

**H^∞ -DESIGN AND THE IMPROVEMENT OF
HELICOPTER HANDLING QUALITIES**

Andrew Yue

Worcester College



Submitted in partial fulfilment of the requirements for the degree of
Doctor of Philosophy at the University of Oxford, Michaelmas 1988

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ABSTRACT

This thesis presents the results of a study into the use of H^∞ -optimization for the design of feedback control laws for improving the handling qualities of a Lynx helicopter. An important improvement to the H^∞ -optimization procedure is the reduction in the number of iterative steps in the γ -iteration before convergence to the optimal γ . Some new algorithms are derived which significantly reduce the computation time for the γ -iteration. Both 2-block and 4-block cases are considered.

Control laws are designed for precise control of pitch and roll attitude, yaw rate and heave velocity. Analysis of the raw helicopter showed the need for a stability augmentation system as the dynamic characteristics of the unaugmented helicopter do not comply with military helicopter handling qualities requirements. Results from current research on helicopter handling qualities were used as guidelines in order to define the required dynamic characteristics. A six-degree of freedom nonlinear simulation was used to analyse the helicopter dynamic time histories.

A possible solution to the problem of incorporating helicopter handling qualities in the design of robust controllers is to use a two-degree of freedom controller structure. This is illustrated using both H_2 and H^∞ -optimization.

A piloted simulation study to assess the effectiveness of advanced control laws was initiated at RAE, Bedford. The trials were carried out in the single seat cockpit flight simulator, at the Flight Research Division and represent the first ever real-time piloted simulation using a H^∞ -controller.

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*To my parents
and Fiona*

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CHAPTER 1

INTRODUCTION

1.1 *Motivation*

This thesis is motivated by the following problem: how to design robust controllers to improve the handling qualities in high performance helicopters in a variety of trimmed and manoeuvring flight conditions.

Handling qualities is the name given to the total set of characteristics that make an aircraft suitable or unsuitable for a particular task. The most significant change affecting handling qualities of fixed wing aircraft has been the adoption of FBW (Fly-By-Wire) control systems with full authority control. Current helicopters on the other hand have relied on low authority stability and control augmentation systems which are mainly effective in small perturbation manoeuvres. In particular the inherent cross coupling effects between the modes controlled by the pilot cause difficulties in handling and limit the desired flight path. This is of paramount concern when considering the tasks involved [STE1]. The requirement then, on any combat helicopter flight control system (FCS) is to stabilize the aircraft and decouple the controlled inputs, thus reducing pilot workload, but still to allow fast and tailored responses to pilot demands.

In an effort to reduce piloting workload, and to improve the pilot's ability to assimilate information and take decisions, helicopter handling qualities are to be improved through the use of active control technology (ACT). "Active Control " signifies that a computer is

directly and dynamically involved in flight control. ACT is expected to improve helicopter fighting potential by providing crisp, tailored handling qualities and carefree manoeuvring.

An important motivation for the use of feedback in control system design is the ability to provide satisfactory performance not only for operations under known conditions but also in the presence of uncertainties. Uncertainty will always be present in one form or another when trying to model an engineering process. For instance errors may be introduced by omitting high frequency dynamics from the model, uncertainty in parameter values and linearization. It is therefore important that any controller designed to give a feedback control system certain desirable properties does so not only to the nominal plant but also for any model taken from a set containing some “neighbourhood” of the nominal model. A controller which possesses this property is called a robust controller.

The term robustness as used in this thesis will refer to the extent to which a model of an open-loop system may be changed from the nominal design model without destabilizing the overall closed-loop system. This assumption assumes that the dynamic compensator is fixed. Real time changes in the compensator structure (gains or other changes) lead to adaptive control systems, a topic that will not be addressed in this thesis. Thus the term robustness refers to the preservation of closed-loop stability in the face of model uncertainty not accounted for in the compensator design.

In general it is not practical to assess the robustness of a system by experimentation alone and therefore robustness analysis tests will be used to determine the extent to which the design model can be perturbed without losing its desirable properties.

1.2 Previous Work and Related Literature

Various approaches exist for the design of multivariable flight control systems, the most popular methods use classical single-input single-output (SISO) techniques to design one

control loop at a time. For example, the Apache AV05 YAH-64 flight control system [ENN1], was designed using single loop techniques, with a proportional plus integral (PI) configuration. This approach can be useful for certain problems, but its capability is severely limited for highly coupled multivariable systems. Furthermore analysis of multivariable feedback systems with SISO techniques can give misleading results.

For single-input, single-output (SISO) systems, the classical frequency domain techniques (Nyquist diagram, Bode plot, Root locus, etc) give procedures that ensure that robust stability and certain desired performance levels are achieved. However these techniques do not readily extend to the multivariable case. For multivariable systems many attempts to provide similar quantifications for these properties have been proposed over the last two decades by MacFarlane and co-workers [MAC1], [MAC2], Rosenbrock [ROS1], [ROS2], Hung [HUN1] and Postlethwaite [POS1]. Although these contributions succeeded in extending many of the classical frequency domain concepts to the multivariable case, they failed to address satisfactorily the issue of robustness. MacFarlane has suggested that other indicators are needed [MAC3].

Many design techniques for MIMO systems such as pole placement, completely neglect the robustness issue in placing poles to obtain a good transient response. The presently available frequency domain MIMO design techniques, Inverse Nyquist Array (INA), [ROS1], Multivariable Root Loci [MAC4] and Characteristic Loci (CL), [MAC5], have the problem that they do not ensure stability for a sufficiently large class of modelling error. By making the system diagonal or near diagonal it is possible to treat the MIMO system as a series of single loop designs. Good stability margins may be obtained in this coordinate frame but not necessarily in the coordinate frame of the input and output of the plant, which is where it is important to have good stability margins. Therefore these methods may not detect small modelling errors that could potentially destabilize the closed loop system. Another drawback to the CL and INA design techniques is that they are limited to square systems.

Alternatives to the frequency domain methods include modal control and linear optimal control. Both of these approaches deal with state space formulations which permit non-square systems. Garrard and Liebst, [GAR1], and Parry, [PAR1], have proposed eigenspace assignment methods for the design of multivariable helicopter flight control systems. Eigenvalue placement is used for stability enhancement and eigenvector shaping is used for modal decoupling. This approach can be useful when design requirements can be easily expressed in terms of desired closed-loop modal characteristics, but in many cases system design requirements cannot be so simply expressed. This is especially true for design requirements associated with tolerance to model uncertainty.

Inspired by the work of Wiener in the 1940's, optimal control has until recently, been concentrated on the minimization of quadratic integral type criteria, either in the frequency domain (Weiner-Hopf optimization) or in the time domain (Linear Quadratic control). The main reason for this interest is the existence of explicit solutions to the optimization problem. In the Linear Quadratic Gaussian (LQG) approach, controllers that minimize a performance index in terms of the integral of a cost function are evaluated. The cost function being a sum of quadratic terms in the states and controls. The controller uses feedback of all the states of the system or estimates of these states if all the states cannot be measured. The Linear Quadratic approach has been used by Broussard et al, [BRO1] and Townsend, [TOW1], in the design of the Chinook CH-47 flight control system.

Control designs resulting from LQG synthesis were initially thought to have very attractive stability robustness properties [SAF1]. Additional research determined that these properties held only for "full state" measurement [DOY1], which is difficult to achieve in practice. Attempts have been made to "recover" these robustness properties using the LQG/LTR design method - LQG synthesis with loop transfer recovery, in which loop shapes of optimal full state regulators or filters are recovered (approximated) at plant inputs or outputs via certain choice of parameters [DOY2], [SAF2], and [STE2]. Unfortunately, this method

is restricted to minimum phase systems and therefore only encompasses a subset of typical engineering problems.

A major deficiency in each of the aforementioned approaches is the lack of guaranteed robustness with respect to model uncertainty eg. there is no natural way to include robustness requirements as design objectives in quadratic integral criteria. Since the work of Doyle and Stein [DOY2], singular value based techniques have emerged as the most popular tools for investigating the robustness of linear multivariable systems, [CRU1], [FRE1], [KOU1], [LEH1], [POS2], [SAF2]. Several considerations contribute to the effectiveness of this technique. First is the direct relationship between the size of the appropriate singular values and such properties as bandwidth. Singular value functions used in statements of design conditions can be thought of as a multi-input multi-output (MIMO) generalization of the classical Bode plots. What distinguishes MIMO from SISO design conditions are the functions used to express transfer function “size”. Singular values replace magnitude, but the underlying concepts governing design remain the same. Secondly, singular value analysis can provide guarantees that system properties are maintained when the plant is subject to certain types of perturbations. Finally, the singular values required for the analysis are easily obtained using widely available and reliable computer software.

However the potential of singular values as a synthesis method for designing robust controllers has not until recently been fully exploited. This has been due to the lack of systematic procedures to perform the loop shaping required. It wasn’t until the early 1980’s that an alternative to the LQG/LTR method was proposed, the H^∞ approach.

Much of the recent work on robustness has focussed on frequency domain characterizations of uncertainty and the use of H^∞ -optimization. Originating from the seminal work of Zames [ZAM1], this approach has succeeded in generalizing several important classical concepts in SISO systems to the multivariable case [FRA1], [FRA2]. In particular the H^∞ approach is more effective than the linear quadratic approach in characterizing robustness

with respect to plant variations [ZAM1], [ZAM2]. Zames argued that the poor robustness properties of LQG could be attributed to the use of the quadratic norm and criticized the use of Gaussian white noise to represent uncertain disturbances as unrealistic. While the quadratic performance index may be a meaningful cost function when the inputs and disturbances are known precisely, in many situations this is not possible in which case it is more meaningful to minimize the maximum error that can occur over the particular class of signals.

The H^∞ -norm is a very natural norm to use in evaluating stability margins in view of the small gain theorem [ZAM4], which guarantees stability for a feedback system if the product of the H^∞ -norms of the frequency response matrices in the feedback loop is less than one. The H^∞ -norm of a transfer matrix being the maximum over all frequencies of its largest singular value. The loop shaping required in order to achieve a robust design is also easier to perform using H^∞ -optimization than with linear quadratic optimal control.

The general problem of simultaneous robust performance and robust stabilization is difficult. Simultaneous minimization of the weighted nominal sensitivity function and weighted complementary sensitivity function has been investigated by Kwakernaak [KWA1] and Foo and Postlethwaite [FOO1]. Helton [HEL1] implicitly combines the robust stabilization and nominal performance requirements by imposing tracking, gain-phase margin and controller bandwidth constraints over different frequency bands for scalar plants. Performance specification is typically prescribed only over a finite frequency interval known as the operating band. This operating band being chosen to reflect a desirable bandwidth of the system and to cover a pertinent region where disturbances should be rejected. Nominal sensitivity over an operating band has been investigated by Zames and Benoussan, [ZAM3], and O'Young and Francis [OYO1], [OYO2].

H^∞ -optimal control is a powerful approach to multivariable feedback design, an approach whose robustness implications are examined in this thesis. Due to the important works of

Sarason [SAR1], Adamjan, Arov and Krein [ADA1], [ADA2], connections were established between operator theory and complex function theory and together with Nevanlinna-Pick interpolation theory [DEL1], this allowed closed-form solutions to the H^∞ problem without recourse to nonlinear programming. The procedure is now well understood and many papers have appeared in the literature outlining the solution using state space based L^∞ optimization theory [CHU1], [CHU2], [DOY3], [FRA3], [FRA4] and [SAF3]. Some applications of the theory can be found in [LIM1], [POS3], [SAF4], [SAF5], [TOM1] and [YUE1], [YUE2], [YUE3].

Despite the growing activity in the development of advanced control laws, there are currently only a few FBW helicopters flying. These include a Bell 205 of Canada's National Aeronautical Establishment, an MBB BO105 of West Germany's DFVLR research agency and NASA's Bell UH-1 and Boeing-Vertol CH-47 rotorcraft. Each is a research machine having a full authority automatic flight control system (AFCS) interfacing with the existing mechanical flight controls. The first attempt to develop a full flight envelope, full authority digital fiber-optic flight control system is the Advanced Digital Optical/Control System program (ADOCS), a demonstrator system being tested on the UH-60 Black Hawk aircraft. A high bandwidth model following control system design is employed to provide task tailored handling qualities [TIS1].

1.3 Contribution And Organization Of This Thesis

This thesis presents a critical assessment of the use of H^∞ - optimization as a tool for designing robust controllers to improve the handling qualities of battlefield helicopters. In the development of suitable control algorithms for future battlefield helicopters handling qualities appropriate to a wide range of tasks are identified and incorporated into the design process. In the context of robustness, amongst the constraints to which control systems must be designed are those associated with the mathematical model. Singular

value analysis is used to identify an acceptable variation in aircraft dynamics for which stability is assured.

The control laws designed will be useful in highlighting any deficiencies which may exist in current helicopter handling qualities criteria. These criteria have evolved from experience on conventional helicopter configurations, since data is not available to formulate design specifications based on any full authority FCS.

A note on the classification of H^∞ -problems is presented. Under certain conditions which are made precise in Theorem 4.1, 4-block problems can degenerate into 2-block problems and 2-block to 1-block. Necessary and sufficient conditions are derived which determine when these situations occur.

Some properties of the 4-block γ -iteration are established and it is shown that it is not possible to attain these results by a simple extension of the proofs given for the 2-block case.

The most computationally demanding part of the solution to the optimization problem lies in the γ -iteration. Some new bounds for the optimal γ are derived for both the 2-block and 4-block cases and these are shown to be much tighter than those attainable through a simple binary search. These algorithms are implemented and the resulting simulations show the benefits of the new algorithms.

One of the first applications of a 4-block H^∞ -optimal controller is presented. Particular attention is paid to the motivation behind the selection of H^∞ -weights. A possible solution to the problem of incorporating helicopter handling qualities criteria in the design of robust controllers is to use a two-degree of freedom controller structure. This is illustrated in chapter 6 using both H_2 and H^∞ -optimization. It is shown that weight selection for H_2 -optimization is similar to the H^∞ -case.

An Attitude Command Attitude Hold (ACAH) controller is designed for the hover and 80 knots flight conditions using H^∞ -optimization. It is shown that these controllers conform to the military handling qualities specifications. These controllers are further evaluated on the Advanced Flight Simulator (AFS) at RAE, Bedford. This is the first time that any real-time piloted simulation has been conducted using an H^∞ -controller.

This thesis is organised into 8 chapters of which this introduction is the first. Chapter 2 explains how current helicopters are controlled, the limitations of existing control systems and the type of augmentation that is needed if handling qualities are to be improved. It also outlines the difficulties involved in trying to quantify design objectives and presents the background on how the control strategies were arrived at. Some preliminary analysis of the unaugmented helicopter is undertaken to establish the difficulties likely to arise in the control system design.

Chapter 3 discusses uncertainty representation and the use of singular values as a measure of stability robustness. Attention is restricted to the class of unstructured uncertainty and it is shown that this class is sufficiently large to incorporate the major uncertainties in the helicopter model. Robustness tests for unstructured uncertainty are presented and are shown to motivate an important class of H^∞ -optimization problems.

Chapter 4 introduces the H^∞ -optimal synthesis theory. The classes of possible problems are presented and their implications towards design and numerical solution are discussed. It is shown that there are no restrictions on the class of plants which are solvable using H^∞ -optimization if the problem is formulated as a well defined 4-block optimization.

Chapter 5 reviews the γ -iteration scheme in H^∞ -optimization. Some new algorithms for the computation of the optimal γ are presented. Current search schemes are based on a bisection method which can be relatively slow if the optimal γ is far away from its initial start point. The new search scheme exploits the properties of a function $f(\gamma)$ to produce

tighter bounds for the optimal γ . The 2-block γ -iteration is discussed first and then these results are extended to the 4-block case. Some simple examples are given to illustrate the advantages and disadvantages of the search scheme.

Chapter 6 presents applications of H^∞ -optimization for the design of helicopter control systems. It is shown that the 2-degree of freedom controller is a useful configuration for incorporating helicopter handling qualities criteria and for stability robustness. A 2-block H^∞ -optimal controller and an H_2 -optimal controller is embedded in a two degree of freedom controller for control over the hover/low speed flight regime. For the forward flight regime a 4-block controller is used. The robustness of the designs is assessed using singular value methods and nonlinear time simulations.

Chapter 7 discusses the problem of handling qualities evaluation and the subjective nature of pilot rating schemes. It is shown that the controllers designed using H^∞ -optimization conform to the military handling qualities specifications. The need for flight testing is discussed and results from piloted simulation trials at RAE, Bedford are presented.

Chapter 8 contains concluding remarks and suggestions for further research.

1.4 Notation, Abbreviations And Definitions

In general, terms will be defined when they are first used. All matrices will be denoted by capital letters, scalars by lower case letters and vectors by underlined lower case letters. In order to simplify notation, the dependence of functions on an independent variable will be omitted when there is no risk of ambiguity.

The following symbols will be employed throughout this thesis.

$\Re$	Real numbers
$\mathbb{C}$	Complex numbers
$j\omega$	Imaginary axis
$\Rightarrow$	Implies
$\Leftarrow$	Is implied by
$\Leftrightarrow$	If, and only if
A^T	Transpose of the matrix A
L_2	Hilbert space of matrix-valued functions which are square integrable on $j\omega$
H_2	The functions in L_2 which are analytic in the open right half plane
H^∞	Hardy space of functions analytic and bounded in the open right half plane
L^∞	Banach space of (essentially) bounded matrix-valued functions
$G(s)$	Nominal plant model
$\tilde{G}(s)$	Perturbed plant model
$K(s)$	Controller transfer function
$E(s)$	Additive plant perturbation
$\Delta_o(s)$	Multiplicative output plant perturbation
$\Delta_i(s)$	Multiplicative input plant perturbation
$\forall$	for all
$\in$	is an element of
$\triangleq$	Equal to by definition
D_D^L	Direct Nyquist D-contour with indentations to the left of imaginary axis poles, and with a clockwise orientation
D_I^L	Inverse Nyquist D-contour with indentations to the left of imaginary axis zeros, and with a clockwise orientation
$\bar{\sigma}[\cdot]$	Maximum singular value of $[\cdot]$
$\underline{\sigma}[\cdot]$	Minimum singular value of $[\cdot]$

$diag[d_1, d_2, \dots, d_n]$	Diagonal matrix with elements $d_1, d_2, \dots, d_n$
$G^*(s)$	$G^T(-s)$
$F_\ell(P, K)$	Linear Fractional Transformation
$\ G\ _2$	$\frac{1}{2\pi} \int_{-\infty}^{+\infty} Trace\{G(j\omega)^* G(j\omega)\} d\omega$
$\ G\ _\infty$	$\sup \bar{\sigma}[G(j\omega)]$

When $\Re$ is used as a prefix, it denotes real-rational. For example $\Re H^\infty$ denotes real-rational functions which are proper (finite at infinity) and stable (no poles in the closed right half plane). The superscripts " n " and " $p \times m$ " (as in $\Re^n$ and $\Re^{p \times m}$) will denote the corresponding n -vectors and $p \times m$ matrices.

Abbreviations

ACAH	Attitude Command Attitude Hold
ACT	Active Control Technology
AFCS	Automatic Flight Control System
AFS	Advanced Flight Simulator
FBW	Fly-By-Wire
FCS	Flight Control System
FDLTI	Finite Dimensional Linear Time Invariant
LQG	Linear Quadratic Gaussian
MIMO	Multi-Input Multi-Output
MTE	Mission Task Elements
RAE	Royal Aircraft Establishment
SCAS	Stability and Control Augmentation System
SISO	Single-Input Single-Output
PIO	Pilot Induced Oscillations

CHAPTER 2

HELICOPTER FLIGHT CONTROL

2.1 *Introduction*

This chapter gives a brief description of how current helicopters are flown and the limitation of existing control systems. The inadequacy of present augmentation systems motivates the need for better control laws if helicopter handling qualities are to be improved. The main difficulty in the design of control laws has been the lack of any specification outlining the desirable vehicle characteristics. However through the use of piloted simulations it has been possible to establish the scope of augmentation needed and to explore the possible control concepts.

Helicopters in hovering flight have neutral static stability with angular displacement and positive static stability with changes in velocity. This means that there are no restoring moments that will tend to bring the airframe back to its original attitude following a disturbance. However any angular displacement from the trim attitude will result in a translational velocity caused by the tilt of the rotor. This creates a dihedral effect, similar to fixed wing aircraft, which restores the aircraft attitude back to its original position.

In forward flight, the helicopter behaves more like a conventional fixed wing aircraft. The combination of tail fin and tail rotor gives weathercock stability about the yaw axis. The helicopter is also neutrally stable about the roll axis for angular displacements. The longitudinal static stability in forward flight depends strongly on the rotor characteristics

and also on the helicopter configuration. Generally, the rotor is statically unstable with regard to changes in forward velocity or angle of attack.

Any form of augmentation system must improve the airframe natural dynamic stability. The difference between forward flight and hover flight behaviour must be taken into account in the system design. This difference is most marked with regard to lateral axis manoeuvres. In the hover flight position, the pilot changes heading using solely tail rotor thrust, whilst keeping the airframe at a level attitude. The augmentation system must therefore increase the deficient static stability about the roll and pitch axes to help the pilot keep the airframe level. However, the helicopter must remain neutrally stable about the yaw axis in order to be able to hover in different wind directions. In forward flight, the pilot uses turn coordination to invoke a change in heading. This involves tilting the helicopter thrust vector by banking the aircraft to counterbalance the centrifugal force generated by the turn rate. The augmentation system must therefore enhance the stiffness about the yaw axis in forward flight to prevent excessive side-slip velocities.

A comprehensive mathematical model of a typical battlefield helicopter has been provided by RAE, Bedford. Using this model, problems associated with the raw helicopter have been established and an initial assessment of the proposed control strategies made. These control concepts will be examined further in chapter 6, where simulation trials are carried out to assess their effectiveness.

No detailed description will be given of the helicopter aeromechanics which may be found in various texts [BRA1], [GEE1], [JOH1], and [SAU1]. Introductions to aircraft stability and control are given in [BAB1], [PAL1] and [PRO1].

2.2 Helicopter Control

Helicopter configurations maybe classified into two main types:

- (i) the single rotor and
- (ii) multiple rotors.

Each type has its unique characteristics, advantages and disadvantages. The single-rotor machine with tail rotor is by far the most common having the advantage of being relatively simple, one rotor, one set of controls and one main transmission. Helicopters with many rotors generally occur in larger machines. For the purposes of this study controller design will be based on the single-rotor type of helicopter.

Helicopter control is achieved by changing rotor blade pitch angle, either collectively to increase or decrease rotor thrust, or cyclically to tilt the rotor tip-path plane and with it the thrust vector. In effect there are four control surfaces available to the pilot; longitudinal and lateral main rotor cyclic for pitch and roll control respectively, and main rotor and tail rotor collective for heave (vertical) and yaw control respectively. The cyclic and collective pitch control schemes are illustrated in figures 2.1 and 2.2 [PAL1]

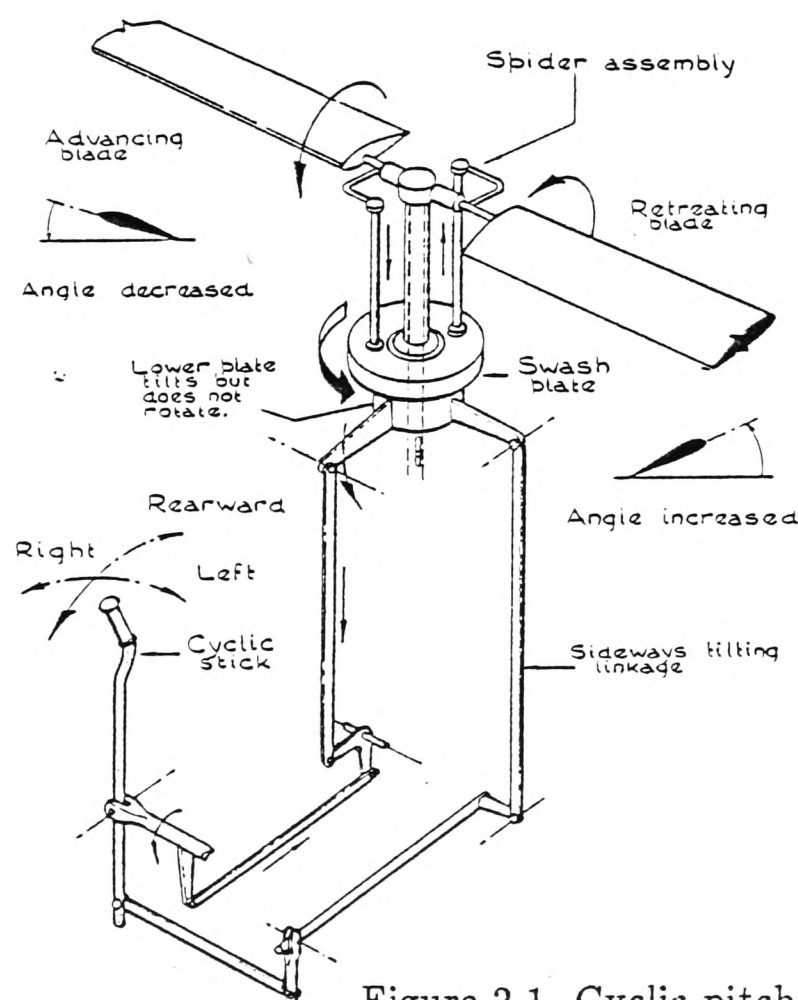


Figure 2.1 Cyclic pitch control system

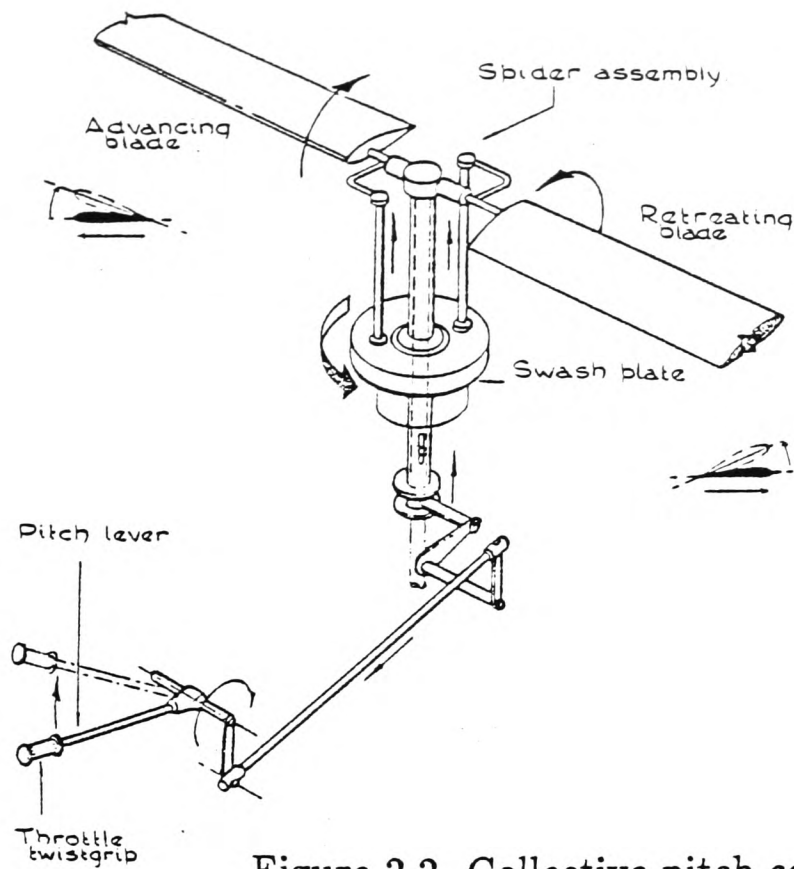


Figure 2.2 Collective pitch control system

In the past, helicopters have been controlled by mechanical links between the pilot's inceptors (sticks and pedals) and the actuators that move the rotor blades. To help the pilot cope with the inherent vehicle instability some form of stability augmentation system is fitted. These systems are usually limited in authority, mechanically, to typically ten percent of the total blade pitch range.

In manoeuvring flight, helicopters with conventional flight control systems test the pilot's skills to the limit. Flying a helicopter requires two hands and two feet. With one hand on the collective lever controlling rotor thrust, the other on the cyclic stick directing that thrust, and both feet on the rudder pedals controlling the anti-torque tail rotor, the pilot literally has his hands full, just flying the aircraft. This is clearly of concern when the pilot is in a battlefield role, when delivery of weapons and survivability are paramount.

In practice, helicopter flight control is complicated by cross-couplings, with perturbations in the pitch, roll and yaw axes caused by changes to main rotor collective pitch. These effects can make the helicopter difficult and fatiguing to fly unaided. The low authority

augmentation systems provide only limited assistance to the pilot and in some cases can be a hindrance. This is because in manoeuvring flight the augmentation system can oppose pilot control inputs, rapidly causing saturation during sudden flight path changes leaving the pilot to contend with the raw aircraft. Pilots often disengage the augmentation system prior to periods of manoeuvring flight. This conflict between pilot inputs and the stability augmentation system can be explained by the augments's inability to discern between disturbances such as gusts, which might cause the vehicle to move from its stable trimmed position and the desire of the pilot to actually do so.

The demand for increased manoeuvrability will lead to the introduction of very unstable bare airframe configurations, demonstrated in recent fixed-wing combat designs. Therefore the next generation of battlefield helicopters will need full time full authority, highly redundant and reliable, stability and control augmentation systems (SCAS). Also as sensors become available that enable the battlefield helicopter to operate in poor visibility and at night, demands on the pilot increase.

With these and future helicopter requirements in mind the aim is to improve helicopter handling qualities by providing:-

- (i) reduced cross-couplings between control axes and
- (ii) improved handling qualities throughout the flight envelope.

2.2.1 Stability augmentation systems

Early stability augmentation systems (SAS) were primarily mechanical. One of the first was Bell Helicopter's stabilizer bar (figure 2.3), which acts as a gyroscope and was installed in the control system so that its motion would produce stabilizing cyclic-pitch inputs if the helicopter was disturbed. Another mechanical device developed somewhat later was

the Hiller servo rotor (figure 2.4), which worked much like the bar, but used air forces to produce the required damping as well as to provide forces for changing cyclic pitch.

Lockheed's rotor system (figure 2.5) used a gyro, coupled by springs to the helicopter and the control system. In this case, fast response to control inputs was obtained by using blades that had no actual flapping hinges but flapped anyway around flexible parts of the hub [PRO1].

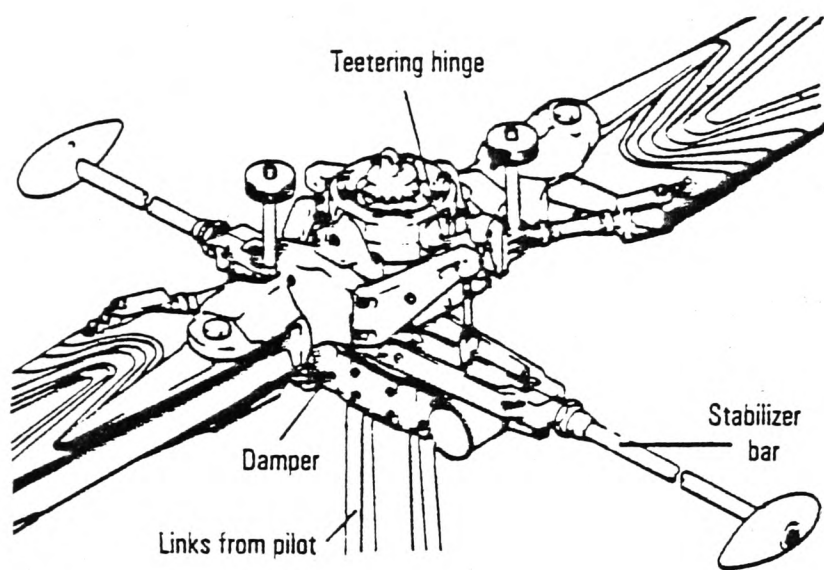


Figure 2.3 Bell stabilizing bar

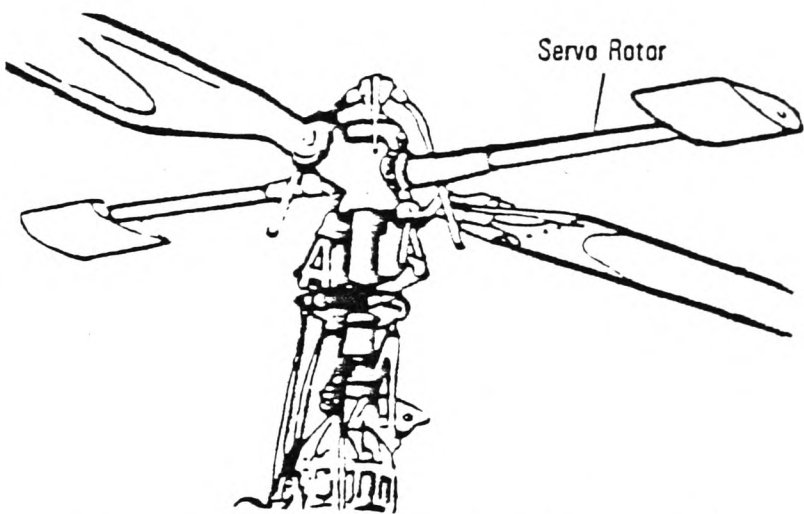


Figure 2.4 Hiller servo rotor system

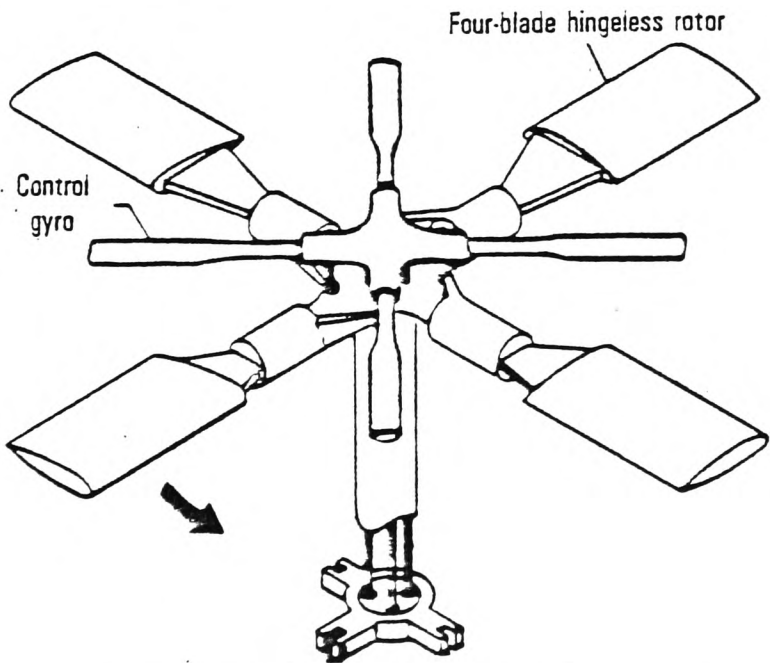


Figure 2.5 Lockheed rotor system

As electronic/hydraulic/mechanical systems became smaller and more realiable, they took

over the job of the mechanical devices, primarily because of their weight advantage. The primary use of the SAS is still to improve handling qualities by damping the pitch and roll motions caused by gusts. This is done by installing small gyros to generate electric signals proportional to pitch and roll rates. These signals are then used to control hydraulic or electric actuators that tilt the swashplate in the right direction to resist the helicopter motion.

Unfortunately, it not only resists motions due to gusts but also resists motions due to deliberate pilot control motions. This results in a helicopter with sluggish control response. To prevent this, the system must be designed to distinguish between the two types of inputs. This is done with a command augmentation system (CAS or a combined SCAS). The CAS uses an electrical signal which is a function of stick motion to cancel out the electrical signal coming from the rate gyros. A similar system can be installed in the tail rotor control system to improve directional and control characteristics.

Present SCAS are limited in authority to typically 10% of the available actuator power. This is really a safety measure, so that if a gyro unit suddenly puts out a signal corresponding to a $1,000^\circ$ per second rate for example, the resulting control motion is limited and can be overridden by the pilot. A more favoured safety scheme is to use dual or even triply independent systems so that the failure of one can be at least partially compensated for by the continuing operation of the other(s).

The use of a computer makes it possible to combine the signals from the various sensors to provide the best handling qualities in various flight conditions. This makes it a Fly-By-Wire (FBW) system with electronic signals generated by cockpit control motions replacing the push rods, cranks and cables used on most helicopters today. These schemes promise some weight saving but will have to be "101%" reliable.

2.3 Design Specifications

It is generally recognised that existing specifications of handling qualities are inadequate for the design of control laws aimed at achieving the best performance for difficult flight tasks and severe environmental conditions. Existing specifications are based on experience with helicopters without full authority FCS and do not therefore, embody the performance benefits of multi-loop feedback control. The difficulties in specifying performance requirements are also compounded by the fact that the control laws and therefore the handling qualities change with flight tasks and environmental conditions. Therefore, handling qualities, control laws and FCS requirements are usually designed for each flight task under the most severe conditions eg. “worst gust”. This leads naturally to the need for a design procedure which is able to handle “worst case” disturbances, model uncertainty, etc easily.

Comparisons of several US military specifications indicate that MIL-F-83300, [MIL1], has clear advantages in its broad coverage of important handling qualities aspects and its systematic structure. Its disadvantages are that it is primarily based on V/STOL data, and explicit helicopter characteristics are only lightly covered. MIL-H-8501A, [MIL2], does specifically address helicopters but is rather sparse in its coverage of many important topics. Walton and Ashkenas, [WAL1], provide a detailed analytical review of these shortcomings. The shortcomings of the current military specification on helicopter handling qualities, MIL-H-8501A, has prompted the major United States helicopter manufacturers to develop advanced test-bed vehicles to demonstrate the control characteristics required of the next generation of battlefield helicopters. This program has been designated as the US Army’s light helicopter program (LHX). The specifications outlined in this program, [HOH1], are ongoing and form the basis of which the control laws are to be designed in this thesis.

The Royal Aircraft Establishment (RAE), Bedford are also involved in refining this specification and through their conceptual simulation studies have identified some key control strategies.

2.4 Conceptual Simulation Studies

Real time piloted simulations have been taking place at RAE, Bedford, primarily to identify desirable handling qualities for the next generation of battlefield helicopters. Simplified “conceptual” models were developed to explore a range of alternate control concepts [BUC1], [TOM2] and [WIN1].

Initial experiments used a relatively complex, nonlinear, helicopter mathematical model, [PAD1], but difficulties arose in the design of control laws for such a detailed model. These problems were due to the need to maintain satisfactory handling characteristics throughout the flight envelope while still suppressing the inherent helicopter cross couplings. The conceptual models therefore had the natural aircraft cross couplings removed. The assumption was that such effects could be suppressed by the flight control system in a “real” FBW helicopter. The optimum control modes identified with the conceptual models are being developed further, using more “realistic” simulations, to determine the acceptable amount of cross-axis coupling, while still attaining “level 1” handling qualities. A vehicle is said to exhibit level 1 handling qualities if it is given a Cooper-Harper pilot rating of 1-3 [COO1]. More discussion of pilot ratings is given in chapter 7.

The conceptual simulations were found to be a powerful tool in answering the fundamental question of what pilots want from a control system. These requirements include precise control over flight path, exercised through attitude control, with low workload stemming from low compensatory control activity, and minimal cross-coupling resulting from controls or manoeuvres. Particular questions are concerned with whether an attitude or rate type of response is preferred.

Pilots feel more in control with a rate system which will stop at an attitude when the stick is centralised. With an attitude system, they are not sure what attitude will result from a given control movement and the rate of change of an attitude change depends on the

speed of movement of the hand. Although a rate system in principle requires more stick movement to effect a given manoeuvre, eg. two movements for a roll and stop versus one with an attitude system, pilots do not consider this to be a serious criticism of a rate system for “agility”. (Agility is the ease with which an aircraft makes rapid, precise changes in manoeuvres.) The main advantage of a rate system is the suitability for a wider range of manoeuvres. Attitude stabilization is useful in minimizing control activity in the “steady phase” of manoeuvres such as turns.

Therefore it was concluded that an attitude command system was preferred in the hover to low speed flight envelope, where precision control tasks were necessary, and a rate command system in the forward flight region, where manoeuvrability was important. These control modes formed the basis in which designs were formulated and are summarized in table 2.1.

Required Response Types

AXIS OF CONTROL	HOVER TO LOW SPEED	FORWARD FLIGHT
Pitch and roll	ACAH + PH	Rate
Yaw	Rate	Rate
Height	Rate	Rate

Table 2.1

Note: ACAH – Attitude Command/Attitude Hold.

PH – Position Hold or “hover hold”.

The results from the conceptual simulations suggested the following control configurations be chosen for detailed control law development.

(1) Hover controller (airspeeds below 40 knots)

(a) Left hand inceptor (collective lever)

Height rate control with optional height hold.

(b) Right hand inceptor (longitudinal and lateral cyclic stick)

Longitudinal displacements proportional to pitch attitude, measured relative to earth fixed axes.

Lateral displacements proportional to roll attitude, measured relative to earth fixed axes.

(c) Pedals

Yaw rate control with heading hold for centrally aligned pedals.

(2) High speed controller (airspeeds above 40 knots)

(a) Left hand inceptor (collective lever)

Height rate control with optional height hold.

(b) Right hand inceptor (longitudinal and lateral cyclic stick)

Longitudinal displacements proportional to pitch rate, measured relative to aerodynamic body axes.

Lateral displacements proportional to roll rate, measured relative to aerodynamic body axes.

(c) Pedals

Yaw rate control with heading hold for centrally aligned pedals.

2.4.1 The use of equivalent systems to quantify handling qualities

The desired handling characteristics are best expressed in terms of simple transfer functions which correspond to “low order equivalent systems”. These transfer functions can then be incorporated in the command model part of a two-degree of freedom controller (see chapter 6).

The concept of equivalent systems involves approximating high order mathematical models of aircraft dynamics with low-order models. These low-order models are equivalent in the sense that they produce a dynamic response to the pilot’s input which is reasonably close to the high order response. One immediate drawback of the equivalent system approach as a specification method is the necessity to quantify a “reasonably close” match between the high order and low order equivalent response. By relating the differences in pilot comments and ratings to analytical differences between the high and low-order responses, allowable levels of mismatch can be defined.

Three methods have been used by various investigators [HOD1], [SMI2] to obtain low order representations of the high order dynamics.

- (1) Selecting a subset of the high order roots for evaluation sometimes called the “dominant root” approach.
- (2) Matching time histories.
- (3) Matching Bode frequency response plots.

Past experience with handling qualities analysis of systems studied has indicated that mismatch is not a strong concern [SMI2]. Nevertheless, this correspondence between mismatch and pilot rating has raised the question as to whether low-order responses are a prerequisite to good control qualities.

High order dynamics often produce large lags at high frequencies. These lags cannot be approximated by simple low-order equivalent parameters and pilots describe these responses as delayed. Therefore, equivalent systems often include a time delay to approximate the phase lag. High order dynamics which reduce to a low order system having a large delay are prone to PIO and loss of control in demanding tasks. Also, the equivalent pure delay only approximates the lagged but smooth initial response onset of the high order continuous system. The question arises as to whether it simulates the high order response with sufficient accuracy.

Nevertheless, the use of equivalent systems is seen as an important link in quantifying pilot's desired responses. Research is currently being conducted at RAE, Bedford [HOU1] to derive equivalent system models which express desired helicopter handling characteristics. It is hoped that when these studies are complete the equivalent systems may be incorporated in the command model part of a two-degree of freedom controller in which the stabilizing controller has been designed using H^∞ -optimization.

2.5 The Helicopter Mathematical Model

A theoretical mathematical model of a single main rotor helicopter has been developed at RAE, Bedford, primarily for flight mechanics studies with application to handling qualities evaluation, [PAD1]. This model is used for both control law design and ground based simulator studies. Particular emphasis is given on modelling the Puma and Lynx helicopters which are research vehicles at RAE, Bedford.

The nonlinear model, “HELISIM”, has 6 degrees of freedom with optional quasi-steady or blade flapping rotor dynamics. The rotor flapping dynamics are difficult to model accurately due to aerodynamic nonlinearities. Actuators are modelled as first order lags with rate and amplitude limits. Validation of the model has been made with flight test data in both steady and manoeuvring conditions, however anomalies still exist and although the model is very complex some degree of uncertainty remains.

HELISIM is configured to run in conjunction with TSIM, [WIN2], [TSI1], an interactive computer aided simulation package for use in flight control system design. The package allows simulation, analysis and design of continuous and discrete control systems based on either linear or nonlinear models of the aircraft dynamics and provides a means of linearizing the equations of motion at any flight condition.

In this thesis, attention will be restricted to designing fixed gain linear controllers which are able to both stabilize the closed loop system and maintain some nominal performance in the two distinct flight envelopes of hover/low speed and forward flight. Since the helicopter model is nonlinear it is necessary to have at least two linearized representations at some nominal operating points. Therefore, the helicopter models were chosen to be representative of the hover flight condition and at 80 knots straight and level flight.

The basic linearized model of the helicopter has 8 states and 4 inputs, and is unstable and non-minimum phase. The non-minimum phase characteristic of the helicopter, limits the achievable performance [FRE2], [OYO1]. If the rotor flapping dynamics is included in the model, a 14 state model is obtained, but the rotor dynamics will be treated as being uncertain and will be left out of the nominal plant description. The actuator dynamics will also be neglected in the linear description of the model to reduce the complexity of the design model. The effect of these unmodelled dynamics is discussed in §2.6.3.

The state space description of the linearized rigid body equations of motion are expressed in the standard form as:-

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \quad (2.1)$$

$$\underline{y} = C\underline{x} + D\underline{u}$$

where the state vector $\underline{x}$ and input vector $\underline{u}$ are :-

$$\underline{x} = \begin{bmatrix} \theta \\ \phi \\ p \\ q \\ r \\ u \\ v \\ w \end{bmatrix} \quad \underline{u} = \begin{bmatrix} \theta_0 \\ \theta_{1s} \\ \theta_{1c} \\ \theta_{0t} \end{bmatrix} \quad (2.2)$$

The state variables and inputs are described in table 2.2 and the measured outputs are shown in table 2.3, it is assumed that all these states are available from appropriate sensors. Appendix 2.1 shows the state-space matrices used in the design.

The plant has only 4 inputs and therefore it is only possible to independently control 4 outputs, however the availability of measured variables in excess of the number of commanded outputs should lead to a design which makes more efficient use of gain. In general it is better to provide the controller with as much information as possible, provided of course that it is reliable.

Instead of using a 'squaring down' procedure to combine all the measured variables to form a new set of outputs (available for feedback) equal to the number of inputs, all of the measured variables will be used for feedback. The ideas are illustrated by figures 2.6 and 2.7. The plant has m inputs and l outputs, where $l \geq m$. The plant output vector $\underline{y}$ can be decomposed into two sub-vectors $\underline{c}$ and $\underline{z}$, where

$$\underline{y} = \begin{bmatrix} \underline{c} \\ \underline{z} \end{bmatrix} \quad (2.3)$$

State Variable and Plant Input Description

STATE	Description	INPUT	Description
θ	Pitch attitude	θ_0	Collective
ϕ	Roll attitude	θ_{1s}	Longitudinal cyclic
p	Roll rate	θ_{1c}	Lateral cyclic
q	Pitch rate	θ_{0t}	Tail rotor collective
r	Yaw rate		
u	Forward velocity		
v	Lateral velocity		
w	Vertical velocity		

Table 2.2

Measured Outputs

Outputs for controller	Description
$\dot{h}$	Heave velocity
p	Roll rate
q	Pitch rate
$\dot{\psi}$	Heading rate
θ	Pitch attitude
ϕ	Roll attitude

Table 2.3

where $\underline{c}$ is an m -dimensional vector of variables to be controlled and $\underline{z}$ is an $(l - m)$ dimensional vector of extra available measurements. One way of utilizing all of the available outputs is to choose a feedback matrix F in the configuration of figure 2.6 such that F has l inputs and m outputs, the choice of F being referred to as a 'squaring down' procedure.

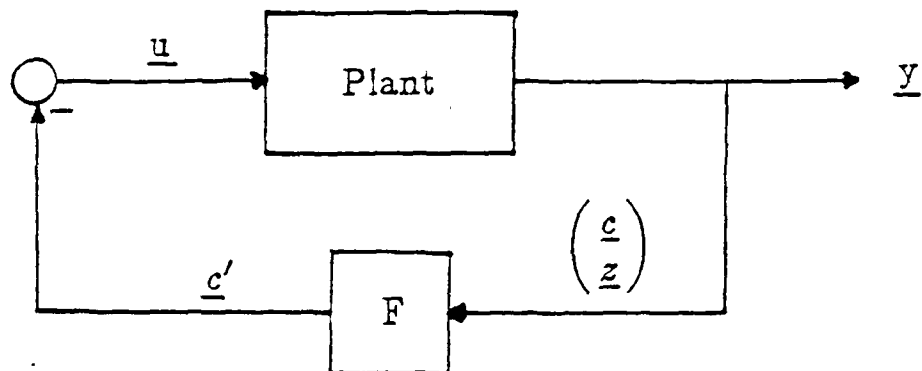


Figure 2.6

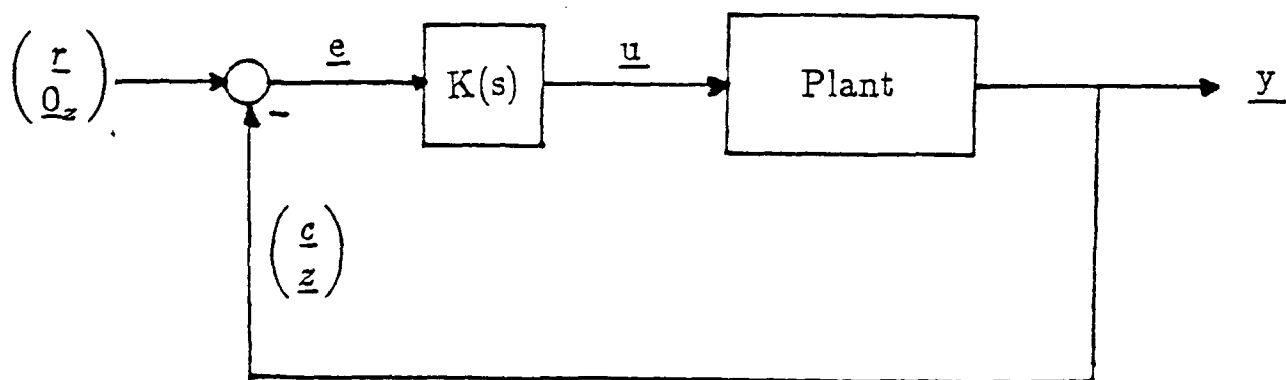


Figure 2.7

However this squaring down process involves the introduction of extra zeros into the corresponding loop transmission transfer function matrix. The location of these zeros vitally affects the stability characteristics of the closed loop system. The placement of these zeros requires dynamical insight and engineering judgement, there being no systematic procedure available. It is therefore proposed instead that all available measurements are fed back directly to the inputs, where zeros are augmented to the input vector to make the number of inputs and outputs equal. A compensator is then designed to dynamically generate the required inputs to the plant. This overcomes the need for a two-stage design procedure such as that proposed in [KOU2].

2.6 Design Objectives

2.6.1 Interaction

The helicopter without augmentation exhibits unsatisfactory response during manoeuvres, [CHA1]. The responses to the collective, longitudinal cyclic, lateral cyclic and pedals are highly coupled and unstable. Pilot workload is high and precise control is difficult.

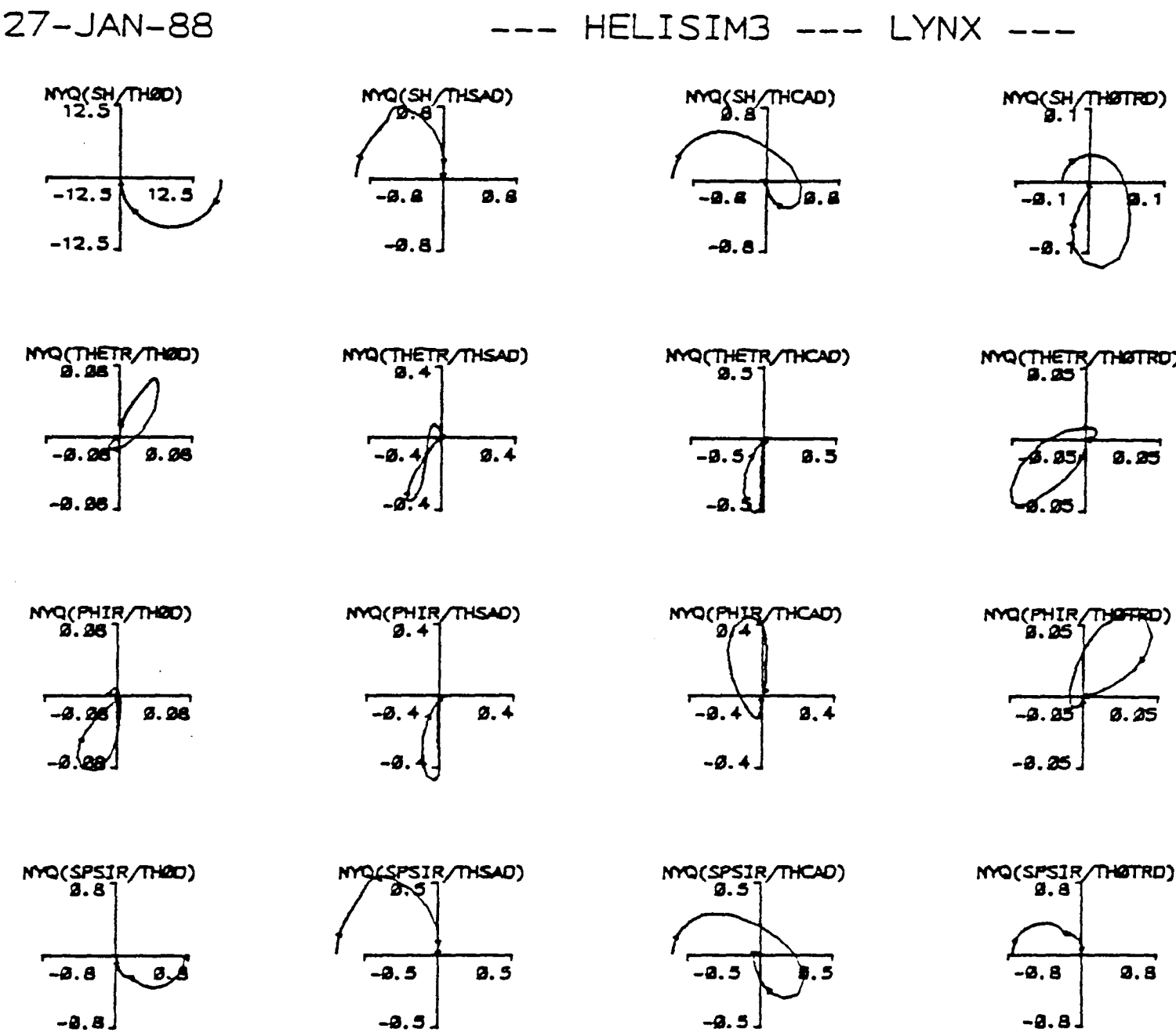


Figure 2.8 Direct Nyquist array of open-loop uncompensated helicopter in the hover flight condition

This interaction is depicted in the open loop direct Nyquist array of figure 2.8 (a key to the HELISIM variables is given in appendix 2.2). This figure shows the heave axis to be weakly coupled, as shown by the dominant Nyquist loci of element (1,1). So reduction of interaction in this loop should not be difficult. The pitch, roll and directional axes are highly coupled with each other, as shown by the large off diagonal elements. These couplings are undesirable and make the helicopter difficult to fly and result in high pilot workload. The amount of interaction varies with speed and attitude; of particular concern is the nonlinear nature of interaction between axes, which makes the selection of linear models both difficult and uncertain.

Specific limits on interaxis coupling for the hover/low speed flight envelope are [HOH1] :-

(a) Collective to Yaw Coupling

The yaw rate response to a step collective input with the directional controller free should not exceed the boundaries specified in figure 2.9.

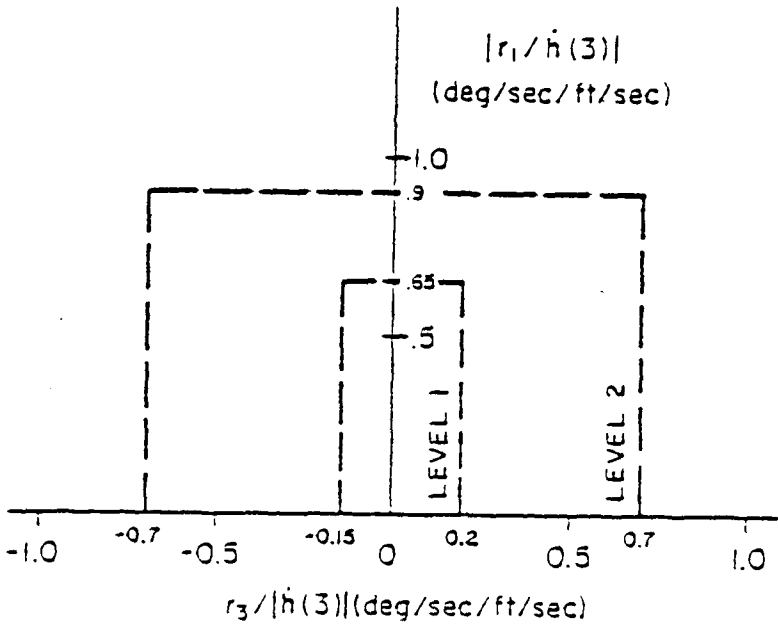
(b) Pitch to Roll and Roll to Pitch Coupling

The pitch (roll) attitude response to a step change in roll (pitch) attitude should not exceed the limits specified in table 2.4.

Limiting Values For Pitch to Roll and Roll to Pitch Coupling

PARAMETER	LEVEL 1	LEVEL 2
$\theta_{peak}/\phi_{step}$ or $\phi_{peak}/\theta_{step}$	± 0.25	± 0.60

Table 2.4



where:

r_1 = first peak (before 3 secs)
or $r(1)$ if no peak occurs before 3 secs.

$$r_3 = \begin{cases} (r(3) - r_1), & \text{for } r_1 > 0; \\ (r_1 - r(3)), & \text{for } r_1 < 0. \end{cases}$$

$r(1)$ and $r(3)$ are yaw rate responses measured at 1 and 3 secs, resp., and $h(3)$ is altitude rate measured at 3 secs., following a step collective input at $t = 0$.

In the unlikely event of more than one peak before 3 secs., the largest (by magnitude) should be designated as r_1 .

Figure 2.9 Collective to Yaw coupling requirements

2.6.2 Performance

The performance objectives are to minimize cross coupling, to achieve improved stability margins, and to meet specifications on time response characteristics required for good handling. Typical control power requirements are shown in table 2.5.

Pitch, Roll and Yaw Control Power

Mission-Task Element	Rate Response Types						Attitude Response Types			
	Minimum Peak Angular Rate at Full Control Deflection (deg/sec)						Minimum Steady Angle at Full Control Deflection (deg)			
	Level 1			Levels 2 and 3			Level 1		Levels 2 and 3	
	q	p	r	q	p	r	θ	ϕ	θ	ϕ
<u>Limited Manoeuvring</u> All MTEs not otherwise specified	±6	± 21	± 9.5	± 3	± 15	± 5	±13	± 13	± 6	± 6
<u>Moderate Manoeuvring</u> Evasive Action Assault Landing	±13	± 50	± 22	± 6	± 21	± 9.5	+20 -30	±60	± 13	± 30
<u>Aggressive Manoeuvring</u> Quickstop Lateral unmask/ remask Rapid pedal turn slalom	±30	± 50	± 60	± 13	± 50	± 22	±30	± 60	+ 20 -30	± 30

Table 2.5

The control power should be sufficient to produce at least the pitch, roll and yaw steady rates (for a rate system) or attitude changes (for an attitude system) shown. The rise times for each controlled output following an abrupt step input should be as small as possible, within the limits of the aircraft, and typically being in the order of less than 1.5 secs for the pitch and roll axes. There should be no objectionable nonlinearities or overshoots in the response.

The criteria used for assessing system performance are based on the proposed US handling qualities specification [HOH1]. The proposed US specification is a major revision of the much older MIL-H-8501A helicopter handling qualities specification [MIL2].

2.6.3 Robustness

Robustness is a primary issue in the design because of model uncertainty (especially due to the omission of high frequency rotor dynamics and actuator dynamics), sensor noise and wind gusts. The maximum achievable bandwidth is determined by the high frequency dynamic characteristics, especially those which are due to the rotor dynamics, and high gain feedback may aggravate these. This problem has been reported by Tischler [TIS1], in the implementation of the ADOCS system in which a 40% reduction of feedback gains was necessary because of unmodelled rotor lead-lag dynamics. Tischler also reports problems in implementing high-bandwidth systems whose designs are based on time-domain methods such as LQG. This was a result of the compensators being too highly tuned to the assumed high-frequency dynamics, so that when the compensators were used on the actual system with additional unmodelled dynamics, unacceptable system sensitivity and instability resulted. Frequency domain techniques such as H^∞ are able to expose the high frequency details more easily and integrate with current proposed handling quality specifications which are based on frequency domain criteria.

By examining the open loop poles of the linearized helicopter model about the trimmed hover position (table 2.6) it is possible to see that the rotor lead-lag dynamics are significant at frequencies above 10 rad/s, indicating that a bandwidth of less than 10 rad/s is desirable if these dynamics are to have little effect.

Open loop Pole positions					
(including rotor dynamics)			(without rotor dynamics)		
No	Real and Imaginary		Real and Imaginary		
1	-15.48	69.67			
2	-15.48	-69.67			
3	-15.85	35.55			
4	-15.85	-35.55			
5	-8.48	10.45			
6	-8.48	-10.45			
7	-12.97		-10.89		- Pure roll mode
8	-2.67		-2.22		- Short period mode
9	-0.335		-0.331		
10	-0.314		-0.314		
11	-0.18	0.16	-0.18	0.16	} Dutch roll mode
12	-0.18	-0.16	-0.18	-0.16	
13	0.25	0.53	0.25	0.53	} Phugoid mode
14	0.25	-0.53	0.25	-0.53	

Table 2.6

By performing an eigenvalue/eigenvector decomposition of the system 'A' matrix it is possible to identify the dominant helicopter modes, however these modes are not so easily discernable as in the fixed wing case because of the strong cross axis couplings. For fixed wing aircraft it is relatively easy to identify the eigenvalues and to relate them to the well known modes such as the Dutch roll and Phugoid motions. The most convenient way to classify them for the helicopter is to separate the short term dynamic responses from the long term responses. This is the method most commonly adopted in the military handling qualities specifications.

From table 2.6 it can be seen that the longitudinal dynamic short term responses are determined by mode 8 and modes 13 and 14. Modes 13 and 14 are an unstable divergent oscillatory motion with a natural frequency of 0.59 rad/s and a divergent damping ratio of -0.43. This motion is due to the natural static instability of the rotor with changes in forward speed and incidence. The short term dynamic response of the helicopter is therefore unstable in this flight case and incidently throughout the flight envelope.

The lateral dynamic short term response can be separated between the yaw and roll axes. The pure roll mode is determined by mode 7. This mode is an aperiodic motion with a time constant of less than 0.1 secs. This fast dynamic response is caused by the rotor which transfers large hub moments and also by the low airframe rotational inertia about the roll axis. The directional short term response about the yaw axis is determined by modes 11 and 12. These modes correspond to a stable oscillatory motion with a natural frequency of 0.24 rad/s and a damping ratio of 0.75. The directional short term response is caused by the stiffness and damping induced by the tail fin and tail rotor. Roughly speaking modes 13 and 14 can be thought of as an unstable phugoid motion and modes 11 and 12 a stable dutch roll.

The change in dynamics of the helicopter with manoeuvring flight can be investigated by considering how the helicopter modes change with speed and turn rate. This was achieved

by use of the Helistab program [SMI1], provided by RAE, Bedford. Helistab computes the trim conditions necessary to maintain equilibrium in a general flight condition eg. descending, turning and sideslipping flight. Figure 2.10 shows the changes in pole positions of the open loop helicopter as the speed increases from 0 to 80 knts and figure 2.11 shows the changes in pole positions as a function of turn rate in both the hover flight condition and at 80 knts straight and level flight.

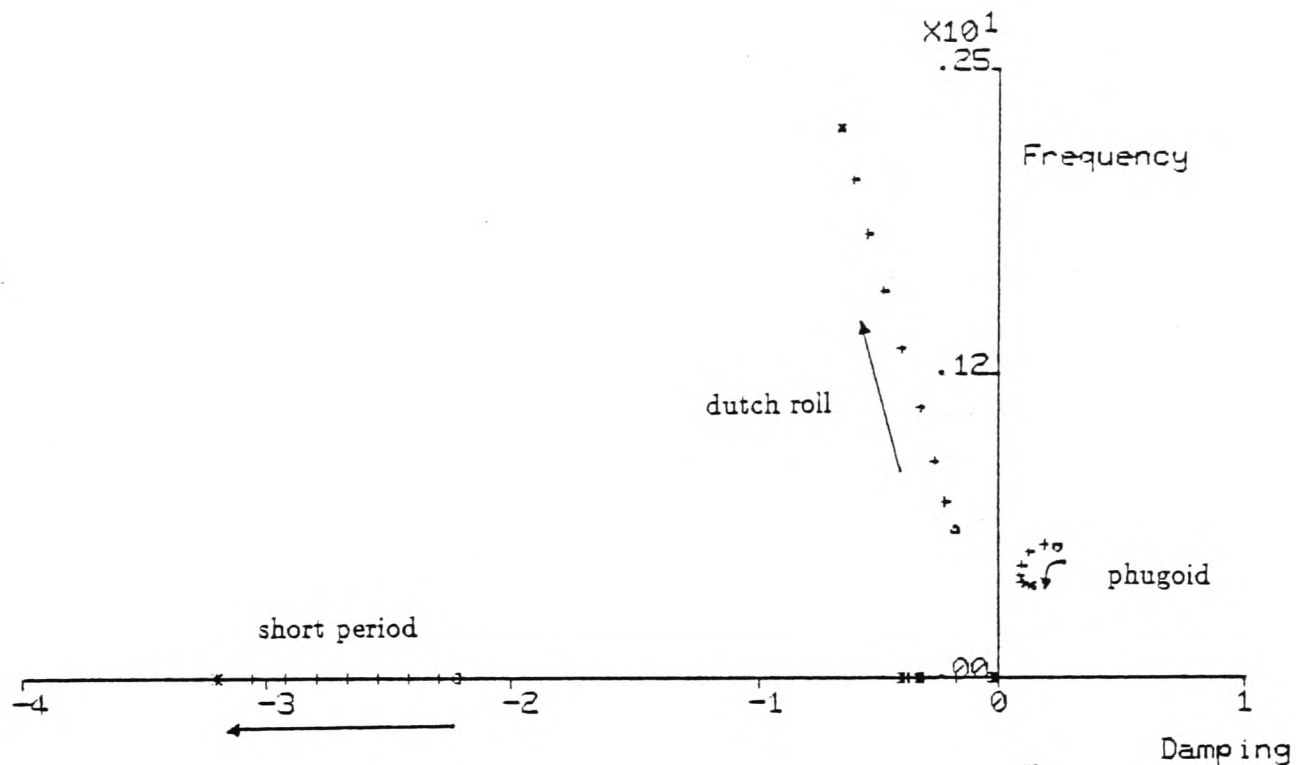


Figure 2.10 Change in pole positions of the helicopter
as speed increases from 0 to 80 knots

These figures show that there is a significant change in dynamics with manoeuvring flight, particularly from the hover position. It can be seen that the helicopter is longitudinally unstable with a divergent short term pitching oscillation. This oscillation becomes more divergent as the turn rate increases. The lateral directional short term response is dynamically stable but poorly damped. The natural frequency and damping ratio of this motion increases with airspeed.

