Analysis on Stochastic Anisotropic Degenerate Parabolic-Hyperbolic Mixed-Type Equations

Peter Ho Cheung Pang

Jesus College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

August 2017

Acknowledgments

"For the wonders that astound us,
For the truths that still confound us,
Most of all that love has found us,
Thanks be to God."

- Fred Pratt Green

This dissertation would not have been possible without the monumental patience and gentle guidance of Gui-Qiang Chen, my supervisor. My parents' sometimes forceful encouragement also gave much-needed impetus to the project, and their support for me generally and through the years cannot be here recounted, much less understated. The work would also not have been without support from the Croucher Foundation, chiefly financially, but the quiet helpfulness of two of the Foundation's administrative staff, who prefer to stay unnamed, have been an especial blessing. No less important were the administrative help that Sandyha Patel, Mott Carter (no longer at the Institute), Monica Finlayson, and Jonathan Whyman at the Mathematical Institute, and Carole Thomas at Jesus College, rendered me, without all of whom navigating the systems of forms and requirements would have been a much more formidable task. I also must thank William Long, Rawan Yaghi, and Alan Jiang at Jesus College, Daniel Navarro at the Queen's College, Chris de L'isle at New College, Spencer Klavan at Magdalen College, and Tobias Barker, Stefano Marchesani, Andi Wang, and Luca Alasio at the Mathematical Institute for their careful readings of the dissertation for mistakes typographical, grammatical, and mathematical – those that remain, remain my own.

Sometimes "emotionally at sea" can be quite a fitting description of one's internal development leading up to this dissertation – at sea in a canoe, no less, where having rowed a mile or a hundred seem not to be very different. I could not have carried through without the prayers and good cheer of the wonderful Eden Bailey, and Sarah

Siaw in particular, but very much also the communities at Just Love, at the Chapel of Jesus College, at the Oxford Community Church, and at a certain film society. To them I am eternally grateful. I am especially thankful for the support Eden was in a very difficult fortnight when a big gap in a proof was discovered shortly before a planned submission. If anyone else were to me lighthouses and charts, these people were very much my stay.

It is important to my mind also to thank a few others perhaps whom I see less often, but on and with whom I have both relied and consulted, and who also have been a source of much comforting encouragement. I refer first to Walter Liu, who when we were still in Toronto, encouraged me to consider coming to Oxford, and have been to me very much a friend whilst we were both here. Without James Colliander I would not have gone past simply considering. He asked me to thank him in theorems. Around the Institute, Jan Kristensen's, Yves Capdeboscq's, Qian Wang's, Gregory Seregin's, Kostos Koumatos's, Sebastian Vollmer's, and Angkana Rüland's friendliness made working there not only not prohibitively frightening, but an outright joy. Jan and Yves in particular have also been kind enough to serve as my examiners in the intermediate examinations of this course, giving me much valuable feedback on my work. Qian has never failed to be kind and full of good advice. I am also unreservedly thankful for a very generous landlady, Felicity Dobbie, and many friendly housemates during my three and a half years at No.20 Summerfield. There cannot be better housemates than Anton Muehlemann and Zuzanna Fiminska. And finally – and I do always have to go back to him to say – thanks, Jeffrey, thank you.

Analysis on Stochastic Anisotropic Degenerate Parabolic-Hyperbolic Equations (archival copy) (Trinity 2017)

Abstract

This dissertation consists chiefly of four parts, which tell different facets in the development of one topic. The first part is an exploration of continuous dependence estimates of stochastically driven degenerate parabolic equations. The second is an extension of work done by Debussche and Vovelle [30] on first order stochastic conservation laws — we extend their results to degenerate parabolic-hyperbolic conservation laws with additive noise, and derive results on the existence and uniqueness of invariant measures. In the third part we explore the long time behaviour of solutions to stochastic degenerate parabolic-hyperbolic conservation laws with *multiplicative* noise, depending non-linearly on the solution itself. The final part considers the existence of invariant measures to a system of one-dimensional compressible Navier-Stokes equations, which is *a priori* degenerate parabolic in its momentum equation.

Contents

List of Symbols

symbol(s)	description
$\mathbf{A}$, $\mathbf{B}$	positive semi-definite matrix-valued functions on $\mathbb{R}$
$\mathbf{A} : \mathbf{B}$	element-wise scalar product of two matrices of the same size
$\mathbf{a}$, $\mathbf{b}$	positive semi-definite square roots of $\mathbf{A}$ and $\mathbf{B}$, resp.
$C_b(\mathfrak{X})$	space of bounded continuous functions on $\mathfrak{X}$
d	(spatial) dimension of ambient space
$\mathbb{E}$	expectation over $(\Omega, \mathcal{P}, \mathbb{P})$ (see below)
$\epsilon(v)$	dynamic viscosity
$\varepsilon, \theta, \vartheta, \rho$	used to denote small numbers, some of which
	are taken to nought in limits
η_ρ, $\tilde{\eta}_\rho$, J_θ	cut-off functions and mollifiers
$\{\mathscr{F}_t\}$	natural filtration of W_t (see below)
H, $\bar{H}$	the Heaviside function and $1 - H$, resp.
$\varkappa$	kinematic viscosity
μ	Lebesgue measure on the d-dimensional torus
$\mathfrak{M}_1(\mathfrak{X})$	space of probability Borel measures on $\mathfrak{X}$
ν, Υ	Borel probability measures on $\mathfrak{X}$
$(\Omega, \mathcal{P}, \mathbb{P})$	a fixed probability space
$\mathcal{P}$	see $(\Omega, \mathcal{P}, \mathbb{P})$
$\mathbb{P}$	see $(\Omega, \mathcal{P}, \mathbb{P})$
$\mathscr{P}_t$	Markov semigroup acting on $C_b(\mathfrak{X})$
$\mathscr{P}_t$	dual semigroup acting on $\mathfrak{M}(\mathfrak{X})$
ϱ	physical denisty
$\mathcal{S}_t$	solution operator of a Cauchy problem
$\mathcal{S}_t^*$	operator dual to $\mathscr{S}_t$
$\fint_{\mathbb{T}^d} f(x)\, \mu(dx)$	average of the integral of f over $\mathcal{T}^d$

symbol(s)	description
σ, τ	coefficient of noise terms $\sigma\partial_t W$ and $\tau\partial_t W$ (Chp.2)
τ	also used as the temporal Fourier variable (Chp. 3, 4)
$\mathcal{T}$	stopping times
$\mathcal{T}^d$	d-dimensional torus
W	a standard Brownian motion on $(\Omega, \mathcal{P}, \mathbb{P}, \mathscr{F}_t)$
$\chi(r, \xi)$ (alter. $\chi_r(\xi)$)	kinetic function
$\mathfrak{X}$,$\mathfrak{Y}$,$\mathfrak{Z}$	Polish spaces

Chapter 1

Introduction

In this chapter we introduce some background literature to the topics we shall discuss in the subsequent chapters. We also provide an overview on the techniques and tools we shall use.

1.1 Background

1.1.1 Continuous Dependence

In the deterministic context, continuous dependence estimates to a whole host of nonlinear equations have been studied by various authors. Classically, Benilan and Crandall [4] considered continuous dependence to

$$\partial_t u + \Delta\varphi(u) = 0.$$

At its heart the question of continuous dependence is about the convergence of solutions u^n to the equation

$$\begin{aligned} \partial_t u^n + \Delta\varphi_n(u^n) =& 0, \\ u^n(0) =& u_0^n \end{aligned}$$

as $\varphi_n \to \varphi_\infty$, and $u_0^n \to u_0^\infty$ in some topology. This question is interesting in a more general context in the study of various physical limits, via the equations that model those physical phenomena such as the inviscid limit in fluid equations. They showed that convergence occurred in $C([0,\infty); L^1(\mathbb{R}^d))$ with different conditions depending on the dimension, d, and on the modulus of continuity of φ_∞ near 0.

Cockburn and Gripenberg considered the more general class of equations

$$\partial_t v = \nabla \cdot (\Phi(v)) + \Delta(\varphi(u))$$

on $\mathbb{R}^d$ in [21], where they derived bounds showing the dependence of solutions $\|u_1(t) - u_2(t)\|$ to t, $\|\Phi_1 - \Phi_2\|$ and $\|\varphi_1 - \varphi_2\|$ under various technical conditions. In particular, they showed under these conditions that

$$\|u_1(\cdot) - u_2(\cdot)\|_{L^1}\Big|_0^t \leq |u_0|_{BV} + t\|\Phi_1' - \Phi_2'\|_{L^\infty} + C(d)\sqrt{t}\|\sqrt{\varphi_1'} - \sqrt{\varphi_2'}\|_{L^\infty}. \tag{1.1}$$

Their results can be easily generalised to the non-isotropic case where $\Delta\varphi$ is replaced by $\nabla \cdot (\mathbf{A}(u) \cdot \nabla u)$, and $\mathbf{A}$ is positive semi-definite. We shall see in Chapter 2 that a very similar continuous dependence estimate can be derived for a corresponding equation with stochastic noise (satisfying some growth and regularity condition) in $L^1(\mathbb{T}^d)$ (or a weighted finite measure space on $\mathbb{R}^d$).

In [10], Chen, Ding, and Karlsen derived some continuous dependence estimates for

$$\partial_t u = \nabla \cdot F(u) + \sigma(u)\partial_t W_t, \tag{1.2}$$

showing that for $t \leq T$,

$$\begin{aligned}\mathbb{E}\left(\|u_1(t) - u_2(t)\|_{L^1(\psi\, dx)}\right) \leq & C_T\mathbb{E}\left(\|u_1(0) - u_2(0)\|_{L^1(\psi\, dx)}\right) + \sqrt{t}\|\psi\|_{L^1(dx)}\|\sigma_1 - \sigma_2\|_{L^\infty} \\ & + t\mathbb{E}\left(|u_1(0)|_{BV}\right)\left(\|F_1' - F_2'\|_{L^\infty} + \|\sigma_1 - \sigma_2\|_{L^\infty}\right),\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\left(\|u_1(t) - u_2(t)\|_{L^1(\psi\, dx)}\right) \\ \leq & C_T\mathbb{E}\left(\|u_1(0) - u_2(0)\|_{L^1(\psi\, dx)}\right) + \sqrt{t}\|\psi\|_{L^1(dx)} \sup_{\xi \neq 0} \frac{|\sigma_1(\xi) - \sigma_2(\xi)|}{|\xi|} \\ & + t\mathbb{E}\left(|u_1(0)|_{BV}\right)\left(\|F_1' - F_2'\|_{L^\infty} + \sup_{\xi \neq 0} \frac{|\sigma_1(\xi) - \sigma_2(\xi)|}{|\xi|}\right),\end{aligned}$$

where $C_T = C(T, \|F'\|_{L^\infty})$, and ψ is some function integrable over $\mathbb{R}^d$ with exponentially decaying tails as $|x| \to \infty$, serving as the weight/density that makes $\psi\, dx$ a finite measure on $\mathbb{R}^d$.

However, the most significant contribution of [10] was their proof of a BV bound,

$$\mathbb{E}(|u(t)|_{BV}) \leq \mathbb{E}(|u_0|_{BV}),$$

for solutions. Of course, as the equation they studied was itself translation invariant, the BV bound follows directly from the L^1 contraction estimate they proved, as Chen pointed out in private communications, and did not rely on the methods they used. Nevertheless application of the BV estimate to develop an existence theory for solutions to this class of equations sparked much subsequent work. Furthermore, their methods were important as we develop in this dissertation a bound of a similar nature for a translation non-invariant class of equations. We show here a fractional BV bound which depends on the continuity of $F(u, x)$ in its second argument, and becomes a BV bound where the quantity in question is continuously differentiable in space (Theorem 8).

The work [10] was one among many built atop a study of Feng and Nualart [38] of the same equation,

$$\partial_t u = \partial_x F(u) + \sigma(u)\partial_t W_t,$$

where the authors developed a well-posedness theory for the equation in one spatial dimension (Feng and Nualart studied a more general noise $\sigma(x, u; z)$, where z is a variable from a Hilbert space). Feng and Nualart used the theory of compensated compactness to prove existence, which depended on the equation being in one spatial dimension. The work by Chen, Ding, and Karlsen discussed previously removed this restriction via the BV estimate they proved. In the existence theory they proposed for the equations in question, Feng and Nualart established the concept of a strong stochastic entropic solution. This is an interesting point because it involved, unusually, a non-adapted stochastic integral, of which [61] gives some explanation – we shall return to this point in the concluding chapter.

Of immediate interest to us is the uniqueness statement shown in [38] via a Kruzhkov variable-doubling technique with which an L^1 contraction inequality was derived. Chen pointed out in private communications that conceptually, one should expect to be able to derive both L^1 contraction estimates and BV bounds from a general continuous dependence estimate, with $F_1 = F_2$ (resp. $F_1(\cdot,\cdot) = F_2(\cdot,\cdot - h)$), and $\sigma_1 = \sigma_2$ in a pair of (1.2). We carry out this programme in Chapter 2 of this dissertation for the degenerate parabolic equations.

We shall discuss invariant measures to stochastically driven equations further on, but in order to set that discussion in context, we briefly describe the development of the subject – the literature cited is in no way exhaustive, and a more complete survey by Chen and Pang is forthcoming [12].

A pioneering work on stochastically driven hyperbolic conservation law was Sinai's paper [98] on space-periodic, Brownian-in-time noise of the Burgers equation on $\mathbb{R}$,

$$\partial_t u + \frac{1}{2}\partial_x u^2 = \varepsilon \partial^2_{xx} u + \partial_x f(x) \partial_t W_t.$$

Sinai showed that an invariant measure exists for the equation, and is supported on the subspace of space-periodic functions. Also quite early on, Holden and Risebro [54] studied scalar conservation laws with multiplicative noise of compact support, constructing a convergent approximation to weak solutions pathwise via a time discretization.

E, Khanin, Mazel, and Sinai [36] worked on the same equation with no constraint that the force should be space-periodic. They showed that almost surely, for each sample path, the corresponding force yielded a unique solution, by framing the problem as a variational problem, and showing that unique minimizers exist globally almost surely. Taking the viscosity $\varepsilon \to 0$ they were able to show that the dissipation provided by the entropy condition alone was enough to preserve uniqueness. From this uniqueness, the existence of an invariant measure follows. We keep this discussion quite short as, even though [36] was important in the development of the subject, we shall not be using techniques developed therein. Nakazawa also studied the inviscid limit earlier on [87], and showed the existence of solutions that are pathwise continuous stochastic processes in the space of bounded variation in the topology of L^1.

The work of Debussche and Vovelle [30], which we do heavily reference, and on which methods we shall develop a significant portion of this dissertation, showed that invariant measures exists, and under further constraints were unique, for equations of the form

$$\partial_t u + \nabla \cdot F(u) = \Phi d\beta,$$

on $\mathbb{T}^n$. Here, Φ is a Hilbert-Schmidt operator from an abstract Hilbert space H to

$L^2(\mathbb{T}^n)$, and β is a cylindrical Wiener process. They relied on velocity averaging ideas to achieve the compactness required for deriving the existence of invariant measures. We give a lengthier discussion on the historical development of velocity averaging in §3.2. [30] further showed the uniqueness of invariant measures by an essentially coupling method. We make further remarks on this in the following section. Following closely the results of [30], Gess and Souganidis [41] showed thatin the Stratonovich case, again with W as a standard Brownian motion,

$$\partial_t u + \nabla \cdot (F(u) \circ dW) = 0$$

also has unique invariant measures. In particular these were the Dirac mass concentrated on the zero function.

We also discuss briefly in Chapter 5 the existence of invariant measures to a system of equations. The well-posedness of compressible fluid equations is a wide field, though in one dimension it is much better understood than in higher dimensions. We give some references and history of the subject in chapter 5 itself. However it bears mentioning here that on the incompressible side, there are some well-known studies of 2D Navier Stokes equations, such as those of Kuksin, and Kuksin and Shirikyan [71, 72]. Still more recently, Hairer and Mattingly [49] proved the existence of a unique invariant measure for the stochastic 2D Navier-Stokes equations. In particular, they introduce an *asymptotic strong Feller* condition to capture a weaker form of the strong Feller condition by which uniqueness of invariant measures can be generally deduced.

1.2 Overview

1.2.1 Kinetic Formulation

This dissertation is much more closely based on the works of Debussche, Hofmanova, and Vovelle [28], and of Debussche and Vovelle [30]. In [28], the three authors provided a well-posedness theory for the stochastic degenerate parabolic equation, on the d-torus, $\mathbb{T}^d$,

$$\partial_t u = -\nabla \cdot F(u) + \nabla \cdot (\mathbf{A}(u) \cdot \nabla u) + \sigma(u)\partial_t W_t.$$

which in some ways is the protagonist of our story. Importantly, the equation is "degenerate" because $\mathbf{A}$ is positive *semi*-definite. In the chapters following, precise continuity and growth conditions will be placed on the coefficients for our purposes. These equations and systems of such equations model various physical processes such as porous media flow and compressible flows [102].

Debussche, Hofmanovà, and Vovelle [] used the kinetic formulation approach, which is probably furthest traceable to an idea of Giga and Miyakawa's [43]. After the work of Vol'pert and Hudjaev [104], [13] was one of the first to make a foray into the question of anisotropic degenerate diffusion [102], and later the same authors [13] used the kinetic formulation to settle questions of uniqueness via L^1-contraction. The idea of the kinetic formulation is concisely as follows. Let the "kinetic function" be defined as follows:

$$\chi(\xi, r) = \begin{cases} 1 & 0 < \xi < r \\ -1 & r < \xi < 0 \\ 0 & \text{else} \end{cases}. \tag{1.3}$$

Using this definition of kinetic functions, we see that for $S \in C^1$ with $S(0) = 0$, we have the representation formula

$$S(r) = \int_0^r S'(\xi)\chi(\xi, r)\, d\xi. \tag{1.4}$$

Then for the (deterministic) equation

$$\partial_t u - \nabla \cdot F(u) = 0,$$

for example, we can find the kinetic formulation by considering the vanishing artifical viscosity limit, multiplying the approximate equation with an additional dissipative $\varepsilon\Delta u$ term against $\Phi'(u)$, where Φ is convex, and satisfies $\Phi(0) = 0$. Representing $\Phi(u)$ by the formula (1.4), we see that

$$\begin{aligned} \Phi'(u^\varepsilon)\partial_t u^\varepsilon =& \Phi'(u^\varepsilon)F'(u^\varepsilon) \cdot \nabla u^\varepsilon + \varepsilon \Phi'(u^\varepsilon)\Delta u^\varepsilon \\ \partial_t \int_0^\infty \Phi(\xi)\chi(\xi, u^\varepsilon)\, d\xi =& \nabla \cdot \int \Phi'(\xi)F'(\xi)\chi(\xi, u^\varepsilon)\, d\xi + \varepsilon\Delta \int_0^\infty \Phi'(\xi)\chi(\xi, u^\varepsilon)\, d\xi \\ & - \varepsilon|\nabla u^\varepsilon|^2 \int_0^\infty \Phi''(\xi)\delta(\xi - u^\varepsilon)\, d\xi. \end{aligned}$$

In the limit $\varepsilon \to 0$, we have the kinetic formulation of this equation, being the weak formulation of

$$\partial_t \chi(\xi, u) = F'(\xi) \cdot \nabla \chi(\xi, u) + \partial_\xi m(\xi, x, t),$$

where m is some non-negative measure called the "kinetic measure", which is the defect measure quantifying the dissipative loss via the artificial viscosity. Where there is a stochastic forcing term $\sigma(u)\partial_t W_t$, there will be Itô correction and noise terms, giving us

$$\partial_t \chi(\xi, u) = F'(\xi) \cdot \nabla \chi(\xi, u) + \partial_\xi m(\xi, x, t) - \partial_\xi p(\xi, x, t) + q(\xi, x, t)\partial_t W, \tag{1.5}$$

where

$$\frac{1}{2}\sigma^2(\xi)\delta(\xi - u^\varepsilon) \to p(\xi, x, t),$$
$$\sigma(\xi)\delta(\xi - u^\varepsilon) \to q(\xi, x, t)$$

are the Itô correction and noise terms, respectively.

This understanding of kinetic functions is closely related to the theory of Young measures. Young measures were first proposed by Young for the calculus of variations [106], and was developed subsequently by Murat [85, 86], Tartar [103], Di Perna [90], Serre [96], Ball and Murat [2], and others, particularly in the theory of compensated compactness. A Young measure can be interpreted as the probability distribution of the values of a limit of a sequence of functions that fluctuate wildly and have no classical limit [37]; so that, if (u_n) were just a bounded sequence in $L^\infty(U, \mathbb{R}^d)$, then there is a subsequence (also (u_n)) and a Borel probability measure ν_x such that for each $\varphi \in C(\mathbb{R}^d)$, the weak* limit of $\varphi(u_n)$ satisfies

$$w^* - \lim_{n\to\infty} \varphi(u_n(x)) = \int \varphi(\xi)\ \nu_x(d\xi).$$

This Borel measure is the Young measure associated to the sequence (u_n).

By considering a sequence of functions u_n bounded in L^1, there is a $u \in L^1_{\text{loc}}$ to which a subsequence of u_n tend in the weak* topology of L^1_{loc}, and a corresponding subsequence $\chi(\cdot, u_n)$ that tend in the weak* topology of L^∞ to some $f(\xi, x) \in L^\infty(\mathbb{R} \times \mathbb{R}^d)$. It holds

[91] (Lemma 2.3.1, Lemma 2.3.3, §2.6) both that, for every Lipschitz S,

$$w^* - \lim_{n\to\infty} S(u_n(x)) = \int_{\mathbb{R}} S'(\xi) f(\xi, x)\, d\xi,$$

and

$$\partial_\xi f(\xi, x) = \delta(\xi) - \nu_x(\xi),$$

where $\nu_x(\xi)$ is a probability measure. This probability is of course the Young measure associated with (u_n).

Another way to understand the kinetic formulation is as in [11], where the authors took $\Phi(u)$ above to be $\Phi(u) = H(\xi)H(u-\xi) - H(-\xi)H(\xi - u)$, H being the Heaviside function, in order to derive heuristically the same kinetic formulation. We carry out this calculation for the stochastic degenerate parabolic equation

$$\partial_t u = -\nabla \cdot F(u) + \nabla \cdot (\mathbf{A}(u) \cdot \nabla u) + \sigma(u)\partial_t W_t \tag{1.6}$$

on $\mathbb{T}^d$ between (2.9) and (2.10), and a more complete description of the theory is found in the section in which these equations are.

Lion, Perthame, and Tadmor discussed the wider applications of the kinetic formulation in their classical paper on the subject [79], particularly to velocity averaging. This is again a point to which we shall return later.

Apart from formula (1.4), another important property of kinetic functions is their ability to represent the L^1 difference, giving us a way to bound this difference and establish continuous dependence in L^1, or L^1-contraction and other similar bounds by way of the expression

$$\int\int |\chi(\xi, u)| + |\chi(\xi, v)| - 2\chi(\xi, u)\chi(\xi, v)\, d\xi\, dx = \int |u - v|\, dx. \tag{1.7}$$

There are a few modifications we make in our calculations in Chapter 2. Following [28], we use the modified kinetic function $\chi(\xi, u) = 1 - H(\xi - u)$, H again being the Heaviside function, instead, so that (1.4) and (1.7) read, respectively,

$$\int S'(\xi)(1 - H(\xi - u))\, d\xi = S(u) - S(-\infty),$$

$$\int \int H(\xi - u)(1 - H(\xi - v))\, d\xi\, dx = \int (u - v)_+\, dx.$$

The second formula in particular will allow us to prove a comparison theorem in place of an L^1 contraction theorem.

Also following [28], and [38], another technique we shall employ is that of variable-doubling of Kruzhkov's [67, 68]. So that, doubling both the spatial and kinetic variables, in fact, we investigate the quantity

$$\iint \iint H(\xi - u(x,t))(1 - H(\zeta - v(y,t)))\phi(x - y)\psi(\xi - \zeta)\, dx\, dy\, d\xi\, d\zeta, \tag{1.8}$$

where ϕ and ψ are approximations to identity. Both the variable doubling technique and the kinetic formulation are strategies to linearize an equation by introducing new independent variables. By introducing new variables, we shall find that continuous dependence estimates, as well as existence arguments can be distilled to a calculus involving the relative rates at which ϕ and ψ tend to the delta function.

In [61], the authors remarked that there are various ways to achieve the results of Feng and Nualart without resorting to the *prima facie* contrived notion of a strong stochastic entropic solution proposed in [38]. Karlsen and Storrøsten pointed out that there are three well posedness theories for the Cauchy problem to the equation (1.2). Feng and Nualart [38] framed their results in the language of the strong stochastic entropic solutions, Debussche and Vovelle [29] used the kinetic formulation we follow in this dissertation, and Bauzet, Vallet, and Wittbold [3] used the framework of entropy solutions. The three different methods of arriving at a well-posedness theory are three ways of capturing the noise-noise interaction when comparing two solutions. [61] points out that [3] avoided the strong entropic solution formulation by comparing entropy solutions to a vanishing viscosity solution, and the analogous notion is captured by the kinetic defect measure in the kinetic formulation approach of [29]. In the kinetic approach, by linearizing the equation further by the introduction of new, kinetic, variables, the interaction in certain cross terms involving noise can be accounted for using these defect measures instead of directly integrated.

Furthermore, [61] modified the Kruzhkov entropy condition to compare a solution not to a constant, but to a general Malliavin differentiable variable, using an anticipating Itô formula. They were able to show that the vanishing viscosity solution is Malliavin differentiable, and use the framework of [3] to suggest where the notion of a strong stochastic entropic solution of [38] arose (Remark 5.1 [61] – although they do not derive the condition completely). It would be interesting to study what theoretical connections there are between kinetic formulations and this Malliavin framework of [61].

1.2.2 Invariant Measures

Turning now to invariant measures of processes described by (1.2) and (1.6), we followed in the footsteps of [30]. An invariant measure may be disintegrated as a tensor product of the probability measure and the stationary measure, and there is a one-to-one correspondence between stationary measures and the invariant measure for Markov invariant measures – this is the Ledrappier-Le Jan-Crauel theorem [74, 57, 22] (also in [35] §10.2). As further described in §3.1, there are two primary ingredients to extraction of a stationary/invariant measure. The first is the compactness of the solution map, and the second is the Krylov-Bogoliubov Theorem (Theorem 12, §3.1). Where $\mathscr{P}_s : C_b(\mathfrak{X}) \to C_b(\mathfrak{X})$ is a Markov semigroup on a Polish Space $\mathfrak{X}$, $\mathscr{P}_s^* : \mathfrak{M}(\mathfrak{X}) \to \mathfrak{M}(\mathfrak{X})$ is the dual semigroup, and Υ is some initial measure, the Krylov-Bogoliubov Theorem gives a stationary measure as a limit of

$$\nu_T = \fint_0^T \mathscr{P}_s^* \Upsilon \, ds, \tag{1.9}$$

as $T \to \infty$, contingent on the tightness of $\{\mathscr{P}_s^* \Upsilon\}$ among other technical conditions. This tightness is implied by the bounded of the temporal averages of norms $\|u\|_{\mathfrak{Z}}$ in spaces $\mathfrak{Z}$ compactly embedded in $\mathfrak{X}$. This requires the compactness of the solution operator.

Whilst [30] showed the compactness of the solution operator to

$$\partial_t u = -\nabla \cdot F(u) + \sigma(x) \partial_t W_t \tag{1.10}$$

on $\mathbb{T}^d$, under the condition that

$$\iota(\varepsilon) = \sup_{|k|^2 + \tau^2 = 1} \text{meas}\{v : |F'(v) \cdot k + \tau| \leq \varepsilon\} \sim \varepsilon^b$$

for some $b > 0$, we extend this analysis to (1.6), and show that we can use an analogous non-degeneracy condition similar to that of [13]. As mentioned in [79], these non-degeneracy conditions are multidimensional genuine non-linearity conditions and related to a non-degeneracy condition of Tartar's [103] in the theory of compensated compactness. That is, the compactness relies essentially on the non-linearity of the equation.

We owe the technique of showing compactness essentially to [30]. This allows us to define a compact subspace $\mathfrak{Z}$ of L^1 of functions with some modulus of continuity. Next, following [30], we re-arranged the equation (1.5) as

$$\partial_t \chi_u - A_{\gamma,\theta}\chi_u = (B_\gamma + \theta \mathrm{Id})\chi_u + \partial_\xi(m_u - p_u) + q_u \partial_t W,$$

with

$$\begin{aligned} B_\gamma =& \gamma(-\Delta)^\alpha, \\ -A_{\gamma,\theta} =& F'(\xi)\cdot\nabla + B_\gamma + \theta\mathrm{Id} + \mathbf{A}(\xi) : \nabla\otimes\nabla. \end{aligned}$$

We have written $\chi_u = \chi_u(\xi, x, t) = \chi(\xi, u(x, t))$, χ being the kinetic function defined in (1.3). Also, we have used the notation $\mathbf{A} : \mathbf{B}$ for $m \times m$ matrices $\mathbf{A} = (\mathbf{a}_{ij})$ and $\mathbf{B} = (\mathbf{b}_{ij})$ to denote the element-wise product $\sum_{1\le i,j\le m} \mathbf{a}_{ij}\mathbf{b}_{ij}$.

Here as before, m_u is the kinetic measure, the vanishing viscosity limit of $\varepsilon|\nabla u^\varepsilon|^2\Xi(\xi - u^\varepsilon)$, p_u is the Itô correction, $\frac{1}{2}\sigma^2(x)\delta(\xi - u)$, and q_u is $\sigma(x)\delta(u-\xi)$.

Now using the mild formulation/Duhamel formula, we can split the solution into the following parts:

$$u = u^0 + u^\flat + M_1 + M_2,$$

where

$$\begin{aligned} u^0 &= \int \mathcal{S}(t)\chi_u(\xi, x, 0)\, d\xi, \\ u^\flat &= \int\int_0^t \mathcal{S}(s)(B_\gamma\chi_u + \theta\chi_u)(\xi, x, t-s)\, ds\, d\xi, \\ \langle M_1(t), \varphi\rangle &= -\int_0^t \int_{\mathbb{T}^d}\int \partial_\xi(\mathcal{S}^*(t-s)\varphi)\, d(m_u + n_u - p_u)(\xi, x, t), \\ \langle M_2(t), \varphi\rangle &= \int_{\mathbb{T}^d} (\mathcal{S}^*(t-s)\varphi)(x, u(x,s))\sigma(x)\, dW_s\, dx. \end{aligned}$$

Using the work on velocity averaging of Bouchut and Desvillettes [9], we estimate the temporal average of the $\mathfrak{Z}$-norm of each of the components.

Lions, Perthame, and Tadmor use the kinetic formulation to recover the velocity averaging results in [31], as well as [45] and [46] where the velocity averaging result was couched in the language of a compactness theorem. The fact that the kinetic formulation is well suited to development of velocity averaging theory was noted in [79], in the authors' discussion of [31]. [11] used a similar kinetic formulation approach to show that periodic solutions decay to the average of the initial data over a period for multidimensional conservation laws. A more recent manifestation of this fact is [102], where new regularity results were proven for quasi-linear second order PDEs. In the deterministic case, as shown in [79] and [11], it was possible to take the space-time Fourier transform of the entire equation in the kinetic formulation,

$$(\tau + F'(\xi) \cdot k)\hat{\chi}_u(\xi, k, \tau) = \partial_\xi \hat{m}(\xi, k, \tau),$$

where k and τ are the Fourier variables corresponding to the periodic space and to time, repsectively, and then divide the entire equation through by $(\tau + F'(\xi) \cdot k)$ in order to perform estimates on $\hat{\chi}_u$ directly. Then by considering different embeddings and using the non-degeneracy condition $\iota(0) = 0$, one can attain some results on the compactness of F. The introduction of the operators $A_{\gamma,\theta}$ and B_γ above allowed us to avoid thorny questions on the Fourier transform of Brownian motion paired with $\delta(\xi - u)$.

To show that the invariant measure is unique, we follow largely in the steps of [30], first showing that solutions enter a given ball finite time, then showing that for solutions starting within that given ball, driven by small enough noise, solutions are themselves small. Next one shows that since the noise is small with positive probability, over a long enough time, all solutions must enter arbitrarily small balls (around one another). The general philosophy behind this framework is explained in §4.3.3.

We briefly describe the idea involved, In a coupling argument we seek an almost finite stopping time, defined to be the first meeting time, to control the probability that two identical processes with different initial distributions will drift apart, showing that eventually, all distributions tend to a unique, invariant distribution. Here we use a relaxed version of the coupling argument, showing first that two processes enter a given ball in finite time almost surely, shuch is the substitute for the traditional first meeting time. From this an increasing sequence of almost surely finites stopping time is contructed. Then it is shown that starting from the given ball, where the noise is small for a fixed duration T, the solutions become small in some norm – as expected from deterministic results on zero-average solutions [11]. Next it is shown that the intervals $[\mathcal{T}, \mathcal{T}+T]$ as $\mathcal{T}$ range over the sequence of increasing stopping times, must eventually coincide with an interval on which the noise is small almost surely, forcing solutions to coincide by L^1 contraction.

The general philosophy behind this framework is further explained in §§3.1.5,4.3.1.

1.2.3 Multiplicative noise

Next we turn to the question of multiplicative noise. Surprisingly little is known about the long time behaviour of solutions with general initial data to equations with multiplicative noise. The equation most intensively studied with multiplicative noise is the Kolmogorov-Petrovskii-Piskunov equation (KPP).

It is given by

$$\begin{aligned}\partial_t u = \nabla\cdot(\mathbf{A}(x,t)\cdot\nabla u) + h(u) + g(u)\partial_t W,\\ u(0) = \varphi \in (\xi_1, \xi_2) \subset \mathbb{R},\end{aligned}$$

where W is a standard Brownian motion. Attention is often restricted to the case in which g and h both vanish at the the two points $\xi_1, \xi_2 \in \mathbb{R}$, and $g, h > 0$ on (ξ_1, ξ_2). In this way the asymptotic size is controlled in L^1.

Chueshov and Villermot [16, 15, 17, 18, 19, 20] in a series of papers showed that solutions to the semilinear equation with $h(u) = sg(u)$, and evolution on the bounded, open domain D with Neuman boundary condition

$$\frac{\partial u}{\partial n} = 0,$$

were bounded in space. By considering moments of averages they showed that the constant functions $u_i = \xi_i$ with $i = 1, 2$ were fixed points whose stability in probability dependend on the values of s.

Bergé and Saussereau [5] refined the results of [20] and showed using monotonicity methods that either the solutions oscillate between equilibrium states or decay exponentially to them. They calculate the Lyapunov exponents exactly for the decay scenario. Using uniform ellipticity, they are also able to show an "exchange of stability" between two components of the global attractor in the oscillatory case.

The chief reason that multiplicative noises complicate the analysis of stochastic PDEs is that one fails to have much control over the norm, except when additional restrictions on the noise and initial conditions are specified. Where the noise has a root, that constant is immediately a fixed point. This cannot be avoided even when working over the non compact domain $\mathbb{R}$, and oftentimes, L^p boundedness relies on the space being compact.

Using other methods, namely, by investigating the spatial moments of the solution directly, possibly in the spirit of [80], we show here that for a class of unbounded *multiplicative* Lipschitz noises with one root,

$$\partial_t u = \nabla \cdot (\mathbf{A}(u) \cdot \nabla u) - \nabla \cdot F(u) + \sigma(u)\partial_t W$$

exhibits decay in the solutions to the root, for any initial condition in $L^1(\mathbb{T}^d)$, with corresponding results for

$$\partial_t u = -\nabla \cdot F(u) + \sigma(u)\partial_t W$$

as a special case.

