Stochastic Analysis and Stochastic PDEs on Fractals

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Dedicated to my parents
for being, to this day, an endless source of support and inspiration

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Abstract

Stochastic analysis on fractals is, as one might expect, a subfield of analysis on fractals. An intuitive starting point is to observe that on many fractals, one can define diffusion processes whose law is in some sense invariant with respect to the symmetries and self-similarities of the fractal. These can be interpreted as fractal-valued counterparts of standard Brownian motion on $\mathbb{R}^d$. One can study these diffusions directly, for example by computing heat kernel and hitting time estimates. On the other hand, by associating the infinitesimal generator of the fractal-valued diffusion with the Laplacian on $\mathbb{R}^d$, it is possible to pose stochastic partial differential equations on the fractal such as the stochastic heat equation and stochastic wave equation. In this thesis we investigate a variety of questions concerning the properties of diffusions on fractals and the parabolic and hyperbolic SPDEs associated with them. Key results include an extension of Kolmogorov's continuity theorem to stochastic processes indexed by fractals, and existence and uniqueness of solutions to parabolic SPDEs on fractals with Lipschitz data.

Contents

Chapter 1

Introduction

The study of random processes on fractals was motivated initially by interest in the physical properties of such spaces – for example, how would one model the spread of heat or other fluids on such a space? What about the evolution of waves? These are certainly pertinent questions from the viewpoint of the natural scientist since nature is rife with structures that exhibit some level of self-similarity, from the shape of a fern leaf to that of a coastline. Moreover, even the mathematician often encounters self-similar structures in perhaps unexpected places; take for example percolation (in the shape of clusters at criticality) or chaos theory (the logistic map). See [39] for a survey, and for more mathematical approaches see [52], [3], [56].

To see how one might model the diffusion of heat on a fractal, consider the relationship between Brownian motion and the heat equation through the Feynman-Kac formula [47]: for $\varphi : \mathbb{R}^d \to \mathbb{R}$ continuous and bounded, the function

$$\begin{aligned} u : [0,\infty) \times \mathbb{R}^d &\to \mathbb{R} \\ (t,x) &\mapsto u(t,x) \end{aligned}$$

satisfies the partial differential equation (PDE)

$$\begin{aligned} \partial_t u &= \frac{1}{2}\Delta u, \\ u(0,x) &= \varphi(x) \end{aligned}$$

if and only if $u(t,x) = \mathbb{E}^x[\varphi(B_t)]$, where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}^n$. Therefore suppose that for $t \geq 0$ we define an operator T_t such that $T_t\varphi(x) = \mathbb{E}^x[\varphi(B_t)]$, known as the semigroup of B. T_t is a semigroup because B is a (time-homogeneous) Markov process, and from its derivative at $t = 0$ we recover the Laplacian on $\mathbb{R}^n$: $\partial_t(T_t\varphi) = \frac{1}{2}\Delta\varphi$. This suggests that if we can define a Markov process on an arbitrary space $\mathcal{M}$, we may derive from it a "Laplacian"

$\Delta_{\mathcal{M}}$, and finally from that a "heat equation" $\partial_t f = \Delta_{\mathcal{M}} f$ on $\mathcal{M}$ for suitable functions $f : [0, \infty) \times \mathcal{M} \to \mathbb{R}$. The first order of business is thus to construct Markov processes on fractals.

We will take as an initial instructive example a characterisation of Brownian motion B on $\mathbb{R}$. We proceed by first defining $(S_n)_{n \geq 0}$ to be the simple symmetric random walk on $\mathbb{Z}$, starting at 0. Let $(S(t))_{t \in [0,\infty)}$ be the linear interpolation of $(S_n)_{n \geq 0}$, and let $S_t^n = 2^{-n} S(4^n t)$. Then Donsker's invariance principle states that

$$S^n \to B$$

in law in the space $C[0, 1]$ of continuous functions on $[0, 1]$. Intuitively, by repeatedly halving the step size of the chain (so that it takes values in $2^{-1}\mathbb{Z}$, then $2^{-2}\mathbb{Z}$ and so on) and quadrupling its speed, we end up with a Markov process on the entire non-discrete space $\mathbb{R}$. Notice importantly that $(2^{-n}\mathbb{Z})_n$ are a nested increasing sequence of subsets such that

$$\mathbb{R} = \overline{\bigcup_{n \geq 0} (2^{-n}\mathbb{Z})}.$$

Many fractals of note, such as the Sierpinski gasket (Figure 1.1) and the Vicsek fractal (Figure 1.2), can be characterised as limits of increasing sequences of graph approximations similar to the way that $2^{-n}\mathbb{Z} \nearrow \mathbb{R}$. Each subsequent approximation is constructed by taking a number $N > 1$ of copies of the previous approximation, contracting them, and rearranging them; this process will be defined rigorously later. We might therefore hope that by defining a simple symmetric Markov process on each of the approximations and speeding them up by the correct factor we can obtain in the limit a Markov process that takes values on the entire fractal. This is indeed possible on a class of fractals known as post-critically finite self-similar sets (p.c.f.s.s. sets) which will be defined later on, and include the Sierpinski gasket and the Vicsek set. It is in fact possible to show that the limiting process is a diffusion – that is, it is a strong Markov process with continuous sample paths.

Once we have a diffusion X on a fractal, we may obtain its semigroup and its infinitesimal generator, both of which respect the same symmetries of the fractal that X does. We call the infinitesimal generator a "Laplacian" on the fractal, as its relationship with X is equivalent to the relationship between the standard Laplacian on $\mathbb{R}^d$ and Brownian motion on $\mathbb{R}^d$ (ignoring a factor of 2). In particular, this allows us to define a class of stochastic partial differential equations (SPDEs) on the fractal. SPDEs can be interpreted as a generalization of stochastic differential equations to infinite-dimensional spaces. The study of these equations is motivated by the desire

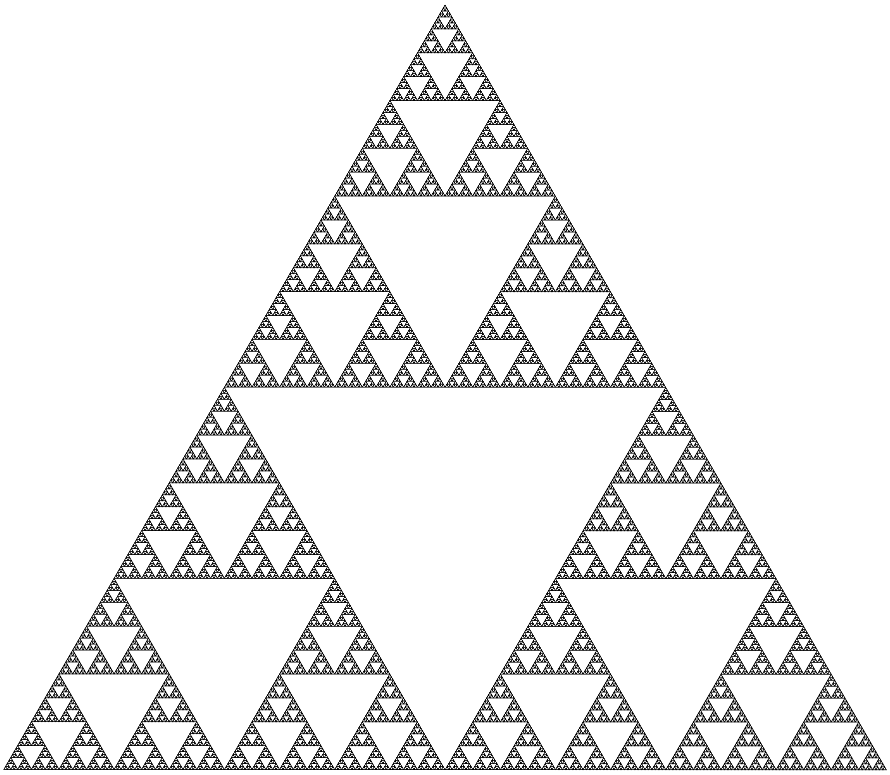

Figure 1.1: The Sierpinski gasket.

to understand physical systems that incorporate notions of random noise. The canonical example, described in [77], is of a stringed instrument left out in a sandstorm. We model an (ideal and frictionless) elastic string as the interval $[0, 1]$. Then its displacement in a direction orthogonal to its length is, after scaling, classically described by a well-known PDE called the wave equation:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x), \quad (t,x) \in [0,T] \times [0,1],$$

with some appropriate initial and boundary conditions. Here f describes the influence of any forces acting on the string which come from external sources. In the case of a sandstorm we assume that the string is randomly struck from all directions by a large number of particles. We assume that each collision is independent of all the others and that the collisions are distributed evenly along the string in space and time. With the central limit theorem in mind, the correct characterization of f should therefore be as some kind of Gaussian process (which, after having subtracted the mean effect of the collisions, can be assumed to be centred).

We thus model the force of the sandstorm on the string as a *space-time white noise*, which at this point can be thought of as a centred Gaussian random measure

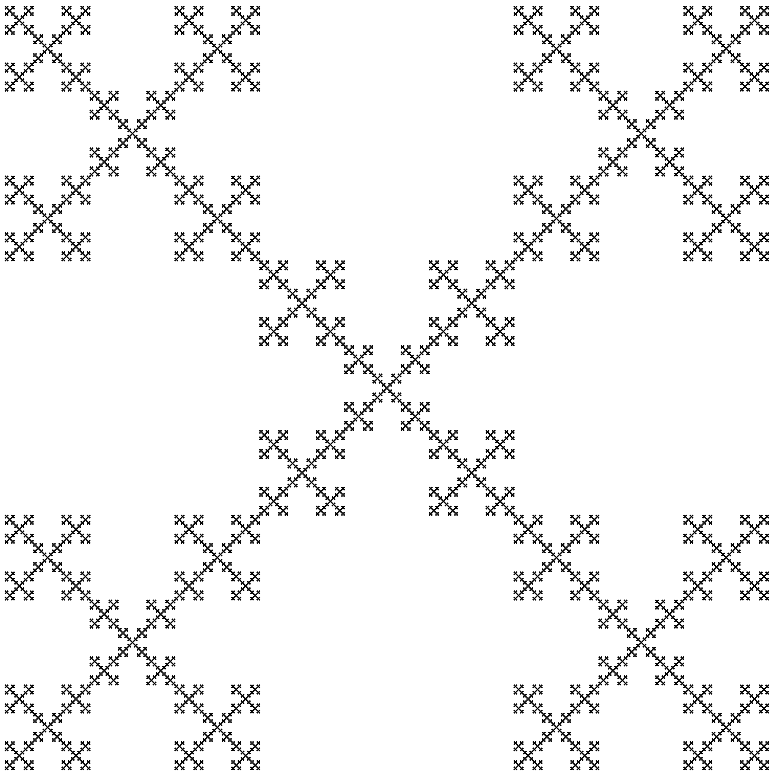

Figure 1.2: The Vicsek fractal.

ξ on $[0,T]\times[0,1]$ with covariance

$$\mathbb{E}\left[\xi(A)\xi(B)\right] = \mathrm{Leb}_2(A\cap B)$$

for Borel measurable $A,B\subseteq[0,T]\times[0,1]$, where Leb_d denotes d-dimensional Lebesgue measure. Using a scaling coefficient $\sigma>0$ to describe the magnitude of the noise, the resulting wave equation is the following SPDE:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \sigma\xi(t,x), \quad (t,x)\in[0,T]\times[0,1].$$

We see that if $A\subseteq[0,1]$ is measurable, then the net impulse from the sandstorm on the subset A of the string in time $t>0$ is the Gaussian random variable

$$\int_0^t\int_A \sigma\xi(s,x)dxds = \sigma\xi([0,t]\times A),$$

and it is easy to show that the process $t\mapsto\sigma\xi([0,t]\times A)$ has the law of a real Brownian motion multiplied by the scaling factor $\sigma\sqrt{\mathrm{Leb}_1(A)}$. Moreover, if $A_1,A_2\subseteq[0,1]$ are measurable and disjoint, then the impulses on these two subsets over any intervals of time are independent. This SPDE is popularly known as the *stochastic wave equation.*

In the remainder of this chapter we review much of the background theory that we will need on analysis and probability on fractals, and we give a brief introduction to the general theory of SPDEs. Every chapter after this one is based on an article which is either published or in the process of being published; the first page of each chapter bears a footnote referring to its original article. Because of this, each chapter is notationally independent of the others. In Chapter 2 we consider a superclass of p.c.f.s.s. sets for which it is possible to relax some symmetry conditions in order to construct a continuum of different diffusions. It is shown that these diffusions tend towards a natural limiting diffusion which exists not on the original fractal, but on a topological quotient of it. The remaining chapters are devoted to developing a theory of SPDEs on p.c.f. fractals. We show that unique solutions exist for general parabolic SPDEs and for the stochastic wave equation, and we prove that under additional minor assumptions these solutions are function-valued and almost surely Hölder continuous. We additionally prove a number of other regularity properties for the solutions, such as moment bounds and limiting distributions. In the appendix we prove two theorems concerning uniform tightness of families of continuous stochastic processes indexed by fractals. These do not fit into any of the existing chapters but are of independent interest. We then give an overview of possible directions for further research based on the results proven in this thesis.

1.1 Dirichlet forms

We quickly review some of the theory compiled in [3, Section 4] on Dirichlet forms; see also [52], [22]. Let F be a locally compact metric space with a countable base, and μ a Radon measure on F. Let $C(F)$ be the space of real-valued continuous functions on F equipped with the uniform norm $\|\cdot\|_\infty$, let $C_0(F) \subseteq C(F)$ be the subspace of functions that approach zero at infinity (which is a Banach space), and let $L^2(F,\mu)$ be the usual Hilbert space with inner product $\langle f, g\rangle_2 = \int_F fg d\mu$.

Definition 1.1.1. Let D be a linear subspace of $L^2(F,\mu)$. A *symmetric form* is a map $\mathcal{E} : D \times D \to \mathbb{R}$ such that

(1). $\mathcal{E}$ is bilinear, symmetric.

(2). $\mathcal{E}(f, f) \geq 0$ for all $f \in D$.

In addition, on D we define for each $a \geq 0$ a bilinear map $\mathcal{E}_a(f, g) := \mathcal{E}(f, g)+a\langle f, g\rangle_2$ and a norm $\|\cdot\|_{\mathcal{E}_a}$ by

$$\|f\|^2_{\mathcal{E}_a} := \|f\|^2_2 + a\mathcal{E}(f, f) = \mathcal{E}_a(f, f).$$

Definition 1.1.2. Let $(\mathcal{E}, D)$ be a symmetric form.

(1). $\mathcal{E}$ is *closed* if $(D, \|\cdot\|_{\mathcal{E}_1})$ is complete. Note that $\mathcal{E}$ is closed if and only if $(D, \|\cdot\|_{\mathcal{E}_a})$ is complete for all $a > 0$.

(2). $(\mathcal{E}, D)$ is *Markov* if for $f \in D$, if $g := (0 \vee f) \wedge 1$ then $g \in D$ and $\mathcal{E}(g, g) \leq \mathcal{E}(f, f)$.

(3). $(\mathcal{E}, D)$ is a *Dirichlet form* if it is closed and Markov and if D is dense in $L^2(F, \mu)$.

Definition 1.1.3. Let $(\mathcal{E}, D)$ be a Dirichlet form.

(1). $\mathcal{E}$ is *regular* if the space $D \cap C_0(F)$ is dense in D with respect to $\|\cdot\|_{\mathcal{E}_1}$, and is dense in $C_0(F)$ with respect to $\|\cdot\|_\infty$.

(2). $\mathcal{E}$ is *local* if $\mathcal{E}(f, g) = 0$ whenever f, g have disjoint support.

(3). $\mathcal{E}$ is *conservative* if $1 \in D$ and $\mathcal{E}(1, 1) = 0$.

(4). $\mathcal{E}$ is *irreducible* if it is conservative and $\mathcal{E}(f, f) = 0$ implies that f is constant.

Definition 1.1.4. Let $(\mathcal{E}, D)$ be a Dirichlet form. For $a > 0$ the *resolvent* $U_a : D \to D$ associated with $\mathcal{E}$ is the linear operator characterized by

$$\mathcal{E}_a(U_a f, g) = \langle f, g \rangle_2$$

for all $f, g \in D$. This defines U_a uniquely because of the completeness of $(D, \|\cdot\|_{\mathcal{E}_a})$ and the Riesz representation theorem. It can be verified that U_a satisfies the resolvent equation and is strongly continuous. It follows by the Hille-Yosida theorem that (U_a) is the resolvent of a semigroup (T_t). Define this to be the semigroup associated with $\mathcal{E}$.

Remark 1.1.5. Using the resolvent, it is possible to reconstruct $\mathcal{E}$ from its semigroup (T_t), implying that the mapping $\mathcal{E} \mapsto (T_t)$ is injective.

We now define the classes of stochastic processes that are naturally associated with Dirichlet forms:

Definition 1.1.6. A strong Markov process $X = (X_t)_{t \in [0,\infty)}$ on F is a *Hunt process* if it has càdlàg sample paths and is quasi-left-continuous. It is a *diffusion* if it is continuous.

The above definitions are sufficient for our purposes. More detailed definitions of Hunt processes exist, see for example [22, Section A.2] or [7, Section 1.9].

Definition 1.1.7. A diffusion X on F is *μ-symmetric* if

$$\langle T_t f, g\rangle_2 = \langle f, T_t g\rangle_2$$

for all bounded and compactly supported $f, g \in L^2(F, \mu)$, where $(T_t)_{t\in[0,\infty)}$ is the semigroup associated with X (so that $T_t f(x) = \mathbb{E}^x f(X_t)$ μ-a.e.). If (T_t) is associated with a Dirichlet form $\mathcal{E}$, then we say that $\mathcal{E}$ is the Dirichlet form associated with X.

Theorem 1.1.8 (Fundamental theorem)**.** *Let $(\mathcal{E}, D)$ be a regular Dirichlet form on $L^2(F, \mu)$. Then there exists a μ-symmetric Hunt process $X = (X_t)_{t\in[0,\infty)}$ on F with Dirichlet form $\mathcal{E}$.*

Moreover, X is a diffusion if and only if $\mathcal{E}$ is local. If $\mathcal{E}$ is conservative then X has infinite lifetime. If $\mathcal{E}$ is irreducible then X is an irreducible process in the sense of [3, Definition 4.4(b)].

Remark 1.1.9. The law of the process X is not a priori uniquely specified by $(\mathcal{E}, D)$. However if all points in F are non-polar for X, that is, if X "hits points in finite time", then X is indeed uniquely specified (see [22, Theorem 4.2.8]). This will be the case for all of the processes that we will be considering.

Definition 1.1.10. For a measurable set $H \subseteq F$, its *capacity* $\mathrm{Cap}(H)$ is given by

$$\mathrm{Cap}(H) = \inf_{f\in\mathcal{A}_H} \|f\|_{\mathcal{E}_1}$$

where $\mathcal{A}_H := \{f \in D : f \geq 1 \ \mu\text{-a.e. on } H\}$.

Definition 1.1.11. Let ν be a Radon measure on F, possibly distinct from μ. Let G be its topological support. For $g \in L^2(F, \nu)$ define

$$\tilde{\mathcal{E}}(g, g) = \inf\{\mathcal{E}(f, f) : f|_G = g\}$$

and

$$\tilde{D} = \{g \in L^2(F, \nu) : \tilde{\mathcal{E}}(g, g) < \infty\}$$

where we use the convention $\inf \emptyset = \infty$. The symmetric form $\tilde{\mathcal{E}}$ is called the *trace* of $\mathcal{E}$ on G, and we write

$$\tilde{\mathcal{E}} = \mathrm{Tr}(\mathcal{E}|G).$$

Remark 1.1.12. Observe that if $H \subseteq G \subseteq F$, then

$$\mathrm{Tr}(\mathrm{Tr}(\mathcal{E}|G)|H) = \mathrm{Tr}(\mathcal{E}|H),$$

and if $f \in D$ then

$$\mathrm{Tr}(\mathcal{E}|G)(f|_G, f|_G) \leq \mathcal{E}(f, f).$$

Theorem 1.1.13 (Trace theorem)**.** *Let X be a μ-symmetric Hunt process on (F, μ) with regular Dirichlet form $(\mathcal{E}, D)$.*

(1). The pair $(\tilde{\mathcal{E}}, \tilde{D})$ is a regular Dirichlet form on $L^2(F, \nu)$.

(2). Suppose in addition $\mathrm{Cap}(\{x\}) > 0$ for all $x \in F$. Then there exists a time-change (τ_t) of X such that if $\tilde{X}$ is the process with $\tilde{X}_t = X_{\tau_t}$ then $\tilde{X}$ is a ν-symmetric Hunt process on (G, ν) with Dirichlet form $(\tilde{\mathcal{E}}, \tilde{D})$. $\tilde{X}$ is known as the trace of X on G.

Remark 1.1.14. This is [22, Theorem 6.2.1] and [3, Theorem 4.17]. In the latter the exact form of the time-change is given.

Theorem 1.1.13 gives us an elegant way of relating a Dirichlet form on the set F to a Dirichlet form on a subset of F. Moreover, the additional assumption given in Theorem 1.1.13(2) will be true in all of the cases that we will be studying.

We might now hope that, if our space F can be characterised as an increasing sequence of subsets $\bigcup F^n$ where each $F^n \subseteq F$ is in some sense "easy" to study, then we can use the trace theorem to glean some information about the process X on F by studying its traces X^n on each of the F^n. We might even hope for some kind of convergence $X^n \to X$. For a certain class of fractals this can in fact be done! The subsets F^n in this case are the natural approximations to the fractal, such as the increasing finite collections of (vertices of) triangles that converge toward the Sierpinski gasket. To do this, we need to first understand how Dirichlet forms behave on finite sets. This is the subject of the next section.

1.1.1 Dirichlet forms on finite sets

We continue the review of theory from [3, Section 4]. Now let F be a finite set and μ a finite measure on F that charges every point (otherwise we just replace F with the support of μ). Thus $L^2(F, \mu) = C(F) = \{f : F \to \mathbb{R}\}$. We use the notation $\mu(x) := \mu(\{x\})$ for $x \in F$.

Definition 1.1.15. A *conductance matrix* A on F is a symmetric matrix $A = (a_{xy})_{x,y \in F}$ such that

- $a_{xy} \geq 0$ for $x \neq y$,
- $\sum_{y \in F} a_{xy} = 0$ for all $x \in F$.

In particular, this means that $a_{xx} = -\sum_{y \neq x} a_{xy}$ for all $x \in F$. The conductance matrix A is associated with an *edge set* $E_A := \{\{x, y\} : a_{xy} > 0\}$ and a Dirichlet form $\mathcal{E}_A$ with domain $D = C(F)$ given by

$$\mathcal{E}_A(f, g) = \frac{1}{2} \sum_{x,y} a_{xy}(f(x) - f(y))(g(x) - g(y)).$$

A is *irreducible* if (F, E_A) is a connected graph.

Notice that if the functions f, g are interpreted as vectors indexed by F then

$$\mathcal{E}_A(f, g) = -f^T A g.$$

The pair (F, A) has the following interpretation in the context of electrical theory: if F is a collection of nodes where each pair of nodes in E_A is connected by a wire, and a_{xy} is the conductance of the wire connecting the pair $\{x, y\} \in E_A$, and the nodes are held at electric potentials given by $f \in D$, then $\mathcal{E}_A(f, f)$ is the power dissipation of the circuit. For this reason, (F, A) is called a *conductance network*.

In fact, conductance matrices characterise all of the (interesting) Dirichlet forms on a finite set:

Proposition 1.1.16 (Characterisation of Dirichlet forms)**.** *If A is a conductance matrix on F, then $\mathcal{E}_A$ is a regular conservative Dirichlet form on $L^2(F, \mu)$. Conversely, if $\mathcal{E}$ is a conservative Dirichlet form on $L^2(F, \mu)$, then there exists a conductance matrix A on F such that $\mathcal{E} = \mathcal{E}_A$.*

Moreover, $\mathcal{E}_A$ is irreducible if and only if A is irreducible.

Every conductance matrix A on F is thus associated with a symmetric Markov process X on F. The generator L of X can be shown to be given by

$$Lf(x) = \mu(x)^{-1} \sum_{y \neq x} a_{yx}(f(y) - f(x)),$$

so if $x \neq y$ then the rate at which X jumps from x to y is $\frac{a_{yx}}{\mu(x)}$.

Remark 1.1.17. We see that the measure μ only affects the transition rates of X; it has no effect on the jump chain of X.

Remark 1.1.18. Let $(Y_n)_{n \geq 0}$ be the jump chain of X, let $G \subseteq F$, let $\mathcal{E}' = \mathrm{Tr}(\mathcal{E}|G)$ and let X' be the Markov process associated with G. Let

$$n_0 = \inf\{n \geq 0 : Y_n \in G\},$$
$$n_{j+1} = \inf\{n \geq n_j : Y_n \in G \setminus \{Y_{n_j}\}\}, \quad j \geq 0$$

be stopping times. Then the discrete process $(Y_{n_j})_{j \geq 0}$ is equal in law to the jump chain of X'. This is a consequence of the trace theorem.

Remark 1.1.19. Having calculated the transition rates of X, it is easy to show that it satisfies the detailed balance equations with stationary measure μ; X is thus indeed symmetric. The converse is also true: suppose X is a symmetric Markov chain on F with generating matrix $Q = (q_{xy})_{x,y \in F}$. There thus exists a measure π that charges every point in F such that for all $x, y \in F$,

$$\pi(x) q_{xy} = \pi(y) q_{yx}.$$

Let $\mu = \pi$ and let A be the conductance matrix such that $a_{xy} = \pi(x) q_{xy}$ for each $x \neq y$ in F. Then X is the Markov process associated with A on (F, μ).

Definition 1.1.20. Let G_0, G_1 be disjoint subsets of F. The *effective resistance* between G_0 and G_1 is given by

$$R(G_0, G_1) = \left(\inf\left\{\mathcal{E}(f,f) : f|_{G_0} = 0, f|_{G_1} = 1\right\}\right)^{-1}.$$

Notice that this is always finite if A is irreducible.

Definition 1.1.21. Define a function $r : F \times F \to [0, \infty)$ by $r(x,x) = 0$ for all $x \in F$ and

$$r(x,y) = R(\{x\}, \{y\})$$

for $x \neq y$. It can be shown that r is a metric on F, known as the *resistance metric.*

Evidently the resistance metric is preserved under traces: if $G \subseteq F$ is a subset with Dirichlet form $\mathcal{E}' = \mathrm{Tr}(\mathcal{E}|G)$ and resistance metric $r' : G \times G \to [0, \infty)$, then we can easily show that

$$r' = r|_{G \times G}.$$

It is this property that makes the resistance metric the natural metric to use on F when working with Dirichlet forms.

1.2 Fractals

In this section we discuss the precise construction of the fractals that we will be studying. It is a review of theory from [3, Section 5] and [52].

Definition 1.2.1. Let (M, d) be a metric space. A map $\psi : M \to M$ is a *similitude* if there exists $a \in (0,1)$ such that $d(\psi(x), \psi(y)) = a d(x,y)$ for all $x, y \in M$. The constant a is called the *contraction factor* of ψ.

Theorem 1.2.2. *Let $N \in \mathbb{N}$ and let $\psi_1, \ldots, \psi_N$ be similitudes on $\mathbb{R}^n$ with contraction factors a_i. For a subset $A \subseteq \mathbb{R}^n$ set*

$$\Psi(A) := \bigcup_{i=1}^{N} \psi_i(A).$$

Then there exists a unique non-empty compact $F \subseteq \mathbb{R}^n$ such that $F = \Psi(F)$.

See [44, Section 3] for a much more general version of this result. The proof uses the Banach fixed-point theorem on the space of non-empty compact subsets of $\mathbb{R}^n$ with the Hausdorff metric. Many well known fractals (for example the Sierpinski gasket or the Koch curve) can be obtained as fixed points using the above theorem. These fractals are thus naturally embedded in Euclidean space. However, since we are hoping eventually to define a new metric (a resistance metric) on F, we need to be able to work with it without having to think about any particular embedding. We therefore use the following more abstract definition:

Definition 1.2.3. Let (F, d) be a compact metric space. Then $(F, (\psi_i)_{1 \leq i \leq N})$ is a *self-similar structure* if

(1). ψ_i are continuous injections with

$$F = \bigcup_{i=1}^{N} \psi_i(F),$$

(2). There exists $\delta > 0$ such that

$$d(\psi_i(x), \psi_i(y)) \leq (1 - \delta) d(x, y)$$

for all $x, y \in F$ and $1 \leq i \leq N$.

Notice that we have weakened the condition on the ψ_i; they no longer need to be strict similitudes.

Let $I := \{1, \ldots, N\}$, and define the *word spaces* $\mathbb{W}_n = I^n$, $\mathbb{W} = I^{\mathbb{N}}$. Endow $\mathbb{W}$ with the usual product topology, and let σ be the left shift operator, which maps $\mathbb{W}$ to $\mathbb{W}$ or $\mathbb{W}_n$ to $\mathbb{W}_{n-1}$. That is, if $w = (w_1, w_2, w_3, \ldots)$ then $\sigma(w) = (w_2, w_3, \ldots)$. For $w = (w_1, \ldots, w_n) \in \mathbb{W}_n$, define $\psi_w := \psi_{w_1} \circ \cdots \circ \psi_{w_n}$. For $A \subseteq F$ let $A_w := \psi_w(A)$. If $n \geq 1$ and $w \in \mathbb{W}$ (or $w \in \mathbb{W}_m$ with $m \geq n$) then define $w|n = (w_1, \ldots, w_n) \in \mathbb{W}_n$.

Lemma 1.2.4. *(1). For each $w \in \mathbb{W}$, there exists $x_w \in F$ such that*

$$\bigcap_{n=1}^{\infty} \psi_{w|n}(F) = \{x_w\}.$$

(2). Let $\pi : \mathbb{W} \to F$ be the mapping given by $\pi(w) := x_w$. Then π is the unique mapping $\pi : \mathbb{W} \to F$ such that $\pi(i \cdot w) = \psi_i(\pi(w))$ for all $w \in \mathbb{W}$ and $i \in I$, where $i \cdot w \in \mathbb{W}$ denotes the concatenation of i with w. Moreover, π is continuous and surjective.

It follows from the above that every word $w \in \mathbb{W}$ is the "address" of a point in F, and every point has at least one address. But can a point in F have more than one address? The answer is yes, but we must place a restriction on how many of these points exist:

Definition 1.2.5. For a self-similar structure $\mathcal{S} = (F, (\psi_i)_{1 \le i \le N})$, let

$$B = B(\mathcal{S}) = \bigcup_{i,j: i \neq j} F_i \cap F_j,$$

and let

$$\Gamma = \pi^{-1}(B)$$

be the *critical set* of $\mathcal{S}$. Let

$$P = \bigcup_{n=1}^{\infty} \sigma^n(\Gamma)$$

be the *post-critical set* of $\mathcal{S}$.

Definition 1.2.6. A self-similar structure $(F, (\psi_i)_{1 \le i \le N})$ is *post-critically finite* if P is finite. A metric space (F, d) is a *post-critically finite self-similar* set, or *p.c.f.s.s.* set, if there exists a post-critically finite self-similar structure $(\psi_i)_{1 \le i \le N}$ on F. An equivalent term for p.c.f.s.s. set is *p.c.f. fractal*.

This definition seems fairly contrived at first glance, but it can be reformulated as follows:

$$P = \pi^{-1}\left(\left\{x \in F : \exists w, v \in \bigcup_{n \ge 1} \mathbb{W}_n,\ w \neq v,\ \psi_w(x) \in F_v\right\}\right).$$

In words, P is the preimage under π of what is in some sense the "boundary" of F. The Sierpinski gasket is a p.c.f.s.s. set with $N = 3$. We see that in this case, $|P| = 3$ and $\pi(P)$ is the set of the three outermost vertices of the gasket.

The next step is to define a Dirichlet form on a p.c.f. fractal in a way that, in some sense, is invariant with respect to the self-similarities of the fractal. Then we may obtain an associated diffusion. This is recapped (in slightly greater generality) in Section 2.2.

1.3 Infinite-dimensional stochastic analysis

In this section we give a brief introduction to the theory of stochastic analysis on infinite-dimensional spaces. Recall our example SPDE, the stochastic wave equation:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \sigma\xi(t,x), \quad (t,x) \in [0,T] \times [0,1].$$

Upon closer inspection, a problem quickly presents itself: what exactly *is* $\xi(t,x)$ for some $(t,x) \in [0,T] \times [0,1]$? For example, suppose the function $(t,x) \mapsto \xi(t,x)$ were continuous. Then we would have

$$\xi(t,x) = \lim_{n\to\infty} (\mathrm{Leb}_2(A_n))^{-1} \int_{A_n} \xi(s,y) ds dy = \lim_{n\to\infty} (\mathrm{Leb}_2(A_n))^{-1} \xi(A_n)$$

for any decreasing sequence of measurable subsets $(A_n) \subseteq [0,T] \times [0,1]$ with finite Lebesgue measure and for which $\bigcap_n A_n = \{(t,x)\}$. However, since $\xi(A_n)$ is a centred Gaussian random variable with variance $\mathrm{Leb}_2(A_n)$, it is easy to see that the sequence of random variables $((\mathrm{Leb}_2(A_n))^{-1}\xi(A_n))_n$ cannot converge, even in law! This is why the stochastic wave equation can only be considered formally, and in fact there are multiple ways to make sense of it. We briefly introduce two of these, specifically the approaches of Walsh [77] and da Prato–Zabczyk [15]. We then discuss the similarities and differences between these two approaches.

1.3.1 The Walsh approach

1.3.1.1 Martingale measures

The core notion of the Walsh theory of SPDEs is the *martingale measure*, developed in [77, Chapter 2]. We will then define space-time white noise as a particularly well-behaved example of a martingale measure. Let E be a Lusin space, that is, a topological space homeomorphic to a Borel subset of a compact metric space. Let $\mathscr{E}$ be the Borel σ-algebra of E. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We start off with a series of definitions:

Definition 1.3.1. Let $\mathscr{A} \subseteq \mathscr{E}$ be an algebra and let $\nu : \mathscr{A} \to L^2(\Omega)$ be a countably additive set function. We say that ν is *σ-finite* if there exists an increasing sequence $(E_n) \in \mathscr{A}$ such that $\bigcup_n E_n = E$ and for all n, $\mathscr{E}|_{E_n} \subseteq \mathscr{A}$ and $\sup\{\mathbb{E}[\nu(A)^2] : A \in \mathscr{E}|_{E_n}\} < \infty$.

Definition 1.3.2. Let $\mathscr{A} \subseteq \mathscr{E}$ be an algebra and let $\nu : \mathscr{A} \to L^2(\Omega)$ be a σ-finite countably additive set function. Let ν_* be the following extension of ν to $\mathscr{E}$:

$$\nu_*(A) = \begin{cases} \lim_{n\to\infty} \nu(A \cap E_n) & \text{if this limit exists in } L^2(\Omega), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then ν_* is called a *σ-finite L^2-valued measure*.

Henceforth we identify all σ-finite countably additive set functions on $(E, \mathscr{E})$ with their respective σ-finite L^2-valued measure extensions.

Definition 1.3.3. Let $\mathscr{A} \subseteq \mathscr{E}$ be an algebra and let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a right-continuous filtration on $(\Omega, \mathcal{F}, \mathbb{P})$. A real-valued process $M = (M_t(A) : t \geq 0, A \in \mathscr{A})$ is a *martingale measure* with respect to $\mathbb{F}$ if

(1). $M_0(A) = 0$ for each $A \in \mathscr{A}$,

(2). M_t is a σ-finite L^2-valued measure for each $t \geq 0$,

(3). $t \mapsto M_t(A)$ is an $\mathbb{F}$-martingale for each $A \in \mathscr{A}$.

Definition 1.3.4. A martingale measure $M = (M_t(A) : t \geq 0, A \in \mathscr{A})$ is *orthogonal* if for all disjoint $A, B \in \mathscr{A}$, the process $t \mapsto M_t(A)M_t(B)$ is a martingale.

Now we define space-time white noise on a general measure space and prove it has all the properties defined above.

Definition 1.3.5. Let E be a Lusin space equipped with a σ-finite Borel measure μ and let $\mathbb{F}$ be a right-continuous filtration. Let $\mathscr{A} = \{A \subseteq E : \mu(A) < \infty\}$. A *space-time white noise* on (E, μ) with respect to $\mathbb{F}$ is a centred Gaussian process $\xi = (\xi_t(A) : t \geq 0, A \in \mathscr{A})$ with covariance given by

$$\mathbb{E}\left[\xi_s(A)\xi_t(B)\right] = (s \wedge t)\mu(A \cap B)$$

such that each $t \mapsto \xi_t(A)$ is an $\mathbb{F}$-martingale.

Note that the existence of space-time white noise on (E, μ) is guaranteed by the Kolmogorov extension theorem. We may just write *space-time white noise on E* if the measure is obvious from context.

Theorem 1.3.6. *Let E be a Lusin space equipped with a σ-finite Borel measure μ and let $\mathbb{F}$ be a right-continuous filtration. Then any space-time white noise ξ on (E, μ) with respect to $\mathbb{F}$ is (extendable to) an orthogonal martingale measure (with respect to $\mathbb{F}$).*

Proof. It is easy to show that each ξ_t is a countably additive set function from $\mathscr{A}$ to $L^2(\Omega)$. The rest is definition chasing using the regularity of Gaussian processes. $\square$

1.3.1.2 Stochastic integrals

Let E be a Lusin space equipped with a σ-finite Borel measure μ and let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a completed filtration. Let ξ be a space-time white noise on (E, μ) with respect to $\mathbb{F}$. We now construct a theory of stochastic integration on (E, μ) with respect to ξ. In fact, the class of processes with respect to which a theory of stochastic integration can be constructed includes all orthogonal martingale measures ([77, Corollary 2.9]). As usual, we start by defining integrals with simple integrands explicitly and then extend this to a wider space using an isometry.

Definition 1.3.7. A process $f : \Omega \times [0, \infty) \times E \to \mathbb{R}$ is *elementary* if it takes the form

$$f(\omega, t, x) = X(\omega)\mathbb{1}_{(a,b]}(t)\mathbb{1}_A(x),$$

where $0 \leq a < b < \infty$, X is bounded and $\mathcal{F}_a$-measurable, and $A \in \mathscr{A} = \{B \subseteq E : \mu(B) < \infty\}$. The process f is *simple* if it is a finite linear combination of elementary functions. Let $\mathscr{S}$ be the vector space of simple processes.

Definition 1.3.8. Let $f \in \mathscr{S}$. It thus has a representation as a finite linear combination of elementary functions

$$f(\omega, s, x) = \sum_{i=1}^{n}\sum_{j=1}^{m} X_{i,j}(\omega)\mathbb{1}_{(a_i,b_i]}(s)\mathbb{1}_{A_j}(x)$$

where the sets $(a_i, b_i]$ are pairwise disjoint and the sets A_j are pairwise disjoint. Its stochastic integral with respect to the space-time white noise ξ is then defined to be

$$\int_0^\infty \int_E f(s,x)\xi(s,x)\mu(dx)ds := \sum_{i=1}^{n}\sum_{j=1}^{m} X_{i,j}(\omega)(\xi_{b_i}(A_j) - \xi_{a_i}(A_j))$$

where, by convention, we suppress all dependence on ω. If $t \geq 0$ then the corresponding integral up to time t is

$$\int_0^t \int_E f(s,x)\xi(s,x)\mu(dx)ds := \int_0^\infty \int_E f(s,x)\mathbb{1}_{[0,t]}(s)\xi(s,x)\mu(dx)ds.$$

Proposition 1.3.9 (Isometry)**.** *Let $f \in \mathscr{S}$ as in the previous definition. Then*

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^\infty \int_E f(s,x)\xi(s,x)\mu(dx)ds\right)^2\right] &= \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbb{E}\left[X_{i,j}^2\right](b_i - a_i)\mu(A_j) \\ &= \int_0^\infty \int_E \mathbb{E}\left[f(s,x)^2\right]\mu(dx)ds.\end{aligned}$$

Proof. Using (conditional) independence. □

The above result establishes an isometry from a subspace of $L^2(\Omega \times [0,\infty) \times E)$ (namely the simple functions $\mathscr{S}$) to $L^2(\Omega)$ (their stochastic integrals). The natural way to extend the definition of stochastic integral to a wider space of integrands would be to find the closure of the subspace $\mathscr{S}$ in $L^2(\Omega \times [0,\infty) \times E)$.

Definition 1.3.10. Let $\mathcal{P}(E)$ be the σ-algebra generated by $\mathscr{S}$, called the *predictable σ-algebra*. Functions measurable with respect to $\mathcal{P}(E)$ are called *predictable*. Let $\mathcal{P}_2(E)$ be the Hilbert space given by

$$\mathcal{P}_2(E) := \left\{ f \in L^2(\Omega \times [0,\infty) \times E) : f \text{ is predictable} \right\}.$$

Proposition 1.3.11. *$\mathscr{S}$ is dense in $\mathcal{P}_2(E)$ (with respect to $L^2(\Omega \times [0,\infty) \times E)$).*

Proof. [77, Proposition 2.3] implies the result when μ is finite. So the result is true when E is replaced by any E_n, for some increasing sequence of measurable subsets $(E_n) \subseteq E$ for which $\mu(E_n) < \infty$ and $\bigcup_n E_n = E$. So it is true when $\mathcal{P}_2(E)$ is replaced by $\mathcal{P}_2^n := \{f \in \mathcal{P}_2(E) : f|_{E_n^c} = 0\}$ (and $\mathscr{S}$ is restricted to $\mathscr{S} \cap \mathcal{P}_2^n$). Finally, $\bigcup_n \mathcal{P}_2^n$ is dense in $\mathcal{P}_2(E)$ by dominated convergence. □

With the above result in mind, we can now extend the definition of stochastic integral: let $f \in \mathcal{P}_2(E)$. Let $(f_n) \in \mathscr{S}$ be a sequence converging to f. By the isometry, the sequence $\int_0^\infty \int_E f_n(s,x)\xi(s,x)\mu(dx)ds$ has a limit in $L^2(\Omega)$ which is independent of the convergent sequence chosen. We define this to be the stochastic integral $\int_0^\infty \int_E f(s,x)\xi(s,x)\mu(dx)ds$. Further extensions can be made, but this is all that we will need.

1.3.2 The da Prato–Zabczyk approach

1.3.2.1 Cylindrical Wiener processes

In [15], da Prato and Zabczyk develop a theory of stochastic integration on a class of abstract infinite-dimensional vector spaces. Central to this is the concept of Wiener processes on such spaces. We seek a analogue to Brownian motion in infinite-dimensional Hilbert-spaces. In particular, we wish to replicate the characteristic property that if B is a d-dimensional standard Brownian motion and $x, y \in \mathbb{R}^d$ and $s, t \geq 0$, then

$$\mathbb{E}\left[\langle B(s), x\rangle_{\mathbb{R}^d} \langle B(t), y\rangle_{\mathbb{R}^d}\right] = (s \wedge t)\langle x, y\rangle_{\mathbb{R}^d}.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Let $\mathcal{H}$ be a separable Hilbert space. Let Q be a symmetric non-negative finite-trace operator on $\mathcal{H}$. By a classical result, there exists a countable complete orthonormal system $(e_k)_{k=1}^{\infty} \in \mathcal{H}$ of eigenvectors of Q with associated eigenvalues $(\lambda_k)_{k=1}^{\infty}$, such that $\lambda_k \geq 0$ for all k and $\sum_{k=1}^{\infty} \lambda_k < \infty$. Let $(B^k)_{k=1}^{\infty}$ be an independent collection of real-valued standard $\mathbb{F}$-Brownian motions and consider the process W^Q on $\mathcal{H}$ defined by

$$W^Q(t) := \sum_{k=1}^{\infty} B_t^k \sqrt{\lambda_k} e_k.$$

Then W^Q is called a *Q-Wiener process* on $\mathcal{H}$ (with respect to $\mathbb{F}$). It is a well-defined Gaussian process on $\mathcal{H}$ since

$$\mathbb{E}\left[\|W^Q(t)\|_{\mathcal{H}}^2\right] = t\sum_{k=1}^{\infty} \lambda_k < \infty.$$

Define $W_h^Q(t) := \langle W^Q(t), h\rangle_{\mathcal{H}}$ for $h \in \mathcal{H}$ and $t \geq 0$. Notice that W^Q satisfies the covariance structure

$$\mathbb{E}\left[W_{h_1}^Q(s) W_{h_2}^Q(t)\right] = (s \wedge t)\langle h_1, Qh_2\rangle_{\mathcal{H}}$$

for $s, t \geq 0$ and $h_1, h_2 \in \mathcal{H}$.

Since we require Q to have finite trace, the above definition precludes $Q = I$, the identity operator on $\mathcal{H}$. We can get around this by defining generalized Wiener processes on $\mathcal{H}$ in a somewhat weaker sense such that they are not necessarily $\mathcal{H}$-valued processes:

Definition 1.3.12. Let Q be a bounded symmetric non-negative operator on the separable Hilbert space $\mathcal{H}$. Let $W^Q = \{W_h^Q : h \in \mathcal{H}\}$ be a collection of real-valued stochastic processes satisfying the following:

(1). The map $h \mapsto W_h^Q$ is linear.

(2). $W_h^Q(t)$ is centred Gaussian for all $h \in \mathcal{H}$ and $t \geq 0$. For $h_1, h_2 \in \mathcal{H}$ and $s, t \geq 0$ it has the covariance structure

$$\mathbb{E}\left[W_{h_1}^Q(s) W_{h_2}^Q(t)\right] = (s \wedge t)\langle h_1, Qh_2\rangle_{\mathcal{H}}.$$

(3). For $h \in \mathcal{H}$, W_h^Q is an $\mathbb{F}$-martingale.

Then W^Q is called a *Q-Wiener process* on $\mathcal{H}$ with respect to $\mathbb{F}$.

Remark 1.3.13. Let Q be a bounded symmetric non-negative operator on $\mathcal{H}$. Then a Q-Wiener process on $\mathcal{H}$ with respect to $\mathbb{F}$ indeed exists; let $(e_k)_{k=1}^{\infty}$ be any orthonormal basis of $\mathcal{H}$ and let $(B^k)_{k=1}^{\infty}$ be an independent collection of real-valued standard $\mathbb{F}$-Brownian motions. Let $Q^{\frac{1}{2}}$ be the non-negative square root of Q, then for $h \in \mathcal{H}$ define

$$W_h^Q(t) = \sum_{k=1}^{\infty} B_t^k \langle h, Q^{\frac{1}{2}} e_k \rangle_{\mathcal{H}}.$$

Then $W^Q = \{W_h^Q : h \in \mathcal{H}\}$ is a Q-Wiener process with respect to $\mathbb{F}$.

Remark 1.3.14. The above definition of Q-Wiener processes extends its previous definition in the following sense: suppose Q has finite trace. If W^Q is a Q-Wiener process in the old sense, then $\{\langle W^Q, h\rangle_{\mathcal{H}} : h \in \mathcal{H}\}$ is a Q-Wiener process in the new sense. If W^Q is a Q-Wiener process in the new sense, then $\sum_{k=1}^{\infty} W_{e_k}^Q e_k$ is a Q-Wiener process in the old sense, where (e_k) is the orthonormal basis of eigenvectors of Q. Moreover, these two "operations" are inverses of each other. For this reason, for finite-trace Q we can identify the two different definitions of a Q-Wiener process on $\mathcal{H}$.

Definition 1.3.15. Let I be the identity operator on $\mathcal{H}$. A *cylindrical Wiener process* on $\mathcal{H}$ is an I-Wiener process on $\mathcal{H}$.

Since we have now defined Wiener processes on $\mathcal{H}$ in a somewhat abstract sense, we need to extend the definition of what it means to transform a Wiener process using a linear operator.

Definition 1.3.16. Let Q be a bounded symmetric non-negative operator on the separable Hilbert space $\mathcal{H}$. Let $\mathcal{G}$ be another separable Hilbert space and let $T : \mathcal{H} \to \mathcal{G}$ be a bounded linear operator. Let W^Q be a Q-Wiener process on $\mathcal{H}$. Then TW^Q is defined to be the TQT^*-Wiener process on $\mathcal{G}$ given by

$$(TW^Q)_g = W_{T^*g}^Q$$

for $g \in \mathcal{G}$, where $T^* : \mathcal{G} \to \mathcal{H}$ is the adjoint of T. It is a Wiener process with respect to the same filtration as that of W^Q.

It can be checked that the above definition is a valid extension, by looking at the case where W^Q is an $\mathcal{H}$-valued process.

1.3.2.2 Stochastic integrals

Let W be a cylindrical Wiener process on the separable Hilbert space $\mathcal{H}$ with respect to some completed filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Let $(e_k)_{k=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. As usual we first define stochastic integrals of simple integrands and then use an isometry to extend this definition. Let $\mathcal{L}_2$ be the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$. That is, it is the space of linear operators T on $\mathcal{H}$ for which the Hilbert-Schmidt norm

$$\|T\|_{\mathrm{HS}} := \left(\sum_{k=1}^{\infty} \|Te_k\|_{\mathcal{H}}^2\right)^{\frac{1}{2}}$$

is finite. The associated inner product is $\langle T_1, T_2\rangle_{\mathrm{HS}} = \sum_{k=1}^{\infty}\langle T_1 e_k, T_2 e_k\rangle_{\mathcal{H}}$. The Hilbert spaces $\mathcal{H}$ and $\mathcal{L}_2$ are assumed to be equipped with their respective Borel σ-algebras $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{L}_2)$.

Remark 1.3.17. It is a simple exercise to show that $\|T^*\|_{\mathrm{HS}} = \|T\|_{\mathrm{HS}}$ and that $\|T\|_{\mathrm{HS}}$ is independent of the orthonormal basis $(e_k)_{k=1}^{\infty}$ chosen. The latter property extends to the inner product $\langle\cdot,\cdot\rangle_{\mathrm{HS}}$ by the parallelogram law.

Definition 1.3.18. A process $\Phi : \Omega \times [0,\infty) \to \mathcal{L}_2$ is *elementary* if it takes the form

$$\Phi(\omega, t) = \mathbb{1}_{(a,b]}(t)T(\omega),$$

where $0 \leq a < b < \infty$, and $T : \Omega \to \mathcal{L}_2$ takes finitely many values and is $\mathcal{F}_a$-measurable. The process Φ is *simple* if it is a finite linear combination of elementary processes. All simple processes can be represented as a finite linear combination of elementary processes

$$\Phi(\omega, t) = \sum_{i=1}^{n} \mathbb{1}_{(a_i,b_i]}(t)T_i(\omega)$$

such that the intervals $(a_i, b_i]$ are pairwise disjoint. Let $\mathscr{S}_{\mathcal{H}}$ be the vector space of simple processes.

Definition 1.3.19. Let $\Phi \in \mathscr{S}_{\mathcal{H}}$ be a simple process as above. Then its stochastic integral with respect to the cylindrical Wiener process W is defined to be

$$\int_0^{\infty} \Phi(s)dW(s) := \sum_{i=1}^{n}(T_iW(b_i) - T_iW(a_i)),$$

where the dependence on ω is suppressed by convention. Notice that if $T \in \mathcal{L}_2$ then TW is a TT^*-Wiener process on $\mathcal{H}$, and TT^* is non-negative symmetric with trace $\|T\|_{\mathrm{HS}}^2 < \infty$. Thus TW is in fact a well-defined $\mathcal{H}$-valued process.

Moreover, the respective stochastic integral up to some finite time $t \geq 0$ is defined to be

$$\int_0^t \Phi(s)dW(s) := \int_0^\infty \Phi(s)\mathbb{1}_{[0,t]}(s)dW(s).$$

Proposition 1.3.20 (Isometry). *If $\Phi \in \mathscr{S}_\mathcal{H}$ as in the previous definition then*

$$\begin{aligned}\mathbb{E}\left[\left\|\int_0^\infty \Phi(s)dW(s)\right\|_\mathcal{H}^2\right] &= \sum_{i=1}^n (b_i - a_i)\mathbb{E}\left[\|T_i\|_{\mathrm{HS}}^2\right] \\ &= \int_0^\infty \mathbb{E}\left[\|\Phi(s)\|_{\mathrm{HS}}^2\right] ds\end{aligned}$$

Proof. Use properties of the processes T_iW and the cylindrical Wiener process W. Since each T_i takes only finitely many values, it helps to write $T_i = \sum_j X_i^j T_i^j$, where each $T_i^j \in \mathcal{L}_2$ is deterministic and each X_i^j is the indicator function of some event in $\mathcal{F}_{a_i}$. □

By the above result we have established an isometry from a subspace of $L^2(\Omega \times [0,\infty); \mathcal{L}_2)$, namely $\mathscr{S}_\mathcal{H}$, to $L^2(\Omega; \mathcal{H})$ (the stochastic integrals).

Definition 1.3.21. Let $\mathcal{P}$ be the σ-algebra on $\Omega \times [0,\infty)$ generated by sets of the form $A \times (s,t]$, where $0 \leq s < t < \infty$ and A is $\mathcal{F}_s$-measurable. In particular, $\mathcal{P}$ is generated by $\mathscr{S}_\mathcal{H}$. $\mathcal{P}$ is called the *predictable σ-algebra.* Functions that are $\mathcal{P}$-measurable are called *predictable.*

Proposition 1.3.22. *$\mathscr{S}_\mathcal{H}$ is dense in the Hilbert space*

$$\{\Phi \in L^2(\Omega \times [0,\infty); \mathcal{L}_2) : \Phi \textit{ is predictable}\}.$$

Proof. Consequence of [15, Proposition 4.22(ii)]. □

Therefore through the isometry we now have a definition for the stochastic integral $\int_0^\infty \Phi(s)dW(s)$ of any predictable $\Phi \in L^2(\Omega \times [0,\infty); \mathcal{L}_2)$.

1.4 Comparison of the two approaches

The two approaches to stochastic integration described in this introduction evidently share a few major similarities, and it turns out that their intersection is significant and encompasses many (and dare I say *most*) significant examples. Let E be a Lusin space equipped with a σ-finite Borel measure μ. We see that $L^2(E,\mu)$ is a separable Hilbert space[1]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a completed filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Then there is a correspondence between space-time white noises on (E, μ) and cylindrical Wiener processes on $L^2(E, \mu)$. Recall that if A is a set then $\mathbb{1}_A$ denotes its indicator function.

(1). If ξ is a space-time white noise on (E, μ) with respect to $\mathbb{F}$, then

$$W := \left\{ \left(\int_0^t \int_E h(x)\xi(s,x)\mu(dx)ds \right)_{t\geq 0} : h \in L^2(E, \mu) \right\}$$

is a cylindrical Wiener process on $L^2(E, \mu)$ with respect to $\mathbb{F}$.

(2). If W is a cylindrical Wiener process on $L^2(E, \mu)$ with respect to $\mathbb{F}$, then ξ given by

$$\xi_t(A) := W_{\mathbb{1}_A}(t), \quad A \in \{B \subseteq E : \mu(B) < \infty\}, \ t \geq 0$$

is a space-time white noise on (E, μ) with respect to $\mathbb{F}$.

This correspondence can be used to translate results from the Walsh world into the da Prato–Zabczyk world and vice-versa.

[1]Indeed, let $(E_n)_n \subseteq E$ be an increasing sequence of measurable subsets such that $\mu(E_n) < \infty$ and $\bigcup_n E_n = E$. Then $\bigcup_n L^2(E_n, \mu)$ is dense in $L^2(E, \mu)$, so it suffices to show that each $L^2(E_n, \mu)$ is separable. By [8, Exercise 13:4.4] it is enough to show that the space $\mathcal{M}_n$ of measurable subsets of E_n (modulo μ-null sets) equipped with symmetric difference metric $(A, B) \mapsto \mu(A \triangle B)$ is separable. Let $\mathcal{O}_n$ be the subspace of $\mathcal{M}_n$ of open subsets of E_n. E_n is Lusin, and Lusin spaces are separable and metrizable. Thus [71, Exercise 2.23] implies that E_n has a countable base for its topology, so $\mathcal{O}_n$ is separable. By [78, Theorem 6.1] μ is regular on E_n, so in particular $\mathcal{O}_n$ is dense in $\mathcal{M}_n$. Thus $\mathcal{M}_n$ is separable.

Chapter 2

Degenerate limits for one-parameter families of non-fixed-point diffusions on fractals

2.1 Introduction

The study of diffusion on fractals has largely focused on constructing and analysing the 'Brownian motion', that is the stochastic process generated by the 'natural' Laplace operator, on a given fractal. For the Sierpinski gasket, the simple symmetric random walk on the natural graph approximation has the property of being decimation invariant, in that it has same law when stopped at its visits to coarser approximations and this enabled the initial detailed analysis of the diffusion and its properties [5]. However the simple symmetric random walk is not the only possible discrete Markov chain on graph approximations to the Sierpinski gasket that can be used to construct a scaling limit. By considering processes invariant under reduced symmetry groups it is possible to construct processes such as the rotationally and scale invariant but non-reversible p-stream diffusions of [55], the not-necessarily scale invariant homogeneous diffusions of [40] and the one that will provide a fundamental example for our work, the asymptotically one-dimensional diffusion of [36]. This process is invariant under reflection in the vertical axis but is not scale invariant and displays local anisotropy but global isotropy.

The construction of a canonical Brownian motion on the Sierpinski gasket was generalized to nested fractals, [62], through to the large class of finitely ramified fractals, the p.c.f. self-similar sets of [52]. In these extensions it became clear that a straightforward approach to the construction of a Brownian motion was available

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through the theory of Dirichlet forms, [21], [59], [50], and the questions of the existence and uniqueness of the Brownian motion could be reduced to the existence and uniqueness of a fixed point for a finite dimensional renormalization map on the cone of Dirichlet forms over the basic cell structure in the fractal, [63], [72]. When seeking to generalize some of the exotic diffusions on the Sierpinski gasket it is easiest to work in the reversible case (which excludes the p-stream and homogeneous diffusions) and use the theory of Dirichlet forms. Using this approach the asymptotically one-dimensional diffusion processes were extended to some nested fractals in [33]. Our first aim here will be to generalize this class further to a sub-class of p.c.f. self-similar fractals.

The construction of a Laplace operator on a finitely ramified fractal through an associated sequence of Dirichlet forms just requires that a compatible sequence of resistance neworks can be constructed on the graph approximations to the fractal [52]. The self-similarity allows this to be reduced to the study of a finite dimensional renormalization map on a cone of discrete irreducible Dirichlet forms. The fixed point problem for this renormalization map was the subject of the work of Sabot, Metz and Pierone, see (among others) [72], [63], [64], [67], [68]. In solving the uniqueness problem for nested fractals [72] and [64] showed that when considering the renormalization map, although it is not in general a strict contraction in Hilbert's projective metric on the cone, iterates of the map converge to a non-degenerate fixed point under some conditions. Essentially one has to examine the map in the neighbourhood of possible degenerate fixed points on the boundary of the cone of irreducible Dirichlet forms and find conditions that ensure there is a move 'away' from degeneracy and hence the map can be iterated toward the non-degenerate fixed point. The construction of what we will call here a one-parameter family of non-fixed point diffusions is then about associating a sequence of resistor networks with the inverse iterates of this renormalization map, within a one-parameter family, so that the corresponding networks resemble the degenerate fixed point on the small scale but resemble the non-degenerate fixed point on the large scale. Using this approach, [36], [33] showed that it was possible to construct a one parameter family of diffusions on the Sierpinski gasket by inverting a one-parameter version of the renormalization map and placing suitable conductances on the graphs G_n which approximate the gasket so that G_0, the graph given by the triangle, had the effective resistance given by $(1, w, w)$. Figure 2.1 shows the configuration of resistors and the renormalization of total resistance $R(w) = (12w^2 + 26w + 12)/(w^3 + 8w^2 + 15w + 6)$ and for the diagonal edges $\gamma(w) = (6w^2 + 4w)/(w^2 + 6w + 3)$ obtained from the renormalization map.

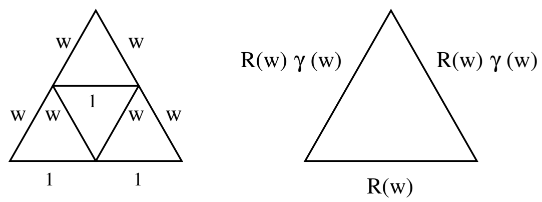

Figure 2.1: The original weighting of the edges of the Sierpinski gasket for the asymptotically one-dimensional process and the renormalization.

Let $\{X_t^{n,w}; t \geq 0\}$ denote the continuous time random walk corresponding to the sequence of resistance networks G_n in which the effective conductance over G_0 is given by the triple of conductors $(1, w, w)$. Then, it was shown in [36], that for any starting weight $0 < w < 1$, as $n \to \infty$,

$$\{X_{6^n t}^{n,w}; t \geq 0\} \to \{X_t^{a,w}; t \geq 0\},$$

weakly in $\mathcal{D}_G[0, \infty)$ (the space of càdlàg functions on the limiting fractal G) where $X^{a,w}$ is the asymptotically one-dimensional diffusion, a continuous, strong Markov process on G. This is called an asymptotically one-dimensional diffusion as the local behaviour of the conductances gives the horizontal line segment an increasing weight relative to the two diagonal segments.

To see the large scale behaviour of this diffusion we can extend the graph G_0 to an infinite Sierpinski gasket graph $\tilde{G}_0$, in which each copy of the basic triangle has conductance $(1, w, w)$, and proceed to construct the limiting asymptotically one-dimensional diffusion $\{\tilde{X}_t^{a,w}; t \geq 0\}$ on the infinite fractal $\tilde{G}$. Then, in [33], it is shown that under the classical scaling for the Brownian motion on $\tilde{G}$ we have, as $n \to \infty$,

$$\{2^{-n} \tilde{X}_{5^n t}^{a,w}; t \geq 0\} \to \{B_t; t \geq 0\},$$

weakly in $C_{\tilde{G}}[0, \infty)$ (the space of continuous functions on $\tilde{G}$), where B is the Brownian motion on $\tilde{G}$. This is a homogenization result not seen in Euclidean space as the geometry of the fractal causes the homogenization. If we think of a diffusing particle, even though it moves much more frequently in a horizontal direction, in order to cross large regions it must make vertical moves and thus on the very large scale it behaves like the Brownian motion.

In [36] it is observed that, even in the situation where there is no fixed point for the renormalization map, this procedure could lead to a diffusion on the $O(1)$ scale which was non-degenerate even though at both small and large scale the diffusion is degenerate. This was illustrated in the case of *abc*-gaskets where the ratios between a, b and c are such that there is no fixed point [38].

It is clear that for the asymptotically one-dimensional diffusion on the Sierpinski gasket on the small scale there is a separation of time scales in that there will be many more horizontal steps than vertical ones. The question of the so-called ultraviolet limit of these processes was raised in [36]. Is there a rescaling of the diffusion process over short time and space scales which will produce a non-trivial object in the short scale limit? In other words does there exist a scaling $\lambda \in (0,1)$ and a non-trivial process X^b such that

$$\{2^n X^{a,w}_{\lambda^n t}; t \geq 0\} \rightarrow \{X^b_t; t \geq 0\}$$

weakly as $n \rightarrow \infty$? Our aim in this chapter is to consider a class of fractals for which there are asymptotically lower-dimensional processes and discuss their short time scaling limits. The degenerate diffusions which arise could be called the ultraviolet limits for the asymptotically lower-dimensional diffusions.

In order to examine the short time scaling limit we show that, by thinking of the fractal as a resistance form, a metric space equipped with a resistance metric, and shorting the high conductance edges, there is Gromov-Hausdorff convergence of the fractals to a limit space. We can then exploit the recent work of [13] (extending that in [14]) to establish that there is weak convergence of the rescaled diffusions to a diffusion process on this limit fractal. This limit fractal is not a p.c.f. self-similar set, but is typically a simple fractal space [66], and the theory for p.c.f. fractals is easily extended to include such limit objects. We note that this limit construction provides a case where we fuse each element of a countable collection of subsets of a space, going beyond the fusing of a single subset or finite pairs of points as discussed in [53], [12], [13].

In our approach to these non-fixed point diffusions we write the conductance matrix in a different form to those used previously in that we fix (in Figure 2.1) what was w to be 1 and let the edge which was 1 have conductance v for $v > 1$. Examples are shown in Figures 2.2 and 2.3. This shows that, for the Sierpinski gasket, as we look over smaller scales the horizontal edge has conductance which goes to infinity and thus in the limit we expect that the horizontal edges will be shorted, leading to an object which is shown on the right side of Figure 2.2. The analysis of the fixed point problem in [72] required the analysis of such a shorted graph. Here we establish a general result about the weak convergence of diffusions on a class of fractals as such a parameter v tends to ∞. We work in the framework of (generalized) p.c.f. self-similar sets but restrict the class to those which support a resistance form, and whose shorted versions also support a resistance form. We will view our sequence of fractals with the resistance metric as metric spaces and embed them along with the limit into a metric space $\mathcal{M}$. We prove the weak convergence in $\mathcal{D}_{\mathcal{M}}[0,\infty)$, the space of càdlàg processes on $\mathcal{M}$.

We can use our main result, Theorem 2.5.7, in the case of the Sierpinski gasket to analyse the short time behaviour of the asymptotically one-dimensional diffusion. We will consider the diffusion over small scales and times and prove the weak convergence to a diffusion on the shorted gasket, the limit of the graphs on the right hand side of Figure 2.2. We will think of the gasket G now as a self-sufficient metric space with a resistance metric R^v determined by the conductance parameter v. Let ψ_1, ψ_2, ψ_3 denote the three similitudes used in defining the Sierpinski gasket and we assume that they have fixed points p_1, p_2 and p_3, respectively. We assume that for G_0, the triangle graph on the points p_i, the edge from p_1 to p_3 is of resistance v and the other two edges are of resistance 1. We write $T_A(X) = \inf\{t > 0 : X_t \in A\}$ for the hitting time of a set A by the process X. We write L_n for the image of the line joining p_1, p_3 in the triangle $\psi_2^n(G_0)$. That is the bottom edge of the triangle with Euclidean size 2^{-n} with top vertex at p_2.

Theorem 2.1.1. *Let $X^{a,v}$ denote the asymptotically one-dimensional diffusion on the metric space (G, R^v) with conductances $(1,1,v)$ on G_0 and $X_0^{a,v} = p_2$ and let X^s be the diffusion on the 'shorted gasket', that is the limit of the graphs shown in Figure 2.2, with $X_0^s = 0$, where 0 denotes the top vertex with only one edge into it and the base vertex is denoted by 1. Then, there exists a constant $\sigma > 0$ such that*

$$\{\psi_2^{-n}(X^{a,v}_{(9/2)^{-n}t}); 0 \leq (9/2)^{-n}t \leq T_{L_n}(X^{a,v})\} \to \{X^s_{\sigma t}; 0 \leq t \leq T_1(X^s)\},$$

weakly in $\mathcal{D}_{\mathcal{M}}[0,\infty)$.

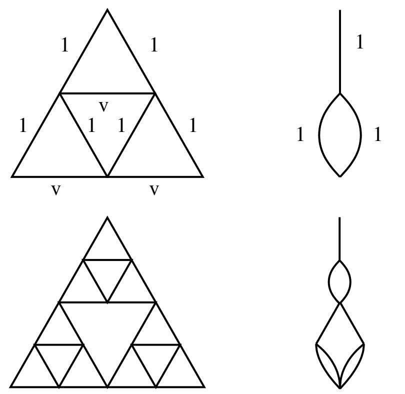

Figure 2.2: The first two levels of the Sierpinski gasket and the corresponding levels of the limit fractal obtained as $v \to \infty$.

In the setting in which we work the limit spaces will have an associated resistance form and it is straightforward to determine some of the properties of the limit diffusion. In particular it is not difficult to see that the spectral dimension for the shorted gasket is $\frac{2\log 3}{\log 9/2}$. In the paper [32] this was shown to be the local spectral dimension (the exponent for the short time asymptotic decay of the on-diagonal heat kernel) for the asymptotically one-dimensional diffusion on the Sierpinski gasket.

The outline of this chapter is as follows. We begin by giving the framework in which we will work in Section 2. As the shorted gasket is not a p.c.f. self-similar set we will need a small extension of the class of p.c.f. self-similar sets to discuss our limit processes. In Section 3 we introduce the class of locally degenerate diffusions on fractals. We then construct the limit spaces that will support our limiting diffusions in Section 4. In order to prove our result we use Gromov-Hausdorff-vague convergence and hence we construct a suitable space in which to embed the sequence of fractals and the limiting space. We do this in Section 5 and establish the weak convergence by showing how to employ the result of [13] (simplifying substantially our original 30 page direct proof of the weak convergence).

2.2 Preliminaries

We slightly generalize the idea of a post critically finite self-similar set [52]. A generalization which goes beyond what we introduce here can be found in [75].

Definition 2.2.1. Fix some $N \in \mathbb{N}$. Let $F = (F, d)$ be a metric space and for $i = 1, \ldots, N$ let $\psi_i : F \to F$ be a function. Then $\mathcal{S} = (F, (\psi_i)_{1\le i\le N})$ is a *self-similar structure* if

(1). (F, d) is compact,

(2). The ψ_i are continuous injections from F to itself with

$$F = \bigcup_{i=1}^{N} \psi_i(F),$$

(3). There exists $\delta > 0$ such that

$$d(\psi_i(x), \psi_i(y)) \le (1-\delta) d(x, y)$$

for all $x, y \in F$ and $1 \le i \le N$.

