \frac{1}{2}\eta$.

So, considering (‡), for all $\nu > N$ we have $\|(\partial_j \phi)(\partial_j(\nabla^2 u_\nu))\|^2 < \eta$.

Therefore $(\partial_j \phi)(\partial_j(\nabla^2 u_\nu)) \rightarrow 0$ in L^2 , and $\{\phi u_\nu\}$ is the required sequence. \square

Lemma 3.8 If $u \in \mathcal{D}(T^*)$, then $T(\phi u) \in L^2$ and there exists a sequence

$\{u_\nu\}$, $u_\nu \in \mathcal{D}$, such that $u_\nu \rightarrow \phi u$ and $Tu_\nu \rightarrow T(\phi u)$ in L^2 , for any $\phi \in \mathcal{D}$.

Proof: Choose R such that $\text{supp } \phi \subsetneq B_R$.

$u \in \mathcal{D}(T^*)$ implies that $\underline{T}u \in L_{loc}^2$, so, by Lemma 3.6, $\underline{T}(\phi u) \in L^2$.

Applying Lemma 3.7 to ϕu instead of u , there exists $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, such that $u_\nu \rightarrow \phi u$ and $\underline{T}u_\nu \rightarrow \underline{T}(\phi u)$ in L^2 , with $\text{supp } u_\nu \subset B_R$. But $\underline{T}v = Tv$ if $\text{supp } v \subset B_R$, so $T(\phi u) \in L^2$ and $Tu_\nu = \underline{T}u_\nu \rightarrow \underline{T}(\phi u) = T(\phi u)$ in L^2 . \square

This is the result referred to at the end of Chapter 1. Before we make use of it in the next lemma, we prove a corollary, which will also be used in the lemma.

Corollary 3.8.1 If $\{u_\nu\}$ is a sequence with the properties in

Lemma 3.8, then

$$\int q|u_\nu|^2 \rightarrow \int q|\phi u|^2 \quad \text{as } \nu \rightarrow \infty \quad (1)$$

Proof: Consider $\int \bar{u}_\nu(Tu_\nu)$:

$$\int \bar{u}_\nu(Tu_\nu) = \int \bar{u}_\nu \nabla^2(\nabla^2 u_\nu) - \int \bar{u}_\nu \nabla \cdot (p \nabla u_\nu) + \int q|u_\nu|^2.$$

$$\text{Therefore } \int q|u_\nu|^2 = \int \bar{u}_\nu(Tu_\nu) - \int |\nabla^2 u_\nu|^2 - \int p|\nabla u_\nu|^2.$$

q is bounded below on the common support of the functions u_ν , say $q \geq k+1$, where k may be negative, and, as $p \in C^1$, $|p|$ is also bounded on this support.

$$\text{Therefore } \int (q-k)|u_\nu|^2 \leq \frac{1}{2} \|Tu_\nu\|^2 + K' \|u_\nu\|_2^2 \quad (2)$$

where K' depends on k and the bound on $|p|$.

We have the convergence of $\{u_\nu\}$ and $\{Tu_\nu\}$ in L^2 and it is part of the proof of Lemma 3.7 (using Lemma 3.3(1)) that we then also have the convergence of $\{u_\nu\}$ in H^2_0 . Hence the left-hand side of (2) is convergent, from which the result follows. \square

We may now proceed to derive the identity stated at the beginning of this chapter, namely

$$\int \phi^4 u Tu - \int q \phi^4 u = \int \nabla^2(\phi^4 u) \nabla^2 u + \int \nabla(\phi^4 u) \cdot p \nabla u$$

for $u \in \mathcal{D}(T^*_0)$ and $\phi \in \mathcal{D}$, with u and ϕ real-valued. (The restriction on u of being real-valued makes calculations slightly simpler and is of no consequence, as will be seen in Chapter 5.)

Had we imposed some sort of integrability condition on the functions $u \in \mathcal{D}(T^*_0)$ under consideration (such as requiring the third-order partial derivatives of u to be absolutely continuous), we could have integrated

$\phi^4 u T u$ directly, to obtain the desired result. However, this would have meant that the size of $\mathcal{D}(T^*)$ was unnecessarily reduced. Instead, by virtue of Lemma 3.8 and its corollary, we may move from any function $u \in \mathcal{D}(T^*)$ to elements u_ν of a sequence $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, and perform the integrations before moving back to u by a limiting process, with $\mathcal{D}(T^*)$ being as large as possible.

Lemma 3.9 If $u \in \mathcal{D}(T^*)$ and $\phi \in \mathcal{D}$, with u and ϕ real-valued, then

$$\int \phi^4 u T u - \int q \phi^4 u^2 = \int \nabla^2 (\phi^4 u) \nabla^2 u + \int \nabla (\phi^4 u) \cdot p \nabla u \quad (1)$$

Proof: As $u \in L^2$ and $T u \in L^2$, the first integral is finite.

By Corol.3.8.1, the second integral is finite.

By Lemma 3.1 and its corollary, $\nabla^2 u \in L^2_{loc}$ and $|\nabla u| \in L^2_{loc}$, and, as $p \in C^1$ and is therefore locally bounded, the remaining two integrals are finite.

From Lemma 3.8, there exists a sequence $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, such that $u_\nu \rightarrow \phi^4 u$ in L^2 and $T u_\nu \rightarrow T(\phi^4 u)$ in L^2 , where R is such that $\text{supp } \phi \subsetneq B_R$. So consider $\int u_\nu (T u) - \int u_\nu q u$:

$$\begin{aligned} \int u_\nu (T u) - \int u_\nu q u &= \int [u_\nu \nabla^2 (\nabla^2 u) - u_\nu \nabla \cdot (p \nabla u)] \\ &= \int (\nabla^2 u_\nu) (\nabla^2 u) + \int \nabla u_\nu \cdot p \nabla u \end{aligned} \quad (2)$$

all integrals being finite by the preceding statements.

With $u_\nu \rightarrow \phi^4 u$ in L^2 and $T u_\nu \rightarrow T(\phi^4 u)$ in L^2 , it is part of the proof of Lemma 3.7 (using Lemma 3.1(1)) that we then also have $u_\nu \rightarrow \phi^4 u$ in H^2_0 . Also, the strong convergence in Corol.3.8.1(1) implies the weak convergence required here. Hence, letting $\nu \rightarrow \infty$ in (2), we obtain the desired result. \square

CHAPTER 4

Throughout this chapter we shall continue to assume that $u \in \mathcal{D}(T_0^*)$ and $\phi \in \mathcal{D}$, with u and ϕ real-valued. From Lemma 3.8, we know that $\phi u \in L^2$ and $T(\phi u) \in L^2$, and so, using Lemma 3.3, we have $\phi u \in H_0^2$. The main result of this chapter is the derivation of an upper bound for the integral $\int (\phi \nabla^2(\phi u))^2$ in terms of ϕ , u , Tu , p and q . The reason for doing this is that in the next chapter we shall derive $\int (\phi \nabla^2(\phi u))^2$ itself as an upper bound for integrals which we shall be considering, and thereby be able to use the bound derived here.

In order to obtain the estimate for $\int (\phi \nabla^2(\phi u))^2$, we shall consider the two integrals on the right-hand side of Lemma 3.9(1) in turn.

Proposition 4.1 If $u \in \mathcal{D}(T_0^*)$ and $\phi \in \mathcal{D}$, with u and ϕ real-valued and $\phi \geq 0$, then

$$\int \nabla^2(\phi^4 u) \nabla^2 u \geq \frac{2}{3} \int (\phi \nabla^2(\phi u))^2 - K \int |\nabla(\phi u)|^2 - K \int (1 + \phi^2) u^2 \quad (1)$$

Proof: We may assume that $|\nabla \phi| \leq K$ and $|\nabla^2 \phi| \leq K$.

$$\begin{aligned} \int \nabla^2(\phi^4 u) \nabla^2 u &= \int \phi^2 [\phi \nabla^2(\phi u) + 6 \nabla \phi \cdot \nabla(\phi u) + (3 \phi \nabla^2 \phi + 6 |\nabla \phi|^2) u] \nabla^2 u \\ &= \int [\phi \nabla^2(\phi u) + 6 \nabla \phi \cdot \nabla(\phi u) + (3 \phi \nabla^2 \phi + 6 |\nabla \phi|^2) u] \phi^2 \nabla^2 u \\ &= \int [\phi \nabla^2(\phi u) + 6 \nabla \phi \cdot \nabla(\phi u) + (3 \phi \nabla^2 \phi + 6 |\nabla \phi|^2) u] \times \\ &\quad \times [\phi \nabla^2(\phi u) - 2 \nabla \phi \cdot \nabla(\phi u) - (\phi \nabla^2 \phi - 2 |\nabla \phi|^2) u] \\ &= \int [(\phi \nabla^2(\phi u))^2 + 4 \phi \nabla^2(\phi u) \nabla \phi \cdot \nabla(\phi u) + \\ &\quad + \phi \nabla^2(\phi u) (2 \phi \nabla^2 \phi + 8 |\nabla \phi|^2) u - 12 (\nabla \phi \cdot \nabla(\phi u))^2 - \\ &\quad - 12 \nabla \phi \cdot \nabla(\phi u) (\phi \nabla^2 \phi) u - (3 \phi^2 (\nabla^2 \phi)^2 - 12 |\nabla \phi|^4) u^2] \\ &\geq \int (\phi \nabla^2(\phi u))^2 - 4 \int |\phi \nabla^2(\phi u)| |\nabla \phi| |\nabla(\phi u)| - \\ &\quad - \int |\phi \nabla^2(\phi u)| |2 \phi \nabla^2 \phi + 8 |\nabla \phi|^2| |u| - \\ &\quad - 12 \int |\nabla \phi|^2 |\nabla(\phi u)|^2 - 12 \int |\nabla \phi| |\nabla(\phi u)| |\phi \nabla^2 \phi| |u| - \\ &\quad - \int |3 \phi^2 (\nabla^2 \phi)^2 - 12 |\nabla \phi|^4| u^2 \end{aligned}$$

$$\begin{aligned}
\int \nabla^2(\phi^4 u) \nabla^2 u &\geq \int (\phi \nabla^2(\phi u))^2 - \frac{1}{6} \int (\phi \nabla^2(\phi u))^2 - K \int |\nabla(\phi u)|^2 - \\
&\quad - \frac{1}{6} \int (\phi \nabla^2(\phi u))^2 - K \int (1 + \phi)^2 u^2 - K \int |\nabla(\phi u)|^2 - \\
&\quad - K \int |\nabla(\phi u)|^2 - K \int \phi^2 u^2 - K \int (1 + \phi^2) u^2 \\
&\geq \frac{2}{3} \int (\phi \nabla^2(\phi u))^2 - K \int |\nabla(\phi u)|^2 - K \int (1 + \phi^2) u^2. \quad \square
\end{aligned}$$

Proposition 4.2 If $u \in \mathcal{D}(T_0^*)$ and $\phi \in \mathcal{D}$, with u and ϕ real-valued

and $\phi \geq 0$, and if $|p\phi^2| \leq c$, then

$$\int \nabla(\phi^4 u) \cdot p \nabla u \geq -Kc \int |\nabla(\phi u)|^2 - Kc \int u^2 \quad (1)$$

Proof: We may assume that $|\nabla\phi| \leq K$.

$$\begin{aligned}
\int \nabla(\phi^4 u) \cdot p \nabla u &= \int p \nabla(\phi^4 u) \cdot \nabla u \\
&= \int p \phi^3 [\nabla(\phi u) + 3u \nabla\phi] \cdot \nabla u \\
&= \int p \phi^2 [\nabla(\phi u) + 3u \nabla\phi] \cdot \phi \nabla u \\
&= \int p \phi^2 [\nabla(\phi u) + 3u \nabla\phi] \cdot [\nabla(\phi u) - u \nabla\phi] \\
&= \int p \phi^2 [|\nabla(\phi u)|^2 + 2u \nabla\phi \cdot \nabla(\phi u) - 3u^2 |\nabla\phi|^2] \\
&\geq -\int |p \phi^2| [|\nabla(\phi u)|^2 + 2|u| |\nabla\phi| |\nabla(\phi u)| + 3u^2 |\nabla\phi|^2] \\
&\geq -c \int |\nabla(\phi u)|^2 - Kc \int |\nabla(\phi u)|^2 - Kc \int u^2 - Kc \int u^2 \\
&\geq -Kc \int |\nabla(\phi u)|^2 - Kc \int u^2. \quad \square
\end{aligned} \quad (2)$$

Proposition 4.3 If $u \in \mathcal{D}(T_0^*)$ and $\phi \in \mathcal{D}$, with u and ϕ real-valued,

then, for any $\varepsilon > 0$,

$$\int |\nabla(\phi u)|^2 \leq \varepsilon \int (\phi \nabla^2(\phi u))^2 + K\varepsilon^{-1} \int u^2 \quad (1)$$

Proof:

$$\begin{aligned}
\int |\nabla(\phi u)|^2 &= \int \nabla(\phi u) \cdot \nabla(\phi u) \\
&= -\int \phi u \nabla^2(\phi u) \\
&\leq \int |u| |\phi \nabla^2(\phi u)| \\
&\leq \varepsilon \int (\phi \nabla^2(\phi u))^2 + K\varepsilon^{-1} \int u^2. \quad \square
\end{aligned}$$

Lemma 4.4 If $u \in \mathcal{D}(T^*)$ and $\phi \in \mathcal{D}$, with u and ϕ real-valued and $\phi \geq 0$, and if $|p\phi^2| \leq c$, then

$$\int (\phi \nabla^2(\phi u))^2 \leq \int \phi^4 (Tu)^2 + K \int (\phi^4 + c^2 - q\phi^4) u^2 \quad (1)$$

Proof: We may assume that $|\nabla\phi| \leq K$, $|\nabla^2\phi| \leq K$ and $c \geq 1$.

From Lemma 3.9(1), Prop.4.1(1) and Prop.4.2(1):

$$\begin{aligned} \int \phi^4 u Tu - \int q\phi^4 u^2 &= \int \nabla^2(\phi^4 u) \nabla^2 u + \int \nabla(\phi^4 u) \cdot p \nabla u \\ &\geq \frac{2}{3} \int (\phi \nabla^2(\phi u))^2 - K \int |\nabla(\phi u)|^2 - K \int (1 + \phi^2) u^2 - \\ &\quad - Kc \int |\nabla(\phi u)|^2 - Kc \int u^2 \\ &\geq \frac{2}{3} \int (\phi \nabla^2(\phi u))^2 - Kc \int |\nabla(\phi u)|^2 - K \int (\phi^2 + c) u^2 \\ &\geq \left(\frac{2}{3} - Kc\varepsilon\right) \int (\phi \nabla^2(\phi u))^2 - K \int (c\varepsilon^{-1} + \phi^2 + c) u^2 \\ &\hspace{15em} \text{(by Prop.4.3(1))} \\ &\geq \frac{1}{2} \int (\phi \nabla^2(\phi u))^2 - K \int (c^2 + \phi^2 + c) u^2 \quad \left(\text{with } \varepsilon = \frac{1}{6}(Kc)^{-1}\right) \\ &\geq \frac{1}{2} \int (\phi \nabla^2(\phi u))^2 - K \int (c^2 + \phi^2) u^2. \end{aligned}$$

$$\begin{aligned} \text{Therefore } \int (\phi \nabla^2(\phi u))^2 &\leq 2 \int \phi^4 u Tu + K \int (c^2 + \phi^2 - q\phi^4) u^2 \\ &\leq \int \phi^4 (Tu)^2 + K \int (\phi^4 + c^2 + \phi^2 - q\phi^4) u^2 \\ &\leq \int \phi^4 (Tu)^2 + K \int (\phi^4 + c^2 - q\phi^4) u^2. \quad \square \end{aligned}$$

This is the desired estimate of $\int (\phi \nabla^2(\phi u))^2$ which we shall use in the next chapter.

If we assume further that $p \geq 0$, then the result of Proposition 4.2 can be amended slightly, and consequently also that of Lemma 4.4. The effect will be that the term c^2 in Lemma 4.4(1) will be replaced by $p\phi^2$ and we shall have an extra term, namely $\int p\phi^2 |\nabla(\phi u)|^2$, on the left-hand side of the inequality. Since c is the bound on $p\phi^2$, and hence c^2 is the bound on $p^2\phi^4$, this means that p itself will appear on the right-hand side of the estimate rather than effectively p^2 . We also note the appearance of $1 + \phi^4$ on the right-hand side of Lemma 4.6(1) as opposed to just ϕ^4 on the right-hand side of Lemma 4.4(1). This is because we

no longer mention an upper bound here for $p\phi^2$ which we formerly assumed to be ≥ 1 and which subsumed the 1 in the proof of Lemma 4.4.

Proposition 4.5 If $u \in \mathcal{D}(T_0^*)$ and $\phi \in \mathcal{D}$, with u and ϕ real-valued and $\phi \geq 0$, and if $p \geq 0$, then

$$\int \nabla(\phi^4 u) \cdot p \nabla u \geq \frac{1}{2} \int p \phi^2 |\nabla(\phi u)|^2 - K \int p \phi^2 u^2 \quad (1)$$

Proof: We may assume that $|\nabla \phi| \leq K$. From Prop. 4.2(2):

$$\begin{aligned} \int \nabla(\phi^4 u) \cdot p \nabla u &= \int p \phi^2 [|\nabla(\phi u)|^2 + 2u \nabla \phi \cdot \nabla(\phi u) - 3u^2 |\nabla \phi|^2] \\ &\geq \int p \phi^2 [|\nabla(\phi u)|^2 - 2|u| |\nabla \phi| |\nabla(\phi u)| - 3u^2 |\nabla \phi|^2] \\ &\geq \int p \phi^2 |\nabla(\phi u)|^2 - \frac{1}{2} \int p \phi^2 |\nabla(\phi u)|^2 - K \int p \phi^2 u^2 - K \int p \phi^2 u^2 \\ &\geq \frac{1}{2} \int p \phi^2 |\nabla(\phi u)|^2 - K \int p \phi^2 u^2. \quad \square \end{aligned}$$

Lemma 4.6 If $u \in \mathcal{D}(T_0^*)$ and $\phi \in \mathcal{D}$, with u and ϕ real-valued and $\phi \geq 0$, and if $p \geq 0$, then

$$\int [(\phi \nabla^2(\phi u))^2 + p \phi^2 |\nabla(\phi u)|^2] \leq \int \phi^4 (Tu)^2 + K \int (1 + \phi^4 + p \phi^2 - q \phi^4) u^2 \quad (1)$$

Proof: We may assume that $|\nabla \phi| \leq K$ and $|\nabla^2 \phi| \leq K$.

From Lemma 3.9(1), Prop. 4.1(1) and Prop. 4.5(1):

$$\begin{aligned} \int \phi^4 u Tu - \int q \phi^4 u^2 &= \int \nabla^2(\phi^4 u) \nabla^2 u + \int \nabla(\phi^4 u) \cdot p \nabla u \\ &\geq \frac{2}{3} \int (\phi \nabla^2(\phi u))^2 - K \int |\nabla(\phi u)|^2 - K \int (1 + \phi^2) u^2 + \\ &\quad + \frac{1}{2} \int p \phi^2 |\nabla(\phi u)|^2 - K \int p \phi^2 u^2 \\ &\geq \frac{1}{2} \int (\phi \nabla^2(\phi u))^2 - K \int (1 + \phi^2) u^2 + \quad (\text{by Prop. 4.3(1)}) \\ &\quad + \frac{1}{2} \int p \phi^2 |\nabla(\phi u)|^2 - K \int p \phi^2 u^2. \quad \text{with } \varepsilon = \frac{1}{6} K^{-1} \end{aligned}$$

$$\begin{aligned} \text{Therefore } \int [(\phi \nabla^2(\phi u))^2 + p \phi^2 |\nabla(\phi u)|^2] &\leq 2 \int \phi^4 u Tu + K \int (1 + \phi^2 + p \phi^2 - q \phi^4) u^2 \\ &\leq \int \phi^4 (Tu)^2 + K \int (\phi^4 + 1 + \phi^2 + p \phi^2 - q \phi^4) u^2 \\ &\leq \int \phi^4 (Tu)^2 + K \int (1 + \phi^4 + p \phi^2 - q \phi^4) u^2. \quad \square \end{aligned}$$

This is the amended estimate in the case $p \geq 0$ which we shall use in the next chapter.

CHAPTER 5

In this chapter, we shall establish the generalisation, referred to in the Introduction, of Theorem 0.3 to the fourth-order partial differential operator under consideration here in Part One, and derive the corollary which is a generalisation of Theorem 0.4 to dimension greater than 1.

If u and $v \in \mathcal{D}(T_0^*)$, then u, v, T_0^*u and $T_0^*v \in L^2$ and so $\int (u\overline{T_0^*v} - \overline{v}T_0^*u)$ is certainly finite, i.e.

$$(u, T_0^*v) - (T_0^*u, v) = L, \quad \text{where } L \text{ is finite.}$$

It is a standard result of Functional Analysis that, given a symmetric operator A , A is e.s.a. iff A^* is symmetric. Now, by Lemma 1.1, T_0 is symmetric and so T_0 is e.s.a. iff T_0^* is symmetric, i.e. iff $L=0$ for any pair of functions u and $v \in \mathcal{D}(T_0^*)$.

However, the proof of Lemma 4.4 is only for a real-valued function u . If $L=0$ for any pair of complex-valued functions u and v , then, a fortiori, $L=0$ for any pair of real-valued functions u and v . But if $L=0$ for any pair of real-valued functions and if $u = u_1 + iu_2$ and $v = v_1 + iv_2$, where u_1, u_2, v_1 and v_2 are real-valued and $i = \sqrt{-1}$, then, because T_0^* is a linear operator with real coefficients:

$$\begin{aligned} (u, T_0^*v) - (T_0^*u, v) &= [(u_1, T_0^*v_1) - (T_0^*u_1, v_1)] \\ &\quad - i[(u_1, T_0^*v_2) - (T_0^*u_1, v_2)] \\ &\quad + i[(u_2, T_0^*v_1) - (T_0^*u_2, v_1)] \\ &\quad - i^2[(u_2, T_0^*v_2) - (T_0^*u_2, v_2)] \\ &= 0. \end{aligned}$$

We have just shown that T_0 is e.s.a. iff $L=0$ for any pair of real-valued functions u and $v \in \mathcal{D}(T_0^*)$. Therefore, in Theorem 5.4, we shall consider $\int (uT_0^*v - vT_0^*u)$ for real-valued functions u and $v \in \mathcal{D}(T_0^*)$.

Before moving on to the theorem, we shall derive estimates for three integrals which will occur in the proof of the theorem. As in Chapter 4, if u and $v \in \mathcal{D}(T^*)$ and $\phi \in \mathcal{D}$, then ϕu and $\phi v \in H_0^2$, and so the estimates will be in terms of first- and second-order derivatives of ϕu and ϕv . In the theorem we shall use Prop.4.3(1) to reduce the terms involving first-order derivatives to ones involving second-order derivatives and ones involving u and v , and then use the estimate given by Lemma 4.4(1).

In the propositions, $\underline{n}(x)$ denotes a vector-valued function of x with real-valued components $n_j(x)$, $1 \leq j \leq n$.

Proposition 5.1 If u and $v \in \mathcal{D}(T^*)$ and $\phi \in \mathcal{D}$, with u , v and ϕ real-valued, and if $|p\phi^2| \leq c$ and $|\underline{n}| \leq K$, then

$$|\int \underline{n} \cdot [\phi^3 p(u\nabla v - v\nabla u)]| \leq Kc \int [|\nabla(\phi u)|^2 + |\nabla(\phi v)|^2 + u^2 + v^2] \quad (1)$$

Proof: We use the identity $\phi \nabla u = \nabla(\phi u) - u \nabla \phi$.

$$\begin{aligned} |\int \underline{n} \cdot [\phi^3 p(u\nabla v - v\nabla u)]| &\leq \int |\underline{n}| \cdot |\phi^3 p(u\nabla v - v\nabla u)| \\ &\leq \int |\underline{n}| |p\phi^2| |u\phi \nabla v - v\phi \nabla u| \\ &\leq Kc \int |u\nabla(\phi v) - v\nabla(\phi u)| \\ &\leq Kc \int [|u| |\nabla(\phi v)| + |v| |\nabla(\phi u)|] \\ &\leq \frac{1}{2} Kc \int [|u|^2 + |\nabla(\phi v)|^2 + |v|^2 + |\nabla(\phi u)|^2] \end{aligned}$$

which gives the desired estimate. \square

Proposition 5.2 If u and $v \in \mathcal{D}(T^*)$ and $\phi \in \mathcal{D}$, with u , v and ϕ real-valued and $\phi \geq 0$, and if $|\underline{n}| \leq K$, then

$$\begin{aligned} |\int \underline{n} \cdot [\phi^3 (\nabla u \nabla^2 v - \nabla v \nabla^2 u)]| &\leq \frac{1}{4} \int [(\phi \nabla^2(\phi u))^2 + (\phi \nabla^2(\phi v))^2] + \\ &\quad + K \int [|\nabla(\phi u)|^2 + |\nabla(\phi v)|^2] + \\ &\quad + K \int (1 + \phi^2)(u^2 + v^2) \end{aligned} \quad (1)$$

Proof: We use the identities $\phi \nabla u = \nabla(\phi u) - u \nabla \phi$ and

$$\phi^2 \nabla^2 u = \phi \nabla^2(\phi u) - 2\nabla \phi \cdot \nabla(\phi u) - (\phi \nabla^2 \phi - 2|\nabla \phi|^2)u.$$

We may assume that $|\nabla \phi| \leq K$ and $|\nabla^2 \phi| \leq K$.

$$\begin{aligned}
& \left| \int_{\underline{n}} [\phi^3 (\nabla u \nabla^2 v - \nabla v \nabla^2 u)] \right| \leq \\
& \leq \int_{\underline{n}} |\phi^3 (\nabla u \nabla^2 v - \nabla v \nabla^2 u)| \\
& \leq \int_{\underline{n}} |(\phi \nabla u)(\phi^2 \nabla^2 v) - (\phi \nabla v)(\phi^2 \nabla^2 u)| \\
& \leq K \int [|(\nabla(\phi u) - u \nabla \phi)(\phi \nabla^2(\phi v) - 2 \nabla \phi \cdot \nabla(\phi v) - (\phi \nabla^2 \phi - 2|\nabla \phi|^2)v) - \\
& \quad - [(\nabla(\phi v) - v \nabla \phi)(\phi \nabla^2(\phi u) - 2 \nabla \phi \cdot \nabla(\phi u) - (\phi \nabla^2 \phi - 2|\nabla \phi|^2)u)] | \\
& \leq K \int [(|\nabla(\phi u)| + |u| |\nabla \phi|)(|\phi \nabla^2(\phi v)| + 2|\nabla \phi| |\nabla(\phi v)| + \\
& \quad + |\phi \nabla^2 \phi - 2|\nabla \phi|^2| |v|) + \\
& \quad + (|\nabla(\phi v)| + |v| |\nabla \phi|)(|\phi \nabla^2(\phi u)| + 2|\nabla \phi| |\nabla(\phi u)| + \\
& \quad + |\phi \nabla^2 \phi - 2|\nabla \phi|^2| |u|)] \\
& \leq K \int [(|\nabla(\phi u)| + |u|)(|\phi \nabla^2(\phi v)| + |\nabla(\phi v)| + (1+\phi)|v|) + \\
& \quad + (|\nabla(\phi v)| + |v|)(|\phi \nabla^2(\phi u)| + |\nabla(\phi u)| + (1+\phi)|u|)] \\
& \leq K \int [|\phi \nabla^2(\phi v)| |\nabla(\phi u)| + |\phi \nabla^2(\phi v)| |u| + |\nabla(\phi v)| |\nabla(\phi u)| + \\
& \quad + |\nabla(\phi v)| |u| + (1+\phi)|v| |\nabla(\phi u)| + (1+\phi)|v| |u| + \\
& \quad + |\phi \nabla^2(\phi u)| |\nabla(\phi v)| + |\phi \nabla^2(\phi u)| |v| + |\nabla(\phi u)| |\nabla(\phi v)| + \\
& \quad + |\nabla(\phi u)| |v| + (1+\phi)|u| |\nabla(\phi v)| + (1+\phi)|u| |v|] \\
& \leq \frac{1}{8} \int [(\phi \nabla^2(\phi v))^2 + (\phi \nabla^2(\phi u))^2] + K \int [|\nabla(\phi u)|^2 + |\nabla(\phi v)|^2] + \\
& \quad + \frac{1}{8} \int [(\phi \nabla^2(\phi v))^2 + (\phi \nabla^2(\phi u))^2] + K \int (u^2 + v^2) + \\
& \quad + K \int [|\nabla(\phi v)|^2 + |\nabla(\phi u)|^2] + K \int [|\nabla(\phi u)|^2 + |\nabla(\phi v)|^2] + \\
& \quad + K \int [|\nabla(\phi v)|^2 + |\nabla(\phi u)|^2] + K \int (u^2 + v^2) + \\
& \quad + K \int (1+\phi)^2 (v^2 + u^2) + K \int [|\nabla(\phi u)|^2 + |\nabla(\phi v)|^2] + \\
& \quad + K \int (1+\phi)^2 (v^2 + u^2) + K \int (1+\phi)^2 (u^2 + v^2) \\
& \leq \frac{1}{4} \int [(\phi \nabla^2(\phi u))^2 + (\phi \nabla^2(\phi v))^2] + \\
& \quad + K \int [|\nabla(\phi u)|^2 + |\nabla(\phi v)|^2] + K \int (1+\phi^2)(u^2 + v^2)
\end{aligned}$$

which is the desired estimate. \square

Unlike the integrals considered in the above two propositions, the one to be considered in the proposition below involves third-order derivatives of u and v , and so we must integrate by parts before we can derive the desired estimate. This requires the existence and boundedness

of the first-order derivatives of $n_j(x)$, $1 \leq j \leq n$.

Proposition 5.3 If u and $v \in \mathcal{D}(T_0^*)$ and $\phi \in \mathcal{D}$, with u, v and ϕ real-valued and $\phi \geq 0$, and if $|\underline{n}| \leq K$ and $|\nabla \cdot \underline{n}| \leq K$, then

$$\begin{aligned} \left| \int \underline{n} \cdot [\phi^3 (u \nabla(\nabla^2 v) - v \nabla(\nabla^2 u))] \right| &\leq \frac{1}{4} \int [(\phi \nabla^2(\phi u))^2 + (\phi \nabla^2(\phi v))^2] + \\ &+ K \int [|\nabla(\phi u)|^2 + |\nabla(\phi v)|^2] + \\ &+ K \int (1 + \phi^2)(u^2 + v^2) \end{aligned} \quad (1)$$

Proof: We use the identities $\phi \nabla u = \nabla(\phi u) - u \nabla \phi$ and $\phi^2 \nabla^2 u = \phi \nabla^2(\phi u) - 2 \nabla \phi \cdot \nabla(\phi u) - (\phi \nabla^2 \phi - 2|\nabla \phi|^2)u$.

We may assume that $|\nabla \phi| \leq K$ and $|\nabla^2 \phi| \leq K$.

$$\begin{aligned} \left| \int \underline{n} \cdot [\phi^3 u \nabla(\nabla^2 v)] \right| &= \left| \int (\underline{n} \phi^3 u) \cdot \nabla(\nabla^2 v) \right| \\ &= \left| \int \nabla \cdot (\underline{n} \phi^3 u) \nabla^2 v \right| \\ &= \left| \int [(\nabla \cdot \underline{n}) \phi^3 u \nabla^2 v + \underline{n} \cdot (\nabla(\phi^3 u)) \nabla^2 v] \right| \\ &\leq \int |(\nabla \cdot \underline{n}) \phi u \phi^2 \nabla^2 v + \underline{n} \cdot (\nabla(\phi^3 u)) \nabla^2 v| \\ &\leq K \int |\phi u \phi^2 \nabla^2 v| + K \int |\nabla(\phi^3 u) \nabla^2 v| \end{aligned} \quad (2)$$

$$\begin{aligned} \text{Now } K \int |\phi u \phi^2 \nabla^2 v| &\leq K \int \phi |u| |\phi \nabla^2(\phi v) - 2 \nabla \phi \cdot \nabla(\phi v) - (\phi \nabla^2 \phi - 2|\nabla \phi|^2)v| \\ &\leq K \int \phi |u| [|\phi \nabla^2(\phi v)| + 2|\nabla \phi| |\nabla(\phi v)| + |\phi \nabla^2 \phi - 2|\nabla \phi|^2| |v|] \\ &\leq K \int \phi |u| [|\phi \nabla^2(\phi v)| + |\nabla(\phi v)| + (1 + \phi)|v|] \\ &\leq \frac{1}{8} \int (\phi \nabla^2(\phi v))^2 + K \int |\nabla(\phi v)|^2 + K \int (1 + \phi)^2 v^2 + K \int \phi^2 u^2 \end{aligned} \quad (3)$$

$$\begin{aligned} \text{and } K \int |\nabla(\phi^3 u) \nabla^2 v| &= K \int \phi^2 [\nabla(\phi u) + 2u \nabla \phi] \nabla^2 v \\ &= K \int [\nabla(\phi u) + 2u \nabla \phi] [\phi^2 \nabla^2 v] \\ &= K \int [\nabla(\phi u) + 2u \nabla \phi] [\phi \nabla^2(\phi v) - 2 \nabla \phi \cdot \nabla(\phi v) - (\phi \nabla^2 \phi - 2|\nabla \phi|^2)v] \\ &\leq K \int [|\nabla(\phi u)| + 2|u| |\nabla \phi|] \times \\ &\quad \times [|\phi \nabla^2(\phi v)| + 2|\nabla \phi| |\nabla(\phi v)| + |\phi \nabla^2 \phi - 2|\nabla \phi|^2| |v|] \\ &\leq K \int [|\nabla(\phi u)| + |u|] [|\phi \nabla^2(\phi v)| + |\nabla(\phi v)| + (1 + \phi)|v|] \\ &\leq \frac{1}{8} \int (\phi \nabla^2(\phi v))^2 + K \int |\nabla(\phi v)|^2 + K \int (1 + \phi)^2 v^2 + \\ &\quad + K \int |\nabla(\phi u)|^2 + K \int u^2 \end{aligned} \quad (4)$$

Substituting from (3) and (4) into (2):

$$\begin{aligned} \left| \int_{\underline{n}} [\phi^3 u \nabla(\nabla^2 v)] \right| &\leq \frac{1}{4} \int (\phi \nabla^2(\nabla v))^2 + K \int [|\nabla(\phi u)|^2 + |\nabla(\phi v)|^2] + \\ &\quad + K \int (1 + \phi)^2 (u^2 + v^2) \\ &\leq \frac{1}{4} \int (\phi \nabla^2(\phi v))^2 + K \int [|\nabla(\phi u)|^2 + |\nabla(\phi v)|^2] + \\ &\quad + K \int (1 + \phi^2)(u^2 + v^2). \end{aligned}$$

Obviously, we have a similar inequality with u and v interchanged, and using these inequalities we obtain:

$$\begin{aligned} \left| \int_{\underline{n}} [\phi^3 (u \nabla(\nabla^2 v) - v \nabla(\nabla^2 u))] \right| &\leq \\ &\leq \left| \int_{\underline{n}} [\phi^3 u \nabla(\nabla^2 v)] \right| + \left| \int_{\underline{n}} [\phi^3 v \nabla(\nabla^2 u)] \right| \\ &\leq \frac{1}{4} \int [(\phi \nabla^2(\phi u))^2 + (\phi \nabla^2(\phi v))^2] + \\ &\quad + K \int [|\nabla(\phi u)|^2 + |\nabla(\phi v)|^2] + K \int (1 + \phi^2)(u^2 + v^2) \end{aligned}$$

which is the desired estimate. \square

We shall now introduce the 'bands' in which conditions on the coefficient functions p and q will be more precise.

Let $\{\Omega_\nu\}$, $\nu = 1, 2, \dots$, be a sequence of compact sets in \mathbb{R}^n , such that the sequence covers all of \mathbb{R}^n as $\nu \rightarrow \infty$. Let $h_\nu(x) = \text{dist}(x, \Omega_\nu)$. For each ν , let Ω_ν have a C^1 boundary and have the property that if h_ν is sufficiently small, say $h_\nu < \delta_\nu$ for some $\delta_\nu > 0$, then $h_\nu \in C^2$ as a function of x . Define a compact set Ω'_ν by $\Omega'_\nu = \{x: h_\nu(x) \leq d_\nu\}$ for some d_ν with $0 < d_\nu < \delta_\nu$, and suppose that this can be done so that $\Omega'_\nu \subset \Omega_{\nu+1}$. Let A_ν be the closed 'annulus' between Ω_ν and Ω'_ν . We shall say that a function $\theta \in C_0^{\mu,+}(A_\nu)$ if it is a function only of h_ν , is real-valued, non-negative, continuous and piece-wise C^μ , and $\text{supp } \theta \subsetneq A_\nu$. For any particular A_ν under consideration, define $V_k = \{x: h_\nu(x) \leq k\}$ and $S_k = \{x: h_\nu(x) = k\}$, with dS being the element of surface area, for $0 \leq k \leq d_\nu$. For $x \in A_\nu$, with $h_\nu(x) = k$, let $\underline{n}(x)$ be the outward normal to the surface S_k . Then $|\underline{n}| = 1$. Suppose that the sequence $\{A_\nu\}$ is such that we have $|\nabla \cdot \underline{n}| \leq K$.

Theorem 5.4 Suppose that conditions $\mathcal{E}(i)$ - $\mathcal{E}(v)$ are satisfied, and that we have a sequence $\{A_\nu\}$ as described above. Let $\theta \in C_0^{2,+}(A_\nu)$ with $|\nabla\theta| \leq K$ and $|\nabla^2\theta| \leq K$. If u and $v \in \mathcal{D}(T^*)$ with u and v real-valued and $\int_{\Omega_\nu} (uT_\circ^*v - vT_\circ^*u) \rightarrow L \neq 0$ (we may assume that $L > 0$), and if (a) $|p\theta^2| \leq c$ (we may assume that $c \geq 1$), and

(b) $q\theta^4 \geq -Q(h_\nu)$,

then, for ν sufficiently large,

$$L \int_0^{\mathbf{d}_\nu} \theta^3(h_\nu) dh_\nu \leq K \int_{A_\nu} [\theta^4 + c^2 + Q][u^2 + v^2 + (Tu)^2 + (Tv)^2] dx \quad (1)$$

Proof: We may assume that u and $v \in \mathcal{D}$, because, as we are interested in u and v only in A_ν , we could consider ψu and ψv , where $\psi \in \mathcal{D}$ and $\psi(x) = 1$ if $x \in A_\nu$, and then work with elements of sequences $\{u_{\nu^i}\}$ and $\{v_{\nu^i}\}$, u_{ν^i} and $v_{\nu^i} \in \mathcal{D}$, as defined in Lemma 3.8, before letting $\nu^i \rightarrow \infty$.

$$\begin{aligned} \int_{V_k} (uT_\circ^*v - vT_\circ^*u) dx &= \int_{V_k} [u\nabla^2(\nabla^2v) - u\nabla \cdot (p\nabla v) + uqv - \\ &\quad - v\nabla^2(\nabla^2u) + v\nabla \cdot (p\nabla u) - vqu] dx \\ &= \int_{V_k} [u\nabla^2(\nabla^2v) - \nabla^2u\nabla^2v - v\nabla^2(\nabla^2u) + \nabla^2v\nabla^2u - \\ &\quad - (u\nabla \cdot (p\nabla v) - v\nabla \cdot (p\nabla u))] dx \\ &= \int_{V_k} [\nabla \cdot (u\nabla(\nabla^2v)) - \nabla u \cdot \nabla(\nabla^2v) - \\ &\quad - \nabla \cdot (\nabla u\nabla^2v) + \nabla u \cdot \nabla(\nabla^2v) - \\ &\quad - \nabla \cdot (v\nabla(\nabla^2u)) + \nabla v \cdot \nabla(\nabla^2u) + \\ &\quad + \nabla \cdot (\nabla v\nabla^2u) - \nabla v \cdot \nabla(\nabla^2u) - \\ &\quad - (\nabla \cdot (up\nabla v) - p\nabla u \cdot \nabla v - \nabla \cdot (vp\nabla u) + p\nabla v \cdot \nabla u)] dx \\ &= \int_{V_k} \nabla \cdot [(u\nabla(\nabla^2v) - v\nabla(\nabla^2u)) - \\ &\quad - (\nabla u\nabla^2v - \nabla v\nabla^2u) - \\ &\quad - p(u\nabla v - v\nabla u)] dx \\ &= \int_{S_k} \underline{n} \cdot [(u\nabla(\nabla^2v) - v\nabla(\nabla^2u)) - (\nabla u\nabla^2v - \nabla v\nabla^2u) - \\ &\quad - p(u\nabla v - v\nabla u)] dS \end{aligned}$$

by an application of Green's Theorem.