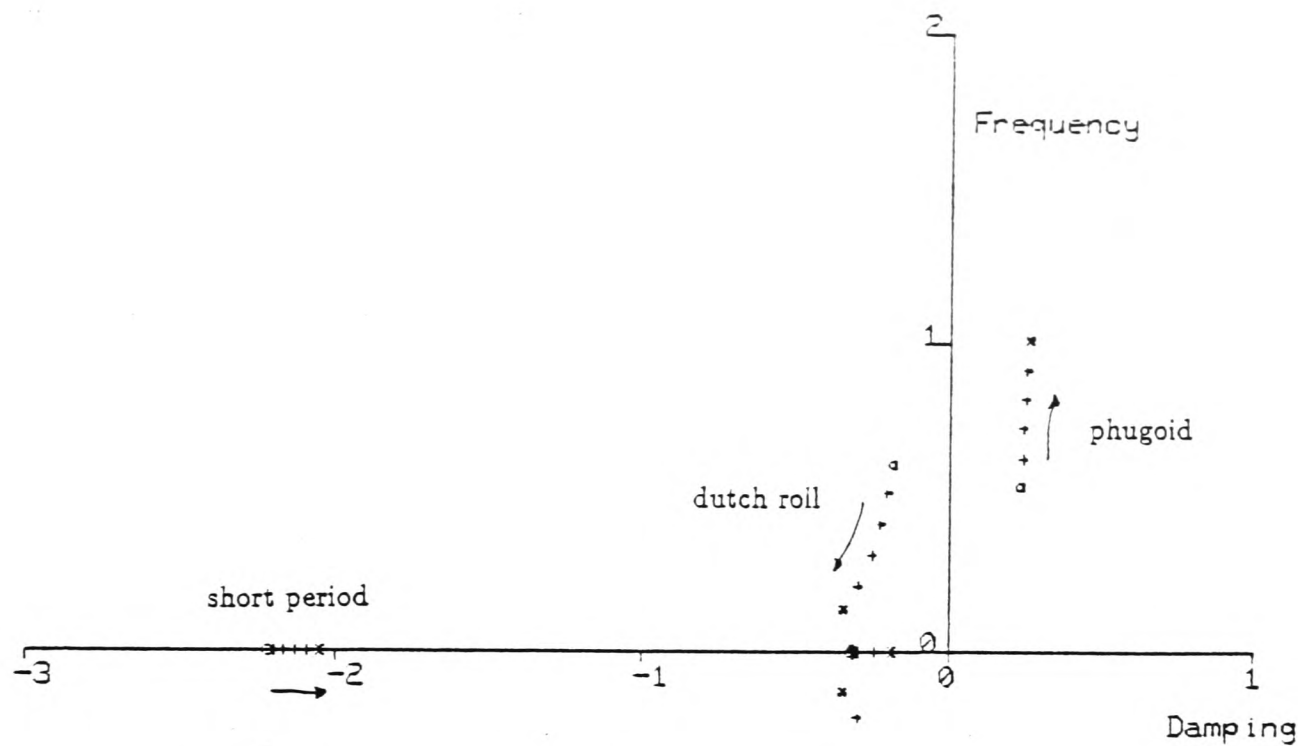


Figure 2.11(a) Change in pole positions of the helicopter as turn rate increases from 0 to -0.5 rad/s (hover case)

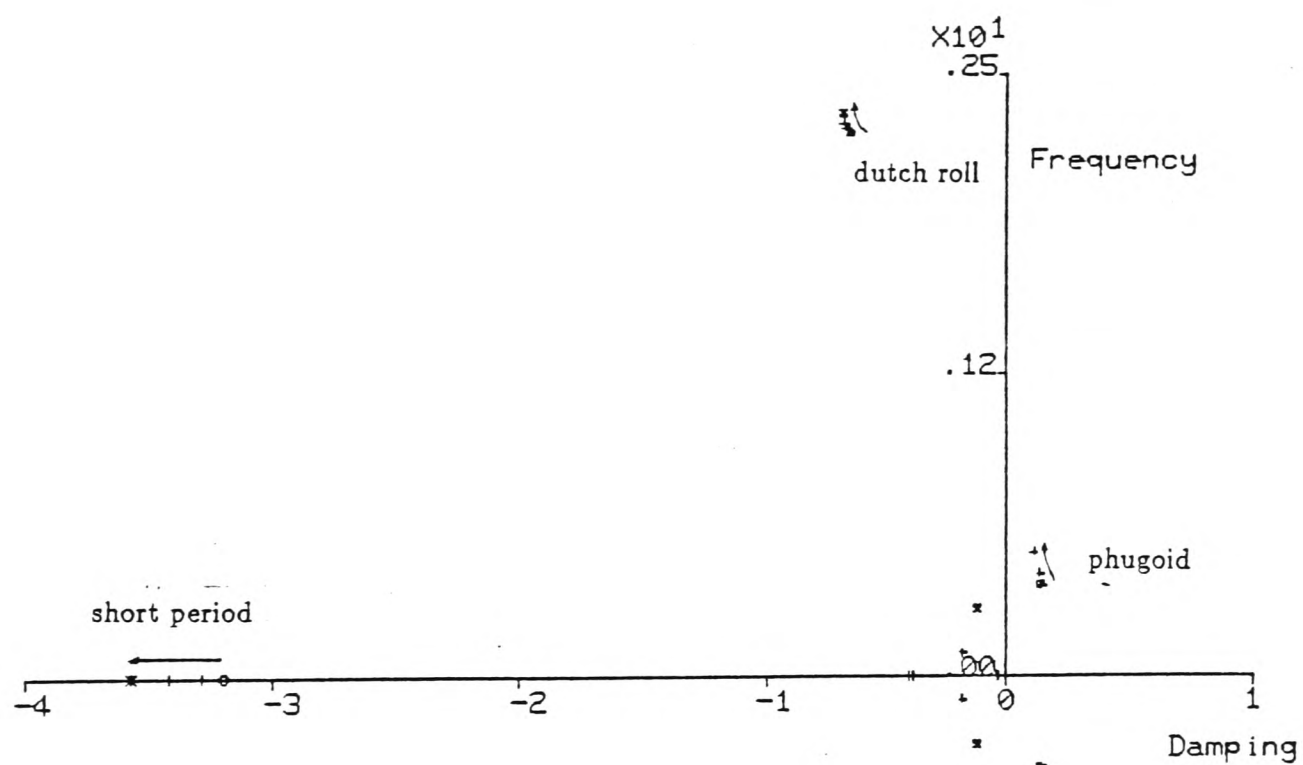


Figure 2.11(b) Change in pole positions of the helicopter as turn rate increases from 0 to -0.5 rad/s (80 knts case)

When turning from the hover (figure 2.11(a)), the dutch roll mode becomes more stable with a decrease in frequency whereas the phugoid becomes slightly more unstable with an increase in frequency. This trend is not necessarily the same at all speed ranges as shown

in figure 2.11(b) where the dutch roll mode becomes slightly more stable, with a small increase in frequency. Here relative stability is judged by the locus of poles traversing either further into the left (stable) region or right (unstable) region of the 's' plane. The roll damping mode at $\simeq -10$ has not been shown for clarity.

It is important that no further unstable poles are introduced by the rotor flapping dynamics and the resultant change in dynamics due to manoeuvring flight. This is of significance when trying to provide characterisations of these unmodelled dynamics. If the final design, based on one operating point, is to be insensitive to these changes it is necessary to have some information indicating where the uncertainty becomes significant. Uncertainty representation will be discussed in Chapter 3.

The effect of the unmodelled actuator dynamics on the closed loop system will become significant when the time constants of the filters are of similar magnitude to the desired closed loop responses. They have the effect of making the responses more sluggish. To eliminate this effect it is necessary to include the actuator dynamics in the original plant description.

2.6.4 Stability

Stability must be one of the most important objectives, for without stability there cannot be performance. In the frequency domain, stability of a system is defined by its transfer function $G(s)$ not having any singularities in the closed right half-plane (CRHP). This is because poles in the CRHP correspond to non-decreasing exponential functions in the time domain. The stability problem then becomes one of determining whether there are any poles of $G(s)$ in the CRHP.

The uncompensated helicopter in the hover flight condition has two poles in the right half-plane (table 2.6), therefore for stability, the generalized Nyquist stability criterion,

requires that the characteristic loci have a net sum of two anti-clockwise encirclements of the critical $-1/k$ point. The characteristic loci of a matrix $G(s)$ are defined as the set of loci in the complex plane traced out by the eigenvalues of $G(s)$ as s traverses the standard Nyquist contour in the clockwise direction; they are discussed in detail in [MAC6].

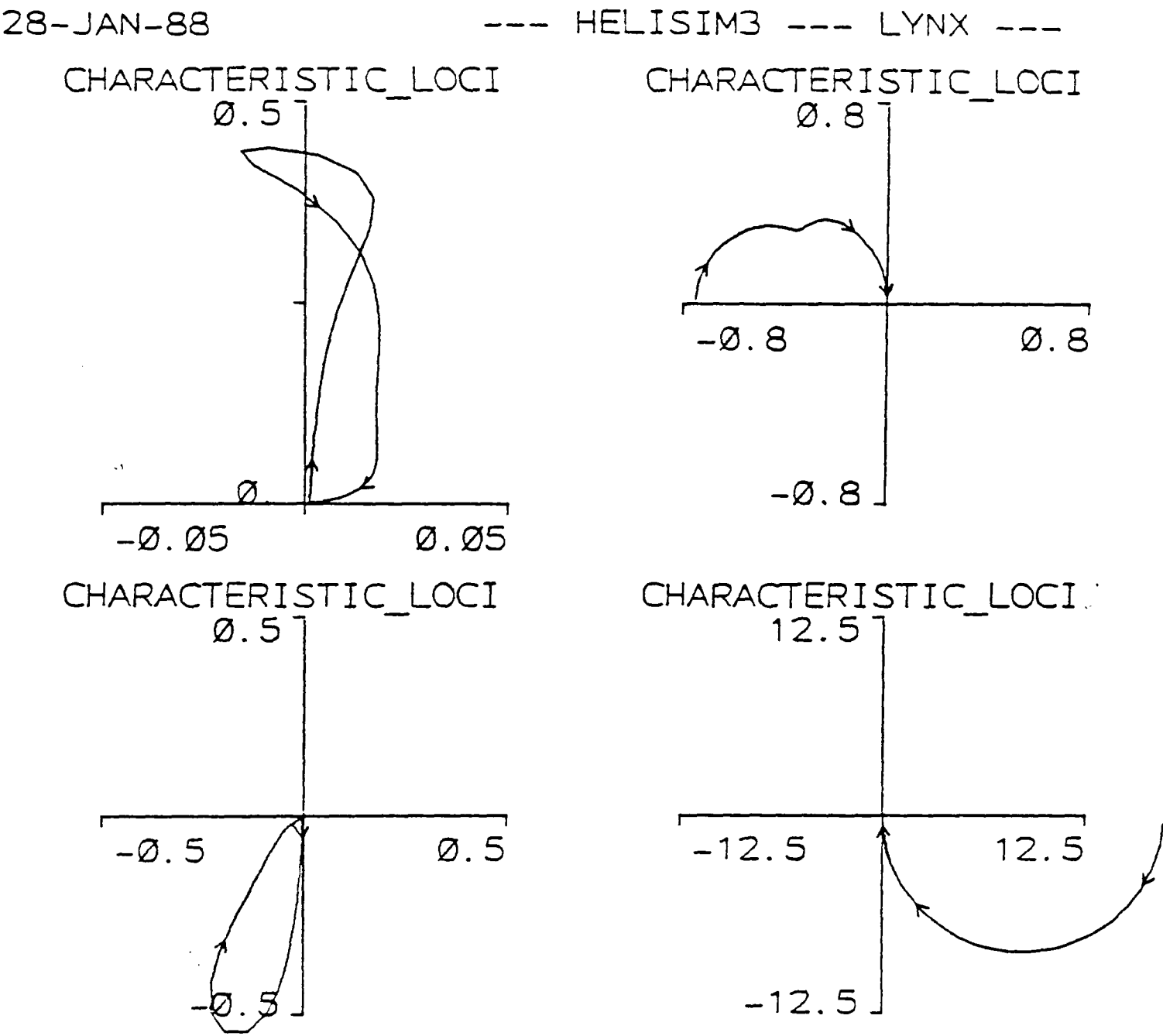


Figure 2.12 Characteristic loci of open-loop uncompensated helicopter

Since the characteristic loci are defined only for square systems, the outputs of the uncompensated system will be taken as the four outputs requiring direct control, namely $\dot{h}, \theta, \phi$ and $\dot{\psi}$. An inspection of the characteristic loci of the uncompensated helicopter,

figure 2.12, shows that there is one clockwise encirclement of the critical $-1/k$ point. Therefore the system cannot be made stable under direct proportional feedback and some form of compensation is necessary.

2.7 Concluding Remarks

This chapter has introduced the helicopter flight control problem and motivated the need for augmentation over the raw helicopter. The helicopter has been shown to be a complex nonlinear multivariable system. Control is difficult due to the rapid and significant changes in dynamics when manoeuvring and therefore care must be taken when designing the control system on the basis of a linear model alone. Design objectives have been shown to be difficult to quantify, from a control point of view, with specifications arising from criteria based on pilots' comments. The proposed US handling qualities specification, [HOH1], is the only design data available and this is far from being completely satisfactory. Specifications are based on SISO types of response and therefore do not reflect the true multivariable nature of the helicopter. This specification is not geared to any particular design technique and therefore whatever method is used, must be flexible enough to incorporate this data.

Although complex mathematical models are available for design, some degree of uncertainty still remains as to their true representation of the actual helicopter. Therefore a design technique is required which is able to encapsulate worst disturbances and unmodelled dynamics and be able to demonstrate compliance in terms of present specifications.

APPENDIX 2.1

STATE SPACE MODELS OF HELICOPTER

A Matrix

-0.1098004d+02	0.0000000d+00	-0.6227829d-01	-0.1189311d+01	0.1500764d-04	0.2361438d+01	0.2115513d+01	0.0000000d+00
-0.1000000d+01	0.0000000d+00	0.0000000d+00	0.6690983d-01	-0.8443195d-06	0.3124583d-01	0.0000000d+00	0.0000000d+00
-0.8498279d-02	0.1587611d-05	-0.3111644d+00	0.4957515d-03	-0.9109505d-06	0.8289166d-03	-0.1063457d-01	-0.1196480d-01
0.1911301d+01	0.0000000d+00	0.2004853d-01	-0.5168961d+00	0.6522590d-05	0.2553959d+00	-0.2019138d+00	-0.2062515d-02
-0.2411826d-04	0.0000000d+00	-0.2529877d-06	-0.4231152d+00	0.5339191d-05	0.9060696d+00	0.2536546d-05	0.2602639d-07
0.5032578d+00	0.0000000d+00	0.7127924d-02	0.7437967d+00	-0.9385796d-05	-0.1756567d+01	-0.1099349d+00	0.4156928d+00
0.6176109d-01	0.1236467d+01	-0.1045912d-01	0.7562032d-01	0.1582106d+01	-0.1042913d+00	-0.2713285d-01	0.9153990d-02
0.1310181d+00	0.1577811d+01	-0.1170052d-01	-0.8877347d-02	-0.1239834d+01	0.6207670d-01	-0.3164287d-01	-0.3207163d-01

B Matrix

-0.1227945d+00	-0.1312919d+00	0.2741718d+01	0.1958942d-01
0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
-0.3361077d+00	0.2353931d-02	0.1933370d-02	-0.1066703d-02
0.2584371d+00	-0.2018792d+00	-0.4470044d+00	-0.2327907d+00
-0.3261157d-05	0.2547466d-05	0.5640644d-05	0.2937531d-05
0.1567702d+00	0.4286827d+00	-0.2128582d+00	-0.1242282d+00
-0.1188708d-01	0.2778698d-01	-0.1983606d-01	0.1242516d-01
-0.1504712d-01	-0.1970843d-01	-0.2773675d-01	0.1572759d-01

C Matrix

0.0000000d+00	0.0000000d+00	-0.3200000d+01	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
-0.2000000d+01	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.0000000d+00	0.0000000d+00	0.0000000d+00	-0.2354021d+01	0.2970484d-04	0.4411189d+01	0.0000000d+00	0.0000000d+00
0.0000000d+00	0.3070178d-03	0.0000000d+00	0.5674763d+01	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.0000000d+00	0.0000000d+00	0.0000000d+00	0.6309383d-04	0.5000000d+01	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.0.00000d+00	0.2000000d+01	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00

D Matrix

0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00

State space matrices for helicopter in the hover flight condition

A Matrix

-0.1328088d+01	0.0000000d+00	0.1482468d+00	0.1599098d+01	-0.0000000d+00	-0.6437764d-01	0.4137436d+00	0.2140905d+00
-0.3609275d-07	0.0000000d+00	-0.1000000d+01	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.1329921d+00	0.0000000d+00	-0.2206046d+01	0.6758482d+00	0.0000000d+00	0.2700248d+00	0.5388271d-01	0.1353701d+00
0.5611290d+00	0.0000000d+00	-0.1786558d+01	-0.1064990d+02	0.0000000d+00	0.4580907d+00	0.1086462d+01	0.0000000d+00
0.3568408d-01	0.0000000d+00	-0.1080901d-08	-0.1000000d+01	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.1071740d+00	0.9471773d-06	0.7775618d+01	-0.3271831d+01	-0.1443620d-05	-0.7749502d+00	-0.4596619d-01	-0.3262838d-01
-0.7264736d+01	-0.2558448d+00	0.2022673d-01	-0.1516092d+00	0.1992420d+01	-0.9734013d-01	-0.2025740d+00	0.5667781d-02
-0.4139726d+01	0.1993688d+01	0.1147677d+00	-0.1618329d+00	0.2556814d+00	-0.6407759d-02	-0.1443331d-01	-0.2092322d-01

B Matrix

-0.3446595d+01	-0.1292752d+01	0.6727126d+01	0.5098350d+01
0.0.00000d+00	0.0.00000d+00	0.0000000d+00	-0.0000000d+00
0.3734149d+01	0.8040693d+01	0.1767571d+01	0.7598846d+00
0.6700075d+00	0.2921469d+01	-0.4283819d+02	-0.3826954d+00
0.0.00000d+00	0.0.00000d+00	0.0000000d+00	-0.0000000d+00
-0.6677505d+01	-0.1345011d+01	-0.1117183d+01	0.5776440d-01
-0.9762202d+00	-0.1732839d+00	0.3796597d+00	-0.3849275d+00
-0.1553142d+00	-0.5606512d+00	0.3363301d-01	-0.4731846d-01

C Matrix

0.6270099d-07	-0.7826456d+00	0.8472672d-07	0.2235170d-06	0.3470400d+00	-0.1000000d+00	0.0000000d+00	0.0000000d+00
0.0000000d+00	0.0000000d+00	0.0000000d+00	0.1250000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
-0.4606661d-01	0.0000000d+00	0.3090859d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
-0.3126989d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.0000000d+00	-0.3125000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00	-0.1250000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00

D Matrix

0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00
0.0000000d+00	0.0000000d+00	0.0000000d+00	0.0000000d+00

State space matrices for helicopter at 80 knts flight condition

APPENDIX 2.2

HELICOPTER STATE VARIABLE DESCRIPTION

COLL	Pilot collective	(—)
LATT	Pilot lateral cyclic	(—)
LONG	Pilot longitudinal cyclic	(—)
PEDAL	Pilot pedals	(—)
P	Roll rate, body axes	(rads/s)
Q	Pitch rate, body axes	(rads/s)
R	Yaw rate, body axes	(rads/s)
PHIR	Bank angle	(rads)
THETR	Pitch attitude	(rads)
PSIR	Heading angle	(rads)
H	Height	(ft)
X	X position of cg	(ft)
Y	Y position of cg	(ft)
UB	Vel. comp. rel. to the air, X body axis	(ft/s)
VB	Vel. comp. rel. to the air, Y body axis	(ft/s)
WB	Vel. comp. rel. to the air, Z body axis	(ft/s)
TH0D	Collective blade angle	(degs)
THCAD	Lateral cyclic blade angle	(degs)
THSAD	Longitudinal cyclic blade angle	(degs)
TH0TRD	Tail rotor collective blade angle	(degs)

CHAPTER 3

ROBUSTNESS ANALYSIS

3.1 *Introduction*

The purpose of this chapter is to give an explanation of how to measure the stability robustness of multivariable feedback systems using singular values of certain frequency response matrices. It is assumed throughout that the models are finite dimensional linear time invariant (FDLTI) systems.

Uncertainty will always be present in any real system due to nonlinearities, ageing, etc. and therefore a mathematical model can never perfectly describe the physical process. The usefulness of the plant model is therefore greatly enhanced if the uncertainty between the model and the real system can be measured or characterized.

The most popular way of doing this is by defining a set of perturbed plant models in the “neighbourhood” of the nominal plant model, the true plant model being assumed to lie within this set. This approach has been used extensively by many research workers and several representations of uncertainty can be found in [CRU1], [DOY2], [FOO2], [LEH1], [POS4], and [SAF2],

These approaches are based on the idea of deforming the Nyquist diagram of the nominal plant into a Nyquist diagram of the actual system without changing the number of encirclements of the critical point required for stability. Although the stability of a MIMO

system can be determined from the multivariable Nyquist diagram, reliable stability margins for the MIMO system cannot, this is in contrast to the SISO case.

In §3.2, different kinds of modelling errors are defined and it is shown in §3.3 that if the magnitude of these errors are bounded appropriately then the feedback system will remain stable despite these modelling errors. §3.3.4 discusses some of the limitations of these robustness tests and §3.6 discusses the design implications. The need for loop shaping in order to satisfy the stability robustness bounds motivates the idea of using H^∞ -optimization.

The conservativeness of the unstructured robustness tests is handled in §3.5. Uncertainty representations which are used for robustness analysis of the final helicopter design are discussed in §3.7 and since singular values are used throughout this chapter, a brief discussion of them is given in Appendix 3.1.

3.2 Uncertainty Representation

Uncertainty representation is the assumed mathematical relationship that exists between the actual plant, the nominal model and the modelling error. Probably the most familiar types of errors are those of absolute and relative errors. *Absolute errors* are additive in nature whereas *relative errors* are multiplicative. Both types of errors can be used to derive robustness tests. However gain and phase margins as usually defined are associated only with relative errors since these margins are multiplicative.

Let $\tilde{G}(s)$ denote the *perturbed* transfer function matrix which represents the actual system and differs from the nominal transfer function $G(s)$ because of the uncertainty in the open loop plant model. The various types of model errors presented will emphasize different aspects between the nominal $G(s)$ and $\tilde{G}(s)$ and thus will give different assessments of the

robustness or stability margins of the system. The general case of $G(s)$ being non square, with more outputs than inputs (which is usually the case), will be treated.

Three commonly used representations of uncertainty for a nominal transfer function $G(s)$ are:

$$\{\tilde{G}(s)\}_E \triangleq \{G(s) + E(s) \mid \bar{\sigma}[E(s)] \leq e(s), \forall s \in D_D^L\} \quad (3.1)$$

$$\{\tilde{G}(s)\}_{\Delta_o} \triangleq \{(I + \Delta_o(s))G(s) \mid \bar{\sigma}[\Delta_o(s)] \leq \delta_o(s), \forall s \in D_D^L\} \quad (3.2)$$

$$\{\tilde{G}(s)\}_{\Delta_i} \triangleq \{G(s)(I + \Delta_i(s)) \mid \bar{\sigma}[\Delta_i(s)] \leq \delta_i(s), \forall s \in D_D^L\} \quad (3.3)$$

where $\bar{\sigma}[\cdot]$ denotes the maximum singular value, and D_D^L is the standard Nyquist D-contour with indentations to the left of imaginary axis poles and with a clockwise orientation. Each of these representations defines a set of perturbed plant models in a neighbourhood of $G(s)$, and it is assumed that one of the perturbed plants $\tilde{G}(s)$ common to each of the sets is the “true” model. Perturbations before and after the plant can be thought of as uncertainties in the actuator and sensor models respectively.

In each of the above representations a bound on a maximum singular value is used to define a set of perturbed plant models. An important question is whether the set defined includes all possible physical perturbations in $G(s)$. If it does not, then the set is clearly inadequate for representing the uncertainty. The following example will help clarify this. Unless $\tilde{G}(s)$ is transformable to $L(s)G(s)$ such that

$$\tilde{G}(s) = L(s)G(s) = (I + \Delta_o(s))G(s) \quad (3.4)$$

then (3.2) will not be an adequate representation of the uncertainty. It is clear that the additive perturbation described by

$$\tilde{G}(s) = G(s) + E(s) \quad (3.5)$$

is always an adequate representation. It turns out, [FOO2], that no matter what the uncertainties actually are they will show up as an additive perturbation and robustness

against this perturbation implies and is implied by, robustness against the actual perturbation. Another interesting point is that if a plant has more outputs than inputs, (which is usually the case), then an input multiplicative type perturbation is generally unsuitable for representing uncertainty, [FOO2]. It is important to characterize the modelling error as accurately as possible if stability robustness tests are to provide any useful information, otherwise overly conservative results may arise.

The multiplicative perturbations are related to the additive error $E(s)$ by

$$\Delta_i(s) = G^\dagger(s)E(s) \quad (3.6)$$

$$\Delta_o(s) = E(s)G^\dagger(s) \quad (3.7)$$

Since the maximum singular value of a matrix is the matrix norm induced by the Euclidean vector norm (Appendix 3.1), it is also a sub-multiplicative norm and therefore,

$$\bar{\sigma}[\Delta_i(s)] = \bar{\sigma}[G^\dagger(s)E(s)] \leq \bar{\sigma}[G^\dagger(s)]\bar{\sigma}[E(s)] \quad (3.8)$$

$$\bar{\sigma}[\Delta_o(s)] = \bar{\sigma}[E(s)G^\dagger(s)] \leq \bar{\sigma}[G^\dagger(s)]\bar{\sigma}[E(s)] \quad (3.9)$$

$G^\dagger(s)$ denotes the pseudo inverse of $G(s)$ such that, if $G(s)$ has more rows than columns, then $G^\dagger(s)G(s) = I$

More conservatism is clearly introduced if (3.8) or (3.9) are used to estimate the maximum singular value of the multiplicative perturbation, indicating that perhaps the additive perturbation is the more useful representation of uncertainty, although the multiplicative type perturbation is a more meaningful characterization, being a representation of a fractional or percentage error in the nominal plant description.

Alternative representations of uncertainty may be derived by measuring the absolute and relative errors between $\tilde{G}^\dagger(s)$ and $G^\dagger(s)$. In the SISO case, this would correspond to measuring the errors on an *inverse* Nyquist diagram in which the inverse loop transfer

functions $G^{-1}(s)$ and $\tilde{G}^{-1}(s)$ are plotted. This then leads to the following sets of uncertainty representations.

$$\{\tilde{G}(s)\}_{\tilde{E}} \triangleq \{(G^\dagger(s) + \tilde{E}(s))^\dagger \mid \bar{\sigma}[\tilde{E}(s)] \leq \tilde{e}(s), \forall s \in D_I^L\} \quad (3.10)$$

$$\{\tilde{G}(s)\}_{\tilde{\Delta}_o} \triangleq \{(I + \tilde{\Delta}_o(s))^{-1}G(s) \mid \bar{\sigma}[\tilde{\Delta}_o(s)] \leq \tilde{\delta}_o(s), \forall s \in D_I^L\} \quad (3.11)$$

$$\{\tilde{G}(s)\}_{\tilde{\Delta}_i} \triangleq \{G(s)(I + \tilde{\Delta}_i(s))^{-1} \mid \bar{\sigma}[\tilde{\Delta}_i(s)] \leq \tilde{\delta}_i(s), \forall s \in D_I^L\} \quad (3.12)$$

D_I^L is the standard inverse Nyquist D-contour with indentations to the left of imaginary axis zeros and with a clockwise orientation.

The main advantage of using unstructured perturbation models is that simple graphical tests can be used to determine robust stability. In the absence of any knowledge of say $E(j\omega)$, a plot of $\bar{\sigma}[E(j\omega)]$ against frequency serves as a useful characterization of the stability robustness of the closed-loop system. However, since the only assumed information about the model uncertainty is described by a simple frequency dependent number, which measures the *size* or *magnitude* of the model error, all directionality information about the uncertainty is lost and conservative results can arise. For example, suppose $E(s)$ in (3.1) is given by

$$E(s) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \quad (3.13)$$

where $E(s)$ is an $n \times n$ matrix. Then $\bar{\sigma}[E(s)] \leq n$ but any $n \times n$ matrix $E'(s)$ whose largest singular value $\leq n$ also belongs to the set of perturbations described by (3.1). The true $E'(s)$ may be

$$E'(s) = \begin{pmatrix} n & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \quad (3.14)$$

which also has $\bar{\sigma}[E'(s)] \leq n$. Thus, the size of the model error does not necessarily give any indication of how the uncertainty is made up.

The *structure* of a modelling error is simply the numerical relationship of the elements of the error matrix $E(s)$, representing the difference between the nominal and perturbed models. Its structure is specified by magnitude and phase relationships between the $e_{ij}(s)$ elements of $E(s)$. Doyle, [DOY4], has introduced the concept of structured singular values to help overcome the conservatism of the unstructured uncertainty representation. This concept relies on the perturbation being represented as a block diagonal matrix whose blocks are each norm bounded. Unfortunately, the computation of the structured singular value, denoted μ , is non trivial and firmly remains an analysis tool, there at present being no synthesis method available which guarantees that the structured singular value robustness test will be satisfied.

This is in contrast to the unstructured uncertainty representation, whereby the appropriate singular values required to guarantee robust stability can be generated using H^∞ -optimization.

3.3 Robustness Tests

In this section, tests that guarantee the stability of a perturbed closed-loop system for different characterizations of model uncertainty (ie. different types of model error) are introduced. These tests use only the magnitude of the modelling error and do not exploit any other characteristics or structure of the model error and hence are based on the *unstructured* part of the model error.

For the purposes of determining stability robustness, the feedback configuration of figure 3.1 will be used, where $G(s) \in \mathbb{R}^{l \times m}$ is a nominal model used for design purposes and $K(s) \in \mathbb{R}^{m \times l}$ is a controller such that the nominal feedback system is stable. $r(s) \in \mathbb{R}^l$ denotes the vector of reference inputs, $e(s) \in \mathbb{R}^l$, the error signal, $u(s) \in \mathbb{R}^m$ the input to the plant and $y(s) \in \mathbb{R}^l$ the vector of outputs. $d(s) \in \mathbb{R}^l$ and $n(s) \in \mathbb{R}^l$ model plant disturbance and measurement signal vectors. For internal stability, there can be no right

half-plane pole-zero cancellations in $G(s)K(s)$ (or $K(s)G(s)$) and similarly for $\tilde{G}(s)K(s)$ (or $K(s)\tilde{G}(s)$), where $\tilde{G}(s)$ represents the perturbed plant model.

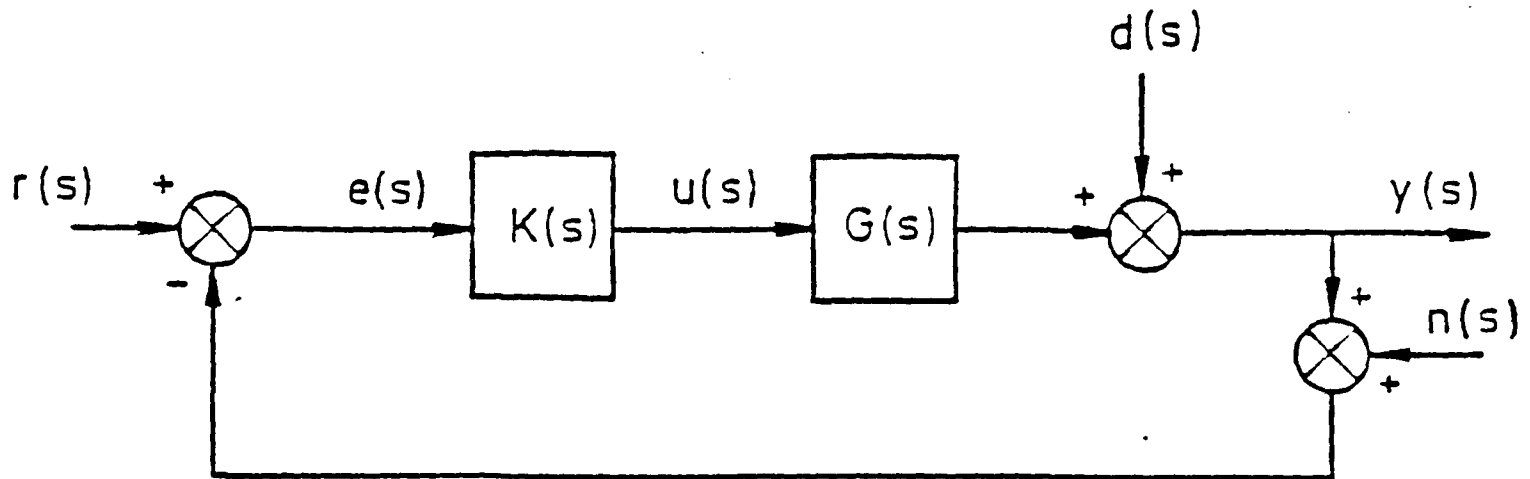


Figure 3.1 Unity feedback system

3.3.1 Direct Nyquist – based tests

The following tests are based on a generalization of Nyquist's stability criterion applied to $\det[I + G(s)K(s)]$. The results are not new and are therefore stated without proof. There will be three robustness tests presented, one for each of the uncertainty representations given in (3.1)-(3.3). All the following tests assume that $G(s)$ and $\tilde{G}(s)$ share the same number of CRHP poles.

Theorem 3.1

The feedback system of figure 3.1 will remain stable when the nominal plant model is replaced by any perturbed plant model taken from the sets $\{\tilde{G}(s)\}_E$, $\{\tilde{G}(s)\}_{\Delta_o}$, $\{\tilde{G}(s)\}_{\Delta_i}$ if

$$(i) \quad \bar{\sigma}[K(s)(I + G(s)K(s))^{-1}] < \frac{1}{e(s)}, \quad \forall s \in D_D^L \quad (3.15)$$

$$(ii) \quad \bar{\sigma}[(I + G(s)K(s))^{-1}G(s)K(s)] < \frac{1}{\delta_o(s)}, \quad \forall s \in D_D^L \quad (3.16)$$

$$(iii) \quad \bar{\sigma}[(I + K(s)G(s))^{-1}K(s)G(s)] < \frac{1}{\delta_i(s)}, \quad \forall s \in D_D^L \quad (3.17)$$

respectively.

Equations (3.16) and (3.17) are the MIMO generalization of the familiar SISO requirement that loop gains be small whenever the magnitude of unstructured uncertainties is large. In fact, whenever $\delta_o(s) \gg 1$, GK is constrained by

$$\bar{\sigma}[G(s)K(s)] < \frac{1}{\delta_o(s)}, \quad \forall s \in D_D^L \text{ such that } \delta_o(s) \gg 1 \quad (3.18)$$

Note that these tests are not conservative stability conditions. If the uncertainties are truly unstructured and any of the above tests are violated, then there exists a perturbation within the sets allowed by expressions (3.1)-(3.3) for which the system is unstable. Hence the stability conditions impose hard limits on the permissible loop gains.

The major disadvantage of the above tests is that, because $\tilde{G}(s)$ and $G(s)$ must have the same number of CRHP poles, they cannot be used when $G(s)$ has any poles whose regions of uncertainty extend across the imaginary axis. There are no restrictions on the movement of zeros across the imaginary axis.

3.3.2 Inverse Nyquist – based tests

The following tests are derived in a similar manner to those above, but using the inverse Nyquist diagram of the image of D_I^L under $\det[I + (G(s)K(s))^{-1}]$. The tests are only applicable when $G(s)$ and $\tilde{G}(s)$ share the same number of CRHP zeros, there being no restriction on the poles.

Theorem 3.2

The feedback system of figure 3.1 will remain stable when $G(s)$ is replaced by $\tilde{G}(s)$ taken from the sets $\{\tilde{G}(s)\}_{\tilde{E}}$, $\{\tilde{G}(s)\}_{\tilde{\Delta}_o}$, $\{\tilde{G}(s)\}_{\tilde{\Delta}_i}$ if

$$(i) \quad \bar{\sigma}[(I + G(s)K(s))^{-1}G(s)] < \frac{1}{\tilde{e}(s)}, \quad \forall s \in D_I^L \quad (3.19)$$

$$(ii) \quad \bar{\sigma}[(I + G(s)K(s))^{-1}] < \frac{1}{\tilde{\delta}_o(s)}, \quad \forall s \in D_I^L \quad (3.20)$$

$$(iii) \quad \bar{\sigma}[(I + K(s)G(s))^{-1}] < \frac{1}{\tilde{\delta}_i(s)}, \quad \forall s \in D_I^L \quad (3.21)$$

respectively.

3.3.3 Interpretation of robustness theorems

Singular value functions used in statements of design play a role much like classical Bode plots. They can be plotted as frequency dependent functions in order to display and analyse features of a multivariable design. The maximum singular value of the inverse return difference transfer function matrix, $[I + G(s)K(s)]^{-1}$, generalizes the classical notion of the distance of the Nyquist locus of a SISO feedback system to the critical (-1) point. For stability it is required that $I + G(s)K(s)$ be large or correspondingly that $[I + G(s)K(s)]^{-1}$ be small. What is required then, is a notion of *size* for MIMO systems, singular values provide such a characterization. They can be thought of as a multivariable generalization of the modulus of a complex number (see Appendix 3.1).

Equation (3.16) is particularly significant with regard to design against uncertainties. $\bar{\sigma}[T(s)]$ is an explicit measure of the stability robustness of the feedback system, where $T(s) \triangleq [I + G(s)K(s)]^{-1}G(s)K(s)$. Stability is guaranteed for all perturbations $\delta_o(s)$ whose maximum singular value falls below it. $\delta_o(s)$ can include gain or phase changes in individual output channels, simultaneous changes in several channels and various other kinds of perturbations. Thus $\bar{\sigma}[T(s)]$ can be thought of as a multivariable generalization of SISO stability margins where $\bar{\sigma}[T(s)]$ measures tolerances for uncertainties at the plant outputs only. Tolerances for uncertainties at the input are not the same and can be analysed by using $\bar{\sigma}[(I + KG)^{-1}KG]$ instead of $\bar{\sigma}[(I + GK)^{-1}GK]$ as in equation (3.19).

When the multiplicative perturbation $[I + \tilde{\Delta}_o(s)]^{-1}$ is used to characterize uncertainty, the stability test is in terms of the maximum singular value of the sensitivity matrix

$S(s) \triangleq [I + G(s)K(s)]^{-1}$. That is the function useful in assessing performance (with respect to disturbance and sensitivity) is also useful for assessing the robust stability of a system.

It follows then, that good (low) sensitivity and robust stability with respect to perturbations of the type $[I + \tilde{\Delta}_o(s)]^{-1}$, are compatible objectives in the sense that making $\bar{\sigma}[S(s)]$ small over some frequency range, reduces sensitivity and admits a larger size of perturbation, $\tilde{\Delta}_o(s)$, which the system can tolerate without becoming unstable. This is provided of course that the perturbation does not cause a change in the number of CRHP zeros.

Similarly making $\bar{\sigma}[T(s)]$ small, restricts the closed loop bandwidth of the system thereby attenuating any high frequency uncertainty and sensor noise. It is obvious that it is not possible to make both S and T simultaneously small because of the relationship

$$S(s) + T(s) = I \quad (3.22)$$

The robustness test (3.15) can be thought of as a constraint on the allowable actuator power, since KS is the transfer function from reference input to controller output (fig. 3.1). Making $\bar{\sigma}[K(s)S(s)]$ small has the dual effect of restricting actuator output and guarding against potentially destabilizing additive type perturbations.

Table 3.1 [LEH1], shows how each error enters the structure of the perturbed system from a block diagram perspective. This table serves to illustrate the symmetry between the different types of perturbation corresponding to the operations of addition, subtraction, multiplication and division.

Perturbed models corresponding to various error criteria and associated stability tests

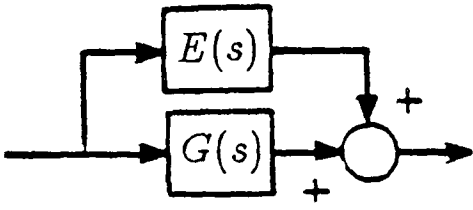
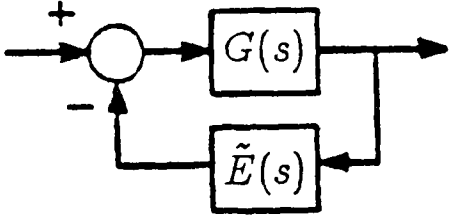
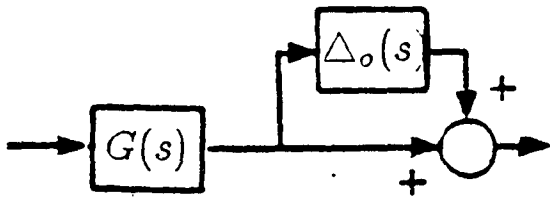
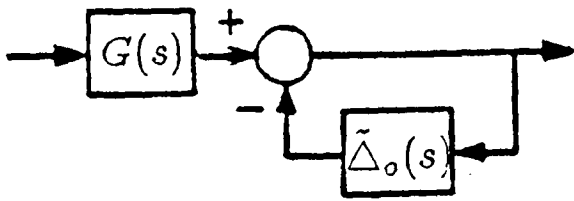
Block diagram of perturbed system	Error criteria perturbed systems and stability test
 (addition)	$E(s) = \tilde{G}(s) - G(s)$ $\tilde{G}(s) = G(s) + E(s)$ $\overline{\sigma}[K(s)S(s)] < 1/\overline{\sigma}[E(s)]$
 (subtraction)	$\tilde{E}(s) = \tilde{G}^\dagger(s) - G^\dagger(s)$ $\tilde{G}(s) = (G^\dagger(s) + \tilde{E}(s))^\dagger$ $\overline{\sigma}[S(s)G(s)] < 1/\overline{\sigma}[\tilde{E}(s)]$
 (multiplication)	$\Delta_o(s) = [\tilde{G}(s) - G(s)]G^\dagger(s)$ $\tilde{G}(s) = [I + \Delta_o(s)]G(s)$ $\overline{\sigma}[T(s)] < 1/\overline{\sigma}[\Delta_o(s)]$
 (division)	$\tilde{\Delta}_o(s) = G(s)[\tilde{G}^\dagger(s) - G^\dagger(s)]$ $\tilde{G}(s) = [I + \tilde{\Delta}_o(s)]^{-1}G(s)$ $\overline{\sigma}[S(s)] < 1/\overline{\sigma}[\tilde{\Delta}_o(s)]$

Table 3.1

3.3.4 Limitations of singular value analysis

Singular value techniques have been criticized for leading to conservative analyses and designs. This is because singular values provide information in terms of guaranteed bounds on tolerable perturbations. If the disturbances or perturbations are assumed to be completely unknown, except where they occur in the loop, then the derived bounds are *tight*. However, plant perturbations and disturbances can be highly structured over some frequency ranges. If this is the case, then singular value analysis on its own, can lead to conservative results. The problem is that the singular value tests of §3.3 for structured uncertainty provide only sufficient conditions, so failure of them does not necessarily imply system instability.

Another drawback is that singular value analysis cannot deal effectively with simultaneous perturbations at different points in the loop. If more than one type of perturbation is present, it is in general not possible to extrapolate separate analysis results for each perturbation and claim that simultaneous perturbations of similar magnitude can be tolerated. Consequently, a system may appear insensitive from separate singular value analyses and yet be extremely sensitive to simultaneous perturbations at different points in the loop.

It is possible to obtain stability robustness bounds for simultaneous perturbations, by reconfiguring the multiple perturbation models into a single perturbation with a block diagonal structure. This structure can then be ignored by replacing the block diagonal perturbation with one matrix perturbation, which then allows standard singular value analysis to be applied. For example, consider figure 3.2 in which the plant is subject to both input and output multiplicative perturbations, this is representative of uncertainties in the actuator and sensor models.

These figures show that the feedback system can be analysed using the techniques of §3.3, provided that P' is thought of as the plant, K' the controller and Δ a multiplicative

perturbation before the plant. Unfortunately, the stability robustness results obtained using this configuration will be in general conservative unless the μ -analysis of Doyle is used [DOY4].

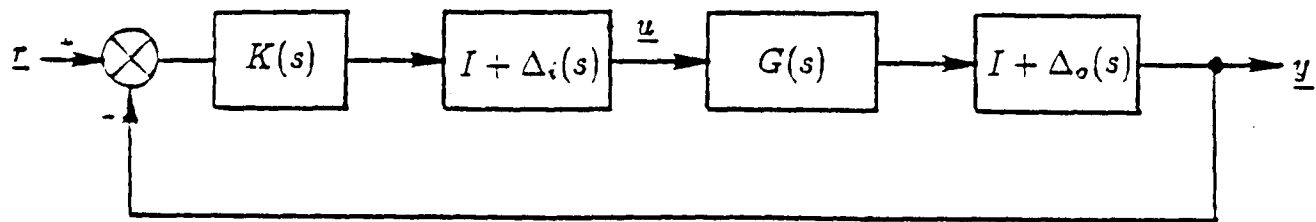
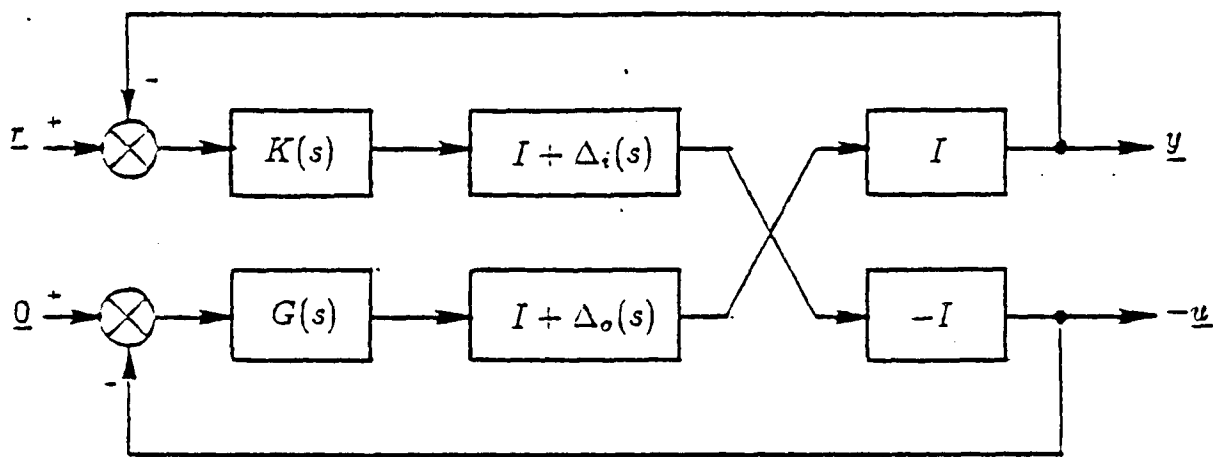
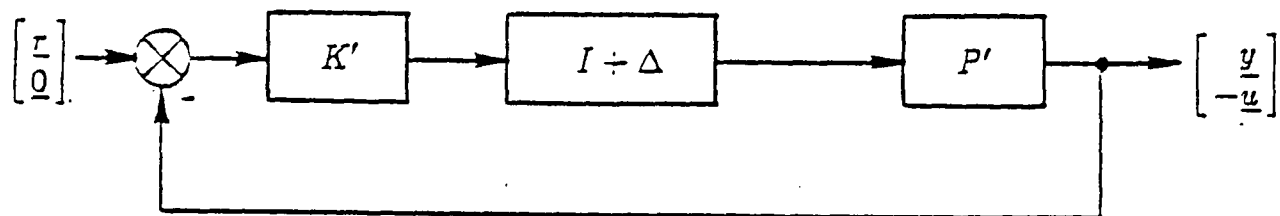


Figure 3.2 Perturbed feedback system

The above feedback configuration can be redrawn as shown in figure 3.3(i) and 3.3(ii).



(i)



(ii)

$$K' = \begin{bmatrix} K & 0 \\ 0 & G \end{bmatrix} \quad \Delta = \begin{bmatrix} \Delta_i & 0 \\ 0 & \Delta_o \end{bmatrix} \quad P' = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

Figure 3.3 The perturbed feedback system of figure 3.2 redrawn

Freudenberg et al, [FRE1], have suggested using the singular value sensitivity function in conjunction with singular value analysis to overcome the above deficiency. However, this approach suffers from being applicable only for small perturbations. Singular values sensitivities can be used to infer properties of a system subject to more general perturbations, but there are no guarantees, the analyses may be conservative or optimistic.

Despite the aforementioned limitations, singular value analysis still provides considerable information about the relative stability of a system. Singular values are a powerful analysis tool for determining stability robustness, however they are not without their shortcomings and as with any analysis tool, one should be aware of the limits and not attempt to extrapolate information beyond those limits. If they are used carefully in conjunction with *other* analysis tools and extensive simulations, they should lead to robust designs.

3.4 Gain And Phase Margins As Measures Of Robustness

In the SISO case, gain and phase margins are still the most widely used technique for assessing the relative stability of a system. However, in a MIMO system, gain or phases in one loop can affect the calculation of the gain and phase in another loop. For this reason, the margins are normally defined with respect to *simultaneous* gain *or* phase changes in every loop as follows.

- (i) The multiloop gain margin is the pair of numbers $gm_l, gm_u \in \mathbb{R}$, which define the largest interval (gm_l, gm_u) , such that when $\ell_i(s) \in \mathbb{R}$, in figure 3.4 satisfy

$$gm_l < \ell_i < gm_u, \quad i = 1, 2, \dots, n \quad (3.23)$$

the closed-loop system is stable.

- (ii) The multiloop phase margin is the pair of numbers $pm_l, pm_u \in \mathbb{R}$ which define the largest interval $(-pm_l, pm_u)$, such that when $\ell_i(s) \in \mathbb{R}$, in figure 3.4 are of the form $e^{j\phi_i(\omega)}$, where $\phi_i(\omega) \in \mathbb{R}$, satisfy

$$-pm_l < \phi_i(\omega) < pm_u, \quad i = 1, 2, \dots, n \quad (3.24)$$

the closed-loop system is stable.

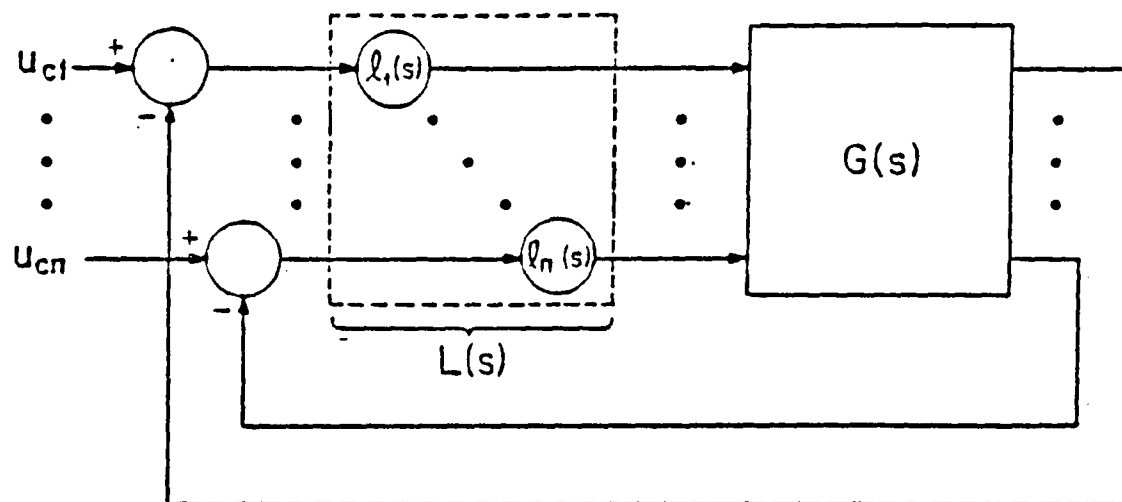


Figure 3.4 Configuration for gain and phase margin definitions

The margins can be thought of as a multiplicative perturbation of the form

$$L(j\omega) = I + \Delta_i(j\omega) = \text{diag}\{\ell_i(\omega)e^{j\phi_i(\omega)}\} \quad (3.25)$$

however they represent only a small class of modelling error which are described by a diagonal $L(j\omega)$, positioned at the plant input.

In [LEH2], Lehtomaki et al derive some useful formulae for calculating the gain and phase margins as

$$Gm \supset \left[\frac{1}{1-\alpha}, \frac{1}{1+\alpha} \right] \quad (3.26)$$

and

$$Pm \supset \left[-\cos^{-1}\left(1 - \frac{\alpha^2}{2}\right), \cos^{-1}\left(1 - \frac{\alpha^2}{2}\right) \right] \quad (3.27)$$

where

$$\alpha = \min_{\omega} \underline{\sigma}[I + K(j\omega)G(j\omega)] \text{ and } \underline{\sigma} \text{ denotes the minimum singular value.}$$

These margins hold for simultaneous gain *or* phase changes in each loop and provide worst case bounds on what the actual stability margins are. They can be thought of as minimum gain or phase margins. The formulae show once again that singular values provide an important role in stability robustness analysis and that they have a direct relationship with classical frequency domain measures.

In [POS4], it is shown that multiloop gain and phase margins in the input channels are not adequate measures of robustness, but that good margins at *both* the input and output of the plant are necessary for robust stability. This is one of the shortcomings of the LQG optimal state-feedback regulator, which is renowned for its guaranteed gain and phase margins of at least $(1/2, \infty)$ and $(-60^\circ, 60^\circ)$ respectively. These margins are guaranteed only at the plant input and therefore the stability margins at the output may be comparatively poor, [POS4].

In practice, stability margins for MIMO systems are often calculated for each feedback loop separately, opening one feedback loop at a time, while keeping the remaining loops closed and determining the gain and phase margins for the resulting SISO systems. This is the basis of the sequential loop closure approach to multivariable design pioneered by Mayne, [MAY1]. A major deficiency of the loop at a time method of determining stability margins, is that one cannot expect model uncertainty to affect one loop at a time.

The main reason for considering gain and phase margins is because current helicopter handling qualities specifications use these measures as a basis for design. For example, the military specification on the design of flight control systems for piloted aircraft [MIL3], requires that feedback control systems must have certain gain or phase margins in order to be acceptable. The robustness issue is thus well understood from the viewpoint of classical

frequency domain techniques applied to single-loop systems, however this is not the case for multi-loop systems. Therefore care must be exercised in using these margins as they are inadequate indicators of robustness. In particular the use of a diagonal uncertainty matrix $L(j\omega)$ may not be adequate to detect any potentially destabilizing cross feed errors such as those reported by Sandell, [SAN1] in the design of the CH-47B helicopter.

3.5 Directionality In Multivariable Systems

Signals in MIMO systems have a *spatial* as well as a frequency distribution. Some sensors are noisier than others, certain actuators might saturate at lower signal levels, disturbances enter particular loops, but not others, etc. This leads naturally to the need for different bandwidths in various loops.

The stability conditions of §3.3 are *tight* if all that is known about the perturbations $\Delta(s)$ say, are their maximum singular values. However, if more is known about $\Delta(s)$, large in one direction, small in another, then the structure of $\Delta(s)$ can be extracted to tighten up the stability tests. The conservatism of the tests is reduced if, conditions can be found that distinguish between, modelling errors that do not destabilize the feedback system, but violate the tests of §3.3 and those that violate the tests, but also destabilize the feedback system.

In the robustness tests of theorem 3.1, the conditions ensuring the stability of the perturbed closed-loop system are inequalities of the form

$$\overline{\sigma}[\Delta(s)] < \underline{\sigma}[F(G(s))] \quad \forall s \in D_L^D \quad (3.28)$$

where $F(G)$ is one of the transformations $(K^{-1} + G)$, $(I + (GK)^{-1})$, $(I + (KG)^{-1})$, corresponding to a perturbation Δ , of the form E , Δ_o or Δ_i , respectively.

Lehtomaki et al [LEH3], have shown that by performing a singular value decomposition (SVD) on $F(G)$ it is possible to obtain a *partial* characterization of the model error structure. The main drawback of this result is that the structural information cannot be determined until the compensator has been designed. Also, it is not very easy to compute the SVD of $F(G(s))$, since it is a function of s , except in simple cases. This limits this analysis procedure to very simple problems and therefore is not of much practical use.

A more effective procedure for handling structured uncertainty is the *Structured Singular Value (SSV) (μ)* test of Doyle [DOY4]. Doyle has shown that a necessary and sufficient condition for robust stability or robust performance in the H^∞ -framework may be formulated as a bound on μ of a specific matrix M . M includes information on the system model, the controller, the model uncertainty and performance specifications.

3.5.1 The structured singular value μ

Let M be a square complex matrix. $\mu(M)$ is defined such that $\mu^{-1}(M)$ is equal to the smallest $\bar{\sigma}(\Delta)$ needed to make $(I + \Delta M)$ singular. i.e

$$\mu^{-1}(M) = \min_{\delta} \{ \delta \mid \det(I + \Delta M) = 0 \text{ for some } \Delta, \bar{\sigma}(\Delta) < \delta \} \quad (3.29)$$

where Δ is a block diagonal perturbation matrix with a given structure

$$\Delta = \begin{pmatrix} \Delta_1 & & \\ & \ddots & \\ & & \Delta_n \end{pmatrix}, \quad \bar{\sigma}(\Delta_i) < \delta \quad \forall i \quad (3.30)$$

$\mu(M)$ depends on both the matrix M and the structure of the perturbations Δ . An equivalent statement of (3.29) is

$$\det(I + \Delta M) \neq 0, \quad \forall \Delta \text{ st } \bar{\sigma}(\Delta) < \delta$$

$$\iff \rho(\Delta M) \leq 1 \iff \mu(M) \leq 1/\delta$$

where ρ denotes the spectral radius and Δ is allowed to be any complex matrix satisfying $\bar{\sigma}(\Delta) < \delta$.

In this formulation Δ is restricted to be stable; unstable perturbations can be factored as ratios of stable perturbations, although these may in general be difficult to find. Recently Foo and Postlethwaite [FOO3], have extended the small- μ test to account for unstable perturbations. Using the concept of arcwise connectivity, they develop robustness tests which can be applied when there are simultaneous and possibly unstable perturbations in the feedback loop.

Any norm-bounded perturbation problem, regardless of structure, can be reconfigured as the block diagonal perturbation problem shown in figure 3.5(i).

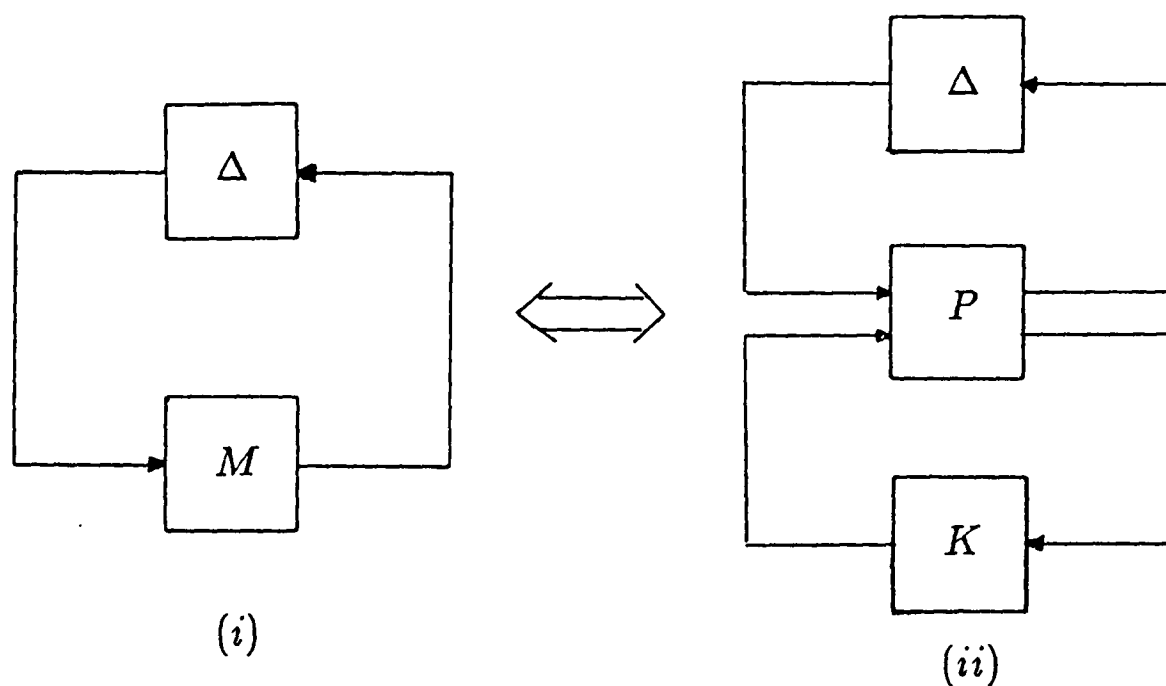


Figure 3.5 Feedback configuration for μ

Assume that M can be written as a linear fractional transformation (LFT) given by

$$M = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \quad (3.31)$$

then figure 3.5(i) is equivalent to figure 3.5(ii). The interconnection structure shown in figure 3.5 is fundamental to the H^∞ synthesis procedure and will be discussed further in chapter 4.

To illustrate how M is formulated, consider the simultaneous input and output perturbation problem of §3.3.4, which is depicted in figure 3.6.

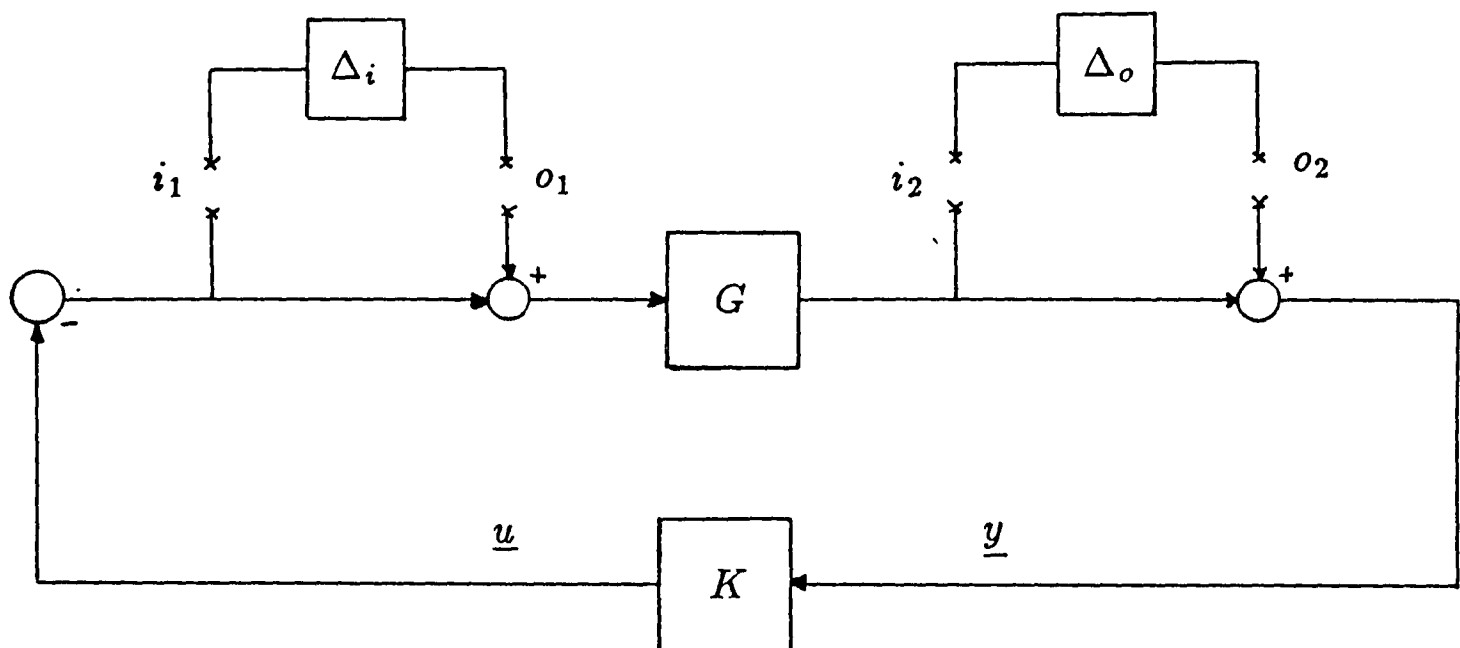


Figure 3.6 Simultaneous input and output perturbation problem

Break the loops due to the perturbations at points i_1, i_2 and o_1, o_2 ; which denote the input and output signals to the perturbations. Then

$$\begin{bmatrix} i_1 \\ i_2 \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 & -I \\ G & 0 & -G \\ G & I & -G \end{bmatrix} \begin{bmatrix} o_1 \\ o_2 \\ u \end{bmatrix} \quad (3.32)$$

$$M = LFT(P, K) = \begin{bmatrix} -(I + KG)^{-1}KG & -(I + KG)^{-1}K \\ (I + GK)^{-1}G & -(I + GK)^{-1}GK \end{bmatrix} \quad (3.33)$$

and $\Delta = \begin{pmatrix} \Delta_i & \\ & \Delta_o \end{pmatrix}$.