Let $I := \{1, \dots, N\}$, and define the word spaces $\mathbb{W}_n = I^n$, $\mathbb{W} = I^{\mathbb{N}}$. Endow $\mathbb{W}$ with the product topology, and let σ be the left shift operator, which maps $\mathbb{W}$ to $\mathbb{W}$ or $\mathbb{W}_n$ to $\mathbb{W}_{n-1}$. That is, if $w = w_1 w_2 w_3 \dots$ then $\sigma(w) = w_2 w_3 \dots$. For $w = w_1 \dots w_n \in \mathbb{W}_n$ we now define

$$\psi_w := \psi_{w_1} \circ \cdots \circ \psi_{w_n}.$$

For $A \subseteq F$ let $A_w := \psi_w(A)$. If $n \geq 1$ and $w \in \mathbb{W}$ (or $w \in \mathbb{W}_m$ with $m \geq n$) then define

$$w|n = w_1 \dots w_n \in \mathbb{W}_n.$$

For each $w \in \mathbb{W}$, there exists $x_w \in F$ such that

$$\bigcap_{n=1}^{\infty} \psi_{w|n}(F) = \{x_w\}.$$

and we write $\pi : \mathbb{W} \to F$ for the mapping $\pi(w) := x_w$. As in [3, Lemma 5.10], we have the following result:

Lemma 2.2.2. *The function π is the unique mapping $\pi : \mathbb{W} \to F$ such that for all $w \in \mathbb{W}$ and $i \in I$,*

$$\pi(i \cdot w) = \psi_i(\pi(w))$$

where $i \cdot w \in \mathbb{W}$ denotes the concatenation of i with w. Moreover, π is continuous and surjective.

Definition 2.2.3. For a self-similar structure $\mathcal{S} = (F, (\psi_i)_{1 \leq i \leq N})$, let

$$B(\mathcal{S}) = \bigcup_{i,j,i \neq j} F_i \cap F_j.$$

This is the set of points that exist in the "overlap" of the images of two distinct ψ_i. Let

$$\Gamma(\mathcal{S}) = \pi^{-1}(B(\mathcal{S}))$$

be the set of words corresponding to $B(\mathcal{S})$. This is called the *critical set* of $\mathcal{S}$. Let

$$P(\mathcal{S}) = \bigcup_{n=1}^{\infty} \sigma^n(\Gamma(\mathcal{S}))$$

be the *post-critical set* of $\mathcal{S}$.

Definition 2.2.4 (Generalized p.c.f.s.s. sets)**.** A self-similar structure $(F, (\psi_i)_{1 \leq i \leq N})$ is called *generalized post-critically finite* if $\pi(P(\mathcal{S}))$ is finite. A metric space (F, d) is a *generalized post-critically finite self-similar* set, or *generalized p.c.f.s.s.* set, if there exists a generalized post-critically finite self-similar structure $(\psi_i)_{1 \leq i \leq N}$ on F.

Now $\pi(P(\mathcal{S}))$ has two equivalent reformulations given below:

$$\pi(P(\mathcal{S})) = \left\{ x \in F : \exists w, v \in \bigcup_{n \geq 1} \mathbb{W}_n,\ w \neq v,\ \psi_w(x) \in F_v \right\}$$
$$= \left\{ x \in F : \exists w \in \bigcup_{n \geq 1} \mathbb{W}_n,\ \psi_w(x) \in B(\mathcal{S}) \right\}.$$

Remark 2.2.5. Notice that Definition 2.2.4 differs from the definition of a p.c.f.s.s. set (see [52]), which has the stronger condition of $P(\mathcal{S})$ itself being finite.

Example 2.2.6. Any p.c.f. self-similar set is clearly a generalized p.c.f.s.s. set. Also the 'shorted gasket' of Figure 2.2 and the Diamond Hierarchical Lattice studied in [35] are examples of generalized, but not strictly, p.c.f.s.s. sets.

Definition 2.2.7. Let $(F, (\psi_i)_{1 \leq i \leq N})$ be a generalized p.c.f.s.s. set. For $n \geq 0$ we set

$$P^{(n)} = \{w \in \mathbb{W} : \sigma^n w \in P(\mathcal{S})\},$$
$$F^n = \pi(P^{(n)}).$$

Any set of the form $\psi_w(F)$, $w \in \mathbb{W}_n$, we call an *n-complex*. Any set of the form $\psi_w(F^0)$, $w \in \mathbb{W}_n$, we call an *n-cell*.

The set F^0 is the "boundary" of F and $(F^n)_{n \geq 0}$ is an increasing sequence of subsets of F where for $n \geq 0$,

$$F^{n+1} = \bigcup_{i=1}^{N} \psi_i(F^n).$$

It can then be easily proven, by the compactness of F and the contraction property of the functions (ψ_i), that

$$F = \overline{\bigcup_{n \geq 0} F^n}. \tag{2.2.1}$$

2.2.1 Measures on F

We first define a Bernoulli measure on $\mathbb{W}$, and then push it forward onto F. Let $\theta = (\theta_1, \dots, \theta_N)$ be a vector such that $\sum_{i=1}^{N} \theta_i = 1$ and $0 < \theta_i < 1$ for each i. Writing $\theta_w := \theta_{w_1}\theta_{w_2}\dots\theta_{w_n}$ for $w \in \mathbb{W}_n$, and defining random variables $\xi_n : \mathbb{W} \to I$ by $\xi_n(w) = w_n$, let $\tilde{\mu}_\theta$ be the unique Borel probability measure on $\mathbb{W}$ satisfying

$$\tilde{\mu}_\theta(\{\xi_1 = w_1, \dots, \xi_n = w_n\}) = \theta_w$$

for any $n \in \mathbb{N}$ and $w \in \mathbb{W}_n$.

Definition 2.2.8. We define a *Bernoulli measure* μ_θ on F to be the pushforward (through our canonical mapping π) of the corresponding Bernoulli measure on $\mathbb{W}$:

$$\mu_\theta := \tilde{\mu}_\theta \circ \pi^{-1}.$$

We also define corresponding measures on each of our approximating sets F^n.

Definition 2.2.9. For a fixed $\theta = (\theta_1, \dots, \theta_N)$ such that $\sum_{i=1}^N \theta_i = 1$ and $0 < \theta_i < 1$ for each i, let μ_n be the measure on F^n given by

$$\mu_n(x) = (\#F^0)^{-1} \sum_{w \in \mathbb{W}_n} \theta_w \mathbb{1}_{F_w^0}(x).$$

Note that μ_n charges every point of F^n and we have [3, Lemma 5.29].

Lemma 2.2.10. *μ_n is a probability measure on F^n, and $\mu_n \to \mu_\theta$ weakly as $n \to \infty$.*

2.2.2 The renormalization map

We first recall the concept of the trace of a Dirichlet form from [3, Chapter 4].

Definition 2.2.11. Let G be a set and $(\mathcal{E}, \mathcal{D})$ a Dirichlet form defined on G and $H \subseteq G$. Define a Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{D}})$ on H by

$$\tilde{\mathcal{E}}(h,h) = \inf \left\{ \mathcal{E}(g,g) : g \in \mathcal{D},\ g|_H = h \right\}.$$

and $\tilde{\mathcal{D}}$ is the set of functions h such that the above is finite. The form $\tilde{\mathcal{E}}$ is called the *trace* of $\mathcal{E}$ on H and is denoted by $\mathrm{Tr}(\mathcal{E}|H)$.

This leads naturally to the notion of *effective resistance* with respect to a Dirichlet form:

Definition 2.2.12. Let G be a set and $(\mathcal{E}, \mathcal{D})$ a Dirichlet form defined on G. Let H_1 and H_2 be disjoint subsets of G. The *(effective) resistance* between H_1 and H_2 is

$$R_{\mathcal{E}}(H_1, H_2) = \left(\inf \left\{ \mathcal{E}(g,g) : g \in \mathcal{D},\ g|_{H_1} = 0,\ g|_{H_2} = 1 \right\} \right)^{-1},$$

with the convention that $0^{-1} = +\infty$. In particular, if $H_i = \{x_i\}$ for $x_i \in G$, $i = 1, 2$, then let $R_{\mathcal{E}}(x_1, x_2) = R_{\mathcal{E}}(\{x_1\}, \{x_2\})$. If this defines a metric on G (after extending it such that $R_{\mathcal{E}}(x, x) = 0$ for all $x \in G$) then we call it the *resistance metric* associated with $(\mathcal{E}, \mathcal{D})$.

In this section we seek to define a Dirichlet form on each of the F^n respectively such that the sequence of Dirichlet forms is "nested", in the sense of taking traces. From this sequence we can construct a Dirichlet form on F as a limit. We follow closely the approach given in Barlow [3].

Let $(F, (\psi_i)_{1\le i\le N})$ be a generalized p.c.f.s.s. set with a Bernoulli measure $\mu = \mu_\theta$. Define $r = (r_1, \dots, r_N)$ to be a *resistance vector* of positive numbers. Each r_i roughly corresponds to the "size" of the subset $F_i \subseteq F$. For $n \ge 0$ let $\mathbb{D}_n$ be the set of conservative Dirichlet forms on F^n. Observe that since F^n is finite, $\mathbb{D}_n$ is in direct correspondence with the set of conductance matrices on F^n. For $\mathcal{E} \in \mathbb{D}_0$ we write

$$\mathcal{E}(f,g) = \frac{1}{2}\sum_{x,y\in F^0} a(x,y)(f(x)-f(y))(g(x)-g(y)) = -f^T A g,$$

where $A = (a(x,y))_{x,y\in F^0}$ is a matrix of conductances with the diagonal terms given by $a(x,x) = -\sum_{y\neq x} a(x,y)$.

Definition 2.2.13. We define the following maps as in [3]:

(1). The *replication* operation $R : \mathbb{D}_0 \to \mathbb{D}_1$ is given by

$$R(\mathcal{E})(f,g) = \sum_{i=1}^{N} r_i^{-1}\mathcal{E}(f\circ\psi_i, g\circ\psi_i).$$

The subset F^1 can be viewed as N copies of F^0. The Dirichlet form $R(\mathcal{E})$ is simply the sum of $\mathcal{E}$ evaluated on each of these copies, weighted by r.

(2). The *trace* operation $T : \mathbb{D}_1 \to \mathbb{D}_0$ is given by

$$T(\mathcal{E}) = \mathrm{Tr}(\mathcal{E}|F^0).$$

(3). The *renormalization map* is $\Lambda = T\circ R : \mathbb{D}_0 \to \mathbb{D}_0$.

Remark 2.2.14. Note that Λ is positively homogeneous: if $c > 0$ then $\Lambda(c\mathcal{E}) = c\Lambda(\mathcal{E})$. However it is not in general linear, because T is non-linear.

The replication operation R can be regarded as a mapping $R : \bigcup_n \mathbb{D}_n \to \bigcup_n \mathbb{D}_n$ such that if $\mathcal{E} \in \mathbb{D}_n$ then $R(\mathcal{E}) \in \mathbb{D}_{n+1}$ with

$$R(\mathcal{E})(f,g) = \sum_{i=1}^{N} r_i^{-1}\mathcal{E}(f\circ\psi_i, g\circ\psi_i).$$

Notice now that if $\mathcal{E} \in \mathbb{D}_0$, then $R^n(\mathcal{E}) \in \mathbb{D}_n$ with

$$R^n(\mathcal{E})(f,g) = \sum_{w \in \mathbb{W}_n} r_w^{-1} \mathcal{E}(f \circ \psi_w, g \circ \psi_w).$$

where $r_w := r_{w_1} \dots r_{w_n}$.

The fixed point problem is to find eigenvectors of Λ, that is, Dirichlet forms $\mathcal{E} \in \mathbb{D}_0$ such that there exists $\rho > 0$ with

$$\Lambda(\mathcal{E}) = \rho^{-1}\mathcal{E}.$$

For such a form, let $\mathcal{E}^{(0)} := \mathcal{E} \in \mathbb{D}_0$ and for $n \geq 1$ put $\mathcal{E}^{(n)} := \rho^n R^n(\mathcal{E}) \in \mathbb{D}_n$. Thus we have a nested sequence: if $m \leq n$ then

$$\mathrm{Tr}(\mathcal{E}^{(n)}|F^m) = \mathcal{E}^{(m)}.$$

Definition 2.2.15. Let $\mathcal{E} \in \mathbb{D}_0$ be an eigenvector of Λ as above. If $0 < r_i\rho^{-1} < 1$ for all $1 \leq i \leq N$, then we say that $(\mathcal{E}, r)$ is a *regular fixed point*. If $\mathcal{E}$ is irreducible (see [3, Definition 4.3]), then we call $\mathcal{E}$ a *non-degenerate fixed point* of Λ.

The term "non-degenerate" corresponds to the irreducibility of the Markov process associated with $\mathcal{E}$. If we were to take a degenerate fixed point and construct Dirichlet forms $\mathcal{E}^{(n)}$ as before, then all of our associated Markov processes would be restricted to only a small part of the fractal. We thus restrict our attention to non-degenerate fixed points.

Example 2.2.16. Let F be the Sierpinski gasket. $F^0 \subseteq F$ is then a set of three points, the outermost vertices of the gasket. Labelling these vertices $F^0 =: \{1,2,3\}$, Λ has a non-degenerate fixed point $\mathcal{E}_A$ where $a_{ij} = 1$ for $i \neq j$, $i, j \in F^0$ and $\rho = \frac{5}{3}$. If instead we let $a_{12} = 1$, $a_{23} = 0$, $a_{31} = 0$ then we see that $\mathcal{E}_A$ is again a fixed point, this time degenerate, with $\rho = 2$.

2.2.3 Fixed-point diffusions

We can now define a Dirichlet form on F, closely following the approach given in [52]. Let $\mathcal{S} = (F, (\psi_i)_{1 \leq i \leq N})$ be a (connected) generalized p.c.f.s.s. set and r a resistance vector. Let $\mu = \mu_\theta$ be a Bernoulli measure on F. Let $r_{\min} = \min_i r_i$ and $r_{\max} = \max_i r_i$, and let $\theta_{\min}$ and $\theta_{\max}$ be defined likewise. For each $n \geq 0$ let μ_n be the measure on F^n given by Definition 2.2.9. Suppose the renormalization map Λ has a non-degenerate regular fixed point $\mathcal{E}^{(0)} \in \mathbb{D}_0$ with eigenvalue ρ^{-1}, $\rho > 0$. Construct the nested sequence of Dirichlet forms $(\mathcal{E}^{(n)})_n$ as

$$\mathcal{E}^{(n)}(f,g) := \rho^n \sum_{w \in \mathbb{W}_n} r_w^{-1} \mathcal{E}^{(0)}(f \circ \psi_w, g \circ \psi_w), \quad f, g \in C(F^n).$$

For each $n \geq 0$, let $X^n = (X_t^n)_{t \geq 0}$ be the Markov process associated with $\mathcal{E}^{(n)}$ on $L^2(F^n, \mu_n)$. From now on we identify real valued functions on F with their restrictions to F^n to simplify notation.

Observe that if f is a real-valued function on F, then the sequence $(\mathcal{E}^{(n)}(f,f))_n$ is non-decreasing by the properties of the trace operator and we can define our limiting form

$$D = \left\{ f \in C(F) : \sup_n \mathcal{E}^{(n)}(f,f) < \infty \right\},$$
$$\mathcal{E}(f,f) = \sup_n \mathcal{E}^{(n)}(f,f), \quad f \in D,$$

where $C(F)$ is the space of real-valued continuous functions on F.

Theorem 2.2.17. *The pair $(\mathcal{E}, D)$ is a regular local irreducible Dirichlet form on $L^2(F, \mu)$. It has an associated resistance metric generating a topology that is equivalent to the existing topology on F.*

Proof. The proof is identical to proofs of similar results in [52]. The resistance metric result is from Theorem 3.3.4. $(\mathcal{E}, D)$ is a local regular Dirichlet form by Theorem 3.4.6. It is irreducible since each $\mathcal{E}^{(n)}$ is irreducible and the union of the F^n is dense in F. It is easy to verify that the form is densely defined given regularity: D is dense in $C(F)$ in the uniform norm. Since $\mu(F) < \infty$, D also is dense in $C(F)$ in the L^2 norm. And $C(F)$ is dense in $L^2(F, \mu)$ by the compactness of F. □

By [22] there therefore exists a μ-symmetric diffusion $X = (X_t)_{t \geq 0}$ on F associated with $(\mathcal{E}, D)$ and $L^2(F, \mu)$. Let $\mathcal{D}_F[0, \infty)$ be the space of càdlàg functions $f : [0, \infty) \to F$. Then the following also holds:

Theorem 2.2.18. *$X^n \to X$ weakly in $\mathcal{D}_F[0, \infty)$. Precisely, if $(x_n)_n$ is a sequence in F such that $x_n \in F^n$ for each n and $x_n \to x \in F$, then*

$$(X^n, \mathbb{P}^{x_n}) \to (X, \mathbb{P}^x)$$

weakly in $\mathcal{D}_F[0, \infty)$.

Proof. By the fact that the sequence $(\mathcal{E}^{(n)})_n$ of Dirichlet forms are compatible, their induced resistance metrics must agree. Therefore if each F^n and F are interpreted as metric spaces equipped with their respective resistance metrics (using Theorem 2.2.17), we see that each F^n is isometrically embedded in F. In particular, by Theorem 2.2.17 the resistance metric on F induces the existing topology on F. Since F is compact and $\bigcup_{n\geq 0} F^n$ is dense in F, the increasing sequence of finite subsets F^n must converge to F in the Hausdorff topology on compact subsets of F equipped with its resistance metric. Taking into account that we also have the weak convergence of Lemma 2.2.10, the result follows directly from [13, Theorem 7.1], since all of the spaces F^n and F are compact (see [13, Remark 1.3(b)]). □

2.3 Non-fixed-point diffusions

The advantage of using a fixed point $\mathcal{E} \in \mathbb{D}_0$ of the renormalization map Λ is that it is easy to produce a nested sequence of Dirichlet forms. We will see in this section that it is in fact possible to consider forms $\mathcal{E} \in \mathbb{D}_0$ that are *not* fixed points, but where $\Lambda(\mathcal{E})$ is sufficiently easy to understand. Now let $\mathcal{S} = (F, (\psi_i)_{1\leq i\leq N})$ be a connected generalized p.c.f.s.s. set and r a resistance vector. Let $\mu = \mu_\theta$ be a Bernoulli measure on F. For each $n \geq 0$ let μ_n be the measure on F^n given by Definition 2.2.9. Let $r_{\min}, r_{\max}, \theta_{\min}, \theta_{\max}$ be defined as before.

Assumption 2.3.1. There exists a one-parameter family $\mathbb{D} \subseteq \mathbb{D}_0$ of forms such that if $\mathcal{E} \in \mathbb{D}$, then $\rho_{\mathcal{E}}\Lambda(\mathcal{E}) \in \mathbb{D}$ for some $\rho_{\mathcal{E}} > 0$. It is such that there exists a parametrisation $\mathbb{D} = (\mathcal{E}_v^{(0)})_{v>0}$ where $\mathcal{E}_v^{(0)}$ only has edges of conductance 1 or v, and there exists at least one edge of conductance v. Call $\mathbb{D}$ a *one-parameter invariant family* with respect to Λ. Additionally, we assume that the family is *asymptotically regular*, that is, there exists $u \in [0, \infty)$ such that

$$r_{\max} \sup_{v>u} \rho^{-1}_{\mathcal{E}_v^{(0)}} < 1.$$

Remark 2.3.2. We specified edges to have conductance 1 or v; the constant 1 here is arbitrary, due to the positive homogeneity of Λ. Other examples can be found in [33].

In addition to the above assumption, we need another technical assumption on the structure of F, in particular its diameter in the resistance metric. For $v > 0$ let R_v be the resistance metric on F^0 associated with $\mathcal{E}_v^{(0)}$. Let $\mathrm{diam}(F^0, R_v)$ be the diameter of F^0 with respect to R_v. We require that F^0 does not "shrink to zero" in

R_v as $v \to \infty$. This can be verified geometrically. Note that in the following result we interpret $(F^0, \mathcal{E}_v^{(0)})$ as a graph with vertex set F^0 and edge set containing (x, y), $x \neq y$ if and only if the $\mathcal{E}_v^{(0)}$-conductance between x and y is strictly positive.

Lemma 2.3.3. *Suppose* $\mathbb{D} = (\mathcal{E}_v^{(0)})_{v>0}$ *is an asymptotically regular one-parameter invariant family with respect to* Λ*. The following are equivalent:*

(1). $\inf_{v>0} \operatorname{diam}(F^0, R_v) > 0$.

(2). The subgraph of $(F^0, \mathcal{E}_v^{(0)})$ *generated by removing all of the edges of conductance* 1 *is disconnected.*

Proof. Suppose (2) holds. Evidently $(F^0, \mathcal{E}_v^{(0)})$ must have at least one edge of conductance 1. On the generated subgraph there exists by assumption $x, y \in F^0$ in distinct connected components. Consider a function f that takes the value 1 on the connected component containing x and takes the value 0 elsewhere. Then for every $v > 0$,

$$R_v(x, y) \geq \frac{(f(x) - f(y))^2}{\mathcal{E}_v^{(0)}(f, f)} \geq \frac{1}{\#\text{edges of conductance 1 in } (F^0, \mathcal{E}_v^{(0)})}$$

which does not depend on v and is positive. So $\inf_{v>0} \operatorname{diam}(F^0, R_v) > 0$.

Now suppose (2) does not hold. Then for any $x, y \in F^0$ there is a path between x and y consisting only of edges of conductance v. Thus

$$R_v(x, y) \leq v^{-1} \cdot \#\text{edges of conductance } v \text{ in } (F^0, \mathcal{E}_v^{(0)}),$$

which only depends on v through the term v^{-1}, as before. Thus $R_v(x, y) \to 0$ as $v \to \infty$. There are only a finite number of pairs $x, y \in F^0$, so $\operatorname{diam}(F^0, R_v) \to 0$ as $v \to \infty$ and so $\inf_{v>0} \operatorname{diam}(F^0, R_v) = 0$. □

The first property in the above lemma is the important one; the second is just a simple geometric way of verifying it.

Definition 2.3.4. We will say that a one-parameter invariant family $\mathbb{D}$ with the first property in the above lemma is *non-vanishing*.

Definition 2.3.5. Define functions $\rho : (0, \infty) \to (0, \infty)$ and $\alpha : (0, \infty) \to (0, \infty)$ such that

$$\Lambda(\mathcal{E}_v^{(0)}) = \rho(v)^{-1} \mathcal{E}_{\alpha(v)}^{(0)}$$

for any $v > 0$.

The functions ρ and α can be computed by the standard method of taking traces of conductance matrices, see [3, Chapter 4]. It follows that ρ and α are rational functions of finite-degree polynomials. The non-vanishing property allows us to get some additional control on these functions.

Proposition 2.3.6. *Suppose that $\mathbb{D} = (\mathcal{E}_v^{(0)})_{v>0}$ is an asymptotically regular non-vanishing one-parameter invariant family with respect to Λ. Then there exists a finite positive limit*

$$\rho_G := \lim_{v\to\infty} \rho(v)$$

such that $\rho_G > r_{\max}$.

Proof. ρ is a positive rational function so as $v \to \infty$ we have $\rho(v) \to \rho_G$ for some $\rho_G \in [0, \infty]$. Recall the replication and trace maps from Definition 2.2.13. Consider $\rho(v)R(\mathcal{E}_v^{(0)}) \in \mathbb{D}_1$. By definition, the trace of this form on F^0 is $\mathcal{E}_{\alpha(v)}^{(0)} \in \mathbb{D}_0$. If $\rho(v) \to \infty$ as $v \to \infty$, then the diameter of F^1 with respect to the resistance metric induced by $\rho(v)R(\mathcal{E}_v^{(0)})$ must tend to 0 as $v \to \infty$. Then we would have $\operatorname{diam}(F^0, R_{\alpha(v)}) \to 0$ which contradicts the non-vanishing property. Finally $\rho_G > r_{\max}$ by the asymptotic regularity property in Assumption 2.3.1. □

Proposition 2.3.7. *Suppose that $\mathbb{D} = (\mathcal{E}_v^{(0)})_{v>0}$ is an asymptotically regular non-vanishing one-parameter invariant family with respect to Λ. The following are equivalent:*

(1). $\lim_{v\to\infty} \alpha(v) = \infty$.

(2). For all $x, y \in F^0$, there exists a path in $(F^0, \mathcal{E}_v^{(0)})$ from x to y consisting only of edges of conductance v if and only if there is a path in $(F^1, R(\mathcal{E}_v^{(0)}))$ from x to y consisting only of edges of conductance of the form $r_i^{-1}v$.

(3). There exist $x, y \in F^0$ such that the $\mathcal{E}_v^{(0)}$-conductance between x and y is v and there is a path in $(F^1, R(\mathcal{E}_v^{(0)}))$ from x to y consisting only of edges of conductance of the form $r_i^{-1}v$.

Moreover, if the above statements hold then there exists $\beta > 0$ such that

$$\lim_{v\to\infty} \frac{\alpha(v)}{v} = \frac{1}{\beta}.$$

Proof. Obviously (2) implies (3) under Assumption 2.3.1.

Suppose (3) holds. Consider $\rho(v)R(\mathcal{E}_v^{(0)}) \in \mathbb{D}_1$. By definition, the trace of this form on F^0 is $\mathcal{E}_{\alpha(v)}^{(0)} \in \mathbb{D}_0$. Since there is a path in $(F^1, \rho(v)R(\mathcal{E}_v^{(0)}))$ from x to y

consisting only of edges of conductance of the form $r_i^{-1}v\rho(v)$, and $\lim_{v\to\infty} v\rho(v) = \infty$ by the previous proposition, it follows that $R_{\alpha(v)}(x,y) \to 0$ as $v \to \infty$. Hence we must have $\lim_{v\to\infty} \alpha(v) = \infty$ and so (1) holds.

Now suppose (1) holds. Let $x, y \in F^0$ such that there exists a path in $(F^0, \mathcal{E}_v^{(0)})$ from x to y consisting only of edges of conductance v. By the assumption of (1) this implies that

$$\lim_{v\to\infty} R_{\alpha(v)}(x,y) = \lim_{v\to\infty} R_v(x,y) = 0.$$

Considering $\rho(v)R(\mathcal{E}_v^{(0)}) \in \mathbb{D}_1$ again, we see that this Dirichlet form only has edges of conductance $r_i^{-1}\rho(v)$ or $r_i^{-1}v\rho(v)$, for $1 \leq i \leq N$. We know from the previous proposition that as $v \to \infty$, $\rho(v) \to \rho_G$ and so $v\rho(v) \to \infty$. Then $R_{\alpha(v)}(x,y) \to 0$ as $v \to \infty$ implies that there is a path in $(F^1, \rho(v)R(\mathcal{E}_v^{(0)}))$ from x to y consisting only of edges of conductance of the form $r_i^{-1}v\rho(v)$. Both of these implications are in fact equivalences, and hence (2) is proven.

Finally, suppose (1), (2), (3) above hold. For $v > 0$ consider the Dirichlet form $v^{-1}\rho(v)R(\mathcal{E}_v^{(0)}) \in \mathbb{D}_1$, which only has edges of conductance $r_i^{-1}v^{-1}\rho(v)$ or $r_i^{-1}\rho(v)$, for $1 \leq i \leq N$. Its trace on F^0 is $v^{-1}\mathcal{E}_{\alpha(v)}^{(0)} \in \mathbb{D}_0$, which only has edges of conductance v^{-1} or $v^{-1}\alpha(v)$. Take $x, y \in F^0$ from (3). We will consider the distance d_v between x and y in the resistance metrics associated with these two Dirichlet forms, which are equal since the trace operator preserves the resistance metric. First we consider d_v with respect to $v^{-1}\rho(v)R(\mathcal{E}_v^{(0)}) \in \mathbb{D}_1$. We know that $\rho(v) \to \rho_G$ as $v \to \infty$, so $v^{-1}\rho(v) \to 0$. By (3) there is a path in $(F^1, v^{-1}\rho(v)R(\mathcal{E}_v^{(0)}))$ from x to y consisting only of edges of conductance of the form $r_i^{-1}\rho(v)$, so

$$d_v = O(\rho(v)^{-1}) = O(\rho_G^{-1}) = O(1) \text{ as } v \to \infty.$$

Now we consider d_v with respect to $v^{-1}\mathcal{E}_{\alpha(v)}^{(0)} \in \mathbb{D}_0$. Obviously $v^{-1} \to 0$ as $v \to \infty$, and by our choice of $x, y \in F^0$ the $v^{-1}\mathcal{E}_{\alpha(v)}^{(0)}$-conductance between x and y is $v^{-1}\alpha(v)$. So similarly we see that

$$d_v = O((v^{-1}\alpha(v))^{-1}) = O(v\alpha(v)^{-1}) \text{ as } v \to \infty.$$

Combining these it follows that

$$\frac{\alpha(v)}{v} = O(1) \text{ as } v \to \infty,$$

and so since α is a positive rational function, it must be that

$$\alpha(v) = \frac{p(v)}{\beta q(v)}$$

where p and q are monic polynomials with $\deg p = \deg q + 1$, and $\beta > 0$. So

$$\lim_{v \to \infty} \frac{\alpha(v)}{v} = \frac{1}{\beta}.$$

□

Assumption 2.3.8. The statements of Proposition 2.3.7 hold with $\beta > 1$.

An asymptotically regular non-vanishing one-parameter invariant family satisfying this assumption will be known as a one-parameter *iterable* family.

For such a family, it follows from Assumptions 2.3.1 and 2.3.8 that if we define

$$v_{\min} = \max\left(\left\{v : \frac{d\alpha}{dv}(v) = 0\right\} \cup \{v : \alpha(v) = v\} \cup \{v : \rho(v) = r_{\max}\} \cup \{0\}\right),$$

then $v_{\min} \in [0, \infty)$ and α is strictly increasing on $(v_{\min}, \infty)$. The inverse α^{-1} of α can then be uniquely defined on $(v_{\min}, \infty)$. It is continuous, strictly increasing, satisfies $\alpha^{-1}(v) > v$ for all $v \in (v_{\min}, \infty)$, and

$$\lim_{v \to \infty} \frac{\alpha^{-1}(v)}{v} = \beta > 1.$$

The Assumption 2.3.8 ensures that all the iterates $\alpha^{-n} := \alpha^{-1} \circ \alpha^{-(n-1)}$ can be defined inductively on the same domain, that is, $\alpha^{-n} : (v_{\min}, \infty) \to (v_{\min}, \infty)$ for all $n \in \mathbb{N}$. Hence the family is "iterable".

Remark 2.3.9. In many cases $v_{\min}$ is the largest (finite) fixed point of α and so $\mathcal{E}^{(0)}_{v_{\min}}$ is a non-degenerate fixed point of Λ. For example, in Figure 2.2 we may take $v_{\min} = 1$.

Henceforth we will be operating under the assumption of iterability of the one-parameter family $\mathbb{D}$, as only under this assumption do the definitions of $v_{\min}$ and α^{-n} make sense.

Definition 2.3.10. For $n \geq 0$, define $\rho_n : (v_{\min}, \infty) \to (v_{\min}, \infty)$ by

$$\rho_n(v) = \prod_{i=1}^{n} \rho(\alpha^{-i}(v)).$$

Now for each $n \geq 1$ define

$$\mathcal{E}^{(n)}_v = \rho_n(v) R^n(\mathcal{E}^{(0)}_{\alpha^{-n}(v)}). \tag{2.3.1}$$

It is simple to show, as in the previous section, that $(\mathcal{E}^{(n)}_v)_n$ is a nested sequence of Dirichlet forms on each of the F^n respectively. As before, we define:

Definition 2.3.11. Let

$$D_v = \left\{ f \in C(F) : \sup_n \mathcal{E}_v^{(n)}(f,f) < \infty \right\},$$
$$\mathcal{E}_v(f,f) = \sup_n \mathcal{E}_v^{(n)}(f,f), \quad f \in D.$$

Theorem 2.3.12. *For $v \in (v_{\min}, \infty)$, the pair $(\mathcal{E}_v, D_v)$ is a regular local irreducible Dirichlet form on $L^2(F,\mu)$. It has an associated resistance metric that is equivalent to the existing topology on F.*

Proof. Again the resistance metric result comes directly from [52, Theorem 3.3.4]. By [51, Theorem 4.1] $(\mathcal{E}_v, D_v)$ is a Dirichlet form on $L^2(F,\mu)$. For regularity and locality we can follow the respective arguments in [52, Theorem 3.4.6]. Finally, the proof that $(\mathcal{E}_v, D_v)$ is irreducible and densely defined is identical to the respective proofs in Theorem 2.2.17. □

Thus we have constructed a one-parameter family of diffusions on F which are not necessarily associated with a fixed point of the renormalization map, as [22, Theorem 7.2.1] gives us

Corollary 2.3.13. *For each $v \in (v_{\min}, \infty)$ there exists a μ-symmetric diffusion $X^v = (X_t^v)_{t\geq 0}$ on F with Dirichlet form $(\mathcal{E}_v, D_v)$ on $L^2(F,\mu)$.*

Now we fix a $v \in (v_{\min}, \infty)$. For each $n \geq 0$, let $X^{v,n} = (X_t^{v,n})_{t\geq 0}$ be the continuous time random walk associated with $\mathcal{E}_v^{(n)}$ on $L^2(F^n, \mu_n)$.

Theorem 2.3.14. *$X^{v,n} \to X^v$ weakly in $\mathcal{D}_F[0,\infty)$. Precisely, if $(x_n)_n$ is a sequence in F such that $x_n \in F^n$ for each n and $x_n \to x \in F$, then*

$$(X^{v,n}, \mathbb{P}^{x_n}) \to (X^v, \mathbb{P}^x)$$

weakly in $\mathcal{D}_F[0,\infty)$.

Proof. This follows by an identical argument to Theorem 2.2.18. □

Although using a different parametrization, this diffusion corresponds to that constructed in [36], [33] for the examples considered in those papers. If we take our parametrization, set $w = v^{-1}$ and multiply all conductances by w, then we end up with the parametrization in the aforementioned papers.

2.4 Construction of a limiting self-similar set

We continue with the set-up of Section 2.3. We have $\mathcal{S} = (F, (\psi_i)_{1\le i\le N})$ a connected generalized p.c.f.s.s. set with a one-parameter iterable family $\mathbb{D}$ (Assumption 2.3.1, Assumption 2.3.8). We aim to investigate the limit as $v \to \infty$ of our family $\mathbb{D}$. For $v > v_{\min}$ and $n \ge 0$, let $G^{v,n} = (F^n, \mathcal{E}_v^{(n)})$ and $\mathcal{G}^v = (F, \mathcal{E}_v)$. Let $A^{v,n} = (a_{xy}^{v,n})_{x,y\in F^n}$ be the conductance matrix associated with $G^{v,n}$.

Let d_{GH} be the *Gromov-Hausdorff distance* on the space of metric spaces, as defined in [9, Definition 7.3.10]. This is in fact a metric on the space of isometry classes of non-empty compact metric spaces ([9, Theorem 7.3.30]). When working with the Gromov-Hausdorff distance on non-empty compact metric spaces, we will always identify a space with its isometry class. We would like to find conditions under which, taking $v \to \infty$, the space $\mathcal{G}^v$ converges in the Gromov-Hausdorff topology to a (possibly different) generalized p.c.f.s.s. set. This will allow us to understand the limiting behaviour of X^v. In this section we will prove the following:

(1). For each $n \ge 0$, $G^{v,n}$ (equipped with its resistance metric) has a limit H^n in the Gromov-Hausdorff topology as $v \to \infty$ (Corollary 2.4.13). The metric on H^n is the resistance metric of some conductance matrix.

(2). Given a geometric assumption on F and $\mathbb{D}$ (Assumption 2.4.14), there exists a generalized p.c.f.s.s. set $\mathcal{S}_* = (F_*, (\varphi_i)_{1\le i\le N})$ such that its sequence of approximating networks (as in (2.2.1)) is $(H^n)_{n\ge 0}$ (Theorem 2.4.28).

(3). The Dirichlet form associated with H^0 is a fixed point of the renormalization map of $\mathcal{S}_*$. We can then define a limiting Dirichlet form and a diffusion on F_* (Theorem 2.4.29).

In fact it will eventually be proven that if F_* is equipped with the resistance metric associated with its Dirichlet form, then this metric space is the Gromov-Hausdorff limit as $v \to \infty$ of $\mathcal{G}^v$.

2.4.1 Preliminary results

We first need some more precise uniform estimates on the maps $\alpha(v)$ and $\rho(v)$ as well as control on the structure of $\mathcal{G}^v$.

Lemma 2.4.1. *Fix a $v_0 > v_{\min}$. Then there exist positive constants k_α, K_α dependent only on v_0 such that for all $m, i \ge 0$ and all $v \ge v_0$,*

$$k_\alpha \le \frac{\alpha^{-m-i}(v)}{\alpha^{-m}(v)\beta^i} \le K_\alpha,$$

Proof. We will first prove that, given $v \geq v_0$, there exist $k'_\alpha(v), K'_\alpha(v) > 0$ such that

$$k'_\alpha(v) \leq \frac{\alpha^{-m}(v)}{v\beta^m} \leq K'_\alpha(v)$$

for all $m \geq 0$. Recall that by Assumption 2.3.8 we have that

$$\lim_{v\to\infty} \frac{\alpha(v)}{v} = \frac{1}{\beta} < 1.$$

Along with the fact that α is a positive rational function of finite-degree polynomials, this implies that there exist $l_1, l_2 > 0$ such that

$$1 - \frac{l_1}{v} \leq \frac{\alpha(v)\beta}{v} \leq 1 + \frac{l_2}{v} \tag{2.4.1}$$

for all $v > v_{\min}$. Since the above lower bound may be negative, we note that we also have constant bounds

$$0 < l_3 \leq \frac{\alpha(v)\beta}{v} \leq l_4$$

for all $v > v_{\min}$ and note that $l_3 < 1$. By the fact that α^{-1} is continuous, $\alpha^{-1}(v) > v$ for all $v \in [v_0, \infty)$ and $\frac{\alpha^{-1}(v)}{v} \to \beta > 1$ as $v \to \infty$, there exists l_5 such that

$$1 < l_5 < \frac{\alpha^{-1}(v)}{v}$$

for all $v \in [v_0, \infty)$. Now write

$$\begin{aligned}
\frac{\alpha^{-m}(v)}{v\beta^m} &= \prod_{j=1}^{m} \left(\frac{\alpha^{-j}(v)}{\alpha^{-(j-1)}(v)\beta} \right) \\
&= \prod_{j=1}^{m} \left(\frac{\alpha(\alpha^{-j}(v))\beta}{\alpha^{-j}(v)} \right)^{-1}.
\end{aligned}$$

We start with the upper bound $K'_\alpha(v)$. We see that

$$\begin{aligned}
\left(\frac{\alpha^{-m}(v)}{v\beta^m} \right)^{-1} &\geq \prod_{j=1}^{m} \max\left\{ 1 - \frac{l_1}{\alpha^{-j}(v)}, l_3 \right\} \\
&\geq \prod_{j=1}^{m} \max\left\{ 1 - \frac{l_1}{v l_5^j}, l_3 \right\} \\
&\geq \prod_{j=1}^{m} \max\left\{ 1 - \frac{l_1}{v_{\min} l_5^j}, l_3 \right\}
\end{aligned}$$

for $v \in [v_0, \infty)$. It suffices to prove that the right-hand side has a strictly positive lower bound over $m \geq 0$. Since for large j we will have $1 - \frac{l_1}{v_{\min} l_5^j} > l_3$, it suffices to consider the asymptotics of

$$\prod_{j=k}^{m} \left(1 - \frac{l_1}{v_{\min} l_5^j} \right),$$

where k is high enough such that $\frac{l_1}{v_{\min} l_5^k} < \frac{1}{2}$, say. Note that this expression is decreasing in m. Taking logarithms and noting that $\log(1+x) \sim x$ for small x, there is a constant $l_6 > 0$ such that

$$\begin{aligned}\log \prod_{j=k}^{m} \left(1 - \frac{l_1}{v_{\min} l_5^j}\right) &= \sum_{j=k}^{m} \log \left(1 - \frac{l_1}{v_{\min} l_5^j}\right) \\ &\geq -l_6 \sum_{j=k}^{m} \frac{l_1}{v_{\min} l_5^j} \\ &\geq -\frac{l_1 l_5 l_6}{v_{\min}(l_5 - 1)} \\ &> -\infty.\end{aligned}$$

It follows that $\left(\frac{\alpha^{-m}(v)}{v\beta^m}\right)^{-1}$ is bounded away from zero uniformly over all m and so there exists some $K'_\alpha(v)$ bounding $\frac{\alpha^{-m}(v)}{v\beta^m}$ from above for all m. For the lower bound, we have that

$$\begin{aligned}\left(\frac{\alpha^{-m}(v)}{v\beta^m}\right)^{-1} &\leq \prod_{j=1}^{m} \left(1 + \frac{l_2}{\alpha^{-j}(v)}\right) \\ &\leq \prod_{j=1}^{m} \exp\left(\frac{l_2}{\alpha^{-j}(v)}\right) \\ &\leq \prod_{j=1}^{m} \exp\left(\frac{l_2}{v l_5^j}\right) \\ &\leq \exp\left(\frac{l_2 l_5}{v(l_5 - 1)}\right),\end{aligned}$$

and so set

$$k'_\alpha(v) = \exp\left(\frac{-l_2 l_5}{v(l_5 - 1)}\right).$$

Now we can prove the result. Let us prove $k_\alpha \leq \frac{\alpha^{-m-i}(v)}{\alpha^{-m}(v)\beta^i}$ first. Setting

$$k''_\alpha(v) = \frac{k'_\alpha(v)}{K'_\alpha(v)},$$

$$K''_\alpha(v) = \frac{K'_\alpha(v)}{k'_\alpha(v)},$$

it is clear that

$$k''_\alpha(v) \leq \frac{\alpha^{-m-i}(v)}{\alpha^{-m}(v)\beta^i} \leq K''_\alpha(v)$$

for all $v \geq v_0$ and $m, i \geq 0$. For the lower bound, our job is to minimize $\frac{\alpha^{-m-i}(v)}{\alpha^{-m}(v)\beta^i}$ over $[v_0, \infty)$. Observe that it is enough to minimize it over the much more manageable interval $[v_0, \alpha^{-1}(v_0)]$ since any higher values of v can then be covered by increasing the value of m. Thus for all $v \in [v_0, \alpha^{-1}(v_0)]$, since $\alpha^{-1}(v)$ is increasing in v,

$$\frac{\alpha^{-m-i}(v)}{\alpha^{-m}(v)\beta^i} \geq \frac{\alpha^{-m-i}(v_0)}{\alpha^{-m}(\alpha^{-1}(v_0))\beta^i} = \frac{\alpha^{-m-i}(v_0)}{\alpha^{-m-1}(v_0)\beta^i} \geq \frac{k''_\alpha(v_0)}{\beta} =: k_\alpha$$

and the proof is complete. The derivation of K_α is essentially the same. □

Lemma 2.4.2. *Fix a $v_0 > v_{\min}$. There exist positive constants k_ρ, K_ρ dependent only on v_0 such that for all $m, i \geq 0$ and all $v \geq v_0$,*

$$k_\rho \leq \frac{\rho_{m+i}(v)}{\rho_m(v)\rho_G^i} \leq K_\rho.$$

Proof. The proof will proceed in much the same way as in the previous lemma. We first prove that there exist $k'_\rho(v), K'_\rho(v) > 0$ such that

$$k'_\rho(v) \leq \frac{\rho_m(v)}{\rho_G^m} \leq K'_\rho(v)$$

for all $v \geq v_0$ and $m \geq 0$. We use the same constants as in the proof of Lemma 2.4.1, and introduce $l_7, l_8 > 0$ satisfying

$$1 - \frac{l_7}{v} \leq \frac{\rho(v)}{\rho_G} \leq 1 + \frac{l_8}{v}$$

for all $v > v_{\min}$, which is possible since $\rho(v)$ is a positive rational function converging to ρ_G as $v \to \infty$. Also as before there exist $l_9, l_{10} > 0$ such that

$$l_9 \leq \frac{\rho(v)}{\rho_G} \leq l_{10}$$

for all $v > v_{\min}$, where $l_9 < 1$. Writing

$$\frac{\rho_m(v)}{\rho_G^m} = \prod_{j=1}^{m} \frac{\rho(\alpha^{-j}(v))}{\rho_G},$$

for the lower bound we see that

$$\frac{\rho_m(v)}{\rho_G^m} \geq \prod_{j=1}^{m} \max\left\{1 - \frac{l_7}{\alpha^{-j}(v)}, l_9\right\}.$$

We bound the right-hand side away from zero by taking logarithms in exactly the same way as in the proof of Lemma 2.4.1. For the upper bound,

$$\frac{\rho_m(v)}{\rho_G^m} \leq \prod_{j=1}^{m} \left(1 + \frac{l_8}{\alpha^{-j}(v)}\right)$$

which we bound by an exponential just as in the proof of Lemma 2.4.1. So we have shown the existence of $k'_\rho(v), K'_\rho(v)$.

As before we now set

$$k''_\rho(v) = \frac{k'_\rho(v)}{K'_\rho(v)},$$
$$K''_\rho(v) = \frac{K'_\rho(v)}{k'_\rho(v)},$$

so that

$$k''_\rho(v) \leq \frac{\rho_{m+i}(v)}{\rho_m(v)\rho_G^i} \leq K''_\rho(v)$$

for all $v \geq v_0$ and $m, i \geq 0$. From the definition we know that

$$\frac{\rho_{m+i}(v)}{\rho_m(v)} = \prod_{j=1}^{i} \rho(\alpha^{-(m+j)}(v)).$$

To do something similar to the above we need to prove some kind of monotonicity property. We will prove that the function $v \mapsto \alpha^{-m}(v)\rho_m(v)$ is non-decreasing for any m. Indeed, consider the conductance network $(F^m, \mathcal{E}_v^{(m)})$, where all edges have had their conductance multiplied by a factor of $(\alpha^{-m}(v)\rho_m(v))^{-1}$. That is, this conductance network has conductances of the form r_w^{-1} or $r_w^{-1}\alpha^{-m}(v)^{-1}$ for $w \in \mathbb{W}_n$. Taking the trace onto F^0, we see that the corresponding conductance network on F^0 has conductances of either $(\alpha^{-m}(v)\rho_m(v))^{-1}$ or $v(\alpha^{-m}(v)\rho_m(v))^{-1}$. Consider the monotonicity law of conductance networks: decreasing some of the edge conductances of the network cannot increase any of its effective conductances. On the other hand, if we were to strictly increase all of the edge conductances of a network, all of its effective conductances would necessarily strictly increase. It follows that, since $\alpha^{-m}(v)^{-1}$ is decreasing in v, $(\alpha^{-m}(v)\rho_m(v))^{-1}$ cannot be increasing in v. So $\alpha^{-m}(v)\rho_m(v)$ is non-decreasing in v.

Now as in Lemma 2.4.1, we can similarly minimize only over $[v_0, \alpha^{-1}(v_0)]$. Using Lemma 2.4.1,

$$\begin{aligned}
\frac{\rho_{m+i}(v)}{\rho_m(v)\rho_G^i} &= \frac{\alpha^{-m}(v)}{\alpha^{-m-i}(v)} \frac{\alpha^{-m-i}(v)\rho_{m+i}(v)}{\alpha^{-m}(v)\rho_m(v)\rho_G^i} \\
&\geq \frac{\alpha^{-m}(v_0)}{\alpha^{-m-i}(\alpha^{-1}(v_0))} \frac{\alpha^{-m-i}(v_0)\rho_{m+i}(v_0)}{\alpha^{-m}(\alpha^{-1}(v_0))\rho_m(\alpha^{-1}(v_0))\rho_G^i} \\
&\geq \frac{k_\alpha \beta^{i-1}}{K_\alpha \beta^{i+1}} \frac{\rho_{m+i}(v_0)\rho(\alpha^{-1}(v_0))}{\rho_{m+1}(v_0)\rho_G^i} \\
&\geq \frac{k_\alpha \rho(\alpha^{-1}(v_0)) k''_\rho(v_0)}{K_\alpha \beta^2 \rho_G}
\end{aligned}$$

and we define the right-hand side to be k_ρ. The derivation of K_ρ is similar. □

Corollary 2.4.3. *For any $v \geq v_0 > v_{\min}$,*

$$\lim_{m\to\infty} \frac{\alpha^{-m}(v)}{v\beta^m} \in [k_\alpha, K_\alpha] \subset (0,\infty),$$

$$\lim_{m\to\infty} \frac{\rho_m(v)}{\rho_G^m} \in [k_\rho, K_\rho] \subset (0,\infty). \tag{2.4.2}$$

Proof. $\frac{v}{\alpha(v)}$ is a positive rational function so is monotone in v for sufficiently large v. Therefore $\frac{\alpha^{-1}(v)}{v}$ is monotone for sufficiently large v. Since $\frac{\alpha^{-1}(v)}{v\beta} \to 1$ as $v \to \infty$, this implies that $\frac{\alpha^{-m}(v)}{v\beta^m}$ is a monotone sequence in m for sufficiently large m. By Lemma 2.4.1 this sequence is bounded in $[k_\alpha, K_\alpha]$ so it must converge in this interval. Similarly for the second statement, using Lemma 2.4.2. □

Lemma 2.4.4. *For any $v_0 > v_{\min}$,*

$$\sup_{v\geq v_0} \operatorname{diam} \mathcal{G}^v < \infty.$$

Proof. An adaptation of [3, Proposition 7.10] will suffice. First of all, observe that

$$c := \sup_{v>v_{\min}} \operatorname{diam} G^{v,1} < \infty.$$

Fix $v \geq v_0$. Let $x \in G^{v,n}$ for some $n \geq 0$. Then there exists $w \in \mathbb{W}_{n-1}$ and $x_1 \in G^{v,n-1}$ such that $x, x_1 \in F_w$. Thus

$$R_v(x, x_1) \leq c r_w \rho_{n-1}(v)^{-1}.$$

Iterating this process, we find $y \in G^{v,0}$ such that

$$R_v(x,y) \leq c \sum_{i=0}^{n-1} r_{\max}^i \rho_i(v)^{-1} \leq \frac{c\rho_G}{k_\rho(\rho_G - r_{\max})}$$

where we have used Lemma 2.4.2. Thus for any $x, y \in \bigcup_{n\geq 0} F^n$,

$$R_v(x,y) \leq \frac{2c\rho_G}{k_\rho(\rho_G - r_{\max})} + c.$$

Then F is the closure of $\bigcup_{n\geq 0} F^n$ so we are done. □

2.4.2 Finite approximations

Definition 2.4.5. Fix $n \geq 0$. For $x, y \in F^n$, if $x = y$ or if there exists a path between x and y in $G^{v,n}$ consisting only of edges with conductance of the form $r_w^{-1}\rho_n(v)\alpha^{-n}(v)$, then we say that $x \sim y$. By Assumption 2.3.8, $\sim$ is independent of n. Thus $\sim$ is an equivalence relation on $\bigcup_{n\geq 0} F^n$.

As $v \to \infty$, the resistance between certain pairs x, y of points in $\mathcal{G}^v$ tends to zero. This suggests that if any Gromov-Hausdorff limit of the spaces $\mathcal{G}^v$ were to exist as $v \to \infty$, then these pairs of points would have to be regarded as the same in that space. The limit space is thus unlikely to have the same underlying set F.

We start by identifying points of our original fractal for which the effective resistance decreases to zero as $v \to \infty$. For each $n \geq 0$ define the conductance network (F_*^n, A^n) as follows: F_*^n is the result of taking F^n and identifying all pairs of points $x, y \in F^n$ for which the effective resistance between x and y in $G^{v,n}$ tends to zero as $v \to \infty$. Equivalently, F_*^n is the set of equivalence classes of F^n under $\sim$. The A^n-conductance between two distinct $y_1, y_2 \in F_*^n$ is defined to be

$$a^n_{y_1 y_2} = \sum_{\substack{x \in y_1 \\ z \in y_2}} \lim_{v \to \infty} a^{v,n}_{xz}.$$

It is thus always an integer multiple of ρ_G^n. The natural measure to use on F_*^n is ν_n, where for $y \in F_*^n$,

$$\nu_n(\{y\}) = \mu_n(y) = \sum_{x \in y} \mu_n(\{x\}).$$

That is, each class is given a mass equal to the sum of the original masses of its constituent elements. Let $\mathcal{E}_*^{(n)}$ be the Dirichlet form associated with (F_*^n, A^n) and let $H^n = (F_*^n, \mathcal{E}_*^{(n)})$. Each $\mathcal{E}_*^{(n)}$ is clearly irreducible.

2.4.3 The projection map

The identification of points through $\sim$ induces a surjective *projection map* $p : F^n \to F_*^n$ for each $n \geq 0$ which takes each point of F^n to the class in F_*^n that contains it. Recall that the equivalence relation $\sim$ is independent of n. This means that if $n > m$ then F_*^m can naturally be thought of as a subset of F_*^n, analogously to how F^m is a subset of F^n. Each element of F_*^m is identified with the element of F_*^n of which it is a subset. Nesting the F_*^n in this way hence preserves the definition of the projection map p, so that p is independent of n. Thus we can extend p to a surjective function

$$p : \bigcup_{n \geq 0} F^n \to \bigcup_{n \geq 0} F_*^n \tag{2.4.3}$$

which agrees with its previous definition when restricted to F^n for any $n \geq 0$.

Remark 2.4.6. Each measure ν_n on F_*^n is the pushforward through p of the respective measure μ_n on F^n.

We now prove a useful topological result, which is similar to results in [65]:

Lemma 2.4.7. *Let E_1, E_2 be topological spaces, and $f : E_1 \times E_2 \to (-\infty, \infty]$ a continuous function. Suppose E_2 is compact. Let $g : E_1 \to (-\infty, \infty]$ be given by $g(x) = \inf_{y \in E_2} f(x, y)$. Then g is a well-defined continuous function.*

Proof. Since E_2 is compact, the infimum is well-defined and moreover for each $x \in E_1$ the infimum is attained by some $y \in E_2$. A sub-basis of the topology on $(-\infty, \infty]$ is

$$\{(-\infty, a) : a \in (-\infty, \infty]\} \cup \{(a, \infty] : a \in (-\infty, \infty)\}.$$

It is enough to show that $g^{-1}(S)$ is open for each set S in the above sub-basis.

If $S = (-\infty, a)$, then

$$\begin{aligned} g^{-1}(S) &= \{x \in E_1 : \exists y \in E_2 : f(x, y) < a\} \\ &= \pi_1(f^{-1}(S)) \end{aligned}$$

where π_1 is the projection map $E_1 \times E_2 \to E_1$, which is an open map. $f^{-1}(S)$ is open, thus $g^{-1}(S)$ is open.

If $S = (a, \infty]$, then since the infimum is always attained,

$$g^{-1}(S) = \{x \in E_1 : \forall y \in E_2, f(x, y) > a\}.$$

If $g^{-1}(S)$ is empty then the proof is done, so we assume that $g^{-1}(S)$ is non-empty. Again we consider $f^{-1}(S)$. Since this is open, then by the definition of the product topology there exist for each $(x, y) \in f^{-1}(S)$ open sets $U_{xy} \in E_1$ and $V_{xy} \in E_2$ such that $(x, y) \in U_{xy} \times V_{xy} \subseteq f^{-1}(S)$. Now we fix $x_0 \in g^{-1}(S)$. This means that $(x_0, y) \in f^{-1}(S)$ for all $y \in E_2$, so the open sets $\{V_{x_0 y} : (x_0, y) \in f^{-1}(S)\}$ cover E_2. E_2 is compact so there is a finite subcover $\{V_{x_0 y_i}\}_i$. Now let

$$U_{x_0} = \bigcap_i U_{x_0 y_i},$$

which is a finite intersection, thus open in E_1. Moreover, $U_{x_0} \times V_{x_0 y_i} \subseteq f^{-1}(S)$ for every i, so

$$U_{x_0} \times E_2 = U_{x_0} \times \bigcup_i V_{x_0 y_i} \subseteq f^{-1}(S).$$

This implies that $U_{x_0} \subseteq g^{-1}(S)$. The set U_{x_0} is open and $x_0 \in U_{x_0}$. The point x_0 was arbitrary, so $g^{-1}(S)$ is open. □

Our analysis of the limit space will be clearer if we define the *preimages* of the Dirichlet forms $\mathcal{E}_*^{(n)}$ on F^n. Let $C(F^n)$ be the space of (continuous) real-valued functions on F^n. For each n, we define a form $\mathcal{E}_\infty^{(n)}$ on the domain

$$C_*(F^n) := \{f \in C(F^n) : x \sim y \Rightarrow f(x) = f(y)\}$$

such that

$$\mathcal{E}_\infty^{(n)}(f,f) = \mathcal{E}_*^{(n)}(f \circ p^{-1}, f \circ p^{-1}).$$

Remark 2.4.8. Note that $C_*(F^n)$ is closed in $C(F^n)$, and that $f \circ p^{-1}$ makes sense here because of the definition of $C_*(F^n)$.

Defining $\mathcal{E}_\infty^{(n)}(f,f) = \infty$ for all $f \in C(F^n) \setminus C_*(F^n)$, we importantly have that for all $f \in C(F^n)$,

$$\mathcal{E}_\infty^{(n)}(f,f) = \lim_{v \to \infty} \mathcal{E}_v^{(n)}(f,f).$$

With this set-up, the pairs $G^{\infty,n} := (F^n, \mathcal{E}_\infty^{(n)})$ and $H^n = (F_*^n, \mathcal{E}_*^{(n)})$ define essentially the same structure, and so this provides an analytical link between the networks $(G^{v,n})_{v>v_{\min}}$ and H^n. This allows us, for example, to prove the following key result:

Proposition 2.4.9. *Let $m < n$. Then the trace of H^n on F_*^m is H^m.*

Proof. Let $f \in C(F_*^m)$. We need to prove that

$$\mathcal{E}_*^{(m)}(f,f) = \inf\{\mathcal{E}_*^{(n)}(g,g) : g \in C(F_*^n),\ g|_{F_*^m} = f\}.$$

Since $\mathcal{E}_\infty^{(n)}(g,g) = \infty$ for all $g \in C(F^n) \setminus C_*(F^n)$,

$$\begin{aligned}
&\inf\{\mathcal{E}_*^{(n)}(g,g) : g \in C(F_*^n),\ g|_{F_*^m} = f\} \\
&\quad = \inf\{\mathcal{E}_\infty^{(n)}(g,g) : g \in C(F^n),\ g|_{F^m} = f \circ p\} \\
&\quad = \inf\{\mathcal{E}_\infty^{(n)}(g,g) : g \in C_*(F^n),\ g|_{F^m} = f \circ p\}.
\end{aligned}$$

Observe that we can restrict the function g to only taking values between $\max_x f(x)$ and $\min_x f(x)$ inclusive, since if we let $g' = (g \vee \min_x f(x)) \wedge \max_x f(x)$, then $\mathcal{E}_\infty^{(n)}(g',g') \leq \mathcal{E}_\infty^{(n)}(g,g)$. This makes the set over which we are taking the infimum compact. We can thus use Lemma 2.4.7 and the fact that $\mathcal{E}_v^{(n)}(g,g)$ is continuous in v:

$$\begin{aligned}
&\inf\{\mathcal{E}_\infty^{(n)}(g,g) : g \in C_*(F^n),\ g|_{F^m} = f \circ p\} \\
&\quad = \inf\{\lim_{v \to \infty} \mathcal{E}_v^{(n)}(g,g) : g \in C_*(F^n),\ g|_{F^m} = f \circ p\} \\
&\quad = \lim_{v \to \infty} \inf\{\mathcal{E}_v^{(n)}(g,g) : g \in C_*(F^n),\ g|_{F^m} = f \circ p\} \\
&\quad = \lim_{v \to \infty} \mathcal{E}_v^{(m)}(f \circ p, f \circ p) \\
&\quad = \mathcal{E}_\infty^{(m)}(f \circ p, f \circ p) \\
&\quad = \mathcal{E}_*^{(m)}(f,f).
\end{aligned}$$

where between lines three and four we have used the compatibility of the sequence of Dirichlet forms $(\mathcal{E}_v^{(n)})_n$. □

The above proposition immediately implies that $(\mathcal{E}_*^{(n)})_{n\geq 0}$ can be taken to be a nested sequence of Dirichlet forms on $\bigcup_{n\geq 0} F_*^n$ (by identifying a function with its restriction to F_*^n). It therefore induces a well-defined resistance metric d on $\bigcup_{n\geq 0} F_*^n$. Recall that R_v is the resistance metric on $\mathcal{G}^v$. We thus have the following result:

Proposition 2.4.10. *Let $n \geq 0$ and $x, y \in F^n$. Then*

$$\lim_{v\to\infty} R_v(x,y) = d(p(x), p(y)).$$

Proof. If $x \sim y$, then $R_v(x,y) \to 0 = d(p(x), p(y))$ as $v \to \infty$. So assume that we do not have $x \sim y$. By Lemma 2.4.7 and the fact that we can take the infimum over a compact set (as in the proof of Proposition 2.4.9),

$$\begin{aligned}
\lim_{v\to\infty} R_v(x,y)^{-1} &= \lim_{v\to\infty} \inf\{\mathcal{E}_v^{(n)}(f,f) : f \in C(F^n),\ f(x) = 1,\ f(y) = 0\} \\
&= \inf\{\lim_{v\to\infty} \mathcal{E}_v^{(n)}(f,f) : f \in C(F^n),\ f(x) = 1,\ f(y) = 0\} \\
&= \inf\{\mathcal{E}_\infty^{(n)}(f,f) : f \in C_*(F^n),\ f(x) = 1,\ f(y) = 0\} \\
&= \inf\{\mathcal{E}_*^{(n)} f,f) : f \in C(F_*^n),\ f(p(x)) = 1,\ f(p(y)) = 0\} \\
&= d(p(x), p(y))^{-1}.
\end{aligned}$$

Here again we have used the fact that $\mathcal{E}_\infty^{(n)}(g,g) = \infty$ for all $g \in C(F^n) \setminus C_*(F^n)$. □

We conclude this section with a small convergence result, showing the link between the resistance metrics of the spaces $G^{v,n}$ and H^n.

Definition 2.4.11 ([9])**.** Let $(M_1, d_1), (M_2, d_2)$ be metric spaces. Let $f : M_1 \to M_2$ be a surjective function. The *distortion* of f is

$$\operatorname{dis} f := \sup_{x_1, x_2 \in M_1} \{|d_1(x_1, x_2) - d_2(f(x_1), f(x_2))|\}.$$

Lemma 2.4.12. *Let $(M_1, d_1), (M_2, d_2)$ be compact metric spaces and let $f : M_1 \to M_2$ be a surjective function. Then*

$$d_{GH}(M_1, M_2) \leq \frac{1}{2} \operatorname{dis} f.$$

Proof. Direct corollary of [9, Theorem 7.3.25]. □

Corollary 2.4.13. *For each $n \geq 0$, $G^{v,n} \to H^n$ as $v \to \infty$ in the Gromov-Hausdorff metric.*

Proof. All of these metric spaces have a finite number of elements, so are compact. Fix $n \geq 0$. The map p is surjective, and by Proposition 2.4.10 its distortion from $G^{v,n}$ to H^n tends to 0 as $v \to \infty$ (as the distortion takes a supremum over a finite number of pairs of elements). The result follows by Lemma 2.4.12. □

2.4.4 The limit set

The natural thing to do now would be to take the F_*^n as an approximating sequence to some generalized p.c.f.s.s. set. We require a further assumption about our set-up, in order to avoid a number of pathological examples:

Assumption 2.4.14. For all $n \geq 0$, $x, y \in G^{v,n}$ and $1 \leq i \leq N$, $x \sim y$ if and only if $\psi_i(x) \sim \psi_i(y)$. This property will be known as $\mathbb{D}$-*injectivity* of the functions ψ_i with respect to the one-parameter family $\mathbb{D}$.

Remark 2.4.15. In fact to verify that Assumption 2.4.14 holds it is enough to verify it in the case $n = 0$. Indeed, assume that $\mathbb{D}$-injectivity does not hold. Then for some n and i there exists $x, y \in G^{v,n}$ with $\psi_i(x) \sim \psi_i(y)$ but not $x \sim y$ (since $x \sim y$ always implies $\psi_i(x) \sim \psi_i(y)$). By definition, there is a path between $\psi_i(x)$ and $\psi_i(y)$ consisting only of edges with conductance of the form $r_w^{-1}\rho_n(v)\alpha^{-n}(v)$, but this path cannot be contained in F_i (otherwise its preimage in ψ_i is a path that implies $x \sim y$). Thus there exist $z_1, z_2 \in G^{v,0}$ such that $\psi_i(z_1)$ and $\psi_i(z_2)$ lie on the path, and $x \sim z_1$ and $y \sim z_2$. So $\psi_i(z_1) \sim \psi_i(z_2)$ but not $z_1 \sim z_2$ and we are reduced to the case $n = 0$.

Henceforth we take Assumption 2.4.14 to be true. The resistance metric d on $\bigcup_{n\geq 0} F_*^n$ is bounded by Lemma 2.4.4 and Proposition 2.4.10. Let F_* be the completion of $\bigcup_{n\geq 0} F_*^n$ with respect to d. Then (F_*, d) is a bounded complete metric space. To define contractions φ_i, recall the original generalized p.c.f.s.s. structure $(F, (\psi_i)_{1\leq i\leq N})$. For $x \in \bigcup_{n\geq 0} F_*^n$ and $1 \leq i \leq N$, the natural definition of φ_i is

$$\varphi_i(x) = p \circ \psi_i \circ p^{-1}(x).$$

This is well-defined and injective because if $x, y \in \bigcup_{n\geq 0} F^n$ then $x \sim y$ if and only if $\psi_i(x) \sim \psi_i(y)$ by Assumption 2.4.14. It's easy to check that

$$\bigcup_{i=1}^{N} \varphi_i\left(F_*^n\right) = F_*^{n+1} \tag{2.4.4}$$

for each $n \geq 0$ and thus that

$$\bigcup_{i=1}^{N} \varphi_i \left(\bigcup_{n\geq 0} F_*^n \right) = \bigcup_{n\geq 0} F_*^n.$$

Definition 2.4.16 (Harmonic extension)**.** Let $v > v_{\min}$ and $n \geq 0$. For a function $f : F^n \to \mathbb{R}$, its *harmonic extension* to $\mathcal{G}^v$ is the unique continuous function $g : F \to \mathbb{R}$ for which

$$\mathcal{E}_v(g, g) = \mathcal{E}_v^{(n)}(f, f).$$

This can be shown to exist by the following argument: by the same reasoning as [52, Lemma 2.2.2], f can be extended uniquely to a function $f_1 : \bigcup_{n\geq 0} F^n \to \mathbb{R}$ for which

$$\mathcal{E}_v^{(m)}(f_1, f_1) = \mathcal{E}_v^{(n)}(f, f)$$

for all $m \geq n$. Then for all $x, y \in \bigcup_{n\geq 0} F^n$ we have that

$$|f_1(x) - f_1(y)|^2 \leq R_v(x, y)\mathcal{E}_v^{(n)}(f, f),$$

so there is a unique continuous extension g of f_1 to F.

Our definition of harmonic extension given above coincides with that given in [52, Section 3.2]. We notice by construction that if the function f is constant on $\psi_w(F^0)$ for some $w \in \mathbb{W}_n$, then its harmonic extension to $\mathcal{G}^v$ must be constant on F_w for any $v > v_{\min}$.

