On the hypothesis of locality: visibility and the limitations of kinetic roughening critical exponents.

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We compare using visibility the usual Kardar-Parisi-Zhang (KPZ) universality class and a fractional Edward-Wilkinson (EW) equation with correlated noise, which share the same kinetic roughening exponents. The first universality class is well known, and its equation given in terms of the usual derivatives, uncorrelated noise, and therefore in intrinsically local, and includes nonlinear terms. However, the second model includes fractional derivatives and correlated noise, both of which are nonlocal. One could be tempted to conclude that both dynamics belong to the same universality class, specifically, to the KPZ universality class. However, this is a wrong result that calls the attention against the indiscriminate application of the theory in some real systems without taking into consideration basic physical assumptions (e.g. locality). Moreover, these examples reveal the necessity of finding new algorithms for detecting characteristics that remain unnoticed to classical scaling analysis, where only the two first moments of the interfaces distribution (mean and variance) are used to classify the dynamics. We show that visibility (and in particular the kinetic roughening exponents of the connectivity interfase) is able to distinguish this two dynamics which would be confused by standard techniques.

Physical systems leave signals (traces) during their evolution. These signals can be either functions of time, giving rise to temporal series, or functions of space, forming static borders or interfaces (surfaces). They can also depend on both time and space as occur in stochastic dynamic surfaces or interfaces [refs]. In all cases, we are handling with a set of data whose analysis, we certainly assume, will provide valuable information about the physical principles that are driven the system dynamics. Statistical approaches seek for correlations to discern essential properties as temporal irreversibility or spatial locality. For non-equilibrium growing interfaces the study of global characteristics enables a classification of their dynamics. This procedure is common in scaling analysis and renormalization group techniques. Under some assumptions, these methodologies are able to classify the dynamics according to a set of critical parameters. However, the computation of the critical exponents of experimental growing interfaces does not enable the rigorous classification of their dynamics and, consequently, to explain the physical processes that are operating microscopically, unless additional information about the system is available.

The theory of kinetic roughening is a key tool to analyse the dynamics of non-equilibrium processes and the critical exponents that define its behaviour are known to characterise the universality class. This analysis has been applied in many papers to determine the dynamics of empirical interfaces (see, e.g. [1–4]). However, this characterization is based on a Taylor expansion and thus, it is intrinsically local. Whereas for local equations (i.e. given in terms of classical derivatives) the set of critical exponents is finite, for fractional equations (which are non local) a continuum of critical exponents might exist (see [5]).

Recently, Brú *et al.* (see [6]) have presented an alternative characterization of dynamic interfaces by universality classes as derived from the dynamics of its associate visibility interfaces which are obtained at each time from the height interface by a natural visibilty algorithm [7].

This methodology is applied to discriminate between two growing interfaces that posses the same classical critical exponents α and β but that, in contrast, the critical exponents of their visibility interfaces are incompatible. Concretely, we construct a particular non-local dynamics in which dynamical coefficients α_{loc} and β are compatible with the KPZ universality class (see [8, 9]). With this comparison, we highlight the essentiality of the locality assumption, that is the hypothesis that the dynamics is intrisicaly local and therefore, it can be described in terms of (Stochastic) Partial Differential Equations.

To have an efficient procedure to distinguish local from non-local dynamics is of great relevance. non-local models appear in many contexts, in particular, fractional Brownian motions (as suggested by Mandelbrot in [10]), Lévy process (see [11]) and others (see [12]). These processes allow the modeling of anomalous diffusion (sub and super diffusion) and transport in heterogeneous media ([13]) and population dynamics ([14]). A particularly important group of non local models are those described in terms of fractional derivatives (see, for instance, [15] and the references therein).

