Centralisers and Amalgams of Saturated Fusion Systems



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For my wife, Rachel and our son, Jude

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Abstract

In this thesis, we mainly address two contrasting topics in the area of saturated fusion systems. The first concerns the notion of a centraliser of a subsystem $\mathcal{E}$ of a fusion system $\mathcal{F}$, and we give new proofs of the existence of such an object in the case where $\mathcal{E}$ is normal in $\mathcal{F}$. The second concerns the development of the theory of 'trees of fusion systems', an analogue for fusion systems of Bass–Serre theory for finite groups. A major theorem finds conditions on a tree of fusion systems for there to exist a saturated completion, and this is applied to construct and classify certain fusion systems over p-groups with an abelian subgroup of index p. Results which do not fall into either of the above categories include a new proof of Thompson's normal p-complement Theorem for saturated fusion systems and characterisations of certain quotients of fusion systems which possess a normal subgroup.

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Introduction

The theory of saturated fusion systems is an important and growing area of mathematical research. It has come into prominence over the course of the last two decades and its origins are usually traced back to the work of Lluís Puig. The structures themselves are very natural objects to consider: for each prime p and finite p-group S, a saturated fusion system on S is a category whose objects are subgroups of S and whose morphisms are certain group monomorphisms. Every finite group G generates a saturated fusion system over a Sylow p-subgroup of G where morphisms are G-conjugation maps. Modular representation theory and algebraic topology also provide examples, coming from blocks of finite groups and certain 'p-complete' spaces respectively. Although saturated fusion systems can be induced by finite groups, this is not always the case and we say that a fusion system is *exotic* if there is no finite group which induces it.

There are two major aims to this thesis. One is a broader understanding of how multiple saturated fusion systems can be 'amalgamated' to produce another one - primarily (but not exclusively) as a means of generating exotic examples. Secondly, we wish to investigate properties of the centraliser of a subsystem inside a fusion system. The first of these aims was inspired by work of Broto, Levi and Oliver, while the second is based on work of Aschbacher.

Recall that a tree of groups consists of a tree $\mathcal{T}$ with an assignment of groups to vertices and edges, and monomorphisms from 'edge groups' to incident 'vertex groups'. The completion of a tree of groups is the free product of all vertex groups modulo relations determined by the monomorphisms. The theory of trees of groups is essentially what is called 'Bass–Serre theory' in the literature and in Chapter 2, we extend this theory to the category of fusion systems (over a fixed prime p) by attaching fusion systems to vertices and edges, and injective morphisms from edge to vertex fusion systems. As in the case for groups, the completion of a 'tree of fusion systems' is a colimit for the natural diagram, and we find conditions which render this the fusion system generated by each of those attached to a vertex.

One naturally obtains a tree of fusion systems from a tree of finite groups by replacing each group in the tree by its fusion systems at the prime p. Two questions arise at this point:

(1) Is the completion of this tree of fusion systems the fusion system of the completion of the underlying tree of groups?

(2) Is it possible to prove that the completion is saturated 'fusion theoretically,' i.e. without reference to the groups themselves?

We answer (1) in the affirmative essentially by applying a straightforward lemma of Robinson concerning conjugacy relations in completions of trees of groups:

Theorem 1 *Let $(\mathcal{T}, \mathcal{G})$ be a tree of finite groups and write $\mathcal{G}_\mathcal{T}$ for the completion of $(\mathcal{T}, \mathcal{G})$. Let $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ be a tree of fusion systems induced by $(\mathcal{T}, \mathcal{G})$ which satisfies (H) so that there exists a completion $\mathcal{F}_\mathcal{T}$ for $(\mathcal{T}, \mathcal{F}, \mathcal{S})$. The following hold:*

(a) $\mathcal{S}(v_*)$ is a Sylow p-subgroup of $\mathcal{G}_\mathcal{T}$.

(b) $\mathcal{F}_{\mathcal{S}(v_*)}(\mathcal{G}_\mathcal{T}) = \mathcal{F}_\mathcal{T}$.

In this theorem, (H) is a condition on $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ which forces the existence of a completion $\mathcal{F}_\mathcal{T}$ on the p-group $\mathcal{S}(v_*)$ associated to a fixed vertex v_* of the underlying tree $\mathcal{T}$. To answer question (2) above, we require the introduction of some more terminology. Alperin's Theorem asserts that conjugacy in a saturated fusion system $\mathcal{F}$ on S is determined by the '$\mathcal{F}$-essential' subgroups and S. Conversely a deep result of Puig asserts that whenever conjugacy in $\mathcal{F}$ is determined by the '$\mathcal{F}$-centric' subgroups, $\mathcal{F}$ is saturated if the saturation axioms hold between such subgroups. We exploit both of these facts in the proof of the following theorem.

Theorem 2 *Let $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ be a tree of fusion systems which satisfies (H) and assume that $\mathcal{F}(v)$ is saturated for each vertex v of $\mathcal{T}$. Write $S := \mathcal{S}(v_*)$ and $\mathcal{F}_\mathcal{T}$ for the completion of $(\mathcal{T}, \mathcal{F}, \mathcal{S})$. Assume that the following hold for each $P \leq S$:*

(a) If P is $\mathcal{F}_\mathcal{T}$-conjugate to an $\mathcal{F}(v)$-essential subgroup or $P = \mathcal{S}(v)$ then P is $\mathcal{F}_\mathcal{T}$-centric.

(b) If P is $\mathcal{F}_\mathcal{T}$-centric then $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P, \mathcal{F})$ is a tree.

Then $\mathcal{F}_\mathcal{T}$ is a saturated fusion system on S.

Theorem 2 in the case where $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ is induced by a tree of groups $(\mathcal{T}, \mathcal{G})$ is [7, Theorem 4.2], and can be deduced from Theorem 2 by applying Theorem 1. The novelty of our approach is the introduction of the graph $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P, \mathcal{F})$ called the 'P-orbit graph' (defined for each $P \leq \mathcal{S}(v_*)$) which gives detailed information about the way in which P 'acts' on $\mathcal{F}_\mathcal{T}$. Condition (a) ensures that conjugacy in $\mathcal{F}_\mathcal{T}$ is determined by the $\mathcal{F}_\mathcal{T}$-centric subgroups and (b) ensures that the saturation axioms hold between such subgroups.

Results like Theorem 2 have already been applied extensively in the literature to classify fusion systems over specific (families of) p-groups. Our next major result provides a new way of doing this by finding conditions which allow one to extend a fusion system by p'-automorphisms. Recall that a subgroup H of a finite group G is *strongly p-embedded* if $H < G$, H contains a Sylow p-subgroup of G and $H \cap H^g$ is a p'-group for each $g \in G \backslash H$. Also, if Q is a subgroup of a fusion system $\mathcal{F}$ then $\mathrm{Out}_\mathcal{F}(Q)$ is the group $\mathrm{Hom}_\mathcal{F}(Q, Q)/\mathrm{Inn}(Q)$.

Theorem 3 *Let $\mathcal{F}_0$ be a saturated fusion system on a finite p-group S. For $1 \leq i \leq m$, let $Q_i \leq S$ be fully $\mathcal{F}_0$-normalised subgroups with $Q_i\varphi \not\leq Q_j$ for each $\varphi \in \mathrm{Hom}_{\mathcal{F}_0}(Q_i, S)$ and $i \neq j$. Set $K_i := \mathrm{Out}_{\mathcal{F}_0}(Q_i)$ and choose $\Delta_i \leq \mathrm{Out}(Q_i)$ so that K_i is a strongly p-embedded subgroup of Δ_i. Write*

$$\mathcal{F} = \langle \{\mathrm{Hom}_{\mathcal{F}_0}(P, S) \mid P \leq S\} \cup \{\Delta_i \mid 1 \leq i \leq m\} \rangle_S.$$

Assume further that for each $1 \leq i \leq m$,

(a) Q_i is $\mathcal{F}_0$-centric (hence $\mathcal{F}$-centric) and minimal (under inclusion) amongst all $\mathcal{F}$-centric subgroups; and

(b) no proper subgroup of Q_i is $\mathcal{F}_0$-essential.

Then $\mathcal{F}$ is saturated.

This will be proved in Section 3.1. In Chapter 3 we apply Theorem 3 to determine 'simple' fusion systems $\mathcal{F}$ over p-groups S with a unique abelian subgroup A at index p. This is a particularly rich class of p-groups since many finite simple groups possess Sylow p-subgroups which lie in this class, and an analysis of their fusion systems reveals that they are almost always simple[1]. The classification of such fusion systems $\mathcal{F}$ is divided into two cases according to whether or not A is $\mathcal{F}$-essential. In both cases,

[1]This follows from unpublished work of Oliver. These observations promote a general philosophy that the theory of (simple) fusion systems is an extension of the theory of simple groups.

Bob Oliver has shown that $\mathcal{F}$ is determined by the action of $\mathrm{Out}_{\mathcal{F}}(S)$ on the groups $Z(S)$ and S/A, and lists the subgroups whose automorphisms generate $\mathcal{F}$. The fact that $\mathcal{F}$ is *uniquely* determined by this action is a consequence of Theorem 3. When A is $\mathcal{F}$-essential, A may be regarded as a module on which its group of $\mathcal{F}$-automorphisms $\mathrm{Aut}_{\mathcal{F}}(A)$ acts. It can be shown that this action has a fairly restrictive structure and this leads to a classification theorem concerning which actions can arise. Here, we will consider only the case where $\mathrm{Aut}_{\mathcal{F}}(A)$ is a bicyclic extension of a classical group of Lie type, but remark that all other families of simple groups can be dealt with similarly. Recall that a fusion system $\mathcal{F}$ on a finite p-group is reduced if it has no normal subgroups, no normal subsystems on S and no normal subsystems of p-power index.

Theorem 4 *Let p be a prime and G be a bicyclic extension of classical group of Lie type with a non-normal Sylow p-subgroup U order p such that $|\mathrm{Aut}_G(U)| = p-1$. Let A be an elementary abelian p-group regarded as a module on which G acts and write $S := A \rtimes U$. If there exists a reduced saturated fusion system $\mathcal{F}$ over S in which A is $\mathcal{F}$-essential then $(\mathrm{Aut}_{\mathcal{F}}(A) = G, p, \dim(A))$ appears on the following list:*

$\mathrm{Aut}_{\mathcal{F}}(A)$	p	$\dim(A)$	G	p	$\dim(A)$
$PSL_3(2)$	3	3	$PSp_6(2)$	5	7
$PSL_3(2).2$	7	3, 5, 7	$PSp_6(2)$	7	7
$2.PSL_3(2).2$	7	4, 6	$2.PSp_6(2)$	5	7
$PSL_4(2)$	5	7	$2.PSp_6(2)$	7	7
$PSL_4(2).2$	5	7	$PSp_4(4).4$	17	18
$PSL_4(2).2$	7	7	$PSU_4(2)$	5	6
$2.PSL_4(2)$	5	7	$PSU_4(2).2$	5	6
$2.PSL_4(2).2$	5	7	$PSU_5(2).2$	11	10
$2.PSL_4(2).2$	7	7, 8	$PSU_3(4).4$	13	12
$6.PSL_3(4).2_1$	5	6	$PSU_3(3).2$	7	6, 7
$6.PSL_3(4).2_1$	7	6	$6_1.PSU_4(3)$	5	6
$PSp_4(2)$	5	5	$6_1.PSU_4(3)$	7	6
$2.PSp_4(2)$	5	5	$6.PSU_4(3).2_2$	5	6
$3.PSp_4(2)$	5	5	$6.PSU_4(3).2_2$	7	6
$6.PSp_4(2)$	5	5			

This will be proved in Section 3.3, using a reduction which relies on the existence of a suitable bound for $\dim(A)$.

The second major focus of this thesis concerns centralisers of subsystems of fusion systems, discussed in Chapter 4. If $\mathcal{F}$ is a saturated fusion system on a finite p-group S and $\mathcal{E}$ is a saturated subsystem of $\mathcal{F}$ on T then an S-centraliser of $\mathcal{E}$ is a subgroup

X of S, maximal with respect to the property that $C_{\mathcal{F}}(X)$ contains $\mathcal{E}$. Here $C_{\mathcal{F}}(X)$ is a fusion system on $C_S(X)$ containing all $\mathcal{F}$-morphisms which extend in $\mathcal{F}$ to ones which act trivially on X. Clearly any such subgroup satisfies $X \leq C_S(T)$, so an $\mathcal{F}$-centraliser of $\mathcal{E}$ should be a saturated fusion system on X contained in $C_{\mathcal{F}}(T)$. The problem with this definition is that X may not be unique, so we need conditions which force this to be the case. It is easy to see that this is equivalent to asking whether the set $C_S(\mathcal{E})$ of elements g of S for which $\mathcal{E}$ is contained in $C_{\mathcal{F}}(\langle g \rangle)$ is a group. By considering examples, one quickly sees that a basic requirement is that $\mathcal{E}$ be 'weakly normal' in $\mathcal{F}$, but this is not always enough, so we look at two ways of tightening this condition:

Theorem 5 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let $\mathcal{E}$ be a weakly normal subsystem of $\mathcal{F}$ on $T \leq S$. Suppose further that at least one of the following holds:*

(a) $\mathcal{E}$ is normal in $\mathcal{F}$; or

(b) there exist linking systems $\mathcal{L}$ and $\mathcal{L}_0$ associated to $\mathcal{F}$ and $\mathcal{E}$ respectively such that $\mathcal{L}_0$ is a weakly normal subsystem of $\mathcal{L}$.

Then $C_S(\mathcal{E})$ is a strongly $\mathcal{F}$-closed subgroup of S and there exists a saturated fusion system $C_{\mathcal{F}}(\mathcal{E})$ on $C_S(\mathcal{E})$ which is of p-power index in $C_{\mathcal{F}}(T)$.

We will prove Theorem 5 in two stages. When $\mathcal{E}$ is normal in $\mathcal{F}$, work of Aschbacher already implies the conclusions of the result hold and in Section 4.2 we supply an improved version of the proof that $C_S(\mathcal{E})$ is a strongly $\mathcal{F}$-closed subgroup. The condition involving 'linking systems' in (b) concerns the association of a certain pair of compatible categories to $\mathcal{F}$ and $\mathcal{E}$ which fill in topological information. In this case, the result follows quickly from the definitions and is shown in Section 4.3. Some remarks concerning the definition of $C_{\mathcal{F}}(\mathcal{E})$ are in order. It is motivated by the following consequence of the Classification of Finite Simple Groups first shown by Gross in [19]: if G is a finite group with $O_{p'}(G) = 1$ and $H \trianglelefteq G$ then

$$O^p(C_G(S \cap H)) \cap S \leq C_G(H)$$

for each Sylow p-subgroup S of G. In the language of fusion systems, this implies that $C_S(\mathcal{E})$ contains the 'hyperfocal subgroup' of $C_{\mathcal{F}}(S \cap H)$ when $\mathcal{F}$ and $\mathcal{E}$ are induced by G and H respectively. Therefore (in this case) the existence of a unique saturated fusion system on $C_S(\mathcal{E})$ contained in $C_{\mathcal{F}}(T)$ follows from a general fact concerning subsystems of 'p-power index'.

Chapter 4 concludes with some general but related questions. The first concerns $\mathcal{F}$-centralisers when $\mathcal{F}$ is induced by a finite group G. The Z^*-Theorem for finite groups says that the centre of $\mathcal{F}$ is exactly $Z(G)$ (assuming $O_{p'}(G) = 1$), so it is natural to ask whether this statement can be generalised to one involving centralisers. In other words if $H \trianglelefteq G$ and $O_{p'}(H) = 1$ is it true that $C_S(\mathcal{E}) = C_S(H)$ for each Sylow p-subgroup S of G where $\mathcal{E}$ is the fusion system of H? We ask this question when H is simple and $C_G(H) = 1$, giving a complete answer in the case where p is odd. We end Chapter 4 with a brief look at 'normalisers' of subsystems.

Chapter 1 is an introductory chapter whose goal is to provide an appropriate setting for this work. We recall a number of relevant notions from group theory, category theory and homotopy theory before constructing fusion and linking systems from scratch. The author would like to draw the reader's attention to some new results which appear in this chapter which do not have an obvious bearing on the rest of the work. In Section 1.3.3 we give a short proof of the analogue of Thompson's normal p-complement Theorem for fusion systems, a version of which has already appeared in the author's masters dissertation. In Section 1.4.1 we give characterisations of the fusion systems of $\mathrm{Aut}_{\mathcal{F}}(Q)$ and $\mathrm{Out}_{\mathcal{F}}(QC_S(Q))$ when Q is a normal subgroup of a saturated fusion system $\mathcal{F}$ on S. The existence of these characterisations confirms a hypothesis of Craven that these systems should be realisable as quotients of $\mathcal{F}$.

Finally, we make some important remarks regarding notation. 'p' will be a prime number throughout. Elements of a finite group G typically act on the right. We reserve the superscript notation x^g for image of the conjugation action $g^{-1}xg$ of G on itself for each $x, g \in G$. Similarly if H is a subgroup of G we write H^g for the group $\{x^g \mid x \in H\}$. We will also frequently write c_g for the group homomorphism induced by conjugation by g, stating the domain where appropriate. Note that group homomorphisms will always act on the right and composition read from left to right. We extend this rule to morphisms in small categories which are not functor categories. Functors between categories will always act on the left and compose from right to left. These two conventions will interact especially often when dealing with linking systems (see Section 1.5).

Chapter 1

Background

This chapter is concerned with developing the theory of fusion systems and related objects from first principles whilst at the same time serving as a reference point for notation and elementary results which will be used throughout this thesis.

Section 1.1 will introduce certain group theoretic and category theoretic notions, many of which will be needed in Section 1.2. There, we introduce the main objects of study in this thesis, saturated fusion systems, starting with the definition of a fusion system and culminating in a proof of Alperin's Fusion Theorem. We develop some more theory in Section 1.3, where we define the 'K-normaliser subsystem' of a fusion system, introduce 'constrained' fusion systems and use these notions to give a new proof of Thompson's normal p-complement theorem for fusion systems. In Section 1.4 we define 'normal subsystems' and 'quotient systems' and introduce 'direct products' of fusion systems. This section also contains two new results, Theorems 1.50 and 1.52, concerning quotients of a fusion system which are always realisable as that of a finite group. In Section 1.5, we introduce 'linking systems' associated to fusion systems and state a fundamental theorem concerning their objects (Theorem 1.64). We end by defining what is meant by a 'normal subsystem' and 'automorphism' of a linking system.'

1.1 Preliminary Notions

1.1.1 Group Theory

Sylow's Theorem and Fusion in Finite Groups

The notation and terminology of this section is based on that found in [18]. Let G be a finite group and p be a prime number. A Sylow p-subgroup S of G is any subgroup

of G whose order is p^m, where $m = v_p(|G|)$. The following is a foundational result in finite group theory due to Sylow.

Theorem 1.1 (Sylow) *Let G be a finite group and p be a prime number. The following hold:*

(a) G possesses a Sylow p-subgroup.

(b) For each pair of Sylow p-subgroups S_1 and S_2 there exists $g \in G$ such that $g^{-1}S_1g = S_2$.

(c) If P is a p-subgroup of G then there exists a Sylow p-subgroup S of G such that $P \leq S$.

Theorem 1.1 implies that Sylow p-subgroups of a group G are unique up to isomorphism and contain representatives for each G-conjugacy class of p-subgroups of G. We may therefore regard a Sylow p-subgroup as a sort of 'group invariant' for G at the prime p. More generally, one can ask how much information can be associated to the pair (G, p) which is uniquely determined by the isomorphism type of G. Moreover, to what extent is G determined by such information? To answer these questions, we introduce the concept of fusion (conjugation) inside a p-group S. Given a triple of finite groups $H \leq K \leq G$, we say that K *controls G-fusion in H* if whenever a pair of elements of H are G-conjugate, they are also K-conjugate. We have the following result in the case where G possesses an abelian Sylow p-subgroup.

Lemma 1.2 *Let G be a finite group with an abelian Sylow p-subgroup S. Then $N_G(S)$ controls G-fusion in S.*

Proof. Suppose $x, y \in S$ are G-conjugate so that there exists $g \in G$ such that $x^g = y$. Since S is abelian, $S^g \leq C_G(x)^g = C_G(x^g) = C_G(y)$, so both S and S^g are Sylow p-subgroups of $C_G(y)$. By Theorem 1.1 (b) there is some $h \in C_G(y)$ such that $S^h = S^g$. But $gh^{-1} \in N_G(S)$ and $x^{gh^{-1}} = y^{h^{-1}} = y$, proving that x and y are $N_G(S)$-conjugate, as required. □

In other words, all the information concerning G-conjugacy of elements in S is provided by $N_G(S)$. Lemma 1.2 is a special case of the following more general result due to Alperin. Recall that a subgroup Q of a Sylow p-subgroup S of G is *extremal* if $N_S(Q)$ is a Sylow p-subgroup of $N_G(Q)$.

Theorem 1.3 (Alperin) *Let G be a finite group and S be a Sylow p-subgroup of G. Let $P, Q \leq S$ and $g \in G$ be such that $P^g = Q$. There exist extremal subgroups $Q_1, \ldots, Q_n$ of S and elements $x_1, \ldots, x_n$ with $x_i \in N_G(Q_i)$ such that*

(i) $g = x_1 x_2 \ldots x_n$;

(ii) $P^{x_1 x_2 \ldots x_i} \leq Q_{i+1}$ *for all* $0 \leq i \leq n-1$.

Proof. This is shown in [11, Theorem 1.29]. □

Alperin's Theorem shows that fusion in a Sylow p-subgroup S of a finite group G is a 'local' phenomenon. By this, we mean that G-conjugacy of subgroups of S is detected by H-conjugacy inside a set $\mathcal{X}$ of subgroups H of G. A minimum value for $|\mathcal{X}|$ measures the 'complexity' of fusion in G so it is natural to ask how small $|\mathcal{X}|$ can be. Lemma 1.2 implies that we may choose $\mathcal{X}$ such that $|\mathcal{X}| = 1$ whenever S is abelian, so one might infer from this that the complexity of fusion is determined by the isomorphism type of S. This is a reasonable hypothesis but in order to test it, we need the concept of a fusion system and therefore suspend our discussion of fusion until Section 1.2.

Finite p-groups and their Automorphisms

Let π be a set of primes. A natural number n is a π-number if $p \in \pi$ for each $p \mid n$. If G is a finite group, we say that G is a *π-group* if $|G|$ is a π-number and that an element $g \in G$ is a *π-element* if $o(g)$ is a π-number. If $\pi = \{p\}$, we will simply write 'p' in place of 'π' and if $\pi = \mathbb{P}\backslash\{p\}$, we write '$p'$' in place of '$\pi$,' where $\mathbb{P}$ denotes the set of all prime numbers. Since fusion systems are objects associated to finite p-groups, some abstract p-group theory will frequently be of use. First, we state an elementary fact concerning p-group actions on sets, from which two more basic facts about p-groups emerge.

Lemma 1.4 *Let P be a p-group and Ω be a set on which P acts. Then*

$$|\operatorname{Fix}_\Omega(P)| \equiv |\Omega| \bmod p,$$

where $\operatorname{Fix}_\Omega(P) := \{\omega \in \Omega \mid \omega^g = \omega \text{ for all } g \in P\}$.

Proof. For each $\omega \in \Omega\backslash\operatorname{Fix}_\Omega(P)$, $\operatorname{stab}_P(\omega) = \{g \in P \mid \omega^g = \omega\} < P$. Hence, by the Orbit-Stabiliser Theorem, p divides the order of $\Omega\backslash\operatorname{Fix}_\Omega(P)$ since it is a union of P-orbits. □

Corollary 1.5 *Let P be a p-group and let Q be a subgroup of P. The following hold:*

(a) If $Q < P$ then $Q < N_P(Q)$.

(b) If $Q \trianglelefteq P$ then $Z(P) \cap Q \neq 1$.

Proof. To prove (a), let $\Omega := P/Q = \{Qg \mid g \in P\}$ and consider the action of Q on Ω by right multiplication of cosets. Observe that $Q \in \mathrm{Fix}_\Omega(Q)$ so that by Lemma 1.4, $|\mathrm{Fix}_\Omega(Q)| \geq p \geq 2$ and there exists $g \notin Q$ such that $Qg \in \mathrm{Fix}_\Omega(Q)$. Equivalently, $Qgx = Qg$ for all $x \in Q$ so $g^{-1}Qg = Q$ and $g \in N_P(Q)\backslash Q$, as needed. To see (b), let $\Omega := Q$ so that P acts on Ω by conjugation. In this case, $\mathrm{Fix}_\Omega(P) = Z(P) \cap Q$ and since $1 \in \mathrm{Fix}_\Omega(P)$, $|\mathrm{Fix}_\Omega(P)| \geq p \geq 2$ by Lemma 1.4 again. □

Recall that an automorphism of a finite group G is a group isomorphism from G to itself. The set of all such maps forms a finite group which we denote by $\mathrm{Aut}(G)$. Given a set of primes π, a *π-automorphism* of G is a π-element of $\mathrm{Aut}(G)$. The following result asserts that a p'-automorphism α of a finite p-group P that acts trivially on successive quotients of an α-invariant normal chain must be trivial.

Lemma 1.6 *Let P be a finite p-group and α be a p'-automorphism of P. If there exists a normal chain*

$$1 = P_0 \trianglelefteq P_1 \trianglelefteq P_2 \trianglelefteq \cdots \trianglelefteq P_m = P$$

which satisfies the relations $P_i\alpha = P_i$ and $[P_i, \alpha] \leq P_{i-1}$ for each $i = 1, \ldots, m$, then $\alpha = \mathrm{Id}_P$.

Proof. Let α be as in the statement of the lemma and suppose we have a sequence of subgroups of P for which the stated relations hold. We proceed by induction on i, noting that $\alpha|_{P_1} = \mathrm{Id}_{P_1}$. Assume that $\alpha|_{P_j} = \mathrm{Id}_{P_j}$ for some $j > 1$ and let $x \in P_{j+1}$. Since $[P_{j+1}, \alpha] \leq P_j$, $x\alpha = xz$ for some $z \in P_j$ so that for any integer $k \geq 0$, $x(\alpha^k) = xz^k$ since $z\alpha = z$. Setting $k = o(\alpha)$, we have $x = xz^{o(\alpha)}$ and $z^{o(\alpha)} = 1$. Since α is a p'-automorphism and z is a p-element this implies that $z = 1$. Hence $x\alpha = x$ and $\alpha|_{P_{j+1}} = \mathrm{Id}_{P_{j+1}}$ since x was arbitrary. The result follows. □

The following is frequently a more useful formulation of Lemma 1.6:

Corollary 1.7 *Let P be a finite p-group and $A \leq \mathrm{Aut}(P)$. Suppose there exists a normal chain*

$$1 = P_0 \trianglelefteq P_1 \trianglelefteq P_2 \trianglelefteq \cdots \trianglelefteq P_m = P$$

of subgroups such that for each $\alpha \in A$, $P_i\alpha = P_i$ for all $0 \leq i \leq m$. The set of all $\beta \in A$ satisfying the relation $[P_i, \beta] \leq P_{i-1}$ for each $i = 1, \ldots, m$ is a normal p-subgroup of A.

Proof. The condition on A means there exists a well-defined homomorphism

$$\Psi : A \longrightarrow \prod_{i=1}^{m} \mathrm{Aut}(P_i/P_{i-1})$$

which sends $\alpha \in A$ to $(\bar{\alpha}_1, \ldots, \bar{\alpha}_m)$ where $\alpha_i = \alpha|_{P_i} \in \mathrm{Aut}(P_i)$ and $\bar{\alpha}_i$ is the induced automorphism of P_i/P_{i-1} for each $1 \leq i \leq m$. The kernel of Ψ is exactly the set

$$\mathrm{Ker}(\Psi) = \{\beta \in A \mid [P_i, \beta] \leq P_{i-1} \text{ for each } i = 1 \ldots m\}$$

and this forms a normal p-subgroup of A by Lemma 1.6, as needed. □

Normal p-subgroups and Strongly p-embedded Subgroups

We recall, for later use, the notation $O_\pi(G)$ to denote the largest normal π-subgroup of a finite group G. It is easy to see that this group is well defined since the product of two normal π-subgroups is again a normal π-subgroup. We also define $O^\pi(G)$ to be the smallest normal subgroup X of G whose quotient G/X is a π-group. Later, when dealing with saturated fusion systems, we will frequently meet 'strongly p-embedded' subgroups which are defined as follows.

Definition 1.8 Let G be a finite group. A subgroup H of G is *strongly p-embedded* if $H < G$, H contains a Sylow p-subgroup of G and $H \cap H^g$ is a p'-group for each $g \in G \backslash H$.

Writing $\mathrm{Syl}_p(G)$ for the set of all Sylow p-subgroups of G, it is a trivial consequence of Sylow's theorem that

$$O_p(G) = \bigcap_{S \in \mathrm{Syl}_p(G)} S.$$

Hence groups which possess a strongly p-embedded subgroup must also have the property that $O_p(G) = 1$. Examples of such groups G include $SL_2(p)$, the set of all 2×2 matrices with entries in $\mathbb{F}_p$ and with determinant 1. The next lemma and its corollary will allow us to construct many similar examples.

Lemma 1.9 *Let G be a finite group and define $H_S := \langle x \in G \mid S \cap S^x \neq 1 \rangle$ for each $S \in \mathrm{Syl}_p(G)$. If $H_S < G$ then H_S is a strongly p-embedded subgroup of G.*

Proof. Firstly $S \in \mathrm{Syl}_p(H_S)$, so it remains to show that $p \nmid |H_S \cap H_S^x|$ whenever $x \in G \backslash H_S$. We prove the contrapositive to this statement. Suppose that $x \in G$ is such that $p \mid |H_S \cap H_S^x|$, so that there exists an element $g \in H_S \cap H_S^x$ of order p. Now, $g, g^{x^{-1}} \in H_S$ and by Sylow's Theorem there exist elements $y, z \in H_S$ such that $g^y, g^{x^{-1}z} \in S$. Since $1 \neq g^y \in S \cap S^{z^{-1}xy}$ we have $z^{-1}xy \in H_S$ and therefore $x \in H_S$, as required. □

Corollary 1.10 *Let G be a finite group with a cyclic Sylow p-subgroup S of order p. If $O_p(G) = 1$ then $N_G(S)$ is a strongly p-embedded subgroup of G.*

Proof. Since $O_p(G) = 1$, $H_S = N_G(S) < G$ for each Sylow p-subgroup S of G. The result now follows from Lemma 1.9. □

Cohomology of Finite Groups

We do not give a full introduction to group cohomology here. We will content ourselves to say that to each group G, G-module M and natural number n one can associate a group $H^n(G, M)$ called the *n-th cohomology group of G* (with coefficients in M), which is unique up to isomorphism. These groups encode fundamental information about G and M. For example, when $n = 2$, $H^2(G, M)$ classifies all equivalence classes of extensions of G by M that give rise to the given action of G on M. One can also say something about extensions of non-abelian finite groups:

Theorem 1.11 *Let G and N be finite groups. Each extension*

$$1 \longrightarrow N \longrightarrow E \longrightarrow G \longrightarrow 1$$

determines a homomorphism $\psi_E : G \longrightarrow \mathrm{Out}(N)$ giving rise to an obstruction in $H^3(G, Z(N))$ which vanishes if and only if such an extension exists. Furthermore, the group $H^2(G, Z(N))$ acts transitively on the set of all such extensions.

Proof. This is shown in [8, Theorems IV.6.6 and IV.6.7]. □

When $H \leq G$, there is a *restriction map*

$$H^n(G, M) \xrightarrow{\mathrm{res}^G_H} H^n(H, M)$$

for each G-module M and $n \geq 0$ (see [8, Section III.9]). Each $g \in G$ induces a well-defined map $H^n(H, M) \xrightarrow{g} H^n(H^g, M)$ and we define $z \in H^*(H, M)$ to be *G-invariant* if

$$\mathrm{res}^H_{H \cap H^g} z = \mathrm{res}^{H^g}_{H \cap H^g} zg$$

for each $g \in G$. The following result characterises G-invariant elements when M is an $\mathbb{F}_p G$-module for some prime p.

Theorem 1.12 *Let G be a finite group, p be prime and M be an $\mathbb{F}_pG$-module. For each $H \leq G$ with $p \nmid |G : H|$ and $n \geq 0$, res^G_H maps $H^n(G, M)$ isomorphically onto the set of G-invariant elements of $H^n(H, M)$.*

Proof. This follows from [8, Theorem III.10.3]. □

As an immediate consequence we have:

Corollary 1.13 *Let G be a finite group and M be an $\mathbb{F}_pG$-module. If H is a strongly p-embedded subgroup of G then res^G_H induces an isomorphism*

$$H^*(G, M) \xrightarrow{res^G_H} H^*(H, M).$$

There is a more general cohomology theory called *functor cohomology*, which one can associate to categories and which returns group cohomology as a special case. Indeed, functor cohomology provides a natural setting in which to define the cohomology of fusion systems. We make no use of this theory here, although it is now an important area of research. We refer the reader to [4, Section III.5.1] for a relevant introduction.

1.1.2 Category Theory

Categories and Functors

A category $\mathcal{C}$ consists of a class $\mathrm{Ob}(\mathcal{C})$ of *objects* and a class $\mathrm{Mor}(\mathcal{C})$ of *morphisms* or *maps* between the objects. Each morphism $\varphi \in \mathrm{Mor}(\mathcal{C})$ has a unique 'source object' $A \in \mathrm{Ob}(\mathcal{C})$ and a unique 'target object' $B \in \mathrm{Ob}(\mathcal{C})$. In this case we say that 'φ is a morphism from A to B'. Given a pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, we write $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for the *hom-class* of all morphisms from A to B. For each $A, B, C \in \mathrm{Ob}(\mathcal{C})$ there exists a binary operation

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C)$$

called *composition of morphisms*. For $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $\psi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, the composition of φ and ψ is written $\varphi \circ \psi$ and denotes "φ followed by ψ." We assume that composition satisfies the following two axioms:

- (associativity) For each $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$ and each $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B), \psi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ and $\rho \in \mathrm{Hom}_{\mathcal{C}}(C, D)$, $(\varphi \circ \psi) \circ \rho = \varphi \circ (\psi \circ \rho)$.
- (identity) For each $A \in \mathrm{Ob}(\mathcal{C})$ there is an *identity morphism* $\mathrm{Id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ which satisfies the relation $\mathrm{Id}_A \circ \varphi = \varphi = \varphi \circ \mathrm{Id}_A$ for each $B \in \mathrm{Ob}(\mathcal{C})$ and $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

Categories are ubiquitous objects in mathematics with obvious examples being the category $\mathfrak{Set}$ of sets and functions between sets, $\mathfrak{Top}$ of topological spaces and continuous maps between spaces and $\mathfrak{Grp}$, the category of groups and group homomorphisms. In this thesis, we will primarily be concerned with *finite categories* defined as those which possess a finite number of objects and morphisms. For example, every finite group G gives rise to a finite category $\mathcal{B}(G)$ which has one object o_G and a morphism $\varphi_g \in \mathrm{Hom}_{\mathcal{B}(G)}(o_G, o_G)$ for each $g \in G$ and where composition in $\mathcal{B}(G)$ induced by composition of elements in G.

Given a pair of categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* F from $\mathcal{C}$ to $\mathcal{D}$ is a map which sends each $A \in \mathrm{Ob}(\mathcal{C})$ to an object $F(A) \in \mathrm{Ob}(\mathcal{D})$ and associates to each morphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ (with $A, B \in \mathrm{Ob}(\mathcal{C})$) a morphism $F(\varphi) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$. Furthermore F satisfies the following two axioms:

- $F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$ for each $A \in \mathrm{Ob}(\mathcal{C})$.
- $F(\varphi \circ \psi) = F(\varphi) \circ F(\psi)$ for each $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $\psi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ with $A, B, C \in \mathrm{Ob}(\mathcal{C})$.

For example, given a pair of finite groups G and H and a group homomorphism $\psi \in \mathrm{Hom}(G, H)$, there exists a functor F_ψ from $\mathcal{B}(G)$ to $\mathcal{B}(H)$ which sends o_G to o_H and φ_g to $\varphi_{g\psi}$ for each $g \in G$. Recall that a functor F from $\mathcal{C}$ to $\mathcal{D}$ is *contravariant* if F is arrow reversing, that is if $F(\varphi) \in \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$ for each $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $F(\varphi \circ \psi) = F(\psi) \circ F(\varphi)$ for each $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $\psi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$.

Natural Transformations and Automorphisms of Small Categories
Given a pair of categories $\mathcal{C}$ and $\mathcal{D}$ and a pair of functors F and G from $\mathcal{C}$ to $\mathcal{D}$, a *natural transformation* η from F to G associates to each $A \in \mathrm{Ob}(\mathcal{C})$ a morphism $\eta_A \in \mathrm{Hom}_{\mathcal{D}}(F(A), G(A))$ such that for all $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ \Big\downarrow{\scriptstyle F(\varphi)} & & \Big\downarrow{\scriptstyle G(\varphi)} \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array}$$

commutes. η is a *natural isomorphism* if η_A is an isomorphism for each $A \in \mathrm{Ob}(\mathcal{C})$, where two objects X and Y of a category $\mathcal{D}$ are *isomorphic* if there exist maps $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(Y, X)$ such that $f \circ g = \mathrm{Id}_X$ and $g \circ f = \mathrm{Id}_Y$. We write $F \simeq G$ if F and G are naturally isomorphic. Denote by $\mathrm{Id}_{\mathcal{C}}$ and $\mathrm{Id}_{\mathcal{D}}$ respectively the identity functors of $\mathcal{C}$ and $\mathcal{D}$, which send objects and morphisms to themselves. A

functor F from $\mathcal{C}$ to $\mathcal{D}$ is an *equivalence* if there exists another functor G from $\mathcal{D}$ to $\mathcal{C}$ such that

$$F \circ G \simeq \mathrm{Id}_{\mathcal{D}} \text{ and } G \circ F \simeq \mathrm{Id}_{\mathcal{C}}.$$

In other words, F and G induce bijections between isomorphism classes of objects.

If $\mathcal{C}$ is finite then the set of all equivalences from $\mathcal{C}$ to $\mathcal{C}$ forms a finite group with identity element $\mathrm{Id}_{\mathcal{C}}$ which we denote by $\mathrm{Aut}(\mathcal{C})$. For example, we see without too much difficulty that $\mathrm{Aut}(\mathcal{B}(G)) \cong \mathrm{Aut}(G)$. In fact, we can also construct the analogue of an *outer automorphism* for categories as follows: $\mathrm{Out}(\mathcal{C})$ is the set of all natural isomorphism classes of equivalences from $\mathcal{C}$ to $\mathcal{C}$. This notation is justified by the fact that elements of $\mathrm{Out}(\mathcal{B}(G))$ are in one-one correspondence with elements of $\mathrm{Out}(G)$ (proving that $\mathrm{Out}(\mathcal{B}(G))$ is a group).

Later, we will need slightly more general classes of functors from a category to itself. If $\mathcal{C}$ and $\mathcal{D}$ are finite categories, we say that $\mathcal{C}$ *is a subcategory of* $\mathcal{D}$ (and write $\mathcal{C} \subseteq \mathcal{D}$) if $\mathrm{Ob}(\mathcal{C}) \subseteq \mathrm{Ob}(\mathcal{D})$ and $\mathrm{Mor}(\mathcal{C}) \subseteq \mathrm{Mor}(\mathcal{D})$. Two elements F and G of $\mathrm{Aut}(\mathcal{C})$ are said to be $\mathcal{D}$*-naturally isomorphic* if there exists a map η from F to G which associates to each $A \in \mathrm{Ob}(\mathcal{C})$ a morphism $\eta_A \in \mathrm{Hom}_{\mathcal{D}}(F(A), G(A))$ such that for all $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ \Big\downarrow{\scriptstyle F(\varphi)} & & \Big\downarrow{\scriptstyle G(\varphi)} \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array}$$

commutes. We denote the set of equivalence classes of elements in $\mathrm{Aut}(\mathcal{C})$ under $\mathcal{D}$-natural isomorphism by $\mathrm{Out}(\mathcal{C}, \mathcal{D})$. Observe that $\mathrm{Out}(\mathcal{C}, \mathcal{C}) = \mathrm{Out}(\mathcal{C})$.

Colimits

Given a pair of categories $\mathcal{C}$ and $\mathcal{D}$, write $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ for the category of functors and natural transformations from $\mathcal{D}$ to $\mathcal{C}$. Let Δ be the functor from $\mathcal{C}$ to $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ defined as follows: for each $X \in \mathrm{Ob}(\mathcal{C})$, $\Delta(X)$ is the *constant functor* which sends each $A \in \mathrm{Ob}(\mathcal{D})$ to X and each morphism $\varphi \in \mathrm{Hom}_{\mathcal{D}}(A, B)$ to Id_X. For each $\varphi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ with $(X, Y \in \mathrm{Ob}(\mathcal{C}))$, $\Delta(\varphi)$ is the *constant natural transformation* η defined by $\eta_A = \varphi$ for all $A \in \mathrm{Ob}(\mathcal{D})$.

Assume that $\mathcal{D}$ is finite and let F be any functor from $\mathcal{D}$ to $\mathcal{C}$. A *colimit* for F is a pair (C, η) where $C \in \mathrm{Ob}(\mathcal{C})$ and η is a natural transformation from F to $\Delta(C)$ with the property that for each other pair (X, ν) where $X \in \mathrm{Ob}(\mathcal{C})$ and ν a natural transformation from F to $\Delta(X)$, there exists a unique map $\nu' \in \mathrm{Hom}_{\mathcal{C}}(C, X)$ such

that $\eta \circ \Delta(\nu') = \nu$. Often, we will suppress mention of the natural transformation η and regard a colimit for F as simply an object of $\mathcal{D}$.

In the case where $\mathcal{C} = \mathfrak{Top}$, we may explicitly compute a colimit for a functor F from $\mathcal{D}$ to $\mathcal{C}$ as follows: let

$$X := \coprod_{A \in Ob(\mathcal{D})} F(A),$$

and define an equivalence relation $\sim$ on X via $x \sim y$ if and only if $y = F(\varphi)(x)$ for some $\varphi \in \mathrm{Hom}_{\mathcal{D}}(A, B)$, where $x \in F(A)$ and $y \in F(B)$. $(X/\sim, \eta)$ is a colimit for F, where η is the obvious natural transformation from F to $\Delta(C)$ defined by maps $\eta_A \in \mathrm{Hom}_{\mathcal{D}}(F(A), C)$ which send elements of $F(A)$ to their class in C. Unfortunately, colimits for F cannot in general be reconstructed from the spaces $F(A)$ and maps $F(\varphi)$ between them. For example, if $\mathcal{D}$ is the category with 3 objects and 2 morphisms represented by the diagram

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ \downarrow{\scriptstyle \varphi} & & \\ C & & \end{array}$$

and F is a functor from $\mathcal{D}$ to $\mathfrak{Top}$ which sends B and C to a point and ψ and φ to constant maps, one sees immediately that a colimit for F is also a point, regardless of where F sends A. In order to rectify this, we need to introduce 'homotopy colimits' and therefore 'simplicial sets'.

Simplicial Sets and their Geometric Realisations

Despite their technical appearance, simplicial sets are simply a convenient way of storing data determined by an algebraic object, usually as a means of representing that object topologically. Start with the category Δ whose objects are the sets $[n] = \{0, \ldots, n\}$ for all $n \geq 0$ and where $\mathrm{Hom}_{\Delta}([i], [j])$ is the set the *order preserving maps* from $[i]$ to $[j]$. In other words,

$$x \leq y \Longrightarrow \varphi(x) \leq \varphi(y) \text{ for all } \varphi \in \mathrm{Hom}_{\Delta}([i], [j]) \text{ and } x, y \in [i].$$

A *simplicial set* is any contravariant (arrow reversing) functor from Δ to $\mathfrak{Set}$, the category of sets and functions between sets.

Let K be a simplicial set. In order to 'geometrically realise' K, we first define a set of points for each $n \geq 1$ as follows. Let

$$\Delta^n := \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_i \leq 1, \sum_{i=0}^{n} t_i = 1\}$$

with vertices labelled by the basis elements $e_0, \ldots, e_n$ of $\mathbb{R}^{n+1}$. Observe that there is a functor $F : \Delta \longrightarrow \mathfrak{Top}$ which sends $[n]$ to Δ^n and each $\phi \in \mathrm{Hom}_\Delta([n],[m])$ to the continuous map $\phi_* : \Delta^n \longrightarrow \Delta^m$ which satisfies the relation $\phi_*(e_i) = e_{\phi(i)}$ for each $0 \leq i \leq n$. Now define the *geometric realisation* $|K|$ of K to be

$$\left(\coprod_{n \geq 0} K([n]) \times \Delta^n \right) / \sim,$$

where for each $\sigma \in K([m])$, $\tau \in \Delta^n$ and $\phi \in \mathrm{Hom}_\Delta([n],[m])$ we have $(\sigma, \phi_*(\tau)) \sim (K(\phi)(\sigma), \tau)$.

The Nerve and Classifying Space of a Category and Homotopy Colimits

Both the classifying space of a category and the homotopy colimit of a functor are obtained from geometrically realising certain simplicial sets. Given a finite category $\mathcal{C}$, the *nerve* of $\mathcal{C}$, $\mathrm{N}(\mathcal{C})$ is the contravariant functor from Δ to $\mathfrak{Set}$ which sends each object $[n]$ in Δ to the set of n-chains in $\mathcal{C}$

$$\{A_0 \xrightarrow{\psi_0} A_1 \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_{n-1}} A_n \mid \psi_i \in \mathrm{Hom}_\mathcal{C}(A_i, A_{i+1}) \text{ for all } 0 \leq i \leq n\},$$

and where $\mathrm{N}(\mathcal{C})(\phi)$ is the map from $\mathrm{N}(\mathcal{C})([m])$ to $\mathrm{N}(\mathcal{C})([n])$ which sends each m-chain to the unique n-chain induced by $\phi \in \mathrm{Hom}_\Delta([n],[m])$ obtained by composing morphisms and inserting identity maps. The *classifying space of* $\mathcal{C}$ is the space $|\mathrm{N}(\mathcal{C})|$.