Proposition 2.4.17. *For each $1 \leq i \leq N$, φ_i can be extended uniquely to an injective contraction from F_* to itself with Lipschitz constant at most $r_{\max}\rho_G^{-1}$, and $(F_*, (\varphi_i)_{1\leq i\leq N})$ is a self-similar structure.*

Proof. Notice (by using (2.3.1) for example) that if $f \in C(F^n)$ for $n \geq 1$ then

$$\begin{aligned} \mathcal{E}_v^{(n)}(f, f) &= \rho(\alpha^{-n}(v))R(\mathcal{E}_{\alpha^{-1}(v)}^{(n-1)})(f, f) \\ &= \rho(\alpha^{-n}(v)) \sum_{i=1}^{N} r_i^{-1}\mathcal{E}_{\alpha^{-1}(v)}^{(n-1)}(f \circ \psi_i, f \circ \psi_i). \end{aligned} \tag{2.4.5}$$

for $v > v_{\min}$. It follows that if $x, y \in \bigcup_{n\geq 0} F_*^n$, then there exists $n \geq 1$ such that $x, y \in F_*^{n-1}$ and so $\varphi_i(x), \varphi_i(y) \in F_*^n$. Therefore, similarly to the proof of Proposition 2.4.9,

$$\begin{aligned} &d(\varphi_i(x), \varphi_i(y))^{-1} = \inf\left\{\mathcal{E}_*^{(n)}(f, f) : f \in C(F_*^n),\ f(\varphi_i(x)) = 0,\ f(\varphi_i(y)) = 1\right\} \\ &= \inf\left\{\lim_{v\to\infty} \mathcal{E}_v^{(n)}(f, f) : f \in C_*(F^n),\ f|_{p^{-1}(\varphi_i(x))} = 0,\ f|_{p^{-1}(\varphi_i(y))} = 1\right\} \\ &= \inf\left\{\lim_{v\to\infty} \mathcal{E}_v^{(n)}(f, f) : f \in C_*(F^n),\ f|_{\psi_i(p^{-1}(x))} = 0,\ f|_{\psi_i(p^{-1}(y))} = 1\right\} \\ &\geq \rho_G r_i^{-1} \inf\left\{\lim_{v\to\infty} \mathcal{E}_{\alpha^{-1}(v)}^{(n-1)}(f, f) : f \in C_*(F^{n-1}),\ f|_{p^{-1}(x)} = 0,\ f|_{p^{-1}(y)} = 1\right\} \\ &= \rho_G r_i^{-1} \inf\left\{\mathcal{E}_*^{(n-1)}(f, f) : f \in C(F_*^{n-1}),\ f(x) = 0,\ f(y) = 1\right\} \\ &= \rho_G r_i^{-1} d(x, y)^{-1} \end{aligned}$$

Thus $d(\varphi_i(x), \varphi_i(y)) \leq r_{\max}\rho_G^{-1} d(x, y)$, and $r_{\max}\rho_G^{-1} < 1$ by Proposition 2.3.6. So the φ_i are all contractions, and so there exists a unique continuous extension of each φ_i

to the whole of F_*, which is a contraction with the same constant. By [52, Theorem 1.1.4], there exists a unique non-empty compact subset K of F_* satisfying

$$K = \bigcup_{i=1}^{N} \varphi_i(K).$$

Take $x \in K$. If $y \in \bigcup_{n\geq 0} F_*^n$ then for each $n \geq 0$ there exists $w \in \mathbb{W}_n$ such that $y \in \varphi_w(F_*)$. Thus $d(y, \varphi_w(x)) < (r_{\max}\rho_G^{-1})^n \operatorname{diam}(F_*)$ and of course $\varphi_w(x) \in K$, so y is a limit point of K, so $y \in K$. So $\bigcup_{n\geq 0} F_*^n \subseteq K$, so by denseness we must have $K = F_*$. Hence F_* is compact and

$$F_* = \bigcup_{i=1}^{N} \varphi_i(F_*).$$

A consequence of compactness is that for any $n \geq 0$, $w \in \mathbb{W}$, we have

$$\overline{\varphi_w\left(\bigcup_{n\geq 0} F_*^n\right)} = \varphi_w(F_*).$$

We now prove that each φ_i is injective on F_*. Suppose $n \geq 0$ and $w, u \in \mathbb{W}_n$ such that $\varphi_w(F_*)$ and $\varphi_u(F_*)$ are disjoint. Thus F_w and F_u are disjoint and in $G^{v,n}$ the two subsets $\psi_w(F^0)$ (which is equal to $F^n \cap F_w$) and $\psi_u(F^0)$ are not connected by any path whose resistance tends to zero as $v \to \infty$. Using Assumption 2.4.14 we can say the same thing about $\psi_{iw}(F^0)$ and $\psi_{iu}(F^0)$ in $G^{v,n+1}$. For each $v > v_{\min}$, take a function f on F^{n+1} that is bounded in $[0,1]$, takes the value 1 on $\psi_{iw}(F^0)$ and the value 0 on $\psi_{iu}(F^0)$ and $f(x) = f(y)$ if $x \sim y$. Then consider its harmonic extension to $\mathcal{G}^v$, also called f. Now f is bounded in $[0,1]$, takes the value 1 on F_{iw} and the value 0 on F_{iu}, so for any $x_1 \in F_{iw}$ and $x_2 \in F_{iu}$,

$$\begin{aligned} R_v(x_1, x_2) &\geq \mathcal{E}_v(f,f)^{-1} = \mathcal{E}_v^{(n+1)}(f,f)^{-1} \\ &\geq r_{\min}^{n+1}\left(\sum_{z_1,z_2\in F^{n+1}:z_1\nsim z_2} \rho_{n+1}(v)(f(z_1)-f(z_2))^2\right)^{-1} \\ &\geq K_\rho^{-1}(r_{\min}\rho_G^{-1})^{n+1}\#\{z_1,z_2\in F^{n+1} : z_1 \nsim z_2\}^{-1} \\ &=: c_{n+1} > 0 \end{aligned}$$

where we have fixed some $v_0 > v_{\min}$ such that $v \geq v_0$ and used Lemma 2.4.2. Note that this bound is independent of v, x_1 and x_2, and indeed also w, u and i. It follows that for $y_1 \in \varphi_{iw}(\bigcup_{n\geq 0} F_*^n)$ and $y_2 \in \varphi_{iu}(\bigcup_{n\geq 0} F_*^n)$ there exist $x_1 \in F_{iw} \cap \bigcup_{n\geq 0} F^n$

and $x_2 \in F_{iu} \cap \bigcup_{n\geq 0} F^n$ such that $p(x_1) = y_1$ and $p(x_2) = y_2$ and so, using Proposition 2.4.10,

$$d(y_1, y_2) = \lim_{v\to\infty} R_v(x_1, x_2) \geq c_{n+1} > 0.$$

Taking closures the same estimate holds for $y_1 \in \varphi_{iw}(F_*)$ and $y_2 \in \varphi_{iu}(F_*)$. Thus $\varphi_{iw}(F_*)$ and $\varphi_{iu}(F_*)$ are disjoint. Finally, if $y_1, y_2 \in F_*$ with $y_1 \neq y_2$ then by the contractive property there exists $n \geq 0$ and $w, u \in \mathbb{W}_n$ such that $y_1 \in \varphi_w(F_*)$ and $y_2 \in \varphi_u(F_*)$ and $\varphi_w(F_*), \varphi_u(F_*)$ are disjoint. Then $\varphi_{iw}(F_*), \varphi_{iu}(F_*)$ are disjoint and so $\varphi_i(y_1) \neq \varphi_i(y_2)$. □

Definition 2.4.18. For $n \geq 0$ and $x \in F$ let

$$D_n^0(x) = \bigcup\{F_w : w \in \mathbb{W}_n,\ F_w \ni x\},$$

$$D_n^1(x) := \bigcup\{F_w : w \in \mathbb{W}_n,\ F_w \cap D_n^0(x) \neq \emptyset\}$$

be *n-neighbourhoods* of x. These are each a sequence of decreasing subsets of F (as n increases). Let

$$\partial D_n^0(x) = (D_n^0(x) \cap F^n) \setminus \{x\},$$

which is the boundary of $D_n^0(x)$ in the sense that any continuous path from an element of $D_n^0(x)$ to an element of $F \setminus D_n^0(x)$ must hit some element of $\partial D_n^0(x)$.

Similarly for $y \in F_*$ let

$$D_n^0(y) = \bigcup\{\varphi_w(F_*) : w \in \mathbb{W}_n,\ \varphi_w(F_*) \ni y\},$$

$$D_n^1(y) := \bigcup\{\varphi_w(F_*) : w \in \mathbb{W}_n,\ \varphi_w(F_*) \cap D_n^0(y) \neq \emptyset\}.$$

Proposition 2.4.19. *The projection map p can be uniquely extended to a surjective function $p : \mathcal{G}^v \to (F_*, d)$ (independently of v) which is continuous for all $v > v_{\min}$.*

Proof. Fix $v > v_{\min}$. Let $\varepsilon > 0$. Then there exists $n \geq 0$ such that

$$2(r_{\max}\rho_G^{-1})^n \operatorname{diam}(F_*) < \varepsilon.$$

It follows that if $y \in F_*$ then $D_n^1(y) \subseteq B(y, \varepsilon)$, the ε-ball about y. Using a method identical to that used in the proof of the previous proposition, we find that there is a constant $c_n > 0$ such that if $w, u \in \mathbb{W}_n$ are such that F_w, F_u are disjoint, then for all $z_1 \in F_w$ and $z_2 \in F_u$, $R_v(z_1, z_2) \geq c_n$. It follows that if $x \in F$ then

$$B(x, c_n) \subseteq D_n^1(x).$$

Let $x \in \bigcup_{n \geq 0} F^n$ and $y = p(x) \in \bigcup_{n \geq 0} F_*^n$. We see that $p(D_n^1(x)) \subseteq D_n^1(y)$, so we may conclude that

$$p(B(x, c_n)) \subseteq B(y, \varepsilon).$$

The choice of c_n did not depend on x, so p is uniformly continuous on $x \in \bigcup_{n \geq 0} F^n$. It is a uniformly continuous function defined on a dense subset of a metric space F and taking values in a complete space F_*, and so has a unique continuous extension to the whole of F. Now let $v_1, v_2 > v_{\min}$ and let $p_1, p_2 : F \to F_*$ be the functions generated by the above method by using $v = v_1, v_2$ respectively. By Theorem 2.3.12 $\mathcal{G}^{v_1}$ and $\mathcal{G}^{v_2}$ have the same topology and so p_1 and p_2 are continuous functions on $\mathcal{G}^{v_1}$ that agree on the dense subset $\bigcup_{n \geq 0} F^n$, and so must agree on the whole of F. So p is independent of v.

It remains to prove the surjectivity of the extended p, so let $y \in F_*$. There exists a decreasing sequence of non-empty sets $(\varphi_{w_n}(F_*))_n$ such that $w_n \in \mathbb{W}_n$ and $y \in \varphi_{w_n}(F_*)$ for all n. Each $\varphi_{w_n}(F_*)$ is compact, hence closed, and

$$\operatorname{diam}(\varphi_{w_n}(F_*)) \to 0$$

by the contractivity of the φ_i. Thus it must be the case that

$$\{y\} = \bigcap_{n=0}^{\infty} \varphi_{w_n}(F_*),$$

and so

$$p^{-1}(\{y\}) = \bigcap_{n=0}^{\infty} p^{-1}(\varphi_{w_n}(F_*))$$

where we note that the right-hand side is again a decreasing sequence of closed and non-empty sets. So $p^{-1}(\{y\})$ must be non-empty. □

Remark 2.4.20. We may now naturally extend the equivalence relation $\sim$ to be defined on the whole of F. For $x, y \in F$, we define $x \sim y$ if and only if $p(x) = p(y)$.

Corollary 2.4.21. *For all $1 \leq i \leq N$, $\varphi_i \circ p = p \circ \psi_i$ on F. In particular, for any $w \in \mathbb{W}_n$,*

$$p(F_w) = \varphi_w(F_*).$$

Proof. The maps $\varphi_i \circ p$ and $p \circ \psi_i$ are continuous and agree on a dense subset of F. □

2.4.4.1 Structural regularity of the projection map

We seek to prove that $\mathcal{S}_* = (F_*, (\varphi_i)_{1\le i\le N})$ is a generalized p.c.f.s.s. set and that its sequence of approximating networks (as in (2.2.1)) is $(F_*^n)_{n\ge 0}$. To do this we need a few more regularity results.

Lemma 2.4.22. *Let $n \ge 0$ and $w \in \mathbb{W}_n$. Let $v_0 > v_{\min}$. If $v \ge v_0$ then taking F_w as a subset of $\mathcal{G}^v$ we have*

$$\operatorname{diam}(F_w, R_v) \le r_{\max}^n \rho_n(v)^{-1} \sup_{v'\ge v_0} \operatorname{diam} \mathcal{G}^{v'}.$$

Proof. Recall from Lemma 2.4.4 that $\sup_{v\ge v_0} \operatorname{diam} \mathcal{G}^v < \infty$. By the construction of the $\mathcal{G}^v$ through the replication map, the function

$$\psi_w : \mathcal{G}^{\alpha^{-n}(v)} \to (F_w, R_v)$$

is a bijective contraction with Lipschitz constant at most $r_{\max}^n \rho_n(v)^{-1}$, which is strictly less than 1 by Assumption 2.3.1. Thus

$$\operatorname{diam}(F_w, R_v) \le r_{\max}^n \rho_n(v)^{-1} \operatorname{diam} \mathcal{G}^{\alpha^{-n}(v)} \le r_{\max}^n \rho_n(v)^{-1} \sup_{v'\ge v_0} \operatorname{diam} \mathcal{G}^{v'}.$$

□

Definition 2.4.23. For $y \in F_*$ define $\mathcal{C}(y)$ to be the closure of the set

$$\left\{ z \in \bigcup_{n\ge 0} F^n : p(z) = y \right\}.$$

Note that by Theorem 2.3.12 all of the resistance metrics R_v induce the same topology on F, so there is no confusion when talking about closures of subsets of F. Clearly if $x \in \mathcal{C}(y)$ then $p(x) = y$, by continuity of p. The purpose of the next result is to prove a converse to this statement.

Proposition 2.4.24. *Let $y \in F_*$. Suppose there exists $z \in \bigcup_{n\ge 0} F^n$ such that $p(z) = y$. Then for all $x \in F$, if $p(x) = y$ then $x \in \mathcal{C}(y)$.*

Proof. Suppose $x \in F$ such that $x \notin \mathcal{C}(y)$. We aim to show that $p(x) \ne y$. If $x \in \bigcup_{n\ge 0} F^n$ then we may immediately conclude that $p(x) \ne y$, so we may assume that $x \notin \bigcup_{n\ge 0} F^n$. Fix a $v > v_{\min}$.

We have that $z \in \mathcal{C}(y)$. The set $\mathcal{C}(y)$ is a non-empty and closed subset of F, thus is compact, so $\varepsilon := \inf_{x'\in\mathcal{C}(y)} R_v(x, x') > 0$ is well-defined. Then by Lemma 2.4.22

there exists $n \geq 0$ such that $D_n^0(x) \subseteq B_v(x, \frac{\varepsilon}{2})$, the open R_v-ball in F with centre x and radius $\frac{\varepsilon}{2}$, and also such that $z \in F^n$. In particular,

$$\left(D_n^0(x) \cap \bigcup_{m \geq 0} F^m \right) \cap \mathcal{C}(y) = \emptyset, \tag{2.4.6}$$

and notice that this is now independent of v. Now consider the indicator function of $\mathcal{C}(y) \cap F^n$ as a function from F^n to $\mathbb{R}$, written as $\mathbb{1}_{\mathcal{C}(y)\cap F^n}$. For $x_1, x_2 \in G^{v,n}$, if $\mathbb{1}_{\mathcal{C}(y)\cap F^n}(x_1) \neq \mathbb{1}_{\mathcal{C}(y)\cap F^n}(x_2)$ then the edge between x_1 and x_2 must either have zero conductance or have conductance of the form $r_w^{-1}\rho_n(v)$, by definition of $\mathcal{C}(y)$. Then by Proposition 2.3.6, for each $v_0 > v_{\min}$,

$$\sup_{v \geq v_0} \mathcal{E}_v^{(n)}(\mathbb{1}_{\mathcal{C}(y)\cap F^n}, \mathbb{1}_{\mathcal{C}(y)\cap F^n}) < \infty.$$

We see by (2.4.6) that $\mathbb{1}_{\mathcal{C}(y)\cap F^n}$ vanishes on $D_n^0(x) \cap F^n$. Therefore for any $v > v_{\min}$, the harmonic extension of $\mathbb{1}_{\mathcal{C}(y)\cap F^n}$ to $\mathcal{G}^v$ takes the value 1 at z and must take the value 0 in all of $D_n^0(x)$, so it follows that for each $v_0 > v_{\min}$ there exists a constant $c > 0$ such that for all $x' \in D_n^0(x)$ and $v \geq v_0$,

$$R_v(x', z) \geq c.$$

Now let $(x_i)_i$ be a sequence in $D_n^0(x) \cap \bigcup_{n \geq 0} F^n$ converging to x. For each x_i we have $d(p(x_i), y) \geq c$ by Proposition 2.4.10. Thus by continuity of p, $d(p(x), y) \geq c$. So $p(x) \neq y$. □

Corollary 2.4.25. *Let $n \geq 0$, $w \in \mathbb{W}_n$ and $x \in F_w$. Suppose there exists $z \in \bigcup_{n \geq 0} F^n$ such that $z \sim x$. Then there exists $z' \in F_w \cap \bigcup_{n \geq 0} F^n$ such that $z' \sim x$.*

Proof. By Proposition 2.4.24 there exists a sequence of $x_i \in \bigcup_{n \geq 0} F^n$ such that $x_i \sim x$ for all i and $x_i \to x$ as $i \to \infty$. Suppose that the conclusion of the present result does not hold, then the sequence (x_i) must accumulate (in a subsequence) in F_v for some $v \in \mathbb{W}_n$, $v \neq w$. This implies that $x \in F_w \cap F_v = \psi_w(F^0) \cap \psi_v(F^0)$ by [52, Proposition 1.3.5(2)], so $x \in F^n$ and we may simply take $z' = x$, which is a contradiction. □

Now we provide a partial extension of Proposition 2.4.10 to points in the space F.

Lemma 2.4.26. *Let $x, y \in F$. Then*

$$\limsup_{v \to \infty} R_v(x, y) \leq d(p(x), p(y)).$$

Proof. Fix a $v_0 > v_{\min}$. Let $(x_i)_i$ and $(y_i)_i$ be sequences in $\bigcup_{n\geq 0} F^n$ such that $x_i \in D_i^0(x)$ and $y_i \in D_i^0(y)$ for each i. Evidently (Lemma 2.4.22) $x_i \to x$ and $y_i \to y$ as $i \to \infty$. Let $\varepsilon > 0$. Then there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$ we have $\sup_{v\geq v_0} R_v(x_i, x) \leq \varepsilon$ and $\sup_{v\geq v_0} R_v(y_i, y) \leq \varepsilon$ (by Lemma 2.4.22) and $d(p(x_i), p(y_i)) \leq d(p(x), p(y)) + \varepsilon$ (by continuity of p). Thus

$$\begin{aligned}\limsup_{v\to\infty} R_v(x,y) &\leq \limsup_{v\to\infty} \left(R_v(x,x_i) + R_v(x_i,y_i) + R_v(y_i,y)\right)\\ &\leq 2\varepsilon + \limsup_{v\to\infty} R_v(x_i,y_i).\end{aligned}$$

Then by Proposition 2.4.10,

$$\limsup_{v\to\infty} R_v(x,y) \leq 2\varepsilon + d(p(x_i),p(y_i)) \leq 3\varepsilon + d(p(x),p(y)).$$

Finally we take $\varepsilon \to 0$ to get the required result. □

Proposition 2.4.27. *Let $x, y \in F$ such that $x \neq y$ and $p(x) = p(y)$. Then there exists $z \in \bigcup_{n\geq 0} F^n$ such that $p(x) = p(z) = p(y)$.*

Proof. If $x \in \bigcup_{n\geq 0} F^n$ or $y \in \bigcup_{n\geq 0} F^n$, then the result instantly holds, so we assume that neither of these is the case. Since $x \neq y$, by Lemma 2.4.22 there exists $n_0 \geq 0$ such that for all $n \geq n_0$ we have $D_n^0(x) \cap D_n^0(y) = \emptyset$. We have $p(x) = p(y)$ so by Lemma 2.4.26, $\lim_{v\to\infty} R_v(x, y) = 0$.

For $n \geq n_0$ and $v > v_{\min}$, we consider the distance $R_v(\partial D_n^0(x), \partial D_n^0(y))$. The sets $\partial D_n^0(x)$ and $\partial D_n^0(y)$ are both subsets of F^n, so it is enough approximate their effective resistance by considering functions from F^n to $\mathbb{R}$. Suppose $v > v_{\min}$, and suppose that the function $f : F^n \to \mathbb{R}$ takes the value 0 on $\partial D_n^0(x)$ and takes the value 1 on $\partial D_n^0(x)$. Since $x, y \notin \bigcup_{n\geq 0} F^n$ we must have $\partial D_n^0(x) = D_n^0(x) \cap F^n$ and $\partial D_n^0(y) = D_n^0(y) \cap F^n$, so the harmonic extension of f to $\mathcal{G}^v$ must take the value 0 at x and the value 1 at y. It follows that

$$R_v(\partial D_n^0(x), \partial D_n^0(y)) \leq R_v(x,y)$$

for all $n \geq n_0$ and $v > v_{\min}$. In particular,

$$\lim_{v\to\infty} R_v(\partial D_n^0(x), \partial D_n^0(y)) = 0$$

for all $n \geq n_0$. This means that for each $n \geq n_0$, on $G^{v,n}$ there exists some $x_n \in \partial D_n^0(x)$ and some $y_n \in \partial D_n^0(y)$ such that there is a path from x_n to y_n consisting only of edges with conductance of the form $r_w^{-1}\rho_n(v)\alpha^{-n}(v)$. Since $D_{n_0}^0(x) \cap D_{n_0}^0(y) = \emptyset$, this path (which depends on n in general) must contain some $x_n' \in \partial D_{n_0}^0(x)$ and some

$y'_n \in \partial D^0_{n_0}(y)$. Since the set $\partial D^0_{n_0}(x) \times \partial D^0_{n_0}(y)$ is finite we may take a subsequence (x_{n_k}, y_{n_k}) of (x_n, y_n) such that $x'_{n_k} =: x' \in \partial D^0_{n_0}(x)$ and $y'_{n_k} =: y' \in \partial D^0_{n_0}(y)$ are constant over k. Therefore

$$x_{n_k} \sim x' \sim y' \sim y_{n_k}$$

for all k. Since $x_{n_k} \in D^0_{n_k}(x)$ we must have $x_{n_k} \to x$ as $k \to \infty$ by Lemma 2.4.22, and likewise $y_{n_k} \to y$. So by the continuity of p,

$$p(x) = p(x') = p(y') = p(y),$$

and $x', y' \in \bigcup_{n \geq 0} F^n$. □

We may now finally prove the target result.

Theorem 2.4.28. *$\mathcal{S}_* = (F_*, (\varphi_i)_{1 \leq i \leq N})$ is a generalized p.c.f.s.s. set and its sequence of approximating networks (as in (2.2.1)) is $(F^n_*)_{n \geq 0}$.*

Proof. We see that

$$\begin{aligned} B(\mathcal{S}_*) &= \bigcup_{i \neq j} (\varphi_i (F_*) \cap \varphi_j (F_*)) \\ &= \bigcup_{i \neq j} (p (F_i) \cap p (F_j)) \end{aligned}$$

and so $p(B(\mathcal{S})) \subseteq B(\mathcal{S}_*)$. If $y \in B(\mathcal{S}_*)$, then let $i \neq j$ be such that $y \in \varphi_i (F_*) \cap \varphi_j (F_*)$. Then there exist $x_i \in F_i$ and $x_j \in F_j$ such that $p(x_i) = y = p(x_j)$. If $x_i = x_j$, then $x_i \in F_i \cap F_j$ and so $y \in p(B(\mathcal{S}))$. On the other hand if $x_i \neq x_j$, then $x_i \sim x_j$ and so by Proposition 2.4.27 there exists $z \in \bigcup_{n \geq 0} F^n$ such that $z \sim x_i \sim x_j$. Then by Corollary 2.4.25 there must exist $x'_i \in F_i \cap \bigcup_{n \geq 0} F^n$ and $x'_j \in F_j \cap \bigcup_{n \geq 0} F^n$ such that

$$x'_i \sim x_i \sim x_j \sim x'_j.$$

Choose $n \geq 0$ such that $x'_i, x'_j \in F^n$. Then in $G^{v,n}$ there must exist a path between x'_i and x'_j consisting only of edges with conductance $r_w^{-1} \rho_n(v) \alpha^{-n}(v)$. Since $i \neq j$, this path must contain some element z of $F_i \cap F_k$ for some $k \in \{1, \dots, N\}$ such that $k \neq i$. So $z \in B(\mathcal{S})$ and $p(z) = y$, so $y \in p(B(\mathcal{S}))$. Thus

$$p(B(\mathcal{S})) = B(\mathcal{S}_*).$$

Now it follows from the surjectivity of p that

$$\begin{aligned}\pi(P(\mathcal{S}_*)) &= \left\{x \in F_* : \exists w \in \bigcup_{n\geq 1} \mathbb{W}_n,\ \varphi_w(x) \in B(\mathcal{S}_*)\right\} \\ &= \left\{p(x) \in F_* : x \in F,\ \exists w \in \bigcup_{n\geq 1} \mathbb{W}_n,\ p \circ \psi_w(x) \in B(\mathcal{S}_*)\right\} \\ &= \left\{p(x) \in F_* : x \in F,\ \exists w \in \bigcup_{n\geq 1} \mathbb{W}_n,\ \exists y \in B(\mathcal{S}),\ \psi_w(x) \sim y\right\} \\ &= p\left(\left\{x \in F : \exists w \in \bigcup_{n\geq 1} \mathbb{W}_n,\ \exists y \in B(\mathcal{S}),\ \psi_w(x) \sim y\right\}\right).\end{aligned} \tag{2.4.7}$$

Recall that

$$\pi(P(\mathcal{S})) = \left\{x \in F : \exists w \in \bigcup_{n\geq 1} \mathbb{W}_n,\ \psi_w(x) \in B(\mathcal{S})\right\},$$

so it is clear that $p(\pi(P(\mathcal{S}))) \subseteq \pi(P(\mathcal{S}_*))$.

Now suppose that x is a member of the subset of F described by the last line of (2.4.7). That is, there exists $n \geq 1$ and $w \in \mathbb{W}_n$ and $y \in B(\mathcal{S})$ such that $\psi_w(x) \sim y$. We observe that $y \in F^1$.

Assume that $x \notin \pi(P(\mathcal{S})) = F^0$. Since $n \geq 1$, if $y \in F_w$ then $y \in \psi_w(F^0)$ so there exists some $z \in F^0$ such that $p(\psi_w(z)) = p(\psi_w(x))$. We now suppose that $y \notin F_w$. By Corollary 2.4.25 there exists $x' \in F_w \cap \bigcup_{n\geq 0} F^n$ such that $x' \sim \psi_w(x) \sim y$. Pick $m \geq n$ such that $x', y \in F^m$. Then there exists a path in $G^{v,m}$ from x' to y consisting only of edges with conductance of the form $r_{w'}^{-1}\rho_m(v)\alpha^{-m}(v)$. Since $x' \in F_w$ and we have assumed that $y \notin F_w$, this path must contain some element of $\psi_w(F^0)$. So there exists some $z \in F^0$ such that $p(\psi_w(z)) = p(\psi_w(x))$.

We have seen that in every case there exists $z \in F^0 = \pi(P(\mathcal{S}))$ such that $p(\psi_w(z)) = p(\psi_w(x))$. So $\varphi_w(p(z)) = \varphi_w(p(x))$ and so $p(z) = p(x)$ by the injectivity of the φ_i (Proposition 2.4.17). We conclude that

$$p\left(\left\{x \in F : \exists w \in \bigcup_{n\geq 1} \mathbb{W}_n,\ \exists y \in B(\mathcal{S}),\ \psi_w(x) \sim y\right\}\right) \subseteq p(\pi(P(\mathcal{S}))),$$

and so

$$\pi(P(\mathcal{S}_*)) = p(\pi(P(\mathcal{S}))) = p(F^0) = F^0_*.$$

This is finite, so $\mathcal{S}_*$ is a generalized p.c.f.s.s. set. Then (2.4.4) implies that $(F^n_*)_{n\geq 0}$ is the associated sequence of approximating networks. □

2.4.5 The limit process

Now let ν be the pushforward measure of μ onto F_* through the continuous function p. It is easy to verify that ν is a Bernoulli measure and that the ν_n are its approximations as given in Definition 2.2.9.

Theorem 2.4.29. *$\mathcal{E}_*^{(0)}$ is a regular non-degenerate fixed point of the renormalization operator Λ_* of $\mathcal{S}_*$ with respect to the resistance vector r. Let*

$$D_* = \left\{ f \in C(F_*) : \sup_n \mathcal{E}_*^{(n)}(f,f) < \infty \right\},$$
$$\mathcal{E}_*(f,f) = \sup_n \mathcal{E}_*^{(n)}(f,g), \quad f,g \in D_*.$$

Then the pair $(\mathcal{E}_*, D_*)$ is a regular local irreducible Dirichlet form on $L^2(F_*, \nu)$.

If $X^{*,n}$ is the Markov process associated with $H^n = (F_*^n, \mathcal{E}_*^{(n)})$ and $L^2(F_*^n, \nu_n)$ for each n, and if X^* is the Markov process associated with $(F_*, \mathcal{E}_*)$ and $L^2(F_*, \nu)$, then $X^{*,n} \to X^*$ weakly in $\mathcal{D}_{F_*}[0,\infty)$.

Proof. Λ_* is defined on the set of conservative Dirichlet forms on F_*^0. Let R_* be the associated replication operator. We see that

$$\begin{aligned} R_*(\mathcal{E}_*^{(0)})(f,g) &= \sum_{i=1}^{N} r_i^{-1} \mathcal{E}_*^{(0)}(f \circ \varphi_i, g \circ \varphi_i) \\ &= \sum_{i=1}^{N} r_i^{-1} \mathcal{E}_\infty^{(0)}(f \circ \varphi_i \circ p, g \circ \varphi_i \circ p) \\ &= \sum_{i=1}^{N} r_i^{-1} \mathcal{E}_\infty^{(0)}(f \circ p \circ \psi_i, g \circ p \circ \psi_i) \\ &= \lim_{v \to \infty} R(\mathcal{E}_v^{(0)})(f \circ p, g \circ p) \end{aligned}$$

where R is the replication operator of $\mathcal{S}$. Then

$$\begin{aligned} \Lambda_*(\mathcal{E}_*^{(0)})(f,f) &= \inf\{R_*(\mathcal{E}_*^{(0)})(g,g) : g \in C(F_*^1),\ g|_{F_*^0} = f\} \\ &= \inf\{\lim_{v \to \infty} R(\mathcal{E}_v^{(0)})(g \circ p, g \circ p) : g \in C(F_*^1),\ g|_{F_*^0} = f\} \\ &= \inf\{\lim_{v \to \infty} R(\mathcal{E}_v^{(0)})(g,g) : g \in C_*(F^1),\ g|_{F^0} = f \circ p\} \end{aligned}$$

Now we observe that for any $g \in C(F^1) \setminus C_*(F^1)$, we have $\lim_{v \to \infty} R(\mathcal{E}_v^{(0)})(g,g) = \infty$. Also we can restrict the set over which we take the infimum to functions taking values in $[\min f, \max f]$. This is a compact set, so we can use Lemma 2.4.7. So

$$
\begin{aligned}
\Lambda_*(\mathcal{E}_*^{(0)})(f,f) &= \lim_{v\to\infty} \inf\{R(\mathcal{E}_v^{(0)})(g,g) : g \in C(F^1),\ g|_{F^0} = f\circ p\} \\
&= \lim_{v\to\infty} \Lambda(\mathcal{E}_v^{(0)})(f\circ p, f\circ p) \\
&= \lim_{v\to\infty} (\rho(v)^{-1}\mathcal{E}_{\alpha(v)}^{(0)}(f\circ p, f\circ p)) \\
&= \rho_G^{-1}\mathcal{E}_\infty^{(0)}(f\circ p, f\circ p) \\
&= \rho_G^{-1}\mathcal{E}_*^{(0)}(f,f)
\end{aligned}
$$

and so $\mathcal{E}_*^{(0)}$ is a regular non-degenerate fixed point of Λ_* with eigenvalue ρ_G^{-1}. It is then simple to verify that

$$
\mathcal{E}_*^{(n)}(f,g) = \rho_G^n \sum_{w\in\mathbb{W}_n} r_w^{-1}\mathcal{E}_*^{(0)}(f\circ\varphi_w, g\circ\varphi_w), \quad f,g\in C(F_*^n)
$$

and so we are able to use Theorems 2.2.17 and 2.2.18. □

2.5 Convergence of processes

We continue with the set-up of the previous sections; we have $\mathcal{S} = (F, (\psi_i)_{1\le i\le N})$ a connected generalized p.c.f.s.s. set with a one-parameter iterable family $\mathbb{D}$ (Assumption 2.3.1, Assumption 2.3.8) satisfying $\mathbb{D}$-injectivity (Assumption 2.4.14). We have the spaces $\mathcal{G}^v = (F, \mathcal{E}_v)$ and the limit space $(F_*, \mathcal{E}_*)$ constructed in the previous section. Recall that $H^n = (F_*^n, \mathcal{E}_*^{(n)})$. Let $\mathcal{H} = (F_*, \mathcal{E}_*)$. Recall that X^v is the diffusion associated with $\mathcal{G}^v$ and X^* is the diffusion associated with $\mathcal{H}$.

2.5.1 The ambient space

Fix a $v_0 > v_{\min}$. We aim to construct a metric space $\mathcal{M} = (E, d)$ such that all of our metric spaces $(\mathcal{G}^v)_{v\ge v_0}$ and $\mathcal{H}$ exist as disjoint subspaces of $\mathcal{M}$ in such a way that $\mathcal{G}^v \to \mathcal{H}$ as $v\to\infty$ in the Hausdorff topology on $\mathcal{M}$. This will mean that we can define all of the X^v and X^* on the same space and so it will be easier to study the relationship between them. The new metric d will agree with the old definition of d as the resistance metric on $\mathcal{H}$, so no confusion will occur. We construct the underlying set E as follows.

Let E be the disjoint union

$$
E = \mathcal{H} \sqcup \bigsqcup_{v\ge v_0} \mathcal{G}^v.
$$

By construction E contains each $\mathcal{G}^v$ and $\mathcal{H}$. Notice that E also contains each $G^{v,n}$ as a subspace of its respective $\mathcal{G}^v$ and likewise contains each H^n as a subspace of $\mathcal{H}$.

We also perform our last extension of the projection map p. We will now have

$$p : \mathcal{M} \to \mathcal{H}$$

where $p(x)$ agrees with the previous definition if $x \in \mathcal{G}^v$ for any $v \geq v_0$, and $p(x) = x$ if $x \in \mathcal{H}$. We also define restrictions of this new p to certain subspaces of E: for $v \geq v_0$ and $n \geq 0$ let the functions $p_{v,n}$ and p_v be given by

$$p_{v,n} = p|_{G^{v,n}} : G^{v,n} \to H^n,$$
$$p_v = p|_{\mathcal{G}^v} : \mathcal{G}^v \to \mathcal{H}.$$

2.5.1.1 Constructing the metric

We start with an extension of Lemma 2.4.22.

Lemma 2.5.1. $\operatorname{diam} \mathcal{H} < \infty$. *Let $n \geq 0$ and $w \in \mathbb{W}_n$. Then*

$$\operatorname{diam}(\varphi_w(F_*)) \leq r_{\max}^n \rho_G^{-n} \operatorname{diam} \mathcal{H}.$$

Proof. $\mathcal{H}$ is a self-similar structure so is compact, so $\operatorname{diam} \mathcal{H} < \infty$. This proof is essentially the same as the proof of Lemma 2.4.22. The function φ_w on $\mathcal{H}$ is a bijective contraction with Lipschitz constant at most $r_{\max}^n \rho_G^{-n}$, which is strictly less than 1 by Assumption 2.3.1. So

$$\operatorname{diam} \varphi_w(F_*) \leq r_{\max}^n \rho_G^{-n} \operatorname{diam} \mathcal{H}.$$

□

Definition 2.5.2. Let δ_v be the Hausdorff metric on non-empty compact subsets of $\mathcal{G}^v$. Let δ_* be the Hausdorff metric on non-empty compact subsets of $\mathcal{H}$.

Corollary 2.5.3. *For all $n \geq 0$,*

$$\sup_{v \geq v_0} \delta_v(G^{v,n}, \mathcal{G}^v) \leq r_{\max}^n \rho_n(v)^{-1} \sup_{v \geq v_0} \operatorname{diam} \mathcal{G}^v$$

and

$$\delta_*(H^n, \mathcal{H}) \leq r_{\max}^n \rho_G^{-n} \operatorname{diam} \mathcal{H}.$$

Proof. We observe that $G^{v,n}$ is a metric subspace of $\mathcal{G}^v$. If $x \in \mathcal{G}^v$ then there exists $w \in \mathbb{W}_n$ such that $x \in F_w$. Then there exists $y \in G^{v,n}$ such that $y \in F_w$ also. Then by Lemma 2.4.22,

$$R_v(x,y) \leq r_{\max}^n \rho_n(v)^{-1} \sup_{v' \geq v_0} \operatorname{diam} \mathcal{G}^{v'}.$$

It follows that

$$\delta_v(G^{v,n}, \mathcal{G}^v) \leq r_{\max}^n \rho_n(v)^{-1} \sup_{v' \geq v_0} \operatorname{diam} \mathcal{G}^{v'}.$$

The second statement is proven similarly using Lemma 2.5.1. □

We now have all the pieces we need to construct d.

Definition 2.5.4 (The metric d on $\mathcal{M}$)**.** If $x, y \in \mathcal{G}^v$ for some $v \geq v_0$ then define $d(x,y) = R_v(x,y)$. On $\mathcal{H}$ likewise we define d to agree with the existing resistance metric (which was also called d).

Let $n_0 = 0$. For each $k \geq 1$, using Corollary 2.5.3, Lemma 2.4.2 and regularity (Assumption 2.3.1) we can choose a $n_k \geq 0$ such that for all $n \geq n_k$,

$$\sup_{v \geq v_0} \delta_v(G^{v,n}, \mathcal{G}^v) \leq \frac{1}{2k},$$
$$\delta_*(H^n, \mathcal{H}) \leq \frac{1}{2k}.$$

Without loss of generality we may assume that the sequence $(n_k)_{k \geq 0}$ is strictly increasing.

Let $c = \sup_{v \geq v_0} \operatorname{dis} p_{v,0}$, which is finite by Proposition 2.4.10 and the fact that the resistance metric R_v restricted to F^0 is continuous in v (using Lemma 2.4.7). Now for each $k \geq 1$ we use the finiteness of F^{n_k} and Proposition 2.4.10 to find a $v_k \geq v_0$ such that for all $v \geq v_k$,

$$\operatorname{dis} p_{v,n_k} \leq \frac{c}{k+1}.$$

Again without loss of generality we may assume that $v_{k+1} \geq v_k + 1$ for all $k \geq 0$.

Let $v \geq v_0$. By the construction of the sequence $(v_k)_{k \geq 0}$ we may pick the unique $k \geq 0$ such that $v_k \leq v < v_{k+1}$. For this k, for each $x \in G^{v,n_k} \subseteq \mathcal{G}^v$ we set

$$d(x, p_{v,n_k}(x)) = \frac{c}{2(k+1)}. \tag{2.5.1}$$

The restriction of d to the spaces $\mathcal{G}^v$ and $\mathcal{H}$ is then constructed naturally around this: For $x \in \mathcal{G}^v$ and $y \in \mathcal{H}$, set

$$d(x,y) = \inf_{x' \in G^{v,n_k}} \left\{ d(x,x') + \frac{c}{2(k+1)} + d(p_{v,n_k}(x'), y) \right\}.$$

The bound on the distortion of p_{v,n_k} ensures that this is still a metric. Finally, for $v_1 \neq v_2$, $x_1 \in \mathcal{G}^{v_1}$, $x_2 \in \mathcal{G}^{v_2}$, we define

$$d(x_1, x_2) = \inf_{y \in \mathcal{H}} \{d(x_1, y) + d(y, x_2)\}.$$

This is indeed a metric. Checking that the triangle inequality holds is a simple but tedious exercise.

2.5.2 Convergence

Definition 2.5.5. For each $v \geq v_0$ let μ^v be the Borel probability measure on $\mathcal{M}$ given by

$$\mu^v(A) = \mu(A \cap \mathcal{G}^v),$$

where μ is interpreted as a measure on $\mathcal{G}^v = (F, R_v)$. That is, μ^v is the measure μ defined on the vth copy of F in E.

Proposition 2.5.6. *$\mathcal{G}^v \to \mathcal{H}$ as $v \to \infty$ in the Hausdorff metric on non-empty compact subsets of $\mathcal{M}$. Additionally, $\mu^v \to \nu$ as $v \to \infty$ weakly as probability measures on $\mathcal{M}$.*

Proof. Let δ be the Hausdorff metric on subsets of $\mathcal{M}$. We take the sequences $(n_k)_{k \geq 0}$ and $(v_k)_{k \geq 0}$ and the constant $c > 0$ from Definition 2.5.4. Then for each $k \geq 0$ we have n_k satisfying

$$\sup_{v \geq v_0} \delta(G^{v,n_k}, \mathcal{G}^v) \leq \frac{1}{2k},$$
$$\delta(H^{n_k}, \mathcal{H}) \leq \frac{1}{2k}.$$

In addition, for all $v > v_k$ we have

$$\delta(G^{v,n_k}, H^{n_k}) \leq \frac{c}{2(k+1)}$$

by (2.5.1). Recall that we constructed the sequence $(v_k)_k$ to be strictly increasing with $\lim_{k \to \infty} v_k = \infty$, and so

$$\lim_{v \to \infty} \delta(\mathcal{G}^v, \mathcal{H}) = 0.$$

We now tackle the weak convergence result. Since ν is the push-forward measure of μ^v with respect to p_v for all $v \geq v_0$, it suffices to prove that

$$\lim_{v \to \infty} \sup_{x \in \mathcal{G}^v} d(x, p(x)) = 0.$$

Let $v \geq v_0$ and $x \in \mathcal{G}^v$. There exists a unique $k \geq 0$ such that $v_k \leq v < v_{k+1}$. Let $w \in \mathbb{W}_{n_k}$ such that $x \in F_w \subseteq \mathcal{G}^v$, and pick some $y \in (G^{v,n_k} \cap F_w) \subseteq \mathcal{G}^v$. Then Corollary 2.4.21 implies that $p(x), p(y) \in \varphi_w(F_*)$. Then by Lemma 2.4.22, Lemma 2.5.1 and the construction of the metric d we have that

$$\begin{aligned} d(x,p(x)) &\leq d(x,y) + d(y,p(y)) + d(p(y),p(x)) \\ &\leq r_{\max}^{n_k} \rho_G^{-n_k} (k_\rho^{-1} \sup_{v' \geq v_0} \operatorname{diam} \mathcal{G}^{v'} + \operatorname{diam} \mathcal{H}) + \frac{c}{2(k+1)}. \end{aligned}$$

Therefore

$$\sup_{x \in \mathcal{G}^v} d(x,p(x)) \leq r_{\max}^{n_k} \rho_G^{-n_k} (k_\rho^{-1} \sup_{v' \geq v_0} \operatorname{diam} \mathcal{G}^{v'} + \operatorname{diam} \mathcal{H}) + \frac{c}{2(k+1)}.$$

Proposition 2.3.6 states that $r_{\max} \rho_G^{-1} < 1$. Lemma 2.4.4 and Lemma 2.5.1 respectively state that $\sup_{v' \geq v_0} \operatorname{diam} \mathcal{G}^{v'} < \infty$ and $\operatorname{diam} \mathcal{H} < \infty$. By the construction of the metric we have that $n_k \to \infty$ and $v_k \to \infty$ as $k \to \infty$. The choice of k is such that $k \to \infty$ as $v \to \infty$. Therefore

$$\lim_{v \to \infty} \sup_{x \in \mathcal{G}^v} d(x,p(x)) = 0.$$

□

We are now in a position to state and prove the main theorem of this chapter. All of the assumptions needed are described at the start of this section.

Theorem 2.5.7. *Let $x \in F_*$ and for each $v \geq v_0$ let $x_v \in p^{-1}(x) \subseteq \mathcal{G}^v$. Then*

$$(X^v, \mathbb{P}^{x_v}) \to (X^*, \mathbb{P}^x)$$

as $v \to \infty$ weakly as random variables in $\mathcal{D}_{\mathcal{M}}[0,\infty)$, the Skorokhod space of càdlàg processes on $\mathcal{M}$.

Proof. It suffices to prove convergence on a countable sequence $(u_m)_{m=0}^\infty$ such that $u_m \geq v_0$ for all m and $\lim_{m\to\infty} u_m = \infty$. Let

$$\mathcal{M}' := \mathcal{H} \sqcup \bigsqcup_{m \geq 0} \mathcal{G}^{u_m} \subset \mathcal{M},$$

in which $\mathcal{H}$ and all the $\mathcal{G}^{u_m}$ are isometrically embedded. This inherits the metric d from $\mathcal{M}$ but is much more manageable, indeed it is compact. We prove sequential compactness. Let $(z_i)_i$ be a sequence in $\mathcal{M}'$. If an infinite number of the z_i lie in a single $\mathcal{G}^{u_m}$ or in $\mathcal{H}$, then all of these individual subspaces are compact so we can take a convergent subsequence. Therefore assume that this is not the case. By taking a subsequence, we can assume that for each i, $z_i \in \mathcal{G}^{u_{m_i}}$ where $(u_{m_i})_i$ is a strictly increasing sequence. Observe that $\lim_{i\to\infty} u_{m_i} = \infty$. For each i, using the sequences $(n_k)_k$ and $(v_k)_k$ and the constant $c > 0$ defined in Definition 2.5.4, pick the unique $k_i \geq 0$ such that $v_{k_i} < u_{m_i} \leq v_{k_i+1}$. The sequence $(k_i)_i$ is thus non-decreasing and increases up to infinity. For each i, pick $w_i \in \mathbb{W}_{n_{k_i}}$ such that $z_i \in F_{w_i}$ and then pick some $y_i \in F_{w_i}^{n_{k_i}} \subseteq \mathcal{G}^{u_{m_i}}$. Then $d(z_i, y_i) \leq \operatorname{diam}(F_{w_i}, R_{u_{m_i}})$ and $d(y_i, p(y_i)) = \frac{c}{2(k_i+1)}$, by the construction of $\mathcal{M}$ in Definition 2.5.4. By Lemma 2.4.2, Lemma 2.4.22 and the fact that $n_{k_i} \to \infty$, we thus have that

$$\lim_{i\to\infty} d(z_i, p(y_i)) \leq \lim_{i\to\infty} \left(k_\rho^{-1} (r_{\max}\rho_G^{-1})^{n_{k_i}} \sup_{v' \geq v_0} \operatorname{diam} \mathcal{G}^{v'} + \frac{c}{2(k_i+1)} \right) = 0.$$

Now $(p(y_i))_i$ is a sequence in $\mathcal{H}$, which is a compact metric space, and so has a subsequential limit. By taking a subsequence, we can assume that $p(y_i) \to z \in \mathcal{H}$. Thus $z_i \to z$ as well, and so we have sequential compactness.

Now to prove convergence of processes we use [13, Theorem 7.1]. The method of proof of weak convergence in Proposition 2.5.6 implies that $x_{u_m} \to x$ as $m \to \infty$ in $\mathcal{M}'$. Then Proposition 2.5.6 and the compactness of the spaces $\mathcal{G}^{u_m}$ and $\mathcal{H}$ immediately implies that $(\mathcal{G}^{u_m}, R_{u_m}, \mu^{u_m}, x_{u_m}) \to (\mathcal{H}, d, \nu, x)$ as $m \to \infty$ in the *spatial Gromov-Hausdorff-vague* topology (see [13, Section 7]) on the compact space $\mathcal{M}'$. Thus the conditions of [13, Theorem 7.1] are satisfied (see [13, Remark 1.3(b)]), and so we conclude that

$$(X^{u_m}, \mathbb{P}^{x_{u_m}}) \to (X^*, \mathbb{P}^x)$$

as $m \to \infty$ as random elements of $\mathcal{D}_{\mathcal{M}'}[0, \infty)$, the Skorokhod space of càdlàg processes on $\mathcal{M}'$. Therefore this convergence also occurs in $\mathcal{D}_{\mathcal{M}}[0, \infty)$. □

2.5.3 The Sierpinski gasket

Finally we give the proof of the result stated in the introduction for the Sierpinski gasket. Calculations with traces give that, for the Sierpinski gasket with conductances as in Figure 2.2,

$$\alpha(v) = (3v^2 + 6v + 1)/(4v + 6) \text{ and } \rho(v) = (3v + 2)/(2v + 1), \qquad (2.5.2)$$

and hence $\rho_G = 3/2$ and $\beta = 4/3$ giving a one-parameter iterable family which is also $\mathbb{D}$-injective. As $v \to \infty$, by Theorem 2.4.28, the Sierpinski gasket becomes the shorted gasket of Figure 2.2.

Proof of Theorem 2.1.1. Recall that we have $X_0^{a,v} = p_2 = (1/2, \sqrt{3}/2)$. We let $L = L_0$ denote the line segment from $p_1 = (0,0)$ to $p_3 = (1,0)$ and let

$$L_n = \psi_2^n(L),$$

denote the line segment which forms the base of the triangle of side 2^{-n}, with top vertex at p_2. Now let $T_{L_n}(X) = \inf\{t > 0 : X_t \in L_n\}$, that is the hitting time of the base of the top triangle. If we set

$$\tilde{X}_t^{v,n} = \psi_2^{-n}(X_t^v)), \ \ 0 \le t \le T_{L_n}(X^v),$$

then we have a process on the Sierpinski gasket, G, run until it hits the base line L, which has the same paths as the process in which the effective conductors on the graph G_0 associated to G are given by $(\rho_n(v), \rho_n(v), \rho_n(v)\alpha^{-n}(v))$. Thus it is the same process in law as $X_t^{\alpha^{-n}(v)}, 0 \le t \le T_L(X^{\alpha^{-n}(v)})$ up to a time change. This time change is $3^n\rho_n(v)$ to take into account the resistance scaling and the scaling in the invariant measure. Hence the law of $\tilde{X}_{t/3^n\rho_n(v)}^{v,n} = X_t^{\alpha^{-n}(v)}$ for $0 \le t \le T_L(X^{\alpha^{-n}(v)})$.

We recall that 1 denotes the base point of the shorted gasket. From (2.5.2) we can apply Theorem 2.5.7 to see that $X^{\alpha^{-n}(v)} \to X^s$ weakly giving the result that

$$\{\psi_2^{-n}(X_{t/3^n\rho_n(v)}^v), \ \ 0 \le t \le T_{L_n}(X^v)/3^n\rho_n(v)\} \to \{X_t^s, \ \ 0 \le t \le T_1(X^s)\},$$

weakly in $\mathcal{D}_{\mathcal{M}}[0,\infty)$.

Finally we observe that these are deterministic time changes and by (2.4.2) we have the existence of a constant $\sigma = \lim_{n\to\infty} 3^n\rho_n(v)/3^n\rho_G^n > 0$. Hence, as our processes are almost surely continuous, we have the weak convergence

$$\{\psi_1^{-n}(X_{t/3^n\rho_G^n}^v), \ \ 0 \le t \le T_{L_n}(X^v)/3^n\rho_G^n\} \to \{X_{\sigma t}^s, \ \ 0 \le t \le T_1(X^s)\}.$$

As $\rho_G = 3/2$ the ultraviolet scaling factor is given by $\lambda = 3\rho_G = 9/2$ as required. □

Remark 2.5.8. We make the following observations:

(1). It is straightforward to extend the analysis here to many fractals based on the d-dimensional tetrahedron. For instance the examples mentioned in [33].

(2). In the case of the (weighted) Vicsek set, with resistance weights $r = (1,1,1,1,s)$ as shown in Figure 2.3, the analysis does not hold. Calculations with traces give that

$$\rho_s(v) = s + \frac{4}{1+v}, \ \ \alpha_s(v) = \frac{sv+2}{s+2}.$$

Thus $\rho_G = s, \beta = 1 + 2/s$ and, even though $\beta > 1$ for all $s > 0$, we can see that this is not asymptotically regular as $\rho_G^{-1} r_{\max} \ge 1$ for any $s > 0$ (the $s = 1$ case was discussed in [33]).

(3). In [37] a version of the asymptotically one-dimensional process is constructed on a scale irregular Sierpinski gasket. This class of fractals was analysed in [31, 4], where an analogue of the fixed point diffusion was constructed. Our approach developed here could be applied to yield a scale irregular 'shorted gasket' as the short time asymptotic limit for the asymptotically one-dimensional process on the scale irregular Sierpinski gasket.

(4). The spectral dimension of the limit fractal for the Sierpinski gasket is $\frac{\log 9}{\log 9/2}$ which is the local spectral dimension of the asympotically one-dimensional diffusion on the Sierpinski gasket [32]. We conjecture that the local spectral dimension for the non-fixed point diffusion should be the spectral dimension for the limiting fractal in general.

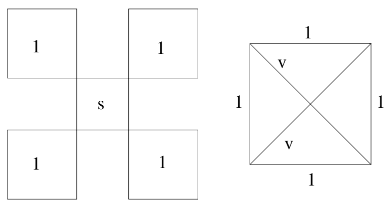

Figure 2.3: The Vicsek set with resistance weights $r = (1, 1, 1, 1, s)$ is not asymptotically regular for any $s > 0$.

Chapter 3

Existence and space-time regularity for stochastic heat equations on p.c.f. fractals

3.1 Introduction

The stochastic heat equation (or SHE) on $\mathbb{R}^n, n \in \mathbb{N}$ is a stochastic partial differential equation which can be expressed formally as

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \Delta u(t,x) + \dot{W}(t,x), \\ u(0,\cdot) &= u_0 \end{aligned}$$

for $(t,x) \in [0,\infty) \times \mathbb{R}^n$, where Δ is the Laplacian on $\mathbb{R}^n$, u_0 is a (sufficiently regular) function on $\mathbb{R}^n$ and $\dot{W}$ is a space-time white noise on $\mathbb{R} \times \mathbb{R}^n$. Written in the differential notation of stochastic calculus this is equivalent to

$$\begin{aligned} du(t) &= \Delta u(t)dt + dW(t), \\ u(0) &= u_0, \end{aligned}$$

where W is a cylindrical Wiener process on $L^2(\mathbb{R}^n)$. A solution to this SPDE is a process $u = (u(t) : t \in [0,T])$ taking values in some space containing $L^2(\mathbb{R}^n)$ that satisfies the above equations in some weak sense; see [15] for details. The SHE on $\mathbb{R}^n$ is one of the prototypical examples of an SPDE and has been widely studied, see for example [17], [23] and [77]. It has two notable properties that are relevant to the present chapter. The first is its so-called "curse of dimensionality". Solutions to the SHE on $\mathbb{R}^n$ are function-valued only in the case $n = 1$; in dimension $n \geq 2$ solutions are forced to take values in a wider space of distributions on $\mathbb{R}^n$, see [77]. Secondly if $n = 1$ and $u_0 = 0$ then the solution is unique and jointly Hölder continuous in space and time, see again [77]. One of the aims of the present chapter is to investigate what

Original article: [29]

happens regarding these two properties in the setting of finitely ramified fractals, which behave in many ways like spaces with dimension between one and two.

The family of spaces that we will be considering is the class of connected *post-critically finite self-similar* (or *p.c.f.s.s.*) sets endowed with *regular harmonic structures*. This family includes many well-known fractals such as the Sierpinski gasket and the Vicsek fractal but not the Sierpinski carpet. The unit interval $[0,1]$ also has several formulations in the language of p.c.f.s.s. sets that belong to this family. Analysis on these sets is a relatively young field which started with the construction of a "Brownian motion" on the Sierpinski gasket in [25], [58] and [5]. This broader theory was then developed and provides a concrete framework where reasonably explicit results can be obtained, see [52] and [3]. Associated with a regular harmonic structure on a p.c.f.s.s. set $(F, (\psi_i)_{i=1}^M)$ is an operator, called the *Laplacian* on F, which is the generator of a "Brownian motion" on F by analogy with the Laplacian on $\mathbb{R}^n$ as the generator of Brownian motion in $\mathbb{R}^n$. We will see that there exists a constant $d_s > 0$ associated with the harmonic structure known as the *spectral dimension*, and it will turn out that the assumption that the harmonic structure is regular implies that $d_s \in [1, 2)$. The existence of a Laplacian allows us to define certain PDEs and SPDEs on F, such as a heat equation and a stochastic heat equation. The former has been studied extensively, see [52, Chapter 5] and further references. The latter is the subject of the present chapter.

For examples of some previous work in this area, in [20] it is shown that on certain fractals a stochastic heat equation can be defined which yields a random-field solution, that is, a solution which is a random map $[0,T] \times F \to \mathbb{R}$. We extend this result in the main theorem of Section 3.4 of the present chapter. In [46] (see also [41]) it is shown that solutions to some nonlinear stochastic heat equations on more general metric measure spaces have Hölder continuous paths when considered as a random map from a "time" set to some space of functions. However in that paper the authors do not consider the Hölder exponents of the solution when considered as a random field, which is what we will do.

The structure of the present chapter is as follows: In the following subsection we describe the precise set-up of the problem and the specific SPDE that we will be studying, and state a theorem which is an important corollary of our main result. In Section 3.2 we recall some useful spectral theory for Laplacians on p.c.f.s.s. sets from [52] and show that (unique) solutions to the SPDE exist as $L^2(F)$-valued stochastic processes. In Section 3.3 we prove generic results analogous to Kolmogorov's continuity theorem for families of random variables indexed by F and by $[0,1] \times F$. In Section

3.4 we show that the resolvent densities associated with the Laplacian are Lipschitz continuous with respect to the resistance metric on F. More importantly we also show that evaluations of solutions to the SPDE at points $(t, x) \in [0, \infty) \times F$ can be done in a well-defined way, which is necessary for us to talk about continuity of these solutions. Section 3.5 contains the main results of the chapter, which use our continuity theorems to establish space-time Hölder continuity of solutions to the SPDE and compute the respective Hölder exponents. Section 3.6 serves as a "coda" of the chapter, where we prove results on the invariant measures and long-time behaviour of the solutions to the SPDE.

3.1.1 Description of the problem

Let $M \geq 2$ be an integer. Let $\mathcal{S} = (F, (\psi_i)_{i=1}^M)$ be a connected p.c.f.s.s. set (see [52]) such that F is a compact metric space and the $\psi_i : F \to F$ are injective strict contractions on F. Let $I = \{1, \ldots, M\}$ and for each $n \geq 0$ let $\mathbb{W}_n = I^n$. Let $\mathbb{W}_* = \bigcup_{n\geq 0} \mathbb{W}_n$ and let $\mathbb{W} = I^{\mathbb{N}}$. We call the sets $\mathbb{W}_n$, $\mathbb{W}_*$ and $\mathbb{W}$ *word spaces* and we call their elements *words*. Note that $\mathbb{W}_0$ is a singleton containing an element known as the *empty word*. Words $w \in \mathbb{W}_n$ or $w \in \mathbb{W}$ will be written in the form $w = w_1 w_2 w_3 \ldots$ with $w_i \in I$ for each i. For a word $w = w_1, \ldots, w_n \in \mathbb{W}_*$, let $\psi_w = \psi_{w_1} \circ \cdots \circ \psi_{w_n}$ and let $F_w = \psi_w(F)$.

If $\mathbb{W}$ is endowed with the standard product topology then there is a canonical continuous surjection $\pi : \mathbb{W} \to F$ given in [3, Lemma 5.10]. Let $P \subset \mathbb{W}$ be the post-critical set of $\mathcal{S}$ (see [52, Definition 1.3.4]), which is finite by assumption. Then let $F^0 = \pi(P)$, and for each $n \geq 1$ let $F^n = \bigcup_{w\in\mathbb{W}_n} \psi_w(F^0)$. Let $F_* = \bigcup_{n=0}^{\infty} F^n$. It is easily shown that $(F^n)_{n\geq 0}$ is an increasing sequence of finite subsets and that F_* is dense in F.

Let the pair $(A_0, \mathbf{r})$ be a regular irreducible harmonic structure on $\mathcal{S}$ such that $\mathbf{r} = (r_1, \ldots, r_M) \in \mathbb{R}^M$ for some constants $r_i > 0$, $i \in I$ (harmonic structures are defined in [52, Section 3.1]). Here *regular* means that $r_i \in (0, 1)$ for all i. Let $r_{\min} = \min_{i\in I} r_i$ and $r_{\max} = \max_{i\in I} r_i$. If $n \geq 0$, $w = w_1, \ldots w_n \in \mathbb{W}_*$ then write $r_w := \prod_{i=1}^n r_{w_i}$. Let $d_H > 0$ be the unique number such that

$$\sum_{i\in I} r_i^{d_H} = 1.$$

Then let μ be the self-similar Borel probability measure on F such that for any $n \geq 0$, if $w \in \mathbb{W}_n$ then $\mu(F_w) = r_w^{d_H}$. In other words, μ is the self-similar measure on F in the sense of [52, Section 1.4] associated with the weights $r_i^{d_H}$ on I. Let $(\mathcal{E}, \mathcal{D})$ be the regular local Dirichlet form on $L^2(F, \mu)$ associated with this harmonic structure, as given by [52, Theorem 3.4.6]. This Dirichlet form is associated with a resistance metric R on F, defined by

$$R(x, y) = \left(\inf\{\mathcal{E}(f, f) : f(x) = 0, f(y) = 1, f \in \mathcal{D}\}\right)^{-1},$$

which generates the original topology on F, by [52, Theorem 3.3.4]. Additionally, let

$$\mathcal{D}_0 = \{f \in \mathcal{D} : f|_{F^0} = 0\}.$$

Then by [52, Corollary 3.4.7], $(\mathcal{E}, \mathcal{D}_0)$ is a regular local Dirichlet form on $L^2(F \backslash F^0, \mu)$.

By [3, Chapter 4], associated with the Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(F, \mu)$ is a μ-symmetric diffusion $X^N = (X_t^N)_{t \geq 0}$ which itself is associated with a C_0-semigroup of contractions $S^N = (S_t^N)_{t \geq 0}$. Let Δ_N be the generator of this diffusion. Likewise associated with $(\mathcal{E}, \mathcal{D}_0)$ we have a μ-symmetric diffusion X^D with C_0-semigroup of contractions S^D and generator Δ_D. The process X^D is similar to X^N, except for the fact that it is absorbed at the points F^0, whereas X^N is reflected. The letters N and D indicate *Neumann* and *Dirichlet* boundary conditions respectively. As a consequence of theory developed in [22, Sections 1.3 and 1.4], the operator $-\Delta_N$ is the non-negative self-adjoint operator associated with the form $(\mathcal{E}, \mathcal{D})$, in the sense that $\mathcal{D} = \mathcal{D}((-\Delta_N)^{\frac{1}{2}})$ and

$$\mathcal{E}(f, g) = \langle (-\Delta_N)^{\frac{1}{2}} f, (-\Delta_N)^{\frac{1}{2}} g \rangle_\mu$$

for all $f, g \in \mathcal{D}$. An analogous result holds with $-\Delta_D$ and $(\mathcal{E}, \mathcal{D}_0)$. This justifies us calling Δ_N the *Neumann Laplacian* and Δ_D the *Dirichlet Laplacian.*

Example 3.1.1. Let $F = [0, 1]$ and take *any* $M \geq 2$. For $1 \leq i \leq M$ let $\psi_i : F \to F$ be the affine map such that $\psi_i(0) = \frac{i-1}{M}$, $\psi_i(1) = \frac{i}{M}$. It follows that $F^0 = \{0, 1\}$. Let $r_i = M^{-1}$ for all $i \in I$ and let

$$A_0 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then all the conditions given above are satisfied. We have $\mathcal{D} = H^1[0, 1]$ and $\mathcal{E}(f, g) = \int_0^1 f'g'$. The associated generators Δ_N and Δ_D are respectively the standard Neumann and Dirichlet Laplacians on $[0, 1]$. In particular, the induced resistance metric R is none other than the standard Euclidean metric. This interpretation of the unit interval as a p.c.f.s.s. set that fits into our set-up will be useful to us later on.

The object of study in the present chapter is the following SPDE on F:

$$\begin{aligned} du(t) &= \Delta_b u(t)dt + (1-\Delta_b)^{-\frac{\alpha}{2}} dW(t), \\ u(0) &= u_0 \in L^2(F,\mu), \end{aligned} \tag{3.1.1}$$

where $b \in \{N, D\}$ and $\alpha \in [0,\infty)$ are parameters and W is a cylindrical Wiener process on $L^2(F,\mu)$. That is, W formally satisfies

$$\mathbb{E}\left[\langle f, W(s)\rangle_{L^2(F,\mu)} \langle W(t), g\rangle_{L^2(F,\mu)}\right] = (s \wedge t)\langle f, g\rangle_{L^2(F,\mu)}$$

for all $s,t \in [0,\infty)$ and $f,g \in L^2(F,\mu)$. Note that W is not an $L^2(F,\mu)$-valued process; to be precise, it takes values in some separable Hilbert space in which $L^2(F,\mu)$ can be continuously embedded (see [15]). The vast majority of results in this chapter hold regardless of the value of b; whenever this is not the case it will be explicitly stated.

The SPDE (3.1.1) in the case $\alpha = 0$ will be called the *stochastic heat equation* or *SHE* for $(A_0, \mathbf{r})$ on F. It is well known (see for example [77]) that the solution to the standard SHE on $[0,1]$ with initial condition $u_0 = 0$ is jointly continuous with Hölder exponents of essentially $\frac{1}{2}$ in space and essentially $\frac{1}{4}$ in time (the meaning of "essentially" is given in Definition 3.2.10). The following extension of this result is a simple consequence of our main result Theorems 3.5.6 and 3.5.7 and was the original motivation for the writing of the present chapter:

Theorem 3.1.2. *Equip F with the resistance metric R. Then for each $b \in \{N, D\}$, the SHE for $(A_0, \boldsymbol{r})$ on F with $u_0 = 0$ has a unique solution $u = (u(t,x))_{(t,x)\in[0,\infty)\times F}$ which is jointly continuous, essentially $\frac{1}{2}$-Hölder continuous in space (i.e. in (F,R)) and essentially $\frac{1}{2}(1-\frac{d_s}{2})$-Hölder continuous in time, where*

$$d_s = \frac{2d_H}{d_H + 1}$$

is the spectral dimension of (F,R).

Note that many p.c.f.s.s. sets F can be embedded into Euclidean space in such a way that R is equivalent to the Euclidean metric up to some exponent. Therefore, for such sets, we can also make sense of the above result with respect to a spatial Euclidean metric, see Remark 3.5.8.

Example 3.1.3. (1). (Interval.) Take $F = [0,1]$ with the Dirichlet form given in Example 3.1.1. Then $d_s = 1$ and the resistance metric is the Euclidean metric, so using the above theorem we obtain the usual well-known Hölder exponents for the SHE on $[0,1]$.

(2). (n-dimensional Sierpinski gasket.) See [52, Example 3.1.5] and [43, Section 3]. The standard harmonic structure on the n-dimensional Sierpinski gasket (for $n \geq 2$) fits into our set-up; it is given by $M = n+1$,

$$A_0 = \begin{pmatrix} -n & 1 & 1 & \cdots & 1 \\ 1 & -n & 1 & \cdots & 1 \\ 1 & 1 & -n & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & -n \end{pmatrix},$$

and $r_i = \frac{n+1}{n+3}$ for all $i \in I$. In fact for $n = 1$ we have the binary decomposition of the unit interval and recover the usual case. For $n = 2$ the diffusion X^N is known as Brownian motion on the Sierpinski gasket and is ubiquitous in the field of analysis on fractals ([25], [58], [5]). We can compute $d_H = \frac{\log(n+1)}{\log(n+3)-\log(n+1)}$ and $d_s = \frac{2\log(n+1)}{\log(n+3)}$. This gives us a family of examples which live naturally in $\mathbb{R}^n$ for any geometric dimension n and where the spectral dimension can be made arbitrarily close to 2 by taking n large. Using the properties of the resistance metric we can have solutions that have arbitrarily small spatial (with respect to the Euclidean metric) and temporal Hölder exponents. See Remark 3.5.8 for further discussion.

3.2 Existence (and uniqueness)

Definition 3.2.1. Henceforth we let $\mathcal{H} = L^2(F, \mu)$. Denote the inner product on $\mathcal{H}$ by $\langle \cdot, \cdot \rangle_\mu$. Let $T > 0$. Following [15], an $\mathcal{H}$-valued predictable process $u = (u(t) : t \in [0, T])$ is a *(mild) solution* to (3.1.1) if

$$u(t) = S^b_t u_0 + \int_0^t S^b_{t-s}(1 - \Delta_b)^{-\frac{\alpha}{2}} dW(s)$$

almost surely for every $t \in [0, T]$. We write $u : [0, T] \to \mathcal{H}$, where we suppress the dependence of u on the underlying probability space. If $T = \infty$ we call the solution *global.*

Remark 3.2.2. Global solutions to (3.1.1) are unique up to versions by definition.

Notice that for any $f \in \mathcal{H}$, u is a solution to (3.1.1) with $u_0 = 0$ if and only if $u + S^b f$ is a solution to (3.1.1) with $u_0 = f$. Thus we can safely assume that $u_0 = 0$, and so we are interested in the properties of the stochastic convolution

$$W^b_\alpha(t) := \int_0^t S^b_{t-s}(1 - \Delta_b)^{-\frac{\alpha}{2}} dW(s). \tag{3.2.1}$$

Observe that if a solution exists for $u_0 = 0$, then it must equal W_α^b up to versions.

The first thing to investigate is the validity of the operator $(1-\Delta_b)^{-\frac{\alpha}{2}}$ in the case $\alpha > 0$. For an operator $\mathcal{A}$ on $\mathcal{H}$, we denote the domain of $\mathcal{A}$ by $\mathcal{D}(\mathcal{A})$. If $\mathcal{A}$ is bounded then let $\|\mathcal{A}\|$ denote its operator norm. The following statements are immediate by standard operator theory (see [69, Theorem VIII.5] and [70, Theorem 12.31]):

Corollary 3.2.3. *For $b \in \{N, D\}$ we have that*

(1). $S_t^b = \exp(t\Delta_b)$ for $t \geq 0$,

(2). S^b can be extended to an analytic semigroup (which we will identify with S^b),

(3). For $\alpha \geq 0$, $(1-\Delta_b)^{-\frac{\alpha}{2}}$ is a bounded linear operator.

Remark 3.2.4. The operator $(1-\Delta_b)^{-\frac{\alpha}{2}}$ is known as a *Bessel potential operator*, see [46], [73].

3.2.1 Spectral theory of Laplacians

Now that we have established the close relationship between Δ_b and $\mathcal{E}$, we may make use of the spectral theory of these Laplacians developed in [52, Chapters 4 and 5]. We summarise the useful definitions and results below:

Definition 3.2.5. The unique real $d_H > 0$ such that

$$\sum_{i \in I} r_i^{d_H} = 1$$

is the *Hausdorff dimension* of (F, R), see [52, Theorem 1.5.7 and Theorem 4.2.1]. The *spectral dimension* of (F, R) is given by

$$d_s = \frac{2d_H}{d_H + 1}.$$

See [52, Theorem 4.1.5 and Theorem 4.2.1].

Remark 3.2.6. (1). The definition of d_s given in [52] is far more general, but the definition above is equivalent for our purposes. We immediately see that $d_s \in (0, 2)$ a priori. Were the harmonic structure $(A_0, \mathbf{r})$ not regular, it would be possible to have $d_s \geq 2$ via its more general definition.

(2). It is possible to show that $d_H \geq 1$. Indeed, by [52, Theorem 1.6.2 and Lemma 3.3.5] we have that

$$\max_{x,y\in F^0} R(x,y) \leq \sum_{i\in I} \max_{x,y\in F^0} R(F_i(x), F_i(y)) \leq \left(\sum_{i\in I} r_i\right) \max_{x,y\in F^0} R(x,y),$$

so that $\sum_{i\in I} r_i \geq 1 = \sum_{i\in I} r_i^{d_H}$ and thus $d_H \geq 1$. It follows that $d_s \in [1, 2)$.