ROUGHNESS

Let $(r_i(t))_{i=1,\cdots,N}$ be a sequence that changes over time and let $\langle s_i\rangle_{i=i_1,\cdots,i_n}$ be the mean of $(s_{i_1},\cdots,s_{i_n})$. We define roughness of length l as the average of the variation over each section of a partition of length l:

$$w(l,t) = \langle V_i(t)\rangle^{\frac{1}{2}}_{i=1,\cdots,N_l} \tag{1}$$

$$V_i(t) = \left\langle (r_j(t) - m_i(t))^2 \right\rangle_{j\in I_i} \tag{2}$$

$$m_i(t) = \langle r_k(t)\rangle_{k\in I_i} \tag{3}$$

where $I_1,\cdots,I_{N_l}$ is a partition of the set $\{1,\cdots,N\}$ that indexes the heights r_i, such that the "length" of every I_i is constant. Notice that a 1D interface can be expressed as (x_i, r_i). If the elements x_i are equidistant and ordered (for example $x_i = d\ i$) then the sets $I_i = \{N_d,\cdots,N_d(i+1)-1\}$ where $dN_d = \lfloor l\rfloor$, the integer part of l. These expressions appear written compactly in the literature (see [1, 2]) as $w(l,t) = \left\langle \langle (h(x,t) - \overline{h}(t))^2\rangle^{\frac{1}{2}}_x \right\rangle$.

The dynamical exponents α and β such that

$$w(l,t) = \begin{cases} t^{\beta} & \text{if } t \ll t_s \\ l^{\alpha_{loc}} & \text{if } t \gg t_s \end{cases} \tag{4}$$

VISIBILITY INTERFACES

Given an interface of heights $(h_i)_{i=1,\cdots,n}$ we can define the visibility as

> i sees j if the line l_{ij} from (i,h_i) to (j,h_j) keeps all h_k below, that is $\forall i < k < j$:
>
> $$h_k < h_j + (h_i - h_j)\frac{j-k}{j-i}.$$

Using this definition, we can define the associated visibility interface as $(k_i)_{i=1,\cdots,n}$ where k_i is the number of points in the basin of visibility of site i. Of course, a dynamical interface $(h_i(t))$ induces a dynamical visibility interface $(k_i(t))$.

KPZ UNIVERSALITY CLASS

We consider the standard KPZ universality class (see [8]) given by the equation:

$$\frac{\partial h}{\partial t} = \Delta h + |\nabla h|^2 + \eta(x,t) \tag{5}$$

were the noise is uncorrelated, in the sense that

$$\langle \eta(x,t)\eta(x',t')\rangle = D\delta(x-x')\delta(t-t'). \tag{6}$$

which is simulated in the standard way....

FRACTIONAL EW MODEL WITH CORRELATED NOISE

We consider an alternative model:

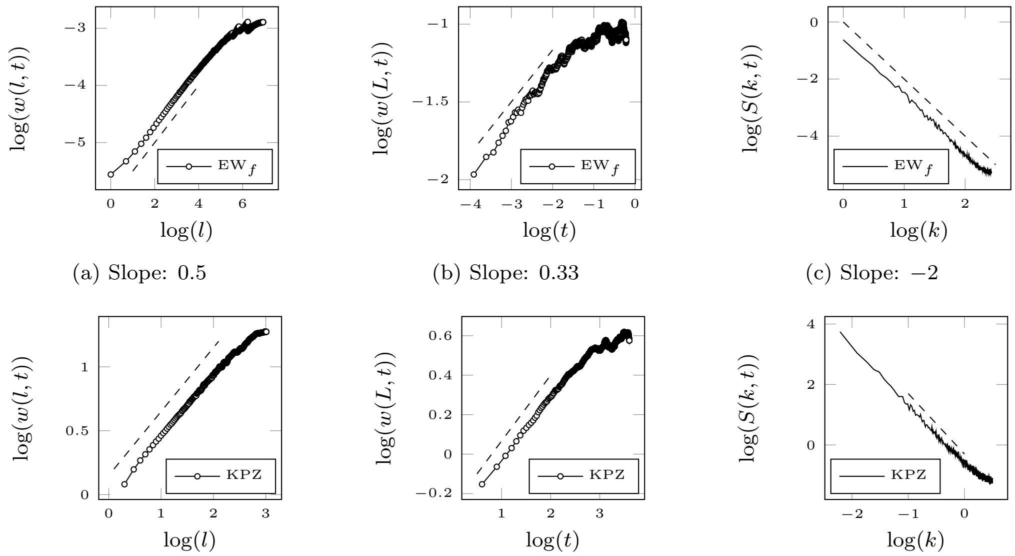
$$\frac{\partial^{\gamma} h}{\partial t^{\gamma}} = \frac{\partial^{\nu} h}{\partial |x|^{\nu}} + \eta(x,t) \tag{7}$$