We now define the homotopy colimit of a functor F from $\mathcal{D}$ to $\mathfrak{Set}$ (or $\mathfrak{Top}$) for a finite category $\mathcal{D}$. We start by defining a contravariant functor $\mathcal{D}_F$ from Δ to $\mathfrak{Set}$ as follows. Let $\mathcal{D}_F([n])$ be the disjoint union

$$\coprod_{A_0 \to \cdots \to A_n} F(A_0),$$

taken over all n-chains

$$A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_n \in \mathrm{N}(\mathcal{D})([n]).$$

Index $\mathcal{D}_F([n])$ by writing $F(A_0)_\chi$ for the set $F(A_0)$ corresponding to a particular n-chain $\chi \in \mathrm{N}(\mathcal{D})([n])$. For each $\varphi \in \mathrm{Hom}_\Delta([n],[m])$ and $\chi \in \mathrm{N}(\mathcal{D})([m])$, $\mathcal{D}_F(\varphi)$ sends $F(A_0)_\chi$ to $F(A_{\varphi(0)})_{N(\mathcal{D})[\varphi](\chi)}$ via $F(A_0) \longrightarrow F(A_{\varphi(0)})$. The *homotopy colimit of* F is the space:

$$\underrightarrow{\mathrm{hocolim}}_\mathcal{D} F := |\mathcal{D}_F|.$$

If $\mathcal{D}$ is the category with 3 objects and 2 morphisms described above and F is a functor from $\mathcal{D}$ to $\mathfrak{Top}$ which sends B and C to a point, then $\underrightarrow{\mathrm{hocolim}}_\mathcal{D} F$ is the *suspension*

of $F(A)$ and dependent on where F sends A.

Homotopy and Classifying Spaces of Groups
For $i = 1, 2$ let X_i be a space and let $f, f' : X_1 \longrightarrow X_2$ be continuous maps from X_1 to X_2. We say that f and f' are *homotopic* (and write $f \simeq f'$) if there exists a continuous map $F : X_1 \times I \longrightarrow X_2$ satisfying

$$F(x, 0) = f(x) \text{ and } F(x, 1) = f'(x) \text{ for all } x \in X_1,$$

where $I = [0, 1]$ is the unit interval. A *homotopy equivalence* from X_1 to X_2 is a continuous map $f : X_1 \longrightarrow X_2$ for which there exists another continuous map $g : X_2 \longrightarrow X_1$ satisfying the identities

$$f \circ g \simeq \mathrm{Id}_{X_2} \text{ and } g \circ f \simeq \mathrm{Id}_{X_1},$$

where Id_{X_i} is the trivial map from X_i to itself. In this situation, X_1 and X_2 are said to be *homotopy equivalent* or to have the same *homotopy type*. The following lemma allows us to compare homotopy colimits of different functors defined on the same category, under certain conditions.

Lemma 1.14 *Let F and F' be two functors from $\mathcal{D}$ to $\mathfrak{Top}$ for a finite category $\mathcal{D}$. If there exists a natural isomorphism $\varphi : F \longrightarrow F'$ which induces a homotopy equivalence $F(C) \xrightarrow{\simeq} F'(C)$ for each object C of $\mathcal{D}$, then hocolim$(F) \simeq$ hocolim(F').*

Proof. This is shown in [16, Proposition IV.1.9] □

We end this section with the definition of a *classifying space* for a finite group G. This is any space X which is homotopy equivalent to the space $BG := |\mathcal{B}(G)|$. It is characterised (up to homotopy) by the following properties:

(i) The fundamental group of X is isomorphic to G.

(ii) There exists a contractible space Y on which G acts freely such that $Y/G \simeq X$.

The usefulness of this construction lies in the fact that for a G-module M and $n \geq 0$, the cohomology groups $H^n(G, M)$ are simply the groups $H^n(X, M)$ where X is any classifying space for G.

1.2 Saturated Fusion Systems

1.2.1 Saturation

From now on, unless otherwise specified, p will denote a fixed prime number. We begin with a precise definition of what is meant by a fusion system. For any group S and $P, Q \leq S$ we write $\mathrm{Hom}_S(P,Q)$ for the set of homomorphisms from P to Q induced by conjugation by elements of S and define $\mathrm{Aut}_S(P) := \mathrm{Hom}_S(P,P)$. Write ι_P^Q for the inclusion morphism in $\mathrm{Hom}_S(P,Q)$ given by conjugation by the identity. For example, $\mathrm{Hom}_S(P,Q) = \{\iota_P^Q\}$ if S is abelian. We write $\mathrm{Inj}(P,Q)$ for the set of monomorphisms from P to Q.

Definition 1.15 Let S be a finite p-group. A *fusion system* or *p-fusion system on* S is a category $\mathcal{F}$ where $\mathrm{Ob}(\mathcal{F}) := \{P \mid P \leq S\}$, and where for each $P, Q \in \mathrm{Ob}(\mathcal{F})$

(a) $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$; and

(b) each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ is uniquely factorisable as an isomorphism $P \longrightarrow P\varphi$ in $\mathcal{F}$ followed by an inclusion $P\varphi \longrightarrow Q$.

Given an arbitrary collection $X_{P,Q} \subseteq \mathrm{Inj}(P,Q)$ of morphisms containing $\mathrm{Hom}_S(P,Q)$ for each $P, Q \leq S$, we can always construct a fusion system $\mathcal{F}$ on S with $X_{P,Q} \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q)$ and where $\mathrm{Hom}_{\mathcal{F}}(P,Q) \setminus X_{P,Q}$ only consists of isomorphisms. In particular, $\mathcal{F}$ is minimal (with respect to the number of morphisms) amongst all fusion systems $\mathcal{G}$ on S with the property that $X_{P,Q} \subseteq \mathrm{Hom}_{\mathcal{G}}(P,Q)$ for each $P, Q \leq S$.

Fusion systems turn out to be far too general objects. For example, there is clearly a fusion system $\mathcal{F}$ on S with $\mathrm{Hom}_{\mathcal{F}}(P,Q) = \mathrm{Inj}(P,Q)$ for each $P, Q \leq S$. The motivation for the saturation axioms, which restrict the number of morphisms that are allowed, comes from studying fusion systems of finite groups. Given a finite group G and a Sylow p-subgroup S of G, let $\mathcal{F}_S(G)$ be the category with $\mathrm{Ob}(\mathcal{F}_S(G)) := \{P \mid P \leq S\}$ and $\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) := \mathrm{Hom}_G(P,Q)$ for each $P, Q \in \mathrm{Ob}(\mathcal{F}_S(G))$. It is not hard to see that $\mathcal{F}_S(G)$ is a fusion system on S. An abstract fusion system $\mathcal{F}$ on a finite p-group S is said to be *realisable* if there exists some finite group G with $S \in \mathrm{Syl}_p(G)$ such that $\mathcal{F} = \mathcal{F}_S(G)$, and *exotic* otherwise.

The following definition collects some important notions to which we will constantly refer, all of which are needed to define saturation.

Definition 1.16 Let $\mathcal{F}$ be a fusion system on a finite p-group S and let $P, Q \leq S$.

(a) $\mathrm{Iso}_{\mathcal{F}}(P,Q)$ denotes the set of isomorphisms (in $\mathcal{F}$) from P to Q and $\mathrm{Aut}_{\mathcal{F}}(P) := \mathrm{Iso}_{\mathcal{F}}(P,P)$.

(b) P and Q are *$\mathcal{F}$-conjugate* whenever $\mathrm{Iso}_{\mathcal{F}}(P,Q) \neq \emptyset$ and $P^{\mathcal{F}}$ denotes the set of all $\mathcal{F}$-conjugates of P.

(c) P is *fully $\mathcal{F}$-normalised* respectively *fully $\mathcal{F}$-centralised* if for each $R \in P^{\mathcal{F}}$, the inequality

$$|N_S(P)| \geq |N_S(R)| \text{ respectively } |C_S(P)| \geq |C_S(R)|$$

holds.

(d) P is *fully $\mathcal{F}$-automised* if $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

(e) For each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$,

$$N_\varphi = N_\varphi^{\mathcal{F}} := \{g \in N_S(P) \mid \varphi^{-1} \circ c_g \circ \varphi \in \mathrm{Aut}_S(P\varphi)\},$$

where $c_g \in \mathrm{Aut}_S(P)$ is the automorphism induced by conjugation by g.

There are numerous (equivalent) ways to define saturated fusion systems and the choice of definition will largely depend on the context. The definition we choose is now fairly classical and is taken from [6, Definition 1.2].

Definition 1.17 Let $\mathcal{F}$ be a fusion system on a finite p-group S. $\mathcal{F}$ is *saturated* if the following hold:

(a) Whenever $P \leq S$ is fully $\mathcal{F}$-normalised, it is fully $\mathcal{F}$-centralised and fully $\mathcal{F}$-automised.

(b) For all $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ such that $P\varphi$ is fully $\mathcal{F}$-centralised, there is $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

Note that by $\bar{\varphi}|_P$ we mean the *restriction* of the map $\bar{\varphi}$ to P defined as the composition of the inclusion map $P \longrightarrow N_\varphi$ with $\bar{\varphi}$. We remark that, given their topological connections (which we discuss later), it is fairly surprising that we can formulate precise algebraic axioms to define saturated fusion systems. As we previously hinted, a natural class of saturated fusion systems is provided by finite groups.

Theorem 1.18 *Let G be a finite group and S be a Sylow p-subgroup of G. The fusion system $\mathcal{F}_S(G)$ is saturated.*

Proof. This is shown in [4, Theorem I.2.3]. □

Example 1.19 Let $\mathcal{F}$ be a saturated fusion system on a finite abelian p-group S and let $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, Q)$, where $P, Q \leq S$. Since $S = PC_S(P) \leq N_\varphi$ and Q is fully $\mathcal{F}$-centralised, φ extends to a morphism $\bar{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(S)$. From this, one reproves Lemma 1.2 by specialising to the case where $\mathcal{F}$ is realisable as the fusion system of a finite group G and applying Theorem 1.18.

The next proposition is applied repeatedly in arguments involving fusion systems. It can also be used to reformulate the saturation axioms in Definition 1.17.

Proposition 1.20 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. The following are equivalent for each $P \leq S$.*

(a) P is fully $\mathcal{F}$-normalised.

(b) P is fully $\mathcal{F}$-centralised and fully $\mathcal{F}$-automised.

(c) For each $Q \leq S$ and $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q, P)$ there exist $\chi \in \mathrm{Aut}_{\mathcal{F}}(P)$ and $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), N_S(P))$ such that $\bar{\varphi}|_Q = \varphi \circ \chi$.

Proof. Since $\mathcal{F}$ is saturated, (a) implies (b). Assume that (b) holds and fix $Q \leq S$ such that there exists $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q, P)$. Since $\mathrm{Aut}_S(Q)^\varphi \leq \mathrm{Aut}_{\mathcal{F}}(P)$ and P is fully $\mathcal{F}$-automised, Sylow's Theorem implies that there is $\chi \in \mathrm{Aut}_{\mathcal{F}}(P)$ such that $\mathrm{Aut}_S(Q)^{\varphi\chi} \leq \mathrm{Aut}_S(P)$. Hence $N_{\varphi\chi} = N_S(Q)$ and $\varphi\chi$ extends to a map $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), S)$ since $\mathcal{F}$ is saturated and P is fully $\mathcal{F}$-centralised. Furthermore $\mathrm{Im}(\bar{\varphi}) \leq N_S(P)$ since $Q\bar{\varphi} = Q\varphi\chi = P$. It remains to prove that (c) implies (a). However if (c) holds then $|N_S(Q)| = |N_S(Q)\bar{\varphi}| \leq |N_S(P)|$ so that P is fully $\mathcal{F}$-normalised, as required. □

We end this section by defining what is meant by a 'morphism' of fusion systems.

Definition 1.21 Let $\mathcal{F}$ and $\mathcal{E}$ be fusion systems on finite p-groups S and T respectively. $\varphi \in \mathrm{Hom}(S, T)$ is a *morphism* from $\mathcal{F}$ to $\mathcal{E}$ if for each $P, R \leq S$ and each $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P, R)$, we have

$$\varphi^{-1} \circ \alpha \circ \varphi \in \mathrm{Hom}_{\mathcal{E}}(P\varphi, R\varphi).$$

Morphisms are occasionally referred to as *fusion-preserving maps*. Clearly any morphism φ from $\mathcal{F}$ to $\mathcal{E}$ determines a functor Φ from $\mathcal{F}$ to $\mathcal{E}$ defined in the obvious way. The *kernel* of φ is simply the subgroup $\mathrm{Ker}(\varphi)$ of S. In the case where $\mathcal{F} = \mathcal{E}$, we write $\mathrm{Aut}(\mathcal{F})$ for the set of all morphisms from $\mathcal{F}$ to itself and the fact that $\mathrm{Aut}(\mathcal{F})$ is a group follows easily from the axioms for a fusion system. Note that this group is *not* to be confused with the group $\mathrm{Aut}(\mathcal{C})$ for a general category $\mathcal{C}$ defined in Section 1.1.2, the latter notation only applying (from now on) to categories which are not fusion systems.

1.2.2 Alperin's Fusion Theorem

We now concern ourselves with 'generation' of saturated fusion systems. A consequence of Theorem 1.3 is that a morphism in a realisable fusion system $\mathcal{F}$ can be written as a product of restrictions of $\mathcal{F}$-automorphisms of certain subgroups. We seek an analogous result for arbitrary saturated fusion systems, starting with some more definitions:

Definition 1.22 Let $\mathcal{F}$ be a fusion system on a finite p-group S and let $P \leq S$.

(a) P is *S-centric* if $C_S(P) = Z(P)$ and P is *$\mathcal{F}$-centric* if Q is S-centric for each $Q \in P^{\mathcal{F}}$.

(b) Write $\mathrm{Out}_{\mathcal{F}}(P) := \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$. P is *$\mathcal{F}$-essential* if P is $\mathcal{F}$-centric and $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup.

(c) P is *$\mathcal{F}$-radical* if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$.

Note that (b) in the above definition makes sense since $\mathrm{Inn}(P) = \mathrm{Aut}_P(P)$ is a normal p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$. Furthermore, each $\mathcal{F}$-essential subgroup is also clearly $\mathcal{F}$-radical. The following lemma concerning $\mathcal{F}$-centric subgroups is extremely useful.

Lemma 1.23 *Let $\mathcal{F}$ be a fusion system on a finite p-group S. The following hold for each $P, Q \leq S$:*

(a) If P is fully $\mathcal{F}$-centralised then $PC_S(P)$ is $\mathcal{F}$-centric.

(b) If $P \leq Q$ and P is $\mathcal{F}$-centric, then Q is $\mathcal{F}$-centric.

Proof. Writing $R = PC_S(P)$, we see immediately that $C_S(R) \leq R$. If $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, S)$ then $R\varphi \leq P\varphi C_S(P\varphi)$ and since P is fully $\mathcal{F}$-centralised, necessarily $R\varphi = P\varphi C_S(P\varphi)$. Now $C_S(R\varphi) \leq R\varphi$ which proves (a). Part (b) is trivial. □

We now introduce some notation which makes precise the notion of a 'generating' set in our context.

Definition 1.24 Let S be a finite p-group and let $\mathcal{C}$ be a collection of injective maps between subgroups of S. Denote by $\langle \mathcal{C} \rangle_S$ the smallest fusion system on S containing all maps which lie in $\mathcal{C}$. If $\mathcal{F}$ is a fusion system on S and $\mathcal{F} = \langle \mathcal{C} \rangle_S$ then the set $\mathcal{C}$ is said to *generate* $\mathcal{F}$.

Observe that whenever $\mathcal{C}$ generates $\mathcal{F}$, each morphism in $\mathcal{F}$ can be written as a composite of restrictions of elements of $\mathcal{C}$. Define a set of subgroups $\mathcal{X}$ of a finite p-group S to be a *conjugation family* for a fusion system $\mathcal{F}$ on S if $\langle \mathrm{Aut}_{\mathcal{F}}(P) \mid P \in \mathcal{X} \rangle_S = \mathcal{F}$. It is a fact that every conjugation family $\mathcal{X}$ can be modified (through the removal of elements) to one containing exactly one representative for each $\mathcal{F}$-conjugacy class of elements of $\mathcal{X}$ (see [12, Proposition 2.10]).

We are now almost in a position to prove Alperin's Theorem for fusion systems which demonstrates the existence of a very useful conjugation family for any saturated fusion system $\mathcal{F}$. It will follow from the next result, shown in [4, Proposition I.3.3], proving that $\mathcal{F}$-automorphisms of a fully $\mathcal{F}$-normalised subgroup which extend non-trivially in $\mathcal{F}$ always generate a strongly p-embedded subgroup.

Lemma 1.25 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let $P < S$ be fully $\mathcal{F}$-normalised. If P is not $\mathcal{F}$-essential then $\mathrm{Aut}_{\mathcal{F}}(P) = H_P$ where*

$$H_P := \langle \alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \mid \exists \bar{P} \leq S \text{ and } \bar{\alpha} \in \mathrm{Hom}_{\mathcal{F}}(\bar{P}, S) \text{ such that } P < \bar{P} \text{ and } \bar{\alpha}|_P = \alpha \rangle.$$

Proof. Since $\mathcal{F}$ is saturated and P is fully $\mathcal{F}$-normalised, each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ extends to an element $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(PC_S(P))$. We may therefore assume that P is S-centric and hence $\mathcal{F}$-centric by Lemma 1.23. We show that

$$H_P = \langle \alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \mid \ \mathrm{Inn}(P) < \mathrm{Aut}_S(P) \cap \mathrm{Aut}_S(P)^{\alpha} \rangle.$$

Suppose $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ satisfies $\mathrm{Inn}(P) < \mathrm{Aut}_S(P) \cap \mathrm{Aut}_S(P)^{\alpha}$. Then $\mathrm{Inn}(P) = \mathrm{Inn}(P)^{\alpha^{-1}} < \mathrm{Aut}_{N_\alpha}(P)$ so that $P < N_\alpha$ and α extends to an element $\bar{\alpha} \in \mathrm{Hom}_{\mathcal{F}}(N_\alpha, S)$ since $\mathcal{F}$ is saturated. In particular, $\alpha \in H_P$. Conversely if there is $\bar{P}$ properly containing P such that α extends to $\bar{\alpha} \in \mathrm{Hom}_{\mathcal{F}}(\bar{P}, S)$ then $\mathrm{Inn}(P) < \mathrm{Aut}_{\bar{P}}(P)^{\alpha} \leq \mathrm{Aut}_S(P) \cap \mathrm{Aut}_S(P)^{\alpha}$ since $\mathrm{Aut}_{\bar{P}}(P)^{\alpha} = \mathrm{Aut}_{\bar{P}\bar{\alpha}}(P) \leq \mathrm{Aut}_S(P)$. This establishes the claim.

Now, $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ (by Proposition 1.20) so we have $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$ since $PC_S(P) = P$. Finally, since $\mathrm{Out}_{\mathcal{F}}(P)$ does not contain a strongly p-embedded subgroup, Lemma 1.9 implies that $H_P/\mathrm{Inn}(P) = \mathrm{Out}_{\mathcal{F}}(P)$ and necessarily $H_P = \mathrm{Aut}_{\mathcal{F}}(P)$, as required. □

Theorem 1.26 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S.*

$$\{S\} \cup \{P \mid P \text{ is fully } \mathcal{F}\text{-normalised and } \mathcal{F}\text{-essential}\}$$

is a conjugation family for $\mathcal{F}$.

Proof. Let $\mathcal{X} := \{S\} \cup \{P \mid P \text{ is fully } \mathcal{F}\text{-normalised and } \mathcal{F}\text{-essential}\}$ and write $\mathcal{C} := \{\mathrm{Aut}_{\mathcal{F}}(P) \mid P \in \mathcal{X}\}$. Define $\mathcal{F}' := \langle \mathcal{C} \rangle_S$ and assume that $\mathcal{F}' \neq \mathcal{F}$. This implies that there are $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ such that $\varphi \notin \mathrm{Mor}(\mathcal{F}')$ and we may choose P such that $|P|$ is as large as possible. Clearly $\mathrm{Aut}_{\mathcal{F}'}(S) = \mathrm{Aut}_{\mathcal{F}}(S)$, so $P < S$ and since φ factors as an $\mathcal{F}$-isomorphism followed by an inclusion (an element of $\mathrm{Mor}(\mathcal{F}')$), we may assume that φ is an isomorphism.

Let R be a fully $\mathcal{F}$-normalised $\mathcal{F}$-conjugate of P so that by Proposition 1.20 there exist maps

$$\psi_1 \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), N_S(R)) \text{ and } \psi_2 \in \mathrm{Hom}_{\mathcal{F}}(N_S(P\varphi), N_S(R))$$

with $P\psi_1 = P\varphi\psi_2 = R$. Since $P, P\varphi < S$, we have $P < N_S(P)$ and $P\varphi < N_S(P\varphi)$ by Corollary 1.5(a). Therefore $\psi_1, \psi_2 \in \mathrm{Mor}(\mathcal{F}')$ by the maximality of $|P|$. Hence, on replacing φ by $\psi_1^{-1} \circ \varphi \circ \psi_2$, we may assume that P is fully $\mathcal{F}$-normalised and $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$. It suffices to show that $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}'}(P)$. If P is not $\mathcal{F}$-essential then by Lemma 1.25, $\mathrm{Aut}_{\mathcal{F}}(P)$ is generated by automorphisms which extend to $\mathcal{F}$-morphisms between subgroups of order strictly larger than P. By assumption each such $\mathcal{F}$-morphism lies in $\mathrm{Mor}(\mathcal{F}')$ and hence $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}'}(P)$ in this case. If P is $\mathcal{F}$-essential then clearly $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}'}(P)$ by the definition of $\mathcal{F}'$. This completes the proof. □

Proving that a fusion system $\mathcal{F}$ is saturated can often be a difficult task, since there may be many subgroups for which the saturation axioms must be checked. Fortunately, this job is occasionally made easier when an $\mathcal{F}$-conjugation family is known to exist. The proof of the next theorem is long and technical so we omit it, but essentially comprises an induction on the index of subgroups.

Theorem 1.27 *Let $\mathcal{F}$ be a fusion system on a finite p-group S. If $\mathcal{F}$-centric subgroups of S form a conjugation family for $\mathcal{F}$ then $\mathcal{F}$ is saturated if (a) and (b) in Definition 1.15 hold for all $\mathcal{F}$-centric subgroups of S.*

Proof. This is shown in [27, Theorem 3.8]. □

We think of Theorem 1.27 as a partial converse to Theorem 1.26. It will be a fundamental tool used in our argument to prove Theorem 2 in Chapter 2.

1.3 Normaliser Subsystems and Constrained Fusion Systems

Usually, one is motivated to prove theorems about fusion systems from observations one makes about finite groups. Indeed, this philosophy has been an inspiration for many of the results in this thesis. This section is concerned with finding analogues for fusion systems of normalisers, normal and central subgroups and p-constrained groups in finite group theory. With these tools at our disposal, we will be able to give a new proof of the analogue for fusion systems of Thompson's normal p-complement Theorem for finite groups in Section 1.3.3.

1.3.1 K-normaliser Subsystems

All proofs of the results in this section can be found in [4, Section I.5]. Let G be a finite group, S be a Sylow p-subgroup of G and $Q \leq S$. For each $K \leq \mathrm{Aut}(Q)$ we define

$$N_G^K(Q) := \{g \in N_G(Q) \mid c_g \in K\},$$

and observe that this defines a family of familiar groups – $N_G(Q)$, $C_G(Q)$, $QC_G(Q)$ and so on – according to the choice of K. If we want to construct saturated fusion systems from these groups, a useful question to ask is:

$$\textit{when is } N_S^K(Q) \textit{ a Sylow } p\textit{-subgroup of } N_G^K(Q)\textit{?}$$

To answer this question, we need a new definition. Let $\mathcal{F}$ be a fusion system on a finite p-group S, Q be a subgroup of S and K be a subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. We say that Q is *fully K-normalised (in $\mathcal{F}$)* if for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$, the inequality

$$|N_S^K(Q)| \geq |N_S^{K^\varphi}(Q\varphi)|$$

holds where K^φ is the group $\varphi^{-1}K\varphi \leq \mathrm{Aut}_{\mathcal{F}}(Q\varphi)$. Therefore, fully Id_Q-normalised and fully $\mathrm{Aut}_{\mathcal{F}}(Q)$-normalised subgroups correspond to fully $\mathcal{F}$-centralised and fully $\mathcal{F}$-normalised subgroups respectively.

Lemma 1.28 *Let G be a finite group, S be a Sylow p-subgroup of G and $Q \leq S$. For each $K \leq \mathrm{Aut}(Q)$, the following are equivalent:*

(a) Q is fully K-normalised in $\mathcal{F}_S(G)$.

(b) $N_S^K(Q)$ is a Sylow p-subgroup of $N_G^K(Q)$.

Proof. This is shown in [4, Proposition I.5.4]. □

In particular, by Theorem 1.18, $\mathcal{F}_{N_S^K(Q)}(N_G^K(Q))$ is a saturated fusion system on $N_S^K(Q)$ whenever Q is fully K-normalised in $\mathcal{F}_S(G)$. If $\mathcal{F}$ is a fusion system on a finite p-group S then a *subsystem* $\mathcal{E}$ of $\mathcal{F}$ on T is a subcategory of $\mathcal{F}$ which is a fusion system on T. The previous lemma suggests that there should exist a saturated subsystem $\mathcal{N}$ of $\mathcal{F}$ on $N_S^K(Q)$ when $\mathcal{F}$ is an *arbitary* saturated fusion system $\mathcal{F}$ on S and Q is fully K-normalised in $\mathcal{F}$. Furthermore $\mathcal{N}$ should have the property that $\mathcal{N} = \mathcal{F}_{N_S^K(Q)}(N_G^K(Q))$ when $\mathcal{F} = \mathcal{F}_S(G)$. We define the *$K$-normaliser fusion system*, $N_{\mathcal{F}}^K(Q)$ as follows. Let the objects of $N_{\mathcal{F}}^K(Q)$ be subgroups of $N_S^K(Q)$ and for each $P, R \leq N_S^K(Q)$ let $\mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(P, R)$ be the set

$$\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, R) \mid \text{ there exists } \bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, RQ) \text{ with } \bar{\varphi}|_P = \varphi \text{ and } \bar{\varphi}|_Q \in K\}.$$

It is immediate that the category $N_{\mathcal{F}}^K(Q)$ is a fusion system on $N_S^K(Q)$. We first observe that $N_{\mathcal{F}}^K(Q)$ behaves as it should in the case where $\mathcal{F} = \mathcal{F}_S(G)$:

Proposition 1.29 *Suppose $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with $S \in Syl_p(G)$ and let $Q \leq S$ and $K \leq \mathrm{Aut}(Q)$. If Q is fully K-normalised in $\mathcal{F}$ then*

$$N_{\mathcal{F}}^K(Q) = \mathcal{F}_{N_S^K(Q)}(N_G^K(Q)).$$

Proof. This is shown in [4, Proposition I.5.4]. □

Hence $N_{\mathcal{F}}^K(Q)$ is saturated if $\mathcal{F} = \mathcal{F}_S(G)$ and Q is fully K-normalised in $\mathcal{F}$ by Lemma 1.28 and Theorem 1.18. In fact, the same statement holds when $\mathcal{F}$ is exotic as a consequence of the following generalisation of Proposition 1.20.

Proposition 1.30 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let $Q \leq S$ and $K \leq \mathrm{Aut}_{\mathcal{F}}(Q)$. The following are equivalent:*

(a) Q is fully K-normalised in $\mathcal{F}$.

(b) Q is fully $\mathcal{F}$-centralised and $\mathrm{Aut}_S(Q) \cap K \in \mathrm{Syl}_p(K)$.

Proof. This is shown in [4, Proposition I.5.2]. □

Theorem 1.31 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let $Q \leq S$ and $K \leq \mathrm{Aut}(Q)$. If Q is fully K-normalised in $\mathcal{F}$ then $N_{\mathcal{F}}^K(Q)$ is saturated.*

Proof. This is shown in [4, Theorem I.5.5]. □

1.3.2 Normal Subgroups and Constrained Fusion Systems

Let $\mathcal{F}$ be a fusion system on a finite p-group S and let $Q \leq S$. Denote by $C_{\mathcal{F}}(Q)$ and $N_{\mathcal{F}}(Q)$ the fusion systems $N_{\mathcal{F}}^{\mathrm{Id}_Q}(Q)$ and $N_{\mathcal{F}}^{\mathrm{Aut}_{\mathcal{F}}(Q)}(Q)$ respectively. These subsystems naturally give rise to definitions for normal and central subgroups in a fusion system as follows:

Definition 1.32 Let $\mathcal{F}$ be a fusion system on a finite p-group S and let $Q \leq S$. Q is *central* in $\mathcal{F}$ if $C_{\mathcal{F}}(Q) = \mathcal{F}$, and *normal* if $N_{\mathcal{F}}(Q) = \mathcal{F}$. Write $Z(\mathcal{F})$ for the largest subgroup of S which is central in $\mathcal{F}$ and $O_p(\mathcal{F})$ for the largest subgroup which is normal in $\mathcal{F}$.

One quickly checks that this definition makes sense by observing that products of subgroups which are central and normal in $\mathcal{F}$ remain central and normal. Slightly weaker than the concept of normality are the concepts of 'weak' and 'strong' $\mathcal{F}$-closure.

Definition 1.33 Let $\mathcal{F}$ be a fusion system on a finite p-group S and let $Q \subseteq S$.

(a) Q is *strongly $\mathcal{F}$-closed* if $g\varphi \in Q$ for each $g \in Q$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle g \rangle, S)$.

(b) Q is *weakly $\mathcal{F}$-closed* if $Q \leq S$ and $Q^{\mathcal{F}} = Q$.

Observe that each normal subgroup of $\mathcal{F}$ is strongly $\mathcal{F}$-closed and that a strongly $\mathcal{F}$-closed subgroup is weakly $\mathcal{F}$-closed. We think of strong closure as a sort of invariance condition for subgroups and we will need the same notion for morphisms shortly. The next two lemmas characterise central elements and normal subgroups.

Lemma 1.34 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. The following are equivalent.*

(a) $x \in Z(\mathcal{F})$.

(b) $x\varphi = x$ whenever $x \in P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$.

In particular $Z(\mathcal{F}) \cap T \leq Z(\mathcal{E})$ for each subsystem $\mathcal{E}$ of $\mathcal{F}$ on $T \leq S$.

Proof. The equivalence of (a) and (b) follows from [4, Theorem I.4.2]. The last statement is an immediate consequence of the first. □

Lemma 1.35 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let Q be a strongly $\mathcal{F}$-closed subgroup of S. If Q is normal in $\mathcal{F}$ then $Q \leq R$ for each $\mathcal{F}$-centric $\mathcal{F}$-radical subgroup R of S. Furthermore, the following are equivalent:*

(a) Q is normal in $\mathcal{F}$.

(b) $Q \leq R$ for each fully $\mathcal{F}$-normalised, $\mathcal{F}$-essential subgroup R of S.

Proof. This is shown in [11, Theorem 4.61]. □

As an immediate application of Lemma 1.35, we prove the following result of Onofrei–Stancu ([25, Lemma 3.7]) concerning the factorisation of morphisms in fusion systems with a normal subgroup. This observation will be key to the proof of Theorem 1.41 in the next section.

Lemma 1.36 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and $Q \leq S$ be normal subgroup of $\mathcal{F}$. Writing $K := \mathrm{Aut}_S(Q)$, we have*

$$\mathcal{F} = \langle \mathrm{Mor}(N_{\mathcal{F}}(QC_S(Q)) \cup \mathrm{Mor}(N_{\mathcal{F}}^K(Q)) \rangle_S.$$

Proof. By Theorem 1.26, it suffices to prove that for each fully $\mathcal{F}$-normalised, $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroup R of S and each $\varphi \in \mathrm{Aut}_{\mathcal{F}}(R)$, there exist elements $\alpha \in \mathrm{Mor}(N_{\mathcal{F}}(QC_S(Q))$ and $\beta \in \mathrm{Mor}(N_{\mathcal{F}}^K(Q))$ such that $\varphi = \alpha \circ \beta$. Fix such a subgroup R, let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(R)$ and notice that by Lemma 1.35, $Q \leq R$. Since $N_{\mathcal{F}}(Q) = \mathcal{F}$, Q is strongly $\mathcal{F}$-closed and $\psi := \varphi|_Q \in \mathrm{Aut}_{\mathcal{F}}(Q)$.

Clearly $R \leq N_\psi$, so that $RQC_S(Q) \leq N_\psi$ and since $\mathcal{F}$ is saturated, there is $\theta \in \mathrm{Hom}_{\mathcal{F}}(RQC_S(Q), S)$ extending ψ. Setting $\alpha := \theta|_R$ and $\beta := (\theta|_R)^{-1} \circ \varphi$, we see immediately that $\alpha \in \mathrm{Mor}(N_{\mathcal{F}}(QC_S(Q))$ (since $\theta|_Q = \psi$) and $\beta \in \mathrm{Mor}(N_{\mathcal{F}}^K(Q))$, since $\beta|_Q = (\theta|_R)^{-1}|_Q \circ \varphi|_Q = \psi^{-1} \circ \psi = \mathrm{Id}_Q \in \mathrm{Aut}_S(Q)$ (and $N_S^K(Q) = S$). Clearly $\varphi = \alpha \circ \beta$ and the lemma is proven. □

We end by defining a particularly important class of fusion systems which are called 'constrained'. A finite group G is *p-constrained* if there exists a normal p-subgroup R of G with $C_G(R) \leq R$ [1]. At this point, we can define an analogous object for fusion systems:

Definition 1.37 A saturated fusion system $\mathcal{F}$ on a finite p-group S is *constrained* if there exists an $\mathcal{F}$-centric subgroup of S which is normal in $\mathcal{F}$.

For example if G is p-constrained and $S \in \mathrm{Syl}_p(G)$ then $\mathcal{F} = \mathcal{F}_S(G)$ is constrained because $O_p(G) \leq O_p(\mathcal{F})$ and $O_p(G)$ is $\mathcal{F}$-centric. In fact, every constrained fusion system arises in this way as the following result shows.

[1] Traditionally, a group G is p-constrained if there exists a normal p-subgroup R of $\bar{G}$ with $C_{\bar{G}}(R) \leq R$ where $\bar{G} := G/O_{p'}(G)$. We relax this definition for our purposes.

Theorem 1.38 *Let $\mathcal{F}$ be a fusion system on a finite p-group S and suppose that there exists an $\mathcal{F}$-centric subgroup R of S which is normal in $\mathcal{F}$. Then there exists a finite group G with $S \in \mathrm{Syl}_p(G)$ and with the properties that*

$$\mathcal{F} = \mathcal{F}_S(G),\ R \trianglelefteq G,\ \mathrm{Out}_G(R) \cong G/R,\ \textit{and } O_{p'}(G) = 1.$$

Proof. This is shown in [4, Theorem I.4.9]. □

We therefore think of constrained fusion systems and p-constrained groups as essentially the same thing, since the existence of one always implies the existence of the other. The proof of Theorem 1.38 can be regarded as a complex exercise in group and functor cohomology. The group G fits into an exact sequence

$$1 \longrightarrow Q \longrightarrow G \longrightarrow \mathrm{Out}_{\mathcal{F}}(Q) \longrightarrow 1,$$

and much of the proof is devoted to showing that $\mathcal{F}$-morphisms between subgroups of S containing Q may be identified with G-conjugation maps, so that the result follows from Lemma 1.35 and then Theorem 1.26.

1.3.3 Application: Thompson's Normal p-complement Theorem

We exhibit the power of some of the techniques we have developed so far to prove a deep result, giving conditions which imply that a fusion system $\mathcal{F}$ on a finite p-group S is a *trivial* fusion system $\mathcal{F}_S(S)$. This topic represents the starting point for the author's interest in fusion systems.

To describe this result, we need some new definitions. If S is a finite p-group, the *Thompson Subgroup* $J(S)$ of S is defined to be the subgroup generated by all abelian p-subgroups of S of order m, where m is the largest order of a maximal abelian subgroup of S. A finite group G has a *normal p-complement* if $O_{p'}(G) = O^p(G)$. Thompson's Theorem asserts that when p is an odd prime, the existence of a normal p-complement can be detected locally.

Theorem 1.39 (Thompson) *Let p be an odd prime and let G be a finite group with Sylow p-subgroup S. If $N_G(J(S))$ and $C_G(Z(S))$ have normal p-complements then so does G.*

Proof. This is shown in [15, Corollary 7.5]. □

Glauberman deduces this result in [15] from the following rather different result which gives a condition for an element of a Sylow p-subgroup of a group to generate a weakly closed subgroup.

Theorem 1.40 *Let p be an odd prime and let G be a finite group with Sylow p-subgroup S. If $x \in S \cap Z(N_G(J(S)))$ then $x^G \cap S = \{x\}$.*

Proof. This is shown in [15, Theorem A]. □

Our goal is an analogue of Theorem 1.39 for a saturated fusion system $\mathcal{F}$, which of course relies on an interpretation of the statement '$\mathcal{F}$ has a normal p-complement.' It is a trivial consequence of Frobenius' Theorem for finite groups (see [21, Theorem 7.2.4]) that a finite group G with Sylow p-subgroup S has a normal p-complement if and only if $\mathcal{F}_S(G) = \mathcal{F}_S(S)$. Thus a natural interpretation of '$\mathcal{F}$ has a normal p-complement' is '$\mathcal{F} = \mathcal{F}_S(S)$'. We can now state the main result of this section.

Theorem 1.41 *Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. If $N_{\mathcal{F}}(J(S)) = C_{\mathcal{F}}(Z(S)) = \mathcal{F}_S(S)$ then $\mathcal{F} = \mathcal{F}_S(S)$.*

The strategy for proving Theorem 1.41 will be the same as the proof for groups: we first prove an analogue of Theorem 1.40 for fusion systems, stated here as Theorem 1.46, and then deduce Theorem 1.41 as a corollary. It is remarkable that this deduction is almost trivial in contrast to the group case where it is not. Our proof of Theorem 1.46 is a simplification of that found in [12, Theorem 4.1]. Essentially our argument relies on the implementation of Lemma 1.36 to streamline the induction argument. In order to describe it, we start by defining a particular class of functors from the category of finite p-groups to itself.

Definition 1.42 A *characteristic p-functor* W is a map from the set of all finite p-groups to itself where for each finite p-group P the following hold:

(a) $W(P)$ char P;

(b) $W(P) \neq 1$ whenever $P \neq 1$; and

(c) for any other finite p-group Q, and isomorphism $\varphi \in \mathrm{Iso}(P, Q)$ we have $W(P)\varphi = W(Q)$.

For example the map $Z(-)$ which sends a p-group to its centre is clearly a characteristic p-functor by Corollary 1.5. In fact, $J(-)$ is also a characteristic p-functor since elements of $\mathrm{Iso}(S, R)$ induce a one-to-one correspondence between the sets of maximal abelian subgroups of largest order in S and R.

Characteristic p-functors are important as they allow us to construct non-trivial conjugation families for fusion systems. In order to describe this construction, we need some more notation. Let S be a finite p-group, P be a subgroup of S and W be a characteristic p-functor. Define $W_1(P) := P$, $S_1(P) := N_S(P)$ and for each integer $i \geq 1$ write

$$W_{i+1}(P) := W(S_i(P)) \text{ and } S_{i+1}(P) := N_S(W_{i+1}(P)).$$

Notice that $S_i(P) = S$ when i is large enough. To see this, suppose that $S_i(P) < S$ so that by Corollary 1.5 (a) $S_i(P) < N_S(S_i(P)) \leq N_S(W_{i+1}(P)) = S_{i+1}(P)$ (where the second inequality follows from the fact that $W_{i+1}(P) = W(S_i(P))$ char $S_i(P)$). We now make the following definition:

Definition 1.43 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let W be a characteristic p-functor. A subgroup $P \leq S$ is *$(\mathcal{F}, W)$ well-placed* if $W_i(P)$ is fully $\mathcal{F}$-normalised for all $i \geq 1$. Denote by $\mathcal{F}^W$ the set of all such subgroups.

$(\mathcal{F}, W)$-well placed subgroups have the following important property.

Lemma 1.44 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let W be a characteristic p-functor. $\mathcal{F}^W$ is a conjugation family for $\mathcal{F}$.*

Proof. Let $P \leq S$ and write $W_i := W_i(P)$ and $S_i := S_i(P)$. It is enough to show that P is $\mathcal{F}$-conjugate to an $(\mathcal{F}, W)$-well placed subgroup. This would imply that each $\mathcal{F}$-conjugacy class contains such a subgroup and we may refine the conjugation family $\{P \mid P \leq S\}$ to the set $\mathcal{F}^W$ [2]. Fix a fully $\mathcal{F}$-normalised $\mathcal{F}$-conjugate R of P. By Proposition 1.20 (c) there exists $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), N_S(R))$ such that $P\varphi = R$. Hence on replacing P by $P\varphi$ if necessary we may assume that $W_1 = W_1(P) = P$ is fully $\mathcal{F}$-normalised.

Now suppose (for an induction) that $W_1, \ldots, W_i$ are fully $\mathcal{F}$-normalised. Pick a fully $\mathcal{F}$-normalised $\mathcal{F}$-conjugate U of W_{i+1}, so that by Proposition 1.20 (c) there exists $\varphi_{i+1} \in \mathrm{Hom}_{\mathcal{F}}(N_S(W_{i+1}), S)$ such that $W_{i+1}\varphi_{i+1} = U$. We show that the groups $W_1\varphi_{i+1}, \ldots, W_i\varphi_{i+1}$ are also all fully $\mathcal{F}$-normalised. To see this, note that for each $1 \leq j \leq i$, $S_j\varphi_{i+1} \leq N_S(W_j)\varphi_{i+1} \leq N_S(W_j\varphi_{i+1})$, and since W_j is fully $\mathcal{F}$-normalised,

$$|N_S(W_j\varphi_{i+1})| \leq |N_S(W_j)| = |S_j| = |S_j\varphi_{i+1}|.$$

[2] Note the remark which follows Definition 1.24 concerning refinement of conjugation families.

This proves that $N_S(W_j)\varphi_{i+1} = N_S(W_j\varphi_{i+1})$ and $W_j\varphi_{i+1}$ is fully $\mathcal{F}$-normalised, as needed.

Since $W_j\varphi_{i+1} = W_j(P\varphi_{i+1})$ for each $j \leq i$, on replacing P by $P\varphi_{i+1}$ we have that each of the subgroups $W_1, \ldots, W_{i+1}$ is fully $\mathcal{F}$-normalised. Hence by induction, and the fact that this sequence of subgroups must terminate, the claim is proven. □

We are now almost ready to prove an analogue of Theorem 1.40 for fusion systems. Before doing so, we record the following lemma which concerns the centre of a p-constrained group.

Lemma 1.45 *Let G be a p-constrained finite group with Sylow p-subgroup S and let $\mathcal{F} := \mathcal{F}_S(G)$. Then $Z(\mathcal{F}) = Z(G) \cap S$.*

Proof. Write $Z := Z(G) \cap S$ and notice that since $C_{\mathcal{F}}(Z) = \mathcal{F}_{C_S(Z)}(C_G(Z)) = \mathcal{F}_S(G) = \mathcal{F}$ (by Proposition 1.29), we have that $Z \leq Z(\mathcal{F})$. Conversely if $x \in Z(\mathcal{F})$, then by Lemma 1.34, we must have $\{x^g \mid g \in G\} \cap S = \{x\}$ for each $g \in G$. Also, since $[x, O_p(G)] \leq [x, S] = 1$ we have $x \in C_G(O_p(G)) = Z(O_p(G))$. Now, $Z(O_p(G)) \trianglelefteq G$ so for each $g \in G$, $x^g \in Z(O_p(G)) \leq S$ and $x^g = x$, proving that $x \in Z$, as required. □

Theorem 1.46 *Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then $Z(N_{\mathcal{F}}(J(S))) \leq Z(\mathcal{F})$.*

Proof. Let $\mathcal{F}$ be a minimal counterexample to the theorem. In other words $\mathcal{F}$ is a fusion system on a finite p-group S with $Z(N_{\mathcal{F}}(J(S))) \not\leq Z(\mathcal{F})$ and $|\mathrm{Mor}(\mathcal{F})|$ is as small as possible.

Choose $x \in Z(N_{\mathcal{F}}(J(S)))$ so that $x \notin Z(\mathcal{F})$. Since $J(-)$ is a characteristic p functor, $\mathcal{F}^J$ is a conjugation family for $\mathcal{F}$ by Lemma 1.44. Since x is not fixed by all $\mathcal{F}$-morphisms, there exists some $U \in \mathcal{F}^J$ containing x and an element $\varphi \in \mathrm{Aut}_{\mathcal{F}}(U)$ such that $x\varphi \neq x$. Furthermore, we may choose U so that $|N_S(U)|$ is maximal subject to this condition.

Since $J(-)$ is a characteristic p-functor and U is $(\mathcal{F}, J)$-well placed, U and $R := J(N_S(U))$ are both fully $\mathcal{F}$-normalised and hence the fusion systems $N_{\mathcal{F}}(U)$ and $N_{\mathcal{F}}(R)$ are both saturated by Theorem 1.31. Also, $N_{N_{\mathcal{F}}(U)}(R)$ is saturated since R is fully $N_{\mathcal{F}}(U)$-normalised ($R \trianglelefteq N_S(U)$).

We claim that $N_{\mathcal{F}}(U) = \mathcal{F}$. Suppose that $N_{\mathcal{F}}(U) \subsetneq \mathcal{F}$ so that by the minimality of $\mathcal{F}$,

$$Z(N_{N_{\mathcal{F}}(U)}(R) \leq Z(N_{\mathcal{F}}(U)).$$

Since $x \notin Z(N_{\mathcal{F}}(U))$ ($\varphi \in \mathrm{Aut}_{N_{\mathcal{F}}(U)}(U)$ does not fix x), also $x \notin Z(N_{N_{\mathcal{F}}(U)}(R))$ and x does not lie in $Z(N_{\mathcal{F}}(R)) \cap N_S(U)$, certainly a subgroup of $Z(N_{N_{\mathcal{F}}(U)}(R))$ by Lemma 1.34. By assumption $x \in Z(N_{\mathcal{F}}(J(S)))$ so we must have that $N_S(U) \neq S$ and $N_S(U) < N_S(N_S(U)) \leq N_S(J(N_S(U)))$, since $J(N_S(U))$ char $N_S(U)$. Since $x \notin Z(N_{\mathcal{F}}(R))$, there is $\psi \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $x\psi \neq x$. This, combined with the facts that $N_S(J(N_S(U))) \in \mathcal{F}^J$ and $|N_S(U)| < |N_S(R)|$ contradicts the maximality condition on U. We must therefore have that $N_{\mathcal{F}}(U) = \mathcal{F}$.