Proposition 3.2.7. *For $b \in \{N, D\}$ the following statements hold:*

There exists a complete orthonormal basis $(\varphi_k^b)_{k=1}^\infty$ of $\mathcal{H}$ consisting of eigenfunctions of the operator $-\Delta_b$. The corresponding eigenvalues $(\lambda_k^b)_{k=1}^\infty$ are non-negative and $\lim_{k\to\infty} \lambda_k^b = \infty$. We assume that they are given in ascending order:

$$0 \leq \lambda_1^b \leq \lambda_2^b \leq \cdots .$$

There exist constants $c_1, c_2, c_3 > 0$ such that if $k \geq 2$ then

$$c_1 k^{\frac{2}{d_s}} \leq \lambda_k^b \leq c_2 k^{\frac{2}{d_s}}$$

and

$$\|\varphi_k^b\|_\infty \leq c_3 |\lambda_k^b|^{\frac{d_s}{4}}.$$

Proof. This is a simple corollary of results in [52, Chapters 4, 5], in particular Theorem 4.5.4 and Lemma 5.1.3. □

Remark 3.2.8. Note that all functions $f \in \mathcal{D}$ must be at least $\frac{1}{2}$-Hölder continuous with respect to the resistance metric since

$$|f(x) - f(y)|^2 \leq \mathcal{E}(f, f) R(x, y)$$

for all $x, y \in F$ (see [3, Proposition 7.18]). Thus it makes sense to consider $\varphi_k^b(x)$ for $x \in F$. The above proposition then implies that $|\varphi_k^b(x)| \leq c_3 |\lambda_k^b|^{\frac{d_s}{4}}$ for all $x \in F$, $k \geq 2$.

Remark 3.2.9. The reason why we require $k \geq 2$ in the above proposition is that we may have $\lambda_1^b = 0$. In this case it follows that $\mathcal{E}(\varphi_1^b, \varphi_1^b) = 0$. By the properties of the resistance metric R, for any distinct $x_1, x_2 \in F$ we have that

$$\frac{|\varphi_1^b(x_1) - \varphi_1^b(x_2)|^2}{R(x_1, x_2)} \leq \mathcal{E}(\varphi_1^b, \varphi_1^b) = 0.$$

It follows that φ_1^b is constant. Since $\|\varphi_1^b\|_\mu = 1$ and μ is a probability measure we conclude that $\varphi_1^b \equiv 1$. This confirms that if 0 is an eigenvalue it must necessarily have multiplicity 1, so we always have $\lambda_2^b > 0$. It also implies that we have $\lambda_1^b = 0$ if and only if $b = N$, since the non-zero constant functions are elements of $\mathcal{D} \setminus \mathcal{D}_0$. In the case that $\lambda_1^b > 0$, we will assume that c_1, c_2, c_3 are chosen such that the estimates in the above proposition hold for $k \geq 1$.

The existence of a complete orthonormal basis of eigenfunctions of Δ_b allows us to write down series representations of elements of $\mathcal{H}$ and operators defined on subspaces of $\mathcal{H}$ in a way analogous to the Fourier series representations of elements of $L^2(0,1)$. For example, an element $f \in \mathcal{H}$ has a series representation

$$f = \sum_{k=1}^{\infty} f_k \varphi_k^b$$

where $f_k = \langle \varphi_k^b, f \rangle_\mu$. Then for any map $\Xi : [0, \infty) \to \mathbb{R}$ we have that the operator $\Xi(-\Delta_b)$ has the representation

$$\Xi(-\Delta_b) f = \sum_{k=1}^{\infty} f_k \Xi(\lambda_k^b) \varphi_k^b,$$

and the domain of $\Xi(-\Delta_b)$ is exactly those $f \in \mathcal{H}$ for which the above expression is in $\mathcal{H}$. In particular

$$S_t^b f = \sum_{k=1}^{\infty} f_k e^{-\lambda_k^b t} \varphi_k^b$$

for all $f \in \mathcal{H}$.

3.2.2 Existence of solution

Recall the expression (3.2.1). If we can show that $W_\alpha^b(t) \in \mathcal{H}$ almost surely for every $t > 0$, then we have a unique global solution of (3.1.1) for $u_0 = 0$, and thus by the discussion after Definition 3.2.1 we have a unique global solution for *any* initial value $u_0 \in \mathcal{H}$. In fact we can do better than that:

Definition 3.2.10. Let (M_1, d_1) and (M_2, d_2) be metric spaces, and let $f : M_1 \to M_2$ be continuous. For $\delta \in (0, 1]$ we say that f is *essentially δ-Hölder continuous* if it is γ-Hölder continuous for every $\gamma < \delta$. That is, for every $\gamma \in (0, \delta)$ there exists a constant ε_γ such that $d_2(f(x), f(y)) \leq \varepsilon_\gamma d_1(x, y)^\gamma$ for all $x, y \in M_1$.

Theorem 3.2.11 (Existence)**.** *For every $\alpha \geq 0$, $b \in \{N, D\}$ and $T \geq 0$ we have that*

$$\mathbb{E}\left[\|W_\alpha^b(T)\|_\mu^2\right] < \infty.$$

In particular for any $\alpha \geq 0$, $b \in \{N, D\}$ and any initial condition $u_0 \in \mathcal{H}$ there exists a unique (up to versions) global solution to (3.1.1)*. There exists an $\mathcal{H}$-continuous version of this solution. If $u_0 = 0$ then this version is essentially $\frac{1}{2}\left(1 \wedge (1 - \frac{d_s}{2} + \alpha)\right)$-Hölder continuous on compact intervals.*

Proof. We refer to the proof of [26, Theorem 5.13]. By Itō's isometry for Hilbert spaces we have that

$$\mathbb{E}\left[\|W_\alpha^b(T)\|_\mu^2\right] = \int_0^T \|(1-\Delta_b)^{-\frac{\alpha}{2}} S_t^b\|_{\mathrm{HS}}^2 dt,$$

where $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm. If there exists $\beta \in (0, \frac{1}{2} + \frac{\alpha}{2})$ such that $\|(1-\Delta_b)^{-\beta}\|_{\mathrm{HS}} < \infty$, then by the spectral decomposition of $(1-\Delta_b)^{\beta - \frac{\alpha}{2}} S_t^b$ we have that

$$\begin{aligned} \|(1-\Delta_b)^{-\frac{\alpha}{2}} S_t^b\|_{\mathrm{HS}} &\leq \|(1-\Delta_b)^{-\beta}\|_{\mathrm{HS}} \|(1-\Delta_b)^{\beta-\frac{\alpha}{2}} S_t^b\| \\ &\leq C'(1 \vee t^{\frac{\alpha}{2}-\beta}) \end{aligned} \tag{3.2.2}$$

for some constant $C' > 0$, and the last expression is square-integrable on the interval $[0, T]$. Therefore finding such a β is sufficient for $W_\alpha^b(t)$ to be square-integrable. We see from Proposition 3.2.7 that

$$\begin{aligned} \|(1-\Delta_b)^{-\beta}\|_{\mathrm{HS}}^2 &= \sum_{k=1}^\infty \|(1-\Delta_b)^{-\beta} \varphi_k^b\|_\mu^2 \\ &= \sum_{k=1}^\infty (1+\lambda_k^b)^{-2\beta} \\ &\leq 1 + c_1 \sum_{k=1}^\infty k^{-\frac{4\beta}{d_s}} \end{aligned}$$

and the final expression is finite for $\beta > \frac{d_s}{4}$. Since we know that $d_s < 2$ we can pick any $\beta \in (\frac{d_s}{4}, \frac{1}{2} + \frac{\alpha}{2}) \neq \emptyset$ to show that $\mathbb{E}\left[\|W_\alpha^b(t)\|_\mu^2\right] < \infty$.

For the continuity results, it follows from (3.2.2) that for any positive $\gamma < \frac{1}{2}(1 \wedge (1 - \frac{d_s}{2} + \alpha))$ we have that

$$\int_0^T t^{-2\gamma} \|(1-\Delta_b)^{-\frac{\alpha}{2}} S_t^b\|_{\mathrm{HS}}^2 dt < \infty.$$

The continuity statements then directly follow from [26, Theorems 5.10 and 5.17]. □

3.3 Some Kolmogorov-type continuity theorems

It is well-known that solutions to the one-dimensional stochastic heat equation are essentially $\frac{1}{4}$-Hölder continuous in time and essentially $\frac{1}{2}$-Hölder continuous in space, so we would like to prove analogous results for our SPDE. It will become clear that the natural "spatial" metric to use on F is the resistance metric R.

The usual method of proving continuity of processes indexed by $\mathbb{R}$ is to use Kolmogorov's continuity theorem, see for example [57, Section 1.4]. Our aim in this section is to prove versions of this theorem for the spaces F and $[0,1] \times F$.

3.3.1 Partitions and neighbourhoods

We introduce some more theory and notation from [52] and develop it further for our purposes.

Definition 3.3.1. If $n \geq 1$ and $w = w_1 \dots w_n \in \mathbb{W}_n$ then let

$$\Sigma_w := \{w' = w'_1 w'_2 \dots \in \mathbb{W} : w'_i = w_i \ \forall i \in \{1, \dots, n\}\}.$$

If $n = 0$ and $w \in \mathbb{W}_0$ then w is the empty word and we set $\Sigma_w := \mathbb{W}$.

Definition 3.3.2. A finite subset $\Lambda \subseteq \mathbb{W}_*$ is a *partition* if $\Sigma_w \cap \Sigma_v = \emptyset$ for any $w \neq v \in \Lambda$ and $\mathbb{W} = \bigcup_{w \in \Lambda} \Sigma_w$. A partition Λ is a *refinement* of a partition Λ' if either $\Sigma_w \subseteq \Sigma_v$ or $\Sigma_w \cap \Sigma_v = \emptyset$ for any $(w, v) \in \Lambda \times \Lambda'$.

Definition 3.3.3. For $a \in (0, 1)$ let

$$\Lambda(a) = \{w : w = w_1 \dots w_m \in \mathbb{W}_*,\ r_{w_1 \dots w_{m-1}} > a \geq r_w\}$$

which is a partition, see [52, Definition 1.5.6]. For $n \geq 1$ let $\Lambda_n = \Lambda(2^{-n})$. Let Λ_0 be the singleton containing the empty word; this is also a partition.

Notice that if $w \in \Lambda(a)$ then $r_{\min} a < r_w \leq a$.

Lemma 3.3.4. *If $n_1 \geq n_2 \geq 0$ then Λ_{n_1} is a refinement of Λ_{n_2}.*

Proof. Let $w \in \Lambda_{n_1}$, $v \in \Lambda_{n_2}$ with $\Sigma_w \cap \Sigma_v \neq \emptyset$. Then we must have either $\Sigma_w \subseteq \Sigma_v$ or $\Sigma_v \subseteq \Sigma_w$ (or both). Suppose it is not the case that $\Sigma_w \subseteq \Sigma_v$. Then there exist $m_2 > m_1 \geq 0$ such that $w \in \mathbb{W}_{m_1}$ and $v \in \mathbb{W}_{m_2}$, and $w_i = v_i$ for all $i \in \{1, \dots, m_1\}$. In particular v is not the empty word, so w is not the empty word (since $n_1 \geq n_2$), so it follows that $m_2 \geq 2$ and $m_1 \geq 1$. But then $n_1, n_2 \geq 1$ so

$$2^{-n_1} \geq r_w = r_{v_1 \dots v_{m_1}} \geq r_{v_1 \dots v_{m_2 - 1}} > 2^{-n_2}$$

which is a contradiction. So $\Sigma_w \subseteq \Sigma_v$. $\square$

The above result in particular implies that if $n_1 \geq n_2 \geq 0$ and $v \in \Lambda_{n_1}$ then there exists a $w \in \Lambda_{n_2}$ such that $F_v \subseteq F_w$.

Definition 3.3.5. For $n \geq 0$ let $F^n_\Lambda = \bigcup_{w\in\Lambda_n} \psi_w(F^0)$.

Obviously $F^n_\Lambda \subseteq F_*$ for all n. By Lemma 3.3.4 and [52, Lemma 1.3.10], $(F^n_\Lambda)_{n\geq 0}$ is an increasing sequence of subsets.

Lemma 3.3.6. $\bigcup_{n\geq 0} F^n_\Lambda = F_*$.

Proof. Let $n \geq 0$ and $x \in F^n$. Recall the canonical continuous surjection $\pi : \mathbb{W} \to F$ and the post-critical set P. By assumption $x \in F^n = \bigcup_{w\in\mathbb{W}_n} \psi_w(F^0) = \bigcup_{w\in\mathbb{W}_n} \psi_w(\pi(P))$, so there exists $w \in \mathbb{W}_n$ and $v \in \Sigma_w$ such that $v_{n+1}v_{n+2}\ldots \in P$ and $\pi(v) = x$. By the definition of P it follows that for all integer $i \geq 0$ we must have that $v_{n+i+1}v_{n+i+2}\ldots \in P$. Now consider the sequence $w^i := v_1 \ldots v_{n+i} \in \mathbb{W}_i$ for $i \geq 0$. It follows that $x \in \psi_{w^i}(F^0)$ for all $i \geq 0$. Also some w^i must be in some Λ_m for $m \geq 1$, since $\lim_{i\to\infty} r_{w^i} = 0$. □

Definition 3.3.7. For $n \geq 0$ and $x, y \in F^n_\Lambda$ let $x \sim_n y$ if there exists $w \in \Lambda_n$ such that $x, y \in F_w$. Then $(F^n_\Lambda, \sim_n)$ can be interpreted as a graph.

Lemma 3.3.8. *Suppose that* $n \geq 0$*,* $w \in \Lambda_n$*,* $v \in \Lambda_{n+1}$ *and* $\Sigma_v \cap \Sigma_w \neq \emptyset$*. If* $w \in \mathbb{W}_{m_1}$ *and* $v \in \mathbb{W}_{m_2}$ *then* $0 \leq m_2 - m_1 < \frac{\log 2 + \log r^{-1}_{\min}}{\log r^{-1}_{\max}}$*. In particular if* $n_* := \left\lceil \frac{\log 2 + \log r^{-1}_{\min}}{\log r^{-1}_{\max}} \right\rceil$ *then* $\psi_v(F^0) \subseteq \psi_w(F^{n_*})$*.*

Proof. By the refinement property (Lemma 3.3.4) we have that $\Sigma_v \subset \Sigma_w$ and so there exist $m_2 \geq m_1 \geq 0$ such that $w \in \mathbb{W}_{m_1}$ and $v \in \mathbb{W}_{m_2}$, and $v_i = w_i$ for all $1 \leq i \leq m_1$. Then by the comment after Definition 3.3.3,

$$r_v > 2^{-(n+1)} r_{\min} \geq \frac{r_{\min}}{2} r_w = \frac{r_{\min}}{2} r_{v_1\ldots v_{m_1}},$$

so

$$\frac{r_{\min}}{2} < r^{m_2-m_1}_{\max}.$$

Thus $m_2 - m_1 < \frac{\log 2 + \log r^{-1}_{\min}}{\log r^{-1}_{\max}}$. □

Lemma 3.3.9. *There exists a constant* $c_g > 0$ *such that if* $n \geq 0$ *and* $w \in \Lambda_n$*, then* $(F^{n+1}_\Lambda \cap F_w, \sim_{n+1})$ *is a connected graph and its graph diameter is at most* c_g*.*

Proof. For $z \in F_\Lambda^{n+1} \cap F_w$, take $\omega \in \pi^{-1}(z) \cap \Sigma_w$ and let $v \in \Lambda_{n+1}$ be such that $\omega \in \Sigma_v$. Then $\Sigma_v \subseteq \Sigma_w$ by the refinement property. By [52, Proposition 1.3.5(2)], $z \in \psi_v(F^0)$. So then by Lemma 3.3.8 we have $z \in \psi_w(F^{n_*})$ for all $z \in F_\Lambda^{n+1} \cap F_w$. Therefore the graph-length of *any* non-self-intersecting path in the graph $(F_\Lambda^{n+1} \cap F_w, \sim_{n+1})$ cannot be greater than $c_g := |F^{n_*}|$. So if we can verify that $(F_\Lambda^{n+1} \cap F_w, \sim_{n+1})$ is connected, we are done.

Consider by the refinement property (Lemma 3.3.4) that we must have $F_w = \bigcup_{v \in \Lambda'} F_v$, where

$$\Lambda' = \{v \in \Lambda_{n+1} : \Sigma_v \subseteq \Sigma_w\} = \{wv : v \in \Lambda''\}$$

for some partition Λ''. With [52, Proposition 1.3.5(2)] in mind, the required connectedness result is thus reduced to showing the following: if a graph structure $\sim$ is defined on Λ'' such that $v \sim v'$ if and only if $F_v \cap F_{v'} \neq \emptyset$, then the graph $(\Lambda'', \sim)$ is connected. This is proven in exactly the same way as [52, Theorem 1.6.2, (3)⇒(1)]. □

Definition 3.3.10. Let $n \geq 0$ and $w \in \mathbb{W}_n$. For $x \in F$ let

$$D_n^0(x) = \bigcup \{F_w : w \in \Lambda_n,\ F_w \ni x\}$$

be the *n-neighbourhood* of x. In addition, let

$$D_n^1(x) = \bigcup \{F_w : w \in \Lambda_n,\ F_w \cap D_n^0(x) \neq \emptyset\}.$$

By [52, Lemma 4.2.3] it must be the case that the quantities $|\{w \in \Lambda_n : F_w \ni x\}|$ and $|\{w \in \Lambda_n : F_w \cap D_n^0(x) \neq \emptyset\}|$ are bounded over all $n \geq 0$ and all $x \in F$. Let

$$c_4 = \max_{n,x} |\{w \in \Lambda_n : F_w \cap D_n^0(x) \neq \emptyset\}|. \tag{3.3.1}$$

In particular, observe that $D_n^0(x) \subseteq D_n^1(x)$, and that if $x, y \in F_\Lambda^n$ with $x \sim_n y$ then $y \in D_n^0(x)$.

Definition 3.3.11. For $x \in F$ and $\varepsilon > 0$ let $B(x, \varepsilon)$ be the closed ball in (F, R) with centre x and radius ε.

The next result shows that the resistance metric R is topologically well-behaved with respect to the structure of the p.c.f.s.s. set F and the partitions Λ_n. Compare similar results obtained in [34, Lemmas 3.2, 3.4].

Proposition 3.3.12 (Homogeneity of resistance metric)**.** *There exist two constants $c_5, c_6 > 0$ such that*

$$B(x, c_5 2^{-n}) \subseteq D_n^1(x) \subseteq B(x, c_6 2^{-n})$$

for all $n \geq 0$ and all $x \in F$.

Proof. For the second inclusion, if $y \in D_n^1(x)$ then there exist $w, v \in \Lambda_n$ such that $x \in F_w$, $y \in F_v$ and $F_w \cap F_v \neq \emptyset$. Then the result is a direct consequence of [3, Proposition 7.18(b)] and the definition of Λ_n.

For the first inclusion, let $\mathcal{D}_h \subseteq \mathcal{D}$ be the set of harmonic functions (see [52, Proposition 3.2.1]) $f \in \mathcal{D}$ for which $f(x) \in \{0,1\}$ for all $x \in F^0$. A harmonic function is completely characterised by the values it takes on F^0 so $|\mathcal{D}_h| = 2^{|F^0|}$. Let

$$c = \max_{f \in \mathcal{D}_h} \mathcal{E}(f,f) > 0.$$

We now take g to be the harmonic extension to $\mathcal{H}$ of the indicator function $\mathbb{1}_{D_n^0(x)}|_{F_\Lambda^n} : F_\Lambda^n \to \mathbb{R}$. Then by self-similarity, if $w \in \Lambda_n$ then the function $g \circ \psi_w$ on F agrees exactly with an element of $\mathcal{D}_h$. Evidently $g(x) = 1$, and if $y \notin D_n^1(x)$ then $g(y) = 0$. Therefore it follows by the definition of the resistance metric and the comment after Definition 3.3.3 that if $y \notin D_n^1(x)$ then

$$\begin{aligned} R(x,y) &\geq \mathcal{E}(g,g)^{-1} \\ &= \left(\sum_{w \in \Lambda_n} r_w^{-1} \mathcal{E}(g \circ \psi_w, g \circ \psi_w) \right)^{-1} \\ &> r_{\min} \left(c_4 2^n c \right)^{-1}, \end{aligned}$$

where c_4 is defined in (3.3.1), and this completes the proof. □

The next result gives bounds on the growth of the cardinality of the sets Λ_n in terms of the Hausdorff dimension d_H.

Proposition 3.3.13 (Cardinality of Λ_n)**.** *For all $n \geq 0$,*

$$2^{d_H n} \leq |\Lambda_n| < r_{\min}^{-d_H} 2^{d_H n}.$$

Proof. For $n \geq 0$ and $v \in \Lambda_n$, by the definition of the measure μ we have that

$$r_{\min}^{d_H} 2^{-d_H n} < \mu(F_v) \leq 2^{-d_H n}.$$

Then summing over all $v \in \Lambda_n$ gives

$$r_{\min}^{d_H} 2^{-d_H n} |\Lambda_n| < 1 \leq 2^{-d_H n} |\Lambda_n|.$$

□

3.3.2 The continuity theorems

Theorem 3.3.14 (First continuity theorem). *Let (E, d_E) be a complete separable metric space. Let $\xi = (\xi_x)_{x \in F}$ be an E-valued process indexed by F and let $C, \beta, \gamma > 0$ such that*

$$\mathbb{E}\left[d_E(\xi_x, \xi_y)^\beta\right] \le CR(x,y)^{d_H+\gamma}$$

for all $x, y \in F$. Then there exists a version of ξ which is almost surely essentially $\frac{\gamma}{\beta}$-Hölder continuous with respect to R.

Proof. The set F is uncountable, but $F_* = \bigcup_{n=0}^\infty F_\Lambda^n$ is countable and dense in F. We may therefore consider the countable set $(\xi_x)_{x \in F_*}$ without issues of measurability. Let $\delta \in (0, \frac{\gamma}{\beta})$ and define the measurable event

$$\Omega_\delta = \left\{ \xi'_x := \lim_{\substack{y \to x \\ y \in F_*}} \xi_y \text{ exists } \forall x \in F \text{ and } x \mapsto \xi'_x \text{ is } \delta\text{-Hölder w.r.t. } (F, R) \right\}.$$

We then define the random variables $\hat{\xi}_x$ for $x \in F$ by

$$\hat{\xi}_x = \begin{cases} \xi'_x & \text{if } \xi \in \bigcap\{\Omega_\delta : \delta \in \mathbb{Q} \cap (0, \frac{\gamma}{\beta})\}, \\ x_0 & \text{otherwise,} \end{cases}$$

for some arbitrary fixed $x_0 \in E$. Then $\hat{\xi} := (\hat{\xi}_x)_{x \in F}$ is measurable and essentially $\frac{\gamma}{\beta}$-Hölder continuous. If $\mathbb{P}[\Omega_\delta] = 1$ for all $\delta \in \mathbb{Q} \cap (0, \frac{\gamma}{\beta})$ then $\hat{\xi}$ is also a version of ξ. This is because $\hat{\xi}_x$ is then the almost-sure limit of $(\xi_y)_{y \in F_*}$ as $y \to x$, so applying Fatou's lemma to the estimate in the statement of this theorem shows that $\hat{\xi}_x = \xi_x$ almost surely. It therefore suffices to show that $\mathbb{P}[\Omega_\delta] = 1$ for all $\delta \in (0, \frac{\gamma}{\beta})$. We define the random variable

$$H_\delta = \sup_{\substack{x,y \in F_* \\ x \neq y}} \frac{d_E(\xi_x, \xi_y)}{R(x,y)^\delta}$$

to be the Hölder norm of ξ restricted to F_*, and we observe that $\Omega_\delta = \{H_\delta < \infty\}$ by the completeness of E. For $n \ge 0$ we also define the random variables

$$K_n = \sup_{\substack{x,y \in F_\Lambda^n \\ x \sim_n y}} d_E(\xi_x, \xi_y).$$

By Proposition 3.3.13,

$$|\{(x,y) \in F_\Lambda^n \times F_\Lambda^n : x \sim_n y\}| \le |F^0|^2 \cdot |\Lambda_n| \le |F^0|^2 r_{\min}^{-d_H} 2^{d_H n}.$$

Then using the Markov inequality and Proposition 3.3.12,

$$\begin{aligned}
\mathbb{P}\left[K_n > 2^{-n\delta}\right] &= \mathbb{P}\left[K_n^\beta > 2^{-n\delta\beta}\right] \\
&\leq \frac{1}{2} \sum_{\substack{x,y \in F_\Lambda^n \\ x \sim_n y}} \mathbb{P}\left[d_E(\xi_x, \xi_y)^\beta > 2^{-n\delta\beta}\right] \\
&\leq \frac{2^{n\delta\beta}}{2} \sum_{\substack{x,y \in F_\Lambda^n \\ x \sim_n y}} \mathbb{E}\left[d_E(\xi_x, \xi_y)^\beta\right] \\
&\leq \frac{C 2^{n\delta\beta}}{2} \sum_{\substack{x,y \in F_\Lambda^n \\ x \sim_n y}} R(x,y)^{d_H+\gamma} \\
&\leq C' 2^{d_H n} 2^{-n(d_H+\gamma-\delta\beta)} \\
&= C' 2^{-n(\gamma-\delta\beta)}
\end{aligned}$$

for some constant $C' > 0$. Now $\delta\beta < \gamma$ so

$$\sum_{n=0}^{\infty} \mathbb{P}\left[K_n > 2^{-n\delta}\right] < \infty,$$

so by the Borel-Cantelli lemma we have that $\limsup_{n\to\infty}(2^{n\delta}K_n) \leq 1$ almost surely. In particular there exists an almost surely finite postive random variable J such that $K_n \leq 2^{-n\delta}J$ for all $n \geq 0$ almost surely.

Now recall the constant c_g from Lemma 3.3.9. Let $x, y \in F_*$ be distinct points, and let m_0 be the greatest integer such that $y \in D^1_{m_0}(x)$ (which exists by Proposition 3.3.12). Then there exists $w, v \in \Lambda_{m_0}$ such that $x \in F_w$, $y \in F_v$ and there exists some $z \in F_w \cap F_v$. In fact by [52, Proposition 1.3.5] and the definition of a partition, we can choose z to be in $\psi_w(F^0) \cap \psi_v(F^0)$ so that in particular $z \in F_\Lambda^{m_0}$. Now if $x \in F_\Lambda^{m_0}$ then it follows by Lemma 3.3.9 that

$$d_E(\xi_x, \xi_z) \leq c_g K_{m_0+1}.$$

Otherwise, there exists $m > m_0$ such that $x \in F_\Lambda^m$ and we construct a finite sequence $(x_i)_{i=0}^{m-m_0}$ such that $x_0 = x$, $x_i \in F_\Lambda^{m-i} \cap D^0_{m-i}(x_{i-1}) \cap F_w$ for $i \geq 1$ and $x_{m-m_0} = z$. This can be done in the following way: assume that we already have $x_{i-1} \in F_\Lambda^{m-(i-1)} \cap F_w$ for some $i \in \{1, \ldots, m-m_0\}$. There exists $w^{i-1} \in \Sigma_w$ such that $\pi(w^{i-1}) = x_{i-1}$. Since Λ_{m-i} is a partition, there exists $v^{i-1} \in \Lambda_{m-i}$ such that $w^{i-1} \in \Sigma_{v^{i-1}}$. Therefore $F_{v^{i-1}} \ni x_{i-1}$, so we may pick x_i to be some element of $\psi_{v^{i-1}}(F^0)$. By the refinement property $\Sigma_{v^{i-1}} \subseteq \Sigma_w$ so we have that $x_i \in F_\Lambda^{m-i} \cap D^0_{m-i}(x_{i-1}) \cap F_w$. If $i = m - m_0$ then necessarily $v^{m-m_0-1} = w$ and we can specifically choose $x_{m-m_0} = z$.

We then have by Lemma 3.3.9 that

$$\begin{aligned} d_E(\xi_x, \xi_z) &\le \sum_{i=1}^{m-m_0} d_E(\xi_{x_i}, \xi_{x_{i-1}}) \\ &\le c_g \sum_{i=1}^{m-m_0} K_{m+1-i} \\ &\le c_g \sum_{n=m_0+1}^{\infty} K_n. \end{aligned}$$

We can make the same estimate for y and z. Therefore we conclude that

$$d_E(\xi_x, \xi_y) \le 2c_g \sum_{n=m_0+1}^{\infty} K_n \le 2c_g J \sum_{n=m_0+1}^{\infty} 2^{-n\delta} = \frac{2c_g 2^{-(m_0+1)\delta}}{1 - 2^{-\delta}} J.$$

Now m_0 was chosen such that $y \notin D^1_{m_0+1}(x)$, so we use Proposition 3.3.12 to conclude that $R(x,y) > c_5 2^{-(m_0+1)}$. Thus we find that

$$\frac{d_E(\xi_x, \xi_y)}{R(x,y)^\delta} \le C'' J$$

for all $x \neq y$ in F_* almost surely, where $C'' > 0$ is a constant. So H_δ is almost surely finite. So $\mathbb{P}[\Omega_\delta] = 1$. □

Remark 3.3.15. Taking $F = [0,1]$ as in Example 3.1.1 and E to be the Hilbert space $\mathbb{R}^n$ we obtain the original Kolmogorov continuity theorem.

We would like the solution to our SPDE to be a (random) map $[0,\infty) \times F \to \mathbb{R}$, so the previous theorem is not quite enough. We now seek to prove a version of it for stochastic processes indexed by $[0,1] \times F$. Let G be the set $[0,1] \times F$ equipped with the natural supremum metric on $\mathbb{R} \times F$ given by

$$R_\infty((s,x),(t,y)) = \max\{|s-t|, R(x,y)\}.$$

Proposition 3.3.16. *Let (E, d_E) be a complete separable metric space. Let $\xi = (\xi_{tx} : (t,x) \in [0,1] \times F)$ be an E-valued process indexed by $[0,1] \times F$ and let $C, \beta, \gamma, \gamma' > 0$ be such that*

$$\begin{aligned} \mathbb{E}\left[d_E(\xi_{tx}, \xi_{ty})^\beta\right] &\le C R(x,y)^{d_H+1+\gamma}, \\ \mathbb{E}\left[d_E(\xi_{sx}, \xi_{tx})^\beta\right] &\le C |s-t|^{d_H+1+\gamma'} \end{aligned}$$

for all $s,t \in [0,1]$ and all $x,y \in F$. Then there exists a version of ξ which is almost surely essentially $\frac{\gamma \wedge \gamma'}{\beta}$-Hölder continuous with respect to $G = ([0,1] \times F, R_\infty)$.

Proof. This proof proceeds in much the same way as in Theorem 3.3.14, so we only give an outline.

For $n \geq 0$ we let

$$G^n = \{k2^{-n} : k = 0, 1, \dots, 2^n\} \times F^n_\Lambda$$

and

$$G_* = \bigcup_{n=0}^{\infty} G^n,$$

then G_* is countable and dense in G. Then for each $n \geq 0$ we define a relation $*_n$ on G^n by $(s,x) *_n (t,y)$ if and only if either ($|s-t| = 2^{-n}$ and $x = y$) or ($s = t$ and $x \sim_n y$). Notice that this implies that if $(s,x) *_n (t,y)$ then $R_\infty((s,x),(t,y)) \leq (c_6 \vee 1)2^{-n}$, by Proposition 3.3.12. Then as before we can define

$$K_n = \sup_{\substack{p,q \in G^n \\ p *_n q}} d_E(\xi_p, \xi_q).$$

Since both $[0,1]$ and (F,R) are bounded, for $\delta \in (0, \frac{\gamma \wedge \gamma'}{\beta})$ this satisfies

$$\begin{aligned}
\mathbb{P}\left[K_n > 2^{-n\delta}\right] &= \mathbb{P}\left[K_n^\beta > 2^{-n\delta\beta}\right] \\
&\leq \frac{1}{2} \sum_{\substack{p,q \in G^n \\ p *_n q}} \mathbb{P}\left[d_E(\xi_p, \xi_q)^\beta > 2^{-n\delta\beta}\right] \\
&\leq \frac{2^{n\delta\beta}}{2} \sum_{\substack{p,q \in G^n \\ p *_n q}} \mathbb{E}\left[d_E(\xi_p, \xi_q)^\beta\right] \\
&\leq 2^{\beta-1} 2^{n\delta\beta} \sum_{\substack{(s,x),(t,y) \in G^n \\ (s,x) *_n (t,y)}} \mathbb{E}\left[d_E(\xi_{sx}, \xi_{tx})^\beta + d_E(\xi_{tx}, \xi_{ty})^\beta\right] \\
&\leq C 2^{\beta-1} 2^{n\delta\beta} \sum_{\substack{(s,x),(t,y) \in G^n \\ (s,x) *_n (t,y)}} \left(|s-t|^{d_H+1+\gamma'} + R(x,y)^{d_H+1+\gamma}\right) \\
&\leq C' 2^{n\delta\beta} \sum_{\substack{(s,x),(t,y) \in G^n \\ (s,x) *_n (t,y)}} R_\infty((s,x),(t,y))^{d_H+1+\gamma\wedge\gamma'} \\
&\leq C'' 2^{d_H n} 2^n 2^{-n(d_H+1+\gamma\wedge\gamma'-\delta\beta)} \\
&= C'' 2^{-n(\gamma\wedge\gamma'-\delta\beta)}.
\end{aligned}$$

So as in Theorem 3.3.14, there exists an almost surely finite positive random variable J such that $K_n \leq 2^{-n\delta} J$ for all $n \geq 0$ almost surely. The sets analogous to $D^0_n(x)$ and $D^1_n(x)$ in Theorem 3.3.14 are given by

$$\hat{D}^0_n(s,x) = ([s_-, s^-] \cap [0,1]) \times D^0_n(x)$$

and

$$\hat{D}_n^1(s,x) = ([s_- - 2^{-n}, s^- + 2^{-n}] \cap [0,1]) \times D_n^1(x)$$

where $s_- = \max\{k2^{-n} : k \in \mathbb{Z},\ k2^{-n} < s\}$, $s^- = \min\{k2^{-n} : k \in \mathbb{Z},\ k2^{-n} > s\}$. Using Proposition 3.3.12 it is simple to verify the analogous result that

$$B_\infty(p,(c_5 \wedge 1)2^{-n}) \subseteq \hat{D}_n^1(p) \subseteq B_\infty(p,(c_6 \vee 2)2^{-n})$$

for all $n \geq 0$ and all $p \in G$, where B_∞ denotes the closed R_∞-balls of G. Now if $(s,x),(t,y) \in G_*$ are distinct points, let m_0 be the greatest integer such that $(t,y) \in \hat{D}_{m_0}^1(s,x)$. Then there exists $w,v \in \Lambda_{m_0}$ and $\tau_1,\tau_2 \in \{k2^{-m_0} : k = 0,1,\ldots,2^{m_0}-1\}$ such that

$$\begin{aligned}(s,x) &\in [\tau_1,\tau_1+2^{-m_0}] \times F_w,\\ (t,y) &\in [\tau_2,\tau_2+2^{-m_0}] \times F_v,\end{aligned}$$

and there exists some

$$(\tau,z) \in [\tau_1,\tau_1+2^{-m_0}] \times F_w \cap [\tau_2,\tau_2+2^{-m_0}] \times F_v.$$

In fact just as in the proof of Theorem 3.3.14 we may pick (τ,z) such that

$$(\tau,z) \in \{\tau_1,\tau_1+2^{-m_0}\} \times \psi_w(F^0) \cap \{\tau_2,\tau_2+2^{-m_0}\} \times \psi_v(F^0) \subset G^{m_0}.$$

We can then estimate $d_E(\xi_{sx},\xi_{ty})$ by constructing suitable finite sequences of points from (s,x) to (τ,z) and from (t,y) to (τ,z), similar to the proof of Theorem 3.3.14.

The rest of the details of the proof are left up to the reader; it suffices to adapt the proof of Theorem 3.3.14, using for example c_g+2 instead of c_g. □

We now extend the previous result to this section's main theorem, which includes spatial and temporal Hölder exponents.

Theorem 3.3.17 (Second continuity theorem)**.** *Let (E,d_E) be a complete separable metric space. Let $\xi = (\xi_{tx} : (t,x) \in [0,1] \times F)$ be an E-valued process indexed by $[0,1] \times F$ and let $C,\beta,\gamma,\gamma' > 0$ be such that*

$$\begin{aligned}\mathbb{E}\left[d_E(\xi_{tx},\xi_{ty})^\beta\right] &\leq CR(x,y)^{d_H+1+\gamma},\\ \mathbb{E}\left[d_E(\xi_{sx},\xi_{tx})^\beta\right] &\leq C|s-t|^{d_H+1+\gamma'}\end{aligned} \tag{3.3.2}$$

for all $s,t \in [0,1]$ and all $x,y \in F$. Then there exists a version $\hat{\xi} = (\hat{\xi}_{tx} : (t,x) \in [0,1] \times F)$ of ξ which satisfies the following:

(1). The map $(t, x) \mapsto \hat{\xi}_{tx}$ *is almost surely essentially* δ_0*-Hölder continuous with respect to* R_∞ *where*

$$\delta_0 = \frac{1}{\beta}(\gamma \wedge \gamma').$$

(2). For every $t \in [0, 1]$ *the map* $x \mapsto \hat{\xi}_{tx}$ *is almost surely essentially* δ_1*-Hölder continuous with respect to* R *where*

$$\delta_1 = \frac{1}{\beta}(1 + \gamma).$$

(3). For every $x \in F$ *the map* $t \mapsto \hat{\xi}_{tx}$ *is almost surely essentially* δ_2*-Hölder continuous with respect to the Euclidean metric where*

$$\delta_2 = \frac{1}{\beta}(d_H + \gamma').$$

Proof. (1) is exactly Proposition 3.3.16. For (2) we fix $t \in [0, 1]$. We see that the space increment estimate is equivalent to

$$\mathbb{E}\left[d_E(\hat{\xi}_{tx}, \hat{\xi}_{ty})^\beta\right] \leq CR(x, y)^{d_H + \beta\delta_1}.$$

Then by Theorem 3.3.14 there exists a version $(\tilde{\xi}_x)_{x \in F}$ of $(\hat{\xi}_{tx})_{x \in F}$ which is almost surely essentially δ_1-Hölder continuous with respect to R. Now using (1), $(\tilde{\xi}_x)_{x \in F}$ and $(\hat{\xi}_{tx})_{x \in F}$ are both almost surely continuous on the separable space F (see for example $F_* \subseteq F$) so we must in fact have that $(\tilde{\xi}_x)_{x \in F} = (\hat{\xi}_{tx})_{x \in F}$ almost surely. We conclude that $(\hat{\xi}_{tx})_{x \in F}$ is almost surely essentially δ_1-Hölder continuous with respect to R. The proof of (3) is conceptually identical – we use the standard Kolmogorov continuity theorem for $[0, 1]$. □

Remark 3.3.18. This time by taking $F = [0, 1]$ as in Example 3.1.1 the above theorem reduces to the original Kolmogorov continuity theorem for $[0, 1]^2$.

Corollary 3.3.19. *Theorem 3.3.17 holds if the interval* $[0, 1]$ *is replaced with* $[0, T]$ *for any* $T > 0$.

Proof. We have that $\xi = (\xi_{tx} : (t, x) \in [0, T] \times F)$ is an E-valued process indexed by $[0, T] \times F$. By taking a linear rescaling $[0, T] \leftrightarrow [0, 1]$ of the time coordinate we transform ξ into an E-valued process indexed by $[0, 1] \times F$ with the same exponents in the continuity estimates (3.3.2). Then we use Theorem 3.3.17 to construct a Hölder continuous version of the rescaled ξ. Finally we reverse the rescaling, which is linear so it preserves Hölder exponents. □

3.4 Pointwise regularity

Before we talk about Hölder continuity of the solution u to (3.1.1) we show that the point evaluations $u(t,x)$ for $(t,x) \in [0,\infty) \times F$ are indeed well-defined random variables. Recall from Proposition 3.2.7 and the subsequent discussion that

$$S_t^b(1-\Delta_b)^{-\frac{\alpha}{2}} f = \sum_{k=1}^{\infty} f_k e^{-\lambda_k^b t}(1+\lambda_k^b)^{-\frac{\alpha}{2}} \varphi_k^b$$

for all $f \in \mathcal{H} = L^2(F,\mu)$, where $f_k = \langle \varphi_k^b, f\rangle_\mu$. Equivalently

$$S_t^b(1-\Delta_b)^{-\frac{\alpha}{2}} = \sum_{k=1}^{\infty} e^{-\lambda_k^b t}(1+\lambda_k^b)^{-\frac{\alpha}{2}} \varphi_k^b \varphi_k^{b*}$$

where $\varphi_k^{b*} \in \mathcal{H}^*$ is the bounded linear functional $f \mapsto \langle \varphi_k^b, f\rangle_\mu$. By Proposition 3.2.7 we have that

$$\sum_{k=1}^{\infty}(1+\lambda_k^b)^{-\alpha} \int_0^t \|e^{-\lambda_k^b(t-s)}\varphi_k^{b*}\|_{\mathrm{HS}}^2 ds = \sum_{k=1}^{\infty} \frac{1-e^{-2\lambda_k^b t}}{2\lambda_k^b(1+\lambda_k^b)^\alpha} \leq C\sum_{k=1}^{\infty} k^{-\frac{2}{d_s}} < \infty,$$

so it follows from Itō's isometry for $\mathcal{H}$-valued stochastic integrals that

$$\begin{aligned} W_\alpha^b(t) &:= \int_0^t S_{t-s}^b(1-\Delta_b)^{-\frac{\alpha}{2}} dW(s) \\ &= \sum_{k=1}^{\infty} \int_0^t e^{-\lambda_k^b(t-s)}\varphi_k^{b*} dW(s)(1+\lambda_k^b)^{-\frac{\alpha}{2}}\varphi_k^b. \end{aligned}$$

For each $k \geq 1$ define the real-valued stochastic process $X^{b,k} = (X_t^{b,k})_{t\geq 0}$ by

$$X_t^{b,k} = \int_0^t e^{-\lambda_k^b(t-s)}\varphi_k^{b*} dW(s) \tag{3.4.1}$$

so we have the series representation

$$W_\alpha^b(t) = \sum_{k=1}^{\infty}(1+\lambda_k^b)^{-\frac{\alpha}{2}} X_t^{b,k}\varphi_k^b. \tag{3.4.2}$$

Evidently $X^{b,k}$ is a centred real continuous Gaussian process. We compute its covariance to be

$$\mathbb{E}\left[X_t^{b,k} X_{t+s}^{b,k}\right] = \frac{e^{-\lambda_k^b s}}{2\lambda_k^b}(1-e^{-2\lambda_k^b t})$$

if $\lambda_k^b > 0$ and we identify $X^{b,k}$ to be a centred Ornstein-Uhlenbeck process with unit volatility and rate parameter λ_k^b. If $\lambda_k^b = 0$ then $X^{b,k}$ is simply a standard Wiener process. It is easy to check that the family $(X^{b,k})_{k=1}^\infty$ is independent.

Remark 3.4.1. We give an alternative view on the series representation (3.4.2). Let u be the solution to (3.1.1) in the case $u_0 = 0$, so that $u = W^b_\alpha$. We take an eigenfunction expansion of (3.1.1):

$$\begin{aligned} d\hat{u}(t,k) &= -\lambda^b_k \hat{u}(t,k)dt + (1+\lambda^b_k)^{-\frac{\alpha}{2}} \varphi^{b*}_k dW(t), \\ \hat{u}(0,k) &= 0 \end{aligned} \tag{3.4.3}$$

for each $k \in \mathbb{N}$, where $\hat{u}(\cdot,k) := \langle \varphi^b_k, u(\cdot)\rangle_\mu$ is a real-valued process. This is analogous to using Fourier methods to solve differential equations on $\mathbb{R}^n$. Now using standard theory we see that $\{\varphi^{b*}_k W\}_{k=1}^\infty$ is a family of independent real-valued standard Wiener processes. It follows that (3.4.3) is just a family of decoupled one-dimensional SDEs, and the solution to the kth SDE can be found to be exactly $(1+\lambda^b_k)^{-\frac{\alpha}{2}} X^{b,k}$.

3.4.1 Resolvent density

Definition 3.4.2. If $\lambda > 0$ then $\mathcal{D}$ can be equipped with the inner product

$$\langle f,g\rangle_\lambda := \mathcal{E}(f,g) + \lambda \langle f,g\rangle_\mu.$$

Since $\mathcal{E}$ is a closed form, this turns $\mathcal{D}$ into a Hilbert space which we denote $\mathcal{D}^\lambda$. Observe that the evaluation maps $\{f \mapsto f(x) : \ x \in F\}$ are continuous linear functionals on $\mathcal{D}^\lambda$, by [3, Proposition 7.16(b)]. We have that $\mathcal{D}_0$ is the intersection of the kernels of $\{f \mapsto f(x) : \ x \in F^0\}$ so it must be closed with respect to $\langle \cdot,\cdot\rangle_\lambda$.

Definition 3.4.3. For $\lambda > 0$ and $b \in \{N, D\}$ let $\rho^b_\lambda : F \times F \to \mathbb{R}$ be the *resolvent density* associated with Δ_b. By [3, Theorem 7.20], ρ^N_λ exists and satisfies the following:

(1). (Reproducing kernel property.) For $x \in F$, $\rho^N_\lambda(x,\cdot)$ is the unique element of $\mathcal{D}$ such that

$$\langle \rho^N_\lambda(x,\cdot), f\rangle_\lambda = f(x)$$

for all $f \in \mathcal{D}$.

(2). (Resolvent kernel property.) For all continuous $f \in \mathcal{H}$ and all $x \in F$,

$$\int_0^\infty e^{-\lambda t} S^N_t f(x) dt = \int_F \rho^N_\lambda(x,y) f(y) \mu(dy).$$

By a density argument it follows that for all $f \in \mathcal{H}$,

$$\int_0^\infty e^{-\lambda t} S^N_t f dt = \int_F \rho^N_\lambda(\cdot,y) f(y) \mu(dy).$$

(3). ρ_λ^N is non-negative (easy to see from (2) and fact that $S_t^N f(x) = \mathbb{E}^x[f(X_t^N)]$), symmetric and bounded. We define (for now) $c_7(\lambda) > 0$ such that

$$c_7(\lambda) \geq \sup_{x,y \in F} \rho_\lambda^N(x,y).$$

(4). (Hölder continuity.) For this same constant $c_7(\lambda)$ we have that for all $x, y, y' \in F$,

$$|\rho_\lambda^N(x,y) - \rho_\lambda^N(x,y')|^2 \leq c_7(\lambda) R(y,y').$$

Using symmetry this Hölder continuity result holds in the first argument as well.

By an identical argument to [3, Theorem 7.20], ρ_λ^D exists and satisfies the analogous results with $(\mathcal{E}, \mathcal{D}_0)$ and S^D. By the reproducing kernel property it follows that for every $x \in F$, $\rho_\lambda^D(x,\cdot)$ must be the $\mathcal{D}^\lambda$-orthogonal projection of $\rho_\lambda^N(x,\cdot)$ onto $\mathcal{D}_0$. We now choose $c_7(\lambda)$ large enough that it does not depend on the value of $b \in \{N, D\}$ for (3) and (4).

The Hölder continuity property of the resolvent densities ρ_λ^b described above is the subject of this section. We seek to strengthen it into Lipschitz continuity.

Definition 3.4.4. Let $B \subseteq F$ be closed and non-empty. Let $g_B : F \times F \to \mathbb{R}$ be the *Green function* on F with boundary B, see [53, Chapter 4]. This is the unique function such that for all $x \in F$, $g_B(x,\cdot) \in \mathcal{D}$, $g_B(x,\cdot)$ vanishes on B and $\mathcal{E}(f, g_B(x,\cdot)) = f(x)$ for all $f \in \mathcal{D}$ vanishing on B.

The properties of the Green function that we require are given in [53, Theorem 4.1]. Note in particular that every Green function is symmetric and uniformly Lipschitz in (F,R); this is the main tool of our proof.

Proposition 3.4.5 (Lipschitz resolvent)**.** *For $\lambda > 0$ and $b \in \{N, D\}$, if $x, y, y' \in F$ then*

$$\left|\rho_\lambda^b(x,y) - \rho_\lambda^b(x,y')\right| \leq 2R(y,y').$$

Proof. Let $\mathbf{1} \in \mathcal{H}$ be the constant function taking the value 1. First of all, observe that for all $\lambda > 0$, $x \in F$ and $b \in \{N, D\}$,

$$\langle \rho_\lambda^b(x,\cdot), \mathbf{1} \rangle_\mu = \int_0^\infty e^{-\lambda t} S_t^b \mathbf{1}(x) dt = \int_0^\infty e^{-\lambda t} \mathbb{P}^x[X_t^b \in F] dt \leq \frac{1}{\lambda}.$$

Fix $\lambda > 0$. We prove the result for $b = D$ first. Let $g_D = g_{F^0}$, the Green function associated with Dirichlet boundary conditions. Then for $x, y \in F$,

$$\rho_\lambda^D(x,y) = \mathcal{E}\left(\rho_\lambda^D(x,\cdot), g_D(y,\cdot)\right) = g_D(y,x) - \lambda\langle\rho_\lambda^D(x,\cdot), g_D(y,\cdot)\rangle_\mu.$$

So if $x, y, y' \in F$ then by [53, Theorem 4.1] and the non-negativity of the resolvent density,

$$\begin{aligned}\rho_\lambda^D(x,y) - \rho_\lambda^D(x,y') &\le |g_D(y,x) - g_D(y',x)| + \lambda\int_F \rho_\lambda^D(x,z)|g_D(y,z) - g_D(y',z)|\mu(dz)\\ &\le R(y,y')\left(1 + \lambda\int_F \rho_\lambda^D(x,z)\mathbf{1}(z)\mu(dz)\right)\\ &\le 2R(y,y').\end{aligned}$$

Doing the same estimate with y, y' interchanged gives the required result. Now for the case $b = N$, fix $x_0 \in F$ an arbitrary point. We see that

$$\rho_\lambda^N(x,y) - \rho_\lambda^N(x,x_0) = \mathcal{E}\left(\rho_\lambda^N(x,\cdot), g_{\{x_0\}}(y,\cdot)\right) = g_{\{x_0\}}(y,x) - \lambda\langle\rho_\lambda^N(x,\cdot), g_{\{x_0\}}(y,\cdot)\rangle_\mu,$$

and the rest of the proof is identical to the $b = D$ case. □

As before, by the symmetry of ρ_λ^b the above Lipschitz continuity property in fact holds in both of its arguments.

3.4.2 Pointwise regularity of solution

We return to the SPDE (3.1.1). The next lemma is based on an argument in [20, Section 7.2].

Lemma 3.4.6. *Let $u : [0,\infty) \to \mathcal{H}$ be the solution to* (3.1.1) *with initial condition $u_0 = 0$. If $g \in \mathcal{H}$ and $t \in [0,\infty)$ then*

$$\mathbb{E}\left[\langle u(t), g\rangle_\mu^2\right] \le \frac{e^{2t}}{2}\int_F\int_F \rho_1^b(x,y)g(x)g(y)\mu(dx)\mu(dy).$$

Proof. Let $g^* \in \mathcal{H}^*$ be the bounded linear functional $f \mapsto \langle f, g\rangle_\mu$. We see by Itō's isometry that

$$\begin{aligned}\mathbb{E}\left[\langle u(t), g\rangle_\mu^2\right] &= \mathbb{E}\left[g^*(u(t))^2\right]\\ &= \int_0^t \|g^*(1-\Delta_b)^{-\frac{\alpha}{2}}S_s^b\|_{\mathrm{HS}}^2 ds\\ &= \int_0^t \|(1-\Delta_b)^{-\frac{\alpha}{2}}S_s^b g\|_\mu^2 ds\end{aligned}$$

where the last equality is a result of the self-adjointness of the operator $(1-\Delta_b)^{-\frac{\alpha}{2}}S^b_s$. We know from the functional calculus for self-adjoint operators that $\|(1-\Delta_b)^{-\frac{\alpha}{2}}\| \leq 1$ so

$$\begin{aligned}
\mathbb{E}\left[\langle u(t),g\rangle_\mu^2\right] &\leq \int_0^t \|S^b_s g\|_\mu^2 ds \\
&\leq e^{2t}\int_0^\infty e^{-2s}\|S^b_s g\|_\mu^2 ds \\
&= e^{2t}\left\langle \int_0^\infty e^{-2s} S^b_{2s} g ds, g\right\rangle_\mu \\
&= \frac{e^{2t}}{2}\left\langle \int_F \rho^b_1(\cdot,y)g(y)\mu(dy), g\right\rangle_\mu \\
&= \frac{e^{2t}}{2}\int_F\int_F \rho^b_1(x,y)g(x)g(y)\mu(dx)\mu(dy).
\end{aligned}$$

□

Definition 3.4.7. For $x \in F$ and $n \geq 0$, define

$$f^x_n = \mu(D^0_n(x))^{-1}\mathbb{1}_{D^0_n(x)},$$

see [20, Section 7.2].

Evidently $f^x_n \in \mathcal{H}$, $\|f^x_n\|_\mu^2 = \mu(D^0_n(x))^{-1} < r_{\min}^{-d_H}2^{d_H n}$ (by the definition of d_H and the comment after Definition 3.3.3) and if $g \in \mathcal{H}$ is continuous then

$$\lim_{n\to\infty}\langle f^x_n, g\rangle_\mu = g(x),$$

by Proposition 3.3.12. We can now state and prove the main theorem of this section.

Theorem 3.4.8 (Pointwise regularity)**.** *Let $u : [0,\infty) \to \mathcal{H}$ be the solution to the SPDE* (3.1.1) *with initial value $u_0 = 0$. Then for all $(t,x) \in [0,\infty)\times F$ the expression*

$$u(t,x) := \sum_{k=1}^\infty (1+\lambda^b_k)^{-\frac{\alpha}{2}} X^{b,k}_t \varphi^b_k(x)$$

is a well-defined real-valued centred Gaussian random variable. There exists a constant $c_9 > 0$ such that for all $x \in F$, $t \in [0,\infty)$ and $n \geq 0$ we have that

$$\mathbb{E}\left[(\langle u(t), f^x_n\rangle_\mu - u(t,x))^2\right] \leq c_9 e^{2t}2^{-n}.$$

Proof. Note that $\varphi^b_k \in \mathcal{D}(\Delta_b)$ for each k, so φ^b_k is continuous and so $\varphi^b_k(x)$ is well-defined. By the definition of $u(t,x)$ as a sum of real-valued centred Gaussian random variables we need only prove that it is square-integrable and that the approximation estimate holds. Let $x \in F$. The theorem is trivial for $t = 0$ so let $t \in (0, \infty)$. By Lemma 3.4.6 we have that

$$\begin{aligned}
&\mathbb{E}\left[\langle u(t), f_n^x - f_m^x \rangle_\mu^2\right] \\
&\leq \frac{e^{2t}}{2} \int_F \int_F \rho_1^b(z_1, z_2)(f_n^x(z_1) - f_m^x(z_1))(f_n^x(z_2) - f_m^x(z_2))\mu(dz_1)\mu(dz_2).
\end{aligned}$$

Then using the definition of f_n^x, Proposition 3.4.5 and Proposition 3.3.12 we have that

$$\begin{aligned}
\mathbb{E}\left[\langle u(t), f_n^x - f_m^x \rangle_\mu^2\right] &\leq \frac{e^{2t}}{2}\left(4c_6 2^{-n} + 4c_6 2^{-m}\right) \\
&= 2e^{2t} c_6 \left(2^{-n} + 2^{-m}\right).
\end{aligned} \tag{3.4.4}$$

Writing u in its series representation (3.4.2) and using the independence of the $X^{b,k}$ and the fact that $\sum_{k=1}^\infty \mathbb{E}\left[(X_t^{b,k})^2\right] < \infty$, this is equivalent to

$$\sum_{k=1}^\infty (1 + \lambda_k^b)^{-\alpha} \mathbb{E}\left[(X_t^{b,k})^2\right] \left(\langle \varphi_k^b, f_n^x \rangle_\mu - \langle \varphi_k^b, f_m^x \rangle_\mu\right)^2 \leq 2e^{2t} c_6 \left(2^{-n} + 2^{-m}\right).$$

It follows that the left-hand side tends to zero as $m, n \to \infty$. By Theorem 3.2.11 we know that

$$\sum_{k=1}^\infty (1 + \lambda_k^b)^{-\alpha} \mathbb{E}\left[(X_t^{b,k})^2\right] \langle \varphi_k^b, f_n^x \rangle_\mu^2 = \mathbb{E}\left[\langle u(t), f_n^x \rangle_\mu^2\right] < \infty$$

for all $x \in F$, $n \geq 0$ and $t \in [0, \infty)$, therefore by the completeness of the sequence space ℓ^2 there must exist a unique sequence $(y_k)_{k=1}^\infty$ such that $\sum_{k=1}^\infty y_k^2 < \infty$ and

$$\lim_{n \to \infty} \sum_{k=1}^\infty \left((1 + \lambda_k^b)^{-\frac{\alpha}{2}} \mathbb{E}\left[(X_t^{b,k})^2\right]^{\frac{1}{2}} \langle \varphi_k^b, f_n^x \rangle_\mu - y_k \right)^2 = 0.$$

Since φ_k^b is continuous we have $\lim_{n\to\infty} \langle \varphi_k^b, f_n^x \rangle_\mu = \varphi_k^b(x)$. Thus by Fatou's lemma we can identify the sequence (y_k). We must have

$$\sum_{k=1}^\infty (1 + \lambda_k^b)^{-\alpha} \mathbb{E}\left[(X_t^{b,k})^2\right] \varphi_k^b(x)^2 < \infty \tag{3.4.5}$$

and

$$\lim_{n \to \infty} \sum_{k=1}^\infty (1 + \lambda_k^b)^{-\alpha} \mathbb{E}\left[(X_t^{b,k})^2\right] \left(\langle \varphi_k^b, f_n^x \rangle_\mu - \varphi_k^b(x)\right)^2 = 0.$$

Equivalently by (3.4.2),

$$\mathbb{E}\left[u(t,x)^2\right] < \infty$$

(so we have proven square-integrability) and

$$\lim_{n\to\infty} \mathbb{E}\left[(\langle u(t), f_n^x\rangle_\mu - u(t,x))^2\right] = 0.$$

In particular by taking $m \to \infty$ in (3.4.4) we have that

$$\mathbb{E}\left[(\langle u(t), f_n^x\rangle_\mu - u(t,x))^2\right] \le 2c_6 e^{2t} 2^{-n}.$$

□

By virtue of the previous theorem it is possible to interpret solutions u to the SPDE (3.1.1) as random maps $u : [0,\infty)\times F \to \mathbb{R}$, where as usual we have suppressed the dependence of u on the underlying probability space. It therefore makes sense to consider issues of continuity of u on $[0,\infty)\times F$.

Remark 3.4.9. We note that [20, Example 7.4] gives a similar result for stochastic heat equations where the operator Δ is the generator for a fractional diffusion in the sense of [3, Section 3] under suitable conditions.

3.5 Hölder regularity

The aim of this section is to use our continuity theorems of Section 3.3 to prove Hölder regularity results for a version of the family defined in Theorem 3.4.8, and then show that this version can be identified with the original solution to (3.1.1). We wish to use Theorem 3.3.17 and Corollary 3.3.19, so we need estimates on the expected spatial and temporal increments of the solution.

3.5.1 Spatial estimate

Proposition 3.5.1. *Let $T > 0$. Let $u = (u(t,x))_{(t,x)\in[0,\infty)\times F}$ be the family defined in Theorem 3.4.8. Then there exists a constant $C_2 > 0$ such that*

$$\mathbb{E}\left[(u(t,x) - u(t,y))^2\right] \le C_2 R(x,y)$$

for all $t \in [0,T]$ and all $x, y \in F$.

Proof. Recall from Theorem 3.4.8 that

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\langle u(t), f_n^x\rangle_\mu - u(t,x)\right)^2\right] = 0,$$

and an analogous result holds for y. Thus by Lemma 3.4.6,

$$\begin{aligned}\mathbb{E}\left[(u(t,x)-u(t,y))^2\right] &= \lim_{n\to\infty}\mathbb{E}\left[\langle u(t), f_n^x - f_n^y\rangle_\mu^2\right]\\ &\le \frac{e^{2t}}{2}\lim_{n\to\infty}\int_F\int_F \rho_1^b(z_1,z_2)(f_n^x(z_1)-f_n^y(z_1))(f_n^x(z_2)-f_n^y(z_2))\mu(dz_1)\mu(dz_2)\\ &= \frac{e^{2t}}{2}\left(\rho_1^b(x,x)-2\rho_1^b(x,y)+\rho_1^b(y,y)\right),\end{aligned}$$

where we have used Proposition 3.4.5, Proposition 3.3.12 and the definition of f_n^x (similarly to the proof of Theorem 3.4.8). Hence again by Proposition 3.4.5,

$$\begin{aligned}\mathbb{E}\left[(u(t,x)-u(t,y))^2\right] &\le \frac{e^{2T}}{2}\left(\rho_1^b(x,x)-\rho_1^b(x,y)+\rho_1^b(y,y)-\rho_1^b(y,x)\right)\\ &\le 2e^{2T}R(x,y).\end{aligned}$$

□

3.5.2 Temporal estimates

For the time estimates we can save ourselves some work by noticing that if $X^{b,k}$ is an Ornstein-Uhlenbeck process then

$$\mathbb{E}\left[(X_s^{b,k}-X_{s+t}^{b,k})^2\right] = \frac{1}{\lambda_k^b}(1-e^{-\lambda_k^b t}) - \frac{e^{-2\lambda_k^b s}}{2\lambda_k^b}(1-e^{-\lambda_k^b t})^2,$$

so that

$$\frac{1}{2\lambda_k^b}(1-e^{-\lambda_k^b t}) \le \mathbb{E}\left[(X_s^{b,k}-X_{s+t}^{b,k})^2\right] \le \frac{1}{\lambda_k^b}(1-e^{-\lambda_k^b t})$$

for any $s,t\in[0,\infty)$. Therefore regardless of whether $X^{b,k}$ is an Ornstein-Uhlenbeck or Wiener process we have that

$$\mathbb{E}\left[(X_s^{b,k}-X_{s+t}^{b,k})^2\right] \le 2\mathbb{E}\left[(X_t^{b,k})^2\right].$$

Now since (using the independence of the $X^{b,k}$)

$$\mathbb{E}\left[(u(s,x)-u(s+t,x))^2\right] = \sum_{k=1}^\infty (1+\lambda_k^b)^{-\alpha}\mathbb{E}\left[(X_s^{b,k}-X_{s+t}^{b,k})^2\right]\varphi_k^b(x)^2,$$

it follows that it suffices to find estimates of the above in the case $s=0$.

We start with a method similar to the proof of [77, Proposition 3.7] which does not quite cover all values of α. First, a lemma:

Lemma 3.5.2. *Consider the sum*

$$\sigma_{ab}(t) = \sum_{k=1}^\infty \left(k^{a-1}\wedge(k^{b-1}t)\right)$$

for $t\in(0,\infty)$, where $a,b\in\mathbb{R}$ are constants. Then the following hold:

(1). If $a, b \geq 0$ then $\sigma_{ab}(t)$ diverges.

(2). If $a \in \mathbb{R}$, $b < 0$ then there exists $C_{a,b} > 0$ such that $\sigma_{ab}(t) \leq C_{a,b}t$ for all t.

(3). If $a < 0$, $b \geq 0$ then there exists $C_{a,b} > 0$ such that $\sigma_{ab}(t) \leq C_{a,b}t^{\frac{-a}{b-a}}$ for all t.

Proof. (1) is obvious. For (2) take $C_{a,b} = \zeta(1-b)$ the Riemann zeta function.

For (3) we must consider two cases depending on the value of b. First assume that $b \in [0,1]$. Then $x \mapsto (x^{a-1} \wedge (x^{b-1}t))$ is a decreasing function on $(0,\infty)$ so

$$\begin{aligned}\sigma_{ab}(t) &\leq \int_0^\infty (x^{a-1} \wedge (x^{b-1}t))dx \\ &= t\int_0^{t^{\frac{-1}{b-a}}} x^{b-1}dx + \int_{t^{\frac{-1}{b-a}}}^\infty x^{a-1}dx \\ &= tb^{-1}t^{\frac{-b}{b-a}} - a^{-1}t^{\frac{-a}{b-a}} \\ &= C_{a,b}t^{\frac{-a}{b-a}}\end{aligned}$$

where we have $C_{a,b} = b^{-1} - a^{-1}$. If $b > 1$ then $x \mapsto (x^{a-1} \wedge (x^{b-1}t))$ is increasing on $[0, t^{\frac{-1}{b-a}}]$ where it is equal to $x^{b-1}t$ and decreasing on $[t^{\frac{-1}{b-a}}, \infty)$ where it is equal to x^{a-1}. Thus

$$k^{a-1} \wedge (k^{b-1}t) \leq k^{a-1} \wedge (t^{-\frac{b-1}{b-a}}t) = k^{a-1} \wedge (k^{1-1}t^{\frac{1-a}{b-a}})$$

for all $k \in \mathbb{N}$, and we are back to the case $a < 0$, $b = 1$. It follows that

$$\sigma_{ab}(t) \leq (1 - a^{-1})\left(t^{\frac{1-a}{b-a}}\right)^{\frac{-a}{1-a}} = (1 - a^{-1})t^{\frac{-a}{b-a}}$$

so we have $C_{a,b} = 1 - a^{-1}$. □

Proposition 3.5.3. *Let $T > 0$. Let $u = (u(t,x))_{(t,x)\in[0,\infty)\times F}$ be the family defined in Theorem 3.4.8. If $\alpha > d_s - 1$ then there exists a constant $C_3 > 0$ such that*

$$\mathbb{E}\left[(u(s,x) - u(t,x))^2\right] \leq C_3|s-t|^{1\wedge(1-d_s+\alpha)}$$

for all $s, t \in [0,T]$ and all $x \in F$.

Proof. We assume that $\lambda_1^b > 0$ to streamline our calculations. The case $\lambda_1^b = 0$ is left as an exercise. By the discussion at the start of this section we may assume that $s = 0$. Fix $x \in F$ and $t \in [0,T]$. Recall the constant c_3 from Proposition 3.2.7. By independence of the $X^{b,k}$ we have that

$$\begin{aligned}\mathbb{E}\left[u(t,x)^2\right] &= \mathbb{E}\left[\left(\sum_{k=1}^{\infty}(1+\lambda_k^b)^{-\frac{\alpha}{2}}X_t^{b,k}\varphi_k^b(x)\right)^2\right]\\ &\leq \sum_{k=1}^{\infty}\frac{\varphi_k^b(x)^2}{(1+\lambda_k^b)^\alpha}\frac{1}{\lambda_k^b}(1-e^{-\lambda_k^b t})\\ &\leq c_3^2\sum_{k=1}^{\infty}\frac{1-e^{-\lambda_k^b t}}{(\lambda_k^b)^{1-\frac{d_s}{2}}(1+\lambda_k^b)^\alpha}\\ &\leq c_3^2\sum_{k=1}^{\infty}\frac{1\wedge(\lambda_k^b t)}{(\lambda_k^b)^{1-\frac{d_s}{2}}(1+\lambda_k^b)^\alpha}\\ &\leq c'\sum_{k=1}^{\infty}\left(k^{1-\frac{2+2\alpha}{d_s}}\wedge(k^{1-\frac{2\alpha}{d_s}}t)\right)\end{aligned}$$

and we are within the scope of Lemma 3.5.2 (as long as $t>0$, though the case $t=0$ is trivial). We find that if $\alpha > d_s - 1$ then the sum converges and there exists $c'' > 0$ such that

$$\mathbb{E}\left[u(t,x)^2\right] \leq c'' t^{1\wedge(1+\alpha-d_s)}$$

for all $x \in F$, $t \in [0,T]$. □

Remark 3.5.4. In the case of Example 3.1.1 with $F = [0,1]$ the above estimate can be made to work for all $\alpha \geq 0$. This is because in this case $d_s = 1$ and the eigenfunctions φ_k^b satisfy $\|\varphi_k^b\|_\infty \leq c$ for some $c > 0$ so we instead have that

$$\mathbb{E}\left[u(t,x)^2\right] \leq c''' \sum_{k=1}^{\infty}\left(k^{-(2+2\alpha)}\wedge(k^{-2\alpha}t)\right).$$

Therefore using Lemma 3.5.2, we have for all $\alpha \geq 0$ that

$$\mathbb{E}\left[(u(s,x)-u(t,x))^2\right] \leq C_3|s-t|^{1\wedge(\frac{1}{2}+\alpha)}.$$

This is the method used in [77].

We now prove an alternative estimate that is weaker for large α but holds for all $\alpha \geq 0$.

Proposition 3.5.5. *Let $T > 0$. Let $u = (u(t,x))_{(t,x)\in[0,\infty)\times F}$ be the family defined in Theorem 3.4.8. Then there exists $C_4 > 0$ such that*

$$\mathbb{E}\left[(u(s,x)-u(t,x))^2\right] \leq C_4|s-t|^{1-\frac{d_s}{2}}$$

for all $s,t \in [0,T]$ and all $x \in F$.