To illustrate the advantage of the SSV test consider the simple case where

$$M = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Delta = \begin{bmatrix} \Delta_1 & \\ & \Delta_2 \end{bmatrix} \quad (3.34)$$

Then the singular value test gives stability for

$$\sup_{\omega} \bar{\sigma}[M(j\omega)] < 1/\delta \quad \text{where} \quad \bar{\sigma}(\Delta) < \delta \quad (3.35)$$

whereas the SSV test gives

$$\begin{aligned} \mu^{-1}(M) &= \min_{\delta} \{ \delta \mid \det(I + \Delta M) = 0 \quad \text{for some } \Delta, \bar{\sigma}(\Delta) < \delta \} \\ &= \min_{\delta} \{ \delta \mid \det \begin{bmatrix} 1 & \Delta_1 m_1 \\ 0 & 1 \end{bmatrix}, |\Delta_1| < \delta \} \end{aligned}$$

But $\det \begin{bmatrix} 1 & \Delta_1 m_1 \\ 0 & 1 \end{bmatrix} \neq 0 \quad \forall \omega$, therefore stability is never lost.

An important consideration in the application of μ is numerical software. The definition of μ given by (3.29) is not very useful for its computation, since the optimization problem is not necessarily convex. The computation of μ is difficult and at present there is no numerical procedure which guarantees the solution of μ for perturbations with more than 3 blocks [FAN1].

3.6 Design Implications

This section examines the basic problems that arise in MIMO FDLTI feedback systems concerning the issues of command following, disturbance rejection, sensitivity, etc. and their relationship with stability robustness. By developing MIMO loop shaping techniques in the frequency domain it is possible to achieve both performance and robust stability.

A standard MIMO feedback configuration is shown in figure 3.1. Simple algebraic manipulations lead to the following:

$$\underline{y}(s) = T(s)\underline{r}(s) + S(s)\underline{d}(s) - T(s)\underline{n}(s) \quad (3.36)$$

$$\underline{e}(s) = [\underline{r}(s) - \underline{y}(s)] = S(s)[\underline{r}(s) - \underline{d}(s)] + T(s)\underline{n}(s) \quad (3.37)$$

$$\underline{u}(s) = K(s)S(s)[\underline{r}(s) - \underline{d}(s)] + K(s)T(s)\underline{n}(s) \quad (3.38)$$

where the notation $S(s) \triangleq [I + G(s)K(s)]^{-1}$ and $T(s) \triangleq [I + G(s)K(s)]^{-1}G(s)K(s)$ has been used throughout.

Equations (3.36)-(3.38) summarize the fundamental benefits and design objectives in feedback control. Specifically, the tracking error $\underline{e}(s)$ is made small by making $S(s)$ small when reference signal and disturbances are present. This gives simultaneously good command following and disturbance rejection. Sensor noise is attenuated by making $T(s)$ small and the magnitude of the control signals is constrained by making $K(s)S(s)$ small. In Appendix 3.1 it is shown that a MIMO transfer function matrix is small if its maximum singular value is small and large if its minimum singular value is large.

The bandwidth of the system is determined by the unity gain crossover frequency, ω_c , where ω_c is the frequency at which $\bar{\sigma}[T(s)] < 1$. The effective bandwidth of the system cannot fall much beyond the frequency ω_c , for which the maximum singular value of the uncertainty is greater than 1. As a result the frequency range over which performance objectives can be met is explicitly constrained by the uncertainties. Thus at frequencies less than ω_c , where $\bar{\sigma}[T(s)] \approx 1$ and $\bar{\sigma}[S(s)] \ll 1$, tolerance to the modelling errors of Theorem 3.2 should be good. At frequencies above ω_c , where $\bar{\sigma}[T(s)] \ll 1$, the robustness tests of (3.16) and (3.17) should be satisfied. This is the frequency range where it is important to have good tolerance to modelling error, since in general, modelling error increases with frequency because of unmodelled high frequency dynamics in the nominal design model.

Recall in §3.3.3 that the robustness tests for unstructured uncertainty are related to the functions $S(s)$ and $T(s)$ required for performance. It has been shown that both performance and stability robustness relate to these transfer function matrices being “small”

over a range of frequencies. This suggests that the maximum singular values of $S(s)$ and $T(s)$ are appropriate quantities for loop-shaping in the frequency domain.

Assuming that the energy of the reference signals and the disturbances lie in the low frequency range, while sensor noise and unmodelled dynamics have energy in the high frequency range. The loop-shaping requirement is to design a compensator $K(s)$ such that if $\bar{\sigma}[S(j\omega)]$ and $\bar{\sigma}[T(j\omega)]$ are plotted versus frequency ω , as illustrated in figure 3.7, $\bar{\sigma}[S(j\omega)]$ is smaller than the low frequency barrier that dictates command following and disturbance rejection, while $\bar{\sigma}[T(j\omega)]$ is smaller than the high frequency barrier that defines insensitivity to high frequency noise and also prevents the excitation of unmodelled high frequency dynamics.

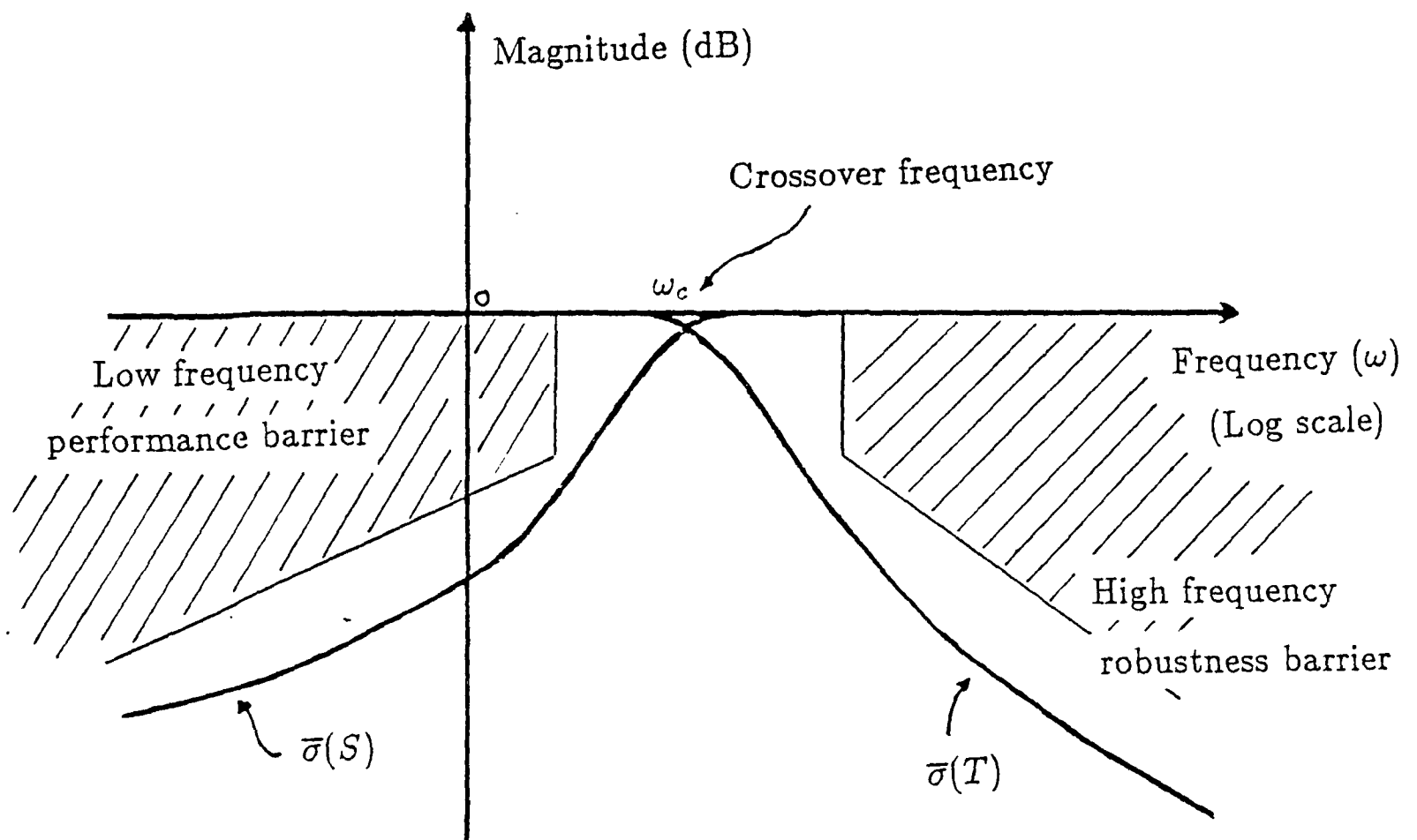


Figure 3.7 Bode-like plot of $\bar{\sigma}[S(s)]$ and $\bar{\sigma}[T(s)]$ required for performance and robustness in control system design

It is evident from figure 3.7 that the severity of the constraints depends on the rate at which $\bar{\sigma}[S(s)]$ and $\bar{\sigma}[T(s)]$ are attenuated. The steeper these functions drop off, the wider

the frequency range over which they are small. However, steep attenuation comes only at the expense of large $\bar{\sigma}[S(s)]$ and large $\bar{\sigma}[T(s)]$ when $\underline{\sigma}[G(s)K(s)]$ and $\bar{\sigma}[G(s)K(s)] \approx 1$. This means that while performance is good at lower frequencies and stability robustness is good at higher frequencies, both are poor near crossover. Therefore there exists a fundamental tradeoff between achievable performance and stability robustness, it is not possible to achieve both at the same frequency.

The shaping of the singular values for a MIMO system are non trivial and what is needed is a systematic computer-aided design procedure that accomplishes this frequency domain loop shaping process in an easy manner while maintaining closed-loop stability. It will be shown that the H^∞ -optimization approach to MIMO control system synthesis provides such a procedure.

The above arguments lead naturally to the following optimization problem: find a stabilizing controller K which minimizes the cost function

$$\min_{\text{stabilizing } K} \left\| \begin{bmatrix} W_1 S W_i \\ W_2 K S W_i \end{bmatrix} \right\|_\infty \quad (3.39)$$

where $\|\cdot\|_\infty \triangleq \sup_\omega \bar{\sigma}[\cdot]$ and W_1, W_2 and W_i are weighting functions used to emphasize (or demphasize) maximum singular values at various frequencies, so that engineering objectives can be incorporated into the optimization procedure. Note that by minimizing $S(s)$, $T(s)$ is indirectly controlled because of the relationship $S(s) + T(s) = I$.

This optimization problem is often referred to as robust stabilization with nominal performance, [GLO1] and is an example of a H^∞ -optimization problem. Similar problems have been proposed by Kwakernaak, [KWA1], [KWA2], for polynomial systems and by Foo, [FOO1], where sensitivity and complementary sensitivity functions are combined. The synthesis of H^∞ -optimal controllers will be discussed in the next chapter.

3.7 Uncertainties In The Helicopter Model

The major approximations associated with the linearized helicopter model are due to both neglected rotor dynamics and neglected nonlinearities in the blade angle actuation system as well as changes in flight condition. Attention will be focused primarily on errors due to neglected rotor dynamics and at a single flight condition in the hover. This allows for a simple illustration of the ideas of the previous sections.

The effect of increases in system order due to unmodelled rotor dynamics is an important area for helicopters, since the rotor system can introduce a significant number of modes into the vehicle dynamics and these can lead to vibration problems or place constraints on sensor positions. The task of identifying regions for which stability is assured is complex, since each element of the open loop transfer function is frequency dependent.

The first step in the design is to estimate the errors due to the neglected rotor dynamics. Elementary dynamic and aerodynamic analyses of rotating airfoils, hinged at the rotor hub, indicate that lift forces will not be transmitted to the hub instantaneously with collective changes in blade angle of attack, but will appear only when the cone angle of the rotor has appropriately changed. The dynamics of the latter have been shown to be damped second-order oscillations with natural frequency equal to rotor speed and damping, by somewhat uncertain aerodynamic effects [HOH2].

Hence, rotor dynamics can be crudely represented by second-order transfer functions

$$g_R(s, \zeta) = \frac{\omega_R^2}{s^2 + 2\zeta\omega_R s + \omega_R^2} \quad (3.40)$$

where ω_R and ζ are uncertain parameters. Due to the fact that collective and differential-collective inputs both involve coning motions of the rotors, one such transfer function will appear in each control channel. The structure of the perturbation rules out uncertain

couplings between tail rotor and the other three inputs but does not account for bounded uncertain couplings between collective, longitudinal cyclic and lateral cyclic.

Since the collective, longitudinal and lateral cyclic controls are all implemented with the main rotor, model uncertainty can be parameterized as a multiplicative perturbation at the four inputs. This function was evaluated for a range of frequencies with ω_R and ζ taken at some nominal values and is depicted in figure 3.8. From this figure it can be seen that the uncertainty level will limit the bandwidth of the closed loop system to about 10 rad/s.

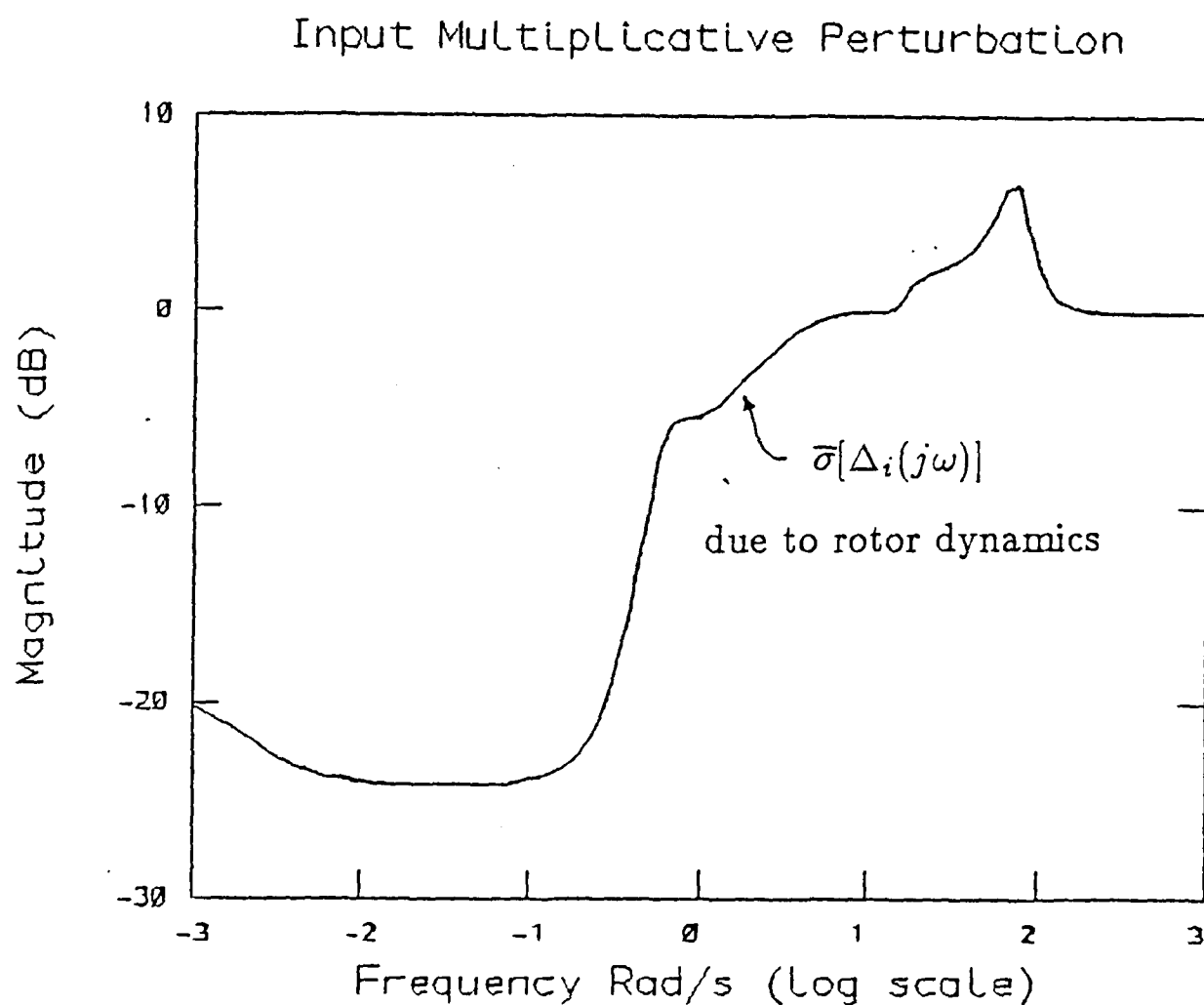


Figure 3.8 Uncertainty due to unmodelled rotor dynamics

An alternative approach would be to explicitly add the rotor dynamics to the nominal model and represent the uncertain damping and frequency by variable parameters. This would eliminate the perturbation due to the first rotor mode and allow for a higher bandwidth controller. Unfortunately, such a high bandwidth controller would require modification and more accurate modelling of the blade angle actuation system as well as additional

modelling of higher frequency rotor dynamics (second and higher harmonics). The benefits of a high bandwidth controller would have to be weighed up against its cost. In any case, even with more accurate modelling, there is still likely to be large unstructured uncertainty, possibly at higher frequencies due to neglected dynamics. Therefore it was opted to only model the rigid body dynamics explicitly and include other effects in the unstructured uncertainty.

In addition to the rotor dynamics, another major source of model uncertainty is the variation of the state matrix A and control matrix B of the linearised model with changes in flight condition. A direct way to compute a bound on the uncertainty due to coefficient variations is to evaluate the loop transfer matrix for a number of representative flight conditions and then compute the maximum (singular value) deviation. Results of this process are shown in figure 3.9, where the flight conditions were taken to be from 0 to 80 knts straight and level flight.

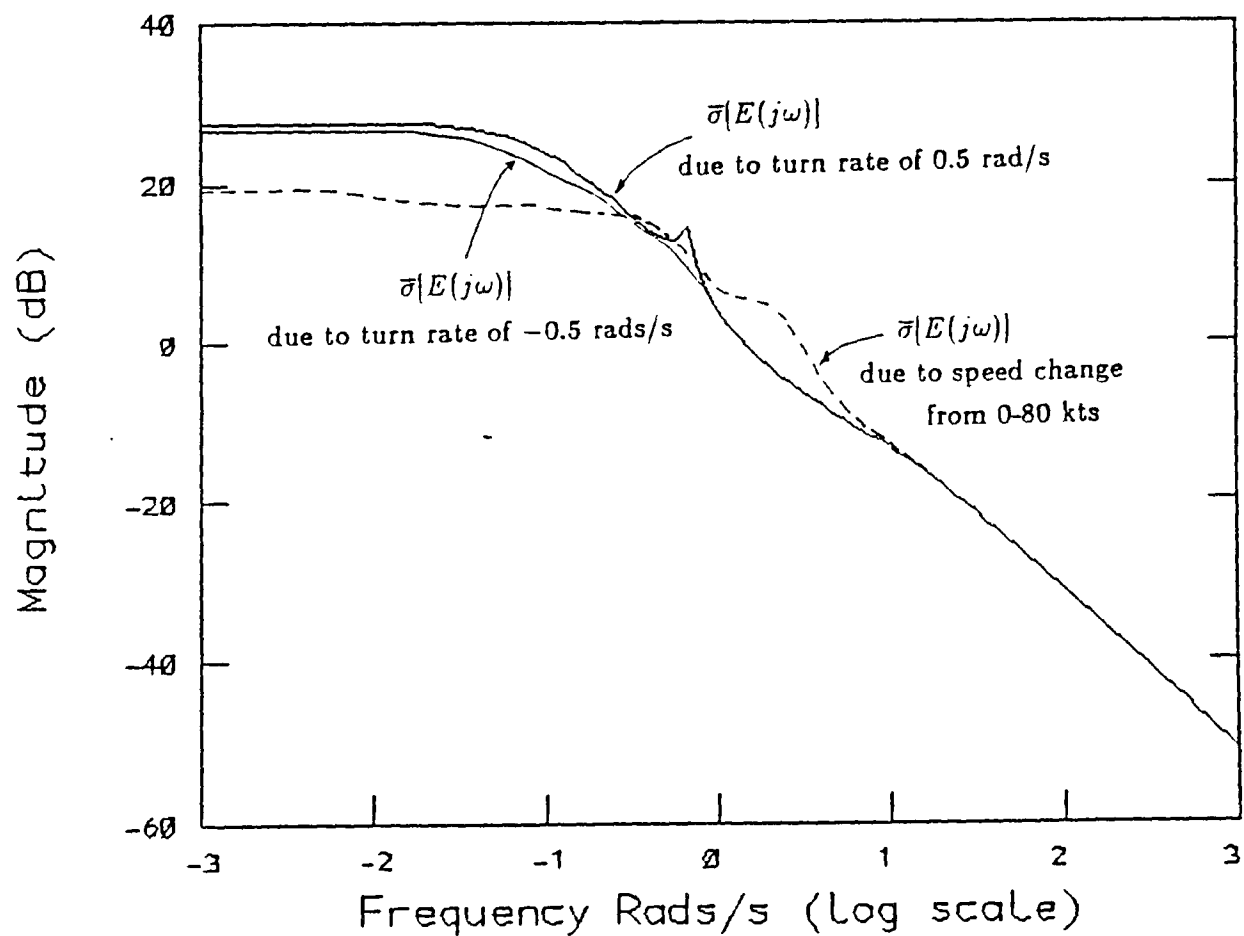


Figure 3.9 Perturbations in helicopter dynamics due to speed and turn rate

Also shown in figure 3.9 are the perturbations due to a ± 0.5 rads/s turn rate from the hover flight condition. The unmodelled dynamics due to changes in speed and turn rate have been represented as an unstructured additive perturbation. These flight conditions were chosen as being typical worst case deviations from the hover. The change in dynamics of the helicopter is most severe from this flight condition.

From figure 3.9 it can be seen that $\bar{\sigma}[E(s)]$ becomes quite large at low frequencies. This is because the helicopter's low frequency modes vary greatly with speed, as demonstrated in figures 2.10 and 2.11. However, since these modes do not appear to cross the $j\omega$ axis, the perturbation due to changes in speed and turn rate from the hover flight condition are stable. This allows the application of the direct Nyquist stability robustness tests of §3.3.

Each control channel of the helicopter also exhibits various nonlinearities which are neglected in the nominal design model. Of these, the rate limit nonlinearity imposes the greatest dynamic constraints on performance. Bounds on this effect can be developed by treating the saturation element as an uncertain component. These bounds will depend on the magnitudes of the arguments of the saturating function and in general will lead to conservative designs because every actuator is unlikely to saturate at the same magnitude.

3.8 Concluding Remarks

A brief study into the representation of uncertainty has been presented. The quantities used to measure the *magnitude* of the uncertainty, namely singular values, are amenable to the development of analysis techniques for assessing stability robustness of the closed-loop system.

A variety of robustness tests have been presented, but it is not clear which is “best” in determining the largest class of model errors that a feedback system will tolerate. Each test

works with a different type of model error, and therefore one test will be better than another if a particular characterization of the uncertainty is a more adequate representation. Postlethwaite and Foo argue it is better to use a combination of tests [POS5].

Although the singular value approach is useful in detecting the near instability of a control system, it is sometimes unnecessarily conservative. This is due to the fact that some of the small perturbations that would theoretically destabilize the closed-loop system never occur in the physical system. A single magnitude bound on matrix perturbations is a worst-case representation. The problem lies in obtaining a true representation of the uncertainty associated with a given model. One of the advantages of the singular value approach is that it highlights the worst type of destabilizing perturbation.

The small- μ test is a powerful tool for robustness analysis but its widespread use is restricted by lack of available software. Different results for μ will be obtained depending on the assumed structure of the problem. For practical problems, the structure of the uncertainty is dictated by the physical system and engineering constraints. The extent to which any structure can be assumed will determine the extent to which conclusions based on any analysis are relevant. The μ synthesis problem does not yet have as complete a solution as the H^∞ synthesis problem.

In general stability margins by themselves are useless unless accompanied with further information about the size of the expected perturbation in the nominal model. However it can be argued that they are desirable if only to guarantee a degree of robustness with respect to uncertainties in the actuators and sensors.

The mathematical results of this chapter are possible because of the assumption of FDLTI models and therefore these results will only apply to linearized models of the helicopter about given operating points and for small perturbations around these points. Of course,

local stability about a number of points does not guarantee global stability, this will have to be demonstrated through nonlinear simulation.

The sensitivity of handling qualities to changes or errors in the mathematical model has not been addressed. This is an important area of research, since in particular phases of flight, adverse interaction between the helicopter and pilot can introduce degradation in response which makes the vehicle difficult to fly. The problem lies in being able to accurately model the pilot who is in essence a very nonlinear self adaptive controller. A detailed study of techniques used to derive pilot models and the intricacies of parameter adjustment can be found in [MCR2]. With appropriate models of the pilot it should be possible to derive stability robustness bounds to establish the source and extent of any sensitivity in handling qualities, due to uncertainties in the nominal helicopter model.

Ultimately, the most effective way to assess system robustness is to use ground based simulations. This approach however, is highly iterative, requiring redesign of control laws around a nonlinear model before any new design can be evaluated through piloted simulation. The procedure is both lengthy and time consuming and entails large amounts of computation and use of resources. What is needed is relatively simple and quick analysis methods which can determine whether a given design is good or bad with respect to the design aims. Singular values go some way into providing such a tool and it is hoped that future design specifications will include them.

APPENDIX 3.1

MATRIX NORMS AND SINGULAR VALUES

This Appendix presents the minimal mathematical background needed to define the singular values of a complex matrix.

Definition

A complex matrix A is *Hermitian* if $A = A^*$ where A^* denote the complex conjugate transpose of A

The following properties are stated without proof.

(P1) All the eigenvalues of a Hermitian matrix are real

(P2) (Rayleigh's quotient) If A is a Hermitian matrix then

$$\min_{\underline{x} \neq 0} \frac{\underline{x}^\top A \underline{x}}{\underline{x}^\top \underline{x}} = \lambda_{\min}(A) \quad (A.1)$$

and

$$\max_{\underline{x} \neq 0} \frac{\underline{x}^\top A \underline{x}}{\underline{x}^\top \underline{x}} = \lambda_{\max}(A) \quad (A.2)$$

where $\lambda_{\min}(A)$ ($\lambda_{\max}(A)$) is the minimum (maximum) eigenvalue of A . The minimum (maximum) of eqs. (A.1) and (A.2) are attained when $\underline{x}$ is the eigenvector of A corresponding to $\lambda_{\min}(A)$ ($\lambda_{\max}(A)$) respectively.

Singular values

The singular values of a matrix $A \in \mathbb{C}^{n \times m}$ are defined as

$$\sigma_i(A) \triangleq \lambda_i^{1/2}(A^* A), \quad \text{for } i = 1, 2, \dots, m \quad (A.3)$$

where $\lambda_i(A^*A)$ denotes the i th eigenvalue of A^*A . Note that A^*A is Hermitian and positive semidefinite; hence, the eigenvalues of A^*A are real and non-negative. If A is nonsingular, A^*A is positive definite, and $\lambda_i(A^*A)$ are positive for all $i = 1, 2, \dots, n$.

The minimum and maximum singular values denoted $\underline{\sigma}$ and $\bar{\sigma}$ respectively, are sometimes equivalently defined in terms of the Euclidean vector norm or spectral norm $\| \cdot \|_2$ where

$$\|\underline{x}\|_2 \triangleq \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{\underline{x}^T \underline{x}}, \quad \underline{x} \in \mathbb{R}^n, \quad i = 1, 2, \dots, n \quad (A.4)$$

and

$$\bar{\sigma}(A) = \max_{\|\underline{x}\|_2 \neq 0} \frac{\|A\underline{x}\|_2}{\|\underline{x}\|_2} = \|A\|_2 \quad (A.5)$$

$$\underline{\sigma}(A) = \min_{\|\underline{x}\|_2 \neq 0} \frac{\|A\underline{x}\|_2}{\|\underline{x}\|_2} = \frac{1}{\|A^{-1}\|_2} \quad (A.6)$$

provided that A^{-1} exists. Thus $\bar{\sigma}(A)$ is simply the spectral norm of A and the spectral norm of A^{-1} is the inverse of $\underline{\sigma}(A)$. It follows from the definitions that

$$\bar{\sigma}(A^{-1}) = \|A^{-1}\|_2 = \frac{1}{\underline{\sigma}(A)} \quad (A.7)$$

and

$$\underline{\sigma}(A^{-1}) = \frac{1}{\|A\|_2} = \frac{1}{\bar{\sigma}(A)} \quad (A.8)$$

The minimum and maximum singular values of a transfer function matrix

Consider a system with input $\underline{u}(s)$, output $\underline{y}(s)$ and transfer matrix $G(s)$ related by

$$\underline{y}(s) = G(s)\underline{u}(s) \quad (A.9)$$

Take the complex conjugate transpose of eq. (A.9) to give

$$\underline{y}^*(s) = \underline{u}^*(s)G^*(s) \quad (A.10)$$

From eqs. (A.9) and (A.10)

$$\underline{y}^*(s)\underline{y}(s) = \underline{u}^*(s)G^*(s)G(s)\underline{u}(s) \quad (\text{A.11})$$

which implies

$$\|\underline{y}(s)\|_2^2 = \|G(s)\underline{u}(s)\|_2^2 \quad (\text{A.12})$$

and

$$\|\underline{y}(s)\|_2 = \|G(s)\underline{u}(s)\|_2 \quad (\text{A.13})$$

Divide both sides by $\|\underline{u}(s)\|_2$ to obtain

$$\frac{\|\underline{y}(s)\|_2}{\|\underline{u}(s)\|_2} = \frac{\|G(s)\underline{u}(s)\|_2}{\|\underline{u}(s)\|_2} \quad (\text{A.14})$$

Equation (A.14) is the analog of the SISO equality

$$\left| \frac{y(s)}{u(s)} \right| = |g(s)| \quad (\text{A.15})$$

The magnitude of the right hand side of eq. (A.14) will depend upon the direction of the vector $\underline{u}(s)$. The largest magnitude that (A.14) can attain can be calculated from $\bar{\sigma}[G(s)]$ according to (A.5) as

$$\max_{\underline{u}(s) \neq 0} \frac{\|G(s)\underline{u}(s)\|_2}{\|\underline{u}(s)\|_2} = \bar{\sigma}[G(s)] \quad (\text{A.16})$$

and the minimum that (A.14) can attain is measured by $\underline{\sigma}[G(s)]$.

It is self evident that $G(s)$ is large for all possible inputs $\underline{u}(s)$ if $\underline{\sigma}[G(s)]$ is large. Also, from Rayleigh's quotient (P2), the minimum is attained when $\underline{u}(s)$ is the right singular vector associated with $\underline{\sigma}[G(s)]$.

Conversely, $G(s)$ is small for all inputs $\underline{u}(s)$ if $\bar{\sigma}[G(s)]$ is small and the maximum is attained when $\underline{u}(s)$ is the right singular vector associated with $\bar{\sigma}[G(s)]$.

CHAPTER 4

H^∞ -OPTIMIZATION

4.1 *Introduction*

This chapter reviews the general H^∞ -optimal synthesis theory developed by Doyle [DOY3], which builds on results by many other researchers. The problem in robust multivariable feedback design is to synthesize a control law which maintains command response and error signals to within specified tolerances despite the effects of uncertainty on the system. Uncertainty may take many forms but among the most significant are noise/disturbance signals and modelling errors. Optimization is a powerful technique for dealing with design problems with conflicting objectives. Controller synthesis will be formulated as a well defined H^∞ -optimization problem. H^∞ -optimization is an algorithm for synthesizing stabilizing compensators which minimize a designer selected performance index with respect to the H^∞ -norm.

For the past few years H^∞ -optimal control has been an important and active area of research. Solutions to the H^∞ -optimization problem have led to computer-aided design packages such as LINF, [CHI1], Stable-H, [POS6], Honey-X developed by John Doyle and C.C. Chu at Caltech and QDES developed by Steven Boyd at Stanford University. These programs employ state space L^∞ theory [DOY3], [CHU1], [SAF3], [SAF6], and Hankel-norm model reduction results of Glover [GLO2].

In this chapter some new results on the classification of H^∞ -problems are presented and an iterative procedure called γ -iteration [DOY3], [CHU1], is introduced to solve the optimization problem. The 2-block γ -iteration is known to be continuous, convex and monotonically decreasing, these results are extended to the 4-block case. In particular it is shown that the proof of the results for the 2-block case cannot be readily extended to the 4-block case.

The motivation for studying 4-block design problems is discussed and it is shown that such formulations have no constraints with regard to the location of $j\omega$ -axis poles and zeros. The two-degree of freedom controller is reviewed and its implications with regard to design and synthesis is discussed.

4.2 Control System Designs As H^∞ -Optimizations

To begin, define the Hardy space H^∞ to consist of all complex-valued functions $G(s)$ of a complex variable s which are analytic and bounded in the open right half-plane. This can be thought of as the space of all stable transfer function matrices. Many practical feedback design problems can be formulated as the minimization of the H^∞ -norm of an appropriately weighted closed-loop transfer function matrix. The minimization of such a norm is now well understood and all the computations can be carried out using state space representations of the transfer function matrices.

The systems are modelled as linear, time invariant and finite dimensional and they operate in continuous time with a state space and transfer function representation in the Laplacian s given by:

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right], \quad G(s) = C(sI - A)^{-1}B + D \quad (4.1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{\ell \times n}$ and $D \in \mathbb{R}^{\ell \times m}$. The system G is said to be stable if the state space matrix A has no eigenvalues in the closed right half-plane. Suppose G is

stable; then the H^∞ -norm of G is defined as

$$\|G\|_\infty = \sup_{\omega} \bar{\sigma}[G(j\omega)] \quad (4.2)$$

where $\bar{\sigma}[G(j\omega)]$ denotes the largest singular value of G at frequency ω .

The compensation configuration depicted in figure 4.1 will be referred to as the Standard Compensator Configuration (SCC). The objective is to design a controller K , for the plant G such that the input/output transfer characteristics from the external input vector $\underline{v}$ to the external output vector $\underline{e}$ are desirable, according to some engineering specifications. The exogenous input $\underline{v}$, typically consists of command signals, disturbances and sensor noise; $\underline{u}$ is the control signal; $\underline{e}$ is the output to be controlled, its components typically being tracking errors, filtered actuator signals, etc.; and $\underline{y}$ is the measured outputs.

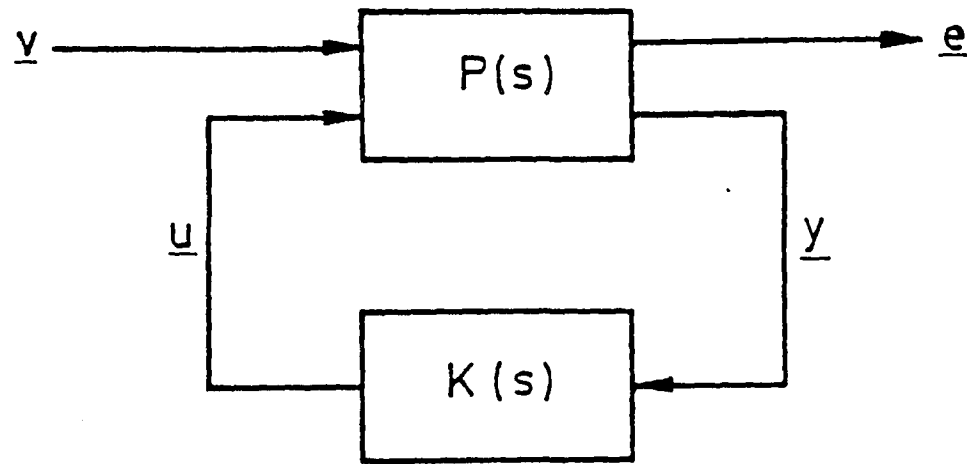


Figure 4.1 Standard Compensator Configuration

Partitioning $P(s)$ in figure 4.1 accordingly, the closed-loop transfer function matrix can be written as the following linear fractional transformation (LFT).

$$\underline{e} = F_\ell(P, K)\underline{v} = (P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21})\underline{v} \quad (4.3)$$

where

$$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \in \mathfrak{R}^{(\ell_1 + \ell_2) \times (m_1 + m_2)} \quad (4.4)$$

and

$$P_{ij} = C_i(sI - A)^{-1}B_j + D_{ij} \text{ is a state space realization of } P_{ij}$$

The H^∞ -synthesis problem is then one of finding a stabilizing K such that the performance measure $\|F_\ell(P, K)\|_\infty$ is minimized.

The SCC is a very powerful configuration in which many control problems can be recast. Depending on the number of signals in the vectors $\underline{v}$, $\underline{e}$, $\underline{u}$, $\underline{y}$, H^∞ -problems can be subdivided into three categories:

- both vectors
- (1) When both vectors $\underline{v}$ and $\underline{y}$ and $\underline{u}$ and $\underline{e}$ are of the same size, this type of problem within the SCC context is denoted as 1-block. In this case both P_{12} and P_{21} are square. An example of this type is the optimal sensitivity problem [ZAM1], [ZAM2].
 - (2) When either the size of $\underline{v}$ is greater than that of $\underline{y}$ or the size of $\underline{e}$ is greater than that of $\underline{u}$, this will be denoted a 2-block problem (either P_{12} or P_{21} non-square). An example of this type is the disturbance attenuation and optimal robustness problem with respect to additive perturbations where it is desired to minimize $\left\| \begin{matrix} W_1 S W_3 \\ W_2 K S W_3 \end{matrix} \right\|_\infty$
 - (3) For the case when both the size of $\underline{v}$ is greater than $\underline{y}$ and the size of $\underline{e}$ is greater than that of $\underline{u}$, this problem is denoted as 4-block (both P_{12} and P_{21} non-square). This may occur for example, when it is desired to attenuate disturbances at both the input and output of the plant.

1-block problems are easiest to solve numerically, while 4-block ones are the hardest. 4-block problems occur most naturally since the number of plant inputs $\underline{u}$ and measurements $\underline{y}$ are normally fixed and less than the corresponding number of external inputs $\underline{v}$ (reference inputs, disturbances, noise, etc) and regulated outputs $\underline{e}$ (error signal, actuator rate and travel limits, plant output, etc). The sizes of the vectors $\underline{v}$, $\underline{e}$, $\underline{u}$ and $\underline{y}$ give necessary but not sufficient conditions to the classification of the problem as 1, 2 or 4-block. In certain

situations to be made precise in §4.3.3, 4-block problems can collapse to 2-block problems and 2-block to 1-block problems.

4.2.1 Weight selections in H^∞ design

Weighting functions are used to emphasize (or de-emphasize) maximum singular values of $F_\ell(P, K)$ given by (4.3), at various frequencies so that engineering objectives can be incorporated in the optimization procedure as:

$$C = \min_{K \text{ stabilizing}} \|W_o F_\ell(P, K) W_i\|_\infty \quad (4.5)$$

where W_o and W_i are weights to be chosen and C is the minimum cost. The engineering requirements are fully specified by the chosen weights. An optimal controller achieving the minimum cost C can be computed automatically within a Computer-Aided-Design (CAD) environment such as Stable-H [POS6], an H^∞ design package developed at the University of Oxford. In general the weights used are, diagonal transfer function matrices, whose elements are constants, low or high pass filters.

The following objectives are influenced by the choice of weights

- (1) Time domain specifications such as actuator travel and rate limits, rise time and damping for closed loop step responses.
- (2) Characterization of achievable performance bounds.
- (3) Robustness to unmodelled dynamics.

4.3 Review Of H^∞ -Optimization Theory

For the existence of a solution to (4.5), the system of figure 4.1 must be well posed; ie the LFT from $\underline{v}$ to $\underline{e}$ exists and is proper. It is assumed that P is real-rational and proper and

K stabilizes P . The system is well posed if and only if the rational matrix $(I - P_{22}K)$ is invertible for every proper real-rational K . A simple sufficient condition for this is that P_{22} be strictly proper. P is stabilizable if and only if its unstable modes are controllable from $\underline{u}$ and observable from $\underline{y}$. Stabilizability of P implies that P and P_{22} share the same unstable modes, so that to stabilize P it is enough to stabilize P_{22} , [FRA4].

The design objective is formulated as an optimization problem and an affine parametrization of the closed-loop system is obtained using Youla's parametrization of all stabilizing controllers [YOU1]. Using observer based stabilization methods and appropriate choices of state feedback and output injection matrices, the H^∞ -optimization formulation is then transformed to an equivalent model matching problem.

Algorithms based on state-space representations can be derived which allow the robust optimal control design to be done completely in an attractive and practical state-space framework. The following sub-sections outline the main steps in the computation.

4.3.1 Parametrization of all stabilizing controllers

In this section it is shown that the SCC of figure 4.1, can be reduced to an equivalent model-matching problem. The procedure is to parametrize via a parameter matrix $Q \in \mathcal{RH}^\infty$, all K 's which stabilize P . The parametrization is due to Youla et al [YOU1], as modified by Desoer et al [DES1].

Suppose that (A, B_2, C_2) in equation (4.4) is stabilizable and detectable. Under these conditions $K(s)$ stabilizes the feedback system in figure 4.1, if and only if it stabilizes $P_{22}(s)$. Such compensators always exist [DOY3], [SAF3]. Introduce coprime factorizations of P and K over the ring $\mathcal{RH}^\infty$ of stable proper real-rational functions. Let

$$P_{22}(s) = N_2 M_2^{-1} = \tilde{M}_2^{-1} \tilde{N}_2 \quad (4.6)$$

be right and left coprime factorizations of $P_{22}(s)$ and

$$\begin{bmatrix} \tilde{X}_2 & -\tilde{Y}_2 \\ -\tilde{N}_2 & \tilde{M}_2 \end{bmatrix} \begin{bmatrix} M_2 & Y_2 \\ N_2 & X_2 \end{bmatrix} = I \quad (4.7)$$

the corresponding Bezout identities. All the matrices in (4.7) belong to $\mathfrak{R}H^\infty$; satisfaction of (4.7) guarantees that N_2, M_2 are right coprime and $\tilde{N}_2, \tilde{M}_2$ are left coprime.

The following formulas parametrize all proper K 's which stabilize P_{22} , [YOU1], [DES1], [VID1]

$$K = (Y_2 - M_2 Q)(X_2 - N_2 Q)^{-1} \quad (4.8)$$

$$= (\tilde{X}_2 - Q\tilde{N}_2)^{-1}(\tilde{Y}_2 - Q\tilde{M}_2), \quad Q \in \mathfrak{R}H^\infty \quad (4.9)$$

The right hand sides of (4.8) and (4.9) constitute a right coprime factorization (rcf) and left coprime factorization (lcf) of K respectively; the inverses exist $\forall Q \in \mathfrak{R}H^\infty$ (because $P_{22}(\infty) = 0$). As Q varies over all stable proper matrices, the formulas generate all possible stabilizing K 's. It is easy to verify that

$$K(I - P_{22}K)^{-1} = (Y_2 - M_2 Q)\tilde{M}_2 \quad (4.10)$$

Hence,

$$\begin{aligned} F_\ell(P, K) &= [P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}] \\ &= [(P_{11} + P_{12}Y_2\tilde{M}_2P_{21}) - (P_{12}M_2)Q(\tilde{M}_2P_{21})] \\ &= [T_{11} - T_{12}QT_{21}] \end{aligned} \quad (4.11)$$

where the $T_{ij}(s) \in \mathfrak{R}H^\infty$ are defined in the obvious way. Equation (4.11) shows that $F_\ell(P, K)$ is parametrized linearly in $Q(s)$. Therefore the SCC reduces to a model-matching problem of finding matrices $Q \in \mathfrak{R}H^\infty$ to minimize $\|T_{11} - T_{12}QT_{21}\|_\infty$. A solution Q to the model-matching problem yields a solution K to the SCC via formulas (4.8) and (4.9).

4.3.2 The general distance problem

Define the model-matching error as

$$\gamma_o \triangleq \inf \{ \|T_{11} - T_{12}QT_{21}\|_\infty : Q \in \mathfrak{RH}^\infty \} \quad (4.12)$$

The infimum in (4.12) is achieved if the ranks of $T_{12}(j\omega)$ and $T_{21}(j\omega)$ are constant $\forall 0 \leq \omega < \infty$, [FRA5].

This condition is equivalent to P_{12} and P_{21} not losing rank on the imaginary axis. In applications these rank conditions hold when the plant G does not have any imaginary axis poles or zeros. Assuming that these rank conditions hold, one is interested in a particular K which results in T_{12} and T_{21} being inner and co-inner respectively; that is $T_{12}^*T_{12} = I$ and $T_{21}T_{21}^* = I$. In general there will be more error signals to be minimized than there are available actuator signals ($\underline{e} > \underline{u}$) which gives rise to T_{12} being a “tall” matrix (more rows than columns) and more external inputs than available measurements ($\underline{v} > \underline{y}$) which gives rise to T_{21} being a “fat” matrix (more columns than rows). For these cases, $T_\perp$ and $\tilde{T}_\perp$ can be found so that $[T_{12} \ T_\perp]$ and $\begin{bmatrix} T_{21} \\ \tilde{T}_\perp \end{bmatrix}$ are square and inner, $T_\perp$ and $\tilde{T}_\perp$ are called complementary inner factors (CIF). These factorizations can be carried out using standard matrix operations on state space representations [DOY3], [CHU1]. The key idea behind the factorizations are the connections between inner functions, algebraic Riccati equations (ARE’s) and spectral factorizations. The stabilizing solution of the ARE’s are needed in order to construct the desired factorizations.

Because the $\|\cdot\|_\infty$ norm is invariant under multiplication by square inner matrices, an alternative expression for (4.11) is

$$\begin{aligned} \|T_{11} - T_{12}QT_{21}\|_\infty &= \left\| [T_{12} \ T_\perp]^* (T_{11} - T_{12}QT_{21}) \begin{bmatrix} T_{21} \\ \tilde{T}_\perp \end{bmatrix}^* \right\|_\infty \\ &= \left\| \begin{bmatrix} T_{12}^*T_{11}T_{21}^* - Q & T_{12}^*T_{11}\tilde{T}_\perp^* \\ T_\perp^*T_{11}T_{21}^* & T_\perp^*T_{11}\tilde{T}_\perp^* \end{bmatrix} \right\|_\infty \end{aligned}$$

$$= \left\| \begin{bmatrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_{\infty} \quad (4.13)$$

where

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} T_{12}^* \\ T_{\perp}^* \end{bmatrix} T_{11} [T_{21}^* \quad \tilde{T}_{\perp}^*] \quad (4.14)$$

Following the state-space approach of Doyle [DOY3], yields an R with all its poles in the open RHP, ie R is completely unstable.

As an aside, it is interesting to note here that if one considers minimization of (4.14) with respect to the H_2 -norm instead of the H^{∞} -norm, the optimal solution is particularly simple since

$$\left\| \begin{bmatrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_2 = \left\{ \|R_{11} - Q\|_2^2 + \left\| \begin{bmatrix} 0 & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_2^2 \right\}^{1/2} \quad (4.15)$$

and the optimal Q is found immediately as $Q = P_{H_2}(R_{11})$.[†] The H^{∞} -norm case is however more complicated.

Equation (4.13) is a multivariable version of the Nehari extension problem, [NEH1] and will be referred to here as the *general distance problem* (GDP). This terminology arises from viewing the optimization (4.13) as given $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \in L^{\infty}$ [‡], find the optimal $Q \in \mathcal{RH}^{\infty}$ such that

$$\gamma_o = \min_{Q \in \mathcal{RH}^{\infty}} \left\| \begin{bmatrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_{\infty} \quad (4.16)$$

is minimized. The minimum norm is the distance

$$\gamma_o = \text{dist} \left(R, \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \right) \quad (4.17)$$

[†] $P_{H_2} : H_2^{\perp} \longrightarrow H_2$

[‡] L^{∞} -Banach space of matrix valued functions which are (essentially) bounded on the imaginary axis

where R is the set of matrix functions of the form

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}, \quad (4.18)$$

This class of problems will be referred to as a “4-block problem” to distinguish from the special cases when $\begin{bmatrix} R_{21} & R_{22} \end{bmatrix}$ or $\begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}$ is identically zero (“2-block problem”) and when $\begin{bmatrix} R_{21} & R_{22} \end{bmatrix}$ and $\begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}$ are zero (“1-block” problem).

The fundamental issue in the GDP is the existence of the optimal solution. The optimal norm of the GDP can be expressed in terms of an induced operator norm or an equivalent eigenvalue problem involving a combination of Hankel and Toeplitz operators, [CHU2], [FEI1]. Although, explicit formulas are given for the smallest achievable norm, it requires the solution of an infinite dimensional eigenvalue problem which is difficult to solve.

4.3.3 A note on the classification of H^∞ problems

2-block (4-block) problems arise when either T_{12} or T_{21} (or both) are non-square. However, these conditions by themselves are not sufficient due to the following theorem.

Theorem 4.1

If T_{11} contains either the inner part of T_{12} or the co-inner part of T_{21} (or both), $R(s)$ in (4.14) reduces to a 2-block (or 1-block) problem. The case T_{21} will be treated, the result for T_{12} follows analogously. The proof requires the following lemma.

Lemma 4.1

Given $P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathfrak{R}^{(\ell_1+\ell_2) \times (m_1+m_2)}$. Consider the case when P_{12} or P_{21} (or both) are non-square and assume that the number of ($\#$) rows (P_{12}) $\geq \#$ columns(P_{12})

$(\ell_1 > m_2)$ and $\# \text{ columns } (P_{21}) \geq \# \text{ rows } (P_{21})$ ($\ell_2 < m_1$), this encompasses most physical systems.

Let $\mathfrak{R}^{\ell \times m}(s)$ denote the ring of $\ell \times m$ matrices whose elements are real-rational functions.

Denote by T_1 and T_2 the following mappings:

$$T_1 : \mathfrak{R}^{\ell_1 \times m_1} \longrightarrow \mathfrak{R}^{\ell_1 \times m_1}, \text{ where } T_1 = \alpha P_{11}, \quad P_{11} \in \mathfrak{R}^{\ell_1 \times m_1}, \alpha \in \mathfrak{R}^{\ell_1 \times \ell_1} \quad (4.19)$$

$$T_2 : \mathfrak{R}^{\ell_2 \times m_1} \longrightarrow \mathfrak{R}^{\ell_1 \times m_1}, \text{ where } T_2 = \beta P_{21}, \quad P_{21} \in \mathfrak{R}^{\ell_2 \times m_1}, \beta \in \mathfrak{R}^{\ell_1 \times \ell_2} \quad (4.20)$$

If $\text{Range}(T_1) \subset \text{Range}(T_2)$ then, $\exists \beta \in \mathfrak{R}^{\ell_1 \times \ell_2}$ such that $\beta P_{21} = P_{11}$. β exists by closure of the ring $\mathfrak{R}^{\ell_1 \times \ell_2}(s)$

Proof

$\hat{\alpha} \in \text{Range}(T_1) \Rightarrow \hat{\alpha} = \alpha P_{11}$, for some $\alpha \in \mathfrak{R}^{\ell_1 \times \ell_1}$. In particular take $\alpha = I_{\ell_1 \times \ell_1}$ then $\hat{\alpha} = P_{11}$. But $\text{Range}(T_1) \subset \text{Range}(T_2) \Rightarrow \exists \beta \in \mathfrak{R}^{\ell_1 \times \ell_2}$ such that $\hat{\alpha} = \beta P_{21} = P_{11}$ $\square$

Corollary 4.1

The condition $\beta P_{21} = P_{11} \Rightarrow \text{Rank} \begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix} = \text{Rank} \begin{bmatrix} \beta \\ I \end{bmatrix} P_{21} < m_1$, since $\ell_1 < m_1$.

Thus, when $P_{11} = \beta P_{21}$, for some $\beta \in \mathfrak{R}^{\ell_1 \times \ell_2}(s)$, the matrix $\begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix}$ is less than full column rank.

However the argument is not reversible ie. $\begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix} < \text{full column rank} \nRightarrow \exists \beta$ such that $\beta P_{21} = P_{11}$

$$\text{eg. take } \begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Proof of Theorem 4.1

Suppose $\exists \beta$ such that lemma 4.1 is satisfied ie. $P_{11} = \beta P_{21}$, then using the realizations for T_{11}, T_{12} and T_{21} from (4.11)

$$\begin{aligned}
 T_{11} &= P_{11} + P_{12}Y\tilde{M}_2P_{21} \\
 &= [\beta\tilde{M}_2^{-1} - P_{12}Y]\tilde{M}_2P_{21} \\
 &= [\beta\tilde{M}_2^{-1} - P_{12}\tilde{Y}]T_{21} \\
 &= \tilde{T}_{11}T_{21}
 \end{aligned} \tag{4.21}$$

$\tilde{M}_2^{-1}$ exists because $\tilde{M}_2^{-1}$ is a lcf of P_{22} and $\tilde{T}_{11} = \beta\tilde{M}_2^{-1} - P_{12}\tilde{Y}$.

$$\begin{aligned}
 \text{Therefore, } R &= \begin{bmatrix} T_{12}^* \\ T_{\perp}^* \end{bmatrix} \tilde{T}_{11}T_{21} \begin{bmatrix} T_{21}^* & \tilde{T}_{\perp}^* \end{bmatrix} \\
 &= \begin{bmatrix} T_{12}^* \\ T_{\perp}^* \end{bmatrix} \begin{bmatrix} \tilde{T}_{11} & 0 \end{bmatrix} = \begin{bmatrix} T_{12}^*\tilde{T}_{11} & 0 \\ T_{\perp}^*\tilde{T}_{11} & 0 \end{bmatrix} \quad \square
 \end{aligned}$$

Hence, the 4-block problem reduces to a 2-block problem. These special cases arise when the error vector $\underline{e}$ is dependent on either the input vector $\underline{v}$ or output vector $\underline{y}$.

Example

Consider the control problem shown in figure 4.2, where it is desired to minimize the error signal and the actuator output subject to the reference input $\underline{r}$ and disturbance input $\underline{d}$.

The input vector $\underline{v} = \begin{bmatrix} \underline{d} \\ \underline{r} \end{bmatrix}$ is greater than the size of $\underline{y}$ and the error vector $\underline{e} = \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix}$ is greater than that of $\underline{u}$ in the SCC of figure 4.1, so at first sight this would appear to be a 4-block problem. However,

$$\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{y} \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 0 & W_1 \\ -W_2G & W_2 & -W_2G \\ \hline -G & I & -G \end{array} \right] \begin{bmatrix} \underline{d} \\ \underline{r} \\ \underline{u} \end{bmatrix} \tag{4.22}$$

$$P_{11} = \begin{bmatrix} 0 \\ W_2 \end{bmatrix} \begin{bmatrix} -G & I \end{bmatrix} = \begin{bmatrix} 0 \\ W_2 \end{bmatrix} P_{21} = \beta P_{21} \quad (4.23)$$

and hence this is really only a 2-block problem, because of theorem 4.1.

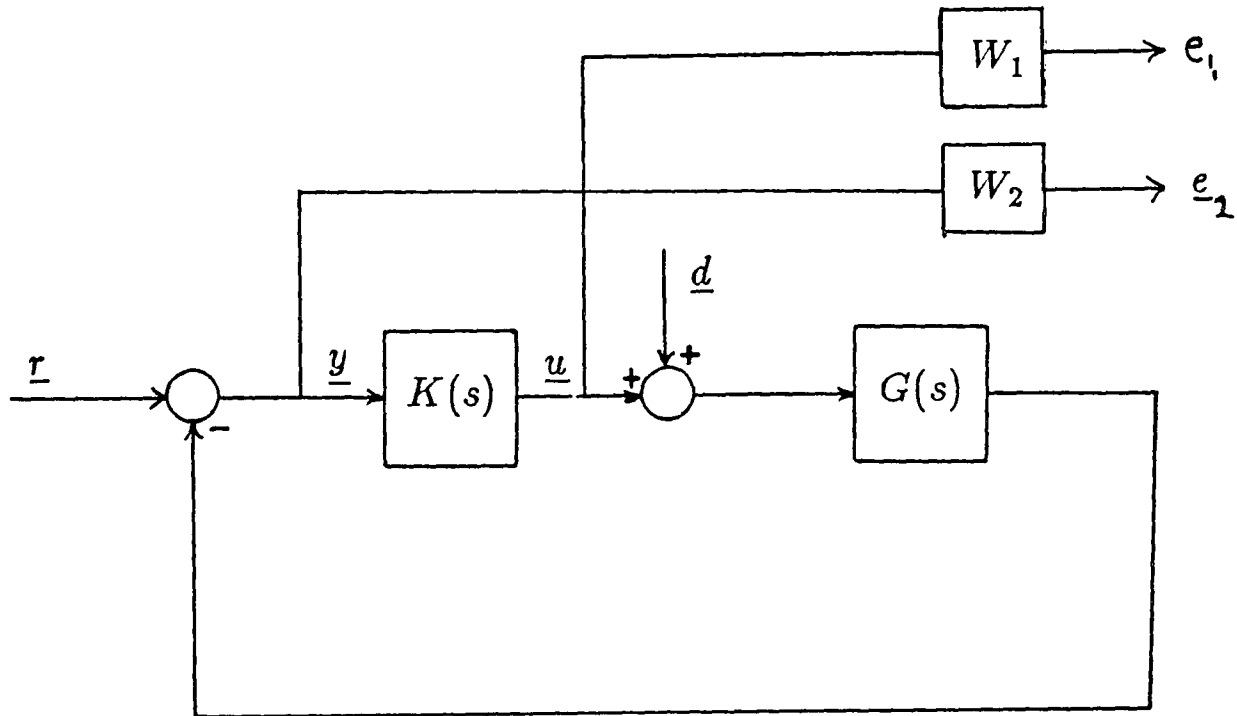


Figure 4.2

4.3.4 Hankel operators and best approximation problems

To present the results it is first necessary to review the Hankel operator and present some definitions. Let L_2 denote the Hilbert space of vector-valued square integrable functions on the imaginary axis. Denote the stable part of L_2 by the Hardy space H_2 and let $H_2^\perp$ denote its orthogonal complement. H_2 is a closed subspace of L_2

Now it is a useful fact that the L^∞ -norm of a matrix $G \in L^\infty$ equals the norm of the corresponding multiplication operator on L_2 ie.

$$\|G\|_\infty = \sup\{\|Gx\|_2 : x \in H_2, \|x\|_2 = 1\} \quad (4.24)$$

For a matrix $G \in H^\infty$, the multiplication operator leaves the subspace H_2 of L_2 invariant,

$$\text{ie. if } x \in H_2 \quad \text{then} \quad Gx \in H_2 \quad (4.25)$$

If $P_{H_2} : H_2 \longrightarrow H_2$ denotes the projection from H_2 onto H_2 , and $P_{H_2^\perp} : H_2 \longrightarrow H_2^\perp$ denotes the orthogonal projection from H_2 onto $H_2^\perp$, then (4.25) is equivalent to the condition that $P_{H_2^\perp} Gx = 0$.

Definition 1 : (Hankel Operator)

The Hankel operator H_G generated by $G \in L^\infty$ is defined as

$$H_G : H_2 \longrightarrow H_2^\perp, \quad \text{such that} \quad H_G f = (P_{H_2^\perp} G) f \quad (4.26)$$

Definition 2 : (Toeplitz Operator)

The Toeplitz operator T_R generated by $G \in L^\infty$ is defined as

$$T_G : H_2 \longrightarrow H_2 \quad \text{such that} \quad T_G f = (P_{H_2} G) f \quad (4.27)$$

Definition 3

The operator $\tilde{T}_G$ generated by $G \in L^\infty$ is defined as

$$\tilde{T}_G : H_2^\perp \longrightarrow H_2^\perp \quad \text{such that} \quad \tilde{T}_G f = (P_{\tilde{H}_2^\perp} G) f \quad (4.28)$$

where $P_{\tilde{H}_2^\perp}$ denotes the projection from $H_2^\perp$ onto $H_2^\perp$.

The existence of the optimal solution to the general distance problem was discussed in § 4.3.2; several approaches have been proposed for its computation. For the 2-block case, the optimal solution to the GDP has been shown to be, [CHU1], [CHU2], [FEI1],

$$\lambda_{\max}^{1/2} [H_{R_{11}^*} H_{R_{11}} + T_{R_{21}^* R_{21}}] \quad (4.29)$$

where $H_{R_{11}}$ and $T_{R_{21}}$ are the Hankel and Toeplitz operators generated by R_{11} and R_{21} respectively and $\lambda_{\max} [H_{R_{11}^*} H_{R_{11}} + T_{R_{21}^* R_{21}}]$ is the maximum eigenvalue (or spectral radius)

of the matrix $[H_{R_{11}}^* H_{R_{11}} + T_{R_{21}}^* R_{21}]$. Unfortunately, the matrix representations of the Hankel and Toeplitz operators are infinite dimensional. However, the spectral radius of an infinite dimensional matrix can be approximated by a truncated finite dimensional matrix of sufficiently large size. This approach has been studied by Jonckheere and Verma [JON1] and Feintuch and Francis [FEI1]. Jonckheere and Juang [JON2], approximate the H^∞ -optimization problem by a suboptimal linear quadratic problem and claim that the suboptimal norm of the linear quadratic problem is a good approximation to the optimal norm.

An alternative approach is the iterative procedure called γ -iteration, which essentially involves guessing a γ and then reducing the problem of finding all Q that give norm less than γ to a standard best (Hankel-norm) approximation problem. This approach has been studied by Doyle [DOY3], Chu [CHU1], [CHU2], and Chang [CHA2], [CHA3].

The best (or Hankel) approximation problem can be stated as follows. For a given matrix function $R_{11} \in L^\infty$, find a function $Q \in \mathfrak{RH}^\infty$ such that $\|R_{11} - Q\|_\infty$ is minimized. This problem has the special property that it reduces to a finite dimensional eigenvalue problem and solutions can be computed with standard matrix algebra. It was first solved by Nehari [NEH1], for general scalar discrete systems and then by Adamjan et al [ADA2], for the multivariable case. Recently, Glover [GLO2], considers a special case of the problem where the function to be approximated is real-rational, ie $R_{11} \in \mathfrak{RL}^\infty$. An efficient and constructive algorithm for the optimal solution can be constructed in terms of a balanced realization of the system. The corresponding controller is then found by back substituting $Q(s)$ into (4.8) or (4.9).

Theorem 4.2 (Best approximation)

Consider the following minimization

$$\gamma_o \triangleq \min_{Q \in \mathfrak{RH}^\infty} \|R_{11} - Q\|_\infty \quad (4.30)$$

then $\gamma_o = \|H_{R_{11}}\|$ and the minimum is achieved by some $Q \in \Re H^\infty$

Proof

$$\begin{aligned} \|R_{11} - Q\|_\infty &= \sup\{\|(R_{11} - Q)x\|_2 : x \in H_2; \|x\|_2 = 1\} \\ &\geq \sup\left\{\left\|P_{H_2^\perp}(R_{11} - Q)x\right\|_2 : x \in H_2; \|x\|_2 = 1\right\} \\ &= \sup\left\{\left\|P_{H_2^\perp}R_{11}x\right\|_2 : x \in H_2; \|x\|_2 = 1\right\} \\ &= \|H_{R_{11}}\| \end{aligned}$$

in fact equality holds [POW1]. $\square$

Corollary 4.2

$$\|H_{R_{11}}\| = \lambda_{\max}^{1/2}[H_{R_{11}}^* H_{R_{11}}] \quad (4.31)$$

4.4 γ -Iteration

The following theorem lies at the heart of the 2-block γ -iteration scheme.

Theorem 4.3

For $\gamma > \gamma_o > \|R_{21}\|_\infty$,

$$\left\| \begin{bmatrix} R_{11} - Q \\ R_{21} \end{bmatrix} \right\|_\infty \leq \gamma \quad (4.32)$$

if and only if

$$\|(R_{11} - Q)M^{-1}\|_\infty \leq 1 \quad (4.33)$$

where M is the spectral factor of the para-hermitian matrix $(\gamma^2 I - R_{21}^* R_{21})$, ie $M^* M = (\gamma^2 I - R_{21}^* R_{21})$

Proof

$$\begin{aligned}
\forall \gamma > \gamma_o > \|R_{21}\|_\infty, \quad & \left\| \begin{bmatrix} R_{11} - Q \\ R_{21} \end{bmatrix} \right\|_\infty \leq \gamma \\
& \iff (R_{11} - Q)^*(R_{11} - Q) + R_{21}^* R_{21} \leq \gamma^2 I \\
& \iff (R_{11} - Q)^*(R_{11} - Q) \leq \gamma^2 I - R_{21}^* R_{21} \\
& \iff \|(R_{11} - Q)u\| \leq \|(\gamma^2 I - R_{21}^* R_{21})^{1/2} u\| \quad \forall u \\
& \iff R_{11} - Q = Y(\gamma^2 I - R_{21}^* R_{21})^{1/2} \quad \text{for some } \|Y\|_\infty \leq 1 \\
& \iff \|(R_{11} - Q)(\gamma^2 I - R_{21}^* R_{21})^{-1/2}\|_\infty \leq 1 \quad \square
\end{aligned}$$

This result means that for $\gamma > \gamma_o > \|R_{21}\|_\infty$, $\exists \tilde{Q} \in \Re H^\infty$ such that $\|[R_{11}M^{-1}]_- - \tilde{Q}\|_\infty \leq 1$ [‡].

It has also been shown by Chu et al [CHU1], [CHU2] that if $\gamma < \gamma_o > \|R_{21}\|_\infty$, then $\|H_{R_{11}M^{-1}}\| > 1$. Therefore, a solution to (4.32) can be obtained by considering the following best approximation formulation

$$\gamma_o = \min_{\tilde{Q} \in \Re H^\infty} \|[R_{11}M^{-1}]_- - \tilde{Q}\|_\infty \quad (4.34)$$

Thus the γ -iteration is basically a method of finding a γ such that $f(\gamma) = 1$, where $f(\gamma) = \|H_{R_{11}M^{-1}}\|$.

The case $\gamma_o = \|R_{21}\|_\infty$ causes the spectral factorization to fail, because the factorization can only be performed when $\gamma^2 I - R_{21}^* R_{21}$ has no zeros on the imaginary axis. Continuing with $\gamma = \gamma_o + \epsilon$, where ϵ is small is a possible solution.

[‡] $[R_{11}M^{-1}]_-$ denotes the antistable part of $R_{11}M^{-1}$ and $\tilde{Q} = QM^{-1}$

When γ_o has been found from the Hankel approximation of the unstable part of $R_{11}M^{-1}$, Q can be recovered as

$$Q = (\tilde{Q} + [R_{11}M^{-1}]_+)M^{\dagger\dagger} \quad (4.35)$$

Then substitution of Q in (4.8) or (4.9) leads to the optimal controller. As a check, the optimal controller should be stabilizing and the maximum singular value of the cost function should be equal to the infinity norm of the model matching error. For the case when Q is optimal, the maximum singular value of the cost function is all pass, ie constant over all frequency.

The following theorem is a generalization of theorem 4.3 to the 4-block case.