Next, by Lemma 1.36,

$$\mathcal{F} = \langle \mathrm{Mor}(N_{\mathcal{F}}(UC_S(U))) \cup \mathrm{Mor}(N_{\mathcal{F}}^K(U)) \rangle_S,$$

where $K := \mathrm{Aut}_S(U)$. Since $x \in Z(N_S(J(S))) \cap U = Z(S) \cap U$, certainly $x \in Z(N_{\mathcal{F}}^K(U))$ and necessarily $x \notin Z(N_{\mathcal{F}}(UC_S(U)))$ by Lemma 1.34. However by assumption we have

$$x \in Z(N_{\mathcal{F}}(J(S))) \leq Z(N_{N_{\mathcal{F}}(UC_S(U))}(J(S))) = Z(N_{N_{\mathcal{F}}(UC_S(U))}(J(N_S(UC_S(U))))),$$

since $U \trianglelefteq S$. Hence by the minimality of $\mathcal{F}$, $\mathcal{F} = N_{\mathcal{F}}(UC_S(U))$.

Since U is fully $\mathcal{F}$-normalised, $UC_S(U)$ is $\mathcal{F}$-centric (by Proposition 1.20 and Lemma 1.23). Hence Theorem 1.38 implies that there exists a p-constrained finite group G with $S \in \mathrm{Syl}_p(G)$ such that $\mathcal{F} = \mathcal{F}_S(G)$, $C_G(UC_S(U)) \leq UC_S(U)$ and $UC_S(U) \trianglelefteq G$. In particular, since $S \leq N_G(J(S))$, $N_G(J(S))$ is p-constrained. By Proposition 1.29, $N_{\mathcal{F}}(J(S)) = \mathcal{F}_S(N_G(J(S)))$ and hence by Lemma 1.45, $x \in Z(N_G(J(S)))$. Finally, by Theorem 1.40 and Lemma 1.34 we have $x \in Z(\mathcal{F})$, a contradiction. □

We now derive Theorem 1.41 as a corollary to Theorem 1.46:

Proof (of Theorem 1.41). By Lemma 1.34, $Z(\mathcal{F}) \leq Z(N_{\mathcal{F}}(J(S)))$, so that by Theorem 1.46 equality holds. Therefore, $Z(\mathcal{F}) = Z(N_{\mathcal{F}}(J(S))) = Z(\mathcal{F}_S(S)) = Z(S)$, and

$$\mathcal{F} = C_{\mathcal{F}}(Z(\mathcal{F})) = C_{\mathcal{F}}(Z(S)) = \mathcal{F}_S(S),$$

as needed. □

1.4 Quotients and Normal Subsystems of Fusion Systems

1.4.1 Quotients

This section is concerned with the analogue for fusion systems of the quotient of a finite group. We use this construction to demonstrate the realisability of certain

quotients of fusion systems with a normal subgroup. We begin with the definition of the quotient of a fusion system.

Definition 1.47 Let $\mathcal{F}$ be a fusion system on a finite p-group S, let Q be a normal subgroup of S and write $\psi : S \longrightarrow S/Q$ for the natural map. For each $P \leq S$ set $\bar{P} := P\psi$. The *quotient* $\bar{\mathcal{F}} := \mathcal{F}/Q$ of $\mathcal{F}$ by Q is the category whose objects are subgroups of $\bar{S}$ and where for each $P, R \leq S$,

$$\mathrm{Hom}_{\bar{\mathcal{F}}}(\bar{P}, \bar{R}) := \{\varphi \in \mathrm{Hom}(\bar{P}, \bar{R}) \mid \psi \circ \varphi \circ \psi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(P, R)\}.$$

Evidence that Definition 1.47 is natural is provided by the following lemma of Stancu whose proof we omit for the sake of brevity.

Lemma 1.48 *Let $H \trianglelefteq G$ be finite groups, let S be a Sylow p-subgroup of G and write $Q := S \cap H$. Then*

$$\mathcal{F}_S(G)/Q = \mathcal{F}_{S/Q}(G/H).$$

Proof. This is shown in [11, Theorem 5.20]. □

It is natural to look for conditions on Q which ensure that $\mathcal{F}/Q$ is saturated. The next result is motivated by the fact that the kernel of a morphism between two saturated fusion systems is a strongly closed subgroup of the domain.

Theorem 1.49 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. If Q is a strongly $\mathcal{F}$-closed subgroup of S then $\mathcal{F}/Q$ is saturated.*

Proof. This is shown in [11, Proposition 5.11]. □

Let G be a finite group, $Q \trianglelefteq G$ and $K \leq \mathrm{Aut}_G(Q)$. We will consider analogues for fusion systems of the identities

$$N_G^K(Q)/C_G(Q) \cong K \text{ and } G/N_G^K(Q) \cong \mathrm{Aut}_G(Q)/K,$$

where in the second case we clearly also require that K is normal in $\mathrm{Aut}_G(Q)$. We begin with an analogue for the first identity.

Theorem 1.50 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Fix a subgroup $Q \leq S$ and a subgroup K of $\mathrm{Aut}_{\mathcal{F}}(Q)$ and assume that Q is fully K-normalised. Then*

$$N_{\mathcal{F}}^K(Q)/C_S(Q) \cong \mathcal{F}_{\mathrm{Aut}_S(Q) \cap K}(K).$$

Proof. First observe that both $N_{\mathcal{F}}^K(Q)/C_S(Q)$ and $\mathcal{F}_{\mathrm{Aut}_S(Q)\cap K}(K)$ are saturated fusion systems. In the first case this follows from Theorem 1.49 since $C_S(Q)$ is strongly $N_{\mathcal{F}}^K(Q)$-closed ([11, Theorem 4.57]) and $N_{\mathcal{F}}^K(Q)$ is saturated by Theorem 1.31. In the second case this follows from Theorem 1.18, since $\mathrm{Aut}_S(Q) \cap K \in \mathrm{Syl}_p(K)$ by Proposition 1.30. Also, since $N_S^K(Q)/C_S(Q) \cong \mathrm{Aut}_S(Q) \cap K$, the conclusion of the theorem makes sense. Write $\mathrm{Aut}_S^K(Q) := \mathrm{Aut}_S(Q) \cap K$. It suffices to check that the functor

$$\Phi : N_{\mathcal{F}}^K(Q) \longrightarrow \mathcal{F}_{\mathrm{Aut}_S^K(Q)}(K)$$

which sends each $P \leq N_S^K(Q)$ to $PC_S(Q)/C_S(Q)$ (as a subgroup of $\mathrm{Aut}_S^K(Q)$) and each morphism $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(P,R)$ to conjugation by $\bar{\varphi}|_Q \in K$ (where $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ,S)$) is well-defined and may be regarded as the functor induced by the natural morphism of fusion systems $N_S^K(Q) \longrightarrow \mathrm{Aut}_S^K(Q)$ with kernel $C_S(Q)$ and image $\mathcal{F}_{\mathrm{Aut}_S^K(Q)}(K)$. For each element g and subgroup P of $N_S^K(Q)$, respectively write $\bar{g}$ and $\bar{P}$ for the image in $\mathrm{Aut}_S^K(Q)$. Observe that we may identify $\bar{g}$ with $c_g \in \mathrm{Aut}_S^K(Q)$. Also observe that for each $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(P,R)$ and $g \in P$, $(\bar{\varphi}|_Q)^{-1} \circ c_g \circ \bar{\varphi}|_Q = c_{g\varphi} = g\bar{\varphi} \in \mathrm{Aut}_S^K(Q)$, so that the image of Φ makes sense. To see that Φ is independent of the choice of morphism $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ,S)$, note that if $\varphi_1, \varphi_2 \in \mathrm{Hom}_{\mathcal{F}}(PQ,S)$ are two morphisms extending $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(P,R)$ then for each $g \in P$,

$$(\varphi_1|_Q)^{-1} \circ c_g \circ \varphi_1|_Q = c_{g\varphi_1} = c_{g\varphi} = c_{g\varphi_2} = (\varphi_2|_Q)^{-1} \circ c_g \circ \varphi_2|_Q.$$

It remains to show that Φ is surjective on morphisms. Let $P \leq N_S^K(Q)$ and $\chi \in K$ and consider the morphism $c_\chi \in \mathrm{Hom}_K(\bar{P}, \mathrm{Aut}_S^K(Q))$. We need to show that there exists a morphism $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(P,R)$ with $\Phi(\varphi) = c_\chi$. Now,

$$N_\chi = \{g \in N_S(Q) \mid \chi^{-1} \circ c_g \circ \chi \in \mathrm{Aut}_S(Q)\}$$

so that by assumption $P \leq N_\chi$. Since $\mathcal{F}$ is saturated and Q is fully $\mathcal{F}$-centralised (by Proposition 1.30), χ extends to a map $\bar{\chi} \in \mathrm{Hom}_{\mathcal{F}}(PQ,S)$. Set $\varphi := \bar{\chi}|_P$ and observe that $\Phi(\varphi) = c_\chi$. This proves that Φ is a well-defined functor from $N_{\mathcal{F}}^K(Q)$ to $\mathcal{F}_{\mathrm{Aut}_S^K(Q)}(K)$ induced by $N_S^K(Q) \longrightarrow \mathrm{Aut}_S^K(Q)$ and completes the proof of the theorem. □

In order to prove an analogue for the second identity above, we will need the following lemma whose proof follows immediately from the definition of the quotient.

Lemma 1.51 *Let $\mathcal{F}$ be a saturated fusion system on a p-group S. If Q is a strongly $\mathcal{F}$-closed subgroup of S then $\mathcal{F}/Q = N_{\mathcal{F}}(Q)/Q$.*

We can now state and prove the desired result.

Theorem 1.52 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let Q be a normal subgroup of $\mathcal{F}$ and let K be a normal subgroup of $Aut_{\mathcal{F}}(Q)$ containing* $\mathrm{Inn}(Q)$. *The following hold.*

(a) $\mathcal{F}/N_S^K(Q) \cong \mathcal{F}_{\mathrm{Out}_S(N_S^K(Q))}(\mathrm{Out}_{\mathcal{F}}(N_S^K(Q)))$.

(b) If Q is S-centric then $\mathcal{F}/N_S^K(Q) \cong \mathcal{F}_{\mathrm{Aut}_S(Q)K/K}(\mathrm{Aut}_{\mathcal{F}}(Q)/K)$.

Proof. We first prove (a). Write $R := N_S^K(Q)$ and $\mathrm{Aut}_S^K(Q) := K \cap \mathrm{Aut}_S(Q)$ and observe that $QC_S(Q) \leq R$ because $\mathrm{Inn}(Q) \leq K$. Since $\mathcal{F} = N_{\mathcal{F}}(Q)$, Q is fully $\mathcal{F}$-normalised and therefore fully $\mathcal{F}$-centralised by Proposition 1.20. Hence Lemma 1.23 implies that $QC_S(Q)$ and R are both $\mathcal{F}$-centric and the fact that $N_S(R) = S$ implies that R is also $N_{\mathcal{F}}(R)$-centric. Therefore, by Theorem 1.38, there exists a unique finite group G satisfying

$$N_{\mathcal{F}}(R) = \mathcal{F}_S(G),\ R \trianglelefteq G,\ \mathrm{Out}_G(R) = G/R,\ \text{and } O_{p'}(G) = 1.$$

By Lemmas 1.48 and 1.51,

$$\mathcal{F}/R = N_{\mathcal{F}}(R)/R = \mathcal{F}_S(G)/R = \mathcal{F}_{S/R}(G/R) \cong \mathcal{F}_{Out_S(R)}(\mathrm{Out}_G(R))$$

and since $\mathrm{Out}_G(R) = \mathrm{Out}_{N_{\mathcal{F}}(R)}(R) = \mathrm{Out}_{\mathcal{F}}(R)$, (a) follows. It remains to prove (b). Assume that Q is S-centric and note that by (a) it will be enough to show that the isomorphisms

$$\mathrm{Aut}_S(Q)K/K \cong \mathrm{Out}_S(R) \text{ and } \mathrm{Aut}_{\mathcal{F}}(Q)/K \cong \mathrm{Out}_{\mathcal{F}}(R)$$

hold. To see the first of these, note that by the isomorphism theorems,

$$\mathrm{Aut}_S(Q)K/K \cong \mathrm{Aut}_S(Q)/\mathrm{Aut}_S^K(Q) \cong S/N_S^K(Q) \cong \mathrm{Out}_S(N_S^K(Q)).$$

The second isomorphism is more subtle. Let $\Phi : \mathrm{Aut}_{\mathcal{F}}(R) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(Q)$ be the restriction map. Since $K \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(Q), R \leq N_\varphi$ for each $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ so that Φ is surjective since $\mathcal{F}$ is saturated. Furthermore Φ restricts to a surjection $\mathrm{Inn}(R) \longrightarrow K$, by the definition of $R = N_S^K(Q)$. Therefore, in order to prove that $\mathrm{Aut}_{\mathcal{F}}(R)/\mathrm{Inn}(R) \cong \mathrm{Aut}_{\mathcal{F}}(Q)/K$ it suffices to show that $\mathrm{Ker}(\Phi) \leq \mathrm{Inn}(R)$. Now,

$$\mathrm{Ker}(\Phi) = \{\gamma \in \mathrm{Aut}_{\mathcal{F}}(R) \mid \gamma|_Q = \mathrm{Id}_Q, [\gamma, R] \leq Q\}$$

is a p-group by Corollary 1.6 applied to the normal chain

$$1 \trianglelefteq Q \trianglelefteq R$$

(here we use that Q is S-centric). Since R is fully $\mathcal{F}$-automised by (Proposition 1.20), we must have $\mathrm{Ker}(\Phi) \leq \mathrm{Aut}_S(R)$. Let L be the preimage in S of $\mathrm{Ker}(\Phi)$ under the natural map $S \longrightarrow \mathrm{Aut}_S(R)$. Clearly $L \leq C_S(Q) \leq N_S^K(Q)$ so that $c_g \in \mathrm{Inn}(R)$ for each $g \in L$. This proves that $\mathrm{Ker}(\Phi) \leq \mathrm{Inn}(R)$ and completes the proof of the theorem. □

1.4.2 Normal Subsystems

It took some time to settle on the right definition of a 'normal' saturated subsystem $\mathcal{E}$ of a fusion system $\mathcal{F}$. It quickly became clear that $\mathcal{E}$-morphisms would need to satisfy an invariance condition when conjugated by $\mathcal{F}$-morphisms but more than this, $\mathcal{F}$-morphisms should factorise in a way which resembles the way in which a group decomposes as a product under the Frattini argument. These two conditions are essentially what constitute the definition of 'weak normality' below. The definition of 'strong normality' adds to this a further condition which is satisfied by most naturally occurring weakly normal subsystems including those induced by a normal subgroup of a group (see Proposition 1.55). Strong normality is now generally considered to be the correct definition because it interacts very well with the class of constrained fusion systems[3]. This is a desirable trait since, by Theorem 1.26, all saturated fusion systems are generated by constrained fusion systems. We start by gradually building up the definition of normality for a subsystem $\mathcal{E}$ of a fusion system $\mathcal{F}$, before listing some examples.

Definition 1.53 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let $\mathcal{E}$ be a subsystem of $\mathcal{F}$ on $T \leq S$.

(a) $\mathcal{E}$ is *$\mathcal{F}$-invariant* if

  (i) T is strongly $\mathcal{F}$-closed;

  (ii) $\mathrm{Aut}_{\mathcal{F}}(T) \leq \mathrm{Aut}(\mathcal{E})$; and

  (iii) for each $P \leq T$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, T)$, there is $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ and $\varphi_0 \in \mathrm{Hom}_{\mathcal{E}}(P, T)$ such that $\varphi = \varphi_0 \circ \alpha$.

(b) $\mathcal{E}$ is *weakly normal* if $\mathcal{E}$ is $\mathcal{F}$-invariant and saturated.

[3]See Theorem 4.2 of Chapter 4.

(c) $\mathcal{E}$ is *(strongly) normal* if $\mathcal{E}$ is weakly normal and each $\alpha \in \mathrm{Aut}_{\mathcal{E}}(T)$ extends to a map $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T))$ such that $\bar{\alpha}|_T = \alpha$ and $[\bar{\alpha}, C_S(T)] \leq Z(T)$.

The definition of $\mathcal{F}$-invariance presented here nicely depicts the 'Frattini condition' for morphisms mentioned above. In fact, conditions (a)(ii) and (a)(iii) can be combined to produce the following equivalent, simpler definition for $\mathcal{F}$-invariance which we occasionally use.

Lemma 1.54 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. A subsystem $\mathcal{E}$ of $\mathcal{F}$ on $T \leq S$ is $\mathcal{F}$-invariant if and only if*

(i) T is strongly $\mathcal{F}$-closed; and

(ii) for each $P \leq T$, $\varphi \in \mathrm{Hom}_{\mathcal{E}}(P, T)$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(T, S)$, we have

$$\psi^{-1}\varphi\psi \in \mathrm{Hom}_{\mathcal{E}}(P\psi, T).$$

Proof. This is shown in [11, Theorem 5.43]. □

From now on, we will refer to $(\mathcal{F}, \mathcal{E})$ as a *(weakly) normal pair* of fusion systems on (S, T) if $\mathcal{E}$ is (weakly) normal in $\mathcal{F}$. Also, if G is a finite group, H is a normal subgroup of G and S is a Sylow p-subgroup of G, we refer to (G, H) as a *normal pair of finite groups* with *Sylow p-subgroup* (S, T) where $T := S \cap H$.

We now consider some natural sources of examples of (weakly) normal pairs of fusion systems $(\mathcal{F}, \mathcal{E})$, starting with normal subgroups of finite groups.

Proposition 1.55 *Let (G, H) be a normal pair of finite groups with Sylow p-subgroup (S, T). Then $(\mathcal{F}_S(G), \mathcal{F}_T(H))$ is a normal pair of fusion systems on (S, T).*

Proof. Define $\mathcal{F} := \mathcal{F}_S(G)$ and $\mathcal{E} := \mathcal{F}_T(H)$, so that $\mathcal{F}$ and $\mathcal{E}$ are both saturated by Theorem 1.18. We will check conditions (a) to (c) in Definition 1.53. Since $H \trianglelefteq G$, T is strongly $\mathcal{F}$-closed and clearly $\alpha \in \mathrm{Aut}(\mathcal{E})$ for each $\alpha \in \mathrm{Aut}_G(T)$ so that (a) parts (i) and (ii) hold. By the Frattini argument, $G = HN_G(T)$ and each morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, T)$ admits the factorisation required to satisfy (a)(iii). It remains to check part (c).

Since $H \trianglelefteq G$, we have

$$[N_H(T), C_S(T)] \leq H \cap C_G(T) = C_H(T) \trianglelefteq N_H(T).$$

In particular, $C_H(T)C_S(T) \trianglelefteq N_H(T)C_S(T)$. Since $C_S(T) \in \mathrm{Syl}_p(C_H(T)C_S(T))$, the Frattini argument implies that

$$N_H(T)C_S(T) = C_H(T)C_S(T)N_{N_H(T)}(C_S(T)).$$

Now fix $g \in N_H(T)$, so that $g = xh$, for $x \in C_H(T)C_S(T) \leq C_{HS}(T)$ and $h \in N_{N_H(T)}(C_S(T))$. Since x acts trivially on T, g and h must act identically on T and therefore on $C_S(T)$. Thus

$$[g, C_S(T)] = [h, C_S(T)] \leq H \cap C_S(T) = Z(T),$$

as needed. □

In the above situation we will say that $(\mathcal{F}_S(G), \mathcal{F}_T(H))$ is a normal pair of fusion *induced* by the normal pair of finite groups (G, H).

Next, we look at normal subsystems of a fusion system with a normal subgroup. It is obvious that Q and $N_G^K(Q)$ are normal subgroups of $N_G(Q)$ when Q is a p-subgroup of some finite group G and K is a normal subgroup of $\mathrm{Aut}(Q)$. We have the following analogous statement for K-normaliser fusion systems:

Theorem 1.56 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, let Q be a normal subgroup of $\mathcal{F}$ and K be a normal subgroup of $\mathrm{Aut}(Q)$. The following hold:*

(a) $(\mathcal{F}, \mathcal{F}_Q(Q))$ is a normal pair fusion systems;

(b) $N_S^K(Q)$ is strongly $\mathcal{F}$-closed; and $(\mathcal{F}, N_{\mathcal{F}}^K(Q))$ is a weakly normal pair of fusion systems; and

(c) if $K = \mathrm{Id}_Q$ or $\mathrm{Inn}(Q) \leq K$ then $(\mathcal{F}, N_{\mathcal{F}}^K(Q))$ is a normal pair of fusion systems.

Proof. Part (a) is a result of Stancu and the weak normality of $\mathcal{F}_Q(Q)$ in $\mathcal{F}$ is shown in [11, Theorem 5.37]. Strong normality is immediate since each $\varphi \in \mathrm{Aut}_Q(Q) \leq \mathrm{Aut}_S(Q)$ satisfies $[\varphi, C_S(Q)] \leq Q \cap C_S(Q) = Z(Q)$ since Q is normal in S. Part (b) is shown in [1, Proposition 1.25(c)]. Part (c) in the case where $K = \mathrm{Id}_Q$ is shown in [11, Proposition 8.8]. We supply a proof when $\mathrm{Inn}(Q) \leq K$. Since Q is fully $\mathcal{F}$-centralised, Lemma 1.23 (a) implies that $QC_S(Q)$ is $\mathcal{F}$-centric and therefore $N_S^K(Q)$ is $\mathcal{F}$-centric by Lemma 1.23 (b). Hence condition (c) in Definition 1.53 is trivially satisfied and the result follows from (b). □

It is somewhat mysterious in the above situation that no proof of normality for the pair $(\mathcal{F}, N_{\mathcal{F}}^K(Q))$ for arbitrary normal subgroups K of $\mathrm{Aut}(Q)$ is known. One suspects such a proof exists, given the truth of the corresponding statement for finite groups[4].

We turn now to consider the analogue for fusion systems of the direct product of two groups. To motivate the definition, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be fusion systems on finite p-groups S_1 and S_2 respectively. Firstly, each subgroup P of $S_1 \times S_2$ is contained in a direct product $P_1 \times P_2$, where P_i is the projection of P onto S_i for $i = 1, 2$. Secondly, given a pair of morphisms (φ_1, φ_2) with $\varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, S_i)$, we obtain a map $\varphi \in \mathrm{Hom}(P_1 \times P_2, S)$ which sends each $(g_1, g_2) \in P_1 \times P_2$ to $(g_1\varphi_1, g_2\varphi_2)$, and hence a map $\varphi|_P \in \mathrm{Hom}(P, S)$. These two observations imply that the following definition for $\mathcal{F}_1 \times \mathcal{F}_2$ is natural.

Definition 1.57 Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be fusion systems on finite p-groups S_1 and S_2 respectively. The fusion system $\mathcal{F}_1 \times \mathcal{F}_2$ on $S_1 \times S_2$ is defined as follows:

$$\mathcal{F}_1 \times \mathcal{F}_2 := \langle (\varphi_1, \varphi_2) \in \mathrm{Hom}(P_1 \times P_2, S) \mid P_i \leq S_i, \varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, S_i) \rangle_{S_1 \times S_2}.$$

Theorem 1.58 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be saturated fusion systems on finite p-groups S_1 and S_2 respectively. Then $\mathcal{F}_1 \times \mathcal{F}_2$ is a saturated fusion system on $S_1 \times S_2$ and $\mathcal{F}_i$ is a normal subsystem of $\mathcal{F}_1 \times \mathcal{F}_2$ for $i = 1, 2$.*

Proof. The saturation of $\mathcal{F}_1 \times \mathcal{F}_2$ is shown in [4, Theorem I.6.6]. Furthermore $\mathcal{F}_i$ is easily seen to be a weakly normal subsystem of $\mathcal{F}_1 \times \mathcal{F}_2$ for $i = 1, 2$. To see that $\mathcal{F}_i$ is also normal, first note that each $\varphi \in \mathrm{Aut}_{\mathcal{F}_i}(S_i)$ extends to a map $\bar{\varphi} \in \mathrm{Aut}_{\mathcal{F}_1 \times \mathcal{F}_2}(S_1 \times S_2)$ which acts trivially on S_{3-i}. Hence

$$[C_{S_1 \times S_2}(S_i), \bar{\varphi}] = [Z(S_i) \times S_{3-i}, \bar{\varphi}] \leq Z(S_i),$$

so that Definition 1.53 (c) is satisfied for the pair $(\mathcal{F}_1 \times \mathcal{F}_2, \mathcal{F}_i)$. □

To end this section, we look at analogues for fusion systems of the normal subgroups $O^p(G)$ and $O^{p'}(G)$ of a finite group G, starting with the former.

Definition 1.59 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. The *hyperfocal subgroup of $\mathcal{F}$* is the group

$$\mathfrak{hyp}(\mathcal{F}) := \langle g^{-1} g\alpha \mid g \in P, \alpha \in O^p(\mathrm{Aut}_{\mathcal{F}}(P)) \rangle.$$

We have the following result concerning $\mathfrak{hyp}(\mathcal{F})$ in the case where $\mathcal{F}$ is realisable.

[4]This follows from Propositions 1.29 and 1.55.

Proposition 1.60 *Let G be a finite group and let S be a Sylow p-subgroup of G. Writing $\mathcal{F} := \mathcal{F}_S(G)$, we have $\mathfrak{hyp}(\mathcal{F}) = O^p(G) \cap S$.*

Proof. This is shown in [26, Section 1.1]. □

Proposition 1.60 shows that there is a saturated fusion system on $\mathfrak{hyp}(\mathcal{F})$ equal to $\mathcal{F}_{O^p(G)\cap S}(O^p(G))$ whenever $\mathcal{F} = \mathcal{F}_S(G)$. One would like to extend this to arbitrary saturated fusion systems. In fact, it is possible to do something slightly better. The following theorem classifies all saturated subsystems of a saturated fusion system $\mathcal{F}$ on subgroups which *contain* $\mathfrak{hyp}(\mathcal{F})$. We do not supply a proof, since the argument is long and technical.

Theorem 1.61 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then for each $R \leq S$ which contains $\mathfrak{hyp}(\mathcal{F})$ there exists a unique saturated fusion system $\mathcal{F}_R$ on R given by*

$$\mathcal{F}_R := \langle \mathrm{Inn}(R), O^p(\mathrm{Aut}_{\mathcal{F}}(P)) \mid P \leq R \rangle_R.$$

Furthermore if R is normal in S then $\mathcal{F}_R$ is normal in $\mathcal{F}$.

Proof. This is shown in [4, Theorem I.7.4]. □

In the above situation, we will term elements of the set $\{\mathcal{F}_R \mid \mathfrak{hyp}(\mathcal{F}) \leq R \leq S\}$ subsystems of *p-power index* in S.

Finally, we briefly consider the subsystem $O^{p'}(\mathcal{F})$ of a saturated fusion system $\mathcal{F}$ on a finite p-group S. To define this object, let $\mathcal{A}$ be the set of all normal subsystems of $\mathcal{F}$ on S ordered by inclusion. In [3, Theorem 1] it shown that for each pair of fusion systems $\mathcal{E}$ and $\mathcal{E}'$ in $\mathcal{A}$ which lie respectively on subgroups T and T' of S say, the fusion system $\mathcal{E} \cap \mathcal{E}'$ on $T \cap T'$ whose morphisms are those in both $\mathcal{E}$ and $\mathcal{E}'$ contains a normal subsystem $\mathcal{E} \wedge \mathcal{E}'$ of $\mathcal{F}$. Furthermore $\mathcal{E} \wedge \mathcal{E}'$ is maximal amongst all normal subsystems of $\mathcal{E}$ and $\mathcal{E}'$ in $\mathcal{A}$. Hence there exists a unique minimal element (with respect to the number of morphisms) of $\mathcal{A}$, called $O^{p'}(\mathcal{F})$.

1.5 Linking Systems

We end this chapter with a brief introduction to linking systems, starting with some remarks concerning their relevance to the theory. Topologists have long been interested in studying the groups $H^n(G, \mathbb{F}_p)$ and $H_n(G, \mathbb{F}_p)$ for G a finite group and p a prime. Since $H^n(G, \mathbb{F}_p) = H^n(BG, \mathbb{F}_p)$, one is often concerned with the structure

of the space BG. The (now proved) Martino–Priddy conjecture states that the p-fusion systems of a pair of finite groups G and H are isomorphic if and only if their classifying spaces have the same mod p-cohomology. Thus it is natural to look for a space $X_{\mathcal{F}}$ to associate to an arbitrary saturated fusion system $\mathcal{F}$ on a finite p-group S which is unique in the sense that the mod-p cohomology of $X_{\mathcal{F}}$ determines the fusion system $\mathcal{F}$ up to isomorphism. The problem of constructing such a space was solved by Broto, Levi and Oliver in [6]. There, the authors define a certain category $\mathcal{L}$ associated to an arbitrary fusion system $\mathcal{F}$ with the property that $|\mathcal{L}|$ and BG have the same mod-p cohomology whenever $\mathcal{L}$ is associated to the fusion system $\mathcal{F}_S(G)$. A number of other properties of $\mathcal{L}$ which hold for groups were checked in the same paper, giving further evidence that the category they had constructed was the correct one. A generalised Martino–Priddy Conjecture was then formulated and this has since been proven by Andrew Chermak in [9] as a consequence of the Classification of Finite Simple Groups.

In this thesis, we will primarily be interested in linking systems because their automorphisms can be used to construct interesting subsystems of fusion systems (see Chapter 4). However, we also mention in Chapter 2 that certain graphs we construct there might be reinterpreted in the language of linking systems.

1.5.1 Definitions and Background Results

We start with the definition of a linking system associated to a saturated fusion system. If S is a finite p-group and $\mathcal{H}$ is a set of subgroups of S, let $\mathcal{T}_{\mathcal{H}}(S)$ denote the category with object set $\mathcal{H}$ and with $\mathrm{Hom}_{\mathcal{T}_{\mathcal{H}}(S)}(P,Q) := \{g \in S \mid g^{-1}Pg \leq Q\}$.

Definition 1.62 Let $\mathcal{F}$ be a fusion system over a finite p-group S. A *linking system associated to* $\mathcal{F}$ is a triple $(\mathcal{L},\, \delta,\, \pi)$ (referred to as '$\mathcal{L}$') where $\mathcal{L}$ is a category with $\mathrm{Ob}(\mathcal{L}) \subseteq \mathrm{Ob}(\mathcal{F})$ and where δ and π are functors

$$\mathcal{T}_{Ob(\mathcal{L})}(S) \xrightarrow{\ \delta\ } \mathcal{L} \xrightarrow{\ \pi\ } \mathcal{F}$$

which satisfy the following conditions:

(a) (i) $\mathrm{Ob}(\mathcal{L})$ is closed under $\mathcal{F}$-conjugacy and taking overgroups in S.

(ii) Each $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroup is contained in $\mathrm{Ob}(\mathcal{L})$.

(iii) Each $P \in \mathrm{Ob}(\mathcal{L})$ is $\mathcal{L}$-isomorphic to a fully $\mathcal{F}$-centralised subgroup of S.

(iv) δ is the identity on objects and π is the inclusion on objects. For each $P, Q \in \mathcal{H}$, write $\delta_{P,Q}(g)$ for the image of the morphism $g \in \mathrm{Hom}_{\mathcal{T}_{Ob(\mathcal{L})}(S)}(P, Q)$ under δ and write $\delta_P(g) := \delta_{P,P}(g)$ when $Q = P$. Similarly, write $\pi_{P,Q}(\psi)$ for the image of $\psi \in \mathrm{Hom}_{\mathcal{L}}(P, Q)$ under π.

(v) Whenever $P, Q \in \mathrm{Ob}(\mathcal{L})$ and P is fully $\mathcal{F}$-centralised, $C_S(P)$ acts freely on $\mathrm{Hom}_{\mathcal{L}}(P, Q)$ via δ_P by left composition and $\pi_{P,Q}$ induces a bijection

$$\mathrm{Hom}_{\mathcal{L}}(P, Q)/C_S(P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}}(P, Q).$$

(b) For each $P, Q \in \mathrm{Ob}(\mathcal{L})$ and $g \in \mathrm{Hom}_{\mathcal{T}_{Ob(\mathcal{L})}(S)}(P, Q)$,

$$\pi_{P,Q}(\delta_{P,Q}(g)) = c_g \in \mathrm{Hom}_{\mathcal{F}}(P, Q).$$

(c) For each $\psi \in \mathrm{Hom}_{\mathcal{L}}(P, Q)$ and $g \in P$ the diagram

$$\begin{array}{ccc} P & \xrightarrow{\psi} & Q \\ \Big\downarrow{\scriptstyle \delta_P(g)} & & \Big\downarrow{\scriptstyle \delta_Q(g\pi(\psi))} \\ P & \xrightarrow{\psi} & Q \end{array}$$

commutes.

We say that $\mathcal{L}$ is a *centric linking system* when $\mathrm{Ob}(\mathcal{L})$ is exactly the set of $\mathcal{F}$-centric objects. Each centric linking system $\mathcal{L}$ may be viewed as a certain 'central extension' of $\mathcal{F}$ and the axioms for $\mathcal{L}$ reflect precisely what this means for objects and morphisms.

Given a pair of linking systems $(\mathcal{L}_1, \delta_1, \pi_1)$ and $(\mathcal{L}_2, \delta_2, \pi_2)$, an isomorphism from $\mathcal{L}_1$ to $\mathcal{L}_2$ is a functor χ from $\mathcal{L}_1$ to $\mathcal{L}_2$ such that $\pi_1 = \pi_2 \circ \chi$ and $\delta_2 = \chi \circ \delta_1$. The key theorem concerning centric linking systems associated to fusion systems is the following result of Chermak whose proof relies on the Classification of Finite Simple Groups.

Theorem 1.63 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then there exists (up to isomorphism) a unique centric linking system associated to $\mathcal{F}$.*

Proof. This is shown in [9]. □

Thus we are guaranteed the existence of at least one linking system associated to a given saturated fusion system. We now look at ways to find more. The next theorem of Oliver provides a construction for doing this, simultaneously placing a bound on the size of $|\mathrm{Ob}(\mathcal{L})|$. To state it, we need a new definition. Given a saturated fusion system $\mathcal{F}$ on a finite p-group S, a subgroup P of S is *$\mathcal{F}$-quasicentric* if $C_{\mathcal{F}}(P)$ is the trivial fusion system $\mathcal{F}_{C_S(P)}(C_S(P))$.

Theorem 1.64 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. The following hold.*

(a) If $\mathcal{L}$ is a linking system associated to $\mathcal{F}$ then each $P \in Ob(\mathcal{L})$ is $\mathcal{F}$-quasicentric.

(b) Let $\mathcal{H} \subseteq Ob(\mathcal{F})$ be a set of subgroups which is closed under $\mathcal{F}$-conjugacy and overgroups in S and contains each $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroup of S. Then for each $\hat{\mathcal{H}} \subseteq Ob(\mathcal{F})$ containing $\mathcal{H}$ and each linking system $\mathcal{L}$ associated to $\mathcal{F}$ with $Ob(\mathcal{L}) = \mathcal{H}$, there exists a unique linking system $\hat{\mathcal{L}}$ containing $\mathcal{L}$ with $Ob(\hat{\mathcal{L}}) = \hat{\mathcal{H}}$.

Proof. Part (a) is shown in [4, Proposition III.4.7] and part (b) is shown in [4, Proposition III.4.8]. □

This result, when combined with Theorem 1.63, shows that we have a large amount of flexibility with the choice of objects in a linking system $\mathcal{L}$ associated to $\mathcal{F}$.

Linking systems have a particularly nice description when they are associated to fusion systems of (possibly infinite) groups. Recall that a group G has a *Sylow p-subgroup* S if S is a finite p-subgroup of G and every other finite p-subgroup P of G is G-conjugate to a subgroup of S. P is said to be *G-quasicentric* if $O^p(C_G(P))$ contains no p-elements. We can now define the category $\mathcal{L}_S^{\mathcal{H}}(G)$ for certain sets $\mathcal{H}$ of subgroups of S.

Definition 1.65 Let G be a group with Sylow p-subgroup S and let $\mathcal{H}$ be a set of G-quasicentric subgroups of S which contains all those which are $\mathcal{F}_S(G)$-centric and $\mathcal{F}_S(G)$-radical. $\mathcal{L}_S^{\mathcal{H}}(G)$ to is the category with $\mathrm{Ob}(\mathcal{L}_S^{\mathcal{H}}(G)) = \mathcal{H}$ and where morphism sets are given by

$$\mathrm{Hom}_{\mathcal{L}_S^{\mathcal{H}}(G)}(P, Q) := N_G(P, Q)/O^p(C_G(P)),$$

for all $P, Q \in \mathcal{H}$. Write $\mathcal{L}_S^c(G) := \mathcal{L}_S^{\mathcal{H}}(G)$ in the case where $\mathcal{H}$ is the set $\mathcal{F}_S(G)$-centric subgroups.

It is not hard to see that $\mathcal{L}_S^{\mathcal{H}}(G)$ is a linking system associated to the fusion system $\mathcal{F}_S(G)$ when G is finite and $\mathcal{H}$ is defined as above. Composition is well-defined since for each $g \in N_G(P, Q)$, $C_G(Q) \leq C_G(P)^g$ so that $O^p(C_G(Q)) \leq O^p(C_G(P))^g$. If G is infinite then we have the following result when $\mathcal{H}$ is the set of all $\mathcal{F}_S(G)$-centric subgroups.

Lemma 1.66 *Let G be a group with Sylow p-subgroup S and let $\mathcal{H}$ be the set of all $\mathcal{F}_S(G)$-centric subgroups. If $H^i(C_G(P)/Z(P);\mathbb{F}_p) = 0$ for each $P \in \mathcal{H}$ and $i = 1, 2$ then $\mathcal{L}^c_S(G)$ is a centric linking system associated to $\mathcal{F}_S(G)$.*

Proof. This is shown in [7, Lemma 1.5]. □

It seems likely that one can replace the condition '$H^i(C_G(P)/Z(P);\mathbb{F}_p) = 0$' in Lemma 1.66 by the condition '$H^i(O^p(C_G(P));\mathbb{F}_p) = 0$' in order to prove a similar result for more general sets of G-quasicentric subgroups. However, this is not really necessary by Theorem 1.64.

We end this section by describing how a linking system may be regarded as the natural analogue of a classifying space of finite group. Recall that a map of spaces $f : X_1 \longrightarrow X_2$ is a *p-equivalence* if f induces isomorphisms

$$H^n(X_2;\mathbb{F}_p) \xrightarrow{f^n} H^n(X_1;\mathbb{F}_p)$$

for all $n \geq 0$. We have the following result.

Theorem 1.67 *Let G be a finite group and S be a Sylow p-subgroup of G. There exists a p equivalence*

$$BG \xrightarrow{f} |\mathcal{L}^c_S(G)|$$

Proof. This is shown in [4, Theorem III.3.2]. □

1.5.2 Automorphisms and Normal Subsystems

Starting with the definition of an 'automorphism' of a linking system $\mathcal{L}$, we proceed to define a 'normal' subsystem $\mathcal{L}_0$ of $\mathcal{L}$ and look closely at the automorphisms induced by the 'action' of $\mathcal{L}$ on $\mathcal{L}_0$. The definitions and results we introduce here will be applied exclusively in Section 4.3 and are motivated by those found in [1, Section 1].

Let $\mathcal{L}$ be a linking system associated to a fusion system $\mathcal{F}$ on a finite p-group S. A functor $\alpha \in \mathrm{Aut}(\mathcal{L})$ is said to be *isotypical* if $\alpha(\delta_P(P)) = \delta_{\alpha(P)}(\alpha(P))$ for each $P \in \mathrm{Ob}(\mathcal{L})$. Also, for each $P, Q \in \mathrm{Ob}(\mathcal{L})$, write ι^Q_P for the morphism $\delta_{P,Q}(1) \in \mathrm{Hom}_{\mathcal{L}}(P, Q)$ and call such a map an *inclusion from P to Q*. We can now define automorphisms of $\mathcal{L}$.

Definition 1.68 Let $\mathcal{F}$ be a fusion system on a finite p-group S and let $\mathcal{L}$ be linking system associated $\mathcal{F}$. An element $\alpha \in \mathrm{Aut}(\mathcal{L})$ is an *automorphism* of $\mathcal{L}$ if α is isotypical and sends inclusions to inclusions.

(a) $\mathrm{Aut}^I_{typ}(\mathcal{L})$ denotes the set of all automorphisms of $\mathcal{L}$.

(b) $\mathrm{Out}_{typ}(\mathcal{L}) \subseteq \mathrm{Out}(\mathcal{L})$ denotes the set of *outer automorphisms* consisting of representatives for the natural isomorphism classes of isotypical equivalences from $\mathcal{L}$ to itself.

Since we shall be interested in finding 'inner automorphisms' of $\mathcal{L}$, it will be important to understand how morphisms in $\mathcal{L}$ restrict and extend. This information is provided by the following proposition, whose proof follows quickly from the definitions.

Proposition 1.69 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let $\mathcal{L}$ be a linking system associated to $\mathcal{F}$.*

(a) For each morphism $\psi \in Hom_{\mathcal{L}}(P,Q)$, and every $P_0, Q_0 \in Ob(\mathcal{L})$ with $P_0 \leq P$, $Q_0 \leq Q$ and $P_0\pi(\psi) \leq Q_0$, there is a unique morphism $\psi|_{P_0,Q_0} \in \mathrm{Hom}_{\mathcal{L}}(P_0,Q_0)$ such that $\iota^P_{P_0} \circ \psi = \psi|_{P_0,Q_0} \circ \iota^Q_{Q_0}$.

(b) Let $P, Q, \bar{P}, \bar{Q} \in Ob(\mathcal{L})$ and $\psi \in \mathrm{Hom}_{\mathcal{L}}(P,Q)$ be such that $P \trianglelefteq \bar{P}$ and $Q \leq \bar{Q}$. There is a morphism $\bar{\psi} \in \mathrm{Hom}_{\mathcal{L}}(\bar{P},\bar{Q})$ such that $\bar{\psi}|_{P,Q} = \psi$ if and only if

$$\forall g \in \bar{P}\ \exists h \in \bar{Q} \text{ such that } \delta_P(g) \circ \psi \circ \iota^{\bar{Q}}_Q = \psi \circ \delta_{Q,\bar{Q}}(h).$$

Furthermore if such a morphism exists, it is unique.

Proof. Parts (a) and (b) are shown in [1, Proposition 1.11 (b) and (e)]. □

We can now introduce the concept of a 'normal subsystem' of a linking system.

Definition 1.70 Let $(\mathcal{F},\mathcal{E})$ be a weakly normal pair of fusion systems on (S,T). We say that $(\mathcal{L},\mathcal{L}_0)$ is a *weakly normal pair of linking systems* associated to $(\mathcal{F},\mathcal{E})$ if the following hold:

(a) $\mathcal{L}$ is a linking system associated to $\mathcal{F}$ and $\mathcal{L}_0$ is a linking system associated to $\mathcal{E}$ with $\mathcal{L}_0 \subseteq \mathcal{L}$ and $\mathrm{Ob}(\mathcal{L}) = \{P \leq S \mid P \cap T \in \mathrm{Ob}(\mathcal{L}_0)\}$. Furthermore the structural functors

$$\mathcal{T}_{Ob(\mathcal{L}_0)}(T) \xrightarrow{\delta_0} \mathcal{L}_0 \xrightarrow{\pi_0} \mathcal{E}$$

are restrictions of the structural functors for $\mathcal{L}$.

(b) (i) For all $P, Q \in \mathrm{Ob}(\mathcal{L}_0)$ and $\psi \in \mathrm{Hom}_{\mathcal{L}}(P,Q)$ there are morphisms $\gamma \in \mathrm{Aut}_{\mathcal{L}}(T)$ and $\psi_0 \in \mathrm{Hom}_{\mathcal{L}_0}(P\pi(\gamma),Q)$ such that $\psi = \gamma|_{P,P\pi(\gamma)} \circ \psi_0$.

(ii) For all $\gamma \in \mathrm{Aut}_{\mathcal{L}}(T)$ and $\psi \in \mathrm{Mor}(\mathcal{L}_0), \gamma^{-1} \circ \psi \circ \gamma \in \mathrm{Mor}(\mathcal{L}_0)$.

We end with the construction of a 'conjugation' map $\mathrm{Aut}_{\mathcal{L}}(T) \longrightarrow \mathrm{Aut}^I_{typ}(\mathcal{L}_0)$ for an arbitrary weakly normal pair $(\mathcal{L}, \mathcal{L}_0)$ of linking systems associated to a weakly normal pair of fusion systems $(\mathcal{F}, \mathcal{E})$ on (S, T). For each $\gamma \in \mathrm{Aut}_{\mathcal{L}}(T)$ let $c_\gamma = c_\gamma|_{\mathcal{L}_0}$ be the functor from $\mathcal{L}_0$ to itself defined by setting $c_\gamma(P) := P\gamma := P\pi(\gamma)$ for $P \in \mathrm{Ob}(\mathcal{L}_0)$ and $c_\gamma(\psi) := (\gamma|_{P,P\gamma})^{-1} \circ \psi \circ (\gamma|_{Q,Q\gamma}) \in \mathrm{Hom}_{\mathcal{L}_0}(P\gamma, Q\gamma)$ for each $\psi \in \mathrm{Hom}_{\mathcal{L}_0}(P, Q)$. (Here we use that $\mathcal{L}_0$ is weakly normal in $\mathcal{L}$).

Lemma 1.71 *Let $(\mathcal{L}, \mathcal{L}_0)$ be a weakly normal pair of linking systems associated to a weakly normal pair of saturated fusion systems $(\mathcal{F}, \mathcal{E})$ on (S, T). Then $c_\gamma|_{\mathcal{L}_0} \in \mathrm{Aut}^I_{typ}(\mathcal{L}_0)$ for each $\gamma \in Aut_{\mathcal{L}}(T)$.*

Proof. By Proposition 1.69 (a), $\iota^Q_P \circ \gamma|_{Q,Q\gamma} = \gamma|_{P,P\gamma} \circ \iota^Q_{P\gamma}$ for each $P, Q \in \mathrm{Ob}(\mathcal{L}_0)$ so that c_γ clearly sends inclusions to inclusions. To see that c_γ is isotypical, note that for each $P \in \mathrm{Ob}(\mathcal{L}_0)$ and $g \in P$, $\delta_P(g) \circ \gamma|_{P,P\gamma} = \gamma|_{P,P\gamma} \circ \delta_{P\gamma}(g\pi(\gamma))$ by Definition 1.62 (c) for $\mathcal{L}_0$, so that $c_\gamma(\delta_P(g)) = \delta_{P\gamma}(g\pi(\gamma))$ and c_γ is isotypical, as needed. □

Chapter 2

Trees of Fusion Systems

Graphs of groups were first considered in the work of Bass and Serre and have since become a prominent tool in geometric group theory. The study of these objects, commonly known as 'Bass–Serre Theory', asserts that trees of groups are in one-to-one correspondence with certain group actions on simplicial trees. We will describe this correspondence in Section 2.1.