We shall use Prop.4.3(1), Lemma 4.4(1), Prop.5.1(1), Prop.5.2(1) and Prop.5.3(1), all of which though stated and proved for functions $\phi \in \mathcal{D}$ still hold for functions $\theta \in C_0^{2,+}(A_\nu)$.

Now for ν sufficiently large:

$$\frac{1}{2}L \leq \int_{V_k} (uT_0^*v - vT_0^*u)dx, \text{ where } V_k \subset \Omega_\nu.$$

$$\text{Therefore } \frac{1}{2}L\theta^3(h_\nu) \leq \theta^3 \int_{S_k} \underline{n} \cdot [(u\nabla(\nabla^2 v) - v\nabla(\nabla^2 u)) - (\nabla u \nabla^2 v - \nabla v \nabla^2 u) - p(u\nabla v - v\nabla u)] dS.$$

$$\text{Therefore } \frac{1}{2}L \int_0^{\theta^3} \theta^3(h_\nu) dh_\nu \leq \int_{A_\nu} \theta^3 \underline{n} \cdot [(u\nabla(\nabla^2 v) - v\nabla(\nabla^2 u)) - (\nabla u \nabla^2 v - \nabla v \nabla^2 u) - p(u\nabla v - v\nabla u)] dx$$

$$\leq \left| \int_{\underline{n}} [\theta^3 (u\nabla(\nabla^2 v) - v\nabla(\nabla^2 u))] \right| + \left| \int_{\underline{n}} [\theta^3 (\nabla u \nabla^2 v - \nabla v \nabla^2 u)] \right| + \left| \int_{\underline{n}} [\theta^3 p(u\nabla v - v\nabla u)] \right| \quad (2)$$

$$\leq \frac{1}{2} \int [(\theta \nabla^2(\theta u))^2 + (\theta \nabla^2(\theta v))^2] + K \int [|\nabla(\theta u)|^2 + |\nabla(\theta v)|^2] + K \int (1 + \theta^2)(u^2 + v^2) + Kc \int [|\nabla(\theta u)|^2 + |\nabla(\theta v)|^2] + Kc \int (u^2 + v^2) \quad \begin{array}{l} \text{(by Prop.5.3(1)} \\ \text{and Prop.5.2(1)} \\ \text{(by Prop.5.1(1)} \end{array}$$

$$\leq \frac{1}{2} \int [(\theta \nabla^2(\theta u))^2 + (\theta \nabla^2(\theta v))^2] + Kc \int [|\nabla(\theta u)|^2 + |\nabla(\theta v)|^2] + K \int (c + \theta^2)(u^2 + v^2) \leq (\frac{1}{2} + Kc\varepsilon) \int [(\theta \nabla^2(\theta u))^2 + (\theta \nabla^2(\theta v))^2] + K \int (c\varepsilon^{-1} + c + \theta^2)(u^2 + v^2) \quad \begin{array}{l} \text{(by Prop.4.3(1)} \\ \text{(with } \varepsilon = \frac{1}{2}(Kc)^{-1} \text{)} \end{array}$$

$$\leq \int [(\theta \nabla^2(\theta u))^2 + (\theta \nabla^2(\theta v))^2] + K \int (c^2 + c + \theta^2)(u^2 + v^2) \leq \int [(\theta \nabla^2(\theta u))^2 + (\theta \nabla^2(\theta v))^2] + K \int (c^2 + \theta^4)(u^2 + v^2)$$

$$\leq \int \theta^4 [(Tu)^2 + (Tv)^2] + K \int (\theta^4 + c^2 - q\theta^4)(u^2 + v^2) + K \int (c^2 + \theta^4)(u^2 + v^2) \quad \text{(by Lemma 4.4(1))}$$

$$\leq K \int [\theta^4 + c^2 + Q][u^2 + v^2 + (Tu)^2 + (Tv)^2]$$

which gives us the desired result. \square

If we review the proof of Lemma 3.8 in light of the above theorem, we see that the key statement in the lemma is $\underline{T}v = Tv$ if $\text{supp } v \subset B_R$, i.e. $\underline{q}(x) = q(x)$ if $x \in B_R$. We now see that, for any particular ν , the proofs still hold if we just assume that $\underline{q}(x) = q(x)$ if $x \in A_\nu$. If we set

$$\underline{q}(x) = 0 \text{ when } x \in \Omega'_\nu / A_\nu \text{ (we may assume that } q^*(|x|) \geq 0),$$

$$\underline{q}(x) = q(x) \text{ when } x \in A_\nu, \text{ and}$$

$$\underline{q}(x) = - \sup_{y \in A_\nu} q^*(|y|) \text{ when } x \notin \Omega'_\nu,$$

we see that conditions $\mathcal{E}(\text{iii})$ - $\mathcal{E}(\nu)$ on q need hold only in A_ν . Similarly, condition $\mathcal{E}(\text{iv})$ on p need hold only in A_ν .

However, with reference to the remark preceding Lemma 3.9, these extra conditions (which are not assumed by Evans in [5]) enable us to work with $\mathcal{D}(T^*_0)$ as large as possible.

One final remark before we prove the corollary to Theorem 5.4: the requirement that $|\nabla \underline{q}| \leq K$ is obviously unnecessary in the 1-dimensional case, and is trivial in the case where the Ω_ν are n -dimensional spheres centred at the origin.

Corollary 5.4.1 Suppose that conditions $\mathcal{E}(\text{i})$ - $\mathcal{E}(\nu)$ are satisfied

and that we have a sequence $\{A_\nu\}$ as in Theorem 5.4, with

$$(a) \liminf_{\nu \rightarrow \infty} d_\nu > 0.$$

Let $B_\nu = \{x: \delta \leq h_\nu(x) \leq d_\nu - \delta\}$ for some $\delta > 0$ with $d_\nu - 2\delta > 0$,

for all ν . For each ν , assume that we have a function

$\theta_\nu \in C^{2,+}_0(A^+_\nu)$ (i.e. we allow $\text{supp } \theta_\nu = A_\nu$), with

$$(b) \theta_\nu \leq K, \quad |\theta'_\nu| \leq K, \quad |\theta''_\nu| \leq K;$$

$$(c) |p\theta^2_\nu| \leq K;$$

$$(d) q\theta^4_\nu \geq -K;$$

where K is independent of ν .

If (e) $\sum_{\nu} \int_{\delta}^{d_{\nu}-\delta} \theta_{\nu}^3(r) dr$ is a divergent infinite series, then

$L=0$ and so T_0 is e.s.a.

Proof: (a) implies the existence of the bands B_{ν} .

Let $\psi_{\nu} \in C_0^{2,+}(A_{\nu})$ with $0 \leq \psi_{\nu} \leq 1$, $\psi_{\nu}(r) = 1$ if $\delta \leq r \leq d_{\nu} - \delta$, and $|\psi'_{\nu}|$ and $|\psi''_{\nu}|$ bounded, the bound depending only on δ .

Let $\theta_{\nu} = \psi_{\nu} \Theta_{\nu}$. Then $\theta_{\nu} \in C_0^{2,+}(A_{\nu})$.

(b) implies that $|\theta'_{\nu}| \leq K$ and $|\theta''_{\nu}| \leq K$, and also $\theta_{\nu} \leq K$.

(c) implies that $|p\theta'_{\nu}| \leq K$, so we take $c=K$ in Theorem 5.4.

(d) implies that $q\theta^4_{\nu} \geq -K$, so we take $Q(h_{\nu})=K$ in Theorem 5.4.

Assume that $L > 0$. Then $L \int_{\delta}^{d_{\nu}-\delta} \theta_{\nu}^3(r) dr \leq L \int_0^{d_{\nu}} \theta_{\nu}^3(r) dr$,

and so, by Theorem 5.4(1):

$$L \int_{\delta}^{d_{\nu}-\delta} \theta_{\nu}^3(r) dr \leq K \int_{A_{\nu}} [u^2 + v^2 + (Tu)^2 + (Tv)^2] dx.$$

This is valid if ν is sufficiently large, say for all $\nu \geq N$. Summing over $\nu \geq N$ gives a finite right-hand side, which contradicts (e).

Therefore $L=0$ and so T_0 is e.s.a. \square

Drawing on an example in Evans [5], we give the example below, generalising the result of Everitt [7] to dimension greater than 1.

Example: In Corol.5.4.1, take $A_{\nu} = \{x: 2\nu\pi \leq |x| \leq (2\nu+1)\pi\}$.

Then $d_{\nu} = \pi$, so take $\delta = \pi/4$, and (a) is satisfied.

Take $\Theta_{\nu}(h_{\nu}(x)) = (2\nu\pi + h_{\nu}(x))^{-1/3} = |x|^{-1/3}$. Then (b) is satisfied.

In A_{ν} , take $|p| \leq K|x|^{2/3}$, with p also satisfying $\mathcal{E}(i)$ and $\mathcal{E}(iv)$. Then (c) is satisfied.

Take $q = K|x|^{\gamma} \sin|x|$ or $q = K \min(-|x|^{4/3}, |x|^{\gamma} \sin|x|)$ for any γ with q satisfying $\mathcal{E}(ii)-\mathcal{E}(v)$. Then, since $q \geq -K|x|^{4/3}$ in A_{ν} , (d) is satisfied.

Since $\int_{\pi/4}^{3\pi/4} (2\nu\pi + r)^{-1} dr = \log \left[\frac{8\nu+3}{8\nu+1} \right]$, (e) is satisfied and T_0 is e.s.a. \square

If, as in Chapter 4, we assume further that $p \geq 0$, we can amend the result of Theorem 5.4, from which we can derive the corollary which is a generalisation of Theorem 0.5 to dimension greater than 1. This entails first amending Proposition 5.1.

Proposition 5.5 If u and $v \in \mathcal{D}(\mathbb{T}_0^*)$ and $\phi \in \mathcal{D}$, with u, v and ϕ real-valued, and if $p \geq 0$ and $|\underline{n}| \leq K$, then

$$\begin{aligned} \left| \int \underline{n} \cdot [\phi^3 p(u \nabla v - v \nabla u)] \right| &\leq \int p \phi^2 [|\nabla(\phi u)|^2 + |\nabla(\phi v)|^2] + \\ &+ K \int p \phi^2 (u^2 + v^2) \end{aligned} \quad (1)$$

Proof: Similar to the proof of Prop. 5.1:

$$\begin{aligned} \left| \int \underline{n} \cdot [\phi^3 p(u \nabla v - v \nabla u)] \right| &\leq K \int p \phi^2 [|u| |\nabla(\phi v)| + |v| |\nabla(\phi u)|] \\ &\leq K \int p \phi^2 u^2 + \int p \phi^2 |\nabla(\phi v)|^2 + \\ &+ K \int p \phi^2 v^2 + \int p \phi^2 |\nabla(\phi u)|^2 \end{aligned}$$

which gives the desired estimate. \square

We can now proceed to the proof of Theorem 5.6, which follows much the same lines as that of Theorem 5.4, the same preamble applying now as it did then except we use Lemma 4.6(1) instead of Lemma 4.4(1). The comparison between the move from the right-hand side of Lemma 4.4(1) to the right-hand side of Lemma 4.6(1) with that from the right-hand side of Theorem 5.4(1) to the right-hand side of Theorem 5.6(1) is clearly very close, i.e. in the latter case $p\theta^2$ replaces c^2 and $1 + \theta^4$ replaces θ^4 ; we no longer mention an upper bound for $p\theta^2$, nor a lower bound for $q\theta^4$ in the theorem.

Theorem 5.6 Suppose that conditions $\mathcal{B}(i)$ - $\mathcal{C}(v)$ are satisfied, and that we have a sequence $\{A_\nu\}$ as described above. Let $\theta \in C_0^{2,+}(A_\nu)$ with $|\nabla \theta| \leq K$ and $|\nabla^2 \theta| \leq K$. If u and $v \in \mathcal{D}(\mathbb{T}_0^*)$ with u and v real-valued and $\int_{\Omega_\nu} (u \mathbb{T}_0^* v - v \mathbb{T}_0^* u) \rightarrow L \neq 0$ (we may assume that $L > 0$),

and if $p \geq 0$ in A_ν , then, for ν sufficiently large,

$$L \int_0^{\mathfrak{d}_\nu} \theta^3(h_\nu) dh_\nu \leq K \int_{A_\nu} [1 + \theta^4 + p\theta^2 - q\theta^4][u^2 + v^2 + (Tu)^2 + (Tv)^2] dx \quad (1)$$

Proof: We shall use Prop.4.3(1), Lemma 4.6(1), Prop.5.1(1), Prop.5.2(1) and Prop.5.5(1), all of which though stated and proved for functions $\phi \in \mathcal{D}$ still hold for functions $\theta \in C_0^{2,+}(A_\nu)$.

From Theorem 5.4(2):

$$\begin{aligned} \frac{1}{2} L \int_0^{\mathfrak{d}_\nu} \theta^3(h_\nu) dh_\nu &\leq \left| \int_{\underline{n}} [\theta^3(u \nabla(\nabla^2 v) - v \nabla(\nabla^2 u))] \right| + \\ &\quad + \left| \int_{\underline{n}} [\theta^3(\nabla u \nabla^2 v - \nabla v \nabla^2 u)] \right| + \\ &\quad + \left| \int_{\underline{n}} [\theta^3 p(u \nabla v - v \nabla u)] \right| \\ &\leq \frac{1}{2} \int [(\theta \nabla^2(\theta u))^2 + (\theta \nabla^2(\theta v))^2] + && \text{(by Prop.5.3(1))} \\ &\quad + K \int [|\nabla(\theta u)|^2 + |\nabla(\theta v)|^2] + && \text{and Prop.5.2(1)} \\ &\quad + K \int (1 + \theta^2)(u^2 + v^2) + \\ &\quad + \int p \theta^2 [|\nabla(\theta u)|^2 + |\nabla(\theta v)|^2] + && \text{(by Prop.5.5(1))} \\ &\quad + K \int p \theta^2 (u^2 + v^2) \\ &\leq \int [(\theta \nabla^2(\theta u))^2 + (\theta \nabla^2(\theta v))^2] + && \text{(by Prop.4.3(1))} \\ &\quad + K \int (1 + \theta^2)(u^2 + v^2) + && \text{with } \varepsilon = \frac{1}{2} K^{-1} \\ &\quad + \int p \theta^2 [|\nabla(\theta u)|^2 + |\nabla(\theta v)|^2] + \\ &\quad + K \int p \theta^2 (u^2 + v^2) \\ &\leq \int [(\theta \nabla^2(\theta u))^2 + p \theta^2 |\nabla(\theta u)|^2 + \\ &\quad + (\theta \nabla^2(\theta v))^2 + p \theta^2 |\nabla(\theta v)|^2] + \\ &\quad + K \int (1 + \theta^2 + p \theta^2)(u^2 + v^2) \\ &\leq \int \theta^4 [(Tu)^2 + (Tv)^2] + && \text{(by Lemma 4.6(1))} \\ &\quad + K \int (1 + \theta^4 + p \theta^2 - q \theta^4)(u^2 + v^2) \\ &\quad + K \int (1 + \theta^2 + p \theta^2)(u^2 + v^2) \\ &\leq K \int [1 + \theta^4 + p \theta^2 - q \theta^4][u^2 + v^2 + (Tu)^2 + (Tv)^2] \end{aligned}$$

which gives us the desired result. \square

The remarks which followed Theorem 5.4 apply equally well here.

Corollary 5.6.1 Suppose that conditions $\mathcal{E}(i)$ - $\mathcal{E}(v)$ are satisfied

and that we have a sequence $\{A_\nu\}$ as in Theorem 5.6, with

$$(a) \liminf_{\nu \rightarrow \infty} d_\nu > 0.$$

For each ν , assume that in A_ν ,

$$(b) p \geq 0 \text{ and either } p \leq Kd_\nu^2 \text{ or } p \leq Kd_\nu^2 |q|^{\frac{1}{2}}, \text{ and}$$

$$(c) q \geq -k,$$

with K and k independent of ν .

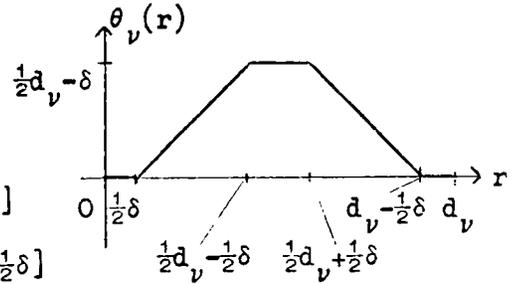
Then $L=0$ and so T_0 is e.s.a.

(The alternative in (b) means that if $p \leq Kd_\nu^2$, we need no further restriction on q , but if p is to grow more rapidly we need a corresponding growth in q .)

Proof: (a) implies the existence of $\delta > 0$ with $d_\nu - 2\delta > 0$ as $\nu \rightarrow \infty$.

For each ν , let $\theta_\nu \in C_0^{2,+}(A_\nu)$ such that

$$\begin{aligned} \theta_\nu(r) &= 0 \text{ if } r \in [0, \frac{1}{2}\delta] \\ &= r - \frac{1}{2}\delta \text{ if } r \in [\frac{1}{2}\delta, \frac{1}{2}d_\nu - \frac{1}{2}\delta] \\ &= \frac{1}{2}d_\nu - \delta \text{ if } r \in [\frac{1}{2}d_\nu - \frac{1}{2}\delta, \frac{1}{2}d_\nu + \frac{1}{2}\delta] \\ &= d_\nu - \frac{1}{2}\delta - r \text{ if } r \in [\frac{1}{2}d_\nu + \frac{1}{2}\delta, d_\nu - \frac{1}{2}\delta] \\ &= 0 \text{ if } r \in [d_\nu - \frac{1}{2}\delta, d_\nu] \end{aligned}$$



So $\theta'_\nu = 0$ or 1 , and $\theta''_\nu = 0$. We have $0 \leq \theta_\nu \leq Kd_\nu$, and also $d_\nu^4 \geq (2\delta)^4$, so $1 \leq Kd_\nu^4$.

Assume that $L > 0$, and suppose that in (b) $p \leq Kd_\nu^2$.

So $p\theta_\nu^2 \leq Kd_\nu^2 d_\nu^2 = Kd_\nu^4$. Also, from (c), $-q\theta_\nu^4 \leq kd_\nu^4$.

$$\text{By Theorem 5.6(1): } L \int_0^{d_\nu} \theta_\nu^3(r) dr \leq Kd_\nu^4 \int_{A_\nu} [u^2 + v^2 + (Tu)^2 + (Tv)^2] dx \quad (1)$$

Now, still assuming that $L > 0$, suppose that in (b) $p \leq Kd_\nu^2 |q|^{\frac{1}{2}}$.

From (c), $|q| \leq q + 2k$, and $(q+k) \geq 0$.

$$\begin{aligned} \text{We have } p\theta_\nu^2 - (q+k)\theta_\nu^4 &\leq Kd_\nu^2 |q|^{\frac{1}{2}} \theta_\nu^2 - (q+k)\theta_\nu^4 \\ &\leq Kd_\nu^4 + \frac{1}{2}|q| \theta_\nu^4 - (q+k)\theta_\nu^4 \\ &\leq Kd_\nu^4 + \frac{1}{2}(q+2k)\theta_\nu^4 - (q+k)\theta_\nu^4 \end{aligned}$$

$$\begin{aligned}
p\theta_\nu^2 - (q+k)\theta_\nu^4 &\leq Kd_\nu^4 + k\theta_\nu^4 - \frac{1}{2}(q+k)\theta_\nu^4 \\
&\leq (K+k)d_\nu^4 - \frac{1}{2}(q+k)\theta_\nu^4 \\
&\leq (K+k)d_\nu^4,
\end{aligned}$$

this inequality holding in A_ν .

By Theorem 5.6(1):

$$\begin{aligned}
L \int_0^{d_\nu} \theta_\nu^3(r) dr &\leq K \int_{A_\nu} [d_\nu^4 + d_\nu^4 + (p\theta_\nu^2 - (q+k)\theta_\nu^4) + k\theta_\nu^4] \times \\
&\quad \times [u^2 + v^2 + (Tu)^2 + (Tv)^2] dx \\
&\leq Kd_\nu^4 \int_{A_\nu} [u^2 + v^2 + (Tu)^2 + (Tv)^2] dx \quad (2)
\end{aligned}$$

From (1) and (2), we see that if $L > 0$ and either of the alternatives in (b) hold (together with (a) and (c)), then we have

$$Ld_\nu^{-4} \int_0^{d_\nu} \theta_\nu^3(r) dr \leq K \int_{A_\nu} [u^2 + v^2 + (Tu)^2 + (Tv)^2] dx \quad (3)$$

As $\nu \rightarrow \infty$, the right-hand side of (3) $\rightarrow 0$.

However, $d_\nu^{-4} \int_0^{d_\nu} \theta_\nu^3(r) dr \geq d_\nu^{-4} \int_{\frac{1}{2}d_\nu + \frac{1}{2}\delta}^{d_\nu - \frac{1}{2}\delta} (d_\nu - \frac{1}{2}\delta - r)^3 dr = \frac{1}{4} \cdot \frac{1}{2^4} (1 - 2\delta/d_\nu)^4 > 0$,
as $\nu \rightarrow \infty$.

This is a contradiction, so $L = 0$ and T_0 is e.s.a. \square

Again drawing on an example in Evans [5], we give the example below, generalising the result of Everitt [6].

Example: In Corol. 5.6.1, take $A_\nu = \{x: a_\nu \leq |x| \leq b_\nu\}$ where $0 < a_\nu < b_\nu$ and with $b_\nu \leq Kd_\nu$; say $a_\nu = \exp(2\nu)$ and $b_\nu = \exp(2\nu + 1)$. Then $d_\nu = \exp(2\nu + 1) - \exp(2\nu)$, and we have $b_\nu \leq Kd_\nu$ and also (a) is satisfied. In A_ν , take $p \geq 0$ such that either $p(x) \leq K|x|^2$ or $p(x) \leq K|x|^2 |q|^{\frac{1}{2}}$, where $q(x) = |x|^\gamma \sin(\pi \log|x|)$, with p and q satisfying conditions $\mathcal{E}(i) - \mathcal{E}(v)$. Then (b) and (c) are satisfied, since if $x \in A_\nu$, $|x| \leq b_\nu \leq Kd_\nu$, and $q \geq 0$. Hence T_0 is e.s.a. \square

PART II

CHAPTER 1

In this second part, we shall generalise the results of Part I to those for a $2m$ -th. order elliptic differential operator ($m \geq 2$) given below. Clearly, for any particular m , $m \leq K$.

We first define operators ℓ_t ($t \geq 0$, t integral) by:

$$\ell_0 = I \text{ (the identity operator);}$$

$$\ell_1 = \nabla \text{ (the vector operator } (\partial_1, \dots, \partial_n) \text{ in } \mathbb{R}^n);$$

$$\text{and } \ell_{t+2} = \nabla^2 \ell_t.$$

Note: ℓ_{2t+1} may be taken as either $\nabla \ell_{2t}$ or $\ell_{2t} \nabla$ as these will be the same for the functions under consideration.

The $2m$ -th. order operator we shall consider is formally given by:

$$\tau u = \sum_{t=0}^m (-1)^t \ell_t \cdot (p_t \ell_t u), \quad \text{with } p_m \equiv 1,$$

where $p_t = p_t(x)$, a real-valued function of $x = (x_1, \dots, x_n)$, for $0 \leq t \leq m$.

τ is formally self-adjoint. As in Part I, we define operators T_0 and T by: $T_0 u = \tau u$, $\mathcal{D}(T_0) = \mathcal{D}$; $Tu = \tau u$, $\mathcal{D}(T) = L^2$.

To ease comparison of results with those of Part I, we put $p_0 = q$, and so τ is given by:

$$\tau u = (-1)^m \ell_{2m} u + \sum_{t=1}^{m-1} (-1)^t \ell_t \cdot (p_t \ell_t u) + qu.$$

Note: Here and throughout, the dot in terms such as $\ell_t \cdot (p_t \ell_t u)$ is to be taken as the operation forming the scalar product of two vectors or the ordinary product of two scalars, whichever is applicable.

The following three conditions on p_t and q are to be assumed to hold throughout Part II:

$$\mathcal{E}(i) \quad p_t \in C^t, \text{ for } 1 \leq t \leq m-1;$$

$$\mathcal{E}(ii) \quad q \in L_{loc}^2;$$

$\mathcal{E}(\text{iii})$ $q(x) \geq -q^*(|x|)$ where $q^*(r)$ is a real-valued, monotonic non-decreasing function of $r \geq 0$ and is locally bounded.

We define the function $\underline{q}(x)$ as in Part I by:

$$\underline{q}(x) = q(x) \text{ if } x \in B_R;$$

$$\underline{q}(x) = q(x) + q^*(|x|) - q^*(R) \text{ if } x \notin B_R.$$

Therefore we have $\underline{q}(x) \geq -q^*(R)$ for all $x \in \mathbb{R}^n$.

We define \underline{r} , \underline{T}_0 and \underline{T} as in Part I.

The following three lemmas are proved as in Part I, and no proof is given here.

Lemma 1.1 T_0 maps \mathcal{D} into L^2 and is symmetric. \square

Lemma 1.2 T maps L^2 into \mathcal{D}' . \square

Lemma 1.3 $\mathcal{D}(T_0^*) = \{u: u \in L^2, Tu \in L^2\}$ and $T_0^*u = Tu$. \square

So, as in Part I, the conditions $\mathcal{E}(B)$, $\mathcal{E}(C)$ and $\mathcal{E}(D)$ are seen to be equivalent to: $\mathcal{E}(A) T_0$ is e.s.a.

As in Part I, Lemmas 1.1, 1.2, 1.3 and the equivalence of $\mathcal{E}(A)$ - $\mathcal{E}(D)$ hold with r replaced by \underline{r} , and the general approach is the same.

CHAPTER 2

We shall state conditions on p_t and q which will render \underline{T}_0 e.s.a. as a corollary of Theorem I.2.1.

As in Part I, for $1 \leq j \leq n$, ∂_j denotes $\partial/\partial x_j$.

So $\ell_{2t} = \prod_{i=1}^t \left\{ \sum_{j_i=1}^n \partial_{j_i}^2 \right\}$ and $\ell_{2t+1} = (\partial_1 \ell_{2t}, \dots, \partial_n \ell_{2t})$.

Let $J_{2t} = \{j_1, j_1, j_2, j_2, \dots, j_t, j_t\}$ and $J_{2t+1} = J_{2t} \cup \{j_{t+1}\}$,

where $1 \leq j_i \leq n$, $1 \leq i \leq t+1$.

Let $S_{r,t} = \{s_1, s_2, \dots, s_r\}$ with $S_{r,t} \subset J_t$ and $S_{/r,t} = \{\sigma_1, \sigma_2, \dots, \sigma_{t-r}\}$

with $S_{/r,t} = J_t / S_{r,t}$ for $0 \leq r \leq t$. Note: $S_{0,t} = \emptyset$ and $S_{/t,t} = \emptyset$.

Let $P[S_{r,t}] = \partial_{s_1} \partial_{s_2} \dots \partial_{s_r}$ and $P[S_{/r,t}] = \partial_{\sigma_1} \partial_{\sigma_2} \dots \partial_{\sigma_{t-r}}$ with $P[\emptyset] = I$.

Then $\ell_{2t} \cdot (p_{2t} \ell_{2t} u) = \sum_{i=1}^t \sum_{j_i=1}^n \sum_{r=0}^{2t} \sum_{\mathcal{C}} (P[S_{r,2t}] p_{2t}) (P[S_{/r,2t}] \ell_{2t} u)$

and $\ell_{2t+1} \cdot (p_{2t+1} \ell_{2t+1} u) = \sum_{i=1}^{t+1} \sum_{j_i=1}^n \sum_{r=0}^{2t+1} \sum_{\mathcal{C}} (P[S_{r,2t+1}] p_{2t+1}) (P[S_{/r,2t+1}] \partial_{j_{t+1}} \ell_{2t} u)$,

for $1 \leq 2t \leq m-1$ and $1 \leq 2t+1 \leq m-1$ respectively.

$\sum_{\mathcal{C}}$ sums over the subsets $S_{r,t}$ of J_t , where, if a particular j_i occurs twice in J_t but only once in $S_{r,t}$, then this subset is counted twice, because we are summing over combinations (hence the " \mathcal{C} ") of r elements of J_t .

Consider the functions $P[S_{r,t}] p_t$ ($1 \leq t \leq m-1$) such that $2t-r$ is constant, say $2t-r = \mu$. As $0 \leq r \leq t$ and $1 \leq t \leq m-1$, we have $1 \leq \mu \leq 2(m-1)$. Index these functions by elements of some finite set $\Lambda(\mu)$, i.e.

$P[S_{r,t}] p_t = q_{\mu,\lambda}(x)$ where $\mu = 2t-r$ and $\lambda \in \Lambda(\mu)$. For any $q_{\mu,\lambda}(x)$ we define $Q_{\mu,\lambda}(D)$ as $P[S_{/r,2t}] \ell_{2t}$ if $q_{\mu,\lambda}(x) = P[S_{r,2t}] p_{2t}$, and as $P[S_{/r,2t+1}] \partial_{j_{t+1}} \ell_{2t}$ if $q_{\mu,\lambda}(x) = P[S_{r,2t+1}] p_{2t+1}$. Then $Q_{\mu,\lambda}(D)$ is an operator of order $\mu < 2m$.

We further define $q_{0,1}(x) = q(x)$, $Q_{0,1}(D) = I$, $\Lambda(0) = \{1\}$, and $P(D) = (-1)^m e_{2m}$,

$$\text{and } L(x,D) = P(D) + \sum_{t=0}^{m-1} (-1)^t \sum_{j_i} \sum_{j_i} \sum_{\mu} \sum_{\lambda} q_{\mu,\lambda}(x) Q_{\mu,\lambda}(D)$$

with \sum_{j_i} to sum over $j_i \in J_t$, \sum_{j_i} to sum over $1 \leq j_i \leq n$, \sum_{μ} to sum over $\mu = 2t - r$ for $0 \leq r \leq t$, and \sum_{λ} to sum over $\lambda \in \Lambda(\mu)$.

We now state condition $\mathcal{E}(iv)$.

$\mathcal{E}(iv)$ $q \in N_{4m}$ and if $4m \leq n$, then $N_{4m,\delta}(q) \rightarrow 0$ as $\delta \rightarrow 0$;

$P[S_{r,t}]p_t \in N_{2(2m-(2t-r))}$ and if $2(2m-(2t-r)) \leq n$, then

$N_{2(2m-(2t-r)),\delta}(P[S_{r,t}]p_t) \rightarrow 0$ as $\delta \rightarrow 0$, for $0 \leq r \leq t$, $1 \leq t \leq m-1$.

We now have as a corollary to Theorem I.2.1:

Corollary 2.1.1 Let q and p_t ($1 \leq t \leq m-1$) satisfy $\mathcal{E}(iv)$.

Then \underline{T}_0 is e.s.a.

Proof: With the notation as above, by $\mathcal{E}(iv)$, $q_{\mu,\lambda} \in N_{2(2m-\mu)}$

and if $2(2m-\mu) \leq n$, then $N_{2(2m-\mu),\delta}(q_{\mu,\lambda}) \rightarrow 0$ as $\delta \rightarrow 0$, for any $\lambda \in \Lambda(\mu)$,

$0 \leq \mu \leq 2(m-1)$. Therefore, by Theorem I.2.1, $L(x,D)$ on \mathcal{D} is e.s.a. in L^2 .

But $L(x,D) = \underline{T}_0$, and so \underline{T}_0 is e.s.a. \square

As in Part I, $\mathcal{E}(iv)$ will be assumed to hold in any lemma requiring \underline{T}_0 to be e.s.a.

CHAPTER 3

Our aim in this chapter is to derive the identity

$$\int \phi^{2m} u \tau u - \int q \phi^{2m} u^2 = \sum_{t=1}^m \int p_t \ell_t(\phi^{2m} u) \cdot \ell_t u, \quad \text{with } p_m = 1,$$

for $u \in \mathcal{D}(T_0^*)$ and $\phi \in \mathcal{D}$, with u and ϕ real-valued. With $m=2$, this is just the identity which we derived in Chapter I.3. Similarly, we wish first to obtain a formal identity for $\tau(\phi u)$. We use the notation introduced in Chapter 2, and shall give full details in order to familiarise the reader with this notation which will be used extensively.

We expand $\ell_t(\phi u)$ as sums of products of the form $(P[S_{r,t}]\phi)(P[S_{/r,t}]u)$ and then extract the term in which $r=0$, i.e. $\phi \ell_t u$. The sum of these (i.e. the $\phi \ell_t u$) will contribute to the term $\phi \tau u$ in the identity.

$$\begin{aligned} \tau(\phi u) &= (-1)^m \ell_{2m}(\phi u) + \sum_{t=1}^{m-1} (-1)^t \ell_t \cdot (p_t \ell_t(\phi u)) + q \phi u \\ &= (-1)^m \ell_{2m}(\phi u) + \sum_{1 \leq 2t < m} \ell_{2t}(p_{2t} \ell_{2t}(\phi u)) - \\ &\quad - \sum_{1 \leq 2t+1 < m} \ell_{2t+1} \cdot (p_{2t+1} \ell_{2t+1}(\phi u)) + q \phi u \end{aligned} \quad (1)$$

$$\begin{aligned} \ell_{2m}(\phi u) &= \sum_{i=1}^m \sum_{j_1=1}^n \sum_{r=0}^{2m} \sum_{\mathcal{E}} (P[S_{r,2m}]\phi)(P[S_{/r,2m}]u) \\ &= \phi \ell_{2m} u + \sum_{i=1}^m \sum_{j_1=1}^n \sum_{r=1}^{2m} \sum_{\mathcal{E}} (P[S_{r,2m}]\phi)(P[S_{/r,2m}]u) \end{aligned} \quad (2)$$

For $1 \leq 2t < m$:

$$\begin{aligned} \ell_{2t} \cdot (p_{2t} \ell_{2t}(\phi u)) &= \ell_{2t}(p_{2t} \{ \sum_{i=1}^t \sum_{j_1=1}^n \sum_{r=0}^{2t} \sum_{\mathcal{E}} (P[S_{r,2t}]\phi)(P[S_{/r,2t}]u) \}) \\ &= \ell_{2t}(p_{2t} \phi \ell_{2t} u) + \\ &\quad + \ell_{2t}(p_{2t} \{ \sum_{i=1}^t \sum_{j_1=1}^n \sum_{r=1}^{2t} \sum_{\mathcal{E}} (P[S_{r,2t}]\phi)(P[S_{/r,2t}]u) \}) \\ &= \sum_{i=1}^t \sum_{j_1=1}^n \sum_{r=0}^{2t} \sum_{\mathcal{E}} (P[S_{r,2t}]\phi)(P[S_{/r,2t}](p_{2t} \ell_{2t} u)) + \\ &\quad + \ell_{2t}(p_{2t} \{ \dots \}) \end{aligned}$$

where we write $\{ \dots \}$ if the term has not changed from the line above, .

$$\begin{aligned}
e_{2t} \cdot (p_{2t} e_{2t}(\phi u)) &= \phi e_{2t}(p_{2t} e_{2t} u) + \\
&+ \sum_{i=1}^t \sum_{j_i=1}^n \sum_{r=1}^{2t} \sum_{\mathcal{C}} (P[S_{r,2t}] \phi) (P[S_{/r,2t}] (p_{2t} e_{2t} u)) + \\
&+ e_{2t}(p_{2t} \{\dots\})
\end{aligned} \tag{3}$$

For $1 \leq 2t+1 < m$:

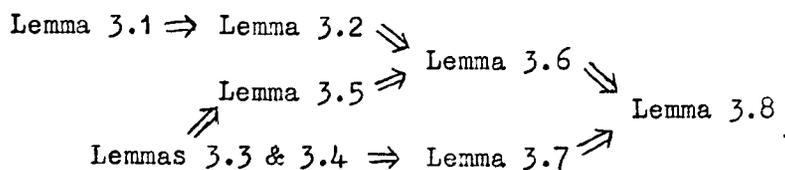
$$\begin{aligned}
e_{2t+1} \cdot (p_{2t+1} e_{2t+1}(\phi u)) &= \\
&= \sum_{j_{t+1}=1}^n \partial_{j_{t+1}} e_{2t}(p_{2t+1} \partial_{j_{t+1}} e_{2t}(\phi u)) \\
&= \sum_{i=1}^{t+1} \sum_{j_i=1}^n \partial_{j_{t+1}} e_{2t}(p_{2t+1} \sum_{r=0}^{\Sigma} \sum_{\mathcal{C}} (P[S_{r,2t+1}] \phi) (P[S_{/r,2t+1}] u)) \\
&= \sum_{j_{t+1}=1}^n \partial_{j_{t+1}} e_{2t}(p_{2t+1} \phi \partial_{j_{t+1}} e_{2t} u) + \\
&+ \left\{ \sum_{i=1}^{t+1} \sum_{j_i=1}^n \sum_{r=1}^{\Sigma} \sum_{\mathcal{C}} \partial_{j_{t+1}} e_{2t}(p_{2t+1} (P[S_{r,2t+1}] \phi) (P[S_{/r,2t+1}] u)) \right\} \\
&= \sum_{i=1}^{t+1} \sum_{j_i=1}^n \sum_{r=0}^{\Sigma} \sum_{\mathcal{C}} (P[S_{r,2t+1}] \phi) (P[S_{/r,2t+1}] (p_{2t+1} \partial_{j_{t+1}} e_{2t} u)) + \\
&+ \{\dots\} \\
&= \phi e_{2t+1} \cdot (p_{2t+1} e_{2t+1} u) + \\
&+ \sum_{i=1}^{t+1} \sum_{j_i=1}^n \sum_{r=1}^{\Sigma} \sum_{\mathcal{C}} (P[S_{r,2t+1}] \phi) (P[S_{/r,2t+1}] (p_{2t+1} \partial_{j_{t+1}} e_{2t} u)) + \\
&+ \{\dots\}
\end{aligned} \tag{4}$$

Substituting from (2), (3) and (4) into (1):

$$\begin{aligned}
r(\phi u) &= \phi r u + (-1)^m \sum_{i=1}^m \sum_{j_i=1}^n \sum_{r=1}^{2m} \sum_{\mathcal{C}} (P[S_{r,2m}] \phi) (P[S_{/r,2m}] u) + \\
&+ \sum_{1 \leq 2t < m} \sum_{i=1}^t \sum_{j_i=1}^n \sum_{r=1}^{2t} \sum_{\mathcal{C}} \{ (P[S_{r,2t}] \phi) (P[S_{/r,2t}] (p_{2t} e_{2t} u)) + \\
&\quad + e_{2t}(p_{2t} (P[S_{r,2t}] \phi) (P[S_{/r,2t}] u)) \} - \\
&- \sum_{1 \leq 2t+1 < m} \sum_{i=1}^{t+1} \sum_{j_i=1}^n \sum_{r=1}^{2t+1} \sum_{\mathcal{C}} \{ (P[S_{r,2t+1}] \phi) (P[S_{/r,2t+1}] (p_{2t+1} \partial_{j_{t+1}} e_{2t} u)) + \\
&\quad + \partial_{j_{t+1}} e_{2t}(p_{2t+1} (P[S_{r,2t+1}] \phi) (P[S_{/r,2t+1}] u)) \}.
\end{aligned}$$

This is the desired identity, and it remains valid with $\underline{\tau}$ in place of τ .