The theorem is as follows:

Theorem 1. *Let F and $\mathbf{A}$ be Hölder continuous with polynomial growth in their arguments, and let the Hölder index γ of $\mathbf{A}$ satisfy $2\gamma > 1$. Let p be a positive even number, and*

$$\varepsilon = \left(1 - \frac{1}{2\sqrt{p}}\right).$$

Suppose there are real numbers λ, c with $|c| > 0$ such that $\sigma : \mathbb{R} \to \mathbb{R}$ is Lipschitz, satisfying

$$|\sigma(u)/c| \leq |u - R|, \qquad |\sigma(u)/c - (u - R)| \leq (1 - \varepsilon)|u - R|.$$

Then a kinetic solution to

$$\partial_t u = -\nabla \cdot F(u) + \nabla \cdot (\mathbf{A}(u) \cdot \nabla u) + \sigma(u)\partial_t W$$

exhibits the following long-time decay:

$$\int |u - R|^p \, dx \to 0.$$

This theorem implies via an L^1 contraction estimate, and a Borel-Cantelli argument a là [30] that there is a unique invariant measure centred at the constant L^1 function R, using a uniqueness framework of Kuksin and Shirikyan [73], and additionally, a decay rate can be derived. The parity of p ensures there is no cancellation effect at work in the integral as it tends to nought.

For typesetting reasons we refer the reader to Fig. 4.2, for a schematic representation of the noise we consider.

1.2.4 Systems of Equation

By revisiting some calculations of Kanel"s [59], using spatial discretization ideas of Hoff [51], we also derived an existence of invariant measure result for the stochastic compressible Navier-Stokes equation in one spatial dimension.

Chapter 2

Continuous Dependence Estimates

2.1 Introduction

Here we prove a continuous dependence estimate in the coefficients and intial data for solutions to equations of the form

$$\partial_t u - \nabla \cdot (\mathbf{A}(u) \cdot \nabla u) + \nabla \cdot F(u,x) = \sigma(u)\partial_t W, \tag{2.1}$$

with evolution on $\mathbb{T}^d$. We derive from this continuous dependence estimate both an L^1-stability property for u and a Nikolskii semi-norm estimate (or fractional BV estimate, defined at (2.49)). We require as a matter of first importance, that $\mathbf{A}$ is positive semi-definite. In such a circumstance, it has also a positive semi-definite square root, which we shall call $\boldsymbol{\alpha}$. The coefficients F and σ have growth and continuity assumptions given later. In the noise term, W_t is a standard (one-dimensional) Brownian motion on the abstract probability space $(\Omega, \mathcal{P}, \mathbb{P})$.

The significance of such a study is three-fold. First, the equation not being translation invariant, one can truly probe the question of a BV bound for solutions with BV initial data, the reason for which the work of Chen, Ding, and Karlsen[10] was so notable – unlike the equations studied there, a BV bound does not follow immediately from the L^1-contraction inequality. In fact, we derive a fractional BV-in-space bound, which depends on the smoothness of $\partial_i F^i$ in its spatial argument, and in the special case that $\partial_i F^i$ is Lipschitz (or nought), we have a BV bound. Secondly, we carry out our analysis as in [28], directly from the definition of a kinetic solution, and not as in [10]. Most importantly we provide a uniform treatment of L^1-boundedness,

BV-boundedness, and L^1-continuous dependence.

The work [10] was one among many results built atop a study of Feng and Nualart [38] of the same equation,

$$\partial_t u = \partial_x F(u) + \sigma(u)\partial_t W_t, \tag{2.2}$$

with evolution in $\mathbb{R}^d$. The authors developed a well-posedness theory for the equation in one spatial dimension (Feng and Nualart studied a more general noise $\sigma(x, u; z)$, where z is a variable from a Hilbert space). Feng and Nualart used the theory of compensated compactness to prove existence, which depended on the dimension being $d = 1$. The work by Chen, Ding, and Karlsen discussed previously removed this restriction via the BV estimate they proved. In the existence theory they proposed for the equations in question, Feng and Nualart established the concept of a strong stochastic entropic solution, which involved, unusually, a non-adapted stochastic integral.

The flow of the logic here is this: First we compare equations

$$\begin{aligned}\partial_t u &= -\nabla \cdot F(u, x) + \nabla \cdot (\mathbf{A}(u) \cdot \nabla u) + \sigma(u)\partial_t W\\ \partial_t v &= -\nabla \cdot G(v, x) + \nabla \cdot (\mathbf{B}(v) \cdot \nabla v) + \tau(v)\partial_t W,\end{aligned}$$

and in setting $F = G$, $\mathbf{A} = \mathbf{B}$, $\sigma = \tau$, derive L^1 and BV bounds. This shall be half the work of proving a stability bound. Next, we use our BV bound to obtain a refined version of the L^1-continuous dependence estimate. Then we use the estimates we have proven to show the existence of solutions.

2.1.1 Plan

In §2.2, we discuss the meaning and origin of the kinetic formulation. In §2.3, we use the kinetic formulation and make preliminary estimates on the difference between the equations. In §2.4, we take $F = G$, $\mathbf{A} = \mathbf{B}$, $\sigma = \tau$, and show that L^1 estimates follow as a corollary of the preliminary estimates. In §2.5, we take $G(\cdot, x) = F(\cdot, x + h)$, and $\mathbf{A} = \mathbf{B}$, $\sigma = \tau$, to derive a fractional BV estimate, noting both that in the translation invariant case, the BV estimate follows trivially from the L^1 (contraction) estimate, and also that if $u(x, t)$ satisfies the equation with flux $F(u, x)$, then $v(x, t) = u(x + h, t)$ satisfies the corresponding equation with flux $F(v, x)$ and all other coefficients and initial conditions the same.

Then using the (fractional) BV estimate in §2.5, we prove a continuous dependence estimate in §2.6. In §2.7, we prove the existence of solutions, showing that a sequence of BV solutions u^ε with degenerate parabolic term $\nabla \cdot ((\mathbf{A}(u^\varepsilon) + \varepsilon \mathrm{Id}) \cdot \nabla u)$ and initial data u_0^ε tends to a limit, and also showing a temporal L^1 continuity estimate as in [10]. The first limit is again a corollary of continuous dependence.

Before we proceed further we note that notationally, we shall use ∇ to denote the material derviative, and ∇^i the substantial derivative in the x^i direction (the ith coordinate of ∇, so that $\nabla^i F(u,x) = \partial_u F(u,x) \nabla^i u + \partial_i F(u,x)$, and $\nabla \cdot F(u,x) = \sum_i \nabla^i F^i$. We shall continue implicitly to sum over repeated indices.

We require the following assumptions on our coefficients:

$$\partial^2_{ui} F \in L^\infty, \tag{2.3}$$

$$|\partial_u F(\xi, x) - \partial_u F(\zeta, x)| \leq C|\xi - \zeta|^{\kappa_1}, \tag{2.4}$$

$$|\partial_i F^i(\xi, x) - \partial_i F(\xi, y)| \leq C|x - y|^{\kappa_2}, \tag{2.5}$$

$$|\sigma(\xi) - \sigma(\xi)| \leq C|\xi - \zeta|^{\lambda}, \tag{2.6}$$

$$\sup_{i,j} |\boldsymbol{\alpha}_{ij}(\xi) - \boldsymbol{\alpha}_{ij}(\zeta)| \leq C|\xi - \zeta|^{\gamma}, \tag{2.7}$$

where $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_{ij})$ is the positive semidefinite square root of $\mathbf{A}$. We shall write γ_α or $\kappa F2$ to specify that these indices are associated with α or with F, respectively, when there is any chance of ambiguity, such as when comparing two equations of the same type, with different coefficients.

We also assume that $\partial_i F^i$ and σ have at most linear growth in ξ, and $\mathbf{a}_{ij}$ has polynomial growth in ξ. Furthermore, we require that $2\gamma > 1$.

2.2 The Kinetic Formulation

First we define the solutions with which we seek to work. Following Lions, Pertham, and Tadmore [79], Chen and Perthame [13], and Debussche, Hofmanova, and Vovelle [28], we take the kinetic formulation approach. Because of spatial dependence in the flux term, the definition of a kinetic solution has to be generalised relative to that used in [28] in order to preserve a "divergence form" structure, as shall be evident below (Definition 4).

Definition 1 (Kinetic Function)**.** For a measure space $(\mathfrak{A}, \Upsilon)$, a mapping $\nu : \mathfrak{A} \to \mathfrak{M}_1(\mathbb{R})$, the set of probability measures on $\mathbb{R}$, is a *Young measure* if for every bounded continuous function $\varphi \in C_b(\mathbb{R})$, $x \mapsto \nu_x(\varphi)$ from $\mathfrak{A}$ to $\mathbb{R}$, is measurable. A Young measure vanishes at infinity if for every $p \geq 1$,

$$\int_{\mathfrak{A}} \int_{\mathbb{R}} |\xi|^p \, \nu_x(d\xi) \, \Upsilon(dx) < \infty.$$

A measurable function $f : \mathbb{R} \times \mathfrak{A} \to [0, 1]$ is a *kinetic function* if there exists a Young measure ν on $\mathfrak{A}$, vanishing at infinity, such that

$$f(\xi, x) = -\nu_x((\xi, \infty)), \tag{2.8}$$

for Υ- almost every $x \in \mathfrak{A}$.

Where $u \in L^p$ for every p, we can take our kinetic functions to be

$$f(\xi, (x, t)) = H(\xi - u(x, t)),$$

where $u(x, t)$ generates the measurable Young measure via the relation

$$\nu_{(x,t)}(d\xi) = -\delta(\xi - u(x, t)) \, d\xi,$$

where H is the Heaviside step function, defined as

$$H(r) = \mathbf{1}_{\{\xi : \xi \geq 0\}}(r).$$

Next we motivate the notion of a "kinetic solution", as a weaker form of a weak solution.

Starting from the approximate equation with artificial viscosity,

$$\partial_t u^\varepsilon + \nabla \cdot F(u^\varepsilon, x) = \nabla \cdot (\mathbf{A}(u^\varepsilon) \cdot \nabla u^\varepsilon) + \varepsilon \Delta u^\varepsilon + \sigma(u^\varepsilon) \partial_t W, \tag{2.9}$$

we can multiply through heuristically by $\partial_u H(\xi - u^\varepsilon)$, using

$$\partial_u H(\xi - u^\varepsilon) = -\partial_\xi H(\xi - u^\varepsilon).$$

We achieve:

$$
\begin{aligned}
\partial_t H(\xi - u^\varepsilon) = & - \partial_u H(\xi - u^\varepsilon)\partial_u F(u^\varepsilon, x) \cdot \nabla u^\varepsilon - \partial_u H(\xi - u^\varepsilon)\partial_i F^i(u^\varepsilon, x) \\
& + \nabla \cdot (\partial_u H(\xi - u^\varepsilon)\mathbf{A}(u^\varepsilon) \cdot \nabla u^\varepsilon) - \mathbf{A}(u^\varepsilon) : (\nabla \partial_u H(\xi - u^\varepsilon)) \otimes \nabla u^\varepsilon \\
& + \varepsilon \Delta H(\xi - u^\varepsilon) - \varepsilon \nabla(\partial_u H(\xi - u^\varepsilon)) \cdot \nabla u^\varepsilon \\
& + \partial_u H(\xi - u^\varepsilon)\sigma(u^\varepsilon)\partial_t W \\
& + \frac{1}{2}\partial^2_{uu} H(\xi - u^\varepsilon)\sigma^2(u^\varepsilon).
\end{aligned}
$$

Here we use the colon to denote element-wise scalar product, so that if $\mathbf{A} = (\mathbf{a}_{ij})$ and $\mathbf{B} = (\mathbf{b}_{ij})$ are $d \times d$ matrices,

$$
\mathbf{A} : \mathbf{B} = \sum_{1 \le i,j, \le d} \mathbf{a}_{ij}\mathbf{b}_{ij}.
$$

This is understood in the sense of distributions, so that for any $\varphi \in C^\infty_{\xi,x} C^\infty([0,T))$, we require that

$$
\begin{aligned}
& - \int_0^T \iint \partial_t \varphi H(\xi - u^\varepsilon)\, d\xi\, dx\, dt - \iint \varphi(\xi, x, 0) H(\xi - u^\varepsilon_0)\, d\xi\, dx \\
= & - \int_0^T \iint \varphi \partial_u H(\xi - u^\varepsilon)(\partial_u F(u^\varepsilon, x) - \partial_u F(\xi, x))\, d\xi \ \cdot \nabla u^\varepsilon\, dx\, dt \\
& - \int_0^T \iint \varphi \partial_u H(\xi - u^\varepsilon)(\partial_i F^i(u^\varepsilon, x) - \partial_i F^i(\xi, x))\, d\xi\, dx\, dt \\
& - \int_0^T \iint \varphi \partial_u F(\xi, x)\partial_u H(\xi - u^\varepsilon) \cdot \nabla u^\varepsilon\, d\xi\, dx\, dt \\
& - \int_0^T \iint \varphi \partial_u H(\xi - u^\varepsilon)\partial_i F^i(\xi, x)\, d\xi\, dx\, dt \\
& - \int_0^T \iint \partial_u H(\xi - u^\varepsilon)(\mathbf{A}(u^\varepsilon) - \mathbf{A}(\xi)) : \nabla \varphi \otimes \nabla u^\varepsilon\, d\xi\, dx\, dt \\
& + \int_0^T \iint \nabla^2 \varphi : \mathbf{A}(\xi) H(\xi - u^\varepsilon)\, d\xi\, dx\, dt \\
& - \int_0^T \iint \varphi \partial^2_{uu} H(\xi - u^\varepsilon)\mathbf{A}(u^\varepsilon) : \nabla u^\varepsilon \otimes \nabla u^\varepsilon\, d\xi\, dx\, dy \\
& + \int_0^T \iint \varphi(\varepsilon \Delta H(\xi - u^\varepsilon) - \varepsilon \partial^2_{uu} H(\xi - u^\varepsilon)|\nabla u^\varepsilon|^2)\, d\xi\, dx\, dt \\
& + \int_0^T \iint \varphi \partial_u H(\xi - u^\varepsilon)\sigma(u^\varepsilon)\, d\xi\, dx\, dW_t
\end{aligned}
$$

$$+\frac{1}{2}\int_0^T \iint \varphi \partial_{uu}^2 H(\xi - u^\varepsilon)\sigma^2(u^\varepsilon)\, d\xi\, dx\, dt.$$

Now using

$$\partial_u H(\xi - u^\varepsilon) = -\delta(\xi - u^\varepsilon) = -\partial_\xi H(\xi - u^\varepsilon),$$

we can write

$$\begin{aligned}
&-\int_0^T \iint \partial_t \varphi H(\xi - u^\varepsilon)\, d\xi\, dx\, dt - \iint \varphi(\xi, x, 0) H(\xi - u_0^\varepsilon)\, d\xi\, dx \\
=&\int_0^T \iint \varphi \partial_u F(\xi, x)\delta(\xi - u^\varepsilon)\, d\xi \ \cdot \nabla u^\varepsilon\, dx\, dt + \int_0^T \iint \varphi \delta(\xi - u^\varepsilon)\partial_i F^i(\xi, x)\, d\xi\, dx\, dt \\
&+\int_0^T \iint \nabla^2 \varphi : \mathbf{A}(\xi) H(\xi - u^\varepsilon)\, d\xi\, dx\, dt + \int_0^T \iint \partial_\xi \varphi \delta(\xi - u^\varepsilon)\mathbf{A}(\xi) : \nabla u^\varepsilon \otimes \nabla u^\varepsilon\, d\xi\, dx\, dy \\
&+\varepsilon \int_0^T \iint \Delta \varphi H(\xi - u^\varepsilon)\, d\xi\, dx\, dt + \varepsilon \int_0^T \iint \partial_\xi \varphi \delta(\xi - u^\varepsilon))|\nabla u^\varepsilon|^2\, d\xi\, dx\, dt \\
&-\int_0^T \iint \varphi \delta(\xi - u^\varepsilon)\sigma(\xi)\, d\xi\, dx\, dW_t \\
&-\frac{1}{2}\int_0^T \iint \partial_\xi \varphi \delta(\xi - u^\varepsilon)\sigma^2(\xi)\, d\xi\, dx\, dt.
\end{aligned}$$

The divergence form structure was very helpful in the analysis of the spatially invariant equation. We contrive to retain this structure with an error. Notice we can write

$$\begin{aligned}
&-\int_0^T \iint \nabla \cdot (H(\xi - u)\partial_u F(\xi, x))\varphi\, dx\, dt \\
=&\int_0^T \iint \varphi \partial_u F(\xi, x)\delta(\xi - u^\varepsilon)\, d\xi \ \cdot \nabla u^\varepsilon\, dx\, dt - \int_0^T \iint \varphi H(\xi - u^\varepsilon)\partial_{ui}^2 F^i(\xi, x)\, d\xi\, dx\, dt \\
=&\int_0^T \iint \varphi \partial_u F(\xi, x)\delta(\xi - u^\varepsilon)\, d\xi \ \cdot \nabla u^\varepsilon\, dx\, dt - \int_0^T \iint \varphi H(\xi - u^\varepsilon)\partial_\xi(\partial_i F^i(\xi, x))\, d\xi\, dx\, dt \\
=&\int_0^T \iint \varphi \partial_u F(\xi, x)\delta(\xi - u^\varepsilon)\, d\xi \ \cdot \nabla u^\varepsilon\, dx\, dt + \int_0^T \iint \varphi \delta(\xi - u^\varepsilon)\partial_i F^i(\xi, x)\, d\xi\, dx\, dt \\
&+\int_0^T \iint \partial_\xi \varphi H(\xi - u^\varepsilon)\partial_i F^i(\xi, x)\, d\xi\, dx\, dt.
\end{aligned}$$

Hence,

$$\int_0^T \iint \varphi \partial_u F(\xi, x)\delta(\xi - u^\varepsilon)\, d\xi \ \cdot \nabla u^\varepsilon\, dx\, dt + \int_0^T \iint \varphi \delta(\xi - u^\varepsilon)\partial_i F^i(\xi, x)\, d\xi\, dx\, dt$$

$$= \int_0^T \iint H(\xi - u^\varepsilon)\partial_u F(\xi, x) \cdot \nabla\varphi \, dx \, dt - \int_0^T \iint \partial_\xi \varphi \, H(\xi - u^\varepsilon)\partial_i F^i(\xi, x) \, d\xi \, dx \, dt,$$

where we have integrated-by-parts in the spatial variable after the final equal sign.

Taking the artificial viscosity to nought, that is, as $\varepsilon \to 0$, we arrive at the *kinetic formulation* of the equation:

$$\begin{aligned} \partial_t H(\xi - u) = & - \nabla \cdot (\partial_u F(\xi, x) H(\xi - u)) + \partial_\xi (H(\xi - u)\partial_i F^i(\xi, x)) \\ & + \mathbf{A}(\xi) : \nabla^2 H(\xi - u) - \delta(\xi - u)\sigma(\xi)\partial_t W - \partial_\xi (m_u + n_u - p_u), \end{aligned} \tag{2.10}$$

where the measures m_u, n_u and p_u are the limit as $\varepsilon \to 0$ of the kinetic dissipation, parabolic defect, and Itô correction measures, respectively,

$$\begin{aligned} \varepsilon\delta(\xi - u^\varepsilon)|\nabla u^\varepsilon|^2 &\to m_u \\ \delta(\xi - u^\varepsilon)\mathbf{A}(\xi) : \nabla u^\varepsilon \otimes \nabla u^\varepsilon &\to n_u \\ \delta(\xi - u^\varepsilon)\sigma^2(\xi) &\to p_u. \end{aligned}$$

However, in the following, $M_u = m_u + n_u$ shall take on the name "kinetic measure", following [28], and in contradistinction to [13], where only m_u is called the "kinetic measure", and n_u shall take on the name "parabolic defect measure".

First it is necessary to define exactly the two kinds of measures mentioned.

Definition 2 (Kinetic Measure)**.** A measure $M : \Omega \to \mathfrak{M}_b^+(\mathbb{R} \times \mathbb{T}^d \times [0, T])$ (where $\mathfrak{M}_b^+$ is the space of non-negative, bounded Radon measures) is a *kinetic measure* provided that

(i) for each $\varphi \in C_0(\mathbb{R} \times \mathbb{T}^d \times [0, T])$, the map $M(\varphi) : \Omega \to \mathbb{R}$ is measurable;

(ii) where $B_R^c \subseteq \mathbb{R}$ is the complement of the ball of radius R,

$$\lim_{R \to \infty} \mathbb{E}(M(B_R^c \times \mathbb{T}^d \times [0, T])) = 0; \tag{2.11}$$

(iii) and for any $\varphi \in C_0(\mathbb{R} \times \mathbb{T}^d)$,

$$\int_{\mathbb{R} \times \mathbb{T}^d \times [0,T]} \varphi(\xi, x) M(\omega; d\xi, dx, ds) \in L^2(\Omega \times [0, T])$$

admits a predictable representative (in the L^2 equivalence classes of functions).

Definition 3 (Parabolic Defect Measure)**.** The *parabolic defect measure* of a certain function u, $n_u : \Omega \to M_b^+(\mathbb{R}\times\mathbb{T}^d\times[0,T])$, is one for which, given $\varphi \in C_0(\mathbb{R}\times\mathbb{T}^d\times[0,T])$,

$$n_u(\varphi) = \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}} \varphi(\xi,x,t) \left| \nabla \cdot \int_0^u \boldsymbol{\alpha}(\zeta)\, d\zeta \right|^2 \delta(\xi - u(x,t))\, d\xi\, dx\, dt. \tag{2.12}$$

Having thus defined the parabolic defect measure, we shall write $m_u = M - n_u$. As our next definition shall specify, the only case which interests us is that in which m_u is a non-negative measure.

Finally, the definition of the kinetic solution is as follows:

Definition 4 (Kinetic Solution)**.** A member

$$u \in L^p(\Omega \times [0,T]; L^p(\mathbb{T}^d)) \cap L^p(\Omega; L^\infty([0,T]; L^p(\mathbb{T}^d)))$$

is a *kinetic solution* to (2.1) , with initial datum u_0, if u satisfies the following,

(i)

$$\nabla \cdot \int_0^u \boldsymbol{\alpha}(\xi)\, d\xi \in L^2(\Omega \times \mathbb{T}^d \times [0,T]),$$

(ii) for any $\varphi \in C_b(\mathbb{R})$, we have the following artificial chain rule, equivalent in $\mathcal{D}'(\mathbb{T}^d)$ and almost everywhere in (t,ω):

$$\nabla \cdot \int_0^u \varphi(\xi)\boldsymbol{\alpha}(\xi)\, d\xi = \varphi(u)\nabla \cdot \int_0^u \boldsymbol{\alpha}(\xi)\, d\xi. \tag{2.13}$$

(iii) writing $H(\xi - u)$ for $H(\xi - u(x,t))$ – there is a kinetic measure M_u such that given the parabolic defect measure, n_u, $M_u \geq n_u$ $\mathbb{P}$-a.e.; and (M_u, n_u) satisfies, for any $\varphi \in C_c^\infty(\mathbb{R}, \mathbb{T}^d \times [0,T))$, almost surely,

$$\begin{aligned}
&- \int_0^T \iint \partial_t\varphi H(\xi - u)\, d\xi\, dx\, dt - \iint \varphi(\xi,x,0)H(\xi - u_0)\, d\xi\, dx \qquad (2.14)\\
&= \int_0^T \iint \nabla\varphi \cdot \partial_u F(\xi,x)H(\xi - u)\, dx\, dt - \int_0^T \iint \partial_\xi\varphi\, \partial_i F^i(\xi,x)H(\xi - u)\, d\xi\, dx\, dt\\
&\quad + \int_0^T \iint \nabla^2\varphi : \mathbf{A}(\xi)H(\xi - u)\, d\xi\, dx\, dt + \int_0^T \iint \partial_\xi\varphi M_u(\xi,x,t)\, d\xi\, dx\, dt
\end{aligned}$$

$$- \frac{1}{2}\int_0^T \int (\partial_\xi \varphi)(u,x,t)\sigma^2(u)\ dx\, dt - \int_0^T \int \varphi(u,x,t)\sigma(u)\ dx\, dW_t.$$

This arises from testing (2.10) with φ, using the artificial chain rule (2.13).

Remark. A remark to be made here is that $\partial_u F(\xi, x)$ in (2.14) above means $(\partial_u F)(\xi, x)$, and this is equivalent to $\partial_\xi(F(\xi, x))$. If we write $\bar{\nabla} = (\nabla, -\partial_\xi)$ the two integrals involving the flux in (2.14) can be expressed as

$$\int_0^T \iint \bar{\nabla}\varphi \cdot \left(\partial_\xi F, \sum_i \partial_{x^i} F^i\right)^T H(\xi - u)\ d\xi\, dx\, dt,$$

showing clearly the divergence structure attained in this formulation of the kinetic solution. Integrating-by-parts in all the spatial and kinetic variables, the integral above is seen to be the negative of

$$\int_0^T \iint \bar{\nabla}\varphi \cdot \left(\partial_\xi F, \sum_i \partial_{x^i} F^i\right)^T \bar{H}(\xi - u)\ d\xi\, dx\, dt,$$

where $\bar{H} = 1 - H$.

2.3 Preparation Estimates

Having defined solution we consider the pair:

$$\partial_t u - \nabla \cdot (\mathbf{A}(u) \cdot \nabla u) + \nabla \cdot F(u, x) = \sigma(u)\partial_t W, \tag{2.15}$$

$$\partial_t u - \nabla \cdot (\mathbf{B}_{ij}(v) \cdot \nabla v) + \nabla \cdot G(v, y) = \tau(v)\partial_t W, \tag{2.16}$$

where $\mathbf{B}$ is positive semi-definite, with positive semi-definite square root $\boldsymbol{\beta}$.

Corresponding to assumptions (2.3) - (2.7) on (2.15) we have,

$$\partial^2_{ui} G \in L^\infty, \tag{2.17}$$

$$|\partial_u G(\xi, x) - \partial_u G(\zeta, x)| \leq C|\xi - \zeta|^{\kappa G1}, \tag{2.18}$$

$$|\partial_i G^i(\xi, x) - \partial_i G(\xi, y)| \leq C|x - y|^{\kappa G2}, \tag{2.19}$$

$$|\tau(\xi) - \tau(\xi)| \leq C|\xi - \zeta|^{\lambda}, \tag{2.20}$$

$$\sup_{i,j} |\boldsymbol{\beta}_{ij}(\xi) - \boldsymbol{\beta}_{ij}(\zeta)| \leq C|\xi - \zeta|^{\gamma_{\boldsymbol{\beta}}}, \tag{2.21}$$

where $(\boldsymbol{\beta}_{ij})$ is the positive semidefinite square root of $\mathbf{B}$. We allow for $\kappa F1, \kappa F2$ to be different from $\kappa G1$ and $\kappa G2$, respectively, and also γ_α to be different from $\gamma_{\boldsymbol{\beta}}$. However, we keep $\lambda_\sigma = \lambda_\tau$ for simplicity in the following.

We employ a Kruzhkov doubling-of-variable technique and attempt to bound the difference of their kinetic solutions, so that u, the kinetic solution to the first equation is understood to take the spatial variable x, and v, the kinetic solution to the second equation is understood to take the spatial variable y.

In the following we assume $u_0, v_0 \in L^p(\Omega, \mathcal{P}, d\mathbb{P}; L^p(\mathbb{T}^d)) \cap L^p(\Omega, \mathcal{P}, d\mathbb{P}; BV(\mathbb{T}^d))$.

The employability of kinetic functions is based on the following observation, that a simple combination of them, heuristically, gives us

$$\int_{\mathbb{R}} H(\xi - u(x,t))(1 - H(\xi - v(y,t)))\; d\xi = (v(y,t) - u(x,t))_+.$$

The manipulations are unjustified as they stand. So we turn to a mollified version. Let $\eta_1 : \mathbb{R} \to \mathbb{R}$ be defined as a smooth, convex function, equal to $(\cdot)_+$ outside $[-1,1] \subseteq \mathbb{R}$, and symmetric about the origin in the sense that $\eta_1'(-r) = 1 - \eta_1'(r)$. We will use this symmetry repeatedly, such as in (2.23) below.

To see that η_1 exists, let $\tilde{J}_1$ be a standard symmetric bump function supported on $[-1,1]$, such as

$$\tilde{J}_1(r) = C \exp\left(\frac{1}{1-r^2}\right),$$

where C is a normalisation constant, so that $\int_{\mathbb{R}} \tilde{J}_1(r)\; dr = 1$. Setting $\eta_1'(r) = \int_{-\infty}^{r} \tilde{J}_1(s)\; ds$ gives $1 - \eta_1'(r) = \eta_1'(-r)$.

Now scaling by ρ in the usual way to get an approximation to $\delta(r)$, thus:

$$\eta_\rho''(r) = \rho^{-1} \eta_1''(r/\rho)$$

, we see on integration that η_ρ' preserves this symmetry. Finally we can set

$$\eta_\rho(r) = \int_{-\infty}^{r} \eta_\rho'(s)\; ds. \tag{2.22}$$

And by the symmetry $1 - \eta_\rho'(r) = \eta_\rho'(-r)$, we see that η_ρ coincides with $(\cdot)_+$ outside

$[-\rho, \rho]$.

We shall now make further use of the convenient symmetry $1-\eta'_\rho(r) = \eta'_\rho(-r)$. Using the definition of the Heaviside function H, and writing $(1 - H(\zeta - v))$ as $\bar{H}(\zeta - v)$, it holds that:

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} H(\xi - u(x,t))\bar{H}(\zeta - v(y,t))\eta''_\rho(\xi - \zeta)\, d\xi\, d\zeta \\ =&\int_{\mathbb{R}} \int_{u(x,t)}^{\infty} \eta''_\rho(\xi - \zeta)\, d\xi\, \bar{H}(\zeta - v(y,t))\, d\zeta \\ =&\int_{-\infty}^{v(y,t)} (1 - \eta'_\rho(u(x,t) - \zeta))\, d\zeta \\ =&\eta_\rho(v(y,t) - u(x,t)) \end{aligned} \tag{2.23}$$

and $\eta_\rho(u-v)$ approximates $(v(y,t)-u(x,t))_+$ as $\rho \to 0$, which integral in space against a mollified instance of $\delta(x - y)$ we seek quantitatively to bound.

Before we proceed to manipulations that shall mould the equation into a form similar to terms in (2.23) above, we shall state a lemma that gives us a way to leverage the definition (2.14) into a more versatile form. This is essentially Proposition 10 of [29], and Proposition 3.1 of [28]. It is also found in [53] as Proposition 3.1 – there is an almost surely time-continuous representative of any possible kinetic solution.

Lemma 2. *Let u be a kinetic solution to (2.1) with initial condition $u(0) = u_0$. There exist representatives $f^{\pm}(t)$ of $H(\xi - u) = \chi_{\xi \geq u}$ that are almost surely left- and right-continuous-in-time. That is, for every $\tau \in [0, T]$, and for all $\psi \in C^2_c(\mathbb{R} \times \mathbb{T}^d)$, almost surely,*

$$\langle f^{\pm}(\tau + \varepsilon), \psi\rangle \to \langle f^{\pm}(\tau), \psi\rangle$$

as $\varepsilon \to 0$. Moreover, $f^+ = f^-$ except on at most a countable subset of $[0, T]$.

Using the definition of kinetic solutions in (2.14), we can manipulate as follows to arrive at bounds for terms in (2.23) above.

As demonstrated in [29], first we derive a version of (2.14) free of the temporal integral by choosing a test function of the form $\varphi(\xi, x, t) = \psi(\xi, x)\alpha(t)$, with,

$$\alpha(t) = \begin{cases} 1 & t \leq s \\ 1 - \frac{t-s}{\varepsilon} & s \leq t \leq s + \varepsilon \\ 0 & t \geq s + \varepsilon \end{cases},$$

as $\varepsilon \to 0$, $-\partial_t \alpha$ approximates the delta function, and from (2.14), we have:

$$
\begin{aligned}
&-\int_0^T \iint \partial_t \alpha H(\xi - u)\psi \, d\xi \, dx \, dt - \iint \psi(\xi, x) H(\xi - u_0) \, d\xi \, dx \\
&= \int_0^T \iint \nabla\psi \cdot \partial_u F(\xi, x) H(\xi - u)\alpha \, dx \, dt - \int_0^T \iint \partial_\xi \psi \, \partial_i F^i(\xi, x) H(\xi - u)\alpha \, d\xi \, dx \, dt \\
&\quad + \int_0^T \iint \alpha \nabla^2 \psi : \mathbf{A}(\xi) H(\xi - u) \, d\xi \, dx \, dt + \int_0^T \iint \alpha \partial_\xi \psi M_u(\xi, x, t) \, d\xi \, dx \, dt \\
&\quad - \frac{1}{2} \int_0^T \int \alpha(t)(\partial_\xi \psi)(u, x)\sigma^2(u) \, dx \, dt - \int_0^T \int \alpha(t)\psi(u, x)\sigma(u) \, dx \, dW_t \\
&\to \int_0^s \iint \nabla\psi \cdot \partial_u F(\xi, x) H(\xi - u) \, dx \, dt - \int_0^s \iint \partial_\xi \psi \, \partial_i F^i(\xi, x) H(\xi - u) \, d\xi \, dx \, dt \\
&\quad + \int_0^s \iint \nabla^2 \psi : \mathbf{A}(\xi) H(\xi - u) \, d\xi \, dx \, dt + \int_0^s \iint \partial_\xi \psi M_u(\xi, x, t) \, d\xi \, dx \, dt \\
&\quad - \frac{1}{2} \int_0^s \int (\partial_\xi \psi)(u, x)\sigma^2(u) \, dx \, dt - \int_0^s \int \psi(u, x)\sigma(u) \, dx \, dW_t.
\end{aligned}
$$

whilst on the left-hand side,

$$
\begin{aligned}
&-\int_0^T \iint \partial_t \alpha H(\xi - u)\psi \, d\xi \, dx \, dt - \iint \psi(\xi, x) H(\xi - u_0) \, d\xi \, dx \\
&\to \iint H(\xi - u(s))\psi \, d\xi \, dx - \iint \psi(\xi, x) H(\xi - u_0) \, d\xi \, dx.
\end{aligned}
$$

For legibility, we shall use the angle brackets to denote integration in the spatial and kinetic variables x and ξ, respectively.