This chapter is concerned with generalising the notion of a tree of groups to a 'tree of fusion systems' on a tree of p-groups. In order to motivate our definitions, we focus first on the case where a tree of fusion systems is 'induced' by a tree of groups. This allows us to prove a key theorem (Theorem 1) relating the fusion system of the group completion G (assuming it exists) with the fusion systems of the vertex-stabilisers of G.

Let S be a p-group, X be a space and f be a map $f : BS \longrightarrow X$. In [7, Theorem 2.1] the authors find conditions on the triple (S, X, f) for there to exist a saturated fusion system $\mathcal{F}$ on S and linking system $\mathcal{L}$ associated to $\mathcal{F}$ such that f extends to a p-equivalence $|\mathcal{L}| \longrightarrow X$. Their result applies when G is a discrete group and X is an $\mathbb{F}_p$-acyclic G-complex (i.e. $H^n(X, \mathbb{F}_p) = 0$ for all $n \geq 0$) with finite isotropy subgroups and finitely many orbits of cells, and S is a Sylow p-subgroup of G. Such groups G arise as completions of trees of groups, where X is taken to be the covering tree. Our goal is to extend the results of [7] to arbitrary trees of fusion systems without reference to group actions or homotopy theory. To do this, we introduce a certain graph determined by an arbitrary finite p-group P called the 'P-orbit graph' which is analogous to a certain quotient of the covering tree associated to a graph of groups. We think of the completion of the tree of fusion systems 'acting' on this graph via automorphisms of P. Theorem 2.18 (Theorem 2) asserts that the completion $\mathcal{F}$ of a tree of fusion systems is saturated provided all essential subgroups of the vertex fusion systems are $\mathcal{F}$-centric and the P-orbit graphs are trees whenever P is $\mathcal{F}$-centric.

The chapter is structured as follows. In Section 2.1 we introduce the necessary definitions and results from Bass-Serre Theory, sketching a proof of the Fundamental Theorem in the process. In Section 2.2 we define trees of fusion systems before considering those that are induced by finite groups and proving Theorem 1 (Theorem 2.13) in Section 2.3. Finally, we prove Theorem 2 in Section 2.4. We remark that this chapter is relatively self contained, relying almost exclusively on the definitions and results recorded in Sections 1.1 and 1.2.

2.1 Trees of Groups

2.1.1 Definitions and Basic Results

We begin by defining what is meant by a 'tree of groups' and the 'completion' of such an object. Furthermore, we prove that the completion always exists. If $\mathcal{G}$ is a simple, undirected graph, write $V(\mathcal{G})$ and $E(\mathcal{G})$ for the sets of vertices and edges of $\mathcal{G}$ respectively. Each edge $e \in E(\mathcal{G})$ is regarded as an unordered pair of vertices, (v, w) say, where $v, w \in V(\mathcal{G})$ and in this case we say that e is an edge from v to w (or from w to v) or that v and w are *incident* to e. Recall that a *tree* $\mathcal{T}$ is simply a graph without any cycles where a *cycle* is an ordered sequence of vertices $v_1, \dots, v_n \in V(\mathcal{T})$ with $v_1 = v_n$ and (v_i, v_{i+1}) an edge of $\mathcal{T}$ for each $1 \leq i \leq n-1$.

Each graph $\mathcal{T}$ gives rise to a category (also called $\mathcal{T}$) where $\mathrm{Ob}(\mathcal{T})$ is the disjoint union of the sets $V(\mathcal{T})$ and $E(\mathcal{T})$ and where for each edge $(v, w) \in E(\mathcal{T})$ there exists a pair of morphisms

$$\begin{array}{ccc} e & \longrightarrow & v \\ \big\downarrow & & \\ w & & \end{array}$$

in $\mathcal{T}$. Thus $\mathrm{Hom}_{\mathcal{T}}(e, v)$ has cardinality 1 for each edge e and vertex v incident to e and we denote the unique morphism in $\mathrm{Hom}_{\mathcal{T}}(e, v)$ by f_{ev}. Writing $\mathfrak{Grp}$ for the category of groups and group homomorphisms, a *tree of groups* is a pair $(\mathcal{T}, \mathcal{G})$ where $\mathcal{T}$ is a tree and $\mathcal{G}$ is a functor from $\mathcal{T}$ (regarded as a category) to $\mathfrak{Grp}$ which sends f_{ev} to an injective group homomorphism from $\mathcal{G}(e)$ to $\mathcal{G}(v)$. The *completion* of a tree of groups $(\mathcal{T}, \mathcal{G})$ is a colimit of $\mathcal{G}$ over $\mathcal{T}$ in the categorical sense. When such an object exists, we denote it by $\mathcal{G}_{\mathcal{T}}$.

Lemma 2.1 *Let $(\mathcal{T}, \mathcal{G})$ be a tree of groups. There exists a colimit $\mathcal{G}_{\mathcal{T}}$ of $\mathcal{G}$ which is unique up to group isomorphism.*

Proof. It suffices to prove existence since uniqueness follows from the universal property of the colimit. One can describe $\mathcal{G}_\mathcal{T}$ via generators and relations as follows. Let the set of generators of $\mathcal{G}_\mathcal{T}$ be the union

$$\bigcup_{v \in V(\mathcal{T})} \mathcal{G}(v) \cup \bigcup_{e \in E(\mathcal{T})} \mathcal{G}(e)$$

of all elements in $\mathcal{G}(e)$ and $\mathcal{G}(v)$. The relations are given by

(a) xyz^{-1} for each $x, y, z \in \mathcal{G}(e)$ or $\mathcal{G}(v)$ such that $xy = z$; and

(b) xy^{-1} for each $x \in \mathcal{G}(e)$ and $y \in \mathcal{G}(v)$ where v is incident to e in $\mathcal{T}$ and $y = x\mathcal{G}(f_{ev})$. □

In order to state the Fundamental Theorem of Bass–Serre Theory in the next section, we will need to introduce group actions on trees. Recall that given a pair of graphs $\mathcal{T}$ and $\mathcal{S}$ a *homomorphism* φ from $\mathcal{T}$ to $\mathcal{S}$ is a function from $V(\mathcal{T})$ to $V(\mathcal{S})$ with the property that whenever (v, w) is an edge of $\mathcal{T}$, $(v\varphi, w\varphi)$ is an edge of $\mathcal{S}$. (Note that in this case φ induces a function from $E(\mathcal{T})$ to $E(\mathcal{S})$). If φ is a bijection and the inverse mapping φ^{-1} from $V(\mathcal{S})$ to $V(\mathcal{T})$ is also a homomorphism then we say that φ is is an *isomorphism* from $\mathcal{T}$ to $\mathcal{S}$. The set of all isomorphisms from a graph $\mathcal{T}$ to itself forms a group which we denote by $\mathrm{Aut}(\mathcal{T})$. Observe that the graph-theoretic and category-theoretic definitions of $\mathrm{Aut}(\mathcal{T})$ coincide. A group G *acts* on a tree $\tilde{\mathcal{T}}$ if there exists a homomorphism

$$G \longrightarrow \mathrm{Aut}(\tilde{\mathcal{T}}).$$

We denote by $\tilde{\mathcal{T}}/G$ the graph with vertex set given by the set of orbits of $V(\tilde{\mathcal{T}})$ under the action of G on $V(\tilde{\mathcal{T}})$ and likewise for the edges.

2.1.2 The Fundamental Theorem of Bass–Serre Theory

The Fundamental Theorem may be stated as follows:

Theorem 2.2 *There is a one-to-one correspondence between trees of groups $(\mathcal{T}, \mathcal{G})$ and pairs $(\tilde{\mathcal{T}}, G)$ where $\tilde{\mathcal{T}}$ is a tree on which a group G acts with $\tilde{\mathcal{T}}/G$ a tree.*

The proof of Theorem 2.2 introduces some important definitions so we sketch it now. Our discussion is based on that found in [29, I.4.5], where the reader will find full proofs of all the results mentioned below.

Firstly, suppose that $\tilde{\mathcal{T}}$ is a tree on which a group G acts. A *fundamental domain* of $\tilde{\mathcal{T}}$ (mod G) is any subgraph $\mathcal{T}$ of $\tilde{\mathcal{T}}$ for which there exists a graph isomorphism from $\mathcal{T}$ to $\tilde{\mathcal{T}}/G$. It is a fact that a fundamental domain $\mathcal{T}$ of $\tilde{\mathcal{T}}$ mod G exists if and only if $\tilde{\mathcal{T}}/G$ is a tree. Write $\mathcal{T}$ for this fundamental domain and let

$$G_v := \{g \in G \mid vg = v\} \text{ and } G_e := \{g \in G \mid eg = e\}$$

be the *stabilisers* of $v \in V(\tilde{\mathcal{T}})$ and $e \in E(\tilde{\mathcal{T}})$ respectively. Note that $G_v \cap G_w \leq G_e$ for each $e = (v, w) \in E(\tilde{\mathcal{T}})$. One obtains a tree of groups $(\mathcal{T}, \mathcal{G})$ where $\mathcal{G}$ is the functor which sends $v \in V(\mathcal{T})$ to G_v and each $e = (v, w) \in E(\mathcal{T})$ to $G_v \cap G_w$ and where the unique morphism $f_{ev} \in \mathrm{Hom}_{\mathcal{T}}(e, v)$ gets sent to the inclusion $G_v \cap G_w \hookrightarrow G_v$ for a vertex v incident to an edge $e = (v, w)$. In this situation, one proves:

Lemma 2.3 *The completion of the tree of groups $(\mathcal{T}, \mathcal{G})$ is isomorphic to G.*

Conversely suppose that one starts with a tree of groups $(\mathcal{T}, \mathcal{G})$ and write $G := \mathcal{G}_{\mathcal{T}}$ for its completion. Let $G/\mathcal{G}(-)$ be the functor from $\mathcal{T}$ to $\mathfrak{Set}$ which sends each v and $e \in \mathrm{Ob}(\mathcal{T})$ respectively to the sets of left cosets $G/\mathcal{G}(v)$ and $G/\mathcal{G}(e)$ and which sends $f_{ev} \in \mathrm{Hom}_{\mathcal{T}}(e, v)$ to the map from $G/\mathcal{G}(e)$ to $G/\mathcal{G}(v)$ given by sending left cosets $g\mathcal{G}(e)$ to $g\mathcal{G}(v)$. (Note that this map is well-defined since we may identify $\mathcal{G}(e)$ as a subgroup of $\mathcal{G}(v)$ in G). Define the *orbit graph* $\tilde{\mathcal{T}}$ to be the homotopy colimit

$$\underrightarrow{\mathrm{hocolim}}_{\mathcal{T}} G/\mathcal{G}(-).$$

Equivalently, we may think of $\tilde{\mathcal{T}}$ as being a graph whose vertices and edges are labelled by the sets $\{g\mathcal{G}(v) \mid g \in G, v \in V(\mathcal{T})\}$ and $\{i\mathcal{G}(e) \mid i \in G, e \in E(\mathcal{T})\}$ respectively and where two vertices $g\mathcal{G}(v)$ and $h\mathcal{G}(w)$ are connected via an edge $i\mathcal{G}(e)$ if and only if $i\mathcal{G}(v) = g\mathcal{G}(v)$ and $i\mathcal{G}(w) = h\mathcal{G}(w)$. This is obviously a graph on which G acts by left multiplication. In fact, more is true:

Lemma 2.4 *Let $(\mathcal{T}, \mathcal{G})$ be a tree of groups with completion $G := \mathcal{G}_{\mathcal{T}}$. Then $\tilde{\mathcal{T}}$ is a tree on which G acts and $\tilde{\mathcal{T}}$ has a fundamental domain mod G isomorphic to $\mathcal{T}$. Furthermore $G_v \cong \mathcal{G}(v)$ and $G_e \cong \mathcal{G}(e)$ for each vertex v and edge e of $\mathcal{T}$.*

This completes the proof sketch of Theorem 2.2.

2.2 Trees of Fusion Systems

In this section, we will carefully define what we mean by a tree of fusion systems and the completion of such an object. We then find a natural condition which ensures that the completion of a tree of fusion systems exists. We warn the reader that for the purposes of this chapter, the use of the symbol '$\mathcal{F}$' has temporarily changed: rather than referring to a category, it now refers to a functor.

2.2.1 Definitions and Basic Results

Recall from Section 1.2.1 that morphisms exist between fusion systems so we may define $\mathfrak{Fus}$, the category of fusion systems on finite p-groups and morphisms between them. Suppose that $\mathcal{F}$ and $\mathcal{E}$ are fusion systems on finite p-groups S and T respectively. A morphism $\alpha \in \mathrm{Hom}(S,T)$ from $\mathcal{F}$ to $\mathcal{E}$ is *injective* if it induces an injective map

$$\mathrm{Hom}_{\mathcal{F}}(P,S) \longrightarrow \mathrm{Hom}_{\mathcal{E}}(P\alpha,T)$$

for each $P \leq S$.

Definition 2.5 A *tree of p-fusion systems* is a triple $(\mathcal{T},\mathcal{F},\mathcal{S})$ where $(\mathcal{T},\mathcal{S})$ is a tree of finite p-groups and $\mathcal{F}$ is a functor from $\mathcal{T}$ to $\mathfrak{Fus}$ such that the following hold:

(a) $\mathcal{F}(v)$ is a fusion system on $\mathcal{S}(v)$ and $\mathcal{F}(e)$ is a fusion system on $\mathcal{S}(e)$; and

(b) $\mathcal{F}$ sends $f_{ev} \in \mathrm{Hom}_{\mathcal{T}}(e,v)$ to an injective morphism from $\mathcal{F}(e)$ to $\mathcal{F}(v)$ with $\mathcal{F}(f_{ev}) = \mathcal{S}(f_{ev})$.

Definition 2.6 Let $(\mathcal{T},\mathcal{F},\mathcal{S})$ be a tree of p-fusion systems. The *completion* of $(\mathcal{T},\mathcal{F},\mathcal{S})$ is a colimit for $\mathcal{F}$.

Of course, we need conditions on $(\mathcal{T},\mathcal{F},\mathcal{S})$ which imply that a colimit for $\mathcal{F}$ exists, since this is no longer guaranteed as it was in the category of groups. Indeed, any colimit for $\mathcal{F}$ must be a fusion system on the completion $S_{\mathcal{T}}$ of $(\mathcal{T},\mathcal{S})$ and this may not be a p-group[1]. The following is a very simple (and natural) condition to impose:

Hypothesis (H)*: There exists a vertex $v_* \in V(\mathcal{T})$ with the property that whenever $v \in V(\mathcal{T})$, $\mathcal{S}(e) \cong \mathcal{S}(v)$ where e is the edge incident to v in the unique minimal path from v to v_*.*

[1]Consider, for example the amalgam $C_2 * C_2 \cong D_\infty$ when $p = 2$.

We will say that a tree of p-fusion systems $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ satisfies (H) if $(\mathcal{T}, \mathcal{S})$ satisfies Hypothesis (H). Let $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ be a tree of p-fusion systems which satisfies (H) and write $S := \mathcal{S}(v_*)$. It is clear that $\mathcal{S}_{\mathcal{T}}$ is a finite group isomorphic to $\mathcal{S}(v_*)$, so that we may view $\mathcal{S}(e)$ and $\mathcal{S}(v)$ as subgroups of $\mathcal{S}(v_*)$ by identifying them with their images in the completion. Also, $\mathcal{S}(v) \cap \mathcal{S}(w) = \mathcal{S}(e)$ in $\mathcal{S}(v_*)$ whenever v and w are vertices of $\mathcal{T}$ incident to e. Furthermore, when v is a vertex incident to an edge e, this identification allows us to embed each fusion system $\mathcal{F}(e)$ as a subsystem of $\mathcal{F}(v)$ by identifying each morphism $\alpha \in \mathrm{Hom}_{\mathcal{F}(e)}(P, \mathcal{S}(e))$ with its image under the functor from $\mathcal{F}(e)$ to $\mathcal{F}(v)$ induced by $f_{ev} \in \mathrm{Hom}_{\mathcal{T}}(e, v)$.

Lemma 2.7 *Let $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ be a tree of p-fusion systems which satisfies (H), set $v_0 := v_*$ and write $S := \mathcal{S}(v_0) = \mathcal{S}_{\mathcal{T}}$. Define*

$$\mathcal{F}_{\mathcal{T}} := \langle \mathrm{Hom}_{\mathcal{F}(v)}(P, \mathcal{S}(v)) \mid P \leq \mathcal{S}(v), v \in V(\mathcal{T}) \rangle_S,$$

the fusion system generated by the $\mathcal{F}(v)$ for each $v \in V(\mathcal{T})$. Then $\mathcal{F}_{\mathcal{T}}$ is a colimit for $\mathcal{F}$ and each $\alpha \in \mathrm{Hom}_{\mathcal{F}_{\mathcal{T}}}(P, S)$ may be written as a composite

$$P = P_0 \xrightarrow{\alpha_0} P_1 \xrightarrow{\alpha_1} P_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} P_n = P\alpha$$

where for $0 \leq i \leq n-1$, $P_i \leq \mathcal{S}(v_{i-1}) \cap \mathcal{S}(v_i)$, $\alpha_i \in \mathrm{Mor}(\mathcal{F}(v_i))$ for some $v_i \in V(\mathcal{T})$ and (v_i, v_{i+1}) is an edge in $\mathcal{T}$.

Proof. The fact that $\mathcal{F}_{\mathcal{T}}$ is a colimit for $\mathcal{F}$ follows immediately from the fact that it is unique (up to an isomorphism of categories) amongst all fusion systems on S which contain $\mathcal{F}(v)$ for each $v \in V(\mathcal{T})$[2]. To see the second statement, clearly any such composite of morphisms lies in $\mathcal{F}_{\mathcal{T}}$. Conversely let

$$P = P_0 \xrightarrow{\alpha_0} P_1 \xrightarrow{\alpha_1} P_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} P_n = P\alpha$$

be a representation of $\alpha \in \mathrm{Hom}_{\mathcal{F}_{\mathcal{T}}}(P, S)$ where $\alpha_i \in \mathrm{Mor}(\mathcal{F}(v_i))$ for $0 \leq i \leq n-1$. If (v_{i-1}, v_i) is not an edge then let η be a minimal path in $\mathcal{T}$ from v_{i-1} to v_i. Since $(\mathcal{T}, S)$ satisfies (H), P_i is contained in $\mathcal{S}(w)$ for each vertex $w \in \eta$ so that by inserting identity morphisms, the above sequence of morphisms can be refined so that (v_{i-1}, v_i) is an edge for each i. This completes the proof of the lemma. □

One observes that by Lemma 2.7, the completion $\mathcal{F}_{\mathcal{T}}$ of a tree of p-fusion systems $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ which satisfies (H) is independent of where $\mathcal{F}$ sends the edges e of $\mathcal{T}$. We will make heavy use of this fact later in the proof of Theorem 2.

[2]More information concerning this characterisation of $\mathcal{F}_{\mathcal{T}}$ is provided by the remarks which follow Definition 1.15 in Section 1.2.

2.2.2 The P-Orbit Graph

We now turn to the definition of a certain graph constructed from a tree of p-fusion systems (now simply referred to as a tree of fusion systems) and an arbitrary finite p-group P. Our discussion culminates in the proof of an important result, Proposition 2.10, which will allow us to describe morphisms in the completion combinatorially.

We begin by introducing some notation. Let $\mathcal{K}$ be a fusion system on a finite p-group S and let P be any other finite p-group. We define an equivalence relation $\sim$ on the set of homomorphisms $\mathrm{Hom}(P,S)$ via $\alpha \sim \beta$ if and only if there is some $\gamma \in \mathrm{Iso}_{\mathcal{K}}(P\alpha, P\beta)$ such that $\alpha \circ \gamma = \beta$ for each $\alpha, \beta \in \mathrm{Hom}(P,S)$. The fact that $\sim$ is an equivalence relation follows from the axioms for a fusion system. Write

$$\mathrm{Rep}(P,\mathcal{K}) := \mathrm{Hom}(P,S)/\sim$$

for the set of all equivalence classes, and for each $\alpha \in \mathrm{Hom}(P,S)$ let $[\alpha]_{\mathcal{K}}$ denote the class of α in $\mathrm{Rep}(P,\mathcal{K})$. If $\hat{\mathcal{K}}$ is a fusion system containing $\mathcal{K}$, set $\mathrm{Rep}_{\hat{\mathcal{K}}}(P,\mathcal{K}) := \mathrm{Hom}_{\hat{\mathcal{K}}}(P,S)/\sim$.

Let $(\mathcal{T},\mathcal{F},\mathcal{S})$ be a tree of fusion systems, and P be a finite p-group. Define $\mathrm{Rep}(P,\mathcal{F}(-))$ to be the functor from $\mathcal{T}$ to $\mathfrak{Set}$ which sends vertices v and edges e of $\mathcal{T}$ respectively to $\mathrm{Rep}(P,\mathcal{F}(v))$ and $\mathrm{Rep}(P,\mathcal{F}(e))$ and which sends $f_{ev} \in \mathrm{Hom}_{\mathcal{T}}(e,v)$ to the map from $\mathrm{Rep}(P,\mathcal{F}(e))$ to $\mathrm{Rep}(P,\mathcal{F}(v))$ given by

$$[\alpha]_{\mathcal{F}(e)} \longmapsto [\alpha \circ \iota_{\mathcal{S}(e)}^{\mathcal{S}(v)}]_{\mathcal{F}(v)}.$$

Observe that this mapping is independent of the choice of α. Using this definition we introduce the following space.

Definition 2.8 Let P be a finite p-group and $(\mathcal{T},\mathcal{F},\mathcal{S})$ be a tree of fusion systems. The *P-orbit graph*, $\mathrm{Rep}(P,\mathcal{F})$ is the homotopy colimit

$$\underrightarrow{\mathrm{hocolim}}_{\mathcal{T}} \mathrm{Rep}(P,\mathcal{F}(-)).$$

Since there are no n-simplices in $\underrightarrow{\mathrm{hocolim}}_{\mathcal{T}} \mathrm{Rep}(P,\mathcal{F}(-))$ for $n \geq 2$, $\mathrm{Rep}(P,\mathcal{F})$ can be described as the geometric realisation of a graph as follows.

Lemma 2.9 *Let P be an arbitrary p-group and $(\mathcal{T},\mathcal{F},\mathcal{S})$ be a tree of fusion systems. Then $R := Rep(P,\mathcal{F})$ may be regarded as a graph with*

$$V(R) = \bigcup_{v \in V(\mathcal{T})} \mathrm{Rep}(P,\mathcal{F}(v)) \text{ and } E(R) = \bigcup_{e \in E(\mathcal{T})} \mathrm{Rep}(P,\mathcal{F}(e)),$$

where $[\alpha]_{\mathcal{F}(v)}$ *and* $[\beta]_{\mathcal{F}(w)}$ *are connected via an edge* $[\gamma]_{\mathcal{F}(e)}$ *if and only if* v *and* w *are both incident to* e *in* $\mathcal{T}$ *and the identities*

$$[\gamma \circ \iota^{\mathcal{S}(v)}_{\mathcal{S}(e)}]_{\mathcal{F}(v)} = [\alpha]_{\mathcal{F}(v)} \text{ and } [\gamma \circ \iota^{\mathcal{S}(v)}_{\mathcal{S}(e)}]_{\mathcal{F}(w)} = [\beta]_{\mathcal{F}(w)}$$

hold.

Proof. This follows immediately from the definition of the homotopy colimit given in Section 1.1.2. □

If $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ is a tree of fusion systems which satisfies (H), then by Lemma 2.7, $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ has a completion which we denote by $\mathcal{F}_{\mathcal{T}}$. Let $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F}(-))$ be the subfunctor of $\mathrm{Rep}(P, \mathcal{F}(-))$ which sends vertices v and edges e of $\mathcal{T}$ respectively to $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F}(v))$ and $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F}(e))$ and which sends f_{ev} to the map from $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F}(e))$ to $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F}(v))$ given by

$$[\alpha]_{\mathcal{F}(e)} \longmapsto [\alpha \circ \iota^{\mathcal{S}(v)}_{\mathcal{S}(e)}]_{\mathcal{F}(v)}.$$

(Note that this map is well-defined since $\mathcal{F}_{\mathcal{T}}$ is closed under composition with inclusion morphisms). Let

$$\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F}) := \varinjlim\nolimits_{\mathcal{T}} \mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F}(-)).$$

We observe that the obvious analogue of Lemma 2.9 holds for $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$, and that (for this reason) $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$ may be embedded as a subgraph of $\mathrm{Rep}(P, \mathcal{F})$ in the obvious way. We end this section with an important result which provides us with a precise description of this embedding.

Proposition 2.10 *Let* $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ *be a tree of fusion systems which satisfies* (H) *and let* $\mathcal{F}_{\mathcal{T}}$ *be its completion. Write* $S := \mathcal{S}(v_*)$ *and fix* $P \leq S$*. The following hold:*

(a) The connected component of the vertex $[\iota^S_P]_{\mathcal{F}(v_*)}$ *in* $\mathrm{Rep}(P, \mathcal{F})$ *is isomorphic to* $Rep_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$*.*

(b) The natural map

$$\Phi_P : \pi_0(\mathrm{Rep}(P, \mathcal{F})) \longrightarrow \mathrm{Rep}(P, \mathcal{F}_{\mathcal{T}})$$

which sends the connected component of a vertex $[\alpha]_{\mathcal{F}(v)}$ *in* $\mathrm{Rep}(P, \mathcal{F})$ *to* $[\alpha \circ \iota^S_{\mathcal{S}(v)}]_{\mathcal{F}_{\mathcal{T}}}$ *is a bijection.*

Proof. Since $\mathcal{T}$ has finitely many vertices and $\mathcal{S}(v)$ is finite for each $v \in V(\mathcal{T})$, the graph $\mathrm{Rep}(P, \mathcal{F})$ contains finitely many vertices and edges. We may identify $\mathcal{F}(v)$ and $\mathcal{F}(e)$ with their images in $\mathcal{F}_\mathcal{T}$ by Lemma 2.7. Set $v_0 := v_*$ and $\alpha_0 := \iota_P^{\mathcal{S}(v_0)}$ and let

$$[\alpha_0]_{\mathcal{F}(v_0)}, [\alpha_1]_{\mathcal{F}(v_1)}, \ldots, [\alpha_n]_{\mathcal{F}(v_n)}$$

be a path in $\mathrm{Rep}(P, \mathcal{F})$ and assume that $[\beta_i]_{\mathcal{F}(e_i)}$ is an edge from $[\alpha_{i-1}]_{\mathcal{F}(v_0)}$ to $[\alpha_i]_{\mathcal{F}(v_1)}$ for each $1 \leq i \leq n$. We need to show that each vertex $[\alpha_i]_{\mathcal{F}(v_i)}$ lies in $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P, \mathcal{F})$. Clearly $[\alpha_0]_{\mathcal{F}(v_0)} \in \mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P, \mathcal{F})$. Assume that $n \geq 1$, and that $[\alpha_i]_{\mathcal{F}(v_i)} \in \mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P, \mathcal{F})$ for some $i < n$. If $[\beta_{i+1}]_{\mathcal{F}(e_{i+1})}$ is an edge from $[\alpha_i]_{\mathcal{F}(v_i)}$ to $[\alpha_{i+1}]_{\mathcal{F}(v_{i+1})}$ then there exist maps

$$\gamma \in \mathrm{Hom}_{\mathcal{F}(v_i)}(P\alpha_i, P\beta_{i+1}) \text{ and } \delta \in \mathrm{Hom}_{\mathcal{F}(v_{i+1})}(P\beta_{i+1}, P\alpha_{i+1})$$

such that $\alpha_i \circ \gamma = \beta_{i+1}$ and $\beta_{i+1} \circ \delta = \alpha_{i+1}$. Hence $\gamma \circ \delta \in \mathrm{Hom}_{\mathcal{F}_\mathcal{T}}(P\alpha_i, P\alpha_{i+1})$ and $[\alpha_{i+1}]_{\mathcal{F}(v_{i+1})} \in \mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P, \mathcal{F})$ and by induction, $[\alpha_i]_{\mathcal{F}(v_i)} \in \mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P, \mathcal{F})$ for all $0 \leq i \leq n$.

Conversely suppose that $\alpha \in \mathrm{Hom}_{\mathcal{F}_\mathcal{T}}(P, \mathcal{S}(v))$ and that $[\alpha]_{\mathcal{F}(v)}$ is a vertex in $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P, \mathcal{F})$. By the definition of $\mathcal{F}_\mathcal{T}$, there exists a path $v_0, v_1, \ldots, v_n = v$ in $\mathcal{T}$ and for $1 \leq i \leq n$, maps $\alpha_i \in \mathrm{Hom}_{\mathcal{F}(v_i)}(P\alpha_{i-1}, P\alpha_i)$ with $\alpha_0 = \iota_P^S$ such that $\alpha = \alpha_0 \circ \cdots \circ \alpha_n$. This implies that there exists a path in $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P, \mathcal{F})$,

$$[\alpha_0]_{\mathcal{F}(v_0)}, [\alpha_0 \circ \alpha_1]_{\mathcal{F}(v_1)} \ldots, [\alpha_0 \circ \alpha_1 \circ \cdots \circ \alpha_{n-1}]_{\mathcal{F}(v_{n-1})}, [\alpha]_{\mathcal{F}(v_n)},$$

and completes the proof of (a).

Next, we prove (b). It suffices to show that two vertices $[\alpha]_{\mathcal{F}(v)}$ and $[\beta]_{\mathcal{F}(w)}$ are connected in $\mathrm{Rep}(P, \mathcal{F})$ if and only if $[\alpha \circ \iota_{\mathcal{S}(v)}^S]_{\mathcal{F}_\mathcal{T}} = [\beta \circ \iota_{\mathcal{S}(w)}^S]_{\mathcal{F}_\mathcal{T}}$ in $\mathrm{Rep}(P, \mathcal{F}_\mathcal{T})$. It is enough to prove this when $[\alpha]_{\mathcal{F}(v)}$ and $[\beta]_{\mathcal{F}(w)}$ are connected via a single edge $[\gamma]_{\mathcal{F}(e)}$.

We prove the 'only if' direction first. Thus we suppose that $[\alpha]_{\mathcal{F}(v)}$ and $[\beta]_{\mathcal{F}(w)}$ are connected via an edge $[\gamma]_{\mathcal{F}(e)}$ in $\mathrm{Rep}(P, \mathcal{F})$ so that there exist $\gamma_1 \in \mathrm{Hom}_{\mathcal{F}(v)}(P\alpha, P\gamma)$ and $\gamma_2 \in \mathrm{Hom}_{\mathcal{F}(w)}(P\beta, P\gamma)$ with

$$P\alpha\gamma_1 = P\gamma = P\beta\gamma_2.$$

Then $\gamma_1\gamma_2^{-1} \in \mathrm{Hom}_{\mathcal{F}_\mathcal{T}}(P\alpha, P\beta)$ and $[\beta \circ \iota_{\mathcal{S}(w)}^S] = [\alpha\gamma_1\gamma_2^{-1} \circ \iota_{\mathcal{S}(w)}^S] = [\alpha \circ \iota_{\mathcal{S}(v)}^S] \in \mathrm{Rep}(P, \mathcal{F}_\mathcal{T})$. Conversely, suppose that $[\alpha \circ \iota_{\mathcal{S}(v)}^S]_{\mathcal{F}_\mathcal{T}} = [\beta \circ \iota_{\mathcal{S}(w)}^S]_{\mathcal{F}_\mathcal{T}} = [\delta]_{\mathcal{F}_\mathcal{T}}$, for some $\delta \in \mathrm{Hom}_{\mathcal{F}_\mathcal{T}}(P, S)$. Then $[\delta^{-1}\alpha]_{\mathcal{F}(v)}$, $[\delta^{-1}\beta]_{\mathcal{F}(v)}$ are vertices in $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P\delta, \mathcal{F})$ and by part (a), this graph is connected so that there must exist a path joining $[\delta^{-1}\alpha]_{\mathcal{F}(v)}$ to $[\delta^{-1}\beta]_{\mathcal{F}(v)}$ in $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P\delta, \mathcal{F})$. This is easily seen to be equivalent to the existence of a path joining $[\alpha]_{\mathcal{F}(v)}$ to $[\beta]_{\mathcal{F}(w)}$ in $\mathrm{Rep}(P, \mathcal{F})$, completing the proof of the lemma. □

2.3 Trees of Group Fusion Systems

In this section we will give a precise description of the relationship between trees of groups $(\mathcal{T}, \mathcal{G})$ and trees of fusion systems $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ by considering what happens when the latter is induced by the former. It will turn out that both the completion and P-orbit graph of $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ have group-theoretic descriptions in this case, which will allow us to give an entirely group theoretic interpretation of Theorem 2, in Section 2.4.

2.3.1 Proof of Theorem 1

We start by giving a precise explanation of the word 'induced' above. The following lemma is a trivial consequence of Sylow's Theorem.

Lemma 2.11 *Let $(\mathcal{T}, \mathcal{G})$ be a tree of finite groups. For any choice of Sylow p-subgroups $\mathcal{S}(v) \in \mathrm{Syl}_p(\mathcal{G}(v))$ and $\mathcal{S}(e) \in \mathrm{Syl}_p(\mathcal{G}(e))$, there exists a tree of finite p-groups $(\mathcal{T}, \mathcal{S})$ and a tree of fusion systems $(\mathcal{T}, \mathcal{F}_{\mathcal{S}(-)}(\mathcal{G}(-)), \mathcal{S}(-))$ induced by $(\mathcal{T}, \mathcal{G})$.*

Proof. Fix a choice of Sylow p-subgroups, $\mathcal{S}(v) \in \mathrm{Syl}_p(\mathcal{G}(v))$ and $\mathcal{S}(e) \in \mathrm{Syl}_p(\mathcal{G}(e))$ for each vertex v and edge e of $\mathcal{T}$. If v is incident to e then by Sylow's Theorem there exists an element $g_{ev} \in \mathcal{G}(v)$ such that $(\mathcal{S}(e)\mathcal{G}(f_{ev}))^{g_{ev}} \leq \mathcal{S}(v)$. Let $\mathcal{S}$ be the functor from $\mathcal{T}$ to $\mathfrak{Grp}$ which sends e and v respectively to $\mathcal{S}(e)$ and $\mathcal{S}(v)$ and $f_{ev} \in \mathrm{Hom}_{\mathcal{T}}(e, v)$ to $\mathcal{G}(f_{ev}) \circ c_{g_{ev}} \in \mathrm{Hom}(\mathcal{S}(e), \mathcal{S}(v))$. Then f_{ev} determines a tree of finite p-groups $(\mathcal{T}, \mathcal{S})$. Now let $\mathcal{F}_{\mathcal{S}(-)}(\mathcal{G}(-))$ be the functor from $\mathcal{T}$ to $\mathfrak{Fus}$ which sends e and v respectively to $\mathcal{F}_{\mathcal{S}(e)}(\mathcal{G}(e))$ and $\mathcal{F}_{\mathcal{S}(v)}(\mathcal{G}(v))$ and f_{ev} to the homomorphism $\Phi_{ev} := \mathcal{G}(f_{ev}) \circ c_{g_{ev}}|_{\mathcal{S}(e)} \in \mathrm{Hom}(\mathcal{S}(e), \mathcal{S}(v))$. Clearly Φ_{ev} is a morphism of fusion systems and hence determines a tree of fusion systems $(\mathcal{T}, \mathcal{F}_{\mathcal{S}(-)}(\mathcal{G}(-)), \mathcal{S}(-))$, as required. □

The proof of Lemma 2.11 shows that there may be many trees of fusion systems $(\mathcal{T}, \mathcal{F}_{\mathcal{S}(-)}(\mathcal{G}(-)), \mathcal{S}(-))$ induced by a tree of groups $(\mathcal{T}, \mathcal{G})$, since a choice for the functor $\mathcal{S}(-)$ is made when Sylow's Theorem is applied. We need to show that this choice does not interfere with the isomorphism type of the completion $\mathcal{F}_{\mathcal{T}}$. To do this, we first isolate precisely how conjugation takes place in $\mathcal{G}_{\mathcal{T}}$ by proving the following lemma of Robinson ([28, Lemma 1]).

Lemma 2.12 *Let $\mathcal{T}$ be a tree consisting of two vertices v and w with a single edge e connecting them. Let $(\mathcal{T}, \mathcal{G})$ be a tree of groups with completion $\mathcal{G}_{\mathcal{T}}$ and write $A := \mathcal{G}(v)$, $B := \mathcal{G}(w)$ and $C := \mathcal{G}(e)$. The following hold:*

(a) *Any product of elements whose successive terms lie alternately in the sets* $A\backslash C$ *and* $B\backslash C$*, lies outside of* C*, and at least one of* A *and* B*.*

(b) *Each element* $g \in G\backslash A$ *may be written as a product* $g = a_0 b_1 a_1 \dots b_s a_s b_\infty$ *where* $a_i \in A\backslash C$ *and* $b_i \in B\backslash C$ *for* $1 \leq i \leq s, a_0 \in A$ *and* $b_\infty \in B$*.*

Consequently, if $X \leq A$*,* $g \in G\backslash A$ *and* $X^g \leq A$ *or* $X^g \leq B$*, then writing* g *as a product* $g = a_0 b_1 a_1 \dots b_s a_s b_\infty$ *as in (b), we have*

$$\langle X_0, X_i, Y_i \mid 1 \leq i \leq s \rangle \leq C$$

where $X_0 = X^{a_0}, Y_1 = X_0^{b_1}, X_1 = Y_1^{a_1}$*, and so on.*

Proof. To see (a), let $w = g_0 \dots g_n$ be a product of elements whose successive terms lie alternately in the sets $A\backslash C$ and $B\backslash C$. Then w is a reduced word in $\mathcal{G}_\mathcal{T}$ so if w lies in C, it is no longer reduced, (being representable by an element of C). Observe that g_0 and g_n dictate where w lies. To see this, note that if $g_0, g_n \in A$ then $w \notin B\backslash C$, since otherwise $w = b \in B\backslash C$ implies that

$$1 = wb^{-1} = g_0, \dots g_n b^{-1} \notin C,$$

a contradiction. Similarly if $g_0, g_n \in B$ then $w \notin A\backslash C$ and in the remaining cases (where g_0 and g_n lie in different sets), $w \notin (A \cup B)\backslash C$. This proves (a). To see (b), note that certainly any element $g \in G\backslash A$ may be written in the stated way, since (by (a)) such a representation allows for all possibilities for the set in which the element g lies. Finally, we prove the last assertion of the lemma. If $b_\infty \in C$ and equals c say, then $X^{gc^{-1}} = (X^g)^{c^{-1}} \in A \cup B$ if and only if $X^g \in A \cup B$. This proves that we may assume without loss of generality that either $b_\infty = 1$ (if $X^g \leq B$) or $b_\infty \in B\backslash C$ (if $X^g \leq A$.) In either case, suppose that there exists $u \in X^{a_0}\backslash C$. Then

$$b_\infty^{-1} a_s^{-1} \dots, b_1^{-1} u b_1 \dots a_s b_\infty$$

lies in A or B by assumption which contradicts (a). We obtain a similar contradiction if $X_0^{b_1}$ is not contained in C. Inductively, we arrive at the stated result. □

We can now apply Lemma 2.12 to prove Theorem 1 which relates the two ways in which one can construct a fusion system from a tree of groups.

Theorem 2.13 *Let* $(\mathcal{T}, \mathcal{G})$ *be a tree of finite groups and write* $\mathcal{G}_\mathcal{T}$ *for the completion of* $(\mathcal{T}, \mathcal{G})$*. Let* $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ *be a tree of fusion systems induced by* $(\mathcal{T}, \mathcal{G})$ *which satisfies* (H) *so that there exists a completion* $\mathcal{F}_\mathcal{T}$ *for* $(\mathcal{T}, \mathcal{F}, \mathcal{S})$*. The following hold:*

(a) $\mathcal{S}(v_*)$ *is a Sylow* p*-subgroup of* $\mathcal{G}_\mathcal{T}$.

(b) $\mathcal{F}_{\mathcal{S}(v_*)}(\mathcal{G}_\mathcal{T}) = \mathcal{F}_\mathcal{T}$.

In particular, $\mathcal{F}_\mathcal{T}$ *is independent of the choice of tree of fusion systems* $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ *induced by* $(\mathcal{T}, \mathcal{G})$.

Proof. We first prove that $S := S(v_*)$ is a Sylow p-subgroup of $G_\mathcal{T}$. Let P be a finite p-subgroup of $\mathcal{G}_\mathcal{T}$ and consider the image $\mathcal{I}$ of $\tilde{\mathcal{T}}^P$ in $\mathcal{T}$ under the composite

$$\tilde{\mathcal{T}} \longrightarrow \tilde{\mathcal{T}}/\mathcal{G}_\mathcal{T} \xrightarrow{\simeq} \mathcal{T}.$$

Since $\tilde{\mathcal{T}}$ is a tree, $\tilde{\mathcal{T}}^P$ is also a tree, so that $\mathcal{I}$ must be connected[3]). Let v be a vertex in $\mathcal{I}$ which is of minimal distance from v_* and assume that $v \neq v_*$. This implies that there is some $g \in \mathcal{G}_\mathcal{T}$ such that $g\mathcal{G}(v) \in (\mathcal{G}_\mathcal{T}/\mathcal{G}(v))^P$ or equivalently such that $g^{-1}Pg \leq \mathcal{G}(v)$. Let e be the edge (v, w) incident to v in the unique minimal path from v to v_*. Since $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ satisfies (H), $p \nmid |\mathcal{G}(v) : \mathcal{G}(e)|$, so by Sylow's Theorem there is $g' \in \mathcal{G}_\mathcal{T}$ such that $g'^{-1}Pg' \leq \mathcal{G}(e)$. Since $\mathcal{G}(e) \leq \mathcal{G}(w)$, we also have $g'\mathcal{G}(w) \in (\mathcal{G}_\mathcal{T}/\mathcal{G}(w))^P$, so that w is a vertex closer to v_* than v in $\mathcal{I}$, a contradiction. Hence $v = v_*$, $g^{-1}Pg \leq \mathcal{G}(v_*)$ and by Sylow's Theorem there is some $g'' \in \mathcal{G}_\mathcal{T}$ with $g''^{-1}Pg'' \leq S$, as needed.

Next we prove that (b) holds. Observe that $\mathcal{F}_\mathcal{T} \subseteq \mathcal{F}_S(\mathcal{G}_\mathcal{T})$ since (by definition) each morphism in $\mathcal{F}_\mathcal{T}$ is a composite of restrictions of morphisms in $\mathcal{F}(v)$, each of which clearly lies in $\mathcal{F}_S(\mathcal{G}_\mathcal{T})$. Hence it remains to prove that $\mathcal{F}_S(\mathcal{G}_\mathcal{T}) \subseteq \mathcal{F}_\mathcal{T}$. We proceed by induction on the number of vertices $n := |V(\mathcal{T})|$ of $\mathcal{T}$, the result being clear in the case where $n = 1$. Suppose that $n > 1$ and fix an extremal vertex, v of $\mathcal{T}$ not equal to v_*. Let $\mathcal{T}'$ be the tree obtained from $\mathcal{T}$ by removing v and the unique edge e incident to v. Then $(\mathcal{T}', \mathcal{G})$ is a graph of groups and $(\mathcal{T}', \mathcal{F}, \mathcal{S})$ is a tree of fusion systems induced by $(\mathcal{T}', \mathcal{G})$ which satisfies (H). Furthermore, $S \in \mathrm{Syl}_p(\mathcal{G}_{\mathcal{T}'})$ and by induction, the completion $\mathcal{F}_{\mathcal{T}'}$ of $(\mathcal{T}', \mathcal{F}, \mathcal{S})$ is the fusion system $\mathcal{F}_S(\mathcal{G}_{\mathcal{T}'})$. Since

$$\mathcal{G}_\mathcal{T} = \mathcal{G}_{\mathcal{T}'} *_{\mathcal{G}(e)} \mathcal{G}(v) \text{ and } \mathcal{F}_\mathcal{T} = \langle \mathcal{F}_{\mathcal{T}'}, \mathcal{F}_{\mathcal{S}(v)}(\mathcal{G}(v)) \rangle_S,$$

it will be enough to show that

$$\mathcal{F}_S(\mathcal{G}_{\mathcal{T}'} *_{\mathcal{G}(e)} \mathcal{G}(v)) \subseteq \langle \mathcal{F}_{\mathcal{T}'}, \mathcal{F}_{\mathcal{S}(v)}(\mathcal{G}(v)) \rangle.$$

[3]See Lemma 2.19.

Let $X \leq S$ and suppose that $\langle X, X^g \rangle \leq S$ with $g \in \mathcal{G}_{\mathcal{T}}$. If $g \in \mathcal{G}_{\mathcal{T}'}$ then we are done, so suppose that $g \notin \mathcal{G}_{\mathcal{T}'}$ and write

$$g = g_{-\infty} h_1 g_1 \dots h_r g_r h_{r+1} g_\infty$$

where $g_{-\infty}, g_\infty \in \mathcal{G}_{\mathcal{T}'}$, $g_i \in \mathcal{G}_{\mathcal{T}'} \backslash \mathcal{G}(e)$ and $h_i \in \mathcal{G}(v) \backslash \mathcal{G}(e)$. Set $X^* := X^{g_{-\infty}}$. Then $\langle X^*, (X^*)^{h_1 g_1 \dots h_r g_r h_{r+1}} \rangle \leq \mathcal{G}_{\mathcal{T}'}$ so that by Lemma 2.12,

$$X^*, (X^*)^{h_1}, (X^*)^{h_1 g_1}, \dots, (X^*)^{h_1 g_1 \dots h_r g_r h_{r+1}} \leq \mathcal{G}(e).$$

By Sylow's Theorem there exists $k_0 \in \mathcal{G}(e)$ such that $(X^*)^{k_0} \leq \mathcal{S}(e)$, $l_1 \in \mathcal{G}(e)$ such that $(X^*)^{h_1 l_1} \leq \mathcal{S}(e)$, $k_1 \in \mathcal{G}(e)$ such that $(X^*)^{h_1 g_1 k_1} \leq \mathcal{S}(e)$, and so on. Furthermore the sequence of elements

$$k_0, k_0^{-1} h_1 l_1, l_1^{-1} g_1 k_1, k_1^{-1} h_2 l_2, \dots, k_r^{-1} h_{r+1} l_{r+1}, l_{r+1}^{-1} g_\infty$$

lie alternately in the groups $\mathcal{G}_{\mathcal{T}'}$ and $\mathcal{G}(v)$ and multiply to give $h_1 g_1 \dots h_r g_r h_{r+1} g_\infty$. Now, for $1 \leq i \leq r$, conjugation from $(X^*)^{h_1 \dots h_i l_i}$ to $(X^*)^{h_1 \dots h_i g_i k_i}$ is carried out in the fusion system of $\mathcal{G}_{\mathcal{T}'}$ and conjugation from $(X^*)^{h_1 \dots h_i g_i k_i}$ to $(X^*)^{h_1 \dots g_i h_{i+1} l_{i+1}}$ is carried out in the fusion system of $\mathcal{G}(v)$. Finally conjugation from $(X^*)^{h_1 \dots h_r g_r h_{r+1} l_{r+1}}$ to $(X^*)^{h_1 \dots h_r g_r h_{r+1} g_\infty}$ is carried out in the fusion system of $\mathcal{G}_{\mathcal{T}'}$. Since $\mathcal{S}(e) = \mathcal{S}(v)$, the result follows. □

We end this section with a short example which illustrates how Theorem 2.13 can be applied to produce fusion systems from group amalgams.