Proof. By the discussion at the start of this section we may assume that $s = 0$. Set

$$c_9' = (c_9 e^{2T}) \vee \frac{T d_H}{r_{\min}^{d_H}}$$

where the constant c_9 is from Theorem 3.4.8. By Theorem 3.4.8 and Itō's isometry (see proof of Lemma 3.4.6) we have that if $n \geq 0$ is an integer then

$$\begin{aligned}\mathbb{E}\left[u(t,x)^2\right] &\leq 2\mathbb{E}\left[\langle u(t), f_n^x\rangle_\mu^2\right] + 2c_9 e^{2t} 2^{-n} \\ &= 2\int_0^t \|(1-\Delta_b)^{-\frac{\alpha}{2}} S_s^b f_n^x\|_\mu^2 ds + 2c_9 e^{2t} 2^{-n} \\ &\leq 2\int_0^t \|(1-\Delta_b)^{-\frac{\alpha}{2}} S_s^b f_n^x\|_\mu^2 ds + 2c_9' 2^{-n}.\end{aligned}$$

By the functional calculus for self-adjoint operators, $\|(1-\Delta_b)^{-\frac{\alpha}{2}} S_s^b\| \leq 1$ for all $s \geq 0$. Thus

$$\begin{aligned}\mathbb{E}\left[u(t,x)^2\right] &\leq 2t\|f_n^x\|_\mu^2 + 2c_9' 2^{-n} \\ &\leq 2r_{\min}^{-d_H} t 2^{d_H n} + 2c_9' 2^{-n}\end{aligned}$$

for all $(t,x) \in [0,T] \times F$ and all integer $n \geq 0$. We assume now that $t > 0$, and our aim is to choose $n \geq 0$ to minimise the above expression. Fixing $t \in (0,T]$, define $g : \mathbb{R} \to [0,\infty)$ by $g(y) = r_{\min}^{-d_H} t 2^{d_H y} + c_9' 2^{-y}$. The function g has a unique stationary point which is a global minimum at

$$y_0 = \frac{1}{(d_H+1)\log 2} \log\left(\frac{r_{\min}^{d_H} c_9'}{d_H t}\right).$$

Since $t \leq T$ we have by the definition of c_9' that $y_0 \geq 0$. Since y_0 is not necessarily an integer we choose $n = \lceil y_0 \rceil$. Then g is increasing in $[y_0, \infty)$ so we have that

$$\mathbb{E}\left[u(t,x)^2\right] \leq 2g(n) \leq 2g(y_0+1).$$

Setting $c_9'' := c_9' \frac{r_{\min}^{d_H}}{d_H}$ and evaluating the right-hand side we see that

$$\begin{aligned}\mathbb{E}\left[u(t,x)^2\right] &\leq 2t r_{\min}^{-d_H} 2^{d_H} \left(\frac{c_9''}{t}\right)^{\frac{d_H}{d_H+1}} + 2c_9' 2^{-1} \left(\frac{c_9''}{t}\right)^{\frac{-1}{d_H+1}} \\ &\leq c_9''' t^{\frac{1}{d_H+1}} \\ &= c_9''' t^{1-\frac{d_s}{2}}\end{aligned}$$

for all $(t,x) \in (0,T] \times F$, where the constant $c_9''' > 0$ is independent of (t,x). This inequality obviously also holds in the case $t = 0$. □

3.5.3 Hölder regularity of solution

We are now ready to prove the Hölder regularity result. It will turn out that the continuous version of $u(t,x)$ can be interpreted as an $\mathcal{H}$-valued process, and is a version of the original $\mathcal{H}$-valued solution to (3.1.1) found in Theorem 3.2.11. Recall R_∞ the natural supremum metric on $\mathbb{R} \times F$ given by

$$R_\infty((s,x),(t,y)) = \max\{|s-t|, R(x,y)\}.$$

Theorem 3.5.6 (Hölder regularity). *Let $u = (u(t,x))_{(t,x)\in[0,\infty)\times F}$ be the family defined in Theorem 3.4.8. Let*

$$\delta_\alpha = \begin{cases} \frac{1}{2}(1-\frac{d_s}{2}), & \alpha \le \frac{d_s}{2}, \\ \frac{1}{2}(1-d_s+\alpha), & \frac{d_s}{2} < \alpha \le d_s, \\ \frac{1}{2}, & \alpha > d_s. \end{cases}$$

Then there exists a version $\tilde{u} = (\tilde{u}(t,x))_{(t,x)\in[0,\infty)\times F}$ of u which satisfies the following:

(1). For each $T > 0$, $\tilde{u}$ is almost surely essentially δ_α-Hölder continuous on $[0,T]\times F$ with respect to R_∞.

(2). For each $t \in [0,\infty)$, $\tilde{u}(t,\cdot)$ is almost surely essentially $\frac{1}{2}$-Hölder continuous on F with respect to R.

Proof. Take $T > 0$ and consider u_T, the restriction of u to $[0,T] \times F$. It is an easily verifiable fact that for every $p \in \mathbb{N}$ there exists a constant $C'_p > 0$ such that if Z is any centred real Gaussian random variable then

$$\mathbb{E}[Z^{2p}] = C'_p \mathbb{E}[Z^2]^p.$$

We also know that u_T is a centred Gaussian process on $[0,T] \times F$ by Theorem 3.4.8.

We will treat the case $\alpha \le \frac{d_s}{2}$, which is precisely the region of values of α for which Proposition 3.5.5 will give us a better temporal Hölder exponent than Proposition 3.5.3. Propositions 3.5.1 and 3.5.5 then give us the estimates

$$\begin{aligned} \mathbb{E}\left[(u_T(t,x)-u_T(t,y))^{2p}\right] &\le C'_p C_2^p R(x,y)^p, \\ \mathbb{E}\left[(u_T(s,x)-u_T(t,x))^{2p}\right] &\le C'_p C_4^p |s-t|^{p(1-\frac{d_s}{2})} \end{aligned} \tag{3.5.1}$$

for all $s,t \in [0,T]$ and all $x,y \in F$. Taking p arbitrarily large and then using Corollary 3.3.19 we get a version $\tilde{u}_T$ of u_T (that is, $\tilde{u}_T$ is a version of u on $[0,T]\times F$) that satisfies the Hölder regularity conditions of the theorem for the given value of T. This works because any two almost surely continuous versions of u_T must coincide almost surely since $[0,T] \times F$ is separable.

If now $T' > T$ and we construct a version $\tilde{u}_{T'}$ of u on $[0,T'] \times F$ in the same way, then $\tilde{u}_{T'}$ must agree with $\tilde{u}_T$ on $[0,T] \times F$ almost surely since both are almost surely continuous on $[0,T] \times F$ which is separable. Therefore let $T = n$ for $n \in \mathbb{N}$ and let Ω' be the almost sure event

$$\Omega' = \bigcap_{n=1}^{\infty} \{\tilde{u}_{n+1} \text{ agrees with } \tilde{u}_n \text{ on } [0,n] \times F\}.$$

Then for $(t,x) \in [0,\infty)\times F$ define $\tilde{u}(t,x) = \tilde{u}_{\lceil t \rceil+1}(t,x)$ on Ω' and $\tilde{u}(t,x) = 0$ otherwise, and we are done.

Now if $\alpha > \frac{d_s}{2}$ then we use the temporal estimate of Proposition 3.5.3 rather than the temporal estimate of Proposition 3.5.5 in (3.5.1). □

Theorem 3.5.7 (Continuous version is version of original solution)**.** *The collection of random variables* $\tilde{u} = (\tilde{u}(t,x))_{(t,x)\in[0,\infty)\times F}$ *constructed in Theorem 3.5.6 is such that* $(\tilde{u}(t,\cdot))_{t\in[0,\infty)}$ *is an* $\mathcal{H}$*-valued process, and moreover* $(\tilde{u}(t,\cdot))_{t\in[0,\infty)}$ *is an* $\mathcal{H}$*-continuous version of the* $\mathcal{H}$*-valued solution to* (3.1.1) *found in Theorem 3.2.11 (with* $u_0 = 0$*).*

Proof. From Theorem 3.5.6, $\tilde{u}$ is almost surely continuous in $[0,\infty) \times F$. Each $\tilde{u}(t,x)$ is a well-defined random variable so by [2, Lemma 4.51], $\tilde{u}$ is jointly measurable. Using continuity again, this implies that $\tilde{u}(t,\cdot) \in \mathcal{H}$ for all $t \in [0,\infty)$ almost surely. We also have that $t \mapsto \tilde{u}(t,\cdot)$ is a continuous function from $[0,\infty)$ to $\mathcal{H}$ for each $\omega \in \Omega$ and that each $\tilde{u}(t,\cdot)$ is a Borel measurable map from Ω to $\mathcal{H}$; the latter follows from the joint continuity of $\tilde{u}$ and the fact that the Borel σ-algebra of $\mathcal{H}$ is generated by the bounded linear functionals on $\mathcal{H}$.

For each $n \geq 1$ and $(t,x) \in [0,\infty) \times F$, define

$$u^{(n)}(t,x) = \sum_{k=1}^{n} (1+\lambda_k^b)^{-\frac{\alpha}{2}} X_t^{b,k} \varphi_k^b(x).$$

Then each $u^{(n)}$ is obviously jointly measurable, and by (3.4.5), $u^{(n)}(t,x) \to \tilde{u}(t,x)$ in $L^2(\mathbb{P})$ for each $(t,x) \in [0,\infty) \times F$. In fact by joint measurability we also have that

$$\begin{aligned}
\mathbb{E}\left[\int_F \left(u^{(n)}(t,x) - \tilde{u}(t,x)\right)^2 \mu(dx)\right] &= \int_F \mathbb{E}\left[\left(u^{(n)}(t,x) - \tilde{u}(t,x)\right)^2\right] \mu(dx) \\
&= \int_F \mathbb{E}\left[\left(\sum_{k=n+1}^{\infty} (1+\lambda_k^b)^{-\frac{\alpha}{2}} X_t^{b,k} \varphi_k^b(x)\right)^2\right] \mu(dx) \\
&= \int_F \sum_{k=n+1}^{\infty} (1+\lambda_k^b)^{-\alpha} \mathbb{E}\left[\left(X_t^{b,k}\right)^2\right] \varphi_k^b(x)^2 \mu(dx)
\end{aligned}$$

where in the last line we have used (3.4.5). Then by Tonelli's theorem,

$$\mathbb{E}\left[\int_F \left(u^{(n)}(t,x) - \tilde{u}(t,x)\right)^2 \mu(dx)\right] = \sum_{k=n+1}^{\infty} (1+\lambda_k^b)^{-\alpha} \mathbb{E}\left[\left(X_t^{b,k}\right)^2\right]$$
$$\to 0$$

as $n \to \infty$, by the fact that $\sum_{k=1}^{\infty} \mathbb{E}[(X_t^{b,k})^2] < \infty$. In particular, this implies that $\langle u^{(n)}(t,\cdot), \varphi_k^b \rangle_\mu \to \langle \tilde{u}(t,\cdot), \varphi_k^b \rangle_\mu$ in $L^2(\mathbb{P})$ as $n \to \infty$, for every $t \in [0,\infty)$ and every $k \geq 1$.

Recall that if $u_0 = 0$ then the solution to (3.1.1) is simply given by the series

$$W_\alpha^b(t) = \sum_{k=1}^{\infty} (1+\lambda_k^b)^{-\frac{\alpha}{2}} X_t^{b,k} \varphi_k^b.$$

It follows that for all $t \in [0,\infty)$ and $n \geq k$ we have that

$$\langle u^{(n)}(t,\cdot), \varphi_k^b \rangle_\mu = (1+\lambda_k^b)^{-\frac{\alpha}{2}} X_t^{b,k} = \langle W_\alpha^b(t), \varphi_k^b \rangle_\mu$$

almost surely. Therefore $\tilde{u}(t,\cdot) = W_\alpha^b(t)$ almost surely for all $t \in [0,\infty)$ and we are done. □

Remark 3.5.8. In [43] it is shown that under some mild conditions on the p.c.f.s.s. set $(F, (\psi_i)_{i=1}^M)$, it can be embedded into Euclidean space in such a way that its resistance metric R is uniformly equivalent to some power of the Euclidean metric. Therefore in this case the conclusion of Theorem 3.5.6 holds with respect to the spatial Euclidean metric, albeit with a different Hölder exponent. An example given in [43, Section 3] is the n-dimensional Sierpinski gasket for $n \geq 2$, see Example 3.1.3(2). This fractal has a natural embedding in $\mathbb{R}^n$, and it is shown that in this case we have a constant $c > 0$ such that

$$c^{-1}|x-y|^{d_w - d_f} \leq R(x,y) \leq c|x-y|^{d_w - d_f}$$

for all $x, y \in F \subseteq \mathbb{R}^n$, where $d_w = \frac{\log(n+3)}{\log 2}$ is the *walk dimension* of the gasket and $d_f = \frac{\log(n+1)}{\log 2}$ is its Euclidean Hausdorff dimension. These fractals all admit function-valued solutions to their respective SHEs but their ambient spaces $\mathbb{R}^n$ do not.

Remark 3.5.9. From Theorem 3.5.6 we see that the operator $(1-\Delta_b)^{-\frac{\alpha}{2}}$ has a smoothing effect on the solution to (3.1.1) as α increases. However the theorem suggests that this does not change the Hölder exponents of the solution until α reaches the value $\frac{d_s}{2}$. On the other hand, recall from Remark 3.5.4 that if $F = [0,1]$ as in Example 3.1.1 then the temporal Hölder exponent of the solution to (3.1.1), viewed as a function of α, is linearly strictly increasing in some neighbourhood of $\alpha = 0$. Intuitively this phenomenon should occur for any F. We therefore conjecture that the Hölder exponent in Theorem 3.5.6(1) is not sharp when $\alpha \in (0, d_s)$, and is in fact equal to the exponent obtained in Theorem 3.2.11 for the solution interpreted as an $\mathcal{H}$-valued process.

We have shown regularity properties of the solution to (3.1.1) in the case $u_0 = 0$, and henceforth we assume that we are dealing with the continuous version of this solution. If we now take an arbitrary initial condition $u_0 = f \in \mathcal{H}$, then obviously the same results may not hold since f may be very rough. We can however prove continuity in *almost* the entire domain $[0, \infty) \times F$.

Theorem 3.5.10. *Let $u : [0, \infty) \times F \to \mathbb{R}$ be the solution to* (3.1.1) *with initial condition $u_0 = f \in \mathcal{H}$. Then u (has a version which) is almost surely continuous in $(0, \infty) \times F$ with respect to R_∞. Moreover, if either*

(1). $b = N$ and f is continuous on F, or

(2). $b = D$ and f is continuous on F with $f|_{F^0} \equiv 0$,

then u (has a version which) is almost surely continuous in $[0, \infty) \times F$ with respect to R_∞.

Proof. Theorem 3.5.6 gives us that the map $(t, x) \mapsto W_\alpha^b(t)(x)$ (has a version which) is almost surely continuous in $[0, \infty) \times F$. We know that if $t > 0$ then S_t^b maps $\mathcal{H}$ into $\mathcal{D}$, and in particular into the space of continuous functions. Thus it makes sense to talk about $S_t^b f(x)$ for $x \in F$. Define

$$u(t, x) := S_t^b f(x) + W_\alpha^b(t)(x),$$

then it suffices to prove continuity of the map $(t, x) \mapsto S_t^b f(x)$. This follows in the same way as the proof of [52, Proposition 5.2.6]. The last two statements in the theorem are immediate corollaries of [52, Proposition 5.2.6]. □

3.6 Invariant measure

We conclude with a brief description of the long-time behaviour of the solutions to (3.1.1). In this section we allow the initial condition u_0 to be an $\mathcal{H}$-valued random variable which is independent of W.

Definition 3.6.1. An *invariant measure* for the SPDE (3.1.1) is a probability measure ν_∞ on $L^2(F,\mu) = \mathcal{H}$ such that if u is the solution to (3.1.1) with random initial condition $u_0 \sim \nu_\infty$ (independent of W) then $u(t) \sim \nu_\infty$ for all $t > 0$.

In the following theorems, let $(Z_k)_{k=1}^\infty$ be a sequence of independent and identically distributed one-dimensional standard Gaussian random variables.

Theorem 3.6.2. *Let $b = D$. Then* (3.1.1) *has a unique invariant measure ν_∞^D and it is given by*

$$\nu_\infty^D = \mathrm{Law}\left(\sum_{k=1}^\infty (1+\lambda_k^D)^{-\frac{\alpha}{2}}(2\lambda_k^D)^{-\frac{1}{2}} Z_k \varphi_k^D\right).$$

If u is a solution to (3.1.1) *in the case $b = D$ then $u(t)$ converges weakly to ν_∞^D as $t \to \infty$ regardless of its initial distribution* $\mathrm{Law}(u_0)$.

Proof. We first show that the definition of ν_∞^D makes sense. We have that

$$\begin{aligned}
\mathbb{E}\left[\sum_{k=1}^\infty \left\|(1+\lambda_k^D)^{-\frac{\alpha}{2}}(2\lambda_k^D)^{-\frac{1}{2}} Z_k \varphi_k^D\right\|_\mu^2\right] &= \mathbb{E}\left[\sum_{k=1}^\infty (1+\lambda_k^D)^{-\alpha}(2\lambda_k^D)^{-1} Z_k^2\right] \\
&\le \sum_{k=1}^\infty (1+\lambda_k^D)^{-\alpha}(2\lambda_k^D)^{-1} \\
&\le \frac{c_1^{-1-\alpha}}{2}\sum_{k=1}^\infty k^{-\frac{2}{d_s}(1+\alpha)} \\
&< \infty,
\end{aligned}$$

so ν_∞^D is indeed a well-defined probability measure on $\mathcal{H}$. Now suppose u has initial distribution $u_0 \sim \nu_\infty^D$. By Definition 3.2.1 and (3.2.1) we have that

$$u(t) = \sum_{k=1}^\infty (1+\lambda_k^D)^{-\frac{\alpha}{2}}\left(e^{-\lambda_k^D t}(2\lambda_k^D)^{-\frac{1}{2}} Z_k + X_t^{D,k}\right)\varphi_k^D,$$

where $(Z_k)_{k=1}^\infty$ and $(X^{D,k})_{k=1}^\infty$ are understood to be independent. Recall that $\lambda_1^D > 0$ by Remark 3.2.9, so for every k, $X^{D,k}$ is a centred Ornstein-Uhlenbeck process with unit volatility and rate parameter λ_k^D. Moreover, $t \mapsto e^{-\lambda_k^D t}(2\lambda_k^D)^{-\frac{1}{2}} Z_k + X_t^{D,k}$ is an Ornstein-Uhlenbeck process with unit volatility, rate parameter λ_k^D and initial distribution given by the law of $(2\lambda_k^D)^{-\frac{1}{2}} Z_k$, which turns out to be exactly its invariant measure (which we leave as an exercise for the reader). Thus the law of $u(t)$ is equal to ν_∞^D for all $t > 0$, so ν_∞^D is an invariant measure. We also see that for all $f \in \mathcal{H}$,

$$\|S_t^D f\|_\mu^2 = \sum_{k=1}^\infty e^{-2\lambda_k^D t} f_k^2 \le e^{-2\lambda_1^D t}\sum_{k=1}^\infty f_k^2 = e^{-2\lambda_1^D t}\|f\|_\mu^2$$

where $f_k = \langle \varphi_k^D, f \rangle_\mu$, so $\lim_{t\to\infty} \|S_t^D\| = 0$. Then the uniqueness and weak convergence results are direct consequences of [15, Theorem 11.20] (or alternatively [26, Proposition 5.23]). □

In the case $b = N$ we do not have nearly as neat a result, but there exists a decomposition of u into two independent processes, one of which has similar invariance properties to the $b = D$ case and the other of which is simply a Brownian motion.

Definition 3.6.3. Let $\mathcal{H}_1 \subseteq \mathcal{H}$ be the space spanned by φ_1^N, which we recall from Remark 3.2.9 to be the constant function $\varphi_1^N \equiv 1$. Let $\mathcal{H}_1^\perp$ be its orthogonal complement. Let $\pi_1 : \mathcal{H} \to \mathcal{H}_1$ be the orthogonal projection onto $\mathcal{H}_1$ and let $\pi_1^\perp : \mathcal{H} \to \mathcal{H}_1^\perp$ be the orthogonal projection onto $\mathcal{H}_1^\perp$.

Let $\star$ denote convolution of measures. For a measure ν on $\mathcal{H}$, let $\pi_1^* \nu$ denote the pushforward of ν with respect to π_1, which is a measure on $\mathcal{H}_1$.

Theorem 3.6.4. *Define the probability measure ν_∞^N on $\mathcal{H}_1^\perp$ by*

$$\nu_\infty^N = \mathrm{Law}\left(\sum_{k=2}^\infty (1+\lambda_k^N)^{-\frac{\alpha}{2}} (2\lambda_k^N)^{-\frac{1}{2}} Z_k \varphi_k^N \right).$$

Let u be a solution to (3.1.1) with $b = N$. Then there exists a one-dimensional standard Wiener process $B = (B(t))_{t\geq 0}$ which is adapted to the filtration generated by W such that B and $u - B$ are independent processes, and if u has initial distribution $u_0 \sim \nu \star \nu_\infty^N$ for some probability measure ν on $\mathcal{H}_1$ then $u(t) - B(t) \sim \nu \star \nu_\infty^N$ for all $t > 0$. Moreover, if u has initial distribution $u_0 \sim \nu_0$ for some probability measure ν_0 on $\mathcal{H}$ then $u(t) - B(t)$ converges weakly to $(\pi_1^* \nu_0) \star \nu_\infty^N$ as $t \to \infty$.

Proof. Recall that $\lambda_1^N = 0$ and that $\lambda_k^N > 0$ for $k \geq 2$. Just as in the $b = D$ case we can prove that ν_∞^N is a well-defined probability measure on $\mathcal{H}_1^\perp$, and indeed on $\mathcal{H}$ as well. We define

$$B(t) := X_t^{N,1} = \int_0^t \varphi_1^{N*} dW(s).$$

From our discussion after (3.4.1) we know that $X^{N,1}$ is a standard one-dimensional Wiener process, and by Remark 3.2.9, $\varphi_1^N \equiv 1$. From the representation (3.4.2) of W_α^b as a sum of independent stochastic processes it is clear that $u - B$ is independent of B.

Now suppose u has initial distribution $u_0 \sim \nu \star \nu_\infty^N$ for some probability measure ν on $\mathcal{H}_1$. The space $\mathcal{H}_1$ is one-dimensional so let Z_0 be a real-valued random variable such that $Z_0 \varphi_1^N$ has law ν. By Definition 4.1.1 and (3.2.1) we can write

$$u(t) = \left(Z_0 + X_t^{N,1} \right) \varphi_1^N + \sum_{k=2}^\infty (1+\lambda_k^N)^{-\frac{\alpha}{2}} \left(e^{-\lambda_k^N t} (2\lambda_k^N)^{-\frac{1}{2}} Z_k + X_t^{N,k} \right) \varphi_k^N,$$

where again Z_0, $(Z_k)_{k=1}^{\infty}$ and $(X^{N,k})_{k=1}^{\infty}$ are understood to be independent. Now we observe that $\pi_1^{\perp} = \mathbb{1}_{(0,\infty)}(-\Delta_N)$ using the functional calculus on self-adjoint operators, and in particular $\pi_1^{\perp}$ commutes with functions of Δ_N, including S_t^N. Then define

$$\pi_1^{\perp} u(t) = u(t) - \left(Z_0 + X_t^{N,1}\right) =: u_1(t),$$

where we have identified scalars $c \in \mathbb{R}$ with their associated constant functions $c\varphi_1^N \in \mathcal{H}$. It is then easily verifiable that $\pi_1^{\perp} W$ is a cylindrical Wiener process on $\mathcal{H}_1^{\perp}$ and that u_1 is the (mild) solution to the SPDE on $\mathcal{H}_1^{\perp}$ given by

$$\begin{aligned} du(t) &= \Delta_N u(t)dt + (1-\Delta_N)^{-\frac{\alpha}{2}} \pi_1^{\perp} dW(t), \\ u(0) &= u_0 \in \mathcal{H}_1^{\perp} \end{aligned} \tag{3.6.1}$$

with $u_0 \sim \nu_\infty^N$. Note that all operators in (3.6.1) commute with $\pi_1^{\perp}$ and so can be identified with their restriction to $\mathcal{H}_1^{\perp}$ for the purposes of the above SPDE. By definition,

$$u_1(t) = \sum_{k=2}^{\infty} (1+\lambda_k^N)^{-\frac{\alpha}{2}} \left(e^{-\lambda_k^N t}(2\lambda_k^N)^{-\frac{1}{2}} Z_k + X_t^{N,k}\right) \varphi_k^N.$$

Now just as in the $b = D$ case, using [15, Theorem 11.20] (or [26, Proposition 5.23]) we find that ν_∞^N is the unique invariant measure for (3.6.1) and that the solution to (3.6.1) converges weakly to ν_∞^N for any initial distribution on $\mathcal{H}_1^{\perp}$. So we have that for all $t > 0$,

$$u(t) - B(t) = Z_0 + u_1(t) \sim \nu \star \nu_\infty^N$$

as required.

We observe that if u has deterministic initial value $u_0 = f \in \mathcal{H}$ then this is equivalent to u_0 being distributed according to the convolution of Dirac measures $\delta_{f_1} \star \delta_{f_2}$ for $f_1 = \pi_1(f) \in \mathcal{H}_1$, $f_2 = \pi_1^{\perp}(f) \in \mathcal{H}_1^{\perp}$. By doing the usual eigenfunction expansion we have that $u(t) = f_1 + X^{N,1} + u_1(t)$ where u_1 is now the solution to (3.6.1) with initial value f_2. Thus $u(t) - B(t) = f_1 + u_1(t)$ which converges weakly to $\delta_{f_1} \star \nu_\infty^N$. Now assume u has an arbitrary initial probability distribution $u_0 \sim \nu_0$ in $\mathcal{H}$. By conditioning first on the value of $\pi_1(u_0) \in \mathcal{H}_1$, then on the value of $\pi_1^{\perp}(u_0) \in \mathcal{H}_1^{\perp}$ and then using the dominated convergence theorem we find that $\mathbb{E}[g(u(t) - B(t))]$ converges to $((\pi_1^* \nu_0) \star \nu_\infty^N)(g)$ for any continuous and bounded function g on $\mathcal{H}$. So we have weak convergence of $u(t) - B(t)$ to $(\pi_1^* \nu_0) \star \nu_\infty^N$. □

Chapter 4

Continuous random field solutions to parabolic SPDEs on p.c.f. fractals

4.1 Introduction

In the study of the analytic properties of fractal sets, there is a large body of work on so called finitely ramified fractals. These are fractals which can be disconnected by the removal of a finite number of points and the Sierpinski gasket is an example. This gives them analytic properties closer to sets in one dimensional Euclidean space than to sets in higher dimensions. Our aim is to investigate a broad class of SPDEs on such finitely ramified sets and for this we will work within the class of p.c.f. self-similar sets as introduced by Kigami, see [52].

Let $(F, (\psi_i)_{i=1}^N)$ be a p.c.f.s.s. set which admits a regular harmonic structure in the sense of [52]. This harmonic structure is associated with a collection of Laplace operators $\{\Delta_b\}_b$ on $L^2(F)$ (with respect to a natural measure on F to be specified later), where b denotes a boundary condition with respect to the space F. Each of these Laplacians has an associated contraction semigroup S^b. We are concerned with stochastic processes $U = (U(t) : t \in [0, T])$ on $L^2(F)$ of the form

$$U(t) = \int_0^t S^b_{t-s}\beta_s ds + \int_0^t S^b_{t-s}\sigma_s dW(s) \tag{4.1.1}$$

for given boundary condition b, where W is a cylindrical Wiener process on $L^2(F)$ and β and σ are processes satisfying minimal regularity conditions for the above integrals to make sense for all $t \in [0, T]$. Processes of this form arise as the mild solutions of parabolic stochastic partial differential equations on $L^2(F)$, for example the stochastic heat equation

$$dU(t) = \Delta_b U(t)dt + dW(t), \quad t \in [0, T] \tag{4.1.2}$$

Original article: [27]

and a version of the parabolic Anderson model

$$\frac{\partial u}{\partial t}(t,x) = \Delta_b u(t,x) + u(t,x)\xi(t,x), \quad (t,x) \in [0,T] \times F, \tag{4.1.3}$$

where ξ is a space-time white noise on F. Note that (4.1.2) is to be interpreted as a stochastic evolution equation on the separable Hilbert space $L^2(F)$ in the sense of da Prato and Zabczyk [15], so its solution is of the form (4.1.1) by definition. On the other hand, (4.1.3) is a parabolic SPDE on F in the sense of Walsh [77]. There is sufficient overlap between these two theories on the space F that even the solution of (4.1.3) can be interpreted as an $L^2(F)$-valued process of the form (4.1.1) – this is explained in detail in Section 4.6. The main question we address is the following: under what conditions does there exist a measurable function $u : \Omega \times [0,T] \times F \to \mathbb{R}$ which is almost surely continuous on $[0,T] \times F$ and such that the process $(u(t,\cdot) : t \in [0,T])$ is a version of U in (4.1.1)?

A well-known result in the vein of continuity of solutions to SPDEs concerns the stochastic heat equation in one Euclidean spatial dimension, see for example [77]. The result is that this equation has a unique mild solution that is essentially $\frac{1}{2}$-Hölder continuous in space and essentially $\frac{1}{4}$-Hölder continuous in time. Here *essentially γ-Hölder continuous* means that the given function is Hölder continuous for every exponent strictly less than γ. This result has since been extended in numerous directions, for example [48], [20], [49], all of which preserve the condition that the underlying space be Euclidean. In Chapter 3 an analogous result is found where the "spatial" Euclidean space $\mathbb{R}$ is replaced with a p.c.f.s.s. set F equipped with a regular harmonic structure which determines a natural metric, called the resistance metric on F. It is shown there that the stochastic heat equation (4.1.2) on F has a unique mild solution, and that this solution is function-valued and (has a version which is) almost surely Hölder continuous in $[0,T] \times F$. We call such a solution a *continuous random field* solution. The aim of the present chapter is to generalise this result; we seek conditions on β and σ such that the process U given by (4.1.1) has a version that is a continuous random field, and moreover we derive the dependence of its Hölder exponents on the properties of β, σ and F. We then apply this result to derive Hölder continuity properties of the solutions of SPDEs on F in the theories of both da Prato–Zabczyk and Walsh. In this way we have a single regularity result that allows us to prove properties of SPDEs in these two different theories.

The main theorem of the present chapter uses a Kolmogorov-type continuity theorem for stochastic processes indexed by $[0,T] \times F$, and extends the result of Chapter 3 in multiple directions; we use general multiplicative drift and diffusion terms and increase the number of boundary conditions that can be considered. Additionally, it can be seen that the unit interval $[0, 1]$ has an interpretation as a p.c.f. fractal and its associated resistance metric is simply the Euclidean metric, so our theorem thus provides an alternative proof for existence of continuous random field versions of SPDEs on $[0, 1]$, one that does not rely on Fourier transforms or on the explicit form of the heat kernel. A significant issue that arises is the fact that the random field version of U that we construct does not, in general, have an explicit expression in terms of known quantities. This contrasts with the case of the stochastic heat equation, the solution of which possesses an explicit representation in terms of the spectrum of Δ_b (Section 3.4). This means that certain nice properties such as joint measurability of the solution do not come cheaply. A special case that we consider is the situation where the diffusion coefficient σ is a time-dependent multiplication operator on $L^2(F)$; in such a situation we find that a somewhat weaker condition on σ is sufficient to obtain the same result. This is the case that corresponds to Walsh-type SPDEs.

Our results cover the version of the parabolic Anderson model over a compact space with time dependent potential as given in (4.1.3). We will investigate this further by considering the question of intermittency – does the solution exhibit tall peaks over infinitely many different scales? This is shown by the exponential growth of the moments in that the function $\lambda(p)$, defined by

$$\lambda(p) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right],$$

should have the property that $p \to \lambda(p)/p$ is strictly increasing on $[2, \infty)$. For a weak form of intermittency it is enough to establish that $\lambda(2) > 0$, [49], and this is what will do for our model in Theorem 4.7.7. We conjecture that, for p.c.f. fractals F, we have $\lambda(p) = \Theta(p^{2+d_H})$ for large p, where d_H is the Hausdorff dimension of F with respect to the resistance metric and can take any value in $[1, \infty)$.

The present chapter is organised in the following way: In Section 4.2 we set up the problem precisely and state the main hypotheses and theorem. In Section 4.3 we construct from the $L^2(F)$-valued process U a candidate collection of random variables $(\tilde{u}(t, x) : (t, x) \in [0, T] \times F)$. Then in Section 4.4 we use the previously mentioned continuity theorem to construct a Hölder continuous version u of $\tilde{u}$, which we identify to be a continuous random field version of U and thus we prove the main theorem. In Sections 4.5 and 4.6 we give applications of our main theorem to two classes of stochastic partial differential equations; these are defined with respect to the SPDE theories developed by da Prato–Zabczyk [15] and Walsh [77] respectively. Finally, in Section 4.7, in the Walsh case, we prove a number of upper and lower bounds on the moments of global solutions under various hypotheses. In particular we establish weak intermittency for the parabolic Anderson model on these fractals.

4.2 Set-up and statement of main result

4.2.1 The fractal

Our set-up is similar to that of Chapter 3. Let $N \geq 2$ be an integer, and let $(F, (\psi_i)_{i=1}^N)$ be a connected compact p.c.f. fractal such that each ψ_i is a strict contraction on F (see [52] for details). Let $I = \{1, \ldots, N\}$, and for $n \geq 0$ let $\mathbb{W}_n = I^n$. Let $\mathbb{W}_* = \bigcup_{n=0}^\infty \mathbb{W}_n$ and let $\mathbb{W} = I^{\mathbb{N}}$. Elements of $\mathbb{W}_*$ are called *words* and the element of the singleton $\mathbb{W}_0$ is known as the *empty word*. For $w = w_1 \ldots w_n \in \mathbb{W}_n$ let $\psi_w = \psi_{w_1} \circ \cdots \circ \psi_{w_n}$ and let $F_w = \psi_w(F)$. There is a canonical continuous surjection $\pi : \mathbb{W} \to F$ given by defining $\pi(w)$ to be the element of the singleton $\bigcap_{m=1}^\infty F_{w_1 \ldots w_m}$ (see [3, Lemma 5.10]). Let P be the post-critical set of $(F, (\psi_i)_{i=1}^N)$ and set $F^0 = \pi(P)$. In view of [52, Proposition 1.3.5(2)], F^0 can be said to be the *boundary* of F.

Let $(A, \mathbf{r})$ be a regular irreducible harmonic structure on F ([52, Section 3.1]). Since it is regular we have that $\mathbf{r} = (r_1, \ldots, r_N)$ with $r_i \in (0, 1)$ for each $i \in I$. Let $r_{\max} = \max_{i \in I} r_i$ and $r_{\min} = \min_{i \in I} r_i$ and for $w \in \mathbb{W}_m$ let $r_w = r_{w_1} \ldots r_{w_m}$. Let $d_H > 0$ be the unique number such that

$$\sum_{i \in I} r_i^{d_H} = 1.$$

Let μ be the self-similar measure on F with weights given by $(r_i^{d_H})_{i \in I}$. Set $\mathcal{H} = L^2(F, \mu)$ with inner product $\langle \cdot, \cdot \rangle_\mu$. Then the harmonic structure $(A, \mathbf{r})$ is associated with a local regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $\mathcal{H}$. This Dirichlet form provides a resistance metric R on F which is compatible with its original topology ([52, Thoerem 3.3.4]), and it can be seen that the constant d_H is in fact the Hausdorff dimension of (F, R) ([52, Theorem 4.2.1]).

Let 2^{F^0} be the power set of F^0. For $b \in 2^{F^0}$ define $\mathcal{D}_b = \{f \in \mathcal{D} : f|_{F^0 \setminus b} = 0\}$. Then $(\mathcal{E}, \mathcal{D}_b)$ is a local regular Dirichlet form on $L^2(F \setminus (F^0 \setminus b), \mu)$ which is naturally associated with a diffusion $X^b = (X_t^b)_{t \geq 0}$, which has semigroup $S^b = (S_t^b)_{t \geq 0}$ and generator Δ_b, see [22]. The latter is known as the *Laplacian*, as $(-\Delta_b)$ is the self-adjoint operator associated with the closed form $(\mathcal{E}, \mathcal{D}_b)$ (see [22, Sections 1.3 and 1.4]). The value of $b \in 2^{F^0}$ represents the boundary condition that we impose on the Laplacian – the operator Δ_b is the Laplacian with Neumann boundary conditions at elements of $b \subseteq F^0$ and Dirichlet boundary conditions at elements of $F^0 \setminus b$. The case $b = F^0$ therefore denotes Neumann boundary conditions and the case $b = \emptyset$ denotes Dirichlet boundary conditions. We may occasionally use the notation $b = N$ instead of $b = F^0$ and likewise we may use $b = D$ instead of $b = \emptyset$.

4.2.2 Preliminaries

We start with a couple of very useful results on the operator Δ_b. Let $\mathcal{L}(\mathcal{H})$ be the Banach space of bounded linear operators on $\mathcal{H}$ with operator norm $\|\cdot\|$. If $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces, let $\mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_2)$ be the Hilbert space of Hilbert-Schmidt operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ with inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_2)}$. If A is a linear operator on $\mathcal{H}$ then we denote the domain of A by $\mathcal{D}(A)$.

Definition 4.2.1. For $\lambda > 0$ let $\mathcal{D}^\lambda$ be the space $\mathcal{D}$ equipped with the inner product

$$\langle \cdot, \cdot \rangle_\lambda := \mathcal{E}(\cdot, \cdot) + \lambda \langle \cdot, \cdot \rangle_\mu.$$

Since $(\mathcal{E}, \mathcal{D})$ is closed, $\mathcal{D}^\lambda$ is a Hilbert space.

Remark 4.2.2. The space $\mathcal{D}$ contains only $\frac{1}{2}$-Hölder continuous functions since by the definition of the resistance metric we have that

$$|f(x) - f(y)|^2 \leq R(x, y)\mathcal{E}(f, f) \tag{4.2.1}$$

for all $f \in \mathcal{D}$ and all $x, y \in F$. We deduce that for each $b \in 2^{F^0}$, $\mathcal{D}_b$ is a subspace of $\mathcal{D}^\lambda$ with finite codimension $|F^0 \setminus b|$. Indeed, $\mathcal{D}_b$ is the intersection of the kernels of the set of evaluation functionals $\{f \mapsto f(x) : x \in F^0 \setminus b\}$, which are linear and continuous and linearly independent on $\mathcal{D}^\lambda$. It follows that $\mathcal{D}_b$ is closed in $\mathcal{D}^\lambda$.

Definition 4.2.3. The unique real $d_H > 0$ such that

$$\sum_{i \in I} r_i^{d_H} = 1$$

is the *Hausdorff dimension* of (F, R), see [52, Theorem 1.5.7].

The *spectral dimension* of (F, R) is given by

$$d_s = \frac{2d_H}{d_H + 1},$$

see [52, Theorem 4.1.5 and Theorem 4.2.1].

Remark 4.2.4. It is possible to show that we must have $d_H \in [1, \infty)$ and thus $d_s \in [1, 2)$, see Remark 3.2.6(2).

Proposition 4.2.5 (Spectral theory)**.** *For $b \in 2^{F^0}$ the following statements hold:*

There exists a complete orthonormal basis $(\varphi_k^b)_{k=1}^\infty$ of $\mathcal{H}$ consisting of eigenfunctions of the operator $-\Delta_b$. The corresponding eigenvalues $(\lambda_k^b)_{k=1}^\infty$ are non-negative and $\lim_{k\to\infty} \lambda_k^b = \infty$. We assume that they are given in ascending order:

$$0 \le \lambda_1^b \le \lambda_2^b \le \cdots .$$

There exist constants $c_0, c_0' > 0$ such that if $k \ge 2$ then

$$c_0 k^{\frac{2}{d_s}} \le \lambda_k^b \le c_0' k^{\frac{2}{d_s}}.$$

Proof. With [52, Lemma 5.1.3] in mind, this is immediate from [52, Proposition A.2.11] and subsequent discussion. □

Remark 4.2.6. We have $k \ge 2$ in the above proposition because of the possibility that $\lambda_1^b = 0$. This occurs if and only if the non-zero constant functions are elements of $\mathcal{D}_b$, if and only if $b = N$. If this is the case, it follows that $\varphi_1^b \equiv 1$ and the eigenvalue 0 has multiplicity 1, so $\lambda_2^b > 0$. See Remark 3.2.9. In the case $b \neq N$ we assume that the constants c_0, c_0' in the proposition above are chosen such that the result in fact holds for $k \ge 1$.

4.2.3 The class of SPDEs

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space that supports a cylindrical Wiener process $W = (W(t))_{t\ge 0}$ on $\mathcal{H}$ (note that we always suppress the dependence of stochastic processes on $\omega \in \Omega$). Let $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ be a filtration satisfying the usual conditions, and let $\mathcal{P}$ be the associated predictable σ-algebra. Let $T > 0$ be fixed. We are concerned with stochastic partial differential equations on $\mathcal{H}$ of the form

$$\begin{aligned} dU(t) &= \Delta_b U(t)dt + \beta_t dt + \sigma_t dW(t) \quad t \in [0,T], \\ U(0) &= 0, \end{aligned} \tag{4.2.2}$$

where $\beta : \Omega \times [0,T] \to \mathcal{H}$ and $\sigma : \Omega \times [0,T] \to \mathcal{L}(\mathcal{H})$ are predictable processes, and $b \in 2^{F^0}$. Our question is not one of existence or uniqueness of U – we simply assume that a predictable $\mathcal{H}$-valued process $U = (U(t))_{t\in[0,T]}$ exists and satisfies (4.2.2). In particular, we assume that U satisfies (4.2.2) in the *mild* sense: for each $t \in [0,T]$,

$$U(t) = \int_0^t S_{t-s}^b \beta_s ds + \int_0^t S_{t-s}^b \sigma_s dW(s) \tag{4.2.3}$$

almost surely. This means that a priori we require β and σ to be at least regular enough such that the two integrals on the right-hand side of the above are well-defined for all $t \in [0,T]$. To be precise, we need the following to hold almost surely for each $t \in [0,T]$:

$$\begin{aligned} \int_0^t \|S^b_{t-s}\beta_s\|_\mu ds < \infty, \\ \int_0^t \|S^b_{t-s}\sigma_s\|^2_{\mathcal{L}_2(\mathcal{H},\mathcal{H})} ds < \infty. \end{aligned} \tag{4.2.4}$$

We will shortly make hypotheses that automatically guarantee (4.2.4), so these do not need to be checked explicitly.

Remark 4.2.7. While (4.2.2) may seem to be an uninteresting equation in that there is no explicit dependence of β and σ on U, it actually encompasses the solutions to a large class of SPDEs. For example, if we take Y to be a mild solution of the SPDE

$$\begin{aligned} dY(t) &= \Delta_b Y(t)dt + f(t,Y(t))dt + g(t,Y(t))dW(t) \quad t \in [0,T], \\ Y(0) &= Y_0 \in \mathcal{H} \end{aligned}$$

where the functions $f : \Omega \times [0,T] \times \mathcal{H} \to \mathcal{H}$ and $g : \Omega \times [0,T] \times \mathcal{H} \to \mathcal{L}(\mathcal{H})$ satisfy appropriate measurability conditions, then setting $\beta_t = f(t,Y(t))$ and $\sigma_t = g(t,Y(t))$ we see that the process U given by $U(t) = Y(t) - S^b_t Y_0$ satisfies (4.2.3).

Definition 4.2.8. A (square-integrable) *random field* (on $[0,T] \times F$) is a function $u : \Omega \times [0,T] \times F \to \mathbb{R}$ such that $u(t,x) : \Omega \to \mathbb{R}$ is a random variable for each $(t,x) \in [0,T] \times F$, and $(u(t,\cdot))_{t\in[0,T]}$ is an $\mathcal{H}$-valued stochastic process. It is a *continuous random field* if u is, in addition, almost surely jointly continuous in $[0,T] \times F$.

Remark 4.2.9. Any continuous random field must be jointly measurable in $\Omega \times [0,T] \times F$ by [2, Lemma 4.51] and the fact that $[0,T] \times F$ is compact.

Definition 4.2.10. Let (M_1,d_1) and (M_2,d_2) be metric spaces and let $\delta \in (0,1]$. A function $f : M_1 \to M_2$ is *essentially δ-Hölder continuous* if for each $\gamma \in (0,\delta)$ there exists $C_\gamma > 0$ such that

$$d_2(f(x),f(y)) \le C_\gamma d_1(x,y)^\gamma$$

for all $x,y \in M_1$.

Define a metric R_∞ on $\mathbb{R} \times F$ by $R_\infty((s,x),(t,y)) = |s-t| \vee R(x,y)$. We now give two separate sets of hypotheses for the behaviour of the processes β and σ and state our main theorem.

Hypothesis 4.2.11. There exist $p > (d_H + 1)^2$ and $K > 0$ such that

$$\mathbb{E}\left[\left(\int_0^T \|\beta_s\|_\mu^2 ds\right)^p\right] \leq K,$$
$$\sup_{s\in[0,T]} \mathbb{E}\left[\|\sigma_s\|^{2p}\right] \leq K.$$

Definition 4.2.12. For a measurable function $f : F \to \mathbb{R}$, let $\mathcal{M}_f$ be the multiplication operator on $\mathcal{H}$ associated with f. That is, $\mathcal{M}_f h$ is the pointwise multiplication $f \cdot h := (x \mapsto f(x)h(x))$ for all $h \in \mathcal{H}$ such that $f \cdot h \in \mathcal{H}$.

Note that for $f : F \to \mathbb{R}$ measurable, $\mathcal{M}_f \in \mathcal{L}(\mathcal{H})$ if and only if

$$\operatorname*{ess\,sup}_{x\in F} |f(x)| := \inf\{c > 0 : \mu(\{x \in F : |f(x)| > c\}) = 0\} < \infty.$$

Additionally if the above inequality holds then

$$\|\mathcal{M}_f\| = \operatorname*{ess\,sup}_{x\in F} |f(x)|,$$

see [11, Theorem II.1.5]. Note also that $\operatorname{ess\,sup}_{x\in F} |f(x)| \leq \sup_{x\in F} |f(x)|$.

Our second set of hypotheses is similar to the first one, but we restrict the space of operators in which the process σ can take values to the set of multiplication operators. This allows us to weaken the integrability condition on σ given in Hypothesis 4.2.11.

Hypothesis 4.2.13. There exists a jointly measurable function $\tilde{\sigma} : \Omega\times[0, T]\times F \to \mathbb{R}$ such that for each $t \in [0, T]$,

$$\sigma_t = \mathcal{M}_{\tilde{\sigma}(t,\cdot)}$$

almost surely. There exist $p > (d_H + 1)^2$ and $K > 0$ such that

$$\mathbb{E}\left[\left(\int_0^T \|\beta_s\|_\mu^2 ds\right)^p\right] \leq K,$$
$$\sup_{(s,x)\in[0,T]\times F} \mathbb{E}\left[|\tilde{\sigma}(s, x)|^{2p}\right] \leq K.$$

The following result shows that we need not check the conditions (4.2.4) if we have assumed either of our above hypotheses:

Proposition 4.2.14. *Assume either Hypothesis 4.2.11 or Hypothesis 4.2.13. Then the conditions* (4.2.4) *are satisfied.*

Proof. Fix $t \in [0, T]$. Assume Hypothesis 4.2.11. We have

$$\int_0^t \|S_{t-s}^b \beta_s\|_\mu ds \leq \int_0^t \|\beta_s\|_\mu ds < \infty$$

almost surely, and

$$\begin{aligned}
\mathbb{E}\left[\int_0^t \|S_{t-s}^b \sigma_s\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^2 ds\right] &\leq \int_0^t \mathbb{E}\left[\|\sigma_s\|^2\right] \|S_{t-s}^b\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^2 ds \\
&= \sup_{s\in[0,T]} \mathbb{E}\left[\|\sigma_s\|^2\right] \int_0^t \sum_{k=1}^\infty \|S_s^b \varphi_k^b\|_\mu^2 ds \\
&= \sup_{s\in[0,T]} \mathbb{E}\left[\|\sigma_s\|^2\right] \sum_{k=1}^\infty \int_0^t e^{-2\lambda_k^b s} ds \\
&< \infty
\end{aligned}$$

where the last inequality follows by Proposition 4.2.5 and the fact that $d_s < 2$.

Now we instead assume Hypothesis 4.2.13. The condition on β is the same as in Hypothesis 4.2.11. Let σ_s^* be the adjoint of σ_s for each $s \in [0, T]$. We note that (bounded) multiplication operators are self-adjoint ([11, Example VI.1.5]), so we have that

$$\begin{aligned}
\mathbb{E}\left[\int_0^t \|S_{t-s}^b \sigma_s\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^2 ds\right] &= \int_0^t \mathbb{E}\left[\|\sigma_s^* S_{t-s}^b\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^2\right] ds \\
&= \int_0^t \mathbb{E}\left[\sum_{k=1}^\infty \|\mathcal{M}_{\tilde{\sigma}(s,\cdot)} S_{t-s}^b \varphi_k^b\|_\mu^2\right] ds \\
&= \sum_{k=1}^\infty \int_0^t e^{-2\lambda_k^b (t-s)} \mathbb{E}\left[\|\mathcal{M}_{\tilde{\sigma}(s,\cdot)} \varphi_k^b\|_\mu^2\right] ds \\
&= \sum_{k=1}^\infty \int_0^t e^{-2\lambda_k^b (t-s)} \int_F \mathbb{E}\left[\tilde{\sigma}(s,x)^2\right] \varphi_k^b(x)^2 \mu(dx) ds \\
&= \sup_{(s,x)\in[0,T]\times F} \mathbb{E}\left[\tilde{\sigma}(s,x)^2\right] \sum_{k=1}^\infty \int_0^t e^{-2\lambda_k^b s} ds \\
&< \infty
\end{aligned}$$

where the last inequality follows by Proposition 4.2.5 and the fact that $d_s < 2$. □

The following is the main result of the present chapter:

Theorem 4.2.15. *Assume either Hypothesis 4.2.11 or Hypothesis 4.2.13. Then the process U given by* (4.2.3) *has a version which is a continuous random field $u : \Omega \times [0, T] \times F \to \mathbb{R}$. In particular:*

(1). The process u is almost surely essentially $\frac{1}{2}\left((d_H+1)^{-1} - \frac{d_H+1}{p}\right)$-Hölder continuous in $[0,T] \times F$ with respect to R_∞,

(2). For each $t \in [0,T]$, $u(t,\cdot)$ is almost surely essentially $\frac{1}{2}\left(1 - \frac{d_H}{p}\right)$-Hölder continuous in F with respect to R,

(3). For each $x \in F$, $u(\cdot,x)$ is almost surely essentially $\frac{1}{2}\left((d_H+1)^{-1} - \frac{1}{p}\right)$-Hölder continuous in $[0,T]$.

Example 4.2.16. We compare this with Theorem 3.5.6, which concerns the stochastic heat equation on a p.c.f. fractal. In this case β and σ are both deterministic and constant, so in Hypothesis 4.2.11 we are able to take $p \to \infty$. In the case $\alpha = 0$, Theorem 3.5.6 and Theorem 4.2.15 then give the same Hölder exponents for the random field solution to the stochastic heat equation (since $(d_H+1)^{-1} = (1 - \frac{d_s}{2})$). However this is not the case for general α, and this is because in the present chapter we do not take into account any potential smoothing properties of the process σ.

4.3 Construction of random field

4.3.1 Resolvent density

We develop and expand some theory from Section 3.4:

Definition 4.3.1. For $\lambda > 0$ and $b \in 2^{F^0}$ let $\rho_\lambda^b : F \times F \to \mathbb{R}$ be the *resolvent density* associated with Δ_b. By [3, Theorem 7.20], ρ_λ^N exists and satisfies the following:

(1). (Reproducing kernel property.) For $x \in F$, $\rho_\lambda^N(x,\cdot)$ is the unique element of $\mathcal{D}_N = \mathcal{D}$ such that

$$\langle \rho_\lambda^N(x,\cdot), f \rangle_\lambda = f(x)$$

for all $f \in \mathcal{D}_N$.

(2). (Resolvent kernel property.) For all continuous $f \in \mathcal{H}$ and all $x \in F$,

$$\int_0^\infty e^{-\lambda t} S_t^N f(x) dt = \int_F \rho_\lambda^N(x,y) f(y) \mu(dy).$$

By a density argument it follows that for all $f \in \mathcal{H}$,

$$\int_0^\infty e^{-\lambda t} S_t^N f dt = \int_F \rho_\lambda^N(\cdot,y) f(y) \mu(dy).$$

(3). ρ_λ^N is symmetric and bounded. We define (for now) $c_\rho(\lambda) > 0$ such that

$$c_\rho(\lambda) \geq \sup_{x,y \in F} |\rho_\lambda^N(x,y)|.$$

(4). (Hölder continuity.) For this same constant $c_\rho(\lambda)$ we have that for all $x, y, y' \in F$,

$$|\rho_\lambda^N(x,y) - \rho_\lambda^N(x,y')|^2 \leq c_\rho(\lambda) R(y,y').$$

Using symmetry this Hölder continuity result holds in the first argument as well.

By an identical argument to [3, Theorem 7.20], ρ_λ^b exists for each $b \in 2^{F^0}$ and satisfies the analogous results with $(\mathcal{E}, \mathcal{D}_b)$ and the semigroup S^b. By the reproducing kernel property it follows that for every $x \in F$, $\rho_\lambda^b(x,\cdot)$ must be the $\mathcal{D}^\lambda$-orthogonal projection of $\rho_\lambda^N(x,\cdot)$ onto $\mathcal{D}_b$. We now choose $c_\rho(\lambda)$ large enough that it does not depend on the value of $b \in 2^{F^0}$ for (3) and (4).

Proposition 4.3.2 (Lipschitz resolvent)**.** *If $\lambda > 0$, $b \in 2^{F^0}$ and $x, x', y \in F$ then*

$$\left|\rho_\lambda^b(x',y) - \rho_\lambda^b(x,y)\right| \leq 2R(x,x').$$

Proof. The cases $b \in \{N, D\}$ are treated in Proposition 3.4.5. All other cases can be proven in the same way as the $b = D$ case, using the Green function with boundary $F^0 \setminus b$. □

We will henceforth be using the resolvent density exclusively in the case $\lambda = 1$, so let $c_\rho = c_\rho(1)$.

4.3.2 A priori estimates

The regularity of the resolvent density leads us to the first a priori estimates on the behaviour of (4.2.3), which make clear the significance of Hypotheses 4.2.11 and 4.2.13:

Proposition 4.3.3. *Let $t \in [0,T]$ and $q \geq 1$.*

(1). For all $b \in 2^{F^0}$ and $h \in \mathcal{H}$ we have that

$$\mathbb{E}\left[\left|\left\langle \int_0^t S_{t-s}^b \beta_s ds, h \right\rangle_\mu\right|^{2q}\right]$$
$$\leq \frac{e^{2qt}}{2^q} \mathbb{E}\left[\left(\int_0^t \|\beta_s\|_\mu^2 ds\right)^q\right] \left(\int_F \int_F \rho_1^b(x,y) h(x) h(y) \mu(dx)\mu(dy)\right)^q.$$

(2). Moreover there exists a constant $C_q > 0$ *depending only on* q *such that if* $\sup_{s\in[0,T]} \mathbb{E}\left[\|\sigma_s\|^{2q}\right] < \infty$ *then for all* $b \in 2^{F^0}$ *and* $h \in \mathcal{H}$ *we have that*

$$\mathbb{E}\left[\left|\left\langle \int_0^t S^b_{t-s}\sigma_s dW(s), h\right\rangle_\mu\right|^{2q}\right]$$
$$\leq C_q e^{2qt} \sup_{s\in[0,t]} \mathbb{E}\left[\|\sigma_s\|^{2q}\right]\left(\int_F\int_F \rho^b_1(x,y)h(x)h(y)\mu(dx)\mu(dy)\right)^q.$$

(3). Finally, for the same constant $C_q > 0$ *the following holds: suppose there exists a jointly measurable function* $\tilde{\sigma} : \Omega \times [0,T]\times F \to \mathbb{R}$ *such that for each* $t \in [0,T]$,

$$\sigma_t = \mathcal{M}_{\tilde{\sigma}(t,\cdot)}$$

almost surely. Then if $\sup_{(s,x)\in[0,T]\times F} \mathbb{E}\left[|\tilde{\sigma}(s,x)|^{2q}\right] < \infty$ *then for all* $b \in 2^{F^0}$ *and* $h \in \mathcal{H}$ *we have that*

$$\mathbb{E}\left[\left|\left\langle \int_0^t S^b_{t-s}\sigma_s dW(s), h\right\rangle_\mu\right|^{2q}\right]$$
$$\leq C_q e^{2qt} \sup_{(s,x)\in[0,t]\times F} \mathbb{E}\left[|\tilde{\sigma}(s,x)|^{2q}\right]\left(\int_F\int_F \rho^b_1(x,y)h(x)h(y)\mu(dx)\mu(dy)\right)^q.$$

Proof. First of all we adapt the proof of Lemma 3.4.6; we see that if $b \in 2^{F^0}$, $t \geq 0$ and $h \in \mathcal{H}$ then

$$\begin{aligned}\int_0^t \|S^b_s h\|^2_\mu ds &\leq e^{2t}\int_0^\infty e^{-2s}\|S^b_s h\|^2_\mu ds\\ &\leq e^{2t}\left\langle \int_0^\infty e^{-2s} S^b_{2s} h ds, h\right\rangle_\mu\\ &= \frac{e^{2t}}{2}\left\langle \int_F \rho^b_1(\cdot,y)h(y)\mu(dy), h\right\rangle_\mu\\ &= \frac{e^{2t}}{2}\int_F\int_F \rho^b_1(x,y)h(x)h(y)\mu(dx)\mu(dy). \qquad (4.3.1)\end{aligned}$$

As a side-effect, this shows that the double integral on the right-hand side is non-negative.

(1). Using Cauchy-Schwarz and Hölder's inequalities we find that

$$\begin{aligned}
\mathbb{E}\left[\left|\left\langle\int_0^t S^b_{t-s}\beta_s ds, h\right\rangle_\mu\right|^{2q}\right] &= \mathbb{E}\left[\left|\int_0^t \left\langle\beta_s, S^b_{t-s}h\right\rangle_\mu ds\right|^{2q}\right] \\
&\le \mathbb{E}\left[\left(\int_0^t \|\beta_s\|_\mu \|S^b_{t-s}h\|_\mu ds\right)^{2q}\right] \\
&\le \mathbb{E}\left[\left(\int_0^t \|\beta_s\|^2_\mu ds\right)^q\right]\left(\int_0^t \|S^b_s h\|^2_\mu ds\right)^q
\end{aligned}$$

and then (4.3.1) implies the first required estimate.

(2). Let h^* be the linear functional on $\mathcal{H}$ given by $f \mapsto \langle f, h\rangle_\mu$. Then

$$\mathbb{E}\left[\left|\left\langle\int_0^t S^b_{t-s}\sigma_s dW(s), h\right\rangle_\mu\right|^{2q}\right] = \mathbb{E}\left[\left|\int_0^t h^* S^b_{t-s}\sigma_s dW(s)\right|^{2q}\right].$$

Let σ_s^* be the adjoint of σ_s for each $s \in [0, T]$. We see that

$$\begin{aligned}
\mathbb{E}\left[\int_0^t \|h^* S^b_{t-s}\sigma_s\|^2_{\mathcal{L}_2(\mathcal{H},\mathbb{R})} ds\right] &= \mathbb{E}\left[\int_0^t \|\sigma_s^* S^b_{t-s}h\|^2_\mu ds\right] \\
&\le \|h\|^2_\mu \int_0^t \mathbb{E}\left[\|\sigma_s\|^2\right] ds \\
&< \infty
\end{aligned}$$

which implies that $t' \mapsto \int_0^{t'} h^* S^b_{t-s}\sigma_s dW(s)$ is a real-valued square-integrable martingale for $t' \in [0, t]$. Then by the Burkholder-Davis-Gundy inequality there exists a constant $C'_q > 0$ depending only on q such that

$$\begin{aligned}
\mathbb{E}\left[\left|\int_0^t h^* S^b_{t-s}\sigma_s dW(s)\right|^{2q}\right]^{\frac{1}{q}} &\le C'_q \mathbb{E}\left[\left(\int_0^t \|h^* S^b_{t-s}\sigma_s\|^2_{\mathcal{L}_2(\mathcal{H},\mathbb{R})} ds\right)^q\right]^{\frac{1}{q}} \\
&= C'_q \mathbb{E}\left[\left(\int_0^t \|\sigma_s^* S^b_{t-s}h\|^2_\mu ds\right)^q\right]^{\frac{1}{q}} \\
&\le C'_q \mathbb{E}\left[\left(\int_0^t \|\sigma_s\|^2 \|S^b_{t-s}h\|^2_\mu ds\right)^q\right]^{\frac{1}{q}} \\
&\le C'_q \int_0^t \mathbb{E}\left[\|\sigma_s\|^{2q}\right]^{\frac{1}{q}} \|S^b_{t-s}h\|^2_\mu ds \\
&\le C'_q \sup_{s\in[0,t]} \mathbb{E}\left[\|\sigma_s\|^{2q}\right]^{\frac{1}{q}} \int_0^t \|S^b_s h\|^2_\mu ds,
\end{aligned}$$

where we have used the Minkowski integral inequality. Take powers of q on both sides, then (4.3.1) implies the second estimate.

(3). Now we assume that there exists a jointly measurable function $\tilde{\sigma} : \Omega \times [0,T] \times F \to \mathbb{R}$ such that for each $t \in [0,T]$,

$$\sigma_t = \mathcal{M}_{\tilde{\sigma}(t,\cdot)}$$

almost surely. We note that (bounded) multiplication operators are self-adjoint ([11, Example VI.1.5]), so we have that

$$\begin{aligned}
\mathbb{E}\left[\int_0^t \|h^* S^b_{t-s}\sigma_s\|^2_{\mathcal{L}_2(\mathcal{H},\mathbb{R})} ds\right] &= \int_0^t \mathbb{E}\left[\left\|\sigma_s^* S^b_{t-s} h\right\|_\mu^2\right] ds \\
&= \int_0^t \mathbb{E}\left[\left\|\mathcal{M}_{\tilde{\sigma}(s,\cdot)} S^b_{t-s} h\right\|_\mu^2\right] ds \\
&= \int_0^t \mathbb{E}\left[\int_F \tilde{\sigma}(s,x)^2 (S^b_{t-s}h)(x)^2 \mu(dx)\right] ds \\
&\leq \sup_{(s,x)\in[0,T]\times F} \mathbb{E}\left[\tilde{\sigma}(s,x)^2\right] \int_0^t \int_F (S^b_s h)(x)^2 \mu(dx) ds \\
&\leq t\|h\|_\mu^2 \sup_{(s,x)\in[0,T]\times F} \mathbb{E}\left[\tilde{\sigma}(s,x)^2\right] \\
&< \infty,
\end{aligned}$$

which as before implies that $t' \mapsto \int_0^{t'} h^* S^b_{t-s}\sigma_s dW(s)$ is a real-valued square-integrable martingale for $t' \in [0,t]$. As in (2) we then have that

$$\mathbb{E}\left[\left|\int_0^t h^* S^b_{t-s}\sigma_s dW(s)\right|^{2q}\right]^{\frac{1}{q}} \leq C'_q \mathbb{E}\left[\left(\int_0^t \|\sigma_s^* S^b_{t-s} h\|_\mu^2 ds\right)^q\right]^{\frac{1}{q}}$$

with the same constant C'_q. We apply the Minkowski integral inequality twice:

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t \|\sigma_s^* S^b_{t-s} h\|_\mu^2 ds\right)^q\right]^{\frac{1}{q}} &\leq \int_0^t \mathbb{E}\left[\left\|\sigma_s^* S^b_{t-s} h\right\|_\mu^{2q}\right]^{\frac{1}{q}} ds \\
&= \int_0^t \mathbb{E}\left[\left\|\mathcal{M}_{\tilde{\sigma}(s,\cdot)} S^b_{t-s} h\right\|_\mu^{2q}\right]^{\frac{1}{q}} ds \\
&= \int_0^t \mathbb{E}\left[\left(\int_F \tilde{\sigma}(s,x)^2 (S^b_{t-s}h)(x)^2 \mu(dx)\right)^q\right]^{\frac{1}{q}} ds \\
&\leq \int_0^t \int_F \mathbb{E}\left[|\tilde{\sigma}(s,x)|^{2q}\right]^{\frac{1}{q}} (S^b_{t-s}h)(x)^2 \mu(dx) ds \\
&\leq \sup_{(s,x)\in[0,t]\times F} \mathbb{E}\left[|\tilde{\sigma}(s,x)|^{2q}\right]^{\frac{1}{q}} \int_0^t \|S^b_s h\|_\mu^2 ds.
\end{aligned}$$

Now we take powers of q on both sides and use (4.3.1) for the required result.

□

4.3.3 Partitions and delta-approximants

In this section we define sequences of functions in $\mathcal{H}$ that approximate the delta functional on F in a controllable way. We then use this sequence to "evaluate" $U(t)$ given in (4.2.3) at points $x \in F$. Much of the material in this section follows the method and results of Section 3.3.1.

Definition 4.3.4. Let $m, n \geq 0$. If $w \in \mathbb{W}_m$ and $v \in \mathbb{W}_n$ then we say that w is a *prefix* of v if $m \leq n$ and $w_i = v_i$ for all $i = 1, \ldots, m$. It is a *proper prefix* if in addition we have that $m < n$. These definitions also make sense if $v \in \mathbb{W}$.

We define a sequence of partitions $(\Lambda_n)_{n=0}^{\infty}$ on $\mathbb{W}_*$ by

$$\Lambda_n = \{w : w = w_1 \ldots w_m \in \mathbb{W}_*,\ r_{w_1 \ldots w_{m-1}} > 2^{-n} \geq r_w\}$$

for $n \geq 1$, and we let $\Lambda_0 = \mathbb{W}_0$, which is also a partition (for the definition of a partition see Definitions 3.3.2 and 3.3.3, or [52, Definitions 1.3.9 and 1.5.6]). These possess the following properties:

Lemma 4.3.5. *For each $n \geq 0$:*

(1). $|\Lambda_n| < \infty$ and $\bigcup_{w \in \Lambda_n} F_w = F$,

(2). If $w \in \Lambda_n$ then there exists a subset $\Lambda' \subseteq \Lambda_{n+1}$ such that $F_w = \bigcup_{v \in \Lambda'} F_v$,

(3). If $w, v \in \Lambda_n$ are distinct then $|F_w \cap F_v| < \infty$,

(4). If $w \in \Lambda_n$ then $r_{\min}^{d_H} 2^{-d_H n} < \mu(F_w) \leq 2^{-d_H n}$.

In addition, if $w \in \mathbb{W}_*$ then there exists some $n \geq 0$ and $\Lambda' \subseteq \Lambda_n$ such that $F_w = \bigcup_{v \in \Lambda'} F_v$.

Proof. (1). Directly from definition of a partition.

(2). By Lemma 3.3.4.

(3). By the definition of a partition combined with [52, Proposition 1.3.5(2)]. If $w, v \in \Lambda_n$ are distinct then $F_w \cap F_v = \psi_w(F^0) \cap \psi_v(F^0)$ which is finite since F is post-critically finite.

(4). By definition of μ and definition of Λ_n.

Finally we prove the last claim. The empty word is in Λ_0 so we can ignore that case. Suppose $w = w_1 \dots w_m \in \mathbb{W}_*$. Pick n such that $r_{w_1 \dots w_{m-1}} > 2^{-n}$, then no element of Λ_n can be a proper prefix of w. By the definition of the map π, for any $x \in F_w$ there exists a $w_x \in \mathbb{W}$ which has w as a prefix such that $\pi(w') = x$. By the definition of a partition there must exist a unique $v_x \in \Lambda_n$ such that v_x is a prefix of w_x. Since v_x is not a proper prefix of w, w must be a prefix of v_x. This implies that $x = \pi(w_x) \in F_{v_x} \subseteq F_w$. Let $\Lambda' = \{v \in \Lambda_n : v = v_x \text{ for some } x \in F_w\}$ be the set of v's that can be obtained in this way. It follows that $F_w = \bigcup_{v \in \Lambda'} F_v$. □

Definition 4.3.6. Let $n \geq 0$ and $w \in \mathbb{W}_n$. For $x \in F$ let

$$D_n^0(x) = \bigcup \{F_w : w \in \Lambda_n,\ F_w \ni x\}$$

be the *n-neighbourhood* of x.

The following is an important structural property of n-neighbourhoods:

Lemma 4.3.7. *There exists a constant $c_2 > 0$ such that if $x, y \in F$ and $y \in D_n^0(x)$ then $R(x, y) \leq c_2 2^{-n}$.*

Proof. Simple corollary of Proposition 3.3.12. □

We now define our family of functions approximating the delta functional, which is identical to those given in Definition 3.4.7.

Definition 4.3.8. For $x \in F$ and $n \geq 0$, define

$$f_n^x = \mu(D_n^0(x))^{-1} \mathbb{1}_{D_n^0(x)}.$$

In particular we note that by Lemma 4.3.5(4), $\|f_n^x\|_\mu^2 = \mu(D_n^0(x))^{-1} \leq r_{\min}^{-d_H} 2^{d_H n}$, and that by Lemma 4.3.7 we have that $\lim_{n \to \infty} \langle f_n^x, g \rangle_\mu = g(x)$ for any continuous $g \in \mathcal{H}$. We are now ready to prove the main result of this section.