Thus do we arrive at

$$
\begin{aligned}
&\langle H(\cdot - u(s)), \psi \rangle - \langle H(\cdot - u_0), \psi \rangle \\
&= \int_0^s \langle \nabla\psi, \partial_u F(\cdot, \cdot) H(\cdot - u) \rangle \, dt - \int_0^s \langle \partial_\xi \psi, \partial_i F^i(\cdot, \cdot) H(\cdot - u) \rangle \, dt \\
&\quad + \int_0^s \langle \nabla^2 \psi : \mathbf{A}(\cdot), H(\cdot - u) \rangle \, dt + M_u(\partial_\xi \psi \times [0, s]) \\
&\quad - \frac{1}{2} \int_0^s \int_{\mathbb{T}^d} \sigma^2(u(x, t))(\partial_\xi \psi)(u(x, t), x) \, dx \, dt \\
&\quad - \int_0^s \int_{\mathbb{T}^d} \psi(u(x, t), x)\sigma(u(x, t)) \, dx \, dW_t.
\end{aligned}
\tag{2.24}
$$

As for the analogous equation for $H(\zeta - v)$, recall as in the derivation of (2.10), and again denoting $1 - H(\zeta - v)$ by $\bar{H}(\zeta - v)$, using

$$-\partial_u \bar{H}(\zeta - v) = \partial_\zeta \bar{H}(\zeta - v) = \delta(\zeta - v),$$

we can write:

$$\begin{aligned}
&- \int_0^T \iint \partial_t \varphi \bar{H}(\zeta - v^\varepsilon)\, d\xi\, dy\, dt - \iint \varphi(\xi, x, 0) \bar{H}(\xi - v_0^\varepsilon)\, d\xi\, dy \\
&= \int_0^T \iint \nabla \varphi \cdot \partial_u G(\zeta, y) \bar{H}(\zeta - v^\varepsilon)\, d\zeta\, dy\, dt - \int_0^T \iint \partial_\zeta \varphi \bar{H}(\zeta - v^\varepsilon) \partial_i G^i(\zeta, y)\, d\zeta\, dy\, dt \\
&\quad + \int_0^T \iint \nabla^2 \varphi : \mathbf{B}(\zeta) \bar{H}(\zeta - v^\varepsilon)\, d\zeta\, dy\, dt - \int_0^T \iint \partial_\zeta \varphi \delta(\zeta - v^\varepsilon) \mathbf{B}(\zeta) : \nabla v^\varepsilon \otimes \nabla v^\varepsilon\, d\zeta\, dy\, dy \\
&\quad + \varepsilon \int_0^T \iint \Delta \varphi \bar{H}(\zeta - v^\varepsilon)\, d\zeta\, dy\, dt - \varepsilon \int_0^T \iint \partial_\zeta \varphi \delta(\zeta - v^\varepsilon)) |\nabla v^\varepsilon|^2\, d\zeta\, dy\, dt \\
&\quad + \int_0^T \iint \varphi \delta(\zeta - v^\varepsilon) \tau(\zeta)\, d\zeta\, dy\, dW_t \\
&\quad + \frac{1}{2} \int_0^T \iint \partial_\zeta \varphi \delta(\zeta - v^\varepsilon) \tau^2(\zeta)\, d\zeta\, dy\, dt.
\end{aligned}$$

Either by directly making the requisite changes in (2.10), or in the limit of the preceding calculation, we arrive at

$$\begin{aligned}
&\langle \bar{H}(\cdot - v(s)), \phi \rangle - \langle \bar{H}(\cdot - v_0), \phi \rangle \qquad (2.25) \\
&= \int_0^s \langle \nabla_y \phi, \partial_u G(v, \cdot) \bar{H}(\cdot - v) \rangle\, dt - \int_0^T \langle \partial_\zeta \phi, \partial_i G^i(v, \cdot) \bar{H}(\cdot - v) \rangle\, dt \\
&\quad + \int_0^s \langle \nabla^2 \phi : \mathbf{B}(\cdot), \bar{H}(\cdot - v) \rangle\, dt - M_v(\partial_\xi \phi \times [0, s \\
&\quad + \frac{1}{2} \int_0^s \int_{\mathbb{T}^d} \tau^2(v(y,t)) (\partial_\xi \phi)(v(y,t), y)\, dy\, dt \\
&\quad + \int_0^s \int_{\mathbb{T}^d} \phi(v(y,t), y) \tau(v(y,t))\, dy\, dW_t.
\end{aligned}$$

Therefore we can find expressions for the left-hand side in (2.23) via (2.24) by choosing test functions as shall be subsequently prescribed.

Again, let η_ρ be chosen as in (2.22). Next we elect to have the product $\psi(\xi, x)\phi(\zeta, y)$ take the form

$$\psi(\xi, x)\phi(\zeta, y) = \eta_\rho''(\xi - \zeta) J_\theta(x - y) = \varphi(\xi, \zeta, x, y),$$

where $J_\theta(\cdot) = \theta^{-d} J_1(\cdot/\theta)$, and $J_1(\cdot)$ is a smooth, symmetric, non-negative, and centred at the origin, supported on $B_1(0)$.

Multiplying (2.23) through by $J_\theta(x-y)$ and integrating in x and y, we have

$$\begin{aligned}&\iint_{\mathbb{T}^d\times\mathbb{T}^d}\int_{\mathbb{R}}\int_{\mathbb{R}} H(\xi - u(x,t))\bar{H}(\zeta - v(y,t))\eta_\rho''(\xi-\zeta)J_\theta(x-y)\, d\xi\, d\zeta\, dx\, dy\\ =&\iint_{\mathbb{T}^d\times\mathbb{T}^d} \eta_\rho(v(y,t)-u(x,t))J_\theta(x-y)\, dx\, dy.\end{aligned}$$

As $\theta \to 0$, right-hand side of tends towards $\int \eta_\rho(u(x,t)-v(x,t))\, dx$. We shall resurrect the name $\varphi = \varphi(\xi,\zeta,x,y)$ and use this to denote the product $\psi(\xi,x)\phi(\zeta,y)$ above.

By such a choice of test function, we have the following usual identities:

$$\begin{aligned}\nabla_x\varphi =& -\nabla_y\varphi,\\ \nabla^2_{xx}\varphi =&\nabla^2_{yy}\varphi,\\ \partial_\xi\varphi =& -\partial_\zeta\varphi.\end{aligned}\tag{2.26}$$

We use ∇^2_{xx} to denote $\nabla_x\otimes\nabla_x$, and ∇^2_{xy} to denote $\nabla_x\otimes\nabla_y$. So $(\nabla^2_{xy})^T = \nabla^2_{yx}$.

2.3.1 The Product

Now we can obtain an expression for the left-hand side of (2.23) as follows:

Lemma 3. *Write dE as a shorthand for $d\xi\, d\zeta\, dx\, dy$. Let u be a kinetic solution to* (2.15) *with initial datum u_0, and let v be a kinetic solution to* (2.16) *with initial datum v_0. Let* (2.15) *and* (2.16) *have coefficients satisfying* (2.3) - (2.7) *and* (2.17) - (2.21), *as well as remarks following* (2.16).

We can express $\eta_\rho(v(y,t)-u(x,t))$ as

$$\begin{aligned}&\iint \eta_\rho(v(y,s)-u(x,s))J_\theta(x-y)\, dx\, dy\\ =&\iint\int_{\mathbb{R}}\int_{\mathbb{R}} H(\xi-u(x,t))\bar{H}(\zeta-v(y,t))\eta_\rho''(\xi-\zeta)J_\theta(x-y)\, dE\\ \le&\mathscr{M} + I_0 + I_a + I_F + I_\sigma,\end{aligned}$$

where $\mathscr{M}$ is a martingale, and

$$
\begin{aligned}
I_0 =& \int H(\xi - u_0(x))\bar{H}(\zeta - v_0(y))\eta''_\rho(\xi-\zeta)J_\theta(x-y)\, dE,\\
I_F =& \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_u F(\xi, x) - \partial_u G(\zeta, y)) \cdot \nabla_x \varphi \, dE\, dt\\
& - \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_i F^i(\xi, x) - \partial_i G^i(\zeta, y))\, \partial_\xi \varphi\, dE\, dt,\\
I_a =& \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\mathbf{A}(\xi) : \nabla^2_{xx}\varphi + \mathbf{B}(\zeta)\nabla^2_{yy}\varphi)\, dE\, dt\\
& - \int_0^s \iint \int \varphi(\xi, v, x, y)\, n_u\, d\xi\, dx\, dy\, dt\\
& - \int_0^s \iint \int \varphi(u, \zeta, x, y)\, n_v\, d\zeta\, dx\, dy\, dt,\\
I_\sigma =& \frac{1}{2}\int_0^s \iint \varphi(u, v, x, y)(\sigma(u(x,t)) - \tau(v(y,t)))^2\, dx\, dy\, dt.
\end{aligned}
$$

Proof. First we re-write (2.24):

$$
\begin{aligned}
& \langle H(\cdot - u(s)), \psi\rangle \\
=& \langle H(\cdot - u_0), \psi\rangle \\
& + \int_0^s \langle \nabla\psi, \partial_u F(\cdot,\cdot)H(\cdot - u)\rangle\, dt - \int_0^s \langle \partial_\xi\psi, \partial_i F^i(\cdot,\cdot)H(\cdot - u)\psi\rangle\, dt\\
& + \int_0^s \langle \nabla^2\psi : \mathbf{A}(\cdot), H(\cdot - u)\rangle\, dt + M_u(\partial_\xi\psi \times [0,s])\\
& - \frac{1}{2}\int_0^s \int_{\mathbb{T}^d} \sigma^2(u(x,t))(\partial_\xi\psi)(u(x,t), x)\, dx\, dt\\
& - \int_0^s \int_{\mathbb{T}^d} \psi(u(x,t), x)\sigma(u(x,t))\, dx\, dW_t\\
=& I_u + f_u + B_u,
\end{aligned}
\tag{2.27}
$$

with

$$
\begin{aligned}
I_u =& \langle H(\cdot - u_0), \psi\rangle\\
f_u =& \int_0^s \langle \nabla\psi, \partial_u F(\cdot,\cdot)H(\cdot - u)\rangle\, dt - \int_0^s \langle \partial_\xi\psi, \partial_i F^i(\cdot,\cdot)H(\cdot - u)\psi\rangle\, dt\\
& + \int_0^s \langle \nabla^2\psi : \mathbf{A}(\cdot), H(\cdot - u)\rangle\, dt + M_u(\partial_\xi\psi \times [0,s])
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2}\int_0^s \int_{\mathbb{T}^d} \sigma^2(u(x,t))(\partial_\xi \psi)(u(x,t),x)\; dx\, dt \\
B_u =& -\int_0^s \int_{\mathbb{T}^d} \psi(u(x,t),x)\sigma(u(x,t))\; dx\, dW_t.
\end{aligned}
$$

Similarly, with

$$
\begin{aligned}
I_v =& \langle \bar{H}(\cdot - v_0), \phi\rangle \\
f_v =& \int_0^s \langle \nabla_y \phi, \partial_u G(v,\cdot)\bar{H}(\cdot - v)\rangle\; dt - \int_0^T \langle \partial_\zeta \phi, \partial_i G^i(v,\cdot)\bar{H}(\cdot - v)\rangle\; dt \\
&+ \int_0^s \langle \nabla^2 \phi : \mathbf{B}(\cdot), \bar{H}(\cdot - v)\rangle\; dt - M_v(\partial_\zeta \phi \times [0,s]) \\
&+ \frac{1}{2}\int_0^s \int_{\mathbb{T}^d} \tau^2(v(y,t))(\partial_\zeta \phi)(v(y,t),y)\; dy\, dt \\
B_v =& \int_0^s \int_{\mathbb{T}^d} \phi(v(y,t),y)\tau(v(y,t))\; dy\, dW_t,
\end{aligned}
$$

we can write

$$
\langle \bar{H}(\cdot - v(s)), \phi\rangle = I_v + f_v + B_v. \tag{2.28}
$$

Let dE be shorthand for $d\xi\, d\zeta\, dx\, dy$. Multiplying (2.27) together with (2.28), we have

$$
\begin{aligned}
&\int H(\xi - u(x,s))\bar{H}(\zeta - v(y,s))\psi(\xi,x)\phi(\zeta,y)\; dE \\
=& \int H(\xi - u(x,s))\bar{H}(\zeta - v(y,s))\varphi\; dE \\
=& I_u I_v + f_u\langle \bar{H}(\cdot - v(s)), \phi\rangle + f_v\langle H(\cdot - u(s)), \psi\rangle - f_u f_v + B_u B_v.
\end{aligned}
$$

Concentrating on the middle three terms, using integration by parts, we can write

$$
\begin{aligned}
f_u\langle \bar{H}(\cdot - v(s)), \phi\rangle =& \int_0^s \langle \bar{H}(\cdot - v(s)), \phi\rangle\; df_u + \int_0^s f_u\; d(\langle \bar{H}(\cdot - v(t)), \phi\rangle) \\
=& \int_0^s \langle \bar{H}(\cdot - v(t)), \phi\rangle\; df_u(t) + \left(\int_0^s f_u\; df_v(t) + \int_0^s f_u\; dB_v(t)\right).
\end{aligned}
$$

Similarly,

$$f_v\langle H(\cdot - u(s)), \psi\rangle = \int_0^s \langle H(\cdot - u(s)), \phi\rangle \; df_v + \left(\int_0^s f_v \; df_u(t) + \int_0^s f_v \; dB_u(t) \right).$$

And finally,

$$f_u f_v = \int_0^s f_u \; df_v + \int_0^s f_v \; df_u.$$

Adding the preceding three calculations together gets us

$$\begin{aligned} &\int H(\xi - u(x,s))\bar{H}(\zeta - v(y,s))\psi(\xi,x)\phi(\zeta,y) \; dE \\ =& I_u I_v + \int_0^s \langle \bar{H}(\cdot - v(t)), \phi\rangle \; df_u + \int_0^s \langle H(\cdot - u(t)), \phi\rangle \; df_v + B_u B_v + \mathscr{M}, \end{aligned} \tag{2.29}$$

where $\mathscr{M}$ denotes a generic martingale term, which has expectation nought.

Now,

$$\begin{aligned} I_u I_v =& \langle H(\cdot - u_0), \psi\rangle \langle \bar{H}(\cdot - v_0), \phi\rangle \\ =& \int H(\xi - u_0(x))\bar{H}(\zeta - v_0(y))\eta_\rho''(\xi - \zeta) J_\theta(x-y) \; dE \end{aligned}$$

Also, recalling that we are using $\varphi = \varphi(\xi, \zeta, x, y)$ to denote $\eta_\rho''(\xi - \zeta)J_\theta(x-y)$, we can write

$$\begin{aligned} \int_0^s \langle \bar{H}(\cdot - v(t)), \phi\rangle \; df_u =& \int_0^s \int \bar{H}(\zeta - v(t))H(\xi - u)\partial_u F(\xi,x) \cdot \nabla_x \varphi \; dE \; dt \\ &- \int_0^s \int \bar{H}(\zeta - v(t))H(\xi - u)\partial_{x^i} F^i(\xi,x) \; \partial_\xi \varphi \; dE \; dt \\ &+ \int_0^s \int \bar{H}(\zeta - v(t))H(\xi - u)\mathbf{A}(\xi) : \nabla_{xx}^2 \varphi \; dE \; dt \\ &+ \int_0^s \int \bar{H}(\zeta - v(t)) \; \partial_\xi \varphi \; M_u \; dE \; dt \\ &- \frac{1}{2}\int_0^s \iint \int \bar{H}(\zeta - v(t))\sigma^2(u(x,t))(\partial_\xi \varphi)(u(x,t),\zeta,x,y) \; d\zeta \, dx \, dy \, dt. \end{aligned}$$

Using the form of the test function φ, we know that $\partial_\xi \varphi = -\partial_\zeta \varphi$. Also, integrating against $\bar{H}(\zeta - v)$ in ζ means integrating from $-\infty$ to v in that variable. This allows us to write the last two lines of the preceding expansion as

$$
\begin{aligned}
&-\int_0^s \iint \int \int_{-\infty}^{v} \partial_\zeta \varphi \, d\zeta \; M_u \, d\xi \, dx \, dy \, dt \\
&+\frac{1}{2}\int_0^s \iint \int_{-\infty}^{v} (\partial_\zeta \varphi)(u(x,t),\zeta,x,y) \, d\zeta \; \sigma^2(u(x,t)) \, dx \, dy \, dt \\
=&-\int_0^s \iint \int \varphi(\xi,v,x,y) \; M_u \, d\xi \, dx \, dy \, dt \\
&+\frac{1}{2}\int_0^s \iint \varphi(u,v,x,y)\sigma^2(u(x,t)) \, dx \, dy \, dt,
\end{aligned}
$$

where $M_u = m_u + n_u$ is the kinetic measure as defined in Definition 2.

Likewise,

$$
\begin{aligned}
&\int_0^s \langle H(\cdot - u(t)), \phi\rangle \, df_v \\
=&\int_0^s \int H(\xi - u(t))\bar{H}(\zeta - v)\partial_u G(\zeta, y) \cdot \nabla_y \varphi \, dE \, dt \\
&-\int_0^T \int H(\xi - u(t))\bar{H}(\zeta - v)\partial_i G^i(\zeta, y) \; \partial_\zeta \varphi \, dE \, dt \\
&+\int_0^s \int H(\xi - u(t))\bar{H}(\zeta - v)\mathbf{B}(\zeta) : \nabla^2_{yy}\varphi \, dE \, dt \\
&-\int_0^s \iint \int \varphi(u,\zeta,x,y) \; M_v \, d\zeta \, dx \, dy \, dt \\
&+\frac{1}{2}\int_0^s \iint \tau^2(v(y,t))\varphi(u,v,x,y) \, dx \, dy \, dt,
\end{aligned}
$$

where again, M_v is the kinetic measure.

Furthermore, by Itô isometry, it holds that

$$
\begin{aligned}
B_u B_v =& \left(-\int_0^s \int_{\mathbb{T}^d} \psi(u(x,t),x)\sigma(u(x,t)) \, dx \, dW_t\right)\left(\int_0^s \int_{\mathbb{T}^d} \phi(v(y,t),y)\tau(v(y,t)) \, dy \, dW_t\right) \\
=&-\int_0^s \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \psi(u(x,t),x)\phi(v(y,t),y)\sigma(u(x,t))\tau(v(y,t)) \, dx \, dy \, dt \\
=&-\int_0^s \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \sigma(u(x,t))\tau(v(y,t))\eta_\rho''(u(x,t) - v(y,t))J_\theta(x-y) \, dx \, dy \, dt \\
=&-\int_0^s \iint \sigma(u)\tau(v)\varphi(u,v,x,y) \, dx \, dy \, dt.
\end{aligned}
$$

Putting these expressions back together in (2.29), we get

$$
\begin{aligned}
&\int H(\xi - u(x,s))\bar{H}(\zeta - v(y,s))\psi(\xi,x)\phi(\zeta,y)\, dE \\
=&\mathscr{M} + \int H(\xi - u_0(x))\bar{H}(\zeta - v_0(y))\eta_\rho''(\xi-\zeta)J_\theta\, dE \\
&+ \int_0^s \int \bar{H}(\zeta - v(t))H(\xi - u)\partial_u F(\xi,x)\cdot\nabla_x\varphi\, dE\, dt \\
&- \int_0^s \int \bar{H}(\zeta - v(t))H(\xi - u)\partial_{x^i}F^i(\xi,x)\, \partial_\xi\varphi\, dE\, dt \\
&+ \int_0^s \int \bar{H}(\zeta - v(t))H(\xi - u)\mathbf{A}(\xi) : \nabla^2_{xx}\varphi\, dE\, dt \\
&- \int_0^s \iint \int \varphi(\xi, v, x, y)\, M_u\, d\xi\, dx\, dy\, dt \\
&+ \frac{1}{2}\int_0^s \iint \varphi(u,v,x,y)\sigma^2(u(x,t))\, dx\, dy\, dt \\
&+ \int_0^s \int H(\xi - u(t))\bar{H}(\zeta - v)\partial_u G(\zeta,y)\cdot\nabla_y\varphi\, dE\, dt \\
&- \int_0^T \int H(\xi - u(t))\bar{H}(\zeta - v)\partial_i G^i(\zeta,y)\, \partial_\zeta\varphi\, dE\, dt \\
&+ \int_0^s \int H(\xi - u(t))\bar{H}(\zeta - v)\mathbf{B}(\zeta) : \nabla^2_{yy}\varphi\, dE\, dt \\
&- \int_0^s \iint \int \varphi(u,\zeta,x,y)\, M_v\, d\zeta\, dx\, dy\, dt \\
&+ \frac{1}{2}\int_0^s \iint \tau^2(v(y,t))\varphi(u,v,x,y)\, dx\, dy\, dt \\
&- \int_0^s \iint \sigma(u)\tau(v)\varphi(u,v,x,y)\, dx\, dy\, dt \\
=&\mathscr{M} + \int H(\xi - u_0(x))\bar{H}(\zeta - v_0(y))\eta_\rho''(\xi-\zeta)J_\theta\, dE \\
&+ \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_u F(\xi,x) - \partial_u G(\zeta,y))\cdot\nabla_x\varphi\, dE\, dt \\
&- \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_i F^i(\xi,x) - \partial_i G^i(\zeta,y))\, \partial_\xi\varphi\, dE\, dt \\
&+ \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\mathbf{A}(\xi) : \nabla^2_{xx}\varphi + \mathbf{B}(\zeta) : \nabla^2_{yy}\varphi)\, dE\, dt \\
&- \int_0^s \iint \int \varphi(\xi, v, x, y)\, n_u\, d\xi\, dx\, dy\, dt \\
&- \int_0^s \iint \int \varphi(u,\zeta,x,y)\, n_v\, d\zeta\, dx\, dy\, dt
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^s \iint \int \varphi(\xi, v, x, y)\, (M_u - n_u)\, d\xi\, dx\, dy\, dt \\
&-\int_0^s \iint \int \varphi(u, \zeta, x, y)\, (M_v - n_v)\, d\zeta\, dx\, dy\, dt \\
&+\frac{1}{2}\int_0^s \iint \varphi(u, v, x, y)(\sigma(u(x,t)) - \tau(v(y,t)))^2\, dx\, dy\, dt \\
=&\mathscr{M} + I_0 + I_F + I_a + I_\sigma \\
&-\int_0^s \iint \int \varphi(\xi, v, x, y)\, (M_u - n_u)\, d\xi\, dx\, dy\, dt \\
&-\int_0^s \iint \int \varphi(u, \zeta, x, y)\, (M_v - n_v)\, d\zeta\, dx\, dy\, dt.
\end{aligned}
$$

Recall from (2.22) that $\eta_\rho : \mathbb{R} \to \mathbb{R}$ is defined as a smooth, convex function, equal to $(\cdot)_+$ outside $B_\rho(0)$, and so that $\eta'_\rho(-r) = 1 - \eta'_\rho(r)$. Notice that since η is convex, $\varphi \geq 0$, and since $M_u - m_u$ and $M_v - n_v$ are also non-negative (see Definition 4), so immediately,

$$
\begin{aligned}
0 \geq -\int_0^s \iint \int &\varphi(\xi, v, x, y)\, (M_u - n_u)\, d\xi\, dx\, dy\, dt \\
&-\int_0^s \iint \int \varphi(u, \zeta, x, y)\, (M_v - n_v)\, d\zeta\, dx\, dy\, dt.
\end{aligned}
$$

This concludes the proof of the lemma.

□

2.3.2 Difference Estimates

From the previous section, bearing in mind that $\mathbb{E}(\mathscr{M}) = 0$, we know that

$$
\mathbb{E}\left(\iint \eta_\rho(v(y,s) - u(x,s)) J_\theta(x-y)\, dx\, dy\right) \leq \mathbb{E}(I_0 + I_F + I_a + I_\sigma).
$$

Therefore we need to estimate the following integrals:

$$
\begin{aligned}
I_0 =& \int H(\xi - u_0(x)) \bar{H}(\zeta - v_0(y)) \eta''_\rho(\xi - \zeta) J_\theta\, dE, \\
I_F =& \int_0^s \int \bar{H}(\zeta - v) H(\xi - u)(\partial_u F(\xi, x) - \partial_u G(\zeta, y)) \cdot \nabla_x \varphi\, dE\, dt, \\
&- \int_0^s \int \bar{H}(\zeta - v) H(\xi - u)(\partial_i F^i(\xi, x) - \partial_i G^i(\zeta, y))\, \partial_\xi \varphi\, dE\, dt,
\end{aligned}
$$

$$
\begin{aligned}
I_a =& \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\mathbf{A}(\xi) : \nabla^2_{xx}\varphi + \mathbf{B}(\zeta) : \nabla^2_{yy}\varphi)\, dE\, dt \\
&- \int_0^s \iint \int \varphi(\xi, v, x, y)\, n_u\, d\xi\, dx\, dy\, dt \\
&- \int_0^s \iint \int \varphi(u, \zeta, x, y)\, n_v\, d\zeta\, dx\, dy\, dt, \\
I_\sigma =& \frac{1}{2} \int_0^s \iint \varphi(u, v, x, y)(\sigma(u(x,t)) - \tau(v(y,t)))^2\, dx\, dy\, dt.
\end{aligned}
$$

We shall refer to these integrals as the *initial term*, the *flux terms*, the *parabolic terms*, and the *Itô correction term*, respectively.

Having now expressions by which we can estimate the left-hand side of (2.23), we proceed so to do. We first estimate what is easy to estimate – the difference arising from the noise.

Lemma 4. *Let u be a kinetic solution to* 2.15)*, and let v be a kinetic solution to* 2.16)*. Let σ and τ both satisfy* 2.6) *with index λ. Furthermore, suppose*

$$
\left(\sup_{r \neq 0} \frac{|\sigma(r) - \tau(r)|^2}{r^{1+\theta}} \right), \|\sigma - \tau\|_{L^\infty(\mathbb{R})} < \infty. \tag{2.30}
$$

The Itô correction terms suffers to be so bounded:

$$
\begin{aligned}
|I_\sigma| \leq \min \Bigg[&\left(\sup_{r \neq 0} \frac{|\sigma(r) - \tau(r)|^2}{r^{1+\theta}} \right) \frac{C\|u_0\|^{1+\theta}_{L^{1+\theta}}}{C_\sigma \rho} e^{C_\sigma s}, \\
&s \left(\frac{\|\sigma - \tau\|^2_{L^\infty(\mathbb{R})}}{\rho} \right) \mu(\mathbb{T}^d) \Bigg] + C_\tau s \rho^{2\lambda - 1} \mu(\mathbb{T}^d).
\end{aligned}
$$

Here μ is the Lebesgue measure.

Proof. We split the proof in to two parts.

Itô Correction Terms I

We can calculate as follows:

$$
\begin{aligned}
I_\sigma =& \frac{1}{2} \int_0^s \iint \varphi(u, v, x, y)(\sigma(u(x,t)) - \tau(v(y,t)))^2\, dx\, dy\, dt \\
=& \frac{1}{2} \int_0^s \iint \varphi(u, v, x, y)(\sigma(u(x,t)) - \tau(u(x,t)) + \tau(u(x,t)) - \tau(v(y,t)))^2\, dx\, dy\, dt \\
\leq & \int_0^s \iint \eta''_\rho(u - v)(\sigma(u) - \tau(u))^2 J_\theta(x - y)\, dx\, dy\, dt
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^s \iint \eta_\rho''(u-v)(\tau(u)-\tau(v))^2 J_\theta(x-y)\,dx\,dy\,dt \\
&\le \left(\sup_{r\neq 0}\frac{|\sigma(r)-\tau(r)|^2}{r^{1+\theta}}\right)\frac{1}{\rho}\int_0^s \int |u|^{1+\theta}\int J_\theta(x-y)\;dy\;dx\;dt \\
&+\int_0^s C_\tau |u-v|^{2\lambda}\eta_\rho''(u-v)J_\theta(x-y)\;dx\;dy\;dt \\
&\le \left(\sup_{r\neq 0}\frac{|\sigma(r)-\tau(r)|^2}{r^{1+\theta}}\right)\frac{C\|u_0\|_{L^{1+\theta}}^{1+\theta}}{C_\sigma\rho}e^{C_\sigma s}+C_\tau s\rho^{2\lambda-1}\mu(\mathbb{T}^d). \qquad (2.31)
\end{aligned}
$$

In the final line of the preceding calculation we used that $\int J_\theta(x-y)\;dy = 1$, and the L^p bound, that in expectation,

$$
\|u(t)\|_{L^p_x}^p \le \|u_0\|_{L^p_x}^p e^{C_\sigma t}
$$

and C_σ is the Lipschitz constant for σ. This *a priori* bound can be proven by testing the function against $\Phi'(u)$, where $\Phi(u)=|u|^p$.

Itô Correction Term II

Alternatively, we have

$$
\begin{aligned}
I_\sigma =&\frac{1}{2}\int_0^s \int_{\mathbb{T}^d\times\mathbb{T}^d}(\sigma(u(x,t))-\tau(v(y,t)))^2\varphi(u(x,t),v(y,t),x,y)\;dx\,dy\,dt \\
\le&\int_0^s \int_{\mathbb{T}^d\times\mathbb{T}^d}((\sigma(u)-\tau(u))^2+(\tau(u)-\tau(v))^2)\eta_\rho''(u-v)J_\theta(x-y)\;dx\,dy\,dt \\
\le&\int_0^s \int_{\mathbb{T}^d\times\mathbb{T}^d}\left(\|\sigma-\tau\|_{L^\infty}^2+\frac{(\tau(u)-\tau(v))^2}{|u-v|^{2\lambda}}\rho^{2\lambda}\right)\eta_\rho''(u-v)J_\theta(x-y)\;dx\,dy\,dt.
\end{aligned}
$$

As above, we have

$$
I_\sigma \le s\left(\frac{\|\sigma-\tau\|_{L^\infty(\mathbb{R})}^2}{\rho}+C_\tau\rho^{2\lambda-1}\right)\mu(\mathbb{T}^d). \qquad (2.32)
$$

□

Remark. Unrelatedly, we see from the proof above that for the Hölder indices λ_σ of σ and λ_τ of τ, as long as $\lambda_\sigma, \lambda_\tau > 1/2$, it does not affect the proof that $\lambda_\sigma \neq \lambda_\tau$. We shall keep things simple and retain $\lambda_\sigma = \lambda_\tau = \lambda$ throughout.

Next we estimate the parabolic terms, using the following lemma as preparation:

Lemma 5. *Let u be a kinetic solution to* (2.15), *and let v be a kinetic solution to* 2.16).

We can express the integral I_a, defined above, as the following sum of differences:

$$\begin{aligned}
I_a = &\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi))(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi)) : \nabla^2_{xx}\varphi \, dE \, dt \\
&+ \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi))(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta)) : \nabla^2_{xx}\varphi \, dE \, dt \\
&+ \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta))(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi)) : \nabla^2_{xx}\varphi \, dE \, dt \\
&+ \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta))(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta)) : \nabla^2_{xx}\varphi \, dE \, dt.
\end{aligned}$$

Proof. Recall that we denoted by the symmetric, semidefinite square root of $\mathbf{A}$ by $\boldsymbol{\alpha}$; in the same way we denote the symmetric, semidefinite square root of $\mathbf{B}$ by $\boldsymbol{\beta}$.

First we write out the parabolic terms, the integrals I_a, again, noting that as discussed in (2.26), $\nabla^2_{xx}\varphi = \nabla^2_{yy}\varphi$,

$$\begin{aligned}
I_a = &\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\mathbf{A}(\xi) : \nabla^2_{xx}\varphi + \mathbf{B}(\zeta) : \nabla^2_{yy}\varphi) \, dE \, dt \\
&- \int_0^s \iint \int \varphi(\xi, v, x, y) \, n_u \, d\xi \, dx \, dy \, dt \\
&- \int_0^s \iint \int \varphi(u, \zeta, x, y) \, n_v \, d\zeta \, dx \, dy \, dt \\
= &\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\mathbf{A}(\xi) - \boldsymbol{\alpha}(\xi)\boldsymbol{\beta}(\zeta) - \boldsymbol{\beta}(\zeta)\boldsymbol{\alpha}(\xi) + \mathbf{B}(\zeta)) : \nabla^2_{xx}\varphi \, dE \, dt \\
&+ \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi)\boldsymbol{\beta}(\zeta) + \boldsymbol{\beta}(\zeta)\boldsymbol{\alpha}(\xi)) : \nabla^2_{xx}\varphi \, dE \, dt \\
&- \int_0^s \iint \int \varphi(\xi, v, x, y) \, n_u \, d\xi \, dx \, dy \, dt - \int_0^s \iint \int \varphi(u, \zeta, x, y) \, n_v \, d\zeta \, dx \, dy \, dt.
\end{aligned}$$

Let us next concern ourselves with the final two lines of the preceding expression. Using $\nabla^2_{xx}\varphi = -\nabla^2_{xy}\varphi = -\nabla^2_{yx}\varphi$, we can move the derivatives in the second line from φ and onto the kinetic functions. Note that $\nabla^2_{xy} = (\nabla^2_{yx})^T$. Using the chain rule (2.13) for kinetic solutions, we have

$$\begin{aligned}
&\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi)\boldsymbol{\beta}(\zeta) + \boldsymbol{\beta}(\zeta)\boldsymbol{\alpha}(\xi)) : \nabla^2_{xx}\varphi \, dE \, dt \qquad (2.33) \\
&= - \int_0^s \int \nabla_x H(\xi - u) \otimes \nabla_y \bar{H}(\zeta - v) : \boldsymbol{\alpha}(\xi)\boldsymbol{\beta}(\zeta)\eta''_\rho(\xi - \zeta)J_\theta(x - y) \, dE \, dt
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^s \int \nabla_y \bar{H}(\zeta - v) \otimes \nabla_x H(\xi - u) : \boldsymbol{\alpha}(\xi)\boldsymbol{\beta}(\zeta)\eta_\rho''(\xi-\zeta) J_\theta(x-y)\; dE\; dt \\
=&\int_0^s \nabla^2_{yx} : \int_\infty^u \boldsymbol{\alpha}(\xi) \int_{-\infty}^v \boldsymbol{\beta}(\zeta)\eta_\rho''(\xi-\zeta)\; d\xi\, d\zeta\; J_\theta(x-y)\; dx\, dy\, dt \\
&+\int_0^s \nabla^2_{yx} : \int_\infty^u \int_{-\infty}^v \boldsymbol{\beta}(\zeta)\boldsymbol{\alpha}(\xi)\eta_\rho''(\xi-\zeta)\; d\xi\, d\zeta\; J_\theta(x-y)\; dx\, dy\, dt \\
=&2\int_0^s \eta_\rho''(u-v)\nabla_x \otimes \nabla_y : \left(\int_0^u \boldsymbol{\alpha}(\xi)\; d\xi\right)\left(\int_0^v \boldsymbol{\beta}(\zeta)\; d\zeta\right) J_\theta(x-y)\; dx\, dy\, dt,
\end{aligned}
$$

as, where a derivative is present,

$$
\nabla \cdot \int_r^u \alpha(\xi)\eta_\rho''(\xi-\zeta)\; d\xi = \nabla \cdot \int_0^u \alpha(\xi)\eta_\rho''(\xi-\zeta)\; d\xi,
$$

for any fixèd r.

Next we require the form (2.12) of the parabolic defect measure, which is

$$
\begin{aligned}
n_u(\varphi) =& \int_0^T \int_{\mathbb{T}^d}\int_{\mathbb{R}} \varphi(t,x,\xi) \left|\nabla \cdot \int_0^u \boldsymbol{\alpha}(\zeta)\; d\zeta\right|^2 \delta(\xi - u(x,t))\; d\xi\, dx\, dt \\
=& \int_0^T \int_{\mathbb{T}^d}\int_{\mathbb{R}} \varphi(t,x,\xi) \left|\nabla \cdot \int \chi(\zeta, u(x,t))\boldsymbol{\alpha}(\zeta)\; d\zeta\right|^2 \delta(\xi - u(x,t))\; d\xi\, dx\, dt.
\end{aligned}
$$

This gives us

$$
\begin{aligned}
&-\int_0^s \iint \varphi(\xi, v(y,t), x, y)\; dy\, n_u(d\xi\, dx, dt) \\
=&-\int_0^s \iint \varphi(\xi, v(y,t), x, y) \left|\nabla_x \cdot \int_0^u \boldsymbol{\alpha}(\zeta)\; d\zeta\right|^2 \delta(\xi - u(x,t))\; d\xi\, dx\, dy\, dt \\
=&-\int_0^s \iint \eta_\rho''(u-v) J_\theta(x-y) \left|\nabla_x \cdot \int_0^u \boldsymbol{\alpha}(\zeta)\; d\zeta\right|^2\; dx\, dy\, dt,
\end{aligned}
$$

and likewise,

$$
\begin{aligned}
&-\int_0^s \iint \varphi(u(x,t), \zeta, x, y)\; dy\, n_v(d\zeta\, dx, dt) \\
=&-\int_0^s \iint \eta_\rho''(u-v) J_\theta(x-y) \left|\nabla_y \cdot \int_0^v \boldsymbol{\beta}(\xi)\; d\xi\right|^2\; dx\, dy\, dt.
\end{aligned}
$$

Therefore we have

$$\begin{aligned}
&\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi)\boldsymbol{\beta}(\zeta) + \boldsymbol{\beta}(\zeta)\boldsymbol{\alpha}(\xi)) : \nabla_{xx}^2 \varphi \; dE \; dt \\
&- \int_0^s \iint \int \varphi(\xi, v, x, y) \; n_u \; d\xi \, dx \, dy \; dt - \int_0^s \iint \int \varphi(u, \zeta, x, y) \; n_v \; d\zeta \, dx \, dy \; dt \\
=&2 \int_0^s \nabla_{xy}^2 : \left(\int_0^u \boldsymbol{\alpha}(\xi) \; d\xi \right) \left(\int_0^v \boldsymbol{\beta}(\zeta) \; d\zeta \right) \varphi(u, v, x, y) \; dx \, dy \, dt \\
&- \int_0^s \iint \left| \nabla_x \cdot \int_0^u \boldsymbol{\alpha}(\zeta) \; d\zeta \right|^2 \varphi(u, v, x, y) \; dx \, dy \, dt \\
&- \int_0^s \iint \left| \nabla_y \cdot \int_0^v \boldsymbol{\beta}(\xi) \; d\xi \right|^2 \varphi(u, v, x, y) \; dx \, dy \, dt \\
\leq& 0.
\end{aligned}$$

Inserting this into (2.33),

$$\begin{aligned}
I_a \leq& \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\mathbf{A}(\xi) - \boldsymbol{\alpha}(\xi)\boldsymbol{\beta}(\zeta) - \boldsymbol{\beta}(\zeta)\boldsymbol{\alpha}(\xi) + \mathbf{B}(\zeta)) : \nabla_{xx}^2 \varphi \; dE \; dt \\
=& \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\zeta))(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\zeta)) : \nabla_{xx}^2 \varphi \; dE \; dt. \qquad (2.34)
\end{aligned}$$

This leaves us one last integral to estimate. Now notice that one part of the integrand can be expanded to read:

$$\begin{aligned}
&(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\zeta))(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\zeta)) \\
=&((\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi)) + (\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta))) \cdot ((\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi)) + (\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta))) \\
=&(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi))(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi)) \\
&+ (\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi))(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta)) \\
&+ (\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta))(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi)) \\
&+ (\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta))(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta)).
\end{aligned}$$

So we can estimate this last and only integral (2.34) thus:

$$\begin{aligned}
&\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\zeta))(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\zeta)) : \nabla_{xx}^2 \varphi \; dE \; dt \qquad (2.35) \\
=& \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi))(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi)) : \nabla_{xx}^2 \varphi \; dE \; dt
\end{aligned}$$

$$+\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi))(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta)) : \nabla^2_{xx}\varphi \, dE \, dt$$
$$+\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta))(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi)) : \nabla^2_{xx}\varphi \, dE \, dt$$
$$+\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta))(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta)) : \nabla^2_{xx}\varphi \, dE \, dt.$$

□

We can estimate the various integrals after (2.35) above essentially by taking uniform bounds of differences between the coefficient functions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, and taking Hölder bounds between differences of the same coefficient function evaluated at different points. We also bound the nth derivatives of J_θ by $\theta^{-1}J_\theta$, and the derivatives of $\eta''_\rho(\xi - \zeta)$ by $\rho^{-1}\eta''_\rho(\xi - \zeta)$. These are written out in the lemma below.