Example 2.14 Let $H \cong \mathrm{Sym}(4)$, $S \cong D_8$ be a Sylow 2-subgroup of H and let V_1 and V_2 be the two non-cyclic non-conjugate subgroups of order 4 in S. Also, let $\mathcal{T}$ be a tree consisting of two vertices v_1 and v_2 with a single edge e connecting them. Form a tree of groups $(\mathcal{T}, \mathcal{G})$ where $\mathcal{G}(v_1) = \mathcal{G}(v_2) = H$ and $\mathcal{G}(e) = S$ and where $V_i \mathcal{G}(f_{ev_i})$ is normal in $\mathcal{G}(v_i)$ but not normal in $\mathcal{G}(v_{3-i})$ for $i = 1, 2$. Let $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ be a tree of fusion systems induced by $(\mathcal{T}, \mathcal{G})$ and observe that $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ satisfies (H) so that it has a completion $\mathcal{F}_{\mathcal{T}}$ by Lemma 2.7. It is well known that the group $PSL_3(2)$ is a faithful, finite completion of $(\mathcal{T}, \mathcal{G})$ (see, for example, [17, Theorem A]) so that by Theorem 2.13 we must have that $\mathcal{F}_{\mathcal{T}}$ is the fusion system of $PSL_3(2)$.

2.3.2 A Characterisation of the P-Orbit Graph

We next consider what can said about the P-orbit graph in the special case where a tree of fusion systems $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ which satisfies (H) is induced by a tree of groups $(\mathcal{T}, \mathcal{G})$. It is shown that like the completion $\mathcal{F}_{\mathcal{T}}$, the isomorphism type of $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$ is independent of the choice of $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ since it may described in terms of the orbit graph $\tilde{\mathcal{T}}$ of $(\mathcal{T}, \mathcal{G})$.

We start by introducing some more notation. If A and B are groups, let $\mathrm{Rep}(A, B)$ denote the set of orbits of the action of $\mathrm{Inn}(B)$ on $\mathrm{Hom}(A, B)$ by right composition. For each $\alpha \in \mathrm{Hom}(A, B)$, let $[\alpha]_B$ denote its class modulo $\mathrm{Inn}(B)$. If A and B are subgroups of some group C, write $\mathrm{Rep}_C(A, B)$ for the set of orbits of the action of $\mathrm{Inn}(B)$ on $\mathrm{Hom}_C(A, B) \cong N_C(A, B)/C_C(A)$ (where $N_C(A, B) = \{g \in C \mid A^g \leq B\}$ and $C_C(A)$ acts on $N_C(A, B)$ via left multiplication).

Now let $(\mathcal{T}, \mathcal{G})$ be a tree of groups and fix a vertex v_* of $\mathcal{T}$. For each $P \leq \mathcal{G}(v_*)$ let $\mathrm{Rep}(P, \mathcal{G}(-))$ be the functor from $\mathcal{T}$ to $\mathfrak{Set}$ which sends vertices v and edges e respectively to $\mathrm{Rep}(P, \mathcal{G}(e))$ and $\mathrm{Rep}(P, \mathcal{G}(e))$ and which sends f_{ev} to the map from $\mathrm{Rep}(P, \mathcal{G}(e))$ to $\mathrm{Rep}(P, \mathcal{G}(v))$ given by

$$[\alpha]_{\mathcal{G}(e)} \longmapsto [\alpha \circ \iota_{\mathcal{G}(e)}^{\mathcal{G}(v)}]_{\mathcal{G}(v)}.$$

Similarly if $\mathcal{G}_{\mathcal{T}}$ is the completion of $(\mathcal{T}, \mathcal{G})$ let $\mathrm{Rep}_{\mathcal{G}_{\mathcal{T}}}(P, \mathcal{G}(-))$ be the functor from $\mathcal{T}$ to $\mathfrak{Set}$ which sends vertices v and edges e respectively to $\mathrm{Rep}_{\mathcal{G}_{\mathcal{T}}}(P, \mathcal{G}(v))$ and $\mathrm{Rep}_{\mathcal{G}_{\mathcal{T}}}(P, \mathcal{G}(e))$ and which sends f_{ev} to the map given by

$$[\alpha]_{\mathcal{G}(e)} \longmapsto [\alpha \circ \iota_{\mathcal{G}(e)}^{\mathcal{G}(v)}]_{\mathcal{G}(v)}.$$

(Observe that this well-defined). Now define:

$$\mathrm{Rep}(P, \mathcal{G}) := \varinjlim_{\mathcal{T}} \mathrm{Rep}(P, \mathcal{G}(-)) \text{ and } \mathrm{Rep}_{\mathcal{G}_{\mathcal{T}}}(P, \mathcal{G}) := \varinjlim_{\mathcal{T}} \mathrm{Rep}_{\mathcal{G}_{\mathcal{T}}}(P, \mathcal{G}(-)),$$

both regarded as graphs whose vertex and edge sets consist of elements of $\mathrm{Rep}(P, \mathcal{G}(e))$ and $\mathrm{Rep}(P, \mathcal{G}(v))$ respectively. The next proposition allows us to compare $\mathrm{Rep}_{\mathcal{G}_{\mathcal{T}}}(P, \mathcal{G})$ with $\tilde{\mathcal{T}}$.

Proposition 2.15 *Let $(\mathcal{T}, \mathcal{G})$ be a tree of finite groups. Fix a vertex v_* of $\mathcal{T}$ and a subgroup $P \leq \mathcal{G}(v_*)$ and write $\mathcal{G}_{\mathcal{T}}$ for the completion of $(\mathcal{T}, \mathcal{G})$. There exists a graph isomorphism*

$$\tilde{\mathcal{T}}^P / C_{\mathcal{G}_{\mathcal{T}}}(P) \xrightarrow{\sim} \mathrm{Rep}_{\mathcal{G}_{\mathcal{T}}}(P, \mathcal{G}),$$

where $\tilde{\mathcal{T}}^P$ is the subgraph of $\tilde{\mathcal{T}}$ fixed under the action of P.

Proof. Define a map

$$\Phi_P : \tilde{\mathcal{T}}^P \longrightarrow \mathrm{Rep}_{\mathcal{G}_\mathcal{T}}(P, \mathcal{G})$$

by sending a vertex $g\mathcal{G}(v)$ to $[c_g]_{\mathcal{G}(v)}$ where $c_g \in \mathrm{Hom}_{\mathcal{G}_\mathcal{T}}(P, \mathcal{G}(v))$ and similarly by sending an edge $g\mathcal{G}(e)$ to $[c_g]_{\mathcal{G}(e)}$ where $c_g \in \mathrm{Hom}_{\mathcal{G}_\mathcal{T}}(P, \mathcal{G}(e))$. Note that this makes sense: if $g\mathcal{G}(v)$ is a vertex in $\tilde{\mathcal{T}}^P$ then for each $x \in P$, $xg\mathcal{G}(v) = g\mathcal{G}(v)$ which is equivalent to $P^g \leq \mathcal{G}(v)$. The same argument works for edges. To see that this map defines a homomorphism of graphs, note that if $g\mathcal{G}(v)$ and $h\mathcal{G}(w)$ are connected via an edge $i\mathcal{G}(e)$ in $\tilde{\mathcal{T}}^P$ then $g\mathcal{G}(v) = i\mathcal{G}(v)$ and $h\mathcal{G}(w) \in i\mathcal{G}(w)$ so that $[c_g]_{\mathcal{G}(v)} = [c_i \circ \iota^{\mathcal{G}(v)}_{\mathcal{G}(e)}]_{\mathcal{G}(v)}$ and $[c_h]_{\mathcal{G}(w)} = [c_i \circ \iota^{\mathcal{G}(v)}_{\mathcal{G}(e)}]_{\mathcal{G}(w)}$. It is clear from the definition that Φ_P is surjective. If $[c_g]_{\mathcal{G}(v)} = [c_h]_{\mathcal{G}(w)}$ then $v = w$ and there is some $r \in \mathcal{G}(v)$ such that $c_g = c_h \circ c_r \in \mathrm{Hom}_{\mathcal{G}_\mathcal{T}}(P, \mathcal{G}(v))$. This is equivalent to the orbits of g and hr under the action of $C_{\mathcal{G}_\mathcal{T}}(P)$ on $N_{\mathcal{G}_\mathcal{T}}(P, \mathcal{G}(v))$ being equivalent so that there is some $x \in C_{\mathcal{G}_\mathcal{T}}(P)$ with $xg = hr$. Hence $h^{-1}xg \in \mathcal{G}(v)$, $h\mathcal{G}(v) = xg\mathcal{G}(v)$ and the vertices $h\mathcal{G}(v)$ and $g\mathcal{G}(v)$ lie in the same $C_{\mathcal{G}_\mathcal{T}}(P)$-orbit, as needed. □

We can now compare the graphs $\mathrm{Rep}(P, \mathcal{G})$ and $\mathrm{Rep}(P, \mathcal{F})$ when $\mathcal{F}$ is a choice of functor $\mathcal{F}_{\mathcal{S}(-)}(\mathcal{G}(-))$ associated to a tree of finite groups $(\mathcal{T}, \mathcal{G})$ (see Lemma 2.11). We need the following simple consequence of Sylow's Theorem.

Lemma 2.16 *Let G be a finite group and let S be a Sylow p-subgroup of G. Then for each p-group P, there exists a bijection*

$$\mathrm{Rep}(P, \mathcal{F}_S(G)) \xrightarrow{\sim} \mathrm{Rep}(P, G)$$

given by the map Ψ which sends $[\alpha]_{\mathcal{F}_S(G)}$ to $[\alpha \circ \iota^G_S]_G$ for each $\alpha \in \mathrm{Hom}(P, S)$.

Proof. Let $\alpha, \beta \in \mathrm{Hom}(P, S)$ and suppose that $[\alpha \circ \iota^G_S]_G = [\beta \circ \iota^G_S]_G$. Then there exists a map $c_g \in \mathrm{Inn}(G)$ such that $\alpha \circ \iota^G_S \circ c_g = \beta \circ \iota^G_S$. This implies that $c_g|_S \in \mathrm{Hom}_{\mathcal{F}_S(G)}(P\alpha, P\beta)$ and $\alpha \circ c_g|_S = \beta$ so that $[\alpha]_{\mathcal{F}_S(G)} = [\beta]_{\mathcal{F}_S(G)}$ and Ψ is injective. To see that Ψ is surjective, let $[\gamma]_G \in \mathrm{Rep}(P, G)$. By Sylow's Theorem there exists $g \in G$ such that $(P\gamma)^g \leq S$. Set $\varphi := \gamma \circ c_g \in \mathrm{Hom}(P, S)$ and observe that $[\varphi \circ \iota^G_S]_G = [\gamma]_G$, as required. □

Corollary 2.17 *Let $(\mathcal{T}, \mathcal{G})$ be a tree of finite groups with completion $\mathcal{G}_\mathcal{T}$ and let $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ be a tree of fusion systems induced by $(\mathcal{T}, \mathcal{G})$. Fix a vertex v_* of $\mathcal{T}$ and a subgroup $P \leq \mathcal{S}(v_*)$. Then there exists a natural isomorphism of functors*

$$\mathrm{Rep}(P, \mathcal{F}(-)) \xrightarrow{\sim} \mathrm{Rep}(P, \mathcal{G}(-)),$$

which induces a homotopy equivalence $\mathrm{Rep}(P,\mathcal{F}) \xrightarrow{\simeq} \mathrm{Rep}(P,\mathcal{G})$. *Furthermore, if* $(\mathcal{T},\mathcal{F},\mathcal{S})$ *satisfies* (H) *and has completion* $\mathcal{F}_{\mathcal{T}}$ *then*

$$\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P,\mathcal{F}) \xrightarrow{\simeq} \mathrm{Rep}_{\mathcal{G}_{\mathcal{T}}}(P,\mathcal{G}),$$

so that $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P,\mathcal{F})$ *is independent of the choice of tree of fusion systems* $(\mathcal{T},\mathcal{F},\mathcal{S})$.

Proof. The existence of a natural isomorphism of functors

$$\eta : \mathrm{Rep}(P,\mathcal{F}(-)) \xrightarrow{\sim} \mathrm{Rep}(P,\mathcal{G}(-))$$

is an immediate consequence of Lemma 2.16. Hence by Lemma 1.14 this natural isomorphism induces a homotopy equivalence

$$\mathrm{hocolim}(\eta) : \mathrm{Rep}(P,\mathcal{F}) \xrightarrow{\simeq} \mathrm{Rep}(P,\mathcal{G}).$$

If $(\mathcal{T},\mathcal{F},\mathcal{S})$ satisfies (H) then the above equivalence sends the vertex $[\iota_P^{\mathcal{S}(v_*)}]_{\mathcal{F}(v_*)}$ to $[\iota_P^{\mathcal{G}(v_*)}]_{\mathcal{G}(v_*)}$. Clearly $[\iota_P^{\mathcal{G}(v_*)}]_{\mathcal{G}(v_*)}$ lies in $\mathrm{Rep}_{\mathcal{G}_{\mathcal{T}}}(P,\mathcal{G})$ (it is the image of the coset $\mathcal{G}(v_*)$ under Φ_P). Since $\mathrm{Rep}_{\mathcal{G}_{\mathcal{T}}}(P,\mathcal{G})$ is connected (being the image under Φ_P of $\tilde{\mathcal{T}}^P$ which is connected[4]), $\mathrm{hocolim}(\eta)$ must restrict to a homotopy equivalence

$$\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P,\mathcal{F}) \xrightarrow{\simeq} \mathrm{Rep}_{\mathcal{G}_{\mathcal{T}}}(P,\mathcal{G})$$

by Proposition 2.10. This completes the proof. □

2.4 The Completion of a Tree of Fusion Systems

In this section, we prove our first main result which gives conditions for a tree of fusion systems $(\mathcal{T},\mathcal{F},\mathcal{S})$ to have a saturated completion $\mathcal{F}_{\mathcal{T}}$:

Theorem 2.18 *Let* $(\mathcal{T},\mathcal{F},\mathcal{S})$ *be a tree of fusion systems which satisfies* (H) *and assume that* $\mathcal{F}(v)$ *is saturated for each vertex* v *of* $\mathcal{T}$. *Write* $S := \mathcal{S}(v_*)$ *and* $\mathcal{F}_{\mathcal{T}}$ *for the completion of* $(\mathcal{T},\mathcal{F},\mathcal{S})$. *Assume that the following hold for each* $P \leq S$.

(a) If P *is* $\mathcal{F}_{\mathcal{T}}$*-conjugate to an* $\mathcal{F}(v)$*-essential subgroup or* $P = \mathcal{S}(v)$ *then* P *is* $\mathcal{F}_{\mathcal{T}}$*-centric.*

(b) If P *is* $\mathcal{F}_{\mathcal{T}}$*-centric then* $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P,\mathcal{F})$ *is a tree.*

Then $\mathcal{F}_{\mathcal{T}}$ *is a saturated fusion system on* S.

[4]This follows from Lemma 2.19.

Conditions (a) and (b) are respectively motivated by Theorems 1.26 and 1.27. The former condition implies that $\mathcal{F}_\mathcal{T}$ is generated by its $\mathcal{F}_\mathcal{T}$-centric subgroups, while the latter ensures that the saturation axioms (Definition 1.17 (a) and (b)) hold for all such subgroups. Thus the saturation of $\mathcal{F}_\mathcal{T}$ will ultimately follow from Theorem 1.27. Theorem 2.18 was inspired by a theorem of Broto, Levi and Oliver ([7, Theorem 4.2]) and we will deduce their result from ours in Corollary 2.22.

2.4.1 Finite Group Actions on Trees

Before embarking on the proof of Theorem 2.18, it will be important to understand the way in which the automiser $\mathrm{Aut}_{\mathcal{F}_\mathcal{T}}(P)$ can act on the P-orbit graph $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P, \mathcal{F})$. Note that we may draw an analogy here with the situation for trees of groups $(\mathcal{T}, \mathcal{G})$ where there exists an action of the completion $\mathcal{G}_\mathcal{T}$ on the orbit tree $\tilde{\mathcal{T}}$.

We start with a short lemma concerning the action of a finite group on a tree. Recall that the *diameter* of a graph $\mathcal{T}$ is the maximum distance between two vertices of $\mathcal{T}$. When $\mathcal{T}$ is a tree, the parity of this number determines the fixed points under group actions on $\mathcal{T}$.

Lemma 2.19 *Let $\mathcal{T}$ be a tree with finitely many vertices and let $1 \neq G \leq \mathrm{Aut}(\mathcal{T})$. Write $n = d(\mathcal{T})$ for the diameter of $\mathcal{T}$. Then G fixes a vertex of $\mathcal{T}$ if n is odd and an edge of $\mathcal{T}$ if n is even. Furthermore, the subgraph $\mathcal{T}^G$ spanned by all vertices and edges of $\mathcal{T}$ fixed by G is also a tree.*

Proof. One usually proves both of these statements by induction on n. Here, we supply a different proof. Fix a maximal path α in $\mathcal{T}$ and an element $g \in G$. Then αg is also a maximal path. If α and αg do not intersect one obtains a contradiction to the maximal length of α by first considering a minimal path from any vertex in α to one in αg, and then traversing the appropriate ray. Furthermore the overlapping region contains the mid-vertex or edge of α according to whether $|\alpha|$ is odd or even respectively. Since g was arbitrary and $|\alpha| = n$, the first part of the lemma now follows.

For the second part, if G fixes a pair of vertices x and y it must permute the paths between them. Since $\mathcal{T}$ is a tree, there is a unique path from x to y. Therefore this must be fixed and in particular $\mathcal{T}^G$ is connected. If G fixes an edge e then it must swap the connected components of $\mathcal{T} \backslash \{e\}$, so that $\mathcal{T}^G = \{e\}$ in this case. □

Next we define two important concepts which play a pivotal role in the proof of Theorem 2.18.

Definition 2.20 Let $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ be a tree of fusion systems which satisfies (H) and write $\mathcal{F}_{\mathcal{T}}$ for the completion of $(\mathcal{T}, \mathcal{F}, \mathcal{S})$.

(a) For each pair of p-groups $P \leq Q$ define the *restriction map* from Q to P:

$$\mathrm{res}^Q_P : \mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(Q, \mathcal{F}) \longrightarrow \mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$$

to be the map which sends a vertex $[\varphi]_{\mathcal{F}(v)}$ in $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(Q, \mathcal{F})$ to $[\varphi|_P]_{\mathcal{F}(v)}$ in $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$.

(b) For each $P \leq \mathcal{S}(v_*)$ the *action* of $\mathcal{F}_{\mathcal{T}}$ on $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$ is given by the group action of $\mathrm{Aut}_{\mathcal{F}_{\mathcal{T}}}(P)$ on $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$ which sends a vertex $[\varphi]_{\mathcal{F}(v)}$ to $[\psi \circ \varphi]_{\mathcal{F}(v)}$ for each $\psi \in \mathrm{Aut}_{\mathcal{F}_{\mathcal{T}}}(P)$.

We quickly check that this definition make sense. Firstly, one observes that res^Q_P defines a homomorphism of graphs. To see this, note that if $[\alpha]_{\mathcal{F}(v)}$ is incident to some edge $[\beta]_{\mathcal{F}(e)}$ in $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(Q, \mathcal{F})$, then there exists $\psi \in \mathrm{Hom}_{\mathcal{F}(v)}(Q\alpha, \mathcal{S}(v))$ such that $\beta \circ \iota^{\mathcal{S}(v)}_{\mathcal{S}(e)} = \alpha \circ \psi$. Hence $\beta|_P \circ \iota^{\mathcal{S}(v)}_{\mathcal{S}(e)} = \alpha|_P \circ \psi|_{P\alpha}$ and $[\alpha|_P]_{\mathcal{F}(v)}$ is incident to $[\beta|_P]_{\mathcal{F}(e)}$ in $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$. Secondly, note the described action of $\mathrm{Aut}_{\mathcal{F}_{\mathcal{T}}}(P)$ on $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$ makes sense: if $[\varphi]_{\mathcal{F}(v)}$ is a vertex of $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$ and $\psi \in \mathrm{Aut}_{\mathcal{F}_{\mathcal{T}}}(P)$, then $\psi \circ \varphi \in \mathrm{Hom}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{S}(v))$ so that $[\psi \circ \varphi]_{\mathcal{F}(v)}$ is also a vertex of $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$.

The following proposition gives conditions on $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ which will allow us to calculate the image of res^Q_P in terms of certain fixed points of the action of $\mathcal{F}_{\mathcal{T}}$ on $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$.

Proposition 2.21 *Let $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ be a tree of fusion systems which satisfies (H) and assume that the following hold:*

(a) $\mathcal{F}(v)$ is saturated for each vertex v of $\mathcal{T}$; and

(b) $\mathcal{F}(e)$ is a trivial fusion system $\mathcal{F}_{\mathcal{S}(e)}(\mathcal{S}(e))$ for each edge e of $\mathcal{T}$.

Let P and Q be p-groups with $P \trianglelefteq Q \leq S(v_*)$ and assume that P is $\mathcal{F}_{\mathcal{T}}$-centric, where $\mathcal{F}_{\mathcal{T}}$ is the completion of $(\mathcal{T}, \mathcal{F}, \mathcal{S})$. If $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$ is a tree, then the image of the restriction homomorphism

$$\mathrm{res}^Q_P : \mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(Q, \mathcal{F}) \longrightarrow \mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$$

is equal to $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})^{\mathrm{Aut}_Q(P)}$.

Proof. Set $K := \mathrm{Aut}_Q(P)$. If $[\alpha]$ is a vertex or edge in $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(Q,\mathcal{F})$ then $[\alpha|_P] \in \mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P,\mathcal{F})^K$ since

$$c_g \circ \alpha|_P = \alpha|_P \circ (\alpha^{-1}|_{P\alpha} \circ c_g \circ \alpha|_P)$$

for each $g \in Q$. It remains to prove that each vertex or edge of $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P,\mathcal{F})^K$ is the image of *some* vertex or edge of $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(Q,\mathcal{F})$. Since $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P,\mathcal{F})$ is a tree, $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P,\mathcal{F})^K$ is also a tree by Lemma 2.19. In particular $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P,\mathcal{F})^K$ is connected. Hence to complete the proof, it will be enough to show that for each edge $[\beta]_{\mathcal{F}(e)}$ in $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P,\mathcal{F})^K$ which is connected to the image $[\alpha|_P]_{\mathcal{F}(v)}$ of some vertex $[\alpha]_{\mathcal{F}(v)}$ under res^Q_P, there exists an edge $[\bar{\beta}]_{\mathcal{F}(e)}$ in $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(Q,\mathcal{F})$ connected to $[\alpha]_{\mathcal{F}(v)}$ whose image under res^Q_P is $[\beta]_{\mathcal{F}(e)}$. Since $[\iota^S_Q]_{\mathcal{F}(v_*)}$ gets sent to $[\iota^S_P]_{\mathcal{F}(v_*)}$ under res^Q_P, we may assume (inductively) that we have chosen e to be the edge incident to v in the unique minimal path from v to v_*. In particular we may assume that $\mathcal{S}(e) = \mathcal{S}(v)$. By assumption, there exists $\psi \in \mathrm{Hom}_{\mathcal{F}(v)}(P\beta, P\alpha)$ such that $\beta \circ \psi = \alpha|_P$. Set $R := N^{K^\beta}_{\mathcal{S}(e)}(P\beta)$. Since $[\beta]_{\mathcal{F}(e)}$ is fixed by K, $K^\beta \leq \mathrm{Aut}_{\mathcal{S}(e)}(P\beta)$ so $\mathrm{Aut}_R(P\beta) = K^\beta$. Now, $\mathrm{Aut}_R(P\beta)^\psi = K^{\beta\psi} = K^\alpha = \mathrm{Aut}_{Q\alpha}(P\alpha)$ and hence

$$Q\alpha \leq N_{\psi^{-1}} = \{g \in N_{\mathcal{S}(v)}(P\alpha) \mid (c_g)^{\psi^{-1}} \in \mathrm{Aut}_S(P\beta)\}.$$

Since $\mathcal{F}(v)$ is saturated and $P\alpha$ is $\mathcal{F}(v)$-centric (and therefore fully $\mathcal{F}(v)$-centralised), ψ^{-1} extends to a map $\rho \in \mathrm{Hom}_{\mathcal{F}(v)}(Q\alpha, \mathcal{S}(v))$. Set $\bar{\beta} := \alpha \circ \rho \in \mathrm{Hom}_{\mathcal{F}(v)}(Q, \mathcal{S}(v))$ so that $\mathrm{res}^Q_P([\bar{\beta}]_{\mathcal{F}(e)}) = [\bar{\beta}|_P]_{\mathcal{F}(e)} = [\beta]_{\mathcal{F}(e)}$ and $[\alpha]_{\mathcal{F}(v)}$ is incident to $[\bar{\beta}]_{\mathcal{F}(e)}$ in $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(Q,\mathcal{F})$. This completes the proof. □

2.4.2 Proof of Theorem 2

We now have all of the tools necessary to prove Theorem 2. We remind the reader that this theorem was originally conjectured based on the corresponding statement for trees of group fusion systems, [7, Theorem 4.2], and the proof in that case may be seen to rely on some fairly deep homotopy theory. No deep results are required in our proof and for this reason we feel the statement is more naturally understood in the context of trees of fusion systems. Nevertheless, we prove [7, Theorem 4.2] (without the statement involving linking systems[5]) as Corollary 2.22 below.

A key step in our proof is the observation that the completion $\mathcal{F}_\mathcal{T}$ of a tree of fusion systems $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ is independent of where $\mathcal{F}$ sends edges e of $\mathcal{T}$, provided $\mathcal{F}_{\mathcal{S}(e)}(\mathcal{S}(e)) \subseteq \mathcal{F}(e) \subseteq \mathcal{F}(v)$ (inside $\mathcal{F}_\mathcal{T}$). This means that we are able to assume that $\mathcal{F}(e)$ is the trivial fusion system $\mathcal{F}_{\mathcal{S}(e)}(\mathcal{S}(e))$.

[5]This is Lemma 2.25 in Section 2.4.3.

Proof (of Theorem 2). Let $(\mathcal{T}, \mathcal{F}', \mathcal{S})$ be the tree of fusion systems obtained from $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ by setting $\mathcal{F}'(v) := \mathcal{F}(v)$ for each vertex v of $\mathcal{T}$ and $\mathcal{F}'(e) := \mathcal{F}_{\mathcal{S}(e)}(\mathcal{S}(e))$ for each edge e of $\mathcal{T}$. Then $(\mathcal{T}, \mathcal{F}', \mathcal{S})$ satisfies (H) and has completion $\mathcal{F}_{\mathcal{T}}$. Also, since any cycle in $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F}')$ is a cycle in $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$, hypothesis (b) in the theorem remains true for $(\mathcal{T}, \mathcal{F}', \mathcal{S})$. Hence we may assume from now on that $(\mathcal{T}, \mathcal{F}, \mathcal{S}) = (\mathcal{T}, \mathcal{F}', \mathcal{S})$.

Since $\mathcal{F}(v)$ is a saturated fusion system on $\mathcal{S}(v)$ for each vertex v in $\mathcal{T}$, by Theorem 1.26

$$\mathcal{F}(v) = \langle \{\mathrm{Aut}_{\mathcal{F}(v)}(P) \mid P \text{ is } \mathcal{F}(v)\text{-essential } \} \cup \{\mathrm{Aut}_{\mathcal{F}(v)}(\mathcal{S}(v))\} \rangle_{\mathcal{S}(v)}.$$

By hypothesis, each $\mathcal{F}(v)$-essential subgroup and each $\mathcal{S}(v)$ is an $\mathcal{F}_{\mathcal{T}}$-centric subgroup so by the definition of $\mathcal{F}_{\mathcal{T}}$ we have

$$\mathcal{F}_{\mathcal{T}} = \langle \mathrm{Aut}_{\mathcal{F}(v)}(P) \mid v \in V(\mathcal{T}), P \text{ is } \mathcal{F}_{\mathcal{T}}\text{-centric} \rangle_S.$$

Hence by Theorem 1.27 it suffices to verify that axioms (a) and (b) in Definition 1.15 hold for $\mathcal{F}_{\mathcal{T}}$-centric subgroups.

First we prove Definition 1.15 (a) holds. Let P be a fully $\mathcal{F}_{\mathcal{T}}$-normalised, $\mathcal{F}_{\mathcal{T}}$-centric subgroup of S. Then since $|C_S(P\varphi)| = |Z(P\varphi)| = |Z(P)|$ for each $\varphi \in \mathrm{Hom}_{\mathcal{F}_{\mathcal{T}}}(P, S)$, P is certainly fully $\mathcal{F}_{\mathcal{T}}$-centralised. It remains to prove that P is fully $\mathcal{F}_{\mathcal{T}}$-automised. Each $\varphi \in \mathrm{Aut}_{\mathcal{F}_{\mathcal{T}}}(P)$ acts on $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$ by sending a vertex $[\alpha]_{\mathcal{F}(v)}$ to $[\varphi \circ \alpha]_{\mathcal{F}(v)}$. Since $\mathrm{Aut}_{\mathcal{F}_{\mathcal{T}}}(P)$ is finite and $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$ is a tree, by Lemma 2.19 there exists a vertex or edge $[\alpha]_{\mathcal{F}(r)}$ in $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$ which is fixed under the action of $\mathrm{Aut}_{\mathcal{F}_{\mathcal{T}}}(P)$ ($r \in V(\mathcal{T}) \cup E(\mathcal{T})$). This means that $\mathrm{Aut}_{\mathcal{F}(r)}(P\alpha) = \mathrm{Aut}_{\mathcal{F}_{\mathcal{T}}}(P)^{\alpha}$.

Choose $\beta \in \mathrm{Hom}_{\mathcal{F}(r)}(P\alpha, S(r))$ so that $P\alpha\beta$ is fully $\mathcal{F}(r)$-normalised. Since $\mathcal{F}(r)$ is saturated, $\mathrm{Aut}_{S(r)}(P\alpha\beta) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}(r)}(P\alpha\beta))$. Now

$$|\mathrm{Aut}_S(P)| = |N_S(P)|/|Z(P)| \geq |N_S(P\alpha\beta)|/|Z(P\alpha\beta)| \geq |\mathrm{Aut}_{S(r)}(P\alpha\beta)|,$$

where the first inequality follows from the fact that P is fully $\mathcal{F}_{\mathcal{T}}$-normalised. Since

$$|\mathrm{Aut}_{\mathcal{F}(r)}(P\alpha\beta)| = |\mathrm{Aut}_{\mathcal{F}(r)}(P\alpha)| = |\mathrm{Aut}_{\mathcal{F}_{\mathcal{T}}}(P)|,$$

we must have $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}_{\mathcal{T}}}(P))$, so that P is fully $\mathcal{F}_{\mathcal{T}}$-automised, as needed.

Next we prove that (b) holds in Definition 1.15. Fix $\varphi \in \mathrm{Hom}_{\mathcal{F}_{\mathcal{T}}}(P, S)$ where P is $\mathcal{F}_{\mathcal{T}}$-centric. We need to show that there is some $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}_{\mathcal{T}}}(N_\varphi, S)$ which extends φ. Set $K := \mathrm{Aut}_{N_\varphi}(P) = \mathrm{Aut}_S(P) \cap \mathrm{Aut}_S(P\varphi)^{\varphi^{-1}}$. If $c_g \in K$ then $c_g \circ \varphi = \varphi \circ c_h$ for some $h \in \mathrm{Aut}_S(P\varphi)$ and this shows that the vertex $[\varphi]_{\mathcal{F}(v_*)}$ in $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F})$ is fixed by the action of K. By Proposition 2.21 applied with $Q = N_\varphi$ there is some

$[\psi]_{\mathcal{F}(v_*)} \in \mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(N_\varphi, \mathcal{F})$ with $[\psi|_P]_{\mathcal{F}(v_*)} = [\varphi]_{\mathcal{F}(v_*)}$. This implies that there is $\rho \in \mathrm{Iso}_{\mathcal{F}(v_*)}(P\varphi, P\psi)$ such that $\psi|_P = \varphi \circ \rho$. Since $\mathcal{F}(v_*)$ is saturated, $\rho^{-1}|_{P\psi} = (\psi|_P)^{-1} \circ \varphi$ extends to a map $\chi \in \mathrm{Hom}_{\mathcal{F}(v_*)}((N_\varphi)\psi, S)$. To see this, observe that $(N_\varphi)\psi \leq N_{\rho^{-1}}$ since for $g \in N_\varphi$, $(c_{g\psi})^{\rho^{-1}} = (c_g)^\varphi \in \mathrm{Aut}_S(P\varphi) = \mathrm{Aut}_S(P\psi\rho^{-1})$. Hence we set $\bar{\varphi} := \psi \circ \chi \in \mathrm{Hom}_\mathcal{F}(N_\varphi, S)$ so that $\bar{\varphi}$ extends φ, as needed. This completes the proof of (b) in Definition 1.15 and hence the proof of the theorem. □

Corollary 2.22 *Let $(\mathcal{T}, \mathcal{G})$ be a tree of groups with completion $\mathcal{G}_\mathcal{T}$ and assume that there exists a vertex v_* of $\mathcal{T}$ for which the following is true: for each $v \in V(\mathcal{T})$ not equal to v_*, $p \nmid |\mathcal{G}(v) : \mathcal{G}(e)|$ where e is the edge incident to v in the unique minimal path from v to v_*. Write $S := \mathcal{S}(v_*)$ and assume that for each $P \leq S$,*

(a) if P is $\mathcal{G}_\mathcal{T}$-conjugate to an $\mathcal{F}_{\mathcal{S}(v)}(\mathcal{G}(v))$-essential subgroup then P is $\mathcal{F}_S(G_\mathcal{T})$-centric; and

(b) if P is $\mathcal{F}_S(G_\mathcal{T})$-centric then $\tilde{\mathcal{T}}^P / C_{\mathcal{G}_\mathcal{T}}(P)$ is a tree.

Then $\mathcal{F}_S(\mathcal{G}_\mathcal{T})$ is saturated.

Proof. Let $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ be a tree of fusion systems induced by $(\mathcal{T}, \mathcal{G})$. The hypothesis of the corollary implies that $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ satisfies (H), so there exists a completion $\mathcal{F}_\mathcal{T}$ of $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ by Lemma 2.7. By Theorem 2.13, $\mathcal{F}_\mathcal{T} \cong \mathcal{F}_S(\mathcal{G}_\mathcal{T})$, so it suffices to verify conditions (a) and (b) in Theorem 2.18 hold. Clearly (a) holds by assumption and (b) holds by Proposition 2.15 and Corollary 2.17. □

2.4.3 Examples

To end this chapter, we consider how Theorem 2.18 can be applied to construct new saturated fusion systems and also make some remarks concerning to what extent a converse holds. By this last statement we mean: given a tree of fusion systems $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ which satisfies (H) with saturated completion $\mathcal{F}_\mathcal{T}$, do hypotheses (a) and (b) in Theorem 2.18 hold? We do not have a complete answer to this question, but present some evidence to support the claim that (b) always holds. First, we state a lemma which will help us verify whether or not (b) holds in Theorem 2.18 directly.

Lemma 2.23 *Let $\mathcal{T}$ be a connected graph and write $n := |V(\mathcal{T})|$ and $m := |E(\mathcal{T})|$. $\mathcal{T}$ is a tree if and only if*

$$m + 1 = n.$$

Proof. The 'only if' direction is an obvious induction on n. Conversely suppose $m+1 = n$ and let $\mathcal{T}'$ be a maximal subtree of $\mathcal{T}$ so that $|V(\mathcal{T}')| = n$ and $|E(\mathcal{T}')| \leq m$. Hence if $\mathcal{T} \neq \mathcal{T}'$, we must have $|E(\mathcal{T}')| \leq n = |V(\mathcal{T}')|$, which implies that $\mathcal{T}'$ is not connected, a contradiction. □

Lemma 2.23 can be applied to demonstrate how the saturation of the fusion system $\mathcal{F}_\mathcal{T}$ in Example 2.14 follows directly as a consequence of Theorem 2.18.

Example 2.24 Let $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ be the tree of fusion systems constructed in Example 2.14. Clearly (a) holds in the statement of Theorem 2.18 since S, V_1 and V_2 are the only $\mathcal{F}_\mathcal{T}$-centric subgroups. It remains to check that (b) holds for these subgroups. Firstly, we show that $|\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(V_i, \mathcal{F}(e))| = 3$. To see this, for $i = 1, 2$ write $Z(S) = \langle z \rangle$, choose x_i such that $V_i = \langle z, x_i \rangle$ and notice that $V_i\varphi = V_i$ for each $\varphi \in \mathrm{Hom}_{\mathcal{F}_\mathcal{T}}(V_i, S)$. Since $\mathrm{Aut}_{\mathcal{F}(e)}(V_i)$ interchanges x_i and x_iz, the class of φ is determined by which of x_i, z and x_iz gets mapped to z under φ yielding exactly 3 classes of embeddings. By similar arguments the facts that $\mathrm{Aut}_{\mathcal{F}(v_i)}(V_i) \cong \mathrm{Sym}(3)$ and $\mathrm{Aut}_{\mathcal{F}(v_{3-i})}(V_i) \cong C_2$ imply that $|\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(V_i, \mathcal{F}(v_i))| = 1$ and $|\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(V_i, \mathcal{F}(v_{3-i}))| = 3$ respectively for $i = 1, 2$. Since $4 = 3+1$, Lemma 2.23 implies that the graph $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(V_i, \mathcal{F})$ is a tree for $i = 1, 2$. In fact, we can see directly that $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(V_i, \mathcal{F})$ is a 3 pointed star. Finally, the graph $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(S, \mathcal{F})$ obviously consists of a single vertex and so the saturation of $\mathcal{F}_\mathcal{T}$ follows from Theorem 2.18.

The previous example is one of a large number of examples of fusion systems which can be constructed using Theorem 2.18. A more general construction is given by Theorem 3 in Chapter 3.

We turn now to consider the sufficiency of the conditions in Theorem 2.18, starting with a useful characterisation of the graph $\mathrm{Rep}_{\mathcal{G}_\mathcal{T}}(P, \mathcal{G})$ for a tree of finite groups $(\mathcal{T}, \mathcal{G})$ with completion $\mathcal{G}_\mathcal{T}$.

Lemma 2.25 *Let $(\mathcal{T}, \mathcal{G})$ be a tree of finite groups and write $\mathcal{G}_\mathcal{T}$ for the completion of $(\mathcal{T}, \mathcal{G})$. Let $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ be a tree of fusion systems induced by $(\mathcal{T}, \mathcal{G})$ and assume that $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ satisfies (H), so that there exists a completion $\mathcal{F}_\mathcal{T}$ for $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ on $S := \mathcal{S}(v_*)$. Let $P \leq \mathcal{S}(v_*)$ be fully $\mathcal{F}_\mathcal{T}$-centralised. There exists a homotopy equivalence*

$$\tilde{\mathcal{T}}^P / C_{\mathcal{G}_\mathcal{T}}(P) \xrightarrow{\simeq} B(C_{G_\mathcal{T}}(P)/Z(P)).$$

Furthermore, if $\tilde{\mathcal{T}}^P / C_{\mathcal{G}_\mathcal{T}}(P)$ is a tree for each $\mathcal{F}_\mathcal{T}$-centric subgroup P then $\mathcal{L}^c_S(\mathcal{G}_\mathcal{T})$ is a centric linking system associated to $\mathcal{F}_\mathcal{T}$.

Proof. Observe that $C_{G_\mathcal{T}}(P)/Z(P)$ acts freely on the contractible space $\tilde{\mathcal{T}}^P/Z(P) \simeq \tilde{\mathcal{T}}^P$ so that $\tilde{\mathcal{T}}^P/C_{\mathcal{G}_\mathcal{T}}(P)$ is a classifying space for $C_{G_\mathcal{T}}(P)/Z(P)$. When P is $\mathcal{F}_\mathcal{T}$-centric, $Z(P)$ is a Sylow p-subgroup of $C_{\mathcal{G}_\mathcal{T}}(P)$. Hence by the Fundamental Theorem of Bass-Serre Theory, this action must have finite vertex and edge stabilisers of p'-order. In particular $H^1(C_{\mathcal{G}_\mathcal{T}}(P)/Z(P); \mathbb{F}_p) = 0$ (since $O^p(C_{\mathcal{G}_\mathcal{T}}(P)/Z(P)) = C_{\mathcal{G}_\mathcal{T}}(P)/Z(P)$). The Mayer–Vietoris spectral sequence for graphs of groups (see [8, VII.9.2]) implies that $H^i(C_{\mathcal{G}_\mathcal{T}}(P)/Z(P); \mathbb{F}_p) = 0$ for all $i > 0$. so by Lemma 1.66, $\mathcal{L}^c_S(\mathcal{G}_\mathcal{T})$ is a centric linking system associated to $\mathcal{F}_\mathcal{T}$. □

The following example, taken from [7, Example 3.5], proves that (b) in Theorem 2.18 is a necessary condition for saturation. Our technique will be to construct a tree of fusion systems $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ satisfying (H) whose completion is not saturated, and use Lemma 2.25 to show that (b) in Theorem 2.18 does not hold.

Example 2.26 Let $p = 2$ and let S be isomorphic to the group $D_8 \times V_4$. Regard $T \cong C_4 \times V_4$ as a subgroup of S via the natural embedding $C_4 \longrightarrow D_8$, and as a subgroup of $H \cong C_4 \times A_4$ via the natural embedding $V_4 \longrightarrow A_4$. The triple (S, T, H) naturally gives rise to a tree of finite groups $(\mathcal{T}, \mathcal{G})$ where $\mathcal{T}$ consists of a single edge $e = (v, w)$, $\mathcal{G}(v) = S$, $\mathcal{G}(w) = H$ and $\mathcal{G}(e) = T$. Let $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ be a a tree of fusion systems induced by $(\mathcal{T}, \mathcal{G})$ and let $\mathcal{F}_\mathcal{T}$ be its completion. Write $G := \mathcal{G}_\mathcal{T}$ and note that S is a Sylow p-subgroup of G and $\mathcal{F}_\mathcal{T} = \mathcal{F}_S(G)$ by Theorem 2.13. Since T is normal in S and H, it is normal in G and

$$G/T \xrightarrow{\sim} S/T * H/T \xrightarrow{\sim} C_2 * C_3 \xrightarrow{\sim} PSL_2(\mathbb{Z})$$

(see [8, Example 3 of Section IX.6] for a proof of this last isomorphism). Now $C_G(T)/T$ is exactly the kernel of the natural map $\Phi : G/T \longrightarrow \mathrm{Out}_G(T)$, and since $\mathrm{Out}_G(T) \cong C_2 \times C_3$ is abelian, $\mathrm{Ker}(\Phi) \geq PSL_2(\mathbb{Z})'$ which is well-known to contain a free subgroup. In particular, $H^1(C_G(T)/Z(T); \mathbb{Z}) \neq 0$ and by Lemma 2.25, $\tilde{\mathcal{T}}^T/C_G(T) \simeq B(C_G(T)/Z(T))$ is not acyclic so that $\tilde{\mathcal{T}}^T/C_G(T) \simeq \mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(T, \mathcal{F})$ is not a tree. Furthermore, $\mathcal{F}_\mathcal{T}$ is not saturated. To see this, observe that $\mathrm{Aut}_{\mathcal{F}_\mathcal{T}}(T) \cong C_2 \times C_3$, while $\mathrm{Aut}_{\mathcal{F}_\mathcal{T}}(S)$ is a 2-group, so that the automorphism φ of order 3 in $\mathrm{Aut}_{\mathcal{F}_\mathcal{T}}(T)$ does not extend to a map $\tilde{\varphi} \in \mathrm{Aut}_{\mathcal{F}_\mathcal{T}}(S)$.

Example 2.26 can be modified to prove that (a) is also a necessary condition in Theorem 2.18 as follows. Replace S by the group $C_8 \times V_4$ and observe that T is $\mathcal{F}(w)$-essential but not $\mathcal{F}_\mathcal{T}$-centric and that $\mathcal{F}_\mathcal{T}$ is not saturated for the same reasons as before. Thus it seems plausible that conditions (a) and (b) in Theorem 2.18 both

hold when $\mathcal{F}_\mathcal{T}$ is the saturated completion of a tree of fusion systems $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ which satisfies (H), but we do not have a proof of this.

Given a saturated fusion system $\mathcal{F}$ on a finite p-group S, a centric linking system $\mathcal{L}$ associated to $\mathcal{F}$ and subgroup P of S there exists a centric linking system $C_\mathcal{L}(P)$ associated to $C_\mathcal{F}(P)$ when P is fully $\mathcal{F}$-centralised (see [6, Proposition 2.5]). The following conjecture should be verifiable in the case where $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ is induced by a tree of groups, loosely as a consequence of Lemma 2.25 and Theorem 1.67. The general case will probably rely on some fairly deep characterisations of $C_\mathcal{L}(P)$, such as [6, Theorem 6.3].

Conjecture 2.27 *Let $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ be a tree of fusion systems which satisfies (H). Write $S := \mathcal{S}(v_*)$ and $\mathcal{F}_\mathcal{T}$ for the completion of $(\mathcal{T}, \mathcal{F}, \mathcal{S})$ and assume that $\mathcal{F}_\mathcal{T}$ is saturated. Let $\mathcal{L}$ is any centric linking system associated to $\mathcal{F}_\mathcal{T}$. For each fully $\mathcal{F}_\mathcal{T}$-centralised subgroup P of S, the following are equivalent:*

(a) $\mathrm{Rep}_{\mathcal{F}_\mathcal{T}}(P, \mathcal{F})$ *is a tree.*

(b) There exists a p-equivalence $|C_\mathcal{L}(P)| \longrightarrow BZ(P)$.