Theorem 4.3.9. *Assume either Hypothesis 4.2.11 or Hypothesis 4.2.13. Then for each $t \in [0, T]$ and $x \in F$, the sequence $(\langle U(t), f_n^x \rangle_\mu)_{n \geq 0}$ is Cauchy in $L^{2p}(\Omega)$ as $n \to \infty$. Define $\tilde{u}(t, x)$ to be the $L^{2p}(\Omega)$-limit of $(\langle U(t), f_n^x \rangle_\mu)_{n \geq 0}$, then there exists a constant c_3 such that for all $(t, x) \in [0, T] \times F$ and all $n \geq 0$,*

$$\mathbb{E}\left[|\langle U(t), f_n^x \rangle_\mu - \tilde{u}(t, x)|^{2p} \right] \leq c_3 2^{-np}.$$

Proof. Recall that $p \geq 1$. By (4.2.3) and Proposition 4.3.3, for $m, n \geq 0$ we have that

$$\begin{aligned}
&\mathbb{E}\left[|\langle U(t), f_m^x\rangle_\mu - \langle U(t), f_n^x\rangle_\mu|^{2p}\right] \\
&= \mathbb{E}\left[|\langle U(t), f_m^x - f_n^x\rangle_\mu|^{2p}\right] \\
&\leq 2^{2p-1}\mathbb{E}\left[\left|\left\langle \int_0^t S_{t-s}^b \beta_s ds, f_m^x - f_n^x \right\rangle_\mu\right|^{2p} + \left|\left\langle \int_0^t S_{t-s}^b \sigma_s dW(s), f_m^x - f_n^x \right\rangle_\mu\right|^{2p}\right] \\
&\leq c_3' \left(\int_F \int_F \rho_1^b(z,y)(f_m^x(z) - f_n^x(z))(f_m^x(y) - f_n^x(y))\mu(dz)\mu(dy)\right)^p,
\end{aligned}$$

where $c_3' = e^{2pT}2^{2p-1}(2^{-p} + C_p)K$. By Lemma 4.3.2 and Lemma 4.3.7 and the definition of f_n^x, we see that

$$\int_F \int_F \rho_1^b(z,y)(f_m^x(z) - f_n^x(z))(f_m^x(y) - f_n^x(y))\mu(dz)\mu(dy) \leq 4c_2(2^{-m} + 2^{-n}).$$

If we let $c_3 = 4^p c_2^p c_3'$ then it follows that

$$\mathbb{E}\left[|\langle U(t), f_m^x\rangle_\mu - \langle U(t), f_n^x\rangle_\mu|^{2p}\right] \leq c_3(2^{-m} + 2^{-n})^p,$$

so $(\langle U(t), f_n^x\rangle_\mu)_{n\geq 0}$ is Cauchy in $L^{2p}(\Omega)$ as $n \to \infty$. If we let $\tilde{u}(t,x)$ be the limit, then immediately we have that

$$\mathbb{E}\left[|\langle U(t), f_n^x\rangle_\mu - \tilde{u}(t,x)|^{2p}\right] \leq c_3 2^{-np}.$$

□

The collection $(\tilde{u}(t,x) : t \in [0,T],\ x \in F)$ constructed in Theorem 4.3.9 is the precursor to our candidate random field. Currently this is nothing more than a collection of unrelated real random variables indexed by $[0,T] \times F$ so we cannot show that $(\tilde{u}(t,\cdot))_{t\in[0,T]}$ is a version of U; there are problems of joint measurability which will be resolved in the next section.

4.4 Continuity of random field

In this section we collect the estimates required to use the continuity theorems, Theorem 3.3.17 and Corollary 3.3.19, on the collection $(\tilde{u}(t,x) : t \in [0,T],\ x \in F)$.

4.4.1 Spatial estimate

Proposition 4.4.1. *Assume either Hypothesis 4.2.11 or Hypothesis 4.2.13. It follows that there exists a constant $c_4 > 0$ such that for all $t \in [0,T]$ and $x, y \in F$,*

$$\mathbb{E}\left[|\tilde{u}(t,x) - \tilde{u}(t,y)|^{2p}\right] \leq c_4 R(x,y)^p.$$

Proof. By Theorem 4.3.9, Proposition 4.3.3 and (4.2.3),

$$\begin{aligned}
\mathbb{E}\left[|\tilde{u}(t,x) - \tilde{u}(t,y)|^{2p}\right] &= \lim_{n\to\infty} \mathbb{E}\left[|\langle U(t), f_n^x - f_n^y\rangle_\mu|^{2p}\right] \\
&\leq c' \lim_{n\to\infty} \left(\int_F \int_F \rho_1^b(z_1,z_2)(f_n^x(z_1) - f_n^y(z_1))(f_n^x(z_2) - f_n^y(z_2))\mu(dz_1)\mu(dz_2)\right)^p
\end{aligned}$$

where $c' > 0$ is independent of x, y, t. Since ρ_1^b is jointly Lipschitz (Lemma 4.3.2) and symmetric we see by Lemma 4.3.7 and the definition of the f_n^x that

$$\begin{aligned}
\lim_{n\to\infty} &\int_F \int_F \rho_1^b(z_1,z_2)(f_n^x(z_1) - f_n^y(z_1))(f_n^x(z_2) - f_n^y(z_2))\mu(dz_1)\mu(dz_2) \\
&= \rho_1^b(x,x) - 2\rho_1^b(x,y) + \rho_1^b(y,y) \\
&\leq 4R(x,y).
\end{aligned}$$

The result follows. □

4.4.2 Temporal estimate

The temporal estimate is slightly more complicated to derive. First we prove a lemma:

Lemma 4.4.2. *Let $S = (S_t)_t$ be a contraction semigroup on a separable Hilbert space generated by a self-adjoint operator. If $t_0, t \in (0,\infty)$ then*

$$\|S_{t_0} - S_{t_0+t}\| \leq \frac{t}{t_0 + t}.$$

Proof. Let L be the generator of S. This operator is non-positive and self-adjoint, and $S_t = e^{Lt}$. By the functional calculus for self-adjoint operators we thus have that

$$\|S_{t_0} - S_{t_0+t}\| \leq \sup_{\lambda\in[0,\infty)} \left(e^{-\lambda t_0} - e^{-\lambda(t_0+t)}\right).$$

The function $\lambda \mapsto e^{-\lambda t_0} - e^{-\lambda(t_0+t)}$ on $[0,\infty)$ is smooth, non-negative, bounded, and vanishes at 0 and infinity so we differentiate to find a maximum. We find that

$$\sup_{\lambda\in[0,\infty)} \left(e^{-\lambda t_0} - e^{-\lambda(t_0+t)}\right) = \left(1 + \frac{t}{t_0}\right)^{-\frac{t_0}{t}} \frac{t}{t_0+t}.$$

Now for all $x \in (0,\infty)$ we have that $(1 + \frac{1}{x})^{-x} \leq 1$ so the result follows. □

Proposition 4.4.3. *Assume either Hypothesis 4.2.11 or Hypothesis 4.2.13. It follows that there exists a constant $c_5 > 0$ such that for all $s, t \in [0,T]$ and $x \in F$,*

$$\mathbb{E}\left[|\tilde{u}(s,x) - \tilde{u}(t,x)|^{2p}\right] \leq c_5 |s-t|^{p(d_H+1)^{-1}}.$$

Proof. Fix $x \in F$ and $s \in [0,T]$, $t > 0$ such that $s + t \in [0,T]$. Then by Theorem 4.3.9, for all $n \geq 0$ we have that

$$\mathbb{E}\left[|\tilde{u}(s+t,x) - \tilde{u}(s,x)|^{2p}\right] \leq 2^{2p-1}\mathbb{E}\left[|\langle U(s+t) - U(s), f_n^x\rangle_\mu|^{2p}\right] + c_3 2^{4p-1} \cdot 2^{-np}. \tag{4.4.1}$$

Now if we let f_n^{x*} be the bounded linear functional on $\mathcal{H}$ given by $h \mapsto \langle h, f_n^x\rangle_\mu$, then by (4.2.3),

$$\begin{aligned}\langle U(s+t) - U(s), f_n^x\rangle_\mu = &\int_0^{s+t} f_n^{x*}\left(S^b_{s+t-t'} - \mathbb{1}_{[0,s]}(t')S^b_{s-t'}\right)\beta_{t'}dt' \\ &+ \int_0^{s+t} f_n^{x*}\left(S^b_{s+t-t'} - \mathbb{1}_{[0,s]}(t')S^b_{s-t'}\right)\sigma_{t'}dW(t').\end{aligned}$$

We deal with these two terms separately. For the β term, both Hypothesis 4.2.11 and Hypothesis 4.2.13 allow for the same calculation: by the self-adjointness of S^b and Hölder's inequality,

$$\begin{aligned}&\mathbb{E}\left[\left|\int_0^{s+t} f_n^{x*}\left(S^b_{s+t-t'} - \mathbb{1}_{[0,s]}(t')S^b_{s-t'}\right)\beta_{t'}dt'\right|^{2p}\right] \\ &= \mathbb{E}\left[\left(\int_0^{s+t}\left|\left\langle \beta_{t'}, \left(S^b_{s+t-t'} - \mathbb{1}_{[0,s]}(t')S^b_{s-t'}\right)f_n^x\right\rangle_\mu\right| dt'\right)^{2p}\right] \\ &\leq \mathbb{E}\left[\left(\int_0^{s+t}\|\beta_{t'}\|_\mu \left\|\left(S^b_{s+t-t'} - \mathbb{1}_{[0,s]}(t')S^b_{s-t'}\right)f_n^x\right\|_\mu dt'\right)^{2p}\right] \\ &\leq \mathbb{E}\left[\left(\int_0^{s+t}\|\beta_{t'}\|_\mu^2 dt'\right)^p\right]\left(\int_0^{s+t}\left\|\left(S^b_{s+t-t'} - \mathbb{1}_{[0,s]}(t')S^b_{s-t'}\right)f_n^x\right\|_\mu^2 dt'\right)^p \\ &\leq \|f_n^x\|_\mu^{2p}\mathbb{E}\left[\left(\int_0^T \|\beta_{t'}\|_\mu^2 dt'\right)^p\right]\left(\int_0^{s+t}\left\|S^b_{s+t-t'} - \mathbb{1}_{[0,s]}(t')S^b_{s-t'}\right\|^2 dt'\right)^p\end{aligned}$$

and then using Lemma 4.4.2,

$$\begin{aligned}&\int_0^{s+t}\left\|S^b_{s+t-t'} - \mathbb{1}_{[0,s]}(t')S^b_{s-t'}\right\|^2 dt' \\ &\qquad = \int_0^s \left\|S^b_{s+t-t'} - S^b_{s-t'}\right\|^2 dt' + \int_s^{s+t}\left\|S^b_{s+t-t'}\right\|^2 dt' \\ &\qquad = \int_0^s \left\|S^b_{t+t'} - S^b_{t'}\right\|^2 dt' + \int_0^t \left\|S^b_{t'}\right\|^2 dt' \\ &\qquad \leq \int_0^s \frac{t^2}{(t'+t)^2}dt' + \int_0^t 1dt' \\ &\qquad \leq 2t.\end{aligned} \tag{4.4.2}$$

It follows that

$$\mathbb{E}\left[\left|\int_0^{s+t} f_n^{x*}\left(S_{s+t-t'}^b - \mathbb{1}_{[0,s]}(t')S_{s-t'}^b\right)\beta_{t'}dt'\right|^{2p}\right] \le 2^p K \|f_n^x\|_\mu^{2p} t^p.$$

For the σ term, we first assume Hypothesis 4.2.11. We use the Burkholder-Davis-Gundy inequality (the validity of which, given Hypothesis 4.2.11, is easy to verify; see proof of Proposition 4.3.3(2)) and see that

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^{s+t} f_n^{x*}\left(S_{s+t-t'}^b - \mathbb{1}_{[0,s]}(t')S_{s-t'}^b\right)\sigma_{t'}dW(t')\right|^{2p}\right]^{\frac{1}{p}}\\
&\le C_p'\mathbb{E}\left[\left(\int_0^{s+t}\left\|f_n^{x*}\left(S_{s+t-t'}^b - \mathbb{1}_{[0,s]}(t')S_{s-t'}^b\right)\sigma_{t'}\right\|_{\mathcal{L}_2(\mathcal{H},\mathbb{R})}^2 dt'\right)^p\right]^{\frac{1}{p}}\\
&= C_p'\mathbb{E}\left[\left(\int_0^{s+t}\left\|\sigma_{t'}^*\left(S_{s+t-t'}^b - \mathbb{1}_{[0,s]}(t')S_{s-t'}^b\right)f_n^x\right\|_\mu^2 dt'\right)^p\right]^{\frac{1}{p}}\\
&\le C_p'\int_0^{s+t}\mathbb{E}\left[\|\sigma_{t'}^*\|^{2p}\right]^{\frac{1}{p}}\left\|\left(S_{s+t-t'}^b - \mathbb{1}_{[0,s]}(t')S_{s-t'}^b\right)f_n^x\right\|_\mu^2 dt'\\
&\le C_p'\sup_{t'\in[0,T]}\mathbb{E}\left[\|\sigma_{t'}\|^{2p}\right]^{\frac{1}{p}}\int_0^{s+t}\left\|\left(S_{s+t-t'}^b - \mathbb{1}_{[0,s]}(t')S_{s-t'}^b\right)f_n^x\right\|_\mu^2 dt'\\
&\le C_p'K^{\frac{1}{p}}\|f_n^x\|_\mu^2\int_0^{s+t}\left\|S_{s+t-t'}^b - \mathbb{1}_{[0,s]}(t')S_{s-t'}^b\right\|^2 dt'\\
&\le 2C_p'K^{\frac{1}{p}}\|f_n^x\|_\mu^2 t
\end{aligned}$$

where we have used the Minkowski integral inequality and the calculation (4.4.2) a second time. If we instead assume Hypothesis 4.2.13 then we get the same estimate by a similar calculation, but use the Minkowski integral inequality twice; see the proof of Proposition 4.3.3. Plugging these values into (4.4.1) we get

$$\mathbb{E}\left[|\tilde{u}(s+t,x)-\tilde{u}(s,x)|^{2p}\right] \le 2^{5p-2}(1+(C_p')^p)K\|f_n^x\|_\mu^{2p}t^p + c_3 2^{4p-1}\cdot 2^{-np}.$$

We now use the fact that $\|f_n^x\|_\mu^2 = \mu(D_n^0(x))^{-1} \le r_{\min}^{-d_H}2^{d_H n}$. If we let $c_1' = 2^{5p-2}(1+(C_p')^{2p})Kr_{\min}^{-pd_H}$ and $c_2' = c_3 2^{4p-1}$ we have that

$$\mathbb{E}\left[|\tilde{u}(s+t,x)-\tilde{u}(s,x)|^{2p}\right] \le c_1' 2^{d_H np}t^p + c_2' 2^{-np}.$$

The final step is to minimize the right-hand side of the above equation over $n \ge 0$, which is similar to the method used to prove Proposition 3.5.5. It is in fact easier to let $c_2'' = c_2' \vee d_H c_1' T^p$ and then minimize $c_1' 2^{d_H np}t^p + c_2'' 2^{-np}$ over $n \ge 0$. To this end, let $f(y) = c_1' 2^{d_H y}t^p + c_2'' 2^{-y}$ be a function on $\mathbb{R}$, then the unique minimum value of f occurs at

$$y_0 := \frac{1}{(d_H+1)\log 2}\log\left(\frac{c_2''}{d_H c_1' t^p}\right) \ge 0.$$

If we set $n = \lceil y_0 p^{-1} \rceil$ then since f is increasing in $[y_0, \infty)$ we get

$$\begin{aligned}
\mathbb{E}\left[|\tilde{u}(s+t,x) - \tilde{u}(s,x)|^{2p}\right] &\leq f(np) \\
&\leq f(y_0 + p) \\
&= c_1' 2^{d_H p} \left(\frac{c_2''}{d_H c_1' t^p}\right)^{\frac{d_H}{d_H+1}} t^p + c_2'' 2^{-p} \left(\frac{c_2''}{d_H c_1' t^p}\right)^{-\frac{1}{d_H+1}} \\
&=: c_1'' t^{\frac{p}{d_H+1}}
\end{aligned}$$

where $c_1'' > 0$ is a constant that does not depend on s, t, x. □

4.4.3 Proof of main theorem

Proof of Theorem 4.2.15. The spatial and temporal estimates in Proposition 4.4.1 and Proposition 4.4.3 respectively combined with the continuity theorem Corollary 3.3.19 together imply that there exists a version $u : \Omega \times [0,T] \times F \to \mathbb{R}$ of the collection $(\tilde{u}(t,x) : t \in [0,T],\ x \in F)$ satisfying the required Hölder continuity properties. We have that $u(t,x)$ is a random variable satisfying $\tilde{u}(t,x) = u(t,x)$ almost surely for each $(t,x) \in [0,T] \times F$. It only remains to show that $(u(t,\cdot))_{t \in [0,T]}$ is a version of U.

We need to show that for each $t \in [0,T]$, $u(t,\cdot) = U(t)$ as elements of $\mathcal{H}$ almost surely. Fix $t \in [0,T]$. Since $u(t,\cdot)$ is almost surely continuous on the compact metric space (F,R) it is a Carathéodory function, so by [2, Lemma 4.51] it must be jointly measurable on $\Omega \times F$. Additionally the compactness of F implies that $u(t,\cdot)$ is almost surely bounded, so we have that $u(t,\cdot) \in \mathcal{H}$ almost surely. Since $\mathcal{D}$ is dense in $\mathcal{H}$, it will suffice to show that $\langle u(t,\cdot), h\rangle_\mu = \langle U(t), h\rangle_\mu$ for all $h \in \mathcal{D}$ almost surely. Furthermore, every element of $\mathcal{D}$ is continuous, and thus can be approximated arbitrarily closely in $\mathcal{H}$ by finite linear combinations of functions of the form $\mathbb{1}_{F_w}$, $w \in \mathbb{W}_*$. Since $\mathbb{W}_*$ is countable, we therefore need only to show that $\langle u(t,\cdot), \mathbb{1}_{F_w}\rangle_\mu = \langle U(t), \mathbb{1}_{F_w}\rangle_\mu$ almost surely for each $w \in \mathbb{W}_*$. To this end, we fix $w \in \mathbb{W}_*$. By the final part of Lemma 4.3.5 and then repeated application of Lemma 4.3.5(2), there exists $n_w \geq 0$ such that for each $n \geq n_w$, there exists a subset $\Lambda_n' \subseteq \Lambda_n$ such that

$$\bigcup_{v \in \Lambda_n'} F_v = F_w.$$

By Lemma 4.3.5(3), for each $v \in \Lambda_n'$ we have that $D_n^0(x) = F_v$ for all but finitely many $x \in F_v$, so the map $x \mapsto f_n^x$ from F_w to $\mathcal{H}$ is measurable. Thus $(\omega, x) \mapsto \langle U(t), f_n^x\rangle_\mu \mathbb{1}_{F_w}(x)$ is jointly measurable. Again by Lemma 4.3.5(3), for each $n \geq n_w$

the set of $y \in F$ such that $y \in F_v \cap F_{v'}$ for two distinct $v, v' \in \Lambda'_n$ is finite. It follows that

$$\int_{F_w} \langle U(t), f_n^x \rangle_\mu \mu(dx) = \sum_{v \in \Lambda'_n} \int_{F_v} \langle U(t), f_n^x \rangle_\mu \mu(dx).$$

Thus we have that

$$\begin{aligned}
\sum_{v \in \Lambda'_n} \int_{F_v} \langle U(t), f_n^x \rangle_\mu \mu(dx) &= \sum_{v \in \Lambda'_n} \mu(F_v)^{-1} \int_{F_v} \langle U(t), \mathbb{1}_{F_v} \rangle_\mu \mu(dx) \\
&= \sum_{v \in \Lambda'_n} \langle U(t), \mathbb{1}_{F_v} \rangle_\mu \\
&= \sum_{v \in \Lambda'_n} \int_{F_v} U(t)(y) \mu(dy) \\
&= \int_{F_w} U(t)(y) \mu(dy).
\end{aligned}$$

We conclude that

$$\int_{F_w} \langle U(t), f_n^x \rangle_\mu \mu(dx) = \langle U(t), \mathbb{1}_{F_w} \rangle_\mu$$

for all $n \geq n_w$. Finally from Jensen's inequality, Tonelli's theorem and Theorem 4.3.9 we have that

$$\begin{aligned}
&\mathbb{E}\left[\left| \int_{F_w} \langle U(t), f_n^x \rangle_\mu \mu(dx) - \langle u(t, \cdot), \mathbb{1}_{F_w} \rangle_\mu \right|^{2p} \right] \\
&\leq \mu(F_w)^{2p-1} \int_{F_w} \mathbb{E}\left[\left| \langle U(t), f_n^x \rangle_\mu - u(t, x) \right|^{2p} \right] \mu(dx) \\
&\leq c_3 \mu(F_w)^{2p} 2^{-np}
\end{aligned}$$

for $n \geq n_w$. Then using Markov's inequality, for every $\varepsilon > 0$ we have the bound

$$\mathbb{P}\left[\left| \int_{F_w} \langle U(t), f_n^x \rangle_\mu \mu(dx) - \langle u(t, \cdot), \mathbb{1}_{F_w} \rangle_\mu \right| \geq \varepsilon \right] \leq c_3 \mu(F_w)^{2p} \varepsilon^{-2p} 2^{-np}$$

for sufficiently large n. Therefore by the Borel-Cantelli lemma,

$$\int_{F_w} \langle U(t), f_n^x \rangle_\mu \mu(dx) \to \langle u(t, \cdot), \mathbb{1}_{F_w} \rangle_\mu$$

as $n \to \infty$ almost surely. So $\langle u(t, \cdot), \mathbb{1}_{F_w} \rangle_\mu = \langle U(t), \mathbb{1}_{F_w} \rangle_\mu$ almost surely and the proof is complete. □

4.5 Application to a class of da Prato–Zabczyk SPDEs

In this section we give an example of how to apply Theorem 4.2.15 to the solutions of a class of SPDEs on p.c.f. fractals of the form

$$\begin{aligned} dY(t) &= \Delta_b Y(t)dt + f(t, Y(t))dt + g(t, Y(t))dW(t), \quad t \in [0, T], \\ Y(0) &= Y_0 \end{aligned} \tag{4.5.1}$$

with $T > 0$ and $b \in 2^{F^0}$. We recall that a *mild solution* of the SPDE (4.5.1) is a predictable $\mathcal{H}$-valued process $Y = (Y_t)_{t\in[0,T]}$ satisfying

$$Y(t) = S_t^b Y_0 + \int_0^t S_{t-s}^b f(s, Y(s))ds + \int_0^t S_{t-s}^b g(s, Y(s))dW(s)$$

almost surely for each $t \in [0, T]$. For a topological space $\mathcal{S}$, let $\mathcal{B}(\mathcal{S})$ be the Borel σ-algebra on $\mathcal{S}$. Let $\mathcal{P}_T$ be the predictable σ-algebra on $\Omega \times [0, T]$ associated with the truncated filtration $(\mathcal{F}_t : 0 \le t \le T)$. The following hypothesis is adapted from [15, Hypothesis 7.2]:

Hypothesis 4.5.1. We make the following assumptions for the SPDE (4.5.1):

(1). Y_0 is an $\mathcal{H}$-valued $\mathcal{F}_0$-measurable random variable such that there exists $q \ge 2$ with $\mathbb{E}[\|Y_0\|_\mu^q] < \infty$.

(2). The function $f : \Omega \times [0, T] \times \mathcal{H} \to \mathcal{H}$ is measurable from $\mathcal{P}_T \otimes \mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$.

(3). The function $g : \Omega \times [0, T] \times \mathcal{H} \to \mathcal{L}(\mathcal{H})$ is measurable from $\mathcal{P}_T \otimes \mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{L}(\mathcal{H}))$.

(4). There exists a constant $C > 0$ such that for all $h, h' \in \mathcal{H}$, $\omega \in \Omega$ and $t \in [0, T]$ we have that

$$\|f(\omega, t, h)\|_\mu + \|g(\omega, t, h)\| \le C(1 + \|h\|_\mu)$$

and

$$\|f(\omega, t, h) - f(\omega, t, h')\|_\mu + \|g(\omega, t, h) - g(\omega, t, h')\| \le C\|h - h'\|_\mu.$$

Definition 4.5.2. As in the proof of [15, Theorem 7.2], let $\mathscr{H}_q$ be the space of $\mathcal{H}$-valued predictable processes $Z = (Z(t))_{t\in[0,T]}$ such that

$$|||Z|||_q := \sup_{t\in[0,T]} \mathbb{E}\left[\|Z(t)\|_\mu^q\right]^{\frac{1}{q}} < \infty.$$

We equip $\mathscr{H}_q$ with the norm $|||\cdot|||_q$, which makes it a Banach space. Define a stochastic-process-valued function $\mathscr{K}$ on $\mathscr{H}_q$ by

$$\mathscr{K}(Z)(t) = S_t^b Y_0 + \int_0^t S_{t-s}^b f(s, Z(s))ds + \int_0^t S_{t-s}^b g(s, Z(s))dW(s).$$

Lemma 4.5.3. *Assume Hypothesis 4.5.1. Then there exist constants $\alpha_1, \alpha_2 > 0$ dependent only on T such that if $Z \in \mathscr{H}_q$ then*

$$|||\mathscr{K}(Z)|||_q \leq \mathbb{E}[\|Y_0\|_\mu^q]^{\frac{1}{q}} + \alpha_1 + \alpha_2 |||Z|||_q.$$

In particular, the function $\mathscr{K}$ maps $\mathscr{H}_q$ into itself.

Proof. Fix $Z \in \mathscr{H}_q$. For $t \in [0, T]$ we see that

$$\begin{aligned}
\mathbb{E}\left[\left\|\int_0^t S_{t-s}^b f(s, Z(s))ds\right\|_\mu^q\right] &\leq \mathbb{E}\left[\left|\int_0^t \|f(s, Z(s))\|_\mu ds\right|^q\right] \\
&\leq t^{q-1}\mathbb{E}\left[\int_0^t \|f(s, Z(s))\|_\mu^q ds\right] \\
&\leq t^{q-1}C^q\mathbb{E}\left[\int_0^t (1 + \|Z(s)\|_\mu)^q ds\right] \\
&\leq 2^{q-1}t^q C^q(1 + |||Z|||_q^q)
\end{aligned}$$

and by [15, Theorem 4.37] there exists a constant $k_q > 0$ such that

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_0^t S_{t-s}^b g(s, Z(s))dW(s)\right\|_\mu^q\right] \\
&\leq k_q\left(\int_0^t \mathbb{E}\left[\|S_{t-s}^b g(s, Z(s))\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^q\right]^{\frac{2}{q}} ds\right)^{\frac{q}{2}} \\
&\leq k_q\left(\int_0^t \mathbb{E}\left[\|g(s, Z(s))\|^q\right]^{\frac{2}{q}} \|S_{t-s}^b\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^2 ds\right)^{\frac{q}{2}} \\
&\leq k_q \sup_{s\in[0,t]} \mathbb{E}\left[\|g(s, Z(s))\|^q\right]\left(\int_0^t \|S_s^b\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^2 ds\right)^{\frac{q}{2}} \\
&\leq 2^{q-1}C^q k_q(1 + |||Z|||_q^q)\left(\int_0^t \|S_s^b\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^2 ds\right)^{\frac{q}{2}} \\
&< \infty.
\end{aligned}$$

The integral of $\|S_s^b\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^2$ is finite by Proposition 4.2.5 and the fact that $d_s < 2$. Finally we note that $\|S_t^b Y_0\|_\mu \leq \|Y_0\|_\mu < \infty$ and we put together all of these estimates to see that there exist constants $\alpha_1, \alpha_2 > 0$ dependent only on T such that

$$|||\mathscr{K}(Z)|||_q \leq \mathbb{E}[\|Y_0\|_\mu^q]^{\frac{1}{q}} + \alpha_1 + \alpha_2 |||Z|||_q.$$

It remains to show that $\mathscr{K}(Z)$ is predictable (that is to say, it has a predictable version). The term $S_t^b Y_0$ causes no trouble as it is $\mathcal{F}_0$-measurable. By [15, Proposition 3.7(ii)], it is enough to show that the convolutions $t \mapsto \int_0^t S_{t-s}^b f(s, Z(s))ds$ and $t \mapsto \int_0^t S_{t-s}^b g(s, Z(s))dW(s)$ are stochastically continuous. These follow in a similar way to the proof of [15, Theorem 5.2(i)]. We treat the g integral first as it is the more difficult of the two and is instructive. We in fact prove L^2 stochastic continuity. For $0 \le t' < t \le T$ we have by Itō's isometry:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t S_{t-s}^b g(s, Z(s))dW(s) - \int_0^{t'} S_{t'-s}^b g(s, Z(s))dW(s)\right)^2\right] \\
&= \mathbb{E}\left[\int_0^t \left\|\left(S_{t-s}^b - \mathbb{1}_{[0,t']}(s) S_{t'-s}^b\right) g(s, Z(s))\right\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^2 ds\right] \\
&\le \int_0^t \left\|S_{t-s}^b - \mathbb{1}_{[0,t']}(s) S_{t'-s}^b\right\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^2 \mathbb{E}\left[\|g(s, Z(s))\|^2\right] ds \\
&\le 2(1 + \|\!|Z|\!\|_2^2) \int_0^t \left\|S_{t-s}^b - \mathbb{1}_{[0,t']}(s) S_{t'-s}^b\right\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^2 ds,
\end{aligned}$$

and then

$$\begin{aligned}
&\int_0^t \left\|S_{t-s}^b - \mathbb{1}_{[0,t']}(s) S_{t'-s}^b\right\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^2 ds \\
&= \int_0^{t'} \left\|S_{t-s}^b - S_{t'-s}^b\right\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^2 ds + \int_{t'}^t \left\|S_{t-s}^b\right\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^2 ds \\
&= \int_0^{t'} \left\|S_{s+t-t'}^b - S_s^b\right\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^2 ds + \int_0^{t-t'} \left\|S_s^b\right\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}^2 ds \\
&\le \sum_{k=1}^\infty \int_0^T e^{-2\lambda_k^b s} \left(e^{-\lambda_k^b (t-t')} - 1\right)^2 ds + \sum_{k=1}^\infty \int_0^{t-t'} e^{-2\lambda_k^b s} ds
\end{aligned}$$

which tends to 0 as $t - t' \searrow 0$ by the dominated convergence theorem and Proposition 4.2.5. Thus we have proven L^2 stochastic continuity for the g integral. Now L^2 stochastic continuity for the f integral follows very similarly; we may even use the fact that $\|\mathcal{A}\| \le \|\mathcal{A}\|_{\mathcal{L}^2(\mathcal{H},\mathcal{H})}$ for all $\mathcal{A} \in \mathcal{L}^2(\mathcal{H},\mathcal{H})$ to end up doing exactly the same calculation as the one above. □

Theorem 4.5.4. *Assume Hypothesis 4.5.1. Then the SPDE* (4.5.1) *has a unique mild solution Y in $\mathscr{H}_q$.*

Let $U(t) = Y(t) - S_t^b Y_0$ for $t \in [0, T]$. If $q > 2(d_H + 1)^2$ then U has a version which is a continuous random field $u : \Omega \times [0, T] \times F \to \mathbb{R}$ and has the following Hölder exponents:

(1). The process u is almost surely essentially $\left(\frac{1}{2}(d_H+1)^{-1} - \frac{d_H+1}{q}\right)$*-Hölder continuous in* $[0,T]\times F$ *with respect to* R_∞*,*

(2). For each $t\in[0,T]$*,* $u(t,\cdot)$ *is almost surely essentially* $\left(\frac{1}{2} - \frac{d_H}{q}\right)$*-Hölder continuous in* F *with respect to* R*,*

(3). For each $x\in F$*,* $u(\cdot,x)$ *is almost surely essentially* $\left(\frac{1}{2}(d_H+1)^{-1} - \frac{1}{q}\right)$*-Hölder continuous in* $[0,T]$*.*

Proof. For $Z_1, Z_2 \in \mathscr{H}_q$, we see that

$$\begin{aligned}&\mathscr{K}(Z_1)(t) - \mathscr{K}(Z_2)(t)\\ &= \int_0^t S^b_{t-s}\left(f(s,Z_1(s)) - f(s,Z_2(s))\right)ds + \int_0^t S^b_{t-s}\left(g(s,Z_1(s)) - g(s,Z_2(s))\right)dW(s).\end{aligned}$$

We proceed in a similar way as in the proof of Lemma 4.5.3:

$$\begin{aligned}&\mathbb{E}\left[\left\|\int_0^t S^b_{t-s}\left(f(s,Z_1(s)) - f(s,Z_2(s))\right)ds\right\|_\mu^q\right]\\ &\le t^{q-1}\mathbb{E}\left[\int_0^t \|f(s,Z_1(s)) - f(s,Z_2(s))\|_\mu^q ds\right]\\ &\le 2^{q-1}t^qC^q|||Z_1 - Z_2|||_q^q\end{aligned}$$

and by [15, Theorem 4.37] there exists a constant $k_q > 0$ such that

$$\begin{aligned}&\mathbb{E}\left[\left\|\int_0^t S^b_{t-s}\left(g(s,Z_1(s)) - g(s,Z_2(s))\right)dW(s)\right\|_\mu^q\right]\\ &\le k_q\left(\int_0^t \mathbb{E}\left[\|S^b_{t-s}\left(g(s,Z_1(s)) - g(s,Z_2(s))\right)\|^q_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}\right]^{\frac{2}{q}} ds\right)^{\frac{q}{2}}\\ &\le k_qC^q\left(\int_0^t \mathbb{E}\left[\|Z_1(s) - Z_2(s)\|_\mu^p\right]^{\frac{2}{q}}\|S^b_{t-s}\|^2_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}ds\right)^{\frac{q}{2}}\\ &\le k_qC^q|||Z_1 - Z_2|||_q^q\left(\int_0^t \|S^b_s\|^2_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}ds\right)^{\frac{q}{2}}.\end{aligned}$$

Now $\int_0^t \|S^b_s\|^2_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}ds = \sum_{i=1}^\infty \int_0^t e^{-2\lambda_i^b s}ds < \infty$ for all $t\in[0,\infty)$, see the proof of Lemma 4.5.3. Therefore by the dominated convergence theorem, the map $t\mapsto \int_0^t \|S^b_s\|^2_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}ds$ is continuous with $\lim_{t\to 0}\int_0^t \|S^b_s\|^2_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}ds = 0$. We thus see that there exists an increasing continuous function $\alpha_3 : [0,\infty)\to[0,\infty)$ independent of Y_0 with $\alpha_3(0) = 0$ and such that

$$|||\mathscr{K}(Z_1) - \mathscr{K}(Z_2)|||_q \le \alpha_3(T)|||Z_1 - Z_2|||_q$$

for all $Z_1, Z_2 \in \mathscr{H}_q$. Therefore using Lemma 4.5.3, if $\alpha_3(T) < 1$ then $\mathscr{K}$ is a strict contraction on $\mathscr{H}_q$ and so the SPDE (4.5.1) has a unique mild solution in $[0,T]$ by the contraction mapping theorem. If this condition does not hold we simply take some $\tilde{T} > 0$ satisfying $\alpha_3(\tilde{T}) < 1$ and solve the SPDE in the intervals $[0,\tilde{T}]$, $[\tilde{T}, 2\tilde{T}]$, and so on (this is possible by Lemma 4.5.3 and the independence of α_3 from Y_0). We have thus proven existence of a unique mild solution Y to (4.5.1) in $\mathscr{H}_q$.

Now we assume $q > 2(d_H + 1)^2$ and set $U = Y - S^b Y_0$. For $t \in [0,T]$, let $\beta_t = f(t, Y(t))$ and $\sigma_t = g(t, Y(t))$. Then we see that (4.2.3) is satisfied. The solution process Y is predictable, so β and σ immediately have the required measurability properties and so it remains to show that Hypothesis 4.2.11 holds. We have that

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^T \|f(s,Y(s))\|_\mu^2 ds\right)^{\frac{q}{2}}\right] &\le C^q \mathbb{E}\left[\left(\int_0^T (1 + \|Y(s)\|_\mu)^2 ds\right)^{\frac{q}{2}}\right] \\ &\le T^q C^q \sup_{s\in[0,T]} \mathbb{E}\left[(1 + \|Y(s)\|_\mu)^q\right] \\ &< \infty\end{aligned}$$

and

$$\sup_{s\in[0,T]} \mathbb{E}\left[\|g(s,Y(s))\|^q\right] \le C^q \sup_{s\in[0,T]} \mathbb{E}\left[(1 + \|Y(s)\|_\mu)^q\right] < \infty$$

so Theorem 4.2.15 can be used with $p = \frac{q}{2}$. □

4.6 Application to a class of Walsh SPDEs

We consider the SPDE

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) &= \Delta_b u(t,x) + f(t, u(t,x)) + g(t, u(t,x))\xi(t,x), \\ u(0,x) &= u_0(x)\end{aligned} \tag{4.6.1}$$

for $(t,x) \in [0,T] \times F$, where $T > 0$ and $b \in 2^{F^0}$, $u_0 : \Omega \times F \to \mathbb{R}$, and f, g are functions from $\Omega \times [0,T] \times F$ to $\mathbb{R}$. We take ξ to be an $\mathbb{F}$-space-time white noise on (F,μ) (in the martingale measure sense of [77]). Without loss of generality we may assume that ξ is the space-time white noise associated with the cylindrical Wiener process W considered previously; that is, for all $h \in \mathcal{H}$ and $t \geq 0$,

$$\int_0^t \int_F h(y)\xi(s,y)\mu(dy)ds = \langle h, W(t)\rangle_\mu.$$

Recall that $\mathcal{P}_T$ is the predictable σ-algebra on $\Omega \times [0,T]$ associated with the truncated filtration $(\mathcal{F}_t : 0 \le t \le T)$. We now define the spaces in which we will look for solutions to (4.6.1).

Definition 4.6.1. Let $q \geq 2$. Let $\mathcal{S}_q$ be the space of processes $v = \{v(x) : x \in F\}$ such that $v : \Omega \times F \to \mathbb{R}$ is measurable from $\mathcal{F}_0 \otimes \mathcal{B}(F)$ into $\mathcal{B}(\mathbb{R})$ and

$$\|v\|_q := \sup_{x \in F} \mathbb{E}\left[|v(x)|^q\right]^{\frac{1}{q}} < \infty.$$

This can be shown to be a Banach space with the norm $\|\cdot\|_q$, if we identify processes v_1, v_2 such that $v_1(x) = v_2(x)$ almost surely for all $x \in F$. Evidently $\mathcal{S}_{q_1} \subseteq \mathcal{S}_{q_2}$ if $q_1 \geq q_2 \geq 2$.

Likewise for $T > 0$ let $\mathcal{S}_{q,T}$ be the space of processes $v = \{v(t,x) : (t,x) \in [0,T] \times F\}$ such that v is predictable (which means that $v : \Omega \times [0,T] \times F \to \mathbb{R}$ is measurable from $\mathcal{P}_T \otimes \mathcal{B}(F)$ into $\mathcal{B}(\mathbb{R})$, see [77]) and such that

$$\|v\|_{q,T} := \sup_{t \in [0,T]} \sup_{x \in F} \mathbb{E}\left[|v(t,x)|^q\right]^{\frac{1}{q}} < \infty.$$

This is likewise a Banach space with the norm $\|\cdot\|_{q,T}$, if we identify processes v_1, v_2 such that $v_1(t,x) = v_2(t,x)$ almost surely for all $(t,x) \in [0,T] \times F$. As before, $\mathcal{S}_{q_1,T} \subseteq \mathcal{S}_{q_2,T}$ if $q_1 \geq q_2 \geq 2$.

Hypothesis 4.6.2. We make the following assumptions. Suppose there exists $q \geq 2$ such that:

(1). $u_0 \in \mathcal{S}_q$.

(2). $f, g : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ are functions which are measurable from $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$ into $\mathcal{B}(\mathbb{R})$ and obey the following Lipschitz and linear growth conditions: There exists a constant $C > 0$ and a non-negative real predictable process $M : \Omega \times [0,T] \to \mathbb{R}$ with

$$\|M\|_{q,T} := \sup_{s \in [0,T]} \mathbb{E}\left[M(s)^q\right]^{\frac{1}{q}} < \infty$$

such that for all $(\omega, t) \in \Omega \times [0,T]$ and all $x, y \in \mathbb{R}$,

$$|f(\omega,t,x) - f(\omega,t,y)| + |g(\omega,t,x) - g(\omega,t,y)| \leq C|x-y|,$$
$$|f(\omega,t,x)| + |g(\omega,t,x)| \leq M(\omega,t) + C|x|.$$

We usually suppress the dependence of f, g and M on ω. Note that we use the same notation $\|\cdot\|_{q,T}$ for M and for elements of $\mathcal{S}_{q,T}$, but the meaning will be clear from context. For the sake of the following definitions we now give a (suboptimal) technical result on the growth of the eigenfunctions φ_k^b. Compare [52, Theorem 4.5.4].

Lemma 4.6.3. *If $b \in 2^{F^0}$, then $\sup_{x\in F}|\varphi_k^b(x)| < \infty$ for all $k \geq 1$ and*

$$\sup_{x\in F}|\varphi_k^b(x)| = O(k^{\frac{1}{d_s}})$$

as $k \to \infty$.

Proof. By [3, Proposition 7.16(b)], there exists $c > 0$ such that if $b \in 2^{F^0}$ and $k \geq 1$ then for all $x \in F$,

$$\varphi_k^b(x)^2 \leq 2 + c\lambda_k^b.$$

Then Proposition 4.2.5 implies the required result. □

Definition 4.6.4 (Heat kernel)**.** Let $b \in 2^{F^0}$. For $(t, x, y) \in (0, \infty) \times F \times F$ let

$$p_t^b(x, y) = \sum_{k=1}^{\infty} e^{-\lambda_k^b t}\varphi_k^b(x)\varphi_k^b(y).$$

This is non-negative and jointly continuous in $(0, \infty) \times F \times F$; see [52, Definition A.2.12]. It is called the *heat kernel* associated with the boundary condition b. Note the obvious symmetry in (x, y). Observe that if $h \in \mathcal{H}$ and $t, x \in (0, \infty) \times F$ then

$$\int_F p_t^b(x, y)h(y)\mu(dy) = \langle p_t^b(x, \cdot), h\rangle_\mu = S_t^b h(x),$$

which in particular implies that p^b is the transition density of the diffusion X^b. Due to the above identity, for $t = 0$ we define

$$\int_F p_0^b(x, y)h(y)\mu(dy) := h(x)$$

(so long as $h(x)$ is well-defined). Thus if $s \geq 0$ and $t > 0$ and $x, y \in F$ then we see that

$$\int_F p_s^b(x, z)p_t^b(z, y)\mu(dz) = p_{s+t}^b(x, y).$$

Definition 4.6.5. A *mild solution* of the SPDE (4.6.1) is a predictable process $\{u(t, x) : (t, x) \in [0, \infty) \times F\}$ such that for each $(t, x) \in [0, \infty) \times F$ we have that

$$\begin{aligned} u(t, x) = &\int_F p_t^b(x, y)u_0(y)\mu(dy) + \int_0^t \int_F p_{t-s}^b(x, y)f(s, u(s, y))\mu(dy)ds \\ &+ \int_0^t \int_F p_{t-s}^b(x, y)g(s, u(s, y))\xi(s, y)\mu(dy)ds \end{aligned} \tag{4.6.2}$$

almost surely.

4.6.1 Existence and uniqueness

We begin with a number of estimates that will be useful in the current and subsequent sections.

Lemma 4.6.6. *The following estimates on the heat kernel p^b hold for any $T > 0$:*

(1). There exists a constant $c_1(T) > 0$ such that for all $(t, x, y) \in (0, T] \times F \times F$ and any $b \in 2^{F^0}$,

$$0 \le p_t^b(x, y) \le c_1(T) t^{-\frac{d_s}{2}}.$$

(2). There exists a constant $c_2(T) > 0$ such that for all $(t, x, x', y) \in (0, T] \times F \times F \times F$ and any $b \in 2^{F^0}$,

$$\left| p_t^b(x, y) - p_t^b(x', y) \right|^2 \le c_2(T) R(x, x') t^{-1-\frac{d_s}{2}}.$$

(3). There exists a constant $c_3(T) > 0$ such that for all $(s, t, x) \in (0, T] \times (0, T] \times F$ with $s \le t$ and any $b \in 2^{F^0}$,

$$\left| p_s^b(x, x) - p_t^b(x, x) \right| \le c_3(T) \left(s^{-\frac{d_s}{2}} - t^{-\frac{d_s}{2}} \right).$$

Proof. (1). By [52, Theorem A.2.16] we see that $0 \le p_t^b(x, y) \le p_t^N(x, y)$ for all $(t, x, y) \in (0, \infty) \times F \times F$ and $b \in 2^{F^0}$. Now by [52, Theorem 5.3.1(1)] we see that there exists $c_1(1) > 0$ such that

$$\sup_{x,y \in F} p_t^N(x, y) \le c_1(1) t^{-\frac{d_s}{2}}$$

for $t \in (0, 1]$. Finally if $T > 1$ then the fact [52, Proposition 5.1.2(1)] that p^N is continuous in the compact set $[1, T] \times F \times F$ (and hence bounded in this set) implies that there exists $c_1(T) > 0$ such that the required result holds.

(2). For $t > 0$ and $x, y \in F$ define $p_t^{b,x}(y) := p_t^b(x, y)$. By (1),

$$\|p_t^{b,x}\|_\mu^2 = \int_F p_t^b(x, y)^2 \mu(dy) = p_{2t}^b(x, x) \le 2^{-\frac{d_s}{2}} c_1(2T) t^{-\frac{d_s}{2}}$$

for all $t \in (0, T]$, $x \in F$ and $b \in 2^{F^0}$. Then we simply follow the method of [34, Lemma 5.2], using [22, Lemma 1.3.3(i)].

(3). For any $x, y \in F$ and $b \in 2^{F^0}$, $p_t^b(x,y)$ is differentiable in t with derivative $\Delta_b p_t^{b,x}(y)$. This can be shown using Lemma 4.6.3 in the same way as the proof of [52, Proposition 5.1.2(4)]. The regularity of $p_t^b(x,y)$ and $\Delta_b p_t^{b,x}(y)$ is enough (again using Lemma 4.6.3) that for $(t,x) \in (0,\infty) \times F$ we can differentiate under the integral thus:

$$\begin{aligned}\frac{\partial}{\partial t} p_t^b(x,x) &= \frac{\partial}{\partial t} \int_F p_{\frac{t}{2}}^b(x,y)^2 \mu(dy) = 2 \int_F p_{\frac{t}{2}}^{b,x}(y) \frac{\partial}{\partial t} p_{\frac{t}{2}}^{b,x}(y) \mu(dy) \\ &= 2 \int_F p_{\frac{t}{2}}^{b,x}(y) \Delta_b p_{\frac{t}{2}}^{b,x}(y) \mu(dy) = -2\mathcal{E}\left(p_{\frac{t}{2}}^{b,x}, p_{\frac{t}{2}}^{b,x}\right) \leq 0,\end{aligned}$$

so $p_t^b(x,x)$ is decreasing in t for any $x \in F$. Then [22, Lemma 1.3.3(i)] and the estimate for $\|p_t^{b,x}\|_\mu^2$ in the proof of (2) imply that there exists $c_3'(T) > 0$ such that

$$-c_3'(T) t^{-1-\frac{d_s}{2}} \leq \frac{\partial}{\partial t} p_t^b(x,x) \leq 0$$

for all $(t,x) \in (0,T] \times F$. Therefore if $s, t \in (0,T]$ with $s \leq t$ then

$$\begin{aligned}\left|p_s^b(x,x) - p_t^b(x,x)\right| &\leq c_3'(T) \int_s^t z^{-1-\frac{d_s}{2}} dz \\ &= c_3'(T) \frac{2}{d_s} \left(s^{-\frac{d_s}{2}} - t^{-\frac{d_s}{2}}\right).\end{aligned}$$

□

Proposition 4.6.7 (Stochastic continuity)**.** *Fix $q \geq 2$ and $T > 0$. Then there exists $c_6 > 0$ such that the following holds: Let $v_0 \in \mathcal{S}_{q,T}$ and define*

$$\begin{aligned}v_1(t,x) &= \int_0^t \int_F p_{t-s}^b(x,y) g(s, v_0(s,y)) \xi(s,y) \mu(dy) ds, \\ v_2(t,x) &= \int_0^t \int_F p_{t-s}^b(x,y) f(s, v_0(s,y)) \mu(dy) ds\end{aligned}$$

for $(t,x) \in [0,T] \times F$. Then v_1 and v_2 are well-defined and for all $s,t \in [0,T]$ and $x, y \in F$,

$$\begin{aligned}\mathbb{E}\left[|v_1(t,x) - v_1(t,y)|^q\right] &\leq c_6(1 + \|v_0\|_{q,T}^q) R(x,y)^{\frac{q}{4}}, \\ \mathbb{E}\left[|v_1(s,x) - v_1(t,x)|^q\right] &\leq c_6(1 + \|v_0\|_{q,T}^q) |s-t|^{\frac{q}{2}(1-\frac{d_s}{2})}, \\ \mathbb{E}\left[|v_2(t,x) - v_2(t,y)|^q\right] &\leq c_6(1 + \|v_0\|_{q,T}^q) R(x,y)^{\frac{q}{4}}, \\ \mathbb{E}\left[|v_2(s,x) - v_2(t,x)|^q\right] &\leq c_6(1 + \|v_0\|_{q,T}^q) |s-t|^{\frac{q}{2}(1-\frac{d_s}{2})}.\end{aligned}$$

Proof. We note that v_1 and v_2 are well-defined by the assumption on v_0 and the regularity of p^b (Lemma 4.6.6(1)) and f, g. We now prove the spatial estimate for v_1.

By the Burkholder-Davis-Gundy inequality, there exists a universal constant $C_q > 0$ such that

$$\begin{aligned}
&\mathbb{E}\left[|v_1(t,x) - v_1(t,x')|^q\right] \\
&= \mathbb{E}\left[\left|\int_0^t \int_F \left(p^b_{t-s}(x,y) - p^b_{t-s}(x',y)\right) g(s, v_0(s,y))\xi(s,y)\mu(dy)ds\right|^q\right] \\
&\leq C_q \mathbb{E}\left[\left|\int_0^t \int_F \left(p^b_{t-s}(x,y) - p^b_{t-s}(x',y)\right)^2 g(s, v_0(s,y))^2 \mu(dy)ds\right|^{\frac{q}{2}}\right] \\
&\leq C_q \left|\int_0^t \int_F \left(p^b_{t-s}(x,y) - p^b_{t-s}(x',y)\right)^2 \mathbb{E}\left[|g(s, v_0(s,y))|^q\right]^{\frac{2}{q}} \mu(dy)ds\right|^{\frac{q}{2}}
\end{aligned}$$

where in the last line we have used the Minkowski integral inequality. Now we have that $|g(s, v_0(s,y))|^q \leq (M(s) + C|v_0(s,y)|)^q \leq 2^{q-1}(M(s)^q + C^q|v_0(s,y)|^q)$ so using Lemma 4.6.6(2),

$$\begin{aligned}
&\mathbb{E}\left[|v_1(t,x) - v_1(t,x')|^q\right] \\
&\leq 2^{q-1} C_q (\|M\|^q_{q,T} + C^q \|v_0\|^q_{q,T}) \left|\int_0^t \int_F \left(p^b_{t-s}(x,y) - p^b_{t-s}(x',y)\right)^2 \mu(dy)ds\right|^{\frac{q}{2}} \\
&\leq 2^{q-1} C_q (\|M\|^q_{q,T} + C^q \|v_0\|^q_{q,T}) \left|\int_0^t \left(p^b_{2(t-s)}(x,x) - 2p^b_{2(t-s)}(x,x') + p^b_{2(t-s)}(x',x')\right) ds\right|^{\frac{q}{2}} \\
&\leq 2^{\frac{q}{4}(5-d_s)} c_2 (2T)^{\frac{q}{4}} C_q (\|M\|^q_{q,T} + C^q \|v_0\|^q_{q,T}) \left(\int_0^t (t-s)^{-\frac{1}{2}-\frac{d_s}{4}} ds\right)^{\frac{q}{2}} R(x,x')^{\frac{q}{4}} \\
&\leq 2^{\frac{q}{4}(5-d_s)} c_2 (2T)^{\frac{q}{4}} C_q (\|M\|^q_{q,T} + C^q \|v_0\|^q_{q,T}) \left(\int_0^T s^{-\frac{1}{2}-\frac{d_s}{4}} ds\right)^{\frac{q}{2}} R(x,x')^{\frac{q}{4}}
\end{aligned}$$

and the integral is finite since $d_s < 2$. Now we take care of the temporal estimate for v_1: Let $t, t' \in [0,T]$ with $t < t'$. Then as before,

$$\begin{aligned}
&\mathbb{E}\left[|v_1(t,x) - v_1(t',x)|^q\right] \\
&\leq C_q \mathbb{E}\left[\left|\int_0^{t'} \int_F \left(p^b_{t-s}(x,y)\mathbb{1}_{\{s<t\}} - p^b_{t'-s}(x,y)\right)^2 g(s, v_0(s,y))^2 \mu(dy)ds\right|^{\frac{q}{2}}\right] \\
&\leq C_q \left|\int_0^{t'} \int_F \left(p^b_{t-s}(x,y)\mathbb{1}_{\{s<t\}} - p^b_{t'-s}(x,y)\right)^2 \mathbb{E}\left[|g(s, v_0(s,y))|^q\right]^{\frac{2}{q}} \mu(dy)ds\right|^{\frac{q}{2}} \\
&\leq 2^{q-1} C_q (\|M\|^q_{q,T} + C^q \|v_0\|^q_{q,T}) \left|\int_0^{t'} \int_F \left(p^b_{t-s}(x,y)\mathbb{1}_{\{s<t\}} - p^b_{t'-s}(x,y)\right)^2 \mu(dy)ds\right|^{\frac{q}{2}} \\
&\leq 2^{q-1} C_q (\|M\|^q_{q,T} + C^q \|v_0\|^q_{q,T}) \left|\int_0^{t'} \int_F \left(p^b_{t-s}(x,y)\mathbb{1}_{\{s<t\}} - p^b_{t'-s}(x,y)\right)^2 \mu(dy)ds\right|^{\frac{q}{2}}.
\end{aligned}$$

The integral in the final line above must be split into the sum of its parts $s < t$ and $s \geq t$. The first part is

$$\begin{aligned}
&\int_0^t \int_F \left(p_{t-s}^b(x,y) - p_{t'-s}^b(x,y)\right)^2 \mu(dy)ds \\
&\qquad = \int_0^t \int_F \left(p_s^b(x,y) - p_{s+t'-t}^b(x,y)\right)^2 \mu(dy)ds \\
&\qquad = \int_0^t \left(p_{2s}^b(x,x) - 2p_{2s+t'-t}^b(x,x) + p_{2(s+t'-t)}^b(x,x)\right) ds \\
&\qquad \leq \int_0^t \left(p_{2s}^b(x,x) - 2p_{2s+t'-t}^b(x,x) + p_{2(s+t'-t)}^b(x,x)\right) ds \\
&\qquad \leq 2^{-\frac{d_s}{2}} c_3(2T) \int_0^t \left(s^{-\frac{d_s}{2}} - (s+t'-t)^{-\frac{d_s}{2}}\right) ds \\
&\qquad = 2^{-\frac{d_s}{2}} \left(1 - \frac{d_s}{2}\right)^{-1} c_3(2T) \left(t^{1-\frac{d_s}{2}} - (t')^{1-\frac{d_s}{2}} + (t'-t)^{1-\frac{d_s}{2}}\right) \\
&\qquad \leq 2^{-\frac{d_s}{2}} \left(1 - \frac{d_s}{2}\right)^{-1} c_3(2T)(t'-t)^{1-\frac{d_s}{2}}
\end{aligned}$$

where we have used Lemma 4.6.6(3). The second part is

$$\begin{aligned}
\int_t^{t'} \int_F p_{t'-s}^b(x,y)^2 \mu(dy)ds &= \int_0^{t'-t} p_{2s}^b(x,x)ds \\
&\leq 2^{-\frac{d_s}{2}} c_1(2T) \int_0^{t'-t} s^{-\frac{d_s}{2}} ds \\
&\leq 2^{-\frac{d_s}{2}} c_1(2T) \left(1 - \frac{d_s}{2}\right)^{-1} (t'-t)^{1-\frac{d_s}{2}}
\end{aligned}$$

where we have used Lemma 4.6.6(1). Together these give us the temporal estimate for v_1.

The respective estimates for v_2 can be found similarly, though they are generally easier as there is no noise to deal with. In particular, we use Jensen's instead of the Burkholder-Davis-Gundy inequality. □

Corollary 4.6.8. *Fix $q \geq 2$ and $T > 0$. Let $v_0 \in \mathcal{S}_{q,T}$ and define v_1, v_2 as in Proposition 4.6.7. Then $v_1, v_2 \in \mathcal{S}_{q,T}$.*

Proof. The estimates found in Proposition 4.6.7 show us that v_1 and v_2 are uniformly stochastically continuous on $[0,T] \times F$. For each fixed $t \in [0,T]$, $v_1(t,\cdot)$ and $v_2(t,\cdot)$ are $\mathcal{F}_t \otimes \mathcal{B}(F)$-measurable, which we can check by approximating the continuous map $(s,x,y) \mapsto p_{t-s}^b(x,y)$ pointwise from below by a sequence of finite positive linear combinations of rectangles $(s,x,y) \mapsto \mathbb{1}_{A_1}(s))\mathbb{1}_{A_2}(x)\mathbb{1}_{A_3}(y)$ such that $A_1 \times A_2 \times A_3 \subseteq [0,t) \times F \times F$ is jointly measurable.

Since v_1 and v_2 are both evidently adapted, a standard argument (see for example [15, Proposition 3.7(ii)]) shows that they have predictable versions. As a rough sketch, for $n \geq 1$ let

$$v_1^n(t,x) = \sum_{i=0}^{2^n-1} v_1\left(\frac{i}{2^n}T, x\right) \mathbb{1}_{(\frac{i}{2^n}T, \frac{i+1}{2^n}T]}(t)$$

for $(t,x) \in [0,T]\times F$. Then each v_1^n is predictable and we can show that $\|v_1^n - v_1\|_{q,T} \to 0$ as $n \to \infty$.

It remains to show that $\|v_i\|_{q,T} < \infty$ for $i = 1, 2$. This is easy to see by setting $s = 0$ in Proposition 4.6.7, since $v_i(0,\cdot) = 0$. □

To prove existence and uniqueness we follow closely the methods of [77, Theorem 3.2] and [49, Theorem 5.5].

Theorem 4.6.9 (Existence and uniqueness). *Assume Hypothesis 4.6.2. Then the SPDE* (4.6.1) *has a unique mild solution u in $\mathcal{S}_{q,T}$.*

Proof. Uniqueness: We may assume without loss of generality that $q = 2$. Suppose that $u_1, u_2 \in \mathcal{S}_{2,T}$ are both mild solutions to (4.6.1). Let $v = u_1 - u_2 \in \mathcal{S}_{2,T}$. For $(t,x) \in [0,T] \times F$ let $G(t) = \sup_{x\in F} \mathbb{E}\left[v(t,x)^2\right]$. Note that G is non-negative and bounded in $[0,T]$ since $v \in \mathcal{S}_{2,T}$. Then for $(t,x) \in [0,T] \times F$ we have by the Burkholder-Davis-Gundy inequality that

$$\begin{aligned}\mathbb{E}\left[v(t,x)^2\right] \leq & 2T\mathbb{E}\left[\int_0^t \int_F p_{t-s}^b(x,y)^2 (f(s,u_1(s,y)) - f(s,u_2(s,y)))^2 \mu(dy)ds\right] \\ & + 2\mathbb{E}\left[\int_0^t \int_F p_{t-s}^b(x,y)^2 (g(s,u_1(s,y)) - g(s,u_2(s,y)))^2 \mu(dy)ds\right]\end{aligned}$$

so that

$$\begin{aligned}\mathbb{E}\left[v(t,x)^2\right] &\leq 2C^2(T+1)\mathbb{E}\left[\int_0^t \int_F p_{t-s}^b(x,y)^2 (u_1(s,y) - u_2(s,y))^2 \mu(dy)ds\right] \\ &\leq 2C^2(T+1)\int_0^t G(s) \int_F p_{t-s}^b(x,y)^2 \mu(dy)ds \\ &= 2C^2(T+1)\int_0^t G(s) p_{2(t-s)}^b(x,x)ds\end{aligned}$$

and thus by Lemma 4.6.6(1), for all $t \in [0,T]$,

$$G(t) \leq 2^{1-\frac{d_s}{2}} C^2(T+1)c_1(2T) \int_0^t G(s)(t-s)^{-\frac{d_s}{2}} ds.$$

Now $d_s < 2$, so by [77, Lemma 3.3] we have that $G(t) = 0$ for all $t \in [0, T]$. This concludes the proof of uniqueness.

Existence: We find a solution in $\mathcal{S}_{q,T}$ by Picard iteration. Let $u_1 = 0 \in \mathcal{S}_{q,T}$, then for $n \geq 1$ we would like to let $u_{n+1} = \{u_{n+1}(t, x) : (t, x) \in [0, T] \times F\}$ be the predictable version (using Proposition 4.6.7) of the process defined by

$$\begin{aligned} u_{n+1}(t,x) = &\int_F p_t^b(x,y)u_0(y)\mu(dy) + \int_0^t \int_F p_{t-s}^b(x,y) f(s, u_n(s,y))\mu(dy)ds \\ &+ \int_0^t \int_F p_{t-s}^b(x,y) g(s, u_n(s,y))\xi(s,y)\mu(dy)ds. \end{aligned} \tag{4.6.3}$$

We need to show that this sequence is well-defined. By Proposition 4.6.7 it will be enough to prove that if $u_n \in \mathcal{S}_{q,T}$ for some n and u_{n+1} is defined as in (4.6.3), then $u_{n+1} \in \mathcal{S}_{q,T}$. We do this by induction on n. Assume that $u_n \in \mathcal{S}_{q,T}$ and define u_{n+1} as in (4.6.3). By Proposition 4.6.7 and Corollary 4.6.8, the second and third term on the right-hand side of (4.6.3) are well-defined elements of $\mathcal{S}_{q,T}$, which leaves us to deal with the first term $\int_F p_t^b(x,y)u_0(y)\mu(dy)$. This is trivially predictable as it is $\mathcal{F}_0$-measurable and, by dominated convergence using Lemma 4.6.6, it is almost surely continuous in $(0, T] \times F$. Finally using Minkowski's integral inequality,

$$\begin{aligned} \mathbb{E}\left[\left|\int_F p_t^b(x,y)u_0(y)\mu(dy)\right|^q\right] &\leq \left(\int_F p_t^b(x,y)\mathbb{E}\left[|u_0(y)|^q\right]^{\frac{1}{q}}\mu(dy)\right)^q \\ &\leq \|u_0\|_q^q \left|\int_F p_t^b(x,y)\mu(dy)\right|^q \\ &\leq \|u_0\|_q^q \end{aligned} \tag{4.6.4}$$

for all $(t, x) \in [0, T] \times F$, showing that the first term of the right-hand side of (4.6.3) is indeed an element of $\mathcal{S}_{q,T}$. So $u_{n+1} \in \mathcal{S}_{q,T}$, and in conclusion $(u_n)_{n=1}^\infty$ is a well-defined sequence of elements of $\mathcal{S}_{q,T}$.

We now show that the sequence $(u_n)_{n=1}^\infty$ is Cauchy. For $n \geq 1$ let $v_n = u_{n+1} - u_n \in \mathcal{S}_{q,T}$. Then by Jensen's and Burkholder-Davis-Gundy inequalities, there exists a constant $C_q > 0$ such that for all $(t, x) \in [0, T] \times F$ we have that

$$\begin{aligned} &\mathbb{E}\left[|v_{n+1}(t,x)|^q\right] \\ &\leq 2^{q-1}T^{\frac{q}{2}}\mathbb{E}\left[\left(\int_0^t \int_F p_{t-s}^b(x,y)^2 \left(f(s,u_{n+1}(s,y)) - f(s,u_n(s,y))\right)^2 \mu(dy)ds\right)^{\frac{q}{2}}\right] \\ &\quad + 2^{q-1}C_q\mathbb{E}\left[\left(\int_0^t \int_F p_{t-s}^b(x,y)^2 \left(g(s,u_{n+1}(s,y)) - g(s,u_n(s,y))\right)^2 \mu(dy)ds\right)^{\frac{q}{2}}\right]. \end{aligned}$$

Using the Lipschitz property of f and g and then Minkowski's integral inequality this implies that

$$\begin{aligned}\mathbb{E}\left[|v_{n+1}(t,x)|^q\right] &\le 2^{q-1}\left(T^{\frac{q}{2}}+C_q\right)C^q\mathbb{E}\left[\left(\int_0^t\int_F p^b_{t-s}(x,y)^2 v_n(s,y)^2\mu(dy)ds\right)^{\frac{q}{2}}\right]\\ &\le 2^{q-1}\left(T^{\frac{q}{2}}+C_q\right)C^q\left(\int_0^t\int_F p^b_{t-s}(x,y)^2\mathbb{E}\left[|v_n(s,y)|^q\right]^{\frac{2}{q}}\mu(dy)ds\right)^{\frac{q}{2}}\end{aligned}$$

Let $H_n(t)=\sup_{x\in F}\mathbb{E}\left[|v_n(t,x)|^q\right]^{\frac{2}{q}}$ for $n\ge 1$ and $t\in[0,T]$. Since $v_n\in\mathcal{S}_{q,T}$, each H_n must be bounded in $[0,T]$. Then the above equation implies that there is a constant $C>0$ such that for all n and (t,x),

$$\begin{aligned}\mathbb{E}\left[|v_{n+1}(t,x)|^q\right]^{\frac{2}{q}} &\le C\int_0^t H_n(s)\int_F p^b_{t-s}(x,y)^2\mu(dy)ds\\ &= C\int_0^t H_n(s)p^b_{2(t-s)}(x,x)ds\\ &\le 2^{-\frac{d_s}{2}}c_1(2T)C\int_0^t H_n(s)(t-s)^{-\frac{d_s}{2}}ds,\end{aligned}$$

where we have used Lemma 4.6.6(1). This implies that

$$H_{n+1}(t)\le 2^{-\frac{d_s}{2}}c_1(2T)C\int_0^t H_n(s)(t-s)^{-\frac{d_s}{2}}ds$$

for $t\in[0,T]$. By [77, Lemma 3.3] and the fact that $d_s<2$, there then exists a constant $C_0>0$ and an integer $k\ge 1$ such that for each $n,m\ge 1$ and $t\in[0,T]$ we have that

$$H_{n+mk}(t)\le\frac{C_0^m}{(m-1)!}\int_0^t H_n(s)(t-s)ds.$$

Thus for each $n\ge 1$, $\sum_{m=0}^\infty H_{n+mk}^{\frac{1}{2}}$ converges uniformly in $[0,T]$. This implies that $\sum_{n=1}^\infty H_n^{\frac{1}{2}}$ also converges uniformly in $[0,T]$ and so the sequence $(u_n)_{n=1}^\infty$ is Cauchy in $\mathcal{S}_{q,T}$. Let $u\in\mathcal{S}_{q,T}$ be the limit of this sequence. Now for each $(t,x)\in[0,T]\times F$ we take the limit $n\to\infty$ in $L^q(\Omega)$ on both sides of (4.6.3). The left-hand side tends to $u(t,x)$, whereas the right-hand side tends to

$$\begin{aligned}&\int_F p^b_t(x,y)u_0(y)\mu(dy)+\int_0^t\int_F p^b_{t-s}(x,y)f(s,u(s,y))\mu(dy)ds\\ &+\int_0^t\int_F p^b_{t-s}(x,y)g(s,u(s,y))\xi(s,y)\mu(dy)ds\end{aligned}$$

by a calculation that by now is routine. Therefore u is a mild solution to the SPDE (4.6.1) defined on $[0,T]\times F$. □

4.6.2 Continuous random field version

Assuming Hypothesis 4.6.2, let $u \in \mathcal{S}_{q,T}$ be the mild solution to (4.6.1) as obtained in Theorem 4.6.9. Using Corollary 4.6.8, let $u^{\text{sto}} = \{u^{\text{sto}}(t,x) : (t,x) \in [0,T] \times F\}$ be the element of $\mathcal{S}_{q,T}$ satisfying

$$\begin{aligned} u^{\text{sto}}(t,x) = &\int_0^t \int_F p^b_{t-s}(x,y) f(s,u(s,y))\mu(dy)ds \\ &+ \int_0^t \int_F p^b_{t-s}(x,y) g(s,u(s,y))\xi(s,y)\mu(dy)ds \end{aligned}$$

almost surely for each $(t,x) \in [0,T] \times F$. So u^{sto} is the "stochastic part" of u and satisfies

$$u^{\text{sto}}(t,x) = u(t,x) - \int_F p^b_t(x,y) u_0(y)\mu(dy)$$

almost surely for each $(t,x) \in [0,T] \times F$. In this section we prove that u^{sto} has a version which is a continuous random field satisfying relatively weak continuity properties, and then we use Theorem 4.2.15 to bootstrap these into stronger Hölder continuity properties.

Lemma 4.6.10. *Let $b \in 2^{F^0}$. Let $h \in \mathcal{H}$. Consider the map from $(0,\infty) \times F$ to $\mathbb{R}$ given by*

$$(t,x) \mapsto \int_F p^b_t(x,y) h(y)\mu(dy).$$

Then for every $0 < T_1 < T_2$, on $[T_1,T_2] \times F$ this map is $\frac{1}{2}$-Hölder continuous with respect to R_∞. Moreover, if $\sup_{x\in F} |h(x)| < \infty$ then

$$\sup_{(t,x)\in(0,\infty)\times F} \left| \int_F p^b_t(x,y) h(y)\mu(dy) \right| \leq \sup_{x\in F} |h(x)|.$$

Proof. Fix $0 < T_1 < T_2$. It is enough to prove that the map is uniformly Hölder continuous in each argument. By Lemma 4.6.6, for any $t \in [T_1,T_2]$ and $x,y \in F$,

$$\left| \int_F (p^b_t(x,z) - p^b_t(y,z)) h(z)\mu(dz) \right| \leq c_2 (T_2)^{\frac{1}{2}} T_1^{-\frac{1}{2}-\frac{d_s}{4}} R(x,y)^{\frac{1}{2}} \int_F |h(z)|\mu(dz).$$

Similarly for any $s,t \in [T_1,T_2]$ with $s < t$ and $x \in F$, the fact that $t' \mapsto p^b_{t'}(x,x)$ is decreasing (evident by definition of p^b) implies that

$$\begin{aligned}
&\left(\int_F (p^b_s(x,z)-p^b_t(x,z))h(z)\mu(dz)\right)^2\\
&\le \int_F (p^b_s(x,z)-p^b_t(x,z))^2\mu(dz)\int_F h(z)^2\mu(dz)\\
&\le \left(p^b_{2s}(x,x)-2p^b_{s+t}(x,x)+p^b_{2t}(x,x)\right)\int_F h(z)^2\mu(dz)\\
&\le \left(p^b_{2s}(x,x)-p^b_{s+t}(x,x)\right)\int_F h(z)^2\mu(dz)\\
&\le C_3(2T_2)\left((2s)^{-\frac{d_s}{2}}-(s+t)^{-\frac{d_s}{2}}\right)\int_F h(z)^2\mu(dz)\\
&\le C_3(2T_2)\left(s^{-\frac{d_s}{2}}-t^{-\frac{d_s}{2}}\right)\int_F h(z)^2\mu(dz).
\end{aligned}$$

Now $t\mapsto t^{-\frac{d_s}{2}}$ is Lipschitz in $[T_1,T_2]$ so we have the required result.