Lemma 6. *Let u be a kinetic solution to* (2.15) *with initial datum u_0, and let v be a kinetic solution to* (2.16) *with initial datum v_0. Let η_ρ and J_θ be as defined just preceding* (2.26)*. Suppose $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy* (2.7)*, with indices γ_α and $\gamma_{\boldsymbol{\beta}}$, respectively.*

Write

$$\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|_{L^\infty} := \sup_{i,j} \|\boldsymbol{\alpha}_{ij} - \boldsymbol{\beta}_{ij}\|_{L^\infty}, \tag{2.36}$$

and suppose this quantity is bounded.

For the parabolic integrals, we have

$$\begin{aligned} |I_a| \le & n\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|^2_{L^\infty}\theta^{-2} \int_0^s \iint \eta_\rho(v-u)J_\theta(x-y)\, dx\, dy\, dt \\ & + C(\boldsymbol{\beta}, d)\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|_{L^\infty}\rho^{\gamma_{\boldsymbol{\beta}}}\theta^{-2} \int_0^s \iint \eta_\rho(v-u)J_\theta\, dx\, dy\, dt \\ & + C(\boldsymbol{\beta}, d)\rho^{2\gamma_{\boldsymbol{\beta}}}\theta^{-2} \int_0^s \iint \eta_\rho(v-u)J_\theta(x-y)uu\, dx\, dy\, dt, \end{aligned} \tag{2.37}$$

where $C(\boldsymbol{\beta}, d) \le n \sup_{ij} |\boldsymbol{\beta}_{ij}|_{C^{\gamma_{\boldsymbol{\beta}}}}$.

For the flux intergrals, we have

$$\begin{aligned} |I_F| \le & C(d)\|\partial_u F - \partial_u G\|_{L^\infty}\theta^{-1} \int_0^s \iint \eta_\rho(v-u)J_\theta(x-y)\, dx\, dy\, dt \\ & + C(d)\|\partial_i F^i - \partial_i G^i\|_{L^\infty}\rho^{-1} \int_0^s \iint \eta_\rho(v-u)J_\theta(x-y)\, dx\, dy\, dt \end{aligned} \tag{2.38}$$

$$
\begin{aligned}
&+ C(d)\|\partial^2_{ui}G\|_{L^\infty} \int_0^s \iint \eta_\rho(v-u)J_\theta(x-y)\, dx\, dy\, dt \\
&+ C_1(G,d)\rho^{\kappa G1}\theta^{-1} \int_0^s \iint \eta_\rho(v-u)J_\theta(x-y)\, dx\, dy\, dt \\
&+ C_2(G,d)\theta^{\kappa G2}\rho^{-1} \int_0^s \iint \eta_\rho(v-u)J_\theta(x-y)\, dx\, dy\, dt,
\end{aligned}
$$

where $C_1(G,d)$ is a multiple of the Hölder norm of $\partial_u G$ in its first variable, and $C_2(G,d)$ is a multiple of the Hölder norm of $\partial_i G$ in its second (spatial) variable, where both these multipliers depend solely on n.

Proof. Parabolic Terms

The integrals constituting the parabolic terms are

$$
\begin{aligned}
&\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi))(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi)) : \nabla^2_{xx}\varphi\, dE\, dt \\
&+ \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi))(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta)) : \nabla^2_{xx}\varphi\, dE\, dt \\
&+ \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta))(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi)) : \nabla^2_{xx}\varphi\, dE\, dt \\
&+ \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta))(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta)) : \nabla^2_{xx}\varphi\, dE\, dt.
\end{aligned}
$$

They can be estimated by invoking either the boundedness of $\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|_{L^\infty}$ or the continuity of $(\boldsymbol{\alpha}_{ik})$ and $(\boldsymbol{\beta}_{ik})$ in (2.7), as follows:

$$
\begin{aligned}
&\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi))(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi)) : \nabla^2_{xx}\varphi\, dE\, dt \\
\leq & n \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|^2_{L^\infty}\theta^{-2}|\nabla^2_{xx}J_\theta(x-y)|\eta''_\rho(\xi - \zeta)\, dE\, dt \\
\leq & n\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|^2_{L^\infty}\theta^{-2} \int_0^s \iint \eta_\rho(v-u)J_\theta(x-y)\, dx\, dy\, dt. \qquad (2.39)
\end{aligned}
$$

Writing $\gamma_{\boldsymbol{\beta}}$ for the lowest Hölder exponent of all $\boldsymbol{\beta}_{ij}$, we have the estimates:

$$
\begin{aligned}
&\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi))(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta)) : \nabla^2_{xx}\varphi\, dE\, dt \\
\leq & n\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|_{L^\infty} \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)\frac{|\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta)|}{|\xi - \zeta|^{\gamma_{\boldsymbol{\beta}}}}\rho^{\gamma_{\boldsymbol{\beta}}}|\nabla^2_{xx}J_\theta|\eta''_\rho(\xi - \zeta)\, dE\, dt \\
\leq & C(\boldsymbol{\beta}, d)\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|_{L^\infty}\rho^{\gamma_{\boldsymbol{\beta}}}\theta^{-2} \int_0^s \iint \eta_\rho(v-u)J_\theta\, dx\, dy\, dt, \qquad (2.40)
\end{aligned}
$$

and similarly,

$$\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta))(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi)) : \nabla^2_{xx}\varphi \, dE \, dt$$
$$\leq C(\boldsymbol{\beta}, d)\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|_{L^\infty}\rho^{\gamma_{\boldsymbol{\beta}}}\theta^{-2}\int_0^s \iint \eta_\rho(v-u)J_\theta \, dx \, dy \, dt.$$

Finally, we have the estimate:

$$\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta))(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta)) : \nabla^2_{xx}\varphi \, dE \, dt.$$
$$\leq C(\boldsymbol{\beta}, d)\rho^{2\gamma_{\boldsymbol{\beta}}}\theta^{-2}\int_0^s \iint \eta_\rho(v-u)J_\theta(x-y) \, dx \, dy \, dt. \tag{2.41}$$

In all the above $C(\boldsymbol{\beta}, d) = n|\boldsymbol{\beta}|_{C^{\gamma_{\boldsymbol{\beta}}}}$.

Flux Terms

The flux terms are

$$\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_u F(\xi, x) - \partial_u G(\zeta, y)) \cdot \nabla_x\varphi \, dE \, dt$$
$$- \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_i F^i(\xi, x) - \partial_i G^i(\zeta, y)) \, \partial_\xi\varphi \, dE \, dt.$$

We can estimate these terms much as we did the parabolic integrals. First we have

$$\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_u F(\xi, x) - \partial_u G(\zeta, y)) \cdot \nabla_x\varphi \, dE \, dt$$
$$= \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_u F(\xi, x) - \partial_u G(\xi, x)) \cdot \nabla_x\varphi \, dE \, dt$$
$$+ \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_u G(\xi, x) - \partial_u G(\zeta, x)) \cdot \nabla_x\varphi \, dE \, dt$$
$$+ \int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_u G(\zeta, x) - \partial_u G(\zeta, y)) \cdot \nabla_x\varphi \, dE \, dt$$
$$\leq C(d)\|\partial_u F - \partial_u G\|_{L^\infty}\theta^{-1}\int_0^s \iint \eta_\rho(v-u)J_\theta(x-y) \, dx \, dy \, dt \tag{2.42}$$
$$+ C_1(G, d)\rho^{\kappa G1}\theta^{-1}\int_0^s \iint \eta_\rho(v-u)J_\theta(x-y) \, dx \, dy \, dt \tag{2.43}$$
$$+ C(d)\|\partial^2_{ui}G\|_{L^\infty}\int_0^s \iint \eta_\rho(v-u)J_\theta(x-y) \, dx \, dy \, dt.$$

And similarly,

$$
\begin{aligned}
&-\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_i F^i(\xi, x) - \partial_i G^i(\zeta, y))\, \partial_\xi \varphi \; dE \; dt \\
=&-\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_i F^i(\xi, x) - \partial_i G^i(\xi, x))\, \partial_\xi \varphi \; dE \; dt \\
&-\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_i G^i(\xi, x) - \partial_i G^i(\zeta, x))\, \partial_\xi \varphi \; dE \; dt \\
&-\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_i G^i(\zeta, x) - \partial_i G^i(\zeta, y))\, \partial_\xi \varphi \; dE \; dt \\
\leq& C(d)\|\partial_i F^i - \partial_i G^i\|_{L^\infty} \rho^{-1} \int_0^s \iint \eta_\rho(v - u) J_\theta(x - y)\; dx\, dy\; dt \qquad (2.44) \\
&+ n\|\partial^2_{ui} G\|_{L^\infty} \int_0^s \iint \eta_\rho(v - u) J_\theta(x - y)\; dx\, dy\; dt \\
&+ C_2(G, d)\theta^{\kappa G2} \rho^{-1} \int_0^s \iint \eta_\rho(v - u) J_\theta(x - y)\; dx\, dy\; dt.
\end{aligned}
$$

In the above, $C_1(G, d)$ is a constant multiple of the Hölder norm of $\partial_u G$ in its first variable, and $C_2(G, d)$ is a constant multiple of the Hölder norm of $\partial_i G$ in its second (spatial) variable, where both these constant multipliers depend only on n.

□

2.3.3 Mollification Estimates

Whilst by consulting Lemmata 4 and 6, we have a bound for

$$\iint \eta_\rho(v(y, s) - u(x, s)) J_\theta(x - y)\; dx\, dy,$$

we shall need to translate this to a bound on

$$\int (v(x, s) - u(x, s))_+ \; dx.$$

In this section we estimate the difference arising from using a mollification.

To this end we turn to the basic inequalities, that

$$(\cdot)_+ \leq \eta_\rho(\cdot) \leq (\cdot)_+ + \eta_\rho(0).$$

From the definition of η_ρ, we also have, $\eta_\rho(0) = C\rho$.

Exploiting the finiteness of the Lebesgue measure on the torus, we have

$$\begin{aligned}
&\iint \eta_\rho(v(y,s)-u(x,s))J_\theta(x-y)\,dx\,dy \\
\leq &\iint \eta_\rho(v(y,s)-v(x,s))J_\theta(x-y)\,dx\,dy \\
&+ \iint \eta_\rho(v(x,s)-u(x,s))J_\theta(x-y)\,dx\,dy \\
\leq &\int (v(x,s)-u(x,s))_+\,dx + C\rho\mu(\mathbb{T}^d) \\
&+ \iint \eta_\rho(v(y,s)-v(x,s))J_\theta(x-y)\,dx\,dy.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int (v(x,s)-u(x,s))_+\,dx &= \iint (v(x,s)-u(x,s))_+ J_\theta(x-y)\,dx\,dy \\
&\leq \iint (v(x,s)-v(y,s))_+ J_\theta(x-y)\,dx\,dy \\
&\quad + \iint (v(y,s)-u(x,s))_+ J_\theta(x-y)\,dx\,dy \\
&\leq \iint (v(x,s)-v(y,s))_+ J_\theta(x-y)\,dx\,dy \\
&\quad + \iint \eta_\rho(v(y,s)-u(x,s))J_\theta(x-y)\,dx\,dy.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left| \iint \eta_\rho(v(y,s)-u(x,s))J_\theta(x-y)\,dx\,dy - \int (v(x,s)-u(x,s))_+\,dx \right| \qquad (2.45)\\
\leq &C\rho\mu(\mathbb{T}^d) + \iint \eta_\rho(v(y,s)-v(x,s))J_\theta(x-y)\,dx\,dy.
\end{aligned}$$

We shall return to bound this quantity after we have attained a bound on the bounded variation seminorm of $v(s)$ in §2.5.

2.4 L^1 Stability Estimate

The main theorem of this section is

Theorem 7 (L^1 Estimate)**.** *Let u, v be kinetic solutions to* (2.15) *with initial datum*

u_0, v_0, repsectively. Suppose the coefficients of (2.15) *satisfy the assumptions* (2.3) *to* (2.7)*. Then we have the L^1 estimate*

$$\mathbb{E}\left(\int (v-u)_+ \, dx\bigg|_s\right) \leq \exp(C(d)\|\partial^2_{ui}F\|_{L^\infty} s)\mathbb{E}\left(\int (v_0-u_0)_+ \, dx\right).$$

Remark. Where F is translation invariant, we have the familiar L^1 contraction —

$$\mathbb{E}\left(\int (v-u)_+ \, dx\bigg|_s\right) \leq \mathbb{E}\left(\int (v_0-u_0)_+ \, dx\right).$$

Proof. Putting $F(\cdot,\cdot) = G(\cdot,\cdot)$ and $\mathbf{A}(\cdot) = \mathbf{B}(\cdot)$ and $\sigma(\cdot) = \tau(\cdot)$, we have from Lemmata 4, and 6

$$\begin{aligned}
|I_F| \leq C(d)\|\partial^2_{ui}F\|_{L^\infty} &\int_0^s \iint \eta_\rho(v-u)J_\theta(x-y)\, dx\, dy\, dt \\
&+ C_1(F,d)\rho^{\kappa_1}\theta^{-1}\int_0^s \iint \eta_\rho(v-u)J_\theta(x-y)\, dx\, dy\, dt \\
&+ C_2(F,d)\theta^{\kappa_2}\rho^{-1}\int_0^s \iint \eta_\rho(v-u)J_\theta(x-y)\, dx\, dy\, dt, \\
|I_a| \leq C(\boldsymbol{\alpha}, d)\rho^{2\gamma}\theta^{-2} &\int_0^s \iint \eta_\rho(v-u)\, dx\, dy\, dt, \\
|I_\sigma| \leq C_\tau s\mu(\mathbb{T}^d)\rho^{2\lambda-1}&.
\end{aligned}$$

Since $C_2(F,d)$ is nought when F is not dependent on the spatial variables directly, we see that in that case, taking $m = \min(\gamma/2, \kappa_1/2)$, and choosing $\rho = \theta^{1/m}$, we have the bound:

$$\begin{aligned}
\iint \eta_\rho(v-u)J_\theta\, dx\, dy\bigg|_0^s \leq &(C_1(F,d) + C(\alpha,d))\theta\int_0^s \iint \eta_\rho(v-u)J_\theta(x-y)\, dx\, dy\, dt \\
&+ C_\tau s\mu(\mathbb{T}^d)\min(1, \rho^{2\lambda-1}).
\end{aligned}$$

An application of Gronwall's inequality and taking ρ, θ to zero at the chosen relative rate yields the L^1-contraction bound.

Putting $\mathbf{B} = \mathbf{A}$, $G(u,x) = F(u, x+h)$, $\tau = \sigma$, from Lemms 6 and 4 , we have

$$\begin{aligned}
|I_a| &\leq C(\boldsymbol{\alpha}, d)\rho^{2\gamma}\theta^{-2}\int_0^s \iint \eta_\rho(v-u)\, dx\, dy\, dt, \\
|I_\sigma| &\leq C_\sigma \rho^{2\lambda-1}\mu(\mathbb{T}^d).
\end{aligned}$$

For the flux bounds we make a small fine-tuning, related to the fact that $F = G$, so that returning to (2.44), we have

$$\begin{aligned}
&-\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_i F^i(\xi, x) - \partial_i G^i(\zeta, y))\, \partial_\xi \varphi \, dE \, dt \\
=&-\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_i F^i(\xi, x) - \partial_i F^i(\xi, y))\, \partial_\xi \varphi \, dE \, dt \\
&-\int_0^s \int \bar{H}(\zeta - v))H(\xi - u)(\partial_i F^i(\xi, y) - \partial_i F^i(\zeta, y))\, \partial_\xi \varphi \, dE \, dt.
\end{aligned}$$

Now, focusing on the first integral after the equal sign in the calculation immediately above,

$$\begin{aligned}
&-\int \bar{H}(\zeta - v)H(\xi - u)(\partial_i F^i(\xi, x) - \partial_i F^i(\xi, y))\, \partial_\xi \varphi \, dE \\
=&\iint \int \int \bar{H}(\zeta - v)\partial_\zeta \eta''_\rho(\xi - \zeta)\, d\zeta \; H(\xi - u)(\partial_i F^i(\xi, x) - \partial_i F^i(\xi, y)) J_\theta(x - y)\, d\xi \, dx \, dy \\
=&\iint \int \eta''_\rho(\xi - v)H(\xi - u)(\partial_i F^i(\xi, x) - \partial_i F^i(\xi, y)) J_\theta(x - y)\, d\xi \, dx \, dy \\
\leq& C_2(F, d)\theta^{\kappa_2} \iint \int \eta''_\rho(\xi - v)H(\xi - u) J_\theta(x - y)\, d\xi \, dx \, dy. \qquad (2.46)
\end{aligned}$$

This remaining integral is bounded because first,

$$\int \eta''_\rho(\xi - v)H(\xi - u)\, d\xi$$

is uniformly bounded by a constant in x, y and as $\rho \to 0$, being 1 or 0 according as v is greater than or less than u, and $1/2$ where $u = v$ because of the symmetry required in the definition of η_ρ. And secondly,

$$\int \int J_\theta(x - y)\, dx \, dy = \mu(\mathbb{T}^d).$$

Hence we arrive at a version of (2.38) adapted to this particular situation where $F = G$:

$$\begin{aligned}
|I_f| \leq& C(d)\|\partial^2_{ui} F\|_{L^\infty} \int_0^s \iint \eta_\rho(v - u) J_\theta(x - y)\, dx \, dy \, dt \qquad (2.47) \\
&+ C_1(F, d)\rho^{\kappa_1}\theta^{-1} \int_0^s \iint \eta_\rho(v - u) J_\theta(x - y)\, dx \, dy \, dt
\end{aligned}$$

$$+ C_2(F,d)s\theta^{\kappa_2}.$$

In fact we notice that as long as we have an improved bound on $\partial_i F(\xi,x) - \partial_i G(\xi,y)$, calculations analogous to the preceding carry through.

Putting (2.47) together with the estimates we already have for $|I_a|$ and $|I_\sigma|$, we see that taking $m = \min(\kappa_1/2, \gamma/2)$, and requiring $\rho = \theta^{1/m}$ as $\rho, \theta \to 0$, we can again apply Gronwall's inequality to

$$\begin{aligned}
&\mathbb{E}\left(\iint \eta_\rho(v-u) J_\theta \, dx\, dy \Big|_0^s \right) \\
\leq &\mathbb{E}\Bigg((C_1(F,d) + C(\boldsymbol{\alpha}, d))\theta \int_0^s \iint \eta_\rho(v-u) J_\theta(x-y)\, dx\, dy\, dt \\
&+ C(d)\|\partial^2_{ui} F\|_{L^\infty} \int_0^s \iint \eta_\rho(v-u) J_\theta(x-y)\, dx\, dy\, dt \\
&+ C_2(F,d)s\theta^{\kappa_2} + C_\sigma s \mu(\mathbb{T}^d)\rho^{2\lambda-1} \Bigg).
\end{aligned}$$

The resulting bound will not be a *contraction*, but it is clear that the growth, if any, results from $\|\partial^2_{ui} F\|_{L^\infty}$.

□

2.5 Bounded Variation Estimate

We apply Lemmata 6 and 4 to the pair

$$\begin{aligned}
\partial_t u =& -\nabla \cdot F(u,x) + \nabla \cdot (\mathbf{A}(u) \cdot \nabla u) + \sigma(u)\partial_t W, \\
\partial_t v =& -\nabla \cdot F(v, x+h) + \nabla \cdot (\mathbf{A}(v) \cdot \nabla v) + \sigma(v)\partial_t W,
\end{aligned}$$

with initial conditions $u(x,0) = u_0(x)$ and $v(x,0) = u_0(x+h)$, respectively, where $u_0 \in BV$, we can derive a BV estimate. And using this BV estimate, we can refine our continuous dependence estimate. We show:

Theorem 8 (Fractional Bounded Variation Estimate)**.** *Let u be a kinetic solution to* (2.15) *with initial datum u_0, and*

$$\mathbb{E}(|u_0|_{BV}) < \infty.$$

Suppose the coefficients of (2.15) *satisfy the assumptions* (2.3) *to* (2.7).

Where $C_2(F,d)$ is a constant multiple of the Hölder norm of $\partial_i F$ in its second (spatial) variable, with the constant dependent only on dimension d, we have the fractional BV bound

$$\mathbb{E}\left(\iint (u(y+h,s)-u(x,s))_+ J_\theta(x-y)\,dx\,dy\Big|_s\right) \tag{2.48}$$
$$\leq \exp\left(C(d)\|\partial^2_{ui}F\|_{L^\infty} s\right)\left[C_2(F,d)s|h|^{\kappa_2} + \mathbb{E}\left(\iint (u_0(y+h)-u_0(x))_+ J_\theta(x-y)\,dx\,dy\right)\right].$$

Proof. By the substitution $z = x + h$ in the second equation, it can be seen that if $u(x,t)$ solves the first equation, $u(z,t) = u(x+h,t)$ solves the second equation.

As in the L^1 stability estimate in the previous section, putting $\mathbf{B} = \mathbf{A}$, $G(u,x) = F(u,x+h)$, $\tau = \sigma$, from Lemmata 6 and 4 , we have

$$\begin{aligned}
&|I_a| \leq C(\alpha,d)\rho^{2\gamma}\theta^{-2}\int_0^s \iint \eta_\rho(v-u)\,dx\,dy\,dt,\\
&|I_\sigma| \leq C_\sigma \rho^{2\lambda-1}\mu(\mathbb{T}^d).
\end{aligned}$$

Again, for the flux bounds we make a small fine-tuning, related to the fact that $G(\cdot,\cdot) = F(\cdot,\cdot+h)$, so that returning to (2.44), and following (2.46), we have

$$\begin{aligned}
&-\int_0^s \int \bar{H}(\zeta-v)H(\xi-u)(\partial_i F^i(\xi,x)-\partial_i G^i(\zeta,y))\,\partial_\xi\varphi\,dE\,dt\\
&-\int_0^s \int \bar{H}(\zeta-v)H(\xi-u)(\partial_i F^i(\xi,x)-\partial_i F^i(\zeta,y+h))\,\partial_\xi\varphi\,dE\,dt\\
\leq& C_2(F,d)s(\theta+|h|)^{\kappa_2}.
\end{aligned}$$

Also, notice that

$$\|\partial_u F - \partial_u G\|_{L^\infty} \leq \|\partial^2_{ui}F\|_{L^\infty}|h|.$$

And hence altogether we have

$$\begin{aligned}
&\mathbb{E}\left(\iint \eta_\rho(u(y+h)-u(x))J_\theta(x-y)\,dx\,dy\Big|_0^s\right)\\
\leq&\mathbb{E}\Bigg(C(\boldsymbol{\alpha},d)\rho^{2\gamma}\theta^{-2}\int_0^s \iint \eta_\rho(v-u)\,dx\,dy\,dt\\
&+C(d)\|\partial^2_{ui}F\|_{L^\infty}(|h|\theta^{-1}+1)\int_0^s \iint \eta_\rho(v-u)J_\theta(x-y)\,dx\,dy\,dt
\end{aligned}$$

$$+ C_1(F,d)\rho^{\kappa_1}\theta^{-1}\int_0^s \iint \eta_\rho(v-u)J_\theta(x-y)\,dx\,dy\,dt$$
$$+ C_2(F,d)s(\theta+|h|)^{\kappa_2} + C_\sigma \rho^{2\lambda-1}\mu(\mathbb{T}^d)\Big).$$

Now we choose $\theta = |h|$, and take $\rho \to 0$.

□

We conclude this section with a few remarks:

Remark. If u_0 is in the fractional BV class with index κ_2, that is, functions of bounded $1/\kappa_2$ variation, then we have the fractional BV bound

$$\mathbb{E}(|u|_{N^{\kappa_2,1}}) \le \exp(C(d)\|\partial^2_{ui}F\|_{L^\infty}s)(C_2(F,d) + |u_0|_{N^{\kappa_2,1}}),$$

where $|\cdot|_{N^{\kappa,1}}$ denotes the bounded $1/\kappa$ variation seminorm (the Nikolskii semi-norm [97]),

$$|u|_{N^{\kappa,1}} = \sup_{|h|>0} \frac{1}{|h|^\kappa}\int |u(x+h)-u(x)|\,dx. \tag{2.49}$$

Remark. If we assume that $\kappa_2 = 1$, we have an actual BV estimate in taking the supremum whilst sending $\theta = |h| \to 0$. In fact, adding to the inequality the corresponding inequality for $(u(y+h)-u(x))_-$, we have

$$\mathbb{E}(|u|_{BV}) \le \exp\big(C(d)\|\partial^2_{ui}F\|_{L^\infty}s\big)(C_2(F,d)s + \mathbb{E}(|u_0|_{BV})). \tag{2.50}$$

Finally, in the translation invariant case, we have $C_2(F,d) = 0$, $\|\partial^2_{ui}F\|_{L^\infty} = 0$, and reduce to the expected simple bound

$$\mathbb{E}(|u|_{BV}) \le \mathbb{E}(|u_0|_{BV}). \tag{2.51}$$

2.6 Continuous Dependence Estimate

A continuous dependence esimate for the equations,

$$\partial_t u + \nabla\cdot F(u,x) - \nabla\cdot(\mathbf{A}(u)\cdot\nabla u) = \sigma(u)\partial_t W,$$
$$\partial_t v + \nabla\cdot G(v,x) - \nabla\cdot(\mathbf{B}(v)\cdot\nabla v) = \tau(v)\partial_t W,$$

is as stated an estimate of the form,

$$\mathbb{E}(\|u(t)-v(t)\|) \leq H(\mathbf{A},\mathbf{B},F,G,\sigma,\tau,u_0,v_0,t)E(\mathbf{A}-\mathbf{B},F-G,\sigma-\tau,u_0-v_0,t),$$

where E tends to nought as any of its arguments tend to zero, and $\|\cdot\|$ is some norm.

To prove the full continuous dependence estimates, we shall use our (fractional) BV estimates to refine both the mollification estimates (2.45) and the estimates in the Lemmas 6 and 4. This will allow us to show that

Theorem 9. *Let u be a kinetic solution to*

$$\partial_t u + \nabla \cdot F(u,x) - \nabla \cdot (\mathbf{A}(u)\cdot \nabla u) = \sigma(u)\partial_t W,$$

on $\mathbb{T}^d$ with initial condition $u_0 \in L^p$, and let v be a kinetic solution to

$$\partial_t v + \nabla \cdot G(v,x) - \nabla \cdot (\mathbf{B}(v)\cdot \nabla v) = \tau(v)\partial_t W,$$

on $\mathbb{T}^d$ with initial condition $v_0 \in BV \cap L^p$.

Let F and G satisfy (2.3) - (2.5) *and* (2.17) - (2.19)*, respectively with coefficients $\kappa F1$ and $\kappa G1$ in place of κ_1, and both with $\kappa 2 = 1$. Let σ,τ satisfy* (2.6)*, And let $(\mathbf{a}_{ij}),(\mathbf{b}_{ij})$ satisfy* (2.7)*, respectively with γ_α and $\gamma_{\boldsymbol{\beta}}$ in place of γ.*

Let $C_1(G,d)$ be the supremum of $\kappa G1$-Hölder seminorm of $\partial_u G(\cdot,x)$ in first variable. Let $C_2(G,d)$ be the supremum of the derivative of $\partial_x G(r,\cdot)$ in its second variable. Let $C(\boldsymbol{\beta},d)$ be the supremum of the $\gamma_{\boldsymbol{\beta}}$ seminorm of $\boldsymbol{\beta}_{ij}$ over pairs (i,j).

Let

$$K(v_0,s) = \int_0^s \exp(C(d)\|\partial^2_{ui}G\|_{L^\infty}t)(C_2(G,d)+|v_0|_{BV})\,dt,$$

and

$$\Lambda_\theta = \left(\sup_{r\neq 0}\frac{|\sigma(r)-\tau(r)|^2}{r^{1+\theta}}\right)\frac{C\|u_0\|^{1+\theta}_{L^{1+\theta}}}{C_\sigma}.$$

Let all the differences below be bounded. Then for free real variables $\rho,\theta>0$, we have the continuous dependence bound:

$$\begin{aligned}&\mathbb{E}\left(\int (v(x,s)-u(x,s))_+\,dx\right)\\ \leq& C\rho\mu(\mathbb{T}^d) + \theta\exp\big(C(d)\|\partial^2_{ui}G\|_{L^\infty}s\big)(C_2(G,d)s + \mathbb{E}(|v_0|_{BV}))\end{aligned}$$

$$
\begin{aligned}
&+\exp\left[\left(C(d)\|\partial_i F^i-\partial_i G^i\|_{L^\infty}\rho^{-1}+C(d)\|\partial_{ui}^2 G\|_{L^\infty}+C_2(G,d)\theta\rho^{-1}\right)s\right]\\
&\cdot\Bigg[\mathbb{E}\left(\iint \eta_\rho(v(y,0)-u(x,0))J_\theta(x-y)\,dx\,dy\right)+\\
&+(C(d)\|\partial_u F-\partial_u G\|_{L^\infty}+C(\boldsymbol{\beta},d)\rho^{2\gamma_\beta}\theta^{-1})K(v_0,s)\\
&+(C(\boldsymbol{\beta},d)\|\sqrt{\mathbf{A}}-\sqrt{\mathbf{B}}\|_{L^\infty}\rho^{\gamma_\beta}\theta^{-1}+n\|\sqrt{\mathbf{A}}-\sqrt{\mathbf{B}}\|_{L^\infty}^2\theta^{-1})K(v_0,s)\\
&+C_1(G,d)\rho^{\kappa G1}K(v_0,s)+C_\tau s\rho^{2\lambda-1}\mu(\mathbb{T}^d)\\
&+\min\left[\frac{\Lambda_\theta}{\rho}e^{C_\sigma s},s\frac{\|\sigma-\tau\|_{L^\infty}^2}{\rho}\mu(\mathbb{T}^d)\right]\Bigg].
\end{aligned}
$$

2.6.1 Refining the Mollification Estimate

Returning to the mollification estimate, (2.45),

$$
\begin{aligned}
&\mathbb{E}\left(\left|\iint \eta_\rho(v(y,s)-u(x,s))J_\theta(x-y)\,dx\,dy-\int(v(x,s)-u(x,s))_+\,dx\right|\right)\\
\le&C\rho\mu(\mathbb{T}^d)+\mathbb{E}\left(\iint \eta_\rho(v(y,s)-v(x,s))J_\theta(x-y)\,dx\,dy\right),
\end{aligned}
$$

we can now properly bound it by assuming that v_0 is of bounded variation.

Again we use the simple inequality:

$$
\eta_\rho(\cdot)\le(\cdot)_++\eta_\rho(0).
$$

Using (2.48), we have

$$
\begin{aligned}
&\mathbb{E}\left(\left|\iint \eta_\rho(v(y,s)-u(x,s))J_\theta(x-y)\,dx\,dy-\int(v(x,s)-u(x,s))_+\,dx\right|\right)\\
\le&C\rho\mu(\mathbb{T}^d)+\mathbb{E}\left(\iint \eta_\rho(v(y,s)-v(x,s))J_\theta(x-y)\,dx\,dy\right)\\
\le&C\rho\mu(\mathbb{T}^d)+\mathbb{E}\left(\iint (v(y,s)-v(x,s))_+J_\theta(x-y)\,dx\,dy\right)\\
\le&C\rho\mu(\mathbb{T}^d)+\mathbb{E}\left(\iint \sup_{|z|\le\theta}|v(y,s)-v(y+z,s)|\,dy\;J_\theta(z)\,dz\right)\\
\le&C\rho\mu(\mathbb{T}^d)+\mathbb{E}\left(|v(s)|_{BV}\theta\right)\\
\le&C\rho\mu(\mathbb{T}^d)+\theta\exp\left(C(d)\|\partial_{ui}^2 G\|_{L^\infty}s\right)(C_2(G,d)s+\mathbb{E}(|v_0|_{BV})).
\end{aligned}
\tag{2.52}
$$

Also, where $s = 0$,

$$\mathbb{E}\left(\left|\iint \eta_\rho(v_0(y) - u_0(x)) J_\theta(x-y)\, dx\, dy - \int (v_0(x) - u_0(x))_+ \, dx\right|\right)$$
$$\leq C\rho\mu(\mathbb{T}^d) + \theta\mathbb{E}(|v_0|_{BV}).$$

2.6.2 Refining the Continuous Dependence Estimate

Next we refine the continuous dependence estimate, using the fact that one solution is of bounded variation, and assuming that $\kappa G2 = 1$.