Chapter 3

Small Modules and Exotic Fusion Systems

The primary goal of this chapter is to apply some unpublished results of Oliver to demonstrate the existence of a large family of saturated fusion systems coming from certain modular representations of finite groups at odd primes p. Given a pair (G, A) where G is a finite group with a cyclic Sylow p-subgroup of order p and A is an $\mathbb{F}_p(G)$-module we show that under certain conditions, one can associate a 'reduced' fusion system $\mathcal{F}$ over a Sylow p-subgroup of $A \rtimes G$ with the property that $\mathrm{Aut}_{\mathcal{F}}(A) = G$. The proof that these fusion systems are saturated is an interesting application of the theory of trees of fusion systems developed in Chapter 2. It follows from Theorem 3, which is a generalisation of a similar result due to Broto, Levi and Oliver, [7, Proposition 5.1], giving conditions enabling one to enlarge a fusion system $\mathcal{F}$ by adding p'-automorphisms of the $\mathcal{F}$-centric subgroups.

It is a fact (which we will not prove here) that almost all of these systems are exotic. However, we are able to verify this statement in certain cases. The method we use for doing so is new to the theory. It turns out that some of the fusion systems possess strongly closed subgroups despite being simple. Work of Flores–Foote in [13] has established that no simple group with a simple fusion system possesses a strongly closed p-subgroup. This means that in certain cases, the exoticity of these fusion systems follows from the fact that any group which realises them can be chosen to be simple.

Since each fusion system is determined by one of the pairs (G, A) defined above, it is interesting to attempt a classification of such pairs. We choose to focus on the case where G is close to simple and using a recent observation of Craven, Magaard and Parker concerning the generation of G, we prove that $\dim(A) < 2p$. In the case where G is a classical group $G_n(q)$ (q a prime power), a polynomial lower bound for $\dim(A)$

can be obtained by appealing to work of Landzuri–Seitz in [22]. This allows us to find a short list of pairs (n, q) for which to check the existence of a suitable representation of $G_n(q)$ or one of its finite bicyclic extensions. The general classification for simple groups is due to be completed in a paper by Bob Oliver.

The chapter is structured as follows. The theory of trees of fusion systems is used to prove Theorem 3 in Section 3.1. In Section 3.2 we describe the essential subgroups of fusion systems over p-groups with abelian subgroup of index p, use this information to reconstruct them and then give a detailed example of an infinite family of simple exotic fusion systems which arise from modules associated to the groups $GL_2(p)$ (p an odd prime). Finally we prove Theorem 4 in Section 3.3, providing a complete list of bicylic extensions of classical groups $G_n(q)$ (other than the groups $A_1(q)$) which possess irreducible modules satisfying our hypotheses.

3.1 Attaching p'-automorphisms to Fusion Systems

In this section, we apply Theorem 2 to prove Theorem 3 which is a general procedure for extending a fusion system by p'-automorphisms of its centric subgroups. Theorem 2 is a generalisation of [7, Proposition 5.1] to arbitrary saturated fusion systems.

3.1.1 Extensions of p-constrained Groups

We begin by appealing to some of the cohomological machinery developed in Chapter 1 to prove a lemma which gives conditions for a p-constrained group to be extended by a p'-group of automorphisms of a normal centric subgroup.

Lemma 3.1 *Let H be a finite group with $O_{p'}(H) = 1$ and let Q be a normal p-subgroup of H with $C_H(Q) \leq Q$. Assume (if possible) that $\Delta \leq \mathrm{Out}(Q)$ is chosen so that $\mathrm{Out}_H(Q)$ is a strongly p-embedded subgroup of Δ. There exists a finite group G containing H as a p'-index subgroup with $Q \trianglelefteq G$ and $\mathrm{Out}_G(Q) \cong \Delta$.*

Proof. Write $K := \mathrm{Out}_H(Q)$. Since K is strongly p-embedded in Δ and $Z(Q)$ may be regarded as an $\mathbb{F}_p(\Delta)$-module, Corollary 1.13 implies that $H^j(\Delta; Z(Q)) \cong H^j(K; Z(Q))$ for each $j > 0$. Since $H^3(K; Z(Q)) = 0$ (H is an extension of Q by K), the above isomorphism implies that there exists a finite group G which fits into a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Q & \longrightarrow & G & \longrightarrow & \Delta & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & Q & \longrightarrow & H & \longrightarrow & K & \longrightarrow & 1, \end{array}$$

by Theorem 1.11. Since $H^2(\Delta; Z(Q)) \cong H^2(K; Z(Q))$ acts freely and transitively on the set of all such extensions of Q by Δ, we may choose G such that $H \leq G$. In particular, $\Delta \cong G/Q = \mathrm{Out}_G(Q)$, as required. □

The following simple example helps to motivate the sorts of situations in which we expect to apply Lemma 3.1:

Example 3.2 Let $Q = C_p \times C_p$ so that $\mathrm{Aut}(Q) \cong GL_2(p)$. Let R be a Sylow p-subgroup of $\mathrm{Aut}(Q)$ and write $K := N_{\mathrm{Aut}(Q)}(R) \cong C_p \rtimes C_{p-1}$. Let H be the group $Q \rtimes K$ so that $C_H(Q) = Q$ and $K = \mathrm{Out}_H(Q) \cong H/Q$. Since K is strongly p-embedded in a subgroup Δ of $\mathrm{Aut}(Q)$ isomorphic to $SL_2(p)$, by Lemma 3.1 there exists a finite group G containing H as a p'-index subgroup such that $G/Q \cong \Delta$.

3.1.2 Proof of Theorem 3

Let $\mathcal{F}_0$ be a saturated fusion system on a finite p-group S. Our goal is to apply Lemma 3.1 to find conditions for there to exist a saturated fusion system $\mathcal{F}$ containing $\mathcal{F}_0$ with the property that $\mathrm{Aut}_{\mathcal{F}_0}(Q)$ is a strongly p-embedded subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$ whenever $Q \leq S$ is $\mathcal{F}$-essential. The technique we use will involve the construction of a star of fusion systems $(\mathcal{T}, \mathcal{F}(-), \mathcal{S}(-))$ with $\mathcal{F}_0$ associated to the central vertex and the fusion systems of certain p'-extensions of constrained models of the $N_{\mathcal{F}_0}(Q)$ associated to the outer vertices. The saturation of the completion of $(\mathcal{T}, \mathcal{F}(-), \mathcal{S}(-))$ will follow from Theorem 2.18.

Theorem 3.3 *Let $\mathcal{F}_0$ be a saturated fusion system on a finite p-group S. For $1 \leq i \leq m$, let $Q_i \leq S$ be fully $\mathcal{F}_0$-normalised subgroups with $Q_i\varphi \not\leq Q_j$ for each $\varphi \in \mathrm{Hom}_{\mathcal{F}_0}(Q_i, S)$ and $i \neq j$. Set $K_i := \mathrm{Out}_{\mathcal{F}_0}(Q_i)$ and choose $\Delta_i \leq \mathrm{Out}(Q_i)$ so that K_i is a strongly p-embedded subgroup of Δ_i. Write*

$$\mathcal{F} = \langle \{\mathrm{Hom}_{\mathcal{F}_0}(P, S) \mid P \leq S\} \cup \{\Delta_i \mid 1 \leq i \leq m\} \rangle_S.$$

Assume further that for each $1 \leq i \leq m$,

(a) Q_i is $\mathcal{F}_0$-centric (hence $\mathcal{F}$-centric) and minimal (under inclusion) amongst all $\mathcal{F}$-centric subgroups, and

(b) no proper subgroup of Q_i is $\mathcal{F}_0$-essential.

Then $\mathcal{F}$ is saturated.

Proof. Since Q_i is fully $\mathcal{F}_0$-normalised and $\mathcal{F}_0$-centric, $N_{\mathcal{F}_0}(Q_i)$ is a constrained fusion system. Hence by Theorem 1.38 there exists a unique finite group H_i with $Q_i \trianglelefteq H_i$, $C_{H_i}(Q_i) \leq Q_i$ and $N_{\mathcal{F}_0}(Q_i) = \mathcal{F}_{N_S(Q_i)}(H_i)$. Since $\mathrm{Out}_{\mathcal{F}_0}(Q_i) = \mathrm{Out}_{H_i}(Q_i)$, Lemma 3.1 implies that there is some finite group G_i containing H_i with $p \nmid |G_i : H_i|$ and which satisfies $\mathrm{Out}_{G_i}(Q_i) = \Delta_i$. Now let $\mathcal{T}$ be the star formed by a central vertex v_0 and m vertices v_i each incident to v_0 via a unique edge e_i for $1 \leq i \leq m$. Let $(\mathcal{T}, \mathcal{F}(-), \mathcal{S}(-))$ be the tree of fusion systems formed by setting $\mathcal{F}(v_0) := \mathcal{F}_0$, $\mathcal{F}(v_i) := \mathcal{F}_{N_S(Q_i)}(G_i)$ and $\mathcal{F}(e_i) := \mathcal{F}_{N_S(Q_i)}(H_i) = N_{\mathcal{F}_0}(Q_i)$ for $1 \leq i \leq m$. Then $(\mathcal{T}, \mathcal{F}(-), \mathcal{S}(-))$ satisfies (H), $\mathcal{F}_{\mathcal{T}} = \mathcal{F}$ by Lemma 2.7 and it suffices to verify that conditions (a) and (b) of Theorem 2.18 hold.

Suppose that $P = \mathcal{S}(v_i)$ or P is $\mathcal{F}$-conjugate to an $\mathcal{F}(v_i)$-essential subgroup. We need to show that P is $\mathcal{F}$-centric. In the first case, since Q_i is $\mathcal{F}$-centric (by assumption), $P = N_S(Q_i)$ is also $\mathcal{F}$-centric by Lemma 1.23 (b). In the second case, there exists some $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ such that $P\varphi \leq \mathcal{S}(v_i)$ is $\mathcal{F}(v_i)$-essential. If $i \neq 0$ then since $Q_i \trianglelefteq \mathcal{F}(v_i)$, Lemma 1.35 implies that $Q_i \leq P\varphi$ so that since Q_i is $\mathcal{F}$-centric also $P\varphi$ (and hence P) is $\mathcal{F}$-centric by Lemma 1.23 (b) again. If $i = 0$ then since $P\varphi$ is not contained in Q_i (by condition (b) in the theorem) for all $1 \leq i \leq m$ and $P\varphi$ is $\mathcal{F}_0$-centric, $P\varphi$ is also $\mathcal{F}$-centric, by the definition of $\mathcal{F}_{\mathcal{T}}$.

It remains to prove that (b) holds in Theorem 2.18. It will be enough to show directly that the connected component of the vertex $[\iota_P^S]_{\mathcal{F}(v_0)}$ in $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F}(-))$ is a tree for each $\mathcal{F}_{\mathcal{T}}$-centric subgroup $P \leq S$. Suppose $[\beta]_{\mathcal{F}(v_0)}$ is connected to $[\beta']_{\mathcal{F}(v_i)}$ via an edge $[\alpha]_{\mathcal{F}(e_i)}$ in $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F}(-))$. If $[\beta']_{\mathcal{F}(v_i)}$ is incident to another edge $[\alpha']_{\mathcal{F}(e_i)}$ then there exists $\rho \in \mathrm{Iso}_{\mathcal{F}(v_i)}(P\alpha, P\alpha')$ such that $\alpha' = \alpha \circ \rho$. Since $Q_i \trianglelefteq \mathcal{F}(v_i)$, ρ extends to a map $\bar{\rho} \in \mathrm{Hom}_{\mathcal{F}(v_i)}((P\alpha)Q_i, (P\alpha')Q_i)$ with $\bar{\rho}|_{Q_i} \in \mathrm{Aut}_{\mathcal{F}(v_i)}(Q_i)$. Set $\chi := [\rho|_{Q_i}] \in \mathrm{Out}_{\mathcal{F}(v_i)}(Q_i) = \Delta_i$ for the class of $\rho|_{Q_i}$ in Δ_i and note that $\chi \notin K_i$, since otherwise $[\alpha]_{\mathcal{F}(e_i)} = [\alpha']_{\mathcal{F}(e_i)}$. Now

$$\chi^{-1} \circ \mathrm{Out}_{(P\alpha)Q_i}(Q_i) \circ \chi = \mathrm{Out}_{(P\alpha')Q_i}(Q_i) \leq K_i$$

and

$$\mathrm{Out}_{(P\alpha)Q_i}(Q_i) \leq K_i \cap \chi \circ K_i \circ \chi^{-1},$$

which is a p'-group by hypothesis. Since Q_i is $\mathcal{F}$-centric, $\mathrm{Out}_{(P\alpha)Q_i}(Q_i) \cong (P\alpha)Q_i/Q_i$ so that $P\alpha \leq Q_i$. But P is $\mathcal{F}$-centric and Q_i is minimal amongst all $\mathcal{F}$-centric subgroups (by assumption) and so $P\alpha = Q_i$. Furthermore since Q_i is not $\mathcal{F}_0$-conjugate to Q_j for $i \neq j$, this must occur for some unique i.

Setting $\beta = \iota_P^S$ in the previous paragraph we see that either $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F}(-))$ is a tree or P is $\mathcal{F}_0$-conjugate to a unique Q_i. Suppose we are in this latter situation

and let $\Gamma_0 \subset \mathrm{Rep}_{\mathcal{F}}(P, \mathcal{T})$ be the connected subgraph consisting of all vertices of the form $[\beta]_{\mathcal{F}(v_0)}$ or $[\beta']_{\mathcal{F}(v_i)}$ and edges of the form $[\alpha]_{\mathcal{F}(e_i)}$. If $[\gamma]_{\mathcal{F}(v_j)}$ is a vertex not in Γ_0 then $j \neq i$ and a minimal path from $[\gamma]_{\mathcal{F}(v_j)}$ to Γ_0 must consist of a single edge $[\delta]_{\mathcal{F}(e_j)}$, otherwise by the argument in the previous paragraph, we would have that P is $\mathcal{F}_0$-conjugate to Q_j, a contradiction (see Figure 3.1 below). Hence Γ_0 is a deformation retract of $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P, \mathcal{F}(-))$ and it is enough to prove that Γ_0 is a tree. In fact we prove that Γ_0 is a star. Suppose that a vertex $[\beta]_{\mathcal{F}(v_0)} \in \Gamma_0$ is incident to two (different) edges $[\alpha]_{\mathcal{F}(e_i)}$ and $[\alpha']_{\mathcal{F}(e_i)}$. The argument in the previous paragraph shows that $\alpha, \alpha' \in \mathrm{Iso}_{\mathcal{F}(e_i)}(P, Q_i)$ and there exists $\rho \in \mathrm{Aut}_{\mathcal{F}(v_i)}(Q_i) = \Delta_i$ such that $\alpha \circ \rho = \alpha'$. Hence $[\alpha]_{\mathcal{F}(e_i)} = [\alpha']_{\mathcal{F}(e_i)}$ and each $[\alpha]_{\mathcal{F}(v_0)} \in \Gamma_0$ is incident to a unique edge. This establishes the claim, and finishes the proof of the theorem. □

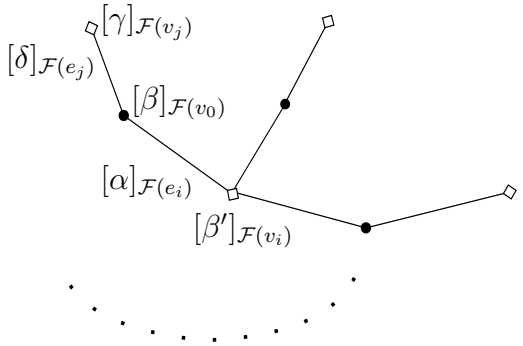


Figure 3.1: Vertices of distance at most two from $[\beta']_{\mathcal{F}(v_i)}$ in Γ for each $i \neq j$.

3.2 Fusion Systems over p-groups with Abelian Subgroup of Index p

After introducing some notation, we sketch a proof of a result which shows that certain fusion systems over p-groups with abelian subgroup of index p are determined by G-modules for G a finite group (Theorem 3.7). In Section 3.2.2 we apply this result in the case where $G = GL_2(p)$ (p an odd prime) to produce an infinite family of simple, exotic fusion systems first discovered by Clelland and Parker in [10].

3.2.1 The Essential Subgroups

This section largely follows [24]. Let S be a non-abelian p-group with a unique abelian subgroup A of index p and let $\mathcal{F}$ be a fusion system on S. Set

$$Z := Z(S), \;\; Z_0 := Z \cap [S,S], \;\; Z_2 := Z_2(S), \quad \text{and} \quad A_0 := Z[S,S],$$

where Z_2 is the *second centre of* S defined by $Z_2/Z(S) := Z(S/Z(S))$. One observes that

$$Z_0 \leq S' \leq A_0 \leq A \quad \text{and} \quad Z_0 \leq Z \leq Z_2 \leq A_0.$$

The following sets of groups will be of particular interest to us:

$$\mathcal{H} := \{Z\langle x\rangle \mid x \in S\backslash A\} \quad \text{and} \quad \mathcal{B} := \{Z_2\langle x\rangle \mid x \in S\backslash A\}.$$

We quote the following from [24]:

Lemma 3.4 *Let S be a non-abelian p-group with a unique abelian subgroup A of index p and let $\mathcal{F}$ be a saturated fusion system on S. Write $\mathcal{E}_\mathcal{F}$ for the set of $\mathcal{F}$-essential subgroups. The following hold:*

(a) $\mathcal{E}_\mathcal{F} \subseteq \{A\} \cup \mathcal{H} \cup \mathcal{B}$.

(b) If $Z_2\langle x\rangle \in \mathcal{E}_\mathcal{F}$ *for* $x \in S\backslash A$ *then* $Z\langle x\rangle \notin \mathcal{E}_\mathcal{F}$.

(c) If $\mathcal{E}_\mathcal{F} \nsubseteq \{A\}$ *then*

(i) $p = |Z_0| = |A/A_0| = |Z_2/Z|$; *and*

(ii) there exists $x \in S\backslash A$ *such that* $\mathrm{Aut}_\mathcal{F}(S)$ *normalises* $A_0\langle x\rangle$.

(d) For each $P \in \mathcal{E}_\mathcal{F}$ *and* $\alpha \in N_{\mathrm{Aut}_\mathcal{F}(P)}(\mathrm{Aut}_S(P))$, α *extends to* $\bar{\alpha} \in \mathrm{Aut}_\mathcal{F}(S)$.

Proof. This is shown in [24, Lemma 2.3]. □

Assume that $\mathcal{E}_\mathcal{F} \nsubseteq \{A\}$ so that $p = |Z_0| = |A/A_0| = |Z_2/Z|$ by Lemma 3.4 (c)(i). By Lemma 3.4 (c)(ii) we may fix $x \in S\backslash A$ so that $\mathrm{Aut}_\mathcal{F}(S)$ normalises $A_0\langle x\rangle$ and we also fix an (arbitrary) element $a \in A\backslash A_0$. For $0 \leq i \leq p-1$, set

$$H_i := \langle Z, xa^i\rangle \quad \text{and} \quad B_i := \langle Z_2, xa^i\rangle,$$

and note that $H_i \cong C_p \times C_p$ and $B_i \cong p_+^{1+2}$ if $|Z| = p$. Let $\mathcal{H}_i$ and $\mathcal{B}_i$ respectively denote the S-conjugacy classes of H_i and B_i. We observe that $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_*$ and $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_*$, where

$$\mathcal{H}_* := \bigcup_{i=1}^{p-1} \mathcal{H}_i \quad \text{and} \quad \mathcal{B}_* := \bigcup_{i=1}^{p-1} \mathcal{B}_i,$$

since $[A : A_0] = p$. Set $\Delta := (\mathbb{Z}/p)^\times \times (\mathbb{Z}/p)^\times$ and construct a map

$$\mu : \mathrm{Aut}(S) \longrightarrow \Delta,$$

by letting

$$\mu(\alpha) = (r, s) \text{ if } \begin{cases} x\alpha \in x^r A, & \text{for } x \in S \backslash A \\ g\alpha = g^s, & \text{for } g \in Z_0. \end{cases}$$

Writing $G := \mathrm{Aut}_{\mathcal{F}}(A)$ and $U := \mathrm{Aut}_S(A)$, we observe that μ may be written as a composition

$$\mu : \mathrm{Aut}(S) \xrightarrow{\mathrm{Res}} N_G(U)/U \xrightarrow{\delta} \Delta,$$

where

$$\delta(\alpha U) = (r, s) \text{ if } \begin{cases} \alpha^{-1}\tau\alpha = \tau^r, & \text{for } \tau \in U \\ g\alpha = g^s, & \text{for } g \in Z_0. \end{cases}$$

Since $\mathcal{F}$ is saturated, Res is surjective and $\mathrm{Out}_{\mathcal{F}}(S)$ is determined completely by the action of G on A. In unpublished work, Bob Oliver uses the image of the map μ to give a precise description of all saturated fusion systems $\mathcal{F}$ over S with the property that $\mathrm{Aut}_{\mathcal{F}}(A)$ is $\mathcal{F}$-essential. To describe this image, we need one more piece of notation. Let δ^* denote the composite of the natural map $\rho : N_G(U) \longrightarrow N_G(U)/U$ with δ and define:

$$\Delta_i := \{(r, r^i) \mid r \in \mathbb{Z}_p^\times\} \leq \Delta, \text{ for each } i \in \mathbb{Z}.$$

Oliver's results in the case where $Z(S)$ has order p are summarised in the following lemma, and due to the technical nature of the arguments involved, we do not supply a proof. The reader is referred to [24, Theorem 2.8], for a detailed description of these fusion systems in the case where A is not $\mathcal{F}$-essential, and many of the arguments used to handle that case carry forward to this.

Lemma 3.5 *Let S be a non-abelian p-group with a unique abelian subgroup A of index p of order p^m and with $|Z(S)| = p$. Let $\mathcal{F}$ be a saturated fusion system on S with $A \notin \mathcal{F}$ in which A is $\mathcal{F}$-essential and write*

$$G := \mathrm{Aut}_{\mathcal{F}}(A), \ U := \mathrm{Aut}_S(A), \quad \textit{and} \quad N := \{\alpha \in N_G(U) \mid \alpha|_Z = \mathrm{Id}_Z\}.$$

If $O^{p'}(\mathcal{F}) = \mathcal{F}$ then one of the following holds:

(i) $\mathcal{E}_{\mathcal{F}} = \mathcal{H}_0 \cup \mathcal{B}_*$ if $m \equiv$ 0 mod p or $\mathcal{E}_{\mathcal{F}} = \mathcal{B}_0 \cup \mathcal{H}_*$ if $m \equiv$ -1 mod p, $\delta^*(N) = \Delta$ and $G = \langle O^{p'}(G), N \rangle$.

(ii) $\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{H}_0 \cup \mathcal{H}_*$, $\delta^*(N) \geq \Delta_{-1}$ and $G = \langle O^{p'}(G), (\Delta_{-1})\delta^{*-1} \rangle$.

(iii) $\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{B}_0 \cup \mathcal{B}_*$, $\delta^*(N) \geq \Delta_0$ and $G = \langle O^{p'}(G), (\Delta_0)\delta^{*-1} \rangle$.

Proof. Omitted. □

Notice that (i), (ii) and (iii) are mutually exclusive and that the pair (G, A) completely determines the isomorphism types of the $\mathcal{F}$-essential subgroups in each case.

Now suppose that we have an *arbitrary* pair (G, A) of finite groups where A is an elementary abelian p-group on which G acts. Assume that G has a Sylow p-subgroup U of order p and set $S := A \rtimes U$. We would like to construct a saturated fusion system $\mathcal{F}$ on S (when $|Z(S) = p|$) with $\mathrm{Aut}_{\mathcal{F}}(A) = G$ and $O^{p'}(\mathcal{F}) = \mathcal{F}$. The next lemma will be used to establish whether or not this latter condition holds.

Lemma 3.6 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and write $\mathcal{E}_{\mathcal{F}}$ for the set of $\mathcal{F}$-essential subgroups. If each $P \in \mathcal{E}_{\mathcal{F}}$ is minimal amongst all $\mathcal{F}$-centric subgroups then $O^{p'}(\mathcal{F}) = \mathcal{F}$ if and only if $\mathrm{Aut}_{\mathcal{F}}(S) = \langle \mathrm{Inn}(S), \mathrm{Aut}_{\mathcal{F}}^{(P)}(S) \mid P \in \mathcal{E}_{\mathcal{F}} \rangle$, where*

$$\mathrm{Aut}_{\mathcal{F}}^{(P)}(S) = \{\alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \mid P\alpha = P, \alpha|_P \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))\}.$$

Proof. This is shown in [24, Lemma 1.4]. □

We can now construct the fusion systems associated to (G, A) described above, essentially by applying Theorem 3.3. The argument we use is based on that found in [24, Theorem 2.8] in the case where A is not $\mathcal{F}$-essential.

Theorem 3.7 *Let G be a finite group with a non-normal Sylow p-subgroup U order p such that $|\mathrm{Aut}_G(U)| = p - 1$. Let A be an elementary abelian p-group of order p^m on which G acts. Write $S := A \rtimes U$ and assume that $Z := Z(S)$ has order p. Assume further that exactly one of the following holds for $N := \{\alpha \in N_G(U) \mid \alpha|_Z = \mathrm{Id}_Z\}$:*

(i) $\delta^*(N) = \Delta$, $m \equiv$ 0 or -1 mod p and $G = \langle O^{p'}(G), N \rangle$;

(ii) $\delta^*(N) \geq \Delta_{-1}$ and $G = \langle O^{p'}(G), (\Delta_{-1})\delta^{*-1} \rangle$; or

(iii) $\delta^*(N) \geq \Delta_0$ and $G = \langle O^{p'}(G), (\Delta_0)\delta^{*-1} \rangle$.

There exists a saturated fusion system $\mathcal{F}$ over S with $\mathrm{Aut}_{\mathcal{F}}(A) = G$ and $O^{p'}(\mathcal{F}) = \mathcal{F}$. Furthermore $\mathcal{E}_{\mathcal{F}} = \mathcal{H}_0 \cup \mathcal{B}_* \cup \{A\}$ or $\mathcal{E}_{\mathcal{F}} = \mathcal{B}_0 \cup \mathcal{H}_* \cup \{A\}$ if (i) holds, and $\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{H}_0 \cup \mathcal{H}_* \cup \{A\}$ or $\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{B}_0 \cup \mathcal{B}_* \cup \{A\}$ if (ii) or (iii) hold respectively.

Proof (sketch). We will deal only with the case where (i) holds for the pair (G, A) since an almost identical argument can be used to deal with the other cases. Following Lemma 3.5, define a set $\hat{\mathcal{E}}$ of subgroups of S as follows:

(a) If $m \equiv 0 \bmod p - 1$, set $\hat{\mathcal{E}} := \{H_0, B_1, \ldots, B_{p-1}\}$.

(b) If $m \equiv -1 \bmod p-1$, set $\hat{\mathcal{E}} := \{B_0, H_1, \dots, H_{p-1}\}$.

For each $i \in \mathbb{Z}$ with $(i,p) = 1$ and each $j \in (\mathbb{Z}/p)^\times$ define $\lambda_i, \nu_j \in \mathrm{Aut}(S)$ by setting:

$$a\lambda_i := a^i, \;\; x\lambda_i := x, \;\; x\nu_j := x^j, \quad \text{and} \quad a\nu_j := a,$$

for all $a \in A$. Then $\Lambda := \langle \lambda_i, \nu_j \mid 0 \le i,j \le p-1 \rangle \le \mathrm{Aut}(S)$ satisfies $\mu(\Lambda) = \Delta$, fixes the groups H_0 and B_0 and permutes the sets $\{H_1, \dots, H_{p-1}\}$ and $\{B_1, \dots, B_{p-1}\}$.

By [24, Lemma 2.6(b)], Λ contains a subgroup of order $p-1$ which normalises each $Q_i \in \hat{\mathcal{E}}$. Define $H := S \rtimes \Lambda$ so that $S \in \mathrm{Syl}_p(H)$. Let $Q_0, \dots, Q_{p-1}$ be fully $\mathcal{F}_S(H)$-normalised representatives for the H-conjugacy classes of elements in $\hat{\mathcal{E}}$ where $Q_i \in \mathcal{H} \cup \mathcal{B}$ and with indices chosen in such a way that Q_i is H-conjugate to B_i or H_i for each i. Also define

$$\Theta_i := \begin{cases} GL_2(p), & \text{if } i = 0; \\ SL_2(p), & \text{otherwise.} \end{cases}$$

We refer to [24, Lemma 2.7] for a precise explanation for why these groups are chosen. Set $Q_\infty := A$ and $\Theta_\infty := G \le \mathrm{Aut}(A)$, and define

$$\mathcal{F} := \langle \{\mathrm{Hom}_{\mathcal{F}_S(H)}(P,S) \mid P \le S\} \cup \{\Theta_0, \dots, \Theta_{p-1}, \Theta_\infty\} \rangle_S.$$

Observe that for each $i \in \{0, \dots, p-1\} \cup \{\infty\}$,

(1) $\mathrm{Out}_H(Q_i) = N_{\Theta_i}(\mathrm{Out}_S(Q_i))$ is strongly p-embedded in Θ_i;

(2) Q_i is $\mathcal{F}_S(H)$-centric and minimal (under inclusion) among all $\mathcal{F}$-centric subgroups of S; and

(3) no proper subgroup of Q_i is $\mathcal{F}_S(H)$-essential.

(1) follows from Corollary 1.10. (2) and (3) are obvious in the cases where $Q_i \cong C_p \times C_p$ or $i = \infty$. When $Q_i \cong p^{1+2}_+$, (2) and (3) follow from the fact that Θ_i acts transitively on the set of index p subgroups of Q_i. To see this, observe that for each such subgroup P there is $\alpha \in \Theta_i$ such that $P\alpha = Z_2 < A$, so that P is neither $\mathcal{F}$-centric nor $\mathcal{F}_S(H)$-essential. Hence by Theorem 3.3 $\mathcal{F}$ is a saturated fusion system with $\mathrm{Aut}_{\mathcal{F}}(A) = G$. Furthermore $\mathcal{E}_{\mathcal{F}} = \mathcal{H}_0 \cup \mathcal{B}_* \cup \{A\}$ or $\mathcal{E}_{\mathcal{F}} = \mathcal{B}_0 \cup \mathcal{H}_* \cup \{A\}$ according to whether (a) or (b) hold respectively, and $O^{p'}(\mathcal{F}) = \mathcal{F}$ by Lemma 3.6. This completes the proof. □

In fact, unpublished work of Bob Oliver shows that the condition that $Z(S)$ have order p in Lemma 3.5 and Theorem 3.7 is not really necessary. A slightly more intricate analysis allows one to gain control over the way in which $Z(S)$ splits as a product and formulate analogous conditions for the action to give rise to a fusion system. This point will be important in Section 3.3.

3.2.2 An Example

We give a detailed description of a family of pairs (G, A) which satisfy the conditions of Theorem 3.7 and give rise to infinitely many simple exotic fusion systems $\mathcal{F}$. Recall that a fusion system $\mathcal{F}$ is *simple* if it possesses no proper normal subsystems. One of the most remarkable things about these fusion systems is that they all possess a non-trivial strongly $\mathcal{F}$-closed subgroup and hence determine a non-trivial quotient fusion system by Theorem 1.49. The following result of Flores–Foote says that this cannot happen in cases where $\mathcal{F}$ is realisable as the fusion system of a finite simple group.

Theorem 3.8 *Let G be a finite simple group with non-abelian Sylow p-subgroup S such that $\mathcal{F} := \mathcal{F}_S(G)$ is simple. There are no proper strongly $\mathcal{F}$-closed subgroups of S.*

Proof. This follows from [4, Theorem II.12.12]: The p-fusion systems of p-Goldschmidt groups are constrained by [4, Theorem II.12.13] (and hence not simple) and the 3-fusion system of $G_2(q)$ with $q \equiv \pm 1 \bmod 9$ is not simple by [3, Lemma 16.11]. □

Therefore, we may prove that a simple fusion system $\mathcal{F}$ with a strongly $\mathcal{F}$-closed subgroup is exotic by applying Theorem 3.8, provided we can show that any group which realises it can be chosen to be simple. We find a test for this property now, recalling first the following standard group theoretic fact concerning minimal normal subgroups.

Lemma 3.9 *Let G be a finite group and L be a non-trivial normal subgroup of G, with $|L|$ minimal. There exist a natural number k and for all $1 \leq i \leq k$ isomorphic simple groups L_i on which G acts transitively such that $L = L_0 \times \ldots \times L_k$.*

Proof. Let K be a minimal normal subgroup of L and notice that $\prod_{g \in G} K^g$ is a normal subgroup of G contained in L, so we have $L = \prod_{g \in G} K^g$ by the minimality of $|L|$. Let M be a direct product of elements in the set $\{K^g \mid g \in G\}$ chosen maximally subject to the condition that $M \trianglelefteq L$. If $M \neq L$ then there exists $g \in G$ such that $K^g \not\leq M$. Since $K^g \cap M \trianglelefteq L$, and K^g is a minimal normal subgroup of L, this implies that $K^g \cap M = 1$. Hence M is properly contained in $M \times K^g \trianglelefteq L$, a contradiction. Thus $L = M$. In particular, any normal subgroup of K is also normal in L and we must have that K is simple, as required. □

Lemma 3.9 allows us to find criteria for a realisable reduced fusion system to be the fusion system of a simple group. A saturated fusion system $\mathcal{F}$ is *reduced* if $O^p(\mathcal{F}) = O^{p'}(\mathcal{F}) = \mathcal{F}$ and $O_p(\mathcal{F}) = 1$. In particular, $\mathcal{F}$ is reduced if it simple.

Lemma 3.10 *Let $\mathcal{F}$ be a saturated fusion system on a non-abelian finite p-group S. Assume that every strongly $\mathcal{F}$-closed subgroup P satisfies the following 3 conditions.*

(a) P is S-centric;

(b) P is not elementary abelian; and

(c) P does not factor as a direct product of subgroups which are permuted transitively by $\mathrm{Aut}_{\mathcal{F}}(P)$.

If $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with $S \in Syl_p(G)$ then G is almost simple. Furthermore, if $\mathcal{F}$ also reduced then G can be chosen to be simple.

Proof. Assume $\mathcal{F}$ satisfies the hypotheses of the lemma and choose G minimally subject to the condition that $\mathcal{F} = \mathcal{F}_S(G)$. Since $\mathcal{F}_S(G) = \mathcal{F}_S(G/O_{p'}(G))$ we must have $O_{p'}(G) = 1$. Let L be a minimal normal subgroup of G. By Lemma 3.9 for each $1 \leq i \leq k$ there exist isomorphic simple groups L_i on which G acts transitively such that $L = L_0 \times \ldots \times L_k$. Define $P := L \cap S \in \mathrm{Syl}_p(L)$ and observe that $P \neq 1$ since $O_{p'}(G) = 1$. If L is abelian, it must be an elementary abelian p-group which contradicts (b) above, so L_i is non-abelian for each i. Also, since G acts transitively on the L_i, it must also act transitively on the factors of $P = L_0 \cap S \times \ldots \times L_k \cap S$. Hence by (c), we must have $k = 1$ and L is a non-abelian simple group. Next we prove that $C_G(L) = 1$. It suffices to show that $S \cap C_G(L) = 1$, since this implies that $C_G(L)$ is a normal p'-subgroup and hence trivial since $O_{p'}(G) = 1$. Assume that $C_G(L) \cap S \neq 1$ and then by (a) we have $S \cap C_G(L) \leq C_S(P) \leq P \leq L$ so that $S \cap C_G(L) = P$. Hence $1 \neq P \trianglelefteq C_G(L) \leq C_G(P)$ and $1 \neq O_p(C_G(L)) \leq O_p(G) \cap C_G(L) \trianglelefteq G$ (since $C_G(L) \trianglelefteq G$). Now,

$$|O_p(G) \cap C_G(L)| \leq |S \cap C_G(L)| \leq |C_S(P)| \leq |P| < |L|,$$

which contradicts the minimality of L, so we must have $C_G(L) \cap S = 1$. We sketch a proof of the final statement. Observe that $G = L \rtimes \mathrm{Out}_G(L)$, so it suffices to prove that $\mathrm{Out}_G(L) = 1$. If $\mathrm{Out}_G(L) \neq 1$ then by the Schreier Conjecture, $\mathrm{Out}_G(L)$ is a non-trivial solvable group and must possess a normal subgroup of index p or at p'-index. The first of these assertions contradicts the assumption that $O^p(\mathcal{F}) = \mathcal{F}$, while the second contradicts the minimality of G. □

We can now use Theorem 3.7 to construct simple fusion systems associated to modules for $GL_2(p)$ and prove that they are exotic.

Example 3.11 Let $V := \mathbb{F}_p \oplus \mathbb{F}_p$ be the natural module for $G := GL_2(p)$. Fix a natural number n which satisfies, $3 \leq n < p$. The action of G on V extends to an action on the symmetric power vector space

$$A := V^{\otimes n-1}/\langle v \otimes w - w \otimes v \rangle$$

of dimension n. Fix a generator

$$x := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

for a Sylow p-subgroup of G and let $\{e_1, e_2\}$ be the standard basis for V. For $0 \leq i \leq n-1$,

$$v_i := e_2 \otimes \cdots \otimes e_2 \otimes e_1 \cdots \otimes e_1 \ (i \text{ copies of } e_2 \text{ and } n-1-i \text{ copies of } e_1)$$

is a basis for A. Since $xe_1 = e_1$ and $xe_2 = e_1 + e_2$, one readily checks that the formula

$$xv_i = \sum_{j=0}^{i} \binom{i}{j} v_j$$

holds. Identify A with an elementary abelian p-group of order p^n generated by the elements $\{a_i\}_{i=0}^{n-1}$, and write $U := \langle x \rangle$. Hence $S := A \rtimes U$ (with the given action of x) has presentation

$$S = \langle a_0, ..., a_{n-1}, x \mid a_i^p = [a_i, a_j] = x^p = 1, a_i^x = \prod_{j=0}^{i} a_j^{\binom{i}{j}} \ \forall \ 0 \leq i, j \leq n-1 \rangle.$$

From this, we see that $Z(S) = \langle a_0 \rangle$ has order p so that by Theorem 3.7 there is a saturated fusion system $\mathcal{F}$ on S with $\mathrm{Aut}_{\mathcal{F}}(A) = G$ and $O^{p'}(\mathcal{F}) = \mathcal{F}$. Since $N_G(U) \cong C_p \rtimes C_{p-1}$ does not act trivially on $Z(S)$ we must have $\delta^*(N_G(U)) = \Delta_{-1}$ so that $\mathcal{E}_{\mathcal{F}} = \{A\} \cup \mathcal{H}_0$ and $\mathrm{Aut}_{\mathcal{F}}(H_0) \cong SL_2(p)$. Therefore, by [24, Lemma 2.7 (a)], $n \equiv -1 \mod p-1$ (in particular $n \geq 2p-3$). To prove that $\mathcal{F}$ is reduced it remains to check that $\mathfrak{hyp}(\mathcal{F}) = S$ and $O_p(\mathcal{F}) = 1$. Since $[A, G] = A$, certainly $A \leq \mathfrak{hyp}(\mathcal{F})$. Also, since there exists $\alpha \in \mathrm{Aut}_{\mathcal{F}}(H_0)$ such that $a_0\alpha = a_0x$, $x = a_0^{-1}a_0x \in \mathfrak{hyp}(\mathcal{F})$ and we must have $\mathfrak{hyp}(\mathcal{F}) = S$. By Lemma 1.35, $O_p(\mathcal{F}) \leq A \cap H_0 = Z(S)$ so that $O_p(\mathcal{F}) = 1$ since $Z(S)$ is a non-normal subgroup of order p. Hence $\mathcal{F}$ is reduced. Next, we determine the strongly $\mathcal{F}$-closed subgroups of S. Let P be such a subgroup and notice that P

contains $Z(S)$ since P is normal in S and $|Z(S)| = p$. Hence $H_0 = \langle a_0, x\rangle \leq P$, since $\mathrm{Aut}_{\mathcal{F}}(H_0) = SL_2(p)$ acts transitively on the index p subgroups of H_0. Therefore,

$$P \geq \langle \mathcal{H}_0 \rangle = A_0 H_0 = \langle a_0, \ldots a_{n-2}, x\rangle$$

and either $P = A_0H_0$ or $P = S$. Clearly S is strongly $\mathcal{F}$-closed. To see that A_0H_0 is strongly $\mathcal{F}$-closed, it suffices to check that it is normalised by $\mathrm{Aut}_{\mathcal{F}}(S) \cong C_{p-1}$, which holds by inspection. Hence A_0H_0 and S are the only strongly $\mathcal{F}$-closed subgroups of S. Next we show that $\mathcal{F}$ is exotic. Both S and A_0H_0 are S-centric and do not split as a direct product so Lemma 3.10 implies that $\mathcal{F}$ is either exotic or the fusion system of a simple group G. In the latter case, A_0H_0 is a proper strongly G-closed subgroup of S which is impossible by Theorem 3.8. Therefore $\mathcal{F}$ is exotic, as required. Finally, we prove that $\mathcal{F}$ is simple. Any proper normal subsystem $\mathcal{E}$ of $\mathcal{F}$ must lie on the group $T := A_0H_0$. When this happens, $\mathcal{H}_0$ splits into p T-conjugacy classes of subgroups (since $[S : T] = p$) and for all $P \in \mathcal{H}_0$, $N_S(P) = Z_2(P) \leq T$ since $Z_2 \leq A_0$. Therefore $\mathrm{Aut}_{\mathcal{E}}(P)$ contains the normal closure $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ of $\mathrm{Aut}_S(P)$ and $P \in \mathcal{E}_{\mathcal{E}}$. Since $n \geq 4$, [24, Lemma 2.7 (a)] implies that $|A_0| = n - 1 \equiv -1 \bmod p - 1$, so that $p - 1$ divides $(n, n + 1)$ and $p = 2$, a contradiction. Hence $\mathcal{F}$ is simple.

A more direct proof that $\mathcal{F}$ is exotic in the above example could have been obtained without reference to Lemma 3.10 by appealing to the fact that $\mathcal{F}$ is simple. This is because any group which realises a simple fusion system can be chosen to be simple. We include Lemma 3.10 because proving that a fusion system is simple is usually harder than finding its strongly closed subgroups.

3.3 Small Modules for Almost Quasisimple Classical Groups

A group G is *almost quasisimple* if there exists a subgroup H of $G/Z(G)$ such that $G/Z(G) \leq \mathrm{Aut}(H)$. In this section we carry out a systematic search for pairs (G, A) which satisfy the hypotheses of Theorem 3.7 when G is an almost quasisimple group. Specifically, we will concentrate on the case where G is a bicyclic extension of a classical group of Lie type. The hypothesis that $Z(S)$ have order p will not be considered[1].

[1]See the remark which follows Theorem 3.7.

3.3.1 Small Modules

We start with an unpublished result of Oliver which says that when $\mathcal{F}$ is a reduced fusion system determined by a pair (G, A) as in Theorem 3.7, the action of $\mathrm{Aut}_S(A)$ on A has a very restricted structure.

Lemma 3.12 *Let S be a non-abelian p-group with a unique elementary abelian subgroup A of index p and let $\mathcal{F}$ be a reduced saturated fusion system on S with $A \in \mathcal{E}_{\mathcal{F}}$. Write $G := \mathrm{Aut}_{\mathcal{F}}(A)$ and $U := \mathrm{Aut}_S(A)$. The following hold:*

(a) U is a non-normal Sylow p-subgroup of G of order p;

(b) $[G, A] = A$ and $|\mathrm{Aut}_G(U)| = p - 1$; and

(c) as an $\mathbb{F}_p(U)$-module A has exactly one non-trivial indecomposable summand.

Proof. Omitted. □

Since we are focussing only on the case where G is close to simple, we define a pair (G, A) to be *small* if the following three conditions hold:

(i) G is (a bicylic extension of) a finite simple group;

(ii) A is an elementary abelian p-group on which G acts irreducibly (i.e. A is an irreducible $\mathbb{F}_p(G)$-module); and

(iii) (a), (b) and (c) are all satisfied in the statement of Lemma 3.12 where U is a fixed Sylow p-subgroup of G.

The next two sections are concerned with classifying all small pairs (G, A) in the case where G is a bicylic extension of classical group of Lie type of dimension n over the field $\mathbb{F}_q$ and A is irreducible.

3.3.2 Bounding the Module Dimension

First, we provide upper and lower bounds for $\dim(A)$ and use these to produce a short list of pairs (n, q) for which the existence of a small module for $a.G_n(q).r$ is possible, where $G_n(q)$ is a classical group of Lie type of dimension n over the field $\mathbb{F}_q$ and $a.G_n(q).r$ is the 'ATLAS notation'. We begin with the upper bound. For this, we appeal to the following unpublished result of Craven, Magaard and Parker on generation of finite simple groups, which will probably be of independent interest.

Theorem 3.13 *Let G be a finite almost quasisimple group and assume that $v_p(|G|) = 1$ for some prime p. There exists a pair (U_1, U_2) of distinct Sylow p-subgroups of G which satisfies $O^{p'}(G) = \langle U_1, U_2 \rangle$.*

Proof. Omitted. □

Corollary 3.14 *For each small pair (G, A), we have $\dim(A) < 2p$.*

Proof. Let (G, A) be small and let U_1 and U_2 be Sylow p-subgroups of G, chosen such that $\langle U_1, U_2 \rangle = G$. Since $A|_{U_i}$ has exactly 1 non-trivial indecomposable summand, we must have $|A : C_A(U_i)| \leq p^{p-1}$ for $i = 1, 2$. Hence

$$|A| = |A : C_A(G)| = |A : C_A(U_1) \cap C_A(U_2)| \leq p^{2p-2},$$

and when A is regarded as an $\mathbb{F}_p(G)$-module we have $\dim(A) < 2p$, as needed. □

It remains to find a lower bound for $\dim(A)$ when (G, A) is small and $G = a.G_n(q).r$. For this, we appeal to the work of Landazuri and Seitz in [22] which provides a lower bound for the degree $l(G, t)$ of a non-trivial modular representation of G of characteristic t where $t \neq \text{char}(\mathbb{F}_q)$. $l(G, t)$ is always a polynomial $g(q)$ in q and their results (for classical groups) are summarised in Figure 3.2.