Theorem 4.4

$$\left\| \begin{bmatrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_{\infty} < \gamma \quad (\forall \gamma > \gamma_o) \quad (4.36)$$

$$\iff \left\| (I - LL^*)^{-1/2} \left[F_{\ell} \left\{ \frac{1}{\gamma}R, \frac{1}{\gamma}R_{22}^* \right\} - \frac{1}{\gamma}Q \right] (I - \tilde{L}^*\tilde{L})^{-1/2} \right\|_{\infty} < 1 \quad (4.37)$$

where $(I - LL^*)^{-1/2}$ = spectral factor of $(I - LL^*)$

$(I - \tilde{L}^*\tilde{L})^{-1/2}$ = spectral factor of $(I - \tilde{L}^*\tilde{L})$

$$L = R_{12}S^{-1}, \quad \tilde{L} = \tilde{S}^{-1}R_{21} \quad (4.38)$$

$$S = (\gamma^2 I - R_{22}^* R_{22})^{1/2}, \quad \tilde{S} = (\gamma^2 I - R_{22} R_{22}^*)^{1/2} \quad (4.39)$$

$$\text{and } F_{\ell} \left\{ \frac{1}{\gamma}R, \frac{1}{\gamma}R_{22}^* \right\} = \frac{1}{\gamma} \{ R_{11} + R_{12}(\gamma^2 I - R_{22}^* R_{22})^{-1} R_{22}^* R_{21} \} \quad (4.40)$$

Proof [CHU1], [CHU2].

$\dagger\dagger [R_{11}M^{-1}]_+$ denotes the stable part of $R_{11}M^{-1}$

4.4.1 Properties of the γ -iteration

Recall for the 2-block γ -iteration

$$\gamma_o = \min_{\tilde{Q} \in \mathcal{RH}^\infty} \| [R_{11}M^{-1}]_- - \tilde{Q} \|_\infty$$

Since R_{11} and M are rational, this operator is finite rank and γ_o can be found as the square root of the largest eigenvalue λ_{max} of the following standard eigenvalue problem [CHU1], [CHU2].

$$(H_{R_{11}M^{-1}})^*(H_{R_{11}M^{-1}})u = \lambda u \quad (4.41)$$

$$\iff H_{R_{11}}^* H_{R_{11}} v = \lambda(\gamma^2 I - T_{R_{21}^* R_{21}})v \quad (4.42)$$

where $v = T_{M^{-1}}u$ and H, T denote the Hankel operator and Toeplitz operators respectively.

Provided that $\gamma > \|T_{R_{21}^* R_{21}}\|^{1/2} = \|R_{21}\|_\infty$ and noting that $H_{R_{11}}^* H_{R_{11}} \geq 0$, $(\gamma^2 I - T_{R_{21}^* R_{21}}) > 0$ it can be shown that the 2-block γ -iteration is convex, by treating (4.42) as a perturbed generalized eigenvalue problem. The following lemma is needed before the corresponding result for the 4-block case is presented.

Lemma 4.2

$$H_{R_{11}M^{-1}} = H_{R_{11}} T_{M^{-1}} \quad (4.43)$$

Proof

$$\begin{aligned} H_{R_{11}M^{-1}} f &= (P_{H_2^\perp} R_{11} M^{-1}) f && \forall f \in H_2 \\ &= (P_{H_2^\perp} R_{11} (P_{H_2} + P_{H_2^\perp}) M^{-1}) f \\ &= (P_{H_2^\perp} R_{11}) (P_{H_2} M^{-1}) f + (P_{H_2^\perp} R_{11}) (P_{H_2^\perp} M^{-1}) f \\ &= (P_{H_2^\perp} R_{11}) (P_{H_2} M^{-1}) f^\dagger \\ &= (H_{R_{11}} T_{M^{-1}}) f \quad \square \end{aligned}$$

$^\dagger P_{H_2^\perp} M^{-1} = 0$ since $M^{-1} \in H^\infty$

Turning attention now to the 4-block case,

$$\left\| \begin{bmatrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_{\infty} < \gamma \quad (\forall \gamma > \gamma_0)$$

$$\iff \left\| \begin{bmatrix} R_{11} - Q & R_{12} \end{bmatrix} N^{-1} \right\|_{\infty} < 1$$

$$\text{where } N \text{ satisfies } N^* N = \left(\gamma^2 I - \begin{bmatrix} R_{21}^* \\ R_{22}^* \end{bmatrix} \begin{bmatrix} R_{21} & R_{22} \end{bmatrix} \right)$$

$$\iff \left\| (I - LL^*)^{-1/2} \left[F_{\ell} \left\{ \frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \right\} - \frac{1}{\gamma} Q \right] (I - \tilde{L}^* \tilde{L})^{-1/2} \right\|_{\infty} < 1$$

$$\iff \left\| M^{-1} F_{\ell} \left\{ \frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \right\} \tilde{M}^{-1} - \tilde{Q} \right\|_{\infty} < 1$$

$$\iff \left\| H_{M^{-1} F_{\ell} \left\{ \frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \right\} \tilde{M}^{-1}} \right\| < 1 \quad (4.44)$$

where $M = (I - LL^*)^{1/2}$, $\tilde{M} = (I - \tilde{L}^* \tilde{L})^{1/2}$, $\tilde{Q} = \frac{1}{\gamma} Q (I - \tilde{L}^* \tilde{L})^{-1/2} \in \mathfrak{R}H^{\infty}$ and L and $\tilde{L}$ are given by equation (4.38).

Since M , $\tilde{M}$ and $F_{\ell} \left(\frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \right)$ are rational, the operator (4.43) is finite rank and γ_0 can be found as the square root of the largest eigenvalue λ_{max} of the following standard eigenvalue problem.

$$\left(H_{M^{-1} F_{\ell} \left(\frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \right) \tilde{M}^{-1}} \right)^* \left(H_{M^{-1} F_{\ell} \left(\frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \right) \tilde{M}^{-1}} \right) u = \lambda u \quad (4.45)$$

Using the definitions for the Hankel and Toeplitz operators given by (4.26) and (4.27) and dropping the argument of $F_{\ell} \left(\frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \right)$ for brevity,

$$\begin{aligned} \left(H_{M^{-1} F_{\ell} \tilde{M}^{-1}} \right) f &= \left(P_{H_2^{\perp}} M^{-1} F_{\ell} \tilde{M}^{-1} \right) f & \forall f \in H_2 \\ &= \left(P_{H_2^{\perp}} M^{-1} F_{\ell} \left(P_{H_2} + P_{H_2^{\perp}} \right) \tilde{M}^{-1} \right) f \end{aligned}$$

$$\begin{aligned}
&= \left(P_{H_2^\perp} M^{-1} F_\ell \right) \left(P_{H_2} \tilde{M}^{-1} \right) f + \left(P_{H_2^\perp} M^{-1} F_\ell \right) \left(P_{H_2^\perp} \tilde{M}^{-1} \right) f \\
&= \left(P_{H_2^\perp} M^{-1} F_\ell \right) T_{\tilde{M}^{-1}} f^\dagger \\
&= \left(P_{H_2^\perp} M^{-1} \left(P_{H_2} + P_{H_2^\perp} \right) F_\ell \right) T_{\tilde{M}^{-1}} f \\
&= \left(P_{H_2^\perp} M^{-1} P_{H_2} \right) \left(P_{H_2} F_\ell T_{\tilde{M}^{-1}} \right) f + \left(P_{H_2^\perp} M^{-1} P_{H_2^\perp} \right) \left(P_{H_2^\perp} F_\ell \right) T_{\tilde{M}^{-1}} f \\
&= \left(\tilde{T}_{M^{-1}} H_{F_\ell} T_{\tilde{M}^{-1}} \right) f^{\dagger\dagger} \tag{4.46}
\end{aligned}$$

Therefore (4.45) becomes $\left(T_{\tilde{M}^{-1}}^* H_{F_\ell}^* \tilde{T}_{M^{-1}}^* \tilde{T}_{M^{-1}} H_{F_\ell} T_{\tilde{M}^{-1}} \right) u = \lambda u$

$$\begin{aligned}
\Rightarrow \left(\tilde{T}_{M^{-1}}^* \tilde{T}_{M^{-1}} \right) v &= \lambda \left(H_{F_\ell}^* \right)^{-1} \left(T_{\tilde{M}^{-1}}^* \right)^{-1} \left(T_{\tilde{M}^{-1}} \right)^{-1} \left(H_{F_\ell} \right)^{-1} v \\
&= \lambda \left(H_{F_\ell}^* \right)^{-1} T_{\tilde{M}}^* T_{\tilde{M}} \left(H_{F_\ell} \right)^{-1} v \\
&= \lambda \left(H_{F_\ell}^* \right)^{-1} T_{\tilde{M}^*} T_{\tilde{M}} \left(H_{F_\ell} \right)^{-1} v \\
&= \lambda \left(H_{F_\ell}^* \right)^{-1} T_{\tilde{M}^* \tilde{M}} \left(H_{F_\ell} \right)^{-1} v \\
&= \lambda \left(H_{F_\ell}^* \right)^{-1} \left(T_{\gamma^2 I - (I - \tilde{L}^* \tilde{L})} \right) \left(H_{F_\ell} \right)^{-1} v \\
&= \lambda \left(H_{F_\ell}^* \right)^{-1} \left(\gamma^2 I - T_{(I - \tilde{L}^* \tilde{L})} \right) \left(H_{F_\ell} \right)^{-1} v \tag{4.47}
\end{aligned}$$

where $v = H_{F_\ell} T_{\tilde{M}^{-1}} u$

The following algebraic properties of the Toeplitz operator have been used throughout, more details can be found in the standard publications [BRO2], [DOU1], [HAL1], [SAR2].

$$T_G^* = T_{G^*}, \quad T_{G^{-1}} = T_G^{-1} \quad \forall G \in L^\infty, \text{ and } \quad T_{G_1 G_2} = T_{G_1} T_{G_2} \text{ if either } G_1^* \text{ or } G_2 \in H^\infty$$

Unfortunately, the eigenvalue problem given by equation (4.47) has no solution, since both the left hand and right hand sides of the equation consist of a Hermitian Toeplitz operator, which has infinite rank and it is well known [BRO2], that such an operator has no point

† $P_{H_2^\perp} \tilde{M}^{-1} = 0$ since $\tilde{M}^{-1} \in H^\infty$ and $P_{H_2} \tilde{M}^{-1} = T_{\tilde{M}^{-1}}$

†† $P_{H_2^\perp} M^{-1} P_{H_2} = 0$, since $M^{-1} \in H^\infty$, $P_{H_2^\perp} F_\ell = H_{F_\ell}$, and $P_{H_2^\perp} M^{-1} P_{H_2^\perp} = \tilde{T}_{M^{-1}}$

spectrum (ie. no eigenvalues) except in the scalar case. Thus, convexity cannot readily be proven by simply following the proof for the 2-block case.

$$\text{Define, } \hat{f}(\gamma) = \| [R_{11} - Q] N^{-1} \|_{\infty} \quad (4.48)$$

$$= \left\| \left[\left(F_{\ell} \left\{ \frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \right\} - \frac{1}{\gamma} Q \right) (I - \tilde{L}^* \tilde{L})^{-1/2} \quad L \right] \right\|_{\infty} \quad (4.49)$$

$$f(\gamma) = \left\| (I - LL^*)^{-1/2} \left(F_{\ell} \left\{ \frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \right\} - \frac{1}{\gamma} Q \right) (I - \tilde{L}^* \tilde{L})^{-1/2} \right\|_{\infty} \quad (4.50)$$

$$= \left\| H_{(I-LL^*)^{-1/2} (F_{\ell} \{ \frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \} - \frac{1}{\gamma} Q) (I - \tilde{L}^* \tilde{L})^{-1/2}} \right\| \quad (4.51)$$

Then it is possible to show the following properties for the 4-block γ -iteration.

Theorem 4.5

$\forall \gamma > \gamma_o$, $\hat{f}(\gamma)$ is a monotonic decreasing function.

Proof

Suppose that $\gamma_2 > \gamma_1 \geq \gamma_o$ are approximations to γ_o at consecutive iterations. Then

$$\hat{f}(\gamma_i) = \| [R_{11} - Q] N_i^{-1} \|_{\infty} \quad i = 1, 2, \quad \text{for some } Q \in \Re H^{\infty}$$

$$\text{where } N_i^* N_i = \gamma_i^2 I - \begin{bmatrix} R_{21}^* \\ R_{22}^* \end{bmatrix} \begin{bmatrix} R_{21} & R_{22} \end{bmatrix}$$

For $\gamma_2 > \gamma_1 (\geq \gamma_o)$

$$N_1^* N_1 - N_2^* N_2 = (\gamma_1^2 - \gamma_2^2) I < 0$$

$$\implies N_1^* N_1 < N_2^* N_2 \quad (4.52)$$

$\forall \gamma_1 \geq \gamma_o$, $0 < \epsilon \leq 1$, $\exists Q$ such that

$$\| [R_{11} - Q \quad R_{12}] N_1^{-1} \|_{\infty} \leq \epsilon$$

$$\Rightarrow (N_1^*)^{-1} \begin{bmatrix} (R_{11} - Q)^* \\ R_{12}^* \end{bmatrix} [R_{11} - Q \quad R_{12}] N_1^{-1} \leq \epsilon^2 I$$

$$\Rightarrow \begin{bmatrix} (R_{11} - Q)^* \\ R_{12}^* \end{bmatrix} [R_{11} - Q \quad R_{12}] \leq \epsilon^2 N_1^* N_1 < \epsilon^2 N_2^* N_2$$

$$\Rightarrow \|[R_{11} - Q \quad R_{12}] N_2^{-1}\|_\infty < \epsilon$$

$$\Rightarrow \hat{f}(\gamma_2) < \hat{f}(\gamma_1) \quad \square$$

Theorem 4.5

$\hat{f}(\gamma)$ is right continuous ie $\lim_{\gamma \rightarrow \gamma_o+} \hat{f}(\gamma)$ exists

Proof

Since $\hat{f}(\gamma) \geq 0$, $\forall \gamma \in [\gamma_o, \infty)$, in particular $\forall \delta > 0$, $\hat{f}(\gamma) \geq 0$, for $\gamma \in [\gamma_1 - \delta, \gamma_1]$, some $\gamma_1 \in [\gamma_o, \infty)$,

Therefore there exists a “ greatest lower bound ” c , for the set $\{\hat{f}(\gamma) \mid \gamma \in [\gamma_1 - \delta, \gamma_1]\}$.

Consequently (according to the definition of the greatest lower bound) $\exists$ a point $\gamma_2 \in [\gamma_1 - \delta, \gamma_1]$, such that

$$c \leq \hat{f}(\gamma_2) < c + \epsilon/2, \quad \forall \epsilon > 0$$

Take $\delta = \gamma_1 - \gamma_2 > 0$, then $\forall \gamma', \gamma'' \in [\gamma_1 - \delta, \gamma_1] = [\gamma_2, \gamma_1]$

$$\begin{aligned} \|\hat{f}(\gamma') - \hat{f}(\gamma'')\| &= \|\hat{f}(\gamma') - \hat{f}(\gamma_2) + \hat{f}(\gamma_2) - \hat{f}(\gamma'')\| \\ &\leq \|\hat{f}(\gamma') - \hat{f}(\gamma_2)\| + \|\hat{f}(\gamma_2) - \hat{f}(\gamma'')\| \\ &\leq [\hat{f}(\gamma_2) - \hat{f}(\gamma')] + [\hat{f}(\gamma_2) - \hat{f}(\gamma'')]^\dagger \\ &\leq c + \epsilon/2 - c + c + \epsilon/2 - c \\ &= \epsilon \quad \square \end{aligned}$$

† because $\hat{f}(\gamma)$ is a monotonic decreasing function

Therefore since γ_1 and γ_2 can be made arbitrarily close to γ_o , $\lim_{\gamma \rightarrow \gamma_o+} \hat{f}(\gamma)$ exists.

Thus it has been shown that $\hat{f}(\gamma)$ is right continuous and monotonically decreasing $\forall \gamma > \gamma_o$. Since $\hat{f}(\gamma) < 1 \iff \gamma > \gamma_o$, it is clear that $\hat{f}(\gamma) \geq 1 \forall \gamma \leq \gamma_o$. However, the iterative scheme uses $f(\gamma)$ not $\hat{f}(\gamma)$, but from Theorem 4.4, $f(\gamma) < 1 \forall \gamma > \gamma_o \implies f(\gamma) \geq 1, \forall \gamma \leq \gamma_o$. Thus as for the 2-block γ -iteration, a search scheme for the 4-block γ -iteration would test for when $f(\gamma) < 1$ and when $f(\gamma) > 1$ in determining the next choice of γ . It has not been possible to establish whether $f(\gamma)$ is continuous or convex, but simulations seem to indicate that convergence is normally attained. More discussion of the iterative schemes is given in chapter 5.

4.4.2 Calculation of bounds

The following upper and lower bounds on γ_o are useful since they are easily computable and have been shown by Chu [CHU1], [CHU2], to have a ratio no greater than $\sqrt{2}$ in the 2-block case and 2 in the 4-block case.

Bounds for the 2-block case will be considered first. It is quite straight forward to show that upper and lower bounds for γ_o are given by

$$\gamma_u = \left\| \begin{bmatrix} \|H_{R_{11}}\| \\ \|R_{21}\|_\infty \end{bmatrix} \right\|_2 \quad (4.53)$$

$$\gamma_\ell = \max \{ \|H_{R_{11}}\|, \|R_{21}\|_\infty \} \quad (4.54)$$

such that

$$\frac{1}{\sqrt{2}}\gamma_o \leq \gamma_\ell \leq \gamma_o \leq \gamma_u \leq \sqrt{2}\gamma_o \quad (4.55)$$

Unfortunately, these bounds are not very useful from a computational point of view because $\|R_{21}\|_\infty$ is difficult to calculate. However, it is possible to compute a lower bound for $\|R_{21}\|_\infty$ as

$$\|R_{21}\|_\infty \geq \max\{H_{R_{21}}, \bar{\sigma}[R_{21}(s=0)], \bar{\sigma}[R_{21}(s=0)]\} \quad (4.56)$$

It is not clear that the inclusion of $H_{R_{21}}$ in the right hand side of equation (4.56) gives rise to a tighter lower bound for $\|R_{21}\|_\infty$. As a result this term is excluded in practice because of the computational effort involved in calculating it.

Since the spectral factorization $M^*M = \gamma^2 I - R_{21}^* R_{21}$ can only be performed when $\gamma^2 I - R_{21}^* R_{21}$ has no $j\omega$ -axis zeros, this implies that $\gamma > \|R_{21}\|_\infty$, when the factorization succeeds. This in turn gives an upper bound for $\|R_{21}\|_\infty$ and thus for γ_u as

$$\gamma_u = \sqrt{\|H_{R_{11}}\|^2 + \gamma^2} \quad (4.57)$$

The bounds for the 4-block γ -iteration are more complicated. Lower and upper bounds are given by

$$\gamma_\ell = \max\{\|H_R\|, \|[R_{21} \ R_{22}]\|_\infty, \left\| \begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix} \right\|_\infty\} \quad (4.58)$$

$$\gamma_u = \min \left\{ \left\| \begin{bmatrix} \|H_{R_{11}}\| \\ \|R_{12}\|_\infty \\ \|[R_{21} \ R_{22}]\|_\infty \end{bmatrix} \right\|_2, \left\| \begin{bmatrix} \|H_{R_{11}}\| \\ \|R_{21}\|_\infty \\ \left\| \begin{bmatrix} R_{21} \\ R_{22} \end{bmatrix} \right\|_\infty \end{bmatrix} \right\|_2 \right\} \quad (4.59)$$

Then

$$\frac{1}{2}\gamma_o \leq \gamma_\ell \leq \gamma_o \leq \gamma_u \leq 2\gamma_o \quad (4.60)$$

Similarly as for the 2-block case an upper bound for γ_u , when all the spectral factorizations are passed is given by

$$\gamma_u = \sqrt{\|H_{R_{11}}\|^2 + 2\gamma^2} \quad (4.61)$$

This is because $(I - LL^*)$ admits a spectral factorization $\iff I - LL^* \geq 0$

$$\Rightarrow I \geq LL^*$$

$$\Leftrightarrow I \geq (R_{12}S^{-1})(R_{12}S^{-1})^*$$

$$\Leftrightarrow I \geq \bar{\sigma}[R_{12}S^{-1}]$$

$$\Leftrightarrow I \geq (R_{12}S^{-1})^*(R_{12}S^{-1})$$

$$\Leftrightarrow S^*S \geq R_{12}^*R_{12}$$

$$\Leftrightarrow \gamma^2 I - R_{22}^*R_{22} \geq R_{12}^*R_{12}$$

$$\Leftrightarrow \gamma^2 I \geq R_{12}^*R_{12} + R_{22}^*R_{22}$$

$$\Leftrightarrow \gamma^2 I \geq \begin{bmatrix} R_{12}^* & R_{22}^* \end{bmatrix} \begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}$$

Similarly $(I - \tilde{L}^* \tilde{L})$ admits a spectral factorization

$$\Leftrightarrow \gamma^2 I \geq \begin{bmatrix} R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} R_{21}^* \\ R_{22}^* \end{bmatrix}$$

Therefore since $\left\| \begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix} \right\|_{\infty}^2 \geq \|R_{12}\|_{\infty}^2$

$$\begin{aligned} \left\| \begin{bmatrix} R_{11} - Q & R_{21} \\ R_{21} & R_{22} \end{bmatrix} \right\|_{\infty} &< \|H_{R_{11}}\|_{\infty}^2 + \|R_{12}\|_{\infty}^2 + \left\| \begin{bmatrix} R_{21} & R_{22} \end{bmatrix} \right\|_{\infty}^2 \\ &< \|H_{R_{11}}\|_{\infty}^2 + 2\gamma^2 \quad \square \end{aligned}$$

The upper bounds given by (4.57) and (4.59) however are generally conservative.

4.5 Motivation For 4-Block Problems

When a plant has $j\omega$ -axis zeros or $j\omega$ -axis poles, as in the case of strictly proper plants which have zeros at $\omega = \infty$ and plants with integrators (poles at $\omega = 0$), algorithms for

solving the H^∞ - optimization may fail. This is because the optimization problem (4.12) cannot be solved if either T_{12} is less than full column rank and/or T_{21} is less than full row rank. The effect of these $j\omega$ -axis zeros is to make the transformation of (4.12) to (4.16) impossible. In the current literature $j\omega$ -axis zeros are often assumed not to exist in the theoretical treatment of H^∞ -optimization problems. However, plants which have $j\omega$ -axis poles or zeros form a large class of physical systems.

Various solutions have been proposed to tackle the problem of $j\omega$ -axis zeros and these include “ $j\omega$ -axis shifting” [XU1] and conformal transformations [SAF7], to redefine the $j\omega$ -axis and hence move the offending points. The main problem with these approaches is that $j\omega$ -axis shifting is performed at the expense of reduced stability margins and conformal transformations are difficult to perform while still retaining insight into the original engineering problem.

O’Young et al [OYO3], have shown that $j\omega$ -axis zeros can be eliminated by augmenting the design specifications to account for the presence of additive noise in the sensor and actuator signals. The effect of this is to make the problem generically 4-block. This is due to the fact that the input vector $\underline{v}$ in the SCC for figure 4.3 now includes a disturbance input and additive noise signals $\underline{n}_a$ and $\underline{n}_s$, thus its size must be greater than that of $\underline{y}$. Similarly, the vector $\underline{e}$ includes signals $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$, hence its size must be greater than that of $\underline{u}$.

The corresponding equations for the SCC of figure 4.3 assuming unit weighting functions are given by

$$\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \\ -\underline{y} \end{bmatrix} = \begin{bmatrix} I & G & 0 & G \\ I & G & I & G \\ 0 & I & 0 & I \\ -\bar{I} & -\bar{G} & -\bar{I} & -\bar{G} \end{bmatrix} \begin{bmatrix} \underline{d} \\ \underline{n}_s \\ \underline{n}_a \\ \underline{u} \end{bmatrix} \quad (4.62)$$

and the corresponding optimization problem is

$$\min_{\text{stabilizing } K} \|W_o F_\ell(P, K) W_i\|_\infty$$

where $W_o = \begin{bmatrix} W_1 & & \\ & W_2 & \\ & & W_3 \end{bmatrix}$ and $W_i = \begin{bmatrix} W_4 & & \\ & W_5 & \\ & & W_6 \end{bmatrix}$

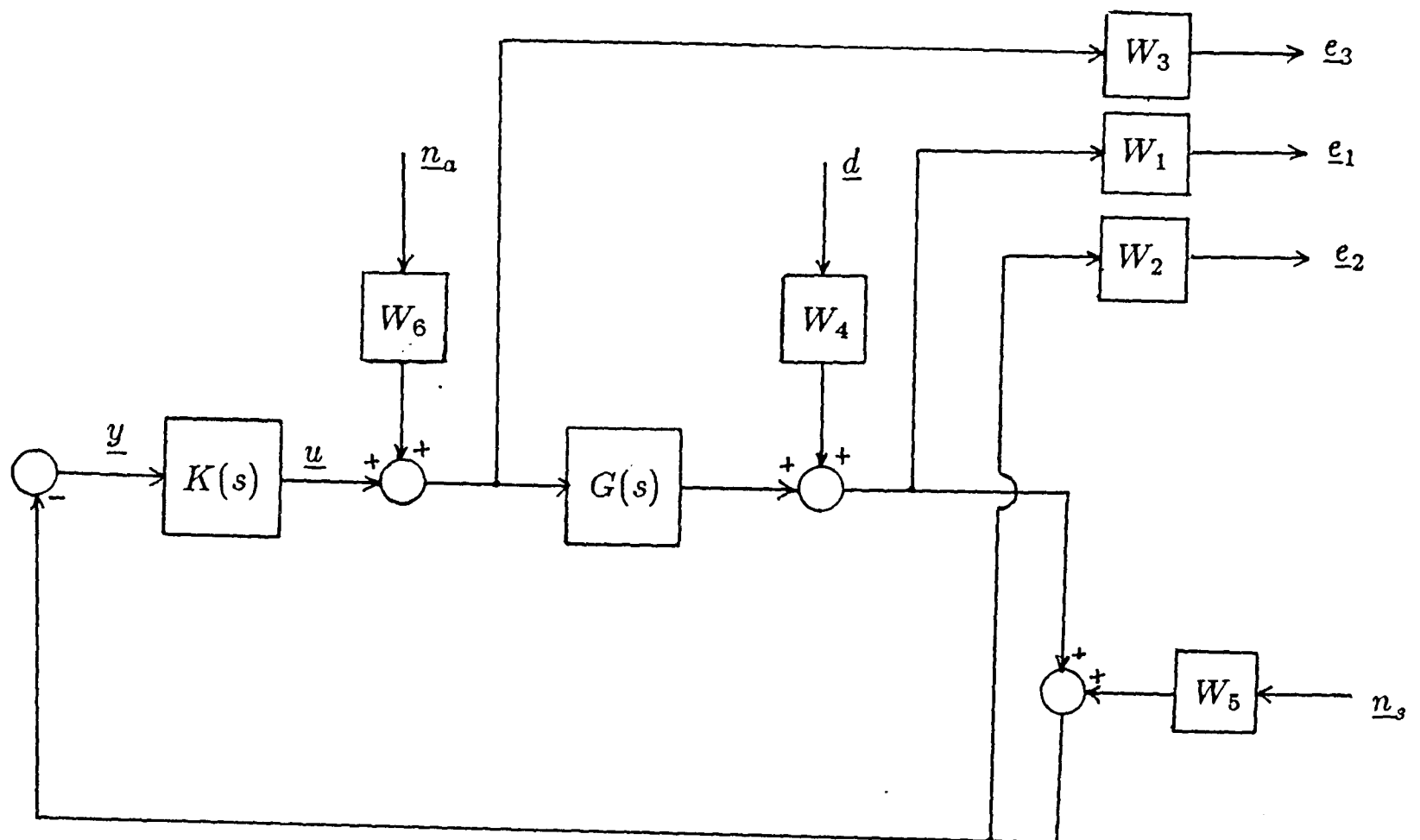


Figure 4.3 Generic 4-block problem

Before proceeding it will be useful to introduce the following lemma

Lemma 4.3

- (i) The zeros of P_{12} = zeros of T_{12}
- (ii) The zeros of P_{21} = zeros of T_{21}

Proof

$$\text{Given, } P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \quad (4.63)$$

A state space realization for $T_{ij}(s) \in \mathcal{RH}^\infty$ is given by [CHU1], [DOY3], [FRA4]

$$T(s) = \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} A + B_2F & -B_2F & B_1 & B_2 \\ 0 & A + HC_2 & B_1 + HD_{21} & 0 \\ \hline C_1 + D_{12}F & -D_{12}F & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array} \right] \quad (4.64)$$

where F and H are chosen such that $(A + B_2F)$ and $(A + HC_2)$ are stable.

For a system described by its system matrix [ROS2], in state space form, the decoupling zeros are the values of s for which the matrices $\begin{bmatrix} sI - A & B \end{bmatrix}$ and $\begin{bmatrix} sI - A \\ C \end{bmatrix}$ are rank deficient. For a system $[A, B, C]$, the decoupling zeros correspond to the uncontrollable and unobservable modes. Hence

$$\begin{aligned} \text{Zeros of } P_{12} &= \text{values of } s \text{ at which } \begin{bmatrix} sI - A & -B_2 \\ C_1 & D_{12} \end{bmatrix} \text{ loses rank} \\ &= \text{values of } s \text{ at which } \begin{bmatrix} sI - A & -B_2 \\ C_1 & D_{12} \end{bmatrix} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix} \text{ loses rank}^\dagger \\ &= \text{values of } s \text{ at which } \begin{bmatrix} sI - A - B_2F & -B_2 \\ C_1 + D_{12}F & D_{12} \end{bmatrix} \text{ loses rank} \\ &= \text{zeros of } T_{12} \quad \square \end{aligned}$$

Similarly zeros of $P_{21} = \text{zeros of } T_{21}$.

It can be seen from (4.62) that P_{12} and P_{21} will always have full column and row ranks respectively. Therefore from lemma 4.3, T_{12} and T_{21} must also have full column and row rank respectively. It is important to consider the effect of any weightings on the input and output signals. Sufficient conditions for T_{12} and T_{21} to be less than full column/row rank respectively are that the weighting functions have no $j\omega$ -axis zeros, including those at infinity. Therefore, a solution to the optimization problem exists, if the weighting functions are bi-proper and contain no imaginary axis zeros. This constraint on the weighting functions is not too restrictive.

[†] Since the zeros are invariant under constant state feedback

Thus one of the main reasons for studying 4-block problems is that there are no constraints on the plant having $j\omega$ -axis poles and zeros in order to lead to a solution to the H^∞ -optimization, provided of course that the problem is sensibly posed. It can be argued that a sensible design specification will always include performance requirements in the presence of additive noise and thus will always result in a 4-block problem.

4.6 Two-Degree of Freedom Controllers

In general, every control problem possesses two natural degrees of freedom corresponding to the availability of an exogenous input $\underline{r}$ and sensor output $\underline{y}$ (figure 4.4). To utilize both degrees of freedom fully, it is necessary [YOU2], for the controller to generate an output

$$\underline{u} = K_1 \underline{r} - K_2 \underline{y} \quad (4.65)$$

where $\underline{u}$ denotes the input to the plant G , and K_1, K_2 are arbitrary transfer matrices subject solely to the constraint that the closed loop system is stable.

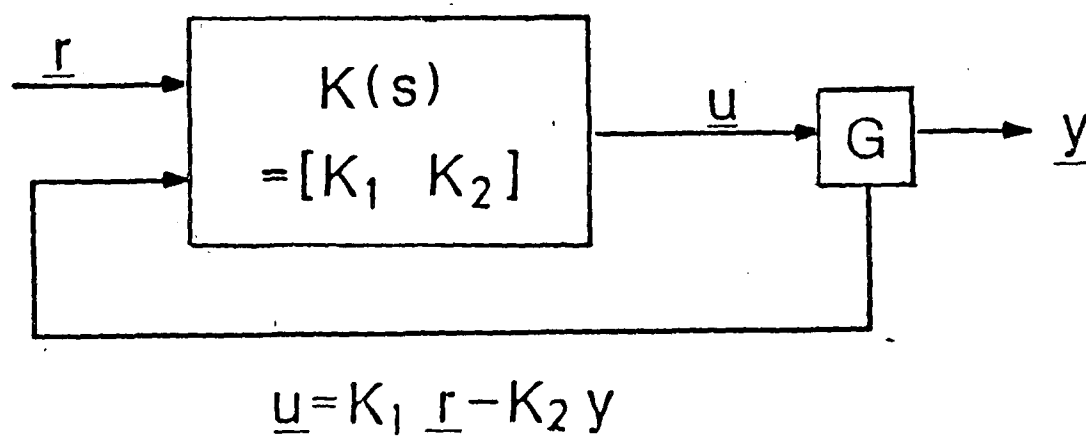


Figure 4.4 General two-degree of freedom compensator

The compensator $K = [K_1 \ K_2]$ is referred to as a two-degree of freedom compensator. It can be shown that K_1 and K_2 are independent [VID1], and therefore it is possible to break the design procedure into two stages:

- (i) Synthesize K_1 for desirable command response.

- (ii) Synthesize K_2 for desirable feedback properties eg. disturbance rejection, stability, sensitivity and robust stability.

Thus this controller structure allows freedom to specify desirable command response and independently to provide good robustness properties. A feasible implementation of the two-degree of freedom compensator is illustrated in figure 4.5 and is due to Moore et al [MOO1].

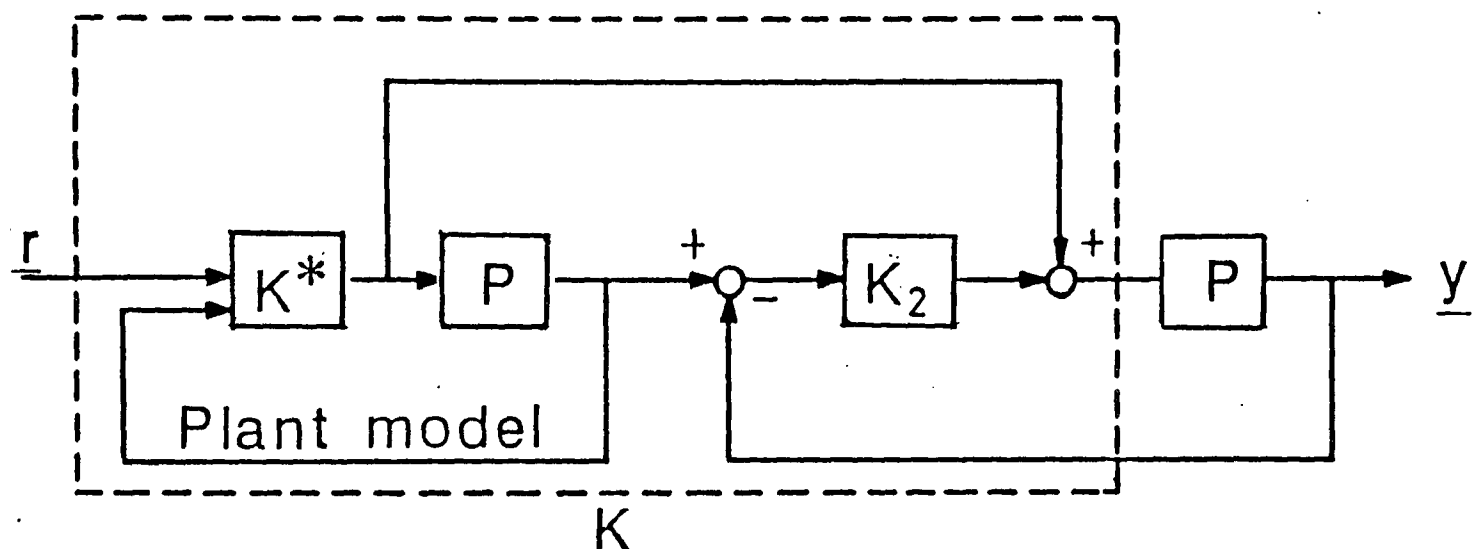


Figure 4.5 Feasible implementation of two-degree of freedom controller

The controller structure consists of a command model, a feedforward link and a stabilization network. The stabilization part provides disturbance rejection and improves the model-following performance in the presence of any unmodelled dynamics. The stabilization bandwidth is generally set as high as possible. The command model may be generated using any design technique or may contain simple transfer functions which represent the desired command response. This capability to set stabilization and command response characteristics is a key advantage of the two-degree of freedom structure.

The synthesis of the controller can be broken into two stages as follows:

(1) Design a controller K^* for the plant G to have “desirable” transfer functions from $\underline{r}$ to $\underline{u}$ and $\underline{r}$ to $\underline{y}$. Denote these by

$$T_1^* = K^*(I + GK^*)^{-1} \text{ and } T_2^* = (I + GK^*)^{-1}GK^* \text{ respectively.}$$

K^* is designed to meet performance requirements for the nominal plant G , it is important to note that K^* may be realized with any design technique eg. LQG, pole placement, etc.

(2) Design K_2 to have “desirable” robustness properties using H^∞ -optimization.

To design a compensator which processes $\underline{r}$ and $\underline{y}$ in an independent manner as (4.65), the corresponding transfer functions from $\underline{r}$ to $\underline{u}$ and $\underline{r}$ to $\underline{y}$ must be

$$T_1 = K_1(I + GK_2)^{-1} \quad \text{and} \quad T_2 = (I + GK_2)^{-1}GK_1 \quad (4.66)$$

It can be shown [MOO1], that necessary and sufficient conditions for model matching are:

$$T_1 = K_1(I + GK_2)^{-1} = K^*(I + GK^*)^{-1} = T_1^* \quad (4.67)$$

$$\text{and } T_2 = (I + GK_2)^{-1}GK_1 = (I + GK^*)^{-1}GK^* = T_2^* \quad (4.68)$$

which implies that $K_1 = T_1^* + K_2T_2^*$, and K_2 is stabilizing.

Thus the only requirement on K_2 is that it is stabilizing as a single-degree of freedom controller. Note however that $T_2^* = T_2$ iff the plant model is identical to the real plant. This of course is never likely to be the case, so to ensure model matching it is necessary also for the controller to be robustly stable to uncertainties in the plant model and also to have a sufficiently high bandwidth to ensure good tracking from $\underline{r}$ to $\underline{y}$.

4.7 Concluding Remarks

This chapter has presented the framework of the H^∞ -optimal synthesis procedure. Some new results regarding the classification of H^∞ -problems has been given and an iterative

procedure to find the optimal solution has been introduced. This will be discussed further in the next chapter.

It was not possible to prove that the 4-block γ -iteration is a convex optimization problem by a mere extension of the results for the 2-block case. It has however been shown that intermediate steps within the 4-block iterative scheme are convex, although this does not guarantee that the overall procedure is. Experience has shown that convergence is always attained. The proof of convexity for the 4-block γ -iteration requires further research.

A feedback configuration has been presented which is generically 4-block in the SCC setting. It has been shown that this configuration has no restriction on the location of the plant poles and zeros and will form the basis of further 4-block design work.

The two-degree of freedom compensator is easily synthesized in the H^∞ framework and provides a powerful tool for integrating helicopter handling qualities criteria and stability robustness.

CHAPTER 5

THE γ -ITERATION IN H^∞ -OPTIMIZATION

5.1 *Introduction*

This Chapter presents an iterative scheme, called γ -iteration [DOY3], to solve the general distance problem. The idea is that by guessing a value for the minimal norm, the distance problem can be simplified to an equivalent best approximation problem which can be solved by existing algorithms. This guess can be iterated to obtain convergence to the optimal norm and optimal Q .

The problem of how to determine the next choice for γ in the iterative solution to the H^∞ -optimization problem will be discussed. The most computationally demanding part of the H^∞ -optimization problem is the computation of the optimal norm γ_o . The γ -iteration will converge more rapidly if good initial upper and lower bounds can be found for γ_o . The convergence rate of the γ -iteration depends on the search scheme employed.

Chu et al [CHU2], have shown that for the 2-block γ -iteration, $f(\gamma)$ is a continuous, convex and strictly monotonically decreasing function. Based on these properties it is possible to compute new upper and lower bounds which are tighter than those obtainable using bisection. Some new properties for $f(\gamma)$ are obtained which lead to a fast search scheme for finding γ_o . Both the 2-block and 4-block γ -iteration schemes will be discussed.

Section 5.2.1 outlines a new algorithm for the 2-block γ -iteration. This algorithm does not require the computation of any L^∞ -norms and therefore represents a numerically attractive

procedure. An example is given to illustrate the convergence properties of the algorithm compared with a simple bisection algorithm.

Results for the 2-block case are extended to the more general 4-block case and the corresponding algorithm is outlined in §5.3.1.

5.2 2-block γ -iteration

This section starts with a description of the 2-block γ -iteration procedure [DOY3], [CHU1], [CHU2].

- (i) Compute $\gamma_\ell = \max\{\|H_{R_{11}}\|, \|R_{21}\|_\infty\}$
- (ii) Choose $\gamma = (1 + \delta)\gamma_\ell$
- (iii) Compute the spectral factor $M = (\gamma^2 I - R_{21}^* R_{21})^{1/2}$ with this γ . If successful continue, otherwise set $\gamma_\ell = \gamma$ and goto (ii).
- (iv) Let $f(\gamma) = \|H_{R_{11}} M^{-1}\|$
 - (a) if $f(\gamma) > 1$, choose a larger γ and goto (iii)
 - (b) if $f(\gamma) < 1$ and $\gamma > \gamma_\ell$, choose a smaller γ and goto (iii)
 - (c) if $f(\gamma) = 1$ and $\gamma \geq \gamma_\ell$, goto (v)
 - (d) if $f(\gamma) \leq 1$ and $\gamma = \gamma_\ell$, goto (v)
- (v) The value of γ is the minimal achievable norm. Find the best approximation of $R_{11} M^{-1}$, denoted by Q_o .

A numerical implementation of this procedure would not be able to test for $f(\gamma) = 1$ exactly. However, the procedure can be used to find a Q which yields a solution which is arbitrarily close to γ_o . The deviation of the corresponding sub-optimal cost function from the optimal cost function is a function of the accuracy in the γ -iteration.

Suppose that Q_1 is a solution to (4.37) and γ_1 is the corresponding value of γ which gives rise to $f(\gamma_1) = 1 \pm \epsilon$, $\forall \epsilon > 0$. Then $f(\gamma_1) = 1 \pm \epsilon$

$$\Rightarrow \|(R_{11} - Q_1)M^{-1}\|_{\infty} = 1 \pm \epsilon$$

$$\Rightarrow (M^{-1})^*(R_{11} - Q_1)^*(R_{11} - Q_1)M^{-1} \leq (1 \pm \epsilon)^2 I \quad \forall \omega$$

$$\Rightarrow (R_{11} - Q_1)^*(R_{11} - Q_1) \leq (1 \pm \epsilon)^2 M^* M = (1 \pm \epsilon)^2 (\gamma_1^2 I - R_{21}^* R_{21})$$

$$\begin{aligned} \text{But } \left\| \begin{bmatrix} R_{11} - Q_1 \\ R_{21} \end{bmatrix} \right\|_{\infty} &= \sup_{\omega} \bar{\sigma}[(R_{11} - Q_1)^*(R_{11} - Q_1) + R_{21}^* R_{21}] \\ &\leq \sup_{\omega} \bar{\sigma}[(1 \pm \epsilon)^2 (\gamma_1^2 I - R_{21}^* R_{21}) + R_{21}^* R_{21}] \\ &= \sup_{\omega} \bar{\sigma}[(1 \pm \epsilon)^2 \gamma_1^2 I + (1 - (1 \pm \epsilon)^2) R_{21}^* R_{21}] \end{aligned}$$

Therefore the cost function depends on the tolerance used to terminate the γ -iteration and $\|R_{21}\|_{\infty}$. The closer ϵ is to zero the closer γ is to γ_o . In general the sub-optimal cost function is not all pass except in the special case when R_{21} is all pass.

The above algorithm for the 2-block γ -iteration is not complete without some method for selecting the next guess for γ . One method of achieving this is through a binary search. Starting from a lower bound γ_{ℓ} and an upper bound γ_u , one can define $\gamma = (\gamma_{\ell} + \gamma_u)/2$ and evaluate $f(\gamma)$. If $f(\gamma) > 1$, then γ_{ℓ} is updated by γ , otherwise γ_u is replaced by γ . This process is repeated until $\|\gamma_u - \gamma_{\ell}\| < \epsilon$, where ϵ is small. This search scheme is simple but the convergence rate is slow. For example if the initial difference between γ_{ℓ} and γ_u is 1, the number of iterations required to make $\|\gamma_u - \gamma_{\ell}\| < \epsilon$ is given by $2^{-n} < \epsilon$ where

$\epsilon < 1 \Rightarrow n > \frac{-\log_{10}\epsilon}{\log_{10}2}$. So for $\epsilon = 10^{-10}$, $n > \frac{10}{\log_{10}2} = 33.2$. Better search schemes can be developed using the properties of $f(\gamma)$.

Firstly recall from §4.4 that

$$\gamma_o = \inf_{Q \in \mathcal{RH}^\infty} \left\| \begin{bmatrix} R_{11} - Q \\ R_{21} \end{bmatrix} \right\|_\infty$$

and

$$f(\gamma) = \|[R_{11} - Q]M^{-1}\|_\infty = \|H_{R_{11}}M^{-1}\|$$

Where $M^*M = \gamma^2 I - R_{21}^* R_{21}$

The results of this section are based on the following theorem.

Theorem 5.1

Define $\mu(\alpha, \gamma) = f(\alpha) \frac{\sqrt{\alpha^2 - b^2}}{\sqrt{\gamma^2 - b^2}}$ for some $\alpha, \gamma > b$, where $b = \|R_{21}\|_\infty$.

$$(i) \ f(\gamma) < \mu(\alpha, \gamma) \text{ if } \gamma < \alpha \quad (5.1)$$

$$(ii) \ f(\gamma) = \mu(\alpha, \gamma) \text{ if } \gamma = \alpha \quad (5.2)$$

$$(iii) \ f(\gamma) > \mu(\alpha, \gamma) \text{ if } \gamma > \alpha \quad (5.3)$$

Proof

$$(i) \ \alpha^2 > \gamma^2$$

$$\Rightarrow \alpha^2(b^2 - R_{21}^* R_{21}) > \gamma^2(b^2 - R_{21}^* R_{21}), \quad \text{for some } b > \|R_{21}\|_\infty$$

(NB. $A > B$ means $A - B$ is positive definite.)

$$\Rightarrow \alpha^2 b^2 I + \gamma^2 R_{21}^* R_{21} > \gamma^2 b^2 I + \alpha^2 R_{21}^* R_{21}$$

$$\Rightarrow (\alpha^2 - b^2)(\gamma^2 I - R_{21}^* R_{21}) > (\gamma^2 - b^2)(\alpha^2 I - R_{21}^* R_{21})$$

$$\Rightarrow (\alpha^2 - b^2)M_\gamma^* M_\gamma > (\gamma^2 - b^2)M_\alpha^* M_\alpha$$

$$\text{where } M_\gamma^* M_\gamma = (\gamma^2 I - R_{21}^* R_{21}) \text{ and } M_\alpha^* M_\alpha = (\alpha^2 I - R_{21}^* R_{21})$$

$$\Rightarrow (\alpha^2 - b^2)I > (\gamma^2 - b^2)(M_\gamma^*)^{-1}M_\alpha^* M_\alpha M_\gamma^{-1}$$

$$\Rightarrow (\alpha^2 - b^2)I > (\gamma^2 - b^2)M_\alpha M_\gamma^{-1}(M_\gamma^*)^{-1}M_\alpha^*$$

$$\Rightarrow (\alpha^2 - b^2)M_\alpha^{-1}(M_\alpha^*)^{-1} > (\gamma^2 - b^2)M_\gamma^{-1}(M_\gamma^*)^{-1}$$

$$\Rightarrow (\alpha^2 - b^2)(R_{11} - Q)M_\alpha^{-1}(M_\alpha^*)^{-1}(R_{11} - Q)^* > (\gamma^2 - b^2)(R_{11} - Q)M_\gamma^{-1}(M_\gamma^*)^{-1}(R_{11} - Q)^*$$

$$\Rightarrow (\alpha^2 - b^2) \|(R_{11} - Q)M_\alpha^{-1}\|_\infty^2 > (\gamma^2 - b^2) \|(R_{11} - Q)M_\gamma^{-1}\|_\infty^2$$

$$\Rightarrow (\alpha^2 - b^2)f^2(\alpha) > (\gamma^2 - b^2)f^2(\gamma)$$

$$\Rightarrow f(\gamma) < f(\alpha) \frac{\sqrt{\alpha^2 - b^2}}{\sqrt{\gamma^2 - b^2}} = \mu(\alpha, \gamma) \quad \square$$

(ii) Obvious.

(iii) Same as (i) with inequalities reversed.

Corollary 5.1

$$\text{If } f(\gamma) < 1 \text{ then } \gamma_o < \sqrt{f^2(\gamma)\gamma^2 + (1 - f^2(\gamma))b^2} \quad (5.4)$$

$$\text{If } f(\gamma) > 1 \text{ then } \gamma_o > \sqrt{f^2(\gamma)\gamma^2 + (1 - f^2(\gamma))b^2} \quad (5.5)$$

where $b = \|R_{21}\|_\infty$

Proof

$$(i) f(\gamma) < 1 \Rightarrow \gamma > \gamma_o$$

$$\Rightarrow f(\gamma) > \mu(\gamma_o, \gamma) = f(\gamma_o) \frac{\sqrt{\gamma_o^2 - b^2}}{\sqrt{\gamma^2 - b^2}}$$

$$\Rightarrow f^2(\gamma)(\gamma^2 - b^2) > \gamma_o^2 - b^2, \quad \text{since } f(\gamma_o) = 1$$

$$\Rightarrow \gamma_o < \sqrt{f^2(\gamma)\gamma^2 + (1 - f^2(\gamma))b^2}$$

$$(ii) f(\gamma) > 1 \Rightarrow \gamma < \gamma_o$$

$$\Rightarrow f(\gamma) < \mu(\gamma_o, \gamma)$$

$$\Rightarrow \gamma_o > \sqrt{f^2(\gamma)\gamma^2 + (1 - f^2(\gamma))b^2} \quad \square$$

Theorem 5.2

Define $\beta(\alpha, \gamma) = f(\alpha) \frac{\sqrt{\alpha^2 - c^2}}{\sqrt{\gamma^2 - c^2}}$, where $c = \inf_{\omega} \underline{\sigma}[R_{21}(j\omega)]$ and $\alpha, \gamma > c$.

$$(i) f(\gamma) < \beta(\alpha, \gamma) \text{ if } \gamma > \alpha \tag{5.6}$$

$$(ii) f(\gamma) = \beta(\alpha, \gamma) \text{ if } \gamma = \alpha \tag{5.7}$$

$$(iii) f(\gamma) > \beta(\alpha, \gamma) \text{ if } \gamma < \alpha \tag{5.8}$$

Proof

$$(i) \gamma^2 > \alpha^2$$

$$\Rightarrow \gamma^2(R_{21}^* R_{21} - c^2 I) > \alpha^2(R_{21}^* R_{21} - c^2 I)$$

$$\Rightarrow \alpha^2 c^2 + \gamma^2 R_{21}^* R_{21} > \gamma^2 c^2 + \alpha^2 R_{21}^* R_{21}$$

$$\Rightarrow (\alpha^2 - c^2)(\gamma^2 I - R_{21}^* R_{21}) > (\gamma^2 - c^2)(\alpha^2 I - R_{21}^* R_{21})$$

proceeding as for the proof of theorem 5.1

$$\Rightarrow (\alpha^2 - c^2)f^2(\alpha) > (\gamma^2 - c^2)f^2(\gamma)$$

$$\Rightarrow f(\gamma) < f(\alpha) \frac{\sqrt{\alpha^2 - c^2}}{\sqrt{\gamma^2 - c^2}} = \beta(\alpha, \gamma) \quad \square$$

(ii) obvious.

(iii) Same as (i) with inequalities reversed.

Corollary 5.2

$$\text{If } f(\gamma) < 1 \text{ then } \gamma_o > \sqrt{f^2(\gamma)\gamma^2 + (1 - f^2(\gamma))c^2} \quad (5.9)$$

$$\text{If } f(\gamma) > 1 \text{ then } \gamma_o < \sqrt{f^2(\gamma)\gamma^2 + (1 - f^2(\gamma))c^2} \quad (5.10)$$

where $c = \inf_{\omega} \underline{\sigma}[R_{21}(j\omega)]$.

Proof

$$(i) f(\gamma) < 1 \Rightarrow \gamma > \gamma_o$$

$$\Rightarrow f(\gamma) < \beta(\gamma_o, \gamma) = f(\gamma_o) \frac{\sqrt{\gamma_o^2 - c^2}}{\sqrt{\gamma^2 - c^2}}$$

$$\Rightarrow \gamma_o > \sqrt{f^2(\gamma)\gamma^2 + (1 - f^2(\gamma))c^2} \quad \square$$

$$(ii) f(\gamma) > 1 \Rightarrow \gamma < \gamma_o$$

$$\Rightarrow f(\gamma) > \beta(\gamma_o, \gamma)$$

$$\Rightarrow \gamma_o < \sqrt{f^2(\gamma)\gamma^2 + (1 - f^2(\gamma))c^2} \quad \square$$

Theorem 5.3

Given $\gamma_e < \gamma < \gamma_u$

$$(i) f(\gamma) < 1 \Rightarrow \gamma_o < \gamma + \frac{[f(\gamma) - 1][\gamma - \gamma_e]}{f(\gamma_e) - f(\gamma)} = \gamma' \quad (5.11)$$

$$(ii) f(\gamma) > 1 \Rightarrow \gamma_o > \gamma + \frac{[f(\gamma) - 1][\gamma - \gamma_e]}{f(\gamma_e) - f(\gamma)} = \gamma' \quad (5.12)$$

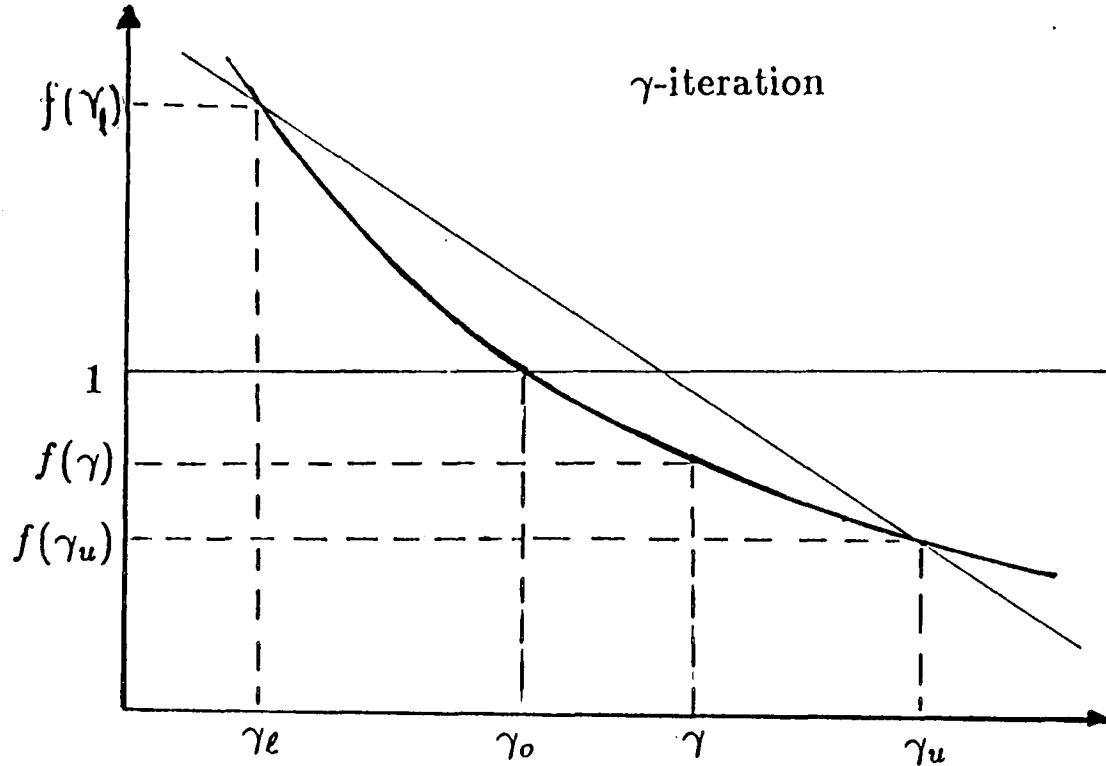
Proof

$$(i) f(\gamma) < 1 \Rightarrow \gamma_e < \gamma_o < \gamma < \gamma_u.$$

Since $f(\gamma)$ is a monotonically decreasing function, $f(\gamma_e) - f(\gamma) > 0$

$\gamma > \gamma_o \Rightarrow f(\gamma) - 1 < 0$ and $\gamma - \gamma_e > 0$. Therefore, $\frac{[f(\gamma) - 1][\gamma - \gamma_e]}{f(\gamma_e) - f(\gamma)} < 0$. Hence $\gamma' < \gamma$

To show $\gamma' > \gamma_o$, requires the convexity of $f(\gamma)$



$$\gamma' = \frac{\gamma f(\gamma_e) - \gamma f(\gamma) + f(\gamma)\gamma - f(\gamma)\gamma_e - \gamma + \gamma_e}{f(\gamma_e) - f(\gamma)}$$

$$\begin{aligned}
&= \gamma \frac{(f(\gamma_\ell) - 1)}{f(\gamma_\ell) - f(\gamma)} + \gamma_\ell \frac{(1 - f(\gamma))}{f(\gamma_\ell) - f(\gamma)} \\
&> \gamma \left(\frac{\gamma_o - \gamma_\ell}{\gamma - \gamma_\ell} \right) + \gamma_\ell \left(1 - \left(\frac{\gamma_o - \gamma_\ell}{\gamma - \gamma_\ell} \right) \right) \\
&= \gamma_o \quad \square
\end{aligned}$$

Therefore $\gamma_o < \gamma' < \gamma$, hence γ' is a tighter upper bound.

(ii) As for (i) above with inequalities reversed.

Corollaries 5.1, 5.2 and theorem 5.3 show a duality between the upper and lower bounds for γ_o which gives rise to easy algorithmic implementation. Theorem 5.3 also has a nice geometric interpretation, which depends only on $f(\gamma)$ being a convex function, thus these bounds carry through immediately to the 4-block case. These inequalities have also been derived by Chang et al [CHA3] and Foo and Postlethwaite [FOO4], however the proofs given here are simpler and enable an easy extension of the results to the 4-block minimization problem. The example given in [CHA3] does not highlight the problem of how to compute $\inf \underline{\sigma}[R_{21}(j\omega)]$.

Now it is not possible to compute $c = \inf \underline{\sigma}[R_{21}(j\omega)]$ exactly so an alternate strategy is to find a lower bound for c (letting c be greater than $\inf \underline{\sigma}[R_{21}(j\omega)]$ will violate the condition of corollary 5.2). This will of course lead to looser bounds than those given by equations (5.9) and (5.10). It is proposed to take $c=0$, which is always a lower bound for $\inf \underline{\sigma}[R_{21}(j\omega)]$.

An algorithm for the 2-block γ -iteration is as follows:

5.2.1 Algorithm for 2-block γ -iteration

- (1) Compute $\gamma_\ell = \max\{\|H_{R_{11}}\|, \bar{\sigma}[R_{21}(s=0)], \bar{\sigma}[R_{21}(s=\infty)]\}$

(2) Choose $\gamma = (1 + \delta)\gamma_\ell$

(3) Compute the spectral factor $M = (\gamma^2 I - R_{21}^* R_{21})^{1/2}$ with this γ .

(4) IF successful THEN

IF $f(\gamma) > 1$ THEN

$$\gamma_\ell = \gamma, \gamma_u = \sqrt{\|H_{R_{11}}\|^2 + \gamma^2}, \quad b = \gamma$$

ELSE

$$\gamma_u = \gamma, \quad b = \gamma$$

END IF

ELSE

$$\gamma_\ell = \gamma \quad \text{GOTO 2}$$

END IF

$$(5) \quad \gamma_{u_1} = \sqrt{f^2(\gamma_u)\gamma_u^2 + (1 - f^2(\gamma_u))b^2}, \quad \gamma_{\ell_1} = f(\gamma_u)\gamma_u$$

IF $f(\gamma_\ell)$ successful THEN

$$\gamma_{u_2} = f(\gamma_\ell)\gamma_\ell$$

$$\gamma_{u_3} = \gamma_u + \frac{[f(\gamma_u) - 1][\gamma_u - \gamma_\ell]}{f(\gamma_\ell) - f(\gamma_u)}$$

$$\gamma_{\ell_2} = \sqrt{f^2(\gamma_\ell)\gamma_\ell^2 + (1 - f^2(\gamma_\ell))b^2}$$

$$\gamma_\ell = \max(\gamma_{\ell_1}, \gamma_{\ell_2}), \quad \gamma_u = \min(\gamma_{u_1}, \gamma_{u_2}, \gamma_{u_3})$$

ELSE

$$\gamma_\ell = \gamma_{\ell_1}, \quad \gamma_u = \gamma_{u_1}$$

END IF

$$(6) \quad \gamma = (\gamma_\ell + \gamma_u)/2$$

IF $\frac{\gamma_u - \gamma_\ell}{\gamma_\ell} < \epsilon$ THEN

$$\gamma_o = \gamma \quad \text{GOTO 10}$$

END IF

(7) IF $f(\gamma)$ not successful THEN

$$\gamma_\ell = \gamma \quad \text{GOTO 6}$$

ELSE

IF $f(\gamma) = 1$, GOTO 10

END IF

$$(8) \quad \beta_1 = f(\gamma)\gamma$$

$$\beta_2 = \sqrt{f^2(\gamma)\gamma^2 + (1 - f^2(\gamma))b^2}$$

IF $f(\gamma_\ell)$ successful THEN

$$\beta_3 = \gamma + \frac{[f(\gamma) - 1][\gamma - \gamma_\ell]}{f(\gamma_\ell) - f(\gamma)}$$

ELSE

$$\beta_3 = \gamma$$

END IF

(9) IF $f(\gamma) > 1$ THEN

$$\gamma_\ell = \max(\beta_2, \beta_3), \quad \gamma_u = \min(\beta_1, \gamma_u)$$

GOTO 6

ELSE

$$\gamma_\ell = \max(\beta_1, \gamma_\ell), \quad \gamma_u = \min(\beta_3, \beta_2)$$

GOTO 6

END IF

(10) STOP

The bounds will be tighter the nearer b is to $\bar{\sigma}[R_{21}]$. The problem lies in how big to choose δ in step 2. If δ is small then b will be a close approximation to $\bar{\sigma}[R_{21}]$. However, it may take many iterations before the spectral factorization in step 3 succeeds.

Example 1

The following example will help illustrate the convergence properties of the above algorithm. A simple feedback control system will be used to generate a 2-block design problem. Given a plant $G(s)$, it is desired to find a controller $K(s)$ such that the closed loop system is internally stable and the H^∞ -norm of $\left\| \begin{bmatrix} W_1 S \\ W_2 K S \end{bmatrix} \right\|_\infty$ is minimized, where $S(s) = (I + GK)^{-1}$ is the sensitivity function.

Note that a 2-block model matching problem has to be solved since there are two error signals but only one available actuator input. Suppose that the plant $G(s)$ and the weighting functions $W_1(s)$ and $W_2(s)$ are given as

$$G = \frac{0.5(s-1)}{(s+2)(s+0.5)}, \quad W_1 = 0.28 \frac{(s+1.24)}{(s+0.07)}, \quad W_2 = 0.5 \frac{(s+0.5)}{(s+2)} \quad (5.13)$$

The plant plus weights has 4 states. The choice of weighting functions for this plant are explained in detail in [TOM1]. Attention will be restricted to the γ -iteration for this particular problem and in particular how the convergence rate of the above algorithm compares with that of a simple bisection method.

It is desired to find

$$\gamma_o \triangleq \min_{Q \in \mathcal{RH}^\infty} \left\| \begin{bmatrix} R_{11} - Q \\ R_{21} \end{bmatrix} \right\|_\infty$$

An accuracy of 1% and $1 \times 10^{-10}\%$ was used in determining the optimal γ , ie. $\frac{(\gamma_u - \gamma_\ell)}{\gamma_\ell} < \epsilon$ where $\epsilon = 0.01$ and 1×10^{-12} . These choices for ϵ serve to illustrate the advantages and

disadvantages of the algorithm. Tables 5.1 and 5.2 show the γ -iteration with $\epsilon=0.01$ for a bisection algorithm and for the algorithm of §5.2.1 respectively.

```

STATUS : Solving a 2-block model-matching problem
STATUS : Doing gamma iteration

Enter required accuracy for the iteration      1.0000000

1  GAMMAL=    0.94276580064843    GAMMAU =    1.13131896077812
   GAMMA =    1.03704238071328    F (GAMMA)=    0.96259186733986

2  GAMMAL=    0.94276580064843    GAMMAU =    1.03704238071328
   GAMMA =    0.98990409068086    F (GAMMA)=    1.01449073882215

3  GAMMAL=    0.98990409068086    GAMMAU =    1.03704238071328
   GAMMA =    1.01347323569707    F (GAMMA)=    0.98782006139799

4  GAMMAL=    0.98990409068086    GAMMAU =    1.01347323569707
   GAMMA =    1.00168866318896    F (GAMMA)=    1.00096706519959

5  GAMMAL=    1.00168866318896    GAMMAU =    1.01347323569707

THE OPTIMAL GAMMA IS =    1.00758094944301
                        F (GAMMA) =    0.99434752113561

The gamma-iteration took    2.800000    CPU seconds

```

Table 5.1: 2-block γ -iteration (bisection algorithm) with $\epsilon = 0.01$

```

STATUS : Solving a 2-block model-matching problem
STATUS : Doing gamma iteration

Enter required accuracy for the iteration      1.0000000

1  GAMMAL=    0.94276580064843    GAMMAU =    1.13131896077812
   GAMMA =    1.03704238071328    F (GAMMA)=    0.96259186733986

2  GAMMAL=    0.99824856176136    GAMMAU =    1.00501760608986

THE OPTIMAL GAMMA IS =    1.00163308392561
                        F (GAMMA) =    1.00102994928754

The gamma-iteration took    2.340000    CPU seconds

```

Table 5.2: 2-block γ -iteration (new algorithm) with $\epsilon = 0.01$

It can be seen from these tables that the bisection algorithm needs more than twice as many iterations before converging to the optimal γ . The computational speed of the two algorithms, as determined by the amount of CPU processing time required, are 2.8 and 2.34 secs for the bisection algorithm and the new algorithm respectively, for this choice of ϵ .

```

STATUS : Solving a 2-block model-matching problem
STATUS : Doing gamma iteration

Enter required accuracy for the iteration      0.10000000e-09

1  GAMMAL=      0.94276580064843      GAMMAU =      1.13131896077812
   GAMMA =      1.03704238071328      F (GAMMA)=      0.96259186733986

2  GAMMAL=      0.99824856176136      GAMMAU =      1.00501760608986
   GAMMA =      1.00163308392561      F (GAMMA)=      1.00102994928754

3  GAMMAL=      1.00253963718894      GAMMAU =      1.00266471520677
   GAMMA =      1.00260217619786      F (GAMMA)=      0.99993468746932

4  GAMMAL=      1.00253963718894      GAMMAU =      1.00254432461516
   GAMMA =      1.00254198090205      F (GAMMA)=      1.00000264578423

5  GAMMAL=      1.00254432427745      GAMMAU =      1.00254432461516
   GAMMA =      1.00254432444630      F (GAMMA)=      0.99999999982419

6  GAMMAL=      1.00254432427745      GAMMAU =      1.00254432429059
   GAMMA =      1.00254432428402      F (GAMMA)=      1.00000000000742

7  GAMMAL=      1.00254432429059      GAMMAU =      1.00254432429059

THE OPTIMAL GAMMA IS =      1.00254432429059
      F (GAMMA) =      1.00000000000000

The gamma-iteration took      6.340000      CPU seconds

```

Table 5.3: 2-block γ -iteration (new algorithm) with $\epsilon = 1 \times 10^{-12}$

The main disadvantage of the proposed algorithm is that it needs $f(\gamma)$, $f(\gamma_\ell)$ and $f(\gamma_u)$ at the pre-iteration stage and $f(\gamma)$ and $f(\gamma_\ell)$ at each iteration. If $f(\gamma)$ is computed using Glover’s method [GLO2], then one Riccati equation (for spectral factorization), one

Sylvester equation (for projection), two Lyapunov equations (for controllability and observability grammians) and an eigenvalue computation are needed in each evaluation, which is computationally very demanding.