The scheme of this chapter is the same as that of Chapter I.3, i.e. to obtain the generalisation of Lemma I.3.8 we shall generalise Lemmas I.3.1-7, the implications being given in the diagram.



In order to maintain the parallel between this chapter and Chapter I.3, the extra results needed for the proofs of the lemmas appear as propositions. Again, we shall make it clear in the proofs when we are assuming \underline{T}_0 to be e.s.a., because not all the proofs require this assumption.

Proposition 3.0.1 Assume that we have a sequence $a(j)$ ($0 \leq j \leq N$)

$$\text{such that } a(j) \leq \varepsilon a(j+1) + K(1 + \varepsilon^{-1})a(j-1) \quad (1)$$

holds for $0 < j < N$ with K independent of j .

Then, with K independent of j , we have, for $0 \leq j < N$,

$$a(j) \leq \varepsilon a(N) + K(1 + \varepsilon^{-1})^{N-1} a(0) \quad (2)$$

Proof: We shall first establish that for $0 \leq j < N$ and K

$$\text{independent of } j: \quad a(j) \leq \varepsilon a(j+1) + K(1 + \varepsilon^{-1})^j a(0) \quad (3)$$

This holds for $j=0$ (trivial) and $j=1$ (put $j=1$ in (1)).

Suppose that (3) holds for $j=0, 1, \dots, s$.

$$\text{Then } a(s+1) \leq \varepsilon' a(s+2) + K(1 + \varepsilon'^{-1})a(s) \quad (\text{by (1)})$$

$$\leq \varepsilon' a(s+2) + K(1 + \varepsilon'^{-1})\{\varepsilon'' a(s+1) + K(1 + \varepsilon''^{-1})^s a(0)\} \quad (\text{by the induction hypothesis})$$

Setting $\varepsilon' = \frac{1}{2}\varepsilon$, $\varepsilon'' = \frac{1}{2}K^{-1}(1 + \varepsilon'^{-1})^{-1}$ we obtain

$$a(s+1) \leq \varepsilon a(s+2) + K(1 + \varepsilon^{-1})^{s+1} a(0).$$

Thus (3) is established by induction.

Now (2) holds for $j = N - 1$ (put $j = N - 1$ in (3)).

Suppose that (2) holds for $j = N - 1, N - 2, \dots, N - s$.

$$\begin{aligned} \text{Then } a^{(N-s-1)} &\leq a^{(N-s)} + Ka(0) \quad (\text{by (3) with } \varepsilon = 1) \\ &\leq \{\varepsilon a^{(N)} + K(1 + \varepsilon^{-1})^{N-1} a(0)\} + Ka(0) \quad (\text{by the induction hypothesis}) \\ &\leq \varepsilon a^{(N)} + K(1 + \varepsilon^{-1})^{N-1} a(0). \end{aligned}$$

Thus (2) is established by induction. \square

Proposition 3.0.2 Assume that we have a sequence $b(i, j)$ ($i \geq 0, j \geq 0, i + j \leq N \leq K$) such that

$$b(i, j) \leq \varepsilon b(i-1, j+1) + K(\varepsilon)b(i-1, j) \quad (1)$$

holds for $i > 0, j \geq 0, i + j \leq N$, with $K(\varepsilon)$ independent of i, j , and

$$b(0, j) \leq \varepsilon b(0, j+1) + K(\varepsilon)b(0, j-1) \quad (2)$$

holds for $0 < j < N$ with $K(\varepsilon)$ independent of j

(i.e. the sequence $b(0, j)$ satisfies Prop. 3.0.1(1)).

Then, with $K(\varepsilon)$ independent of j , we have, for $0 \leq j < N$,

$$b(N-j, j) \leq \varepsilon b(0, N) + K(\varepsilon)b(0, 0) \quad (3)$$

Proof: We shall first establish that for $i > 0, j \geq 0, i + j \leq N$

and $K(\varepsilon)$ independent of i, j :

$$b(i, j) \leq \varepsilon b(0, i+j) + K(\varepsilon) \sum_{\kappa=0}^{i-1} b(0, j+\kappa) \quad (4)$$

This holds for $i = 1, j \geq 0, i + j \leq N$ (put $i = 1$ in (1)).

Suppose that (4) holds for $i = 1, 2, \dots, s, j \geq 0, i + j \leq N$.

Then $b(s+1, j) \leq \varepsilon' b(s, j+1) + K(\varepsilon') b(s, j)$ (by (1))

$$\begin{aligned} &\leq \varepsilon' \{b(0, j+s+1) + K \sum_{\kappa=0}^{s-1} b(0, j+1+\kappa)\} + \\ &\hspace{15em} (\text{by the induction hypothesis}) \\ &\quad + K(\varepsilon') \{b(0, j+s) + K \sum_{\kappa=0}^{s-1} b(0, j+\kappa)\} \\ &\leq \varepsilon b(0, j+s+1) + K(\varepsilon) \sum_{\kappa=0}^s b(0, j+\kappa) \quad (\text{by setting } \varepsilon' = \varepsilon). \end{aligned}$$

Thus (4) is established by induction.

From (2), $b(0, j)$ satisfies Prop.3.0.1(1), and so, by Prop.3.0.1(2),
for $\kappa < N - j$: $b(0, j+\kappa) \leq \varepsilon' b(0, N) + K(\varepsilon') b(0, 0)$.

$$\text{Therefore } \sum_{\kappa=0}^{N-j-1} b(0, j+\kappa) \leq \sum_{\kappa=0}^{N-j-1} \{ \varepsilon' b(0, N) + K(\varepsilon') b(0, 0) \}$$

$$\leq K \{ \varepsilon' b(0, N) + K(\varepsilon') b(0, 0) \} \quad (\text{because } N - j \leq N \leq K) \quad (5)$$

$$\text{From (4), } b(N-j, j) \leq \varepsilon'' b(0, N) + K(\varepsilon'') \sum_{\kappa=0}^{N-j-1} b(0, j+\kappa)$$

$$\leq \varepsilon'' b(0, N) + K(\varepsilon'') K \{ \varepsilon' b(0, N) + K(\varepsilon') b(0, 0) \} \quad (\text{by (5)})$$

$$\leq \varepsilon b(0, N) + K(\varepsilon) b(0, 0) \quad (\text{by suitable choice of } \varepsilon' \text{ and } \varepsilon'').$$

Thus (3) is established. \square

Proposition 3.0.3 Let $\phi \in \mathcal{D}$, $\lambda \geq 0$, $\mu \geq 0$, $\lambda + \mu \leq \sigma$.

$$\text{Then } \left\| \sum_{i=1}^{\lambda} s_i \sum_{s_i=1}^n \partial_{s_1} \partial_{s_2} \dots \partial_{s_\lambda} \ell_{2\mu} \phi \right\| \leq K (\| \ell_{2\sigma} \phi \| + \| \phi \|) \quad (1)$$

Proof: We have $\| \phi \|_\rho \leq K (\| P(D)\phi \|_0 + \| \phi \|_0)$ for any $\phi \in \mathcal{D}$,

for any symmetric elliptic operator $P(D)$ of order ρ with real-coefficients
(see Lemma A2.2). We take $P(D) = \ell_{2\sigma}$, $\rho = 2\sigma$.

$$\text{Therefore } \| \phi \|_{2\sigma} \leq K (\| \ell_{2\sigma} \phi \| + \| \phi \|) \quad (2)$$

$$\text{Now } \left\| \sum_{i=1}^{\lambda} s_i \sum_{s_i=1}^n \partial_{s_1} \partial_{s_2} \dots \partial_{s_\lambda} \ell_{2\mu} \phi \right\| \leq \| \phi \|_{2\mu+\lambda}$$

$$\leq \| \phi \|_{2(\mu+\lambda)}$$

$$\leq \| \phi \|_{2\sigma}$$

$$\leq K (\| \ell_{2\sigma} \phi \| + \| \phi \|) \quad (\text{by (2)}).$$

Thus (1) is established, where the left-hand side of (1) is just $\| \ell_{2\mu} \phi \|$
when $\lambda = 0$. \square

Proposition 3.0.4 Let $u \in L^2$, $\ell_{2\sigma} u \in L^2_{loc}$, $\Omega \subset \Omega'$, $\lambda \geq 0$, $\mu \geq 0$, $\lambda + \mu \leq \sigma \leq K$.

$$\text{Then } \left\| \sum_{i=1}^{\lambda} s_i \sum_{s_i=1}^n \partial_{s_1} \partial_{s_2} \dots \partial_{s_\lambda} \ell_{2\mu} u \right\|_\Omega \leq K (\| \ell_{2\sigma} u \|_{\Omega'} + \| u \|_{\Omega'}) \quad (1)$$

(If $\lambda = 0$, then the left-hand side of (1) is just $\| \ell_{2\mu} u \|_\Omega$.)

Proof: Let $\psi \in \mathcal{D}$, with ψ real-valued, $0 \leq \psi(x) \leq 1$, $\psi(x) = 1$ if $x \in \Omega$, $\psi(x) = 0$ if $x \notin \Omega'$, $|\nabla \psi| \leq K$, $|\nabla^2 \psi| \leq K$.

Assume that $\lambda \neq 0$.

$$\begin{aligned} \left\| \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \partial_{s_1} \partial_{s_2} \dots \partial_{s_{\lambda}} e_{2\mu} u \right\|_{\Omega}^2 &\leq \int \left| \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \psi^{2\mu+\lambda} \partial_{s_1} \partial_{s_2} \dots \partial_{s_{\lambda}} e_{2\mu} u \right|^2 \\ &\leq \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \int \left| \psi^{2\mu+\lambda} \partial_{s_1} \partial_{s_2} \dots \partial_{s_{\lambda}} e_{2\mu} u \right|^2 \end{aligned} \quad (2)$$

$$\begin{aligned} \sum_{s_{\lambda}=1}^n \int \left| \psi^{2\mu+\lambda} \partial_{s_1} \partial_{s_2} \dots \partial_{s_{\lambda}} e_{2\mu} u \right|^2 &= \sum_{s_{\lambda}=1}^n \int \psi^{4\mu+2\lambda} (\partial_{s_1} \partial_{s_2} \dots \partial_{s_{\lambda}} e_{2\mu} u) (\partial_{s_1} \partial_{s_2} \dots \partial_{s_{\lambda}} e_{2\mu} \bar{u}) \\ &= - \sum_{s_{\lambda}=1}^n \int \partial_{s_{\lambda}} (\psi^{4\mu+2\lambda} \partial_{s_1} \dots \partial_{s_{\lambda}} e_{2\mu} u) (\partial_{s_1} \dots \partial_{s_{\lambda-1}} e_{2\mu} \bar{u}) \\ &= - \sum_{s_{\lambda}=1}^n (4\mu+2\lambda) \int \psi^{4\mu+2\lambda-1} (\partial_{s_{\lambda}} \psi) (\partial_{s_1} \dots \partial_{s_{\lambda}} e_{2\mu} u) (\partial_{s_1} \dots \partial_{s_{\lambda-1}} e_{2\mu} \bar{u}) \\ &\quad - \sum_{s_{\lambda}=1}^n \int \psi^{4\mu+2\lambda} (\partial_{s_1} \dots \partial_{s_{\lambda-1}} \partial_{s_{\lambda}}^2 e_{2\mu} u) (\partial_{s_1} \dots \partial_{s_{\lambda-1}} e_{2\mu} \bar{u}) \\ &\leq K \int \left(\sum_{s_{\lambda}=1}^n \left| \psi^{2\mu+\lambda} \partial_{s_1} \dots \partial_{s_{\lambda}} e_{2\mu} u \right|^2 \right)^{\frac{1}{2}} \left| \psi^{2\mu+\lambda-1} \partial_{s_1} \dots \partial_{s_{\lambda-1}} e_{2\mu} u \right| + \\ &\quad + \int \left| \psi^{2\mu+\lambda+1} \partial_{s_1} \dots \partial_{s_{\lambda-1}} e_{2(\mu+1)} u \right| \left| \psi^{2\mu+\lambda-1} \partial_{s_1} \dots \partial_{s_{\lambda-1}} e_{2\mu} u \right| \\ &\leq \frac{1}{2} \sum_{s_{\lambda}=1}^n \int \left| \psi^{2\mu+\lambda} \partial_{s_1} \dots \partial_{s_{\lambda}} e_{2\mu} u \right|^2 + K \int \left| \psi^{2\mu+\lambda-1} \partial_{s_1} \dots \partial_{s_{\lambda-1}} e_{2\mu} u \right|^2 + \\ &\quad + \frac{1}{2} \varepsilon \int \left| \psi^{2\mu+\lambda+1} \partial_{s_1} \dots \partial_{s_{\lambda-1}} e_{2(\mu+1)} u \right|^2 + \\ &\quad + K \varepsilon^{-1} \int \left| \psi^{2\mu+\lambda-1} \partial_{s_1} \dots \partial_{s_{\lambda-1}} e_{2\mu} u \right|^2. \end{aligned}$$

$$\begin{aligned} \text{So } \sum_{s_{\lambda}=1}^n \int \left| \psi^{2\mu+\lambda} \partial_{s_1} \dots \partial_{s_{\lambda}} e_{2\mu} u \right|^2 &\leq \varepsilon \int \left| \psi^{2\mu+\lambda+1} \partial_{s_1} \dots \partial_{s_{\lambda-1}} e_{2(\mu+1)} u \right|^2 + \\ &\quad + K(1 + \varepsilon^{-1}) \int \left| \psi^{2\mu+\lambda-1} \partial_{s_1} \dots \partial_{s_{\lambda-1}} e_{2\mu} u \right|^2 \end{aligned} \quad (3)$$

Summing over $1 \leq s_i \leq n$ for $1 \leq i \leq \lambda-1$ and writing

$$\sum_{i=1}^{\lambda} \sum_{s_i=1}^n \int \left| \psi^{2\mu+\lambda} \partial_{s_1} \dots \partial_{s_{\lambda}} e_{2\mu} u \right|^2 = b(\lambda, \mu),$$

$$(3) \text{ gives } b(\lambda, \mu) \leq \varepsilon b(\lambda-1, \mu+1) + K(1 + \varepsilon^{-1}) b(\lambda-1, \mu) \quad (4)$$

for $\lambda > 0$, $\mu \geq 0$, $\lambda + \mu \leq \sigma \leq K$. Therefore $b(\lambda, \mu)$ satisfies Prop. 3.0.2(1).

We write $\int \left| \psi^{2\mu} e_{2\mu} u \right|^2 = b(0, \mu)$ and now consider $b(0, \mu)$ for $\mu > 0$.

$$\begin{aligned}
b(0, \mu) &= \int |\psi^{2\mu} e_{2\mu} u|^2 \\
&= \int \psi^{4\mu} (e_{2\mu} u) (e_{2\mu} \bar{u}) \\
&= \int \nabla^2 (\psi^{4\mu} e_{2\mu} u) (e_{2(\mu-1)} \bar{u}) \\
&= \int [\psi^{4\mu} \nabla^2 e_{2\mu} u + \nabla^2 (\psi^{4\mu}) e_{2\mu} u + 2\nabla (\psi^{4\mu}) \cdot \nabla e_{2\mu} u] e_{2(\mu-1)} \bar{u} \\
&= \int \psi^{4\mu} (e_{2(\mu+1)} u) (e_{2(\mu-1)} \bar{u}) + 4\mu(4\mu-1) \int \psi^{4\mu-2} |\nabla \psi|^2 (e_{2\mu} u) (e_{2(\mu-1)} \bar{u}) + \\
&\quad + 4\mu \int \psi^{4\mu-1} \nabla^2 \psi (e_{2\mu} u) (e_{2(\mu-1)} \bar{u}) + 8\mu \int \psi^{4\mu-1} (\nabla \psi \cdot \nabla e_{2\mu} u) (e_{2(\mu-1)} \bar{u}) \\
&\leq \int |\psi^{2(\mu+1)} e_{2(\mu+1)} u| |\psi^{2(\mu-1)} e_{2(\mu-1)} u| + \\
&\quad + K \int |\psi^{2\mu} e_{2\mu} u| |\psi^{2(\mu-1)} e_{2(\mu-1)} u| + \\
&\quad + K \int \left(\sum_{j=1}^n |\psi^{2\mu+1} \partial_j e_{2\mu} u|^2 \right)^{\frac{1}{2}} |\psi^{2(\mu-1)} e_{2(\mu-1)} u| \\
&\leq \varepsilon' b(0, \mu+1) + K\varepsilon''^{-1} b(0, \mu-1) + \varepsilon'' b(0, \mu) + K\varepsilon'''^{-1} b(0, \mu-1) + \\
&\quad + \varepsilon''' b(1, \mu) + K\varepsilon'''^{-1} b(0, \mu-1) \\
&\leq \varepsilon' b(0, \mu+1) + \varepsilon'' b(0, \mu) + K(\varepsilon'^{-1} + \varepsilon''^{-1} + \varepsilon'''^{-1}) b(0, \mu-1) + \\
&\quad + \varepsilon''' \{b(0, \mu+1) + Kb(0, \mu)\} \quad (\text{by (4) with } \varepsilon = 1) \\
&\leq (\varepsilon' + \varepsilon''') b(0, \mu+1) + (\varepsilon'' + K\varepsilon''') b(0, \mu) + K(\varepsilon'^{-1} + \varepsilon''^{-1} + \varepsilon'''^{-1}) b(0, \mu-1) \\
&\leq \frac{1}{2}\varepsilon b(0, \mu+1) + \frac{1}{2}b(0, \mu) + K(1 + \varepsilon^{-1}) b(0, \mu-1)
\end{aligned}$$

by setting $\varepsilon''' = (1/4)\min(K^{-1}, \varepsilon)$, $\varepsilon'' = 1/4$, $\varepsilon' = \varepsilon/4$.

Therefore $b(0, \mu) \leq \varepsilon b(0, \mu+1) + K(1 + \varepsilon^{-1}) b(0, \mu-1)$.

This holds for $0 < \mu < \sigma$ and so $b(0, \mu)$ satisfies Prop.3.0.2(2).

Therefore, Prop.3.0.2(3) holds for $b(\lambda, \mu)$.

By (2), with $\lambda \neq 0$:

$$\begin{aligned}
\left\| \sum_{i=1}^{\lambda} \sum_{s_1=1}^n \partial_{s_1} \dots \partial_{s_{\lambda}} e_{2\mu} u \right\|_{\Omega}^2 &\leq b(\lambda, \mu) \\
&\leq b(0, \mu+\lambda) + Kb(0, 0) \quad (\text{by Prop.3.0.2(3)} \\
&\quad \text{with } \varepsilon = 1) \quad (5)
\end{aligned}$$

If $\lambda = 0$, the left-hand side is $\|e_{2\mu} u\|_{\Omega}^2 \leq \|\psi^{2\mu} e_{2\mu} u\|^2 = b(0, \mu)$.

Therefore, (5) holds in the case $\lambda = 0$ as well.

So, by Prop.3.0.1(2), with $\varepsilon = 1$ to cover the case $\lambda = 0$ and $\mu = \sigma$:

$$\begin{aligned} \left\| \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \partial_{s_1} \dots \partial_{s_{\lambda}} e_{2\mu} u \right\|_{\Omega}^2 &\leq b(0, \sigma) + Kb(0, 0) \\ &\leq \|e_{2\sigma} u\|_{\Omega}^2 + K\|u\|_{\Omega}^2, \end{aligned}$$

which implies (1). \square

Proposition 3.0.5 Let $\phi \in \mathcal{D}$ with $\text{supp } \phi \subset \Omega \subset \Omega'$; $u \in L^2$ and $e_{2s} u \in L^2_{\text{loc}}$ for any particular s with $0 \leq s \leq m-2$; $1 \leq t \leq m-1$. Then

$$| \int \phi e_t \cdot (p_t e_t u) | \leq K(\|e_{2(m-1-s)} \phi\| + \|\phi\|)(\|e_{2s} u\|_{\Omega'} + K\|u\|_{\Omega'}) \quad (1)$$

Proof: We first consider t (in (1)) even. For $1 \leq 2t \leq m-1$:

$$\begin{aligned} | \int \phi e_{2t} (p_{2t} e_{2t} u) | &= \left| \int \sum_{i=1}^t \sum_{j=1}^n \sum_{r=0}^{2t} \sum_{\xi} \phi(P[S_{r,2t}](e_{2t} u))(P[S_{r,2t}]p_{2t}) \right| \\ &\leq K \left| \int \sum_{r=0}^{2t} \sum_{i=1}^r \sum_{s_i=1}^n \sum_{\xi} \phi P[S_{r,2t}](e_{2t} u) \right| \quad (\text{because the derivatives of } p_{2t} \text{ are bounded on } \Omega) \end{aligned}$$

Now $P[S_{r,2t}]$ may be rewritten (with rearrangement of the subscripts) as

$$\partial_{s_1} \dots \partial_{s_{\lambda}} \partial_{s_{\lambda+1}}^2 \dots \partial_{s_{(r+\lambda)/2}}^2 \quad \text{where } s_i \neq s_j \text{ if } i \neq j,$$

with $\lambda \equiv r \pmod{2}$ and $0 \leq \lambda \leq \min(r, 2t-r)$.

$$(\text{Note that } \sum_{i=\lambda+1}^r \sum_{s_i=1}^n \partial_{s_1} \dots \partial_{s_{\lambda}} \partial_{s_{\lambda+1}}^2 \dots \partial_{s_{(r+\lambda)/2}}^2 = \partial_{s_1} \dots \partial_{s_{\lambda}} e_{r-\lambda}.)$$

Let $\Lambda(r) = \{\lambda: \lambda \equiv r \pmod{2}, 0 \leq \lambda \leq \min(r, 2t-r)\}$.

\sum_{Λ} will denote the sum over $\lambda \in \Lambda(r)$.

So we now have:

$$\begin{aligned} | \int \phi e_{2t} (p_{2t} e_{2t} u) | &\leq K \left| \int \sum_{r=0}^{2t} \sum_{\Lambda} \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \phi \partial_{s_1} \dots \partial_{s_{\lambda}} e_{r-\lambda} (e_{2t} u) \right| \\ &= K \left| \int \sum_{r=0}^{2t} \sum_{\Lambda} \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \phi \partial_{s_1} \dots \partial_{s_{\lambda}} e_{2t+r-\lambda} u \right| \quad (2) \end{aligned}$$

Our object is to recast the summands in (2) in the form

$$(\partial_{s_1} \dots \partial_{s_{\lambda'}} e_{2\mu'} \phi) (\partial_{s_1} \dots \partial_{s_{\lambda''}} e_{2\mu''} u)$$

with $\lambda' \geq 0$, $\mu' \geq 0$, $\lambda' + \mu' \leq (m-1-s)$ and $\lambda'' \geq 0$, $\mu'' \geq 0$, $\lambda'' + \mu'' \leq s$,

in order that we may use the results of Props. 3.0.3 and 3.0.4.

If $2t+r+\lambda \leq 2s$, we rewrite (2) as

$$K \left| \int \sum_{r=0}^{2t} \sum_{\Lambda} \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \phi \partial_{s_1} \dots \partial_{s_{\lambda}} e_{2\mu} u \right| \quad \text{where } \mu + \lambda \leq s \quad (3a)$$

If $2t+r+\lambda > 2s$ and $\lambda \leq s$, we integrate (2) by parts to obtain

$$K \left| \int_{r=0}^{2t} \sum_{\Lambda} \sum_{i=1}^{\lambda} \sum_{s_i=1}^n (e_{2t+r-\lambda-2\mu} \phi) (\partial_{s_1} \dots \partial_{s_{\lambda}} e_{2\mu} u) \right| \text{ where } \lambda + \mu = s \quad (3b)$$

Then $\mu = s - \lambda \geq 0$ and $2\mu = 2(s - \lambda) < 2t + r - \lambda$.

So $0 \leq 2t + r - \lambda - 2\mu = 2t + r + \lambda - 2s \leq 2t + r + \min(r, 2t-r) - 2s$

$$= 2t + \min(2r, 2t) - 2s \leq 2(2t - s) \leq 2(m - 1 - s).$$

If $2t+r+\lambda > 2s$ and $\lambda > s$, we integrate (2) by parts to obtain

$$K \left| \int_{r=0}^{2t} \sum_{\Lambda} \sum_{i=1}^{\lambda} \sum_{s_i=1}^n (\partial_{s_{\nu+1}} \dots \partial_{s_{\lambda}} e_{2t+r-\lambda} \phi) (\partial_{s_1} \dots \partial_{s_{\nu}} u) \right| \text{ where } \nu = s \quad (3c)$$

Then $0 \leq 2t + r - \lambda + 2(\lambda - \nu) = 2t + r + \lambda - 2s \leq 2(m - 1 - s)$ as above.

So, in view of (3a,b,c), we see that (2) is equal to

$$K \left| \int_{r=0}^{2t} \sum_{\Lambda} \sum_{i=1}^{\lambda} \sum_{s_i=1}^n (\partial_{s_{\nu+1}} \dots \partial_{s_{\lambda}} e_{2t+r-\lambda-2\mu} \phi) (\partial_{s_1} \dots \partial_{s_{\nu}} e_{2\mu} u) \right| \quad (3)$$

with $0 \leq 2\mu \leq 2t + r - \lambda$, $0 \leq \nu \leq \lambda$, $\mu + \nu \leq s$ and $(2t + r - \lambda - 2\mu) + 2(\lambda - \nu) \leq 2(m - 1 - s)$.

$$\begin{aligned} \text{Now (3)} &\leq K \sum_{r=0}^{2t} \sum_{\Lambda} \int \left| \sum_{i=\nu+1}^{\lambda} \sum_{s_i=1}^n \partial_{s_{\nu+1}} \dots \partial_{s_{\lambda}} e_{2t+r-\lambda-2\mu} \phi \right| \left| \sum_{i=1}^{\nu} \sum_{s_i=1}^n \partial_{s_1} \dots \partial_{s_{\nu}} e_{2\mu} u \right| \\ &\leq K \sum_{r=0}^{2t} \sum_{\Lambda} \left\| \sum_{i=\nu+1}^{\lambda} \sum_{s_i=1}^n \partial_{s_{\nu+1}} \dots \partial_{s_{\lambda}} e_{2t+r-\lambda-2\mu} \phi \right\| \left\| \sum_{i=1}^{\nu} \sum_{s_i=1}^n \partial_{s_1} \dots \partial_{s_{\nu}} e_{2\mu} u \right\|_{\Omega} \\ &\leq K \sum_{r=0}^{2t} \sum_{\Lambda} (\|e_{2(m-1-s)} \phi\| + \|\phi\|) (\|e_{2s} u\|_{\Omega} + \|u\|_{\Omega}) \quad (4) \\ &\quad \text{(by Props. 3.0.3(1) and 3.0.4(1))} \end{aligned}$$

This implies (1) for the case t (in (1)) even, since the summands in (4) are independent of r and λ , and we have a bounded number of them.

We now consider t (in (1)) odd. For $1 \leq 2t+1 \leq m-1$:

$$\begin{aligned} |f \phi e_{2t+1} \cdot (p_{2t+1} e_{2t+1} u)| &= \left| \int_{j_0=1}^n \phi e_{2t} \partial_{j_0} (p_{2t+1} \partial_{j_0} e_{2t} u) \right| \\ &\leq \left| \int_{j_0=1}^n \phi e_{2t} (p_{2t+1} \partial_{j_0}^2 e_{2t} u) \right| + \\ &\quad + \left| \int_{j_0=1}^n \phi e_{2t} ((\partial_{j_0} p_{2t+1}) (\partial_{j_0} e_{2t} u)) \right| \\ &= |f \phi e_{2t} (p_{2t+1} e_{2(t+1)} u)| + \left| \int_{j_0=1}^n \phi e_{2t} ((\partial_{j_0} p_{2t+1}) (\partial_{j_0} e_{2t} u)) \right| \end{aligned}$$

$$\begin{aligned}
& |f \phi e_{2t+1} \cdot (p_{2t+1} e_{2t+1} u)| \leq \\
& \leq \left| \int_{i=1}^t \sum_{j_i=1}^n \sum_{r=0}^{2t} \sum_{\mathcal{C}} \phi (P[S_{r,2t}](e_{2(t+1)} u)) (P[S_{/r,2t}] p_{2t+1}) \right| + \\
& + \left| \int_{i=0}^t \sum_{j_i=1}^n \sum_{r=0}^{2t} \sum_{\mathcal{C}} \phi (P[S_{r,2t}](\partial_{j_0} e_{2t} u)) (P[S_{/r,2t}](\partial_{j_0} p_{2t+1})) \right| \\
& \leq K \left| \int_{r=0}^{2t} \sum_{i=1}^r \sum_{j_i=1}^n \sum_{\mathcal{C}} \phi P[S_{r,2t}](e_{2(t+1)} u) \right| + \quad (\text{because the derivatives} \\
& + K \left| \int_{r=0}^{2t} \sum_{i=0}^r \sum_{j_i=1}^n \sum_{\mathcal{C}} \phi P[S_{/r,2t}](\partial_{j_0} e_{2t} u) \right| \quad (\text{of } p_{2t+1} \text{ are bounded on } \Omega) \quad (5)
\end{aligned}$$

Now, the first term on the right-hand side of (5) is bounded by

$$\begin{aligned}
& K \left| \int_{r=0}^{2t} \sum_{\Lambda} \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \phi \partial_{s_1} \dots \partial_{s_{\lambda}} e_{2(t+1)+r-\lambda} u \right| \quad \text{as in (2)} \\
& \leq K \left| \int_{r=0}^{2t} \sum_{\Lambda} \sum_{i=1}^{\lambda} \sum_{s_i=1}^n (\partial_{s_{\nu+1}} \dots \partial_{s_{\lambda}} e_{2(t+1)+r-\lambda-2\mu} \phi) (\partial_{s_1} \dots \partial_{s_{\nu}} e_{2\mu} u) \right| \quad \text{as in (3)} \quad (6)
\end{aligned}$$

with $0 \leq 2\mu \leq 2(t+1) + r - \lambda$, $0 \leq \nu \leq \lambda$, $\mu + \nu \leq s$ and

$$(2(t+1) + r - \lambda - 2\mu) + 2(\lambda - \nu) \leq 2(m-1-s).$$

[To see that the last inequality is valid, as in (3a,b,c), either the left-hand side = 0, or $\mu + \nu = s$. If $\mu + \nu = s$, then

$$\begin{aligned}
(2(t+1) + r - \lambda - 2\mu) + 2(\lambda - \nu) &= 2(t+1) + r + \lambda - 2s \\
&\leq 2(t+1) + \min(2r, 2t) - 2s \\
&\leq 2(2t+1-s) \leq 2(m-1-s).]
\end{aligned}$$

$$\text{So (6)} \leq K \sum_{r=0}^{2t} \sum_{\Lambda} (\|e_{2(m-1-s)} \phi\| + \|\phi\|) (\|e_{2s} u\|_{\Omega} + \|u\|_{\Omega}) \quad \text{as in (4)} \quad (7)$$

Now, writing $j_0 = s_{\lambda+1}$, the second term on the right-hand side of (5) \leq

$$\begin{aligned}
& K \left| \int_{r=0}^{2t} \sum_{\Lambda} \sum_{i=1}^{\lambda+1} \sum_{s_i=1}^n \phi \partial_{s_1} \dots \partial_{s_{\lambda+1}} e_{2t+r-\lambda} u \right| \quad \text{as in (2)} \\
& \leq K \left| \int_{r=0}^{2t} \sum_{\Lambda} \sum_{i=1}^{\lambda+1} \sum_{s_i=1}^n (\partial_{s_{\nu+1}} \dots \partial_{s_{\lambda+1}} e_{2t+r-\lambda-2\mu} \phi) (\partial_{s_1} \dots \partial_{s_{\nu}} e_{2\mu} u) \right| \quad \text{as in (3)} \quad (8)
\end{aligned}$$

with $0 \leq 2\mu \leq 2t + r - \lambda$, $0 \leq \nu \leq \lambda + 1$, $\mu + \nu \leq s$ and

$$(2t + r - \lambda - 2\mu) + 2(\lambda + 1 - \nu) \leq 2(m-1-s).$$

[To see that the last inequality is valid, as above, take the case $\mu + \nu = s$.

$$\begin{aligned} \text{Then } (2t+r-\lambda-2\mu) + 2(\lambda+1-\nu) &\leq 2t+r+\lambda+2-2s \\ &= 2(t+1)+r+\lambda-2s \leq 2(m-1-s) \quad \text{as above.} \end{aligned}$$

$$\text{So (8)} \leq K \sum_{r=0}^{2t} \sum_{\Lambda} (\|\ell_{2(m-1-s)}\phi\| + \|\phi\|) (\|\ell_{2s}u\|_{\Omega} + \|u\|_{\Omega}) \quad \text{as in (4)} \quad (9)$$

Substituting from (7) and (9) into (5) we see that (1) holds for the case t (in (1)) odd, just as it did for t (in (1)) even. Thus (1) is established. \square

In the following proposition, we prove the induction step required in the proof of Lemma II.3.1. The proof of the proposition is similar to that of Lemma I.3.1.

Proposition 3.0.6 Let $u \in L^2$, $\underline{T}u \in L_{1\text{loc}}^2$, and $\ell_{2s}u \in L_{1\text{loc}}^2$ for any particular s with $0 \leq s \leq m-2$. Then $\ell_{2(s+1)}u \in L_{1\text{loc}}^2$.

Proof: Let $\phi \in \mathcal{D}$ with $\text{supp } \phi \subset \Omega \subset \Omega'$ (Ω, Ω' compact), and consider

$$(-1)^m \int (\ell_{2(m-1-s)} + i)\phi \ell_{2(s+1)}u, \text{ where } i = \sqrt{-1}.$$

$$\begin{aligned} (-1)^m \int (\ell_{2(m-1-s)} + i)\phi \ell_{2(s+1)}u &= (-1)^m \int \phi \ell_{2(s+1)}u + (-1)^m \int \ell_{2(m-1-s)}\phi \ell_{2(s+1)}u \\ &= (-1)^m \int \ell_2\phi \ell_{2s}u + (-1)^m \int \phi \ell_{2m}u \\ &= (-1)^m \int \ell_2\phi \ell_{2s}u + \int \phi \underline{T}u - \int \phi \underline{Q}u - \\ &\quad - \sum_{t=1}^{m-1} (-1)^t \int \phi \ell_t \cdot (p_t \ell_t u). \end{aligned}$$

$$\begin{aligned} \text{So } \left| \int (\ell_{2(m-1-s)} + i)\phi \ell_{2(s+1)}u \right| &\leq \\ &\leq \int |\ell_2\phi| |\ell_{2s}u| + \int |\phi| |\underline{T}u| + \int |\phi| |\underline{Q}u| + \sum_{t=1}^{m-1} \left| \int \phi \ell_t \cdot (p_t \ell_t u) \right| \\ &\leq \|\ell_2\phi\| \|\ell_{2s}u\|_{\Omega} + \|\phi\| \|\underline{T}u\|_{\Omega} + K(\Omega) (\|\phi\| + \|\ell_2\phi\|) \|u\|_{\Omega} + \\ &\quad + K(\|\ell_{2(m-1-s)}\phi\| + \|\phi\|) (\|\ell_{2s}u\|_{\Omega} + \|u\|_{\Omega}), \end{aligned} \quad (1)$$

since $\|\phi \underline{Q}\| \leq \sup |\phi| \|\underline{Q}\|_{\Omega} \leq K(\Omega) (\|\phi\| + \|\ell_2\phi\|)$ as in Lemma I.3.1, and the estimate of $\left| \int \phi \ell_t \cdot (p_t \ell_t u) \right|$ is from Prop.3.0.5, the right-hand side of Prop.3.0.5(1) being independent of t , and $m \leq K$.

$$\|\ell_2\phi\| \leq K(\|\ell_{2(m-1-s)}\phi\| + \|\phi\|) \text{ by Prop.3.0.3(1), } (2 \leq 2(m-1-s)),$$

and $\|u\|_{\Omega}$, $\|u\|_{\Omega'}$, $\|\ell_{2s}u\|_{\Omega}$, $\|\ell_{2s}u\|_{\Omega'}$, and $\|\underline{T}u\|_{\Omega}$ are all bounded by K .

Therefore, (1) implies

$$|f(\ell_{2(m-1-s)} + i)\phi \ell_{2(s+1)}u| \leq K(u, \Omega, \Omega') (\|\ell_{2(m-1-s)}\phi\| + \|\phi\|) \quad (2)$$

$$\begin{aligned} \text{Now } \|\ell_{2(m-1-s)}\phi\| &= \|F(\ell_{2(m-1-s)}\phi)\| = \|\xi^{2(m-1-s)}F\phi\| \\ &\leq \|(\xi^{2(m-1-s)} + i)F\phi\| \\ &= \|F((\ell_{2(m-1-s)} + i)\phi)\| \\ &= \|(\ell_{2(m-1-s)} + i)\phi\|, \end{aligned}$$

$$\text{and } \|\phi\| = \|F\phi\| \leq \|(\xi^{2(m-1-s)} + i)F\phi\| = \|(\ell_{2(m-1-s)} + i)\phi\|.$$

So, (2) implies

$$|f(\ell_{2(m-1-s)} + i)\phi \ell_{2(s+1)}u| \leq K(u, \Omega, \Omega') \|(\ell_{2(m-1-s)} + i)\phi\| \quad (3)$$

For any $t > 0$, ℓ_{2t} as an operator from \mathcal{D} to L^2 is e.s.a., and therefore $(\ell_{2t} + i)\mathcal{D}$ is dense in L^2 . Take $t = m - 1 - s$. Consider any $v \in L^2$ with $\text{supp } v$ bounded, say $\text{supp } v \subset \Omega^{\circ}$. Then there exists a sequence $\{\phi_{\nu}\}$, $\phi_{\nu} \in \mathcal{D}$, such that $(\ell_{2t} + i)\phi_{\nu} \rightarrow v$ in L^2 . We must show that the functions in this sequence have common support.

For some $d > 0$, let $\Omega^1 = \{x: \text{dist}(x, \Omega^{\circ}) \leq d\}$, $\Omega^2 = \{x: \text{dist}(x, \Omega^1) \leq d\}$, $\Omega^3 = \{x: \text{dist}(x, \Omega^2) \leq d\}$. Let $\psi(x) \in \mathcal{D}$ be a real-valued function such that $0 \leq \psi \leq 1$, $\psi(x) = 1$ if $x \in \Omega^1$, $\psi(x) = 0$ if $x \notin \Omega^2$, and the first $2t$ derivatives of ψ are bounded by K , independent of Ω° . Then $\psi\phi_{\nu} \in \mathcal{D}$.

$$\text{Let } \phi \in \{\phi_{\nu}\} \text{ such that } \|(\ell_{2t} + i)\phi - v\| \leq \eta \quad (4)$$

We shall show that $\|(\ell_{2t} + i)(\psi\phi) - v\| < \varepsilon$.

$$\begin{aligned} \|(\psi-1)(\ell_{2t} + i)\phi\| &\leq \|(\psi-1)(\ell_{2t} + i)\phi\|_{/\Omega^1} \quad \text{because } \psi = 1 \text{ in } \Omega^1 \\ &\leq \|(\ell_{2t} + i)\phi\|_{/\Omega^1} \quad \text{because } 0 \leq \psi \leq 1 \\ &= \|(\ell_{2t} + i)\phi - v\|_{/\Omega^1} \quad \text{because } v = 0 \text{ outside } \Omega^{\circ} \subset \Omega^1 \\ &\leq \eta \quad (\text{by (4)}) \end{aligned} \quad (5)$$

$$\ell_{2t}(\psi\phi) = \psi\ell_{2t}\phi + \sum_{i=1}^t \sum_{j=1}^n \sum_{r=1}^{2t} \sum_{\xi} (P[S_{r,2t}]\psi)(P[S_{/r,2t}]\phi)$$

as the term for $r=0$ has been extracted to give us the term $\psi\ell_{2t}\phi$.