Since $\mathbb{E}(|v(s)|_{BV})$ is bounded, we can refine the estimates in Lemmas 6 and 4. Let $P \in L^\infty$ be some generic placeholder. Wherever there is a negative power of θ, an estimate was made of the form

$$\begin{aligned}
&\int H(\xi - u)\bar{H}(v - \zeta)P(\xi) \cdot \nabla_x J_\theta(x-y)\eta''_\rho(\xi - \zeta)\, dE \\
\leq &H(\xi - u)\bar{H}(v - \zeta)\|P\|_{L^\infty}|\nabla_x J_\theta(x-y)|\eta''_\rho(\xi - \zeta)\, dE \\
\leq &\|P\|_{L^\infty}\theta^{-1}\int H(\xi - u)\bar{H}(v - \zeta)J_\theta(x-y)\eta''_\rho(\xi - \zeta)\, dE \\
= &\|P\|_{L^\infty}\theta^{-1}\iint \eta_\rho(v - u)J_\theta(x-y)\, dx\, dy.
\end{aligned}$$

Using our control of $|v(s)|_{BV}$, we can refine some of these estimates to

$$\begin{aligned}
&\int H(\xi - u)\bar{H}(v - \zeta)P(\xi) \cdot \nabla_x J_\theta(x-y)\eta''_\rho(\xi - \zeta)\, dE \\
= &-\int H(\xi - u)\bar{H}(v - \zeta)P(\xi) \cdot \nabla_y J_\theta(x-y)\eta''_\rho(\xi - \zeta)\, dE \\
= &-\iint\iint H(\xi - u)\delta(v - \zeta)(\nabla v)P(\xi)J_\theta(x-y)\eta''_\rho(\xi - \zeta)\, d\zeta\, d\xi\, dx\, dy \\
\leq &\|P\|_{L^\infty}\int\int \eta'_\rho(v - u)J_\theta(x-y)\, dx|\nabla v|\, dy \\
\leq &\|P\|_{L^\infty}|v(t)|_{BV}.
\end{aligned}$$

As ∇v is a measure in y (and not in ζ), the product $\delta(\zeta - v)\nabla v$ makes sense. We have used the boundedness of η'_ρ. That is, we can replace

$$\theta^{-1}\iint \eta_\rho(v - u)J_\theta(x-y)\, dx\, dy,$$

with

$$|v(t)|_{BV},$$

avoiding an application of Gronwall's inequality, and an exponential penalization in time here (which shall come from the estimate on $|v(t)|_{BV}$ instead).

In particular, each of the estimates (2.42), (2.43), (2.39), (2.40), and (2.41), respectively,

(2.42):

$$\begin{aligned}&\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_u F(\xi, x) - \partial_u G(\xi, x)) \cdot \nabla_x \varphi \, dE \, dt \\ \leq& C(d)\|\partial_u F - \partial_u G\|_{L^\infty}\theta^{-1} \int_0^s \iint \eta_\rho(v-u)J_\theta(x-y) \, dx \, dy \, dt,\end{aligned}$$

(2.43):

$$\begin{aligned}&\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_u G(\xi, x) - \partial_u G(\zeta, x)) \cdot \nabla_x \varphi \, dE \, dt \\ \leq& C_1(G, d)\rho^{\kappa G1}\theta^{-1} \int_0^s \iint \eta_\rho(v-u)J_\theta(x-y) \, dx \, dy \, dt,\end{aligned}$$

(2.39):

$$\begin{aligned}&\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi))(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi)) : \nabla^2_{xx}\varphi \, dE \, dt \\ \leq& n\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|^2_{L^\infty}\theta^{-2} \int_0^s \iint \eta_\rho(v-u)J_\theta(x-y) \, dx \, dy \, dt,\end{aligned}$$

(2.40):

$$\begin{aligned}&\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi))(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta)) : \nabla^2_{xx}\varphi \, dE \, dt \\ \leq& C(\boldsymbol{\beta}, d)\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|_{L^\infty}\rho^{\gamma_\beta}\theta^{-2} \int_0^s \iint \eta_\rho(v-u)J_\theta \, dx \, dy \, dt,\end{aligned}$$

and

(2.41):

$$\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta))(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta)) : \nabla^2_{xx}\varphi \; dE \; dt$$
$$\leq C(\boldsymbol{\beta}, d)\rho^{2\gamma_{\boldsymbol{\beta}}}\theta^{-2} \int_0^s \iint \eta_\rho(v - u)J_\theta(x - y) \; dx \, dy \; dt,$$

can be re-written respectively as

(2.42'):

$$\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_u F(\xi, x) - \partial_u G(\xi, x)) \cdot \nabla_x \varphi \; dE \; dt$$
$$\leq C(d)\|\partial_u F - \partial_u G\|_{L^\infty} \int_0^s |v(t)|_{BV} \; dt,$$

(2.43'):

$$\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\partial_u G(\xi, x) - \partial_u G(\zeta, x)) \cdot \nabla_x \varphi \; dE \; dt$$
$$\leq C_1(G, d)\rho^{\kappa G1} \int_0^s |v(t)|_{BV} \; dt,$$

(2.39'):

$$\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi))(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi)) : \nabla^2_{xx}\varphi \; dE \; dt$$
$$\leq n\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|^2_{L^\infty}\theta^{-1} \int_0^s |v(t)|_{BV} \; dt,$$

(2.40'):

$$\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\alpha}(\xi) - \boldsymbol{\beta}(\xi))(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta)) : \nabla^2_{xx}\varphi \; dE \; dt$$
$$\leq C(\boldsymbol{\beta}, d)\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|_{L^\infty}\rho^{\gamma_{\boldsymbol{\beta}}}\theta^{-1} \int_0^s |v(t)|_{BV} \; dt,$$

and

(2.41'):

$$\int_0^s \int \bar{H}(\zeta - v)H(\xi - u)(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta))(\boldsymbol{\beta}(\xi) - \boldsymbol{\beta}(\zeta)) : \nabla^2_{xx}\varphi \; dE \; dt$$
$$\leq C(\boldsymbol{\beta}, d)\rho^{2\gamma_\beta}\theta^{-1} \int_0^s |v(t)|_{BV} \; dt.$$

From the bound on $\mathbb{E}(|v(t)|_{BV})$ in Theorem 8, let us denote bound on the temporal integral by $K(v_0, s)$, where

$$K(v_0, s) = K(v_0, s, G, \theta, \rho, d) \geq \int_0^s |v(t)|_{BV} \; dt.$$

Finally, recalling Lemmata 6 and 4, along with the preceding remark, we have the bound:

$$\begin{aligned}
&\mathbb{E}\left(\iint \eta_\rho(v - u)J_\theta(x - y) \; dx \, dy \Big|_0^s \right) \qquad (2.53)\\
\leq &(C(d)\|\partial_u F - \partial_u G\|_{L^\infty} + C(\boldsymbol{\beta}, d)\rho^{2\gamma_\beta}\theta^{-1})K(v_0, s)\\
&+ (C(\boldsymbol{\beta}, d)\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|_{L^\infty}\rho^{\gamma_\beta}\theta^{-1} + n\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|^2_{L^\infty}\theta^{-1})K(v_0, s)\\
&+ C(d)\|\partial_i F^i - \partial_i G^i\|_{L^\infty}\rho^{-1} \int_0^s \iint \eta_\rho(v - u)J_\theta(x - y) \; dx \, dy \; dt\\
&+ C(d)\|\partial^2_{ui} G\|_{L^\infty} \int_0^s \iint \eta_\rho(v - u)J_\theta(x - y) \; dx \, dy \; dt\\
&+ C_1(G, d)\rho^{\kappa G1}K(v_0, s) + C_2(G, d)\theta\rho^{-1} \int_0^s \iint \eta_\rho(v - u)J_\theta(x - y) \; dx \, dy\\
&+ \min\left[\left(\sup_{r \neq 0} \frac{|\sigma(r) - \tau(r)|^2}{r^{1+\theta}} \right) \frac{C\|u_0\|^{1+\theta}_{L^{1+\theta}}}{C_\sigma \rho} e^{C_\sigma s}, s\frac{\|\sigma - \tau\|^2_{L^\infty}}{\rho}\mu(\mathbb{T}^d) \right]\\
&+ C_\tau s\rho^{2\lambda - 1}\mu(\mathbb{T}^d).
\end{aligned}$$

We have taken $\kappa G2$ to be unity as required for the BV bound. Of course, we could have chosen to take u to be the solution with BV initial data. This is a minor complication, which can be overlooked by restricting our attention to $\kappa G2 = \kappa F2 = 1$.

Because it is quite cumbersome, we shall in this section denote by Λ_θ the quantity

$$\Lambda_\theta = \left(\sup_{r \neq 0} \frac{|\sigma(r) - \tau(r)|^2}{r^{1+\theta}} \right) \frac{C\|u_0\|^{1+\theta}_{L^{1+\theta}}}{C_\sigma}. \qquad (2.54)$$

2.6.3 Continuous Dependence Estimate

In this subsection, we prove the full continuous dependence estimate. First we demonstrate the estimate for the translation-invariant case, then we state prove the entire theorem.

Translation Invariant Case

Let us take a pause and show that in the-translation invariant case, the continuous dependence estimate reverts to the expected one (as suggested by (1.1), also in [21]). In the translation invariant case, $C_2(G,d)$, $\|\partial_i F^i - \partial_i G^i\|_{L^\infty}$, and $\|\partial^2_{ui} G\|_{L^\infty}$ are nought. Also, from (2.50), we can take

$$K(v_0, s) = s\mathbb{E}(|v_0|_{BV}).$$

Furthermore, since setting these quantities to nought in the estimate above, there is no need to apply the Gronwall inequality, which leaves us free to apply the mollification estimate, (2.52), immediately. Hence, we have

$$\begin{aligned}
&\mathbb{E}\left(\int (v(y,\cdot) - u(x,\cdot))_+ \, dx \Big|_0^s\right) \\
\leq & C\rho\mu(\mathbb{T}^d) + 2\theta\mathbb{E}(|v_0|_{BV}) \\
&+ (C(d)\|\partial_u F - \partial_u G\|_{L^\infty} + C(\boldsymbol{\beta}, d)\rho^{2\gamma_\beta}\theta^{-1} + C_1(G,d)\rho^{\kappa G1})s\mathbb{E}(|v_0|_{BV}) \\
&+ (C(\boldsymbol{\beta}, d)\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|_{L^\infty}\rho^{\gamma_\beta}\theta^{-1} + n\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|^2_{L^\infty}\theta^{-1})s\mathbb{E}(|v_0|_{BV}) \\
&+ \min\left[\frac{\Lambda_\theta}{\rho} e^{C_\sigma s}, s\frac{\|\sigma - \tau\|^2_{L^\infty}}{\rho}\mu(\mathbb{T}^d)\right] + C_\tau s\rho^{2\lambda - 1}\mu(\mathbb{T}^d).
\end{aligned}$$

For simplicity let us make the estimate

$$\min\left[\frac{\Lambda_\theta}{\rho} e^{C_\sigma s}, s\frac{\|\sigma - \tau\|^2_{L^\infty}}{\rho}\mu(\mathbb{T}^d)\right] \leq s\frac{\|\sigma - \tau\|^2_{L^\infty}}{\rho}\mu(\mathbb{T}^d).$$

All that is left is a judicious choice of ρ and θ. To this end we set

$$\rho = \theta = \|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|_{L^\infty}\sqrt{s} + \|\sigma - \tau\|_{L^\infty}\sqrt{s}.$$

Since

$$\theta^{-1} \leq \frac{1}{\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|\sqrt{s}}, \; \rho^{-1} \leq \frac{1}{\|\sigma - \tau\|_{L^\infty}\sqrt{s}},$$

we have

$$
\begin{aligned}
&\mathbb{E}\left(\int (v(x,\cdot)-u(x,\cdot))_+ \, dx\Big|_0^s\right) \qquad (2.55)\\
\le & C\rho(1+\mathbb{E}(|v_0|_{BV}))\\
&+(C(d)\|\partial_u F-\partial_u G\|_{L^\infty}+C(\boldsymbol{\beta},d)\rho^{2\gamma_{\boldsymbol{\beta}}}\theta^{-1}+C_1(G,d)\rho^{\kappa G1})s\mathbb{E}(|v_0|_{BV})\\
&+(C(\boldsymbol{\beta},d)\|\sqrt{\mathbf{A}}-\sqrt{\mathbf{B}}\|_{L^\infty}\rho^{\gamma_{\boldsymbol{\beta}}}\theta^{-1}+n\|\sqrt{\mathbf{A}}-\sqrt{\mathbf{B}}\|_{L^\infty}^2\theta^{-1})s\mathbb{E}(|v_0|_{BV})\\
&+s\frac{\|\sigma-\tau\|_{L^\infty}^2}{\rho}\mu(\mathbb{T}^d)+C_\tau s\rho^{2\lambda-1}\mu(\mathbb{T}^d)\\
\le & C\rho(1+\mathbb{E}(|v_0|_{BV}))\\
&+(C(d)\|\partial_u F-\partial_u G\|_{L^\infty}+C(\boldsymbol{\beta},d)\rho^{2\gamma_{\boldsymbol{\beta}}-1}+C_1(G,d)\rho^{\kappa G1})s\mathbb{E}(|v_0|_{BV})\\
&+(C(\boldsymbol{\beta},d)\rho^{\gamma_{\boldsymbol{\beta}}}+n\|\sqrt{\mathbf{A}}-\sqrt{\mathbf{B}}\|_{L^\infty})\sqrt{s}\mathbb{E}(|v_0|_{BV})\\
&+\sqrt{s}\|\sigma-\tau\|_{L^\infty}\mu(\mathbb{T}^d)+C_\tau s\rho^{2\lambda-1}\mu(\mathbb{T}^d).
\end{aligned}
$$

Full Continuous Dependence

In the non-translation invariant case, we do not make immediate use of the mollification estimate, (2.52), but take Grownwall estimates first. Applying the Gronwall estimate to (2.53), we arrive at

$$
\begin{aligned}
&\mathbb{E}\left(\iint \eta_\rho(v-u)J_\theta(x-y)\,dx\,dy\Big|_s\right)\\
\le & \exp\left[\left(C(d)\|\partial_i F^i-\partial_i G^i\|_{L^\infty}\rho^{-1}+C(d)\|\partial^2_{ui}G\|_{L^\infty}+C_2(G,d)\theta\rho^{-1}\right)s\right]\\
&\cdot\Bigg[\mathbb{E}\left(\iint \eta_\rho(v(y,0)-u(x,0))J_\theta(x-y)\,dx\,dy\right)+\\
&+(C(d)\|\partial_u F-\partial_u G\|_{L^\infty}+C(\boldsymbol{\beta},d)\rho^{2\gamma_{\boldsymbol{\beta}}}\theta^{-1})K(v_0,s)\\
&+(C(\boldsymbol{\beta},d)\|\sqrt{\mathbf{A}}-\sqrt{\mathbf{B}}\|_{L^\infty}\rho^{\gamma_{\boldsymbol{\beta}}}\theta^{-1}+n\|\sqrt{\mathbf{A}}-\sqrt{\mathbf{B}}\|_{L^\infty}^2\theta^{-1})K(v_0,s)\\
&+C_1(G,d)\rho^{\kappa G1}K(v_0,s)+C_\tau s\rho^{2\lambda-1}\mu(\mathbb{T}^d)\\
&+\min\left[\frac{\Lambda_\theta}{\rho}e^{C_\sigma s}, s\frac{\|\sigma-\tau\|_{L^\infty}^2}{\rho}\mu(\mathbb{T}^d)\right]\Bigg]
\end{aligned}
$$

Now we apply the mollification estimate (2.52), arriving at

$$
\begin{aligned}
&\mathbb{E}\left(\int (v(x,s)-u(x,s))_+\,dx\right)\\
\le & C\rho\mu(\mathbb{T}^d)+\theta\exp\left(C(d)\|\partial^2_{ui}G\|_{L^\infty}s\right)(C_2(G,d)s+\mathbb{E}(|v_0|_{BV}))
\end{aligned}
$$

$$
\begin{aligned}
&+ \exp\left[\left(C(d)\|\partial_i F^i - \partial_i G^i\|_{L^\infty}\rho^{-1} + C(d)\|\partial^2_{ui} G\|_{L^\infty} + C_2(G,d)\theta\rho^{-1}\right) s\right] \\
&\cdot \left[\mathbb{E}\left(\iint \eta_\rho(v(y,0) - u(x,0)) J_\theta(x-y)\, dx\, dy\right) + \right. \\
&+ (C(d)\|\partial_u F - \partial_u G\|_{L^\infty} + C(\boldsymbol{\beta}, d)\rho^{2\gamma_{\boldsymbol{\beta}}}\theta^{-1}) K(v_0, s) \\
&+ (C(\boldsymbol{\beta}, d)\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|_{L^\infty}\rho^{\gamma_{\boldsymbol{\beta}}}\theta^{-1} + n\|\sqrt{\mathbf{A}} - \sqrt{\mathbf{B}}\|^2_{L^\infty}\theta^{-1}) K(v_0, s) \\
&+ C_1(G,d)\rho^{\kappa G 1} K(v_0, s) + C_\tau s \rho^{2\lambda - 1}\mu(\mathbb{T}^d) \\
&\left. + \min\left[\frac{\Lambda_\theta}{\rho} e^{C_\sigma s}, s\frac{\|\sigma - \tau\|^2_{L^\infty}}{\rho}\mu(\mathbb{T}^d)\right]\right].
\end{aligned}
$$

This proves our theorem.

Remark. Some remarks are in order. First, there is no reason that σ and τ should not satisfy continuity conditions with different indices λ_σ and λ_τ, in which case λ in the theorem statement above should read λ_τ.

More importantly the generalization $\mathbf{A}(u, x)$ behaves differently and additional difficulties present themselves. In particular, in considering the BV estimate, to get a sense of the calculations, one might take the ith derivative of the entire equation (at the bulk, non-kinetic level) and test it against $\eta'_\rho(\partial_i u)$. One cannot easily propose an assumption on $\partial_i \mathbf{A}(u, x)$ by which to bound the terms

$$\int \eta''_\rho(\partial_i u)\partial_i \mathbf{A}(u,x) : \nabla u \otimes \nabla(\partial_i u)\, dx,$$

because inevitably, second derivatives appear in the estimates.

2.7 Existence

2.7.1 Convergence in ε

Let u_0^ε be a collection of initial conditions that tend to u_0 almost everywhere, almost surely.

We show here that there is convergence in the solutions u^ε of the approximate equations. From the continuous dependence estimates we have that the kinetic solutions to

$$\partial_t u^\varepsilon = -\nabla \cdot F(u^\varepsilon, x) + \nabla \cdot ((\mathbf{A}(u^\varepsilon) + \varepsilon \mathbf{Id})\nabla \cdot u^\varepsilon) + \sigma(u^\varepsilon)\partial_t W,$$

and

$$\partial_t u^{\varepsilon'} = -\nabla \cdot F(u^{\varepsilon'}, x) + \nabla \cdot ((\mathbf{A}(u^{\varepsilon'}) + \varepsilon' \mathbf{Id})\nabla \cdot u^{\varepsilon'}) + \sigma(u^{\varepsilon'})\partial_t W,$$

satisfy

$$\begin{aligned}
&\mathbb{E}\left(\int (u^{\varepsilon'}(x,s) - u^{\varepsilon}(x,s))_+ \, dx\right) \\
\leq& C\rho\mu(\mathbb{T}^d) + \theta \exp\big(C(d)\|\partial^2_{ui}F\|_{L^\infty} s\big)(C_2(F,d)s + \mathbb{E}(|u_0^{\varepsilon'}|_{BV})) \\
&+ \exp\left[\left(C(d)\|\partial^2_{ui}F\|_{L^\infty} + C_2(F,d)\theta\rho^{-1}\right) s\right] \\
&\cdot \left[\mathbb{E}\left(\iint \eta_\rho(u^{\varepsilon'}(y,0) - u^{\varepsilon}(x,0))J_\theta(x-y)\, dx\, dy\right) + \right. \\
&+ (C(\alpha,d)\rho^{2\gamma_\alpha}\theta^{-1})K(u_0^{\varepsilon'}, s) \\
&+ (C(\alpha,d)|\sqrt{\varepsilon} - \sqrt{\varepsilon'}|\rho^{\gamma_\alpha}\theta^{-1} + n|\sqrt{\varepsilon} - \sqrt{\varepsilon'}|^2\theta^{-1})K(u_0^{\varepsilon'}, s) \\
&+ C_1(F,d)\rho^{\kappa F 1}K(u_0^{\varepsilon'}, s) + C_\sigma s \rho^{2\lambda - 1}\mu(\mathbb{T}^d).
\end{aligned}$$

We see that we need $\rho, \theta \to 0$ as $\varepsilon, \varepsilon' \to 0$, as well as

$$\begin{aligned}
\theta\rho^{-1} &< \infty \\
\rho^{2\gamma_\alpha}\theta^{-1} &\to 0 \\
|\sqrt{\varepsilon} - \sqrt{\varepsilon'}|\rho^{\gamma_\alpha}\theta^{-1} &\to 0 \\
|\sqrt{\varepsilon} - \sqrt{\varepsilon'}|^2\theta^{-1} &\to 0.
\end{aligned}$$

Since we have assumed $2\gamma_\alpha > 1$, simply take

$$\rho = |\sqrt{\varepsilon} - \sqrt{\varepsilon'}|, \theta = |\sqrt{\varepsilon} - \sqrt{\varepsilon'}|.$$

Temporal L^1-continuity

Following [10], we show that

$$\mathbb{E}\left(\int_0^{T-\Delta t}\int (u(t+\Delta t, x) - u(t,x))_+ \, dx\, dt\right) \leq C_T(\Delta t)^{1/3}$$

as $\Delta t \to 0$.

Let us define the temporal difference

$$w(\cdot,t) = u(\cdot,t+\Delta t) - u(\cdot,t).$$

From the definition of the kinetic solution, for a test function $\varphi(\xi,x;t)$, we have the equality:

$$\begin{aligned}
&\langle \varphi, \bar{H}(\xi - u(t+\Delta t))\rangle - \langle \varphi, \bar{H}(\xi - u(t))\rangle \\
&= \int_t^{t+\Delta t} \langle \nabla\varphi, \partial_u F \bar{H}(\cdot - u(s)\rangle \, ds - \int_t^{t+\Delta t} \langle \partial_\xi \varphi, \partial_i F^i \bar{H}(\cdot - u(s))\rangle \, ds \\
&\quad + \int_t^{t+\Delta t} \langle \nabla^2 \varphi : \mathbf{A}, \bar{H}(\cdot - u(s))\rangle \, ds - M_u(\partial_\xi \varphi \times [t, t+\Delta t]) \\
&\quad + \frac{1}{2}\int_t^{t+\Delta t} \int_{\mathbb{T}^d} \sigma^2(u(x,s))(\partial_\xi \varphi)(u(x,s),x;t) \, dx \, ds \\
&\quad + \int_t^{t+\Delta t} \int_{\mathbb{T}}^d \varphi(u(x,s),x)\sigma(u(x,s)) \, dx \, dW_s.
\end{aligned}$$

where as in (2.24), the angle brackets represent integral in the spatial and kinetic variables. As before $\bar{H} = 1 - H$, and H is the Heaviside function.

We now choose a test function convex in the kinetic variable ξ, so that we can avail ourselves of the sign of the defect measures in the effort to estimate the left-hand side. We retain the positive part function in favour of the sign function used by [10].

Nevertheless inspired by [10], we use the test function:

$$\varphi(\xi,x,t) = (J_\theta * (\operatorname{sgn}(w(\cdot,t)))_+)(x)\eta_\rho'(\xi - u(t)),$$

where J_θ is again an approximation to the delta function centred at the origin, that is smooth, non-negative, supported on $B_\theta(0)$, and has unit mass. Let $\eta_\rho : \mathbb{R} \to \mathbb{R}$ continue to be as in the construction given in (2.22). Notice that here, $\Phi(\xi,x,t)$ for which $\partial_\xi \Phi = \varphi$, is convex in the kinetic variable.

Integrating above in time from nought to $T - \Delta t$, we have the expression:

$$\begin{aligned}
&\int_0^{T-\Delta t} \langle \varphi, \bar{H}(\xi - u(t+\Delta t))\rangle \, dt - \int_0^{T-\Delta t} \langle \varphi, \bar{H}(\xi - u(t))\rangle \, dt \\
&= \int_0^{T-\Delta t} \int_t^{t+\Delta t} \langle \nabla\varphi, \partial_u F \bar{H}(\cdot - u(s)\rangle \, ds \, dt - \int_0^{T-\Delta t} \int_t^{t+\Delta t} \langle \partial_\xi \varphi, \partial_i F^i \bar{H}(\cdot - u(s))\rangle \, ds \, dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{T-\Delta t}\int_t^{t+\Delta t}\langle \nabla^2\varphi : \mathbf{A}, \bar{H}(\cdot - u(s))\rangle \, ds\, dt - \int_0^{T-\Delta t} M_u(\partial_\xi\varphi \times [t, t+\Delta t])\, dt \\
&+\frac{1}{2}\int_0^{T-\Delta t}\int_t^{t+\Delta t}\int_{\mathbb{T}^d} \sigma^2(u(x,s))(\partial_\xi\varphi)(u(x,s),x;t)\, dx\, ds\, dt \\
&+\int_0^{T-\Delta t}\int_t^{t+\Delta t}\int_{\mathbb{T}}^{d} \varphi(u(x,s),x)\sigma(u(x,s))\, dx\, dW_s\, dt \\
&+\int_0^{T-\Delta t}\langle \varphi, \bar{H}(\xi - u(t+\Delta t)) - |w(x,t)|\rangle\, dt.
\end{aligned}
$$

Notice that though the test function depends on $u(t+\Delta t)$, in the stochastic integral above, one can integrate first in s, so all integrals are well-defined either in the sense of Lebesgue-Stieljes or more generally in the sense of Itô, and adapted.

On the right-hand side, as can be seen from the presence of $\eta_\rho'(\xi - u(t))$ in the definition of φ, as $\rho \to 0$, we expect $\langle \varphi, H(\xi - u(t))\rangle \to 0$. We have the following estimate:

$$
\left|\int_0^{T-\Delta t}\langle \varphi, \bar{H}(\xi - u(t))\rangle\, dt\right| \leq C_T\rho.
$$

For the left-hand side, first we mention that as remarked previously,

$$
M_u(\partial_\xi\varphi \times [t, t+\Delta t]) \geq 0.
$$

We proceed to analyse the remaining parts of the left-hand side.

Flux Terms:

Assuming that $\partial_i F^i$ has linear growth in its kinetic argument, we have an L^p estimate on u, and so,

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^{T-\Delta t}\int_t^{t+\Delta t}\langle \nabla\varphi, \partial_u F\bar{H}(\cdot - u(s)\rangle\, ds\, dt\right) \\
&-\mathbb{E}\left(\int_0^{T-\Delta t}\int_t^{t+\Delta t}\langle \partial_\xi\varphi, \partial_i F^i\bar{H}(\cdot - u(s))\rangle\, ds\, dt\right) \\
\leq& C_T(\Delta t/\theta + \Delta t/\rho).
\end{aligned}
$$

Parabolic Term:

Assuming polynomial growth in the entries of $\mathbf{A}$, we have the esimate:

$$\mathbb{E}\left(\int_0^{T-\Delta t}\int_t^{t+\Delta t}\langle\nabla^2\varphi:\mathbf{A},\bar{H}(\cdot-u(s))\rangle\,ds\,dt\right)$$
$$\leq C_T\Delta t/\theta^2.$$

Itô Correction Term:

Assuming linear growth in σ, we have an L^p estimate on u, so we also have the estimate:

$$\frac{1}{2}\mathbb{E}\left(\int_0^{T-\Delta t}\int_t^{t+\Delta t}\int_{\mathbb{T}^d}\sigma^2(u(x,s))(\partial_\xi\varphi)(u(x,s),x;t)\,dx\,ds\,dt\right)\leq C_T\Delta t/\rho.$$

Noise term:

Using the Burkholder-Davis-Gundy inequality, and the L^p estimate on u, we have the bound:

$$\mathbb{E}\left(\int_0^{T-\Delta t}\int_t^{t+\Delta t}\int_{\mathbb{T}}^d\varphi(u(x,s),x)\sigma(u(x,s))\,dx\,dW_s\,dt\right)\leq C_T\sqrt{\Delta t}$$

Mollification term: Using the fractional-BV estimate we derived, and supposing that $\partial_i F$ were κ_2-Hölder in its second (spatial) argument, we have as in Chen-Ding-Karlsen,

$$\mathbb{E}\left(\int_0^{T-\Delta t}\langle\varphi,\bar{H}(\xi-u(t+\Delta t))\rangle-(w(x,t))_+\,dt\right)$$
$$\leq\mathbb{E}\left(\int_0^{T-\Delta t}\int J_\theta(x-y)|w(x,t)-x(y,t)|\,dx\,dy\,dt\right)$$
$$\leq\mathbb{E}\left(\int_0^{T}\int J(\mathbf{z})\int|u(x,t)-u(x-\theta\mathbf{z},t)|\,dx\,dt\,d\mathbf{z}\right)$$
$$\leq C_T\theta^{\kappa_2}.$$

Conclusion:

Taking $\rho=\theta^2$, and $\theta=(\Delta t)^\alpha$, we have the bound:

$$\mathbb{E}\left(\int_0^{T-\Delta t}|w(x,t)|\,dx\,dt\right)\lesssim(\Delta t)^{2\alpha}+(\Delta t)^{1-\alpha}+(\Delta t)^{1-2\alpha}+(\Delta t)^{1/2}+(\Delta t)^{\kappa_2\alpha}.$$

This allows us to optimize the inequality to

$$\mathbb{E}\left(\int_0^{T-\Delta t} |w(x,t)|\, dx\, dt\right) \lesssim (\Delta t)^{\kappa_2/(2+\kappa_2)}. \tag{2.56}$$

Remark. In [10], the authors suggested that the optimal bound for the first-order conservation law should be $(\Delta t)^{1/2}$. Supposing we have a BV bound in place of a fractional BV bound, on the torus their suggestion holds true. In the second-order case, the presence of the second derivative gives us another power of θ^{-1}, which under optimization, in the presence of a spatial BV bound, gives us a bound of $C(\Delta t)^{1/3}$.

2.7.2 Conclusion of the Existence Argument

In conclusion we follow [10] and show that we have proven the following existence theorem,

Theorem 10 (Existence in $L^p \cap N^{\kappa,1}$)**.** *Let all the assumptions* (2.3) - (2.7) *hold, then there exists a kinetic solution of the stochastic anisotropic degenerate parabolic-hyperbolic equation such that the fractional BV bound* (2.48) *holds.*

Proof. For any fixed ε, we can mollify u_0 into $u_0^\varepsilon \in C^\infty$ so that for any s, $\mathbb{E}(\|u\|_{H^s}^2)$ is bounded for any s, and

$$\mathbb{E}\left(\|u_0^\varepsilon\|_{L^p} + |u_0^\varepsilon|_{N^{\kappa,1}}\right) \le \mathbb{E}\left(\|u_0\|_{L^p} + |u_0|_{N^{\kappa,1}}\right) < \infty.$$

Then, as in [10], using the arguments of Section 4 of Feng and Nualart [38], along with the convergence results in this section, we can conclude that there is a convergent subsequence $u^\varepsilon(x,t)$ that converges almost everywhere almost surely to $u(x,t)$. □

Remark. [28], a different path to existence was taken, using martingale solutions and the Krylov-Gyöngy mechanism whereby a weak (martingale) solution together with pathwise uniqueness ensures strong existence, though a form of parabolic approximation was still used (which in fact was alluded to in the final sections of [30]).

Chapter 3

Invariant Measure for the Anisotropic Degenerate Parabolic-Hyperbolic Equation with Additive Noise

In this chapter we shall extend the work of Debussche and Vovelle in [30] to the degenerate parabolic case, and show that as in the case of the first-order equation, the process associated to the degenerate parabolic equation also has unique invariant measures. We follow their schema closely.

The equation we consider is

$$\partial_t u + \nabla \cdot F(u) = \nabla^T \cdot (\mathbf{A}(u) \cdot \nabla u) + \sigma(x)\partial_t W,$$

The evolution occurs over the torus $\mathbb{T}^d$.

To begin with we assume that $\sigma(x)$ is bounded on $\mathbb{T}^d$. In [30], Debussche and Vovelle used a Hilbert-Schmidt operator in place of a simply bounded noise, taking $\sigma(x) = \sum_k g_k(x)e_k$, and $W = \sum_k e_k\beta_k$, a cylindrical Wiener process, where β_k are independent Brownian motions, and (e_k) are a complete orthonormal system in a Hilbert space. In the analysis below the difference between using a cylindrical Wiener process or a one dimensional Wiener driving noise is of little account. We elect to keep the calculation simple and legible by taking $\sigma(x)\partial_t W$ as our driving noise, where dW_t is simply white noise. Nevertheless, as in [30], we assume that

$$\int_{\mathbb{T}^d} \sigma(x)\, dx = 0,$$

and that

$$|\sigma(x)|^2 \leq D_0, \tag{3.1}$$

$$|\sigma(x) - \sigma(y)| \leq D_1 |x - y|. \tag{3.2}$$

This assumption ensures that the average of the solution over the torus, $\int u\, dx = 0$. This is important in order to rule out the drift of solutions off to infinity in norm even though they may smoothen out to a constant, as expected in the deterministic case [11]. We shall make further remarks on this in the following chapter.

3.1 Mechanism for Extracting an Invariant Measure

In this section we introduce the notion of an invariant measure.

3.1.1 Motivation

The notion of invariant measures on a dynamical system is quite direct. Let $(\mathfrak{X}, \Sigma, \mu)$ be a measure space, and let $S : \mathfrak{X} \to \mathfrak{X}$ be a map. System $(\mathfrak{X}, \Sigma, \mu, S)$ is a measure-preserving system if $\mu(S^{-1}A) = \mu(A)$ for any $A \in \Sigma$, and then μ is called an invariant measure of map S.

On a random dynamical system (RDS), there is an added layer of complexity.

We follow the standard definitions in [1]; see [23] for further references on RDSs and [39, 40] in a specifically parabolic SPDE context.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\theta_t : \Omega \to \Omega$ be a collection of probability-preserving maps. A *measurable RDS* on a measurable space $(\mathfrak{X}, \Sigma)$ over quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ is a map:

$$\varphi : \mathbb{R} \times \Omega \times \mathfrak{X} \to \mathfrak{X}$$

satisfying the following:

(i) Measurability: that φ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \Sigma$-$\Sigma$ measurable, and,

(ii) Cocycle property: that $\varphi(t,\omega) = \varphi(t,\omega,\cdot) : \mathfrak{X} \to \mathfrak{X}$ is a cocyle over θ:

$$\begin{aligned}\varphi(0,\omega) =&\mathrm{id}_{\mathfrak{X}},\\ \varphi(t+s,\omega) =&\varphi(s,\theta_t\omega)\varphi(t,\omega).\end{aligned}$$

We think of $\Omega \times \mathfrak{X} \to \Omega$ as a fibre bundle with fibres $\mathfrak{X}$. On the bundle, we have the *skew product* defined as $\Theta = (\theta, \varphi)$. Then, again following [1], we can define the invariant measure:

Definition 5 (Invariant Measure)**.** An *invariant measure* on a random dynamical system φ over θ is a probability measure μ on $(\Omega \times \mathfrak{X}, \mathcal{F} \otimes \Sigma)$ satisfying

$$(\Theta_t)_*\mu = \mu,\ \pi_\Omega\mu = \mathbb{P}.$$

Any probability measure μ on $\Omega \times \mathfrak{X}$ admits a disintegration,

$$\mu(d\omega, du) = \nu_\omega(du)\mathbb{P}(d\omega).$$

A measure ν_ω is *stationary* if

$$\varphi(t,\omega)_*\nu_\omega = \nu_{\theta_t\omega}.$$

A *Markov* invariant measure is an invariant measure for which map $\omega \mapsto \nu_\omega(\Gamma)$ is $\mathscr{F}_0 - \mathcal{B}(\mathbb{R})$ measurable for any $\Gamma \in \mathcal{B}(\mathfrak{X})$.

The disintegration of measures is unique, and there is a one-to-one correspondence between a Markov invariant measure and a stationary measure [22, 74, 57]. Associated with an invariant measure is a random attracting set, which is generalized from the deterministic context [25, 24]. In the context of dissipative PDEs perturbed by noise, it can be shown that the Hausdorff dimension of an attracting set is finite by estimating by using global Lyapunov exponents, such as in [27].

3.1.2 Existence Machineries

There are several approaches to establish the existence of invariant measures. One approach is the Krylov-Bogoliubov mechanism as we are going to discuss here. Another approach is via Khasminskii's theorem ([94], §8.1 of [81]). Both of them are based on the compactness property provided by the Prohorov's theorem [95].