For the last claim, if $\sup_{x\in F}|h(x)|=C$ then

$$\left|\int_F p^b_t(x,y)h(y)\mu(dy)\right|\le C\int_F p^b_t(x,y)\mu(dy)\le C$$

for all $(t,x)\in(0,\infty)\times F$. □

Our first result on u^{sto} uses directly the stochastic continuity results of the previous section.

Proposition 4.6.11. *Assume Hypothesis 4.6.2 with $q>2(d_H+1)^2$. Then there exists $\gamma\in(0,1]$ such that u^{sto} has a version $\tilde{u}^{\text{sto}}$ which is predictable and γ-Hölder continuous on $[0,T]\times F$ with respect to R_∞ almost surely.*

Proof. Given the stochastic continuity estimates in Proposition 4.6.7 and the fact that $d_H\ge 1$ (from Remark 3.2.6(2), the condition $q>2(d_H+1)^2$ is precisely what is needed for us to be able to use Theorem 3.3.17 to construct an almost surely Hölder continuous version of u^{sto} on $[0,T]\times F$ for any $T>0$. It is adapted, so its continuity immediately implies that it is predictable. □

Let $C(F)$ be the Banach space of continuous functions from F to $\mathbb{R}$ equipped with uniform norm $\|\cdot\|_\infty$. We use the above two results to construct a process in this space which is "almost" a version of u.

Corollary 4.6.12. *Assume Hypothesis 4.6.2 with $q>2(d_H+1)^2$. Then there exists a predictable $C(F)$-valued process $(\tilde{u}(t,\cdot),t\in[0,T])$ such that $u(0,\cdot)=0$, and $\tilde{u}(t,x)=u(t,x)$ almost surely for $(t,x)\in(0,T]\times F$. Moreover, $t\mapsto\tilde{u}(t,\cdot)$ is continuous from $(0,T]$ to $C(F)$ almost surely.*

Proof. Define $\tilde{u}(t,x)$ as follows: for $t=0$ let $\tilde{u}(0,x)=0$ for all $x \in F$. For all $(t,x) \in (0,T] \times F$ let

$$\tilde{u}(t,x) = \int_F p_t^b(x,y)u_0(y)\mu(dy) + \tilde{u}^{\text{sto}}(t,x),$$

Where $\tilde{u}^{\text{sto}}$ is as defined in Proposition 4.6.11. We therefore obviously have $\tilde{u}(t,x) = u(t,x)$ almost surely for $(t,x) \in (0,T] \times F$. Hypothesis 4.6.2 implies that $u_0 \in \mathcal{H}$ almost surely, so Lemma 4.6.10 and Proposition 4.6.11 imply that $\tilde{u}$ is almost surely Hölder continuous in $[T_1, T] \times F$ with respect to R_∞ for any $T_1 \in (0,T)$. Thus $t \mapsto \tilde{u}(t,\cdot)$ is well-defined as a $C(F)$-valued process and is almost surely continuous from $(0,T]$ to $C(F)$. It is therefore almost surely left-continuous on $[0,T]$. This combined with its adaptedness implies that it is predictable. □

Lemma 4.6.13. *Assume Hypothesis 4.6.2 with $q > 2(d_H+1)^2$. Let $\tilde{u}$ be as given in Corollary 4.6.12. Let $\beta = \{\beta(t) : t \in [0,T]\}$ be the $\mathcal{H}$-valued process given by $\beta(t)(x) = f(t, \tilde{u}(t,x))$ for $x \in F$, and let $\sigma = \{\sigma(t) : t \in [0,T]\}$ be the $\mathcal{L}(\mathcal{H})$-valued process given by $\sigma(t) = \mathcal{M}_{(x \mapsto g(t,\tilde{u}(t,x)))}$. Then β and σ are predictable processes on their respective spaces $\mathcal{H}$ and $\mathcal{L}(\mathcal{H})$.*

Proof. Recall that $t \mapsto \tilde{u}$ is a predictable $C(F)$-valued process. To prove this result we show that the processes β and σ are compositions of suitably measurable functions of $\tilde{u}$.

To show that β is predictable, let $\bar{f} : \Omega \times [0,\infty) \times \mathcal{H} \to \mathcal{H}$ be given by $\bar{f}(\omega,t,h) = (x \mapsto f(\omega,t,h(x)))$. For fixed $(\omega,t) \in \Omega \times [0,\infty)$ this is continuous in h since f is Lipschitz in x. For fixed $h \in \mathcal{H}$ this is predictable; indeed for any closed $\mathcal{H}$-ball $\bar{B}(h',\varepsilon) \in \mathcal{B}(\mathcal{H})$,

$$\left\{(\omega,t) : f(\omega,t,h(\cdot)) \in \bar{B}(h',\varepsilon)\right\} = \left\{(\omega,t) : \int_F (f(\omega,t,h(x)) - h'(x))^2 \mu(dx) \le \varepsilon\right\}$$

which is predictable because $(\omega,t,x) \mapsto f(\omega,t,h(x))$ is predictable. Therefore by [2, Lemma 4.51], $\bar{f}$ is $\mathcal{P}_T \otimes \mathcal{B}(\mathcal{H})$-measurable. So

$$\beta(t) = \bar{f}(t, \iota(\tilde{u}(t,\cdot))),$$

where $\iota : C(F) \hookrightarrow \mathcal{H}$ is the continuous inclusion map. This is now clearly a predictable $\mathcal{H}$-valued process.

Now we deal with σ, which is slightly different. let $\bar{g} : \Omega \times [0,\infty) \times C(F) \to C(F)$ be given by $\bar{g}(\omega,t,\eta) = (x \mapsto g(\omega,t,\eta(x)))$. For fixed $(\omega,t) \in \Omega \times [0,\infty)$ this is continuous in η since g is Lipschitz in x. For fixed $\eta \in C(F)$ this is predictable;

indeed, let $\{x_i\}_i$ be a countable dense subset of F. Then for any closed $C(F)$-ball $\bar{B}(\eta', \varepsilon) \in \mathcal{B}(C(F))$,

$$\{(\omega, t) : g(\omega, t, \eta(\cdot)) \in \bar{B}(\eta', \varepsilon)\} = \bigcap_i \{(\omega, t) : |g(\omega, t, \eta(x_i)) - \eta'(x_i)| \leq \varepsilon\},$$

which is predictable because $(\omega, t, x) \mapsto g(\omega, t, h(x))$ is predictable. Therefore by [2, Lemma 4.51], $\bar{g}$ is $\mathcal{P}_T \otimes \mathcal{B}(C(F))$-measurable. Now consider the function $\eta \mapsto \mathcal{M}_\eta$ from $C(F)$ to $\mathcal{L}(\mathcal{H})$. This is a continuous linear map since $\|\mathcal{M}_\eta\| = \|\eta\|_\infty$ for all $\eta \in C(F)$. So

$$\sigma(t) = \mathcal{M}_{\bar{g}(t, \hat{u}(t, \cdot))}$$

which is now clearly a predicable $\mathcal{L}(\mathcal{H})$-valued process. □

We may now prove our Hölder regularity result for u^{sto}:

Theorem 4.6.14. *Assume Hypothesis 4.6.2 with $q > 2(d_H + 1)^2$. Then u^{sto} has a version $\tilde{u}^{\text{sto}}$ with the following Hölder continuity properties:*

(1). $\tilde{u}^{\text{sto}}$ is almost surely essentially $\left(\frac{1}{2}(d_H + 1)^{-1} - \frac{d_H+1}{q}\right)$-Hölder continuous in $[0, T] \times F$ with respect to R_∞,

(2). For each $t \in [0, T]$, $\tilde{u}^{\text{sto}}(t, \cdot)$ is almost surely essentially $\left(\frac{1}{2} - \frac{d_H}{q}\right)$-Hölder continuous in F with respect to R,

(3). For each $x \in F$, $\tilde{u}^{\text{sto}}(\cdot, x)$ is almost surely essentially $\left(\frac{1}{2}(d_H + 1)^{-1} - \frac{1}{q}\right)$-Hölder continuous in $[0, T]$.

Proof. We take $\tilde{u}^{\text{sto}}$ to be the almost surely continuous version of u^{sto} as defined in Proposition 4.6.11. Recall $\tilde{u}$ as defined in Corollary 4.6.12; we see that

$$\begin{aligned}\tilde{u}^{\text{sto}}(t, x) = &\int_0^t \int_F p^b_{t-s}(x, y) f(s, \tilde{u}(s, y)) \mu(dy) ds \\ &+ \int_0^t \int_F p^b_{t-s}(x, y) g(s, \tilde{u}(s, y)) \xi(s, y) \mu(dy) ds\end{aligned}$$

almost surely for each $(t, x) \in [0, T] \times F$. This holds since the fact that $\tilde{u}$ does not agree almost surely with u for $t = 0$ has no effect on integrals or stochastic integrals. Define an $\mathcal{H}$-valued predictable process $U = (U(t) : t \in [0, T])$ by $U(t) = \tilde{u}^{\text{sto}}(t, \cdot)$, and let β and σ be defined as in Lemma 4.6.13. Using the connections between the space-time white noise ξ and the cylindrical Wiener process W we can rewrite the definition of $\tilde{u}^{\text{sto}}$ on the level of $\mathcal{H}$:

$$U(t) = \int_0^t S^b_{t-s} \beta(s) ds + \int_0^t S^b_{t-s} \sigma(s) dW(s)$$

almost surely for each $t \in [0,T]$. This equation can be verified by taking $\langle \cdot, h_i \rangle_\mu$ on both sides where h_i are the elements of an orthonormal basis $\{h_i\}_i$ of $\mathcal{H}$ and then using (stochastic) Fubini's theorem [77, Theorem 2.6]. Now we see that

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T \|\beta(s)\|_\mu^2 ds\right)^{\frac{q}{2}}\right] &= \mathbb{E}\left[\left(\int_0^T \int_F f(s,\tilde{u}(s,x))^2 \mu(dx)ds\right)^{\frac{q}{2}}\right] \\
&\leq T^{\frac{q}{2}-1}\mathbb{E}\left[\int_0^T \int_F |M(s) + C\tilde{u}(s,x)|^q \mu(dx)ds\right] \\
&\leq T^{\frac{q}{2}} \sup_{(s,x)\in[0,T]\times F} \mathbb{E}\left[|M(s) + C\tilde{u}(s,x)|^q\right] \\
&< \infty.
\end{aligned}$$

In addition, $(\omega, s, x) \mapsto g(\omega, s, \tilde{u}(\omega, s, x))$ is evidently jointly measurable and

$$\begin{aligned}
\sup_{(s,x)\in[0,T]\times F} \mathbb{E}\left[|g(s,\tilde{u}(s,x))|^q\right] &\leq \sup_{(s,x)\in[0,T]\times F} \mathbb{E}\left[|M(s) + C\tilde{u}(s,x)|^q\right] \\
&< \infty.
\end{aligned}$$

Therefore Hypothesis 4.2.13 holds for $p = \frac{q}{2}$. Thus by applying Theorem 4.2.15 there exists a function $\hat{u} : \Omega \times [0,T] \times F$ such that $\hat{u}(t,x)$ is a random variable for each $(t,x) \in [0,T] \times F$, $\hat{u}$ is almost surely continuous in $[0,T] \times F$ with the required Hölder exponents given in the statement of the present theorem, and $\hat{u}(t,\cdot) = U(t) = \tilde{u}^{\mathrm{sto}}(t,\cdot)$ (in the sense of elements of $\mathcal{H}$) almost surely for each $t \in [0,T]$. We now show that $\hat{u}(t,x) = \tilde{u}^{\mathrm{sto}}(t,x)$ almost surely for each $(t,x) \in [0,T] \times F$. But this is clear: $\hat{u}(t,\cdot) = \tilde{u}^{\mathrm{sto}}(t,\cdot)$ in $\mathcal{H}$ almost surely, and both are almost surely continuous in F, so we must have that $\hat{u}(t,x) = \tilde{u}^{\mathrm{sto}}(t,x)$ almost surely for all $x \in F$. Now since $[0,T] \times F$ is separable, any two almost surely continuous versions of u^{sto} must be indistinguishable on their domains of definition. Therefore $\hat{u} = \tilde{u}^{\mathrm{sto}}$ in $[0,T] \times F$ almost surely and the proof is complete. □

4.7 Bounds on global solutions and intermittency

We seek upper and lower moment bounds on the global solutions to Walsh SPDEs on fractals. A corollary of these results is that a certain class of these SPDEs, which includes a version of the parabolic Anderson model, exhibits *intermittency*, see [49] and further references.

Lemma 4.7.1. *There exist* $c_7, c_8 > 0$ *such that*

$$c_7\left(1 + t^{-\frac{d_s}{2}}\right) \leq p_t^N(x,x) \leq c_8\left(1 + t^{-\frac{d_s}{2}}\right)$$

for all $(t,x) \in (0,\infty) \times F$.

Proof. Recall that

$$p_t^N(x,y) = \sum_{k=1}^{\infty} e^{-\lambda_k^N t}\varphi_k^N(x)\varphi_k^N(y).$$

Therefore by [52, Theorem 4.5.4] and the fact that $\lambda_1^N = 0 < \lambda_2^N$ and $\varphi_1^N \equiv 1$, the map $x \mapsto p_t^N(x,x)$ must converge to 1 uniformly as $t \to \infty$. Then [52, Theorem 5.3.1] implies the result. □

In particular, for all $b \in 2^{F^0}$, we have that $p_t^b(x,x) \leq c_8 \left(1 + t^{-\frac{d_s}{2}}\right)$ for all $(t,x) \in [0,\infty) \times F$.

Definition 4.7.2. Let $\kappa : [0,\infty) \to [0,\infty)$ be the function such that $\kappa(0) = 0$ and for $\alpha > 0$,

$$\kappa(\alpha)\int_0^\infty e^{-\alpha t}\left(1 + t^{-\frac{d_s}{2}}\right) dt = 1.$$

Indeed, direct computation yields

$$\kappa(\alpha) = \frac{\alpha}{1 + \alpha^{\frac{d_s}{2}}\Gamma(1 - d_s/2)} \tag{4.7.1}$$

where Γ is the Gamma function. The following properties are easily verified:

(1). κ is continuous in $[0,\infty)$ and continuously differentiable in $(0,\infty)$ with positive first derivative.

(2). $\kappa(\alpha) = \Theta(\alpha)$ as $\alpha \to 0$ and $\kappa(\alpha) = \Theta(\alpha^{1-\frac{d_s}{2}})$ as $\alpha \to \infty$.

The above two properties imply that κ is a strictly increasing bijection on $[0,\infty)$. Its first derivative is given by

$$\kappa'(\alpha) = \frac{1 + \alpha^{\frac{d_s}{2}}\Gamma(2 - d_s/2)}{(1 + \alpha^{\frac{d_s}{2}}\Gamma(1 - d_s/2))^2}.$$

Corollary 4.7.3. *For $\alpha > 0$ and $b \in 2^{F^0}$,*

$$\kappa(\alpha)\int_0^\infty e^{-\alpha t} p_t^b(x,x) dt \leq c_8.$$

Additionally in the case $b = N$,

$$c_7 \leq \kappa(\alpha)\int_0^\infty e^{-\alpha t} p_t^N(x,x) dt.$$

Proof. Directly from Lemma 4.7.1 and subsequent discussion. □

4.7.1 Upper moment bound

Let $b \in 2^{F^0}$ and consider the following SPDE on F for time $t \in [0,\infty)$:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \Delta_b u(t,x) + f(t,u(t,x)) + g(t,u(t,x))\xi(t,x), \\ u(0,x) &= u_0(x). \end{aligned} \tag{4.7.2}$$

Hypothesis 4.7.4. We make the following assumptions:

(1). $u_0 : F \to \mathbb{R}$ is measurable and bounded.

(2). $f, g : \Omega \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$ are functions which are measurable from $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ to $\mathcal{B}(\mathbb{R})$. There exists a constant $C > 0$ such that for all $(\omega, t) \in \Omega \times [0,\infty)$ and all $x, y \in \mathbb{R}$,

$$\begin{aligned} |f(\omega,t,x) - f(\omega,t,y)| + |g(\omega,t,x) - g(\omega,t,y)| &\le C|x-y|, \\ |f(\omega,t,x)| + |g(\omega,t,x)| &\le C(1+|x|). \end{aligned}$$

We may now find an upper bound for the moments of the solution to (4.7.2). Compare [49, Theorem 5.5].

Theorem 4.7.5 (General moment upper bound). *Assume Hypothesis 4.7.4. Let u be the solution to* (4.7.2)*. Then there exist $c_9, c_{10} > 0$ such that for all $p \ge 1$ and all $(t,x) \in [0,\infty) \times F$,*

$$\mathbb{E}\left[|u(t,x)|^p\right]^{\frac{1}{p}} \le c_9 \exp\left(c_{10} p^{1+d_H} t\right).$$

Proof. Let $p \ge 2$ and $\alpha > 0$. Let $(t,x) \in [0,\infty) \times F$. We see that

$$\begin{aligned} e^{-\alpha t}\mathbb{E}\left[|u(t,x)|^p\right]^{\frac{1}{p}} \le & e^{-\alpha t}\left|\int_F p_t^b(x,y)u_0(y)\mu(dy)\right| \\ &+ \mathbb{E}\left[\left|\int_0^t \int_F e^{-\alpha t} p_{t-s}^b(x,y) f(s,u(s,y))\mu(dy)ds\right|^p\right]^{\frac{1}{p}} \\ &+ \mathbb{E}\left[\left|\int_0^t \int_F e^{-\alpha t} p_{t-s}^b(x,y) g(s,u(s,y))\xi(s,y)\mu(dy)ds\right|^p\right]^{\frac{1}{p}}, \end{aligned}$$

which implies

$$\begin{aligned} e^{-\alpha t}\mathbb{E}\left[|u(t,x)|^p\right]^{\frac{1}{p}} \le & \sup_{x \in F}|u_0(x)| \\ &+ \mathbb{E}\left[\left|\int_0^t \int_F e^{-\alpha t} p_{t-s}^b(x,y) f(s,u(s,y))\mu(dy)ds\right|^p\right]^{\frac{1}{p}} \\ &+ 2\sqrt{p}\,\mathbb{E}\left[\left|\int_0^t \int_F e^{-2\alpha t} p_{t-s}^b(x,y)^2 g(s,u(s,y))^2\mu(dy)ds\right|^{\frac{p}{2}}\right]^{\frac{1}{p}}, \end{aligned}$$

where we have used [49, Theorem B.1]. We treat the two integrals on the right-hand side separately. Firstly,

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \int_F e^{-\alpha t} p_{t-s}^b(x,y) f(s,u(s,y))\mu(dy)ds\right|^p\right]^{\frac{1}{p}} \\
&\leq \int_0^t \int_F e^{-\alpha t} p_{t-s}^b(x,y) \mathbb{E}\left[|f(s,u(s,y))|^p\right]^{\frac{1}{p}} \mu(dy)ds \\
&\leq C \int_0^t \int_F e^{-\alpha t} p_{t-s}^b(x,y) \left(1 + \mathbb{E}\left[|u(s,y)|^p\right]^{\frac{1}{p}}\right) \mu(dy)ds \\
&\leq C \left(te^{-\alpha t} + \int_0^t \int_F e^{-\alpha t} p_{t-s}^b(x,y) \sup_{y' \in F} \left(\mathbb{E}\left[|u(s,y')|^p\right]^{\frac{1}{p}}\right) \mu(dy)ds\right) \\
&\leq \frac{C}{e\alpha} + C \int_0^t e^{-\alpha(t-s)} \sup_{y \in F} \left(e^{-\alpha s} \mathbb{E}\left[|u(s,y)|^p\right]^{\frac{1}{p}}\right) ds \\
&\leq \frac{C}{\alpha} \left(1 + \sup_{(s,y) \in [0,t] \times F} \left(e^{-\alpha s} \mathbb{E}\left[|u(s,y)|^p\right]^{\frac{1}{p}}\right)\right),
\end{aligned}$$

and secondly

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \int_F e^{-2\alpha t} p_{t-s}^b(x,y)^2 g(s,u(s,y))^2 \mu(dy)ds\right|^{\frac{p}{2}}\right]^{\frac{1}{p}} \\
&\leq \left(\int_0^t \int_F e^{-2\alpha t} p_{t-s}^b(x,y)^2 \mathbb{E}\left[|g(s,u(s,y))|^p\right]^{\frac{2}{p}} \mu(dy)ds\right)^{\frac{1}{2}} \\
&\leq C \left(\int_0^t \int_F e^{-2\alpha t} p_{t-s}^b(x,y)^2 \sup_{y' \in F} \left(1 + \mathbb{E}\left[|u(s,y')|^p\right]^{\frac{1}{p}}\right)^2 \mu(dy)ds\right)^{\frac{1}{2}} \\
&\leq C \left(\int_0^t e^{-2\alpha(t-s)} p_{2(t-s)}^b(x,x) \sup_{y \in F} \left(e^{-\alpha s} + e^{-\alpha s} \mathbb{E}\left[|u(s,y)|^p\right]^{\frac{1}{p}}\right)^2 ds\right)^{\frac{1}{2}} \\
&\leq C \sup_{(s,y) \in [0,t] \times F} \left(e^{-\alpha s} + e^{-\alpha s} \mathbb{E}\left[|u(s,y)|^p\right]^{\frac{1}{p}}\right) \left(\int_0^t e^{-2\alpha(t-s)} p_{2(t-s)}^b(x,x) ds\right)^{\frac{1}{2}} \\
&\leq \frac{C}{\sqrt{2}} \sup_{(s,y) \in [0,t] \times F} \left(1 + e^{-\alpha s} \mathbb{E}\left[|u(s,y)|^p\right]^{\frac{1}{p}}\right) \left(\int_0^\infty e^{-\alpha s} p_s^b(x,x) ds\right)^{\frac{1}{2}} \\
&\leq C \sqrt{\frac{c_8}{2\kappa(\alpha)}} \left(1 + \sup_{(s,y) \in [0,t] \times F} \left(e^{-\alpha s} \mathbb{E}\left[|u(s,y)|^p\right]^{\frac{1}{p}}\right)\right).
\end{aligned}$$

Putting these estimates together we see that for all $(t,x) \in [0,\infty) \times F$,

$$\begin{aligned}
&e^{-\alpha t} \mathbb{E}\left[|u(t,x)|^p\right]^{\frac{1}{p}} \\
&\quad \leq \sup_{x \in F} |u_0(x)| + C \left(\frac{1}{\alpha} + \sqrt{\frac{2c_8 p}{\kappa(\alpha)}}\right) \left(1 + \sup_{(s,y) \in [0,t] \times F} \left(e^{-\alpha s} \mathbb{E}\left[|u(s,y)|^p\right]^{\frac{1}{p}}\right)\right).
\end{aligned}$$

Now we pick α such that $C\left(\frac{1}{\alpha}+\sqrt{\frac{2c_8 p}{\kappa(\alpha)}}\right)\leq \frac{1}{2}$. By the asymptotic properties of κ, it is possible to find $c_{10}>0$ such that the choice

$$\alpha=\alpha(p):=c_{10}p^{(1-\frac{d_s}{2})^{-1}}$$

works for all $p\geq 2$, and note that $(1-\frac{d_s}{2})^{-1}=1+d_H$. Thus for all $p\geq 2$,

$$\sup_{(s,x)\in[0,t]\times F}\left(e^{-c_{10}p^{1+d_H}s}\mathbb{E}\left[|u(s,x)|^p\right]^{\frac{1}{p}}\right)\leq 2\sup_{x\in F}|u_0(x)|+1.$$

Let $c_9=2\sup_{x\in F}|u_0(x)|+1$. Thus for all $p\geq 2$ and $(t,x)\in[0,\infty)\times F$,

$$\mathbb{E}\left[|u(t,x)|^p\right]^{\frac{1}{p}}\leq c_9\exp\left(c_{10}p^{1+d_H}t\right).$$

The analogous result for $p\in[1,2)$ follows via Jensen's inequality (and a minor adjustment of the value of c_{10}). □

4.7.2 Weak intermittency

Consider the following SPDE on F for time $t\in[0,\infty)$:

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x)&=\Delta_N u(t,x)+g(u(t,x))\xi(t,x),\\ u(0,x)&=u_0(x).\end{aligned}\tag{4.7.3}$$

Hypothesis 4.7.6. We make the following assumptions:

(1). $u_0:F\to\mathbb{R}$ is measurable, bounded and non-negative.

(2). $g:\mathbb{R}\to\mathbb{R}$ is Lipschitz: there exists a constant $C>0$ such that for all $x,y\in\mathbb{R}$,

$$\begin{aligned}|g(x)-g(y)|&\leq C|x-y|,\\ |g(x)|&\leq C(1+|x|).\end{aligned}$$

Theorem 4.7.7 (Limiting second moment lower bound)**.** *There exists a constant $c_{11}>0$ such that the following holds: Assume Hypothesis 4.7.6. Let u be the solution to* (4.7.3)*, and let $I(t)=\inf_{x\in F}\mathbb{E}\left[u(t,x)^2\right]$ for $t\in[0,\infty)$. Then*

$$\liminf_{t\to\infty}\left(\exp\left(-c_{11}\kappa^{-1}(L_g^2)t\right)I(t)\right)\geq\inf_{x\in F}u_0(x)^2,$$

where

$$L_g=\inf_{z\in\mathbb{R}\setminus\{0\}}\left|\frac{g(z)}{z}\right|.$$

Proof. This is similar to the proof of [49, Theorem 7.8]. Notice that since u_0 is non-negative, for all $x \in F$ we have that

$$\int_F p_t^N(x,y)u_0(y)\mu(dy) \geq \inf_{y\in F} u_0(y) \int_F p_t^N(x,y)\mu(dy) = \inf_{y\in F} u_0(y).$$

Using the mild formulation of the solution we see that

$$\begin{aligned}
&\mathbb{E}\left[u(t,x)^2\right] \\
&= \left(\int_F p_t^N(x,y)u_0(y)\mu(dy)\right)^2 + \mathbb{E}\left[\left(\int_0^t \int_F p_{t-s}^N(x,y)g(u(s,y))\xi(s,y)\mu(dy)ds\right)^2\right] \\
&\geq \inf_{x\in F} u_0(x)^2 + \int_0^t \int_F p_{t-s}^N(x,y)^2\mathbb{E}\left[g(u(s,y))^2\right]\mu(dy)ds \\
&\geq \inf_{x\in F} u_0(x)^2 + \int_0^t \int_F L_g^2 p_{t-s}^N(x,y)^2\mathbb{E}\left[u(s,y)^2\right]\mu(dy)ds
\end{aligned}$$

for all $(t,x) \in [0,\infty) \times F$. It follows by Lemma 4.7.1 that

$$\begin{aligned}
I(t) &\geq \inf_{x\in F} u_0(x)^2 + \int_0^t \int_F L_g^2 p_{t-s}^N(x,y)^2 I(s)\mu(dy)ds \\
&= \inf_{x\in F} u_0(x)^2 + \int_0^t L_g^2 p_{2(t-s)}^N(x,x)I(s)ds \\
&\geq \inf_{x\in F} u_0(x)^2 + \int_0^t 2^{-\frac{d_s}{2}} c_7 L_g^2 \left(1 + (t-s)^{-\frac{d_s}{2}}\right) I(s)ds.
\end{aligned}$$

Now if $L_g = 0$ then the result is clear. Henceforth we assume that $L_g > 0$. Recall that the function κ is strictly increasing and bijective on $[0,\infty)$. Let

$$\alpha = \kappa^{-1}\left(2^{-\frac{d_s}{2}} c_7 L_g^2\right) > 0,$$

then we see that

$$e^{-\alpha t}I(t) \geq e^{-\alpha t}\inf_{x\in F} u_0(x)^2 + \int_0^t \kappa(\alpha)e^{-\alpha(t-s)}\left(1 + (t-s)^{-\frac{d_s}{2}}\right)e^{-\alpha s}I(s)ds$$

for all $t \in [0,\infty)$. Since

$$\int_0^\infty \kappa(\alpha)e^{-\alpha s}\left(1 + s^{-\frac{d_s}{2}}\right)ds = 1,$$

it follows that $t \mapsto e^{-\alpha t}I(t)$ is a non-negative supersolution (in the sense of [49, Definition 7.10]) to the renewal-type equation

$$f(t) = e^{-\alpha t}\inf_{x\in F} u_0(x)^2 + \int_0^t \kappa(\alpha)e^{-\alpha(t-s)}\left(1 + (t-s)^{-\frac{d_s}{2}}\right)f(s)ds. \tag{4.7.4}$$

By classical renewal theory [19, Theorem XI.1.2], (4.7.4) has a unique non-negative bounded solution $f : [0, \infty) \to \mathbb{R}$ for which

$$\lim_{t\to\infty} f(t) = \frac{\int_0^\infty e^{-\alpha t} \inf_{x\in F} u_0(x)^2 dt}{\kappa(\alpha) \int_0^\infty te^{-\alpha t} \left(1 + t^{-\frac{d_s}{2}}\right) dt}.$$

We compute the upper integral and use an elementary result on derivatives of Laplace transforms [19, XIII.2(ii)] on the lower integral. This gives

$$\begin{aligned}
\lim_{t\to\infty} f(t) &= \frac{\alpha^{-1} \inf_{x\in F} u_0(x)^2}{\kappa(\alpha) \cdot \frac{\kappa'(\alpha)}{\kappa(\alpha)^2}} \\
&= \frac{1 + \alpha^{\frac{d_s}{2}}\Gamma(1 - d_s/2)}{1 + \alpha^{\frac{d_s}{2}}\Gamma(2 - d_s/2)} \inf_{x\in F} u_0(x)^2 \\
&= \frac{1 + \alpha^{\frac{d_s}{2}}\Gamma(1 - d_s/2)}{1 + \alpha^{\frac{d_s}{2}}(1 - d_s/2)\Gamma(1 - d_s/2)} \inf_{x\in F} u_0(x)^2,
\end{aligned}$$

and it follows that

$$\lim_{t\to\infty} f(t) \geq \inf_{x\in F} u_0(x)^2$$

for any $\alpha > 0$. Now observe: by Theorem 4.7.5 there exists $\beta \geq 0$ such that

$$\sup_{t\in[0,\infty)} \left(e^{-\beta t}e^{-\alpha t}I(t)\right) < \infty.$$

Therefore by [24, Lemma A.2], $e^{-\alpha t}I(t) \geq f(t)$ for all $t \in [0, \infty)$. Finally, there can easily be found a constant $c_{11} > 0$ such that

$$2^{-\frac{d_s}{2}} c_7\kappa(x) \geq \kappa(c_{11}x)$$

for all $x \geq 0$. Taking $x = \kappa^{-1}(L_g^2)$ this implies that $\alpha \geq c_{11}\kappa^{-1}(L_g^2)$. So

$$\exp(-c_{11}\kappa^{-1}(L_g^2)t)I(t) \geq f(t)$$

for all $t \in [0, \infty)$. The result follows. □

Corollary 4.7.8 (Weak intermittency)**.** *Assume Hypothesis 4.7.6. Suppose that*

$$\inf_{z\in\mathbb{R}\setminus\{0\}} \left|\frac{g(z)}{z}\right| > 0,$$
$$\inf_{x\in F} u_0(x) > 0.$$

Then the solution to (4.7.3) *is weakly intermittent in the sense of [49, Definition 7.7].*

Proof. Theorem 4.7.7 directly implies the required result. □

Chapter 5

The damped stochastic wave equation on p.c.f. fractals

5.1 Introduction

The aim of this chapter is to investigate the properties of some hyperbolic stochastic partial differential equations (SPDEs) on finitely ramified fractals. In one dimension [77] motivated this problem as understanding the behaviour of a guitar string in a sandstorm. That is, we have a one-dimensional string which is forced by white noise at every point in time and space and are interested in the "music" – the properties of the resulting waves induced in the string. In the fractal setting we may think of the vibrations of a fractal drum in a sandstorm. For a two-dimensional drum, it is known that the solutions to the stochastic wave equation are no longer functions and thus it is of interest to see what happens in the case of finitely ramified fractals which behave analytically as objects with dimension between one and two. As yet the theory for the behaviour of waves propagating through a fractal is much less developed than that for the diffusion of heat and we will not discuss such deterministic waves. Instead we consider the regularity properties of the waves starting from rest and arising from forcing by white noise, which are easier to capture, as it is the noise and its smoothing via the Laplacian which are crucial to understanding the behaviour of the waves.

The damped stochastic wave equation on $\mathbb{R}^n$, $n \geq 1$ is the SPDE given by

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t,x) &= -2\beta\frac{\partial u}{\partial t}(t,x) + \Delta u(t,x) + \xi(t,x), \\ u(0,\cdot) &= \frac{\partial u}{\partial t}(0,\cdot) = 0, \end{aligned} \tag{5.1.1}$$

where $\beta \geq 0$, $\Delta = \Delta_x$ is the Laplacian on $\mathbb{R}^n$ and ξ is a space-time white noise on $[0,\infty) \times \mathbb{R}^n$, where we interpret $x \in \mathbb{R}^n$ as space and $t \in [0,\infty)$ as time. The equation

Original article: [30]

(5.1.1) can equivalently be written as a system of stochastic evolution equations in the following way:

$$\begin{aligned} du(t) &= \dot{u}(t)dt, \\ d\dot{u}(t) &= -2\beta\dot{u}(t)dt + \Delta u(t)dt + dW(t), \\ u(0) &= \dot{u}(0) = 0 \in L^2(\mathbb{R}^n), \end{aligned}$$

where W is a cylindrical Wiener process on $L^2(\mathbb{R}^n)$, and the solution u and its (formal) derivative $\dot{u}$ are processes taking values in some space of functions on $\mathbb{R}^n$. Here we have used instead the differential notation of stochastic calculus, and one should not presume any a priori relationship between u and $\dot{u}$. The damped stochastic wave equation (SWE) was introduced in [10] in the case $n = 1$, and a unique solution was found via a Fourier transform. If $\beta = 0$, there is no damping, and this is the stochastic wave equation. The solution then has a neat characterisation given in [77, Theorem 3.1] as a rotated modified Brownian sheet in $[0, \infty) \times \mathbb{R}$, and this immediately implies that it is jointly Hölder continuous in space and time for any Hölder exponent less than $\frac{1}{2}$. These properties, however, do not carry over into spatial dimensions $n \geq 2$. Indeed, for $n \geq 2$ a solution to (5.1.1) still exists, but it is not function-valued. It is necessary to expand our space beyond $L^2(\mathbb{R}^n)$ to include certain distributions in order to make sense of the solution. This is related to the fact that n-dimensional Brownian motion has local times if and only if $n = 1$, see [20] and further references. There is thus a distinct change in the behaviour of the SPDE (5.1.1) between dimensions $n = 1$ and $n \geq 2$. One of the aims of the present chapter is to investigate the behaviour of the SPDE in the case that dimension (appropriately interpreted) is in the interval $[1, 2)$. When does a function-valued solution exist, and if it does, what are its space-time Hölder exponents? To answer these questions we introduce a class of fractals.

The theory of analysis on fractals started with the construction of a symmetric diffusion on the two-dimensional Sierpinski gasket in [25], [58] and [5], which is now known as *Brownian motion on the Sierpinski gasket*. The field has grown quickly since then; see [52] and [3] for analytic and probabilistic introductions respectively. In [52] it is shown that a certain class of fractals, known as *post-critically finite self-similar* (or *p.c.f.s.s.*) sets with *regular harmonic structures*, admit operators Δ akin to the Laplacian on $\mathbb{R}^n$. This class includes many well-known fractals such as the n-dimensional Sierpinski gasket (for $n \geq 2$) and the Vicsek fractal, though not the Sierpinski carpet. The operators Δ generate symmetric diffusions on their respective fractals in the same way that the Laplacian on $\mathbb{R}^n$ is the generator of Brownian motion on $\mathbb{R}^n$, and we therefore refer to them also as "Laplacians", see [3]. In particular, the existence of a Laplacian Δ on a given fractal F allows us to formulate PDEs analogous to the heat equation and the wave equation on F. The heat equation on F has been widely studied, see [52, Chapter 5] and many other papers showing results such as sub-Gaussian decay of the heat kernel. It is possible in the same way to formulate certain SPDEs on these fractals; for example the stochastic heat equation (Chapter 3) and, the subject of the present chapter, the damped stochastic wave equation on F. The *spectral dimension* d_s, defined as the exponent for the asymptotic scaling of the eigenvalue counting function of Δ, for any of these fractals satisfies $d_s < 2$, and is the correct definition of dimension to use when investigating the analytic properties of the SPDE. Since all of our fractals are compact, we can use spectral methods to vastly simplify the problem and find a solution explicitly in terms of the eigenvalues and eigenfunctions of the Laplacian.

Previous work on hyperbolic PDEs and SPDEs on fractals is sparse. The wave equation was first introduced in [58]. Since then, there have been two strands of work, either focusing on bounded or on unbounded fractals. In the case of bounded fractals [16] gave strong evidence that there would be infinite propagation speed for the deterministic wave equation (see also [61]) and [42] showed existence and uniqueness for a non-linear wave equation. For the unbounded case there is work by [60] and [74] discussing the long time behaviour of waves on manifolds with large scale fractal structure and on fractals themselves.

In [20] it is mentioned that the stochastic heat equation on certain fractals has a so-called "random-field" solution as long as the Hausdorff dimension of the fractal is less than 2. The stochastic wave equation is studied elsewhere in that paper but an analogous result is not given. In Chapter 3 the stochastic heat equation on p.c.f.s.s. sets with regular harmonic structures is shown to have continuous function-valued solutions, as the spectral dimension is less than 2, and its spatial and temporal Hölder exponents are computed; this can be seen to be the direct predecessor of the present chapter and is the source of many of the ideas that we use in the following sections.

The structure of the present chapter is as follows: In the next subsection we set up the problem, state the precise SPDE to be solved and summarise the main results of the chapter. In Section 5.2 we make precise the definition of a solution to the damped stochastic wave equation and prove the existence of a unique solution u in the form of an L^2-valued process. We show that it is a solution in both a "mild" sense and a "weak" sense. Then, in Section 5.3, we show that this solution is Hölder continuous in L^2 and that the point evaluations $u(t, x)$ are well-defined random variables. The latter is a necessary condition for us to be able to consider matters of continuity in space and time. In Section 5.4 we utilise a Kolmogorov-type continuity theorem for fractals proven in Chapter 3 to deduce the spatial and temporal Hölder exponents of the solution u. In Section 5.5 we give results that describe the long-time behaviour of the solutions for any given set of parameters, in particular whether or not they eventually settle down into some equilibrium measure.

5.1.1 Description of the problem

We use an identical set-up to Chapter 3. Let $M \geq 2$ be an integer. Let $(F, (\psi_i)_{i=1}^M)$ be a connected p.c.f.s.s. set (see [52]) such that F is a compact metric space and the $\psi_i : F \to F$ are injective strict contractions on F. Let $I = \{1, \dots, M\}$ and for each $n \geq 0$ let $\mathbb{W}_n = I^n$. Let $\mathbb{W}_* = \bigcup_{n \geq 0} \mathbb{W}_n$ and let $\mathbb{W} = I^{\mathbb{N}}$. We call the sets $\mathbb{W}_n$, $\mathbb{W}_*$ and $\mathbb{W}$ *word spaces* and we call their elements *words*. Note that $\mathbb{W}_0$ is a singleton containing an element known as the *empty word*. We use the notation $w = w_1 w_2 w_3 \dots$ with $w_i \in I$ for words $w \in \mathbb{W}_* \cup \mathbb{W}$. For a word $w = w_1, \dots, w_n \in \mathbb{W}_*$, let $\psi_w = \psi_{w_1} \circ \cdots \circ \psi_{w_n}$ and let $F_w = \psi_w(F)$. If w is the empty word then ψ_w is the identity on F.

If $\mathbb{W}$ is endowed with the standard product topology then there is a canonical continuous surjection $\pi : \mathbb{W} \to F$ given in [3, Lemma 5.10]. Let $P \subset \mathbb{W}$ be the post-critical set of $(F, (\psi_i)_{i=1}^M)$, which is finite by assumption. Then let $F^0 = \pi(P)$, and for each $n \geq 1$ let $F^n = \bigcup_{w \in \mathbb{W}_n} \psi_w(F^0)$. Let $F_* = \bigcup_{n=0}^{\infty} F^n$. It is easily shown that $(F^n)_{n \geq 0}$ is an increasing sequence of finite subsets and that F_* is dense in F.

Let the pair $(A_0, \mathbf{r})$ be a regular irreducible harmonic structure on $(F, (\psi_i)_{i=1}^M)$ such that $\mathbf{r} = (r_1, \dots, r_M) \in \mathbb{R}^M$ for some constants $r_i > 0$, $i \in I$ (harmonic structures are defined in [52, Section 3.1]). Here *regular* means that $r_i \in (0,1)$ for all i. Let $r_{\min} = \min_{i \in I} r_i$ and $r_{\max} = \max_{i \in I} r_i$. If $n \geq 0$, $w = w_1, \dots w_n \in \mathbb{W}$ then write $r_w := \prod_{i=1}^n r_{w_i}$. Let $d_H > 0$ be the unique real number such that

$$\sum_{i \in I} r_i^{d_H} = 1.$$

Then let μ be the Borel regular probability measure on F such that for any $n \geq 0$, if $w \in \mathbb{W}_n$ then $\mu(F_w) = r_w^{d_H}$. In other words, μ is the self-similar measure on F in the sense of [52, Section 1.4] associated with the weights $r_i^{d_H}$ on I. Let $(\mathcal{E}, \mathcal{D})$ be the regular local Dirichlet form on $L^2(F, \mu)$ associated with this harmonic structure, as given by [52, Theorem 3.4.6]. This Dirichlet form is associated with a resistance metric R on F, defined by

$$R(x,y) = \left(\inf\{\mathcal{E}(f,f) : f(x) = 0, f(y) = 1, f \in \mathcal{D}\}\right)^{-1},$$

which generates the original topology on F, by [52, Theorem 3.3.4]. Now let $2^{F^0} = \{b : b \subseteq F^0\}$ be the power set of F^0. Let $\mathcal{D}_{F^0} = \mathcal{D}$, and for proper subsets $b \in 2^{F^0}$ let

$$\mathcal{D}_b = \{f \in \mathcal{D} : f|_{F^0 \setminus b} = 0\}.$$

Then similarly to [52, Corollary 3.4.7], $(\mathcal{E}, \mathcal{D}_b)$ is a regular local Dirichlet form on $L^2(F \setminus (F \setminus b), \mu)$. If $b = F^0$ then we may equivalently write $b = N$, and if $b = \emptyset$ then we may equivalently write $b = D$, see Chapter 3. The letters N and D indicate *Neumann* and *Dirichlet* boundary conditions respectively, and all other values of b indicate a *mixed* boundary condition. Intuitively, b gives the subset of F^0 of points that are free to move under the influence of the SPDE, whereas the remaining elements of F^0 are fixed at the value 0.

Let $b \in 2^{F^0}$. By [3, Chapter 4], associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}_b)$ on $L^2(F, \mu)$ is a μ-symmetric diffusion $X^b = (X^b_t)_{t \geq 0}$ on F which itself is associated with a C_0-semigroup of contractions $S^b = (S^b_t)_{t \geq 0}$ on $L^2(F, \mu)$. Let Δ_b be the generator of this diffusion. If $b = N$ then X^N has infinite lifetime, by [3, Lemma 4.10]. On the other hand, if b is a proper subset of F^0 then the process X^b has the law of a version of X^N which is killed at the points $F^0 \setminus b$, by [22, Section 4.4]. Our notation is identical to that of Chapter 4.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The *damped stochastic wave equation* that we consider in the present chapter is the SPDE (system) given by

$$\begin{aligned} du(t) &= \dot{u}(t)dt, \\ d\dot{u}(t) &= -2\beta \dot{u}(t)dt + \Delta_b u(t)dt + dW(t), \\ u(0) &= \dot{u}(0) = 0 \in L^2(F, \mu), \end{aligned} \tag{5.1.2}$$

where $\beta \geq 0$ the *damping coefficient* and $b \in 2^{F^0}$ the *boundary conditions* are parameters, and $W = (W(t))_{t \geq 0}$ is a $\mathbb{P}$-cylindrical Wiener process on $L^2(F, \mu)$. That is, W satisfies

$$\mathbb{E}\left[\langle f, W(s)\rangle_{L^2(F,\mu)} \langle W(t), g\rangle_{L^2(F,\mu)}\right] = (s \wedge t)\langle f, g\rangle_{L^2(F,\mu)}$$

for all $s, t \in [0, \infty)$ and $f, g \in L^2(F, \mu)$. We would like the solution process $u = (u(t))_{t \geq 0}$ to be $L^2(F, \mu)$-valued, however it is not clear whether or not the same should be required of the first-derivative process $\dot{u} = (\dot{u}(t))_{t \geq 0}$. This will be clarified in the following section.

The main results of the present chapter (Theorems 5.2.11, 5.3.9 and 5.4.5) can be roughly paraphrased as follows:

Theorem 5.1.1. *Equip F with its resistance metric R. The SPDE* (5.1.2) *has a unique solution which is a stochastic process $u = (u(t,x) : (t,x) \in [0,\infty) \times F)$, which is almost surely jointly continuous in $[0,\infty) \times F$. For each $t \in [0,\infty)$, $u(t,\cdot)$ is almost surely essentially $\frac{1}{2}$-Hölder continuous in (F,R). For each $x \in F$, $u(\cdot,x)$ is almost surely essentially $(1-\frac{d_s}{2})$-Hölder continuous in the Euclidean metric, where $d_s \in [1,2)$ is the spectral dimension of (F,R).*

The precise meaning of *essentially* is given in Section 5.3. We see that the Hölder exponents given in the above theorem agree with the case "$F = \mathbb{R}$" described in the introduction – there we have $d_s = 1$, and the solution is a rotation of a modified Brownian sheet so it has essential Hölder exponent $\frac{1}{2}$ in every direction. Of course $\mathbb{R}$ is not compact so it doesn't exactly fit into our set-up, but we get a similar result by considering the interval $[0,1]$ instead, see Example 3.1.1.

Example 5.1.2 (Hata's tree-like set)**.** See [52, Figure 1.4] for a diagram. This p.c.f. fractal takes a parameter $c \in \mathbb{C}$ such that $|c|, |1-c| \in (0,1)$, with $F^0 = \{c, 0, 1\}$, as described in [52, Example 3.1.6]. It has a collection of regular harmonic structures given by

$$A_0 = \begin{pmatrix} -h & h & 0 \\ h & -(h+1) & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

with $\mathbf{r} = (h^{-1}, 1 - h^{-2})$ for $h \in (1,\infty)$, and these all fit into our set-up. In the introduction to [77] the stochastic wave equation on the unit interval is said to describe the motion of a guitar string in a sandstorm (as long as we specify Dirichlet boundary conditions). Likewise, by taking $b = \{c, 1\}$ in our tree-like set, we are "planting" it at the point 0 so the associated stochastic wave equation approximately describes the motion of a tree in a sandstorm.

For more examples see Example 3.1.3.

Remark 5.1.3. The resistance metric R is not a particularly intuitive metric on F. However, many fractals have a natural embedding in Euclidean space $\mathbb{R}^n$, and subject to mild conditions on F it can be shown that R is equivalent to some positive power of the Euclidean metric, see [43]. An example is the n-dimensional Sierpinski gasket described in Example 3.1.3 with $n \geq 2$. In [43, Section 3] it is shown that there exists a constant $c > 0$ such that

$$c^{-1}|x-y|^{d_w - d_f} \leq R(x,y) \leq c|x-y|^{d_w - d_f}$$

for all $x, y \in F \subseteq \mathbb{R}^n$, where $d_w = \frac{\log(n+3)}{\log 2}$ is the *walk dimension* of the gasket and $d_f = \frac{\log(n+1)}{\log 2}$ is its Euclidean Hausdorff dimension. It follows that the above theorem holds (with a different spatial Hölder exponent) if R is replaced with a Euclidean metric on F.

5.2 Existence and uniqueness of solution

In this section we will make explicit the meaning of a *solution* to the SPDE (5.1.2), and show that such a solution exists and is unique.

Definition 5.2.1. Henceforth let $\mathcal{H} = L^2(F, \mu)$ and denote its inner product by $\langle \cdot, \cdot \rangle_\mu$. Moreover, for $\lambda > 0$ let $\mathcal{D}^\lambda$ be the space $\mathcal{D}$ equipped with the inner product

$$\langle \cdot, \cdot \rangle_\lambda := \mathcal{E}(\cdot, \cdot) + \lambda \langle \cdot, \cdot \rangle_\mu.$$

Since $(\mathcal{E}, \mathcal{D})$ is closed, $\mathcal{D}^\lambda$ is a Hilbert space.

Remark 5.2.2. The space $\mathcal{D}$ contains only $\frac{1}{2}$-Hölder continuous functions since by the definition of the resistance metric we have that

$$|f(x) - f(y)|^2 \leq R(x, y)\mathcal{E}(f, f) \tag{5.2.1}$$

for all $f \in \mathcal{D}$ and all $x, y \in F$. Therefore, since $\mathcal{D}_b$ is the intersection of the kernels of the continuous linear functionals $\{f \mapsto f(x) : x \in F^0 \setminus b\}$, it is a closed subset of any $\mathcal{D}^\lambda$ and has finite codimension $|F^0 \setminus b|$.

Definition 5.2.3. The unique real $d_H > 0$ such that

$$\sum_{i \in I} r_i^{d_H} = 1$$

is the *Hausdorff dimension* of (F, R), see [52, Theorem 1.5.7].

The *spectral dimension* of (F, R) is given by

$$d_s = \frac{2d_H}{d_H + 1},$$

see [52, Theorem 4.1.5 and Theorem 4.2.1]. Note by Remark 3.2.6 that $d_H \in [1, \infty)$ and $d_s \in [1, 2)$.

If A is a linear operator on $\mathcal{H}$ then we denote the domain of A by $\mathcal{D}(A)$. If A is bounded, then let $\|A\|$ be its operator norm. By Proposition 4.2.5, for each $b \in 2^{F^0}$ there exists an orthonormal basis $(\varphi_k^b)_{k=1}^\infty$ of $\mathcal{H}$, where the associated eigenvalues $(\lambda_k^b)_{k=1}^\infty$ are assumed to be in increasing order. In particular any element $f \in \mathcal{H}$ has a series representation

$$f = \sum_{k=1}^{\infty} f_k \varphi_k^b$$

where $f_k = \langle \varphi_k^b, f \rangle_\mu$. Then for any function $\Xi : [0, \infty) \to \mathbb{R}$, the map $\Xi(-\Delta_b)$ is a well-defined self-adjoint operator from $\mathcal{D}(\Xi(-\Delta_b))$ to $\mathcal{H}$ and has the representation

$$\Xi(-\Delta_b) f = \sum_{k=1}^{\infty} f_k \Xi(\lambda_k^b) \varphi_k^b,$$

where the domain $\mathcal{D}(\Xi(-\Delta_b)))$ is the subspace of $\mathcal{H}$ of exactly those f for which the above expression is in $\mathcal{H}$. In fact the operator $\Xi(-\Delta_b)$ is densely defined since $\varphi_k^b \in \mathcal{D}(\Xi(-\Delta_b)))$ for all k. This theory is known as the *functional calculus* for linear operators, see [69, Theorem VIII.5].

In particular, if $\alpha \geq 0$ then $(1-\Delta_b)^{\frac{\alpha}{2}}$ is an invertible operator on $\mathcal{H}$, and its inverse $(1-\Delta_b)^{-\frac{\alpha}{2}}$ is a bounded operator on $\mathcal{H}$ which is a bijection from $\mathcal{H}$ to $\mathcal{D}((1-\Delta_b)^{\frac{\alpha}{2}})$.

Definition 5.2.4. Let $\alpha \geq 0$ be a real number and $b \in 2^{F^0}$. The bounded operator $(1-\Delta_b)^{-\frac{\alpha}{2}}$ is called a *Bessel potential operator*, compare [73], [46]. Let $\mathcal{H}_b^{-\alpha}$ be the closure of $\mathcal{H}$ with respect to the inner product given by

$$(f, g) \mapsto \langle (1-\Delta_b)^{-\frac{\alpha}{2}} f, (1-\Delta_b)^{-\frac{\alpha}{2}} g \rangle_\mu.$$

$\mathcal{H}_b^{-\alpha}$ is called a *Sobolev space* or a *Bessel potential space*, compare [73], [1, Section 1.2], [76, Section 2.2.2].

Remark 5.2.5. Recall that $\mathcal{D}((1-\Delta_b)^{\frac{\alpha}{2}})$ is dense in $\mathcal{H}$. It follows that the operator $(1-\Delta_b)^{\frac{\alpha}{2}} : \mathcal{D}((1-\Delta_b)^{\frac{\alpha}{2}}) \to \mathcal{H}$ extends to an isomorphism from $\mathcal{H}$ to $\mathcal{H}_b^{-\alpha}$ characterised by

$$(1-\Delta_b)^{\frac{\alpha}{2}} \left(\sum_{k=1}^{\infty} f_k \varphi_k^b \right) = \sum_{k=1}^{\infty} (1+\lambda_k^b)^{\frac{\alpha}{2}} f_k \varphi_k^b.$$

It is easy to see from the above equation that if $b \in 2^{F^0}$ then $\left((1+\lambda_k^b)^{\frac{\alpha}{2}} \varphi_k^b \right)_{k=1}^\infty$ is a complete orthonormal basis of $\mathcal{H}_b^{-\alpha}$.

5.2.1 Solution to the SPDE

Let $\oplus$ denote direct sum of Hilbert spaces. Let $\alpha \geq 0$. The SPDE (5.1.2) can be recast as a first-order SPDE on the Hilbert space $\mathcal{H} \oplus \mathcal{H}_b^{-\alpha}$ given by

$$\begin{aligned} dU(t) &= \mathcal{A}_{b,\beta} U(t) dt + d\mathcal{W}(t), \\ U(0) &= 0 \in \mathcal{H} \oplus \mathcal{H}_b^{-\alpha}, \end{aligned} \tag{5.2.2}$$

where

$$\mathcal{A}_{b,\beta} := \begin{pmatrix} 0 & 1 \\ \Delta_b & -2\beta \end{pmatrix}$$

is a densely defined operator on $\mathcal{H} \oplus \mathcal{H}_b^{-\alpha}$ with $\mathcal{D}(\mathcal{A}_{b,\beta}) = \mathcal{D}\left(\Delta_b^{(1-\frac{\alpha}{2})\vee 0}\right) \oplus \mathcal{H}$ and

$$\mathcal{W} := \begin{pmatrix} 0 \\ W \end{pmatrix}.$$

There is a precise definition of a solution to evolution equations of the form (5.2.2) which is given in [15, Chapter 5], so we can now finally define the notion of a solution to the second-order SPDE (5.1.2). Note that it is still not clear what value of α should be picked.

Definition 5.2.6. Let $T \in (0, \infty]$. An $\mathcal{H}$-valued predictable process $u = (u(t))_{t=0}^T$ is a *solution* to the SPDE (5.1.2) if there exists $\alpha \geq 0$ and an $\mathcal{H}_b^{-\alpha}$-valued predictable process $\dot{u} = (\dot{u}(t))_{t=0}^T$ such that

$$U := \begin{pmatrix} u \\ \dot{u} \end{pmatrix}$$

is an $\mathcal{H} \oplus \mathcal{H}_b^{-\alpha}$-valued weak solution to the SPDE (5.2.2) in the sense of [15, Chapter 5]. If $T = \infty$ then it is a *global* solution.

Admittedly, the above definition is lacking as it is very abstract and unintuitive. In Theorem 5.2.11 we prove that solutions to (5.1.2) also satisfy a property which is analogous to the concept of weak solution as defined in [15, Chapter 5], and is much more instructive.

Definition 5.2.7. For $\lambda \geq 0$ and $\beta \geq 0$, let $V_\beta(\lambda, \cdot) : [0, \infty) \to \mathbb{R}$ be the unique solution to the second-order ordinary differential equation

$$\begin{aligned} &\frac{d^2 v}{dt^2} + 2\beta \frac{dv}{dt} + \lambda v = 0, \\ &v(0) = 0, \ \frac{dv}{dt}(0) = 1. \end{aligned} \tag{5.2.3}$$

Explicitly,

$$V_\beta(\lambda,t)=\begin{cases}(\beta^2-\lambda)^{-\frac{1}{2}}e^{-\beta t}\sinh\left((\beta^2-\lambda)^{\frac{1}{2}}t\right) & \lambda<\beta^2,\\ te^{-\beta t} & \lambda=\beta^2,\\ (\lambda-\beta^2)^{-\frac{1}{2}}e^{-\beta t}\sin\left((\lambda-\beta^2)^{\frac{1}{2}}t\right) & \lambda>\beta^2.\end{cases}$$

For fixed λ and β, this function is evidently smooth in $[0,\infty)$. Let $\dot V_\beta(\lambda,\cdot)=\frac{dV_\beta}{dt}(\lambda,\cdot)$. Compare [10, Lemma 2.1].

Remark 5.2.8. The different forms of V_β correspond respectively to the motion of overdamped, critically damped and underdamped oscillators.

Lemma 5.2.9. *Let $\alpha=1$. Then for each $\beta\geq 0$ and $b\in 2^{F^0}$, the operator $\mathcal{A}_{b,\beta}$ generates a quasicontraction semigroup $\mathcal{S}^{b,\beta}=(\mathcal{S}_t^{b,\beta})_{t\geq 0}$ on $\mathcal{H}\oplus\mathcal{H}_b^{-1}$ such that $\|\mathcal{S}_t^{b,\beta}\|\leq e^{\frac{t}{2}}$ for all $t\geq 0$. Moreover, the right column of $\mathcal{S}_t^{b,\beta}$ is given by*

$$\mathcal{S}_t^{b,\beta}=\left(\begin{array}{cc}\cdot & V_\beta(-\Delta_b,t)\\ \cdot & \dot V_\beta(-\Delta_b,t)\end{array}\right).$$

Proof. Recall that

$$\mathcal{A}_{b,\beta}=\left(\begin{array}{cc}0 & 1\\ \Delta_b & -2\beta\end{array}\right).$$

If $f\in\mathcal{D}(\Delta_b^{\frac{1}{2}})$, $g\in\mathcal{H}$ then

$$\begin{aligned}&\left\langle\mathcal{A}_{b,\beta}\left(\begin{array}{c}f\\ g\end{array}\right),\left(\begin{array}{c}f\\ g\end{array}\right)\right\rangle_{\mathcal{H}\oplus\mathcal{H}_b^{-1}}\\ &=\langle g,f\rangle_\mu+\langle\Delta_b(1-\Delta_b)^{-\frac{1}{2}}f,(1-\Delta_b)^{-\frac{1}{2}}g\rangle_\mu-2\beta\|(1-\Delta_b)^{-\frac{1}{2}}g\|_\mu^2\\ &=\langle(1-\Delta_b)(1-\Delta_b)^{-1}f,g\rangle_\mu+\langle\Delta_b(1-\Delta_b)^{-1}f,g\rangle_\mu-2\beta\|(1-\Delta_b)^{-\frac{1}{2}}g\|_\mu^2\\ &=\langle f,(1-\Delta_b)^{-1}g\rangle_\mu-2\beta\|(1-\Delta_b)^{-\frac{1}{2}}g\|_\mu^2\\ &\leq\frac{1}{2}\|f\|_\mu^2+\frac{1}{2}\|(1-\Delta_b)^{-1}g\|_\mu^2-2\beta\|(1-\Delta_b)^{-\frac{1}{2}}g\|_\mu^2\end{aligned}$$

where in the last line we have used the Cauchy-Schwarz inequality. Now $\|(1-\Delta_b)^{-\frac{1}{2}}\|\leq 1$ by the functional calculus. It follows that

$$\begin{aligned}&\left\langle\left(\mathcal{A}_{b,\beta}-\frac{1}{2}\right)\left(\begin{array}{c}f\\ g\end{array}\right),\left(\begin{array}{c}f\\ g\end{array}\right)\right\rangle_{\mathcal{H}\oplus\mathcal{H}_b^{-1}}\\ &\leq-\frac{1}{2}\left(\|(1-\Delta_b)^{-\frac{1}{2}}g\|_\mu^2-\|(1-\Delta_b)^{-1}g\|_\mu^2\right)-2\beta\|(1-\Delta_b)^{-\frac{1}{2}}g\|_\mu^2\\ &\leq 0,\end{aligned}$$

which implies that the operator $\mathcal{A}_{b,\beta} - \frac{1}{2}$ is dissipative. Moreover, it can be easily checked that the operator $\lambda - \mathcal{A}_{b,\beta}$ is invertible for any $\lambda > 0$ with bounded inverse

$$(\lambda - \mathcal{A}_{b,\beta})^{-1} = \begin{pmatrix} 2\beta + \lambda & 1 \\ \Delta_b & \lambda \end{pmatrix} (\lambda(\lambda + 2\beta) - \Delta_b)^{-1}.$$

It follows by the Lumer–Phillips theorem for reflexive Banach spaces [18, Corollary II.3.20] that $\mathcal{A}_{b,\beta} - \frac{1}{2}$ generates a contraction semigroup on $\mathcal{H} \oplus \mathcal{H}_b^{-1}$. It follows that $\mathcal{A}_{b,\beta}$ generates a quasicontraction semigroup $\mathcal{S}^{b,\beta} = (\mathcal{S}_t^{b,\beta})_{t\geq 0}$ on $\mathcal{H} \oplus \mathcal{H}_b^{-1}$ such that $\|\mathcal{S}_t^{b,\beta}\| \leq e^{\frac{t}{2}}$ for all $t \geq 0$.

To construct the semigroup $\mathcal{S}$, we first observe that $\mathcal{H} \oplus \mathcal{H}_b^{-1}$ has a complete orthonormal basis given by

$$\left\{ \begin{pmatrix} \varphi_k^b \\ 0 \end{pmatrix} : k \in \mathbb{N} \right\} \cup \left\{ \begin{pmatrix} 0 \\ (1 + \lambda_k^b)^{\frac{1}{2}} \varphi_k^b \end{pmatrix} : k \in \mathbb{N} \right\},$$

and that all of the elements of this basis are in $\mathcal{D}(\mathcal{A}_{b,\beta})$. By a density argument, it suffices to compute how $\mathcal{A}_{b,\beta}$ affects the elements of this basis. For $k \geq 1$ we see that

$$\mathcal{A}_{b,\beta} \begin{pmatrix} \varphi_k^b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\lambda_k^b & -2\beta \end{pmatrix} \begin{pmatrix} \varphi_k^b \\ 0 \end{pmatrix},$$
$$\mathcal{A}_{b,\beta} \begin{pmatrix} 0 \\ (1 + \lambda_k^b)^{\frac{1}{2}} \varphi_k^b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\lambda_k^b & -2\beta \end{pmatrix} \begin{pmatrix} 0 \\ (1 + \lambda_k^b)^{\frac{1}{2}} \varphi_k^b \end{pmatrix}.$$

Therefore to compute the semigroup $\mathcal{S}^{b,\beta}$ it will suffice to take a simple matrix exponential. We see that

$$\exp\left[\begin{pmatrix} 0 & 1 \\ -\lambda_k^b & -2\beta \end{pmatrix} t \right] = \begin{pmatrix} \cdot & V_\beta(\lambda_k^b, t) \\ \cdot & \dot{V}_\beta(\lambda_k^b, t) \end{pmatrix},$$

where the left column of the matrix is not computed as it is not important. It follows that the semigroup generated by $\mathcal{A}_{b,\beta}$ takes the form

$$\mathcal{S}_t^{b,\beta} = \begin{pmatrix} \cdot & V_\beta(-\Delta_b, t) \\ \cdot & \dot{V}_\beta(-\Delta_b, t) \end{pmatrix}.$$

□

Proposition 5.2.10. *Let $\alpha = 1$. Then for each $\beta \geq 0$ and $b \in 2^{F^0}$ there is a unique global $\mathcal{H} \oplus \mathcal{H}_b^{-1}$-valued weak solution U to the SPDE* (5.2.2) *given by*

$$U(t) = \begin{pmatrix} \int_0^t V_\beta(-\Delta_b, t-s) dW(s) \\ \int_0^t \dot{V}_\beta(-\Delta_b, t-s) dW(s) \end{pmatrix}.$$

In particular, it is a centred Gaussian process and has an $\mathcal{H}\oplus\mathcal{H}_b^{-1}$-continuous version.

Proof. Following [15, Section 5.1.2], we define the stochastic convolution

$$W_\beta^b(t) := \int_0^t \mathcal{S}_{t-s}^{b,\beta} d\mathcal{W}(t) = \int_0^t \mathcal{S}_{t-s}^{b,\beta} \iota_2 dW(t)$$

for $t \geq 0$, where $\iota_2 : \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}_b^{-1}$ is the (bounded linear) map $f \mapsto \begin{pmatrix} 0 \\ f \end{pmatrix}$. For $a \in [0, 1)$ we wish to show that

$$\int_0^T t^{-a} \left\| \mathcal{S}_t^{b,\beta} \iota_2 \right\|^2_{\mathrm{HS}(\mathcal{H} \to \mathcal{H} \oplus \mathcal{H}_b^{-1})} dt < \infty$$

for all $T > 0$, where $\| \cdot \|_{\mathrm{HS}(\mathcal{H} \to \mathcal{H} \oplus \mathcal{H}_b^{-1})}$ denotes the Hilbert-Schmidt norm of operators from $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}_b^{-1}$. We have that

$$\begin{aligned}
\int_0^T t^{-a} \left\| \mathcal{S}_t^{b,\beta} \iota_2 \right\|^2_{\mathrm{HS}(\mathcal{H} \to \mathcal{H} \oplus \mathcal{H}_b^{-1})} dt &= \int_0^T t^{-a} \sum_{k=1}^\infty \left\| \mathcal{S}_t^{b,\beta} \iota_2 \varphi_k^b \right\|^2_{\mathcal{H} \oplus \mathcal{H}_b^{-1}} dt \\
&= \sum_{k=1}^\infty \int_0^T t^{-a} \left\| \begin{pmatrix} V_\beta(-\Delta_b, t)\varphi_k^b \\ \dot{V}_\beta(-\Delta_b, t)\varphi_k^b \end{pmatrix} \right\|^2_{\mathcal{H} \oplus \mathcal{H}_b^{-1}} dt \\
&= \sum_{k=1}^\infty \int_0^T t^{-a} V_\beta(\lambda_k^b, t)^2 dt + \sum_{k=1}^\infty \int_0^T t^{-a} (1 + \lambda_k^b)^{-1} \dot{V}_\beta(\lambda_k^b, t)^2 dt,
\end{aligned}$$

and we treat the above two sums separately.

Now $t \mapsto t^{-a} V_\beta(\lambda_k^b, t)^2$ is always integrable in $[0, T]$ so the only thing that can go wrong is the sum. Since there are only finitely many k such that $\lambda_k^b \leq \beta^2$, it suffices to consider the case $\lambda_k^b > \beta^2$. In this case we have that

$$\begin{aligned}
\int_0^T t^{-a} V_\beta(\lambda_k^b, t)^2 dt &= (\lambda_k^b - \beta^2)^{-1} \int_0^T t^{-a} e^{-2\beta t} \sin^2 \left((\lambda_k^b - \beta^2)^{\frac{1}{2}} t \right) dt \\
&\leq (\lambda_k^b - \beta^2)^{-1} (1 - a)^{-1} T^{1-a}.
\end{aligned}$$

It follows that

$$\sum_{k=1}^\infty \int_0^T t^{-a} V_\beta(\lambda_k^b, t)^2 dt \leq \sum_{k:\lambda_k^b \leq \beta^2} \int_0^T t^{-a} V_\beta(\lambda_k^b, t)^2 dt + \frac{T^{1-a}}{1-a} \sum_{k:\lambda_k^b > \beta^2} (\lambda_k^b - \beta^2)^{-1}$$

which is finite by Proposition 4.2.5. We use a similar method for the $\dot{V}_\beta$ sum. Taking $a = 0$, it thus follows from [15, Theorem 5.4] that the SPDE (5.2.2) has a unique global solution $U = (U(t))_{t=0}^\infty$ in $\mathcal{H} \oplus \mathcal{H}_b^{-1}$ given by

$$U(t) = W_\beta^b(t) = \begin{pmatrix} \int_0^t V_\beta(-\Delta_b, t - s) dW(s) \\ \int_0^t \dot{V}_\beta(-\Delta_b, t - s) dW(s) \end{pmatrix}.$$

It is a Gaussian process in $\mathcal{H} \oplus \mathcal{H}_b^{-1}$ by [15, Theorem 5.2]. As a stochastic integral of a cylindrical Wiener process, it is centred. Moreover, taking $a \in (0, 1)$ we see that this U has an $\mathcal{H} \oplus \mathcal{H}_b^{-1}$-continuous version by [15, Theorem 5.11]. □

Theorem 5.2.11 (Solution to (5.1.2))**.** *There exists a unique global solution u to the SPDE* (5.1.2)*. It is a centred Gaussian process on $\mathcal{H}$ given by*

$$u(t) = \int_0^t V_\beta(-\Delta_b, t-s)dW(s).$$

Moreover, u is the unique $\mathcal{H}$-valued process which satisfies the following "weak solution" property: For all $h \in \mathcal{D}(\Delta_b)$, the function $t \mapsto \langle u(t), h\rangle_\mu$ satisfies $\langle u(0), h\rangle_\mu = 0$, is continuous in $[0,\infty)$, and is continuously differentiable in $(0,\infty)$ with

$$\frac{d}{dt}\langle u(t), h\rangle_\mu = \int_0^t \langle u(s), \Delta_b h\rangle_\mu ds - 2\beta\langle u(t), h\rangle_\mu + \int_0^t \langle h, dW(s)\rangle_\mu.$$

Proof. Existence is given directly by Proposition 5.2.10, and yields the required centred Gaussian process u as a solution which is continuous in $\mathcal{H}$, with its associated $\dot{u}$ continuous in $\mathcal{H}_b^{-1}$. Now note that our construction of $\mathcal{S}^{b,\beta}$ in Lemma 5.2.9 was independent of the value of α. That is, for any $\alpha \geq 0$ such that $\mathcal{A}_{b,\beta}$ generates a C_0-semigroup on $\mathcal{H} \oplus \mathcal{H}_b^{-\alpha}$, that semigroup must be $\mathcal{S}^{b,\beta}$. This ensures uniqueness.