If this taking of limits is not justifiable, we may proceed as follows:

With $t = m - i - s$, consider $(-1)^m f(\ell_{2t} + i)\phi \cdot (\ell_2 + i)\ell_{2s}u$ to obtain

$$|f(\ell_{2t} + i)\phi \cdot (\ell_2 + i)\ell_{2s}u| \leq K(u, \Omega, \Omega') \|(\ell_{2t} + i)\phi\|,$$

from which it follows that $(\ell_2 + i)\ell_{2s}u$ is a continuous linear functional on a dense subspace of the dual space of L^2_{loc} .

As in Part I, this can be extended to $f \in L^2_{loc}$ with

$$f(\ell_{2t} + i)\phi \cdot (\ell_2 + i)\ell_{2s}u = f(\ell_{2t} + i)\phi \cdot f \quad (\dagger)$$

and if $(\ell_2 + i)g = f$, then $g \in L^2_{loc}$ and so $\ell_{2s}g \in L^2_{loc}$.

So from (\dagger) we have $f(\ell_{2t} + i)\phi \cdot (\ell_2 + i)(\ell_{2s}u - g) = 0$,

and therefore we obtain $f(\ell_2 + i)(\ell_{2t} + i)\phi \cdot (\ell_{2s}u - g) = 0 \quad (\ddagger)$

As in Lemma A2.2, we can show that for $w \in L^2$

$$\|w\| \leq K \|(\ell_2 + i)(\ell_{2t} + i)w\|_{-2(t+1)},$$

and hence $(\ell_2 + i)(\ell_{2t} + i)$ is 1-1 from L^2 to $H^{-2(t+1)}_0$.

As in Lemma A1.1, it follows that $(\ell_2 + i)(\ell_{2t} + i)\mathfrak{D}$ is dense in L^2 .

Thus, from (\ddagger) , $\ell_{2s}u - g = 0$, and so $\ell_2(\ell_{2s}u) \in L^2_{loc}$. \square

Now,
$$\left\| \sum_{i=1}^t \sum_{j=1}^n \sum_{r=1}^{2t} \sum_{\xi} (P[S_{r,2t}] \psi)(P[S_{/r,2t}] \phi) \right\| \leq$$

$$\leq K \sum_{r=1}^{2t} \sum_{\lambda} \left\| \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \partial_{s_1} \dots \partial_{s_{\lambda}} \ell_{2t-r-\lambda} \phi \right\|_{\Omega^2/\Omega^1}$$

using the notation from Prop.3.0.5
as $P[S_{r,2t}] \psi = 0$ for $r \neq 0$, $x \notin \Omega^2/\Omega^1$
and $P[S_{r,2t}] \psi$ is bounded for $x \in \Omega^2/\Omega^1$.

$$\leq K \sum_{r=1}^{2t} \sum_{\lambda} (\| \ell_{2t} \phi \|_{\Omega^3/\Omega^0} + \| \phi \|_{\Omega^3/\Omega^0})$$

by Prop.3.0.4 applied to ϕ instead of u
as $2t-r+\lambda \leq 2t-r+\min(r, 2t-r) \leq 2t-0 = 2t$.

$$\leq K (\| \ell_{2t} \phi \|_{\Omega^3/\Omega^0} + \| \phi \|_{\Omega^3/\Omega^0})$$

as summands independent of r, λ .

$$\leq K \| (\ell_{2t} + i) \phi \|_{\Omega^3/\Omega^0}$$

as above

$$\leq K \| (\ell_{2t} + i) \phi - v \|$$

similar to above

$$\leq K \eta \quad (\text{by (4)}) \tag{6}$$

So
$$\| (\ell_{2t} + i)(\psi \phi - \phi) \| \leq \| (\psi - 1)(\ell_{2t} + i) \phi \| +$$

$$+ \left\| \sum_{i=1}^t \sum_{j=1}^n \sum_{r=1}^{2t} \sum_{\xi} (P[S_{r,2t}] \psi)(P[S_{/r,2t}] \phi) \right\|$$

$$\leq \eta + K \eta \quad (\text{from (5) and (6)}) \tag{7}$$

and so
$$\| (\ell_{2t} + i)(\psi \phi) - v \| \leq \| (\ell_{2t} + i) \phi - v \| + \| (\ell_{2t} + i)(\psi \phi - \phi) \|$$

$$\leq \eta + (K+1)\eta \quad (\text{from (4) and (7)})$$

$$< \varepsilon \quad (\text{by setting } \eta < \varepsilon(K+2)^{-1}).$$

We have therefore shown that $(\ell_{2t} + i)(\psi \phi_v) \rightarrow v$ in L^2 with $\text{supp } \psi \phi_v \subset \Omega^2$.

So, from (3), $|fv \ell_{2(s+1)} u| \leq K(u, \Omega^0) \|v\|$, where the infimum of the K 's for which this is valid for v 's in L^2 having the same support, Ω , is

$$\| \ell_{2(s+1)} u \|_{\Omega}. \text{ Therefore } \ell_{2(s+1)} u \in L_{loc}^2. \quad \square \quad \text{See facing page.}$$

Lemma 3.1 If $u \in L^2$ and $\underline{T}u \in L_{loc}^2$, then $\ell_{2(m-1)} u \in L_{loc}^2$.

Proof: The proof is by induction.

$$u \in L^2 \subset L_{loc}^2, \text{ so } \ell_0 u (=u) \in L_{loc}^2.$$

Prop.3.0.6 gives us that $\ell_{2s} u \in L_{loc}^2$ implies $\ell_{2(s+1)} u \in L_{loc}^2$, for $0 \leq s \leq m-2$.

Therefore, by induction, $\ell_{2(m-1)} u \in L_{loc}^2. \quad \square$

Corollary 3.1.1 If $u \in L^2$ and $\underline{T}u \in L^2_{loc}$, then, for $\mu + \lambda \leq m - 1$,

$$\sum_{i=1}^{\lambda} \sum_{s_i=1}^n \partial_{s_1} \dots \partial_{s_{\lambda}} \ell_{2\mu} u \in L^2_{loc}.$$

Proof: Immediate from Prop.3.0.4(1) and Lemma 3.1. \square

Lemma 3.2 If $u \in L^2$ and $\underline{T}u \in L^2_{loc}$, then $\underline{T}(\phi u) \in H_0^{-1}$ for any $\phi \in \mathcal{D}$.

Proof: Let $\phi \in \mathcal{D}$ with $\text{supp } \phi \subset \Omega$, and ϕ and its first $2m$ derivatives bounded by K . We shall show that all the terms in the expansion of $\underline{T}(\phi u)$ (which we restate here for convenience) are in H_0^{-1} .

$$\begin{aligned} \underline{T}(\phi u) = & \phi \underline{T}u + (-1)^m \sum_{i=1}^m \sum_{j_i=1}^n \sum_{r=1}^{2m} \sum_{\mathcal{C}} (P[S_{r,2m}] \phi) (P[S_{/r,2m}] u) + \\ & + \sum_{1 \leq 2t < m} \sum_{i=1}^t \sum_{j_i=1}^n \sum_{r=1}^{2t} \sum_{\mathcal{C}} \{ (P[S_{r,2t}] \phi) (P[S_{/r,2t}] (P_{2t} \ell_{2t} u)) + \\ & \quad + \ell_{2t} (P_{2t} (P[S_{r,2t}] \phi) (P[S_{/r,2t}] u)) \} - \\ & - \sum_{1 \leq 2t+1 < m} \sum_{i=1}^{t+1} \sum_{j_i=1}^n \sum_{r=1}^{2t+1} \sum_{\mathcal{C}} \{ (P[S_{r,2t+1}] \phi) (P[S_{/r,2t+1}] (P_{2t+1} \partial_{j_{t+1}} \ell_{2t} u)) + \\ & \quad + \partial_{j_{t+1}} \ell_{2t} (P_{2t+1} (P[S_{r,2t+1}] \phi) (P[S_{/r,2t+1}] u)) \}. \end{aligned}$$

$\underline{T}u \in L^2_{loc}$ and so $\phi \underline{T}u \in L^2 \subset H_0^{-1}$.

Let $\psi \in H_0^1$. Consider $1 \leq r < 2m$; so for $\lambda \in \Lambda(r)$, $0 \leq \lambda \leq \min(r, 2m-r)$,

$\lambda \equiv r \pmod{2}$:

$$\begin{aligned} | \langle \psi, \sum_{i=1}^m \sum_{j_i=1}^n (P[S_{r,2m}] \phi) (P[S_{/r,2m}] u) \rangle | = \\ = \left| \int \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \psi (\partial_{s_1} \dots \partial_{s_{\lambda}} \ell_{r-\lambda} \phi) (\partial_{s_1} \dots \partial_{s_{\lambda}} \ell_{2\mu} u) \right| \quad \text{where } 2\mu = 2m - r - \lambda \quad (1) \end{aligned}$$

If $\lambda = 0$, then $r \geq 2$ and so $\mu \leq m - 1$. Hence, by Corol.3.1.1, $\| \ell_{2\mu} u \|_{\Omega} \leq K$, independent of ψ , r and ϵ , and so (1) $\leq K \| \psi \| \leq K \| \psi \|_1$.

If $\lambda \neq 0$, then we integrate (1) by parts to obtain

$$\begin{aligned} (1) = & \left| \int \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \partial_{s_1} \{ \psi (\partial_{s_1} \dots \partial_{s_{\lambda}} \ell_{r-\lambda} \phi) \} (\partial_{s_1} \dots \partial_{s_{\lambda-1}} \ell_{2\mu} u) \right| \\ \leq & K \| \psi \|_1 \left\| \sum_{i=1}^{\lambda-1} \sum_{s_i=1}^n \partial_{s_1} \dots \partial_{s_{\lambda-1}} \ell_{2\mu} u \right\|_{\Omega}. \end{aligned}$$

$$2\mu + 2(\lambda - 1) = 2m - r + \lambda - 2 \leq 2m - 2 - r + \min(r, 2m - r) \leq 2(m - 1).$$

So, by Corol.3.1.1, $\left\| \sum_{i=1}^{\lambda-1} \sum_{s_i=1}^n \partial_{s_1} \dots \partial_{s_{\lambda-1}} \ell_{2\mu} u \right\|_{\Omega} \leq K$, independent of ψ , r and ϵ .

Now considering $r = 2m$:

$$\begin{aligned} \left| \langle \psi, \sum_{i=1}^m \sum_{j_i=1}^n (P[S_{2m, 2m}] \phi) (P[S_{/2m, 2m}] u) \rangle \right| &= \left| \langle \psi, (\ell_{2m} \phi) u \rangle \right| \\ &\leq K \|\psi\| \|u\| \\ &\leq K \|\psi\|_1, \quad K \text{ independent of} \\ &\quad \psi \text{ and } m. \end{aligned}$$

Therefore, $(-1)^m \sum_{i=1}^m \sum_{j_i=1}^n \sum_{r=1}^{2m} \sum_{\mathcal{C}} (P[S_{r, 2m}] \phi) (P[S_{/r, 2m}] u) \in H_0^{-1}$.

The rest of the terms are in fact in L^2 . This seems reasonable when one considers that the highest order of derivatives of u involved is $2m - 3$. However, it is necessary to check that they are of the right form in order to apply Corollary 3.1.1. We first note that p_t and its first t derivatives and ϕ and its first $2m$ derivatives are all bounded on Ω by K , and so we need only demonstrate that the derivatives of u involved are in L^2_{loc} .

The form of the derivatives of u in

$$\begin{aligned} &\sum_{1 \leq 2t < m} \sum_{i=1}^t \sum_{j_i=1}^n \sum_{r=1}^{2t} \sum_{\mathcal{C}} (P[S_{r, 2t}] \phi) (P[S_{/r, 2t}] (p_{2t} \ell_{2t} u)) \\ \text{is } &\sum_{i=1}^{2t-r-a} \sum_{s_i=1}^n P[S_{/r+a, 2t}] \ell_{2t} u \text{ for some } 0 \leq a \leq 2t - r, 1 \leq r \leq 2t, 1 \leq 2t \leq m - 1, \\ &= \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \partial_{s_1} \dots \partial_{s_{\lambda}} \ell_{(2t-r-a-\lambda)+2t} u \end{aligned} \quad (2)$$

where $\lambda \equiv 2t - r - a \pmod{2}$ and $0 \leq \lambda \leq \min(2t - r - a, r, 2t - r)$.

$$\begin{aligned} \text{Now } ((2t - r - a - \lambda) + 2t) + 2\lambda &\leq 4t - r - a + \min(2t - r - a, r, 2t - r) \\ &\leq 4t - r + \min(r, 2t - r) \leq 4t + \min(0, 2t - 2r) \leq 2(m - 1). \end{aligned}$$

So, by Corol.3.1.1, (2) is in L^2_{loc} .

The form of the derivatives of u in

$$\sum_{1 \leq 2t < m} \sum_{i=1}^t \sum_{j_i=1}^n \sum_{r=1}^{2t} \sum_{\mathcal{C}} \ell_{2t} (p_{2t} (P[S_{r,2t}] \phi) (P[S_{/r,2t}] u))$$

$$\begin{aligned} \text{is } & \sum_{i=1}^b \sum_{s_i=1}^n \sum_{j=1}^{2t-r} \sum_{s_j=1}^n P[S'_{b,2t}] P[S_{/r,2t}] u \quad \text{for some } 0 \leq b \leq 2t, 1 \leq r \leq 2t, 1 \leq 2t \leq m-1, \\ & = \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \sum_{j=1}^{\mu} \sum_{s_j=1}^n \partial_{s_1} \cdots \partial_{s_{\lambda}} \partial_{s_1} \cdots \partial_{s_{\mu}} \ell_{(b-\lambda)+(2t-r-\mu)} u \end{aligned} \quad (3)$$

where $\lambda \equiv b \pmod{2}$, $0 \leq \lambda \leq \min(b, 2t-b)$, $\mu \equiv 2t-r \pmod{2}$ and $0 \leq \mu \leq \min(r, 2t-r)$.

$$\begin{aligned} \text{Now } ((b-\lambda) + (2t-r-\mu)) + (2\lambda+2\mu) & \leq 2t-r + \min(r, 2t-r) + b + \min(b, 2t-b) \\ & \leq 2t + \min(0, 2t-2r) + \min(2b, 2t) \\ & \leq 4t \leq 2(m-1). \end{aligned}$$

So, by Corol.3.1.1, (3) is in L^2_{loc} .

The form of the derivatives of u in

$$\sum_{1 \leq 2t+1 < m} \sum_{i=1}^{t+1} \sum_{j_i=1}^n \sum_{r=1}^{2t+1} \sum_{\mathcal{C}} (P[S_{r,2t+1}] \phi) (P[S_{/r,2t+1}] (p_{2t+1} \partial_{j_{t+1}} \ell_{2t} u))$$

$$\begin{aligned} \text{is } & \sum_{j_{t+1}=1}^n \sum_{i=1}^{2t+1-r-c} \sum_{s_i=1}^n P[S_{/r+c,2t+1}] \partial_{j_{t+1}} \ell_{2t} u \quad \text{for some } 0 \leq c \leq 2t+1-r, \\ & 1 \leq r \leq 2t+1, 1 \leq 2t+1 \leq m-1. \end{aligned}$$

$$\text{This is either } \sum_{j_{t+1}=1}^n \sum_{i=1}^{2t-r-a} \sum_{s_i=1}^n P[S_{/r+a,2t}] \partial_{j_{t+1}} \ell_{2t} u \quad \text{for some } 0 \leq a \leq 2t-r,$$

$$\text{or } \sum_{j_{t+1}=1}^n \sum_{i=1}^{2t-r-a} \sum_{s_i=1}^n P[S_{/r+a,2t}] \partial_{j_{t+1}}^2 \ell_{2t} u \quad 0 \leq r \leq 2t, 1 \leq 2t+1 \leq m-1,$$

depending on whether $j_{t+1} \in S_{r+c,2t+1}$ or not.

$$\text{So we obtain } \sum_{j_{t+1}=1}^n \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \partial_{s_1} \cdots \partial_{s_{\lambda}} \partial_{j_{t+1}} \ell_{(2t-r-a-\lambda)+2t} u \quad (4')$$

$$\text{or } \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \partial_{s_1} \cdots \partial_{s_{\lambda}} \ell_{(2t-r-a-\lambda)+2+2t} u \quad (4'')$$

where $\lambda \equiv 2t-r-a \pmod{2}$ and $0 \leq \lambda \leq \min(2t-r-a, r, 2t-r)$.

$$\begin{aligned} \text{Now } ((2t-r-a-\lambda) + 2t) + (2\lambda+2) & = ((2t-r-a-\lambda) + 2+2t) + 2\lambda \\ & = 2(2t+1) - r - a + \lambda \\ & \leq 2(2t+1) \leq 2(m-1) \text{ as in (2)}. \end{aligned}$$

So, by Corol.3.1.1, (4°) and $(4'')$ are in L^2_{loc} .

The form of the derivatives of u in

$$1 \leq 2t+1 \leq m \quad \sum_{i=1}^{t+1} \sum_{j_i=1}^n \sum_{r=1}^{2t+1} \sum_{\epsilon} \partial_{j_{t+1}} \ell_{2t} (P_{2t+1} (P[S_{r,2t+1}] \phi) (P[S_{/r,2t+1}] u))$$

$$\text{is } \sum_{i=1}^d \sum_{s'_i=1}^n \sum_{j=1}^{2t+1-r} \sum_{s_j=1}^n P[S'_{d,2t+1}] P[S_{/r,2t+1}] u \quad \text{for some } 0 \leq d \leq 2t+1,$$

$$1 \leq r \leq 2t+1, \quad 1 \leq 2t+1 \leq m-1.$$

$$\text{This is either } \sum_{i=1}^b \sum_{s'_i=1}^n \sum_{j=1}^{2t-r} \sum_{s_j=1}^n P[S'_{b,2t}] P[S_{/r,2t}] u$$

$$\text{or } \sum_{j_{t+1}=1}^n \sum_{i=1}^b \sum_{s'_i=1}^n \sum_{j=1}^{2t-r} \sum_{s_j=1}^n P[S'_{b,2t}] P[S_{/r,2t}] \partial_{j_{t+1}} u \quad \text{for some } 0 \leq b \leq 2t,$$

$$0 \leq r \leq 2t, \quad 1 \leq 2t+1 \leq m-1,$$

$$\text{or } \sum_{j_{t+1}=1}^n \sum_{i=1}^b \sum_{s'_i=1}^n \sum_{j=1}^{2t-r} \sum_{s_j=1}^n P[S'_{b,2t}] P[S_{/r,2t}] \partial_{j_{t+1}}^2 u$$

depending on whether $j_{t+1} \in S_{r,2t+1} \cap S'_{d,2t+1}$,

or $j_{t+1} \in S_{/r,2t+1} \cup S'_{d,2t+1}$ but $j_{t+1} \notin S_{r,2t+1} \cap S'_{d,2t+1}$,

or $j_{t+1} \in S_{/r,2t+1} \cap S'_{d,2t+1}$.

$$\text{So we obtain } \sum_{i=1}^{\lambda} \sum_{s'_i=1}^n \sum_{j=1}^{\mu} \sum_{s_j=1}^n \partial_{s'_1} \dots \partial_{s'_\lambda} \partial_{s_1} \dots \partial_{s_\mu} \ell_{(b-\lambda)+(2t-r-\mu)} u \quad (5')$$

$$\text{or } \sum_{j_{t+1}=1}^n \sum_{i=1}^{\lambda} \sum_{s'_i=1}^n \sum_{j=1}^{\mu} \sum_{s_j=1}^n \partial_{s'_1} \dots \partial_{s'_\lambda} \partial_{s_1} \dots \partial_{s_\mu} \partial_{j_{t+1}} \ell_{(b-\lambda)+(2t-r-\mu)} u \quad (5'')$$

$$\text{or } \sum_{i=1}^{\lambda} \sum_{s'_i=1}^n \sum_{j=1}^{\mu} \sum_{s_j=1}^n \partial_{s'_1} \dots \partial_{s'_\lambda} \partial_{s_1} \dots \partial_{s_\mu} \ell_{(b-\lambda)+(2t-r-\mu)+2} u \quad (5''')$$

where $\lambda \equiv b \pmod{2}$, $0 \leq \lambda \leq \min(b, 2t-b)$, $\mu \equiv 2t-r \pmod{2}$ and $0 \leq \mu \leq \min(r, 2t-r)$.

$$\text{Now } ((b-\lambda) + (2t-r-\mu)) + (2\lambda+2\mu) < ((b-\lambda) + (2t-r-\mu)) + (2\lambda+2\mu+2)$$

$$= ((b-\lambda) + (2t-r-\mu) + 2) + (2\lambda+2\mu)$$

$$\leq 4t+2 \leq 2(m-1) \text{ as in (3).}$$

So, by Corol.3.1.1, $(5')$, $(5'')$ and $(5''')$ are in L^2_{loc} .

So, as $L^2 \subset H_0^{-1}$, we have all of the terms in the expansion of $\underline{T}(\phi u)$ in H_0^{-1} , and hence $\underline{T}(\phi u) \in H_0^{-1}$. \square

As in Part I, Lemma 3.3 makes use of Gårding's inequality.

Lemma 3.3 If $u \in L^2$ and $\underline{T}u \in H_0^{-m}$, then $u \in H_0^m$, and for any $\varepsilon > 0$

$$(c - \varepsilon) \|u\|_m^2 \leq (k + q^*(R)) \|u\|^2 + K(\varepsilon) \|\underline{T}u\|_{-m}^2 \quad (1)$$

($c > 0$ and k are constants arising from Gårding's inequality applied to the operator $\underline{T}_0 - q$.)

Proof: The proof exactly parallels that of Lemma I.3.3, with H_0^m and H_0^{-m} replacing H_0^2 and H_0^{-2} respectively, together with the concomitant norms. It is assumed that \underline{T}_0 is e.s.a. \square

As in Part I, we now need another condition on q :

$\mathcal{E}(v) N_{4-\alpha}(q)$ is locally bounded, for some $\alpha > 0$ with $n < 4 - \alpha < \infty$.

The condition is the same as in Part I, as is the lemma for which it is required, and which we restate here for convenience.

Lemma 3.4 If $u \in H_{0,loc}^2$ and q satisfies $\mathcal{E}(v)$, then $qu \in L_{loc}^2$, and

$$\|qu\|_{\Omega'} \leq K \|u\|_{2,\Omega'}$$

for any compact set Ω , where Ω' is Ω extended in width by $d > 0$, and K depends on Ω , d and n .

Proof: See Appendix 3. \square

$\mathcal{E}(v)$ will be assumed to hold in any lemma depending on Lemma 3.4.

Lemma 3.5 If $u \in L^2$ and $\underline{T}(\phi u) \in H_0^{-\mu}$ ($0 \leq \mu \leq m$), then $\phi u \in H_0^{2m-\mu}$, for any $\phi \in \mathcal{D}$.

Proof: $H_0^{-\mu} \subset H_0^{-m}$, so $\underline{T}(\phi u) \in H_0^{-m}$. Also, $\phi u \in L^2$.

Therefore, by Lemma 3.3 applied to ϕu instead of u , $\phi u \in H_0^m \subset H_0^2$, i.e. $u \in H_{0,loc}^2$. So, by Lemma 3.4, $qu \in L_{loc}^2$, and hence $q\phi u \in L^2$.

For any $\psi \in \mathcal{D}$ and any s

$$\|\psi\|_{s+\rho} \leq K(\|P(D)\psi\|_s + \|\psi\|_s)$$

where $P(D)$ is an elliptic operator of order ρ with real coefficients (see Lemma A2.2). Taking $P(D) = (-1)^m \ell_{2m}$ and hence $\rho = 2m$, we have

$$\|\psi\|_{s+2m} \leq K(\|\ell_{2m}\psi\|_s + \|\psi\|_s) \quad (1)$$

$$\text{Now } \underline{T}(\phi u) = (-1)^m \ell_{2m}(\phi u) + \sum_{t=1}^{m-1} (-1)^t \ell_t \cdot (p_t \ell_t(\phi u)) + \underline{q}\phi u \quad (2)$$

$\underline{T}(\phi u) \in H_0^{-\mu}$, $\underline{q}\phi u \in L^2 = H_0^0$, and $\phi u \in H_0^m$ so $(-1)^t \ell_t \cdot (p_t \ell_t(\phi u)) \in H_0^{m-2t}$

$$\text{and } \sum_{t=1}^{m-1} (-1)^t \ell_t \cdot (p_t \ell_t(\phi u)) \in H_0^{m-2(m-1)} = H_0^{-(m-2)}.$$

So, considering (2), $(-1)^m \ell_{2m}(\phi u) \in H_0^{-\lambda_1}$, where $\lambda_1 = \max(\mu, 0, m-2)$.

Also, $\phi u \in H_0^{-\lambda_1}$. So putting $s = -\lambda_1$ in (1) gives us

$$\|\psi\|_{2m-\lambda_1} \leq K(\|\ell_{2m}\psi\|_{-\lambda_1} + \|\psi\|_{-\lambda_1}),$$

and since \mathcal{D} is dense in $H_0^{2m-\lambda_1}$, we have $\phi u \in H_0^{2m-\lambda_1}$.

If $\lambda_1 = \mu$, we have the result; otherwise $\lambda_1 = m-2$ and we repeat the process on the new assumption that $\phi u \in H_0^{m+2}$ to obtain

$$\sum_{t=1}^{m-1} (-1)^t \ell_t \cdot (p_t \ell_t(\phi u)) \in H_0^{-(m-4)} \text{ and hence } (-1)^m \ell_{2m}(\phi u) \in H_0^{-\lambda_2},$$

where $\lambda_2 = \max(\mu, 0, m-4)$, and similarly $\phi u \in H_0^{2m-\lambda_2}$.

We iterate to obtain $\phi u \in H_0^{2m-\lambda_r}$, where $\lambda_r = \max(\mu, 0, m-2r)$, and stop when $\lambda_r = \mu$, thus obtaining the desired result. \square

Lemma 3.6 If $u \in L^2$ and $\underline{T}u \in L_{10c}^2$, then $\underline{T}(\phi u) \in L^2$ for any $\phi \in \mathcal{D}$.

Proof: Let $\phi \in \mathcal{D}$. By Lemma 3.2, $\underline{T}(\phi u) \in H_0^{-1}$. Then, by Lemma 3.5 (with $\mu = 1$), $\phi u \in H_0^{2m-1}$.

We shall show that all the terms in the expansion of $\underline{T}(\phi u)$ are in L^2 .

(c.f. Lemma 3.2)

In view of the proof of Lemma 3.2, we need show only that

$$(-1)^m \sum_{i=1}^m \sum_{j_1=1}^n \sum_{r=1}^{2m} \sum_{\mathcal{C}} (P[S_{r,2m}] \phi) (P[S_{r,2m}] u) \in L^2 \quad (1)$$

because we showed there that the other terms are in L^2 . But (1) is immediate now that we have $\phi u \in H_0^{2m-1}$ for any $\phi \in \mathcal{D}$. \square

Lemma 3.7 If $u \in L^2$ and $\underline{T}u \in L^2$ with $\text{supp } u \subsetneq B_R$, then there exists a sequence $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, with $\text{supp } u_\nu \subset B_R$, such that $u_\nu \rightarrow u$ and $\underline{T}u_\nu \rightarrow \underline{T}u$ in L^2 .

Proof: \underline{T}_0 is e.s.a. and so there exists such a sequence if we ignore the restriction: $\text{supp } u_\nu \subset B_R$.

Let $\phi \in \mathcal{D}$ with $\text{supp } \phi \subset B_R$, $\phi = 1$ on $\text{supp } u$, and ϕ and its first $2m$ derivatives bounded by K . Then $\phi u_\nu \in \mathcal{D}$ and $\text{supp } \phi u_\nu \subset B_R$.

We have $\phi u_\nu \rightarrow \phi u = u$ in L^2 and also $\phi \underline{T}u_\nu \rightarrow \phi \underline{T}u = \underline{T}u$ in L^2 since $\text{supp } \underline{T}u \subset \text{supp } u$.

We shall show that the rest of the terms in the expansion of $\underline{T}(\phi u_\nu)$ converge to 0 in L^2 , in which case $\{\phi u_\nu\}$ will be the required sequence.

Now $\underline{T}u_\nu \rightarrow \underline{T}u$ in L^2 implies that $\underline{T}u_\nu \rightarrow \underline{T}u$ in H_0^{-m} , so we apply Lemma 3.3(1) to $u_\nu - u$ to obtain $u_\nu \rightarrow u$ in H_0^m . So we have $D^\alpha u_\nu \rightarrow D^\alpha u$ in L^2 for $0 \leq |\alpha| \leq m$.

Assume that $d > 0$ is such that $\text{supp } u \subset B_{R-d}$. We may assume then that $\phi(x) = 1$ if $|x| \leq R-d$ and $\phi(x) = 0$ if $|x| \geq R$ with ϕ real-valued and $0 \leq \phi \leq 1$. Let $\chi(x) = 1$ if $R-d \leq |x| \leq R$ and $\chi(x) = 0$ otherwise. Note that $\chi u = 0$, and that $\chi = 0$ implies that $D^\alpha \phi = 0$ for $|\alpha| > 0$.

Let v be an arbitrary element of the sequence $\{u_\nu\}$.

We first consider the terms

$$(-1)^m \sum_{i=1}^m \sum_{j_i=1}^n \sum_{r=1}^{2m} \sum_{\mathcal{C}} (P[S_{r,2m}] \phi) (P[S_{r,2m}] v)$$

in the expansion of $\underline{T}(\phi v)$. For $1 \leq r \leq 2m$

$$\begin{aligned} & \left\| \sum_{i=1}^m \sum_{j_i=1}^n (P[S_{r,2m}] \phi) (P[S_{r,2m}] v) \right\| \leq \\ & \leq K \left\| \sum_{i=1}^{2m-r} \sum_{s_i=1}^n \partial_{s_1} \dots \partial_{s_{2m-r}} (\chi v) \right\| \quad \text{(because as } r > 0, P[S_{r,2m}] \phi = 0 \\ & \quad \text{outside } \text{supp } \chi, \text{ and } |P[S_{r,2m}] \phi| \leq K) \\ & \leq K \left\| \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \partial_{s_1} \dots \partial_{s_{\lambda}} \ell_{2m-r-\lambda} (\chi v) \right\| \end{aligned}$$

where $\lambda \equiv 2m - r \pmod{2}$ and $0 \leq \lambda \leq \min(r, 2m-r)$,

and so $2m - r + \lambda \leq 2m - r + \min(r, 2m - r) = 2m + \min(0, 2m - 2r) \leq 2m$.

Similar to Prop.3.0.4, with $\text{supp } \psi \supset \text{supp } \chi$, writing

$$\sum_{i=1}^{\lambda} \sum_{s_i=1}^n \int |\psi^{2\mu+\lambda} \partial_{s_1} \dots \partial_{s_\lambda} \ell_{2\mu}(\chi v)|^2 = b(\lambda, \mu)$$

we obtain, for $\lambda \neq 0$,

$$\begin{aligned} \left\| \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \partial_{s_1} \dots \partial_{s_\lambda} \ell_{2m-r-\lambda}(\chi v) \right\|^2 &\leq b(\lambda, \frac{1}{2}(2m-r-\lambda)) \\ &\leq \epsilon' b(0, \frac{1}{2}(2m-r+\lambda)) + K(\epsilon') b(0, 0) \\ &\leq \epsilon' b(0, m) + K(\epsilon') b(0, 0). \end{aligned}$$

If $\lambda = 0$, then r must be even, and as $r \geq 1$ we have $r \geq 2$, and so

$$\left\| \ell_{2m-r}(\chi v) \right\|^2 \leq b(0, \frac{1}{2}(2m-r)) \leq \epsilon' b(0, m) + K(\epsilon') b(0, 0).$$

Therefore we have a result slightly stronger than that given by Prop.3.0.4(1),

$$\text{namely, } \left\| \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \partial_{s_1} \dots \partial_{s_\lambda} \ell_{2m-r-\lambda}(\chi v) \right\|^2 \leq \epsilon' \left\| \ell_{2m}(\chi v) \right\|^2 + K(\epsilon') \left\| \chi v \right\|^2 \quad (1)$$

$$\text{So } \left\| (-1)^m \sum_{i=1}^m \sum_{j_i=1}^n \sum_{r=1}^{2m} \sum_{\mathcal{C}} (P[S_{r, 2m}] \phi)(P[S_{/r, 2m}] v) \right\|^2 \leq \epsilon'' \left\| \ell_{2m}(\chi v) \right\|^2 + K(\epsilon'') \left\| \chi v \right\|^2$$

by choice of ϵ' , since the summands are independent of r and \mathcal{C} from (1).

In view of the proof of Lemma 3.2 (in which we showed that the derivatives of u in the other summed terms are of the form

$$\partial_{s_1} \dots \partial_{s_\lambda} \ell_{2\mu} u, \text{ with } \mu + \lambda \leq m - 1,$$

and by Prop.3.0.4(1), the norm of the other summed terms in the expansion of $\mathbb{T}(\phi v)$ is bounded by $K(\left\| \ell_{2(m-1)}(\chi v) \right\| + \left\| \chi v \right\|)$, which by (1) is bounded by $\epsilon'' \left\| \ell_{2m}(\chi v) \right\| + K(\epsilon'') \left\| \chi v \right\|$, by choice of ϵ' .

If we write $[\Sigma]v = \mathbb{T}(\phi v) - \phi \mathbb{T}v$ (i.e. $[\Sigma]v$ is all the summed terms in the expansion of $\mathbb{T}(\phi v)$), we have just shown, by choice of ϵ'' , that

$$\left\| [\Sigma]v \right\| \leq \epsilon \left\| \ell_{2m}(\chi v) \right\| + K(\epsilon) \left\| \chi v \right\| \quad (2)$$

For $1 \leq 2t \leq m - 1$, as the derivatives of p_{2t} are bounded on $\text{supp } \chi$,

$$\left\| \ell_{2t}(p_{2t} \ell_{2t}(\chi v)) \right\| \leq K \sum_{r=0}^{2t} \sum_{\mathcal{C}} \left\| \sum_{i=1}^r \sum_{s_i=1}^n \partial_{s_1} \dots \partial_{s_r} \ell_{2t}(\chi v) \right\|$$

$$\|e_{2t}(p_{2t}e_{2t}(\chi v))\| \leq K \sum_{r=0}^{2t} \sum_{\lambda} \left\| \sum_{i=1}^{\lambda} s_{i=1}^n \partial_{s_1} \cdots \partial_{s_{\lambda}} e_{2t+r-\lambda}(\chi v) \right\|,$$

where $\lambda \equiv r \pmod{2}$ and $0 \leq \lambda \leq \min(r, 2t-r)$,

and so $2t+r+\lambda \leq 2t+r+\min(r, 2t-r) \leq 2t+\min(2r, 2t) \leq 4t \leq 2(m-1)$.

$$\text{So, by (1), } \|e_{2t}(p_{2t}e_{2t}(\chi v))\| \leq \varepsilon'' \|e_{2m}(\chi v)\| + K(\varepsilon'') \|\chi v\| \quad (3)$$

For $1 \leq 2t+1 \leq m-1$, as the derivatives of p_{2t+1} are bounded on $\text{supp } \chi$,

$$\begin{aligned} \|e_{2t+1} \cdot (p_{2t+1} e_{2t+1}(\chi v))\| &\leq \left\| \sum_{j_0=1}^n e_{2t}((\partial_{j_0} p_{2t+1})(\partial_{j_0} e_{2t}(\chi v))) \right\| + \\ &\quad + \|e_{2t}(p_{2t+1} e_{2(t+1)}(\chi v))\| \\ &\leq K \sum_{r=0}^{2t} \sum_{\lambda} \left\| \sum_{j_0=1}^n \sum_{i=1}^r \sum_{s_i=1}^n \partial_{s_1} \cdots \partial_{s_r} \partial_{j_0} e_{2t}(\chi v) \right\| + \\ &\quad + K \sum_{r=0}^{2t} \sum_{\lambda} \left\| \sum_{i=1}^r \sum_{s_i=1}^n \partial_{s_1} \cdots \partial_{s_r} e_{2(t+1)}(\chi v) \right\| \\ &\leq K \sum_{r=0}^{2t} \sum_{\lambda} \left\| \sum_{j_0=1}^n \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \partial_{s_1} \cdots \partial_{s_{\lambda}} \partial_{j_0} e_{2t+r-\lambda}(\chi v) \right\| + \\ &\quad + K \sum_{r=0}^{2t} \sum_{\lambda} \left\| \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \partial_{s_1} \cdots \partial_{s_{\lambda}} e_{2(t+1)+r-\lambda}(\chi v) \right\| \end{aligned}$$

where $\lambda \equiv r \pmod{2}$ and $0 \leq \lambda \leq \min(r, 2t-r)$,

and so $2t+r+\lambda+2 \leq 4t+2 \leq 2(m-1)$, as above.

$$\text{So, by (1), } \|e_{2t+1} \cdot (p_{2t+1} e_{2t+1}(\chi v))\| \leq \varepsilon'' \|e_{2m}(\chi v)\| + K(\varepsilon'') \|\chi v\| \quad (4)$$

So, by (3) and (4) and choice of ε'' ,

$$\sum_{t=1}^{m-1} \|e_t \cdot (p_t e_t(\chi v))\| \leq \frac{1}{2} \|e_{2m}(\chi v)\| + K \|\chi v\| \quad (5)$$

$$\text{Now } \underline{T}v = (-1)^m e_{2m} v + \sum_{t=1}^{m-1} (-1)^t e_t \cdot (p_t e_t v) + \underline{q}v,$$

$$\text{and so } \|e_{2m}(\chi v)\| \leq \|\underline{T}(\chi v)\| + \sum_{t=1}^{m-1} \|e_t \cdot (p_t e_t(\chi v))\| + \|\underline{q}\chi v\|,$$

$$\text{and so } \|e_{2m}(\chi v)\| \leq K(\|\underline{T}(\chi v)\| + \|\underline{q}\chi v\| + \|\chi v\|) \quad (\text{by (5)}) \quad (6)$$

all norms being finite because $\underline{T}(\chi v) \in L^2$, $\chi v \in L^2$ and, by Lemma 3.4, $\underline{q}\chi v \in L^2$.

$$\text{Now } \|\underline{T}(\chi v)\| \leq \|\underline{T}(\chi(v-u))\| + \|\underline{T}(\chi u)\| = \|\chi \underline{T}(v-u)\| + 0 \leq K,$$

$$\text{and } \|\chi v\| \leq \|\chi(v-u)\| + \|\chi u\| \leq \|v-u\| \leq K,$$

and, by Lemma 3.4, $\|\underline{q}\chi v\| \leq K\|\chi v\|_2 \leq K\|\chi(v-u)\|_2 + K\|\chi u\|_2 \leq K$,
 in each case, K being independent of whichever element of $\{u_\nu\}$ v is.