First we recall that a sequence of probability measures $\{\lambda_n\}$ on a measure space $\mathfrak{X}$ is *tight* if for every $\varepsilon > 0$, there is a compact set $K_\varepsilon \subseteq \mathfrak{X}$ for which, uniformly in n,

$$\lambda_n(\mathfrak{X}\backslash K_\varepsilon) \leq \varepsilon.$$

Then we can state Prohorov's theorem as

Lemma 11 (Prohorov's theorem)**.** *A tight sequence of probability measures λ_n is weak* compact in the space of* probability *measures – that is, there exists a probability measure λ and a subsequence (still denoted) λ_n such that $\lambda_n \overset{*}{\rightharpoonup} \lambda$.*

Proof. Following [35], notice that by the Stone-Weierstraß Theorem, for every compact $K \subseteq X$, $C(K) = C_b(K)$ is separable. Let D be a countable dense subset of $C(K)$. Then,

$$\left\{\left\{\int f\lambda_n\right\}_{f\in D}\right\}_{n\geq 1}$$

is a sequence in the countable Cartesian product

$$\prod_{f\in D}[\inf(f), \sup(f)].$$

As its factors are metrizable, L is metrizable in the product topology. By Tychonoff's theorem, it is compact in the product topology. By sequential compactness, there is a subsequence (λ_{n_k}) such that

$$\int f\lambda_{n_k}$$

converges for every $f \in D$. By the density of D, this can be extended to $C(K)$.

It remains to extend $C(K)$ to $C_b(X)$ and to show the limit is a probability measure. This calls for tightness.

For each $r = 1, 2, \cdot$, let K_r be the compact set for which $\lambda_n(K_r) > 1 - 1/r$ for every n. Now let $D(r)$ be a countable dense subset of $C(K_r)$. Then

$$\left\{\left\{\int f\lambda_n\right\}_{f\in D(r),\ r\geq 1}\right\}_{n\geq 1}$$

is a sequence in the countable Cartesian product

$$\prod_{r \geq 1} \prod_{f \in D(r)} [\inf(f), \sup(f)].$$

In the product topology there is again convergent subsequence,

$$\int_{K_r} f \lambda_{n_k},$$

for every $f \in C_b(X)$.

The limit of this subsequence is a linear functional L. By the Riesz(-Markov-Kakutani) Representation Theorem, there is a measure λ such that $L(f) = \int f d\lambda$. To see that λ is a probability measure, we need but take $f \equiv 1$.

□

Let $S_t : \mathfrak{X} \to \mathfrak{X}$ be the solution operator of a well-posed Cauchy problem, where $\mathfrak{X}$ is a complete, separable, metric space. Let $\mathscr{P}_t$ be the process associated with S_t, that is, for any $\varphi \in C_b(\mathfrak{X})$, $\mathscr{P}_t \varphi = \varphi \circ S_t$.

Furthermore, a process is said to be *Feller* if for every $\varphi \in C_b(\mathfrak{X})$, it holds that the map

$$(t, x) \in [0, \infty] \times \mathfrak{X} \mapsto \mathscr{P}_t \varphi(x)$$

is continuous.

The Krylov-Bogoliubov Theorem was proven first in [8], and is as follows:

Theorem 12 (Krylov-Bogoliubov theorem)**.** *Let $\mathscr{P}$ be a Feller semigroup, and let μ be a probability measure on $\mathfrak{X}$ for which $\{\mathscr{P}_t^* \mu\}$ is tight. Then there exists an invariant measure for $\mathscr{P}_t$.*

In fact, the invariant measure generated by this theorem is the weak* limit of

$$\nu_T = \frac{1}{T} \int_0^T \mathscr{P}_t^* \mu \, dt.$$

Thus, the tightness of $\{\nu_T\}$ is the key, which is ensured by the tightness assumption of $(\mathscr{P}_t^* \mu)$.

By the Prohorov theorem, the tight sequence has a weakly converging subsequence (still denoted as) (ν_t) for t ranging over a unbounded subset of $\mathbb{R}$. Let its limit be ν_*.

Then

$$
\begin{aligned}
\int_{\mathfrak{X}} \varphi(u)\mathscr{P}_s^*\nu_*(du) &= \int_{\mathfrak{X}} (\mathscr{P}_s\varphi)(u)\nu_*(du) \\
&= \lim_{t\to\infty} \frac{1}{t}\int_0^t \int_{\mathfrak{X}} (\mathscr{P}_s\varphi)(u)(\mathscr{P}_\tau^*\mu)(du)\ d\tau \\
&= \lim_{t\to\infty} \frac{1}{t}\int_0^t \int_{\mathfrak{X}} \varphi(u)(\mathscr{P}_{\tau+s}^*\mu)(du)\ d\tau \\
&= \lim_{t\to\infty} \frac{1}{t}\int_0^t \int_{\mathfrak{X}} \varphi(u)(\mathscr{P}_\tau^*\mu)(du)\ d\tau \\
&\quad + \lim_{t\to\infty} \frac{1}{t}\int_t^{t+s} \int_{\mathfrak{X}} \varphi(u)(\mathscr{P}_\tau^*\mu)(du)\ d\tau \\
&\quad - \lim_{t\to\infty} \frac{1}{t}\int_0^s \int_{\mathfrak{X}} \varphi(u)(\mathscr{P}_\tau^*\mu)(du)\ d\tau.
\end{aligned}
$$

We require the Feller property to execute the second equality, as $\mathscr{P}_s\varphi$ has to remain continuous. With this, the second and third summation after the last equality tend to zero in the limit $t \to \infty$, as s is fixed.

3.1.3 Philosophy

We now illustrate the mechanism in which the Krylov-Bogoliubov theorem is usually applied.

Let $\mathfrak{X}$ be a Banach space of initial condition, from which the solution map $S_t : \mathfrak{X} \to \mathfrak{X}$ takes its value, and let $\mathfrak{Y}$ be a Banach space such that there is a compact embedding $\mathfrak{Y} \hookrightarrow \mathfrak{X}$. Suppose S_t were compact for $t > 0$ so that

$$S_t(\mathfrak{X}) \subseteq \mathfrak{Y} \qquad \text{for almost all } t.$$

Let $K_R = \{x \in \mathfrak{X} : \|x\|_{\mathfrak{Y}} \leq R\}$. Since $\mathfrak{Y} \hookrightarrow \mathfrak{X}$, K_R is compact in $\mathfrak{X}$. Let f be any suitable function satisfying $f(\cdot) \geq \|\cdot\|_{\mathfrak{Y}}$.

In order to show the tightness of $\{\nu_T\}$, we apply the Markov inequality to f. The suitability is defined by the justifiability of the following calculation:

If $f(\cdot) \geq \|\cdot\|_{\mathfrak{Y}}$, then $f(u) \leq R$ implies that $\|u\|_{\mathfrak{Y}} \leq R$, which in turn implies that $u \in K_R$. Therefore, if $u \in \mathfrak{X} \setminus K_R$, then $f(u) > R$.

The bound sought is then the following: For every ε, we can find R such that

$$\begin{aligned}\nu_T(\mathfrak{X} \setminus K_R) &\leq \nu_T(\{f(u) > R\}) \\ &\leq \frac{1}{R} \int_{\mathfrak{X}} f(u) \nu_T(du) \\ &= \frac{1}{RT} \int_0^T \int_{\mathfrak{X}} (\mathscr{P}_t f)(u) \mu(du)\, dt \\ &\leq \varepsilon.\end{aligned}$$

If $\mu = \delta_{u_0}$, then

$$\frac{1}{RT} \int_0^T \int_{\mathfrak{X}} (\mathscr{P}_t f)(u) \mu(du)\, dt = \frac{1}{RT} \int_0^T (\mathscr{P}_t f)(u_0)\, dt = \frac{1}{RT} \int_0^T f(u(t))\, dt.$$

Therefore, if the temporal average is bounded:

$$\frac{1}{T} \int_0^T f(u(t))\, dt < C, \tag{3.3}$$

then a compact set K_R has been found such that $\nu_T(\mathfrak{X} \setminus K_R) \leq C/R$, and $\{\nu_T\}$ are tight. This implies the existence of an invariant measure. Our efforts in §3.2 will be to show this bound for the degenerate parabolic-hyperbolic equation.

It is important to point out that, where the initial condition is u_0, and the solution is $u(t)$, the conditions of the Krylov-Bogoliubov Theorem are satisfied if there is an increasing collection of compact sets K_m and finite times t_m such that

$$\sup_{t \geq t_m} \mathbb{P}(\{u(t) \notin K_m\} | u(0) = u_0) \to 0 \text{ as } m \to \infty.$$

We shall make use of this condition to invoke the Krylov-Bogoliubov machinery.

All the above can be found neatly summarised also in §2.5.1 of [73].

Now S_t in our problem is Feller, by L^1-contraction. We shall show that L^1 balls are mapped into sets of uniformly equicontinuous functions.

3.1.4 Methods for Establishing Uniqueness

It is well known that the invariant measures of a map form a convex set in the probability space on X. By the Krein-Milman theorem, the convex set is the closure of convex combinations of its extreme points. These extreme points μ happen to be the ergodic measures, which are characterised as the property that, for a measurable subset $A \subseteq \mathfrak{X}$,

$$\mu((\mathcal{S}^{-1}A)\Delta A) = 0 \longleftrightarrow \mu(A) = 0 \text{ or } \mu(A) = 1,$$

where $A\Delta B := (A \setminus B) \cup (B \setminus A)$

Ergodic measures heuristically carve up the solution space into essentially disjoint subsets, since any two ergodic measures of a process either coincide or are singular with respect to one another. This is a simple consequence of the property stated above.

It also follows from the extremal property of ergodic measures that, if there are more than one invariant measure, then there are at least two ergodic measures.

One common condition used to guarantee the uniqueness of an invariant measure is the following:

Definition 6 (Strong Feller Property)**.** A Markov transition semigroup $\mathscr{P}_t$ is strong Feller at time t if $\mathscr{P}_t\varphi$ is continuous for every bounded measurable $\varphi : \mathfrak{X} \to \mathbb{R}$.

The strong Feller property guarantees the uniqueness of invariant measures [34, 64] (see also [94], Theorem 5.2.1, and [82] and the references therein).

The strong Feller property always holds for transition semigroups of processes associated with nonlinear stochastic evolution equations with Lipschitz nonlinear coefficients and nondegenerate diffusion, for example, see [92].

3.1.5 Coupling Method

The coupling method is a powerful tool in the probability theory introduced in Doeblin-Fortet[33, 32] (also see [77] for a reference), which can be used to show the uniqueness of invariant measures.

The general argument proceeds as follows: Let X_t be a Markov process with initial distribution λ, and let Y_t be an independent copy of that process but with an initial distribution that is an invariant measure π. Then the first meeting time $\mathcal{T}$ is a stopping time, and the process defined by

$$Z_t = \begin{cases} X_t, & t < \mathcal{T}, \\ Y_t, & t \geq \mathcal{T} \end{cases}$$

is also a copy of X_t by the strong Markov property.

Write $f_*\nu := \nu \circ f^{-1}$ as the pushforward measure (sometimes also denoted $f\sharp\nu$). Using the definition of Z_t, we can write

$$\begin{aligned}\mathscr{P}_t^*\lambda - \pi &= (Z_t)_*\mathbb{P} - (Y_t)_*\mathbb{P} \\ &= (\mathbf{1}_{\{t<\mathcal{T}\}}Z_t)_*\mathbb{P} + (\mathbf{1}_{\{t\geq\mathcal{T}\}}Z_t)_*\mathbb{P} - (\mathbf{1}_{\{t<\mathcal{T}\}}Y_t)_*\mathbb{P} - (\mathbf{1}_{\{t\geq\mathcal{T}\}}Y_t)_*\mathbb{P} \\ &= (\mathbf{1}_{\{t<\mathcal{T}\}}Z_t)_*\mathbb{P} - (\mathbf{1}_{\{t<\mathcal{T}\}}Y_t)_*\mathbb{P}.\end{aligned}$$

Then the total variation norm of $\mathscr{P}_t^*\lambda - \pi$ can be estimates as follows:

$$\begin{aligned}|\mathscr{P}_t^*\lambda - \pi|_{TV} &\leq \int (\mathbf{1}_{\{t<\mathcal{T}\}}Z_t)_*\mathbb{P}(du) + \int (\mathbf{1}_{\{t<\mathcal{T}\}}Y_t)_*\mathbb{P}(du) \\ &= \mathbb{P}(\{t < \mathcal{T}\}).\end{aligned}$$

If $\mathcal{S}$ can be shown to be almost surely finite, then, as $t \to \infty$, we see that $\mathscr{P}_t^*\lambda \to \pi$, and there is only one invariant measure.

In our application below, $\mathcal{S}$ will be slightly modified to be the time of entry into a small ball.

First, we show in §3.3.2 that two solutions u and v enter a given ball in finite time, almost surely. This is a stopping time. From this, by the strong Markov property, we construct a sequence of increasing, almost surely finite stopping times in (3.27), which are spaced at least T apart, for some $T > 0$ later to be fixed.

Then we will show in 3.3.3 that, for $T > 0$ well chosen, if a solution starts within the same given ball and the noise is uniformly small in $W_x^{1,\infty}$ over a duration of length T, then the temporal average of $\|u(t)\|_{L_x^1}$ over that temporal interval can be taken to be smaller than some ε. Since the noise is $\sigma(x)W$, the uniform smallness in $W_x^{1,\infty}$ over an interval $[\mathcal{T}, \mathcal{T} + T]$ depends entirely on the size of W.

We see that, for $T > 0$, the probability that the change in noise remains small between $[\mathcal{T}, \mathcal{T} + T]$ is strictly positive and, by the strong Markov property, we can replace $\mathcal{T}$ with any other stopping time, for example the one in the sequence constructed, spaced at least T apart. Using the L^1–contraction, we will show finally in §3.3.5 that the probability that the difference between two solutions remain large for all intervals $[\mathcal{T}, \mathcal{T} + T]$ with $\mathcal{T}$ in the sequence of increasing stopping times is bounded by the probability that over all such sequences the noise is large in $W^{1,\infty}$. This must be vanishingly small, as the probability is strictly less than unity on each individual sequence.

3.2 Existence of Invariant Measures

Consider the kinetic formulation of the equation

$$\partial_t u + \nabla \cdot F(u) = \nabla^T \cdot (\mathbf{A}(u) \cdot \nabla u) + \sigma(x)\partial_t W, \tag{3.4}$$

where $\mathbf{A}$ is symmetric, positive semi-definite, and σ has zero average over $\mathbb{T}^d$.

This is given by

$$\partial_t \chi_u + (F'(\xi) \cdot \nabla - \mathbf{A}(\xi) : \nabla \otimes \nabla)\chi_u = \partial_\xi(m_u + n_u - p_u) + \sigma(x)\delta(\xi - u)\partial_t W.$$

In order to handle the measures $m_u + n_u - p_u$ and q_u, we need to regularize the operators as in [30], but by adding $\gamma(-\Delta)^2\alpha + \theta\mathrm{Id}$ to each side:

$$\begin{aligned}\partial_t \chi_u &+ (F'(\xi) \cdot \nabla - \mathbf{A}(\xi) : \nabla \otimes \nabla + (\gamma(-\Delta)^\alpha) + \theta\mathrm{Id})\chi_u \\ &= \gamma(-\Delta)^\alpha \chi_u + \partial_\xi(m_u + n_u - p_u) + \sigma(x)\delta(\xi - u)\partial_t W.\end{aligned}$$

There are very specific reasons to include these regularizing operators. They are present in order that the measure $\sigma(x)\delta(\xi - u)$ can be estimated. The spatial regularization $(-\Delta)^\alpha$ gives us a spatial bound, and $\theta\mathrm{Id}$ gives us temporal decay.

Denoting by $\mathcal{S}(t)$ the semigroup

$$\begin{aligned}\mathcal{S}(t)f(x) =& e^{(-F'(\xi)\cdot\nabla + \mathbf{A}(\xi):\nabla\otimes\nabla - \gamma(-\Delta)^\alpha + \theta\mathrm{Id})t} f \\ =& e^{-\theta t}(e^{t\mathbf{A}(\xi):\nabla\otimes\nabla - t\gamma(-\Delta)^\alpha} f)(x - F'(\xi)t).\end{aligned}$$

Expressing the solution in the mild formulation,

$$\begin{aligned}\chi_u =& \mathcal{S}(t)\chi_u(\xi, x, 0) + \int_0^t \mathcal{S}(s)(\gamma(-\Delta)^\alpha - \theta\mathrm{Id})\chi_u(\xi, x, t - s)\, ds \\ &+ \int_0^t \mathcal{S}(t - s)\partial_\xi(m_u + n_u - p_u)(s)\, ds + \int_0^t \mathcal{S}(t - s)\sigma(x)\delta(\xi - u(s))\, dW_s,\end{aligned}$$

leads us to the decomposition,

$$u = u^0 + u^\flat + M_1 + M_2, \tag{3.5}$$

where,

$$\begin{aligned}
u^0(x,t) &= \int \mathcal{S}(t)\chi_u(\xi,x,0)\, d\xi, \\
u^\flat(x,t) &= \int \int_0^t \mathcal{S}(s)(\gamma(-\Delta)^\alpha - \theta \mathrm{Id})\chi_u(\xi,x,t-s)\, ds\, d\xi, \\
\langle M_1, \varphi\rangle &= \int_0^t \iint \mathcal{S}^*(s)\varphi\partial_\xi(m_u + n_u - p_u)(t-s)\, ds\, d\xi\, dx, \\
\langle M_2, \varphi\rangle &= \int_0^t \iint \mathcal{S}^*(t-s)\varphi\sigma(x)\delta(\xi - u(s))\, d\xi\, dx\, dW_s.
\end{aligned}$$

The compactness argument is pinned on an averaging lemma, in the spirit of velocity averaging-in-space. Velocity averaging is a technique whereby a genuine nonlinearity condition, or a non-degeneracy condition on the nonlinearity, is shown to imply improved fractional regularity of solutions, which with the compactness of the solution operator are two pages of the same leaf. The history of velocity averaging is short but fruitful, with contributions from many authors,

Golse, Perthame, Sentis [46] and Golse, Lions, Perthame, Sentis [45] proved velocity averaging results for $u_t + v \cdot \nabla u = f(x,v)$ for $f \in L^p(dx \otimes d\mu(v))$, showing that $\int u(x,v)d\mu(v)$ is in some Sobolev or Marcinkiewicz space. Di Perna, Lions, Meyer, [31] used Littlewood-Paley theory to extend regularity results into Besov spaces. Lions, Perthame Tadmor [79] applied velocity averaging techniques and kinetic formulation of equations to one another, developing genuine nonlinearity conditions under which higher regularity can be expected via velocity averaging in *nonlinear* transport equations. They also showed in [80] the limits of velocity averaging in the case of systems of isentropic gas dynamics equations, where a purely kinetic formulation also cannot easily be formulated. Bézard [7] extended the results of [31] to potential spaces. Gérard and Golse [44] considered velocity averaging problems for more general equations of the form $P(x,v,D_x,D_v)u(x,v) = f(x,v)$ under a transversality condition. DeVore and Petrova [105] using a wavelet approach, established the optimality of some averages in a Besov space. Golse and S. Raymond [47] extended the results of [45] in L^1 to any spatial dimensions. Jabin and Perthame [55] used a mixture of Fourier and real-variable methods to establish new averaging lemmas, and C. De Lellis and M. Westdickenberg [75] showed the optimality of these by considering Burger's equation with finite entropy dissipation. Further using real variable methods involving the Riesz and Radon transforms, Jabin and Vega [56] obtained some optimal regularity results in the time-independent case.

Tadmore, Tao [102] collected many previous results in the stationary (time-independent) framework, and showed that there is a general harmonic analysis viewpoint from which to understand velocity averaging; they also provide new regularity results for anisotropic nonlinear degenerate diffusion, which we did not have time to incorporate here. The following averaging argument follows in spirit from that of [9], used heavily in [30], but differs in its application. It also bears mentioning that Lions, Perthame, and Souganidis [78] also developed a class of averaging lemma for stochastic equations of the form, $u_t + \dot{B}(t) \circ v \cdot \nabla u = f(v, x, t)$, where $\dot{B}(t)$ is white-in-time noise. We shall not be using these here.

Since the regularizations $\gamma(-\Delta)^\alpha + \theta \mathrm{Id}$ were added to estimate measures arising from the stochastic term, we see that $u^\flat$ also exists for this reason. The existence of $u^\flat$ requires a non-degeneracy condition with a rate of decay, unlike in the deterministic case in [14] where it is enough that it tends to nought. This non-degeneracy condition is given, in the style of [14], and requires

$$\sup_{\substack{n \in \mathbb{R}^n \\ \tau \in \mathbb{R}}} \int \frac{(4\pi^2 \mathbf{A}(\xi) : n \otimes n + \gamma |n|^{2\alpha} + \theta)}{(2\pi F'(\xi) \cdot n + \tau)^2 + (4\pi^2 \mathbf{A}(\xi) : n \otimes n + \gamma |n|^{2\alpha} + \theta)^2} \, d\xi = O((\gamma^r \vee 1)|n|^{-2\alpha - b}) \tag{3.6}$$

for some positive $b > 0$ possibly dependent on α as $|n| \to 0$,

Let us explain why this makes sense. Dividing the top and bottom of the integrand by $|n|^2$, and splitting the integral into $\{\xi : \mathbf{A}(\xi) : \hat{n} \otimes \hat{n} = 0\}$ and its complement, where $\hat{n} = n/|n|$, we see that where the integral is finite,

$$\begin{aligned} &\int_{\{\xi : \mathbf{A}(\xi) : \hat{n} \otimes \hat{n} = 0\}} \frac{(\gamma |n|^{2\alpha} + \theta)}{(2\pi F'(\xi) \cdot n + \tau)^2 + (\gamma |n|^{2\alpha} + \theta)^2} \, d\xi \\ &= \int_{\{\xi : \mathbf{A}(\xi) : \hat{n} \otimes \hat{n} = 0\}} \frac{|n|^{-1}(\gamma |n|^{2\alpha - 1} + \theta |n|^{-1})}{(2\pi F'(\xi) \cdot \hat{n} + \tau/|n|)^2 + (\gamma |n|^{2\alpha - 1} + \theta |n|^{-1})^2} \, d\xi, \end{aligned}$$

and when non-linearity is such that, for example, that $|F'(\xi)| \gtrsim |\xi|^c$ with $1/2 < c < 1$, as $|\xi| \to \infty$, we have that the above is bounded by

$$(\gamma|n|^{2\alpha-1} + \theta|n|^{-1})^{1/c-1}|n|^{-1} \int \frac{r^{1/c-1}}{r^2+1}\, dr \sim C\gamma^{1/c-1}|n|^{(2\alpha-1)(1/c-1)-1}.$$

On the other hand, writing $\gamma|n|^{2\alpha-1} + \theta|n|^{-1}$ as $\omega(n) = \omega$,

$$\begin{aligned}
&\int_{\{\xi:\mathbf{A}(\xi):\hat{n}\otimes\hat{n}>0\}} \frac{(4\pi^2\mathbf{A}(\xi) : n\otimes n + \gamma|n|^{2\alpha} + \theta)}{(2\pi F'(\xi)\cdot n + \tau)^2 + (4\pi^2\mathbf{A}(\xi) : n\otimes n + \gamma|n|^{2\alpha} + \theta)^2}\, d\xi \\
&= \int_{\{\xi:\mathbf{A}(\xi):\hat{n}\otimes\hat{n}>0\}} \frac{(4\pi^2\mathbf{A}(\xi) : \hat{n}\otimes\hat{n} + \omega|n|^{-1})}{(2\pi F'(\xi)\cdot \hat{n} + \tau/|n|)^2 + (4\pi^2|n|\mathbf{A}(\xi) : \hat{n}\otimes\hat{n} + \omega)^2}\, d\xi,
\end{aligned}$$

and where the non-linearity is such that additionally, $|\mathbf{A}(\xi) : \hat{n}\otimes\hat{n}| \sim |\xi|^d$, with

$$2c - d - 1 > 2s > 0,$$

(so that the integrand is integrable), then the integral can be bounded by

$$\begin{aligned}
&\int_0^\infty \frac{\omega(|\xi|^d\omega^{-1} + |n|^{-1})}{\omega^2(\xi^{2c}\omega^{-2} + |n|^2\xi^{2d} + 1}\, d\xi \\
&= \int_0^\infty \frac{r^d\omega^{-1+d} + |n|^{-1}}{r^{2c}\omega^{-2+2c} + |n|^2 r^{2d}\omega^{-2+2d} + 1}\, dr,
\end{aligned}$$

where we used the substitution $r = \xi/\omega$. We can split this integral up into parts according as $r^{2c}\omega^{-2+2c}$ or $|n|^2 r^{2d}\omega^{-2+2d}$ is greater. These correspond to respective parts where r is greater than or smaller than

$$K = |n|^{1/(c-d)}\omega^{-1} \sim \gamma^{-1}|n|^{1-2\alpha+1/(c-d)}.$$

Therefore,

$$\begin{aligned}
&\int_{\{\xi:\mathbf{A}(\xi):\hat{n}\otimes\hat{n}>0\}} \frac{(4\pi^2\mathbf{A}(\xi) : n\otimes n + \gamma|n|^{2\alpha} + \theta)}{(2\pi F'(\xi)\cdot n + \tau)^2 + (4\pi^2\mathbf{A}(\xi) : n\otimes n + \gamma|n|^{2\alpha} + \theta)^2}\, d\xi \\
&\leq \int_{r<K} \frac{r^d\omega^{-1+d} + |n|^{-1}}{|n|^2 r^{2d}\omega^{-2+2d} + 1}\, dr + \int_{r>K} \frac{r^d\omega^{-1+d} + |n|^{-1}}{r^{2c}\omega^{-2+2c} + 1}\, dr \\
&\leq \int_{r\omega^{1-1/d}<K\omega^{1-1/d}} \omega^{1/d-1} \frac{(r\omega^{1-1/d})^d + |n|^{-1}}{|n|^2(r\omega^{1-1/d})^{2d} + 1}\, d(r\omega^{1-1/d}) + C\int_{r>K} \frac{r^d\omega^{-1+d}}{r^{2c}\omega^{-2+2c}}\, dr.
\end{aligned}$$

As $|n| \to \infty$, $K \to \infty$. These two integrals can be respectively bound by

$$\begin{aligned}
&\int_{r\omega^{1-1/d}<K\omega^{1-1/d}} \omega^{1/d-1} \frac{(r\omega^{1-1/d})^d + |n|^{-1}}{|n|^2 (r\omega^{1-1/d})^{2d} + 1}\, d(r\omega^{1-1/d}) \\
\leq& C \int_{r\omega^{1-1/d}<K\omega^{1-1/d}} \omega^{1/d-1} \frac{(r\omega^{1-1/d})^d}{|n|^2 (r\omega^{1-1/d})^{2d}}\, d(r\omega^{1-1/d}) \\
\lesssim& |n|^{-2+(1-d)/(c-d)} \\
\lesssim& |n|^{-2s/(c-d)},
\end{aligned}$$

and

$$C \int_{r>K} \frac{r^d \omega^{-1+d}}{r^{2c}\omega^{-2+2c}}\, dr = C\omega^{1+d-2c} K^{1+d-2c} = C|n|^{(1+d-2c)/(c-d)} \leq C|n|^{-2s/(c-d)}.$$

so that,

$$\begin{aligned}
&\int_{\{\xi:\mathbf{A}(\xi):\hat{n}\otimes\hat{n}>0\}} \frac{(4\pi^2 \mathbf{A}(\xi) : n \otimes n + \gamma|n|^{2\alpha} + \theta)}{(2\pi F'(\xi)\cdot n + \tau)^2 + (4\pi^2 \mathbf{A}(\xi) : n\otimes n + \gamma |n|^{2\alpha} + \theta)^2}\, d\xi \\
=& O((\gamma^{1/c-1} \vee 1)|n|^{-2s/(c-d) \vee ((2\alpha-1)(1/c-1)-1)}). \qquad (3.7)
\end{aligned}$$

Apart from the non-degeneracy condition (3.6), we also require that

$$|F''(\xi)| \lesssim |\xi| + 1, \qquad |\mathbf{A}'(\xi)| \lesssim |\xi| + 1. \qquad (3.8)$$

for the existence of invariant measures, and

$$|F''(\xi)| \lesssim 1, \qquad |\mathbf{A}'(\xi)| \lesssim 1 \qquad (3.9)$$

for uniqueness.

The main theorem of this chapter is as follows:

Theorem 13. *Let* $\mathbf{A}, F$ *satisfy the non-degeneracy condition (3.7)*[1]*, and also (3.8). As-*

---

[1] *(post-viva addendum: This has since been improved to*

$$\begin{aligned}
\eta(\varepsilon_1, \varepsilon_2) =& \sup_{\substack{\tau\in\mathbb{R} \\ \hat{n}\in S^{d-1}}} \left(\iint \frac{\partial^2}{\partial a \partial b}\left[\frac{2(a+1)}{(a+1)^2 + b^2}\right] \cdot \mu(\{\xi : \mathbf{A}(\xi) : \hat{n}\otimes\hat{n} \leq \varepsilon_1|\} \cap \{\xi : |F'(\xi)\cdot\hat{n} + \tau| \leq \varepsilon_2\})\, da\, db \right) \\
\leq& c_1 \varepsilon_1^{b_1} \varepsilon_2^{b_2}
\end{aligned}$$

for some fixed $c_1 > 0$, and $1 > b_1, b_2 \geq 0$, and $|b_1 + b_2 - 1| < 1$, as $\varepsilon_1, \varepsilon_2 \to 0$.)

sume also the boundedness and Lipschitz conditions (3.1) and (3.2) on the noise. Then there exists an invariant measure for the anisotropic degenerate parabolic-hyperbolic equation (3.4) in $L^1(\mathbb{T}^d)$.

Furthermore where the addition conditions (3.9) are imposed, the invariant measure is unique.

3.2.1 Analysis of u^0

Taking the Fourier transform in x and integrating ξ, we have

$$\begin{aligned}
\hat{u}^0(n,t) &= \int \mathcal{S}(t)\hat{\chi}_u(\xi,n,0)\ d\xi \\
\int_0^T |\hat{u}^0(n,t)|^2\ dt &= \int_0^T |\mathcal{S}(t)\hat{\chi}_u(\xi,n,0)\ d\xi|^2\ dt \\
&\le \int_{-\infty}^{\infty} \left| \int e^{-2\pi i F'(\xi)\cdot nt-(4\pi^2\mathbf{A}(\xi):n\otimes n+\gamma|n|^{2\alpha}+\theta)|t|}\hat{\chi}_u(\xi,n,0)\ d\xi \right|^2\ dt \\
&= \int_{-\infty}^{\infty} \left| \int \mathcal{F}\left(e^{-2\pi i F'(\xi)\cdot n\cdot-(4\pi^2\mathbf{A}(\xi):n\otimes n+\gamma|n|^{2\alpha}+\theta)|\cdot|}\right)\hat{\chi}_u(\xi,n,0)\ d\xi \right|^2\ d\tau,
\end{aligned}$$

where $\mathcal{F}$ is the Fourier transform in time taking $t \rightsquigarrow \tau$, and the meanings of γ and θ have changed slightly to include multiplies of 2π. This is the Plancherel theorem.

The Fourier transform is given by

$$\mathcal{F}\left(e^{-2\pi i F'(\xi)\cdot n\cdot-(4\pi^2\mathbf{A}(\xi):n\otimes n+\gamma|n|^{2\alpha}+\theta)|\cdot|}\right) = \frac{4\pi^2\mathbf{A}(\xi):n\otimes n+\gamma|n|^{2\alpha}+\theta}{(2\pi F'(\xi)\cdot n+\tau)^2+(4\pi^2\mathbf{A}(\xi):n\otimes n+\gamma|n|^{2\alpha}+\theta)^2},$$

and has the integral,

$$\int_{\mathbb{R}} \frac{4\pi^2\mathbf{A}(\xi):n\otimes n+\gamma|n|^{2\alpha}+\theta}{(2\pi F'(\xi)\cdot n+\tau)^2+(4\pi^2\mathbf{A}(\xi):n\otimes n+\gamma|n|^{2\alpha}+\theta)^2}\ d\tau = \int_{\mathbb{R}} \frac{1}{1+\tau^2}\ d\tau.$$

By Cauchy-Schwarz,

$$\begin{aligned}
\int_0^T |\hat{u}^0(n,t)|^2\ dt &\le \int_{-\infty}^{\infty} \left| \int \mathcal{F}\left(e^{-2\pi i F'(\xi)\cdot n\cdot-(4\pi^2\mathbf{A}(\xi):n\otimes n+\gamma|n|^{2\alpha}+\theta)|\cdot|}\right)\hat{\chi}_u(\xi,n,0)\ d\xi \right|^2\ d\tau \\
&\le \int_{-\infty}^{\infty} \int \left| \mathcal{F}\left(e^{-2\pi i F'(\xi)\cdot n\cdot-(4\pi^2\mathbf{A}(\xi):n\otimes n+\gamma|n|^{2\alpha}+\theta)|\cdot|}\right) \right|\ d\xi \\
&\quad \cdot \int \left| \mathcal{F}\left(e^{-2\pi i F'(\xi)\cdot n\cdot-(4\pi^2\mathbf{A}(\xi):n\otimes n+\gamma|n|^{2\alpha}+\theta)|\cdot|}\right) \right| |\hat{\chi}_u(\xi,n,0)|^2\ d\xi\ d\tau
\end{aligned}$$

$$\le \sup_\tau \int \frac{4\pi^2\mathbf{A}(\xi):n\otimes n+\gamma|n|^{2\alpha}+\theta}{(2\pi F'(\xi)\cdot n+\tau)^2+(4\pi^2\mathbf{A}(\xi):n\otimes n+\gamma|n|^{2\alpha}+\theta)^2}\,d\xi$$
$$\cdot\int\frac{1}{1+\tau^2}\,d\tau\int|\hat{\chi}_u(\xi,n,0)|^2\,d\xi$$
$$\le C|n|^{-2\alpha-b}\|\hat{\chi}(\cdot,n,0)\|^2_{L^2_\xi}.$$

Summing up in $n\in\mathbb{Z}^d\backslash\{0\}$, along with the property that u^0 has zero spatial average, gives us [88],

$$\int_0^T \|u^0(t)\|_{H^{\alpha+b/2}}\,dt \le C\|u(0)\|_{L^1_x}. \tag{3.10}$$

3.2.2 Analysis of $u^\flat$

The analysis of $u^\flat$ proceeds similarly. Again, taking the spatial Fourier transform leads to

$$\hat{u}^\flat(n,t)=\int\int_0^t e^{(-2\pi iF'(\xi)\cdot n-4\pi^2\mathbf{A}(\xi):n\otimes n-\gamma|n|^{2\alpha})s}\gamma|n|^{2\alpha}\hat{\chi}_u(\xi,n,t-s)\,ds\,d\xi,$$

so that,

$$\begin{aligned}
&\int_0^T|\hat{u}^\flat(n,t)|^2\,dt\\
=&\int_0^T\left|\int_0^t\int e^{(-2\pi iF'(\xi)\cdot n-4\pi^2\mathbf{A}(\xi):n\otimes n-\gamma|n|^{2\alpha}-\theta)s}(\gamma|n|^{2\alpha}+\theta)\hat{\chi}_u(\xi,n,t-s)\,d\xi\,ds\right|^2\,dt\\
=&\int_0^T\left|\int_0^t e^{-(\gamma|n|^{2\alpha}+\theta)s}\int e^{(-2\pi iF'(\xi)\cdot n-4\pi^2\mathbf{A}(\xi):n\otimes n-\gamma|n|^{2\alpha}/2-\theta/2)s}(\gamma|n|^{2\alpha}+\theta)\hat{\chi}_u(\xi,n,t-s)\,d\xi\,ds\right|^2\,dt\\
\le&\int_0^T\int_0^T\mathbf{1}_{\{t-s\ge0\}}(\gamma|n|^{2\alpha}+\theta)\left|\int e^{(-2\pi iF'(\xi)\cdot n-4\pi^2\mathbf{A}(\xi):n\otimes n-\gamma|n|^{2\alpha}/2-\theta/2)s}\hat{\chi}_u(\xi,n,t-s)\,d\xi\right|^2\,ds\,dt\\
\le&\int_0^T\int_{-\infty}^\infty(\gamma|n|^{2\alpha}+\theta)\left|\int\mathcal{F}\left(e^{(-2\pi iF'(\xi)\cdot n\cdot-(4\pi^2\mathbf{A}(\xi):n\otimes n+\gamma|n|^{2\alpha}/2+\theta/2)|\cdot|)}\right)\hat{\chi}_u(\xi,n,\mathbf{t})\,d\xi\right|^2\,ds\,d\mathbf{t},
\end{aligned}$$

where again, $\mathcal{F}$ is the Fourier transform between the variables $s\rightsquigarrow\tau$.