where $\frac{\partial^{\gamma}}{\partial t^{\gamma}}, 0 < \gamma \leq 1$ is the fractional derivative in the Caputo sense, $\frac{\partial^{\nu}}{\partial |x|^{\nu}} = (-\Delta)^{\frac{\nu}{2}}, 1 < \nu \leq 2$ is the fractional Laplacian (or the Riesz form of the fractional derivative) and $\eta(x,t)$ is a correlated noise

$$\langle \eta(x,t)\eta(x',t')\rangle \sim |x-x'|^{2\rho-1}|t-t'|^{2\varphi-1} \tag{8}$$

It is very important to remark that fractional derivatives $0 < \gamma 1$ and Laplacian $1 < \nu < 2$ are non-local, they cannot be defined in terms of standard local limits. We will focus in the case $\gamma = 1$ (the usual derivative) and $1 < \nu < 2$. The fractional laplacian can be define in terms of integrals, via Fourier transform (in $\mathbb{R}^n$) or in the operational sense. The last, may be written shortly in the following way: the fractional Laplace is the operator is the unique that shares the eigenfunctions of standard laplacian with eigenvalues $\lambda_n[(-\Delta)^{\frac{\nu}{2}}] = (\lambda_n[-\Delta])^{\frac{\nu}{2}}$.

FIG. 1. Scaling analysis of the height interfaces of the Kardar-Parisi-Zhang model and the non-local fractional EW with correlated noise.



(a) Slope: 0.5

(b) Slope: 0.33

(c) Slope: -2

(d) Slope $= 0.4953 \pm 0.004$

(e) Slope $= 0.322 \pm 0.003$

(f) Slope $= -2.00586 \pm 0.03546$

The scaling of these interfases in known ([16]) in particular the critical exponents are:

$$\alpha = \nu + \frac{\nu}{2\gamma}(2\rho - 1) + \frac{2\rho - d}{2}, \quad (9)$$

$$\beta = \gamma + \frac{\gamma}{2\nu}(2\rho - 1) + \frac{2\varphi - 1}{2} \quad (10)$$

We will choose the following exponents

$$\gamma = 1, \quad \nu = \frac{3}{2} \quad \rho = \frac{1}{8} \quad \varphi = \frac{1}{12} \quad (11)$$

which yield the same critical exponents as the KPZ universality class.

Simulation

In order to generate signals we construct the correlated noise correlating the fourier transform of white noise as in [17]. For the space discretization $\mathbf{x} = (0, \cdots, h(N-1))$ where h is the space step, $\mathbf{u} = (u_1, \cdots, u_N)$ and we consider the space semidiscretized problem

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}^{\frac{\nu}{2}}\mathbf{u} + \eta \quad (12)$$

were $\mathbf{A}$ is the finite element discretization of the second derivative with periodic boundary conditions. That is $\mathbf{A} = \frac{1}{h^2}\left(\text{tridiag}(-1, 2, -1) + \mathbf{B}\right)$ and $B_{ij} = 0$ except for $B_{1,N} = B_{N,1} = -1$ (see [18]) (this is the reasonable approach if we consider the operator definition of the fractional laplacian). For the time discretization we consider a standard implicit Euler scheme.