Using these bounds in (6), we obtain

$$\|\ell_{2m}(\chi v)\| \leq K, \text{ independent of } \nu.$$

Using these bounds in (2), we obtain

$$\|[\Sigma]v\| \leq K\varepsilon + K(\varepsilon)\|v-u\|$$

i.e. $\|[\Sigma]u_\nu\| \leq K\varepsilon + K(\varepsilon)\|u_\nu - u\| \leq K\varepsilon + K(\varepsilon)\varepsilon'$, if ν is sufficiently large.

We choose $\varepsilon = \frac{1}{2}K^{-1}\eta$ and then pick N such that, for all $\nu \geq N$, ε' is such that $K(\varepsilon)\varepsilon' < \frac{1}{2}\eta$. Hence $\|[\Sigma]u_\nu\| < \eta$ for all $\nu \geq N$.

So, from the definition of $[\Sigma]u_\nu$, $\underline{T}(\phi u_\nu) \rightarrow \phi \underline{T}u_\nu \rightarrow \phi \underline{T}u = \underline{T}u$ in L^2 ,
 and therefore $\{\phi u_\nu\}$ is the required sequence. \square

Lemma 3.8 If $u \in \mathcal{D}(T^*)$ and $\phi \in \mathcal{D}$, then $T(\phi u) \in L^2$ and there exists a
 sequence $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, such that $u_\nu \rightarrow \phi u$ and $Tu_\nu \rightarrow T(\phi u)$ in L^2 .

Proof: $u \in \mathcal{D}(T^*)$, i.e. $u \in L^2$ and $Tu \in L^2$.

Now $\underline{T}u = Tu + (\underline{q} - q)u$. $Tu \in L^2$; $(\underline{q} - q)$ is locally bounded and so
 $(\underline{q} - q)u \in L^2_{loc}$; therefore $\underline{T}u \in L^2_{loc}$, and by Lemma 3.6, $\underline{T}(\phi u) \in L^2$.

Choose R such that $\text{supp } \phi \subsetneq B_R$.

Applying Lemma 3.7 to ϕu instead of u , there exists $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, such that
 $u_\nu \rightarrow \phi u$ and $\underline{T}u_\nu \rightarrow \underline{T}(\phi u)$ in L^2 , with $\text{supp } u_\nu \subset B_R$. But $\underline{T}v = Tv$ if $\text{supp } v \subset B_R$, so
 $T(\phi u) \in L^2$ and $Tu_\nu = \underline{T}u_\nu \rightarrow \underline{T}(\phi u) = T(\phi u)$ in L^2 . \square

Corollary 3.8.1 If $\{u_\nu\}$ is a sequence with the properties in

Lemma 3.8, then

$$\int q|u_\nu|^2 \rightarrow \int q|\phi u|^2 \quad \text{as } \nu \rightarrow \infty \quad (1)$$

Proof: With $p_m = 1$, we have

$$\int \bar{u}_\nu(Tu_\nu) = \int \sum_{t=1}^m (-1)^t \bar{u}_\nu \ell_t \cdot (p_t \ell_t u_\nu) + \int q|u_\nu|^2$$

Therefore $\int q|u_\nu|^2 = \int \bar{u}_\nu(Tu_\nu) - \sum_{t=1}^m \int p_t |\ell_t u_\nu|^2$.

q is bounded below on the common support of the functions u_ν , say $q \geq k+1$, where k may be negative, and, as $p_t \in C^t$, $|p_t|$ is also bounded on this support.

$$\text{Therefore } \int (q-k)|u_\nu|^2 \leq \frac{1}{2} \|Tu_\nu\|^2 + K' \|u_\nu\|_m^2 \quad (2)$$

where K' depends on k and the bounds on the p_t .

We have the convergence of $\{u_\nu\}$ and $\{Tu_\nu\}$ in L^2 and it is part of the proof of Lemma 3.7 (using Lemma 3.3(1)) that we then also have the convergence of $\{u_\nu\}$ in H_0^m . Hence the left-hand side of (2) is convergent, from which the result follows. \square

Now that we have the generalisation of Lemma I.3.8 and its corollary, we may proceed to derive the identity stated at the beginning of this chapter, namely

$$\int \phi^{2m} u Tu - \int q \phi^{2m} u^2 = \sum_{t=1}^m \int p_t \ell_t(\phi^{2m} u) \cdot \ell_t u, \quad \text{with } p_m = 1,$$

for $u \in \mathcal{D}(T_0^*)$ and $\phi \in \mathcal{D}$, with u and ϕ real-valued.

As before, had we assumed, for instance, that the partial derivatives of u of order $2m-1$ were absolutely continuous, then this result could have been obtained directly. Instead, we make use of the sequence $\{u_\nu\}$ and a limiting process, with $\mathcal{D}(T_0^*)$ being as large as possible, as before.

Lemma 3.9 If $u \in \mathcal{D}(T_0^*)$ and $\phi \in \mathcal{D}$, with u and ϕ real-valued, then

$$\int \phi^{2m} u Tu - \int q \phi^{2m} u^2 = \sum_{t=1}^m \int p_t \ell_t(\phi^{2m} u) \cdot \ell_t u, \quad \text{with } p_m = 1 \quad (1)$$

Proof: As $u \in L^2$ and $Tu \in L^2$, the first integral is finite.

By Corol.3.8.1, the second integral is finite.

As $p_t \in C^t$ and is therefore locally bounded, by Corol.3.1.1, the remaining integrals are finite.

From Lemma 3.8, there exists a sequence $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, such that

$u_\nu \rightarrow \phi^{2m}u$ in L^2 and $Tu_\nu \rightarrow T(\phi^{2m}u)$ in L^2 , where R is such that $\text{supp } \phi \not\subseteq B_R$.

So consider $\int u_\nu(Tu) - \int u_\nu q u$:

$$\begin{aligned} \int u_\nu(Tu) - \int u_\nu q u &= \int u_\nu \sum_{t=1}^m (-1)^t \ell_t \cdot (p_t \ell_t u) \\ &= \sum_{t=1}^m \int p_t (\ell_t u_\nu) \cdot (\ell_t u) \end{aligned} \quad (2)$$

all integrals being finite by the preceding statements.

With $u_\nu \rightarrow \phi^{2m}u$ in L^2 and $Tu_\nu \rightarrow T(\phi^{2m}u)$ in L^2 , it is part of the proof of Lemma 3.7 (using Lemma 3.3(1)) that we then also have $u_\nu \rightarrow \phi^{2m}u$ in H^m_0 ,

and so $\int p_t (\ell_t u_\nu) \cdot (\ell_t u) \rightarrow \int p_t (\ell_t \phi^{2m}u) \cdot (\ell_t u)$ for $1 \leq t \leq m$.

Also, the strong convergence in Corol.3.8.1(1) implies the weak convergence required here. Hence, letting $\nu \rightarrow \infty$ in (2), we obtain the desired result. \square

CHAPTER 4

In this chapter we continue to assume that \underline{T}_0 is e.s.a. and that $u \in \mathcal{D}(T_0^*)$ and $\phi \in \mathcal{D}$, with u and ϕ real-valued. We further assume that $\phi \geq 0$ and that the first m derivatives of ϕ are bounded by K . Then we have, from Lemma 3.3, that $\phi u \in H_0^m$. Our aim is to derive the inequality

$$\int |\phi^{m-1} \ell_m(\phi u)|^2 \leq K \int \phi^{2m} (Tu)^2 + K \int [\phi^{2m} + c^m - q\phi^{2m}] u^2$$

where $|p_t \phi^{2(m-t)}| \leq c$ for $1 \leq t \leq m-1$, with $c \geq 1$.

We shall use this inequality in the next chapter as in Part I, and when $m=2$, this is just the inequality given in Lemma I.4.4(1).

We shall obtain the result by first proving the two inequalities:

$$\int |\phi^{m-1} \ell_m(\phi u)|^2 \leq K \int |\phi^m \ell_m u|^2 + K \int (1 + \phi)^{2(m-1)} u^2$$

$$\text{and } \int |\phi^m \ell_m u|^2 \leq K \sum_{t=1}^m \int p_t \ell_t(\phi^{2m} u) \cdot \ell_t u + K \int [\phi^{2(m-1)} + c^m] u^2, \quad \text{with } p_m = 1,$$

and then use the result of Lemma 3.9.

The reason for the approach that we adopt in this chapter is that whilst we have information about the product ϕu (i.e. $\phi u \in H_0^m$), it is easier to work with integrals of the form $\int |\phi^t \ell_t u|^2$ rather than $\int |\phi^{t-1} \ell_t(\phi u)|^2$. Therefore we first show that the following inequality holds for arbitrary $t \geq 1$, with the first t derivatives of ϕ bounded by K :

$$\int |\phi^t \ell_t u|^2 \leq K \int |\phi^{t-1} \ell_t(\phi u)|^2 + K \int (1 + \phi)^{2(t-1)} u^2$$

(see Lemma 4.1).

A note on the numbering of the results: The lemmas are numbered 4.1, 4.2 etc.; the propositions are numbered 4.0.1, 4.0.2, 4.0.3, 4.2.4, 4.2.5 etc., i.e. Prop.4.2.5 is the fifth proposition and is between Lemmas 4.2 and 4.3.

Definition: For $t \geq 0$, $\lambda \geq 0$, $\mu \geq 0$, $\rho \geq 0$, we define

$$\begin{aligned}\alpha_\rho(t, u, \phi) &= \int |\phi^{t+\rho} \ell_t u|^2 \\ \beta_\rho(\lambda, \mu, u, \phi) &= \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \int |\phi^{2\mu+\lambda+\rho} \partial_{s_1} \dots \partial_{s_\lambda} \ell_{2\mu} u|^2 \\ \gamma_\rho(t, u, \phi) &= \int |\phi^{t-1+\rho} \ell_t(\phi u)|^2\end{aligned}$$

When $u \in L^2$ and $\phi \in \mathcal{D}$ are understood, we just write $\alpha_\rho(t)$, $\beta_\rho(\lambda, \mu)$ and $\gamma_\rho(t)$. We note that $\alpha_\rho(0) = \gamma_\rho(0) [= \int (\phi^\rho u)^2]$, $\beta_\rho(0, t) = \alpha_\rho(2t)$ and $\beta_\rho(1, t) = \alpha_\rho(2t+1)$. When $\rho = 0$, we shall omit the subscript.

The function β arises naturally in our calculations and the first few propositions determine bounds for β in terms of the function α .

Proposition 4.0.1 If $\lambda > 0$, $\mu \geq 0$, $\lambda + \mu \leq K$ and $0 \leq \rho \leq K$, then

$$\beta_\rho(\lambda, \mu) \leq \varepsilon \beta_\rho(\lambda-1, \mu+1) + K(1 + \varepsilon^{-1}) \beta_\rho(\lambda-1, \mu) \quad (1)$$

If $1 \leq t \leq K$ and $0 \leq \rho \leq K$, then

$$\alpha_\rho(t) \leq \varepsilon \alpha_\rho(t+1) + K(1 + \varepsilon^{-1}) \alpha_\rho(t-1) \quad (2)$$

Proof: With $\rho = 0$, (1) is proved in Prop.3.0.4 (see

Prop.3.0.4(4), where we did not use the assumption $\psi \leq 1$). With $\rho > 0$,

(1) is proved exactly the same (see, for example, the integration below).

Putting $\lambda = 1$ in (1):

$$\begin{aligned}\beta_\rho(1, \mu) &\leq \varepsilon \beta_\rho(0, \mu+1) + K(1 + \varepsilon^{-1}) \beta_\rho(0, \mu) \\ \text{i.e. } \alpha_\rho(2\mu+1) &\leq \varepsilon \alpha_\rho(2\mu+2) + K(1 + \varepsilon^{-1}) \alpha_\rho(2\mu).\end{aligned}$$

Therefore (2) is established for $t = 2\mu+1 \geq 1$.

For $t = 2\mu$, $1 \leq \mu \leq K$:

$$\begin{aligned}\alpha_\rho(2\mu) &= \int |\phi^{2\mu+\rho} \ell_{2\mu} u|^2 \\ &= \int \phi^{4\mu+2\rho} (\ell_{2\mu} u)(\ell_{2\mu} u) \\ &= \int \phi^{4\mu+2\rho} (\ell_{2\mu} u) \nabla \cdot (\ell_{2\mu-1} u) \\ &= -\int \nabla [\phi^{4\mu+2\rho} (\ell_{2\mu} u)] \cdot (\ell_{2\mu-1} u)\end{aligned}$$

$$\begin{aligned}
\alpha_\rho(2\mu) &= -\int [\phi^{4\mu+2\rho} \nabla(\ell_{2\mu} u) + (4\mu+2\rho)\phi^{4\mu+2\rho-1} (\nabla\phi)(\ell_{2\mu} u)] \cdot (\ell_{2\mu-1} u) \\
&\leq \int |\phi^{4\mu+2\rho}(\ell_{2\mu+1} u) \cdot (\ell_{2\mu-1} u)| + K \int |\phi^{4\mu+2\rho-1}(\ell_{2\mu} u)(\ell_{2\mu-1} u)| \\
&\leq \frac{1}{2}\varepsilon \int |\phi^{2\mu+1+\rho} \ell_{2\mu+1} u|^2 + \frac{1}{2}\varepsilon^{-1} \int |\phi^{2\mu-1+\rho} \ell_{2\mu-1} u|^2 + \\
&\quad + \frac{1}{2} \int |\phi^{2\mu+\rho} \ell_{2\mu} u|^2 + K \int |\phi^{2\mu-1+\rho} \ell_{2\mu-1} u|^2 \\
&\leq \frac{1}{2}\varepsilon \alpha_\rho(2\mu+1) + \frac{1}{2}\alpha_\rho(2\mu) + K(1+\varepsilon^{-1})\alpha_\rho(2\mu-1)
\end{aligned}$$

Therefore $\alpha_\rho(2\mu) \leq \varepsilon \alpha_\rho(2\mu+1) + K(1+\varepsilon^{-1})\alpha_\rho(2\mu-1)$.

So (2) is also established for $t=2\mu \geq 2$. \square

Proposition 4.0.2 If $C \geq 0$, $0 \leq \rho \leq K$ and $0 \leq \kappa < N$, then

$$\alpha_\rho(\kappa) \leq \varepsilon(1+C)^{2(\kappa-N)} \alpha_\rho(N) + K(1+\varepsilon^{-1})^{N-1} (1+C)^{2\kappa} \alpha_\rho(0) \quad (1)$$

and if $\lambda \geq 2$, $\mu \geq 0$, $2(\lambda+\mu)-1 \leq N \leq K$, then

$$\beta_\rho(\lambda, \mu) \leq \varepsilon(1+C)^{2(2\lambda+\mu-N)} \alpha_\rho(N) + K(\varepsilon)(1+C)^{2(2\lambda+\mu)} \alpha_\rho(0) \quad (2)$$

[The reason that the latter inequality is only for $\lambda \geq 2$ is

that $\beta_\rho(0, \mu) = \alpha_\rho(2\mu)$ and $\beta_\rho(1, \mu) = \alpha_\rho(2\mu+1)$, i.e. we already have a bound on β in terms of α .]

Proof: As the proof is identical for any $\rho \geq 0$, we put $\rho=0$

and omit the subscript.

(1) is proved similar to Prop.3.0.1(2). Indeed, by Prop.4.0.1(2), we may apply Prop.3.0.1(3) to $\alpha(\kappa)$ to obtain (for $0 \leq \kappa < N$)

$$\alpha(\kappa) \leq \varepsilon' \alpha(\kappa+1) + K(1+\varepsilon'^{-1})^\kappa \alpha(0).$$

Setting $\varepsilon' = \varepsilon(1+C)^{-2}$, we obtain

$$\alpha(\kappa) \leq \varepsilon(1+C)^{-2} \alpha(\kappa+1) + K(1+\varepsilon^{-1})^\kappa (1+C)^{2\kappa} \alpha(0) \quad (3)$$

So (1) holds for $\kappa=N-1$ (put $\kappa=N-1$ in (3)).

Suppose that (1) holds for $\kappa=N-1, N-2, \dots, N-s$.

Then $\alpha(N-s-1) \leq (1+C)^{-2} \alpha(N-s) + K(1+C)^{2(N-s-1)} \alpha(0)$ by (3) with $\varepsilon=1$
 $\leq (1+C)^{-2} \{ \varepsilon(1+C)^{-2s} \alpha(N) + K(1+\varepsilon^{-1})^{N-1} (1+C)^{2(N-s)} \alpha(0) \} +$
 $+ K(1+C)^{2(N-s-1)} \alpha(0)$ by the induction hypothesis

$$\alpha(N-s-1) \leq \varepsilon(1+C)^{-2(s+1)}\alpha(N) + K(1+\varepsilon^{-1})^{N-1}(1+C)^{2(N-s-1)}\alpha(0).$$

Therefore (1) is established by induction.

We prove (2) also by induction.

From Prop.4.0.1(1) with $\lambda=2$ and $\varepsilon=\varepsilon'(1+C)^{-2}$:

$$\begin{aligned} \beta(2,\mu) &\leq \varepsilon'(1+C)^{-2}\beta(1,\mu+1) + K(1+\varepsilon'^{-1})(1+C)^2\beta(1,\mu) \\ &= \varepsilon'(1+C)^{-2}\alpha(2\mu+3) + K(\varepsilon')(1+C)^2\alpha(2\mu+1) \end{aligned} \quad (4)$$

Now if $2\mu+3=N$, (4) gives us

$$\begin{aligned} \beta(2,\mu) &\leq \varepsilon'(1+C)^{-2}\alpha(N) + K(\varepsilon')(1+C)^2\alpha(N-2) \\ &\leq \varepsilon'(1+C)^{-2}\alpha(N) + K(\varepsilon')(1+C)^2\{\varepsilon''(1+C)^{-4}\alpha(N) + K(\varepsilon'')(1+C)^{2(N-2)}\alpha(0)\} \\ &\quad \text{by (1)} \\ &\leq \{\varepsilon' + \varepsilon''K(\varepsilon')\}(1+C)^{-2}\alpha(N) + K(\varepsilon',\varepsilon'')(1+C)^{2(N-1)}\alpha(0) \end{aligned}$$

which gives (2) for $\lambda=2$, $2\mu+3=N$, by choice of $\varepsilon',\varepsilon''$.

If $2\mu+3 < N$, then from (4) with $\varepsilon'=1$ we have

$$\begin{aligned} \beta(2,\mu) &\leq (1+C)^{-2}\alpha(2\mu+3) + K(1+C)^2\alpha(2\mu+1) \\ &\leq (1+C)^{-2}\{\frac{1}{2}\varepsilon(1+C)^{2(2\mu+3-N)}\alpha(N) + K(\varepsilon)(1+C)^{2(2\mu+3)}\alpha(0)\} + \\ &\quad \text{by (1)} \\ &\quad + K(1+C)^2\{\varepsilon''(1+C)^{2(2\mu+1-N)}\alpha(N) + K(\varepsilon'')(1+C)^{2(2\mu+1)}\alpha(0)\} \\ &\leq \{\frac{1}{2}\varepsilon + K\varepsilon''\}(1+C)^{2(2\mu+2-N)}\alpha(N) + K(\varepsilon,\varepsilon'')(1+C)^{2(2\mu+2)}\alpha(0). \end{aligned}$$

So (2) holds for $\lambda=2$ generally, by choice of ε'' .

Suppose that (2) holds for $\lambda=2,3,\dots,s$, $\mu \geq 0$.

Then $\beta(s+1,\mu) \leq \beta(s,\mu+1) + K\beta(s,\mu)$ by Prop.4.0.1(1) with $\varepsilon=1$

$$\begin{aligned} &\leq \frac{1}{2}\varepsilon(1+C)^{2(2\mu+2+s-N)}\alpha(N) + K(\varepsilon)(1+C)^{2(2\mu+2+s)}\alpha(0) + \\ &\quad + \frac{1}{2}\varepsilon(1+C)^{2(2\mu+s-N)}\alpha(N) + K(\varepsilon)(1+C)^{2(2\mu+s)}\alpha(0) \quad \text{by the induction} \\ &\quad \text{hypothesis} \end{aligned}$$

which implies that (2) holds for $\lambda=s+1$, since $1 \leq (1+C)^4$.

Therefore (2) is established by induction. \square

The results of the above proposition can be strengthened as we indicate below.

Prop.4.0.2(1), when written out showing more details is:

$$\alpha_{\rho}(\kappa) \leq \varepsilon(1+C)^{2(\kappa-N)} \int |\phi^{N+\rho} \ell_N u|^2 + K(\varepsilon)(1+C)^{2\kappa} \int \phi^{2\rho} u^2.$$

We may put $C = \phi$ and take the terms involving C inside the integrals to obtain:

$$\alpha_{\rho}(\kappa) \leq \varepsilon \int |(1+\phi)^{\kappa-N} \phi^{N+\rho} \ell_N u|^2 + K(\varepsilon) \int (1+\phi)^{2\kappa} \phi^{2\rho} u^2.$$

If we further put $\rho = N-1-\kappa$ (≥ 0) and note that $\phi \leq 1+\phi$, we obtain:

$$\begin{aligned} \alpha_{N-1-\kappa}(\kappa) &\leq \varepsilon \int |(1+\phi)^{\kappa-N} (1+\phi)^{N-1-\kappa} \phi^N \ell_N u|^2 + \\ &\quad + K(\varepsilon) \int (1+\phi)^{2\kappa} (1+\phi)^{2(N-1-\kappa)} u^2 \\ &= \varepsilon \int |(1+\phi)^{-1} \phi^N \ell_N u|^2 + K(\varepsilon) \int (1+\phi)^{2(N-1)} u^2 \\ &\leq \varepsilon \int |\phi^N \ell_N u|^2 + K(\varepsilon) \int (1+\phi)^{2(N-1)} u^2 \quad \text{as } (1+\phi)^{-1} \leq 1. \end{aligned}$$

Therefore, writing $\int (1+\phi)^{2\sigma} u^2$ as $\alpha_{\sigma}'(0, u,)$, we have

$$\alpha_{N-1-\kappa}(\kappa) \leq \varepsilon \alpha(N) + K(\varepsilon) \alpha_{N-1}'(0).$$

Operating in a similar fashion on Prop.4.0.2(2), we obtain:

$$\beta_{N-1-2\mu-\lambda}(\lambda, \mu) \leq \varepsilon \alpha(N) + K(\varepsilon) \alpha_{N-1}'(0).$$

So we have

Proposition 4.0.2' If $0 \leq \kappa < N$, then

$$\alpha_{N-1-\kappa}(\kappa) \leq \varepsilon \alpha(N) + K(\varepsilon) \alpha_{N-1}'(0) \quad (1)$$

and if $\lambda \geq 2$, $\mu \geq 0$, $2(\lambda + \mu) - 1 \leq N \leq K$, then

$$\beta_{N-1-2\mu-\lambda}(\lambda, \mu) \leq \varepsilon \alpha(N) + K(\varepsilon) \alpha_{N-1}'(0) \quad (2)$$

Proof: See the note above. \square

Proposition 4.0.3 If $0 \leq \rho \leq K$, and in (1) $t > 0$ and in (2) $t \geq 0$, then

$$\begin{aligned} \int \left| \sum_{i=1}^t \sum_{j_i=1}^n \sum_{r=0}^{2t-1} \sum_{\mathcal{C}} \phi^{2t-1+\rho} (P[S_{r,2t}]u) (P[S_{/r,2t}] \phi) \right|^2 &\leq \\ &\leq \varepsilon \alpha_{\rho}(2t) + K(\varepsilon) \alpha_{2t-1+\rho}'(0) \quad (1) \end{aligned}$$

$$\begin{aligned} \text{and } \int \left| \sum_{i=1}^{t+1} \sum_{j_i=1}^n \sum_{r=0}^{2t} \sum_{\mathcal{C}} \phi^{2t+\rho} (P[S_{r,2t+1}]u) (P[S_{/r,2t+1}] \phi) \right|^2 &\leq \\ &\leq \varepsilon \alpha_{\rho}(2t+1) + K(\varepsilon) \alpha_{2t+\rho}'(0) \quad (2) \end{aligned}$$

Proof: We prove the results for $\rho = 0$, as the cases when $\rho > 0$ then follow simply.

$$\begin{aligned} & \int \left| \sum_{i=1}^t \sum_{j_i=1}^n \sum_{r=0}^{2t-1} \sum_{\xi} \phi^{2t-1} (P[S_{r,2t}]u)(P[S_{/r,2t}]\phi) \right|^2 \leq \\ & \leq K \sum_{r=0}^{2t-1} \sum_{\xi} \int \left| \sum_{i=1}^t \sum_{j_i=1}^n \phi^{2t-1} (P[S_{r,2t}]u)(P[S_{/r,2t}]\phi) \right|^2 \\ & \leq K \sum_{r=0}^{2t-1} \sum_{\xi} \int \left| \sum_{i=1}^r \sum_{j_i=1}^n \phi^{2t-1} P[S_{r,2t}]u \right|^2 \quad \text{as the derivatives of } \phi \text{ are} \\ & \leq K \sum_{r=0}^{2t-1} \sum_{\Lambda} \int \left| \sum_{i=1}^{\lambda} \sum_{s_i=1}^n \phi^{2t-1} \partial_{s_1} \dots \partial_{s_{\lambda}} e_{r-\lambda} u \right|^2 \quad \text{bounded} \end{aligned}$$

where $\Lambda(r) = \{\lambda: \lambda \equiv r \pmod{2}, 0 \leq \lambda \leq \min(r, 2t-r)\}$

$$\begin{aligned} & \leq K \sum_{r=0}^{2t-1} \sum_{\Lambda} \beta_{2t-1-r}(\lambda, \frac{1}{2}(r-\lambda)) \\ & \leq K \sum_{r=0}^{2t-1} [\beta_{2t-1-r}(0, \frac{1}{2}r) + \beta_{2t-1-r}(1, \frac{1}{2}(r-1)) + \sum_{\lambda \geq 2} \beta_{2t-1-r}(\lambda, \frac{1}{2}(r-\lambda))] \\ & \leq K \sum_{r=0}^{2t-1} [\alpha_{2t-1-r}(r) + \alpha_{2t-1-r}(r) + \sum_{\lambda \geq 2} \{\epsilon \alpha(2t) + K(\epsilon) \alpha'_{2t-1}(0)\}] \end{aligned}$$

by Prop.4.0.2'(2) with $N = 2t$,

since $r + \lambda - 1 \leq \min(2r, 2t) - 1 \leq 2t - 1$;

so the sum is independent of λ ;

also, $r < 2t$ so we may apply Prop.4.0.2'(1) ($N = 2t$)

$$\leq K \sum_{r=0}^{2t-1} [\epsilon \alpha(2t) + K(\epsilon) \alpha'_{2t-1}(0)].$$

This implies (1) by choice of ϵ since the sum is now independent of r .

We proceed similarly with the left-hand side of (2) to obtain:

$$\begin{aligned} & \int \left| \sum_{i=1}^{t+1} \sum_{j_i=1}^n \sum_{r=0}^{2t} \sum_{\xi} \phi^{2t} (P[S_{r,2t+1}]u)(P[S_{/r,2t+1}]\phi) \right|^2 \leq \\ & \leq K \sum_{r=0}^{2t} \sum_{\Lambda} \beta_{2t-r}(\lambda, \frac{1}{2}(r-\lambda)) \end{aligned}$$

where $\Lambda(r) = \{\lambda: \lambda \equiv r \pmod{2}, 0 \leq \lambda \leq \min(r, 2(t+1)-r)\}$.

We again split the sum into $\lambda = 0$, $\lambda = 1$, $\lambda \geq 2$, and

with $N = 2t+1$ we apply Prop.4.0.2'(1) since $r < 2t+1$

and Prop.4.0.2'(2) since $r + \lambda - 1 \leq \min(2r, 2(t+1)) - 1$

$\leq 2t+1$.

Thus we obtain:

$$K \sum_{r=0}^{2t} \sum_{\lambda} \beta_{2t-r}(\lambda, \frac{1}{2}(r-\lambda)) \leq K \sum_{r=0}^{2t} [\varepsilon \alpha(2t+1) + K(\varepsilon) \alpha'_{2t}(0)].$$

This implies (2) by choice of ε since the sum is now independent of r . \square

Lemma 4.1 If $t > 0$ and $0 \leq \rho \leq K$, then

$$\int |\phi^{t+\rho} e_{2t} u|^2 \leq K \int |\phi^{t-1+\rho} e_t(\phi u)|^2 + K \int (1 + \phi)^{2(t-1+\rho)} u^2 \quad (1)$$

Proof: We prove the result for $\rho = 0$.

$$\phi e_{2t} u = e_{2t}(\phi u) - \sum_{i=1}^t \sum_{j_i=1}^n \sum_{r=0}^{2t-1} \sum_{\phi} (P[S_{r,2t}]u)(P[S_{/r,2t}]\phi) \quad (2)$$

Multiplying through by ϕ^{2t-1} , squaring, using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, and integrating, we obtain:

$$\begin{aligned} \int |\phi^{2t} e_{2t} u|^2 &\leq 2 \int |\phi^{2t-1} e_{2t}(\phi u)|^2 + \\ &+ 2 \int \left| \sum_{i=1}^t \sum_{j_i=1}^n \sum_{r=0}^{2t-1} \sum_{\phi} \phi^{2t-1} (P[S_{r,2t}]u)(P[S_{/r,2t}]\phi) \right|^2 \end{aligned}$$

Therefore $\alpha(2t) \leq 2\gamma(2t) + 2\varepsilon\alpha(2t) + K(\varepsilon)\alpha'_{2t-1}(0)$ by Prop.4.0.3(1).

Putting $\varepsilon = 1/4$ establishes (1) in the case t (in (1)) is even.

$$\phi \partial_{j_{t+1}} e_{2t} u = \partial_{j_{t+1}} e_{2t}(\phi u) - \sum_{i=1}^t \sum_{j_i=1}^n \sum_{r=0}^{2t} \sum_{\phi} (P[S_{r,2t+1}]u)(P[S_{/r,2t+1}]\phi) \quad (3)$$

Multiplying through by ϕ^{2t} , squaring, summing over $1 \leq j_{t+1} \leq n$, using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, and integrating, we obtain:

$$\begin{aligned} \int |\phi^{2t+1} e_{2t+1} u|^2 &\leq 2 \int |\phi^{2t} e_{2t+1}(\phi u)|^2 + \\ &+ 2 \int \left| \sum_{i=1}^{t+1} \sum_{j_i=1}^n \sum_{r=0}^{2t} \sum_{\phi} \phi^{2t} (P[S_{r,2t+1}]u)(P[S_{/r,2t+1}]\phi) \right|^2 \end{aligned}$$

Therefore $\alpha(2t+1) \leq 2\gamma(2t+1) + 2\varepsilon\alpha(2t+1) + K(\varepsilon)\alpha'_{2t}(0)$ by Prop.4.0.3(2).

Putting $\varepsilon = 1/4$ establishes (1) in the case t (in (1)) is odd. \square

Lemma 4.2 If $t > 0$ and $0 \leq \rho \leq K$, then

$$\int |\phi^{t-1+\rho} e_t(\phi u)|^2 \leq K \int |\phi^{t+\rho} e_t u|^2 + K \int (1 + \phi)^{2(t-1+\rho)} u^2 \quad (1)$$

Proof: We prove the result for $\rho = 0$.

In Lemma 4.1(2) and (3) we take the summed term to the other side of the equation and then operate exactly as we did there to obtain:

$$\gamma(t) \leq 2\alpha(t) + 2\varepsilon\alpha(t) + K(\varepsilon)\alpha'_{t-1}(0)$$

which gives (1) by choosing $\varepsilon=1$. \square

In view of Lemma 4.1(1), knowing that $u \in L^2$ and $\phi \in H^m_0$, $\alpha_\rho(t)$ is finite for $0 \leq t \leq m$, $0 \leq \rho \leq K$.

The usefulness of Lemmas 4.1 and 4.2 when taken together is seen when we note that now $K' \leq \alpha_\rho(t) + \alpha'_{t-1+\rho}(0) \leq K''$ implies that we have $K^{-1}K' \leq \gamma_\rho(t) + \gamma'_{t-1+\rho}(0) \leq KK''$, for $t \geq 0$, $0 \leq \rho \leq K$.

Proposition 4.2.4 If $0 < t \leq m$ and $0 \leq \rho \leq K$, then

$$\int \left| \sum_i \sum_{j_i=1}^n \sum_{r=0}^{t-1} \sum_{\mathcal{C}} \phi^{t-2m+\rho} (P[S_{r,t}]u)(P[S_{/r,t}]\phi^{2m}) \right|^2 \leq \varepsilon\alpha_\rho(t) + K(\varepsilon)\alpha'_{t-1+\rho}(0) \quad (1)$$

[\sum_i denotes $\sum_{i=1}^\mu$ if $t=2\mu$ and $\sum_{i=1}^{\mu+1}$ if $t=2\mu+1$]

Proof: We prove the result for $\rho=0$.

We first show that for $0 < t \leq m$, $0 \leq r \leq t-1$

$$\begin{aligned} \left| \sum_{j_i \notin S_{r,t}} \sum_{j_i=1}^n \phi^{t-2m-r} (P[S_{/r,t}]\phi^{2m}) \right| &\leq K(1 + \phi^{t-1-r}) & (2) \\ \left| \sum_{j_i \notin S_{r,t}} \sum_{j_i=1}^n \phi^{t-2m-r} (P[S_{/r,t}]^2\phi^{2m}) \right| &\leq K \left| \phi^{t-2m-r} \sum_{\iota=1}^{t-r} \phi^{2m-\iota} \right| & (\text{derivatives of } \phi \text{ bounded}) \\ &\leq K \left| \sum_{\iota=1}^{t-r} \phi^{t-r-\iota} \right| \\ &\leq K(1 + \phi)^{t-r-1} & (\text{as } t-r-\iota \geq 0 \text{ for } 1 \leq \iota \leq t-r) \\ &\leq K(1 + \phi^{t-r-1}). \end{aligned}$$

$$\begin{aligned} \text{Now } \int \left| \sum_i \sum_{j_i=1}^n \sum_{r=0}^{t-1} \sum_{\mathcal{C}} \phi^{t-2m} (P[S_{r,t}]u)(P[S_{/r,t}]\phi^{2m}) \right|^2 &\leq \\ &\leq K \sum_{r=0}^{t-1} \sum_{\mathcal{C}} \int \left| \sum_i \sum_{j_i=1}^n \phi^r (P[S_{r,t}]u) \phi^{t-2m-r} (P[S_{/r,t}]\phi^{2m}) \right|^2 \\ &\leq K \sum_{r=0}^{t-1} \sum_{\mathcal{C}} \int \left| \sum_{i=1}^r \sum_{j_i=1}^n (1 + \phi^{t-1-r}) \phi^r (P[S_{r,t}]u) \right|^2 & \text{by (2)} \end{aligned}$$

$$\begin{aligned}
& K \sum_{r=0}^{t-1} \sum_{\xi} \int \left| \sum_{i=1}^r \sum_{j_i=1}^n (1 + \phi^{t-1-r}) \phi^r (P[S_{r,t}]u) \right|^2 \leq \\
& \leq K \sum_{r=0}^{t-1} \sum_{\Lambda} [\beta_{t-1-r}(\lambda, \frac{1}{2}(r-\lambda)) + \beta(\lambda, \frac{1}{2}(r-\lambda))] \quad \text{as in Prop.4.0.3} \\
& \leq \varepsilon \alpha(t) + K(\varepsilon) \alpha'_{t-1}(0) \quad \text{as in Prop.4.0.3. } \square
\end{aligned}$$

In the proof above, we obtain the same inequality for the β term as for the β_{t-1-r} term by choosing $\rho=0$ in the note following Prop.4.0.2 and noting that $(1+\phi)^{r-t} \leq (1+\phi)^{-1}$ for $0 \leq r \leq t-1$.