By the same manipulations we have

$$\int_0^T|\hat{u}^\flat(n,t)|^2\,dt$$

$$
\begin{aligned}
\leq & \int_0^T \int_{-\infty}^{\infty} (\gamma|n|^{2\alpha} + \theta) \int \mathcal{F}\left(e^{(-2\pi i F'(\xi)\cdot n \cdot - (4\pi^2 \mathbf{A}(\xi):n\otimes n + \gamma |n|^{2\alpha}/2 + \theta/2)|\cdot|}\right) \, d\xi \\
& \cdot \int \mathcal{F}\left(e^{(-2\pi i F'(\xi)\cdot n \cdot - (4\pi^2 \mathbf{A}(\xi):n\otimes n + \gamma |n|^{2\alpha}/2 + \theta/2)|\cdot|}\right) |\hat{\chi}_u(\xi, n, \mathbf{t})|^2 \, d\xi \, ds \, d\mathbf{t} \\
\leq & \sup_{\tau} \int \frac{(4\pi^2 \mathbf{A}(\xi) : n \otimes n + \gamma |n|^{2\alpha} + \theta)(\gamma |n|^{2\alpha} + \theta)}{(2\pi F'(\xi)\cdot n + \tau)^2 + (4\pi^2 \mathbf{A}(\xi) : n\otimes n + \gamma|n|^{2\alpha} + \theta)^2} \, d\xi \\
& \cdot \int \frac{1}{1+\tau^2} \, d\tau \int_0^T \int |\hat{\chi}_u(\xi, n, \mathbf{t})|^2 \, d\xi \, d\mathbf{t} \\
\leq & C|n|^{-b} \int_0^T \|\hat{\chi}(\cdot, n, t)\|^2_{L^2_\xi} \, dt.
\end{aligned}
$$

Again, using the zero-spatial average property, and summing up in $n \in \mathbb{Z}^d$,

$$
\int_0^T \|u(t)\|^2_{H^{b/2}_x} \, dx \leq C \int_0^T \|u(t)\|_{L^1_x} \, dx. \tag{3.11}
$$

Next we turn to the analysis of the measures M_1 and M_2. In this we shall follow [30] quite closely, as the only difference is the parabolic defect measure, which has the same sign as the kinetic dissipation measure, and the magnitude of the kinetic dissipation measure is never invoked in [30].

3.2.3 Analysis of M_1

Recall first that

$$
\langle M_1, \varphi \rangle = \int_0^t \iint \mathcal{S}^*(s)\varphi \partial_\xi (m_u + n_u - p_u)(t-s) \, ds \, d\xi \, dx.
$$

From

$$
\mathcal{S}(t) f(x) = e^{-\theta t}(e^{t\mathbf{A}(\xi):\nabla\otimes\nabla - t\gamma(-\Delta)^\alpha} f)(x - F'(\xi)t),
$$

we can see that

$$
\begin{aligned}
\partial_\xi(\mathcal{S}^*(t-s)h(\xi,x)) = & (t-s)F''(\xi)\cdot\nabla(\mathcal{S}^*(t-s)h) + \mathbf{A}'(\xi) : \nabla^2(\mathcal{S}^*(t-s)h) \\
& - \mathcal{S}^*(t-s)\partial_\xi h,
\end{aligned} \tag{3.12}
$$

so that integrating-by-parts, we have

$$\begin{aligned}
&\langle M_1, \varphi\rangle \\
=&-\int_0^t \iint \partial_\xi(\mathcal{S}^*(s)\varphi)(m_u+n_u-p_u)(t-s)\, d\xi\, dx\, ds \\
=&\int_0^t \iint (t-s)F''(\xi)\cdot\nabla(\mathcal{S}^*(t-s)\varphi)(m_u+n_u-p_u)(t-s)\, d\xi\, dx\, ds \\
&+\int_0^t \iint \mathbf{A}'(\xi) : \nabla^2(\mathcal{S}^*(t-s)\varphi)(m_u+n_u-p_u)(t-s)\, d\xi\, dx\, ds.
\end{aligned} \tag{3.13}$$

Following [30], we prove a total variation estimate.

Lemma 14. *Let $u : \mathbb{T}^d \times [0,T] \times \Omega$ be a solution with initial datum u_0. Let $\psi \in C_c(\mathbb{R})$ be any compactly supported continuous function, and $\Psi = \int_0^s \int_0^r \psi(t)\, dt\, dr$. Then,*

$$u\mathbb{E}\left(\int_{\mathbb{T}^d\times[0,T]\times\mathbb{R}} \psi(\xi)\; d|m_u+n_u-p_u|(\xi,x,t)\right) \leq D_0\mathbb{E}(\|\psi(u)\|_{L^1_{x,t}}) + \mathbb{E}(\|\Psi(u_0)\|_{L^1_x}).$$

Proof. The proof is the same as that found in [30], and involves bounding $|m_u+n_u-p_u| \leq m_u+n_u+p_u$, so that

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T \iint \psi(\xi)|m_u+n_u-p_u|(d\xi,\, dx,\, dt)\right) \\
\leq&\mathbb{E}\left(\int_0^T \iint \psi(\xi)(m_u+n_u-p_u)(d\xi,\, dx,\, dt)\right) + 2\mathbb{E}\left(\int_0^T \iint \psi(\xi)p_u(d\xi,\, dx,\, dt)\right) \\
=&\mathbb{E}\left(\int \Psi(u)\; dx\Big|_T^0\right) + \mathbb{E}\left(\int_0^T \int \sigma^2(x)\psi(u)\; dx\; dt\right),
\end{aligned}$$

using the equation. Now, using the non-negativity of Ψ, we have

$$\mathbb{E}\left(\int_0^T \iint \psi(\xi)|m_u+n_u-p_u|(d\xi,dx,dt)\right) \leq \mathbb{E}\left(\int \Psi(u_0)\; dx\right)+\mathbb{E}\left(\int_0^T \int D_0\psi(u)\; dx\; dt\right).$$

□

This estimate is quite crude as one doesn't take the cancellation betwen the measures m_u+n_u and p_u, which are both non-negative, into account. But seeing as there is no ready way to quantify m_u+n_u, this is the best possible at the moment.

In addition to a total variation estimate, we also require the kernel estimate

$$\left\|(-\Delta)^{\beta/2}e^{(\mathbf{A}:\nabla\otimes\nabla-\gamma(-\Delta)^{\alpha/2})t}\right\|_{L^p\to L^q}\leq C(\gamma t)^{-d/\alpha\left(p^{-1}-q^{-1}\right)-\beta/\alpha}.$$

The reason that this is no improvement over the estimate for the operator less $e^{t\mathbf{A}:\nabla\otimes\nabla}$ is that we have not specified how degenerate $\mathbf{A}$ is — it may well be simply the zero matrix.

It is the use of this kernel estimate that necessitated the inclusion of the regularizations $\gamma(-\Delta)^\alpha+\theta\mathrm{Id}$.

By Young's convolution inequality, we can then estimate as follows:

$$\begin{aligned}\|(-\Delta)^{\beta/2}\nabla(\mathcal{S}^*(t-s)\varphi)\|_{L^\infty_x}\leq& C(\gamma(t-s))^{-(\beta+1)/(2\alpha)-d(1/(2\alpha)-1/(2\alpha p'))}e^{-\delta(t-s)}\|\varphi\|_{L^{p'}_x}\\ \|(-\Delta)^{\beta/2}\nabla^2(\mathcal{S}^*(t-s)\varphi)\|_{L^\infty_x}\leq& C(\gamma(t-s))^{-(\beta+2)/(2\alpha)-d(1/(2\alpha)-1/(2\alpha p'))}e^{-\delta(t-s)}\|\varphi\|_{L^{p'}_x}.\end{aligned}$$

Importantly the constants C are independent of γ and θ.

Inserting these estimates into (3.13), we have the estimate

$$\begin{aligned}&\mathbb{E}\left(\int_0^T\langle(-\Delta)^{\beta/2}M_1,\varphi\rangle\,dt\right)\\ =&\mathbb{E}\bigg[\int_0^T\int_0^t\iint[(-\Delta)^{\beta/2}F''(\xi)\cdot\nabla(\mathcal{S}^*(t-s)\varphi)(m_u+n_u-p_u)(t-s)\,d\xi\,dx\,ds\,dt\\ &+\int_0^T\int_0^t\iint(-\Delta)^{\beta/2}\mathbf{A}'(\xi):\nabla^2(\mathcal{S}^*(t-s)\varphi)](m_u+n_u-p_u)(t-s)\,d\xi\,dx\,ds\,dt\bigg]\\ \leq&\mathbb{E}\left(\int_0^T\int\|(-\Delta)^{\beta/2}\nabla(\mathcal{S}^*(t-s)\varphi)\|_\infty(t-s)|F''(\xi)||m_u+n_u-p_u|(d\xi,\,dx,\,ds)\,dt\right)\\ &+\mathbb{E}\left(\int_0^T\int\|(-\Delta)^{\beta/2}\nabla(\mathcal{S}^*(t-s)\varphi)\|_\infty|\mathbf{A}'(\xi)||m_u+n_u-p_u|(d\xi,\,dx,\,ds)\,dt\right),\end{aligned}$$

Let

$$\mu=-\frac{\beta+d+1}{2\alpha}+\frac{d}{2\alpha p'}.$$

By the presence of the factor $e^{-\theta(t-s)}$, we can bound the outer temporal integral as well, arriving at

$$\mathbb{E}\left(\int_0^T\langle(-\Delta)^{\beta/2}M_1,\varphi\rangle\,dt\right)$$

$$
\begin{aligned}
\leq & \int_0^T \gamma^{-1}(\gamma\tau)^{\mu+1}e^{-\theta\tau}\, d\tau\, \mathbb{E}\left(\int_{\mathbb{R}\times\mathbb{T}^d\times[0,T]} \|\varphi\|_{L^p}|F''(\xi)||m_u+n_u-p_u|(d\xi,\, dx,\, ds)\right) \\
& + \int_0^T (\gamma\tau)^{\mu-1/(2\alpha)}e^{-\theta\tau}\, d\tau\, \mathbb{E}\left(\int_{\mathbb{R}\times\mathbb{T}^d\times[0,T]} \|\varphi\|_{L^p}|\mathbf{A}'(\xi)||m_u+n_u-p_u|(d\xi,\, dx,\, ds)\right) \\
\leq & \gamma^{-2}\left(\frac{\gamma}{\theta}\right)^{\mu+2}|\Gamma(\mu+2)|\, \mathbb{E}\left(\int_{\mathbb{R}\times\mathbb{T}^d\times[0,T]} \|\varphi\|_{L^p}|F''(\xi)||m_u+n_u-p_u|(d\xi,\, dx,\, ds)\right) \\
& + \gamma^{-1}\left(\frac{\gamma}{\theta}\right)^{\mu+1-1/2\alpha}|\Gamma\left(\mu+1-1/(2\alpha)\right)|\, \mathbb{E}\left(\int_{\mathbb{R}\times\mathbb{T}^d\times[0,T]} \|\varphi\|_{L^p}|\mathbf{A}'(\xi)||m_u+n_u-p_u|(d\xi,\, dx,\, ds)\right).
\end{aligned}
$$

By duality, the total variation estimate, and the sublinearity of F'' and $\mathbf{A}'$, we have the estimate

$$
\begin{aligned}
& \mathbb{E}(\|M_1\|_{L^1_t W^{\beta,p'}_x}) \\
\leq & \gamma^{-2}\left(\frac{\gamma}{\theta}\right)^{\mu+2}|\Gamma(\mu+2)|\, \mathbb{E}\left(\int_{\mathbb{R}\times\mathbb{T}^d\times[0,T]} |F''(\xi)||m_u+n_u-p_u|(d\xi,\, dx,\, ds)\right) \\
& + \gamma^{-1}\left(\frac{\gamma}{\theta}\right)^{\mu+1-1/2\alpha}|\Gamma\left(\mu+1-1/(2\alpha)\right)|\, \mathbb{E}\left(\int_{\mathbb{R}\times\mathbb{T}^d\times[0,T]} |\mathbf{A}'(\xi)||m_u+n_u-p_u|(d\xi,\, dx,\, ds)\right) \\
\leq & C\left[\gamma^{-2}\left(\frac{\gamma}{\theta}\right)^{\mu+2}|\Gamma(\mu+2)| + \gamma^{-1}\left(\frac{\gamma}{\theta}\right)^{\mu+1-1/2\alpha}|\Gamma\left(\mu+1-1/(2\alpha)\right)|\right] \\
& \cdot\left(1+\int_0^T \mathbb{E}(\|u(t)\|_{L^1_x})\, dt + \mathbb{E}\|u_0\|^3_{L^3}\right). \qquad (3.14)
\end{aligned}
$$

Here we choose γ and θ such that

$$
C\left[\gamma^{-2}\left(\frac{\gamma}{\theta}\right)^{\mu+2}|\Gamma(\mu+2)| + \gamma^{-1}\left(\frac{\gamma}{\theta}\right)^{\mu+1-1/2\alpha}|\Gamma\left(\mu+1-1/(2\alpha)\right)|\right] \leq \frac{1}{4}.
$$

3.2.4 Analysis of M_2

$$
\langle M_2, \varphi\rangle = \int_0^t \iint \varphi\mathcal{S}(t-s)\sigma(x)\delta(\xi-u(s))\, d\xi\, dx\, dW_s.
$$

We again invoke the kernel estimate. In fact it is here that the kernel estimate becomes indispensable. In the stochastic setting, with a forcing term given by $\sigma(x)\delta(\xi-u(x,t))\partial_t W$, which does not easily lend itself to the spacetime Fourier transform, one cannot simply take the Fourier transform on both sides, so that on the left-side one has (as we do),

$$(i(\tau + F'(\xi) \cdot n) + \mathbf{A}(\xi) : n \otimes n)\hat{\chi},$$

and $(i(\tau + F'(\xi) \cdot n) + \mathbf{A}(\xi) : n \otimes n)$ can simply be divided out, with a certain genuine non-linearity/ non-degeracy condition [11, 79, 102]. One has to find a way to handle the forcing term. In this we follow [30].

Expanding the effect of the semigroup, we have

$$\int_0^t \int_{\mathbb{T}^d} e^{-\theta(t-s)} \varphi e^{(\mathbf{A}(\xi):\nabla\otimes\nabla - \gamma(-\Delta)^\alpha)(t-s)} \sigma(x - F'(u(s))(t-s))\, dx; dW_s.$$

Since σ is bounded in $\mathbb{R}$, we see that $\sigma(\cdot - F'(u(\cdot, s))(t-s))$ is bounded in x.

The kernel estimate then gives

$$\begin{aligned}
&\|e^{-\theta(t-s)} \varphi e^{(\mathbf{A}(u):\nabla\otimes\nabla - \gamma(-\Delta)^\alpha)(t-s)} \sigma(\cdot - F'(u(\cdot, s))(t-s))\|_{H^\beta} \\
&\le C(\gamma(t-s))^{\beta/(2\alpha)} \|e^{(\mathbf{A}(u):\nabla\otimes\nabla(t-s) - 2\pi i F'(u)\cdot\nabla} \sigma(\cdot)\|_{L^2} \\
&\le C(\gamma(t-s))^{\beta/(2\alpha)} \|\sigma\|_{L^2},
\end{aligned}$$

just as in [30], and in the same way,

$$\mathbb{E}\left(\left\| \int_0^t \int \mathcal{S}(t-s)\sigma(x)\delta(\xi - u(x,s))\, d\xi\, dW_s \right\|_{H^\beta}^2 \right) \le C D_0 \gamma^{-\beta/\alpha} \theta^{\beta/\alpha - 1} \left| \Gamma\left(1 - \beta/\alpha\right) \right|.$$

Hence we have

$$\mathbb{E}\left(\|M_2\|_{H^\beta}^2 \right) \le C D_0 \gamma^{-\beta/\alpha} \theta^{\beta/\alpha - 1} \left| \Gamma\left(1 - \beta/\alpha\right) \right|.$$

3.2.5 Conclusion of the existence argument

There is some $W^{s,q}$ into which $H^{b/2}$ and H^β embed.

From (3.10) and (3.11), we have

$$\mathbb{E}(\|u^0 + u^\flat + M_2\|_{L_t^2 W_x^{s,q}}^2) \le \|u(0)\|_{L^1} + \|u\|_{L^1([0,T], L_x^1)} + CT,$$

and Jensen's inequality,

$$\frac{1}{T}\mathbb{E}^2(\|u^0 + u^\flat + M_2\|_{L_t^1 W_x^{s,q}}) \le \mathbb{E}(\|u^0 + u^\flat + M_2\|_{L_t^2 W_x^{s,q}}^2)$$

$$\mathbb{E}^2(\|u^0 + u^\flat + M_2\|_{L^1_t W^{s,q}_x}) \leq T(\|u(0)\|_{L^1} + \|u\|_{L^1([0,T],L^1_x)} + CT)$$
$$\mathbb{E}(\|u^0 + u^\flat + M_2\|_{L^1_t W^{s,q}_x}) \leq C(\|u(0)\|_{L^1_x} + T) + \frac{1}{4}\|u\|_{L^1([0,T],L^1_x)},$$

by Young's inequality.

From (3.14), we further have

$$\mathbb{E}\left(\|M_1\|_{L^1([0.T],W^{\beta,p'}_x)}\right) \leq \frac{1}{4}(1 + \mathbb{E}(\|u\|_{L^1([0,T],L^1_x)}) + \mathbb{E}(\|u_0\|^3_{L^3_x}).$$

Now choosing (β, p') such that $W^{\beta,p'} \hookrightarrow W^{s,q}$ allows us to write

$$\mathbb{E}\left(\|u\|_{L^1([0,T],W^{s,q}_x)}\right) \leq C(\alpha,\beta,\gamma,\theta,d,D_0)(1 + \|u(0)\|^3_{L^3_x} + T) + \frac{1}{2}\|u\|_{L^1([0,T],L^1_x)}.$$

By the continuous embedding $W^{s,q}_x \hookrightarrow L^1_x$,

$$\mathbb{E}\left(\|u\|_{L^1([0,T],W^{s,q}_x)}\right) \leq C(\alpha,\beta,\gamma,\theta,d,D_0)(1 + \|u(0)\|^3_{L^1_x} + T), \tag{3.15}$$

and we have used the inequality:

$$\|u(0)\|_{L^1(\mathbb{T}^d)} \leq \left(1 + \|u(0)\|^3_{L^1(\mathbb{T}^d)}\right) \leq \left(1 + C\|u(0)\|^3_{L^3(\mathbb{T}^d)}\right).$$

Since $W^{s,q}$ is compactly embedded in L^1 for $q \geq 1$, by the Krylov-Bogoliubov mechanism (§3.1.3), an invariant measure exists.

3.3 Uniqueness of Invariant Measure

Next we turn to the more central issue of recurrence – in this we follow section 4.1 of [30] closely. In this we follow §4 of [30] quite closely, but we have a more general discussion on uniqueness of invariant measures in§4.3.2. We shall show first that solutions enter a fixed ball in almost surely finite time. In the next section, we shall show that solutions enter an arbitrarily small ball if the noise is small enough.

First we prove the lemma,

Lemma 15. *There is a radius κ, dependent on the initial conditions, and an almost surely finite stopping time τ for which a solution enters $B_\kappa(0) \subseteq L^1(\mathbb{T}^d)$ in time τ.*

3.3.1 Proof

Let $u(t)$ and $v(t)$ be solutions with initial conditions u_0 and v_0, respectively.

We partition time by defining the time steps,

$$\begin{aligned} t_0 =& 0 \\ t_{j+1} =& t_j + r_j, \end{aligned}$$

where the durations r_j are constants to be determined.

In all the rest of this subsection, we follow almost exactly the development in §4.1 of [30].

Now define the events

$$A_j = \left\{ \inf_{s\in[t_l,t_{l+1}]} (\|u(s)\|_{L^1_x} + \|v(s)\|_{L^1_x}) \geq 2\kappa,\ l = 0, \ldots, j-1 \right\}. \tag{3.16}$$

We fix the constant κ later.

By the Markov property, as well as by Markov's inequality,

$$\begin{aligned} &\mathbb{P}\left(\inf_{s\in[t_j,t_{j+1}]} (\|u(s)\|_{L^1_x} + \|v(s)\|_{L^1_x}) \geq 2\kappa \middle| \mathscr{F}_{t_j} \right) \\ \leq& \mathbb{P}\left(\frac{1}{r_j} \int_{t_j}^{t_j+r_j} (\|u(s)\|_{L^1_x} + \|v(s)\|_{L^1_x})\, ds \geq 2\kappa \middle| \mathscr{F}_{t_j} \right) \\ \leq& \mathbb{E}\left(\frac{1}{2r_j\kappa} \int_{t_j}^{t_j+r_j} (\|u(s)\|_{L^1_x} + \|v(s)\|_{L^1_x})\, ds \middle| \mathscr{F}_{t_j} \right). \end{aligned}$$

Now this expectation can be bounded by (3.15), which reads:

$$\frac{1}{2} \fint_0^T \mathbb{E}\left(\|u(s)\|_{L^1(\mathbb{T}^d)} \right)\, ds \leq C(\alpha,\beta,\gamma,\theta,d,D_0)(1 + \|u(0)\|^3_{L^3_x} + T). \tag{3.17}$$

By the temporal translation invariance of the equation, we also have

$$\mathbb{E}\left(\fint_{t_j}^{t_j+r_j} \|u(s)\|_{L^1(\mathbb{T}^d)}\, ds \middle| \mathscr{F}_{t_j} \right) \leq 2C\left(1 + \frac{1}{r_j} + \frac{1}{r_j}\|u(t_j)\|^3_{L^3_x} \right). \tag{3.18}$$

With the corresponding estimate for v, we can write

$$\mathbb{E}\left(\fint_{t_j}^{t_j+r_j} \|u(s)\|_{L^1(\mathbb{T}^d)} + \|v(s)\|_{L^1(\mathbb{T}^d)}\, ds \middle| \mathscr{F}_{t_j} \right) \le \frac{\kappa}{r_j} \left(r_j + 1 + \|u(t_j)\|_{L^3_x}^3 + \|v(t_j)\|_{L^3_x}^3 \right). \tag{3.19}$$

We identify this "κ" with the one in (3.16). In the following, κ will not change from line to line.

In order to use (3.18), we turn to estimating $\|u(t_j)\|_{L^3}^3$.

Testing the equation against Φ', where $\Phi(r) = |r|^3$ and integrating in space, we get

$$\|u(t)\|_{L^3}^3 \le \|u_0\|_{L^3}^3 + 3\int_0^t \int \operatorname{sgn}(u)|u|^2 \sigma(x)\, dx\, dW_s + 3\int_0^t \int |u|\sigma^2(x)\, dx\, ds. \tag{3.20}$$

Hence, appealing to (3.19),

$$\begin{aligned}
&\mathbb{P}\left(\inf_{s\in[t_j,t_{j+1}]} (\|u(s)\|_{L^1_x} + \|v(s)\|_{L^1_x}) \ge 2\kappa \middle| \mathscr{F}_{t_j} \right) \\
\le& \mathbb{E}\left(\frac{1}{2r_j\kappa} \int_{t_j}^{t_j+r_j} \|u(s)\|_{L^1_x} + \|v(s)\|_{L^1_x}\, ds \middle| \mathscr{F}_{t_j} \right) \\
\le& \frac{1}{2r_j} \left(r_j + 1 + \|u(t_j)\|_{L^3_x}^3 + \|v(t_j)\|_{L^3_x}^3 \right) \\
\le& \frac{1}{2r_j} \left[r_j + 1 + \|u_0\|_{L^3}^3 + 3\int_0^{t_j} \int \operatorname{sgn}(u)|u|^2\sigma(x)\, dx\, dW_s + 3\int_0^{t_j} \int |u|\sigma^2(x)\, dx\, ds \right] \\
&+ \frac{1}{2r_j} \left[\|v_0\|_{L^3}^3 + 3\int_0^{t_j} \int \operatorname{sgn}(v)|v|^2\sigma(x)\, dx\, dW_s + 3\int_0^{t_j} \int |v|\sigma^2(x)\, dx\, ds \right] \\
\le& \frac{1}{2r_j} \left[r_j + 1 + \|u_0\|_{L^3}^3 + \|v_0\|_{L^3}^3 + 3D_0^2 \int_0^{t_j} \|u(s)\|_{L^1}\, ds + 3D_0^2 \int_0^{t_j} \|v(s)\|_{L^1}\, ds \right] \\
&+ \frac{1}{2r_j} \left[3\int_0^{t_j} \int \operatorname{sgn}(u)|u|^2\sigma(x)\, dx\, dW_s + 3\int_0^{t_j} \int \operatorname{sgn}(v)|v|^2\sigma(x)\, dx\, dW_s \right].
\end{aligned}$$

Following [30], we choose r_j to satisfy two criteria – the first of which is

$$\frac{1}{2r_j}(\|u_0\|_{L^3_x}^3 + \|v_0\|_{L^3_x}^3 + 1) \le \frac{1}{8}. \tag{3.21}$$

Next we multiply through by χ_{A_j} and take expectations so that

$$\begin{aligned}\mathbb{P}(A_{k+1}) \leq & \frac{5}{8}\mathbb{P}(A_j) + \frac{3D_0^2}{2r_j}\mathbb{E}\left[\int_0^{t_j} \|u(s)\|_{L^1}\, ds\, \chi_{A_j} + \int_0^{t_j} \|v(s)\|_{L^1}\, ds\, \chi_{A_j}\right] \\ & + \frac{1}{2r_j}\mathbb{E}\left[3\int_0^{t_j}\int \operatorname{sgn}(u)|u|^2\sigma(x) + \operatorname{sgn}(v)|v|^2\sigma(x)\, dx\, dW_s\, \chi_{A_j}\right].\end{aligned}$$

Using the Cauchy-Schwarz inequality we have

$$\begin{aligned}& \frac{3D_0^2}{2r_j}\mathbb{E}\left[\int_0^{t_j} \|u(s)\|_{L^1}\, ds\, \chi_{A_j} + \int_0^{t_j} \|v(s)\|_{L^1}\, ds\, \chi_{A_j}\right] \\ \leq & \frac{C}{r_j}\mathbb{E}^{1/2}\left[\left(\int_0^{t_j} \|u(s)\|_{L^1} + \|v(s)\|_{L^1}\, ds\right)^2\right]\mathbb{P}^{1/2}(A_j) \\ \leq & \frac{C}{r_j^2}\mathbb{E}\left[\left(\int_0^{t_j} \|u(s)\|_{L^1} + \|v(s)\|_{L^1}\, ds\right)^2\right] + \frac{1}{16}\mathbb{P}(A_j).\end{aligned} \tag{3.22}$$

With the Itô isometry, we also have

$$\begin{aligned}& \frac{1}{2r_j}\mathbb{E}\left[3\int_0^{t_j}\int \operatorname{sgn}(u)|u|^2\sigma(x)\, dx\, dW_s\, \chi_{A_j} + 3\int_0^{t_j}\int \operatorname{sgn}(v)|v|^2\sigma(x)\, dx\, dW_s\, \chi_{A_j}\right] \\ \leq & \frac{C}{r_j}\left[\mathbb{E}\left(\int_0^{t_j} \|u(s)\|_{L^2}^4 + \|v(s)\|_{L^2}^4\right)\right]^{1/2}\mathbb{P}^{1/2}(A_j) \\ \leq & \frac{C}{r_j^2}\left[\mathbb{E}\left(\int_0^{t_j} \|u(s)\|_{L^2}^4 + \|v(s)\|_{L^2}^4\right)\right] + \frac{1}{16}\mathbb{P}(A_j).\end{aligned}$$

By the Itô formula, we have the following bounds to the above:

$$\begin{aligned}\mathbb{E}\left(\|u(s)\|_{L_x^2}^2\right) \leq & \mathbb{E}\left(\|u_0\|_{L_x^2}^2 + \int_0^s\int \sigma^2(x)\, dx\, dr\right) \\ \leq & \mathbb{E}\left(\|u_0\|_{L_x^2}^2\right) + D_0^2\mu(\mathbb{T}^d)t.\end{aligned} \tag{3.23}$$

By a repeated application of Itô's formula, we further get

$$\begin{aligned}\mathbb{E}\left(\|u(s)\|_{L_x^2}^4\right) \leq & \mathbb{E}\left(\|u_0\|_{L_x^2}^4 + 6D_0^2\mu(\mathbb{T}^d)\int_0^s \|u(r)\|_{L_x^2}^2\, dr\right) \\ \leq & \mathbb{E}\left(\|u_0\|_{L_x^2}^4 + 6D_0^2\mu(\mathbb{T}^d)\|u_0\|_{L_x^2}^2 t\right) + 6D_0^4\mu^2(\mathbb{T}^d)t^2 \\ \leq & \mathbb{E}\left(2\|u_0\|_{L_x^2}^4\right) + 15D_0^4\mu^2(\mathbb{T}^d)t^2,\end{aligned} \tag{3.24}$$

and a corresponding bound holds for v. Adding the bound above to that corresponding bound for v, and integrating in time we have

$$\mathbb{E}\left(\int_0^{t_j} \|u(s)\|_{L^2}^4 + \|v(s)\|_{L^2}^4\right) \leq C\left(t_j\mathbb{E}\left(|u_0\|_{L_x^2}^4 + \|v_0\|_{L_x^2}^4\right) + t_j^3\right).$$

Again, $\mu(\mathbb{T}^d)$ is the Lebesgue measure of $\mathbb{T}^d$.

Also, returning to (3.22), using Jensen's inequality and (3.23), we similarly have

$$\mathbb{E}\left[\left(\int_0^{t_j} \|u(s)\|_{L^1} + \|v(s)\|_{L^1}\, ds\right)^2\right] \leq C\left(t_j^2\mathbb{E}\left(\|u_0\|_{L_x^2}^2 + \|v_0\|_{L_x^2}^2\right) + t_j^3\right).$$

Bringing the previous few calculations together we have

$$\begin{aligned}\mathbb{P}(A_{k+1}) \leq& \frac{3}{4}\mathbb{P}(A_j) + \frac{C}{r_j^2}\left(t_j^2\mathbb{E}\left(\|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2\right) + t_j^3\right) \\ &+ \frac{C}{r_j^2}\left(t_j\mathbb{E}\left(|u_0\|_{L_x^2}^4 + \|v_0\|_{L_x^2}^4\right) + t_j^3\right) \\ \leq& \frac{3}{4}\mathbb{P}(A_j) + \frac{C}{r_j^2}\left[\mathbb{E}\left(\|u_0\|_{L_x^2}^4 + \|v_0\|_{L_x^2}^4\right)t_j + t_j^3\right].\end{aligned}$$

Now atop one criterion for the choices of r_j given in (3.21), we stipulate another — that

$$\frac{C}{r_j^2}\left[\mathbb{E}\left(\|u_0\|_{L_x^2}^4 + \|v_0\|_{L_x^2}^4\right)t_j + t_j^3\right] \leq \left(\frac{3}{4}\right)^j.$$

These criteria are compatible with one another, and these choices yield the telescoping bound:

$$\mathbb{P}(A_{j+1}) \leq \frac{3}{4}\mathbb{P}(A_j) + \frac{3}{4},$$

so that,

$$\mathbb{P}(A_j) \leq j\left(\frac{3}{4}\right)^{j-1}.$$

Now the Borel-Cantelli Lemma implies that

$$j_0 = \inf\left\{j \geq 0 : \inf_{s\in[t_l,t_{l+1}]}(\|u(s)\|_{L_x^1} + \|v(s)\|_{L_x^1}) \leq 2\kappa\right\}$$

is almost surely finite. Which in turn implies that the stopping time

$$\tau^{u,v}(B_{2\kappa}(0)) = \inf\{t \geq 0 : \|u(t)\|_{L^1} + \|v(t)\|_{L^2} \leq 2\kappa\} \tag{3.25}$$

is almost surely finite, by the bound

$$\tau^{u,v}(B_{2\kappa}(0)) \leq t_{j_0+1}.$$

Remark. It follows that the recursively defined sequence of stopping times, with $\tau_0 = 0$, and

$$\tau_l = \inf\{t \geq \tau_{l-1} + T : \|u(t)\|_{L^1} + \|v(t)\|_{L^1} \leq 2\kappa\} \tag{3.26}$$

are also almost surely finite.

3.3.2 Uniqueness I: Finite Time to Enter a Ball

To show uniqueness we first show that solutions enter a certain ball in L^1_x in finite time almost surely. Then we show that solutions starting on a fixed ball enter arbitrarily small balls if the noise is sufficiently small in $W^{1,\infty}$. This allows us to conclude, since the noise is sufficiently small for any given duration with positive probability, that any pair of balls enter an arbitrarily small ball of one another. This is the property of recurrence discussed in §3.1.5, and implies the uniqueness of invariant measure. In showing recurrence we follow §4 of [30] quite closely.

The following lemma is proved in exactly the same way as it was in [30], via a Borel-Cantelli argument.

Lemma 16. *There is a radius κ, dependent on the initial conditions, and an almost surely finite stopping time $\mathcal{T}$ for which a solution enters $B_\kappa(0) \subseteq L^1(\mathbb{T}^d)$ in time $\mathcal{T}$.*

The proof uses the coupling method, where v is another solution to the same equation with initial condition $v(0) = v_0$. It furnishes us with the recursively defined sequence of stopping times, with $\mathcal{T}_0 = 0$, and,

$$\mathcal{T}_l = \inf\{t \geq \mathcal{T}_{l-1} + T : \|u(t)\|_{L^1} + \|v(t)\|_{L^1} \leq 2\kappa\} \tag{3.27}$$

which are also almost surely finite.

3.3.3 Uniqueness II: Bounds with small noise

Finally we prove the lemma,

Lemma 17. *For any $\varepsilon > 0$, there are $T > 0$ and $\eta > 0$ such that, for initial conditions u_0 satisfying,*

$$\|u_0\|_{L^1_x} \leq 2\kappa,$$

and noise satisfying,

$$\sup_{t\in[0,T]} \|\sigma W\|_{W^{1,\infty}} \leq \eta,$$

one has,

$$⨍_0^T \|u(t)\|_{L^1_x}\, dt \leq \varepsilon,$$

where we have used the symbol $⨍$ to denote the averaged integral.

Our estimates differ somewhat from [30] in that we use a kernel estimate on $v_F^\sharp + v_A^\sharp$ instead of velocity averaging techniques, because we need to handle the extra derivative. Of course, this method can also be applied to the first order case, and eliminate the need to estimate the average $⨍ v^\sharp\, dx$ that needed to be done in [30].

Following [30], Where u is a solution to

$$\partial_t u + \nabla\cdot F(u) + \nabla\cdot(\mathbf{A}(u)\cdot\nabla u) = \sigma(x)\partial_t W,$$

with initial condition $u(0) = u_0$, let $\tilde{u}$ be the solution to the same equation with initial condition $\tilde{u}_0$ for which

$$\|u_0 - \tilde{u}_0\|_{L^1} \leq \frac{\varepsilon}{8},\ \|\tilde{u}_0\|_{L^2} \leq C\kappa\varepsilon^{-d/2},$$

which can be found by convolving u_0 with a mollifying kernel. Here "κ" is the "κ" of Lemma 16.