Thus although the bounds for γ_ℓ and γ_u are much tighter for the proposed algorithm, this is offset by the greater computational demand at each iteration. As a result the algorithm will only be significantly better when the accuracy of γ is required to be very small. For the case when $\epsilon = 1 \times 10^{-12}$ the optimal γ converged in just 7 iterations (Table 5.3) while a bisection algorithm converged in 38 (Table A5.1, Appendix 5.1), more than 5 times as many iterations. The corresponding CPU processing times were 6.34 secs and 16.38 secs respectively, a factor of 2.6 times faster.

The timing of the algorithms was performed on a SUN workstation 3/140 with floating point accelerator. To minimize fluctuations in CPU time due to multi users, these timings were done during quiet periods in the day when there were no other users. The timing tests were conducted several times and averages were taken of the execution times.

5.3 4-block γ -iteration

For the 4-block problem,

$$\gamma_o = \inf_{Q \in \mathbb{R}^{H \times H}} \left\| \begin{bmatrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty = \inf_{Q \in \mathbb{R}^{H \times H}} \left\| \begin{bmatrix} R_1(Q) \\ R_2 \end{bmatrix} \right\|_\infty \quad (5.14)$$

where $R_1(Q) = [R_{11} - Q \quad R_{12}]$ and $R_2 = [R_{21} \quad R_{22}]$

It was shown in §4.4, that

$$\left\| \begin{bmatrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty < \gamma \quad (\forall \gamma > \gamma_o)$$

$$\iff \| [R_{11} - Q] N^{-1} \|_\infty < 1, \quad \text{where } N^* N = \left(\gamma^2 I - \begin{bmatrix} R_{21}^* \\ R_{22}^* \end{bmatrix} [R_{21} \quad R_{22}] \right)$$

$$\iff \left\| \begin{bmatrix} \left(F_\ell \left\{ \frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \right\} - \frac{1}{\gamma} Q \right) (I - \tilde{L}^* \tilde{L})^{-1/2} & L \end{bmatrix} \right\|_\infty < 1$$

$$\iff \left\| (I - LL^*)^{-1/2} \left[F_\ell \left\{ \frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \right\} - \frac{1}{\gamma} Q \right] (I - \tilde{L}^* \tilde{L})^{-1/2} \right\|_\infty < 1$$

Define, $\hat{f}(\gamma) = \| [R_{11} - Q] N^{-1} \|_\infty$

$$= \left\| \begin{bmatrix} \left(F_\ell \left\{ \frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \right\} - \frac{1}{\gamma} Q \right) (I - \tilde{L}^* \tilde{L})^{-1/2} & L \end{bmatrix} \right\|_\infty$$

and

$$\begin{aligned} f(\gamma) &= \left\| (I - LL^*)^{-1/2} \left[F_\ell \left\{ \frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \right\} - \frac{1}{\gamma} Q \right] (I - \tilde{L}^* \tilde{L})^{-1/2} \right\|_\infty \\ &= \left\| H_{(I - LL^*)^{-1/2} [F_\ell \{ \frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \} - \frac{1}{\gamma} Q] (I - \tilde{L}^* \tilde{L})^{-1/2}} \right\| \end{aligned}$$

It can be shown in a similar manner to the 2-block case that when $f(\gamma) = 1 \pm \epsilon$, $\forall \epsilon > 0$ and for some $Q \in \Re H^\infty$,

$$\left\| \begin{bmatrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty \leq \sup_\omega \bar{\sigma} [(1 \pm \epsilon)^2 \gamma^2 + (1 - (1 \pm \epsilon)^2) [R_{21} \quad R_{22}]]$$

Hence, the sub-optimal cost function for the 4-block case is not all pass except for the special case when $[R_{21} \quad R_{22}]$ is all pass. Also, since $\| [R_{21} \quad R_{22}] \|_\infty \geq \| R_{21} \|_\infty$, the 4-block γ -iteration is more sensitive to the tolerance used in terminating the iteration than for the 2-block case.

The 4-block minimization problem can be considered as a 2-block minimization consisting of $R_1(Q)$ and R_2 defined above. The following theorems are then a straightforward extension of theorems 5.1 and 5.2 to the 4-block case.

Theorem 5.4

Define $\mu(\alpha, \gamma) = \hat{f}(\alpha) \frac{\sqrt{\alpha^2 - b^2}}{\sqrt{\gamma^2 - b^2}}$ for some $\alpha, \gamma > b$, where $b = \| [R_{21} \quad R_{22}] \|_\infty$.

$$(i) \quad \hat{f}(\gamma) < \mu(\alpha, \gamma) \text{ if } \gamma < \alpha \tag{5.15}$$

$$(ii) \hat{f}(\gamma) = \mu(\alpha, \gamma) \text{ if } \gamma = \alpha \quad (5.16)$$

$$(iii) \hat{f}(\gamma) > \mu(\alpha, \gamma) \text{ if } \gamma > \alpha \quad (5.17)$$

Proof

The proofs are as for theorem 5.1 with $R_{11} - Q$ replaced by $[R_{11} - Q \quad R_{12}]$ and R_{21} replaced by $[R_{21} \quad R_{22}]$ $\square$

Corollary 5.4

$$\text{If } f(\gamma) < 1 \text{ then } \gamma_o < \sqrt{\hat{f}^2(\gamma)\gamma^2 + (1 - \hat{f}^2(\gamma))b^2} \quad (5.18)$$

$$\text{If } f(\gamma) > 1 \text{ then } \gamma_o > \sqrt{\hat{f}^2(\gamma)\gamma^2 + (1 - \hat{f}^2(\gamma))b^2} \quad (5.19)$$

Proof

$$(i) f(\gamma) < 1 \Rightarrow \gamma > \gamma_o$$

$$\Rightarrow \hat{f}(\gamma) > \mu(\gamma_o, \gamma) = \hat{f}(\gamma_o) \frac{\sqrt{\gamma_o^2 - b^2}}{\sqrt{\gamma^2 - b^2}}$$

$$\Rightarrow \hat{f}^2(\gamma)(\gamma^2 - b^2) > \gamma_o^2 - b^2, \quad \text{since } \hat{f}(\gamma_o) = 1$$

$$\Rightarrow \gamma_o < \sqrt{\hat{f}^2(\gamma)\gamma^2 + (1 - \hat{f}^2(\gamma))b^2} \quad \square$$

(ii) Same as (i) with inequalities reversed

Theorem 5.5

$$\text{Define } \beta(\alpha, \gamma) = \hat{f}(\alpha) \frac{\sqrt{\alpha^2 - c^2}}{\sqrt{\gamma^2 - c^2}}, \quad \text{for some } \alpha, \gamma > c,$$

$$\text{where } c = \inf_{\omega} \underline{\sigma}([R_{21}(j\omega) \quad R_{22}(j\omega)]).$$

$$(i) \hat{f}(\gamma) < \beta(\alpha, \gamma) \text{ if } \gamma > \alpha \quad (5.20)$$

$$(ii) \hat{f}(\gamma) = \beta(\alpha, \gamma) \text{ if } \gamma = \alpha \quad (5.21)$$

$$(iii) \hat{f}(\gamma) > \beta(\alpha, \gamma) \text{ if } \gamma < \alpha \quad (5.22)$$

Proof

The proofs are as for theorem 5.2 with $R_{11} - Q$ replaced by $\begin{bmatrix} R_{11} - Q & R_{12} \end{bmatrix}$ and R_{21} replaced by $\begin{bmatrix} R_{21} & R_{22} \end{bmatrix}$ $\square$

Corollary 5.5

$$\text{If } f(\gamma) < 1 \text{ then } \gamma_o > \sqrt{\hat{f}^2(\gamma)\gamma^2 + (1 - \hat{f}^2(\gamma))c^2} \quad (5.23)$$

$$\text{If } \hat{f}(\gamma) > 1 \text{ then } \gamma_o < \sqrt{\hat{f}^2(\gamma)\gamma^2 + (1 - \hat{f}^2(\gamma))c^2} \quad (5.24)$$

where $c = \inf_{\omega} \underline{\sigma}[R_{21}(j\omega)]$.

Proof

$$(i) f(\gamma) < 1 \Rightarrow \gamma > \gamma_o$$

$$\Rightarrow f(\gamma) < \beta(\gamma_o, \gamma) = \hat{f}(\gamma_o) \frac{\sqrt{\gamma_o^2 - c^2}}{\sqrt{\gamma^2 - c^2}}$$

$$\Rightarrow \gamma_o > \sqrt{\hat{f}^2(\gamma)\gamma^2 + (1 - \hat{f}^2(\gamma))c^2} \quad \square$$

$$(ii) f(\gamma) > 1 \Rightarrow \gamma < \gamma_o$$

$$\Rightarrow \hat{f}(\gamma) > \beta(\gamma_o, \gamma)$$

$$\Rightarrow \gamma_o < \sqrt{\hat{f}^2(\gamma)\gamma^2 + (1 - \hat{f}^2(\gamma))c^2} \quad \square$$

Corollaries 5.4 and 5.5 bound γ_o in terms of $\hat{f}(\gamma)$, however at each iteration $f(\gamma)$ is computed, not $\hat{f}(\gamma)$, therefore it is necessary to estimate $\hat{f}(\gamma)$ from $f(\gamma)$. Now,

$$\left\| \left[\left(F_{\ell} \left\{ \frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \right\} - \frac{1}{\gamma} Q \right) (I - \tilde{L}^* \tilde{L})^{-1/2} \quad L \right] \right\|_{\infty} = \hat{f}(\gamma) \quad (5.25)$$

$$\iff \left\| \left(\hat{f}(\gamma)^2 I - LL^* \right)^{-1/2} \left[F_\ell \left\{ \frac{1}{\gamma} R, \frac{1}{\gamma} R_{22}^* \right\} - \frac{1}{\gamma} Q \right] (I - \tilde{L}^* \tilde{L})^{-1/2} \right\|_\infty \leq 1 \quad (5.26)$$

Note the left hand side of equation (5.26) is $f(\gamma)$ with $\hat{f}(\gamma) = 1$. Thus evaluating $f(\gamma)$ from $\hat{f}(\gamma)$ is equivalent to taking the guess $\hat{f}(\gamma) = 1$. Also, since (5.25) is a 2-block γ -iteration, corollaries 5.1 and 5.2 can be used to bound $\hat{f}(\gamma)$ as

$$f^2(\gamma) + (1 - f^2(\gamma))\underline{\sigma}(L) < \hat{f}^2(\gamma) < f^2(\gamma) + (1 - f^2(\gamma))\bar{\sigma}(L) \quad \text{for } f(\gamma) < 1 \quad (5.27)$$

$$f^2(\gamma) + (1 - f^2(\gamma))\underline{\sigma}(L) < \hat{f}^2(\gamma) < f^2(\gamma) + (1 - f^2(\gamma))\underline{\sigma}(L) \quad \text{for } f(\gamma) > 1 \quad (5.28)$$

Once again it is not possible to evaluate $\underline{\sigma}(L)$ and $\bar{\sigma}(L)$ exactly. A crude way is to try and compute the singular values of L over a sufficiently large frequency range and then to take the minimum and maximum values as an approximation to $\underline{\sigma}(L)$ and $\bar{\sigma}(L)$. This is a good approximation provided the function L is well behaved, however this procedure can be quite slow. Another approach is to use upper and lower bounds for $\bar{\sigma}(L)$ and $\underline{\sigma}(L)$, since $\bar{\sigma}(L) \leq \max\{L(s=0), L(s=\infty)\}$ and $\underline{\sigma}(L) \geq 0$. However the resultant bounds will not be as tight. For the purpose of testing the tightness of the theoretical bounds, it is proposed to use the former method. As for the 2-block case, c will be approximated by 0. An algorithm for the 4-block γ -iteration is given in §5.3.1.

One important point to note in the 4-block γ -iteration is that the optimal γ is reached when $\hat{f}(\gamma) = 1$ and not necessarily when $f(\gamma)=1$. This is shown by equations (5.25) and (5.26). This situation normally arises when the optimal γ is close to the lower bound. To illustrate this, consider the state space description of $R(s)$ given in table 5.4

```

A=[ 0.001008  0.000000  0.003993  0.027748  0.000028 -0.004019
    -0.999734  0.000000  1.132848  0.280142  0.000284 -0.133120
     1.000199 -1.000000  2.599360 -0.280491 -0.000272 -0.099100
     0.000000  0.000000  0.000000  0.001000  0.000000  0.000000
     0.000000  0.000000  0.000000 -0.471346  0.024318  0.975781
     0.000000  0.000000  0.000000  0.471346 -1.024318  2.524219]

B=[ -0.014988 -0.000698 -0.007985
    -1.544408 -0.006991 -0.265694
     1.405274  0.007131 -0.198724
     0.000000 -0.002546  0.000848
     0.707107  0.001830 -0.121889
    -0.707107 -0.004155 10.096975]

C=[ -0.001000  0.000000 -0.500000  0.000752 -0.000024  0.500024
    -0.134384  1.095053 -3.948923 -0.002804 -0.000003 -0.000497
    -0.613276 -0.099287  0.408881 -2.803413 -0.002773  0.002699]

D=[ -0.002128 -0.000300  0.999999
    -1.414141  0.000071 -0.001000
     0.072125  0.070711  0.000072]

```

Table 5.4: State space description of $R(s)$

Using the above $R(s)$ and doing the γ -iteration yields the results given in table 5.5. It can be seen from table 5.5 that γ has converged within 1% of the optimal γ yet $f(\gamma) \ll 1$. In fact for this particular choice of $R(s)$, $f(\gamma)$ never seems to reach 1, even when the difference between γ_u and γ_ℓ is taken to be very small. For example if the difference is taken to be $1 \times 10^{-10}\%$, $f(\gamma)$ still converges to 0.105. Of course it is not clear whether $f(\gamma) \longrightarrow 1$ in the limit as $\epsilon \longrightarrow 0$ where $\gamma = (\gamma_u + \gamma_\ell)/2$ and $\frac{(\gamma_u - \gamma_\ell)}{\gamma_\ell} < \epsilon$.

STATUS : Solving a 4-block model-matching problem
STATUS : Doing gamma iteration

Enter required accuracy for the iteration		0.10000000	
1	GAMMAL= 7.46094846152301 GAMMA = 7.64747217306109	GAMMAU = 7.83399588459916 F (GAMMA)= 0.10209320028749	
2	GAMMAL= 7.46094846152301 GAMMA = 7.55421031729205	GAMMAU = 7.64747217306109 F (GAMMA)= 0.10354597677031	
3	GAMMAL= 7.46094846152301 GAMMA = 7.50757938940753	GAMMAU = 7.55421031729205 F (GAMMA)= 0.10432499846961	
4	GAMMAL= 7.46094846152301 GAMMA = 7.48426392546527	GAMMAU = 7.50757938940753 F (GAMMA)= 0.10474541007745	
5	GAMMAL= 7.46094846152301 GAMMA = 7.47260619349414	GAMMAU = 7.48426392546527 F (GAMMA)= 0.10497574351605	
6	GAMMAL= 7.46094846152301 GAMMA = 7.46677732750858	GAMMAU = 7.47260619349414 F (GAMMA)= 0.10510467093505	
THE OPTIMAL GAMMA IS = 7.46677732750858			
F (GAMMA) = 0.10510467093505			

Table 5.5: 4-block γ -iteration for $R(s)$

5.3.1 Algorithm for 4-block γ -iteration

- (1) Compute $\gamma_\ell = \max\{\|H_R\|, \|[R_{21} \quad R_{22}]\|_\infty, \left\|\begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}\right\|_\infty, \}$
 (where the infinity norms can be approximated as in the 2-block case)
- (2) Choose $\gamma = (1 + \delta)\gamma_\ell$
- (3) compute the spectral factors $S = (\gamma^2 I - R_{22}^* R_{22})^{1/2}$ and $\tilde{S} = (\gamma^2 I - R_{22} R_{22}^*)^{1/2}$ using this γ .
- IF successful THEN
- Let $L = R_{12} S^{-1}$ and $\tilde{L} = \tilde{S}^{-1} R_{21}$

ELSE

$\gamma_\ell = \gamma$ GOTO 2.

END IF

(4) Find the spectral factors $(I - LL^*)^{1/2}$ and $(I - \tilde{L}^* \tilde{L})^{1/2}$ using this γ .

IF successful THEN

IF $f(\gamma) > 1$ THEN

$\gamma_\ell = \gamma, \quad \gamma_u = \sqrt{\|H_{R_{11}}\|^2 + 2\gamma^2}, \quad b = \gamma$

ELSE

$\gamma_u = \gamma, \quad b = \gamma$

END IF

ELSE

$\gamma_\ell = \gamma$ GOTO 2

END IF

(5) $f_1(\gamma) = f(\gamma) + (1 - f^2(\gamma))\underline{\sigma}(L)$

$f_2(\gamma) = f(\gamma) + (1 - f^2(\gamma))\overline{\sigma}(L)$

(6) $\gamma_{u_1} = \sqrt{f_2^2(\gamma_u)\gamma_u^2 + (1 - f_2^2(\gamma_u))b^2}$

$\gamma_{\ell_1} = f_1(\gamma_u)\gamma_u$

IF $f(\gamma_\ell)$ successful THEN

$\gamma_{u_2} = f_1(\gamma_\ell)\gamma_\ell$

$\gamma_{\ell_2} = \sqrt{f_2^2(\gamma_\ell)\gamma_\ell^2 + (1 - f_2^2(\gamma_\ell))b^2}$

$\gamma_\ell = \max(\gamma_{\ell_1}, \gamma_{\ell_2}), \quad \gamma_u = \min(\gamma_{u_1}, \gamma_{u_2})$

ELSE

$$\gamma_\ell = \max(\gamma_\ell, \gamma_{\ell_1}), \quad \gamma_u = \min(\gamma_u, \gamma_{u_1})$$

END IF

(7) $\gamma = (\gamma_u + \gamma_\ell)/2$

IF $\frac{(\gamma_u - \gamma_\ell)}{\gamma_\ell} < \epsilon$ THEN

$$\gamma_o = \gamma \quad \text{GOTO 11}$$

END IF

(8) IF $f(\gamma)$ not successful THEN

$$\gamma_\ell = \gamma \quad \text{GOTO 7}$$

ELSE

$$\text{IF } f(\gamma) = 1, \text{ GOTO 11}$$

END IF

(9) $\beta_1 = f_1(\gamma)\gamma$

$$\beta_2 = \sqrt{f_2^2(\gamma)\gamma^2 + (1 - f_2^2(\gamma))b^2}$$

(10) IF $f(\gamma) > 1$ THEN

$$\gamma_\ell = \max(\gamma_\ell, \beta_2), \quad \gamma_u = \min(\beta_1, \gamma_u) \quad \text{GOTO 7}$$

ELSE

$$\gamma_\ell = \max(\beta_1, \gamma_\ell), \quad \gamma_u = \min(\beta_2, \gamma_u) \quad \text{GOTO 7}$$

END IF

(11) STOP

Example 2

The feedback control system shown in figure 4.4 will be used to generate a 4-block design problem. Given a plant $G(s)$, it is desired to find a controller $K(s)$ such that the closed loop system is internally stable and the H^∞ -norm of

$$\left\| \begin{bmatrix} W_1(I + PK)^{-1}W_4 & W_1(I + PK)^{-1}PW_5 & -W_1PK(I + PK)^{-1}W_6 \\ W_2(I + PK)^{-1}W_4 & W_2(I + PK)^{-1}PW_5 & W_2(I + PK)^{-1}W_6 \\ -W_3K(I + PK)^{-1}W_4 & W_3(I + KP)^{-1}W_5 & -W_3K(I + PK)^{-1}W_6 \end{bmatrix} \right\|_\infty$$

is minimized.

Note that a 4-block model matching problem has to be solved since there are three error signals and three external input signals but only one available actuator input and one measurement signal. The three external inputs in figure 4.4 are configured to represent output plant disturbance, sensor noise and actuator noise respectively. To illustrate a 4-block design, a simple SISO plant with an integrator will be used. The model-matching problem cannot be solved for such a plant in the 2-block case without the need for some form of $j\omega$ -axis shifting.

Take the plant $G(s) = \frac{1}{s(s+2)}$. Nominal design specifications are:

- (1) Good performance and disturbance rejection at low frequency. Since the plant has a free integrator it should be possible to achieve zero steady state error for a step input.
- (2) Minimize the amplitude and rate of the control signal.
- (3) Minimize the effect of high frequency sensor noise.

Guidelines for the selection of the weighting functions $W_i, i = 1, \dots, 6$ are:

W_1 : A first order high gain low-pass filter is used to ensure that output 1 can be controlled accurately with good disturbance attenuation up to about 1 rad/s. W_1 approximates

integral action hence it forces the resulting closed loop system to have almost perfect steady state error.

W_2 : A constant of 0.1 is used to attenuate the effect of sensor noise.

W_3 : A first order high pass filter is used to constrain the actuator movement.

$W_4 - W_6$: These are chosen as constants with W_4 taken as 1 to represent a unit disturbance and W_5 and W_6 taken as 0.01 to represent the noise level.

The plant plus weights has 4 states. Note that making the weightings on the external inputs due to sensor noise and actuator noise small, has the resultant effect of making the 4-block problem an inherent 2-block problem. It is important to keep the state dimension of the weights to a minimum since the controller will in general be of the same order as the plant plus weights [LIM2], [LIM3], [POS7].

The weighting functions chosen for this particular design are:

$$W_1 = 0.56 \frac{(s + 1.78)}{(s + 0.01)}, \quad W_2 = 0.1, \quad W_3 = 0.5 \frac{(s + 0.0158)}{(s + 5)}, \quad W_4 = 1, \quad W_5 = W_6 = 0.01$$

It is desired to find

$$\gamma_o \triangleq \min_{Q \in \mathcal{RH}^\infty} \left\| \begin{bmatrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty$$

As for the 2-block γ -iteration an accuracy of 1% and $1 \times 10^{-10}\%$ was used in determining the optimal γ . Tables 5.6 and 5.7 show the γ -iteration with $\epsilon=0.01$ for a bisection algorithm and for the algorithm of §5.3.1 respectively.

STATUS : Solving a 4-block model-matching problem
STATUS : Doing gamma iteration

```

Enter required accuracy for the iteration      1.0000000

1  GAMMAL=      1.07019401723966   GAMMAU  =      1.28423282068759
   GAMMA  =      1.17721341896363   F (GAMMA)=      0.89873679160533

2  GAMMAL=      1.07019401723966   GAMMAU  =      1.17721341896363
   GAMMA  =      1.12370371810164   F (GAMMA)=      0.95031981543164

3  GAMMAL=      1.07019401723966   GAMMAU  =      1.12370371810164
   GAMMA  =      1.09694886767065   F (GAMMA)=      0.97861672813160

4  GAMMAL=      1.07019401723966   GAMMAU  =      1.09694886767065
   GAMMA  =      1.08357144245516   F (GAMMA)=      0.99343569180487

5  GAMMAL=      1.07019401723966   GAMMAU  =      1.08357144245516
   GAMMA  =      1.07688272984741   F (GAMMA)=      1.00112465439770

6  GAMMAL=      1.07688272984741   GAMMAU  =      1.08357144245516

THE OPTIMAL GAMMA IS =      1.07688272984741
      F (GAMMA)  =      1.00112465439770

The gamma-iteration took   49.14001   CPU seconds

```

Table 5.6: 4-block γ -iteration (bisection algorithm) with $\epsilon = 0.01$

STATUS : Solving a 4-block model-matching problem
STATUS : Doing gamma iteration

```

Enter required accuracy for the iteration      1.0000000

1  GAMMAL=      1.07019401723966   GAMMAU  =      1.28423282068759
   GAMMA  =      1.17721341896363   F (GAMMA)=      0.89873679160533

2  GAMMAL=      1.07188416669619   GAMMAU  =      1.07942973241857

THE OPTIMAL GAMMA IS =      1.07565694955738
      F (GAMMA)  =      1.00870923904819

The gamma-iteration took   35.30000   CPU seconds

```

Table 5.7: 4-block γ -iteration (new algorithm) with $\epsilon = 0.01$

It can be seen from these tables that the bisection algorithm needs more than twice as many iterations before converging to the optimal γ . The amount of CPU processing time required for the two algorithms was 49.14 and 35.3 secs respectively, for this choice of ϵ . This particular example illustrates how good the bounds are for the optimal gamma as calculated in the pre-iteration stage of the new algorithm.

```

STATUS : Solving a 4-block model-matching problem
STATUS : Doing gamma iteration

Enter required accuracy for the iteration      0.10000000e-09

1  GAMMAL=      1.07019401723966    GAMMAU  =      1.28423282068759
   GAMMA  =      1.17721341896363    F (GAMMA)=      0.89873679160533

2  GAMMAL=      1.07188416669619    GAMMAU  =      1.07942973241857
   GAMMA  =      1.07565694955738    F (GAMMA)=      1.00232472606544

3  GAMMAL=      1.07607777773750    GAMMAU  =      1.07813497783249
   GAMMA  =      1.07710637778500    F (GAMMA)=      1.00079372933392

4  GAMMAL=      1.07724736974204    GAMMAU  =      1.07795360531629
   GAMMA  =      1.07760048752916    F (GAMMA)=      1.00014179395700

5  GAMMAL=      1.07762551020196    GAMMAU  =      1.07775190905915
   GAMMA  =      1.07768870963055    F (GAMMA)=      1.00008150469640

6  GAMMAL=      1.07770307658957    GAMMAU  =      1.07775190905915
   GAMMA  =      1.07772749282436    F (GAMMA)=      0.99994393938018

7  GAMMAL=      1.07770307658957    GAMMAU  =      1.07771761672200
   GAMMA  =      1.07771034665578    F (GAMMA)=      1.00045564194882

8  GAMMAL=      1.07779065784202    GAMMAU  =      1.07771761672200

THE OPTIMAL GAMMA IS =      1.07775413728201
      F (GAMMA)  =      1.00045564194882

The gamma-iteration took      110.1200    CPU seconds

```

Table 5.8: 4-block γ -iteration (new algorithm) with $\epsilon = 1 \times 10^{-12}$

However although the bounds for γ_ℓ and γ_u are much tighter for the new algorithm, this is offset by the greater computational demand at each iteration. Hence as for the 2-block case this algorithm will be significantly better when the accuracy of γ is required to be very small. For the case when $\epsilon = 1 \times 10^{-12}$ the optimal γ converged in just 8 iterations

(Table 5.8) while a bisection algorithm converged in 39 (Table A5.2, Appendix 5.1), nearly 5 times as many iterations. The corresponding CPU processing times were 110.12 secs and 369.36 secs respectively, a factor of 3.4 times faster.

Using the above weights the controller for this example is

$$K = 2.21 \frac{(s + 2.11)(s + 5.02)}{(s^2 + 5.52s + 9.94)}$$

Note that the order of the controller is the same as for the plant, this is consistent with the findings of [POS7]. Figure 5.1 shows the resulting closed-loop step response. This example has been used merely to illustrate the 4-block γ -iteration, the performance is by no means the best that can be achieved.

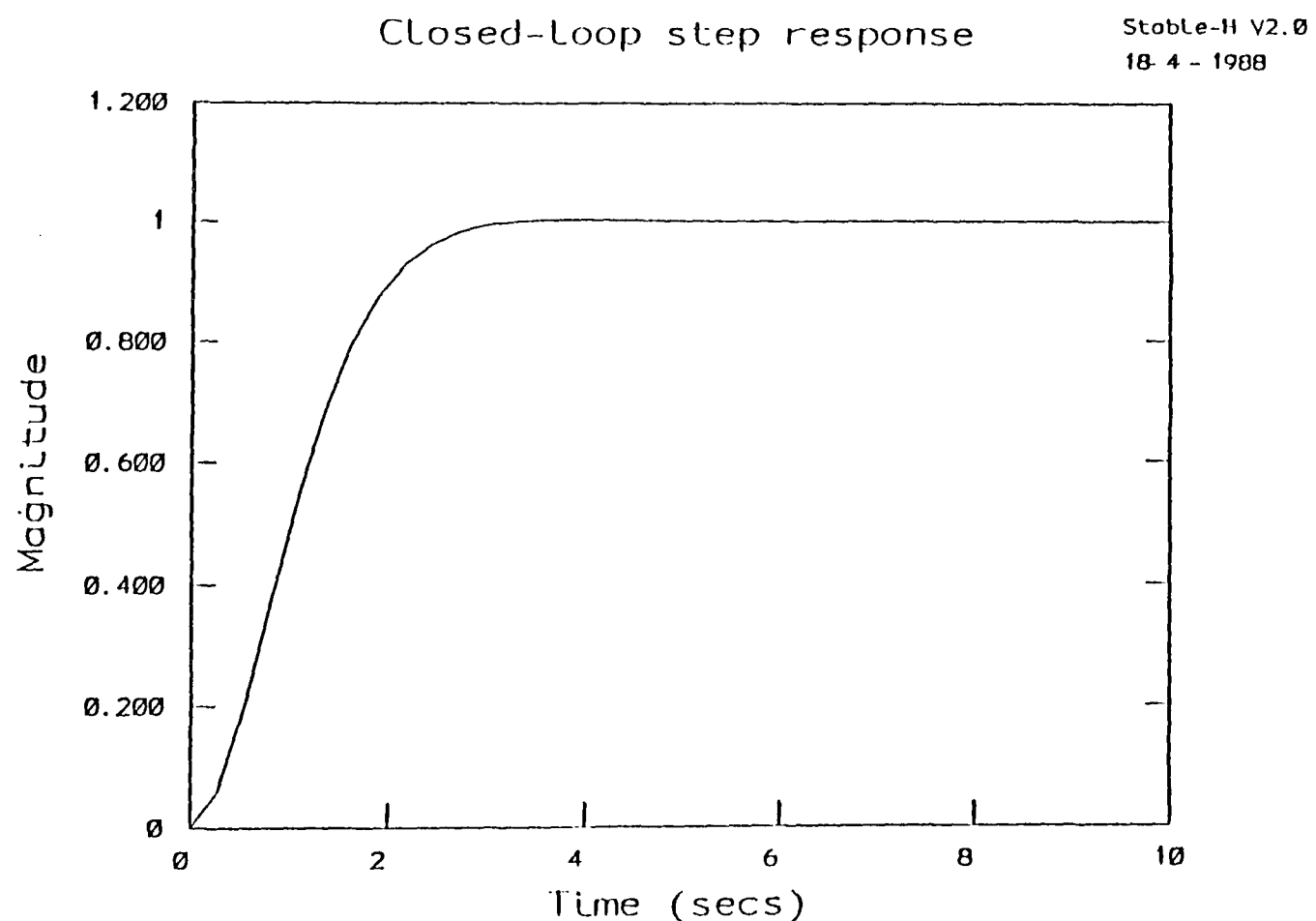


Figure 5.1

5.4 Concluding Remarks

Some new bounds for the optimal γ have been derived for the 2-block and 4-block configurations in the SCC. These new bounds have been shown to be much tighter than those obtained by bisection, resulting in fewer iterations before convergence to the optimal gamma. However each iteration is computationally more demanding because of the need to compute $f(\gamma)$ at several values of γ . The algorithms presented are easily implemented and do not require the need to compute the L^∞ norm of any functions. Simple approximations to such norms have been shown to be adequate. These new algorithms are clearly better when γ is required to be within a very small percentage of the optimal γ .

Some simple SISO examples have been presented to illustrate the main properties of the algorithms. In the case of the 4-block γ -iteration, a plant with an integrator has been shown to be solvable using H^∞ - optimization. This confirms the feedback structure presented in §4.5. In the development of the 4-block iterative algorithms, no assumption has been made regarding the convexity of $f(\gamma)$. From experimentation, γ always seems to converge to a minimum, so it would appear that $f(\gamma)$ is indeed a convex function, although there is no proof of this.

The bounds derived for the 2-block γ -iteration which use knowledge of the convexity of $f(\gamma)$, can easily be incorporated into the 4-block iterative scheme. However since these bounds require $f(\gamma_\ell)$ as well as $f(\gamma)$ at each iteration, the reduction in the number of iterations before convergence, is offset by the greater computational demand. Therefore since the 4-block iterative scheme is very CPU intensive, it was decided that there would not be any significant advantage in incorporating the convexity bounds (5.11) and (5.12).

APPENDIX 5.1

γ -ITERATION RESULTS

STATUS : Solving a 2-block model-matching problem
STATUS : Doing gamma iteration

Enter required accuracy for the iteration 0.10000000e-09

1	GAMMAL=	0.94276580064843	GAMMAU =	1.13131896077812
	GAMMA =	1.03704238071328	F (GAMMA)=	0.96259186733986
2	GAMMAL=	0.94276580064843	GAMMAU =	1.03704238071328
	GAMMA =	0.98990409068086	F (GAMMA)=	1.01449073882215
3	GAMMAL=	0.98990409068086	GAMMAU =	1.03704238071328
	GAMMA =	1.01347323569707	F (GAMMA)=	0.98782006139799
4	GAMMAL=	0.98990409068086	GAMMAU =	1.01347323569707
	GAMMA =	1.00168866318896	F (GAMMA)=	1.00096706519959
5	GAMMAL=	1.00168866318896	GAMMAU =	1.01347323569707
	GAMMA =	1.00758094944301	F (GAMMA)=	0.99434752113561
6	GAMMAL=	1.00168866318896	GAMMAU =	1.00758094944301
	GAMMA =	1.00463480631599	F (GAMMA)=	0.99764565494320
7	GAMMAL=	1.00168866318896	GAMMAU =	1.00463480631599
	GAMMA =	1.00316173475247	F (GAMMA)=	0.99930343433900
8	GAMMAL=	1.00168866318896	GAMMAU =	1.00316173475247
	GAMMA =	1.00242519897072	F (GAMMA)=	1.00013451630047
9	GAMMAL=	1.00242519897072	GAMMAU =	1.00316173475247
	GAMMA =	1.00279346686160	F (GAMMA)=	0.99971879220761
10	GAMMAL=	1.00242519897072	GAMMAU =	1.00279346686160
	GAMMA =	1.00260933291616	F (GAMMA)=	0.99992660844416
11	GAMMAL=	1.00242519897072	GAMMAU =	1.00260933291616
	GAMMA =	1.00251726594344	F (GAMMA)=	1.00003055091586
12	GAMMAL=	1.00251726594344	GAMMAU =	1.00260933291616
	GAMMA =	1.00256329942980	F (GAMMA)=	0.99997857681639
13	GAMMAL=	1.00251726594344	GAMMAU =	1.00256329942980
	GAMMA =	1.00254028268662	F (GAMMA)=	1.00000456315016
14	GAMMAL=	1.00254028268662	GAMMAU =	1.00256329942980
	GAMMA =	1.00255179105821	F (GAMMA)=	0.99999156980429
15	GAMMAL=	1.00254028268662	GAMMAU =	1.00255179105821
	GAMMA =	1.00254603687241	F (GAMMA)=	0.99999806643248

16	GAMMAL=	1.00254028268662	GAMMAU =	1.00254603687241
	GAMMA =	1.00254315977951	F (GAMMA) =	1.00000131478013
17	GAMMAL=	1.00254315977951	GAMMAU =	1.00254603687241
	GAMMA =	1.00254459832596	F (GAMMA) =	0.99999969060351
18	GAMMAL=	1.00254315977951	GAMMAU =	1.00254459832596
	GAMMA =	1.00254387905274	F (GAMMA) =	1.00000050269112
19	GAMMAL=	1.00254387905274	GAMMAU =	1.00254459832596
	GAMMA =	1.00254423868935	F (GAMMA) =	1.00000009664714
20	GAMMAL=	1.00254423868935	GAMMAU =	1.00254459832596
	GAMMA =	1.00254441850766	F (GAMMA) =	0.99999989362528
21	GAMMAL=	1.00254423868935	GAMMAU =	1.00254441850766
	GAMMA =	1.00254432859850	F (GAMMA) =	0.99999999513620
22	GAMMAL=	1.00254423868935	GAMMAU =	1.00254432859850
	GAMMA =	1.00254428364393	F (GAMMA) =	1.00000004589167
23	GAMMAL=	1.00254428364393	GAMMAU =	1.00254432859850
	GAMMA =	1.00254430612122	F (GAMMA) =	1.00000002051393
24	GAMMAL=	1.00254430612122	GAMMAU =	1.00254432859850
	GAMMA =	1.00254431735986	F (GAMMA) =	1.00000000782507
25	GAMMAL=	1.00254431735986	GAMMAU =	1.00254432859850
	GAMMA =	1.00254432297918	F (GAMMA) =	1.00000000148063
26	GAMMAL=	1.00254432297918	GAMMAU =	1.00254432859850
	GAMMA =	1.00254432578884	F (GAMMA) =	0.99999999830842
27	GAMMAL=	1.00254432297918	GAMMAU =	1.00254432578884
	GAMMA =	1.00254432438401	F (GAMMA) =	0.99999999989453
28	GAMMAL=	1.00254432297918	GAMMAU =	1.00254432438401
	GAMMA =	1.00254432368160	F (GAMMA) =	1.00000000068758
29	GAMMAL=	1.00254432368160	GAMMAU =	1.00254432438401
	GAMMA =	1.00254432403280	F (GAMMA) =	1.00000000029105
30	GAMMAL=	1.00254432403280	GAMMAU =	1.00254432438401
	GAMMA =	1.00254432420841	F (GAMMA) =	1.00000000009279
31	GAMMAL=	1.00254432420841	GAMMAU =	1.00254432438401
	GAMMA =	1.00254432429621	F (GAMMA) =	0.99999999999366
32	GAMMAL=	1.00254432420841	GAMMAU =	1.00254432429621
	GAMMA =	1.00254432425231	F (GAMMA) =	1.00000000004322
33	GAMMAL=	1.00254432425231	GAMMAU =	1.00254432429621
	GAMMA =	1.00254432427426	F (GAMMA) =	1.00000000001844
34	GAMMAL=	1.00254432427426	GAMMAU =	1.00254432429621
	GAMMA =	1.00254432428523	F (GAMMA) =	1.00000000000605
35	GAMMAL=	1.00254432428523	GAMMAU =	1.00254432429621
	GAMMA =	1.00254432429072	F (GAMMA) =	0.99999999999985

```

36  GAMMAL=      1.00254432428523      GAMMAU =      1.00254432429072
    GAMMA =      1.00254432428798      F (GAMMA)=      1.000000000000295

37  GAMMAL=      1.00254432428798      GAMMAU =      1.00254432429072
    GAMMA =      1.00254432428935      F (GAMMA)=      1.000000000000140

38  GAMMAL=      1.00254432428935      GAMMAU =      1.00254432429072

THE OPTIMAL GAMMA IS =      1.00254432429004
      F (GAMMA) =      1.000000000000063

```

The gamma-iteration took 16.38000 CPU seconds

Table A5.1: 2-block γ -iteration (bisection algorithm) with $\epsilon = 1 \times 10^{-12}$

```

STATUS : Solving a 4-block model-matching problem
STATUS : Doing gamma iteration

```

```

Enter required accuracy for the iteration      0.10000000e-09

1  GAMMAL=      1.07019401723966      GAMMAU =      1.28423282068759
   GAMMA =      1.17721341896363      F (GAMMA)=      0.89873679160533

2  GAMMAL=      1.07019401723966      GAMMAU =      1.17721341896363
   GAMMA =      1.12370371810164      F (GAMMA)=      0.95031981543164

3  GAMMAL=      1.07019401723966      GAMMAU =      1.12370371810164
   GAMMA =      1.09694886767065      F (GAMMA)=      0.97861672813160

4  GAMMAL=      1.07019401723966      GAMMAU =      1.09694886767065
   GAMMA =      1.08357144245516      F (GAMMA)=      0.99343569180487

5  GAMMAL=      1.07019401723966      GAMMAU =      1.08357144245516
   GAMMA =      1.07688272984741      F (GAMMA)=      1.00112465439770

6  GAMMAL=      1.07688272984741      GAMMAU =      1.08357144245516
   GAMMA =      1.08022708615128      F (GAMMA)=      0.99703516099067

7  GAMMAL=      1.07688272984741      GAMMAU =      1.08022708615128
   GAMMA =      1.07855490799934      F (GAMMA)=      0.99911476307519

8  GAMMAL=      1.07688272984741      GAMMAU =      1.07855490799934
   GAMMA =      1.07771881892338      F (GAMMA)=      1.00016240796875

9  GAMMAL=      1.07771881892338      GAMMAU =      1.07855490799934
   GAMMA =      1.07813686346136      F (GAMMA)=      0.99956574584443

10 GAMMAL=      1.07771881892338      GAMMAU =      1.07813686346136
   GAMMA =      1.07792784119237      F (GAMMA)=      0.99975612233524

11 GAMMAL=      1.07771881892338      GAMMAU =      1.07792784119237
   GAMMA =      1.07782333005787      F (GAMMA)=      0.99992184804743

12 GAMMAL=      1.07771881892338      GAMMAU =      1.07782333005787
   GAMMA =      1.07777107449062      F (GAMMA)=      0.99988873713339

```

13	GAMMAL=	1.07771881892338	GAMMAU =	1.07777107449062
	GAMMA =	1.07774494670700	F (GAMMA) =	0.99985671306973
14	GAMMAL=	1.07771881892338	GAMMAU =	1.07774494670700
	GAMMA =	1.07773188281519	F (GAMMA) =	1.00003563404285
15	GAMMAL=	1.07773188281519	GAMMAU =	1.07774494670700
	GAMMA =	1.07773841476109	F (GAMMA) =	1.00016474042836
16	GAMMAL=	1.07773841476109	GAMMAU =	1.07774494670700
	GAMMA =	1.07774168073405	F (GAMMA) =	1.00010359585739
17	GAMMAL=	1.07774168073405	GAMMAU =	1.07774494670700
	GAMMA =	1.07774331372052	F (GAMMA) =	1.00003065374762
18	GAMMAL=	1.07774331372052	GAMMAU =	1.07774494670700
	GAMMA =	1.07774413021376	F (GAMMA) =	1.00011612307781
19	GAMMAL=	1.07774413021376	GAMMAU =	1.07774494670700
	GAMMA =	1.07774453846038	F (GAMMA) =	1.00004310181385
20	GAMMAL=	1.07774453846038	GAMMAU =	1.07774494670700
	GAMMA =	1.07774474258369	F (GAMMA) =	1.00010745005699
21	GAMMAL=	1.07774474258369	GAMMAU =	1.07774494670700
	GAMMA =	1.07774484464535	F (GAMMA) =	0.99992300555636
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	GAMMA =	1.07774479361452	F (GAMMA) =	1.00010648874064
23	GAMMAL=	1.07774479361452	GAMMAU =	1.07774484464535
	GAMMA =	1.07774481912993	F (GAMMA) =	0.99998315343320
24	GAMMAL=	1.07774479361452	GAMMAU =	1.07774481912993
	GAMMA =	1.07774480637222	F (GAMMA) =	0.99991714014113
25	GAMMAL=	1.07774479361452	GAMMAU =	1.07774480637222
	GAMMA =	1.07774479999337	F (GAMMA) =	1.00001893259369
26	GAMMAL=	1.07774479999337	GAMMAU =	1.07774480637222
	GAMMA =	1.07774480318280	F (GAMMA) =	1.00002865684443
27	GAMMAL=	1.07774480318280	GAMMAU =	1.07774480637222
	GAMMA =	1.07774480477751	F (GAMMA) =	1.00003038814080
28	GAMMAL=	1.07774480477751	GAMMAU =	1.07774480637222
	GAMMA =	1.07774480557487	F (GAMMA) =	0.99999154215157
29	GAMMAL=	1.07774480477751	GAMMAU =	1.07774480557487
	GAMMA =	1.07774480517619	F (GAMMA) =	0.99986550596842
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	GAMMA =	1.07774480507652	F (GAMMA) =	0.99997954930185
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	GAMMA =	1.07774480502669	F (GAMMA) =	0.99998774999240

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	GAMMA =	1.07774480498931	F (GAMMA)=	0.99998038010840
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	GAMMA =	1.07774480498308	F (GAMMA)=	0.99987992737332
36	GAMMAL=	1.07774480497685	GAMMAU =	1.07774480498308
	GAMMA =	1.07774480497997	F (GAMMA)=	1.00000343608612
37	GAMMAL=	1.07774480497997	GAMMAU =	1.07774480498308
	GAMMA =	1.07774480498152	F (GAMMA)=	0.99980481225093
38	GAMMAL=	1.07774480497997	GAMMAU =	1.07774480498152
	GAMMA =	1.07774480498074	F (GAMMA)=	0.99990821937873
39	GAMMAL=	1.07774480497997	GAMMAU =	1.07774480498074
	THE OPTIMAL GAMMA IS =	1.07774480498074		
	F (GAMMA) =	0.99990821937873		

The gamma-iteration took 369.3600 CPU seconds

Table A5.2: 4-block γ -iteration (bisection algorithm) with $\epsilon = 1 \times 10^{-12}$

CHAPTER 6

H^∞ -OPTIMAL DESIGN FOR A LYNX HELICOPTER

6.1 *Introduction*

This chapter presents applications of H^∞ -optimization for the design of control systems for a Lynx helicopter (figure 6.1). The work illustrates how practical problems may be formulated in the H^∞ -framework. Particular attention is paid to the presence of model uncertainty and the motivation behind the selection of H^∞ -weights.

A nonlinear model of the Lynx helicopter was used to produce numerical models at particular operating points as discussed in chapter 2. The uncertainties in the model were represented as unstructured norm bounded perturbations. The linear controllers resulting from the design were then implemented in a full nonlinear simulation of the helicopter including both rotor and actuator dynamics. Nonlinearities are hard to incorporate into control designs. One way of addressing the uncertainty associated with nonlinearities is to approximate them by norm bounded perturbations of linear components. This is currently an area of research.

A quasi-steady flapping motion has been assumed in the modelling of the blade dynamics. The dynamic modes of motion of the rotor have been assumed to be of significantly high frequency so as not to couple with the airframe dynamics. The lowest frequency mode in the multi-blade coordinate system is the regressing flap mode which responds at about 1 Hz. This mode may couple with the fuselage roll subsidence mode or with some control system modes. Therefore it is important that the modes of the controller be kept small.



Figure 6.1 The RAE Lynx Research Helicopter

Input-output scaling of the nominal model is an important aspect in the design process. This is particularly so for multivariable systems where signals have a directional as well as a frequency distribution. Certain actuators might saturate at lower signal levels, some sensors are noisier than others, so it is important to emphasize the relative importance of each signal.

The control laws were developed to provide decoupling, desensitization and stability augmentation in the face of aircraft modelling uncertainty. The control laws are analysed with frequency and time responses and stability robustness assessed using singular value techniques.

Since the helicopter control system is intended to demonstrate advanced rotorcraft system performance and handling, it is logical to use the proposed LHX handling qualities specification [HOH1] as a guide to assess the response characteristics in this case study. An attitude-command/attitude-hold (ACAH) system was chosen, since in the simulation studies, it was considered to be a requirement for satisfactory handling qualities, especially in the hover/low speed flight region.

It is desired that the controller be robust and that nominal performance satisfies the handling qualities requirements outlined in the proposed airworthiness design standard. To meet both these aims a two-degree of freedom controller structure can be adopted. To illustrate the H^∞ -design technique a 2-block model-matching formulation is used for the hover/low speed controller and a 4-block formulation for the forward flight controller.

6.2 Hover/Low Speed Controller

It is desired that pilot longitudinal stick commands correspond to pitch attitude, lateral stick to roll attitude, collective to heave velocity and pedal position to yaw rate. A

two-degree of freedom controller will be designed in which the command model is generated using H_2 -optimization and the stabilizing controller using H^∞ -optimization. The command model will reflect the desired responses to pilot inputs and therefore the optimization will be posed as a tracking problem. The inner loop controller will be designed to be robustly stable to unmodelled dynamics. The inner loop design will be treated first.

6.2.1 Design of H^∞ -optimal controller

H^∞ performance specifications are given in terms of bounded power. As the $\|\cdot\|_\infty$ norm is defined as a supremum over frequency it is possible to include frequency dependent weights at the inputs and outputs. This is necessary in order to appropriately specify performance. The selection of the weights determines the characteristics of the controller.

Care must be taken to ensure that the problem is well posed. In particular the transfer function between $\underline{v}$ and $\underline{y}$ (P_{12}) and between $\underline{u}$ and $\underline{e}$ (P_{21}) should be invertible at all frequencies in the SCC of figure 4.1. The physical interpretation of these constraints is that if P_{12} loses rank this would lead to unconstrained controller action and if P_{21} loses rank this would correspond to an error being uncontrollable.

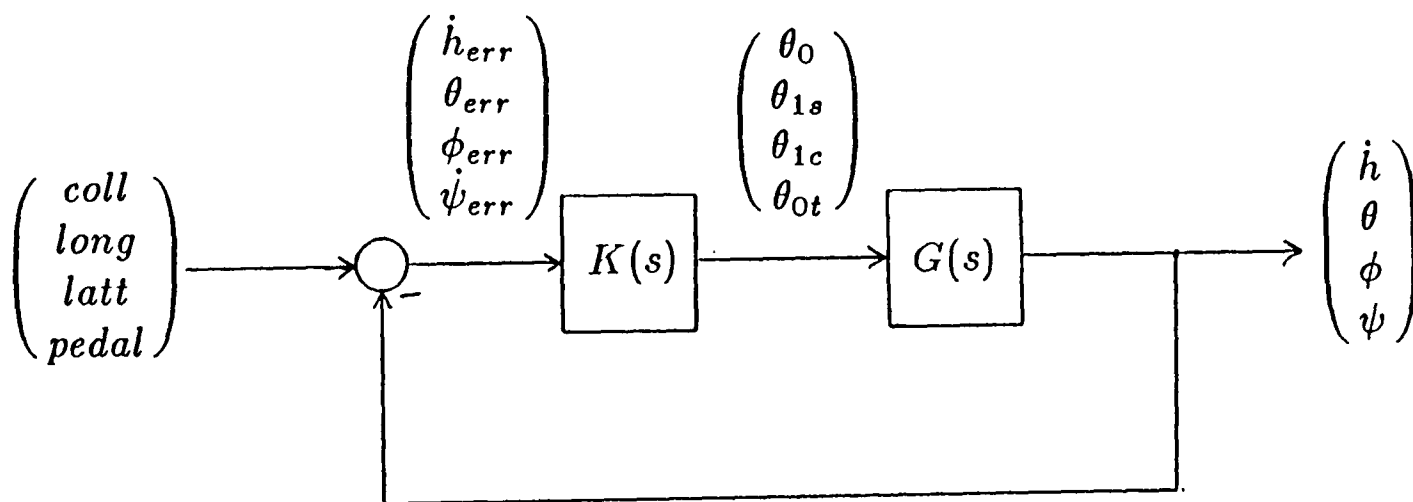


Figure 6.2 Attitude feedback loop

The H^∞ -design procedure will produce a controller K , which fits the feedback structure shown in figure 6.2. The H^∞ synthesis framework requires that outputs be quantities that are kept small. For example tracking errors and actuator movement. Therefore the problem in figure 6.2 is reformulated in the required structure shown in figure 6.3.

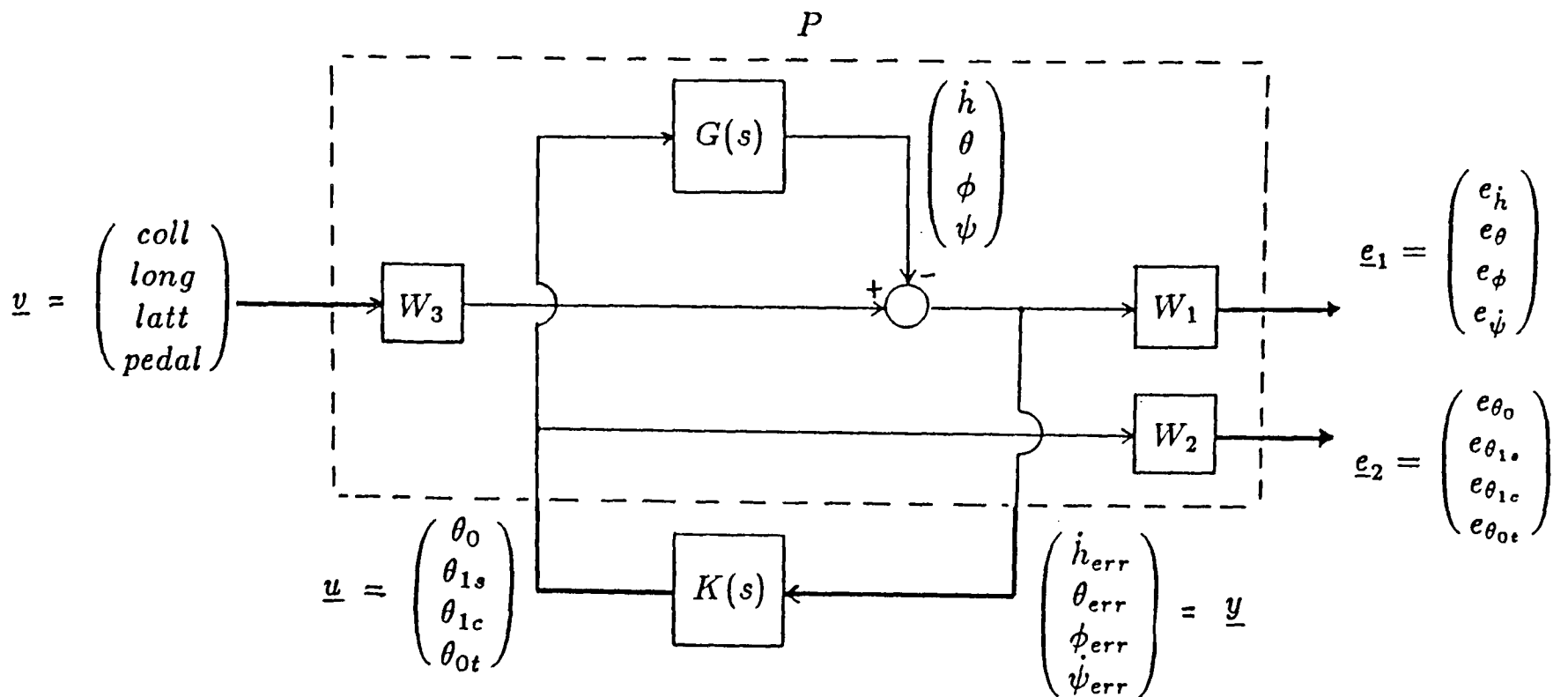


Figure 6.3 Interconnection structure

The linearized model of the helicopter was taken about the hover flight condition. Details of the helicopter model and the control strategies are discussed in chapter 2. The outputs chosen for control were scaled as in table 6.1. The effect of these scalings is such that a unit step input on each of the pilot inceptors would result in the responses given in table 6.1. For example a unit step command in collective would result in a response of 10 ft/s in heave velocity. These scalings ensure that each output channel is of equal importance. The choice of scaling requires some engineering insight into what is physically possible in the real system.

The open-loop frequency response of the linearized helicopter model is shown in figure 6.4. From this figure it can be seen that there are singular values at -45 dB and -47 dB which means that closed-loop tracking will be very poor in these channels. Also the difference in magnitude between the smallest and largest singular value is about 65 dB, resulting in a condition number of 1778. Therefore the plant is close to being singular at low frequency. The design implication of this is that any attempt to provide compensation at low frequency by inverting the plant will result in erroneous results.

Output Scalings

Outputs	Scaling
$\dot{h}$	10 ft/s
p	0.5 rad/s
q	0.2 rad/s
$\dot{\psi}$	0.5 rad/s
θ	0.2 rads
ϕ	0.5 rads

Table 6.1

In particular any loop shaping approach which inverts the nominal plant and then multiplies by the desired loop shape to obtain the controller will lead to both a very complicated controller and one which is not very robust. This is because if the controller is based on the inverse of the plant then it must essentially have the same condition number. A high condition number controller will perform badly in the presence of any plant input uncertainty. In practice, pseudo-inverse precompensation cannot be exactly realized without unstable pole-zero cancellation, but it may be possible to use a constant approximation of it.

The performance criterion chosen in the design was

$$\left\| \begin{bmatrix} W_1(I + GK)^{-1}W_3 \\ W_2K(I + GK)^{-1}W_3 \end{bmatrix} \right\|_{\infty} \quad (6.1)$$

W_1 , W_2 and W_3 are weighting functions to be chosen. There is no loss of generality in assuming that the weighting functions are both stable and minimum phase, since it is the magnitude of the frequency response which is to be shaped. The key feature of the design procedure is that the largest singular value of the frequency response of (6.1) is independent of frequency. In the attitude control design the sensitivity term $(I + GK)^{-1}$ is used to reduce the frequency tracking deviations and also to attenuate any system disturbances. The term $K(I + GK)^{-1}$ is used to improve the system robustness to additive perturbations (§3.3.1) and also to reduce the amplitude of the control signals to the actuators.

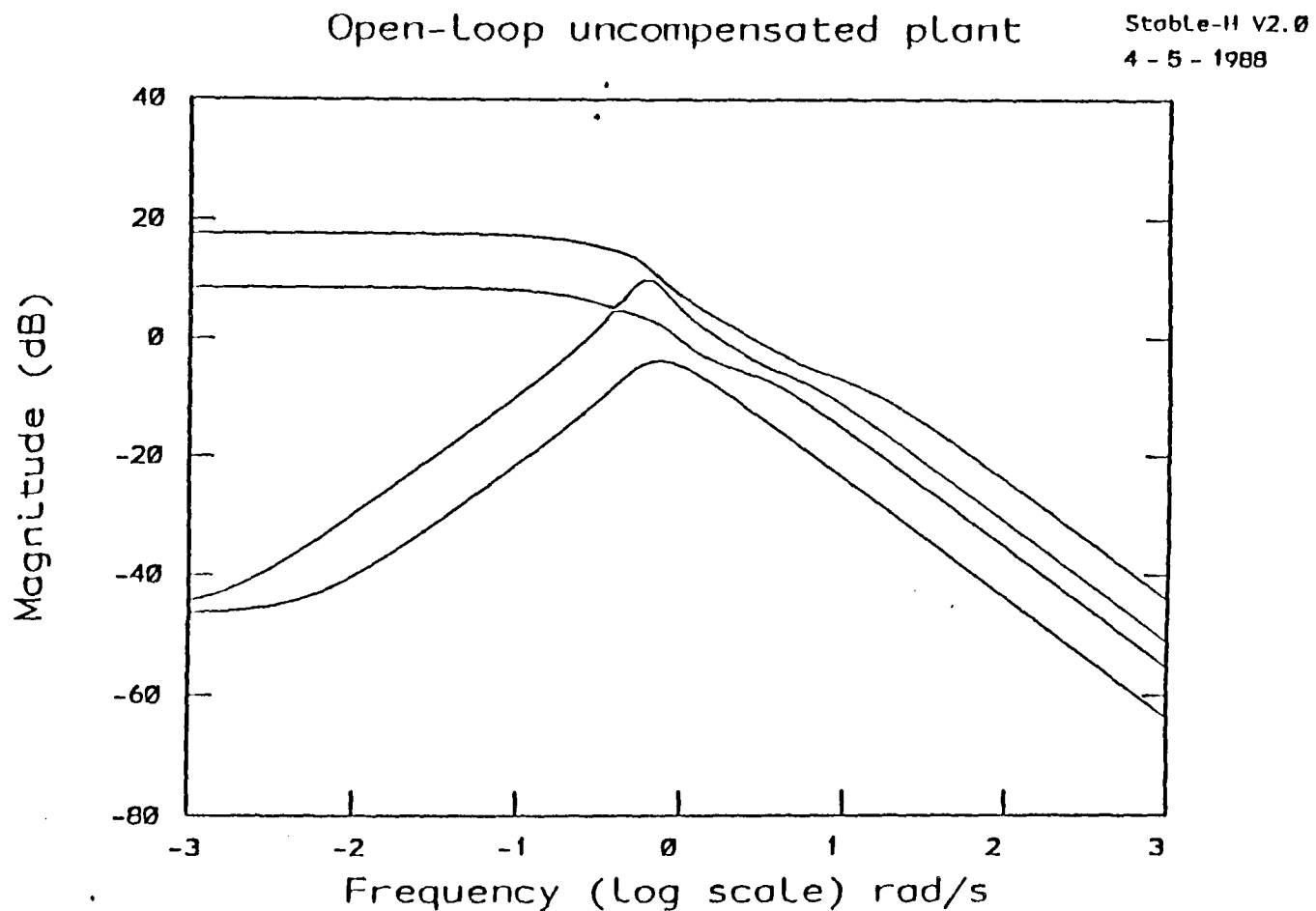


Figure 6.4 Singular value plot of uncompensated helicopter in the hover position

Initial studies showed that it was not possible to obtain good control of the outputs $\dot{h}$, $\dot{\psi}$, θ and ϕ without information from the pitch rate and roll rate sensors, therefore these outputs were added.

6.2.1.1 Weighting function selection

In choosing the weighting functions it is convenient to assume the problem is normalized to a value of 1. The weights are then obtained iteratively. In practice the procedure is to find input weights W_i and output weight W_o such that $\|W_o F_\ell(P, K) W_i\|_\infty \approx 1$. The rationale for this is that if $\|W_o F_\ell(P, K) W_i\|_\infty > 1$, then performance is not achieved and if $\|W_o F_\ell(P, K) W_i\|_\infty \ll 1$ then the performance specified by the weights is in some sense conservative.

The design problem and specifications give a general shape for the weighting functions. The desired frequency responses will be in inverse proportion to the magnitudes of the weights if they are chosen to have disjoint frequency ranges.

Selection of $W_1(s)$

To ensure good tracking accuracy in each of the controlled outputs $\dot{h}$, $\dot{\psi}$, θ and ϕ , integral action is required and this specifies a $1/s$ shape for the weights. There is no point in controlling the rates and angular positions to less than the resolution of the sensors, this allows the weights to be levelled off at low frequencies. In tuning the weights it was found that a finite attenuation in these channels was useful in reducing overshoot. Therefore high gain low pass filters were used to ensure that these outputs could be controlled accurately with good disturbance attenuation up to 6 rad/s. $W_1(s)$ will dominate in the frequency range 0 to 10 rad/s. The presence of unmodelled rotor lead-lag dynamics beyond 10 rad/s makes it impractical to widen the frequency range of W_1 .

With only 4 plant inputs, no attempt was made to control directly the extra rate outputs p and q at low frequencies, but second order band-pass filters were used on each of these variables to reject disturbances and cross coupling effects in the frequency range 4 to 7 rad/s.

Selection of $W_2(s)$

First order high pass filters are used with a cut off frequency of 10 rad/s to limit the system bandwidth and also to limit the magnitudes of the poles of the controller. The actuator penalty weight W_2 is increased at higher frequencies in order to inhibit fast actuator movement. This is done in order that any digital implementation of the controller will not be an issue. Simulations showed that allowing the actuators to be unconstrained resulted in the controller having some poles situated in the extreme left half plane, resulting in a very large dynamic range over which the controller must operate. This is clearly not physically realizable.

A low frequency gain of -100db was used so that W_1 clearly dominates at low frequencies.

Selection of $W_3(s)$

The diagonal weighting function W_3 was chosen to be a constant matrix with a weight of 0.1 on each of the (fictitious) rate input demands and 1 on each of the remaining inputs. The reduced weighting on the rates (which are not directly controlled) is chosen so that some disturbance rejection is obtained on these outputs without them significantly affecting the cost function (6.1). That is the primary aim of W_3 is to force good tracking in $\dot{h}$, $\dot{\psi}$, $\dot{\theta}$ and $\dot{\phi}$.