Also in that note, we defined $\alpha'_{t-1+\rho}(0) = \int (1+\phi)^{2(t-1+\rho)} u^2$ as an upper bound for $\int (1+\phi)^{2(t-1)} \phi^{2\rho} u^2$. Writing $\alpha''_{t-1,\rho}(0) = \int (1+\phi)^{2(t-1)} \phi^{2\rho} u^2$ we see that in Prop.4.2.4(1), $\alpha'_{t-1+\rho}(0)$ may be replaced by $\alpha''_{t-1,\rho}(0)$, and so we have

Proposition 4.2.4'' If $0 < t \leq m$ and $0 \leq \rho \leq K$, then

$$\begin{aligned}
& \int \left| \sum_{i=1}^n \sum_{j_i=1}^n \sum_{r=0}^{t-1} \sum_{\xi} \phi^{t-2m+\rho} (P[S_{r,t}]u) (P[S_{r,t}] \phi^{2m}) \right|^2 \leq \\
& \leq \varepsilon \alpha_{\rho}(t) + K(\varepsilon) \alpha''_{t-1,\rho}(0) \quad (1)
\end{aligned}$$

Proof: See the note above. \square

Proposition 4.2.5 If $1 \leq t \leq m-1$ and $|p_t \phi^{2(m-t)}| \leq c$ with $c \geq 1$, then

$$\int p_t \ell_t(\phi^{2m}u) \cdot \ell_t u \geq -\varepsilon \alpha(m) - K(\varepsilon) c^m \alpha(0) - Kc \alpha'_{t-1}(0) \quad (1)$$

Proof: We first establish

$$\int |\phi^{t-2m} \ell_t(\phi^{2m}u)|^2 \leq K \alpha(t) + K \alpha'_{t-1}(0) \quad (2)$$

Similar to the proof of Lemmas 4.1 and 4.2 we obtain

$$\begin{aligned}
\int |\phi^{t-2m} \ell_t(\phi^{2m}u)|^2 & \leq 2 \int |\phi^t \ell_t u|^2 + \\
& + 2 \int \left| \sum_{i=1}^n \sum_{j_i=1}^n \sum_{r=0}^{t-1} \sum_{\xi} \phi^{t-2m} (P[S_{r,t}]u) (P[S_{r,t}] \phi^{2m}) \right|^2 \\
& \quad \text{(with } \sum_{\xi} \text{ as in Prop.4.2.4(1))} \\
& \leq 2\alpha(t) + 2\{\varepsilon \alpha(t) + K(\varepsilon) \alpha'_{t-1}(0)\} \quad \text{by Prop.4.2.4(1)} \\
& \quad \text{with } \rho=0
\end{aligned}$$

which establishes (2) by choosing $\varepsilon=1$.

$$\begin{aligned}
\text{Now } \int p_t \ell_t(\phi^{2m}u) \cdot \ell_t u &= \int [p_t \phi^{2(m-t)}] [\phi^{t-2m} \ell_t(\phi^{2m}u)] \cdot [\ell^t \phi_t u] \\
&\geq -c \int |\phi^{t-2m} \ell_t(\phi^{2m}u)| |\phi^t \ell_t u| \\
&\geq -\frac{1}{2}c \{ \int |\phi^{t-2m} \ell_t(\phi^{2m}u)|^2 + \int |\phi^t \ell_t u|^2 \} \\
&\geq -\frac{1}{2}c \{ K\alpha(t) + K\alpha'_{t-1}(0) + \alpha(t) \} \quad (\text{by (2)}) \\
&\geq -Kc \{ \alpha(t) + \alpha'_{t-1}(0) \} \\
&\geq -Kc\epsilon' \alpha(m) - Kc(1 + \epsilon'^{-1})^{m-1} \alpha(0) - (\text{by Prop.4.0.2(1) with} \\
&\quad C=0, \rho=0, N=m) \\
&\quad - Kc\alpha'_{t-1}(0) \\
&\geq -\epsilon\alpha(m) - K(\epsilon)c^m \alpha(0) - Kc\alpha'_{t-1}(0) \quad (\text{with } \epsilon' = (Kc)^{-1} \epsilon)
\end{aligned}$$

which gives (1). \square

Proposition 4.2.6

$$\int \ell_m(\phi^{2m}u) \cdot \ell_m u \geq \frac{1}{2}\alpha(m) - K\alpha'_{m-1}(0) \quad (1)$$

Proof:

$$\begin{aligned}
\int \ell_m(\phi^{2m}u) \cdot \ell_m u &= \int [\phi^{-m} \ell_m(\phi^{2m}u)] \cdot [\phi^m \ell_m u] \\
&= \int [\phi^m \ell_m u + \sum_{\mathbf{i}} \sum_{j_i=1}^n \sum_{r=0}^{m-1} \sum_{\mathcal{C}} \phi^{-m} (P[S_{r,m}]u)(P[S_{r,m}] \phi^{2m})] \cdot [\phi^m \ell_m u] \\
&\quad (\text{where the summed term is read as the appropriate} \\
&\quad \text{vector if } m \text{ is odd, and } \sum_{\mathbf{i}} \text{ is treated accordingly}) \\
&\geq \int |\phi^m \ell_m u|^2 - \int \sum_{\mathbf{i}} \sum_{j_i=1}^n \sum_{r=0}^{m-1} \sum_{\mathcal{C}} \phi^{-m} (P[S_{r,m}]u)(P[S_{r,m}] \phi^{2m}) | |\phi^m \ell_m u| \\
&\quad (\text{with } \sum_{\mathbf{i}} \text{ now as in Prop.4.2.4(1)}) \\
&\geq \alpha(m) - \frac{1}{4} \int |\phi^m \ell_m u|^2 - K \int \sum_{\mathbf{i}} \sum_{j_i=1}^n \sum_{r=0}^{m-1} \sum_{\mathcal{C}} \phi^{-m} (P[S_{r,m}]u)(P[S_{r,m}] \phi^{2m}) |^2 \\
&\geq \alpha(m) - (1/4)\alpha(m) - K \{ \epsilon\alpha(m) + K(\epsilon)\alpha'_{m-1}(0) \} \quad (\text{by Prop.4.2.4(1)} \\
&\quad \text{with } \rho=0, t=m)
\end{aligned}$$

which implies (1) by choosing $\epsilon = (1/4)K^{-1}$. \square

Lemma 4.3 If $|p_t \phi^{2(m-t)}| \leq c$ with $c \geq 1$, for $1 \leq t \leq m-1$, and $p_m = 1$,

$$\text{then } \int |\phi^m \ell_m u|^2 \leq K \sum_{t=1}^m \int p_t \ell_t(\phi^{2m}u) \cdot \ell_t u + K \int [\phi^{2(m-1)} + c^m] u^2 \quad (1)$$

Proof:

$$\begin{aligned} \sum_{t=1}^m \int p_t \ell_t(\phi^{2m}u) \cdot \ell_t u &= \int \ell_m(\phi^{2m}u) \cdot \ell_m u + \sum_{t=1}^{m-1} \int p_t \ell_t(\phi^{2m}u) \cdot \ell_t u \\ &\geq \frac{1}{2}\alpha(m) - K\alpha'_{m-1}(0) - \quad (\text{by Prop.4.2.6(1)}) \\ &\quad - \sum_{t=1}^{m-1} \{ \varepsilon\alpha(m) + K(\varepsilon)c^m\alpha(0) + Kc\alpha'_{t-1}(0) \} \quad (\text{by Prop.4.2.5(1)}) \\ &\geq \left\{ \frac{1}{2} - (m-1)\varepsilon \right\} \alpha(m) - K\alpha'_{m-1}(0) - K(\varepsilon)c^m\alpha(0) - Kc\alpha'_{m-2}(0) \end{aligned}$$

because $\alpha'_{t-1}(0) = \int (1+\phi)^{2(t-1)} u^2 \leq \int (1+\phi)^{2(m-2)} u^2 = \alpha'_{m-2}(0)$ for $1 \leq t \leq m-1$, and the sum is then independent of t .

Choosing $\varepsilon = (1/4)(m-1)^{-1}$ gives us:

$$\sum_{t=1}^m \int p_t \ell_t(\phi^{2m}u) \cdot \ell_t u \geq \frac{1}{4} \int |\phi^m \ell_m u|^2 - K \int [(1+\phi)^{2(m-1)} + c^m + c(1+\phi)^{2(m-2)}] u^2$$

which implies (1) since $c(1+\phi)^{2(m-2)} \leq Kc^{m-1} + K(1+\phi)^{2(m-1)}$, and $(1+\phi)^{2(m-1)} \leq K(1+\phi^{2(m-1)})$ and $c \geq 1$. \square

Lemma 4.4 If $|p_t \phi^{2(m-t)}| \leq c$ with $c \geq 1$, for $1 \leq t \leq m-1$, then

$$\int |\phi^{m-1} \ell_m(\phi u)|^2 \leq K \int \phi^{2m} (\tau u)^2 + K \int [\phi^{2m} + c^m - q\phi^{2m}] u^2 \quad (1)$$

Proof: With $p_m = 1$:

$$\begin{aligned} \int |\phi^{m-1} \ell_m(\phi u)|^2 &\leq K \int |\phi^m \ell_m u|^2 + K \int (1+\phi)^{2(m-1)} u^2 \quad (\text{by Lemma 4.2(1): } \rho=0, t=m) \\ &\leq K \sum_{t=1}^m \int p_t \ell_t(\phi^{2m}u) \cdot \ell_t u + K \int [\phi^{2(m-1)} + c^m + (1+\phi)^{2(m-1)}] u^2 \\ &\quad (\text{by Lemma 4.3(1)}) \end{aligned}$$

$$\begin{aligned} &\leq K \int \phi^{2m} \tau u - K \int q \phi^{2m} u^2 + \quad (\text{by Lemma 3.9(1)}) \\ &\quad + K \int [\phi^{2(m-1)} + c^m] u^2 \quad (\text{as in Lemma 4.3}) \\ &\leq K \int \phi^{2m} (\tau u)^2 + K \int [\phi^{2m} + \phi^{2(m-1)} + c^m - q\phi^{2m}] u^2 \end{aligned}$$

which implies (1) since $\phi^{2(m-1)} \leq K(1+\phi^{2m})$ and $c \geq 1$. \square

This is the desired inequality which we referred to at the beginning of the chapter and which we shall use in the next chapter. If we put $m=2$ in Lemma 4.4(1) we obtain Lemma I.4.4(1):

$$\int (\phi \nabla^2(\phi u))^2 \leq K \int \phi^4 (\tau u)^2 + K \int [\phi^4 + c^2 - q\phi^4] u^2.$$

As in Part I, the further assumption that $p_t \geq 0$ allows modification of Lemma 4.4(1) (after first modifying Proposition 4.2.5(1)). However, to obtain a reasonable generalisation of Lemma I.4.6 to the case in which $m > 2$, the condition on p_t has to be slightly stronger, i.e. either $p_t \equiv 0$ or p_t has a strictly positive lower bound, on $\text{supp } \phi$. We shall state the condition generally for $1 \leq t \leq m-1$, although it obviously may be relaxed for $t=1$ to $p_1 \geq 0$ as in Part I.

Proposition 4.4.7 If, for $1 \leq t \leq m-1$, either $p_t \equiv 0$ or $k_t c'_t \leq p_t \leq c'_t$

for some $0 < k_t \leq 1$, then

$$\int p_t \ell_t(\phi^{2m} u) \cdot \ell_t u \geq \frac{1}{2} \int p_t \phi^{2(m-t)} |\phi^t \ell_t u|^2 - K \int p_t \phi^{2(m-t)} (1 + \phi)^{2(t-1)} u^2 \quad (1)$$

Proof: The proof is similar to that of Prop. 4.2.6 where $p_m = c'_m = 1$.

If $p_t \equiv 0$, then (1) is trivial, so we consider $k_t c'_t \leq p_t \leq c'_t$.

We shall write $\sum_i(\dots)$ for the summed term which will be the same

throughout.

$$\begin{aligned} \int p_t \ell_t(\phi^{2m} u) \cdot \ell_t u &= \int p_t \phi^{-m} \ell_t(\phi^{2m} u) \cdot \phi^m \ell_t u \\ &= \int p_t [\phi^m \ell_t u + \sum_i \sum_{j_i=1}^n \sum_{r=0}^{t-1} \sum_{\xi} \phi^{-m} (P[S_{r,t}] u) (P[S_{r,t}] \phi^{2m}) \cdot \phi^m \ell_t u] \\ &\quad \text{(where the summed term is a vector if } t \text{ is odd,} \\ &\quad \text{and } \sum_i \text{ is treated accordingly, as in Prop. 4.2.6)} \end{aligned}$$

$$\geq \int p_t |\phi^m \ell_t u|^2 - \int p_t |\sum_i(\dots)| |\phi^m \ell_t u|$$

(with \sum_i now as in Prop. 4.2.4(1))

$$\begin{aligned} &\geq \int p_t |\phi^m \ell_t u|^2 - \frac{1}{4} \int p_t |\phi^m \ell_t u|^2 - K \int p_t |\sum_i(\dots)|^2 \\ &\geq \frac{3}{4} \int p_t |\phi^m \ell_t u|^2 - K c'_t \int |\sum_i(\dots)|^2 \\ &\geq \frac{3}{4} \int p_t |\phi^m \ell_t u|^2 - K c'_t \{ \epsilon \alpha_{m-t}(t) + K(\epsilon) \alpha_{t-1, m-t}''(0) \} \end{aligned}$$

(by Prop. 4.2.4''(1) with $\rho = m-t$)

$$= \int (\frac{3}{4} p_t - K c'_t \epsilon) |\phi^m \ell_t u|^2 - K(\epsilon) c'_t \int \phi^{2(m-t)} (1 + \phi)^{2(t-1)} u^2$$

Choosing $\varepsilon = (1/4)k_t K^{-1}$, we have $Kc_t' \varepsilon = (1/4)k_t c_t' \leq (1/4)p_t$, and so

$$\int p_t \ell_t(\phi^{2m} u) \cdot \ell_t u \geq \frac{1}{2} \int p_t |\phi^m \ell_t u|^2 - K \int c_t' \phi^{2(m-t)} (1+\phi)^{2(t-1)} u^2$$

which implies (1) as $c_t' \leq k_t^{-1} p_t = K p_t$. \square

Lemma 4.5 If, for $1 \leq t \leq m-1$, either $p_t \equiv 0$ or $k_t c_t' \leq p_t \leq c_t'$ for some

$0 < k_t \leq 1$, and $p_m = 1$, then

$$\begin{aligned} \sum_{t=1}^m \int p_t \phi^{2(m-t)} |\phi^t \ell_t u|^2 &\leq K \sum_{t=1}^m \int p_t \ell_t(\phi^{2m} u) \cdot \ell_t u + \\ &+ K \sum_{t=1}^m \int p_t \phi^{2(m-t)} (1+\phi)^{2(t-1)} u^2 \end{aligned} \quad (1)$$

Proof: Prop.4.4.7(1) gives us (for $1 \leq t \leq m-1$):

$$\frac{1}{2} \int p_t \phi^{2(m-t)} |\phi^t \ell_t u|^2 \leq \int p_t \ell_t(\phi^{2m} u) \cdot \ell_t u + K \int p_t \phi^{2(m-t)} (1+\phi)^{2(t-1)} u^2 \quad (2)$$

Prop.4.2.6(1) gives us (2) with $t=m$. Therefore summing (2) for

$1 \leq t \leq m$ establishes (1). \square

Lemma 4.6: If, for $1 \leq t \leq m-1$, either $p_t \equiv 0$ or $k_t c_t' \leq p_t \leq c_t'$ for some

$0 < k_t \leq 1$, and $p_m = 1$, then

$$\begin{aligned} \sum_{t=1}^m \int p_t \phi^{2(m-t)} |\phi^t \ell_t u|^2 &\leq K \int \phi^{2m} (Tu)^2 + \\ &+ K \sum_{t=1}^{m-1} \int [1 + \phi^{2m} + p_t \phi^{2(m-t)} (1+\phi)^{2(t-1)} - q \phi^{2m}] u^2 \end{aligned} \quad (1)$$

Proof:

$$\sum_{t=1}^m \int p_t \phi^{2(m-t)} |\phi^t \ell_t u|^2 \leq K \sum_{t=1}^m \int p_t \ell_t(\phi^{2m} u) \cdot \ell_t u + K \sum_{t=1}^m \int p_t \phi^{2(m-t)} (1+\phi)^{2(t-1)} u^2$$

(by Lemma 4.5(1))

$$= K \int \phi^{2m} u T u - K \int q \phi^{2m} u^2 + \quad (\text{by Lemma 3.9(1)})$$

$$+ K \sum_{t=1}^m \int p_t \phi^{2(m-t)} (1+\phi)^{2(t-1)} u^2$$

$$\leq K \int \phi^{2m} (Tu)^2 +$$

$$+ K \sum_{t=1}^{m-1} \int [1 + \phi^{2m} + p_t \phi^{2(m-t)} (1+\phi)^{2(t-1)} - q \phi^{2m}] u^2$$

as the term in the sum which is dependent on t is, for $t=m$, $(1+\phi)^{2(m-1)}$,

and $(1+\phi)^{2(m-1)} \leq K(1+\phi^{2m})$. Thus (1) is established. \square

This is the desired generalisation of Lemma I.4.6 to the case $m \geq 2$.

Putting $m=2$ in Lemma 4.6 and using the relaxed condition on p_1 , the right-hand side of the inequality becomes the same as the right-hand side of Lemma I.4.6(1). The difference in the left-hand sides is that where we had $\int(\phi \nabla^2(\phi u))^2$ we now have $\int(\phi^2 \nabla^2 u)^2$; however, it is clear from Lemmas 4.1(1) and 4.2(1) (with $\rho=0$ and $t=m=2$) that this makes no difference, because the additional term $(\int(1+\phi)^2 u^2)$ is bounded by terms which are already present on the right-hand side of the inequality $(\int(1+\phi^4)u^2)$.

The term $p\phi^2$ in the right-hand side of Lemma I.4.6(1) (which replaces the term c^2 in Lemma I.4.4(1), where c is the bound on $p\phi^2$) might suggest that we should have a term $p_t \phi^{2(m-t)}$ here (replacing the term c^m in Lemma 4.4(1), where c is the bound on $p_t \phi^{2(m-t)}$) instead of having the term $p_t \phi^{2(m-t)}(1+\phi)^{2(t-1)}$, which is clearly the same for $t=1$. This was not found to be possible, nor even desirable, as the inequality as it stands is sufficient to obtain the generalisation of Corollary I.5.6.1 to the case $m \geq 2$.

CHAPTER 5

In this chapter, we shall generalise Theorems I.5.4 and I.5.6 to the case $m \geq 2$.

As in Part I, we shall consider $\int (uT_0^*v - vT_0^*u)$ for real-valued functions u and $v \in \mathcal{D}(T_0^*)$, because we have the result that, given T_0 is symmetric (Lemma 1.1) and with $L = (u, T_0^*v) - (T_0^*u, v)$, T_0 is e.s.a. iff $L=0$ for any pair of real-valued functions u and $v \in \mathcal{D}(T_0^*)$. (See Chapter I.5.)

The theorem proceeds along similar lines to that in Part I, beginning with an application of Green's Theorem. We shall first derive identities for the terms appearing in $(uT_0^*v - vT_0^*u)$ in order to do this. (Note that $T_0^*u = \tau u$ with $u \in \mathcal{D}(T_0^*)$.)

Many of the formulas with which we shall be dealing have terms which are symmetric (or antisymmetric) in u and v . The writing of $+u \leftrightarrow v$ (or $-u \leftrightarrow v$) after a term indicates that a similar term with u and v interchanged is to be added (or subtracted).

Proposition 5.1 For $t > 0$:

$$u(\ell_{2t}(p_{2t}\ell_{2t}v)) - u \leftrightarrow v = \sum_{r=0}^{t-1} \nabla \cdot [(\ell_{2r}u)(\ell_{2(t-r)-1}(p_{2t}\ell_{2t}v)) - u \leftrightarrow v - (\ell_{2r+1}u)(\ell_{2(t-r-1)}(p_{2t}\ell_{2t}v)) + u \leftrightarrow v] \quad (1)$$

$$\begin{aligned} u(\ell_{2t+1}(p_{2t+1}\ell_{2t+1}v)) - u \leftrightarrow v &= \\ &= \sum_{r=0}^{t-1} \nabla \cdot [(\ell_{2r}u)(\ell_{2(t-r)-1}(\nabla \cdot (p_{2t+1}\ell_{2t+1}v))) - u \leftrightarrow v - (\ell_{2r+1}u)(\ell_{2(t-r-1)}(\nabla \cdot (p_{2t+1}\ell_{2t+1}v))) + u \leftrightarrow v] + \\ &+ \nabla \cdot [p_{2t+1}((\ell_{2t}u)(\ell_{2t+1}v) - u \leftrightarrow v)] \end{aligned} \quad (2)$$

[We may in fact put $t=0$ in (2), in which case the summed term is taken as 0.]

Proof: Consider $(\ell_{2r}u)(\ell_{2(t-r)}w)$ for $0 \leq r < t$:

$$\begin{aligned}
 (\ell_{2r}u)(\ell_{2(t-r)}w) &= (\ell_{2r}u)(\nabla \cdot \ell_{2(t-r)-1}w) \\
 &= \dot{\nabla} \cdot ((\ell_{2r}u)(\ell_{2(t-r)-1}w)) - (\nabla \ell_{2r}u) \cdot (\ell_{2(t-r)-1}w) \\
 &= \nabla \cdot ((\ell_{2r}u)(\ell_{2(t-r)-1}w)) - (\ell_{2r+1}u) \cdot (\nabla \ell_{2(t-r)-1}w) \\
 &= \nabla \cdot ((\ell_{2r}u)(\ell_{2(t-r)-1}w)) - \nabla \cdot ((\ell_{2r+1}u)(\ell_{2(t-r)-1}w)) + \\
 &\quad + (\nabla \cdot \ell_{2r+1}u)(\ell_{2(t-r)-1}w) \\
 &= \nabla \cdot ((\ell_{2r}u)(\ell_{2(t-r)-1}w)) - \nabla \cdot ((\ell_{2r+1}u)(\ell_{2(t-r)-1}w)) + \\
 &\quad + (\ell_{2(r+1)}u)(\ell_{2(t-r)-1}w)
 \end{aligned}$$

Summing for $0 \leq r \leq t-1$ and cancelling terms which appear on both sides:

$$u(\ell_{2t}w) = \sum_{r=0}^{t-1} \nabla \cdot [(\ell_{2r}u)(\ell_{2(t-r)-1}w) - (\ell_{2r+1}u)(\ell_{2(t-r)-1}w)] + (\ell_{2t}u)w \quad (3)$$

We put $w = p_{2t} \ell_{2t} v$ in (3) and then subtract $u \leftrightarrow v$. The unsummed term on the right-hand side of (3) is now the same for $u \leftrightarrow v$ and therefore cancels. Thus (1) is established.

We now put $w = \nabla \cdot (p_{2t+1} \ell_{2t+1} v)$ in (3). The unsummed term on the right-hand side of (3) then becomes:

$$\begin{aligned}
 (\ell_{2t}u) \nabla \cdot (p_{2t+1} \ell_{2t+1} v) &= \nabla \cdot ((\ell_{2t}u)(p_{2t+1} \ell_{2t+1} v) - (\nabla \ell_{2t}u) \cdot (p_{2t+1} \ell_{2t+1} v)) \\
 &= \nabla \cdot (p_{2t+1} (\ell_{2t}u)(\ell_{2t+1} v)) - p_{2t+1} (\nabla \ell_{2t}u) \cdot (\ell_{2t+1} v) \quad (4)
 \end{aligned}$$

If we now subtract $u \leftrightarrow v$ from (3) and note that the second term on the right-hand side of (4) cancels, we establish (2). (Note that $\ell_{2t} \nabla \cdot = \ell_{2t+1} \cdot$)

Putting $t=0$ in (3) and taking the summed term as 0, we have the trivial identity: $uw = uw$. Therefore, still with $w = \nabla \cdot (p_{2t+1} \ell_{2t+1} v)$ and now also with $t=0$, it is clear from (4) (with $t=0$) that (2) holds in the case $t=0$ where the summed term is taken as 0. \square

The effect of the application of Green's Theorem in the theorem will be to replace the initial ∇ in Proposition 5.1(1) and (2) by a particular vector-valued function $\underline{n}(x)$ having real-valued components $n_j(x)$, as in Part I. Again, we prepare for the theorem by deriving bounds for the integrals that will arise. These bounds will be either other integrals for which we already have bounds in terms of which we wish to state the theorem (e.g. $\chi(m) = \int |\phi^{m-1} \ell_m(\phi u)|^2$ and see Lemma 4.4(1)) or ones readily convertible to such terms (e.g. $\alpha(t) = c \int |\phi^t \ell_t u|^2$ for $1 \leq t \leq m-1$).

As in Chapter 4, we give these preliminary results as propositions, in which we continue to assume that u (and now also v) $\in \mathcal{D}(T_0^*)$, $\phi \in \mathcal{D}$, with u, v and ϕ real-valued, $\phi \geq 0$, and the first m derivatives of ϕ are bounded by K . We further assume that \underline{n} and the first m derivatives of its components are bounded by K .

The notation introduced in Chapter 2 was adequate when we were considering products of two functions, but as we shall be considering products of three functions it needs slight amendment.

For particular t and r (with $0 \leq t \leq \frac{1}{2}m$, $0 \leq r < t$) define

$$J = J_{2(t-r-1)} = \{j_1, j_1, j_2, j_2, \dots, j_{t-r-1}, j_{t-r-1}\}, \quad J' = J \cup \{j\} \quad \text{and} \quad J'' = J' \cup \{j_0\}.$$

$$\text{Define } S_{+\mu} = \{s_1, s_2, \dots, s_\mu\} \subset J \quad \text{for } 0 \leq \mu \leq 2(t-r-1),$$

$$S_{-\kappa} = \{\sigma_1, \sigma_2, \dots, \sigma_\kappa\} = J/S_{+2(t-r-1)-\kappa} \quad \text{for } 0 \leq \kappa \leq 2(t-r-1),$$

$$S_{/(\mu+\kappa)} = J/(S_{+\mu} \cup S_{-\kappa}) \quad \text{for } 0 \leq \mu + \kappa \leq 2(t-r-1).$$

(i.e. for some ordering of the elements of J , $S_{+\mu}$ contains the first μ elements, $S_{-\kappa}$ contains the last κ elements, and $S_{/(\mu+\kappa)}$ contains the rest.)

With J replaced by J' or J'' , and $2(t-r-1)$ replaced by $2(t-r)-1$ or $2(t-r)$ (i.e. the number of elements in J, J', J''), we obtain definitions of $S'_{+\mu}$, $S'_{-\kappa}$ and $S'_{/(\mu+\kappa)}$ or of $S''_{+\mu}$, $S''_{-\kappa}$ and $S''_{/(\mu+\kappa)}$.

When $S_\lambda = \{s_1, s_2, \dots, s_\lambda\}$ we define $P[S_\lambda]$ as the partial differential operator $\frac{\partial}{s_1} \frac{\partial}{s_2} \dots \frac{\partial}{s_\lambda}$.

With $S_\lambda = S_{//(\mu+\kappa)}$ or $S_{/(\mu+\kappa)}$ or $S_{//(\mu+\kappa)}$ we have $\lambda \leq 2(t-r) \leq 2t \leq m$, and so $|P[S_\lambda]n_j| \leq K$, because the derivatives of n_j are bounded. (†)

Also, because the derivatives of ϕ are bounded, with $0 \leq \mu + \kappa \leq 2(t-r-1)$:

$$\begin{aligned} \phi^{2(t-r-1)-(2m-1)-\mu} |P[S_{-\kappa}] \phi^{2m-1}| &\leq K \phi^{2(t-r-1)-(2m-1)-\mu} \sum_{0 \leq \iota \leq \kappa} \phi^{2m-1-\iota} \\ &\leq K \sum_{0 \leq \iota \leq \kappa} \phi^{2(t-r-1)-(\mu+\iota)} \\ &\leq K(1+\phi)^{2(t-r-1)-\mu} \\ &\leq K(1+\phi^{2(t-r-1)-\mu}) \end{aligned} \quad (\dagger)$$

because $2(t-r-1) - (\mu + \iota) \geq 2(t-r-1) - (\mu + \kappa) \geq 0$.

$$\text{Similarly } \phi^{2(t-r)-1-(2m-1)-\mu} |P[S'_{-\kappa}] \phi^{2m-1}| \leq K(1+\phi^{2(t-r)-1-\mu}) \quad (\dagger')$$

for $0 \leq \mu + \kappa \leq 2(t-r)-1$,

$$\text{and } \phi^{2(t-r)-(2m-1)-\mu} |P[S''_{-\kappa}] \phi^{2m-1}| \leq K(1+\phi^{2(t-r)-\mu}) \quad \text{for } 0 \leq \mu + \kappa \leq 2(t-r). \quad (\dagger'')$$

We shall use these results without further reference in the proposition below.

Proposition 5.2 If $0 < t \leq \frac{1}{2}m$ in (1) and (2) and $0 \leq t < \frac{1}{2}m$ in (3) and (4),

$0 \leq r \leq t-1$, and $0 \leq \rho \leq K$, then

$$\int \left| \sum_{j=1}^n \phi^{2t-2m+\rho} \frac{\partial^{\rho}}{j} \ell_{2(t-r-1)}(\phi^{2m-1} n_j \ell_{2r} u) \right|^2 \leq K[\alpha_\rho(2t) + \alpha'_{2t-1+\rho}(0)] \quad (1)$$

$$\int \left| \sum_{j=1}^n \phi^{2t-2m+\rho} \ell_{2(t-r-1)}(\phi^{2m-1} n_j \frac{\partial^{\rho}}{j} \ell_{2r} u) \right|^2 \leq K[\alpha_\rho(2t) + \alpha'_{2t-1+\rho}(0)] \quad (2)$$

$$\begin{aligned} \int \left| \sum_{j_0=1}^n \sum_{j=1}^n \phi^{2t+1-2m+\rho} \frac{\partial^{\rho}}{j_0} \frac{\partial^{\rho}}{j} \ell_{2(t-r-1)}(\phi^{2m-1} n_j \ell_{2r} u) \right|^2 &\leq \\ &\leq K[\alpha_\rho(2t+1) + \alpha'_{2t+\rho}(0)] \end{aligned} \quad (3)$$

$$\begin{aligned} \int \left| \sum_{j_0=1}^n \sum_{j=1}^n \phi^{2t+1-2m+\rho} \frac{\partial^{\rho}}{j} \ell_{2(t-r-1)}(\phi^{2m-1} n_{j_0} \frac{\partial^{\rho}}{j_0} \ell_{2r} u) \right|^2 &\leq \\ &\leq K[\alpha_\rho(2t+1) + \alpha'_{2t+\rho}(0)] \end{aligned} \quad (4)$$

Proof: (Compare with the proof of Prop.4.2.4 and Prop.4.0.3.)

We shall prove the results for $\rho=0$.

In the proof $\sum_{\mathbf{j}}$ denotes the sum over any $j_i \in J'$ from 1 to n in the formula. Similarly for $\sum_{\mathbf{j}''}$. $\sum_{\mathbf{M}}$ denotes the sum over μ in the given range. $\sum_{\mathbf{C}}$ denotes the sum over combinations as before. $\sum_{\mathbf{\Lambda}}$ denotes the sum over λ in the given range.

The left-hand side of (1) is equal to

$$\begin{aligned}
 & \int \left| \sum_{\mathbf{j}}, \sum_{\mathbf{M}} \sum_{\mathbf{C}} \phi^{2t-2m} (P[S'_{+\mu}] e_{2r} u) (P[S'_{/(\mu+\kappa)}] n_j) (P[S'_{-\kappa}] \phi^{2m-1}) \right|^2 \leq \\
 & \leq K \int \left| \sum_{\mathbf{j}}, \sum_{\mathbf{M}} \sum_{\mathbf{C}} \phi^{2r+\mu} (P[S'_{+\mu}] e_{2r} u) \phi^{2(t-r)-1-(2m-1)-\mu} (P[S'_{-\kappa}] \phi^{2m-1}) \right|^2 \\
 & \leq K \int \left| \sum_{\mathbf{j}}, \sum_{\mathbf{M}} \sum_{\mathbf{C}} (1 + \phi^{2(t-r)-1-\mu}) \phi^{2r+\mu} (P[S'_{+\mu}] e_{2r} u) \right|^2 \\
 & \leq K \sum_{\mathbf{j}}, \sum_{\mathbf{M}} \sum_{\mathbf{\Lambda}} \int \left| (1 + \phi^{2(t-r)-1-\mu}) \phi^{2r+\mu} \partial_{s_1} \dots \partial_{s_\lambda} e_{\mu-\lambda} e_{2r} u \right|^2 \\
 & \quad (\text{where } 0 \leq \mu \leq 2(t-r)-1, \lambda \equiv \mu \pmod{2} \text{ and } 0 \leq \lambda \leq \min(\mu, 2(t-r)-\mu)) \\
 & \leq K \sum_{\mathbf{M}} \sum_{\mathbf{\Lambda}} \{ \beta_{2(t-r)-1-\mu}(\lambda, r+\frac{1}{2}(\mu-\lambda)) + \beta(\lambda, r+\frac{1}{2}(\mu-\lambda)) \} \quad (\text{similar to Prop.4.2.4}) \\
 & \quad (\text{we may ignore the } \beta \text{ term as in Prop.4.2.4 because} \\
 & \quad (1 + \phi)^{\mu-2(t-r)} \leq (1 + \phi)^{-1} \text{ for the given range of } \mu) \\
 & \leq K \sum_{\mathbf{M}} \{ \beta_{2(t-r)-1-\mu}(0, r+\frac{1}{2}\mu) + \beta_{2(t-r)-1-\mu}(1, r+\frac{1}{2}(\mu-1)) + \\
 & \quad + \sum_{\lambda \geq 2} \beta_{2(t-r)-1-\mu}(\lambda, r+\frac{1}{2}(\mu-\lambda)) \} \\
 & \leq K \sum_{\mathbf{M}} \{ 2\alpha_{2(t-r)-1-\mu}(2r+\mu) + \sum_{\lambda \geq 2} [\varepsilon\alpha(2t) + K(\varepsilon)\alpha'_{2t-1}(0)] \} \\
 & \quad (\text{by Prop.4.0.2'(2) with } N=2t, \text{ since } 2r+\mu+\lambda-1 \leq 2r+\min(2\mu, 2(t-r))-1 \leq \\
 & \quad \leq 2t-1; \text{ so the sum is independent of } \lambda; \text{ also } 2r+\mu < 2t \text{ so we may} \\
 & \quad \text{apply Prop.4.0.2'(1) with } N=2t) \\
 & \leq K \sum_{\mathbf{M}} \{ \varepsilon\alpha(2t) + K(\varepsilon)\alpha'_{2t-1}(0) \}
 \end{aligned}$$

which implies (1) by choosing $\varepsilon=1$, since the sum is now independent of μ .

The left-hand side of (2) is equal to

$$\begin{aligned}
& \int | \sum_{\mathcal{J}} \sum_{\mathcal{M}} \sum_{\mathcal{C}} \phi^{2t-2m} (P[S_{+\mu}] \partial_j^\ell \ell_{2r} u) (P[S_{/(\mu+\kappa)}] n_j) (P[S_{-\kappa}] \phi^{2m-1}) |^2 \leq \\
& \leq K \int | \sum_{\mathcal{J}} \sum_{\mathcal{M}} \sum_{\mathcal{C}} \phi^{2r+1+\mu} (P[S_{+\mu}] \partial_j^\ell \ell_{2r} u) \phi^{2(t-r-1)-(2m-1)-\mu} (P[S_{-\kappa}] \phi^{2m-1}) |^2 \\
& \leq K \int | \sum_{\mathcal{J}} \sum_{\mathcal{M}} \sum_{\mathcal{C}} (1 + \phi^{2(t-r-1)-\mu}) \phi^{2r+1+\mu} (P[S_{+\mu}] \partial_j^\ell \ell_{2r} u) |^2 \\
& \leq K \sum_{\mathcal{J}} \sum_{\mathcal{M}} \sum_{\mathcal{A}} \int | (1 + \phi^{2(t-r-1)-\mu}) \phi^{2r+1+\mu} \partial_{s_1} \dots \partial_{s_\lambda} \ell_{\mu-\lambda} \ell_{2r} u |^2 \\
& \quad (\text{where } 0 \leq \mu \leq 2(t-r-1), \lambda \equiv \mu \pmod{2} \text{ and } 0 \leq \lambda \leq \min(\mu, 2(t-r-1)-\mu)) \\
& \leq K \sum_{\mathcal{M}} \sum_{\mathcal{A}} \{ \beta_{2(t-r-1)-\mu}(\lambda+1, r+\frac{1}{2}(\mu-\lambda)) + \beta(\lambda+1, r+\frac{1}{2}(\mu-\lambda)) \} \\
& \quad (\text{and ignoring the } \beta \text{ term as above because } (1 + \phi)^{\mu-2(t-r)+1} \leq (1 + \phi)^{-1}) \\
& \leq K \sum_{\mathcal{M}} \{ \beta_{2(t-r-1)-\mu}(1, r+\frac{1}{2}\mu) + \sum_{\lambda \geq 1} \beta_{2(t-r-1)-\mu}(\lambda+1, r+\frac{1}{2}(\mu-\lambda)) \} \\
& = K \sum_{\mathcal{M}} \{ \alpha_{2(t-r-1)-\mu}(2r+\mu+1) + \sum_{\lambda \geq 1} \beta_{2(t-r-1)-\mu}(\lambda+1, r+\frac{1}{2}(\mu-\lambda)) \} \\
& \quad (\text{with } N=2t, \text{ we again apply Prop.4.0.2'(1) as } 2r+\mu+1 < 2t, \text{ and} \\
& \quad \text{Prop.4.0.2'(2) since } 2r+\mu+\lambda+2-1 \leq 2r+\min(2\mu, 2(t-r-1))+1 \leq 2t-1) \\
& \leq K \sum_{\mathcal{M}} \{ \epsilon \alpha(2t) + K(\epsilon) \alpha'_{2t-1}(0) \} \quad (\text{as above})
\end{aligned}$$

which implies (2) by choosing $\epsilon=1$, as above.

The left-hand side of (3) is equal to

$$\begin{aligned}
& \int | \sum_{\mathcal{J}''} \sum_{\mathcal{M}} \sum_{\mathcal{C}} \phi^{2t+1-2m} (P[S_{+\mu}''] \ell_{2r} u) (P[S_{/(\mu+\kappa)}''] n_j) (P[S_{-\kappa}''] \phi^{2m-1}) |^2 \leq \\
& \leq K \int | \sum_{\mathcal{J}''} \sum_{\mathcal{M}} \sum_{\mathcal{C}} \phi^{2r+\mu} (P[S_{+\mu}''] \ell_{2r} u) \phi^{2(t-r)-(2m-1)-\mu} (P[S_{-\kappa}''] \phi^{2m-1}) |^2 \\
& \leq K \int | \sum_{\mathcal{J}''} \sum_{\mathcal{M}} \sum_{\mathcal{C}} (1 + \phi^{2(t-r)-\mu}) \phi^{2r+\mu} (P[S_{+\mu}''] \ell_{2r} u) |^2 \\
& \leq K \sum_{\mathcal{J}''} \sum_{\mathcal{M}} \sum_{\mathcal{A}} \int | (1 + \phi^{2(t-r)-\mu}) \phi^{2r+\mu} \partial_{s_1} \dots \partial_{s_\lambda} \ell_{\mu-\lambda} \ell_{2r} u |^2 \\
& \quad (\text{where } 0 \leq \mu \leq 2(t-r), \lambda \equiv \mu \pmod{2} \text{ and } 0 \leq \lambda \leq \min(\mu, 2(t-r+1)-\mu)) \\
& \leq K \sum_{\mathcal{M}} \sum_{\mathcal{A}} \{ \beta_{2(t-r)-\mu}(\lambda, r+\frac{1}{2}(\mu-\lambda)) + \beta(\lambda, r+\frac{1}{2}(\mu-\lambda)) \} \\
& \quad (\text{we ignore the } \beta \text{ term as above because } (1 + \phi)^{\mu-2(t-r)-1} \leq (1 + \phi)^{-1})
\end{aligned}$$

As when dealing with (1), we split the sum into $\lambda=0$, $\lambda=1$, $\lambda \geq 2$, and now with $N=2t+1$ we apply Prop.4.0.2'(1) since here $2r+\mu < 2t+1$, and we apply Prop.4.0.2'(2) since here $2r+\mu+\lambda-1 \leq 2r+\min(2\mu, 2(t-r+1))-1 \leq 2t+1$.

Thus we obtain

$$K \sum_M \sum_\Lambda \{\beta_{2(t-r)-\mu}(\lambda, r+\frac{1}{2}(\mu-\lambda))\} \leq K \sum_M \{\varepsilon\alpha(2t+1) + K(\varepsilon)\alpha'_{2t}(0)\}$$

which implies (3) by choosing $\varepsilon=1$, as above.

The left-hand side of (4) is equal to

$$\begin{aligned} & \int \left| \sum_{J''} \sum_M \sum_C \phi^{2t+1-2m} (P[S'_{+\mu}]_{j_0} e_{2r} u) (P[S'_{/(\mu+\kappa)}]_{j_0}) (P[S'_{-\kappa}] \phi^{2m-1}) \right|^2 \leq \\ & \leq K \int \left| \sum_{J''} \sum_M \sum_C \phi^{2r+1+\mu} (P[S'_{+\mu}]_{j_0} e_{2r} u) \phi^{2(t-r)-1-(2m-1)-\mu} (P[S'_{-\kappa}] \phi^{2m-1}) \right|^2 \\ & \leq K \int \left| \sum_{J''} \sum_M \sum_C (1 + \phi^{2(t-r)-1-\mu}) \phi^{2r+1+\mu} (P[S'_{+\mu}]_{j_0} e_{2r} u) \right|^2 \\ & \leq K \sum_{J''} \sum_M \sum_\Lambda \int \left| (1 + \phi^{2(t-r)-1-\mu}) \phi^{2r+1+\mu} \partial_{s_1} \dots \partial_{s_\lambda} e_{\mu-\lambda} e_{2r} u \right|^2 \end{aligned}$$

(where $0 \leq \mu \leq 2(t-r)-1$, $\lambda \equiv \mu \pmod{2}$ and $0 \leq \lambda \leq \min(\mu, 2(t-r)-\mu)$)

$$\leq K \sum_M \sum_\Lambda \{\beta_{2(t-r)-1-\mu}(\lambda+1, r+\frac{1}{2}(\mu-\lambda)) + \beta(\lambda+1, r+\frac{1}{2}(\mu-\lambda))\}$$

(we ignore the β term as above because $(1+\phi)^{\mu-(t-r)} \leq (1+\phi)^{-1}$)

As when dealing with (2), we split the sum into $\lambda=0$, $\lambda \geq 1$ (i.e. $\lambda+1=1$, $\lambda+1 \geq 2$), and now with $N=2t+1$ we apply Prop.4.0.2'(1) since here $2r+\mu+1 < 2t+1$, and we apply Prop.4.0.2'(2) since here $2r+\mu+\lambda+2-1 \leq 2r+\min(2\mu, 2(t-r))+1 \leq 2t+1$. Thus we obtain

$$K \sum_M \sum_\Lambda \{\beta_{2(t-r)-1-\mu}(\lambda+1, r+\frac{1}{2}(\mu-\lambda))\} \leq K \sum_M \{\varepsilon\alpha(2t+1) + K(\varepsilon)\alpha'_{2t}(0)\}$$

which implies (4) by choosing $\varepsilon=1$, as above.