Since a Sylow p-subgroup U of G has order p, there exists a unique natural number d (bounded above by a linear function of n) such that $p \mid \Phi_d(q)$. It is a standard fact that d and f (where $q = q_0^f$ and q_0 is prime) provide an upper bound the order of $\text{Aut}_G(U)$, and its precise value depends on the isomorphism type of G. This information is also provided in Figure 3.2. Now, $|\text{Aut}_G(U)| = p - 1$ whenever (G, A) is small so we have

$$g(q) \leq \dim(A) < 2p = 2(p-1) + 2 = 2 \cdot |\text{Aut}_G(U)| + 2 \leq 2 \cdot h(n, f) + 2,$$

where $h(n, f)$ is some *linear* function of n and f which depends on G. The exact values for $h(n, f)$ are provided by Figure 3.3, where the valid pairs (n, q) which the above inequality determines are also listed. Notice that there can exist no small pairs in the case where G is of orthogonal type.

$a.G_n(q).r$	$l(G,t) \geq$	Exceptions: $(n,q,t) \in$	$\lvert \mathrm{Aut}_G(U) \rvert \leq$
$a.PSL_n(q).r$, $n > 2$	$q^{n-1} - 1$	$\{(3,2,2),(3,4,2)\}$	$2df$
$a.PSp_{2n}(q).r$, $n \geq 2$	$\begin{cases} \frac{1}{2}(q^n - 1), & q \text{ odd} \\ \frac{1}{2}q^{n-1}(q^{n-1} - 1)(q-1), & q \text{ even} \end{cases}$	$\{(2,2,2),(3,2,2)\}$	$\begin{cases} 2df & d \text{ odd} \\ df & d \text{ even} \end{cases}$
$a.PSU_n(q).r$, $n > 2$	$\begin{cases} q(q^{n-1} - 1)/(q+1), & n \text{ odd} \\ (q^n - 1)/(q+1), & n \text{ even} \end{cases}$		$\begin{cases} 2df & d \equiv 1,3 \bmod 4 \\ df/2 & d \equiv 2 \bmod 4 \\ df & d \equiv 0 \bmod 4 \end{cases}$
$a.PSO^+_{2n}(q)'.r$, $n > 3$	$\begin{cases} q^{n-2}(q^{n-1} - 1), & q = 2,3,5 \\ (q^{n-1} - 1)(q^{n-2} + 1), & \text{otherwise} \end{cases}$	$\{(4,2,2),(4,3,3)\}$	$\begin{cases} 2df & d \text{ odd} \\ df & d \text{ even} \end{cases}$
$a.PSO^-_{2n}(q)'.r$, $n > 3$	$(q^{n-1} + 1)(q^{n-2} - 1)$	$\{(4,2,2)\}$	$\begin{cases} 2df & d \text{ odd} \\ df & d \text{ even} \end{cases}$
$a.PSO_{2n+1}(q)'.r$, $n > 2$, q odd	$q^{2(n-1)}$	$\{(3,3,3)\}$	$\begin{cases} 2df & d \text{ odd} \\ df & d \text{ even} \end{cases}$

Figure 3.2: Lower bounds for the dimensions of irreducible representations of the classical groups.

G	$h(n,f)$	(n,q) s.t. $g(q) < 2h(n,f)+2$
$a.PSL_n(q).r$	$2nf$	$\{(3,2),(4,2),(5,2),(3,4),(3,3)\}$
$a.Sp_{2n}(q).r$	$4nf$	$\{(2,2),(3,2),(2,4),(2,3),(3,3)\}$
$a.PSO^{\epsilon}_{2n}(q)'.r$	$2(n-1)f$	$\{(3,2)\}$
$a.PSO_{2n+1}(q)'.r$	$2nf$	$\emptyset$
$a.PSU_n(q).r$	$2nf$	$\{(4,2),(5,2),(3,4),(3,3),(4,3)\}$

Figure 3.3: List of pairs (n,q) for which the existence of a small pair is possible.

3.3.3 Proof of Theorem 4

We are now in a position to prove Theorem 4. There are only a handful of classical groups which could possibly provide us with examples of small pairs. We begin our search by listing all pairs (G,A) with the properties that

(1) $G = a.G_n(q).r$ is a bicyclic extension of a classical group of dimension n over the field $\mathbb{F}_q$ determined by a pair (n,q) in Figure 3.3; and

(2) A is an irreducible $\mathbb{F}_p(G)$-module with $\dim(A) < 2p$ for some prime p with the property that $v_p(|G|) = 1$.

The tables contained in [5] allow us to complete this analysis, and Figures 3.5, 3.6 and 3.7 provides a complete list for $(G,p,\dim(A))$ in the linear, symplectic and unitary cases respectively. Notice that modules are listed according to their 'Galois isomorphism-type,' so for example we write '$8,8$' in column 3 to indicate the presence of two classes of non-isomorphic modules of dimension 8. It thus remains to check conditions (b) and (c) in Lemma 3.12 for each of the pairs (G,A) determined by these figures. In each case, this is a fairly straightforward computer calculation that can be carried out using the the MAGMA algebra system. Details of our results are included in columns 4 and 5 of Figures 3.5, 3.6 and 3.7, where in column 5 the dimensions of the indecomposable summands of $A|_U$ are given. From this information, we obtain the following theorem.

Theorem 3.15 *Let (G,A) be a small pair with $G = m.G_n(q).r$ a bicyclic extension of a classical group of Lie type, and A an irreducible $\mathbb{F}_p(G)$-module for some prime p with $v_p(|G|) = 1$. Then $(G,p,\dim(A))$ appears in Figure 3.4.*

G	p	$\dim(A)$	G	p	$\dim(A)$
$PSL_3(2)$	3	3	$PSp_6(2)$	5	7
$PSL_3(2).2$	7	$3, 5, 7$	$PSp_6(2)$	7	7
$2.PSL_3(2).2$	7	$4, 6$	$2.PSp_6(2)$	5	7
$PSL_4(2)$	5	7	$2.PSp_6(2)$	7	7
$PSL_4(2).2$	5	7	$PSp_4(4).4$	17	18
$PSL_4(2).2$	7	7	$PSU_4(2)$	5	6
$2.PSL_4(2)$	5	7	$PSU_4(2).2$	5	6
$2.PSL_4(2).2$	5	7	$PSU_5(2).2$	11	10
$2.PSL_4(2).2$	7	$7, 8$	$PSU_3(4).4$	13	12
$6.PSL_3(4).2_1$	5	6	$PSU_3(3).2$	7	$6, 7$
$6.PSL_3(4).2_1$	7	6	$6_1.PSU_4(3)$	5	6
$PSp_4(2)$	5	5	$6_1.PSU_4(3)$	7	6
$2.PSp_4(2)$	5	5	$6.PSU_4(3).2_2$	5	6
$3.PSp_4(2)$	5	5	$6.PSU_4(3).2_2$	7	6
$6.PSp_4(2)$	5	5			

Figure 3.4: List of small (irreducible) modules for the classical groups.

G	p	$\dim(A)$	$\lvert \mathrm{Aut}_G(U) \rvert$	$A\lvert_U$
$PSL_3(2)$	3	3	2	3
$PSL_3(2).r, r = 1, 2$	7	$3, 5, 7$	$3r$	$3, 5, 7$
$2.PSL_3(2).r, r = 1, 2$	7	$4, 6$	$3r$	$4, 6$
$PSL_4(2).r, r = 1, 2$	5	7	4	$5 + 1 + 1$
$PSL_4(2).r, r = 1, 2$	7	7	$3r$	7
$2.PSL_4(2).r, r = 1, 2$	5	$7, 8$	4	$5 + 1 + 1, 4 + 4$
$2.PSL_4(2).r, r = 1, 2$	7	$7, 8$	$3r$	$7, 7 + 1$
$PSL_5(2).r, r = 1, 2$	31	29	5	29
$2.PSL_3(4).r, r = 1, 2_1, 2_2, 2_3$	7	10	$3r$	$7 + 3$
$4_1.PSL_3(4).r, r = 1, 2_3$	7	10	$3r$	$7 + 3$
$4_2.PSL_3(4).r, r = 1, 2_3$	7	10	$3r$	$7 + 3$
$6.PSL_3(4).r, r = 1, 2_1$	5	6	$2r$	$5 + 1$
$6.PSL_3(4).r, r = 1, 2_1$	7	6	$3r$	6
$6.PSL_3(4).r, r = 2_2, 2_3$	7	12	$3r$	$6 + 6$
$PSL_3(3).r, r = 1, 2$	13	$11, 13, 16$	$3r$	$11, 13, 13 + 3$

Figure 3.5: Small module candidates for $a.PSL_n(q).r$ $(n > 2)$.

G	p	$\dim(A)$	$\lvert \mathrm{Aut}_G(U) \rvert$	$A\lvert_U$
$PSp_4(2)$	5	5	4	5
$PSp_4(2).r, r = 1, 2$	5	8	4	$5 + 3$
$2.PSp_4(2)$	5	$5, 8, 8$	4	$5, 5 + 3, 4 + 4$
$3.PSp_4(2)$	5	$5, 6, 8$	4	$5, 3 + 3, 5 + 3$
$6.PSp_4(2)$	5	$5, 8$	4	$5, 5 + 3$
$PSp_6(2)$	5	7	4	$5 + 1 + 1$
$PSp_6(2)$	7	7	6	7
$2.PSp_6(2)$	5	$7, 8$	4	$5 + 1 + 1, 4 + 4$
$2.PSp_6(2)$	7	$7, 8$	6	$7, 4 + 4$
$PSp_4(4).r, r = 1, 2, 4$	17	18	$4r$	$17 + 1$
$PSp_6(3)$	7	13	6	$7 + 6$
$PSp_6(3)$	13	13	6	13
$2.PSp_6(3)$	13	14	6	$13 + 1$

Figure 3.6: Small module candidates for $a.PSp_n(q).r$ $(n \geq 2)$.

G	p	$\dim(A)$	$\lvert \mathrm{Aut}_G(U) \rvert$	$A\lvert_U$
$PSU_4(2).r, r = 1, 2$	5	6	4	$5 + 1$
$2.PSU_4(2).r, r = 1, 2$	5	$6, 8$	4	$5 + 1, 4 + 4$
$PSU_5(2).r, r = 1, 2$	11	10	$5r$	10
$PSU_3(4).r, r = 1, 2, 4$	13	12	$3r$	12
$PSU_3(3).r, r = 1, 2$	7	$6, 7$	$3r$	$6, 7$
$6_1.PSU_4(3)$	5	6	4	$5 + 1$
$6_1.PSU_4(3)$	7	6	3	6
$6_1.PSU_4(3).2_1$	7	12	6	$7 + 5$
$6.PSU_4(3).2_2$	5	6	4	$5 + 1$
$6.PSU_4(3).2_2$	7	6	6	6

Figure 3.7: Small module candidates for $a.PSU_n(q).r$ $(n > 2)$.

Chapter 4

Centralisers of Subsystems of Fusion Systems

This chapter is concerned with developing (as far as possible) the notion of a 'centraliser' of a subsystem $\mathcal{E}$ inside a fusion system $\mathcal{F}$ on a finite p-group S. As we shall see, fairly restrictive conditions need to placed upon $\mathcal{E}$ in order to be able to define such an object. That said, when it exists, we show that many of the properties of centralisers which hold for finite groups also hold for fusion systems.

Given a saturated fusion system $\mathcal{F}$ on a finite p-group S and a subsystem $\mathcal{E}$ of $\mathcal{F}$ on $T \leq S$, we define

$$C_S(\mathcal{E}) := \{g \in S \mid \mathcal{E} \subseteq C_{\mathcal{F}}(\langle g \rangle)\}$$

to be the *S-centraliser of $\mathcal{E}$*. We observe immediately that $C_S(\mathcal{E}) \subseteq C_S(T)$, and (by Lemma 1.34) that $C_S(\mathcal{F}) = Z(\mathcal{F})$. Of course, there is no reason at this stage to think that $C_S(\mathcal{E})$ is a group. Indeed, in Section 4.3 we give an example which proves that this is not always the case, even when $\mathcal{E}$ is a weakly normal subsystem of $\mathcal{F}$.

When $\mathcal{E}$ is normal it turns out that $C_S(\mathcal{E})$ is always a group, since it can be characterised in terms of certain constrained fusion systems. This characterisation is due to Aschbacher and appears in [3, Chapter 6]. We present it in Section 4.2 (Theorem 4.15), omitting certain technical details which involve local characterisations of normality. Some sections of the argument have been improved, most notably the proof that $C_S(\mathcal{E})$ is strongly $\mathcal{F}$-closed (Theorem 4.13). The $\mathcal{F}$-centraliser $C_{\mathcal{F}}(\mathcal{E})$ has a very nice definition which we describe in Section 4.2.2. Using a certain consequence of the Classification of Finite Simple Groups, Theorem 4.16, we prove that the hyperfocal subgroup $\mathfrak{hyp}(C_{\mathcal{F}}(T))$ of $C_{\mathcal{F}}(T)$ is always contained in $C_S(\mathcal{E})$ when $(\mathcal{F}, \mathcal{E})$ is induced by a normal pair (G, H) of finite groups. This implies that there is a unique subsystem on $C_S(\mathcal{E})$ contained in $C_{\mathcal{F}}(T)$ which we denote by $C_{\mathcal{F}}(\mathcal{E})$. Aschbacher has already handled the exotic case, constructing $C_{\mathcal{F}}(\mathcal{E})$ using his theory of normal maps

(see [2, Section 7]). His results do imply that $\mathfrak{hyp}(C_{\mathcal{F}}(T)) \leq C_S(\mathcal{E})$ whenever $(\mathcal{F}, \mathcal{E})$ is a normal pair on (S, T) (Theorem 4.19), but the author feels that there should be a short, direct proof of this statement despite having been unable to find one.

When $\mathcal{E}$ is weakly normal, there are very simple (topological) conditions which one can impose in order to guarantee that $C_S(\mathcal{E})$ is a group, involving the association of linking systems to $\mathcal{E}$ and $\mathcal{F}$. This extra machinery allows one to identify $C_S(\mathcal{E})$ as the set of elements which live in the kernel of the 'conjugation' action described in Section 1.5.2 (Theorem 4.22), making the both the facts that it is a group and strongly $\mathcal{F}$-closed plain. Since T is $\mathcal{F}$-quasicentric under these hypotheses, $C_{\mathcal{F}}(\mathcal{E})$ is a trivial fusion system in this case. The use of this approach is blighted by the fact that we do not yet have a good understanding of when weakly normal linking systems exist. In other words, the necessary obstruction theory is yet to be developed.

We end the chapter somewhat speculatively by considering some open problems. Firstly, we ask whether $C_S(\mathcal{E})$ is always equal to $C_S(H)$ when the normal pair $(\mathcal{F}, \mathcal{E})$ is induced by a normal pair (G, H) of finite groups with $O_{p'}(H) = 1$. The case where $G = H$ is exactly the Z^*-Theorem for finite groups. When p is odd and $O_p(H) = 1$, this question has an affirmative answer (Theorem 4.27), but when $p = 2$ much less is clear. Secondly, we look briefly at the prospect of defining *normalisers* of subsystems.

4.1 Constrained Fusion Systems

We begin by developing the machinery required to prove that $C_S(\mathcal{E})$ is a group (Theorem 4.15) whenever $T \leq S$ are finite p-groups and $(\mathcal{F}, \mathcal{E})$ is a strongly normal pair of fusion systems on (S, T).

4.1.1 The Subsystems $\mathcal{E}(U)$ and $\mathcal{D}(U)$

Let $\mathcal{F}$ be a fusion system on a finite p-group S and let $\mathcal{E}$ be a saturated subsystem of $\mathcal{F}$ on $T \leq S$ such that T strongly $\mathcal{F}$-closed. We introduce two fundamental subsystems of $\mathcal{F}$, first considered by Aschbacher, which will play prominent roles in Section 4.2. For each fully $\mathcal{F}$-normalised subgroup U of T, write $V := UC_T(U)$ and $\bar{V} := VC_S(V)$. Define

$$\mathcal{D}(U) := N_{N_{\mathcal{F}}(\bar{V})}(U) \text{ and } \mathcal{E}(U) := N_{N_{\mathcal{E}}(V)}(U).$$

The next lemma implies that $\mathcal{E}(U)$ and $\mathcal{D}(U)$ may be regarded as the largest canonically definable constrained fusion systems on $N_T(U)$ and $N_S(U)$ contained in $\mathcal{E}$ and $\mathcal{F}$ respectively.

Lemma 4.1 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let $\mathcal{E}$ be a saturated subsystem of $\mathcal{F}$ on $T \leq S$ such that T is strongly $\mathcal{F}$-closed. For each fully $\mathcal{F}$-normalised subgroup U of T, the following hold:*

(a) $\mathcal{D}(U)$ is a fusion system on $N_S(U)$ and $\mathcal{E}(U)$ is a fusion system on $N_T(U)$;

(b) $\mathcal{D}(U)$ and $\mathcal{E}(U)$ are both saturated, and V and $\bar{V}$ are normal $\mathcal{D}(U)$- and $\mathcal{E}(U)$-centric subgroups of $\mathcal{D}(U)$ and $\mathcal{E}(U)$ respectively;

(c) $\mathcal{E}(U)$ is a subsystem of $\mathcal{D}(U)$; and

(d) if $(\mathcal{F}, \mathcal{E})$ is a weakly normal pair then so is $(\mathcal{D}(U), \mathcal{E}(U))$.

Proof. We first prove (a). Since T is strongly $\mathcal{F}$-closed, $V = UC_T(U)$ is a normal subgroup of $N_S(U)$. Now $\bar{V} = VC_S(V) \trianglelefteq N_S(U)$ so that $N_S(U) \leq N_S(\bar{V})$ and $N_{N_S(\bar{V})}(U) = N_S(U) \cap N_S(\bar{V}) = N_S(U)$. Clearly $N_T(U) \leq N_T(V)$ so that $N_{N_T(V)}(U) = N_T(V) \cap N_T(U) = N_T(U)$, proving (a).

To see (b), first notice that we may write

$$\mathcal{D}(U) = N_{\mathcal{F}}^{J}(\bar{V}) \text{ and } \mathcal{E}(U) = N_{\mathcal{E}}^{K}(V)$$

where $J := \{\alpha \in \mathrm{Aut}_{\mathcal{F}}(\bar{V}) \mid \alpha|_U \in \mathrm{Aut}(U)\}$ and $K := \{\alpha \in \mathrm{Aut}_{\mathcal{E}}(V) \mid \alpha|_U \in \mathrm{Aut}(U)\}$. Hence to prove that $\mathcal{D}(U)$ and $\mathcal{E}(U)$ are saturated, it suffices by Theorem 1.31 to show that $\bar{V}$ is fully J-normalised and V is fully K-normalised. Since T is strongly $\mathcal{F}$-closed,

$$N_S^{J^\phi}(\bar{V}\phi) = N_S(U\phi) \text{ and } N_T^{K^\psi}(V\psi) = N_T(U\psi) = N_S(U\psi) \cap T$$

for each $\phi \in \mathrm{Hom}_{\mathcal{F}}(\bar{V}, S)$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(V, S)$. Hence both of these statements follow from the fact that U is fully $\mathcal{F}$-normalised. The second part of (b) is clear.

Next we prove (c). Clearly $N_T(U) \leq N_S(U)$ so it suffices to show that each morphism in $\mathcal{E}(U)$ is also a morphism in $\mathcal{D}(U)$. To this end, let $P \leq N_T(U)$ and $\varphi \in \mathrm{Hom}_{\mathcal{E}(U)}(P, N_T(U))$. We need to show that φ extends to a map $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(P\bar{V}, S)$ with $\bar{\varphi}|_U \in \mathrm{Aut}(U)$, since any map with this property clearly also fixes $\bar{V}$. By definition, φ extends to a map $\tilde{\varphi} \in \mathrm{Hom}_{\mathcal{E}}(PV, N_T(U))$ with $\tilde{\varphi}|_U \in \mathrm{Aut}(U)$. Hence we may replace P by PV and φ by $\tilde{\varphi}$ if necessary and assume that $V \leq P$.

Suppose the identity

$$\mathrm{Aut}_{C_S(V)}(P) \leq O_p(\mathrm{Aut}_{N_{\mathcal{F}}(U)}(P))$$

holds. By Proposition 1.20 (c) there is some $\beta \in \mathrm{Hom}_{N_{\mathcal{F}}(U)}(N_{N_S(U)}(P\varphi), N_S(U))$ such that $P' := P\varphi\beta$ is fully $N_{\mathcal{F}}(U)$-normalised. P' is also fully $N_{\mathcal{F}}(U)$-automised so that $O_p(\mathrm{Aut}_{N_{\mathcal{F}}(U)}(P')) \leq \mathrm{Aut}_{N_S(U)}(P')$. Now

$$\mathrm{Aut}_{C_S(V)}(P)^{\varphi\beta} \leq O_p(\mathrm{Aut}_{N_{\mathcal{F}}(U)}(P))^{\varphi\beta} = O_p(\mathrm{Aut}_{N_{\mathcal{F}}(U)}(P')) \leq \mathrm{Aut}_{N_S(U)}(P').$$

This means that $C_S(V) \leq N_{\varphi\beta}$ so there is some $\psi \in \mathrm{Hom}_{N_{\mathcal{F}}(U)}(PC_S(V), N_S(U))$ such that $\psi|_P = \varphi \circ \beta|_{P\varphi}$. Set $\bar{\varphi} := \psi \circ \beta^{-1} \in \mathrm{Hom}_{N_{\mathcal{F}}(U)}(PC_S(V), N_S(U))$. Since $\bar{\varphi}|_P = \varphi$, $\bar{\varphi}$ is a map with the required properties.

It remains to prove that $\mathrm{Aut}_{C_S(V)}(P) \leq O_p(\mathrm{Aut}_{N_{\mathcal{F}}(U)}(P))$. Since $P \leq N_T(U)$ and T is strongly $\mathcal{F}$-closed, $[P, C_S(V)] \leq C_S(V) \cap T = C_T(V) = Z(V) \leq P$, since V is T-centric. Hence $C_S(V) \leq N_{N_S(U)}(P)$ and $\mathrm{Aut}_{C_S(V)}(P)$ must centralise the factors in the $\mathrm{Aut}_{N_{\mathcal{F}}(U)}(P)$ invariant chain

$$1 \leq Z(V) \leq P.$$

Thus Corollary 1.7 implies that $\mathrm{Aut}_{C_S(V)}(P) \leq O_p(\mathrm{Aut}_{N_{\mathcal{F}}(U)}(P))$, as needed.

Finally, we prove that (d) holds. To do this, we use the characterisation of weak normality provided by Lemma 1.54. Let $\varphi \in \mathrm{Hom}_{\mathcal{E}(U)}(P, N_T(U))$ and $\alpha \in \mathrm{Hom}_{\mathcal{D}(U)}(N_T(U), N_S(U))$. There exist elements

$$\bar{\varphi} \in \mathrm{Hom}_{\mathcal{E}}(PV, T) \text{ and } \bar{\alpha} \in \mathrm{Hom}_{\mathcal{F}}(N_T(U)VC_S(V), S),$$

(where $V = UC_T(U)$) extending φ and α respectively and such that $U\bar{\varphi} = U = U\bar{\alpha}$. Hence $\bar{\alpha}^{-1}|_{P\alpha V} \circ \bar{\varphi} \circ \bar{\alpha}|_{PV}$ lies in $\mathrm{Hom}_{\mathcal{E}}(P\alpha V, N_T(U))$ (since $(\mathcal{F}, \mathcal{E})$ is a weakly normal pair) and extends $\alpha^{-1} \circ \varphi \circ \alpha \in \mathrm{Hom}_{\mathcal{E}(U)}(P\alpha, N_T(U))$, as needed. □

One of the major advantages in working with normal pairs of fusion systems $(\mathcal{F}, \mathcal{E})$ is that they behave very well in the case where $\mathcal{F}$ and $\mathcal{E}$ are both constrained, as the next result shows. Recall that by Theorem 1.38, a constrained fusion system $\mathcal{F}$ is realisable as the fusion system of a p-constrained group G.

Theorem 4.2 *Let $T \leq S$ be finite p-groups and $(\mathcal{F}, \mathcal{E})$ be a normal pair of saturated fusion systems on (S, T). Assume further that $\mathcal{F}$ is constrained. Let G be p-constrained group with $O_{p'}(G) = 1$ which realises $\mathcal{F}$. Then there exists a p-constrained normal subgroup H of G which realises $\mathcal{E}$.*

Proof. This is shown in [2, Theorem 1]. □

Since the converse to Theorem 4.2 holds by Proposition 1.55, the lattice of normal subsystems of $\mathcal{F}$ is in one to one correspondence with the lattice of normal subgroups of G. This has a large number of applications to the theory. For example, we can now strengthen Lemma 4.1 (d) to prove an analagous statement for normal pairs $(\mathcal{F}, \mathcal{E})$.

Theorem 4.3 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let $\mathcal{E}$ be a saturated subsystem of $\mathcal{F}$ on $T \leq S$ such that T is strongly $\mathcal{F}$-closed. Let U be a fully $\mathcal{F}$-normalised subgroup of T. If $(\mathcal{F}, \mathcal{E})$ is a normal pair then so is $(\mathcal{D}(U), \mathcal{E}(U))$.*

Proof. By Lemma 4.1, $(\mathcal{D}(U), \mathcal{E}(U))$ is a weakly normal pair so it remains to check that condition (c) in Definiton 1.53 holds. Let U be a counterexample with $n := |T : U|$ chosen as small as possible. If $n = 1$ then $U = T$, $(\mathcal{D}(T), \mathcal{E}(T)) = (N_{\mathcal{F}}(TC_S(T)), N_{\mathcal{E}}(T))$ and since $\mathrm{Aut}_{\mathcal{E}(T)}(T) = \mathrm{Aut}_{\mathcal{E}}(T)$ and $\mathrm{Aut}_{\mathcal{D}(T)}(TC_S(T)) = \mathrm{Aut}_{\mathcal{F}}(TC_S(T))$ the fact that $(\mathcal{D}(T), \mathcal{E}(T))$ is a normal pair follows from the fact that $(\mathcal{F}, \mathcal{E})$ is. Hence we may assume that $n > 1$. Write $R := N_T(U)$ and fix an element $\alpha \in \mathrm{Aut}_{\mathcal{E}(U)}(R)$. Let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, T)$ be chosen so that $R\varphi$ is fully $\mathcal{F}$-normalised and φ extends to a map $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_S(R), N_S(R\varphi))$ by Proposition 1.20 (c). Write $\beta := \varphi^{-1} \circ \alpha \circ \varphi \in \mathrm{Aut}_{\mathcal{F}}(R\varphi)$, so β extends to an element $\bar{\beta} \in \mathrm{Aut}_{\mathcal{F}}(R\varphi C_S(R\varphi))$ (since $\mathcal{F}$ is saturated). Since T is strongly $\mathcal{F}$-closed, $(R\varphi C_S(R\varphi) \cap T)\beta = R\varphi C_T(R\varphi)$ which implies $\bar{\beta} \in \mathrm{Mor}(\mathcal{D}(R\varphi))$.

By the minimality of n, $(\mathcal{D}(R\varphi), \mathcal{E}(R\varphi))$ is a normal pair of fusion systems so that by Theorem 4.3 there exists a normal pair $(G(R\varphi), H(R\varphi))$ of finite groups which realises $(\mathcal{D}(R\varphi), \mathcal{E}(R\varphi))$. Since $U \leq T$ is fully $\mathcal{F}$-normalised, it is also fully $\mathcal{E}$-normalised so that $UC_T(U)$ and $R = N_T(U)$ are $\mathcal{E}$-centric by Lemma 1.23. Hence $R\varphi$ is a normal $\mathcal{E}(R\varphi)$-centric subgroup, so that $C_H(R\varphi) = Z(R\varphi)$. Since $\beta \in \mathrm{Mor}(\mathcal{E}(R\varphi))$ there is $h \in H(R\varphi)$ such that $\beta = c_h|_{R\varphi}$. We have

$$[h, C_S(R\varphi)] \leq C_H(R\varphi) = Z(R\varphi),$$

so $\bar{\beta} = c_h|_{R\varphi C_S(R\varphi)}$ satisfies $[\bar{\beta}, C_S(R\varphi)] \leq Z(R\varphi)$. Define

$$\bar{\alpha} := \bar{\varphi}|_{RC_S(R)} \circ \bar{\beta} \circ \varphi^{-1}|_{R\varphi C_S(R\varphi)}$$

and note that $\bar{\alpha}$ extends α and satisfies $[\bar{\alpha}, C_S(R)] \leq Z(R)$. In particular $\mathcal{E}(U)$ is a normal subsystem of $\mathcal{D}(U)$, a contradiction. □

Combining Theorems 4.2 and 4.3 yields the following result.

Corollary 4.4 *Let $T \leq S$ be finite p-groups and $(\mathcal{F}, \mathcal{E})$ be a normal pair of saturated fusion systems on (S, T). Let U be a $\mathcal{F}$-normalised subgroup of T. There exist finite groups $G(U)$ and $H(U)$ with $H(U) \trianglelefteq G(U)$ such that*

$$\mathcal{D}(U) = \mathcal{F}_{N_S(U)}(G(U)) \text{ and } \mathcal{E}(U) = \mathcal{F}_{N_T(U)}(H(U)).$$

We will end this section by quoting a deep theorem of Michael Aschbacher which says that the set of $\mathrm{Aut}_{\mathcal{F}}(T)$-conjugates of fully $\mathcal{F}$-normalised, T-centric subgroups form a conjugation family for $\mathcal{E}$ when $T \leq S$ are finite p-groups and $(\mathcal{F}, \mathcal{E})$ is a normal pair of fusion systems on (S, T). To give this result some perspective, we first prove that the set of subgroups to which it refers is a refinement of the set of all $\mathcal{E}$-centric subgroups, which we know form a conjugation family by Alperin's Theorem.

Lemma 4.5 *Let $T \leq S$ be finite p-groups and let $(\mathcal{F}, \mathcal{E})$ be a weakly normal pair of saturated fusion systems on (S, T). The following hold:*

(a) If R is a fully $\mathcal{F}$-normalised and T-centric subgroup of S then $C_T(R\varphi) \leq R\varphi$ for all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, S)$.

(b) $R \leq T$ is $\mathcal{E}$-centric if and only if $C_T(R\varphi) \leq R\varphi$ for all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, S)$.

Proof. Let R be a fully $\mathcal{F}$-normalised and T-centric subgroup of S and let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, S)$. Since R is also fully $\mathcal{F}$-centralised by Proposition 1.20, $\psi := \varphi^{-1}|_{R\varphi}$ extends to a map $\bar{\psi} \in \mathrm{Hom}_{\mathcal{F}}(R\varphi C_S(R\varphi), S)$ since $\mathcal{F}$ is saturated. Since T is strongly $\mathcal{F}$-closed, we have

$$|C_T(R\varphi)| = |C_T(R\varphi)\bar{\psi}| \leq |C_T(R)| = |Z(R)| = |Z(R\varphi)|,$$

proving (a).

We prove the 'if' direction for part (b) first. Let $R \leq T$ and suppose that $C_T(R\varphi) \leq R\varphi$ for all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, S)$. Certainly $C_T(R\varphi) \leq R\varphi$ for all $\varphi \in \mathrm{Hom}_{\mathcal{E}}(R, T)$, so that R is $\mathcal{E}$-centric. Conversely suppose that R is $\mathcal{E}$-centric and pick some $R_0 \leq S$ and $\varphi \in \mathrm{Iso}_{\mathcal{F}}(R, R_0)$ such that R_0 is fully $\mathcal{F}$-normalised. T is strongly $\mathcal{F}$-closed so $R_0 \leq T$ and since $\mathcal{E}$ is $\mathcal{F}$-invariant there exist $\varphi_0 \in \mathrm{Hom}_{\mathcal{E}}(R, T)$ and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ such that $\varphi = \varphi_0 \circ \alpha$. Clearly $R\varphi_0$ is $\mathcal{E}$-centric and since $\mathrm{Aut}(\mathcal{E})$ permutes the set of $\mathcal{E}$-centric subgroups also $R_0 = R\varphi_0\alpha$ is $\mathcal{E}$-centric. Hence by (a), $C_T(R\psi) \leq R\psi$ for all $\psi \in \mathrm{Hom}_{\mathcal{F}}(R, S)$ and this completes the proof. □

Aschbacher's result is stated below. We omit a proof due to the technical and lengthy nature of the arguments involved.

Theorem 4.6 (Aschbacher) *Let $T \leq S$ be finite p-groups and $(\mathcal{F}, \mathcal{E})$ be a normal pair of saturated fusion systems on (S, T). Then*

$$\mathcal{E} = \langle \mathrm{Aut}_{\mathcal{E}(U)}(U)^{\varphi|_U} \mid U \text{ is fully } \mathcal{F}\text{-normalised and } T\text{-centric}, \ \varphi \in \mathrm{Aut}_{\mathcal{F}}(T) \rangle_T.$$

Proof. This is shown in [2, Theorem 3]. □

4.1.2 The Subsystem $\mathcal{D}(U, X)$

In the proof of Theorem 4.15, it will be important to keep track of subgroups X which centralise $\mathcal{E}$ and fully $\mathcal{F}$-normalised T-centric subgroups U of S simultaneously. It is unfortunate that for this purpose, the subsystems $\mathcal{E}(U)$ and $\mathcal{D}(U)$ will not suffice. To overcome this problem, we adapt the definition of $\mathcal{D}(U)$ as follows.

Definition 4.7 Let $T \leq S$ be finite p-groups and $(\mathcal{F}, \mathcal{E})$ be a weakly normal pair of fusion systems on (S, T). Suppose that $X \leq C_S(T)$ satisfies $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$. Then for each fully $\mathcal{F}$-normalised, T-centric subgroup U of T, write $U_X := UXC_S(UX)$ and set

$$\mathcal{D}(U, X) := N_{\mathcal{F}}^{L}(U_X),$$

where $L := \{\varphi \in \mathrm{Aut}_{\mathcal{F}}(U_X) \mid \varphi|_{UX} \in \mathrm{Aut}_{\mathcal{F}}(UX)\}$.

Observe that $\mathcal{D}(U, 1) = \mathcal{D}(U)$ when U is T-centric. The next result shows that $\mathcal{E}(U)$ is a subsystem of $\mathcal{D}(U, X)$ and provides us with an analogue of Lemma 4.1 for the pair $(\mathcal{D}(U, X), \mathcal{E}(U))$.

Theorem 4.8 *Let $T \leq S$ be finite p-groups and $(\mathcal{F}, \mathcal{E})$ be a weakly normal pair of fusion systems on (S, T). Suppose that $X \leq C_S(T)$ satisfies $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$. Then for each fully $\mathcal{F}$-normalised, T-centric subgroup U of T for which UX is fully $N_{\mathcal{F}}(U)$-normalised the following hold:*

(a) $\mathcal{D}(U, X)$ is a saturated subsystem of $N_{\mathcal{F}}(U)$ on $N_S(UX)$ and U_X is a normal $\mathcal{D}(U, X)$-centric subgroup of $\mathcal{D}(U, X)$;

(b) $\mathcal{E}(U)$ is a subsystem of $\mathcal{D}(U, X)$; and

(c) if $(\mathcal{F}, \mathcal{E})$ is a normal pair then so is $(\mathcal{D}(U, X), \mathcal{E}(U))$.

Proof. Observe that by the Dedekind Identity,

$$U \leq T \cap UX \leq T \cap U_X \leq T \cap UC_S(U) = U(T \cap C_S(U)) = U.$$

We first prove (a). By definition, each element of $\mathrm{Mor}(\mathcal{D}(U,X))$ extends to a map ψ which leaves UX invariant. Since T is strongly closed,

$$U\psi = (UX)\psi \cap T = UX \cap T = U,$$

so that $\mathcal{D}(U,X) \subseteq N_{\mathcal{F}}(U)$. Also, $U_X \trianglelefteq N_S(UX)$ so we must have

$$N^L_{N_S(U)}(U_X) = N_S(U_X) \cap N_S(UX) = N_S(UX).$$

To see that $\mathcal{D}(U,X)$ is saturated, it suffices (by Theorem 1.31) to show that U_X is fully L-normalised in $N_{\mathcal{F}}(U)$. Let $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}(U)}(U_X, N_S(U))$. Then $N^L_{N_S(U)}(U_X) = N_S(UX)$ and $N^{L^\varphi}_{N_S(U)}(U_X\varphi) = N_{N_S(U)}((UX)\varphi)$ so saturation follows from the fact that UX is fully $N_{\mathcal{F}}(U)$-normalised. It is clear that U_X is a normal $\mathcal{D}(U,X)$-centric subgroup of $\mathcal{D}(U,X)$ and this completes the proof of (a).

Next we prove (b). Let $P \leq N_T(U)$ and $\varphi \in \mathrm{Hom}_{\mathcal{E}(U)}(P, N_T(U))$. We need to show that φ extends to a map $\bar{\varphi}$ such that $[UX, \bar{\varphi}] \leq UX$. By Corollary 4.4,

$$(\mathcal{D}(U), \mathcal{E}(U)) = (\mathcal{F}_{N_S(U)}(G(U)), \mathcal{F}_{N_T(U)}(H(U))$$

for some normal pair $(G(U), H(U))$ of finite groups. Hence there is $g \in H(U)$ such that $\varphi = c_g \in \mathrm{Hom}_{G(U)}(P, N_T(U))$ and

$$[UX, \varphi] = [UX, c_g] \leq UC_S(U) \cap H(U) = UC_S(U) \cap T = U \leq UX,$$

as required.

An identical argument to that which proves Lemma 4.1 (d) shows that the pair $(\mathcal{D}(U,X), \mathcal{E}(U))$ is weakly normal. More precisely, consider a pair of morphisms $\varphi \in \mathrm{Hom}_{\mathcal{E}(U)}(P, N_T(U))$ and $\alpha \in \mathrm{Hom}_{\mathcal{D}(U,X)}(N_T(U), N_S(UX))$. There exist elements

$$\bar{\varphi} \in \mathrm{Hom}_{\mathcal{E}}(PU, T) \text{ and } \bar{\alpha} \in \mathrm{Hom}_{\mathcal{F}}(N_T(U)U_X, S)$$

extending φ and α respectively such that $U\bar{\varphi} = U = U\bar{\alpha}$. Hence $\bar{\alpha}^{-1}|_{P\alpha U} \circ \bar{\varphi} \circ \bar{\alpha}|_{PU}$ lies in $\mathrm{Hom}_{\mathcal{E}}(P\alpha U, N_T(U))$ and extends $\alpha^{-1} \circ \varphi \circ \alpha \in \mathrm{Hom}_{\mathcal{E}(U)}(P\alpha, N_T(U))$, as needed. Finally, observe that

$$[N_{H(U)}(N_T(U)), C_S(N_T(U))] \leq H(U) \cap C_S(N_T(U)) = C_{H(U)}(N_T(U)) = Z(N_T(U)),$$

and since there is $g \in N_{H(U)}(N_T(U))$ such that $\varphi = c_g$ for each $\varphi \in \mathrm{Aut}_{\mathcal{E}(U)}(N_T(U))$, we must have that $(\mathcal{D}(U,X), \mathcal{E}(U))$ is a normal pair. This completes the proof of (c). $\square$

Theorem 4.8 naturally gives rise to the following analogue of Corollary 4.4 for the pair $(\mathcal{D}(U,X),\mathcal{E}(U))$.

Corollary 4.9 *Let $T \leq S$ be finite p-groups and $(\mathcal{F},\mathcal{E})$ be a weakly normal pair of fusion systems on (S,T). Suppose that $X \leq C_S(T)$ satisfies $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$. Then for each fully $\mathcal{F}$-normalised, T-centric subgroup U of T for which UX is fully $N_{\mathcal{F}}(U)$-normalised there exist finite groups $G(U,X)$ and $H(U,X)$ with $H(U,X) \trianglelefteq G(U,X)$ such that*

$$\mathcal{D}(U,X) = \mathcal{F}_{N_S(UX)}(G(U,X)) \text{ and } \mathcal{E}(U) = \mathcal{F}_{N_T(U)}(H(U,X)).$$

Furthermore, $H(U)$ is a subgroup of $G(U,X)$ and

$$C_{G(U,X)}(H(U)) \cap N_S(UX) = C_{G(U,X)}(H(U,X)) \cap N_S(UX).$$

Proof. The first statement is an immediate consequence of Theorem 4.2. The second is shown in [3, Lemma 6.3.5]. □

4.2 Centralisers of Normal Subsystems

4.2.1 The S-centraliser $C_S(\mathcal{E})$

Let $T \leq S$ be finite p-groups and $(\mathcal{F},\mathcal{E})$ be a normal pair of fusion systems on (S,T). We are now almost in a position to prove that the set

$$C_S(\mathcal{E}) := \{g \in C_S(T) \mid \mathcal{E} \subseteq C_{\mathcal{F}}(\langle g \rangle)\})$$

is (1) a group and (2) strongly $\mathcal{F}$-closed. We prove (2) in Theorem 4.13 first and use this to deduce (1) in Theorem 4.15. The first step is to gain some control over the way in which groups which centralise $\mathcal{E}$ behave under $\mathcal{F}$-conjugation. The next lemma shows that the obstruction to $C_S(\mathcal{E})$ being strongly $\mathcal{F}$-closed is that not all maps defined on subgroups of $C_S(\mathcal{E})$ extend to maps on T.

Lemma 4.10 *Let $T \leq S$ be finite p-groups and $(\mathcal{F},\mathcal{E})$ be a weakly normal pair of fusion systems on (S,T). Let $X \leq C_S(T)$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(X,S)$. The following hold:*

(a) If φ extends to a map $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(XT,S)$ then $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$ if and only if $\mathcal{E} \subseteq C_{\mathcal{F}}(X\varphi)$.

(b) If $X\varphi$ is fully $\mathcal{F}$-centralised then $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$ if and only if $\mathcal{E} \subseteq C_{\mathcal{F}}(X\varphi)$. In particular, if $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$ then there is some fully $\mathcal{F}$-normalised $\mathcal{F}$-conjugate, Y of X such that $\mathcal{E} \subseteq C_{\mathcal{F}}(Y)$.

(c) If $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$ then $N_T(X\varphi) = C_T(X\varphi)$. In particular, $T \cap (X\varphi C_T(X\varphi))\gamma = C_T(X\varphi)\gamma$ for each $\gamma \in \mathrm{Hom}_{\mathcal{F}}(X\varphi C_T(X\varphi), S)$.

Proof. First we prove (a). Let $\alpha \in \mathrm{Hom}_{\mathcal{E}}(P, T)$. If α is a morphism in $C_{\mathcal{F}}(X)$ then α extends to a map $\bar{\alpha} \in \mathrm{Hom}_{\mathcal{F}}(PX, TX)$ with $\bar{\alpha}|_X = \mathrm{Id}_X$. Hence $\bar{\varphi}^{-1} \circ \bar{\alpha} \circ \bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}((PX)\bar{\varphi}, S)$ extends $\beta := \bar{\varphi}|_{P\varphi}^{-1} \circ \alpha \circ \bar{\varphi}|_{P\alpha}$ and restricts to the identity on $X\varphi$ so that β is a morphism in $C_{\mathcal{F}}(X\varphi)$. Now $\varphi|_T \in \mathrm{Aut}_{\mathcal{F}}(T) \leq \mathrm{Aut}(\mathcal{E})$ (since $\mathcal{E}$ is weakly normal in $\mathcal{F}$), so β is also a morphism in $\mathcal{E}$ and $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$ if and only if $\mathcal{E} \subseteq C_{\mathcal{F}}(X\varphi)$, as needed.

Next, we prove (b). Note that since $\mathcal{F}$ is saturated, there is $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ extending φ. Since $T \leq C_S(X) \leq N_\varphi$, the first statement follows from (a). If $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$ and $\alpha \in \mathrm{Hom}_{\mathcal{F}}(X, S)$ is chosen so that $X\alpha$ is fully $\mathcal{F}$-normalised, then by Proposition 1.20 $X\alpha$ is also fully $\mathcal{F}$-centralised. Thus the second statement follows from the first.

Finally, let Y be a fully $\mathcal{F}$-normalised $\mathcal{F}$-conjugate of $X\varphi$ satisfying $\mathcal{E} \subseteq C_{\mathcal{F}}(Y)$. By Proposition 1.20, there exists a map $\beta \in \mathrm{Hom}_{\mathcal{F}}(N_S(X\varphi), N_S(Y))$ such that $X\varphi\beta = Y$. Now $N_T(X\varphi)\beta \leq N_T(Y) = C_T(Y)$ and on composing both sides of this equation with β^{-1}, the first statement in (c) follows. The second statement follows immediately from the first since T is strongly $\mathcal{F}$-closed. □

Thus, a consequence of Lemma 4.10 (a) is that $C_S(\mathcal{E})$ is strongly $\mathcal{F}$-closed whenever T is a normal subgroup of $\mathcal{F}$. To handle the general case, we will need to proceed inductively over the index of $C_T(X)$ for each subgroup X which centralises $\mathcal{E}$. To do this, we will show that all such subgroups X centralise all T-centric, fully $\mathcal{F}$-normalised subgroups U. The first step is to find a condition which ensures that X centralises $H(U)$ for any given pair (X, U).

Definition 4.11 Let $T \leq S$ be finite p-groups and $(\mathcal{F}, \mathcal{E})$ be a normal pair of fusion systems on (S, T). For each $X, U \leq S$, we say that (X, U) *satisfies* † if

(i) $X \leq C_S(T)$ and $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$;

(ii) U is fully $\mathcal{F}$-normalised and T-centric; and

(iii) for all $Y \in X^{\mathcal{F}}$ with $|C_T(Y)| > |U|$, $\mathcal{E} \subseteq C_{\mathcal{F}}(Y)$.

The next result implies that X centralies $H(U)$ if (X, U) satisfies †. Since the argument given in [3] is fairly long and technical we will not reproduce it here.

Lemma 4.12 *Let $T \leq S$ be finite p-groups and $(\mathcal{F}, \mathcal{E})$ be a normal pair of fusion systems on (S, T). If (X, U) satisfies † then $X \leq C_{N_S(U)}(H(U))$.*

Proof. This is shown in [3, Theorem 6.5]. □

We are finally in a position to show that $C_S(\mathcal{E})$ is strongly $\mathcal{F}$-closed. The argument we present, while based on that found in [3, Theorem 6.6], is cleaner and more direct.

Theorem 4.13 *Let $T \leq S$ be finite p-groups and let $(\mathcal{F}, \mathcal{E})$ be a normal pair of saturated fusion systems on (S, T). Then $C_S(\mathcal{E})$ is strongly $\mathcal{F}$-closed.*

Proof. In what follows, write $\mathcal{C}$ for the set of all subgroups X of S which satisfy $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$. It will be enough to show that $\mathcal{C}$ is closed under $\mathcal{F}$-conjugacy. To this end, assume that there is some $Y \in \mathcal{C}$ such that $Y^{\mathcal{F}} \nsubseteq \mathcal{C}$. We may choose Y to be fully $\mathcal{F}$-normalised by Lemma 4.10 (b). We claim that there exists a morphism $\gamma \in \mathrm{Hom}_{\mathcal{F}}(Y, S)$ and a subgroup R of T such that $(Y\gamma, R)$ satisfies †.