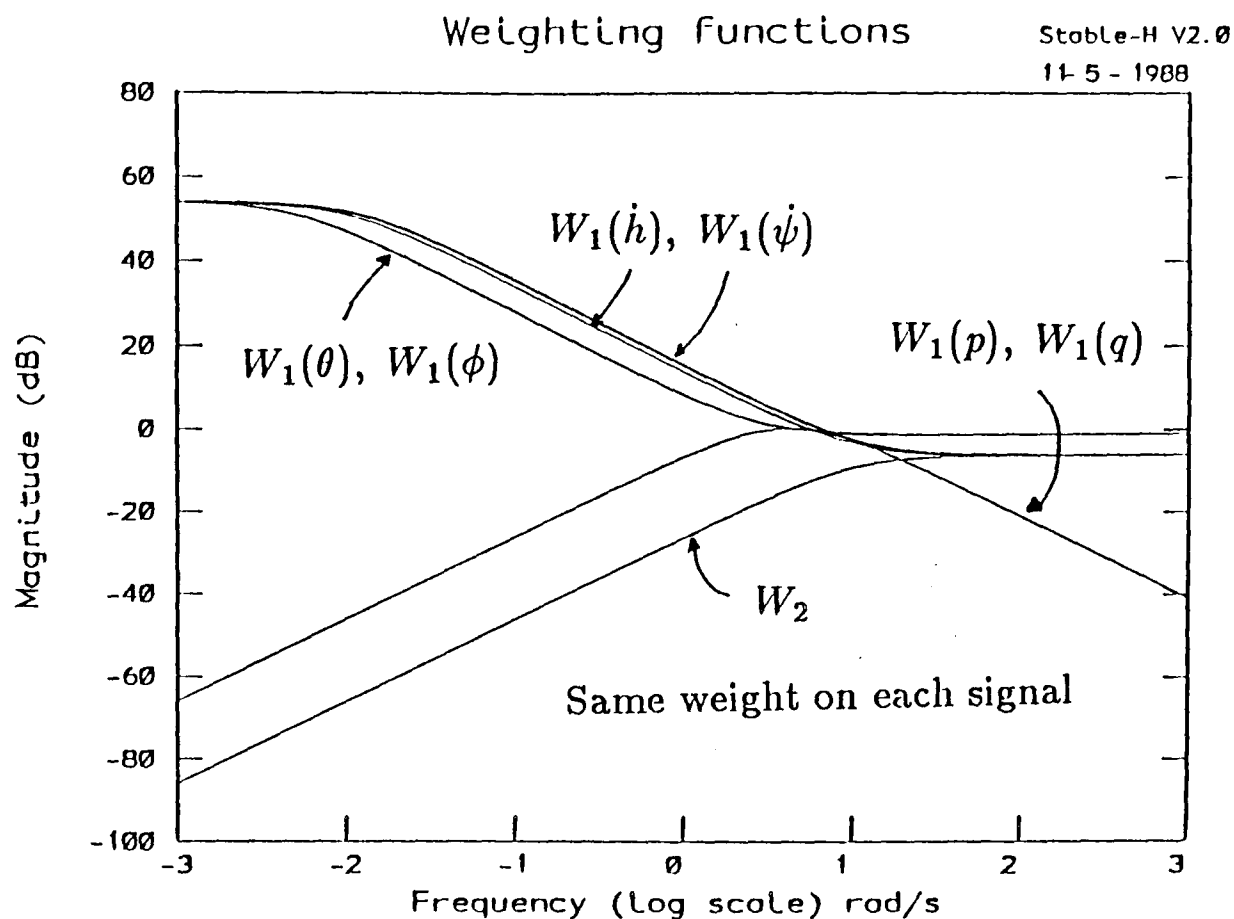
The design weights W_1 , W_2 , and W_3 are given by

$$W_1 = \text{diag} \left\{ 0.5 \frac{(s+12)}{(s+0.012)}, \frac{2s}{(s+4)(s+4.5)}, \frac{2s}{(s+4)(s+4.5)}, \right. \\ \left. 0.5 \frac{(s+10)}{(s+0.01)}, 0.89 \frac{(s+2.81)}{(s+0.005)}, 0.89 \frac{(s+2.81)}{(s+0.005)} \right\}$$

$$W_2 = \text{diag} \left\{ 0.5 \frac{(s+0.0001)}{(s+10)}, 0.5 \frac{(s+0.0001)}{(s+10)}, 0.5 \frac{(s+0.0001)}{(s+10)}, 0.5 \frac{(s+0.0001)}{(s+10)} \right\}$$

$$W_3 = \text{diag}\{1.0, 0.1, 0.1, 1.0, 1.0, 1.0\}$$

W_1 and W_2 are plotted in figure 6.5



6.2.1.2 Analysis of the controller

The design process leads to a controller with as many states as the interconnection structure of figure 6.3. In this case 20 states arising from the following: plant (8) and performance weights (12). The controller initially had 39 states, but balancing and truncation of the small Hankel singular values [TOM3], gives an 18 state controller without any significant change in the singular value plots of the optimal cost function.

The frequency response of the controller is shown in figure 6.6. It can be seen that the controller has high gain at low frequency for good tracking and low gain at high frequency for robustness. This is consistent with the specification of the performance weight $W_1(s)$. An interesting characteristic of H^∞ -controllers is that they are always proper (for well defined problems) and so there is at least one singular value of the controller which remains constant at high frequency.

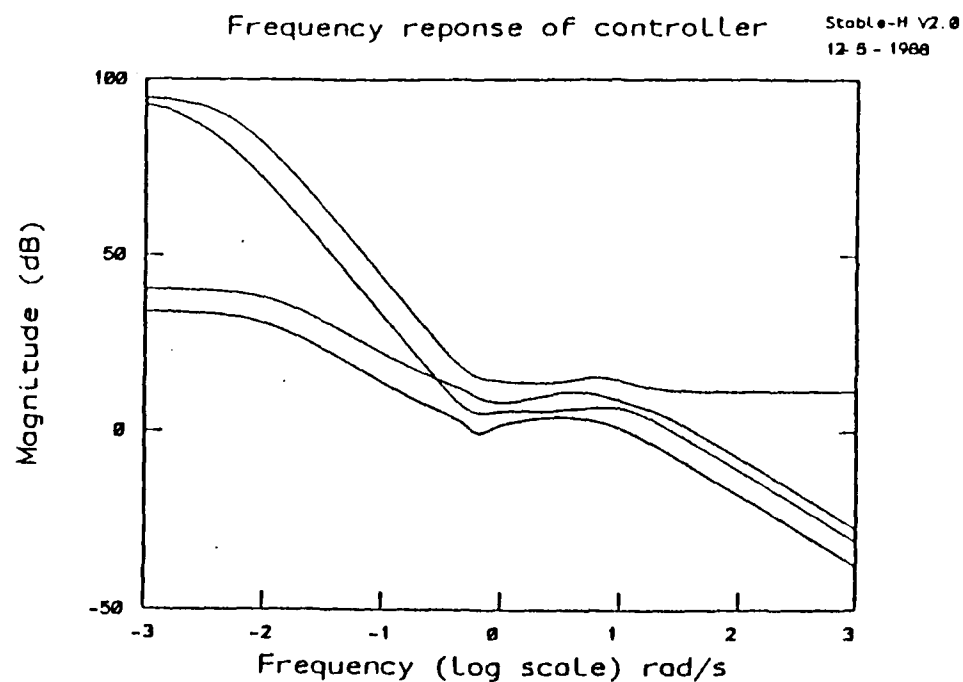


Figure 6.6 Singular values of controller

It is interesting to note that the controller has a condition number which is of the same order as the plant. * This would seem to suggest that to achieve good closed loop tracking performance in every channel for an ill conditioned plant at low frequency, requires a corresponding controller which is also ill conditioned at low frequency. This seems intuitively correct because if there is a large separation in the singular values at low frequency, some very small and some large, then in order to achieve good closed-loop tracking in every loop, requires an increase of gain in each loop proportionally. Therefore the gain in those loops which have a very small DC value will need to be increased proportionally greater than for those loops that have better DC gain. This is precisely what is indicated by the singular value plot of the controller.

Now if one looks at the corresponding loop transfer function GK (figure 6.7), then it can be seen that the controller has been successful in achieving high gain in each of the four controlled loops for good tracking and that the singular values are much closer together than for the open loop plant, which indicates that the compensated system is numerically better conditioned. Therefore although the open loop plant and controller are poorly

* It is important to stress here that the conditioning of the controller is a result of demanding equal gains in each loop and is not a result of plant inversion.

conditioned, the compensated system is not. Of course the conditioning of the controller is a direct result of requiring good tracking in every channel. A better conditioned controller is achieved at the expense of performance.

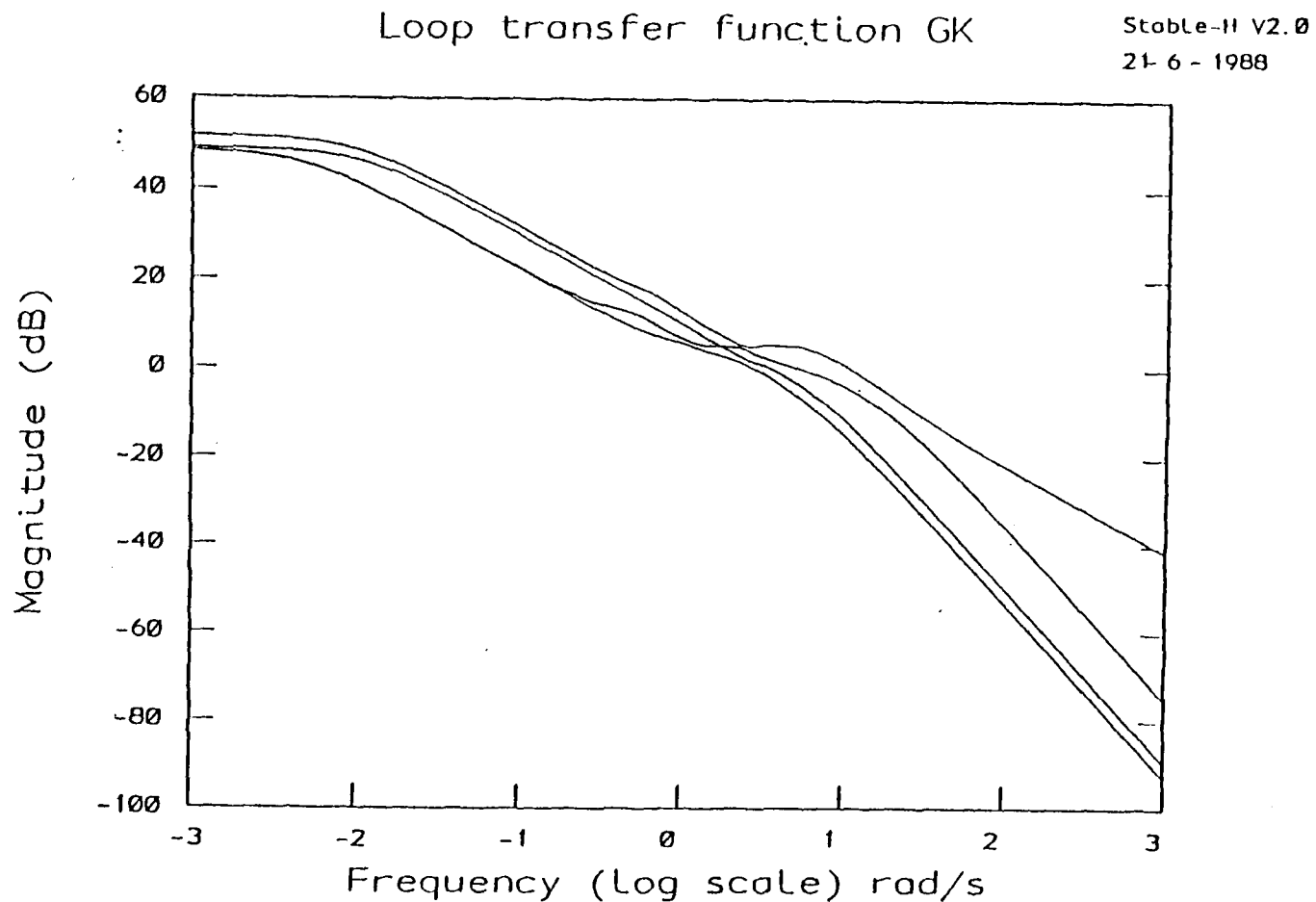


Figure 6.7 Singular values of compensated system

The following sub-sections will summarize the analyses of the helicopter control system and will concentrate on linear analysis, although extensive nonlinear simulations was used to verify the control laws.

6.2.1.3 Sensitivity

A plot of the singular values of $(I + GK)^{-1}$ is shown in figure 6.8. For disturbance attenuation at the plant output and for tracking performance this function is required to be small over the operating bandwidth. The four singular values which are small at low frequency correspond to the controlled outputs and the other two are a result of the inherent interaction between the aircraft attitudes and angular rates.

The largest singular value of $(I + GK)^{-1}$ also determines the size of the smallest unstructured output inverse multiplicative perturbation $\tilde{\delta}_o$ that could destabilize the system (§3.3.2). For good stability margins at the plant output $\bar{\sigma}[(I + GK)^{-1}]$ should be less than 1. From figure 6.8 it can be seen that $\bar{\sigma}[(I + GK)^{-1}] = 6 \text{ dB} = 1/\tilde{\delta}_o$, at $\omega = 0.75$ rad/s. This implies that the smallest perturbation (within the class given by equation 3.11) that will destabilize the system = 0.5. Using the formulae given by equations (3.26) and (3.27) this value of $\tilde{\delta}_o$ guarantees a gain margin of $\left(\frac{1}{1-\tilde{\delta}_o}, \frac{1}{1+\tilde{\delta}_o}\right)$ and a phase margin of $\pm \cos^{-1} \left(1 - \frac{\tilde{\delta}_o^2}{2}\right)$. Thus there is a guaranteed gain margin of (2.00, 0.67) and a phase margin of $\pm 29^\circ$ in every output channel.

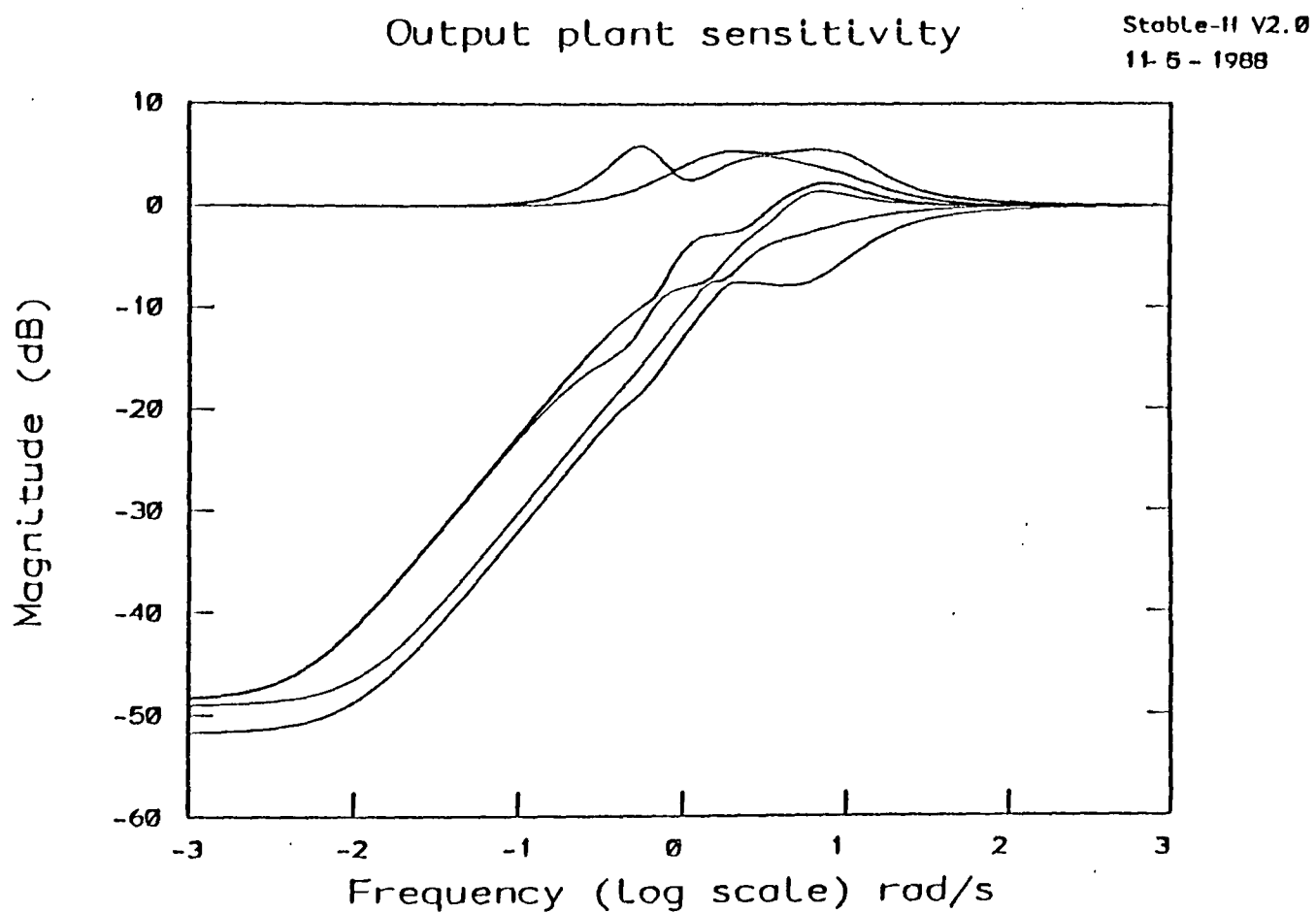


Figure 6.8 Singular values of $(I + GK)^{-1}$

However one normally associates gain and phase margins at the input to the plant. These can be determined by considering the largest singular value of $(I + KG)^{-1}$ which can also be used to calculate the smallest unstructured input inverse multiplicative perturbation $\tilde{\delta}_i$ that could destabilize the system (§3.3.2). A plot of the singular values of $(I + KG)^{-1}$ is

shown in figure 6.9. From this figure it can be seen that $\bar{\sigma}[(I + KG)^{-1}] = 4$ dB at $\omega = 0.8$ rad/s. This implies that the smallest perturbation (within the class given by equation 3.10) that will destabilize the system is 0.63. This gives rise to a guaranteed gain margin of (2.71, 0.61) and phase margin of $\pm 36.8^\circ$.

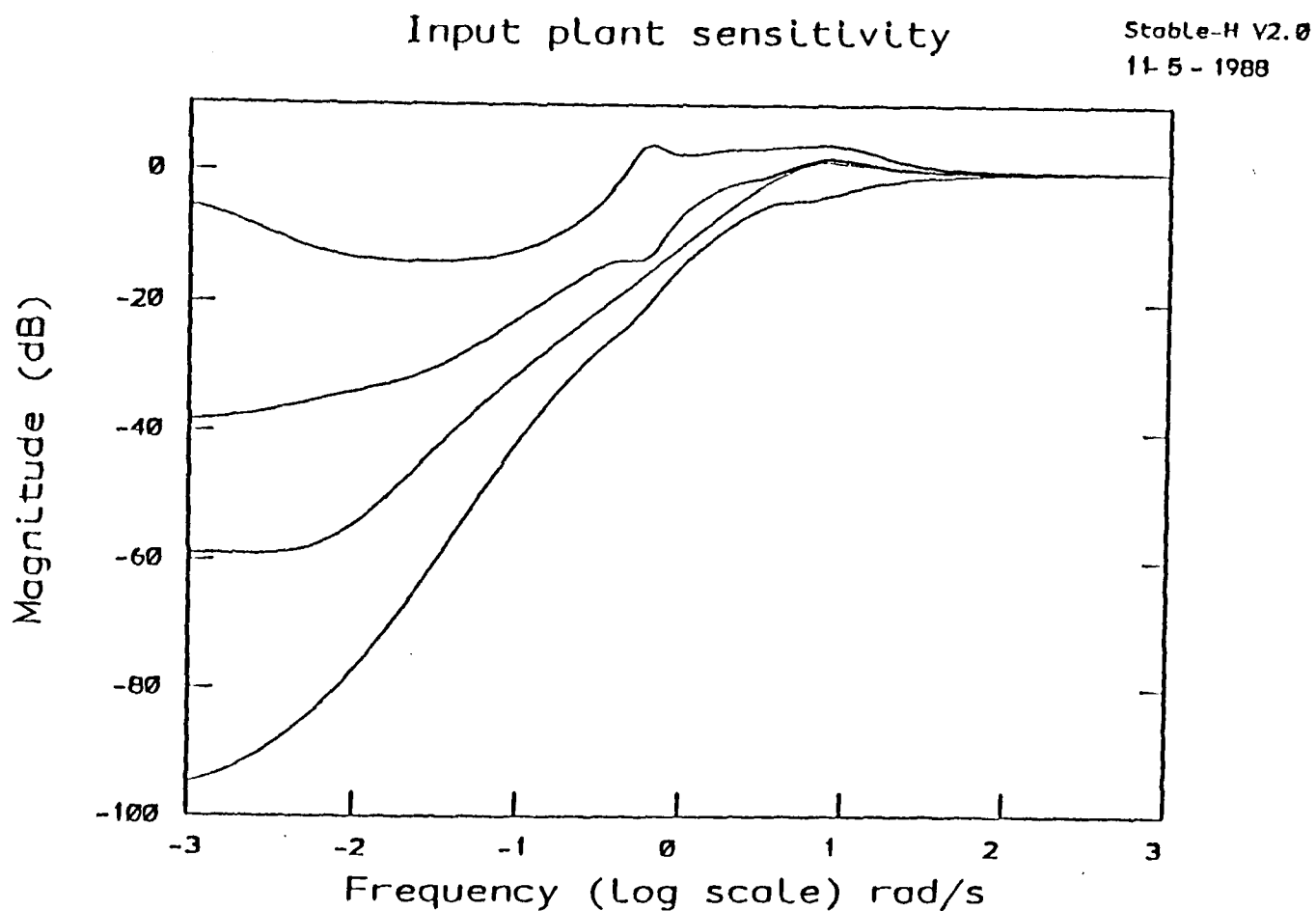


Figure 6.9 Singular values of $(I + KG)^{-1}$

At first sight the gain margins given above seem to be poor, this is because the gain margins calculated using singular values are generally conservative. Any peaks in the sensitivity function will cause $(I + GK)^{-1}$ or $(I + KG)^{-1}$ to be greater than 1. It is hard to remove these peaks without losing performance and is a consequence of the system being non-minimum phase. For non-minimum phase systems it is not possible to attain good sensitivity at some frequencies without the sensitivity being poor at other frequencies [OYO1].

Another multivariable gain margin can be determined by an examination of the characteristic locii of the open loop compensated system. Recall from chapter 2 that the

Figure 6.10 shows the characteristic loci of $KG(s)$. From this figure it can be seen that the compensated system has the correct number of anti-clockwise encirclements, so that under unity feedback this system is closed loop stable. A closer examination shows that stability is lost if the gain is reduced to 0.36 or if the gain is increased to 6.7, giving an overall gain margin of (6.7,0.36) which suggests that the margins given above using singular value analysis are conservative.

Note however that the characteristic loci margins hold only for simultaneous gain changes in each loop. The Nyquist stability criterion only determines stability with respect to a single gain change common to all loops and gives no indication of robustness with respect to arbitrary simultaneous gain or phase variations in any number of loops. Therefore the gain margin is quite good if one uses its interpretation in the same way as the SISO case. Although this is not a good indicator of stability robustness a robust system must necessarily have good stability margins.

6.2.1.4 Complementary sensitivity

A plot of the singular values of $(I + KG)^{-1}KG$ is shown in figure 6.11. $\bar{\sigma}[(I + KG)^{-1}KG]$ determines the size of the smallest unstructured input multiplicative perturbation that could destabilize the system. Figure 6.11 and equation (3.17) suggest that the closed-loop system is stable for all $\delta_i < 0.45$. The unity gain crossover frequency is 8 rad/s which is within the desired value of 10 rad/s.

Figure 6.12 shows the rotor dynamics modelled as an unstructured input multiplicative perturbation (§3.7) and superimposed is $(\bar{\sigma}[(I + KG)^{-1}KG])^{-1}$. From this figure it can be seen that the robustness test of equation (3.17) is satisfied for all frequencies and therefore the closed-loop system should be stable with the introduction of the extra rotor degrees of freedom.

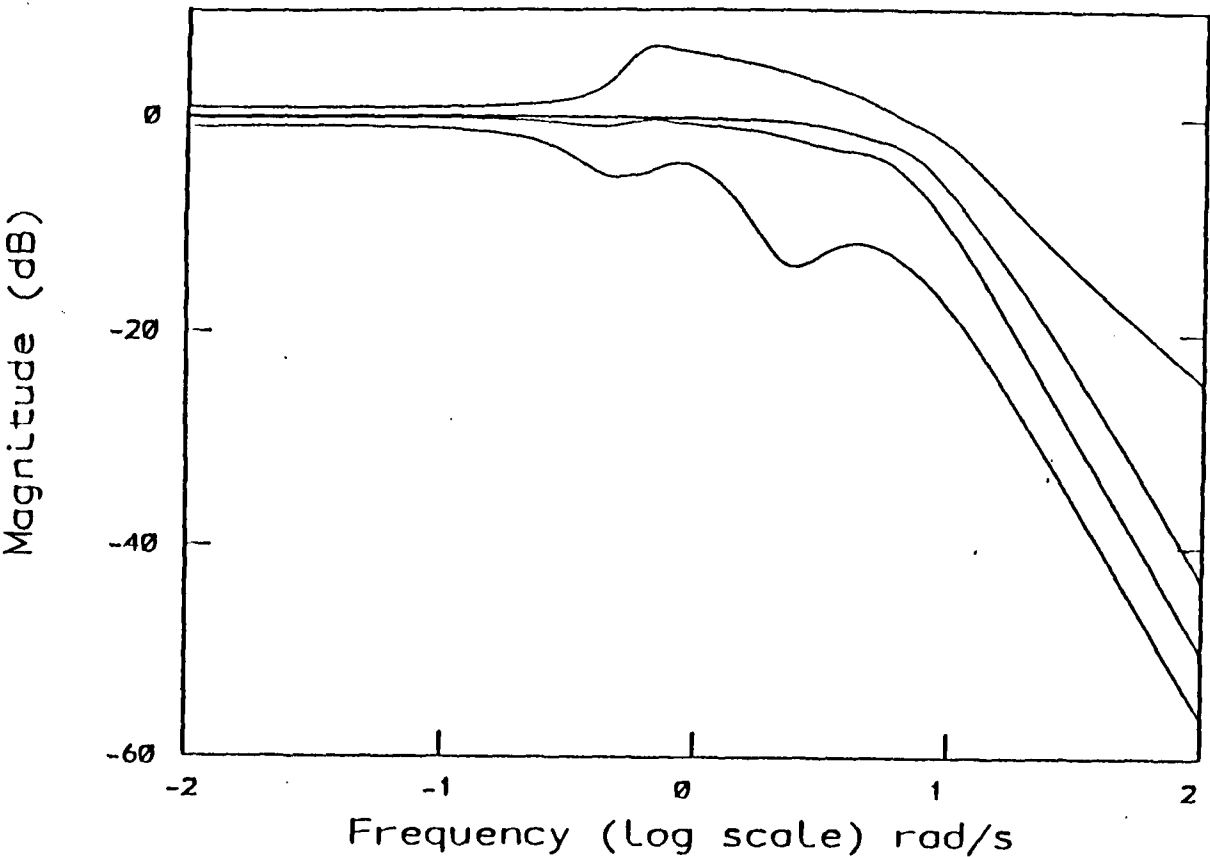


Figure 6.11 Singular values of $(I + KG)^{-1}KG$

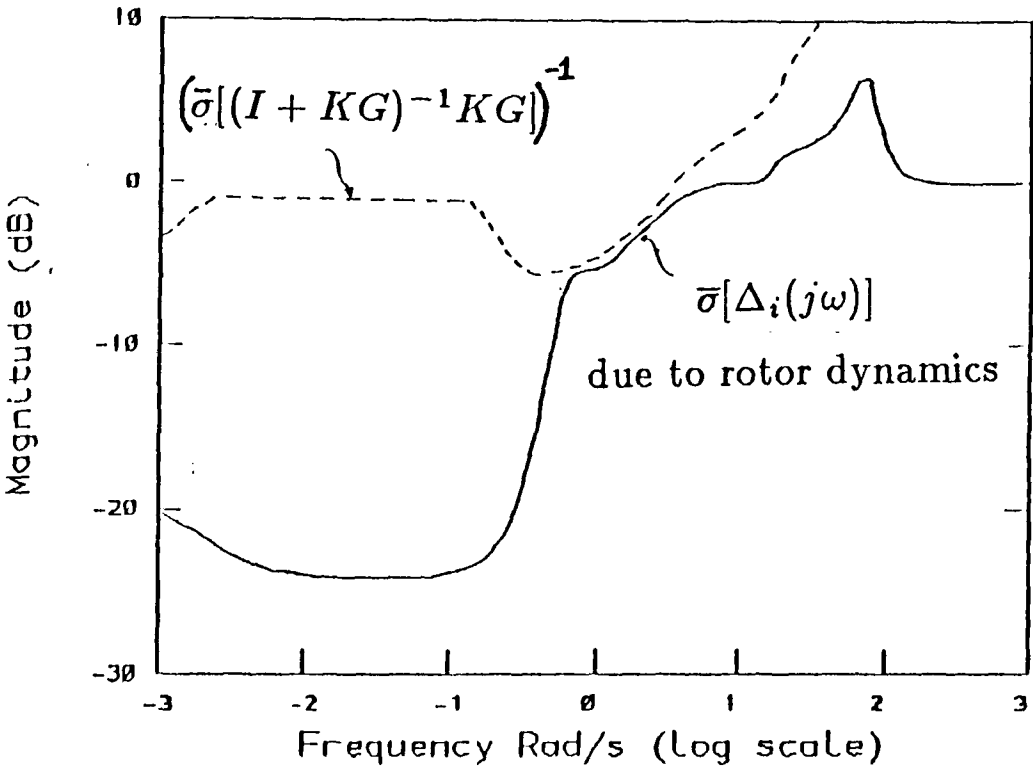


Figure 6.12 Singular value robustness test

For good tracking the closed-loop transfer function $T \triangleq GK(I + GK)^{-1}$ must be approximately 1 at low frequency, but since $T = GKS$ where $S = (I + GK)^{-1}$, this implies that KS must be equal to the inverse of the plant G at low frequency. Figure 6.13 shows the singular values of KS and comparing these with figure 6.4, it is indeed the case.

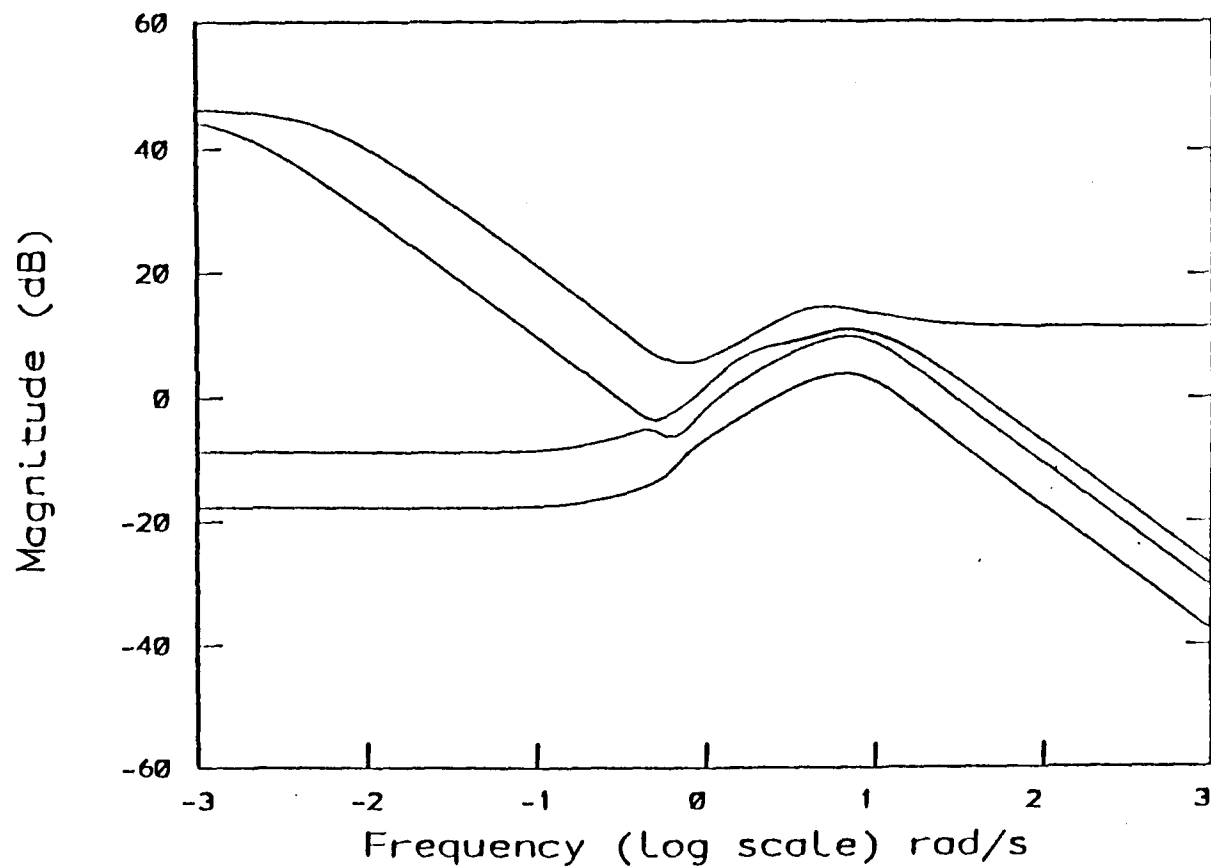


Figure 6.13 Singular values of $K(I + GK)^{-1}$

$\bar{\sigma}[KS(s)]$ is also useful for determining the smallest unstructured additive perturbation that could destabilize the system. In chapter 3 the uncertainty of the helicopter dynamics due to changes in flight condition was represented as an unstructured additive perturbation. From figure 3.9 $\bar{\sigma}[E(s)] = 25$, equation (3.15) therefore suggests that for stability $\bar{\sigma}[KS(s)] = 0.04$, which represents virtually no control movement; ie the most robust control is open-loop control ! This stability robustness test is clearly very conservative and is of no practical use.

6.2.2 Design of H_2 -optimal controller

The H_2 -norm is probably the most popular performance measure in optimal control theory due to its mathematical tractability and ease of solution. In the quadratic minimization approach, exogenous signals are usually assumed to be Gaussian white noise and the

performance requirements are internal stability and minimization of the mean square value of some signal.

The command model part of the two-degree of freedom controller will be designed to achieve closed loop tracking performance. An appropriate performance measure might be to minimize $\|\underline{r} - \underline{y}\|_2$ where $\underline{r}$ is the reference inputs and $\underline{y}$ is the measured outputs. This cost function will have the effect of minimizing the mean square tracking error. Unfortunately this cost function tends to lead to improper controllers and therefore a more sensible cost function is

$$\left(\|\underline{r} - \underline{y}\|_2^2 + \|\rho \underline{u}\|_2^2 \right)^{1/2} \quad (6.2)$$

where ρ is a positive scalar weighting factor which penalizes the control energy. Note that (6.2) is equal to the H_2 norm of $\begin{bmatrix} \underline{r} - \underline{y} \\ \rho \underline{u} \end{bmatrix}$ and an equivalent cost function is that given by

$$\left\| \begin{bmatrix} W_1(I + GK)^{-1}W_3 \\ W_2K(I + GK)^{-1}W_3 \end{bmatrix} \right\|_2 \quad (6.3)$$

This cost function can be interpreted as a constraint on the weighted energy of the sensitivity function $S(s)$ and the controller output. The energy spectrum $\|S(j\omega)\|_2^2$ is weighted by the factor $\|W_1(j\omega)\|_2^2$. So for example if $W_1(j\omega)$ is large on a certain frequency band and small off it then (6.3) would generate a class of signals having their energy concentrated in that band.

The problem addressed here is different from that solved by LQG. In LQG the objective is to optimally reset the states of the plant to a fixed known equilibrium state in the presence of Gaussian white noise [IEEE1]. The task of the controller here is to optimally reset $\underline{y}$ to a new level dictated by $\underline{r}$. Accordingly that part of the cost reflecting tracking accuracy has been imposed directly on $S(j\omega)$ and does not necessarily involve all the state variables although these are kept indirectly within bounds by an appropriate weighting on the actuator outputs.

6.2.2.1 Weighting function selection

It is not surprising that since the cost functions (6.3) and (6.1) are almost identical, the weightings for the solution of (6.1) give similar responses when used in the cost function (6.3). As for the H^∞ -case the desired frequency responses will be in inverse proportion to the magnitudes of the weights.

The philosophy behind the choice of weights is similar to the H^∞ -case. Since there are no constraints on the bandwidth of the command model, this is allowed to increase sufficiently high to meet the required speed of responses given in §2.6. A diagonal matrix of constants was used to minimize the control effort and to ensure a proper controller.

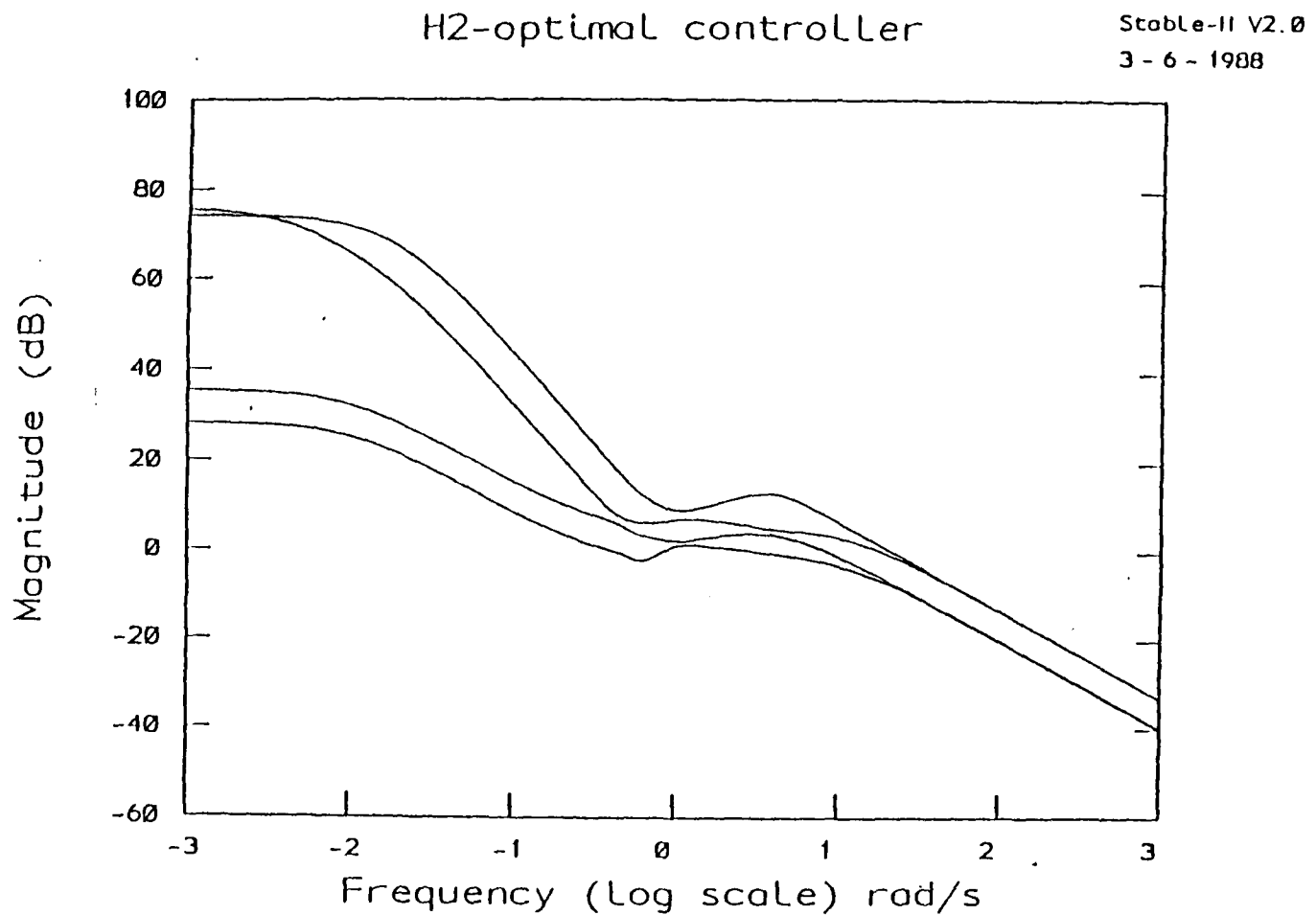
The design weights W_1 , W_2 , and W_3 are given by

$$W_1 = \text{diag} \left\{ 1.58 \frac{(s + 3.16)}{(s + 0.01)}, \frac{2s}{(s + 4)(s + 4.5)}, \frac{2s}{(s + 4)(s + 4.5)}, \right. \\ \left. 1.58 \frac{(s + 3.16)}{(s + 0.01)}, 2.81 \frac{(s + 3.56)}{(s + 0.02)}, 2.81 \frac{(s + 3.56)}{(s + 0.02)} \right\}$$

$$W_2 = \text{diag}\{0.5, 0.5, 0.5, 0.5\}$$

$$W_3 = \text{diag}\{1.0, 0.1, 0.1, 1.0, 1.0, 1.0\}$$

In contrast to H^∞ , the H_2 -optimization problem is much easier to solve, since there is no need for an iterative search to find the optimal controller (§4.3.2). The optimal controller is of the same dimension as the plant plus dynamic weights. It was possible to further reduce the dimension of the controller from 16 states to 13 states using the minimal realization routine of [TOM3]. The H_2 -optimal controller is shown in figure 6.14. Note that this controller is strictly proper in contrast to the H^∞ controller.



There is no state inflation involved in solving the H_2 -problem, this is in contrast to the H^∞ -case. One naturally asks then is there any advantage of using H^∞ -optimization? If the exogenous signals are fixed and have a known fixed power spectrum, then minimization with respect to the H_2 -norm is better, in such situations the use of the H^∞ -norm leads to conservative results. However, there are many practical problems in which the exogenous signals are unknown but belong to prescribed sets, problems involving such general disturbance sets cannot easily be handled in the H_2 -setting.

6.2.2.2 Time simulations

A linear simulation of the nominal closed-loop system command responses with the H_2 -controller are shown in figures 6.15-6.18. In each case the input was a unit step command. From these figures it can be seen that interaction is suppressed and the rise time of the responses are in the region of 1.5 secs, consistent with the requirements shown in table 2.4.

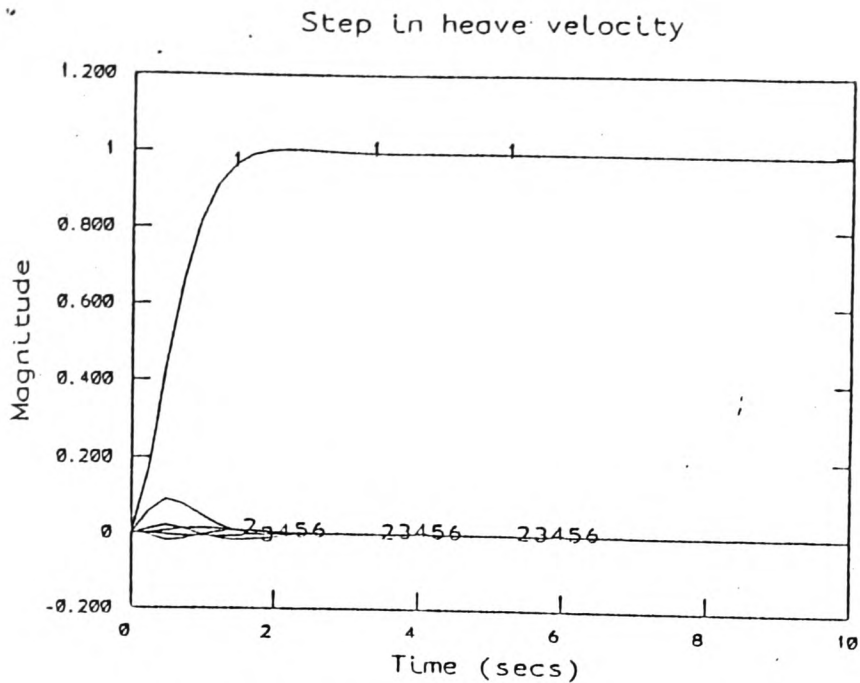


Figure 6.15 Step command in main rotor collective

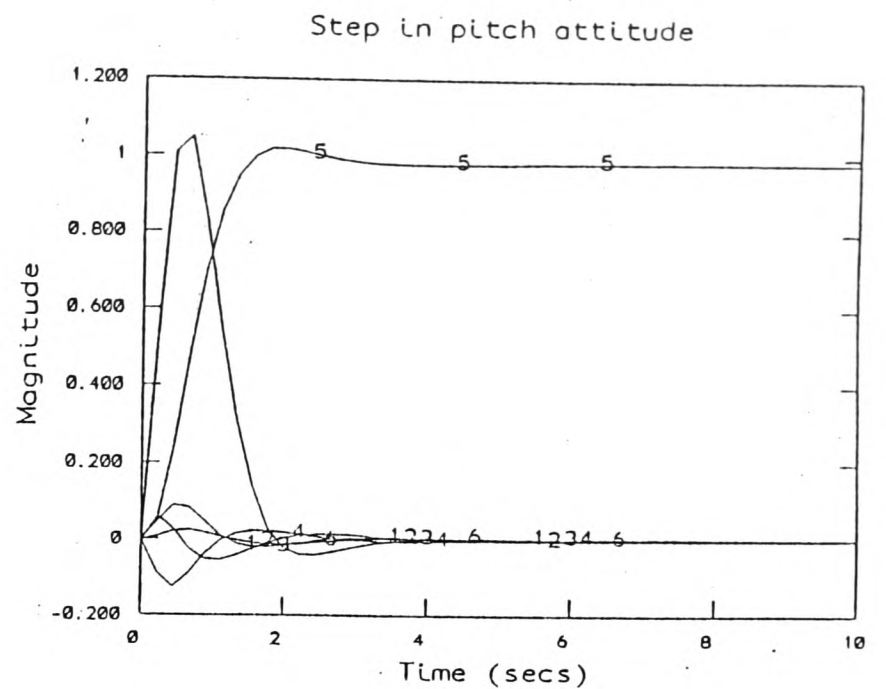


Figure 6.16 Step command in longitudinal cyclic

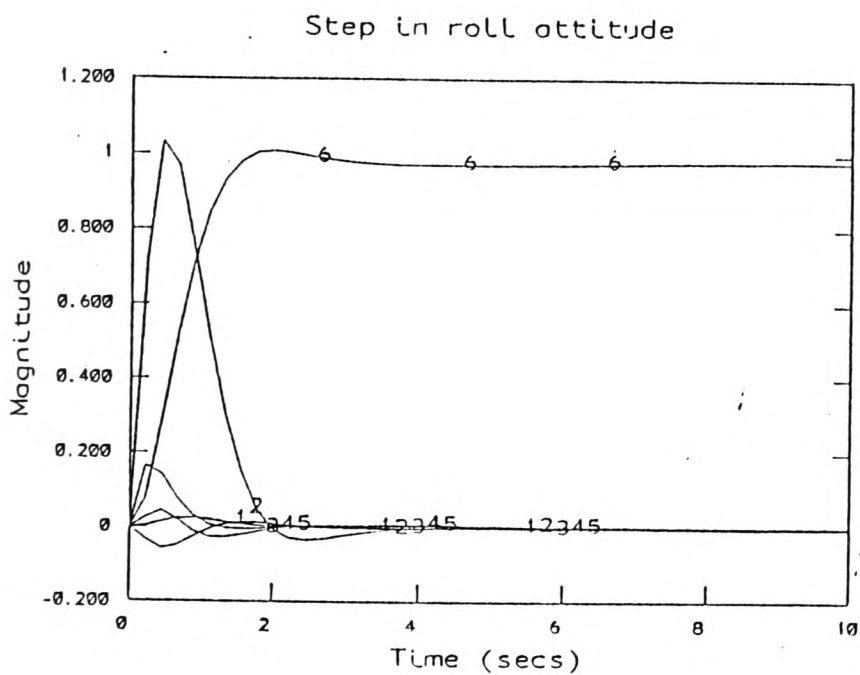


Figure 6.17 Step command in lateral cyclic

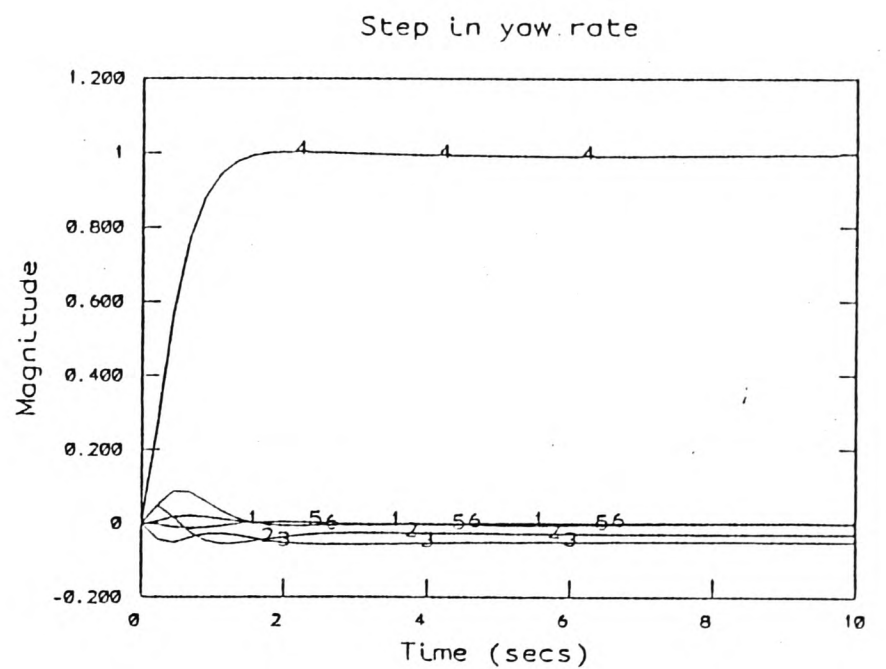


Figure 6.18 Step command in tail rotor collective

The H_2 -controller was then imbedded into the command model part of a two-degree of freedom controller to produce the transfer functions T_1^* and T_2^* as discussed in §4.6. The H^∞ -controller designed in §6.2.1 is used in the stabilization network. This two-degree of freedom controller has a state dimension of 34 after some state reduction using truncated balanced realizations.

The two-degree of freedom controller was then simulated in the TSIM environment [TSI1], using the nonlinear model HELISIM [PAD1], provided by RAE, Bedford. This nonlinear model contained rotor flapping dynamics and actuator dynamics which were initially omitted from the nominal linearized description of the plant. The following effects were observed as a result of including rotor tip path plane dynamics into the helicopter model.

- (i) The time constant associated with the roll subsidence mode is increased.
- (ii) Neither the Dutch roll mode nor the Phugoid is changed significantly.
- (iii) Longitudinal and lateral flapping contribute significantly to the pitch rate and roll rate equations respectively.
- (iv) The aircraft is now very sensitive to roll gain.

Models of the actuators included blade angle deflection and rate limits. The following values were used in the simulation.

Blade angle limits:

$$-5^\circ < \text{collective blade angle} < 20.3^\circ$$

$$-15.7^\circ < \text{longitudinal cyclic blade angle} < 7.5^\circ$$

$$-7.5^\circ < \text{lateral cyclic blade angle} < 7.5^\circ$$

$$-8.5^\circ < \text{tail rotor blade angle} < 33.5^\circ$$

Blade angle rates:

$$-24^\circ/s < \text{collective blade rate} < 24^\circ/s$$

$$-24^\circ/s < \text{longitudinal cyclic blade rate} < 24^\circ/s$$

$$-16^\circ/s < \text{lateral cyclic blade rate} < 16^\circ/s$$

$$-70^{\circ}/s < \text{tail rotor blade rate} < 70^{\circ}/s$$

No detailed models of the actuators were available, the above values were taken as being typically representative of an helicopter actuator blade system.

Figures 6.19-6.22 show the corresponding nonlinear time responses to the four pilot inputs together with the actuator blade responses. These time responses are not significantly different from the linear responses even with the addition of the extra rotor dynamics which indicates the robustness of the design to these unmodelled dynamics.

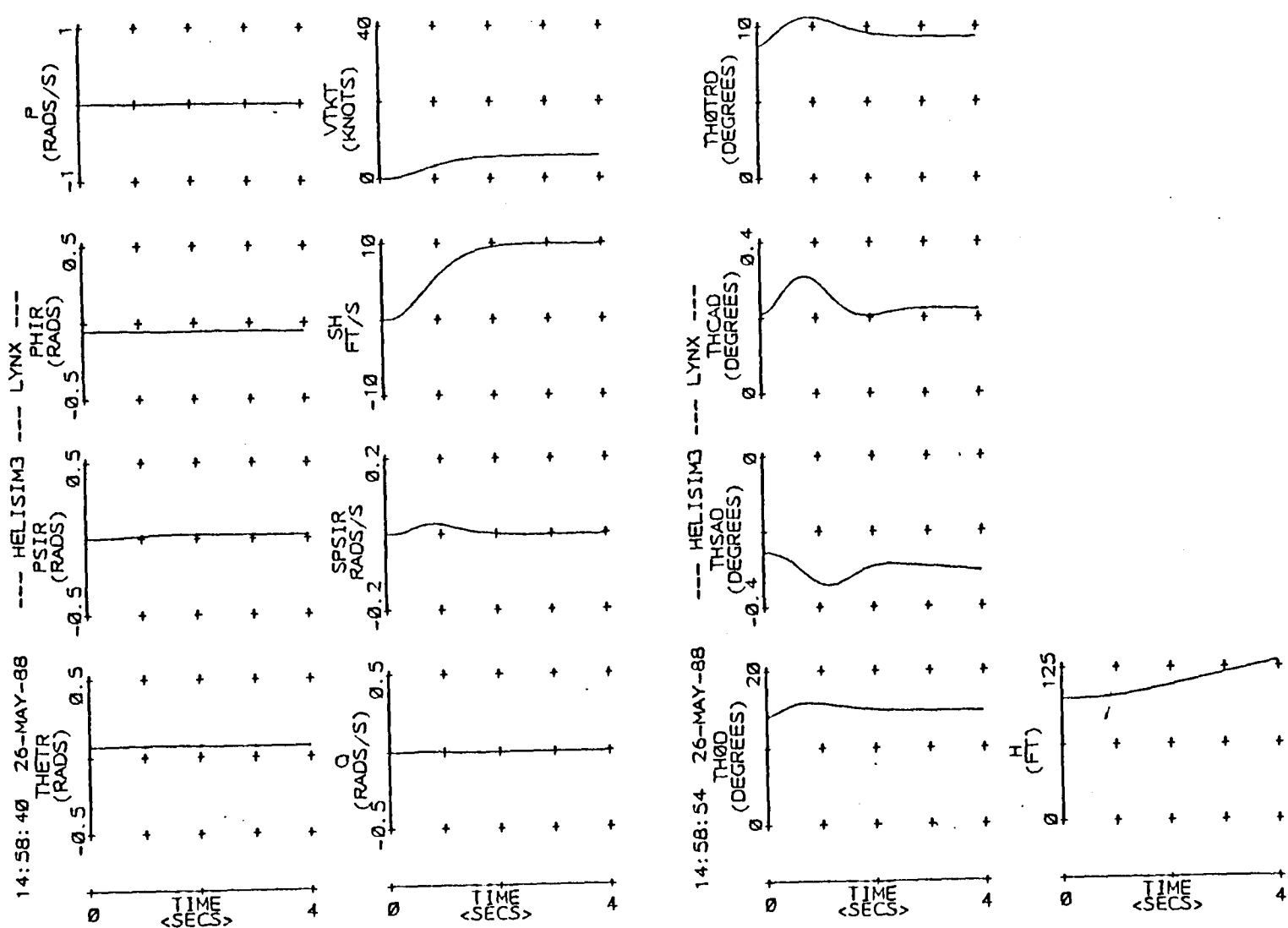


Figure 6.19 Step command of 10 ft/s in heave velocity

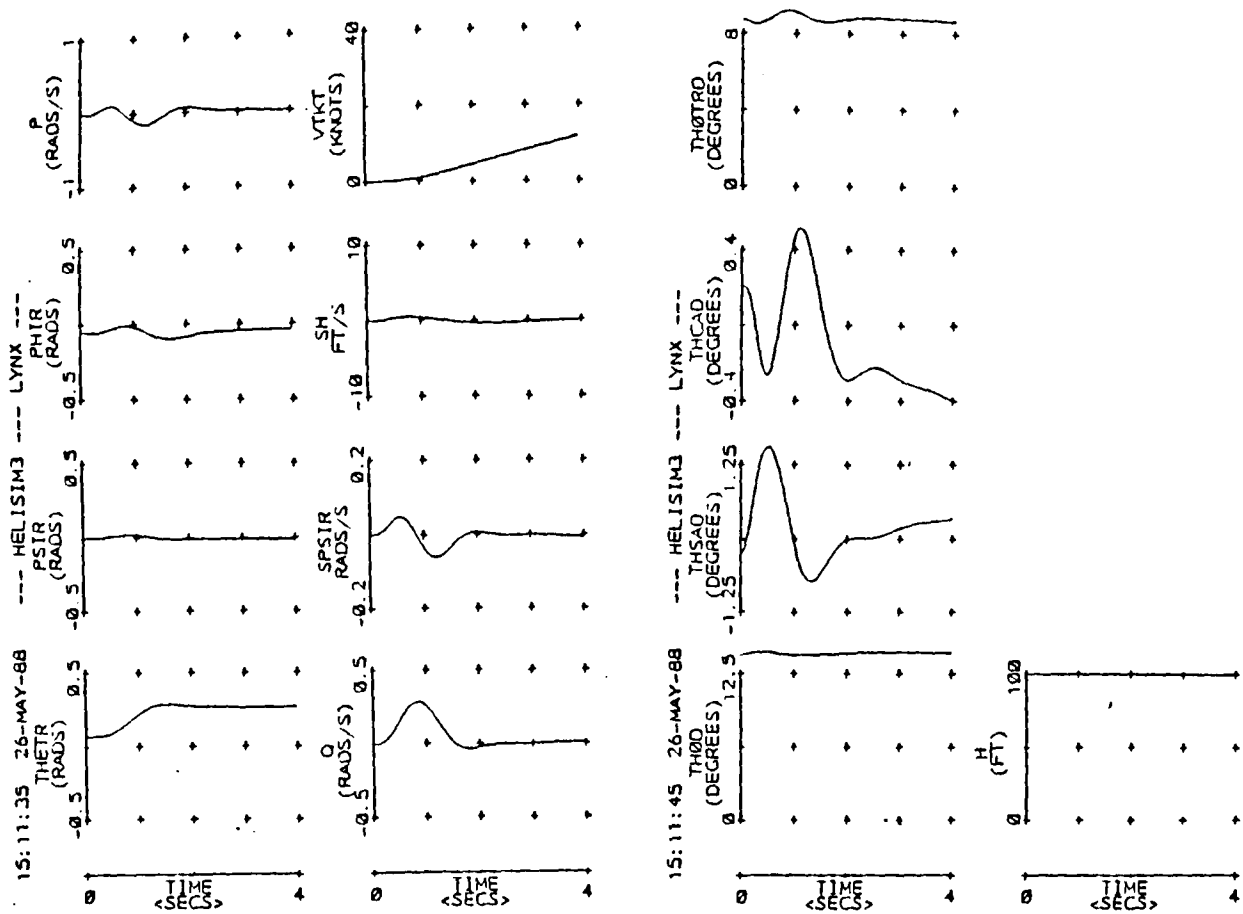


Figure 6.20 Step command of 0.2 rads in pitch attitude

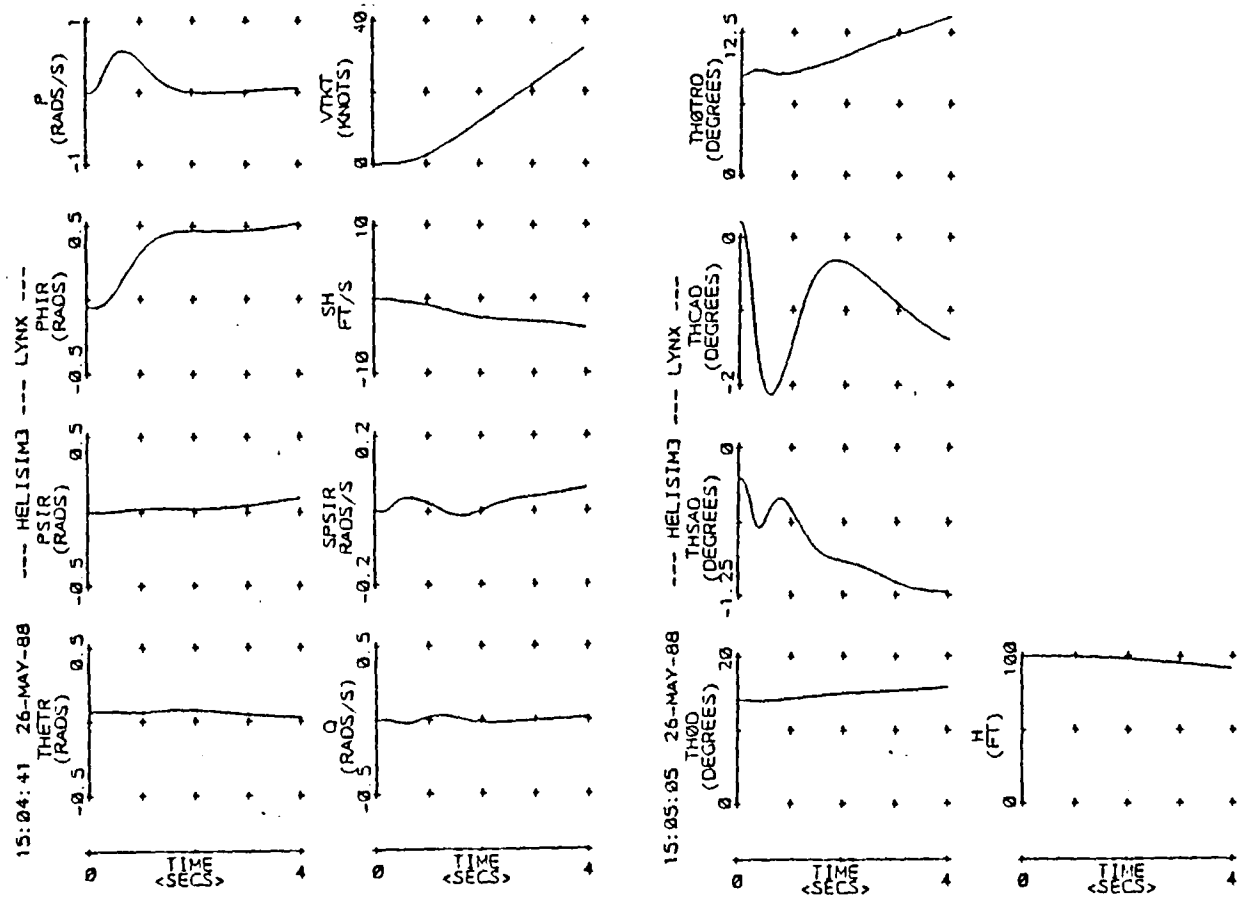


Figure 6.21 Step command of 0.5 rads in roll attitude

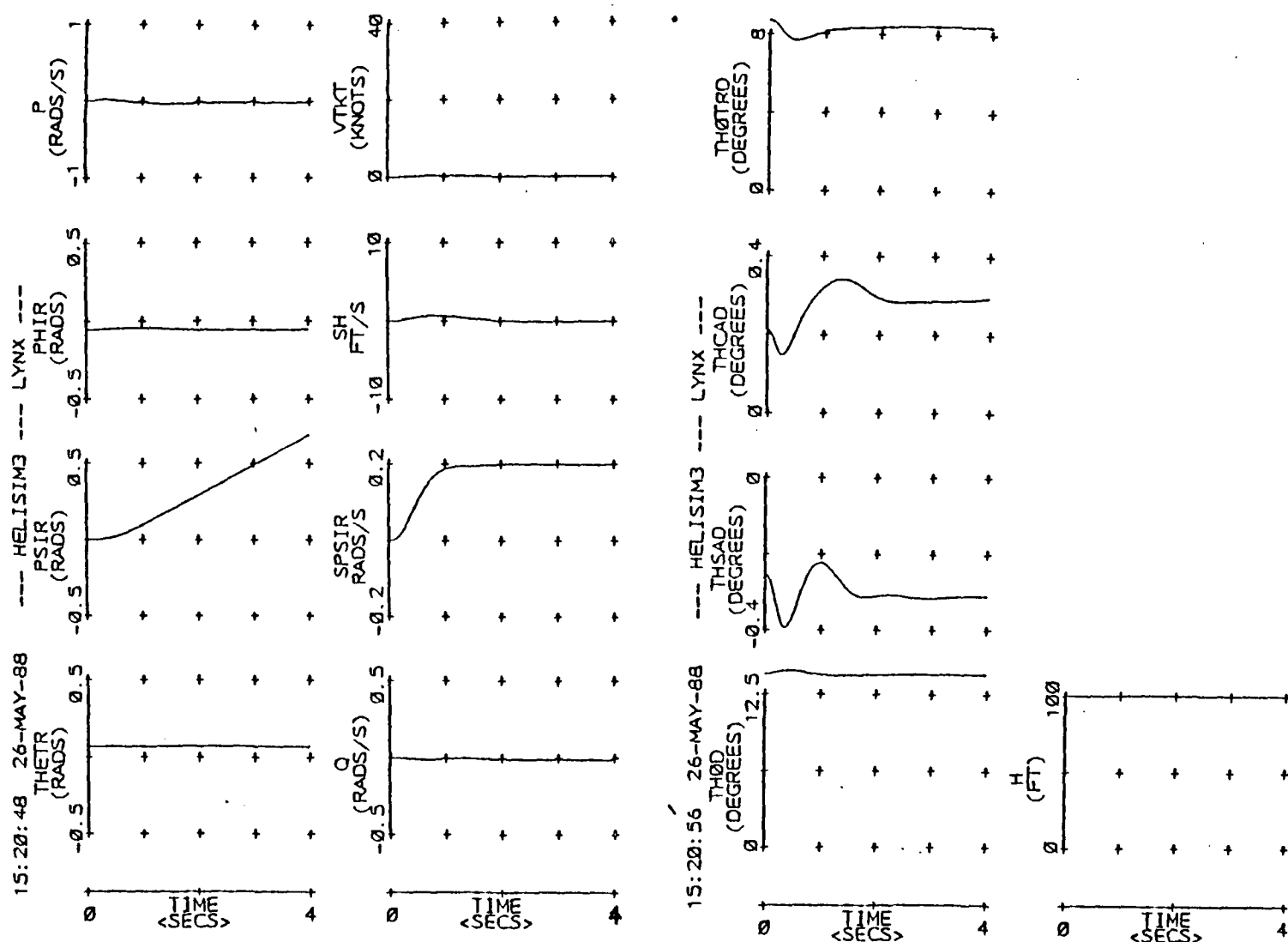


Figure 6.22 Step command of 0.2 rad/s in heading rate

The greatest concern is the change in dynamics of the helicopter with manoeuvring flight. Recall that the nominal design model is linearized about the hover flight condition and therefore any change in the characteristics of the helicopter with speed represents unmodelled dynamics. This is seen quite readily in figure 6.21, which shows a step command of 30° of bank angle. As the helicopter banks, the change in the thrust vector causes the helicopter to have a resultant lateral velocity. The change in speed from 0 to 40 knots represents a considerable change in the dynamic characteristics of the helicopter.

The difficulty experienced by the pilot in achieving coordinated control is typically illustrated by considering the helicopters' response to a step command in longitudinal cyclic. To achieve a transition from hover to forward flight requires forward cyclic with a corresponding increase in collective or height is lost due to the change in the thrust vector. As

more collective is applied there needs to be a change in pedal position to offset the extra torque by the main rotor. As the helicopter starts to move there is an asymmetric increase in the blade lift (greater on the advancing side of the disc) so the helicopter tends to pitch nose up and possibly roll to one side. This needs further forward cyclic and lateral cyclic to compensate for nose up tilting and rolling. As the helicopter gains speed, lift and drag from the fuselage and tail will reduce engine torque effects which requires further pedal action. It is not difficult to see then why pilots find helicopters fatiguing to fly without some form of augmentation and how easy it is to introduce pilot induced oscillations due to over correcting.