Thus the inequalities (1)-(4) are all established. \square

As with Prop.4.2.4(1), we may replace $\alpha'_{2t-1+\rho}(0)$ and $\alpha'_{2t+\rho}(0)$ by $\alpha''_{2t-1,\rho}(0)$ and $\alpha''_{2t,\rho}(0)$ respectively in Prop.5.2.

Proposition 5.2"

Prop.5.2(1) and (2) hold when we replace $\alpha'_{2t-1+\rho}(0)$ by $\alpha''_{2t-1,\rho}(0)$ (1,2)

Prop.5.2(3) and (4) hold when we replace $\alpha'_{2t+\rho}(0)$ by $\alpha''_{2t,\rho}(0)$ (3,4)

Proof: See the note preceding Prop.4.2.4". \square

To make the forthcoming proofs less cumbersome, we define

$$A'(t,r) = \int_{\underline{n}} \phi^{2m-1} [(\ell_{2r} u)(\ell_{2(t-r)-1}(p_{2t} \ell_{2t} v))] \quad \text{for } 0 < t \leq \frac{1}{2}m$$

$$A''(t,r) = \int_{\underline{n}} \phi^{2m-1} [(\ell_{2r+1} u)(\ell_{2(t-r-1)}(p_{2t} \ell_{2t} v))] \quad \text{and } 0 \leq r < t$$

$$B'(t,r) = \int_{\underline{n}} \phi^{2m-1} [(\ell_{2r} u)(\ell_{2(t-r)-1}(\nabla \cdot (p_{2t+1} \ell_{2t+1} v)))] \quad \text{for } 0 < t < \frac{1}{2}m$$

$$B''(t,r) = \int_{\underline{n}} \phi^{2m-1} [(\ell_{2r+1} u)(\ell_{2(t-r-1)}(\nabla \cdot (p_{2t+1} \ell_{2t+1} v)))] \quad \text{and } 0 \leq r < t$$

$$\text{and } C(t) = \int_{\underline{n}} \phi^{2m-1} [p_{2t+1}(\ell_{2t} u)(\ell_{2t+1} v)] \quad \text{for } 0 \leq t < \frac{1}{2}m.$$

The integrals involved here clearly have a very close connection with the terms in the identities given in Proposition 5.1.

We also put $c_m = 1$ and $c_t = c$ ($1 \leq t < m$), where $|p_t \phi^{2(m-t)}| \leq c_t$ with $c_t \geq 1$.

Proposition 5.3 If $A = A'$ or A'' , and $B = B'$ or B'' , then

$$|A(t,r)| \leq Kc_{2t} [\alpha(2t,u) + \alpha'_{2t-1}(0,u) + \alpha(2t,v)] \quad (1'), (1'')$$

$$|B(t,r)| \leq Kc_{2t+1} [\alpha(2t+1,u) + \alpha'_{2t}(0,u) + \alpha(2t+1,v)] \quad (2'), (2'')$$

$$\text{and } |C(t)| \leq Kc_{2t+1} [\alpha(2t,u) + \alpha(2t+1,v)] \quad (3)$$

Proof:

$$\begin{aligned} |A'(t,r)| &= \left| \int_{\underline{n}} \phi^{2m-1} (\ell_{2r} u)(\ell_{2(t-r)-1}(p_{2t} \ell_{2t} v)) \right| \\ &= \left| \int [\ell_{2(t-r)-1} \cdot (\phi^{2m-1} \underline{n} \ell_{2r} u)] [p_{2t} \ell_{2t} v] \right| \\ &= \left| \int \left[\sum_{j=1}^n \phi^{2(t-m)} \partial_j \ell_{2(t-r)-1} (\phi^{2m-1} n_j \ell_{2r} u) \right] [p_{2t} \phi^{2(m-2t)}] [\phi^{2t} \ell_{2t} v] \right| \\ &\leq \int \left| \sum_{j=1}^n \phi^{2(t-m)} \partial_j \ell_{2(t-r)-1} (\phi^{2m-1} n_j \ell_{2r} u) \right| | p_{2t} \phi^{2(m-2t)} | | \phi^{2t} \ell_{2t} v | \\ &\leq c_{2t} \int \left| \sum_{j=1}^n \phi^{2(t-m)} \partial_j \ell_{2(t-r)-1} (\phi^{2m-1} n_j \ell_{2r} u) \right| | \phi^{2t} \ell_{2t} v | \end{aligned}$$

$$\begin{aligned}
(|A'(t,r)|) &\leq Kc_{2t} [\int \left| \sum_{j=1}^n \phi^{2(t-m)} \partial_j \ell_{2(t-r-1)} (\phi^{2m-1} n_j \ell_{2r} u) \right|^2 + \int |\phi^{2t} \ell_{2t} v|^2] \\
&\leq Kc_{2t} [\alpha(2t,u) + \alpha'_{2t-1}(0,u) + \alpha(2t,v)] \quad (\text{by Prop.5.2(1) with } \rho=0)
\end{aligned}$$

which establishes (1').

$$\begin{aligned}
|A''(t,r)| &= \left| \int \underline{n} \cdot \phi^{2m-1} (\ell_{2r+1} u) (\ell_{2(t-r-1)} (p_{2t} \ell_{2t} v)) \right| \\
&= \left| \int \left[\sum_{j=1}^n \ell_{2(t-r-1)} (\phi^{2m-1} n_j \partial_j \ell_{2r} u) \right] [p_{2t} \ell_{2t} v] \right| \\
&\leq \int \left| \sum_{j=1}^n \phi^{2(t-m)} \ell_{2(t-r-1)} (\phi^{2m-1} n_j \partial_j \ell_{2r} u) \right| |p_{2t} \phi^{2(m-2t)}| |\phi^{2t} \ell_{2t} v| \\
&\leq Kc_{2t} [\int \left| \sum_{j=1}^n \phi^{2(t-m)} \ell_{2(t-r-1)} (\phi^{2m-1} n_j \partial_j \ell_{2r} u) \right|^2 + \int |\phi^{2t} \ell_{2t} v|^2] \\
&\leq Kc_{2t} [\alpha(2t,u) + \alpha'_{2t-1}(0,u) + \alpha(2t,v)] \quad (\text{by Prop.5.2(2) with } \rho=0)
\end{aligned}$$

which establishes (1'').

$$\begin{aligned}
|B'(t,r)| &= \left| \int \underline{n} \cdot \phi^{2m-1} (\ell_{2r} u) (\ell_{2(t-r)-1} (\nabla \cdot (p_{2t+1} \ell_{2t+1} v))) \right| \\
&= \left| \int \sum_{j_0=1}^n \left[\sum_{j=1}^n \partial_{j_0} \partial_j \ell_{2(t-r-1)} (\phi^{2m-1} n_j \ell_{2r} u) \right] [p_{2t+1} \partial_{j_0} \ell_{2t} v] \right| \\
&\leq \int \left| \sum_{j_0=1}^n \sum_{j=1}^n \phi^{2t+1-2m} \partial_{j_0} \partial_j \ell_{2(t-r-1)} (\phi^{2m-1} n_j \ell_{2r} u) \right| |p_{2t+1} \phi^{2(m-2t-1)}| \times \\
&\quad \times |\phi^{2t+1} \ell_{2t+1} v| \\
&\leq Kc_{2t+1} [\int \sum_{j_0=1}^n \sum_{j=1}^n \phi^{2t+1-2m} \partial_{j_0} \partial_j \ell_{2(t-r-1)} (\phi^{2m-1} n_j \ell_{2r} u)]^2 + \\
&\quad + \int |\phi^{2t+1} \ell_{2t+1} v|^2] \\
&\leq Kc_{2t+1} [\alpha(2t+1,u) + \alpha'_{2t}(0,u) + \alpha(2t+1,v)] \quad (\text{by Prop.5.2(3) with } \rho=0)
\end{aligned}$$

which establishes (2').

$$\begin{aligned}
|B''(t,r)| &= \left| \int \underline{n} \cdot \phi^{2m-1} (\ell_{2r+1} u) (\ell_{2(t-r-1)} (\nabla \cdot (p_{2t+1} \ell_{2t+1} v))) \right| \\
&= \left| \int \sum_{j=1}^n \left[\sum_{j_0=1}^n \partial_j \ell_{2(t-r-1)} (\phi^{2m-1} n_{j_0} \partial_{j_0} \ell_{2r} u) \right] [p_{2t+1} \partial_j \ell_{2t} v] \right| \\
&\leq \int \left| \sum_{j=1}^n \sum_{j_0=1}^n \phi^{2t+1-2m} \partial_j \ell_{2(t-r-1)} (\phi^{2m-1} n_{j_0} \partial_{j_0} \ell_{2r} u) \right| |p_{2t+1} \phi^{2(m-2t-1)}| \times \\
&\quad \times |\phi^{2t+1} \ell_{2t+1} v|
\end{aligned}$$

$$\begin{aligned}
(|B''(t,r)|) &\leq Kc_{2t+1} \left[\int_{j=1}^n \int_{j_0=1}^n \phi^{2t+1-2m} \partial_j^2 \ell_{2(t-r-1)} (\phi^{2m-1} n_{j_0} \partial_{j_0} \ell_{2r} u) \right]^2 + \\
&\quad + \int |\phi^{2t+1} \ell_{2t+1} v|^2 \\
&\leq Kc_{2t+1} [\alpha(2t+1,u) + \alpha_{2t}^i(0,u) + \alpha(2t+1,v)] \quad (\text{by Prop.5.2(4) with } \rho=0)
\end{aligned}$$

which establishes (2").

$$\begin{aligned}
|C(t)| &= \left| \int \underline{n} \cdot \phi^{2m-1} p_{2t+1}(\ell_{2t} u)(\ell_{2t+1} v) \right| \\
&= \left| \int \underline{n} \cdot [p_{2t+1} \phi^{2(m-2t-1)}] [\phi^{2t} \ell_{2t} u] [\phi^{2t+1} \ell_{2t+1} v] \right| \\
&\leq Kc_{2t+1} \left[\int |\phi^{2t} \ell_{2t} u|^2 + \int |\phi^{2t+1} \ell_{2t+1} v|^2 \right] \\
&\leq Kc_{2t+1} [\alpha(2t,u) + \alpha(2t+1,v)]
\end{aligned}$$

which establishes (3). \square

We shall now introduce the 'bands' in which conditions on the coefficient functions p_t and q will be more precise.

In fact, the sequence $\{A_\nu\}$ described in Chapter I.5 will suffice, with slight modification. For each ν , we now require Ω_ν to be such that $h_\nu \in C^m$ (instead of $h_\nu \in C^2$) if $h_\nu < \delta_\nu$ for some $\delta_\nu > 0$, where $h_\nu(x) = \text{dist}(x, \Omega_\nu)$. We again define $\underline{n}(x)$ to be the outward normal to the surface S_k , so we have $|\underline{n}| = 1$. We now suppose that the sequence $\{A_\nu\}$ is such that the first m derivatives of the components of \underline{n} are bounded by K (instead of $|\nabla \cdot \underline{n}| \leq K$). (We might have expected to require that only the first $m-1$ derivatives be bounded. This is insufficient only for the establishing of Prop.5.2(3) and hence of Prop.5.3(2'), i.e. the bound on B' . The corresponding term in the case $m=2$ is 0, and hence the requirement is weaker in that case. That the corresponding term is 0 becomes apparent when we see that the need for a bound on B' comes from the occurrence of the first summed term in Prop.5.1(2); when $m=2$, the identity corresponding to Prop.5.1(2) is obtained by putting $t=0$ there, in which case the summed term is taken as 0.)

Theorem 5.4 Suppose that conditions $\mathcal{C}(i)$ - $\mathcal{C}(v)$ are satisfied, and that we have a sequence $\{A_\nu\}$ as described above. Let $\theta \in C_0^{m,+}(A_\nu)$ with the first m derivatives of θ bounded by K . If u and $v \in \mathcal{D}(T_0^*)$ with u and v real-valued and $\int_{\Omega_\nu} (uT_0^*v - vT_0^*u) \rightarrow L \neq 0$ (we may assume that $L > 0$), and if

$$(a) \quad |p_t \theta^{2(m-t)}| \leq c \text{ for } 1 \leq t \leq m-1 \text{ with } c \geq 1, \text{ and}$$

$$(b) \quad q\theta^{2m} \geq -q(h_\nu),$$

then, for ν sufficiently large,

$$L \int_0^d \nu \theta^{2m-1}(h_\nu) dh_\nu \leq K \int_{A_\nu} [\theta^{2m} + c^m + q][u^2 + v^2 + (Tu)^2 + (Tv)^2] dx \quad (1)$$

Proof: As in Theorem I.5.4 we may assume that u and $v \in \mathcal{D}$.

With $p_m = 1$ and $p_0 = q$, we have

$$\begin{aligned} \int_{V_k} (uT_0^*v - vT_0^*u) dx &= \int_{V_k} \sum_{t=0}^m (-1)^t (u \ell_t \cdot (p_t \ell_t v) - v \ell_t \cdot (p_t \ell_t u)) \\ &= \int_{V_k} \left[\sum_{0 \leq t \leq \frac{1}{2}m} \{u \ell_{2t} (p_{2t} \ell_{2t} v) - u \leftrightarrow v\} - \right. \\ &\quad \left. - \sum_{0 \leq t < \frac{1}{2}m} \{u \ell_{2t+1} \cdot (p_{2t+1} \ell_{2t+1} v) - u \leftrightarrow v\} \right] \\ &= \int_{V_k} \nabla \cdot \left\{ \sum_{0 \leq t \leq \frac{1}{2}m} \sum_{r=0}^{t-1} \{(\ell_{2r} u)(\ell_{2(t-r)-1} (p_{2t} \ell_{2t} v)) - \right. \\ &\quad \left. - (\ell_{2r+1} u)(\ell_{2(t-r-1)} (p_{2t} \ell_{2t} v))\} - \right. \\ &\quad \left. - \sum_{0 \leq t < \frac{1}{2}m} \sum_{r=0}^{t-1} \{(\ell_{2r} u)(\ell_{2(t-r)-1} (\nabla \cdot (p_{2t+1} \ell_{2t+1} v))) - \right. \\ &\quad \left. - (\ell_{2r+1} u)(\ell_{2(t-r-1)} (\nabla \cdot (p_{2t+1} \ell_{2t+1} v)))\} - \right. \\ &\quad \left. - \sum_{0 \leq t < \frac{1}{2}m} \{p_{2t+1} (\ell_{2t} u)(\ell_{2t+1} v)\} \right\} - u \leftrightarrow v dx \\ &\quad \text{(by Prop.5.1(1) and (2))} \\ &= \int_{S_k} \underline{n} \cdot \{ \dots \} dS \quad (2) \end{aligned}$$

(i.e. \underline{n} replaces the initial $\nabla \cdot$ in the formula) by an application of Green's Theorem.

We shall use Prop.4.0.2, Lemma 4.1, Lemma 4.4 and Prop.5.3, all of which though stated and proved for $\phi \in \mathcal{D}$ still hold for $\theta \in C_0^{m,+}(A_\nu)$.

Now for ν sufficiently large:

$$\frac{1}{2}L \leq \int_{V_k} (u^T \nu - \nu^T u) dx, \quad \text{where } V_k \subset \Omega_\nu.$$

Therefore $\frac{1}{2}L \theta^{2m-1}(h_\nu) \leq \theta^{2m-1} \int_{S_k} \underline{n} \cdot \{ \dots \} dS$, by (2).

$$\begin{aligned} \text{Therefore } \frac{1}{2}L \int_0^d \nu \theta^{2m-1}(h_\nu) dh_\nu &\leq \int_{A_\nu} \theta^{2m-1} \underline{n} \cdot \{ \dots \} dx \\ &= \left\{ \sum_{0 < t \leq \frac{1}{2}m} \sum_{r=0}^{t-1} [A'(t, r; \theta) - A''(t, r; \theta)] - \right. \\ &\quad \left. - \sum_{0 < t < \frac{1}{2}m} \sum_{r=0}^{t-1} [B'(t, r; \theta) - B''(t, r; \theta)] - \right. \\ &\quad \left. - \sum_{0 \leq t < \frac{1}{2}m} C(t; \theta) \right\} + u \leftrightarrow v \quad (3) \\ &\leq \left\{ \sum_{0 < t < \frac{1}{2}m} \sum_{r=0}^{t-1} Kc [\alpha(2t, u) + \alpha'_{2t-1}(0, u)] + \right. \\ &\quad \left. + \sum_{0 \leq t < \frac{1}{2}(m-1)} \sum_{r=0}^{t-1} Kc [\alpha(2t+1, u) + \alpha'_{2t}(0, u)] + \right. \\ &\quad \left. + K\alpha(m, u) \right\} + u \leftrightarrow v \quad (\text{by Prop. 5.3}) \\ &\leq \left\{ K\alpha(m, u) + Kc \sum_{0 < t < m} \alpha(t, u) + Kc \sum_{0 < t < m} \alpha'_{t-1}(0, u) \right\} + u \leftrightarrow v \\ &\quad (\text{because the sum is independent of } r) \\ &\leq \left\{ K\alpha(m, u) + Kc [\varepsilon \alpha(m, u) + K(1 + \varepsilon^{-1})^{m-1} \alpha(0, u)] + \right. \\ &\quad \left. + Kc \alpha'_{m-2}(0, u) \right\} + u \leftrightarrow v \\ &\quad (\text{by Prop. 4.0.2(1) with } C=0, \rho=0, N=m, \\ &\quad \text{and as } \alpha'_{t-1}(0) \leq \alpha'_{m-2}(0) \text{ for } 1 \leq t \leq m-1, \\ &\quad \text{the sum now being independent of } t) \\ &\leq \left\{ K\alpha(m, u) + Kc^m \alpha(0, u) + Kc^{m-1} \alpha(0, u) + K\alpha'_{m-1}(0, u) \right\} + u \leftrightarrow v \\ &\quad (\text{by choosing } \varepsilon = c^{-1}, \text{ and because, as in} \\ &\quad \text{Lemma 4.3, } c\alpha'_{m-2}(0) \leq K[c^{m-1} \alpha(0) + \alpha'_{m-1}(0)]) \\ &\leq \left\{ K\chi(m, u) + K\alpha'_{m-1}(0, u) + Kc^m \alpha(0, u) \right\} + u \leftrightarrow v \\ &\quad (\text{by Lemma 4.1(1) with } t=m, \rho=0, \text{ and as } c \geq 1) \end{aligned}$$

$$\begin{aligned}
\left(\frac{1}{2}L\int_0^{d_\nu} \theta^{2m-1}(h_\nu)dh_\nu\right) &\leq \{Kf\theta^{2m}(Tu)^2 + Kf[\theta^{2m} + c^m - q\theta^{2m}]u^2 + \\
&\quad + Kf[(1+\theta)^{2(m-1)} + c^m]u^2\} + u \leftrightarrow v \\
&\quad \text{(by Lemma 4.4(1))} \\
&\leq \{Kf\theta^{2m}(Tu)^2 + Kf[\theta^{2m} + c^m + Q]u^2\} + u \leftrightarrow v \\
&\quad \text{(because } (1+\theta)^{2(m-1)} \leq K(1+\theta^{2m}) \text{ and } c \geq 1, \\
&\quad \text{and } -q\theta^{2m} \leq Q)
\end{aligned}$$

which implies (1). \square

The remarks following Theorem I.5.4 apply aqally well here, i.e. with $\mathcal{E}(i)$ $p_t \in C^t$, for $1 \leq t \leq m-1$, and $\mathcal{E}(ii)$ $q \in L^2_{loc}$, the other conditions need hold only in A_ν and $\mathcal{D}(T^*)$ is as large as possible.

Also the requirement on the derivatives of \underline{u} is unnecessary in the 1-dimensional case, and is easily satisfied in the case where the Ω_ν are n -dimensional spheres centred at the origin, as in Part I.

The corollary following Theorem I.5.4 needs only slight amendment to give the corollary here:

Corollary 5.4.1 Suppose that conditions $\mathcal{E}(i)$ - $\mathcal{E}(v)$ are satisfied

and that we have a sequence $\{A_\nu\}$ as in Theorem 5.4, with

$$(a) \liminf_{\nu \rightarrow \infty} d_\nu > 0.$$

Let $B_\nu = \{x: \delta \leq h_\nu(x) \leq d_\nu - \delta\}$ for some $\delta > 0$ with $d_\nu - 2\delta > 0$,

for all ν . For each ν , assume that we have a function

$\theta_\nu \in C_0^{m,+}(A_\nu^+)$ (i.e. we allow $\text{supp } \theta_\nu = A_\nu$), with

$$(b) |\theta_\nu^{(t)}| \leq K, \text{ for } 0 \leq t \leq m;$$

$$(c) |p_t \theta_\nu^{2(m-t)}| \leq K, \text{ for } 1 \leq t \leq m-1;$$

$$(d) q\theta_\nu^{2m} \geq -K;$$

where K is independent of ν .

If (e) $\sum \int_{\delta}^{d_{\nu}-\delta} \theta_{\nu}^{2m-1}(r) dr$ is a divergent infinite series, then

$L=0$ and so T_0 is e.s.a.

Proof: (a) implies the existence of the bands B_{ν} .

Let $\psi_{\nu} \in C_0^{m,+}(A_{\nu})$ with $0 \leq \psi_{\nu} \leq 1$, $\psi_{\nu}(r) = 1$ if $\delta \leq r \leq d_{\nu} - \delta$, and the first m derivatives of ψ_{ν} bounded, the bound depending only on δ .

Let $\theta_{\nu} = \psi_{\nu} \Theta_{\nu}$. Then $\theta_{\nu} \in C_0^{m,+}(A_{\nu})$.

(b) implies that $|\theta_{\nu}^{(t)}| \leq K$, for $0 \leq t \leq m$; in particular, $\theta_{\nu} \leq K$.

(c) implies that $|p_t \theta_{\nu}^{2(m-t)}| \leq K$, for $1 \leq t \leq m-1$, so we take $c=K$ in

Theorem 5.4.

(d) implies that $q\theta_{\nu}^{2m} \geq -K$, so we take $Q(h_{\nu})=K$ in Theorem 5.4.

Assume that $L > 0$. Then $L \int_{\delta}^{d_{\nu}-\delta} \theta_{\nu}^{2m-1}(r) dr \leq L \int_0^{d_{\nu}} \theta_{\nu}^{2m-1}(r) dr$,

and so, by Theorem 5.4(1):

$$L \int_{\delta}^{d_{\nu}-\delta} \theta_{\nu}^{2m-1}(r) dr \leq K \int_{A_{\nu}} [u^2 + v^2 + (Tu)^2 + (Tv)^2] dx.$$

This is valid if ν is sufficiently large; say for all $\nu \geq N$. Summing over $\nu \geq N$ gives a finite right-hand side, which contradicts (e).

Therefore $L=0$ and so T_0 is e.s.a. \square

Putting $m=2$ gives Corollary I.5.4.1 and the example we give below gives, with $m=2$, the example illustrating Corollary I.5.4.1.

Example: In Corol.5.4.1, take $A_{\nu} = \{x: 2\nu\pi \leq |x| \leq (2\nu+1)\pi\}$.

Then $d_{\nu} = \pi$, so take $\delta = \pi/4$, and (a) is satisfied.

Take $\Theta_{\nu}(h_{\nu}(x)) = (2\nu\pi + h_{\nu}(x))^{-1/(2m-1)} = |x|^{-1/(2m-1)}$. Then (b) is satisfied.

In A_{ν} , take $|p_t| \leq K|x|^{2(m-t)/(2m-1)}$, with p_t also satisfying $\mathcal{E}(i)$ and $\mathcal{E}(iv)$, for $1 \leq t \leq m-1$. Then (c) is satisfied.

Take $q = K|x|^{\gamma} \sin|x|$ or $q = K \min(-|x|^{2m/(2m-1)}, |x|^{\gamma} \sin|x|)$ for any γ with q

satisfying $e(ii)-e(v)$. Then, since $q \geq -K|x|^{2m/(2m-1)}$ in A_ν , (d) is satisfied.

Since $\int_{\pi/4}^{3\pi/4} (2\nu\pi+r)^{-1} dr = \log[(8\nu+3)/(8\nu+1)]$, (e) is satisfied, and

so T_0 is e.s.a. \square

As in Chapter 4 we now assume further that $p_t \geq 0$, which allows modification of Theorem 5.4. We first modify Proposition 5.3.

We assume that $p_m = c'_m = 1$, and, for $1 \leq t \leq m$, that $k_t c'_t \leq p_t \leq c'_t$ for some $0 < k_t \leq 1$. (If for some $1 \leq t \leq m-1$ we have $p_t \equiv 0$ we may take $c'_t = 0$, but the results are then trivial anyway.)

Proposition 5.5 If $A = A'$ or A'' , and $B = B'$ or B'' , then

$$|A(t,r)| \leq K f_{p_{2t}} \phi^{2(m-2t)} [|\phi^{2t} e_{2t} u|^2 + |\phi^{2t} e_{2t} v|^2] + K f_{p_{2t}} \phi^{2(m-2t)} (1+\phi)^{2(2t-1)} u^2 \quad (1'), (1'')$$

$$|B(t,r)| \leq K f_{p_{2t+1}} \phi^{2(m-2t-1)} [|\phi^{2t+1} e_{2t+1} u|^2 + |\phi^{2t+1} e_{2t+1} v|^2] + K f_{p_{2t+1}} \phi^{2(m-2t-1)} (1+\phi)^{2(2t)} u^2 \quad (2'), (2'')$$

$$\text{and } |C(t)| \leq K f_{p_{2t+1}} \phi^{2(m-2t-1)} [|\phi^{2t} e_{2t} u|^2 + |\phi^{2t+1} e_{2t+1} v|^2] \quad (3)$$

Proof: We first establish

$$|A(t,r)| \leq K c'_{2t} [\alpha_{m-2t}(2t,u) + \alpha''_{2t-1,m-2t}(0,u) + \alpha_{m-2t}(2t,v)] \quad (4)$$

$$|B(t,r)| \leq K c'_{2t+1} [\alpha_{m-2t-1}(2t+1,u) + \alpha''_{2t,m-2t-1}(0,u) + \alpha_{m-2t-1}(2t+1,v)] \quad (5)$$

These are simple modifications of Prop.5.3(1) using Prop.5.2"(1) and (2) with $\rho = m - 2t$, and of Prop.5.3(2) using Prop.5.2"(3) and (4) with $\rho = m - (2t + 1)$ respectively. We derive (4) for $A = A''$ as an example.

$$\begin{aligned} |A''(t,r)| &= \left| \int \underline{n} \cdot \phi^{2m-1} (e_{2r+1} u) (e_{2(t-r-1)} (p_{2t} e_{2t} v)) \right| \\ &= \left| \int \left[\sum_{j=1}^n e_{2(t-r-1)} (\phi^{2m-1} n_j \partial_j e_{2r} u) \right] p_{2t} e_{2t} v \right| \end{aligned}$$

$$\begin{aligned}
(|A''(t, r)|) &\leq c'_{2t} \int \left| \sum_{j=1}^n \phi^{-m} \ell_{2(t-r-1)} (\phi^{2m-1} n_j \partial_j^{\ell} u) \right| |\phi^m \ell_{2t} v| \\
&\leq K c'_{2t} \left[\int \left| \sum_{j=1}^n \phi^{-m} \ell_{2(t-r-1)} (\phi^{2m-1} n_j \partial_j^{\ell} u) \right|^2 + \int |\phi^m \ell_{2t} v|^2 \right] \\
&\leq K c'_{2t} [\alpha_{m-2t}(2t, u) + \alpha_{2t-1, m-2t}''(0, u) + \alpha_{m-2t}(2t, v)]
\end{aligned}$$

by Prop. 5.2''(2) with $\rho = m-2t$, which establishes (4) for $A = A''$.

Now, for $1 \leq t \leq m$:

$$c'_t \alpha_{m-t}(t, u) = \int c'_t |\phi^m \ell_t u|^2 \leq k^{-1} \int p_t \phi^{2(m-t)} |\phi^t \ell_t u|^2 \quad (6)$$

$$\text{and } c'_t \alpha_{t-1, m-t}''(0, u) = \int c'_t \phi^{2(m-t)} (1 + \phi)^{2(t-1)} u^2 \leq k^{-1} \int p_t \phi^{2(m-t)} (1 + \psi)^{2(t-1)} u^2 \quad (7)$$

Substituting both (6) and (7) in (4) and (5) establishes (1) and (2) respectively.

$$\begin{aligned}
|C(t)| &= \left| \int n \cdot \phi^{2m-1} p_{2t+1} (\ell_{2t} u) (\ell_{2t+1} v) \right| \\
&\leq K \int p_{2t+1} \phi^{2(m-2t-1)} |\phi^{2t} \ell_{2t} u| |\phi^{2t+1} \ell_{2t+1} v| \\
&\leq K \int p_{2t+1} \phi^{2(m-2t-1)} [|\phi^{2t} \ell_{2t} u|^2 + |\phi^{2t+1} \ell_{2t+1} v|^2]
\end{aligned}$$

establishing (3). \square

Again the comparison between the move from Lemma 4.4(1) to Lemma 4.6(1) with that from Theorem 5.4(1) to Theorem 5.6(1) is very close.

Theorem 5.6 Suppose that conditions 3(i)-3(v) are satisfied, and

that we have a sequence $\{A_\nu\}$ as described above. Let $\theta \in C_0^{m,+}(A_\nu)$ with the first m derivatives of θ bounded by K . If u and $v \in \mathcal{D}(T_0^*)$ with u and v real-valued and $\int_{\Omega_\nu} (u T_0^* v - v T_0^* u) \rightarrow L \neq 0$ (we may assume that $L > 0$), and if, for $1 \leq t \leq m-1$, we have in A_ν

$$k_t c'_t \leq p_t \leq c'_t \text{ for some } 0 < k_t \leq 1 \text{ independent of } \nu,$$

then, for ν sufficiently large,

$$\begin{aligned}
L \int_0^d \nu \theta^{2m-1}(h_\nu) dh_\nu &\leq K \sum_{t=1}^{m-1} \int_{A_\nu} [1 + \theta^{2m} + p_t \theta^{2(m-t)} (1 + \theta)^{2(t-1)} - q \theta^{2m}] \times \\
&\quad \times [u^2 + v^2 + (Tu)^2 + (Tv)^2] dx \quad (1)
\end{aligned}$$

Proof: We shall use Lemma 4.6 and Prop.5.5, both of which though stated and proved for $\phi \in \mathcal{D}$ still hold for $\theta \in C_0^{m,+}(A_\nu)$.

From Theorem 5.4(3):

$$\begin{aligned}
\frac{1}{2}L \int_{\nu}^d \theta^{2m-1}(h_\nu) dh_\nu &\leq \left\{ \sum_{0 < t \leq \frac{1}{2}m} \sum_{r=0}^{t-1} [A'(t,r;\theta) - A''(t,r;\theta)] - \right. \\
&\quad - \sum_{0 < t \leq \frac{1}{2}m} \sum_{r=0}^{t-1} [B'(t,r;\theta) - B''(t,r;\theta)] - \\
&\quad \left. - \sum_{0 \leq t \leq \frac{1}{2}m} C(t;\theta) \right\} - u \leftrightarrow v \\
&\leq \left\{ \sum_{0 < t \leq \frac{1}{2}m} \sum_{r=0}^{t-1} K \left[\int_{P_{2t}} \theta^{2(m-2t)} |\theta^{2t} \ell_{2t} u|^2 + \right. \right. \\
&\quad \left. \left. + \int_{P_{2t}} \theta^{2(m-2t)} (1+\theta)^{2(2t-1)} u^2 \right] + \right. \\
&\quad \left. + \sum_{0 \leq t \leq \frac{1}{2}m} \sum_{r=0}^{t-1} K \left[\int_{P_{2t+1}} \theta^{2(m-2t-1)} |\theta^{2t+1} \ell_{2t+1} u|^2 + \right. \right. \\
&\quad \left. \left. + \int_{P_{2t+1}} \theta^{2(m-2t-1)} (1+\theta)^{2(2t)} u^2 \right] + \right. \\
&\quad \left. + K \int |\theta^m \ell_m u|^2 \right\} + u \leftrightarrow v \quad (\text{by Prop.5.5}) \\
&\leq \left\{ K \sum_{t=1}^m \int_{P_t} \theta^{2(m-t)} |\theta^t \ell_t u|^2 + \right. \\
&\quad \left. + K \sum_{t=1}^{m-1} \int_{P_t} \theta^{2(m-t)} (1+\theta)^{2(t-1)} u^2 \right\} + u \leftrightarrow v \\
&\quad (\text{because the sum is independent of } r) \\
&\leq \left\{ K \int \theta^{2m} (Tu)^2 + \right. \\
&\quad \left. + K \sum_{t=1}^{m-1} \int [1 + \theta^{2m} + p_t \theta^{2(m-t)} (1+\theta)^{2(t-1)} - q \theta^{2m}] u^2 \right\} + \\
&\quad + u \leftrightarrow v \quad (\text{by Lemma 4.6(1)})
\end{aligned}$$

which implies (1). \square

Noting the relaxation of the condition on p_1 to $p_1 \geq 0$ in A_ν , putting $m=2$ in the above result gives us Theorem I.5.6. The other remarks which follow Lemma 4.6 and, as in Part I, those which follow Theorem 5.4 apply equally well here.

Corollary 5.6.1 Suppose that conditions $\mathcal{E}(i)$ - $\mathcal{E}(v)$ are satisfied

and that we have a sequence $\{A_\nu\}$ as in Theorem 5.6, with

(a) $\liminf_{\nu \rightarrow \infty} d_\nu > 0$.

For each ν , assume that in A_ν ,

(b) $p_t \equiv 0$ or $k_t' d_\nu^2 \leq p_t \leq K d_\nu^2$, or

(b') $p_t \equiv 0$ or $k_t' d_\nu^2 |q|^{1/\kappa} \leq p_t \leq K d_\nu^2 |q|^{1/\kappa}$, with $\kappa \geq m/(m-t) > 1$,

for some $0 < k_t' \leq K$, for $1 \leq t \leq m-1$; and

(c) $q \geq -k$,

with K , k and k_t' independent of ν .

Then $L = 0$ and so T_0 is e.s.a.

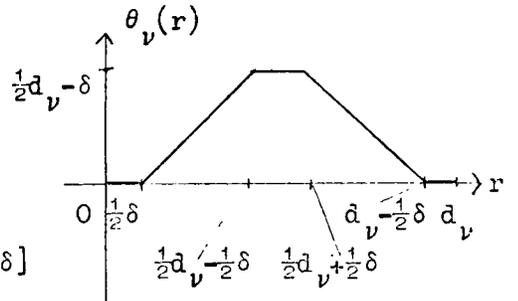
Proof: $p_t \equiv 0$ trivialises the calculations so we shall assume

that $p_t \neq 0$.

(a) implies the existence of $\delta > 0$ with $d_\nu - 2\delta > 0$ as $\nu \rightarrow \infty$.

For each ν , let $\theta_\nu \in C_0^{m,+}(A_\nu)$ such that

$$\begin{aligned} \theta_\nu(r) &= 0 \quad \text{if } r \in [0, \frac{1}{2}\delta] \\ &= r - \frac{1}{2}\delta \quad \text{if } r \in [\frac{1}{2}\delta, \frac{1}{2}d_\nu - \frac{1}{2}\delta] \\ &= \frac{1}{2}d_\nu - \delta \quad \text{if } r \in [\frac{1}{2}d_\nu - \frac{1}{2}\delta, \frac{1}{2}d_\nu + \frac{1}{2}\delta] \\ &= d_\nu - \frac{1}{2}\delta - r \quad \text{if } r \in [\frac{1}{2}d_\nu + \frac{1}{2}\delta, d_\nu - \frac{1}{2}\delta] \\ &= 0 \quad \text{if } r \in [d_\nu - \frac{1}{2}\delta, d_\nu] \end{aligned}$$



So $\theta_\nu' = 0$ or 1, and $\theta_\nu^{(t)} = 0$ for $t > 1$. We have $0 \leq \theta_\nu \leq K d_\nu$, and also $d_\nu > 2\delta$, so $1 \leq K d_\nu$ and $1 \leq K d_\nu^{2m}$.

Assume that $L > 0$ and alternative (b) holds.

So $p_t \theta_\nu^{2(m-t)} (1 + \theta_\nu)^{2(t-1)} \leq K d_\nu^2 d_\nu^{2(m-t)} d_\nu^{2(t-1)} = K d_\nu^{2m}$, for $1 \leq t \leq m-1$.

Also, from (c), $-q \theta_\nu^{2m} \leq k d_\nu^{2m}$.

By Theorem 5.6(1):
$$L \int_0^{d_\nu} \theta_\nu^{2m-1}(r) dr \leq K d_\nu^{2m} \int_{A_\nu} [u^2 + v^2 + (Tu)^2 + (Tv)^2] dx \quad (1)$$

(as the sum is now independent of t)

Now, still assuming that $L > 0$, suppose that (b') holds.

From (c), $|q| \leq q + 2k$, and $(q + k) \geq 0$.

For $1 \leq t \leq m-1$, we have in A_ν

$$\begin{aligned} p_t \theta_\nu^{2(m-t)} (1 + \theta_\nu)^{2(t-1)} &\leq K d_\nu^2 |q|^{1/\kappa} \theta_\nu^{2(m-t)} d_\nu^{2(t-1)} \\ &= K |q|^{1/\kappa} \theta_\nu^{2m/\kappa} \theta_\nu^{2(m-t-m/\kappa)} d_\nu^{2t} \\ &\leq K (|q| \theta_\nu^{2m})^{1/\kappa} (d_\nu^{2m})^{(\kappa-1)/\kappa} \\ &\leq \frac{1}{2} |q| \theta_\nu^{2m} + K d_\nu^{2m} \end{aligned}$$

where we have used the inequality $|a| |b|^{\kappa-1} \leq \varepsilon |a|^\kappa + K(\varepsilon) |b|^\kappa$, valid for $\kappa > 1$.

$$\begin{aligned} \text{So } p_t \theta_\nu^{2(m-t)} (1 + \theta_\nu)^{2(t-1)} - (q+k) \theta_\nu^{2m} &\leq K d_\nu^{2m} + \frac{1}{2} |q| \theta_\nu^{2m} - (q+k) \theta_\nu^{2m} \\ &\leq K d_\nu^{2m} + \frac{1}{2} (q+2k) \theta_\nu^{2m} - (q+k) \theta_\nu^{2m} \\ &\leq K d_\nu^{2m} + k \theta_\nu^{2m} - \frac{1}{2} (q+k) \theta_\nu^{2m} \\ &\leq (K+k) d_\nu^{2m}, \end{aligned}$$

this inequality holding in A_ν .