Amongst all elements of $Y^{\mathcal{F}}$ which do not lie in $\mathcal{C}$, let X be chosen so that $n := |T : C_T(X)|$ is as small as possible. Since Y is fully $\mathcal{F}$-normalised, by Proposition 1.20 there is some $\alpha \in \mathrm{Hom}_{\mathcal{F}}(XC_T(X), S)$ with $X\alpha = Y$. Hence, by Lemma 4.10 (a) we must have $n > 1$. Set $Q := C_T(X)\alpha$ and $W := YQ$.

By Proposition 1.20, we may choose $\gamma \in \mathrm{Hom}_{\mathcal{F}}(N_S(W), N_S(W\gamma))$ such that $W\gamma$ is fully $\mathcal{F}$-normalised. Since $Y \in \mathcal{C}$, T centralises Y and since $N_T(Q) \leq N_S(W)$ we must have $N_T(Q)\gamma \leq C_T(Y\gamma)$. Lastly, $Q < T$ implies that

$$|C_T(X)| = |Q| < |N_T(Q)| \leq |C_T(Y\gamma)|,$$

so that by the minimality of n, $Y\gamma \in \mathcal{C}$.

Since $W\gamma$ is fully $\mathcal{F}$-normalised, we may choose $\beta \in \mathrm{Hom}_{\mathcal{F}}(N_S(XP), N_S(W\gamma))$ so that $(XP)\beta = W\gamma$. Define

$$\varphi := \beta^{-1} \circ \alpha \circ \gamma \in \mathrm{Aut}_{\mathcal{F}}(W\gamma),$$

noting that $X\beta\varphi = Y\gamma$. We show that $X\beta$ does not centralise $N_T(W\gamma)$. First observe that X does not centralise $N_T(XC_T(X))$. To see this, since $C_T(X) < T$ we have $XC_T(X) < N_{XT}(XC_T(X))$ and there exist $x \in X$ and $t \in T$ such that $xt \in N_{XT}(XC_T(X))\backslash XC_T(X)$. Clearly $x \in XC_T(X)$ so we must have $t \in T\backslash XC_T(X)$ and in particular $C_T(X) \leq XC_T(X) < N_T(XC_T(X))$. Now suppose that $X\beta \leq C_S(N_T(W\gamma))$. This implies

$$X \leq C_S(N_T(W\gamma))\beta^{-1} \leq C_S(N_T(W\gamma)\beta^{-1})$$

and since $N_T(XP)\beta \leq N_T((XP)\beta) = N_T(W\gamma)$ we must have $C_S(N_T(W\gamma)\beta^{-1}) \leq C_S(N_T(XP))$ which delivers the required contradiction.

Set $R := Q\gamma C_T(Q\gamma)$. Proposition 1.20 implies that there is $\rho \in \mathrm{Hom}_{\mathcal{F}}(N_S(R), S)$ such that $R\rho$ is fully $\mathcal{F}$-normalised. By Lemma 4.10 (c), $N_S(W\gamma) \leq N_S(Q\gamma) \leq N_S(R)$ so that $N_S(W\gamma)\rho \leq N_S(W\gamma\rho)$ and $W\gamma\rho$ is fully $\mathcal{F}$-normalised since $W\gamma$ is. Thus on replacing γ by $\gamma \circ \rho$ if necessary, we may assume that R is fully $\mathcal{F}$-normalised. Hence by Theorem 1.31, $N_{\mathcal{F}}(R)$ is saturated which means that there is some $\rho' \in \mathrm{Hom}_{N_{\mathcal{F}}(R)}(N_{N_S(R)}(RY\gamma), N_S(R))$ such that $(RY\gamma)\rho'$ is fully $N_{\mathcal{F}}(R)$-normalised by Proposition 1.20. Since $RY\gamma = Q\gamma C_T(Q\gamma)Y\gamma = W\gamma C_T(Q\gamma)$, we have $N_S(W\gamma) \leq N_{N_S(R)}(RY\gamma)$ so $N_S(W\gamma)\rho' \leq N_S(W\gamma\rho')$ and $W\gamma\rho'$ is fully $\mathcal{F}$-normalised. Thus on replacing γ by $\gamma\circ\rho'$ if necessary, we may assume that $RY\gamma$ is fully $N_{\mathcal{F}}(R)$-normalised. In particular, we have shown that (by the minimality of n) the pair $(Y\gamma, R)$ satisfies †.

Since $W\gamma$ is fully $\mathcal{F}$-normalised, there is $\psi \in \mathrm{Aut}_{\mathcal{F}}(W\gamma C_S(W\gamma))$ which extends φ. Also, $R\psi = R$ by Lemma 4.10 (c) (note that $R \leq C_S(W\gamma)$ and $W\gamma\psi = W\gamma$). Therefore ψ leaves $W\gamma R = Y\gamma R$ invariant and then also $Y\gamma RC_S(Y\gamma R)$ invariant. This shows that $\psi \in \mathrm{Mor}(\mathcal{D}(R, Y\gamma))$. Since $RY\gamma$ is fully $N_{\mathcal{F}}(R)$-normalised, $\mathcal{D}(R, Y\gamma)$ is saturated and there exist finite groups $G = G(R, Y\gamma)$ and $H = H(R, Y\gamma)$ which respectively realise $\mathcal{D}(R, Y\gamma)$ and $\mathcal{E}(R, Y\gamma)$ by Corollary 4.9. By Lemma 4.12, $Y\gamma \leq C_G(H(R)) = C_G(H)$ and we may also choose $g \in G$ such that $c_g|_V = \psi|_V$. Finally,

$$Y\gamma = X\beta\varphi = X\beta\psi = (X\beta)^g \leq C_{N_S(RY\gamma)}(H)$$

and since $H \trianglelefteq G$, we have $X\beta \leq C_{N_S(RY\gamma)}(H) \leq C_S(N_T(W\gamma))$ which implies that $N_T(W\gamma) \leq C_T(X\beta)$, a contradiction. □

We now combine Lemma 4.12 with Theorems 4.6 and 4.13 to prove that $C_S(\mathcal{E})$ is a group. It will be useful to make the following definition, in order to render the result more explicit.

Definition 4.14 Let $T \leq S$ be finite p-groups and $(\mathcal{F}, \mathcal{E})$ be a normal pair of saturated fusion systems on (S, T). Let $\mathcal{U}$ be the set of all fully $\mathcal{F}$-normalised, T-centric subgroups of S. Define

$$X(\mathcal{E}) := \bigcap_{\varphi \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T))} \left(\bigcap_{U \in \mathcal{U}} C_{N_S(U)}(H(U)) \right) \varphi.$$

Theorem 4.15 *Let $T \leq S$ be finite p-groups and $(\mathcal{F}, \mathcal{E})$ be a normal pair of saturated fusion systems on (S, T). Then $C_S(\mathcal{E}) = X(\mathcal{E})$ and in particular, $C_S(\mathcal{E})$ is a group.*

Proof. First we prove that $C_S(\mathcal{E}) \leq X$, where $X := X(\mathcal{E})$. Let $R \leq C_S(T)$ be such that $\mathcal{E} \subseteq C_{\mathcal{F}}(R)$. By Theorem 4.13, $(R\varphi, U)$ satisfies † for all fully $\mathcal{F}$-normalised T-centric subgroups U of T and each $\varphi \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T))$. Hence by Lemma 4.12, $R\varphi \leq C_{G(U)}(H(U))$ and in particular, $R \leq X$. Conversely note that $\mathrm{Aut}_{H(U)}(U)^{\varphi} = \mathrm{Aut}_{H(U)}(U\varphi) \leq \mathrm{Aut}_{C_{G(U)}(X)}(U\varphi)$ for each fully $\mathcal{F}$-normalised T-centric subgroup U of T and $\varphi \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T))$. Since $\mathcal{E}$ is generated by the groups $\mathrm{Aut}_{H(U)}(U)^{\varphi}$ by Theorem 4.6, we must have $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$, as needed. □

4.2.2 The $\mathcal{F}$-centraliser $C_{\mathcal{F}}(\mathcal{E})$

We now consider the '$\mathcal{F}$-centraliser' $C_{\mathcal{F}}(\mathcal{E})$ of a normal subsystem $\mathcal{E}$ of a fusion system $\mathcal{F}$. To motivate its definition, we record the following result of Gross which follows as a consequence of the Classification of Finite Simple Groups.

Theorem 4.16 *Let G be a finite group with Sylow p-subgroup S. Assume that $O_{p'}(G) = 1$ and write $C := C_{\mathrm{Aut}(G)}(S) = \{\varphi \in \mathrm{Aut}(G) \mid \varphi|_S = \mathrm{Id}_S\}$. There exists a normal p'-subgroup C_1 of C and an abelian p-subgroup C_2 of C such that $C = C_1C_2$. Furthermore, if p is odd then $C_2 \trianglelefteq C$ and $C_2 = \mathrm{Aut}_{Z(S)}(G)$ whenever $O_p(G) = 1$.*

Proof. This is shown in [19, Theorem A] when p is odd and in [14] when $p = 2$. □

Theorem 4.16 has a large number of corollaries. For example, it is frequently used in the proof of the Z_p^*-Theorem (Theorem 4.26). Another consequence is the following lower bound for $C_G(H)$ when (G, H) is a normal pair of finite groups.

Corollary 4.17 *Let $T \leq S$ be finite p-groups and (G, H) be a normal pair of finite groups with Sylow p-subgroup (S, T). If $O_{p'}(H) = 1$ then $O^p(C_G(T)) \cap S \leq C_G(H)$.*

Proof. Let $C_{\mathrm{Aut}_G(H)}(T)$ be the image of $C_G(T)$ under the natural map

$$\Phi : C_G(T) \longrightarrow \mathrm{Aut}_G(H).$$

Theorem 4.16 implies that there exists a normal p'-subgroup C_1 and a p-subgroup C_2 of $C_{\mathrm{Aut}(H)}(T)$ such that $C_{\mathrm{Aut}_G(H)}(T) \leq C_1C_2$. In particular, $O^p(C_{\mathrm{Aut}_G(H)}(T))$ is a p'-group. Now,

$$(O^p(C_G(T)) \cap S)C_G(H)/C_G(H) \leq O^p(C_G(T))C_G(H)/C_G(H) \leq O^p(C_G(T)/C_G(H)),$$

and since this latter group is a p'-group, we must have $O^p(C_G(T)) \cap S \leq C_G(H)$, as needed. □

If $(\mathcal{F}, \mathcal{E})$ is a normal pair of fusion systems on (S, T) induced by a normal pair (G, H) of finite groups, one quickly checks that $C_S(H) \leq C_S(\mathcal{E})$. To see this, write $X := C_S(H)$ so that $\mathcal{E} = \mathcal{F}_T(H) \subseteq \mathcal{F}_{C_S(X)}(C_G(X))$, and this latter fusion system is equal to $C_{\mathcal{F}}(X)$ by Proposition 1.29. We apply this observation to prove the following result.

Theorem 4.18 *Let $T \leq S$ be finite p-groups and $(\mathcal{F}, \mathcal{E})$ be a normal pair of fusion systems on (S, T) induced by a normal pair of finite groups (G, H). Then*

$$\mathfrak{hyp}(C_{\mathcal{F}}(T)) \leq C_S(\mathcal{E}) \leq C_S(T).$$

In particular, there exists a unique saturated fusion system $C_{\mathcal{F}}(\mathcal{E})$ on $C_S(\mathcal{E})$ contained in $C_{\mathcal{F}}(T)$.

Proof. By Proposition 1.29, $C_{\mathcal{F}}(T) = \mathcal{F}_{C_S(T)}(C_G(T))$ and so Proposition 1.60 implies that $O^p(C_G(T)) \cap S = \mathfrak{hyp}(C_{\mathcal{F}}(T))$. We have already seen that $C_S(H) \leq C_S(\mathcal{E})$ so that by Corollary 4.17, $\mathfrak{hyp}(C_{\mathcal{F}}(T)) \leq C_S(\mathcal{E})$. It remains to verify the last statement, which is an immediate consequence of Theorem 1.61. □

Theorem 4.18 implies that we can sensibly conjecture that the identity $\mathfrak{hyp}(C_{\mathcal{F}}(T)) \leq C_S(\mathcal{E})$ holds when $(\mathcal{F}, \mathcal{E})$ is an arbitrary normal pair of fusion systems. In [2, Chapter 6], Aschbacher constructs $C_{\mathcal{F}}(\mathcal{E})$ using a 'normal map' - essentially by repeating the argument which proves Theorem 1.61 in a special case. From this, we *can* deduce this identity, but this is somewhat dissatisfying; the author feels that a short proof should exist despite being unable to find one.

Theorem 4.19 *Let $T \leq S$ be finite p-groups and $(\mathcal{F}, \mathcal{E})$ be a normal pair of fusion systems on (S, T). There exists a normal subsystem $C_{\mathcal{F}}(\mathcal{E})$ of $\mathcal{F}$ on $C_S(\mathcal{E})$ contained in $C_{\mathcal{F}}(T)$ of p-power index. In particular, $\mathfrak{hyp}(C_{\mathcal{F}}(T)) \leq C_S(\mathcal{E})$.*

Proof. Adopting the subscript notation of Theorem 1.61, by [3, Theorem 6.12] there exists a saturated fusion system $\mathcal{C} := C_{\mathcal{F}}(\mathcal{E})$ on $C_S(\mathcal{E})$ with $\mathrm{Aut}_{\mathcal{C}}(P) = C_{\mathcal{F}}(T)_{C_S(\mathcal{E})}$. Hence by Theorem 1.61 we must have $\mathfrak{hyp}(C_{\mathcal{F}}(T)) \leq C_S(\mathcal{E})$, since there can be at most one fusion system on $C_S(\mathcal{E})$ contained in $C_{\mathcal{F}}(T)$ of p-power index. □

Observe that in the case where $\mathcal{E} = \mathcal{F}_T(T)$, Theorem 4.19 follows from Theorem 1.56 parts (a) and (c) with $K = \mathrm{Id}_T$. The general case is fairly surprising since it implies that the subsystem $\mathcal{E}$ cannot be larger than $C_{\mathcal{F}}(\mathfrak{hyp}(C_{\mathcal{F}}(T)))$.

4.3 Centralisers of Weakly Normal Subsystems

We conclude this chapter by with a different approach to the theory of centralisers of subsystems of fusion systems which relies on that developed in Section 1.5.2. It is shown that a weakly normal pair $(\mathcal{L}, \mathcal{L}_0)$ of linking systems associated to a given pair $(\mathcal{F}, \mathcal{E})$ of weakly normal subsystems on (S, T) renders the existence, uniqueness and strong closure of $C_S(\mathcal{E})$ trivial.

4.3.1 $C_S(\mathcal{E})$ as the Kernel of an Action

Before stating and proving our main results (Theorems 4.22 and 4.25), we motivate them with an example which proves that there exist weakly normal pairs $(\mathcal{F}, \mathcal{E})$ of fusion systems on (S, T) for which $C_S(\mathcal{E})$ is not a group. Thus, a generalisation of Theorem 4.15 to the case where $(\mathcal{F}, \mathcal{E})$ is weakly normal is too much to ask for.

Example 4.20 Let $H := H_1 \times H_2 \times H_3$ with $H_i \cong A_4$ for $1 \leq i \leq 3$ be a direct product of three copies of the alternating group on 4 letters. For $1 \leq i \leq 3$ pick $S_i \in \mathrm{Syl}_2(H_i)$ so that $S_1 \times S_2 \times S_3 \in \mathrm{Syl}_2(H)$ and let $X_i = \langle x_i \rangle$ be a group of order 3 which acts on S_i in such a way that $H_i = S_i \rtimes X_i$ for each i. Set $X := \langle x_1 x_2, x_1 x_3 \rangle \cong C_3 \times C_3$, $G := SX$ and let

$$\mathcal{F} := \mathcal{F}_S(G),\ \mathcal{F}_1 := \mathcal{F}_{S_1 S_2}(\langle S_1, S_2, x_1 x_2 \rangle) \text{ and } \mathcal{F}_2 := \mathcal{F}_{S_1 S_3}(\langle S_1, S_3, x_1 x_3 \rangle).$$

Clearly $\mathcal{F}_i$ is normal in $\mathcal{F}$ for $i = 1, 2$ and hence the fusion system $\mathcal{E} := \mathcal{F}_1 \cap \mathcal{F}_2$ on S_1, whose morphisms consist of those in both $\mathcal{F}_1$ and $\mathcal{F}_2$, is an $\mathcal{F}$-invariant subsystem on S_1 (intersections of $\mathcal{F}$-invariant subsystems are $\mathcal{F}$-invariant). Furthermore $\mathcal{E} = \mathcal{F}_{S_1}(H_1)$ so $\mathcal{E}$ is saturated by Theorem 1.18 and hence weakly normal. However, $\mathcal{E}$ is not normal in $\mathcal{F}$. To see this, let $\varphi \in \mathrm{Aut}_{\mathcal{E}}(S_1)$ be the map induced by conjugation by x_1 so that φ extends to maps $\varphi_1 = c_{x_1 x_2}$ and $\varphi_2 = c_{x_1 x_3} \in \mathrm{Aut}_{\mathcal{F}}(S)$. Observe that φ_i does not act trivially on $C_S(S_1)/Z(S_1) = S/S_1 = S_2 \times S_3$ for $i = 1, 2$, proving the claim.

Now, for $i = 2, 3$, $\mathcal{E} \subseteq C_{\mathcal{F}}(S_i) = \mathcal{F}_{4-i} \times \mathcal{F}_{S_i}(S_i)$. Note that $\mathcal{E}$ is not contained in $C_{\mathcal{F}}(S_2 \times S_3) = \mathcal{F}_S(S)$ so there is no unique maximal subgroup of S which centralises $\mathcal{E}$. In particular, $C_S(\mathcal{E})$ is not a group.

We see in the above example that any linking system $\mathcal{L}$ associated to $\mathcal{F}$ has a unique object S (since S is abelian) so that $S_1 \notin \mathrm{Ob}(\mathcal{L})$ and there can exist no weakly normal pair $(\mathcal{L}, \mathcal{L}_0)$ of linking systems associated to $(\mathcal{F}, \mathcal{E})$. This is always the case for a weakly normal pair $(\mathcal{F}, \mathcal{E})$ of fusion systems when $C_S(\mathcal{E})$ is not a group, and we

turn now to a proof of this fact which follows from the existence of a certain exact sequence into which $C_S(\mathcal{E})$ fits. To construct this sequence, we will need the following generalisation of $\mathrm{Out}_{typ}(\mathcal{L})$.

Definition 4.21 Let $(\mathcal{L}, \mathcal{L}_0)$ be a pair of linking systems (associated to some pair of fusion systems). The set $\mathrm{Out}_{typ}(\mathcal{L}_0, \mathcal{L})$ is the set of equivalence classes of $\mathrm{Aut}_{typ}(\mathcal{L}_0)$ under $\mathcal{L}$-natural isomorphism.

Our motivation for proving the next result came from [1, Lemma 1.14], which follows by specialising to the case $\mathcal{E} = \mathcal{F}$.

Theorem 4.22 *Let $T \leq S$ be finite p-groups, $(\mathcal{F}, \mathcal{E})$ be a weakly normal pair of fusion systems on (S, T) and $(\mathcal{L}, \mathcal{L}_0)$ be a weakly normal pair of linking systems associated $(\mathcal{F}, \mathcal{E})$. The sequence*

$$1 \longrightarrow C_S(\mathcal{E}) \xrightarrow{\delta_S} \mathrm{Aut}_{\mathcal{L}}(T) \xrightarrow{\gamma \mapsto c_\gamma} \mathrm{Aut}^I_{typ}(\mathcal{L}_0) \longrightarrow \mathrm{Out}_{typ}(\mathcal{L}_0, \mathcal{L}) \longrightarrow 1$$

is exact. In particular, $C_S(\mathcal{E})$ is a group, $C_S(\mathcal{E})$ is strongly $\mathcal{F}$-closed and $C_{\mathcal{F}}(\mathcal{E}) = \mathcal{F}_{C_S(\mathcal{E})}(C_S(\mathcal{E}))$ is saturated.

Proof. By [1, Lemma 1.14], $\mathrm{Aut}^I_{typ}(\mathcal{L}_0)$ is a group and so $C_S(\mathcal{E})$ and $\mathrm{Out}_{typ}(\mathcal{L}_0, \mathcal{L})$ are both groups provided the sequence is exact. First we show that the natural homomorphism from $\mathrm{Aut}^I_{typ}(\mathcal{L}_0)$ to $\mathrm{Out}_{typ}(\mathcal{L}_0, \mathcal{L})$ is onto. This amounts to showing that each $\alpha \in \mathrm{Aut}_{typ}(\mathcal{L}_0)$ is $\mathcal{L}$-naturally isomorphic to an isotypical equivalence which sends inclusions to inclusions. However, by [1, Lemma 1.14] any such α is $\mathcal{L}_0$-naturally isomorphic to an isotypical equivalence, establishing the claim.

Now suppose that $\alpha \in \mathrm{Aut}^I_{typ}(\mathcal{L}_0)$ is in the kernel of the map from $\mathrm{Aut}^I_{typ}(\mathcal{L}_0)$ to $\mathrm{Out}_{typ}(\mathcal{L}_0, \mathcal{L})$. Then α is $\mathcal{L}$-naturally isomorphic to the identity functor via some set of maps η consisting of isomorphisms $\eta_P \in \mathrm{Iso}_{\mathcal{L}}(P, \alpha(P))$ and for each $\psi \in \mathrm{Hom}_{\mathcal{L}_0}(P, Q)$, $\eta_P \circ \alpha(\psi) = \psi \circ \eta_Q$. Since α is isotypical $\alpha(\iota^T_P) = \iota^T_{\alpha(P)}$ so that $\eta_P = \eta_T|_{P,\alpha(P)}$ and this shows that α is conjugation by $\eta_T \in \mathrm{Aut}_{\mathcal{L}}(T)$. Conversely, if $\gamma \in \mathrm{Aut}_{\mathcal{L}}(T)$ then c_γ is $\mathcal{L}$-naturally isomorphic to $\mathrm{Id}_{\mathcal{L}_0}$ via the set of maps $\{\gamma|_{P,P\pi(\gamma)}\}_{P \in Ob(\mathcal{L}_0)}$.

Next we show that for each $\gamma \in \mathrm{Aut}_{\mathcal{L}}(T)$, $c_\gamma = \mathrm{Id}_{\mathcal{L}_0}$ if and only if $\gamma \in \delta_S(C_S(\mathcal{E}))$. If $c_\gamma = \mathrm{Id}_{\mathcal{L}_0}$ then since $\gamma^{-1} \circ \delta_T(g) \circ \gamma = \delta_T(g)$ for all $g \in S$, we must have $\pi(\gamma) = \mathrm{Id}_T$ by Definition 1.62 (c) for $\mathcal{L}$ and the fact that δ_T is injective. Hence by Definition 1.62 (a)(v) for $\mathcal{L}$, there is some $a \in C_S(T)$ such that $\gamma = \delta_T(a)$. Now, for each $P, Q \in \mathrm{Ob}(\mathcal{L}_0)$ and $\psi \in \mathrm{Mor}_{\mathcal{L}_0}(P, Q)$, $\psi \circ \delta_Q(a) = \delta_P(a) \circ \psi$ so that by Proposition 1.69 (b), there is some $\bar{\psi} \in \mathrm{Mor}_{\mathcal{L}}(\langle P, a\rangle, \langle Q, a\rangle)$ such that $\bar{\psi}|_{P,Q} = \psi$ (note that

$a \in C_S(T) \leq C_S(P)$). By Definition 1.62 (c) for $\mathcal{L}$ again and the injectivity of δ, $\pi(\bar{\psi})(a) = a$. Hence each morphism in $\mathcal{E}$ extends to one which fixes a and $a \in C_S(\mathcal{E})$. Conversely if $\gamma = \delta_S(a)$ for some $a \in C_S(\mathcal{E})$ then each $\psi \in \mathrm{Mor}(\mathcal{L}_0)$ extends to some $\bar{\psi}$ with $\pi(\bar{\psi})(a) = a$. By Definition 1.62 (c) for $\mathcal{L}$, $\bar{\psi}$ commutes with $\delta_S(a)$ and we have $c_\gamma(\psi) = \psi$. Hence $c_\gamma = \mathrm{Id}_{\mathcal{L}_0}$, as required.

It remains to prove the last statement. First we show that $C_S(\mathcal{E})$ is strongly $\mathcal{F}$-closed. Let $a \in C_S(\mathcal{E})$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ with $a \in P$. Let $\psi \in \mathrm{Hom}_{\mathcal{L}}(P, S)$ be such that $\pi(\psi) = \varphi$ and note that by Definition 1.62 (c) for $\mathcal{L}$, we have $\psi^{-1} \circ \delta_P(a) \circ \psi = \delta_S(a\varphi)$. This implies that $c_{\delta_S(a\varphi)} = c_\psi^{-1} \circ \mathrm{Id}_{\mathcal{L}_0} \circ c_\psi = \mathrm{Id}_{\mathcal{L}_0}$ (since $c_{\delta_P(a)} = \mathrm{Id}_{\mathcal{L}_0}$) and $a\varphi \in C_S(\mathcal{E})$ as needed. Finally, observe that since T is $\mathcal{F}$-quasicentric, $C_{\mathcal{F}}(T)$ is a trivial fusion system on $C_S(T)$, so $\mathfrak{hyp}(C_{\mathcal{F}}(T)) = 1 \leq C_S(\mathcal{E})$ and $C_{\mathcal{F}}(T)_{C_S(\mathcal{E})} = \mathcal{F}_{C_S(\mathcal{E})}(C_S(\mathcal{E}))$ is saturated by Theorem 1.18. □

Next, we show that the exact sequence of Theorem 4.22 has an analogue for normal pairs (G, H) of finite groups. Furthermore, when $(\mathcal{F}, \mathcal{E})$ is induced by (G, H) these two sequences are comparable. We introduce two new pieces of notation: if G is a finite group, H is a normal subgroup of G and $S \in \mathrm{Syl}_p(G)$,

$$\mathrm{Aut}(G, S) := \{\alpha \in \mathrm{Aut}(G) \mid S\alpha = S\} \text{ and } \mathrm{Out}(H, G) := \mathrm{Aut}(H)/\mathrm{Aut}_G(H).$$

Consider the following elementary consequence of the Frattini argument.

Lemma 4.23 *Let H be a finite group with $O_{p'}(H) = 1$ and let T be a Sylow p-subgroup of H. Then $\mathrm{Aut}(H) = \mathrm{Aut}(H, T)\,\mathrm{Inn}(H)$.*

Proof. Since $O_{p'}(H) = 1$, $Z(H) \leq T$ and $T/Z(H)$ may be identified with a Sylow p-subgroup R of $\mathrm{Inn}(H)$. By the Frattini argument, $N_{\mathrm{Aut}(H)}(R)\mathrm{Inn}(H) = \mathrm{Aut}(H)$. The result now follows from the fact that $N_{\mathrm{Aut}(H)}(R) = \mathrm{Aut}(H, T)$. □

The following lemma provides a way of comparing the automorphism groups of G and $\mathcal{L}$ when $\mathcal{L}$ is a linking system associated to the fusion system of G.

Lemma 4.24 *Let G be a finite group with Sylow p-subgroup S and let $\mathcal{L} = \mathcal{L}_S^{\mathcal{H}}(G)$ be a linking system associated $\mathcal{F}_S(G)$. There exists a homomorphism*

$$\mathrm{Aut}(G, S) \xrightarrow{\tilde{\kappa}_G^{\mathcal{H}}} \mathrm{Aut}_{typ}^I(\mathcal{L})$$

where for each $\beta \in \mathrm{Aut}(G, S)$, $\tilde{\kappa}_G^{\mathcal{H}}(\beta)$ is the functor which sends $P \in Ob(\mathcal{L})$ to $P\beta$ and the class $[a] \in \mathrm{Hom}_{\mathcal{L}}(P, Q)$ of the element $a \in N_G(P, Q)$ to $[a\beta]$.

Proof. This is clear. □

We can now state and prove the second main result of this section.

Theorem 4.25 *Let $T \leq S$ be finite p-groups and (G, H) be a normal pair of finite groups with Sylow p-subgroup (S, T) and assume that $O_{p'}(H) = 1$. There exists an exact sequence*

$$1 \longrightarrow C_G(H) \xrightarrow{\iota_{C_G(H)}^{N_G(T)}} N_G(T) \xrightarrow{g \mapsto c_g|_H} \mathrm{Aut}(H,T) \longrightarrow \mathrm{Out}(H,G) \longrightarrow 1.$$

Furthermore if $(\mathcal{F}, \mathcal{E})$ is induced by (G, H), $\mathcal{L} = \mathcal{L}_S^{\mathcal{H}}(G)$ and $\mathcal{L}_0 = \mathcal{L}_T^{\mathcal{H}_0}(H)$ where $\mathcal{H}$ and $\mathcal{H}_0$ are sets of objects chosen so that $(\mathcal{L}, \mathcal{L}_0)$ is a weakly normal pair of linking systems associated to $(\mathcal{F}, \mathcal{E})$, then the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & C_S(\mathcal{E}) & \longrightarrow & \mathrm{Aut}_{\mathcal{L}}(T) & \longrightarrow & \mathrm{Aut}_{typ}^{I}(\mathcal{L}_0) & \longrightarrow & \mathrm{Out}_{typ}(\mathcal{L}_0, \mathcal{L}) \longrightarrow 1 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \tilde{\kappa}_H^{\mathcal{H}_0}} & & \uparrow{\scriptstyle \kappa_H^{\mathcal{H}_0}} \\
1 & \longrightarrow & C_G(H) & \longrightarrow & N_G(T) & \longrightarrow & \mathrm{Aut}(H,T) & \longrightarrow & \mathrm{Out}(H,G) \longrightarrow 1
\end{array}$$

commutes.

Proof. We prove the first sequence is exact. Note that each $g \in N_G(T)$ which lies in the kernel of the map $[g \mapsto c_g|_H]$ also lies in $C_G(H)$ and conversely $c_g|_H = \mathrm{Id}_H$ whenever $g \in C_G(H)$. To see exactness at $\mathrm{Aut}(H,T)$, first note that the natural map from $\mathrm{Aut}(H,T)$ to $\mathrm{Out}(H,G)$ is onto since $\mathrm{Aut}(H) = \mathrm{Aut}(H,T)\mathrm{Inn}(H) = \mathrm{Aut}(H,T)\mathrm{Aut}_G(H)$ by Lemma 4.23. On the other hand, $\mathrm{Aut}(H,T) \cap \mathrm{Aut}_G(H)$ is exactly the image in $\mathrm{Aut}(H,T)$ of $N_G(T)$, as needed. It remains to check that $\tilde{\kappa}_H^{\mathcal{H}_0}$ induces a well-defined map

$$\kappa_H^{\mathcal{H}_0} : \mathrm{Out}(H,G) \longrightarrow \mathrm{Out}(\mathcal{L}_0, \mathcal{L})$$

which sends the class of $\beta \in \mathrm{Aut}(H)$ to the class of $\tilde{\kappa}_H^{\mathcal{H}_0}(\beta)$. This is immediate since $\tilde{\kappa}_H^{\mathcal{H}_0}$ sends $c_g|_H \in \mathrm{Aut}(H,T)$ to $c_{[g]} \in \mathrm{Aut}_{typ}^{I}(\mathcal{L}_0)$ for each $g \in N_G(T)$, proving that $\kappa_H^{\mathcal{H}_0}$ is well-defined. The result now follows from Theorem 4.22. □

4.3.2 Centralisers of Realisable Fusion Systems and the Z^* Theorem

We will use some of the results just proved to study centralisers of realisable subsystems in more detail. The Z_p^*-Theorem for finite groups can be stated in terms of fusion systems as follows.

Theorem 4.26 *Let p be an odd prime and G be a finite group with $O_{p'}(G) = 1$. Then for each Sylow p-subgroup S of G, $Z(\mathcal{F}_S(G)) = Z(G)$.*

Proof. This is a consequence of [20, Theorem D]. □

Observe that Lemma 1.45 implies that Theorem 4.26 holds in the case where G is p-constrained. The general case relies on the Classification of Finite Simple Groups. We would like to generalise Theorem 4.26 to a statement which involves centralisers. Since a normal pair (G, H) of finite groups with Sylow p-subgroup (S, T) induces a normal pair of fusion systems $(\mathcal{F}, \mathcal{E})$ (Proposition 1.55) and $C_S(H) \leq C_S(\mathcal{E})$, a natural generalisation might be the following.

Conjecture 4.27 *Let p be odd, $T \leq S$ be finite p-groups and $(\mathcal{F}, \mathcal{E})$ be a normal pair of fusion systems on (S, T) induced by a normal pair (G, H) of finite groups. If $O_{p'}(H) = 1$ then $C_S(\mathcal{E}) = C_S(H)$.*

Evidence for the truth of this conjecture is provided by the following corollary to Theorem 4.16 when p is odd.

Theorem 4.28 *Let p be odd, $T \leq S$ be finite p-groups and $(\mathcal{F}, \mathcal{E})$ be a normal pair of fusion systems on (S, T) induced by a normal pair (G, H) of finite groups. If $O_{p'}(H) = O_p(H) = 1$ then $C_S(\mathcal{E}) = 1$.*

Proof. Each $g \in C_S(\mathcal{E})$ induces a p-automorphism of H and hence by Theorem 4.16 must lie in $Z(T)$. In particular, $g \in Z(\mathcal{E})$. By Theorem 4.26, $Z(\mathcal{E}) = Z(H)$ and since $O_p(H) = O_{p'}(H) = 1$, $Z(H) = 1$ so $g = 1$, as required. □

In particular, Conjecture 4.27 holds in the case where p is odd, H is simple and $C_G(H) = 1$. It would be very interesting to know if Theorem 4.28 holds for arbitrary primes or even for arbitrary pairs of saturated fusion systems[1]. The following application of Theorem 4.25 answers the former question in the case where $(G, H) = (\mathrm{Sym}(n), \mathrm{Alt}(n))$. It seems plausible that this method can be used to handle other families of 'almost simple' pairs of finite groups at the prime 2.

Proposition 4.29 *Let $p = 2$, $T \leq S$ be finite p-groups and $(\mathcal{F}, \mathcal{E})$ be a normal pair of fusion systems on (S, T) induced by the normal pair $(\mathrm{Sym}(n), \mathrm{Alt}(n))$ for some $n \geq 5$. There exists a weakly normal pair $(\mathcal{L}, \mathcal{L}_0)$ of linking systems associated to $(\mathcal{F}, \mathcal{E})$ and $C_S(\mathcal{E}) = 1$.*

[1]Here, the condition '$O_p(H) = 1$' is replaced by '$O_p(\mathcal{E}) = 1$'.

Proof. The existence of a weakly normal pair $(\mathcal{L}, \mathcal{L}_0) = (\mathcal{L}^c_S(G), \mathcal{L}^c_T(H))$ of linking systems associated to $(\mathcal{F}, \mathcal{E})$ is a standard fact. For example $\mathcal{L}$ can be constructed as an extension of $\mathcal{L}_0$ by combining [23, Theorem 9] and [1, Proposition 4.8]. Hence it suffices to prove that $O^p(C_G(T)) = 1$ since this implies that $\mathrm{Aut}_{\mathcal{L}}(T) = N_G(T)$ and the result will follow from the exact sequences of Theorem 4.25 and the fact that $C_G(H) = 1$. In fact, we will show that $C_G(T)$ is a 2-group. To this end, let $\mathrm{supp}(\sigma) := \{i \in \mathbf{n} \mid i\sigma \neq i\}$ denote the support of each permutation $\sigma \in G$, where $\mathbf{n} := \{1, \ldots, n\}$. Also define $\mathrm{supp}(Y) := \bigcup_{\sigma \in Y} \mathrm{supp}(\sigma)$ for each $Y \leq G$. Let

$$E := \langle (1,2)(3,4), (1,3)(2,4) \rangle \text{ and } Q := \langle E^G \rangle.$$

If E_1 and E_2 are G-conjugates of E, then exactly one of the following is true:

(i) $E_1 = E_2$;

(ii) $\mathrm{supp}(E_1) \cap \mathrm{supp}(E_2) = \emptyset$; or

(iii) $\langle E_1, E_2 \rangle$ is not a 2-group.

Since T is strongly G-closed case (iii) does not occur, so we must have that $Q = Q_1 \times \cdots \times Q_k$ where $k = [n/4]$ and the Q_i are pairwise commuting subgroups which are G-conjugate to E. Each $\sigma \in C_G(Q)$ centralises Q_i and therefore fixes $\mathrm{supp}(Q_i)$ for each $1 \leq i \leq k$. Moreover, by considering the shape of permutations in Q we see that $\sigma \in Q$. Since $Q \leq T$, we have $C_G(T) \leq C_G(Q) \leq Q$ is a 2-group, as needed. □

4.3.3 Normalisers of Subsystems of Fusion Systems

We end this chapter with a discussion of a topic which originally prompted the author to consider centralisers, namely normalisers of fusion systems. The notion of a normaliser would be highly desirable as it would create new opportunities to argue inductively inside a fusion system. The existence of normalisers seems to be connected with the existence of pairs of linking systems, and for this reason we feel that it is appropriate to consider them at this point.

We begin by discussing the S-normaliser of a subsystem of a fusion system $\mathcal{F}$ on S. Assume that $\mathcal{E}$ is a fusion subsystem of $\mathcal{F}$ on T. Let $N_S(\mathcal{E})$ be the group $N_S^K(T)$ where $K := \mathrm{Aut}(\mathcal{E}) \leq \mathrm{Aut}(T)$. We say that $\mathcal{E}$ is *fully $\mathcal{F}$-normalised* if T is fully $\mathrm{Aut}(\mathcal{E})$-normalised. Note that by Theorem 1.31, $N_{\mathcal{F}}^{\mathrm{Aut}(\mathcal{E})}(T)$ is saturated if $\mathcal{E}$ is fully $\mathcal{F}$-normalised. Next, we define the S-normaliser of a linking subsystem. Let $(\mathcal{L}, \mathcal{L}_0)$

be a pair of linking systems associated to a pair of fusion systems $(\mathcal{F}, \mathcal{E})$ on (S, T). By this, we mean that Definition 1.70 (a) is satisfied for the pair $(\mathcal{L}, \mathcal{L}_0)$. $N_S(\mathcal{L}_0)$ is the set of all $g \in N_S(T)$ with $c_{\delta_S(g)}|_T \in \mathrm{Aut}^I_{typ}(\mathcal{L}_0)$.

We would like to compare the groups $N_S(\mathcal{L}_0)$ and $N_S(\mathcal{E})$, and to do this we need a new map. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group and let $\mathcal{L}$ be a linking system associated to $\mathcal{F}$. Define:

$$\mu_{\mathcal{L}} : \mathrm{Aut}^I_{typ}(\mathcal{L}) \longrightarrow \mathrm{Aut}(S)$$

by setting $\alpha\mu_{\mathcal{L}} := \delta_S \circ \alpha_S \circ \delta_S^{-1}$ for each $\alpha \in \mathrm{Aut}^I_{typ}(\mathcal{L})$. We have the following lemma concerning the image of $\mu_{\mathcal{L}}$:

Lemma 4.30 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group and let $\mathcal{L}$ be a linking system associated to $\mathcal{F}$. Then* $\mathrm{Im}(\mu_{\mathcal{L}}) \leq \mathrm{Aut}(\mathcal{F})$.

Proof. This is shown in [23, Proposition 6]. □

Using Lemma 4.30, we discover the following relationship between $N_S(\mathcal{L}_0)$ and $N_S(\mathcal{E})$:

Lemma 4.31 *Let $(\mathcal{L}, \mathcal{L}_0)$ be a pair of linking systems associated to a pair of fusion systems $(\mathcal{F}, \mathcal{E})$ on (S, T). For each $g \in N_S(\mathcal{L}_0), (c_{\delta_S(g)})\mu_{\mathcal{L}_0} = c_g \in \mathrm{Aut}_S(T)$. In particular, $N_S(\mathcal{L}_0) \leq N_S(\mathcal{E})$.*

Proof. Observe that each $\gamma \in \mathrm{Aut}_{\mathcal{L}}(T)$ such that $c_\gamma \in \mathrm{Aut}^I_{typ}(\mathcal{L}_0)$, $(c_\gamma)\mu_{\mathcal{L}_0} = \pi(\gamma)$ by axiom (c) in Definition 1.62 for $\mathcal{L}$. In particular, for each $g \in N_S(\mathcal{L}_0)$ we have $(c_{\delta_S(g)})\mu_{\mathcal{L}_0} = \pi(\delta_S(g)) = c_g \in \mathrm{Aut}_S(T)$ by axiom (b) in Definition 1.62 for $\mathcal{L}$. The relation $N_S(\mathcal{L}_0) \leq N_S(\mathcal{E})$ now follows from Lemma 4.30 applied with $(\mathcal{E}, \mathcal{L}_0)$ in place of $(\mathcal{F}, \mathcal{L})$. □

Since the map $\mu_{\mathcal{L}}$ is not always an isomorphism, we expect that the opposite inclusion $N_S(\mathcal{E}) \leq N_S(\mathcal{L}_0)$ will not hold in general. Which of the groups $N_S(\mathcal{L}_0)$ and $N_S(\mathcal{E})$ is the 'correct' definition for the S-normaliser is unclear, although in the case where $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{E} = \mathcal{F}_T(H)$ for some finite groups $H \leq G$, by Theorem 4.25 there exists a map

$$\kappa_H : \mathrm{Out}_{typ}(\mathcal{L}^c_T(H)) \longrightarrow \mathrm{Out}(H)$$

which is conjecturally always an isomorphism whenever $\mathcal{E}$ is *tame* (see [1, Section 2.2]). This suggests that $N_S(\mathcal{L}_0)$ more closely resembles the group $N_G(H) \cap S$ than $N_S(\mathcal{E})$ in this case.

We now consider $\mathcal{F}$-normalisers of $\mathcal{E}$. A strong (weak) $\mathcal{F}$-normaliser of $\mathcal{E}$ is a subsystem of $\mathcal{F}$ on $N_S(\mathcal{E})$ which is maximal among all subsystems of $\mathcal{F}$ containing $\mathcal{E}$ as a (weakly) normal subsystem. In fact, strong $\mathcal{F}$-normalisers are not necessarily unique, even if $\mathcal{E}$ is fully $\mathcal{F}$-normalised. In Example 4.20, it is straightforward to check that the fusion systems $\mathcal{F}_{S_1S_3}(\langle x_1x_2, x_3\rangle)$ and $\mathcal{F}_{S_1S_2}(\langle x_1x_3, x_2\rangle)$ both contain $\mathcal{E}$ as a normal subsystem and are maximal with respect to this property. This is already a large setback since, as we have seen, most naturally occurring weakly normal subsystems are normal. We are therefore reduced to studying weak normalisers of $\mathcal{E}$ in $\mathcal{F}$. Define

$$N_{\mathcal{F}}(\mathcal{E}) := \langle \mathcal{E}, N_{\mathcal{F}}^{\mathrm{Aut}(\mathcal{E})}(T)\rangle_{N_S(\mathcal{E})},$$

where $N_S(\mathcal{E})$ is defined as above. Clearly T is strongly $N_{\mathcal{F}}(\mathcal{E})$-closed and since

$$\mathrm{Aut}_{N_{\mathcal{F}}(\mathcal{E})}(T) \leq \langle \mathrm{Aut}_{\mathcal{E}}(T), \mathrm{Aut}(\mathcal{E})\rangle = \mathrm{Aut}(\mathcal{E}),$$

$\mathcal{E}$ is invariant under all $N_{\mathcal{F}}(\mathcal{E})$-automorphisms of T. We say that an $\mathcal{F}$-morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,T)$ is *normalising* if it can be written as a product $\varphi_0 \circ \alpha$ where $\alpha \in \mathrm{Aut}(\mathcal{E}) \cap \mathrm{Aut}_{\mathcal{F}}(T)$ and $\varphi_0 \in \mathrm{Hom}_{\mathcal{E}}(P,T)$. Clearly each normalising morphism lies in $N_{\mathcal{F}}(\mathcal{E})$, so we may view $N_{\mathcal{F}}(\mathcal{E})$ as the largest possible fusion system on $N_S(\mathcal{E})$ inside which $\mathcal{E}$ is weakly normal.

In general, of course, $N_{\mathcal{F}}(\mathcal{E})$ will not be saturated since it is too large, so we look for conditions which force the existence of saturated subsystems of $N_{\mathcal{F}}(\mathcal{E})$. As an example, consider the case where $p \nmid |\mathrm{Aut}(\mathcal{E}) \cap \mathrm{Aut}_{\mathcal{F}}(T) : \mathrm{Aut}_{\mathcal{E}}(T)|$ and define the fusion system $\hat{\mathcal{E}} \subseteq N_{\mathcal{F}}(\mathcal{E})$ by setting

$$\mathrm{Hom}_{\hat{\mathcal{E}}}(P,T) := \mathrm{Hom}_{\mathcal{E}}(P,T) \cdot \mathrm{Aut}(\mathcal{E}) \cap \mathrm{Aut}_{\mathcal{F}}(T).$$

By [11] Theorem 7.57, $\hat{\mathcal{E}}$ is a saturated fusion system on T, and each morphism is normalising, but it need not be a fusion system on $N_S(\mathcal{E})$. For example, let $(G,H) =$ (Sym(10), Alt(10)) and let $(\mathcal{F},\mathcal{E})$ be a pair of 2-fusion systems on (S,T) induced by (G,H). By [1, Lemma 4.6], $\mathrm{Aut}_{\mathcal{E}}(T) = \mathrm{Aut}(\mathcal{E})$, so that $\hat{\mathcal{E}} = \mathcal{E}$ is not a fusion system on $N_S(\mathcal{E})$ which is clearly equal to S.

The author feels that there should be methods for dealing with other situations like the one just described, possibly by involving linking systems. For example, there may exist an analogue for fusion systems of the sequence

$$1 \longrightarrow H \longrightarrow N_G(H) \longrightarrow \mathrm{Out}_G(H) \longrightarrow 1$$

for a pair $H \leq G$ of finite groups with $C_G(H) \leq H$. In general, it seems that in contrast to $\mathcal{F}$-centralisers, $\mathcal{F}$-normalisers, if they exist, are fairly badly behaved. The correct definition seems to be largely dependent on the context.

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