No actuator limits were reached with the manoeuvres performed, indicating that the design has been successful in limiting the actuator power needed. However if these limits are reduced, to force limiting to occur, then instability results due to integral wind-up. To offset this, some form of anti-wind up device is necessary. It is hard to encapsulate this nonlinearity in the design of the compensator so that performance is maintained even when actuator limits are reached. The best that can be achieved is to constrain the actuator power needed to perform a manoeuvre but it is accepted this may lead to a possible deterioration in performance.

6.3 Forward Flight Controller

In the forward flight envelope, the preferred control system is a rate command system. However, it has been pointed out in [TOM1] that there are difficulties in controlling rates due to the presence of transmission zeros at the origin. It is therefore proposed to indirectly control rates by using an attitude command system as for the hover/low speed controller. By making the speed of response sufficiently fast this should meet the desired requirement of manoeuvrability. Another advantage of adopting the same controlled outputs throughout the flight envelope is that the “blending” of the high and low speed

controllers should be much simpler. This overcomes the need for a transition region, where there is a switching of attitude to rate control and vice-versa.

The feedback configuration shown in figure 6.23 will be used to generate a 4-block H^∞ design problem. The corresponding equations for the SCC of figure 6.23, assuming unit weighting functions, are given by

$$\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{y} \end{bmatrix} = \begin{bmatrix} I & 0 & \vdots & G \\ 0 & 0 & \vdots & I \\ -I & -I & \vdots & -G \end{bmatrix} \begin{bmatrix} \underline{r} \\ \underline{n} \\ \underline{u} \end{bmatrix} \quad (6.4)$$

The performance criterion is

$$\left\| \begin{bmatrix} W_1(I + GK)^{-1}W_3 & W_1GK(I + GK)^{-1}W_3 \\ W_2K(I + GK)^{-1}W_4 & W_2K(I + GK)^{-1}W_4 \end{bmatrix} \right\|_\infty \quad (6.5)$$

where W_1 , W_2 , W_3 and W_4 are weighting functions to be chosen.

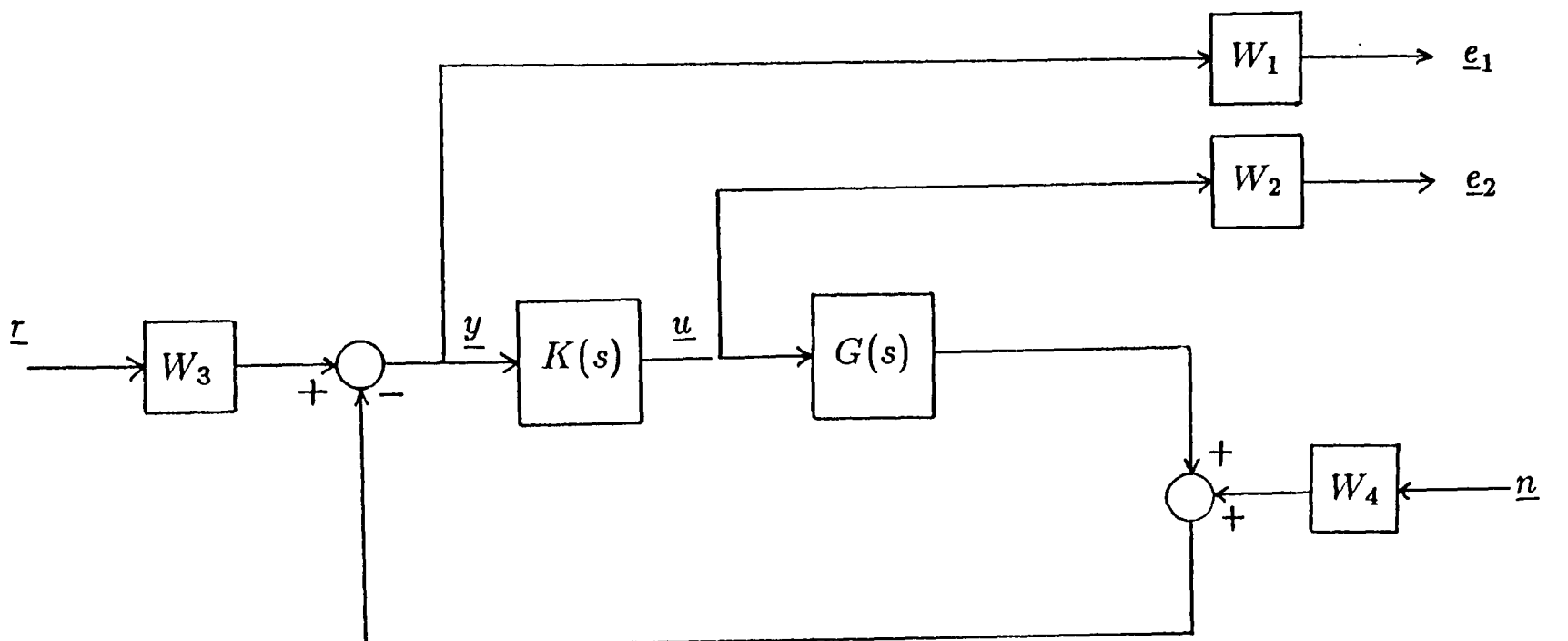


Figure 6.23 4-block configuration

The configuration in figure 6.23 is similar to figure 6.2, except that there is an extra external input $\underline{n}$ which is used to model sensor noise. This is a 4-block H^∞ -design problem, since

there are two error signals and two external input signals but only one actuator input and one measurement signal. This configuration was chosen to represent the simplest 4-block problem which is well posed.

The linearized model of the helicopter was taken at 80 knts straight and level flight. The scaling of the outputs was the same as for the hover/low speed controller. The open-loop frequency response of the linearized helicopter model at 80 knts is shown in figure 6.24.

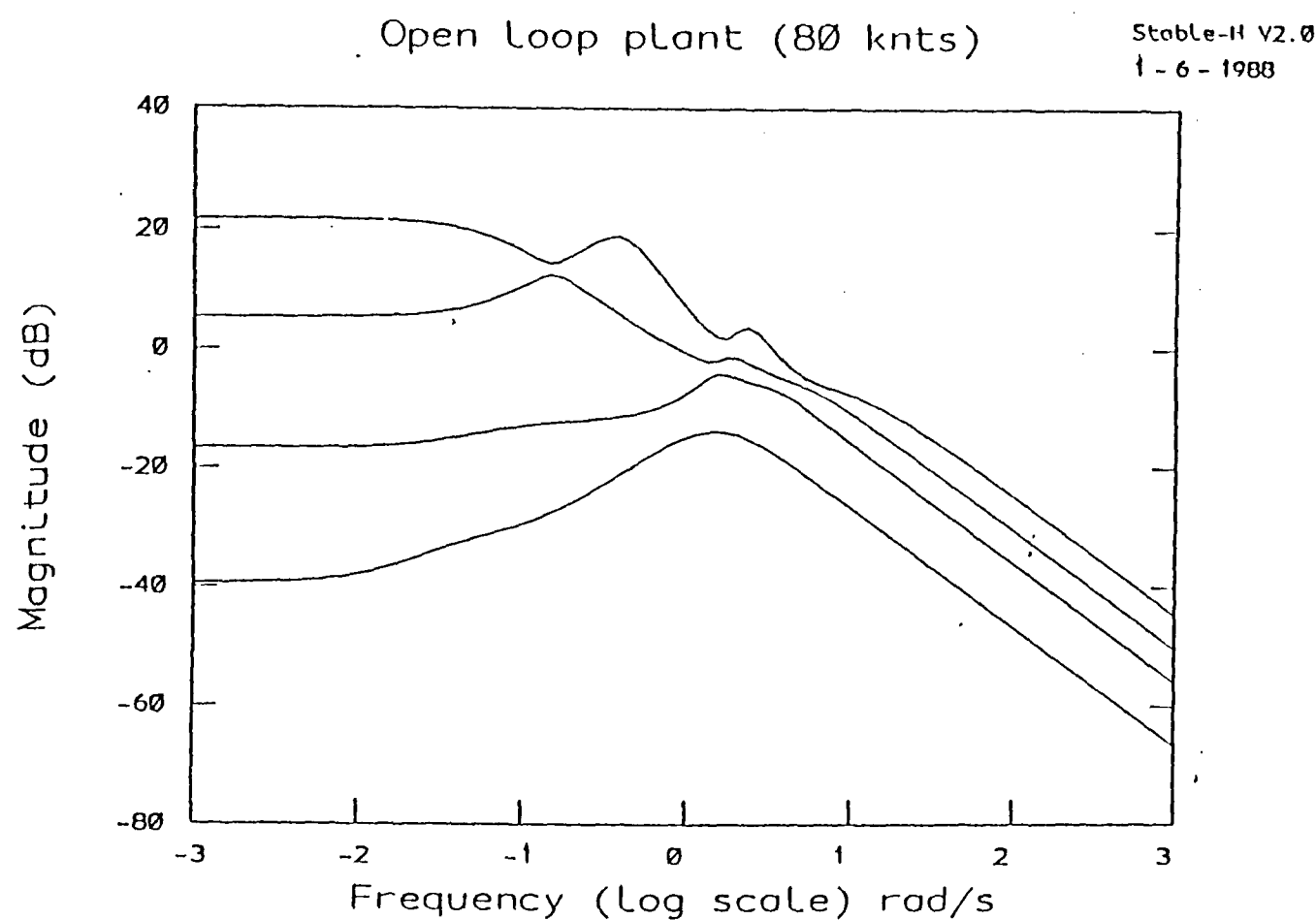


Figure 6.24 Singular value plot of uncompensated helicopter at 80 knts

Comparing figures 6.24 and 6.4 the most significant difference between the helicopter models is at low frequency. For the helicopter at 80 knts there is a much more even distribution of the singular values with each being spaced at approximately 20 dB apart, whereas in the hover position there are two singular values at less than -40 dB and two greater than 10 dB. Therefore one would expect control of the helicopter at 80 knts to be easier than

for the hover case, there being more gain in each loop at low frequency. The helicopter at 80 knts has also a slighter higher bandwidth and therefore responses to pilot inputs should be faster than for the hover case.

6.3.1 Weighting function selection

The weights are obtained iteratively in a manner similar to the two-block case. The procedure is to find input weights W_i and output weights W_o such that $\|W_o F_\ell(P, K) W_i\|_\infty \approx 1$. The weights are best chosen from an input-output point of view. The weights W_i are chosen to represent the shape of the external inputs and the weights W_o are chosen to reflect the desired output response.

The weights W_1 and W_2 are chosen in exactly the same way as for the 2-block case. W_3 is chosen to model the reference inputs and therefore a value of 1 is used for each of the four pilot inputs. W_4 is used to model high frequency sensor noise and therefore a high pass filter with a gain of 0.001 above 20 rad/s is used to characterize sensor noise.

The design weights W_1 , W_2 , W_3 and W_4 are

$$W_1 = \text{diag} \left\{ 0.89 \frac{(s + 5.62)}{(s + 0.01)}, \frac{s}{(s + 5)(s + 5.5)}, \frac{s}{(s + 5)(s + 5.5)}, \right. \\ \left. 0.89 \frac{(s + 2.81)}{(s + 0.005)}, 0.89 \frac{(s + 2.81)}{(s + 0.005)}, 0.89 \frac{(s + 2.81)}{(s + 0.005)} \right\}$$

$$W_2 = \text{diag} \left\{ 0.5 \frac{(s + 0.0001)}{(s + 10)}, 0.5 \frac{(s + 0.0001)}{(s + 10)}, 0.5 \frac{(s + 0.0001)}{(s + 10)}, 0.5 \frac{(s + 0.0001)}{(s + 10)} \right\}$$

$$W_3 = \text{diag}\{1.0, 0.1, 0.1, 1.0, 1.0, 1.0\}$$

$$W_4 = \text{diag} \left\{ 0.001 \frac{s}{(s + 20)}, 0.001 \frac{s}{(s + 20)}, 0.001 \frac{s}{(s + 20)}, 0.001 \frac{s}{(s + 20)} \right\}$$

W_1 , W_2 and W_4 are plotted in figure 6.25

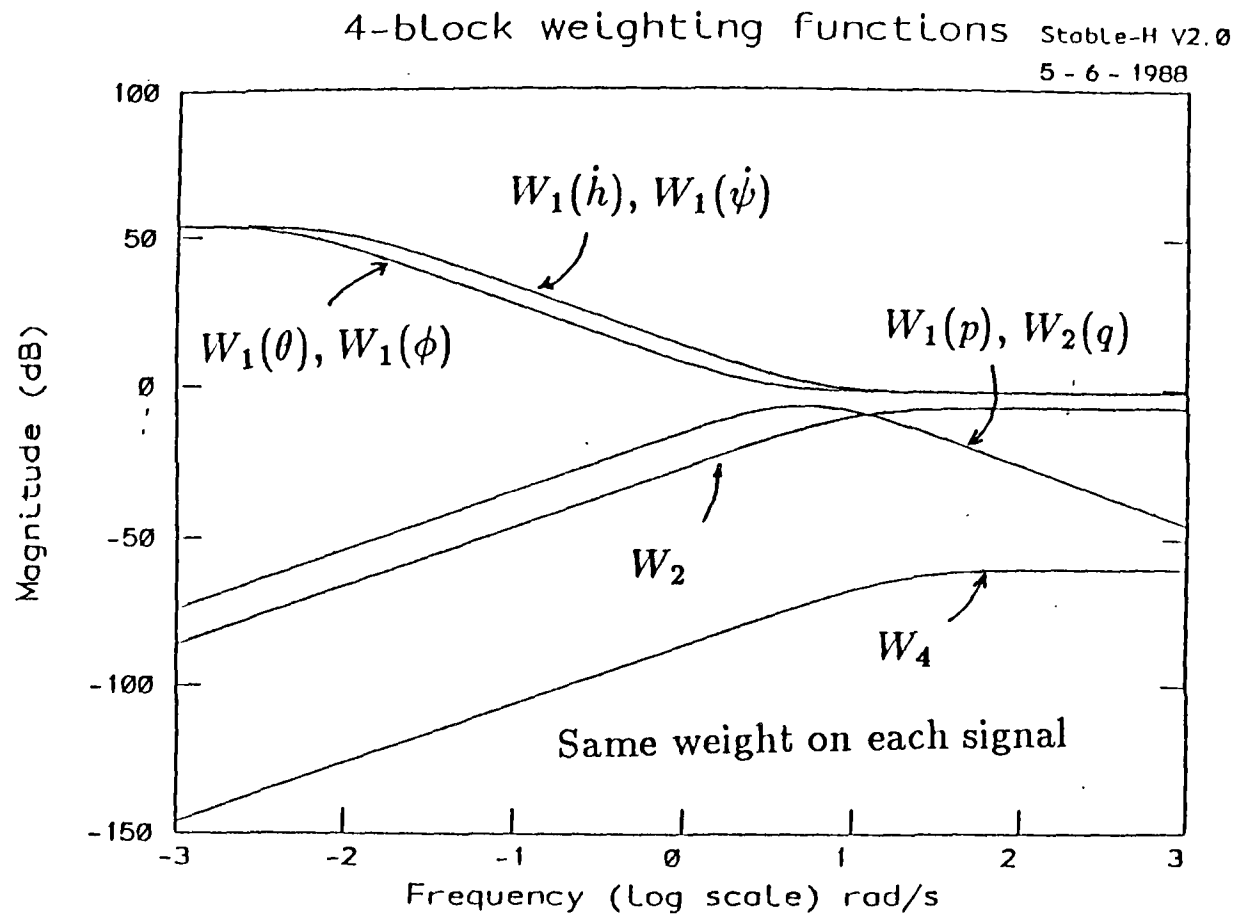


Figure 6.25 Performance weightings W_1 , W_2 and W_4

This leads to an interconnection structure with 26 states consisting of the plant (8 states) and weights (18 states).

6.3.2 Analysis of the controller

The design process leads to a controller with 101 states, but balancing and truncation of the small Hankel singular values, gives an 18 state controller without any significant change in the singular value plots of the optimal cost function. The computation of the 4-block optimal controller is non-trivial, a typical design takes about 51 CPU minutes on a VAX workstation 2000. The current algorithms are prone to numerical problems due to state inflation. It is extremely difficult to realise a sensible controller unless the design problem is well posed and the weighting functions are chosen correctly. The frequency response of the controller is shown in figure 6.26.

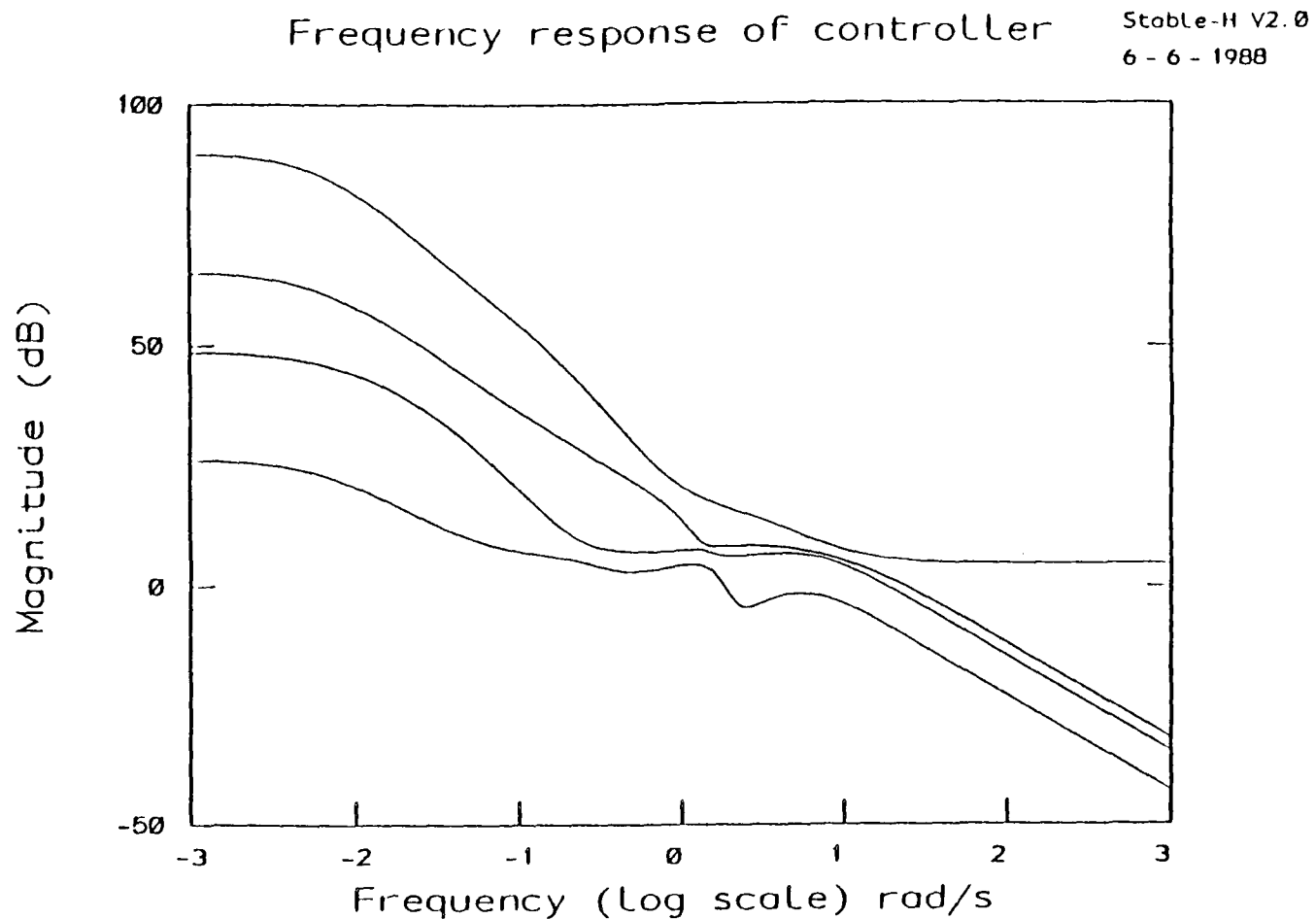


Figure 6.26 Singular values of 4-block controller

Figure 6.27 shows the frequency response of the loop transfer function GK . From this figure it can be seen that the singular values are much more compact than for the open loop plant. Therefore the loop transfer function is numerically better conditioned than for the plant alone. The controller has also been successful in achieving high gain at low frequency and the bandwidth of the augmented system is less than 10 rad/s for robustness.

6.3.3 Sensitivity

A plot of the singular values of $(I + GK)^{-1}$ is shown in figure 6.28. From figure 6.28 it can be seen that $\bar{\sigma}[(I + GK)^{-1}] = 6.4 \text{ dB} = 1/\tilde{\delta}_o$, at $\omega=0.4 \text{ rad/s}$. This implies that the smallest perturbation (within the class given by equation 3.11) that will destabilize the system = 0.48. This value of $\tilde{\delta}_o$ guarantees a gain margin of (1.92,0.68) and a phase margin of $\pm 28^\circ$ in every output channel.

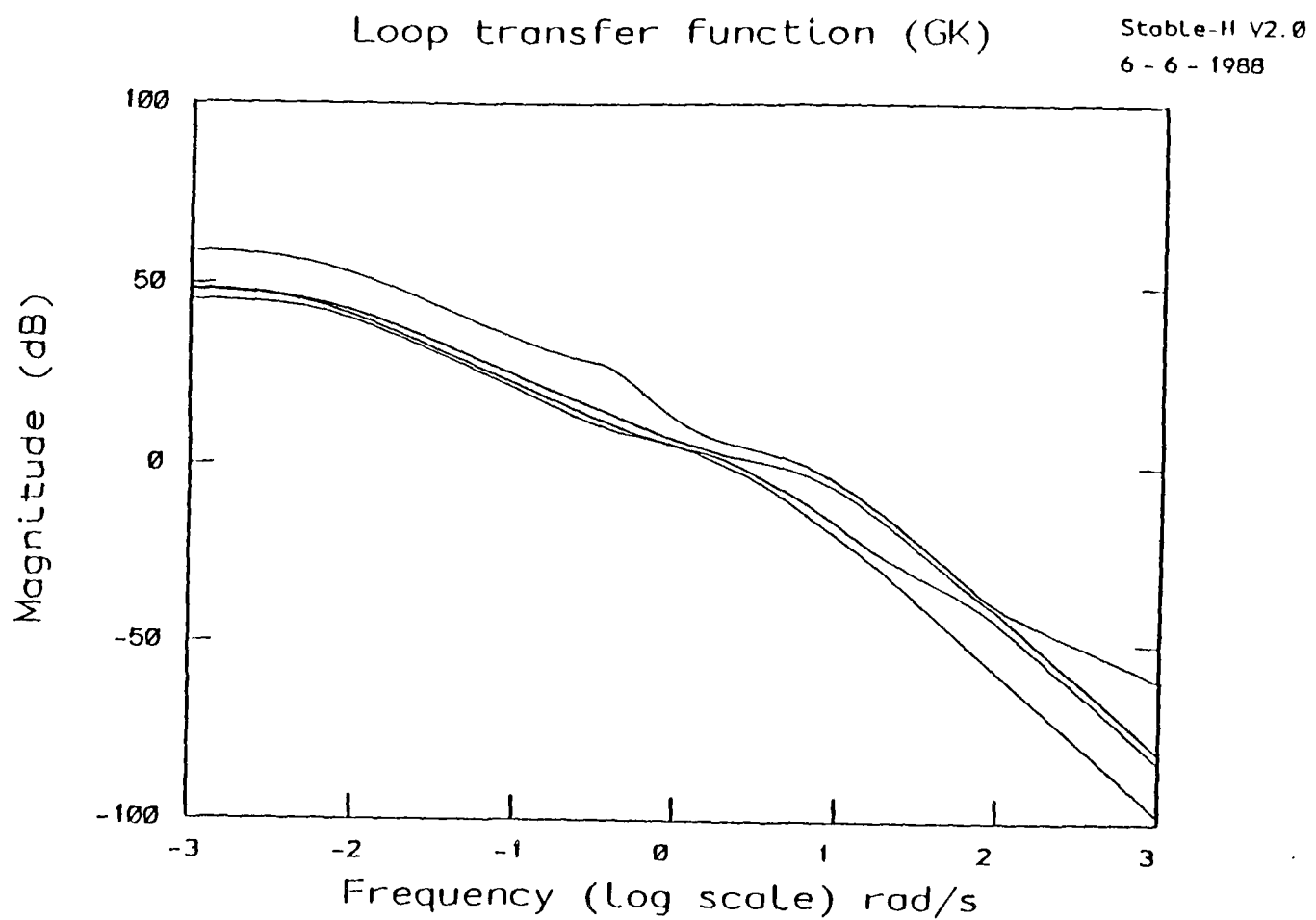


Figure 6.27 Singular values of loop transfer function GK

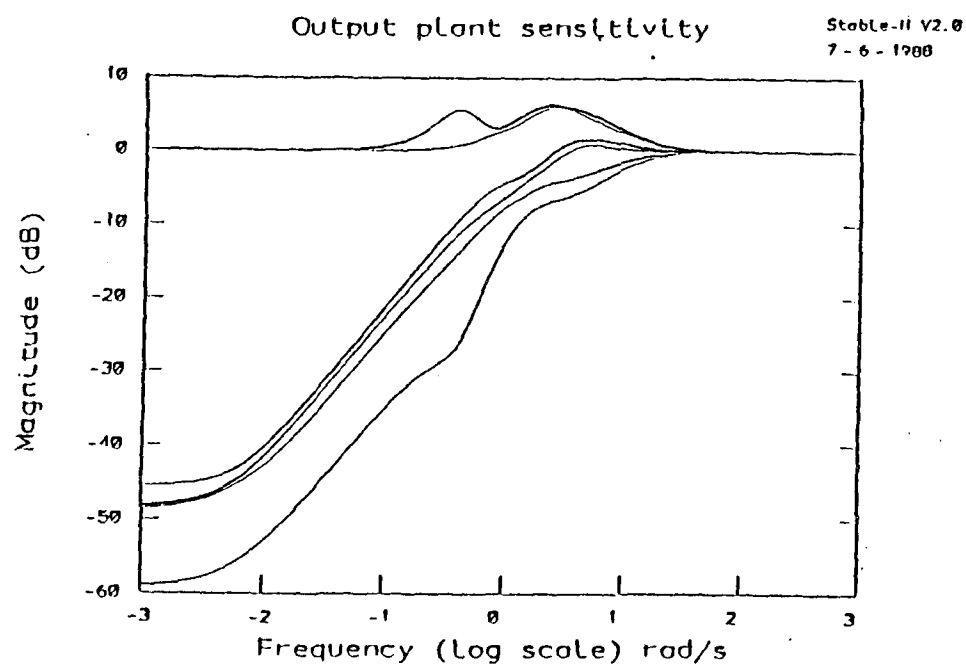


Figure 6.28 Singular values of $(I + GK)^{-1}$

The gain and phase margins at the plant input can be determined by considering the largest singular value of $(I + KG)^{-1}$. From figure 6.29 it can be seen that $\bar{\sigma}[(I + KG)^{-1}] \approx 7$ dB at $\omega = 0.4$ rad/s. This implies that the smallest perturbation (within the class given

by equation 3.10) that will destabilize the system is 0.45. This gives rise to a guaranteed gain margin of (1.82,0.69) and phase margin of $\pm 26^\circ$.

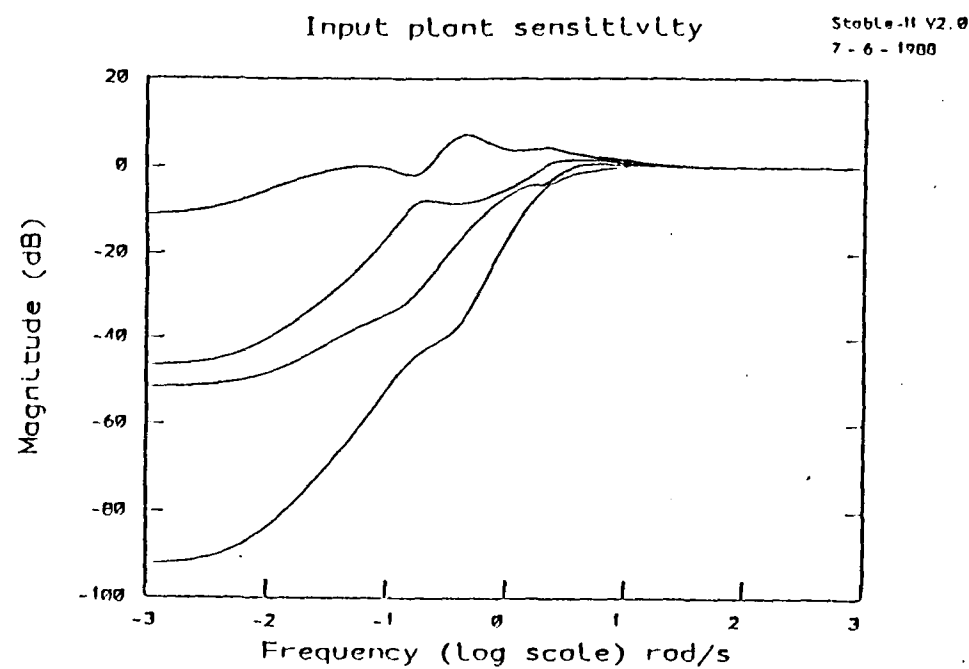


Figure 6.29 Singular values of $(I + KG)^{-1}$

A plot of the complementary sensitivity function $(I + KG)^{-1}KG$ is shown in figure 6.30. This suggests with equation (3.17) that the closed-loop system is stable for all $\delta_i < 0.36$. The unity gain crossover frequency is 6 rad/s which is within the desired value of 10 rad/s.

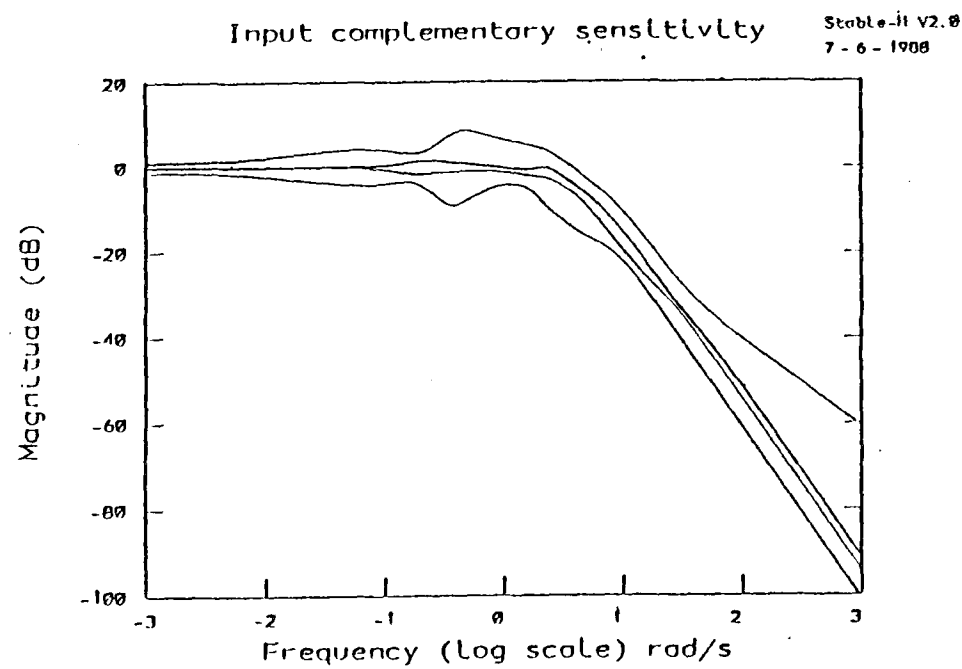


Figure 6.30 Singular values of $(I + KG)^{-1}KG$

6.3.4 Time simulations

This controller was then simulated using the nonlinear helicopter model including rotor dynamics. Figures 6.31-6.36 show the nonlinear time responses to the four pilot inputs. It can be seen that the controller is stabilizing and the responses are decoupled. A requirement for the augmented helicopter is that it must be possible to achieve a pitch attitude of $+30^\circ$ and -10° in less than 1.5 secs from a forward speed of 45 knots and a roll attitude of $\pm 30^\circ$ in less than 1.5 secs. There must be no objectionable nonlinearities or overshoots in the response. It can be seen from figures 6.33-6.36 that this has been achieved.

Note from figure 6.33 that stability is maintained even when there is a 38% change in the trim speed in attaining a pitch attitude of 30° . The nonlinear nature of the helicopter can be seen by comparing figures 6.35 and 6.35 which show the responses to a $+0.5$ rad and -0.5 rad step command in roll attitude. The responses are clearly nonlinear.

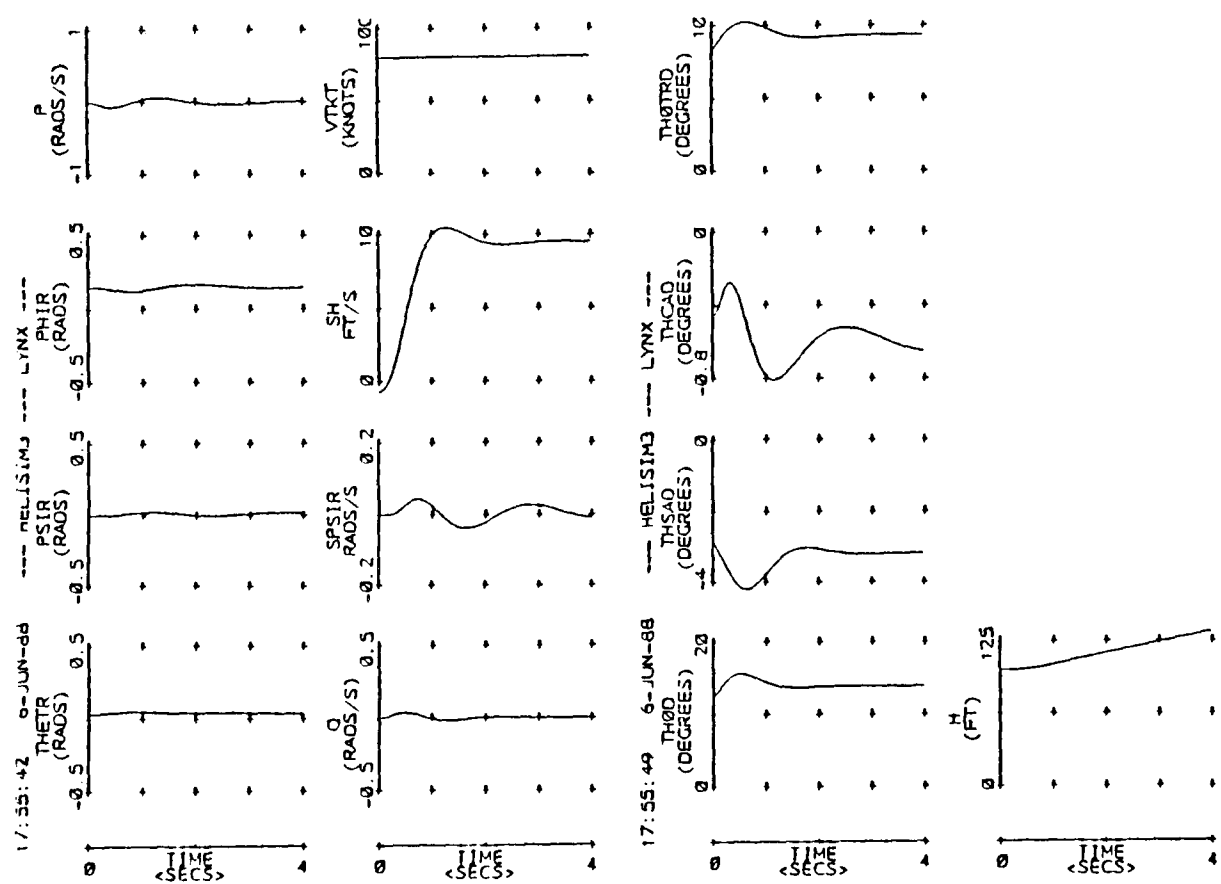


Figure 6.31 Step command of 10 ft/s in heave velocity

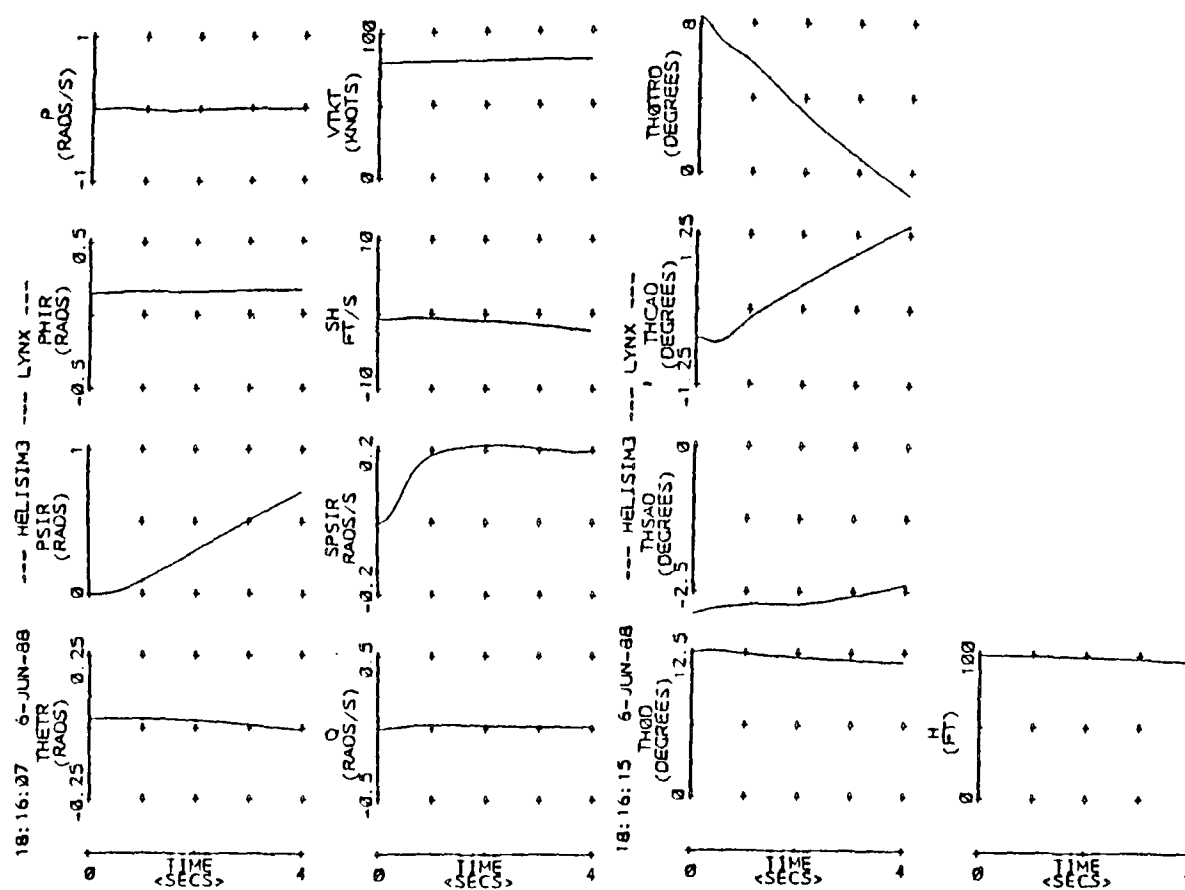


Figure 6.32 Step command of 0.2 rad/s in heading rate

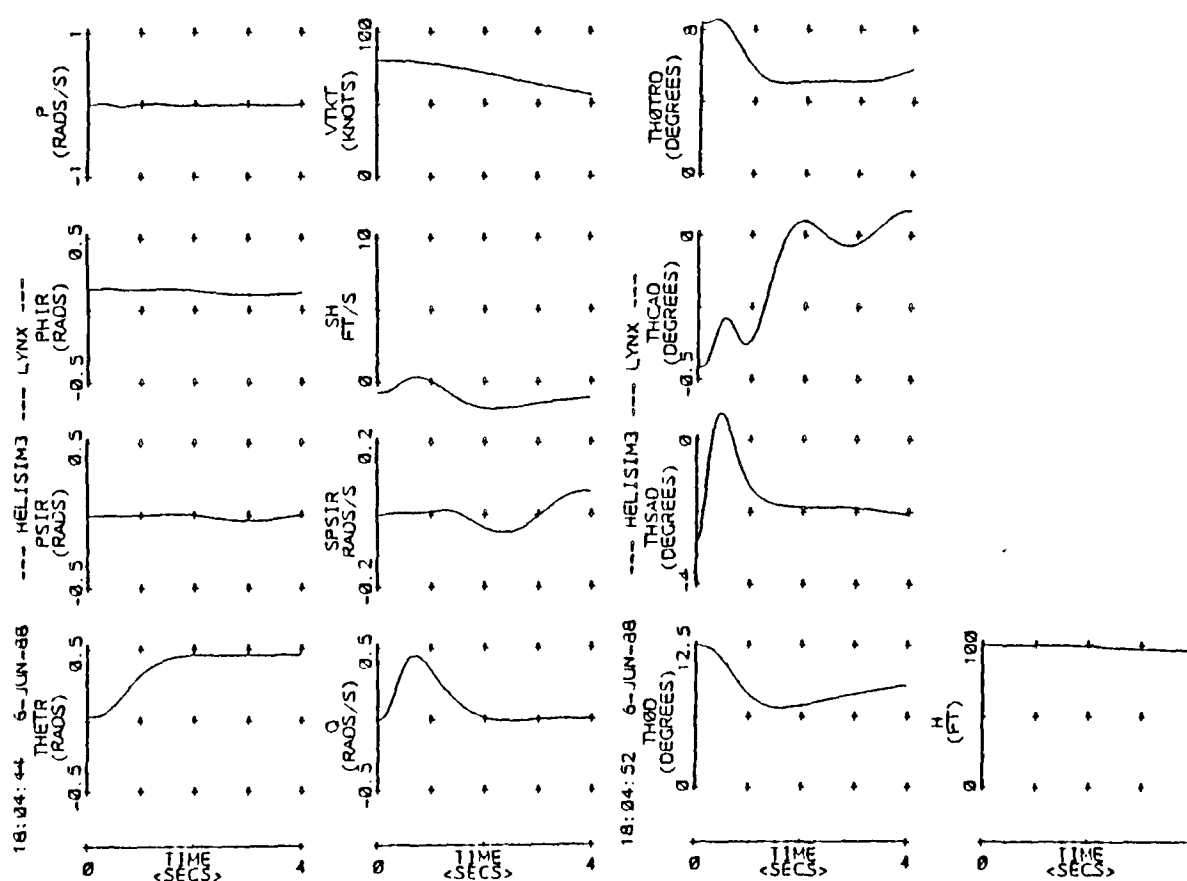


Figure 6.33 Step command of 0.4 rads in pitch attitude

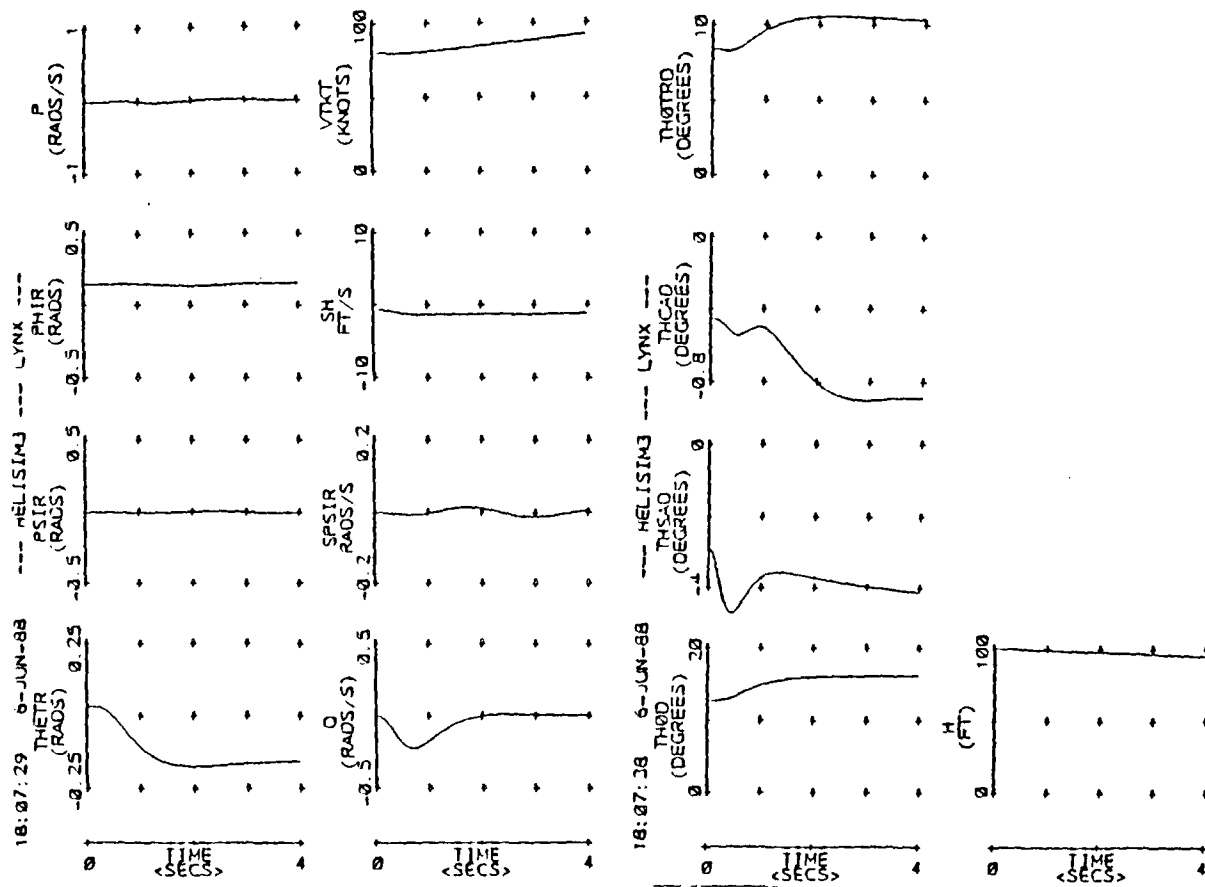


Figure 6.32 Step command of -0.2 rads in pitch attitude

Figure 6.34 Step command of -0.2 rads in pitch attitude

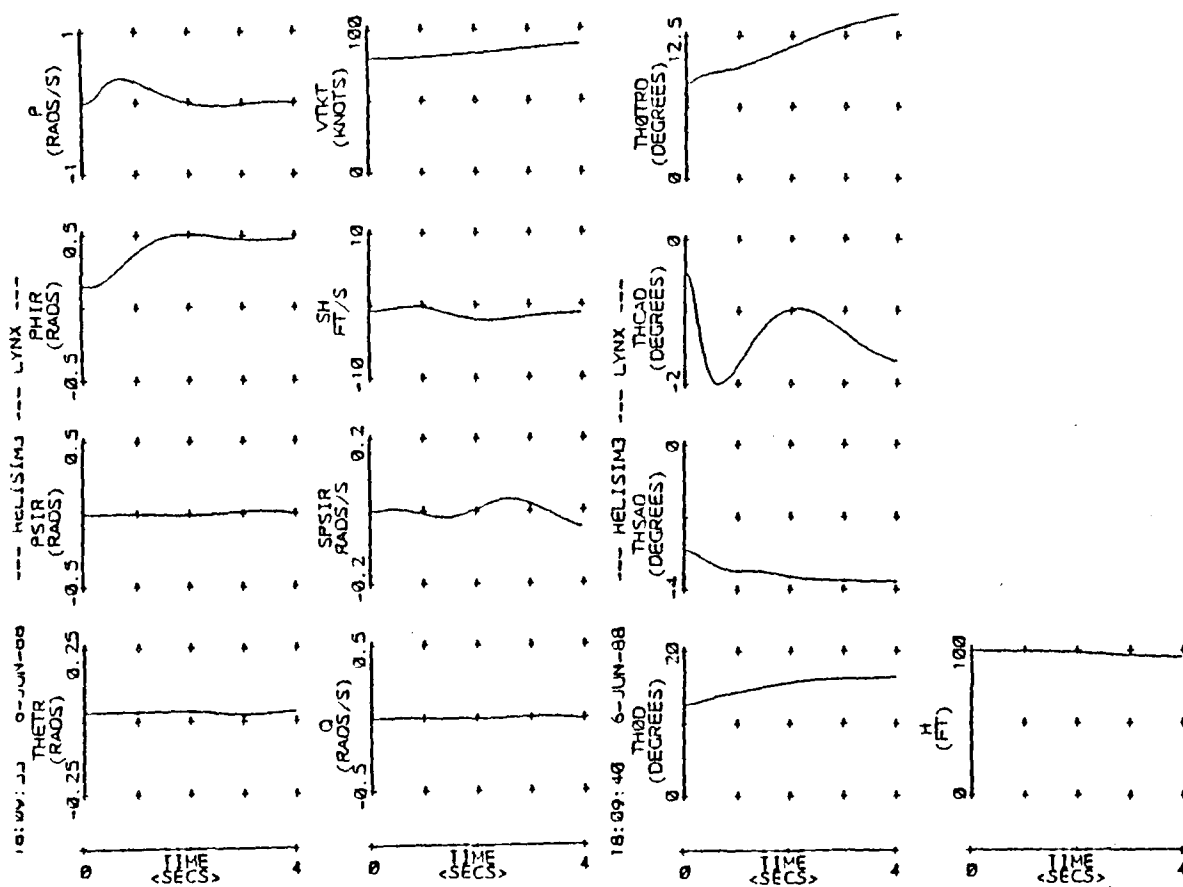


Figure 6.35 Step command of 0.5 rads in roll attitude

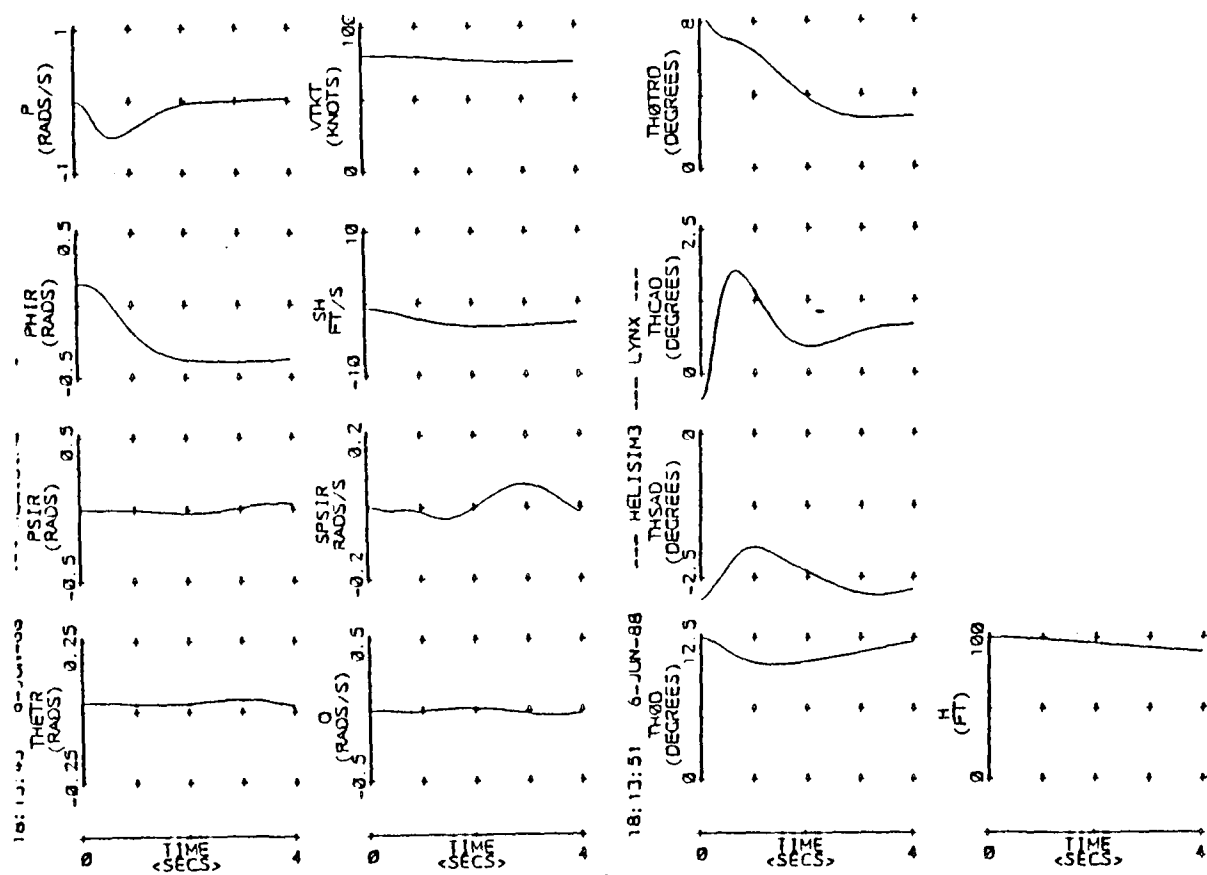


Figure 6.36 Step command of -0.5 rads in roll attitude

With the rotorcraft initially trimmed directionally, it is a requirement of the augmented aircraft, that it should be possible to maintain straight flight over a speed range of $\pm 30\%$ of the trimmed speed or ± 20 knots whichever is less. Figures 6.37 and 6.38 show the aircrafts' attitude and position when flying at 60 and 100 knots respectively. The initial aircraft model was trimmed at 80 knots straight and level flight. It is evident from these figures that the height loop is the most difficult to control. It does not appear possible to maintain control of height using a fixed gain controller throughout the flight envelope. One obvious solution is to increase the gain in that loop, however this can only be achieved with a resultant increase in bandwidth which is detrimental to the system robustness.

There is also a loss in height due to changes in bank angle from the trimmed straight and level position. This is due to a change in the direction of the thrust vector. As the helicopter banks, there is a natural tendency for the aircraft to weathercock into the turn due to its directional stability and for the nose to dip. This phenomenon is greater, the

larger the bank angle demand. This interaction is difficult to suppress since it cannot easily be predicted by a linear model.

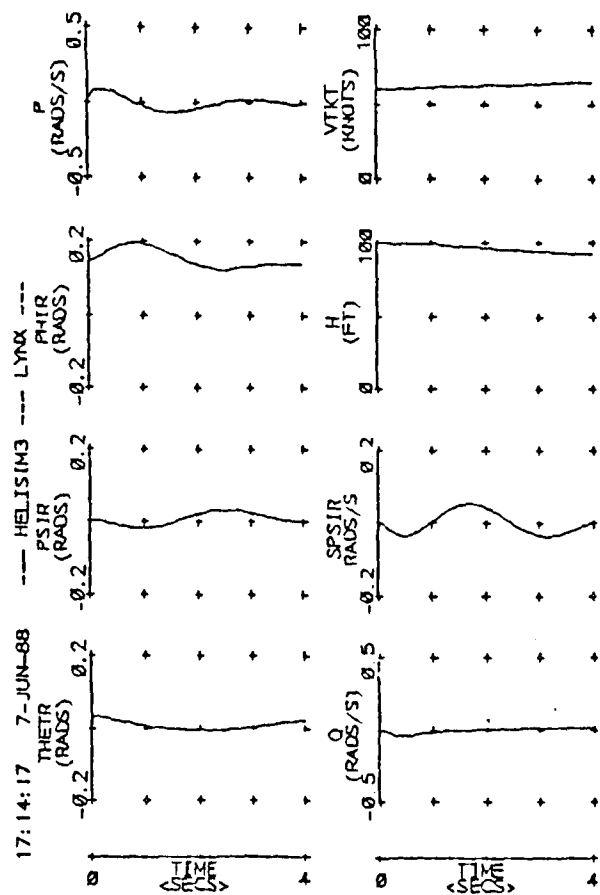


Figure 6.37 Helicopter flight position at 60 knots

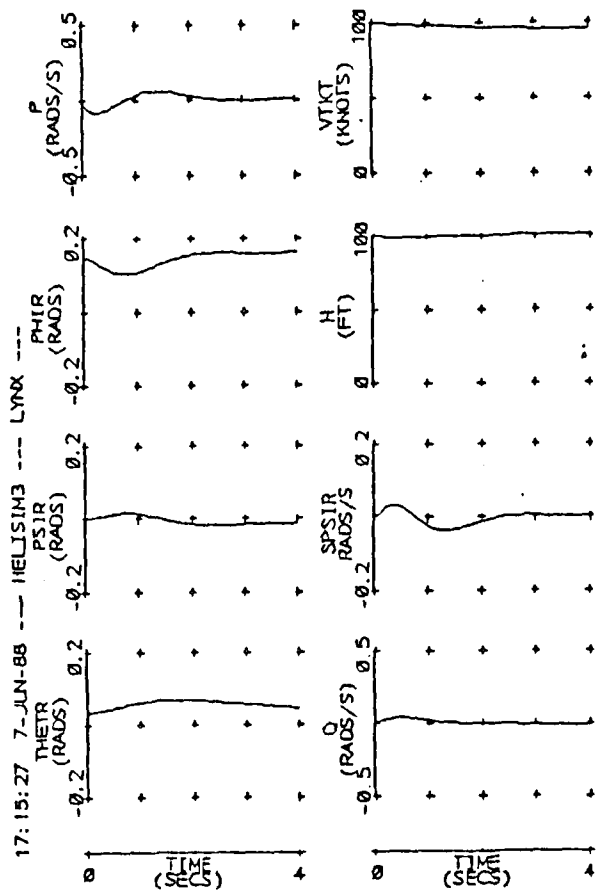


Figure 6.38 Helicopter flight position at 100 knots

6.4 Concluding Remarks

This chapter has presented applications of H^∞ -optimization for the design of helicopter control systems. Both 2-block and 4-block H^∞ -problems have been considered. The main factor limiting more complicated 4-block H^∞ -designs is computing power and numerical conditioning of the problem. For the 4-block configuration presented, a typical design took about 51 CPU minutes, this clearly is unacceptable for interactive design. Some new results from Doyle et al [DOY5] and Limbeer et al [LIM4] may circumvent this problem.

The Lynx helicopter was represented by the mathematical model HELISIM. The inclusion of rotor lead-lag dynamics in the nonlinear model had previously compromised other control law designs. In particular the design put forward by Westland Helicopters for the

Lynx, had to have its control law gains significantly reduced to achieve adequate stability margins and feedforward terms included to improve the controlled response characteristics. This proved not to be the case for the H^∞ -controllers designed in this chapter.

The two-degree of freedom controller is a useful tool for incorporating handling qualities criteria and stability robustness. However, because of the inclusion of a plant model, the resultant controller tends to be of quite high order. The main reason for considering the configuration of figure 4.5 is that there is no restriction on the controller being stable. The conventional configuration with a pre-filter outside the feedback loop, does not admit an unstable precompensator. From the point of view of using the precompensator to shape desired responses for good handling qualities, this may not be too restrictive. Although this control set up is much simpler, the specification of the parameters for the precompensator needs to be very accurate, since small errors in its specification will propagate directly into the system. This is not the case for the feedback configuration of figure 4.5.

Singular values have been the main tool for analysing robustness. Assessment of the control systems using this technique indicate that the designs have relatively poor stability margins. However simulations show this is not the case, which indicates that the analysis is conservative. Singular values on their own are not really good enough indicators of whether the design is robust. The structured singular value of Doyle [DOY4], overcomes the conservativeness problem but its synthesis does not yet have a complete solution.

The nonlinear time responses lie within the specifications given in [HOH1], however it is doubtful whether a fixed gain controller will be able to operate throughout the flight envelope. The problem of scheduling H^∞ controllers is not easy. Since the optimization procedure is unconstrained, there is no restriction on the controller structure. So that a controller designed at one operating point may be totally different in structure to one at another operating point. It is not clear that a simple scheduling in gains would be appropriate, however with the advent of more powerful computers this does not exclude

the possibility of having different types of controller at different operating points... The problem lies in being able to switch them in a smooth continuous manner.

CHAPTER 7

HANDLING QUALITIES EVALUATION

7.1 *Introduction*

This chapter discusses the problem of how to determine whether a flight vehicle exhibits satisfactory handling qualities. The difficulty of trying to define suitable rating scales is discussed and a summary of the existing techniques is presented.

A piloted simulation study to assess the effectiveness of advanced control laws was initiated as part of an RAE/Oxford University research programme to study robust control laws for helicopters. This chapter describes the first phase of the study which took place at RAE, Bedford during August to September 1988. During this study handling characteristics were assessed at flight speeds around the hover and at 80 knts.

The controllers designed in chapter 6 are assessed against the specifications from the proposed airworthiness design standard [HOH1]. Having established that the controlled responses are within the “right ball park”, these controllers are further evaluated under simulated flight on a fixed based motion simulator. The aim of the exercise was to obtain an assessment of the control laws by performing a series of tasks. The control laws were developed with the aim of being able to realize them practically with current hardware technology. Practical considerations in the scheduling of the trials restricted the amount of time individual pilots flew the simulator.

7.2 Pilot Rating Techniques

The suitability of a manually controlled vehicle to serve its intended purpose is ultimately assessed by a series of judgements. Perhaps the most difficult being the evaluation of the vehicle's handling qualities, which play such a key role in the overall suitability of the vehicle, and yet have been in the past perplexing even to define satisfactorily. Unlike for example performance, reliability, maintainability, durability and structural integrity, for which targets can be set, specifications defined and compliance measured by appropriate testing, handling qualities belong very much to a subjective discipline, with pilots differing in some cases quite widely, in their opinion of the helicopter's handling qualities. Very often, the pilot's comments are significantly biased by previous experience of the aircraft type, role flown or simply piloting skill. With the absence of any quantitative assessment, pilots frequently resort to adjectives such as "sluggish", "responsive" or "crisp" to describe their pleasure or displeasure with a particular machine, comments which, if of a negative nature and without quantitative detail, can make improvement of the design difficult.

In the process of measuring and evaluating pilot/vehicle performance, it is necessary, as one facet of the investigation to measure operator opinion. These subjective measures are in fact the ultimate evaluation of the system and consequently are foremost in the designer's mind throughout vehicle development. Unfortunately, the current connections between pilot ratings, pilot behaviour, and vehicle characteristics are, at best, highly qualitative. This situation has not improved as vehicles and associated pilot/vehicle handling qualities considerations have steadily increased in complexity, for then the difficulties with existing rating scales and subjective measures become still more obscure.

The problem is in transforming the subjective assessment into engineering parameters. The most successful method to date has been the Cooper-Harper pilot rating scale (figure 7.1), in which the pilot can classify the desirable and unsatisfactory handling aspects on a points system, scaled from 1 to 10, where 1 represents the most satisfactory qualities. A

rating of 10 represents major and unacceptable system deficiencies, where control may be lost during part of the flight envelope [COO1].

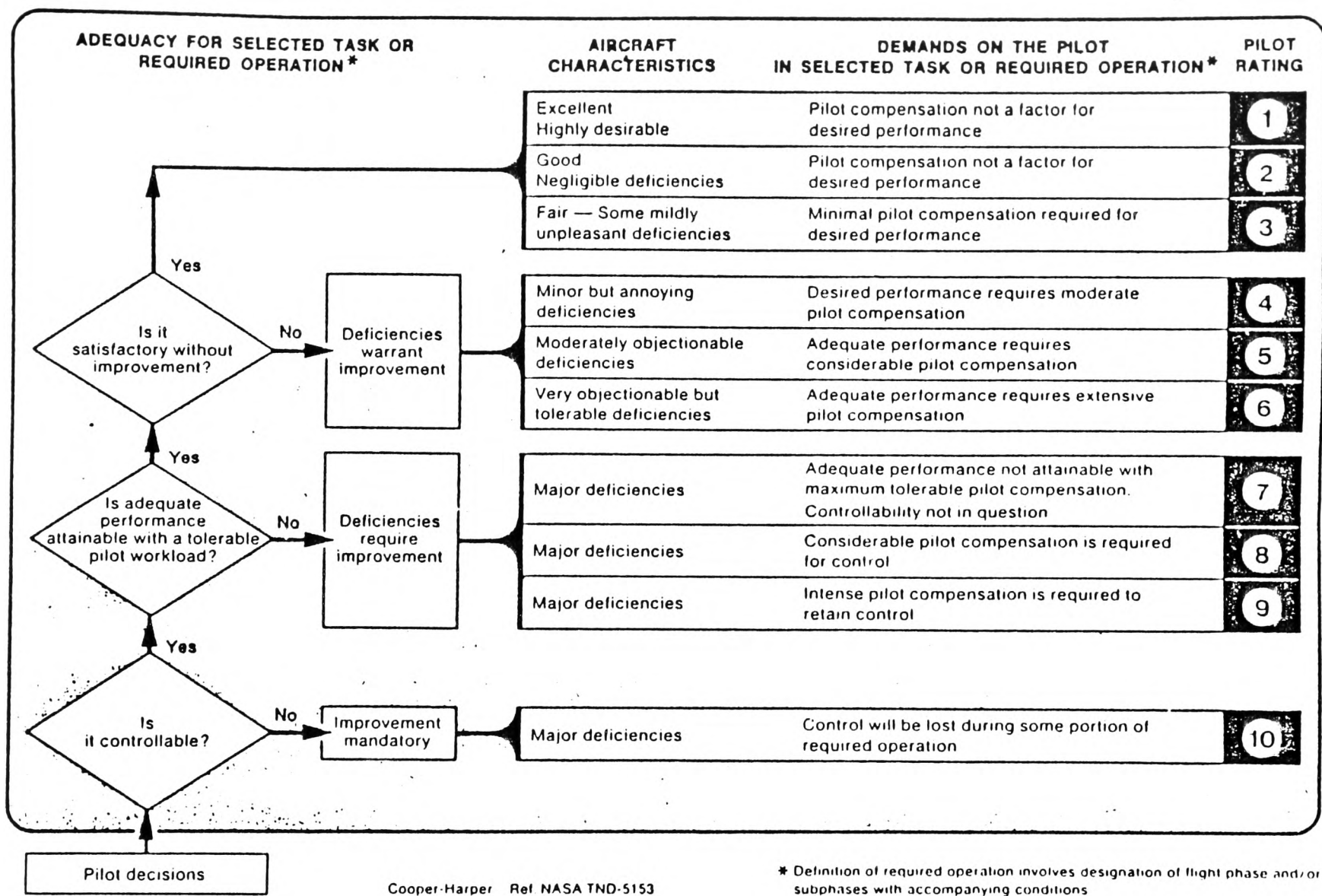


Figure 7.1 Cooper-Harper Handling Qualities Rating Scales

7.3 Bases For Rating Scales

Pilot evaluation is intended to meet two objectives:

- (i) to provide an overall assessment of the suitability of the vehicle in its intended use and
- (ii) to provide information on deficiencies that may exist in the vehicles characteristics which interfere in its intended use.

The first objective requires that the rater be able to express his subjective impression of the handling qualities of the vehicle in performing the required manoeuvres. This “impression” is the sum total of all the physical factors which contribute to the handling qualities of the vehicle. Since there is no common physical measure which integrates all of the factors, a scale must in part at least be in subjective terms.