By Theorem 5.6(1):

$$\begin{aligned} L \int_0^{d_\nu} \theta_\nu^{2m-1}(r) dr &\leq K \sum_{t=1}^{m-1} \int_{A_\nu} [d_\nu^{2m} + d_\nu^{2m} + \{p_t \theta_\nu^{2(m-t)} (1 + \theta_\nu)^{2(t-1)} - (q+k) \theta_\nu^{2m}\} + k \theta_\nu^{2m}] \times \\ &\quad \times [u^2 + v^2 + (Tu)^2 + (Tv)^2] dx \\ &\leq K d_\nu^{2m} \int_{A_\nu} [u^2 + v^2 + (Tu)^2 + (Tv)^2] dx \end{aligned} \quad (2)$$

From (1) and (2), we see that if $L > 0$ and either (b) or (b') holds together with (a) and (c), then we have

$$L d_\nu^{-2m} \int_0^{d_\nu} \theta_\nu^{2m-1}(r) dr \leq K \int_{A_\nu} [u^2 + v^2 + (Tu)^2 + (Tv)^2] dx \quad (3)$$

As $\nu \rightarrow \infty$, the right-hand side of (3) $\rightarrow 0$.

$$\begin{aligned} \text{However, } d_\nu^{-2m} \int_0^{d_\nu} \theta_\nu^{2m-1}(r) dr &\geq d_\nu^{-2m} \int_{\frac{1}{2}d_\nu}^{d_\nu - \frac{1}{2}\delta} (d_\nu - \frac{1}{2}\delta - r)^{2m-1} dr \\ &= (2m)^{-1} 2^{-2m} (1 - 2\delta/d_\nu)^{2m} \\ &> 0, \text{ as } \nu \rightarrow \infty. \end{aligned}$$

This is a contradiction, so $L = 0$ and T_0 is e.s.a. \square

The example we give is a simple modification of that given after Corollary I.5.6.1.

Example: In Corol. 5.6.1, take $A_\nu = \{x: a_\nu \leq |x| \leq b_\nu\}$ where $0 < a_\nu < b_\nu$, with $kd_\nu \leq a_\nu$ and $b_\nu \leq Kd_\nu$; say $a_\nu = s^{2\nu}$ and $b_\nu = s^{2\nu+1}$ for some $s > 1$. So $d_\nu = s^{2\nu}(s-1) \leq k^{-1}a_\nu$, and $b_\nu = s^{2\nu+1} \leq K(1-1/s)s^{2\nu+1} = K(s^{2\nu+1} - s^{2\nu}) = Kd_\nu$, and also (a) is satisfied.

In A_ν , take p_t such that either $k'_t|x|^2 \leq p_t \leq K|x|^2$, or $k'_t|x|^2|q|^{(m-t)/m} \leq p_t \leq K|x|^2|q|^{(m-t)/m}$, for some $0 < k'_t \leq K$, for $1 \leq t \leq m-1$, where $q(x) = |x|^\delta \sin(\pi \log_s |x|)$, with p_t and q satisfying conditions $\mathcal{E}(i) - \mathcal{G}(v)$.

Then (b) (or (b')) and (c) are satisfied, since if $x \in A_\nu$, $kd_\nu \leq a_\nu \leq |x| \leq b_\nu \leq Kd_\nu$, with $k \leq (s-1)^{-1} < s(s-1)^{-1} \leq K$, and $q \geq 0$.

Hence T_0 is e.s.a. \square

CONCLUSION

In Part I, we considered a particular case of the operator formally given by

$$\tau u = \nabla^2(r(x)\nabla^2 u) - \nabla \cdot (p(x)\nabla u) + q(x)u \quad (1)$$

Expressing q as $q_1 + q_2$, we took $q_2 \equiv 0$. In an addendum, we shall reintroduce q_2 satisfying the conditions in Theorem 0.2.

Also, we took $r(x)$ to be constant; in fact we assumed that $r(x) \equiv 1$. It transpires that the proofs in which we had to assume that this coefficient is constant will still hold if we assume that $r(x) \equiv 1$ in A_ν (cf. the remark following Theorem I.5.4). Thus, with $r \in C^2$ and $r(x) = 1$ in A_ν , we were effectively considering the operator τ given by (1).

We may consider the term $\nabla \cdot (p(x)\nabla u)$ to be the term given by $\sum_{j,k=1}^n \frac{\partial}{\partial x_j} (p_{jk}(x) \frac{\partial u}{\partial x_k})$ when we take the $n \times n$ real symmetric matrix $\{p_{jk}(x)\}$ to be $p(x)I$, where I is the $n \times n$ identity matrix. In view of the proof of Theorem 0.3 (see [4]), it seems very reasonable to suppose that had we taken $\{p_{jk}(x)\}$ as a matrix more general than $p(x)I$, the results in Part I would not differ in essence, but only in complexity.

We can also read $r(x)$ as, in some sense, an $n^2 \times n^2$ real symmetric matrix which is the $n^2 \times n^2$ identity matrix in A_ν , without affecting the remarks above.

Similarly, in Part II, we could have considered the coefficient functions $p_t(x)$, for $0 \leq t \leq m$, to be, roughly speaking, $n^t \times n^t$ real symmetric matrices. Again, the results would presumably differ only in complexity, not in essence.

We conclude by restating briefly the examples of essentially self-adjoint operators (with $p_m = 1$) which were given, the conditions to hold in a suitably chosen sequence $\{A_\nu\}$ and assuming that $\mathcal{E}(i) - \mathcal{E}(v)$ hold.

$$\begin{aligned} \S 1. \quad (a) \quad q &\geq -K|x|^{2m/(2m-1)}; \\ (b) \quad |p_t| &\leq K|x|^{2(m-t)/(2m-1)}. \end{aligned}$$

In particular,

when $m = 1$ the conditions become

$$q \geq -K|x|^2;$$

and when $m = 2$ the conditions become

$$\begin{aligned} q &\geq -K|x|^{4/3}, \\ |p| &\leq K|x|^{2/3}. \end{aligned}$$

$$\S 2. \quad (a) \quad q \geq -K;$$

$$\text{either (b')} \quad k_t|x|^2 \leq p_t \leq K|x|^2,$$

$$\text{or (b'')} \quad k_t|x|^2|q|^{(m-t)/m} \leq p_t \leq K|x|^2|q|^{(m-t)/m},$$

where $k_1 \geq 0$ and otherwise $k_t > 0$.

In particular,

when $m = 2$ the conditions become

$$q \geq -K,$$

$$p \geq 0,$$

$$\text{and either } p \leq K|x|^2 \text{ or } p \leq K|x|^2|q|^{1/2}.$$

ADDENDUM

As in Kato [14] and Eastham, Evans and McLeod [4] we can give further flexibility to the coefficient function q by supposing that $q = q_1 + q_2$.

Some of the conditions which were imposed on q in the main text will still be required to be satisfied by q here, but others will be required to be satisfied only by q_1 .

We require that $q_1 \in L^2_{loc}$ and $q_1(x) \geq -q_1^*(|x|)$ as before, and now we require that q_2 satisfies (C3) of [14] (see Theorem 0.2), i.e.

$$q_2 \in L^2_{loc} \text{ with } \int_{|x| \leq r} |q_2(x)|^2 dx \leq (Kr^S)^2 \quad (1 \leq r < \infty) \text{ for some } K \text{ and } s,$$

$$\text{and } N_{2, \delta, y}(q_2) \rightarrow 0 \text{ as } \delta \rightarrow 0, \text{ uniformly in } y;$$

$$\text{or, if } n \geq 5, q_2 \in L^{\frac{1}{2}n}.$$

We may therefore use some of Kato's results.

$$\text{We define } \underline{q}_1(x) = q_1(x) \text{ if } x \in B_R,$$

$$\text{and } \underline{q}_1(x) = q_1(x) + q_1^*(|x|) - q_1^*(R) \text{ if } x \notin B_R,$$

and set $\underline{q} = \underline{q}_1 + q_2$ (i.e. we need not alter q_2). We have $\underline{q} \in L^2_{loc}$.

We still require \underline{q} to satisfy ϵ (iv) in order to obtain \underline{T}_0 e.s.a.

Certain amendments have to be made to various proofs in order that the lemmas, etc., remain valid.

In Lemma I.3.1 and Proposition II.3.0.6: we must consider $\int |q_2 \phi u|$.

Kato shows that $\|q_2 u\|_{L^1(\Omega)} \leq K(q_2, \Omega) \|u\|_{\Omega}$, and so we have

$$\int |q_2 \phi u| \leq \sup |\phi| \|q_2 u\|_{L^1(\Omega)} \leq K(\Omega) \sup |\phi| \|u\|_{\Omega}$$

and we may proceed as before. \square

In Lemma 3.3: Prop.1 of [14] is

$$\int |q_2| |u|^2 \leq \varepsilon \|u\|_1^2 + K(\varepsilon) \|u\|^2 \quad \text{for } u \in H_0^1.$$

So for $u \in \mathcal{D}$, we have

$$\begin{aligned} \Re(\underline{T}_0 u, u) &\geq c \|u\|_m^2 - k \|u\|^2 + (q_1 u, u) + (q_2 u, u) \\ &\geq c \|u\|_m^2 - \varepsilon \|u\|_1^2 - (k + q_1^*(R) + K(\varepsilon)) \|u\|^2 \\ &\geq \frac{1}{2} c \|u\|_m^2 - (k + q_1^*(R) + K) \|u\|^2 \end{aligned}$$

and we may proceed as before. \square

We still require q to satisfy $\mathcal{E}(v)$ in order that we may use the result of Lemma 3.4.

In Corollary 3.8.1: we replace q by q_1 . \square

Corollary 3.8.2 If $\{u_\nu\}$ is a sequence with the properties in

Lemma 3.8, then

$$\int q_2 |u_\nu|^2 \rightarrow \int q_2 |\phi u|^2 \quad \text{as } \nu \rightarrow \infty.$$

Proof: By Prop.1 of [14]:

$$\int q_2 |u_\nu|^2 \leq \varepsilon \|u_\nu\|_1^2 + K(\varepsilon) \|u_\nu\|^2.$$

As in Corol.3.8.1, we have $u_\nu \rightarrow \phi u$ in H_0^1 and the result follows. \square

Thus Lemma 3.9 remains valid.

In Theorem I.5.4: we require $q_1 \theta^4 \geq -Q(h_\nu)$, with no further restriction

on q_2 .

Proof: From the penultimate line of Theorem I.5.4 we have

$$L \int_0^d \theta^3(h_\nu) dh_\nu \leq K f \theta^4 [(Tu)^2 + (Tv)^2] + K f [\theta^4 + c^2 - q \theta^4] [u^2 + v^2] \quad (1)$$

By Prop.1 of [14]:

$$-\int q_2 \theta^4 u^2 \leq \eta \|\theta^2 u\|_1^2 + K(\eta) \|\theta^2 u\|^2;$$

by Prop.I.4.3 and Lemma I.4.4:

$$\|\theta^2 u\|_1^2 + \|\theta^2 v\|_1^2 \leq K\Phi_\nu(u, v)$$

where $\Phi_\nu(u, v)$ denotes the right-hand side of (1).

$$\text{So } -\int q_2 \theta^4 [u^2 + v^2] \leq K\eta\Phi_\nu(u, v) + K(\eta)[\|\theta^2 u\|_1^2 + \|\theta^2 v\|_1^2] \quad (2)$$

We now choose η sufficiently small and take the term involving q_2 in $\Phi_\nu(u, v)$ to the left-hand side of (2), thereby obtaining an estimate for $-\int q_2 \theta^4 [u^2 + v^2]$, which when substituted into (1) gives the desired result. \square

In Theorem I.5.6: we replace q by q_1 , the result remaining valid as with Theorem I.5.4. \square

Theorems II.5.4 and II.5.6 are amended similarly. \square

If q is continuous, as in the examples, then all the conditions on q_2 are satisfied by setting $q_2 = 0$ as in the main text.

APPENDIX 1

With T_0 , T and T^* as in the main text:

Lemma A1.1 T_0 is essentially self-adjoint iff $(T + \zeta)$ is 1-1 from L^2 to \mathcal{D}' for every complex ζ such that $\Re \zeta \neq 0$.

Proof: With ζ as above:

T_0 is essentially self-adjoint iff $(T_0 + \zeta)\mathcal{D}$ is dense in L^2 .

Assume T_0 is essentially self-adjoint.

Suppose $u \in L^2$ and $(T + \zeta)u = 0$.

Then $\langle (T + \zeta)u, \phi \rangle = 0$ for all $\phi \in \mathcal{D}$.

Integration by parts gives us $\langle u, (T_0 + \zeta)\phi \rangle = 0$ for all $\phi \in \mathcal{D}$.

As $(T_0 + \zeta)\mathcal{D}$ is dense in L^2 , this implies that $u = 0$.

Therefore $(T + \zeta)$ is 1-1 from L^2 to \mathcal{D}' .

Now assume $(T + \zeta)$ is 1-1 from L^2 to \mathcal{D}' .

Suppose that $(T_0 + \zeta)\mathcal{D}$ is not dense in L^2 .

Therefore there exists $u \in L^2$, $u \neq 0$, such that

$$\langle u, (T_0 + \zeta)\phi \rangle = 0 \quad \text{for all } \phi \in \mathcal{D}.$$

Integration by parts, treating $(T + \zeta)u$ as a distribution, gives us

$$\langle (T + \zeta)u, \phi \rangle = 0 \quad \text{for all } \phi \in \mathcal{D}.$$

Therefore $(T + \zeta)u = 0$. Therefore $u = 0$ as $(T + \zeta)$ is 1-1.

This is a contradiction, so $(T_0 + \zeta)\mathcal{D}$ is dense in L^2 .

Therefore T_0 is essentially self-adjoint. \square

Lemma A1.2 $u \in \mathcal{D}(T^*)$ implies that there exists a sequence $\{u_\nu\}$,

$u_\nu \in \mathcal{D}$, such that $u_\nu \rightarrow u$ and $Tu_\nu \rightarrow Tu$ in L^2 iff T_0 is essentially self-adjoint.

Proof: Assume that if $u \in \mathcal{D}(T^*)$ then there exists a sequence $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, such that $u_\nu \rightarrow u$ and $Tu_\nu \rightarrow Tu$ in L^2 , i.e.

if $u \in \mathcal{D}(T^*)$ then there exists $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, such that $u_\nu \rightarrow u$ in L^2

and $T_0 u_\nu$ converges in L^2 (as $T\phi = T_0\phi$ if $\phi \in \mathcal{D}$).

Therefore, with \bar{T}_0 denoting the closure of T_0 , $u \in \mathcal{D}(\bar{T}_0)$.

Therefore $\mathcal{D}(T_0^*) \subset \mathcal{D}(\bar{T}_0)$.

But T_0 is symmetric and therefore $T_0 \subset T_0^*$, and so $\bar{T}_0 \subset \overline{T_0^*} = T_0^*$.

Therefore $\mathcal{D}(\bar{T}_0) \subset \mathcal{D}(T_0^*)$; therefore $\bar{T}_0 = T_0^*$, i.e. \bar{T}_0 is self-adjoint, and so T_0 is essentially self-adjoint.

Now assume $\bar{T}_0 = T_0^*$.

Let $u \in \mathcal{D}(\bar{T}_0)$. Then there exists $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, such that $u_\nu \rightarrow u$ and $\bar{T}_0 u_\nu \rightarrow \bar{T}_0 u$ in L^2 .

But $\bar{T}_0 = T_0^*$, and therefore $u \in \mathcal{D}(T_0^*)$ and $T_0^* u_\nu \rightarrow T_0^* u$ in L^2 .

But $T_\nu v = T_0^* v$ for $v \in \mathcal{D}(T_0^*)$, and therefore $Tu_\nu \rightarrow Tu$ in L^2 . \square

APPENDIX 2

We give some details of the proof of Theorem I.2.1 and we assume that $P(D)$, $q_j(x)$ and $Q_j(D)$ are as in the theorem. For complete details, see Schechter [17].

Lemma A2.1 makes use of Young's Lemma and of mollifiers, which were introduced by Friedrichs.

Young's Lemma: $\|u * v\|_{L^2} \leq \|u\|_{L^1} \|v\|_{L^2}$

where $u * v$ is the convolution of u and v . \square

Mollifiers: Define $j(x) = c \exp\{(|x|^2 - 1)^{-1}\}$ if $|x| < 1$
 $= 0$ if $|x| \geq 1$

where c is chosen such that $\int j(x) = 1$.

Set $j_\nu(x) = \nu^n j(\nu x)$ for $\nu = 1, 2, \dots$

and, for $f \in L^2$, define $J_\nu f(x) = \int j_\nu(x-y)f(y)dy$ $\nu = 1, 2, \dots$

i.e. $J_\nu f = j_\nu * f$.

The operator J_ν is a mollifier.

The two properties we shall use are

$$\|J_\nu f\|_0 \leq \|f\|_0$$

and $\|J_\nu f - f\|_0 \rightarrow 0$ as $\nu \rightarrow \infty$. \square

Lemma A2.1 \bar{P} is self-adjoint.

Proof: Let $u \in \mathcal{D}(P^*)$, i.e. there exists $v \in L^2$ such that

$$(u, P(D)\phi) = (v, \phi) \quad \text{for all } \phi \in \mathcal{D}.$$

Letting J_ν be the mollifier defined above, and setting $\phi(x) = j_\nu(y-x)$ we obtain $P(D)J_\nu u = J_\nu v$.

Now $J_\nu v \rightarrow v$ in L^2 as $\nu \rightarrow \infty$, and so $P(D)J_\nu u \rightarrow v$ in L^2 as $\nu \rightarrow \infty$.

We want to show that $u \in \mathcal{D}(\bar{P})$, so we need a sequence $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, such that $u_\nu \rightarrow u$ in L^2 and $P(D)u_\nu$ converges in L^2 .

For each $\rho > 0$, define

$$u_\rho(x) = u(x) \quad \text{if } |x| \leq \rho; \quad u_\rho(x) = 0 \quad \text{if } |x| > \rho.$$

Now $J_\nu u_\rho \in \mathcal{D}$, being the convolution of a function in \mathcal{D} and a function in L^1 with bounded support.

Given $\epsilon > 0$, choose N such that for all $\nu > N$

$$\|J_\nu u - u\| < \frac{1}{2}\epsilon \quad \text{and} \quad \|P(D)J_\nu u - v\| < \frac{1}{2}\epsilon.$$

Now $\|J_\nu(u_\rho - u)\| \leq \|u_\rho - u\|$

and $\|P(D)J_\nu(u_\rho - u)\| \leq \|P(D)j_\nu\|_{L^1} \|u_\rho - u\|$, with $\|P(D)j_\nu\|_{L^1}$ being finite for each ν .

Now $\|u_\rho - u\| \rightarrow 0$ as $\rho \rightarrow \infty$, so for each $\nu > N$ pick $\rho = \rho(\nu)$ such that

$$\|u_\rho - u\| < \frac{1}{2}\epsilon \quad \text{and} \quad \|P(D)j_\nu\|_{L^1} \|u_\rho - u\| < \frac{1}{2}\epsilon.$$

Now set $u_\nu = J_\nu u_{\rho(\nu)}$. We have $u_\nu \in \mathcal{D}$,

$$\|u_\nu - u\| = \|J_\nu u_{\rho(\nu)} - u\| \leq \|J_\nu(u_{\rho(\nu)} - u)\| + \|J_\nu u - u\| < \epsilon,$$

and $\|P(D)u_\nu - v\| = \|P(D)J_\nu u_{\rho(\nu)} - v\|$
 $\leq \|P(D)J_\nu(u_{\rho(\nu)} - u)\| + \|P(D)J_\nu u - v\| < \epsilon.$

So $\{u_\nu\}$ is the required sequence and $\mathcal{D}(P^*) \subset \mathcal{D}(\bar{P})$.

But as \bar{P} is symmetric, $\bar{P} \subset P^*$.

This gives us $\bar{P} = P^*$, i.e. \bar{P} is self-adjoint. \square

Lemma A2.2 $\mathcal{D}(\bar{P}) = H^{\mu}_0$.

Proof: Clearly, for any s and for all $\phi \in \mathcal{D}$,

$$\|P(D)\phi\|_s \leq K \|\phi\|_{\mu+s} \quad \text{and} \quad \|\phi\|_s \leq K \|\phi\|_{\mu+s}$$

and so $\|P(D)\phi\|_0 + \|\phi\|_0 \leq K \|\phi\|_\mu$

Let $P_\mu(D)$ denote the principal part of $P(D)$.

Then $\|P_\mu(D)\phi\|_s = \left\| \sum_{|\alpha|=\mu} a_\alpha D^\alpha \phi \right\|_s$
 $= \left\| (1+|\xi|)^s \sum_{|\alpha|=\mu} a_\alpha (i\xi)^\alpha \mathcal{F}\phi(\xi) \right\|_0$
 $= \left\| (1+|\xi|)^s \sum_{|\alpha|=\mu} a_\alpha \xi^\alpha \mathcal{F}\phi(\xi) \right\|_0$
 $\geq C \left\| (1+|\xi|)^s |\xi|^\mu \mathcal{F}\phi(\xi) \right\|_0 \quad \text{with } C > 0 \text{ as } P(D) \text{ is elliptic.}$

Now $\|\phi\|_{s+\mu} = \left\| (1+|\xi|)^{s+\mu} \mathcal{F}\phi(\xi) \right\|_0$
 $= \left\| (1+|\xi|)^s (1+|\xi|)^{\mu} \mathcal{F}\phi(\xi) \right\|_0$
 $\leq \left\| (1+|\xi|)^s |\xi|^\mu \mathcal{F}\phi(\xi) \right\|_0 + K \left\| (1+|\xi|)^s (1+|\xi|)^{\mu-1} \mathcal{F}\phi(\xi) \right\|_0$

$$\begin{aligned} &\leq C^{-1} \|P_\mu(D)\phi\|_s + K \|\phi\|_{s+\mu-1} \\ &\leq K(\|P(D)\phi\|_s + \|(P(D)-P_\mu(D))\phi\|_s + \|\phi\|_{s+\mu-1}) \\ &\leq K(\|P(D)\phi\|_s + \|\phi\|_{s+\mu-1}) \end{aligned}$$

as $(P(D)-P_\mu(D))$ is an operator of order at most $\mu-1$.

Replacing s by $s-1$:

$$\|\phi\|_{s+\mu-1} \leq K(\|P(D)\phi\|_{s-1} + \|\phi\|_{s+\mu-2})$$

and substituting back, noting that $\|P(D)\phi\|_{s-1} \leq \|P(D)\phi\|_s$, we obtain

$$\|\phi\|_{s+\mu} \leq K(\|P(D)\phi\|_s + \|\phi\|_{s+\mu-2}).$$

We iterate to obtain:

$$\|\phi\|_{s+\mu} \leq K(\|P(D)\phi\|_s + \|\phi\|_s)$$

Therefore $\|\phi\|_\mu \leq K(\|P(D)\phi\|_0 + \|\phi\|_0) \leq K' \|\phi\|_\mu$ for all $\phi \in \mathcal{D}$.

As \mathcal{D} is dense in H^{μ}_0 , this implies that $\mathcal{D}(\bar{P}) = H^{\mu}_0$. \square

Before we proceed to Lemma A2.3, we need some preliminary results.

Let $f = F^{-1}(1+|\xi|)^s F\phi$. Then $\|f\|_0 = \|\phi\|_s$, and also

$$Ff = (1+|\xi|)^s F\phi, \quad F\phi = (1+|\xi|)^{-s} Ff.$$

Set $G_s(x) = F^{-1}(1+|\xi|)^{-s}$. Then $F\phi = FG_s Ff$ and $\phi = G_s^* f$.

Schechter shows that

- (a) $G_s(x) = O(|x|^{s-n})$ as $|x| \rightarrow 0$, for $0 < s < n$;
- (b) $G_n(x) = O(-\log|x|)$ as $|x| \rightarrow 0$;
- (c) $G_s(x) = O(1)$ as $|x| \rightarrow 0$, for $s > n$;
- (d) $G_s(x) = O(|x|^{-\kappa})$ as $|x| \rightarrow \infty$, for $s > 0$, $\kappa = 1, 2, \dots$

It is clear that the conditions imposed on the functions $q_j(x)$ are derived from the properties (a), (b) and (c) of $G_s(x)$ (see Propositions 5 and 6 below).

The following propositions are required:

Proposition 1: For $s > 0$, $G_s(x) \in C^\infty$ for $x \neq 0$, and for any α

$$D^\alpha G_s(x) = O(|x|^{-\kappa}) \text{ as } |x| \rightarrow \infty, \kappa = 1, 2, \dots$$

Corollary 2: If $v \in C^\infty_B$ and $v = 0$ near $x = 0$, then $vG_s \in S$.

Proposition 3: If $s > 0$ and $t > 0$, then $G_s * G_t = G_{s+t}$.

Proposition 4: $G_s(x) > 0$ for $s > 0$ and $x \neq 0$.

Proposition 5: For each $s > 0$, there exists $C > 0$, such that

$$G_{2s}(x) \leq C\omega_{2s}(x) \quad \text{for } 0 < |x| < 1.$$

Proposition 6: If $s > 0$ and $q \in N_{2s}$, then there exists $C(s,n)$ such that

$$\int |q(x-y)|^2 G_{2s}(x) dx \leq CN_{2s}(q).$$

Proposition 7: If $s > 0$ and $\int |q(x-y)|^2 G_{2s}(x) dx \leq C_0$, then

$$\|\bar{q}G_{2s} * (q\phi)\|_0 \leq C_0 \|\phi\|_0 \quad \text{for all } \phi \in \mathcal{D}.$$

Proposition 8: For all $\phi \in \mathcal{D}$, if $\|\bar{q}G_{2s} * (q\phi)\|_0 \leq C_0 \|\phi\|_0$, then

$$\|q\phi\|_{-s}^2 \leq C_0 \|\phi\|_0^2.$$

Proposition 9: For all $\phi \in \mathcal{D}$, if $\|q\phi\|_{-s}^2 \leq C_0 \|\phi\|_0^2$, then

$$\|q\phi\|_0^2 \leq C_0 \|\phi\|_s^2.$$

Most of these results are immediate. We prove Proposition 6.

Proof: (of Prop.6) From Prop.5 (which is immediate from the properties (a),(b) and (c) of $G_s(x)$) we have:

$$\int_{|x| < 1} |q(x-y)|^2 G_{2s}(x) dx \leq CN_{2s}(q),$$

and so we consider $\int_{|x| > 1} |q(x-y)|^2 G_{2s}(x) dx$.

From property (d) of $G_s(x)$, there exists K such that

$$G_{2s}(x) \leq K/|x|^{n+1} \quad \text{for all } |x| > 1.$$

Therefore $\int_{|x| > 1} |q(x-y)|^2 G_{2s}(x) dx \leq K \sum_{\kappa=1}^{\infty} \int_{\kappa < |x| < \kappa+1} |q(x-y)|^2 dx / \kappa^{n+1}$.

Now there exists $C(n)$, depending only on n , such that each shell can be covered by $C\kappa^{n-1}$ spheres of radius 1.

Therefore $\int_{\kappa < |x| < \kappa+1} |q(x-y)|^2 dx \leq C\kappa^{n-1} \int_{|x| < 1} |q(x-y)|^2 dx$.

But $\int_{|x| < 1} |q(x-y)|^2 dx \leq \int_{|x| < 1} |q(x-y)|^2 \omega_{2s}(x) dx \leq N_{2s}(q)$.

Therefore $\int_{|x| > 1} |q(x-y)|^2 G_{2s}(x) dx \leq KCN_{2s}(q) \sum_{\kappa=1}^{\infty} \kappa^{-2} \leq CN_{2s}(q)$. \square

Lemma A2.3 $H_0^\mu \subset \mathcal{D}(q_j \bar{Q}_j)$ for each j .

Proof: Let $qQ(D)$ be any one of the $q_j Q_j(D)$, i.e. $Q(D)$ is an operator of order $j < \mu$ and $q \in N_{2(\mu-j)}$.

Then $\|qQ(D)\phi\|_0 \leq K \|Q(D)\phi\|_{\mu-j}$ for all $\phi \in \mathcal{D}$, by Props.6-9.

Also $\|Q(D)\phi\|_{\mu-j} \leq K \|\phi\|_\mu$ for all $\phi \in \mathcal{D}$.

Let $u \in H_0^\mu$. Then there exists $\{u_\nu\}$, $u_\nu \in \mathcal{D}$, such that $u_\nu \rightarrow u$ in H_0^μ . Therefore $\{Q(D)u_\nu\}$ converges in $H_0^{\mu-j}$ and $u \in \mathcal{D}(\bar{Q})$.

Therefore $\{qQ(D)u_\nu\}$ converges in L^2 and $\bar{Q}u \in \mathcal{D}(q)$.

Therefore $u \in \mathcal{D}(q\bar{Q})$, i.e. $H_0^\mu \subset \mathcal{D}(q\bar{Q})$. \square

Stages (I) and (II) of the proof of Theorem I.2.1 are now done, and we move on to stage (III) which requires a preliminary lemma and proposition.

Lemma A2.4 Assume that $q \in N_{2s}$, $\phi \in \mathcal{D}$ and $\delta > 0$. Then

$$\|q\phi\|_0^2 \leq K(s,n)N_{2s,\delta}(q) \|\phi\|_s^2 + C(s,n,\delta)N_{2s}(q) \|\phi\|_0^2. \quad (\dagger)$$

Proof: (\dagger) is equivalent to

$$\|q(G_s^* f)\|_0^2 \leq KN_{2s,\delta}(q) \|f\|_0^2 + CN_{2s}(q) \|G_s^* f\|_0^2 \quad (\dagger\dagger)$$

Let $\psi \in \mathcal{D}$ such that $0 \leq \psi(x) \leq 1$, $\psi(x) = 0$ if $|x| > 1/2$

and $\psi(x) = 1$ if $|x| < 1/4$. Set $\psi_\delta(x) = \psi(x/\delta)$.

Define $G_{s,\delta}(x) = \psi_\delta(x)G_s(x)$, $\tilde{G}_{s,\delta}(x) = (1-\psi_\delta(x))G_s(x)$

and $H(x) = G_{s,\delta}^* G_{s,\delta}$.

We observe that $G_s = G_{s,\delta} + \tilde{G}_{s,\delta}$, and so we bound $\|q(G_{s,\delta}^* f)\|_0^2$ by the first term in the inequality, and $\|q(\tilde{G}_{s,\delta}^* f)\|_0^2$ by the second.

$$\begin{aligned} \text{Now } H(x) &= \int \psi_\delta(x-y)G_s(x-y)\psi_\delta(y)G_s(y)dy \\ &\leq \int G_s(x-y)G_s(y)dy = G_{2s}(x) \quad \text{using Props.3 and 4.} \end{aligned}$$

If $|x| > \delta$, then either $|y| > \frac{1}{2}\delta$ or $|x-y| > \frac{1}{2}\delta$.

Therefore $\psi_\delta(x-y)\psi_\delta(y) = 0$ for $|x| > \delta$.

Therefore $H(x) \leq K(s,n)\omega_{2s}(x)$ if $|x| < \delta$, and $H(x) = 0$ if $|x| > \delta$,

using Prop.5.

If u and $v \in \mathcal{D}$,

$$\begin{aligned} \|G_{s,\delta}^* (qv)\|_0^4 &= (G_{s,\delta}^* (qv), G_{s,\delta}^* (qv))^2 \\ &= (H^* (qv), qv)^2 \\ &= (\iint H(x-y)q(y)v(y)\bar{q}(x)\bar{v}(x)dx dy)^2 \\ &\leq (\iint H(x-y)|q(x)|^2|v(y)|^2dx dy) \times \\ &\quad (\iint H(x-y)|q(y)|^2|v(x)|^2dx dy). \end{aligned}$$

But $\int |q(x)|^2 H(x-y)dx \leq KN_{2s,\delta}(q)$,

and therefore $\|G_{s,\delta}^* (qv)\|_0^2 \leq KN_{2s,\delta}(q) \|v\|_0^2$.

$$\begin{aligned} |(q(G_{s,\delta}^* u), v)|^2 &= |(u, G_{s,\delta}^* (\bar{q}v))|^2 \\ &\leq \|u\|_0^2 \|G_{s,\delta}^* (\bar{q}v)\|_0^2 \\ &\leq KN_{2s,\delta}(q) \|u\|_0^2 \|v\|_0^2 \end{aligned}$$

since $N_{\alpha,\delta}(\bar{q}) = N_{\alpha,\delta}(q)$ generally.

Since this is true for all $v \in \mathcal{D}$, we have

$$\|q(G_{s,\delta}^* u)\|_0^2 \leq KN_{2s,\delta} \|u\|_0^2 \tag{†}$$

Now $\tilde{G}_{s,\delta} \in S$ by Corol.2. In particular, there exists $C(s,n,\delta)$ such that $(1+|\xi|)^{4s} |F\tilde{G}_{s,\delta}|^2 \leq C$. Therefore, by Prop.9,

$$\begin{aligned} \|q(\tilde{G}_{s,\delta}^* u)\|_0^2 &\leq CN_{2s}(q) \|\tilde{G}_{s,\delta}^* u\|_s^2 \\ &= CN_{2s}(q) \int (1+|\xi|)^{2s} |F\tilde{G}_{s,\delta}|^2 |Fu|^2 d\xi \\ &\leq CN_{2s}(q) \int (1+|\xi|)^{-2s} |Fu|^2 d\xi \\ &= CN_{2s}(q) \|G_{s,\delta}^* u\|_0^2 \end{aligned} \tag{††}$$

Combining (†) and (††) establishes (††). \square

Proposition 10 For all $\phi \in \mathcal{D}$, if $0 < t < s$, then

$$\|\phi\|_t \leq \epsilon \|\phi\|_s + K(\epsilon) \|\phi\|_0 \tag{*}$$

Proof: $(1+|\xi|)^{2t} = (1+|\xi|)^s (1+|\xi|)^{2t-s}$

$$\leq \frac{1}{2}\eta (1+|\xi|)^{2s} + \frac{1}{2}\eta^{-1} (1+|\xi|)^{4t-2s}.$$

Repeat for the second term on the right-hand side (the exponent reducing by increasing multiples of $s-t$) until the exponent is ≤ 0 . Then $(1+|\xi|)^{2t} \leq \epsilon^2 (1+|\xi|)^{2s} + K(\epsilon)^2$. Multiplying through by $|F\phi|^2$ and integrating establishes (*). \square

Lemma A2.5 We have, for $a < 1$ and for all $u \in H_0^\mu$,

$$\Sigma \|q_j \bar{Q}_j u\|_0 \leq a \|\bar{P}u\|_0 + b \|u\|_0.$$

Proof: By (†) (Lemma A2.4), if $q \in N_{2s}$ and $\phi \in \mathcal{D}$, then

$$\|q\phi\|_0^2 \leq KN_{2s, \delta}(q) \|\phi\|_s^2 + CN_{2s}(q) \|\phi\|_0^2.$$

If $2s \leq n$, then take δ small enough so that $KN_{2s, \delta}(q) < \epsilon^2$.

Then $\|q\phi\|_0 \leq \epsilon \|\phi\|_s + K(\epsilon) \|\phi\|_0$.

If $2s > n$, then there exists t such that $n < 2t < 2s$ and $q \in N_{2t}$,

and so $\|q\phi\|_0^2 \leq CN_{2t}(q) \|\phi\|_t^2$ (by Prop.9)

so $\|q\phi\|_0 \leq \epsilon \|\phi\|_s + K(\epsilon) \|\phi\|_0$ (by (*) Prop.10)

Therefore, if $q \in N_{2s}$ and $\phi \in \mathcal{D}$,

$$\|q\phi\|_0 \leq \epsilon \|\phi\|_s + K(\epsilon) \|\phi\|_0.$$

Therefore $\|q_j Q_j(D)\phi\|_0 \leq \epsilon \|Q_j(D)\phi\|_{\mu - \mu_j} + K(\epsilon) \|Q_j(D)\phi\|_0$

$$\leq \epsilon K \|\phi\|_\mu + K(\epsilon) \|\phi\|_{\mu_j}$$

$$\leq \epsilon K \|\phi\|_\mu + \epsilon' K(\epsilon) \|\phi\|_\mu + K(\epsilon, \epsilon') \|\phi\|_0 \quad \text{by (*)}$$

$$\leq \epsilon'' \|\phi\|_\mu + K(\epsilon'') \|\phi\|_0$$

But $\|\phi\|_\mu \leq K(\|P(D)\phi\|_0 + \|\phi\|_0)$, as in Lemma A2.2.

Therefore $\|q_j Q_j(D)\phi\|_0 \leq \eta \|P(D)\phi\|_0 + K(\eta) \|\phi\|_0$

and $\Sigma \|q_j Q_j(D)\phi\|_0 \leq a \|P(D)\phi\|_0 + b \|\phi\|_0$ with $a < 1$.

Since \mathcal{D} is dense in H_0^μ , this establishes the result. \square

This concludes the proof of Theorem I.2.1.

APPENDIX 3

Here we give the proof of Lemma I.3.4, due to Ikebe and Kato [12].

Lemma A3.1 If $u \in H_{0,loc}^2$ and $N_{4-\alpha}(q)$ is locally bounded for some $\alpha > 0$ with $n > 4-\alpha > 0$, then $qu \in L_{loc}^2$, and for any compact set Ω

$$\|qu\|_{\Omega} \leq K \|u\|_{2,\Omega'}$$

where Ω' is Ω extended in width by $d > 0$, and K depends on Ω , d and n .

Proof: Ikebe and Kato show that for any $u \in H_{0,loc}^2$:

$$u(x) = G_d^{\circ}(-\nabla^2 u) + K_d^{\circ}u \quad \text{almost everywhere} \quad (*)$$

where ∇^2 is interpreted as strong, and d is a fixed positive constant.

G_d° and K_d° are integral operators with kernels $G_d^{\circ}(x,y)$ and $K_d^{\circ}(x,y)$.

$G_d^{\circ}(x,y)$ has a singularity at $y=x$ like Green's function: $G_d^{\circ}(x,y) = O(|x-y|^{2-n})$, while $K_d^{\circ}(x,y)$ is smooth. They can be made to vanish for $|x-y| \geq d > 0$.

Let $x \in \Omega$. Then, by (*),

$$\begin{aligned} |u(x)|^2 &\leq K \int_{|x-y| \leq d} |\nabla^2 u(y)| |x-y|^{2-n} dy + K \int_{|x-y| \leq d} |u(y)| dy \\ &\leq K \int_{|x-y| \leq d} |x-y|^{\alpha-n} dy \int_{|x-y| \leq d} |\nabla^2 u(y)|^2 |x-y|^{4-\alpha-n} dy + \\ &\quad + K \int_{|x-y| \leq d} dy \int_{|x-y| \leq d} |u(y)|^2 dy \\ &\leq Kd^{\alpha} \int_{|x-y| \leq d} |\nabla^2 u(y)|^2 \omega_{4-\alpha}(x-y) dy + Kd^n \int_{|x-y| \leq d} |u(y)|^2 dy. \end{aligned}$$

Now $q \in L_{loc}^2$, therefore

$$\begin{aligned} \|qu\|_{\Omega}^2 &= \int_{\Omega} |q(x)|^2 |u(x)|^2 dx \\ &\leq Kd^{\alpha} \int_{\Omega} |q(x)|^2 \left(\int_{|x-y| \leq d} |\nabla^2 u(y)|^2 \omega_{4-\alpha}(x-y) dy \right) dx + \\ &\quad + Kd^n \int_{\Omega} |q(x)|^2 \left(\int_{|x-y| \leq d} |u(y)|^2 dy \right) dx \\ &\leq Kd^{\alpha} \int_{\Omega'} |\nabla^2 u(y)|^2 \left(\int_{|x-y| \leq d} |q(x)|^2 \omega_{4-\alpha}(x-y) dx \right) dy + \\ &\quad + Kd^n \int_{\Omega'} |u(y)|^2 \left(\int_{|x-y| \leq d} |q(x)|^2 dx \right) dy \end{aligned}$$

$$\begin{aligned}
&\leq Kd^\alpha \int_{\Omega} |\nabla^2 u(y)|^2 \left(\int_{|z| \leq 1} |q(z-y)|^2 \omega_{4-\alpha}(z) dz \right) dy + \\
&\quad + Kd^n \|u\|_{\Omega}^2, \\
&\leq Kd^\alpha \sup_{\Omega} N_{4-\alpha}(q) \|u\|_{2,\Omega}^2 + Kd^n \|u\|_{2,\Omega}^2, \\
&\leq K(\Omega, d, n) \|u\|_{2,\Omega}^2,
\end{aligned}$$

which establishes the result. \square

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