It can be checked directly that the adjoint of the operator $\mathcal{A}_{b,\beta}$ is given by

$$\mathcal{A}_{b,\beta}^* = \begin{pmatrix} 0 & (1-\Delta_b)^{-1}\Delta_b \\ 1-\Delta_b & -2\beta \end{pmatrix},$$

with domain $\mathcal{D}(\mathcal{A}_{b,\beta}^*) = \mathcal{D}(\Delta_b^{\frac{1}{2}}) \oplus \mathcal{H} = \mathcal{D}(\mathcal{A}_{b,\beta})$. By the definition of weak solution in [15, Chapter 5] for (5.2.2) we see that for all $f \in \mathcal{D}(\Delta_b^{\frac{1}{2}})$ and $g \in \mathcal{H}$ and $t \in [0,\infty)$,

$$\begin{aligned}&\langle u(t), f\rangle_\mu + \langle \dot{u}(t), g\rangle_{\mathcal{H}_b^{-1}} \\ &= \int_0^t \Big(\langle u(s), (1-\Delta_b)^{-1}\Delta_b g\rangle_\mu + \langle \dot{u}(s), (1-\Delta_b)f - 2\beta g\rangle_{\mathcal{H}_b^{-1}}\Big)\, ds + \int_0^t \langle g, dW(s)\rangle_{\mathcal{H}_b^{-1}}.\end{aligned} \tag{5.2.4}$$

Take $g = 0$ and $f \in \mathcal{D}(\Delta_b^{\frac{1}{2}})$ in (5.2.4). Then by the fact that $\dot{u}$ is continuous in $\mathcal{H}_b^{-1}$ and the fundamental theorem of calculus, the function $t \mapsto \langle u(t), f\rangle_\mu$ is continuously differentiable in $(0,\infty)$ with

$$\frac{d}{dt}\langle u, f\rangle_\mu = \langle \dot{u}, (1-\Delta_b)f\rangle_{\mathcal{H}_b^{-1}}.$$

Note in particular that the right-hand side of the above equation is equal to $\langle \dot{u}, f\rangle_\mu$ if $\dot{u} \in \mathcal{H}$. Now in (5.2.4) we take $f = 0$ and let $g = (1-\Delta_b)h$ for some $h \in \mathcal{D}(\Delta_b)$, which gives

$$\begin{aligned}&\langle \dot{u}(t), (1-\Delta_b)h\rangle_{\mathcal{H}_b^{-1}} \\ &= \int_0^t \Big(\langle u(s), \Delta_b h\rangle_\mu - 2\beta\langle \dot{u}(s), (1-\Delta_b)h\rangle_{\mathcal{H}_b^{-1}}\Big)\, ds + \int_0^t \langle (1-\Delta_b)h, dW(s)\rangle_{\mathcal{H}_b^{-1}},\end{aligned}$$

which is equivalent to

$$\frac{d}{dt}\langle u(t),h\rangle_\mu = \int_0^t \langle u(s),\Delta_b h\rangle_\mu ds - 2\beta\langle u(t),h\rangle_\mu + \int_0^t \langle h, dW(s)\rangle_\mu.$$

Thus u satisfies the required "weak" property. It remains to prove that u uniquely satisfies this property among all $\mathcal{H}$-valued processes, so let $\bar{u}$ be a process also satisfying the property and let $v = u - \bar{u}$. Let $v_k(t) = \langle v(t), \varphi_k^b\rangle_\mu$ for $k \geq 1$, $t \in [0,\infty)$. Then v_k can be seen to satisfy the ordinary differential equation

$$\begin{aligned}\frac{d^2 v_k}{dt^2} &= -\lambda_k^b v_k - 2\beta\frac{dv_k}{dt},\\ v_k(0) &= \frac{dv_k}{dt}(0) = 0.\end{aligned}$$

The unique solution to this ODE is $v_k = 0$ for every k, which implies $u = \bar{u}$. □

Now that we have our solution u to (5.1.2) given by Theorem 5.2.11, we show that it has a nice eigenfunction decomposition. Let $u_k = \langle\varphi_k^b u\rangle_\mu$ for $k \geq 1$. We see that

$$u_k(t) = \int_0^t V_\beta(\lambda_k^b, t-s)\langle\varphi_k^b, dW(s)\rangle_\mu,$$

and it can be easily shown that $(\langle\varphi_k^b, W\rangle_\mu)_{k=1}^\infty$ is a sequence of independent standard real Brownian motions.

Definition 5.2.12 (Series representation of solution)**.** Let $\beta \geq 0$ and $b \in 2^{F^0}$. For $k \geq 0$ let $Y_k^{b,\beta} = (Y_k^{b,\beta}(t))_{t\geq 0}$ be the centred real-valued Gaussian process given by

$$Y_k^{b,\beta}(t) = \int_0^t V_\beta(\lambda_k^b, t-s)\langle\varphi_k^b, dW(s)\rangle_\mu.$$

The family $(Y_k^{b,\beta})_{k=1}^\infty$ is clearly independent, and if u is the solution to (5.1.2) for the given values of β and b then

$$u(t) = \sum_{k=1}^\infty Y_k^{b,\beta}(t)\varphi_k^b. \tag{5.2.5}$$

Remark 5.2.13. By Theorem 5.2.11, the real-valued process $Y_k^{b,\beta}$ satisfies the following stochastic integro-differential equation:

$$\begin{aligned}y'(t) &= -2\beta y(t) - \lambda_k^b\int_0^t y(s)ds + \int_0^t \langle\varphi_k^b, dW(s)\rangle_\mu,\\ y(0) &= 0,\end{aligned}$$

and it is easily shown to be the unique solution.

Remark 5.2.14 (Non-zero initial conditions)**.** For a moment we consider the SPDE

$$\begin{aligned} du(t) &= \dot{u}(t)dt, \\ d\dot{u}(t) &= -2\beta\dot{u}(t)dt + \Delta_b u(t)dt + dW(t), \\ u(0) &= f, \ \dot{u}(0) = g. \end{aligned} \tag{5.2.6}$$

This is simply the SPDE (5.1.2) with possibly non-zero initial conditions. We can characterise the solutions of this SPDE using the deterministic damped wave equation

$$\begin{aligned} du(t) &= \dot{u}(t)dt, \\ d\dot{u}(t) &= -2\beta\dot{u}(t)dt + \Delta_b u(t)dt, \\ u(0) &= f, \ \dot{u}(0) = g, \end{aligned} \tag{5.2.7}$$

which is studied in [16], [42], [61] in the case $\beta = 0$. Let u be the unique solution to (5.1.2) given in Theorem 5.2.11. Then it is clear that a process $\tilde{u}$ solves (5.2.6) if and only if $\tilde{u} - u$ solves (5.2.7). Thus understanding the stochastic wave equation with general initial conditions on a fractal is equivalent to understanding the deterministic wave equation on that fractal.

5.3 Regularity of solution

5.3.1 L^2-Hölder continuity

The first regularity property of the solution $u = (u(t))_{t=0}^{\infty}$ to (5.1.2) that we will consider is Hölder continuity in $\mathcal{H}$, when u is interpreted as a function $u : \Omega \times [0,\infty) \to \mathcal{H}$.

Proposition 5.3.1. *Let $u : \Omega \times [0,\infty) \to \mathcal{H}$ be the solution to the SPDE* (5.1.2)*. For every $T > 0$ there exists a constant $C > 0$ such that*

$$\mathbb{E}\left[\|u(s) - u(s+t)\|_\mu^2\right] \le Ct^{2-d_s}$$

for all $s, t \ge 0$ such that $s, s+t \in [0,T]$.

Proof. By Itō's isometry for Hilbert spaces,

$$\begin{aligned} &\mathbb{E}\left[\|u(s) - u(s+t)\|_\mu^2\right] \\ &= \mathbb{E}\left[\left\| \int_0^{s+t} \left(V_\beta(-\Delta_b, s+t-t') - V_\beta(-\Delta_b, s-t')\mathbb{1}_{\{t' \le s\}}\right) dW(t') \right\|_\mu^2\right] \\ &= \int_0^{s+t} \left\| V_\beta(-\Delta_b, s+t-t') - V_\beta(-\Delta_b, s-t')\mathbb{1}_{\{t' \le s\}} \right\|_{\mathrm{HS}(\mathcal{H})}^2 dt', \end{aligned}$$

where $\|\cdot\|_{\mathrm{HS}(\mathcal{H})}$ denotes the Hilbert-Schmidt norm for operators from $\mathcal{H}$ to itself. It follows that

$$\begin{aligned}
&\mathbb{E}\left[\|u(s)-u(s+t)\|_\mu^2\right] \\
&= \int_0^s \|V_\beta(-\Delta_b,t+t')-V_\beta(-\Delta_b,t')\|_{\mathrm{HS}(\mathcal{H})}^2\,dt' + \int_0^t \|V_\beta(-\Delta_b,t')\|_{\mathrm{HS}(\mathcal{H})}^2\,dt' \\
&= \sum_{k=0}^\infty \int_0^s \left(V_\beta(\lambda_k^b,t+t')-V_\beta(\lambda_k^b,t')\right)^2 dt' + \sum_{k=0}^\infty \int_0^t \left(V_\beta(\lambda_k^b,t')\right)^2 dt'
\end{aligned} \tag{5.3.1}$$

and we treat each of the above two sums separately. Notice that by Proposition 4.2.5 there are only finitely many k such that $\lambda_k^b \le \beta^2$.

We consider the first sum of (5.3.1), and we first look at the case $\lambda_k^b > \beta^2$. Then using standard facts about the Lipschitz coefficients of the functions exp and sin in $[0,T]$ we see that

$$\begin{aligned}
&\int_0^s \left(V_\beta(\lambda_k^b,t+t')-V_\beta(\lambda_k^b,t')\right)^2 dt' \\
&= (\lambda_k^b-\beta^2)^{-1}\int_0^s \left(e^{-\beta(t+t')}\sin\left((\lambda_k^b-\beta^2)^{\frac12}(t+t')\right) - e^{-\beta t'}\sin\left((\lambda_k^b-\beta^2)^{\frac12}t'\right)\right)^2 dt' \\
&\le (\lambda_k^b-\beta^2)^{-1}\int_0^s \left((\beta+(\lambda_k^b-\beta^2)^{\frac12})t\wedge 2\right)^2 dt' \\
&\le 4T\frac{\lambda_k^b t^2\wedge 1}{\lambda_k^b-\beta^2}.
\end{aligned}$$

We get a similar result in the case $\lambda_k^b \le \beta^2$, that is, a term of order $O(t^2)$. In this case the dependence of this term on k is unimportant as there are only finitely many k such that $\lambda_k^b \le \beta^2$. There therefore exists a constant $C'>0$ such that

$$\sum_{k=0}^\infty \int_0^s \left(V_\beta(\lambda_k^b,t+t')-V_\beta(\lambda_k^b,t')\right)^2 dt' \le C't^2 + 4T\sum_{k:\lambda_k^b>\beta^2}\frac{\lambda_k^b t^2\wedge 1}{\lambda_k^b-\beta^2}.$$

Using Proposition 4.2.5, there therefore exists $C''>0$ such that

$$\sum_{k=0}^\infty \int_0^s \left(V_\beta(\lambda_k^b,t+t')-V_\beta(\lambda_k^b,t')\right)^2 dt' \le C''\left(t^2+\sum_{k=1}^\infty k^{-\frac{2}{d_s}}\wedge t^2\right).$$

Then by Lemma 3.5.2, there exists a $C'''>0$ such that

$$\sum_{k=0}^\infty \int_0^s \left(V_\beta(\lambda_k^b,t+t')-V_\beta(\lambda_k^b,t')\right)^2 dt' \le C'''t^{2-d_s}.$$

Now for the second sum of (5.3.1), again we first look at the case $\lambda_k^b > \beta^2$. Using Lipschitz coefficents and the fact that $V_\beta(\lambda_k^b, 0) = 0$ we have that

$$\begin{aligned}\int_0^t \left(V_\beta(\lambda_k^b, t')\right)^2 dr &= (\lambda - \beta^2)^{-1} \int_0^t e^{-2\beta t'} \sin^2\left((\lambda - \beta^2)^{\frac{1}{2}} t'\right) dt' \\ &\le (\lambda_k^b - \beta^2)^{-1} \int_0^t \left((\beta + (\lambda_k^b - \beta^2)^{\frac{1}{2}}) t' \wedge 1\right)^2 dt' \\ &\le 4(\lambda_k^b - \beta^2)^{-1} \int_0^t \left(\lambda_k^b (t')^2 \wedge 1\right) dt' \\ &\le 4t \frac{\lambda_k^b t^2 \wedge 1}{\lambda_k^b - \beta^2}.\end{aligned}$$

In the case $\lambda_k^b \le \beta^2$ we get as usual a similar result, of order $O(t^3)$, and its dependence on k is unimportant as there are only finitely many. Using the same method as for the first sum of (5.3.1) we see that there exists a $C'''' > 0$ such that

$$\sum_{k=0}^{\infty} \int_0^t \left(V_\beta(\lambda_k^b, t')\right)^2 dt' \le C'''' t^{3-d_s}.$$

Plugging the estimates into (5.3.1) finishes the proof. □

Definition 5.3.2. Let (M_1, d_1) and (M_2, d_2) be metric spaces and let $\delta \in (0, 1]$. A function $f : M_1 \to M_2$ is *essentially δ-Hölder continuous* if for each $\gamma \in (0, \delta)$ there exists $C_\gamma > 0$ such that

$$d_2(f(x), f(y)) \le C_\gamma d_1(x, y)^\gamma$$

for all $x, y \in M_1$.

Theorem 5.3.3 (L^2-Hölder continuity)**.** *Let $u : \Omega \times [0, \infty) \to \mathcal{H}$ be the solution to the SPDE* (5.1.2)*. Then there exists a version $\tilde{u}$ of u such that the following holds: for all $T > 0$, the restriction of $\tilde{u}$ to $\Omega \times [0, T]$ is almost surely essentially $(1 - \frac{d_s}{2})$-Hölder continuous as a function from $[0, T]$ to $\mathcal{H}$.*

Proof. Fix $T > 0$. This is a simple application of Kolmogorov's continuity theorem. It is a consequence of Fernique's theorem [15, Theorem 2.7] that for each $p \in \mathbb{N}$ there exists a constant $K_p > 0$ such that if Z is a Gaussian random variable on some separable Banach space B then

$$\mathbb{E}\left[\|Z\|_B^{2p}\right] \le K_p \mathbb{E}\left[\|Z\|_B^2\right]^p,$$

see also [26, Proposition 3.14]. Since u is a Gaussian process, Proposition 5.3.1 gives us that

$$\mathbb{E}\left[\|u(s) - u(t)\|_\mu^{2p}\right] \le K_p C^p |s - t|^{p(2-d_s)}$$

for all $s, t \in [0, T]$, for all $p \in \mathbb{N}$. Then by taking p arbitrarily large and using Kolmogorov's continuity theorem, the result follows. Note that any two continuous versions of u must be indistinguishable, which allows us to extend the construction of $\tilde{u}$ on any given finite time interval $[0, T]$ to the whole interval $[0, \infty)$. □

5.3.2 Pointwise regularity

Let u be the solution to (5.1.2) in Theorem 5.2.11. Henceforth we assume that u is the $\mathcal{H}$-continuous version constructed in Theorem 5.3.3. We currently have u as an $\mathcal{H}$-valued process, so in this section we will show that the "point evaluations" $u(t, x)$ for $(t, x) \in [0, \infty) \times F$ can be defined in such a way that they make sense as real-valued random variables. This will allow us to interpret u as a function from $\Omega \times [0, \infty) \times F$ to $\mathbb{R}$, and is necessary for us to be able to talk about continuity of u in space and time.

Definition 5.3.4. For $\lambda > 0$ and $b \in 2^{F^0}$ let $\rho_\lambda^b : F \times F \to \mathbb{R}$ be the *resolvent density* associated with Δ_b, exactly as in Section 4.3.1.

Lemma 5.3.5. *Let $\beta \geq 0$ and $\lambda \geq 0$. If $\alpha > 0$ then*

$$\int_0^\infty e^{-2\alpha t} V_\beta(\lambda, t)^2 dt = \frac{1}{4(\alpha + \beta)(\alpha^2 + 2\alpha\beta + \lambda)}$$

Proof. Can be computed explicitly using (complex) integration in each of the cases $\lambda < \beta^2$, $\lambda = \beta^2$ and $\lambda > \beta^2$ using the definition of V_β. □

Lemma 5.3.6. *Let $u : [0, \infty) \to \mathcal{H}$ be the solution to the SPDE* (5.1.2). *If $g \in \mathcal{H}$ and $t \in [0, \infty)$ then*

$$\mathbb{E}\left[\langle u(t), g \rangle_\mu^2\right] \leq \frac{e^{2(\sqrt{\beta^2+1}-\beta)t}}{4\sqrt{\beta^2 + 1}} \int_F \int_F \rho_1^b(x, y) g(x) g(y) \mu(dx) \mu(dy).$$

Proof. Let $g^* \in \mathcal{H}^*$ be the bounded linear functional $f \mapsto \langle f, g \rangle_\mu$. We see by Itō's isometry that

$$\begin{aligned}
\mathbb{E}\left[\langle u(t), g \rangle_\mu^2\right] &= \mathbb{E}\left[g^*(u(t))^2\right] \\
&= \int_0^t \|g^* V_\beta(-\Delta_b, s)\|_{\mathrm{HS}}^2 ds \\
&= \int_0^t \|V_\beta(-\Delta_b, s) g\|_\mu^2 ds
\end{aligned}$$

where the last equality is a result of the self-adjointness of the operator $V_\beta(-\Delta_b, s)$. If we let $g_k = \langle \varphi_k^b, g\rangle_\mu$ for $k \geq 1$ then for any $\alpha > 0$ we have that

$$\begin{aligned}
\mathbb{E}\left[\langle u(t), g\rangle_\mu^2\right] &= \sum_{k=1}^\infty g_k^2 \int_0^t V_\beta(\lambda_k^b, s)^2 ds \\
&\leq e^{2\alpha t} \sum_{k=1}^\infty g_k^2 \int_0^\infty e^{-2\alpha s} V_\beta(\lambda_k^b, s)^2 ds \\
&= e^{2\alpha t} \sum_{k=1}^\infty g_k^2 \frac{1}{4(\alpha+\beta)(\alpha^2 + 2\alpha\beta + \lambda_k^b)} \\
&= \frac{e^{2\alpha t}}{4(\alpha+\beta)} \left\langle (\alpha^2 + 2\alpha\beta - \Delta_b)^{-1} g, g\right\rangle_\mu \\
&= \frac{e^{2\alpha t}}{4(\alpha+\beta)} \int_F \int_F \rho^b_{\alpha^2+2\alpha\beta}(x,y) g(x) g(y) \mu(dx)\mu(dy),
\end{aligned}$$

where we have used Lemma 5.3.5. Finally we pick $\alpha = \sqrt{\beta^2+1} - \beta$ so that $\alpha^2 + 2\alpha\beta = 1$ and the proof is complete. □

For $x \in F$ and $\varepsilon > 0$ let $B(x, \varepsilon)$ be the closed R-ball in F with centre x and radius ε.

Lemma 5.3.7 (Neighbourhoods)**.** *There exists a constant $c_5 > 0$ such that the following holds: If $x \in F$ and $n \geq 0$ then there exists a subset $D_n^0(x) \subseteq F$ such that $\mu(D_n^0(x)) > r_{\min}^{d_H} 2^{-d_H n}$ and*

$$x \in D_n^0(x) \subseteq B(x, c_5 2^{-n}).$$

Proof. The $D_n^0(x)$ we want is defined in Definition 3.3.10. The result $D_n^0(x) \subseteq B(x, c_5 2^{-n})$ then follows from Proposition 3.3.12. The result on $\mu(D_n^0(x))$ is due to the fact that by definition, $F_w \subseteq D_n^0(x)$ for some $w \in \mathbb{W}_*$ such that $r_w > r_{\min} 2^{-n}$. □

Definition 5.3.8. For $x \in F$ and $n \geq 0$, define

$$f_n^x = \mu(D_n^0(x))^{-1} \mathbb{1}_{D_n^0(x)}.$$

Evidently $f_n^x \in \mathcal{H}$, $\|f_n^x\|_\mu^2 = \mu(D_n^0(x))^{-1} < r_{\min}^{-d_H} 2^{d_H n}$ by the above Lemma and if $g \in \mathcal{H}$ is continuous then

$$\lim_{n\to\infty} \langle f_n^x, g\rangle_\mu = g(x),$$

by the above lemma.

We can now state and prove the main theorem of this section, compare Theorem 3.4.8.

Theorem 5.3.9 (Pointwise regularity)**.** *Let $u : [0,\infty) \to \mathcal{H}$ be the solution to the SPDE* (5.1.2)*. Then for all $(t,x) \in [0,\infty) \times F$ the expression*

$$u(t,x) := \sum_{k=1}^{\infty} Y_k^{b,\beta}(t)\varphi_k^b(x)$$

is a well-defined real-valued centred Gaussian random variable. There exists a constant $c_6 > 0$ such that for all $x \in F$, $t \in [0,\infty)$ and $n \geq 0$ we have that

$$\mathbb{E}\left[(\langle u(t), f_n^x\rangle_\mu - u(t,x))^2\right] \leq c_6 e^{2(\sqrt{\beta^2+1}-\beta)t}2^{-n}.$$

Proof. Note that $\varphi_k^b \in \mathcal{D}(\Delta_b)$ for each k, so φ_k^b is continuous and so $\varphi_k^b(x)$ is well-defined. By the definition of $u(t,x)$ as a sum of real-valued centred Gaussian random variables we need only prove that it is square-integrable and that the approximation estimate holds. Let $x \in F$. The theorem is trivial for $t = 0$ so let $t \in (0,\infty)$. By Lemma 5.3.6 we have that

$$\begin{aligned}&\mathbb{E}\left[\langle u(t), f_n^x - f_m^x\rangle_\mu^2\right] \\ &\leq \frac{e^{2(\sqrt{\beta^2+1}-\beta)t}}{4\sqrt{\beta^2+1}} \int_F\int_F \rho_1^b(z_1,z_2)(f_n^x(z_1) - f_m^x(z_1))(f_n^x(z_2) - f_m^x(z_2))\mu(dz_1)\mu(dz_2).\end{aligned}$$

Then using the definition of f_n^x, Proposition 4.3.2 and Lemma 5.3.7 we have that

$$\begin{aligned}\mathbb{E}\left[\langle u(t), f_n^x - f_m^x\rangle_\mu^2\right] &\leq \frac{e^{2(\sqrt{\beta^2+1}-\beta)t}}{4\sqrt{\beta^2+1}}\left(8c_5 2^{-n} + 8c_5 2^{-m}\right) \\ &= \frac{2c_5 e^{2(\sqrt{\beta^2+1}-\beta)t}}{\sqrt{\beta^2+1}}\left(2^{-n} + 2^{-m}\right).\end{aligned} \tag{5.3.2}$$

Writing u in its series representation (5.2.5) and using the independence of the $Y_k^{b,\beta}$, it follows that

$$\sum_{k=1}^{\infty} \mathbb{E}\left[Y_k^{b,\beta}(t)^2\right]\left(\langle\varphi_k^b, f_n^x\rangle_\mu - \langle\varphi_k^b, f_m^x\rangle_\mu\right)^2 \leq \frac{2c_5 e^{2(\sqrt{\beta^2+1}-\beta)t}}{\sqrt{\beta^2+1}}\left(2^{-n} + 2^{-m}\right).$$

Thus the left-hand side of the above equation tends to zero as $m, n \to \infty$. The solution u is an $\mathcal{H}$-valued Gaussian process so we know that

$$\sum_{k=1}^{\infty} \mathbb{E}\left[Y_k^{b,\beta}(t)^2\right]\langle\varphi_k^b, f_n^x\rangle_\mu^2 = \mathbb{E}\left[\langle u(t), f_n^x\rangle_\mu^2\right] < \infty$$

for all $x \in F$, $n \geq 0$ and $t \in [0,\infty)$, therefore by the completeness of the sequence space ℓ^2 there must exist a unique sequence $(y_k)_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} y_k^2 < \infty$ and

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\left(\mathbb{E}\left[Y_k^{b,\beta}(t)^2\right]^{\frac{1}{2}}\langle\varphi_k^b, f_n^x\rangle_\mu - y_k\right)^2 = 0.$$

Since φ_k^b is continuous we have $\lim_{n\to\infty}\langle\varphi_k^b, f_n^x\rangle_\mu = \varphi_k^b(x)$. Thus by Fatou's lemma we can identify the sequence (y_k); we must have

$$\sum_{k=1}^{\infty}\mathbb{E}\left[Y_k^{b,\beta}(t)^2\right]\varphi_k^b(x)^2 < \infty$$

and

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\mathbb{E}\left[Y_k^{b,\beta}(t)^2\right]\left(\langle\varphi_k^b, f_n^x\rangle_\mu - \varphi_k^b(x)\right)^2 = 0.$$

Equivalently by (5.2.5),

$$\mathbb{E}\left[u(t,x)^2\right] < \infty$$

(so we have proven square-integrability) and

$$\lim_{n\to\infty}\mathbb{E}\left[\left(\langle u(t), f_n^x\rangle_\mu - u(t,x)\right)^2\right] = 0.$$

In particular by taking $m\to\infty$ in (5.3.2) we have that

$$\mathbb{E}\left[\left(\langle u(t), f_n^x\rangle_\mu - u(t,x)\right)^2\right] \leq \frac{2c_5 e^{2(\sqrt{\beta^2+1}-\beta)t}}{\sqrt{\beta^2+1}}2^{-n}.$$

□

We can now interpret our solution u as a so-called "random field" solution $u : \Omega\times[0,\infty)\times F\to\mathbb{R}$. However, the relationship between the random field solution and the original $\mathcal{H}$-valued solution is still rather unclear. We discuss this in the next section.

5.4 Space-time Hölder continuity

Now that we have the interpretation of the solution u to (5.1.2) as a function $u : \Omega\times[0,\infty)\times F\to\mathbb{R}$, we can prove results about its continuity in time and space. In particular, we show that it has a Hölder continuous version which is also a version of the original $\mathcal{H}$-valued solution found in Theorem 5.2.11.

5.4.1 Spatial estimate

The spatial continuity of u is the same as for the stochastic heat equation, see Section 3.5.1.

Proposition 5.4.1. *Let $T > 0$. Let $u : \Omega \times [0,T] \times F \to \mathbb{R}$ be (the restriction of) the solution to the SPDE* (5.1.2)*. Then there exists a constant $C_1 > 0$ such that*

$$\mathbb{E}\left[(u(t,x) - u(t,y))^2\right] \leq C_1 R(x,y)$$

for all $t \in [0,T]$ and all $x, y \in F$.

Proof. Recall from Theorem 5.3.9 that

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\langle u(t), f_n^x\rangle_\mu - u(t,x)\right)^2\right] = 0,$$

and an analogous result holds for y. Thus by Lemma 5.3.6,

$$\begin{aligned}
\mathbb{E}\left[(u(t,x) - u(t,y))^2\right] &= \lim_{n\to\infty} \mathbb{E}\left[\langle u(t), f_n^x - f_n^y\rangle_\mu^2\right] \\
&\leq \frac{e^{2(\sqrt{\beta^2+1}-\beta)t}}{4\sqrt{\beta^2+1}} \lim_{n\to\infty} \int_F \int_F \rho_1^b(z_1,z_2)(f_n^x(z_1) - f_n^y(z_1))(f_n^x(z_2) - f_n^y(z_2))\mu(dz_1)\mu(dz_2) \\
&= \frac{e^{2(\sqrt{\beta^2+1}-\beta)t}}{4\sqrt{\beta^2+1}} \left(\rho_1^b(x,x) - 2\rho_1^b(x,y) + \rho_1^b(y,y)\right),
\end{aligned}$$

where we have used the continuity of the resolvent density, Lemma 5.3.7, and the definition of f_n^x (similarly to the proof of Theorem 5.3.9). Hence by Proposition 4.3.2,

$$\begin{aligned}
\mathbb{E}\left[(u(t,x) - u(t,y))^2\right] &\leq \frac{e^{2(\sqrt{\beta^2+1}-\beta)T}}{4\sqrt{\beta^2+1}} \left(\rho_1^b(x,x) - \rho_1^b(x,y) + \rho_1^b(y,y) - \rho_1^b(y,x)\right) \\
&\leq \frac{e^{2(\sqrt{\beta^2+1}-\beta)T}}{\sqrt{\beta^2+1}} R(x,y).
\end{aligned}$$

□

5.4.2 Temporal estimate

Lemma 5.4.2. *We have the following estimates on V_β and $\dot{V}_\beta$:*

(1). Let $\beta \geq 0$ and $t \geq 0$. Then

$$\sup_{\lambda\geq 0} |V_\beta(\lambda,t)| = \begin{cases} \beta^{-1}e^{-\beta t}\sinh(\beta t) & \beta > 0, \\ t & \beta = 0. \end{cases}$$

In particular, $\sup_{\lambda\geq 0} |V_\beta(\lambda,t)|$ is $O(t)$ as $t \to 0$.

(2). Let $\beta \geq 0$ and $T \geq 0$. Then

$$\sup_{0\leq t\leq T}\sup_{\lambda\geq 0}|\dot{V}_\beta(\lambda,t)| \leq e^{\beta T}.$$

Proof. It is easy, if somewhat tedious, to prove that V_β and $\dot{V}_\beta$ are both continuous in λ for fixed $t \geq 0$. Note that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 = \lim_{x\to 0}\frac{\sinh x}{x}$$

and

$$\sup_{x\in\mathbb{R}\setminus\{0\}}\left|\frac{\sin x}{x}\right| = 1 = \inf_{x\in\mathbb{R}\setminus\{0\}}\left|\frac{\sinh x}{x}\right|.$$

For (1), assume that $t > 0$ (otherwise the result is trivial). We have that

$$\begin{aligned}
\sup_{\lambda>\beta^2}|V_\beta(\lambda,t)| &= te^{-\beta t}\sup_{\lambda>\beta^2}\left|\left((\lambda-\beta^2)^{\frac{1}{2}}t\right)^{-1}\sin\left((\lambda-\beta^2)^{\frac{1}{2}}t\right)\right| \\
&= te^{-\beta t} \\
&= |V_\beta(\beta^2,t)|,
\end{aligned}$$

so we need only consider the case $\lambda \leq \beta^2$. If $\beta = 0$ then this directly implies the result. Suppose now that $\beta > 0$. The function $x \mapsto \frac{\sinh x}{x}$ is positive and increasing when x is positive so by continuity we have that

$$\begin{aligned}
\sup_{\lambda\geq 0}|V_\beta(\lambda,t)| &= \sup_{\lambda\leq\beta^2}|V_\beta(\lambda,t)| \\
&= te^{-\beta t}\sup_{\lambda\leq\beta^2}\left(\left((\beta^2-\lambda)^{\frac{1}{2}}t\right)^{-1}\sinh\left((\beta^2-\lambda)^{\frac{1}{2}}t\right)\right) \\
&= te^{-\beta t}(\beta t)^{-1}\sinh(\beta t) \\
&= \beta^{-1}e^{-\beta t}\sinh(\beta t)
\end{aligned}$$

which is the required result.

Now for (2), assume that $T > 0$, otherwise the result is trivial. We have

$$\begin{aligned}
&\sup_{0\leq t\leq T}\sup_{\lambda>\beta^2}|\dot{V}_\beta(\lambda,t)| \\
&\quad= \sup_{0\leq t\leq T}\sup_{\lambda>\beta^2}\left|e^{-\beta t}\cos\left((\lambda-\beta^2)^{\frac{1}{2}}t\right) - \beta(\lambda-\beta^2)^{-\frac{1}{2}}e^{-\beta t}\sin\left((\lambda-\beta^2)^{\frac{1}{2}}t\right)\right| \\
&\quad\leq 1 + \sup_{0\leq t\leq T}\left|\beta te^{-\beta t}\right| \\
&\quad\leq 1 + \beta T
\end{aligned}$$

and

$$
\begin{aligned}
&\sup_{0\le t\le T}\sup_{\lambda<\beta^2}|\dot{V}_\beta(\lambda,t)| \\
&\qquad= \sup_{0\le t\le T}\sup_{\lambda<\beta^2}\left|e^{-\beta t}\cosh\left((\beta^2-\lambda)^{\frac{1}{2}}t\right) - \beta(\beta^2-\lambda)^{-\frac{1}{2}}e^{-\beta t}\sinh\left((\beta^2-\lambda)^{\frac{1}{2}}t\right)\right| \\
&\qquad\le \cosh(\beta T) + \beta T\sup_{0\le t\le T}\sup_{\lambda<\beta^2}\left(\left((\beta^2-\lambda)^{\frac{1}{2}}t\right)^{-1}\sinh\left((\beta^2-\lambda)^{\frac{1}{2}}t\right)\right) \\
&\qquad\le \cosh(\beta T)+\sinh(\beta T) = e^{\beta T}
\end{aligned}
$$

and $\sup_{0\le t\le T}|\dot{V}_\beta(\beta^2,t)| = \sup_{0\le t\le T}\left|e^{-\beta t}-\beta t e^{-\beta t}\right| \le 1+\beta T$. Finally we note that the inequality $1+\beta T\le e^{\beta T}$ holds. □

We can now give the temporal estimate. Here we see the effect of the extra time derivative compared to the stochastic heat equation in Proposition 3.5.5.

Proposition 5.4.3. *Let $T>0$. Let $u:\Omega\times[0,T]\times F\to\mathbb{R}$ be (the restriction of) the solution to the SPDE* (5.1.2)*. Then there exists $C_2>0$ such that*

$$\mathbb{E}\left[(u(s,x)-u(s+t,x))^2\right]\le C_2 t^{2-d_s}$$

for all $s,t\ge 0$ such that $s,s+t\le T$ and all $x\in F$.

Proof. Let $c_6' := 8c_6 e^{2(\sqrt{\beta^2+1}-\beta)T}$, where c_6 is from Theorem 5.3.9. By Theorem 5.3.9 we have that if $n\ge 0$ is an integer then

$$\mathbb{E}\left[(u(s,x)-u(s+t,x))^2\right]\le 2\mathbb{E}\left[\langle u(s)-u(s+t),f_n^x\rangle_\mu^2\right]+c_6'2^{-n}. \tag{5.4.1}$$

Then Itō's isometry for Hilbert spaces (see also proof of Lemma 5.3.6) gives us that

$$
\begin{aligned}
&\mathbb{E}\left[\langle u(s)-u(s+t),f_n^x\rangle_\mu^2\right] \\
&\quad=\mathbb{E}\left[\left\langle\int_0^{s+t}\left(V_\beta(-\Delta_b,s+t-t')-V_\beta(-\Delta_b,s-t')\mathbb{1}_{\{t'\le s\}}\right)dW(t'),f_n^x\right\rangle_\mu^2\right] \\
&\quad=\int_0^{s+t}\left\|\left(V_\beta(-\Delta_b,s+t-t')-V_\beta(-\Delta_b,s-t')\mathbb{1}_{\{t'\le s\}}\right)f_n^x\right\|_\mu^2 dt' \\
&\quad\le\|f_n^x\|_\mu^2\int_0^{s+t}\left\|V_\beta(-\Delta_b,s+t-t')-V_\beta(-\Delta_b,s-t')\mathbb{1}_{\{t'\le s\}}\right\|^2 dt'.
\end{aligned}
$$

Recall that $\|f_n^x\|_\mu^2 < r_{\min}^{-d_H} 2^{d_H n}$. Using the functional calculus we see that

$$\begin{aligned}
&\int_0^{s+t} \left\| V_\beta(-\Delta_b, s+t-t') - V_\beta(-\Delta_b, s-t')\mathbb{1}_{\{t' \leq s\}} \right\|^2 dt' \\
&= \int_0^s \|V_\beta(-\Delta_b, s+t-t') - V_\beta(-\Delta_b, s-t')\|^2 dt' + \int_s^{s+t} \|V_\beta(-\Delta_b, s+t-t')\|^2 dt' \\
&= \int_0^s \|V_\beta(-\Delta_b, t+t') - V_\beta(-\Delta_b, t')\|^2 dt' + \int_0^t \|V_\beta(-\Delta_b, t')\|^2 dt' \\
&\leq \int_0^s \sup_{\lambda \geq 0} \left(V_\beta(\lambda, t+t') - V_\beta(\lambda, t')\right)^2 dt' + \int_0^t \sup_{\lambda \geq 0} V_\beta(\lambda, t')^2 dt' \\
&\leq t^2 T \sup_{0 \leq t' \leq T} \sup_{\lambda \geq 0} \dot{V}_\beta(\lambda, t')^2 + \int_0^t \sup_{\lambda \geq 0} V_\beta(\lambda, t')^2 dt',
\end{aligned}$$

where in the last line we have used the mean value theorem. Therefore by using Lemma 5.4.2 there exists $c > 0$ such that

$$\int_0^{s+t} \left\| V_\beta(-\Delta_b, s+t-t') - V_\beta(-\Delta_b, s-t')\mathbb{1}_{\{t' \leq s\}} \right\|^2 dt' \leq ct^2$$

for all $s, t \geq 0$ such that $s, s+t \leq T$. Letting $c' = 2r_{\min}^{-d_H} c$ and plugging this into (5.4.1) we have that

$$\mathbb{E}\left[(u(s,x) - u(s+t,x))^2\right] \leq c' t^2 2^{d_H n} + c_6' 2^{-n}.$$

for all $s, t \geq 0$ such that $s, s+t \leq T$ and all $x \in F$. In fact, defining

$$c_6'' := c_6' \vee d_H c' T^2,$$

we have that

$$\mathbb{E}\left[(u(s,x) - u(s+t,x))^2\right] \leq c' t^2 2^{d_H n} + c_6'' 2^{-n} \tag{5.4.2}$$

as well. This estimate will turn out to be easier to work with.

We assume now that $t > 0$, and our aim is to choose $n \geq 0$ to minimise the expression on the right of (5.4.2). Fixing $t \in (0, T]$, define $g : \mathbb{R} \to [0, \infty)$ by $g(y) = c' t^2 2^{d_H y} + c_6'' 2^{-y}$. The function g has a unique stationary point which is a global minimum at

$$y_0 = \frac{1}{(d_H + 1)\log 2} \log\left(\frac{c_6''}{d_H c' t^2}\right).$$

Since $t \leq T$ we have by the definition of c_6'' that $y_0 \geq 0$. Since y_0 is not necessarily an integer we choose $n = \lceil y_0 \rceil$. Then g is increasing in $[y_0, \infty)$ so we have that

$$\mathbb{E}\left[(u(s,x) - u(s+t,x))^2\right] \leq g(n) \leq g(y_0 + 1).$$

Setting $c_6''' := \frac{c_6''}{d_H c'}$ and evaluating the right-hand side we see that

$$\begin{aligned}\mathbb{E}\left[(u(s,x)-u(s+t,x))^2\right] &\leq c' t^2 2^{d_H} \left(\frac{c_6'''}{t^2}\right)^{\frac{d_H}{d_H+1}} + c_6' 2^{-1} \left(\frac{c_6'''}{t^2}\right)^{\frac{-1}{d_H+1}} \\ &\leq c_6'''' t^{\frac{2}{d_H+1}} \\ &= c_6'''' t^{2-d_s}\end{aligned}$$

for all $s \geq 0$, $t > 0$ such that $s, s+t \leq T$ and all $x \in F$, where the constant $c_6'''' > 0$ is independent of s, t, x. This inequality obviously also holds in the case $t = 0$. □

5.4.3 Hölder continuity

We are now ready to prove the main result of this chapter.

Definition 5.4.4. Let R_∞ be the metric on $\mathbb{R} \times F$ given by

$$R_\infty((s,x),(t,y)) = |s-t| \vee R(x,y).$$

Theorem 5.4.5 (Space-time Hölder continuity). *Let $u : \Omega \times [0,\infty) \times F \to \mathbb{R}$ be the solution to the SPDE* (5.1.2)*. Let $\delta = 1 - \frac{d_s}{2}$. Then there exists a version $\tilde{u}$ of u which satisfies the following:*

(1). For each $T > 0$, $\tilde{u}$ is almost surely essentially $(\frac{1}{2} \wedge \delta)$-Hölder continuous on $[0,T] \times F$ with respect to R_∞.

(2). For each $t \in [0,\infty)$, $\tilde{u}(t,\cdot)$ is almost surely essentially $\frac{1}{2}$-Hölder continuous on F with respect to R.

(3). For each $x \in F$, $\tilde{u}(\cdot,x)$ is almost surely essentially δ-Hölder continuous on $[0,T]$ with respect to the Euclidean metric.

Moreover, the collection of random variables $\tilde{u} = (\tilde{u}(t,x))_{(t,x)\in[0,\infty)\times F}$ is such that $(\tilde{u}(t,\cdot))_{t\in[0,\infty)}$ is an $\mathcal{H}$-valued process, and moreover $(\tilde{u}(t,\cdot))_{t\in[0,\infty)}$ is an $\mathcal{H}$-continuous version of the $\mathcal{H}$-valued solution to (5.1.2) *found in Theorem 5.2.11.*

Proof. Take $T > 0$ and consider u_T, the restriction of u to $[0,T] \times F$. It is a well-known fact that for every $p \in \mathbb{N}$ there exists a constant $C_p' > 0$ such that if Z is any centred real Gaussian random variable then

$$\mathbb{E}[Z^{2p}] = C_p' \mathbb{E}[Z^2]^p.$$

We know that u_T is a centred Gaussian process on $[0,T] \times F$ by Theorem 5.3.9. Propositions 5.4.1 and 5.4.3 then give us the estimates

$$\mathbb{E}\left[(u_T(t,x) - u_T(t,y))^{2p}\right] \leq C'_p C_1^p R(x,y)^p,$$
$$\mathbb{E}\left[(u_T(s,x) - u_T(t,x))^{2p}\right] \leq C'_p C_2^p |s-t|^{p(2-d_s)}$$

for all $s,t \in [0,T]$ and all $x,y \in F$. The existence of a version $\tilde{u}$ with the required Hölder continuity properties then follows in the same way as in Theorem 3.5.6. Then using Theorem 5.3.9 and the series representation of u, the rest of the present theorem follows in the same way as in Theorem 3.5.7. □

5.5 Convergence to equilibrium

We conclude this chapter with a brief discussion of the long-time behaviour of the solution u to the SPDE (5.1.2). We are interested in whether the solution "settles down" as $t \to \infty$ to some equilibrium measure. Intuitively, we expect this to be the case when the damping constant β is positive. However the undamped case $\beta = 0$ is less clear. In this case there is no dissipation of energy, so is the rate of increase of energy quantifiable? Note that in this section we use the term "weak convergence" in the probabilistic sense, *not* in the functional analytic sense.

We treat the undamped case first. Throughout this section we will use the interpretation of the solution $u : \Omega \times [0,\infty) \to \mathcal{H}$ as an $\mathcal{H}$-valued process. Recall the series representation of u,

$$u = \sum_{k=1}^{\infty} Y_k^{b,\beta} \varphi_k^b,$$

given in (5.2.5).

Theorem 5.5.1 ($\beta = 0$)**.** *Let u be the solution to the SPDE* (5.1.2) *with* $\beta = 0$.

(1). If $b \neq N$ then $t^{-\frac{1}{2}} u(t)$ has a non-trivial weak limit in $\mathcal{H}$ as $t \to \infty$.

(2). If $b = N$ then $t^{-\frac{1}{2}} u(t)$ has no weak limit in $\mathcal{H}$ as $t \to \infty$. However $u - Y_1^{N,\beta}\varphi_1^N$ and $Y_1^{N,\beta}\varphi_1^N$ are independent $\mathcal{H}$-valued processes and $t^{-\frac{1}{2}}\left(u(t) - Y_1^{N,\beta}(t)\varphi_1^N\right)$ has a non-trivial weak limit in $\mathcal{H}$ as $t \to \infty$.

Proof. Let $(\zeta_k)_{k=1}^{\infty}$ be an independent and identically distributed sequence of real standard Gaussian random variables. We start with (1), so that $\lambda_1^b > 0$. For each $t \in [0,\infty)$ let

$$\bar{u}(t) = \sum_{k=1}^{\infty} (2\lambda_k^b)^{-\frac{1}{2}} \left(t - (4\lambda_k^b)^{-\frac{1}{2}} \sin\left((4\lambda_k^b)^{\frac{1}{2}} t\right)\right)^{\frac{1}{2}} \zeta_k \varphi_k^b.$$

It can be easily checked that $\bar{u}(t)$ is a well-defined $\mathcal{H}$-valued random variable with the same law as $u(t)$ for each $t \in [0, \infty)$. Now let

$$u_\infty = \sum_{k=1}^{\infty} (2\lambda_k^b)^{-\frac{1}{2}} \zeta_k \varphi_k^b,$$

so that u_∞ is also a well-defined $\mathcal{H}$-valued random variable. It is then simple to check using dominated convergence (see Proposition 4.2.5) that

$$\lim_{t\to\infty} \mathbb{E}\left[\| t^{-\frac{1}{2}} \bar{u}(t) - u_\infty \|_\mu^2 \right] = 0,$$

so in particular $t^{-\frac{1}{2}} \bar{u}(t) \to u_\infty$ weakly as $t \to \infty$. Therefore the same weak convergence holds for $t^{-\frac{1}{2}} u(t)$.

We now tackle (2). The issue that forces us to consider this case separately is that $\lambda_1^N = 0$, so the variance of $\langle t^{-\frac{1}{2}} u(t), \varphi_1^N \rangle_\mu$ tends to infinity as $t \to \infty$. We deal with this by subtracting off the offending component, which is exactly $Y_1^{N,\beta} \varphi_1^N$. It is clearly independent of $u - Y_1^{N,\beta} \varphi_1^N$ by (5.2.5). Now $\lambda_k^N > 0$ for all $k \geq 2$, so similar to (1) we let

$$\bar{u}(t) = \sum_{k=2}^{\infty} (2\lambda_k^N)^{-\frac{1}{2}} \left(t - (4\lambda_k^N)^{-\frac{1}{2}} \sin\left((4\lambda_k^N)^{\frac{1}{2}} t \right) \right)^{\frac{1}{2}} \zeta_k \varphi_k^N,$$

which has the same law as $u(t) - Y_1^{N,\beta}(t) \varphi_1^N$, and

$$u_\infty = \sum_{k=2}^{\infty} (2\lambda_k^N)^{-\frac{1}{2}} \zeta_k \varphi_k^N.$$

As with (1) we conclude that $t^{-\frac{1}{2}} \left(u(t) - Y_1^{N,\beta}(t) \varphi_1^N \right) \to u_\infty$ weakly as $t \to \infty$. □

We now tackle the damped case $\beta > 0$. It turns out that we must split this again into two subcases: $b \neq N$ and $b = N$.

Theorem 5.5.2 ($\beta > 0$). *Let u be the solution to the SPDE* (5.1.2) *with $\beta > 0$.*

(1). If $b \neq N$ then $u(t)$ has a non-trivial weak limit as $t \to \infty$.

(2). If $b = N$ then $u(t)$ has no weak limit as $t \to \infty$. However $u - Y_1^{N,\beta} \varphi_1^N$ and $Y_1^{N,\beta} \varphi_1^N$ are independent $\mathcal{H}$-valued processes, and $\left(u(t) - Y_1^{N,\beta}(t) \varphi_1^N \right)$ has a non-trivial weak limit as $t \to \infty$.

Proof. We do case (1) first. Observe that if $\beta > 0$ and $b \in 2^{F^0} \setminus \{N\}$ then $V_\beta(\lambda, t)$ decays exponentially as $t \to \infty$ for any $\lambda \geq 0$. It follows that

$$\int_0^\infty V_\beta(\lambda, s)^2 ds < \infty \tag{5.5.1}$$

for all $\lambda > 0$, and so by Itō's isometry we may define

$$Z_k^{b,\beta}(t) = \int_0^t V_\beta(\lambda_k^b, s)\langle \varphi_k^b, dW(s)\rangle_\mu$$

for each $t \in [0, \infty]$ and $k \geq 1$, which is an $\mathcal{H}$-valued random variable. In the case $t \in [0, \infty)$ this evidently has the same law as $Y_k^{b,\beta}(t)$. Then for each $t \in [0, \infty)$ let

$$\hat{u}(t) = \sum_{k=1}^\infty Z_k^{b,\beta}(t)\varphi_k^b$$

and

$$u_\infty = \sum_{k=1}^\infty Z_k^{b,\beta}(\infty)\varphi_k^b.$$

It is clear that $\bar{u}(t)$ is an $\mathcal{H}$-valued random variable with the same law as $u(t)$, for all $t \in [0, \infty)$. Now for any $t \in [0, \infty)$ we have by Itō's isometry that

$$\begin{aligned}
\mathbb{E}\left[\|\hat{u}(t) - u_\infty\|_\mu^2\right] &= \sum_{k=1}^\infty \mathbb{E}\left[\left(Z_k^{b,\beta}(t) - Z_k^{b,\beta}(\infty)\right)^2\right] \\
&= \sum_{k=1}^\infty \int_t^\infty V_\beta(\lambda_k^b, s)^2 ds \qquad (5.5.2) \\
&= \sum_{k:\lambda_k^b \leq \beta^2} \int_t^\infty V_\beta(\lambda_k^b, s)^2 ds + \sum_{k:\lambda_k^b > \beta^2} \int_t^\infty V_\beta(\lambda_k^b, s)^2 ds.
\end{aligned}$$

We treat each of these terms separately. As we mentioned in Proposition 5.2.10, there are only finitely many k such that $\lambda_k^b \leq \beta^2$, see Proposition 4.2.5. Then by (5.5.1) we have that

$$\begin{aligned}
\sum_{k:\lambda_k^b \leq \beta^2} \int_t^\infty V_\beta(\lambda_k^b, s)^2 ds < \infty, \quad t \geq 0, \\
\lim_{t\to\infty} \sum_{k:\lambda_k^b \leq \beta^2} \int_t^\infty V_\beta(\lambda_k^b, s)^2 ds = 0.
\end{aligned}$$

Now for the the $\{k : \lambda_k^b > \beta^2\}$ sum we need to do some estimates. Our assumption that $\beta > 0$ allows us to improve on the estimates of Proposition 5.2.10:

$$\begin{aligned}
\sum_{k:\lambda_k^b > \beta^2} \int_t^\infty V_\beta(\lambda_k^b, s)^2 ds &= \sum_{k:\lambda_k^b > \beta^2} \frac{1}{\lambda_k^b - \beta^2} \int_t^\infty e^{-2\beta s} \sin^2\left((\lambda_k^b - \beta^2)^{\frac{1}{2}} s\right) ds \\
&\leq \sum_{k:\lambda_k^b > \beta^2} \frac{1}{\lambda_k^b - \beta^2} \int_t^\infty e^{-2\beta s} ds \\
&= \frac{1}{2\beta} e^{-2\beta t} \sum_{k:\lambda_k^b > \beta^2} \frac{1}{\lambda_k^b - \beta^2}.
\end{aligned}$$

By Proposition 4.2.5 the infinite sum above converges, so we have by dominated convergence that

$$\sum_{k:\lambda_k^b > \beta^2} \int_t^\infty V_\beta(\lambda_k^b, s)^2 ds < \infty, \quad t \geq 0,$$

$$\lim_{t\to\infty} \sum_{k:\lambda_k^b > \beta^2} \int_t^\infty V_\beta(\lambda_k^b, s)^2 ds = 0.$$

Setting $t = 0$ in (5.5.2), we have now proven that

$$\mathbb{E}\left[\|u_\infty\|_\mu^2\right] < \infty,$$

and so u_∞ is a well-defined $\mathcal{H}$-valued random variable. From (5.5.2) we have also proven that

$$\lim_{t\to\infty} \mathbb{E}\left[\|\hat{u}(t) - u_\infty\|_\mu^2\right] = 0.$$

In particular this implies that $\hat{u}(t) \to u_\infty$ weakly as $t \to \infty$. Since $u(t)$ has the same law as $\hat{u}(t)$ for all t, this implies that $u(t) \to u_\infty$ weakly as $t \to \infty$.

In (2), we have the issue that $\lambda_1^N = 0$ so $V_\beta(\lambda_1^N, \cdot)$ is not square-integrable, which precludes $u(t)$ from having a weak limit. We get around this issue by simply subtracting the associated term of the series representation of u, leaving only the square-integrable terms. We we still have $\lambda_k^N > 0$ for all $k \geq 2$, so by Itō's isometry we may define

$$Z_k^{N,\beta}(\infty) := \int_0^\infty V_\beta(\lambda_k^N, s)\varphi_k^{N*} dW(s)$$

for $k \geq 2$. From the series representation (5.2.5) of u, observe that $Y_1^{N,\beta}(t)\varphi_1^N$ is simply the component of $u(t)$ associated with the eigenfunction φ_1^N, so that

$$u(t) - Y_1^{N,\beta}(t)\varphi_1^N = \sum_{k=2}^\infty Y_k^{N,\beta}(t)\varphi_k^N,$$

and the independence result is clear. For each t we then define

$$Z_k^{N,\beta}(t) = \int_0^t V_\beta(\lambda_k^N, s)\varphi_k^{N*} dW(s)$$

and

$$\hat{u}(t) = \sum_{k=2}^{\infty} Z_k^{N,\beta}(t)\varphi_k^N,$$

so that $\hat{u}(t)$ has the same law as $u(t) - Y_1^{N,\beta}(t)\varphi_1^N$. The proof proceeds from here in the same way as in the proof of (1) – we show that

$$\mathbb{E}\left[\|u_\infty\|_\mu^2\right] < \infty$$

and

$$\lim_{t\to\infty} \mathbb{E}\left[\|\hat{u}(t) - u_\infty\|_\mu^2\right] = 0$$

which imply the result. □

Appendix

Two tightness theorems for fractals

Let $(F, (\psi_i)_{i=1}^N)$ be a connected p.c.f. fractal with a regular harmonic structure $(A, \mathbf{r})$, and let R be the associated resistance metric (see [52]). Let $(\Lambda_n : n \geq 0)$ be the partitions defined in Definition 3.3.3, with associated vertex sets $F_\Lambda^n = \bigcup_{w \in \Lambda_n} \psi_w(F^0)$. For $x \in F$ and $n \geq 0$, let $D_n^1(x)$ be the neighbourhood of x defined in Definition 3.3.10. Let $d_H > 0$ be the Hausdorff dimension of (F, R); recall that d_H is precisely the unique real number satisfying

$$\sum_i r_i^{d_H} = 1.$$

Let $C(F) = C(F; \mathbb{R})$ be the separable Banach space of continuous functions from (F, R) to $\mathbb{R}$ equipped with supremum norm. The following two theorems are based on similar results in [45, Section I.4] and [57, Section 1.4].

Theorem A.1. *For each $m \in \mathbb{N}$, let $\xi_m = (\xi_m(x))_{x \in F}$ be an $\mathbb{R}$-valued process indexed by F which is continuous on F. Suppose there exists $K > 0$ such that the following hold:*

(1). There exists $\gamma > 0$ and $x_0 \in F$ such that

$$\sup_{m \in \mathbb{N}} \mathbb{E}\left[|\xi_m(x_0)|^\gamma\right] \leq K,$$

(2). There exist $\alpha, \beta > 0$ such that

$$\sup_{m \in \mathbb{N}} \mathbb{E}\left[|\xi_m(x) - \xi_m(y)|^\alpha\right] \leq K R(x, y)^{d_H + \beta}$$

for all $x, y \in F$.

Then $\{\xi_m\}_{m \in \mathbb{N}}$ is a tight family of $C(F)$-valued random variables.

Proof. Our argument is based on the proof of Theorem 3.3.14, a fractal version of Kolmogorov's continuity theorem. For $m \in \mathbb{N}$ and $\varepsilon > 0$ let

$$\omega_m(\varepsilon) := \sup\{|\xi_m(x) - \xi_m(y)| : x, y \in F,\ R(x,y) \le \varepsilon\}$$

be the modulus of continuity of ξ_m. Our first task is to show that for all $\varepsilon' > 0$,

$$\lim_{\varepsilon\to 0} \sup_{m\in\mathbb{N}} \mathbb{P}\left[\omega_m(\varepsilon) > \varepsilon'\right] = 0.$$

For $n \ge 0$ and $m \in \mathbb{N}$ define

$$K_n^m = \sup_{\substack{x,y\in F_\Lambda^n \\ x\sim_n y}} |\xi_m(x) - \xi_m(y)|.$$

Fix a $\delta \in (0, \frac{\beta}{\alpha})$. By a calculation in the proof of Theorem 3.3.14 we see that assumption (2) of the theorem implies that there exists a constant $C > 0$ such that for all n and m,

$$\mathbb{P}\left[K_n^m > 2^{-n\delta}\right] \le C 2^{-n(\beta-\delta\alpha)}.$$

The fact that C is independent of m, n is key. Since $\beta > \delta\alpha$, it follows that

$$\lim_{n\to\infty} \sup_{m\in\mathbb{N}} \mathbb{P}\left[\bigcup_{n'\ge n} \left\{K_{n'}^m > 2^{-n'\delta}\right\}\right] = 0.$$

Fix $\varepsilon', \varepsilon'' > 0$. Then we pick $n = n_0$ such that:

(1).
$$c_g \frac{2^{1-(n_0+1)\delta}}{1-2^{-\delta}} \le \varepsilon',$$
where $c_g > 0$ is defined in Lemma 3.3.9,

(2).
$$\sup_{m\in\mathbb{N}} \mathbb{P}\left[\bigcup_{n>n_0} \left\{K_n^m > 2^{-n\delta}\right\}\right] \le \varepsilon''.$$

Then for $m \in \mathbb{N}$ define the event

$$\Omega_0^m = \bigcap_{n>n_0} \left\{K_n^m \le 2^{-n\delta}\right\},$$

so that $\mathbb{P}\left[\Omega_0^m\right] \ge 1 - \varepsilon''$. Now pick $\varepsilon > 0$ such that if $x, y \in F$ with $R(x,y) \le \varepsilon$ then $y \in D_{n_0}^1(x)$ (this is possible by Proposition 3.3.12).

Let $x, y \in F$ such that $R(x,y) \leq \varepsilon$. Then $y \in D^1_{n_0}(x)$ so in the same way as in the proof of Theorem 3.3.14 we see that for $m \in \mathbb{N}$,

$$|\xi_m(x) - \xi_m(y)| \leq 2c_g \sum_{n=n_0+1}^{\infty} K_n^m.$$

Therefore under the event Ω_0^m it must be the case that

$$|\xi_m(x) - \xi_m(y)| \leq 2c_g \sum_{n=n_0+1}^{\infty} 2^{-n\delta} = 2c_g \frac{2^{-(n_0+1)\delta}}{1-2^{-\delta}} \leq \varepsilon'.$$

This is true for all pairs $x, y \in F$ satisfying $R(x,y) \leq \varepsilon$ so it follows that

$$\Omega_0^m \subseteq \{\omega_m(\varepsilon) \leq \varepsilon'\}.$$

Therefore we have found $\varepsilon > 0$ such that

$$\sup_{m \in \mathbb{N}} \mathbb{P}\left[\omega_m(\varepsilon) > \varepsilon'\right] \leq \varepsilon''.$$

Note that the expression on the left-hand side of the above inequality evidently decreases as ε decreases. Let us take stock: we have so far proven that for all $\varepsilon' > 0$,

$$\lim_{\varepsilon \to 0} \sup_{m \in \mathbb{N}} \mathbb{P}\left[\omega_m(\varepsilon) > \varepsilon'\right] = 0.$$

Using assumption (1) of the theorem and Markov's inequality it is easy to see that

$$\lim_{n \to \infty} \sup_{m \in \mathbb{N}} \mathbb{P}\left[|\xi_m(x_0)| > n\right] = 0.$$

Then a routine argument using the Arzelà-Ascoli theorem on the compact space F proves the required tightness result, see [45, Theorem I.4.2] or [6, Theorem 5.2]. □

Let R_∞ be the metric on $\mathbb{R} \times F$ given by $R_\infty((s,x),(t,y)) = |s-t| \vee R(x,y)$. Let $C([0,1] \times F) = C([0,1] \times F; \mathbb{R})$ be the separable Banach space of continuous functions from $([0,1] \times F, R_\infty)$ to $\mathbb{R}$ equipped with supremum norm.

Theorem A.2. *For each $m \in \mathbb{N}$, let $\xi_m = (\xi_m(t,x))_{(t,x) \in [0,1] \times F}$ be an $\mathbb{R}$-valued process indexed by $[0,1] \times F$ which is continuous on $[0,1] \times F$. Suppose there exists $K > 0$ such that the following hold:*

(1). There exists $\gamma > 0$ and $x_0 \in F$ such that

$$\sup_{m \in \mathbb{N}} \mathbb{E}\left[|\xi_m(0,x_0)|^\gamma\right] \leq K,$$

(2). There exist $\alpha, \beta > 0$ such that

$$\sup_{m\in\mathbb{N}} \mathbb{E}\left[|\xi_m(t,x) - \xi_m(t,y)|^\alpha\right] \le KR(x,y)^{1+d_H+\beta},$$
$$\sup_{m\in\mathbb{N}} \mathbb{E}\left[|\xi_m(s,x) - \xi_m(t,x)|^\alpha\right] \le K|s-t|^{1+d_H+\beta}$$

for all $x, y \in F$.

Then $\{\xi_m\}_{m\in\mathbb{N}}$ is a tight family of $C([0,1]\times F)$-valued random variables.

Proof. Same method as Theorem A.1. See Proposition 3.3.16 for the analogous continuity theorem. □

Directions for further research

The results in this thesis lead naturally to further questions and promising avenues for research, which I have not had the opportunity to investigate to an appreciable extent due to time constraints. A selection of the most interesting problems are as follows:

Non-fixed-point diffusions with multidimensional renormalization functions

Suppose that in Chapter 2 we replace Assumption 2.3.1 with the following:

Assumption A.3. There exists a family $\mathbb{D} \subseteq \mathbb{D}_0$ of forms such that if $\mathcal{E} \in \mathbb{D}$, then $\rho_{\mathcal{E}}\Lambda(\mathcal{E}) \in \mathbb{D}$ for some $\rho_{\mathcal{E}} > 0$. It is such that there exists a parametrisation $\mathbb{D} = (\mathcal{E}^{(0)}_{v_1,v_2})_{v_1,v_2>0}$ where $\mathcal{E}^{(0)}_{v_1,v_2}$ only has edges of conductance 1 or v_1 or v_2, and there exists at least one edge each of conductances v_1 and v_2. Call $\mathbb{D}$ a *two-parameter invariant family* with respect to Λ.

We could assume instead an n-parameter family, but $n = 2$ will suffice to illustrate this generalization. Note that we have not included any assumption about asymptotic regularity, as it is not immediately evident what form it should take. Lemma 2.3.3 still holds, and thus the definition of *non-vanishing* remains the same. We now define multidimensional renormalization functions $\rho : (0,\infty)^2 \to (0,\infty)$ and $\alpha : (0,\infty)^2 \to (0,\infty)^2$ such that

$$\Lambda(\mathcal{E}^{(0)}_{v_1,v_2}) = \rho(v_1,v_2)^{-1}\mathcal{E}^{(0)}_{\alpha(v_1,v_2)}.$$

This is where the easy generalizations end. Consider Propositions 2.3.6 and 2.3.7. What would we mean by taking a limit $(v_1, v_2) \to \infty$? There are many ways of "going to infinity" in $(0,\infty)^2$ and so there is no a priori guarantee that $\rho(v_1, v_2)$ and $\alpha(v_1, v_2)$ will have unique limits. After this immediately comes the challenge of showing that α is invertible in some region of $(0,\infty)^2$ such that $\alpha^{-n}(v_1, v_2)$ is well-defined for all $n \in \mathbb{N}$. If one can surmount all of these technical hurdles, one arrives at the real meat of the problem: what is the behaviour of $\alpha^{-n}(v_1, v_2)$ as $n \to \infty$, given (v_1, v_2)? This is a question in the realm of two-dimensional dynamical systems, and is thus far more complicated than the one-dimensional case.

Comparison theorems for Walsh SPDEs

Consider the following pair of Walsh SPDEs on a fractal F up to time $T > 0$ indexed by $i = 1, 2$:

$$\begin{aligned} \frac{\partial u^i}{\partial t}(t,x) &= \Delta u^i(t,x) + f(t, u^i(t,x)) + g(t, u^i(t,x))\xi(t,x), \\ u^i(0,x) &= u_0^i(x), \end{aligned} \tag{A.1}$$

where Δ is some Laplacian on F, ξ is a space-time white noise on F, and f, g, u_0^1, u_0^2 are suitable functions. Assume that $u_0^1(x) \le u_0^2(x)$ for all $x \in F$. What additional assumptions do we need for the two SPDEs to have unique solutions for which $u^1(t,x) \le u^2(t,x)$ almost surely for each $(t,x) \in [0,T] \times F$? Results of this form, known as comparison theorems, are common for stochastic (ordinary) differential equations, and Kotelenez [54] gives such a result for a class of SPDEs on $\mathbb{R}$. I am confident that an argument based on Kotelenez's proof can be made to prove a similar comparison theorem for SPDEs on fractals.

Conjecture A.4. *We make the following assumptions:*

(1). $u_0^i : F \to \mathbb{R}$ *is measurable and bounded for* $i = 1, 2$, *and* $u_0^1(x) \le u_0^2(x)$ *for all* $x \in F$.

(2). $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$ *are measurable functions which obey the following Lipschitz and linear growth conditions: There exists a constant* $C > 0$ *such that for all* $t \in [0,T]$ *and all* $x, y \in \mathbb{R}$,

$$\begin{aligned} |f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| &\le C|x - y|, \\ |f(t,x)| + |g(t,x)| &\le C(1 + |x|). \end{aligned}$$

Let $u^1, u^2 : \Omega \times [0,T] \times F \to \mathbb{R}$ *be the solutions to the SPDEs* (A.1). *Then* $u^1(t,x) \le u^2(t,x)$ *almost surely for each* $(t,x) \in [0,T] \times F$.

Lower moment bound for the parabolic Anderson model

Recall Theorem 4.7.5, in which we prove that a certain class of Walsh SPDEs on fractals admit a general upper bound on the pth moments of their solutions. This result is based on [49, Proposition 5.8] which concerns SPDEs on $\mathbb{R}$. On the other hand, for the *parabolic Anderson model* on $\mathbb{R}$ given by

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \Delta u(t,x) + u(t,x)\xi(t,x), \quad (t,x) \in [0,\infty) \times F, \\ u(0,\cdot) &= 1, \end{aligned} \tag{A.2}$$

[49, Theorem 6.4] gives the reverse result – a lower bound for the moments of its solution. The proof is not trivially generalizable to fractals as it uses the Itō calculus, but I nevertheless believe that an analogous statement holds.

Conjecture A.5. *Let F be a p.c.f. fractal with a regular harmonic structure. Let R and Δ be the associated resistance metric and Laplacian. Consider the parabolic Anderson model* (A.2) *on F, where ξ is a space-time white noise on F. Let u be its solution. Then there exist constants $a_1, a_2 > 0$ such that for all $p \geq 1$ and all $(t,x) \in [0,\infty) \times F$,*

$$\mathbb{E}\left[|u(t,x)|^p\right]^{\frac{1}{p}} \geq a_1 \exp\left(a_2 p^{1+d_H} t\right)$$

where d_H is the Hausdorff dimension of (F,R).

Hyperbolic SPDEs

In Chapter 5 we show that the damped stochastic wave equation on a fractal F has a unique solution which is Hölder continuous. Naturally the next question to ask is whether this result can be generalized to a wider class of hyperbolic SPDEs. While a relatively simple extension to da Prato–Zabczyk SPDEs seems possible using the techniques of Chapter 4, Walsh SPDEs are perhaps slightly trickier to deal with. Consider a general hyperbolic Walsh SPDE

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x) + f\left(t, u(t,x), \frac{\partial u}{\partial t}(t,x)\right) + g\left(t, u(t,x), \frac{\partial u}{\partial t}(t,x)\right)\xi(t,x), \\ u(0,x) &= u_0(x),\ \frac{\partial u}{\partial t}(0,x) = u_1(x), \end{aligned}$$

where Δ is some Laplacian on F, ξ is a space-time white noise on F, and f, g, u_0, u_1 are suitable functions. The term $\frac{\partial u}{\partial t}(t,x)$ is immediately worrying. Is it well-defined? Recall from Chapter 5 that there is no precedent for the first time derivative of the solution to even be function-valued at any given time $t > 0$. One workaround would be to simply preclude the functions f and g from depending on $\frac{\partial u}{\partial t}(t,x)$, though this is avoiding the problem rather than dealing with it.

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