The second objective requires that the rater be able to provide information on specific problem areas to aid the experimenter or designer in solving the problems. Thus, a language is required which is unambiguous to as large a population as possible to minimize training requirements and to maximize repeatability.

The disadvantage of an engineering, or scientific, language is the high level of training required for selective and repeatable ratings. McDonnell [MCD1], carried out a survey among pilots and engineers and obtained almost unanimous agreement on this point. Similar results were obtained on the definitions of handling qualities based on frequency domain characteristics, eg. trimmability, manoeuvrability. It was concluded that a pilot would have a difficult time remembering and interpreting the distinction between the frequency domain terms. The ability to assess vehicle and system parameters depends heavily on training and also a variety of manoeuvres to be performed. Even then it is doubtful that a pilot could consistently determine the frequency and time response parameters. Past work has shown that performance is very often not correlated with the pilot’s opinion [ASH1], [ASH2] and [MCR1].

7.4 Difficulties With Existing Scales

Difficulties experienced with the use of existing scales can be divided into four convenient categories:

- (i) Extrapolation of the simulated task to the real flight situation. The necessity of using simple simulations give rise to the problem of extrapolating the simulation to the actual flight situation. Interpretation of the display, and agreement between the experimenter and pilot on the objectives of the evaluation, are the important factors here.
- (ii) The alternate mission character. Some scales allow for a change in mission should the pilot be unable to carry out the primary mission (eg. landing the aircraft in the event of stability augments failure). This is perhaps possible for actual flight testing, but becomes increasingly difficult to structure as the simulation is simplified.
- (iii) Verbal descriptions and phrases. Incomplete and ambiguous scale category descriptors result in an undesirable arbitrariness in the calibration between real and simulated flight, thereby causing evaluation to be nearly a “black art” and lacking in good repeatability and consistency.
- (iv) A scale’s qualitative character. Data relating subjective measures with vehicle and operator parameters are far from complete. Past experience has shown that the pilot is occasionally unable to articulate the primary causes of his discontent.

The criteria which should be used in selecting possibilities for a scale are:

- (a) the language be as natural and unambiguous to the rater as possible so that little analysis by the pilot is required during the rating situation and
- (b) the language be as descriptive of piloting problems as possible.

7.5 Comparison Of Augmented Lynx With Proposed Handling Qualities Requirements

The specifications outlined in the proposed airworthiness design standard are designed to define the minimum response that will allow accomplishment of the Mission-Task Element with level 1 handling qualities. Experience has shown that there are a large number of response characteristics that are important for good handling. However it is not practical to attempt to formulate every conceivable response to show compliance with the dynamic requirements. It is proposed instead, to specify certain fundamental response characteristics which are important in each axis and to show compliance with the necessary specification. Those specifications which are obviously not yet applicable have been ignored such as those dependent on the aircraft flight control computer, hardware and software failure states and specifications defining control forces or breakout forces.

The information and data used to compare the augmented Lynx helicopter with the proposed MIL-H-8501 revision was derived by computer simulation of flight manoeuvres. The version of HELISIM held by Oxford does not correctly provide for wind representation, hence those handling qualities requirements calling for wind effects could not be applied at this time.

Use of a mathematical model introduces the limitation that results are dependent upon the accuracy of the model and ultimately require confirmation by flight tests. For the majority of handling qualities requirements, compliance could be demonstrated through time responses and frequency response analysis.

This section demonstrates that the basic response characteristics required for the ACAH system described in chapter 6, §6.2 is met. To illustrate this it is necessary to quote the relevant sections from the design specification [HOH1].

7.5.1 Attitude hold

“If an attitude hold is required, the pitch (roll) attitude response to a pulse longitudinal (lateral) controller input shall return to within $\pm 10\%$ of the peak in less than 10 secs for level 1. For level 2, there shall be no tendency for pitch (roll) attitude to diverge following the pulse input. The duration of the pulse should be as short as possible as indicated in figure 7.2.”

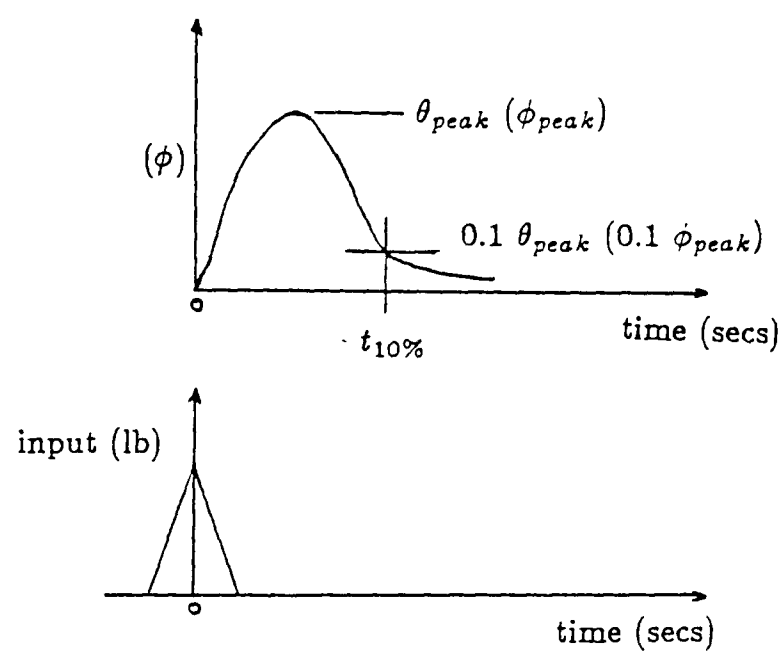


Figure 7.2 Illustration of pulse response for attitude hold requirement

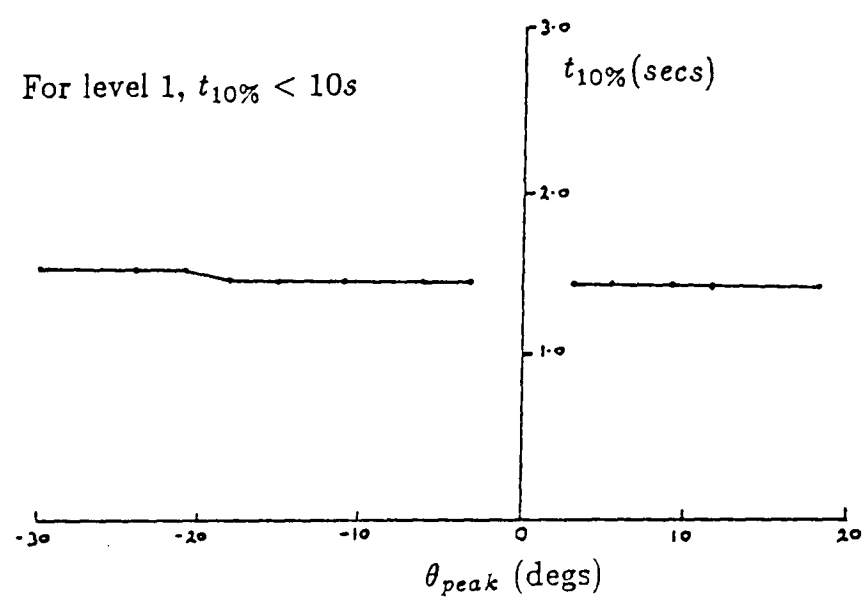


Figure 7.3 Pitch attitude response to a pulse longitudinal controller input

Figures 7.3 and 7.4 show the responses for the augmented helicopter. It can be seen that the requirements are easily met for level 1 handling qualities. These figures also suggest that the the pitch and roll attitude response to a pulse input are almost linear.

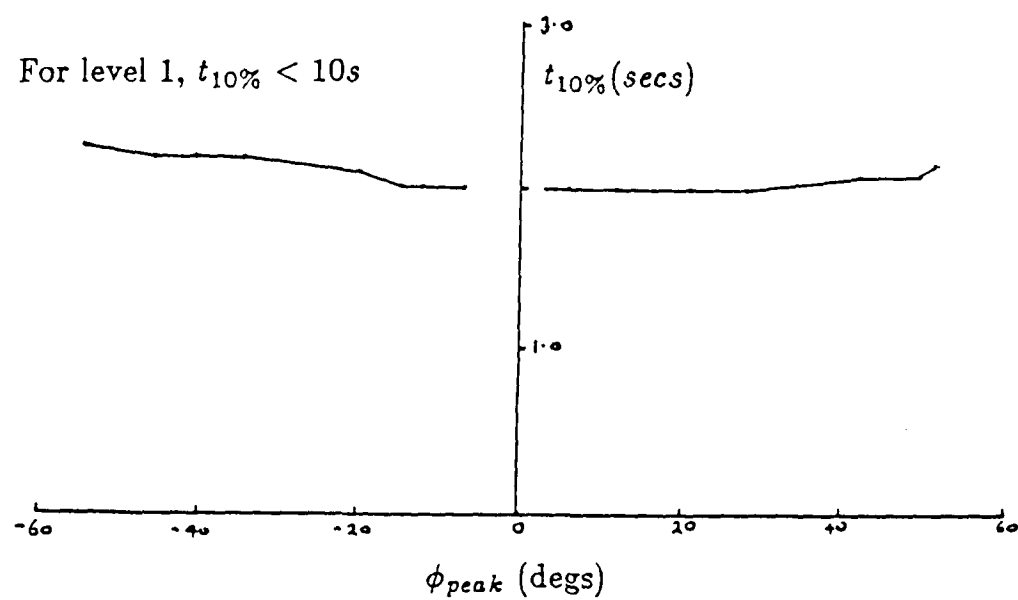


Figure 7.4 Roll attitude response to a pulse lateral controller input

7.5.2 Heading hold

“If heading hold is required, the heading response to a pulse input of the control actuator shall return to within $\pm 10\%$ of peak in less than 10 secs for level 1. For level 2, there shall be no tendency for heading to diverge following the pulse input.”

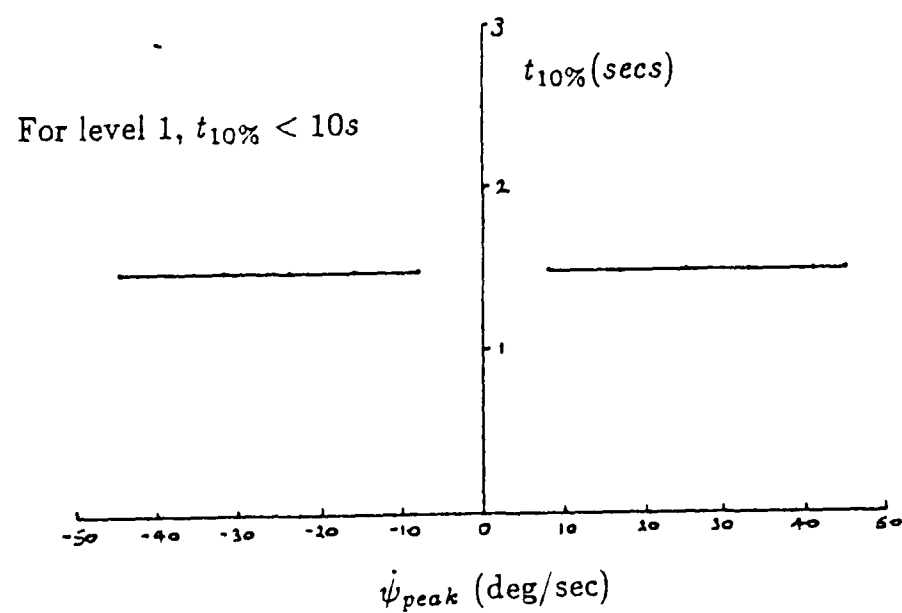


Figure 7.5 Heading response to a pulse pedal input

The heading hold response is shown in figure 7.5 and once again it can be seen that this requirement is not too stringent.

7.5.3 Height response to collective controller

“The rise time ($t_{R_{50}}$) in altitude rate following an abrupt step input of the collective controller shall be within 2.8 secs for level 1 and 4 secs for level 2. The time delay ($t_{R_{25}}$) and response shaping parameter $\frac{(t_{R_{50}} - t_{R_{25}})}{(t_{R_{75}} - t_{R_{50}})}$ shall meet the boundaries specified in figure 7.6.” $t_{R_{50}}$ denotes the time required for the response to a step input to reach 50% of its final value. Similarly for $t_{R_{25}}$ and $t_{R_{75}}$. The above times assume a useable cue environment (UCE) of ≤ 2 , if this figure is degraded to 3, then the required rise time $t_{R_{50}}$, decreases to 0.9 secs. The UCE is merely an indication of pilot visibility.

The altitude rate response is linear with step collective input. The resultant response for the augmented system is superimposed on figure 7.6 and is within the level 1 region.

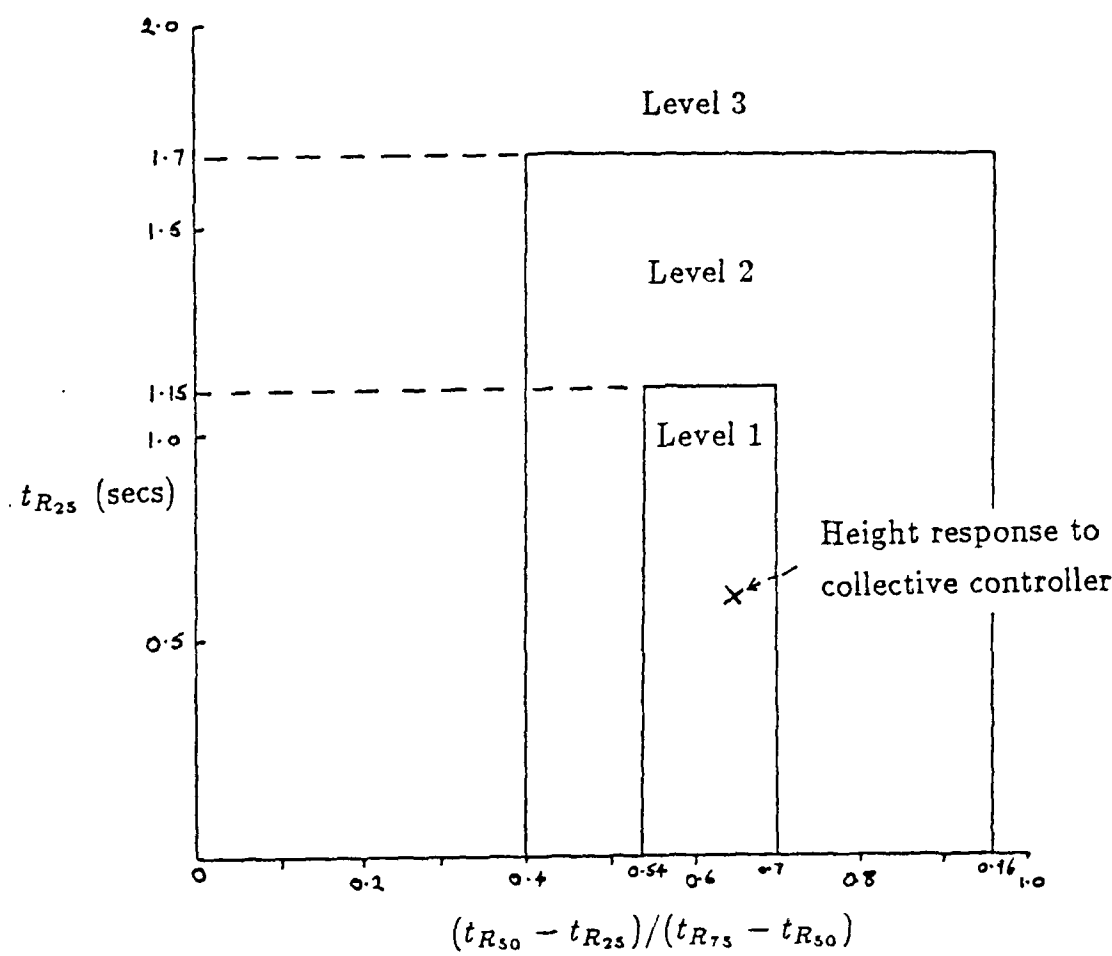


Figure 7.6 Limits on shape of vertical axis response

7.5.4 Large – amplitude heading changes

“The ratio of peak yaw rate to change in heading, $r_{pk}/\Delta\psi$, shall exceed the limit specified in figure 7.7. This requirement applies for control inputs large enough to produce at least a $\pm 10^\circ$ heading change, up to a heading change of 40° .”

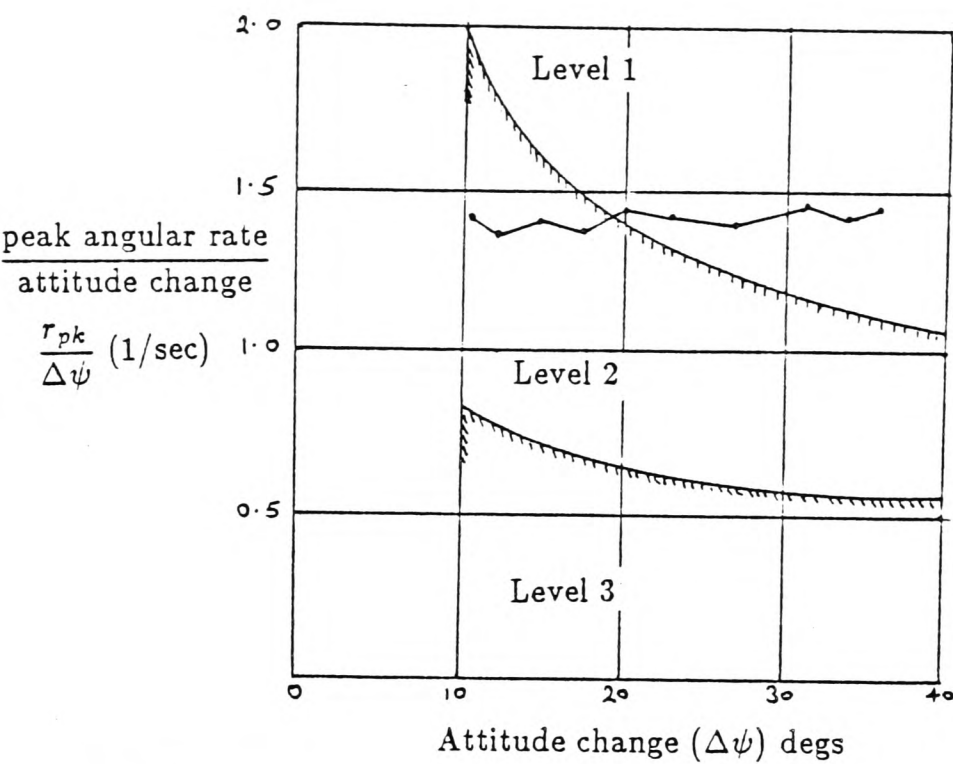


Figure 7.7 Heading response limits for large control inputs

It can be seen from figure 7.7 that more angular rate per change in attitude is required for small changes in heading. This is a translation of the pilot requiring faster responses during tight manoeuvres and slower responses during more gentle manoeuvres. Since the controller has not been designed to provide nonlinear responses, the handling qualities rating degrades to level 2 for attitude angles less than 20° . It is easy to push the line drawn on figure 7.7 higher so that it lies above the level 1 curve for all attitude changes. However this would mean that the controlled response would be unnecessarily great for large heading changes. To achieve the specification would require a scheduling of controllers which would be able to provide fast responses for small angular changes and slower responses for large angular changes.

7.5.5 Interaxis coupling

Figure 7.8 shows the collective-to-yaw coupling. This requirement was discussed in §2.6.1. It can be seen that this coupling is nonlinear and the level 1 region is exceeded for altitude rate demands of greater than 30 ft/s.

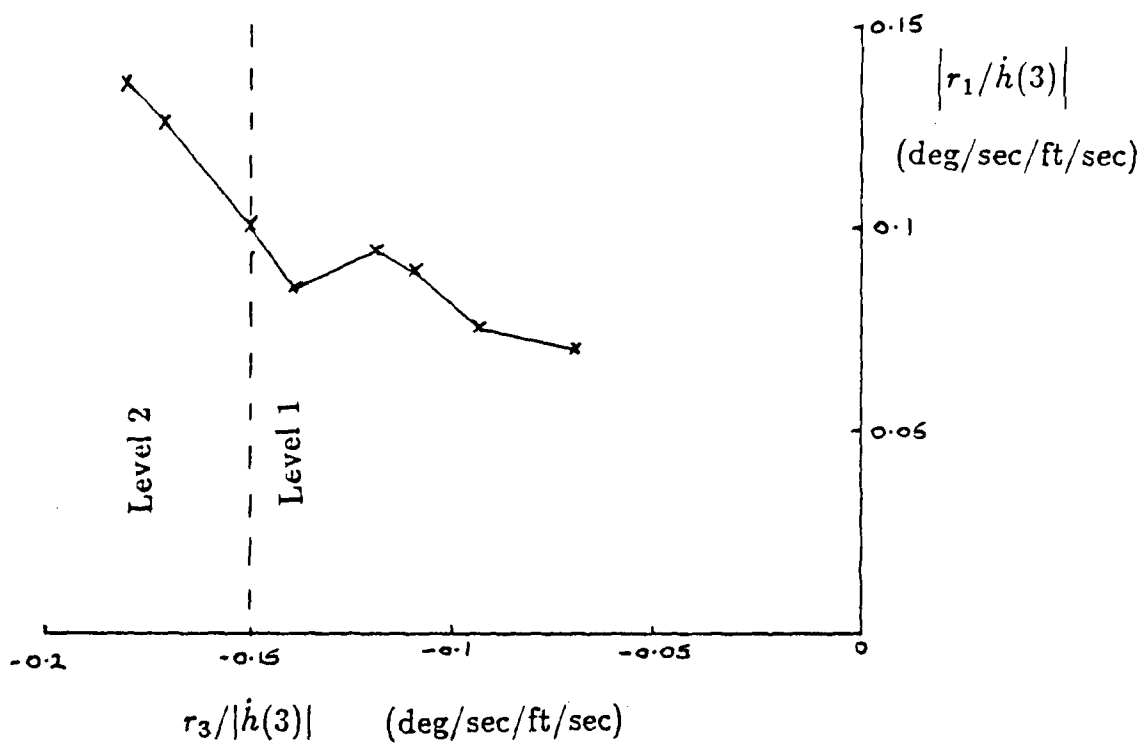


Figure 7.8 Collective to yaw coupling

The pitch-to-roll and roll-to-pitch coupling is shown in figures 7.9 and 7.10. These responses all lie within the level 1 region.

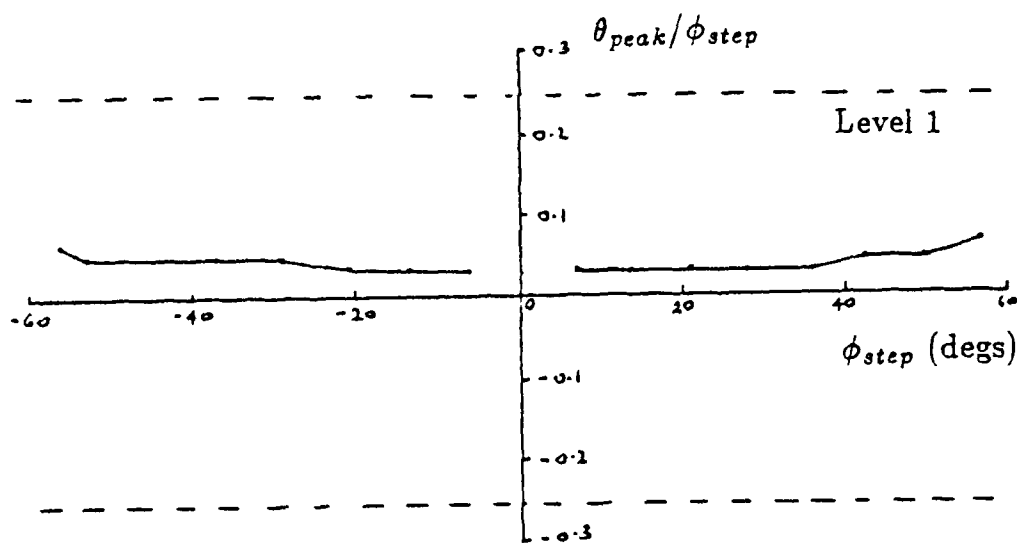


Figure 7.9 Pitch-to-roll coupling

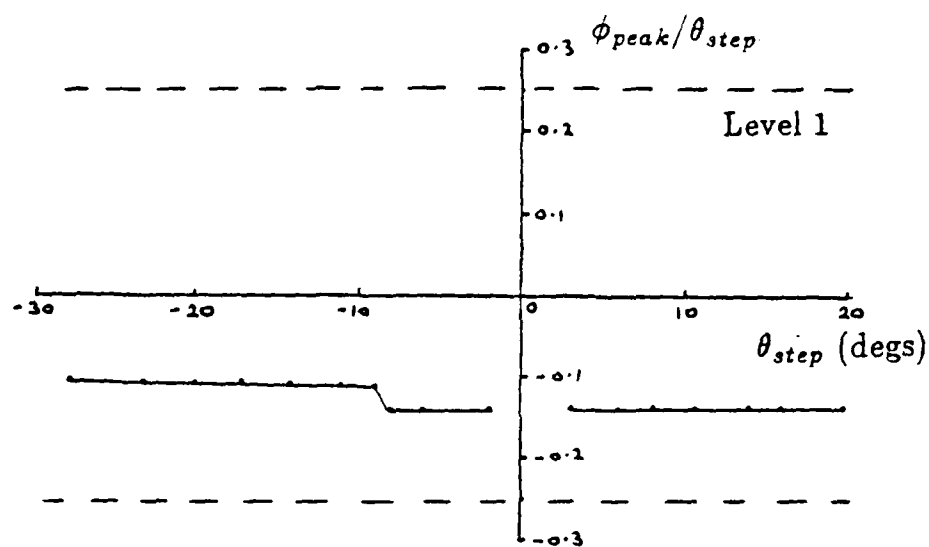


Figure 7.10 Roll-to-pitch coupling

7.6 Flight Test Requirements

In addition to showing numerical compliance with parameter limits and boundaries, it is also necessary to conduct a series of qualitative inflight handling qualities tests. The need for precisely defined tasks to promote aggressive closed loop flight tracking has led to the creation of a set of ground marked courses at RAE, Bedford. These include a series of concentric circular tasks, a spiral and triple bend course. The total scheme is illustrated in figure 7.11, and consists of a winding “serpent” course, designed to evaluate primarily lateral-directional handling qualities and a series of hurdles placing emphasis on longitudinal handling qualities. Identical tasks have been marked on the simulator model terrain to aid control strategy validation work. The circles provide an essentially “stationary” type of task, at least in low wind conditions; the spiral a progressive non stationary task and the triple bend, a combination of stationary and rapid transition (roll reversal) elements.

Detailed questionnaires have been designed to help the pilot describe the interaction between task cues, control strategy and task performance. The validity of the simulations are tested both qualitatively, through general pilot impressions and quantitatively by comparison with

flight test data. The evaluation pilot has no prior knowledge of the configuration which he will fly. He will fly 3 (or more if desired) complete approaches and landing patterns for each evaluation and then evaluate the handling qualities using the Cooper-Harper rating scale and pilot comment card. The procedure is to assign a pilot rating immediately after the task is completed, make the comments using the comment card and finally revise the rating if desired.

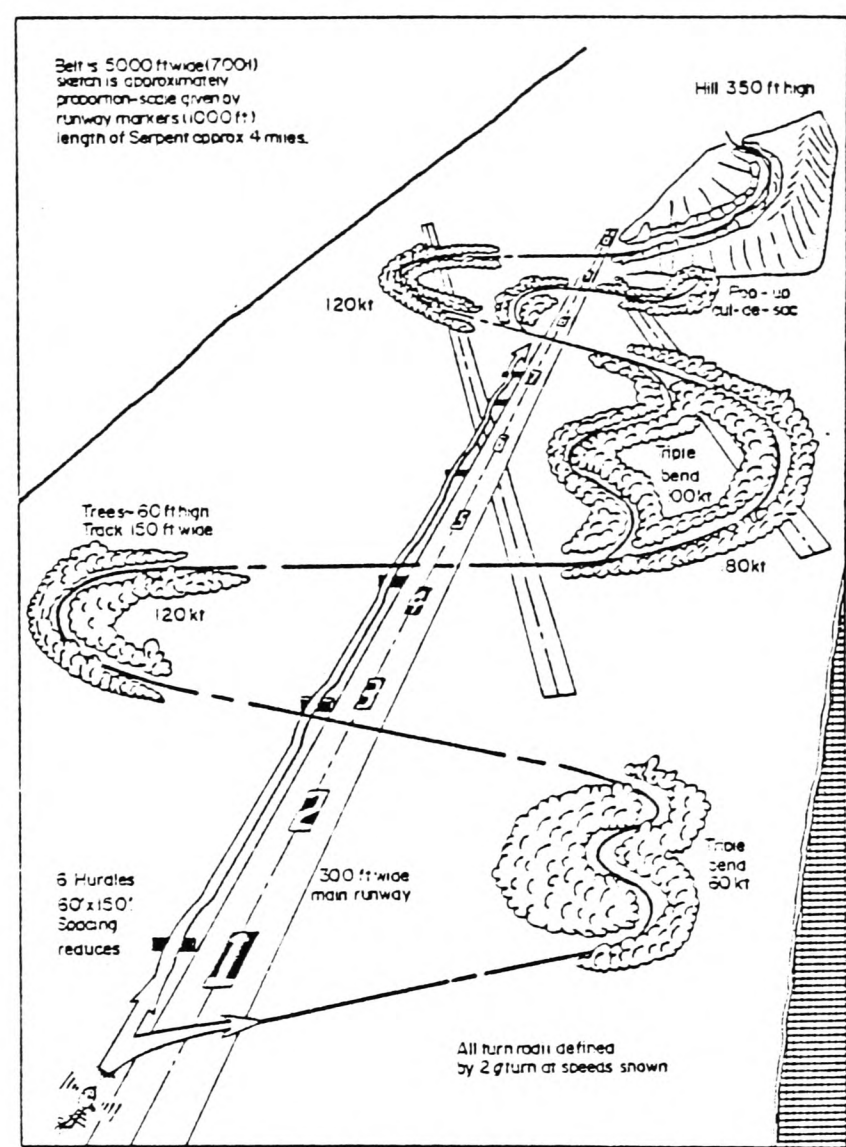


Figure 7.11 Sketch of agility course - the “Serpent” and hurdles

7.7 Simulator Characteristics

The trials were carried out in the single seat cockpit flight simulator, at the Flight Research Division, RAE, Bedford. A photograph of the small motion system is shown in figure 7.12.

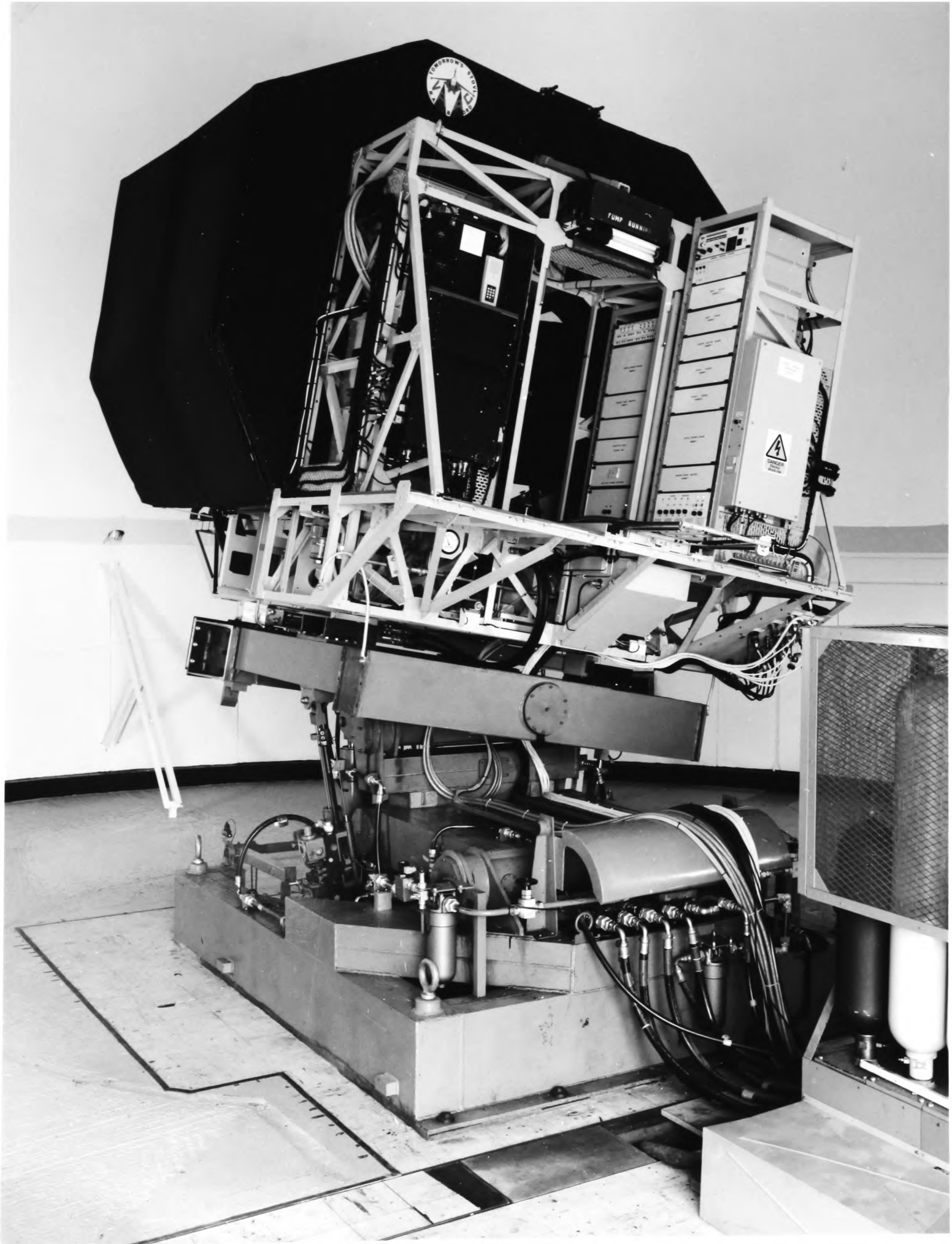


Figure 7.12 The small motion simulator at RAE, Bedford

A view of the outside world was produced by a closed circuit television system, in which a TV camera tracked across a 700:1 scale model landscape, carried on a continuous belt, in response to position and attitude signals from the computer. Motion cues were provided in pitch, roll, heave and yaw, and vibration was simulated as a function of “g” and speed. Audio cues included engine roar, turbine whine and rotor flapping. A number of specific features exist on the terrain belt to define courses for pilots to fly as part of the assessment task. The collimated terrain image gave the pilot an effective field of view of 47° horizontally and 35° vertically.

A Gould Concept-32 computer provides the real-time simulation environment of the Advanced Flight Simulator (AFS). The Concept-32 network consists of 4 processors, linked by shared memory (figure 7.13). The Desk Management Processor (DMP) provides control of the simulator and generates tabular and graphic displays at the control desk. The Linkage Management Processor (LMP) provides communication to all simulator cueing systems, instruments, etc. The Primary Modelling Processor (PMP) executes the user’s aircraft model to provide an operating simulation. The Secondary Modelling Processor (SMP) may be used in conjunction with the PMP to provide an intelligent interface to any actual aircraft components that maybe connected to the simulator. The only language which is currently supported on the AFS processors is Fortran-77.

Aircraft models for the AFS are operated in a modelling environment known as SESAME (a System of Equations for the Simulation of Aircraft in a Modular Environment) [TOM4]. Essentially, SESAME requires an aircraft model in the form of software modules to calculate the total forces and moments on the aircraft, as functions of aircraft states and control inputs. From such a model, SESAME provides a real-time simulation by computing linear and angular accelerations; integrating to give velocities and displacements. SESAME is also used for communicating with the simulator cueing systems (motion, visual, sound, cockpit displays) to provide simulation control and operation.

LIGHT SIMULATOR

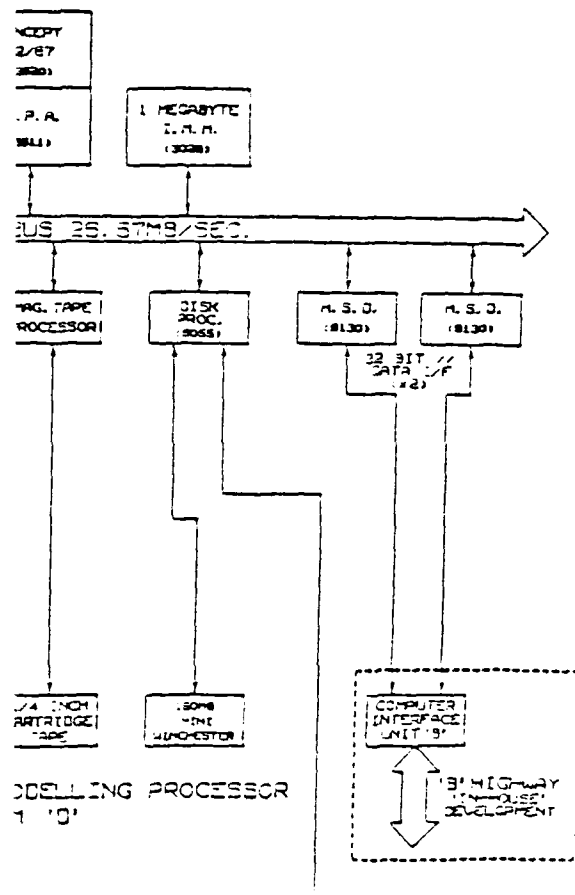
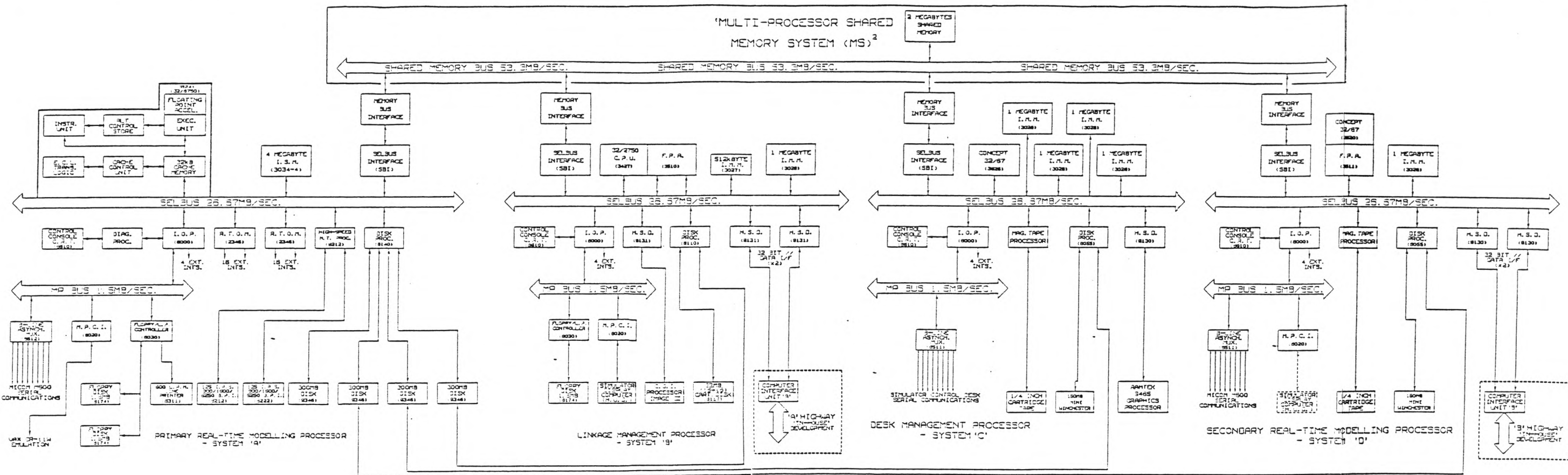


Figure 7.13 Concept-32 network

At present, the AFS is tied to a single, fixed, frame rate of 50 Hz (20 millisecond frame rate). To integrate from accelerations to velocities the Adam-Bashforth second-order algorithm is used and to integrate from velocities to positions the Trapezoidal algorithm is used. There is currently no automatic way of explicitly varying these algorithms.

No assumptions have been made regarding the physical aspects of the inceptors eg. whether the cyclic inceptor should be a central stick or a side arm controller. It is assumed that a means of independent input into each of the 4-axes - collective, roll, pitch and yaw is



available. In each of the pitch, roll and yaw inceptors there is a central deadband of 2.5% of the total stick travel. The collective inceptor is a displacement stick with linear scaling, there is no deadband. The 4 pilot inceptors were scaled so that a maximum displacement in either the collective, pitch, roll or yaw axes demanded 30 ft/s heave velocity, 30° of pitch attitude, 60° of roll attitude or 60°/s of heading rate respectively.

The 3 angular rate gyros - roll, pitch and yaw, are modelled as first order lags with a very small time constant and the 3 accelerometers - longitudinal, lateral and normal are modelled as second order. Attitude information is derived by integrating the outputs of the angular rate gyros through an attitude transformation. Height information is derived from the normal, lateral and longitudinal accelerometers, using estimates of attitude to transform the body axis accelerations into an earth referenced set.

The simulator cockpit was modified from a single-seat Hawk aircraft, with a subset of the Hawk instrumentation, pedals and a helicopter collective lever. The cockpit instrumentation comprised of an airspeed indicator, an attitude indicator, a torque meter and an altimeter. The cockpit seat was a Martin-Baker ejection seat developed to provide a Lynx-like vibration cue. The vibration was at a constant frequency of 22 Hz, with the amplitude scheduled with airspeed and normal acceleration.

A Head-Up Display (HUD) was used during the simulations to compensate for the lack of visual cues. A HUD is not advocated for future battlefield helicopters, since it is more important for the pilot to be able to look around during many battlefield scenarios.

7.8 Simulation Trials

Helicopter control law studies were scheduled for the AFS during August and September 1988. Before any trials work on the motion simulator could be conducted, it was necessary first to test the controller off-line. All control law design had been carried out using

the helicopter mathematical model HELISIM, however this software was not configured for real-time use. A subset of the HELISIM code was used to generate the Rationalized Helicopter Model (RHM), which is the model used for real-time helicopter simulation. Two versions of this model exist at RAE, Bedford; one which runs under the TSIM environment on a VAX 11/785 computer and the other on the Concept Gould-32 computer. The former is used to verify control law design for the RHM and the latter for real-time use.

The RHM was conceived to be identical to the HELISIM model, however during the trials, this was shown not to be the case. Identical blade angles gave rise to different dynamic responses between the two models. Also since the RHM is configured to run in a real-time environment, the aircraft equations of motion could not be easily linearized. Therefore it was not possible to derive a linear model of the RHM and to use linear analysis techniques to compare the RHM and HELISIM models. This linearization problem also meant that pole-zero analysis and frequency response techniques could not be used with confidence. It turns out that a different version of SESAME is used for the simulator model.

No universal interface existed for the implementation of controllers to the RHM. Therefore some programming effort was required to write a controller module which could accept any standard state space matrices. The controller module was written in standard ANSI FORTRAN 77 with variables passed through Global Common as opposed to arguments in subroutines, to save execution time. Global Common is an extension of the standard named common facility which creates Common blocks which reside in shared memory and thus are accessible to all the AFS processors. This stage of the trial took approximately 3 weeks to complete.

One of the major concerns in implementing H^∞ -controllers in real-time, was whether there would be sufficient frame time available to execute all the differential equations. Previous to this exercise, the controller had always used double precision arithmetic and 4th order Runge-Kutta integration with a 15 ms step size. This choice of frame time was to

facilitate the inclusion of high frequency rotor flapping dynamics in the HELISIM model. Rotor flapping dynamics are not included in the RHM because of the limited frame time of 20 ms.

The only user available integration routine for the RHM was rectangular. Unfortunately, this proved to be inadequate, since drifting in the states could readily be seen. This behaviour however, was not seen when using the HELISIM model, with the same integration type and step size. The discrepancies between the RHM and HELISIM models have yet to be fully resolved, a harmonization of models is clearly needed. Due to lack of time it was not possible to investigate the simplest integration routine that would work adequately for the controller. A 4th order Runge-Kutta integration proved to be too computationally demanding and when implemented with the controller, over framed by 3 ms. Instead a 2nd order Runge-Kutta was used with single precision arithmetic and this proved successful. To give an indication of the computation times, the helicopter model took about 8 ms of processing time and the controller 6 ms. Thus the controller still required quite a high percentage of processing time relative to the model.

Having verified that the augmented (RHM) helicopter exhibited responses similar to those seen using the HELISIM model, the next stage in the procedure was to transfer the controller module to the Gould computer. Two distinct processors are used to run aircraft models, the PMP and the SMP. The SMP is primarily used for development work. Facilities for file manipulation and editing are available on the Gould, and a package called SMV is provided which overlays the native MPX operating system, and emulates the VMS operating system of the VAX computer. Other facilities available include PRISM (Page Retrieval and Interaction for Simulation Management). PRISM is a Fortran program which allows users to display pages of Global Common variables and to interact with the real-time model by inspecting and changing variables.

A real-time model image is created by means of the macro RTLINK. On the SMP, control of the aircraft is by means of a screen of tabulated data provided by PRISM. Only very stable or highly augmented aircraft can be flown successfully for any period of time in this environment. Running the model this way confirms that the real-time image has been created and allows the stability of the model to be observed.

Having established that the augmented helicopter was stable using PRISM on the SMP, the next stage was to continue testing using the more sophisticated environment provided by the PMP. This allowed the model to be flown in a more representative manner using joysticks provided at the control desk and through the use of a HUD, desk displays, visual and sound cues.

At this stage of the study, it was established that both the high and low speed controllers were stabilizing over a very wide speed range. For the hover/low speed controller, stability was maintained for speeds up to about 150 knts. The forward flight controller was stabilizing down to about 20 knts with stability being maintained at all speeds above this. Performance however, degraded especially for large amplitude inputs outside the intended operating range of the controllers.

During this period of the evaluation it was decided that the control strategy for the forward flight controller warranted change. The inability to perform co-ordinated turns at high speeds, represented a major deficiency in the system. This deficiency was a direct result of controlling heading rate; it was concluded that sideslip control would have been better. However the control strategy at the hover/low speed region seemed natural and therefore all further evaluation was concentrated in this flight envelope.

7.8.1 Piloted Trials

Lt. Cmdr R.I. Horton of the Royal Navy was the test pilot who flew the sorties during

the piloted trials. Due to constraints on the simulator facility and pilot availability, it was only possible to conduct piloted trials on one day. These trials took place on the 27th September 1988 at RAE, Bedford. Prior to any flying, the pilot was briefed on the control strategy and the purpose of the trial. In addition, it was helpful to talk the pilot through the control strategy and demonstrate the control features at the start.

The trial was monitored via the control desk (figure 7.14) and instructions were passed to the pilot via an intercom. The trial was conducted with motion and sound; and forces on the control inceptors were generated by a feel force system. The pedals were centre sprung so that “feet off” would result in a zero heading rate demand. This facility was not available for the collective. The centre spring feature was to play a large roll in the trials. All conversation over the intercom was recorded and the simulator trial was also videoed. Problems with the chart recorders, limited the amount of useable information available from this media.

The pilot flew four sorties over a period of 3 hours. The first sortie was devoted to familiarization with the simulator and control strategy, and for practising a series of flying tasks. In the other sorties, the pilot was asked to assess the system performance using Cooper-Harper ratings. During the familiarization sortie, the pilot found the control strategy quite natural and so it took only a short time before he was confident enough to perform tasks.

Tasks

A series of tasks requiring the pilot to fly in the hover/low speed flight envelope and based on existing features of the terrain model were devised. The tasks were as follows:

Task 1 - Sidesteps

Task 2 - Quick hops

Task 3 - Free flight - low level, low speed, general manoeuvring



Figure 7.14 The AFS control desk

7.8.1.1 Results from task 1

The sidestep manoeuvre is intended to test the lateral axis response and to highlight any roll to pitch cross-couplings. The sidesteps were performed with a bank angle of 10° and 20° over lateral distances of 50 ft, 100 ft, 150 ft and 300 ft. The distances were chosen to test the helicopters response to bank angle hold, with changes in lateral velocity. It was found that roll to pitch coupling became significant when the lateral velocity exceeded about 40 knts, below this speed the couplings were quite acceptable. Heading also started to wander as the length of the step increased and problems were also experienced during roll reversals. Compared with the real aircraft the sidestep task was easier to perform, typical tolerances for the real aircraft in pitch attitude would be $\pm 5^\circ - 10^\circ$ and $\pm 5^\circ - 10^\circ$ in heading; the augmented helicopter was below these tolerances. The main criticism was the pitch axis coupling and the inability to maintain heading accurately.

The difficulty experienced with heading hold was attributed to the pedals not being centred properly so that there was always a small demand in heading rate as a result of leaving the pedals free. This of course is more prominent in a rate system as opposed to an attitude system and perhaps some form of heading hold is required.

The sidestep manoeuvre is difficult to perform well on the real aircraft and the pilot commented that it was one of the few controls systems he had flown that was able to perform this task well. Overall he rated the short sidesteps with Level 1 handling qualities with a Cooper-Harper rating (CHR) of 3 and Level 2 for the longer steps with a CHR of 4. The workload was significantly reduced with the controller implemented.

As a comparison, the pilot tried to perform the same manoeuvres without the controller but with autostabs in. However, the pilot did not consider the model as representative of the Lynx and deemed the helicopter unflyable. It was pointless trying to do a comparative study as the pilot said he would have to rate any manoeuvres with a CHR of 10 as he

considered there would be some time during flight where he would lose control of the aircraft.

Figure 7.15 shows the amount of control activity required to perform the sidestep manoeuvres with the H^∞ controller. It can be seen that the sidesteps have been performed with only movement from the lateral cyclic stick and very little need for any adjustments from the other inceptors. It was not possible to give a direct comparison of the control activity for the raw aircraft plus autostabs, but to give an indication of how difficult it is to fly, figure 7.16 shows the stick positions required to keep the helicopter in a trimmed position. It can be seen from this figure that there is a large amount of control activity and hence large pilot workload.

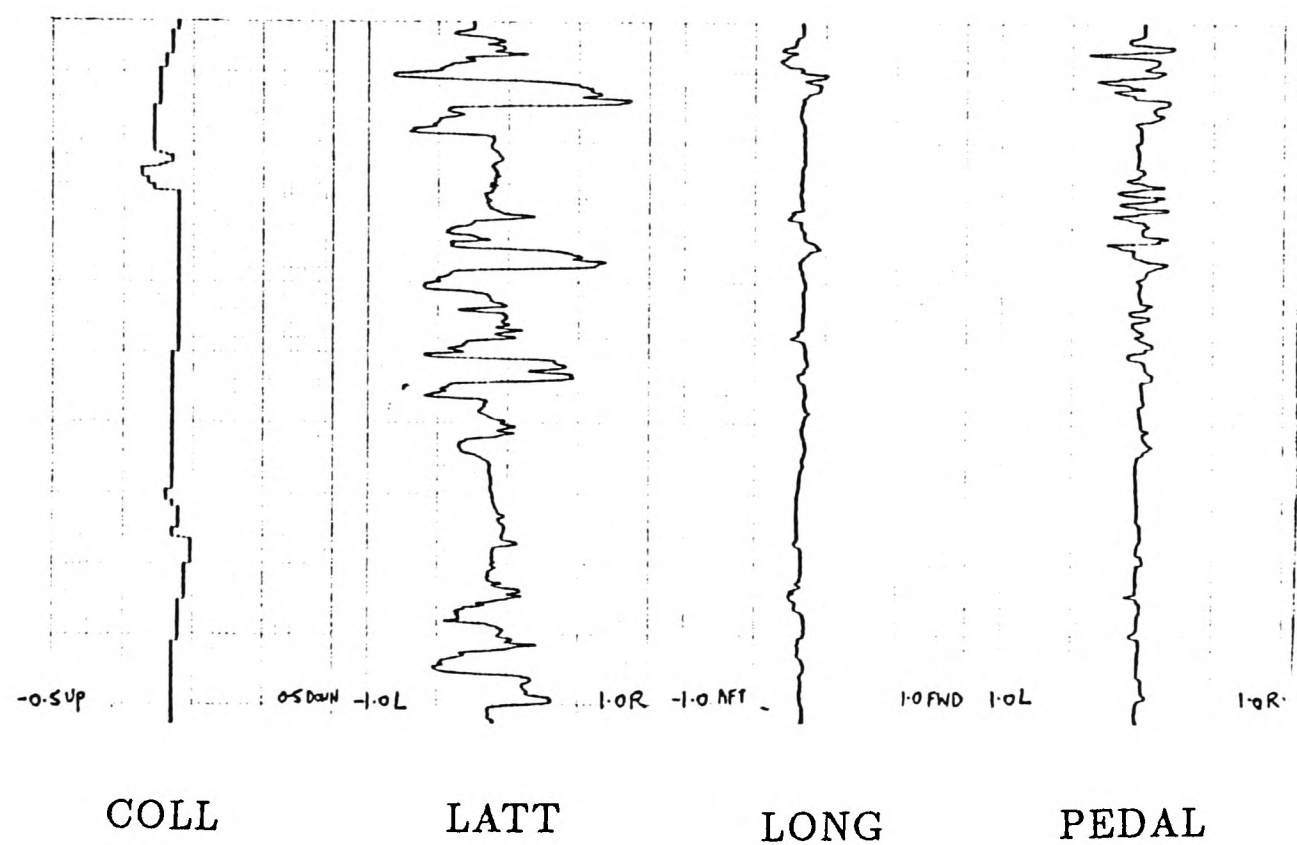


Figure 7.15 Control activity for sidestep manoeuvres with H^∞ -controller

The time scales for figures 7.15 and 7.16 are the same.

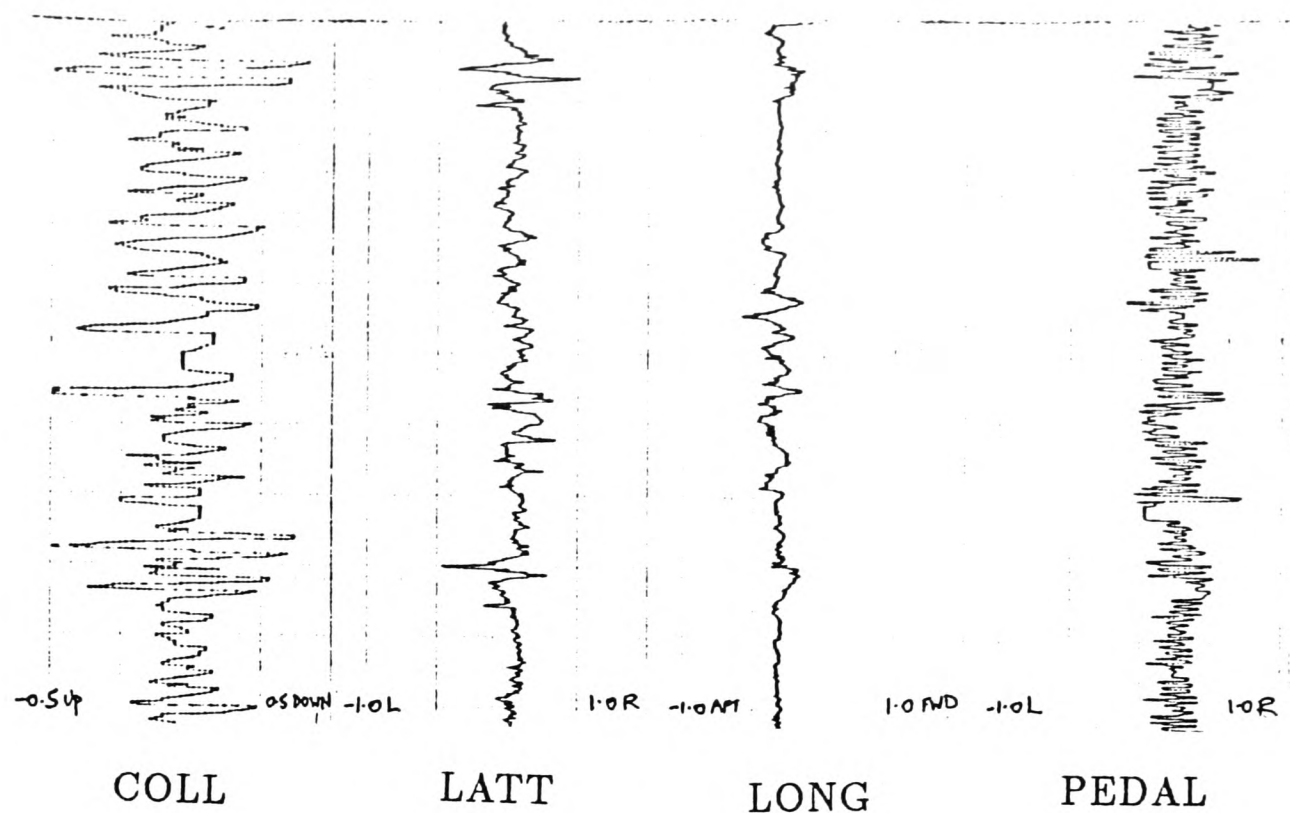


Figure 7.16 Control activity required to trim raw helicopter plus autostabs

7.8.1.2 Results from task 2

The quick hop manoeuvre is intended to test the longitudinal axis response. Starting from the hover flight position the pilot would pitch the nose of the helicopter down, to fly forward a certain distance and then pitch the nose of the helicopter up to decelerate. The pilot was assigned tasks of 5° , 10° and 15° pitch down attitude, which he flew between two hedges roughly 1000 ft apart. The 5° and 10° attitude changes were relatively easy, with the helicopter dropping back into the hover after decelerating. The helicopter started to yaw the greater the pitch attitude demand, nevertheless the pilot gave these tasks level 1 handling qualities with a CHR of 3. At 15° nose down pitch attitude the rating degraded to a CHR of 4, since the workload was slightly increased. At attitude changes greater than this the pilot could not see, when pitching up, due the lack of visual cues. The pilot felt that the basic handling in the primary axis was good apart from the off-axis coupling in yaw. The above tasks could be performed to within $\pm 1^\circ$ tolerance of the pitch angle

required. At the 15° nose down attitude task, the helicopter was attaining speeds up to 40 knts.

In order to test the ability of the controller to maintain pitch attitude at higher speed, the quick hop manoeuvres were performed over a distance of 2000 ft. For this task, distance markers on the runway were used. For a 5° nose down pitch attitude over 2000 ft, height control proved not to be a problem and heading was maintained within $\pm 1^\circ$. However pilot workload was increased a little so a CHR of 4 was given for this task. For a 10° pitch attitude demand, the high nose up position required to decelerate the helicopter caused the pilot to lose all visual cues. It was therefore not possible to rate this task, due to the limitations of the simulator.

7.8.1.3 Results from task 3

As a test for the controller, the pilot flew the Serpent course, which is probably the most difficult task on the terrain belt. This course is used to test a vehicle's manoeuvrability through "aggressive" flying. During the Westland simulator trials which were conducted in 1987, no pilot managed to complete this course without practicing it about 10 times. Lt Cmdr Horton flew this course successfully first time. During this task speeds of up to 60 knts were being attained. It has to be said however, that the pilot had to work quite hard during tight turns.

The pilot then flew a Nap of the Earth (NOE) course to assess the overall manoeuvrability of the augmented helicopter. Height proved to be the hardest to control during the NOE course, the pitch axis response felt rather underdamped and flying at 60-70 knts introduced large cross-axis couplings in yaw. The difficulty in controlling height was attributed to the lack of cues in the vertical axis. None of the pilots who flew this course during the Westland trials felt that it was possible without height hold and speed hold.

7.9 Concluding Remarks

Handling qualities have been defined and a rationale for their ratings established. A large amount of thought and wisdom has gone into these original scales which demonstrates why subsequent improvement has been so difficult.

Comparison of the augmented Lynx helicopter with the handling qualities design specification gave generally level 1 handling qualities using computer simulation. Where requirements were quantified, comparisons were drawn at the hover and at a number of airspeeds within the flight region applicable. It is not clear however, whether a computer model used for demonstrating compliance, is of a sufficient standard to yield dependable results. For example, the helicopter model does not yet incorporate certain hardware characteristics such as detailed actuator modelling, which if included might degrade the results obtained. Therefore it is important to assess the sensitivity of the handling qualities results to variations in the level of complexity of the aircraft.

Although the control laws behaved as predicted in off-line simulation, the use of pilots and real-time simulation was important in identifying areas in which the control laws needed further development, but which had not been highlighted in off-line simulation.

Due to the inability to centre the collective and pedals exactly, small rates were constantly being demanded in these axes when controls were left “hands and feet off”. Therefore slight adjustments were constantly having to be made in these channels. This unfortunately made the assessment of how well the controller was holding height rate and heading rate difficult.

Overall the pilot commented that there were not many systems that he had flown which would allow sidesteps or quickhops to be performed with so much aggression or accuracy. There were very good responses to turns in the hover and height control was easier especially during accelerations with nose down pitch. This allowed more accurate tracking

because the pilot did not need to compensate for height which in turn allowed him to fly more accurately. The pilot felt that it was a “good control system”, he felt confident when flying the NOE course at low level. He commented that he had never taken any system so low or so fast round that particular course. At high speed the controller was still more pleasant to fly than the basic Lynx and is certainly flyable. However a rate system is needed for up and away flight.

The Cooper-Harper results from the simulation exercise need to be interpreted with careful consideration of the simulator characteristics as they impinge greatly on the assessment of the control laws. The simulator lacks fundamental qualities required to assess the adequacy of the control laws in many manoeuvres. These include greater motion and a wider field of view. A poor simulation tends to mask any advantages/disadvantages of the control laws. It is difficult to compare the real Lynx with the augmented Lynx in the simulator. What is needed is an airborne ACT Lynx, where control laws can be tested in the real system.

CHAPTER 8

CONCLUSIONS

8.1 Summary

This thesis has presented the design of a full authority flight control system for a Lynx helicopter using H^∞ -optimization. The dynamic responses of the augmented helicopter were evaluated against military helicopter handling qualities specifications. It was found that the helicopter showed much better dynamic characteristics than those required by the specifications.

In the context of robustness, amongst the constraints to which control systems must be designed are those associated with the mathematical model. Traditionally, the approach to uncertainties in the mathematical model has been to build in suitable levels of gain and phase margin into the control system. In this thesis, singular values have been used to identify an acceptable variation in aircraft dynamics for which stability is assured. It is probably true that minimax worst-case designs cannot be completely avoided if analytic models for plant uncertainty are not available.

The development of some new algorithms to perform the γ -iteration has greatly improved the speed of convergence to the optimal solution, for the H^∞ -procedure. One of the first 4-block model-matching problems has been presented and it is shown that it is possible to realize a H^∞ -controller using real-time software. Results from the first ever real-time piloted simulation indicate that the control laws have been successful in reducing pilot workload.

Recent results by Doyle et al [DOY5] have, however, superseded the above algorithms.

8.2 Recommendations For Further Research

The dynamic compensator produced by the H^∞ -optimal control technique cannot be implemented directly on the helicopter since there are some practical constraints which need to be overcome.

- (i) The H^∞ -design process produces controllers in which the state matrices are not necessarily “sparse”. This is easily seen if one designs a controller for a diagonal multi-variable system, the controller is not necessarily diagonal. Therefore, in general the controller may be more complex than necessary. Hence an interconnection of the controller with the actual helicopter flight control system would lead to a large and complex system. It is not clear whether all the terms in the state matrices play a major role or whether some terms can be ignored because they have negligible effect. It is not easy to determine this without a clear understanding of the controller structure and which elements perform what task.

The interpretation of the controller is always going to be a major problem with any optimal synthesis method where the designer only has a “handle” on the plant input-output behaviour. This stumbling block needs to be overcome if optimal control techniques are to play a major role in the forthcoming generation of aircraft flight control systems.

- (ii) It will be necessary to have as a minimum, a scheduling of controller gains with forward speed, in order to produce consistent and coherent dynamics throughout the flight envelope. The augmented helicopter must also encapsulate the difference between hover/low speed flight and forward flight behaviour which necessitates different control strategies. Therefore a discontinuity in flight control laws exists around the 40 knots speed region. This blending of control laws is a major concern and attention needs to be drawn to this problem.

- (iii) A prime concern of the automatic flight system is its behaviour under loss of sensor information. A total failure of the system would lead to a dynamic response similar to the bare airframe, which is unacceptable with regard to handling qualities. Single component failure is more likely. In such cases the system will appear to have a different configuration and the acceptability of the degraded system needs to be evaluated.

Much of this work has concentrated on the robust stability problem and some attention needs to be focussed on robust performance. That is the preservation of performance levels under perturbations; this can be dealt with by the μ analysis of Doyle. The μ synthesis problem arises when a H^∞ performance requirement is desired for a system with plant uncertainty in the form of structured perturbations. This problem does not yet have a complete solution.

Recent results in the development of the H^∞ -synthesis method have shown that the order of the optimal controller is equal to the size of the plant plus dynamic weights. Although this is much smaller than researchers originally thought, the order of the controller can still be quite large depending on the complexity of the plant model. This has overriding implications on implementation where it is desirable to keep the controller as simple as possible. Therefore the development of procedures for producing robust low order compensators from robust high order compensators would be useful. To produce a robust low order compensator, the loop transfer matrices corresponding to the high and reduced order compensators must be nearly the same. The basic appeal of the classical multivariable frequency domain design techniques such as Characteristic loci and Inverse Nyquist Array is that in many cases they produce lower order and simpler compensators than the H^∞ -procedure.

The control laws designed in this thesis need to be tested in a more complex modelling environment than is currently available. The current model of the helicopter contains no

engine dynamics or realistic actuator dynamics. The need for better simulation facilities were highlighted during the piloted trials. The large motion simulator being currently built at RAE, Bedford may help to alleviate some of the problems.

Some research effort needs to be directed to an investigation into the use of a side-arm multi-axis controller to replace the conventional helicopter inceptors. This is perhaps more concerned with the ergonomics of the helicopter equipment. But such a side-arm controller will require control laws to totally decouple each axis, since there is very little movement available in such an inceptor. Pilots' like the ability to sit up in the aircraft and not to have to stretch to operate the controls, which can make the helicopter fatiguing to fly and limit its handling qualities.

Although only one day was available for piloted simulation studies, the results from that trial were encouraging. They showed that it was feasible to use an H^∞ -controller for improving the handling qualities of a typical combat helicopter. It is envisaged in the ACT Lynx programme that the real aircraft will carry an onboard computer which will be used for management of the control system. At present it is not known what processing power will be available and this might limit the complexity of the controllers that can be used. Far and away the most difficult task before any control scheme can be implemented on the real aircraft is to demonstrate airworthiness. This is a complex procedure whereby all software must be shown to be reliable and loss of any sensors or failure of any components is not detrimental to the safety of the vehicle.

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