RESULTS

Roughness of the height interfaces

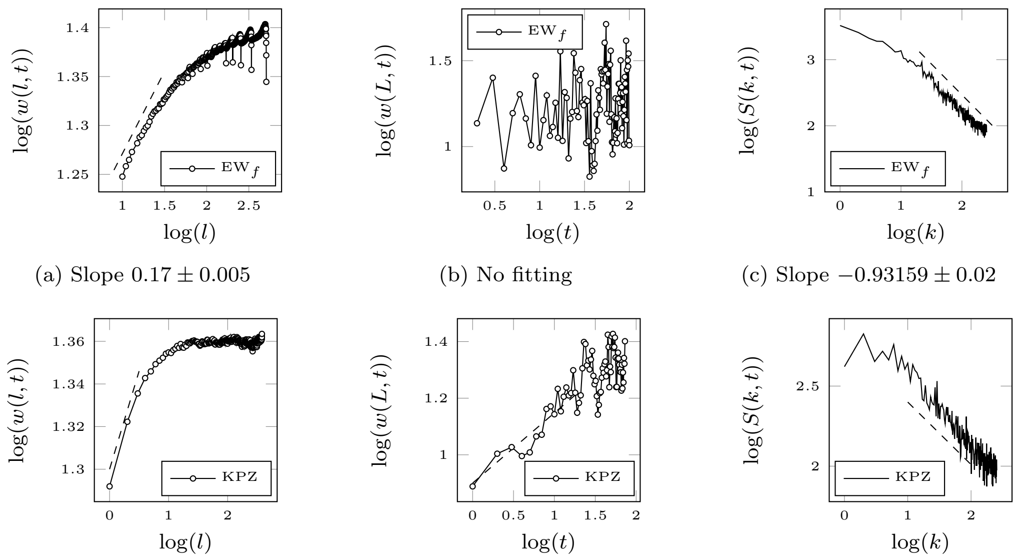
We compute numerically the roughness of the heights both in the simulated KPZ and fractional EW with correlated noise (see Figure 1). For the simulated KPZ we obtain $\alpha_{loc} = 0.500 \pm 0.001$, $\beta = 0.322 \pm 0.003$ which is compatible with the analytic values $\alpha_{loc} = 1/2$, $\beta = 1/3$ (see [8]). For the non-local model with parameter $\gamma = 0.3$ we obtain dynamical exponents $\alpha_{loc} = 0.48 \pm 0.02$, $\beta = 0.36 \pm 0.01$ which are also compatible with the KPZ universality class.

Spectral analysis of the height interfaces

As a supplementary analysis we produce the power spectrum as a Fourier transform. It is known ([4, 19]) that (EXPLAIN α_{glob})

$$S(k, t) = k^{-(2\alpha_{glob}+1)} s\left(kt^{\frac{1}{z}}\right) \quad (13)$$

FIG. 2. Scaling analysis of the visibility interfaces corresponding to the KPZ and fractional EW with correlated noise simulations, respectively.



(a) Slope 0.17 ± 0.005 (b) No fitting (c) Slope -0.93159 ± 0.02

(d) Slope 0.092 ± 0.007 (e) Slope 0.30 ± 0.03 (f) Slope -0.498 ± 0.0218

Roughness of Visibility interface

It is known that the critical exponents of the visibility interface also characterize universality classes (see [6]), at least for the known local universality classes. For the simulated profiles we obtain:

	KPZ	Fractional EW with c.n.
α_{loc}	0.092 ± 0.007	0.17 ± 0.005
β	0.3 ± 0.03	-
α_{glob}	-0.25 ± 0.02	-0.035 ± 0.001

TABLE I. Roughening exponents of the visibility interfases. The results differ slightly from [6] due to the different choice of boundary condition (here periodic, there none was taken)

The surprising result is that the fractional EW with correlated noise does not scale in terms of the variance. Hence, both dynamics do not belong to the same universality class.

CONCLUSIONS

There exist two different dynamics that produce compatible exponents in terms of the usual roughness, but such that the critical exponents of the visibility interfaces are different. In consequence, critical exponents are not able to distinguish whether the processes operating in the interface growth are local or non-local. However, the roughness of the visibility interfaces enables, at least in this case, this discrimination. Therefore, we can conclude that the visibility algorithm is a very useful methodology to check the local character of the microscopic processes that are operating during the interface growth.

ACKNOWLEDGEMENTS

The authors would like to thank Prof. Luis Vázquez for fruitful discussion. The research of D. Gómez-Castro was supported by a FPU grant from the Spanish goverment.

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