Let us consider the difference between the solution and the noise $v = \tilde{u} - \sigma(x)W$, which is a kinetic solution to

$$\partial_t v = -\nabla\cdot F(v + \sigma(x)W) + \nabla\cdot(\mathbf{A}(v + \sigma(x)W)\cdot\nabla(v + \sigma(x)W)).$$

The kinetic formulation for this equation can be derived as in (2.10):

$$\begin{aligned}
&\partial_t \chi_v + F'(\xi)\cdot\nabla\chi_v - \mathbf{A}(\xi):\nabla^2\chi_v \qquad (3.28)\\
=&(F'(\xi) - F'(\xi+\sigma(x)W))\cdot\nabla\chi_v - \nabla\cdot((\mathbf{A}(\xi)-\mathbf{A}(\xi+\sigma(x)W))\cdot\nabla\chi_v)\\
&- F'(\xi+\sigma(x)W)\delta(\xi-v)\cdot\nabla\sigma(x)W + \nabla\cdot(\mathbf{A}(\xi+\sigma(x)W)\delta(\xi-v)\cdot\nabla\sigma(x)W)\\
&- \partial_\xi(\delta(\xi-v)\mathbf{A}(\xi+\sigma(x)W):\nabla\sigma(x)W\otimes\nabla\sigma(x)W)\\
&+ \partial_\xi(m_u + N_u).
\end{aligned}$$

A notable difference here is that the parabolic defect measure is not (as perhaps might be expected) the limit of

$$\delta(\xi - (v^\varepsilon + \sigma(x)W))\mathbf{A}(\xi) : \nabla(v^\varepsilon + \sigma W)\otimes\nabla(v^\varepsilon + \sigma(x)W),$$

but rather the limit of

$$\begin{aligned}
N_u^\varepsilon =&\delta(\xi - v^\varepsilon)\mathbf{A}(\xi+\sigma(x)W):\nabla v^\varepsilon\otimes\nabla v^\varepsilon \qquad (3.29)\\
&+ \delta(\xi - v^\varepsilon)\mathbf{A}(\xi+\sigma(x)W):\nabla(\sigma(x)W)\otimes\nabla(\sigma(x)W)\\
&+ \delta(\xi - v^\varepsilon)\mathbf{A}(\xi+\sigma(x)W):\nabla v^\varepsilon\otimes\nabla\sigma(x)W.
\end{aligned}$$

The asymmetry in the cross term in failing to contain both $\nabla v\otimes\nabla\sigma(x)W$ and $\nabla\sigma(x)W\otimes\nabla v$ arises from the fact that the convex entropy used is rightly $\Phi(v)$ instead of $\Phi(v+\sigma W)$. One of the key insights in [13] was that using the symmetry and non-negativity of $\mathbf{A}$, $\mathbf{A}$ can be written as the square of another symmetric, positive semi-definite matrix, and (3.29) is non-negative. The limit of N_u^ε is the non-negative parabolic defect measure N_u.

As before we insert regularizing operators on both sides, being $\gamma(-\Delta)^\alpha + \theta\mathrm{Id}$.

Again, we can decompose the solution into various components as follows:

$$\langle v(t),\varphi\rangle = \langle v^0 + v^\flat + v_F^\sharp + v_A^\sharp + M_F + M_A + M_1 + M_2, \varphi\rangle,$$

with,

$$v^0(x) = \int \mathcal{S}(t)\chi_v(\xi,x,0)\,d\xi,$$

$$
\begin{aligned}
v^\flat(x) &= \int\int_0^t \mathcal{S}(s)(B_\gamma + \theta \mathrm{Id})\chi_v(\xi, x, t-s)\; ds\, d\xi,\\
v^\sharp_F(x) &= \int\int_0^t \mathcal{S}(t-s)(F'(\xi) - F'(\xi + \sigma(x)W)) \cdot \nabla \chi_v(\xi, x, s)\; ds\, d\xi,\\
v^\sharp_A(x) &= -\int\int_0^t \mathcal{S}(t-s)\nabla \cdot ((\mathbf{A}(\xi) - \mathbf{A}(\xi + \sigma(x)W)) \cdot \nabla \chi_v(\xi, x, s))\; ds\, d\xi,\\
\langle M_F, \varphi\rangle &= -\int\int_0^t F'(v + \sigma(x)W) \cdot \nabla\sigma(x)W(\mathcal{S}^*(t-s)\varphi)(x, v(x,s))\; ds\, dx,\\
\langle M_A, \varphi\rangle &= -\int\int_0^t (\mathbf{A}(v + \sigma(x)W) : \nabla\sigma(x)W \otimes \nabla(\mathcal{S}^*(t-s)\varphi)(v(x,s), x)\; ds\, dx,\\
\langle M_1, \varphi\rangle &= -\int\int_0^t \int \partial_\xi(\mathcal{S}^*(t-s)\varphi) d(m_u + N_u)(\xi, x, s),\\
\langle M_2, \varphi\rangle &= \int\int_0^t \partial_\xi(\mathcal{S}^*(t-s)\varphi)(v(x,s), x)\mathbf{A}(v + \sigma(x)W) : \nabla\sigma(x)W \otimes \nabla\sigma(x)W\; d\xi\, ds\, dx.
\end{aligned}
$$

Now we estimate these integrals one after another, which we do with slight variations on[30].

We have the familar

$$\int_0^T \|v^0(t)\|^2_{H^\alpha_x}\; dt \le C\gamma^r \|u_0\|_{L^1_x},$$

and

$$\int_0^T \|v^\flat(t)\|^2_{L^2_x}\; dt \le C\gamma^{r+1}\int_0^T \|v(t)\|_{L^1_x}\; dt$$

from velocity averaging arguments, where $|r| < 1$ (we see that there is an extra power of γ in the second estimate from thos arguments, no matter what r might be).

These imply that

$$\fint_0^T \|v^0\|_{L^1_x}\; dt \le C T^{-1/2}\gamma^{r/2}\|u_0\|^{1/2}_{L^1_x} \tag{3.30}$$

$$\fint_0^T \|v^\flat(t)\|_{L^1_x}\; dt \le C\gamma^{(r+1)/2}\left(\fint_0^T \|v(t)\|_{L^1_x}\; dt\right)^{1/2}. \tag{3.31}$$

*A*nalysis of $v^\sharp_\cdot$

Next for $v^\sharp_F$ and $v^\sharp_A$, we use the fact that

$$(F'(\xi) - F'(\xi + \sigma(x)W)) \cdot \nabla\chi_v(\xi, x, s) = \nabla \cdot ((F'(\xi) - F'(\xi + \sigma(x)W))\chi_v(\xi, x, s))$$

$$
\begin{aligned}
&\qquad - (F''(\xi + \sigma(x)W) \cdot \nabla\sigma(x))\chi_v(\xi, x, s)W \\
(\mathbf{A}(\xi) - \mathbf{A}(\xi + \sigma(x)W) \cdot \nabla\chi_v(\xi, x, s) =& \nabla \cdot ((\mathbf{A}(\xi) - \mathbf{A}(\xi + \sigma(x)W))\chi_v(\xi, x, s)) \\
&- (\mathbf{A}'(\xi + \sigma(x)W) \cdot \nabla\sigma(x))\chi_v(\xi, x, s)W.
\end{aligned}
$$

In [30], $v^\sharp_\cdot$ were taken care of by velocity averaging methods. It seems we can apply the kernel estimates for the same purpose. Let $\varphi \in L^2$ be any test function. Let $\langle \cdot, \cdot \rangle$ be the pairing in L^2.

$$
\begin{aligned}
\langle v^\sharp_F(t), \varphi \rangle &= \iint \int_0^t \varphi \mathcal{S}(t-s) \nabla \cdot ((F'(\xi) - F'(\xi + \sigma(x)W))\chi_v(\xi, x, s)) \, ds \, d\xi \, dx \\
&\quad - \iint \int_0^t \varphi \mathcal{S}(t-s)((F''(\xi + \sigma(x)W) \cdot \nabla\sigma(x)W)\chi_v(\xi, x, s)) \, ds \, d\xi \, dx \\
&= \iint \int_0^t \nabla(\mathcal{S}^*(t-s)\varphi) \cdot ((F'(\xi) - F'(\xi + \sigma(x)W))\chi_v(\xi, x, s)) \, ds \, d\xi \, dx \\
&\quad - \iint \int_0^t \mathcal{S}^*(t-s)\varphi((F''(\xi + \sigma(x)W)) \cdot \nabla\sigma(x)W\chi_v(\xi, x, s)) \, ds \, d\xi \, dx
\end{aligned}
$$

And likewise,

$$
\begin{aligned}
\langle v^\sharp_A, \varphi \rangle &= \iint \int_0^t \varphi \mathcal{S}(t-s) \nabla^2 : ((\mathbf{A}(\xi) - \mathbf{A}(\xi + \sigma(x)W))\chi_v(\xi, x, s)) \, ds \, d\xi \, dx \\
&\quad - \iint \int_0^t \varphi \mathcal{S}(t-s) \nabla \cdot (\mathbf{A}'(\xi + \sigma(x)W) \cdot \nabla\sigma(x)\chi_v(\xi, x, s))W \, ds \, d\xi \, dx \\
&= \iint \int_0^t \nabla^2(\mathcal{S}^*(t-s)\varphi) : ((\mathbf{A}(\xi) - \mathbf{A}(\xi + \sigma(x)W))\chi_v(\xi, x, s)) \, ds \, d\xi \, dx \\
&\quad - \iint \int_0^t (\nabla(\mathcal{S}^*(t-s)\varphi) \otimes \nabla\sigma(x)W) \cdot (\mathbf{A}'(\xi + \sigma(x)W)\chi_v(\xi, x, s)) \, ds \, d\xi \, dx
\end{aligned}
$$

We have the estimates:

$$
\begin{aligned}
&\int_0^T \int_0^t \|\nabla(\mathcal{S}^*(t-s)\varphi) \cdot ((F'(\cdot) - F'(\cdot + \sigma(\cdot)W))\chi_v(\cdot, \cdot, s))\|_{L^1_{x,\xi}} \, ds \, dt \\
\le &\int_0^T \int_0^t \|\nabla(\mathcal{S}^*(t-s)\varphi)\|_{L^\infty_{x,\xi}} \|F'(\cdot) - F'(\cdot + \sigma(\cdot)W)\|_{L^\infty_{x,\xi}} \|\chi_v(\cdot, \cdot, s)\|_{L^1_{x,\xi}} \, ds \, dt \\
\le &\int_0^T \int_0^t \|\nabla\mathcal{S}^*(t-s)\|_{L^2 \to L^\infty} \|\varphi\|_{L^2} \|F'(\cdot) - F'(\cdot + \sigma(\cdot)W)\|_{L^\infty_{x,\xi}} \|\chi_v(\cdot, \cdot, s)\|_{L^1_{x,\xi}} \, ds \, dt
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_0^T \int_0^t e^{\theta(t-s)}(\gamma t)^{-(d+2)/(4\alpha)} \|v(s)\|_{L^1_x}\, ds\, dt \\
&\le c\eta \|\varphi\|_{L^2} \sup_{s\in[0,T]} \int_0^T e^{\theta(t-s)}(\gamma t)^{-(d+2)/(4\alpha)}\, dt \int_0^T \|v(s)\|_{L^1_x}\, ds \\
&\le c\eta \|\varphi\|_{L^2} \gamma^{-(d+2)/(4\alpha)} \theta^{(d+2)/(4\alpha)-1} \int_0^\infty e^{-t} t^{-(d+2)/(4\alpha)}\, dt \int_0^T \|v(s)\|_{L^1_x}\, ds; \\
&= c\eta \|\varphi\|_{L^2} \gamma^{-(d+2)/(4\alpha)} \theta^{(d+2)/(4\alpha)-1} |\Gamma(1-(d+2)/(4\alpha))| \int_0^T \|v(s)\|_{L^1_x}\, ds;
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^T \int_0^t \|\mathcal{S}^*(t-s)\varphi \nabla\sigma(x) W \cdot ((F''(\cdot + \sigma(\cdot)W))\chi_v(\cdot,\cdot,s))\|_{L^1_{x,\xi}}\, ds\, dt \\
&\le \int_0^T \int_0^t \|\mathcal{S}^*(t-s)\varphi\|_{L^\infty_{x,\xi}} \|F''(\cdot + \sigma(\cdot)W)\|_{L^\infty_{x,\xi}} \|\sigma W\|_{W^{1,\infty}} \|\chi_v(\cdot,\cdot,s)\|_{L^1_{x,\xi}}\, ds\, dt \\
&\le c\eta \|\varphi\|_{L^2} \gamma^{-d/(4\alpha)} \theta^{d/(4\alpha)-1} |\Gamma(1-d/(4\alpha))| \int_0^T \|v(s)\|_{L^1_x}\, ds;
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^T \int_0^t \|\nabla^2(\mathcal{S}^*(t-s)\varphi) \cdot ((\mathbf{A}(\cdot) - \mathbf{A}(\cdot + \sigma(\cdot)W))\chi_v(\cdot,\cdot,s))\|_{L^1_{x,\xi}} \\
&\le \int_0^T \int_0^t \|\nabla^2(\mathcal{S}^*(t-s)\varphi)\|_{L^\infty_{x,\xi}} \|\mathbf{A}(\cdot) - \mathbf{A}(\cdot + \sigma(\cdot)W)\|_{L^\infty_{x,\xi}} \|\chi_v(\cdot,\cdot,s)\|_{L^1_{x,\xi}} \\
&\le c\eta \|\varphi\|_{L^2} \gamma^{-(d+4)/(4\alpha)} \theta^{(d+4)/(4\alpha)-1} |\Gamma(1-(d+4)/(4\alpha))| \int_0^T \|v(s)\|_{L^1_x}\, ds
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^T \int_0^t \|(\nabla(\mathcal{S}^*(t-s)\varphi) \otimes \nabla\sigma(\cdot)W) : (\mathbf{A}'(\cdot + \sigma(\cdot)W))\chi_v(\cdot,\cdot,s)\|_{L^1_{x,\xi}} \\
&\le \int_0^T \int_0^t \|\nabla(\mathcal{S}^*(t-s)\varphi)\|_{L^\infty_{x,\xi}} \|\sigma W\|_{W^{1,\infty}} \|\mathbf{A}'(\cdot + \sigma(\cdot)W)\|_{L^\infty_{x,\xi}} \|\chi_v(\cdot,\cdot,s)\|_{L^1_{x,\xi}} \\
&\le c\eta \|\varphi\|_{L^2} \gamma^{-(d+2)/(4\alpha)} \theta^{(d+2)/(4\alpha)-1} |\Gamma(1-(d+2)/(4\alpha))| \int_0^T \|v(s)\|_{L^1_x}\, ds
\end{aligned}
$$

Now by (3.9) we assumed that

$$
|F''(\xi)| \lesssim 1, \qquad |\mathbf{A}'(\xi)| \lesssim 1,
$$

and we also had $\|\sigma(x)W\|_{W^{1,\infty}} \leq \eta$, so we used the estimates (the second from the first by the Poincaré-Wirtinger inequality, since $\int_{\mathbb{T}^d} \sigma \, dx = 0$),

$$
\begin{aligned}
|F''(\xi + \sigma(x)W) \cdot \nabla \sigma W| &\leq c\eta \qquad |\mathbf{A}'(\xi + \sigma(x)W) \cdot \nabla \sigma W| \leq c\eta \\
|F'(\xi) - F'(\xi + \sigma(x)W)| &\leq c\eta \qquad |\mathbf{A}(\xi) - \mathbf{A}(\xi + \sigma(x)W)| \leq c\eta.
\end{aligned}
$$

Putting these estimate back in to bound $\|v^\sharp_\cdot(t)\|_{L^2} = \sup_{\|\varphi\|_{L^2}=1} \langle v^\sharp_\cdot(t), \varphi \rangle$,

$$
\begin{aligned}
&\int_0^T \|v^\sharp_A(t) + v^\sharp_F(t)\|_{L^2} \, dt \\
\leq & c\eta \Big(\gamma^{-(d+2)/(4\alpha)} \theta^{(d+2)/(4\alpha)-1} |\Gamma(1 - (d+2)/(4\alpha))| \\
&+ \gamma^{-d/(4\alpha)} \theta^{d/(4\alpha)-1} |\Gamma(1 - d/(4\alpha))| \\
&+ \gamma^{-(d+4)/(4\alpha)} \theta^{(d+4)/(4\alpha)-1} |\Gamma(1 - (d+4)/(4\alpha))| \Big) \int_0^T \|v(t)\|_{L^1} \, dt
\end{aligned}
\tag{3.32}
$$

Analysis of M_F and M_2

For M_F and M_2, we use the kernel estimate and $\|\sigma W\|_{W^{1,\infty}} \leq \eta$ again. This gives us

$$
\begin{aligned}
\langle M_F, \varphi \rangle &\leq \int_0^t \|F'(v + \sigma W)\|_{L^1_x} \|\mathcal{S}\varphi\|_{L^\infty_x} \|\nabla \sigma W\|_{L^\infty_x} \, ds \\
\langle M_A, \varphi \rangle &\leq \int_0^t \|\mathbf{A}(v + \sigma W)\|_{L^1_x} \|\nabla(\mathcal{S}\varphi)\|_{L^\infty_x} \|\nabla \sigma W\|_{L^\infty_x} \, ds.
\end{aligned}
$$

Now, by (3.8),

$$
\begin{aligned}
\|F'(v + \sigma W)\|_{L^1} &\leq C(1 + \|v(t)\|_{L^1_x} + \|\sigma\|_{L^1_x} W) \\
\|\mathbf{A}(v + \sigma W)\|_{L^1} &\leq C(1 + \|v(t)\|_{L^1_x} + \|\sigma\|_{L^1_x} W)
\end{aligned}
$$

These give us

$$
\begin{aligned}
&\int_0^T \|M_F(t) + M_A(t)\|_{L^1} \, dt \\
\leq & C\eta \int_0^T \int_0^t (1 + \|v(s)\|_{L^1_x})(e^{-\theta(t-s)} + e^{-\theta(t-s)}(\gamma(t-s))^{-1/(2\alpha)}) \, ds \, dt
\end{aligned}
\tag{3.33}
$$

$$\leq C\eta(\theta^{-1} + \gamma^{-1/(2\alpha)}\theta^{1/(2\alpha)-1}|\Gamma(1-1/(2\alpha))|)\int_0^T (1 + \|v(s)\|_{L^1_x})\,ds$$

*A*nalysis of M_A

We have that

$$\langle M_2, \varphi\rangle = \int\int_0^t \partial_\xi(\mathcal{S}^*(t-s)\varphi)(v(x,s),x)\mathbf{A}(v+\sigma(x)W) : \nabla\sigma(x)W \otimes \nabla\sigma(x)W\;ds\,dx.$$

We use the fact, explained in (3.12), that

$$(\partial_\xi \mathcal{S}\varphi)(v(x,s),x) = (t-s)F''(v(x,s))\cdot\nabla(\mathcal{S}^*(t-s)\varphi) + \mathbf{A}'(v(x,s)) : \nabla^2(\mathcal{S}^*(t-s)\varphi).$$

By (3.9) we assumed

$$|F''(\xi)| \lesssim 1, \qquad |\mathbf{A}'(\xi)| \lesssim 1.$$

Again we also have

$$\|\mathbf{A}(v+\sigma(x)W)\|_{L^1} \leq C(1 + \|v(s)\|_{L^1} + \|\sigma\|_{L^1}W).$$

Finally using the kernel estimate,

$$\begin{aligned}
|\langle M_2,\varphi\rangle| \leq& C\int_0^t (t-s)\|F''\|_{L^\infty}\|\nabla S^*(t-s)\varphi\|_{L^\infty}\|\nabla\sigma W\|^2_{L^\infty}(1+\|v(s)\|_{L^1}+\|\sigma\|_{L^1}W)\;ds\\
&+ C\int_0^t (t-s)\|\mathbf{A}'\|_{L^\infty}\|\nabla^2 S^*(t-s)\varphi\|_{L^\infty}\|\nabla\sigma W\|^2_{L^\infty}(1+\|v(s)\|_{L^1}+\|\sigma\|_{L^1}W)\;ds\\
\leq& c\eta^2\int_0^t (t-s)\|\nabla S^*(t-s)\|_{L^\infty\to L^\infty}\|\varphi\|_{L^\infty}(1+\|v(s)\|_{L^1}+\|\sigma\|_{L^1}W)\;ds\\
&+ c\eta^2\int_0^t (t-s)\|\nabla^2 S^*(t-s)\|_{L^\infty\to hL^\infty}\|\varphi\|_{L^\infty}(1+\|v(s)\|_{L^1}+\|\sigma\|_{L^1}W)\;ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^T \|M_2(t)\|_{L^1}\;dt\\
\leq& c\eta^2\int_0^T\int_0^t (t-s)\|\nabla S^*(t-s)\|_{L^\infty\to L^\infty}(1+\|v(s)\|_{L^1}+\|\sigma\|_{L^1}W)\;ds\,dt
\end{aligned}$$

$$
\begin{aligned}
&+ c\eta^2 \int_0^T \int_0^t (t-s)\|\nabla^2 S^*(t-s)\|_{L^\infty \to hL^\infty}(1 + \|v(s)\|_{L^1} + \|\sigma\|_{L^1} W)\, ds\, dt \\
\le & c\eta^2 \int_0^T \int_0^t (t-s)(\gamma(t-s))^{-1/(2\alpha)} e^{-\theta(t-s)}(1 + \|v(s)\|_{L^1} + \|\sigma\|_{L^1} W)\, ds\, dt \\
&+ c\eta^2 \int_0^T \int_0^t (t-s)(\gamma(t-s))^{-1/\alpha} e^{-\theta(t-s)}(1 + \|v(s)\|_{L^1} + \|\sigma\|_{L^1} W)\, ds\, dt,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T \|M_2(t)\|_{L^1}\, dt \qquad (3.34) \\
\le & c\eta^2 \Big(\gamma^{-1/(2\alpha)}\theta^{1/(2\alpha)-2}|\Gamma(2 - 1/(2\alpha))| + \gamma^{-1/\alpha}\theta^{1/\alpha - 2}|\Gamma(2 - 1/\alpha)|\Big) \cdot \int_0^T (1 + \|v(s)\|_{L^1})\, ds.
\end{aligned}
$$

3.3.4 Analysis of M_1

Finally for the kinetic measure, M_1, we use the total variation estimate again.

First, with $\varphi \in L^\infty_x$,

$$
\begin{aligned}
|\langle M_1, \varphi\rangle| =& \left| \iint \int_0^t \int \partial_\xi (\mathcal{S}^*(t-s)\varphi) d(m_u + N_u)(\xi, x, s) \right| \\
=& \left| \iint \int_0^t (t-s) F''(\xi) \cdot \nabla(\mathcal{S}^*(t-s)\varphi) + \mathbf{A}'(\xi) : \nabla^2(\mathcal{S}^*(t-s)\varphi) d(m_u + N_u) \right| \\
\le & c\|\varphi\|_{L^\infty} \iint \int_0^t \gamma^{-1/(2\alpha)}(t-s)^{1-1/(2\alpha)} e^{-\theta(t-s)} d|m_u + N_u| \\
&+ c\|\varphi\|_{L^\infty} \iint \int_0^t \gamma^{-1/\alpha}(t-s)^{1-1/\alpha} e^{-\theta(t-s)} d|m_u + N_u| \\
\int_0^T \|M_1(t)\|_{L^1}\, dt \le & c\gamma^{-1/(2\alpha)}\theta^{1/(2\alpha)-2}|\Gamma(2 - 1/(2\alpha))| \int_0^T \iint d|m_u + N_u| \\
&+ c\gamma^{-1/\alpha}\theta^{1/\alpha-2}|\Gamma(2 - 1/\alpha)| \int_0^T \iint d|m_u + N_u|.
\end{aligned}
$$

Following [30], and as in Lemma 14 we test the equation (3.28) against ξ, to find

$$
\begin{aligned}
&\frac{1}{2}\|v(t)\|_{L^2}^2 + |m_u + N_u|(\mathbb{R} \times \mathbb{T}^d \times [0,t]) \\
\le & \frac{1}{2}\|\tilde{u}_0\|_{L^2}^2 + \left| \int_0^t \iint \xi(F'(\xi) - F'(\xi + \sigma(x)W)) \cdot \nabla \chi_v\, d\xi\, dx\, ds \right| \\
&+ \left| \int_0^t \int v F'(v + \sigma(x)W) \cdot \nabla\sigma(x) W\, dx\, ds \right|
\end{aligned}
$$

$$
\begin{aligned}
&+ \left| \int_0^t \int \mathbf{A}(v + \sigma(x)W) : \nabla\sigma W \otimes \nabla\sigma W \, dx\, ds \right| \\
\leq& \frac{1}{2}\|\tilde{u}_0\|_{L^2}^2 + \left| \int_0^t \iint \xi F''(\xi + \sigma(x)W)) \cdot \nabla(\sigma(x)W)\chi_v \, d\xi\, dx\, ds \right| \\
&+ c\eta \left| \int_0^t \int v(1 + |v| + |\sigma(x)W|) \, dx\, ds \right| \\
&+ c\eta^2 \left| \int_0^t \int (1 + |v| + |\sigma(x)W|) \, dx\, ds \right| \\
\leq& \frac{1}{2}\|\tilde{u}_0\|_{L^2}^2 + c(\eta + \eta^2) \int_0^t (1 + \|v(s)\|_{L^2}^2) \, ds.
\end{aligned}
$$

By Gronwall's inequality, we have

$$
|m_u + N_u| \leq Ce^{c\eta t}(\|\tilde{u}_0\|_{L^2}^2 + 1) \leq Ce^{c\eta t}(\kappa^2\varepsilon^{-d} + 1).
$$

Therefore,

$$
\begin{aligned}
&\int_0^T \|M_1(t)\|_{L^1} \, dt \qquad (3.35)\\
\leq& c(\gamma^{-1/(2\alpha)}\theta^{1/(2\alpha)-2}|\Gamma(2 - 1/(2\alpha))| + \gamma^{-1/\alpha}\theta^{1/\alpha-2}|\Gamma(2 - 1/\alpha)|)e^{c\eta T}(\kappa^2\varepsilon^{-d} + 1).
\end{aligned}
$$

Closing the Estimates

First we set $\alpha \leq 1/2$ so that the instances of $|\Gamma|$ are never evaluated at a negative integer, where it is infinite. Having ensured that all the values of $|\Gamma|$ are non-infinte, and there being finitely many instances of Γ in the estimates (3.30) - (3.35) above, we can write those estimates as

$$
\begin{aligned}
\fint_0^T \|v^0(t)\|_{L^1} \, dt \leq& CT^{-1/2}\gamma^{r/2}\|u_0\|_{L^1}^{1/2} \\
\fint_0^T \|v^\flat(t)\|_{L^1} \, dt \leq& C\gamma^{(r+1)/2} \left(\fint_0^T \|v\|_{L^1} \, dt \right)^{1/2} \leq C\gamma^{(r+1)/2} \fint_0^T (1 + \|v\|_{L^1}) \, dt, \\
\fint_0^T \|v_F^\sharp + v_A^\sharp\|_{L^1} \, dt \leq& C\eta\big(\gamma^{-(d+2)/(4\alpha)}\theta^{(d+2)/(4\alpha)-1} \\
&+ \gamma^{-d/(4\alpha)}\theta^{d/(4\alpha)-1} + \gamma^{-(d+4)/(4\alpha)}\theta^{(d+4)/(4\alpha)-1}\big) \fint_0^T \|v(t)\|_{L^1} \, dt, \\
\fint_0^T \|M_F(t) + M_A(t)\|_{L^1} \, dt \leq& C\eta \left(\theta^{-1} + \gamma^{-1/(2\alpha)}\theta^{1/(2\alpha)-1} \right) \fint_0^T (1 + \|v(t)\|_{L^1}) \, dt,
\end{aligned}
$$

$$\fint_0^T \|M_2(t)\|_{L^1}\, dt \le C\eta^2 \left(\gamma^{-1/(2\alpha)}\theta^{1/(2\alpha)-2} + \gamma^{-1/\alpha}\theta^{1/\alpha-2}\right) \fint_0^T (1+\|v(t)\|_{L^1})\, dt,$$
$$\fint_0^T \|M_1\|_{L^1}\, dt \le \frac{C}{T}\left(\gamma^{-1/(2\alpha)}\theta^{1/(2\alpha)-2} + \gamma^{-1/\alpha}\theta^{1/\alpha-2}\right) e^{c\eta T}(\kappa^2\varepsilon^{-d}+1).$$

Together we have

$$\begin{aligned}\fint_0^T \|v(t)\|_{L^1}\, dt \le & C_0 T^{-1/2}\gamma^{r/2}\|u_0\|_{L^1}^{1/2} + C_1\gamma^{(r+1)/2}\fint_0^T (1+\|v\|_{L^1})\, dt \\ & + C_2(\gamma,\theta)(\eta+\eta^2)\left(1+\fint_0^T \|v(s)\|_{L^1}\, ds\right) + C_3(\gamma,\theta)T^{-1}e^{c\eta T}(\kappa^2\varepsilon^{-d}+1).\end{aligned}$$

We can choose γ, θ, T, and η in that order so that first, so that, for some q to be determined,

$$C_1\gamma^{(r+1)/2} \le q\varepsilon.$$

For $\alpha < 1/4$, we see that every θ has positive power above except int the estimate of $\|M_F + M_A\|_{L^1_{t,x}}$. So we choose θ such that

$$C_2(\gamma,\theta) < 1.$$

Next we choose T such that

$$C_0 T^{-1/2}\|u_0\|_{L^1}^{1/2} + C_3(\gamma,\theta)T^{-1}(\kappa^2\varepsilon^{-d}+1) \le q\varepsilon.$$

Finally we choose η such that

$$C_2(\gamma,\theta)(\eta+\eta^2) \le q\varepsilon,$$

and,

$$c\eta T \le q\varepsilon.$$

By taking q sufficiently small, we have

$$\fint_0^T \|v(t)\|_{L^1} dt \le \frac{\varepsilon}{4}, \qquad \fint_0^T \|\tilde{u}(t)\|_{L^1}\, dt \le \frac{3\varepsilon}{8},$$

and finally,

$$\fint_0^T \|u(t)\|_{L^1}\, dt \le \varepsilon/2.$$

This proves Lemma 17.

3.3.5 Uniqueness III: Conclusion

We follow [30] quite closely again in the conclusion.

Let u_0^1 and u_0^2 be in L_x^1 and for a given $\varepsilon > 0$, let $\tilde{u}_0^1$ and $\tilde{u}_0^2$ in L_x^3 such that $\|u_0^i - \tilde{u}_0^i\|_{L_x^1} \le \varepsilon/4$. Denote their corresponding solutions by, $u^1, u^2, \tilde{u}^1$, and $\tilde{u}^2$, respectively. Let us now put $\tilde{u}^1$ and $\tilde{u}^2$ in place of u and v in §3.3.2, and associate to them the sequence of stopping times constructed recursively in (3.27),

$$\mathcal{T}_l = \inf\{t \ge \mathcal{T}_{l-1} + T : \|\tilde{u}^1(t)\|_{L^1} + \|\tilde{u}^2(t)\|_{L^1} \le 2\kappa\}$$

By choosing T and η as above, we have by L^1 contraction (for additive noise there is almost sure L^1-contraction)

$$\begin{aligned} &\mathbb{P}\left(\fint_{\mathcal{T}_l}^{\mathcal{T}_l+T} \|u^1(s) - u^2(s)\|_{L_x^1}\, ds \le \varepsilon \middle| \mathscr{F}_{\mathcal{T}_l}\right) \\ \ge &\mathbb{P}\left(\fint_{\mathcal{T}_l}^{\mathcal{T}_l+T} \|\tilde{u}^1(s) - \tilde{u}^2(s)\|_{L^1}\, ds \le \varepsilon/2 \middle| \mathscr{F}_{\mathcal{T}_l}\right) \\ \ge &\mathbb{P}\left(\sup_{t\in[\mathcal{T}_l,\mathcal{T}_l+T]} \|\sigma W(t) - \sigma W(\mathcal{T}_l)\|_{W_x^{1,\infty}} \le \eta \middle| \mathscr{F}_{\mathcal{T}_l}\right). \end{aligned}$$

Since $\eta > 0$ and σ is Lipschitz, we can denote the positive probability of the event as λ. By the strong Markov property, we know it does not change with l.

This allows us to write

$$\mathbb{P}\left(\fint_{\mathcal{T}_l}^{\mathcal{T}_l+T} \|u^1(s) - u^2(s)\|_{L_x^1}\, ds \ge \varepsilon \text{ for } l = l_0, l_0+1, \dots, l_0+k\right) \le (1-\lambda)^k,$$

so that

$$\mathbb{P}\left(\lim_{l\to\infty} \fint_{\mathcal{T}_l}^{\mathcal{T}_l+T} \|u^1(s) - u^2(s)\|_{L_x^1}\, ds \ge \varepsilon\right)$$

$$
\begin{aligned}
=&\mathbb{P}\left(\exists l_0 \forall l \geq l_0 : \fint_{\mathcal{T}_l}^{\mathcal{T}_l+T} \|u^1(s) - u^2(s)\|_{L^1_x} \, ds \geq \varepsilon\right) \\
=&0
\end{aligned}
$$

This limit exists as $t \mapsto \|u^1(s) - u^2(s)\|_{L^1_x}$ is non-increasing, by the L^1 contraction property. And by this same property,

$$
\mathbb{P}\left(\lim_{t\to\infty} \|u^1(t) - u^2(t)\|_{L^1_x} \geq \varepsilon\right) = 0.
$$

Therefore, almost surely,

$$
\lim_{t\to\infty} \|u^1(t) - u^2(t)\|_{L^1_x} = 0,
$$

which implies the uniqueness of the invariant measure.

Chapter 4

Remarks on various cases of multiplicative noises

In this Chapter we remark on the case of multiplicative noises. In the case of general multiplicative noises, with general initial data, there is no bound on the spatial average. As we saw in §3.2 , this creates a major hurdle for anlayses. McKean[84], Øksendal, Våge, and Zhao [69, 70], Chueshov and Vuillermot [20], among others, studied the KPP equation, which is:

$$\begin{aligned} du(x,t) =& \mathrm{div}(k(x,t)\nabla u(x,t) + sg(u))\ dt + g(u)dW \\ u(x,0) =& \varphi(x) \in (u_0, u_1), \end{aligned}$$

where $\mathrm{div}(k\nabla)$ is uniformly elliptic, W is a standard Brownian motion, and while $g : \mathbb{R} \to \mathbb{R}$ has roots u_0 and u_1, and is positive, twice continuously differentiable between these roots. This keeps the spatial average within the range (u_0, u_1), and the archetypal noise of such form is $g(u) = u(1-u)$. [20] in particular derived different stability conditions for u_0 and u_1 as attracting or repelling (stable or unstable) fixed points as s varies.

Here we derive a theory for the long-time behaviour of solutions to equations

$$\partial_t u = -\nabla \cdot F(u) + (cu + \lambda)\partial_t W$$

$$\partial_t u = -\nabla \cdot F(u) + \nabla \cdot (\mathbf{A}(u) \cdot \nabla u) + (cu + \lambda)\partial_t W$$

with a class of unbounded, Lipschitz noise with one root, without any constraint on initial conditions. Here as in previous chapters, $\mathbf{A}$ is positive semi-definite, and W is a standard Brownian motion. A well-posedness theory for these equations with Hölder continuous coefficients F and $\mathbf{A}$ with polynomial growth can be found in [28]. The evolution occurs on $\mathbb{T}^d$.

We show that the root is always an attracting fixed point, using methods that depart from both Debussche and Vovelle in [30], and the authors named in the previous paragraph. Of course, another difference is that we are working with degenerate parabolic (or first order) conservation laws. We do this in two steps. First we consider linear noises in §4.1. After that we consider Lipschitz noises that are close to a linear noise near the root of the linear noise §4.2.

4.1 Unbounded Noise: the Multiplicative Linear Case

First we consider the linear case $\sigma(u) = cu + \lambda$ in which c and λ are deterministic (i.e., do not depend on the variable ω). Recall that the chief barrier to tackling multiplicative equations was the possibility that their spatial averages were uncontrollable, and may increase without bound. This leads us to investigate the behaviour of the spatial average. We shall call the difference between a function and its spatial average the "spatial fluctuation". Whether for the first or the second order equation

$$\partial_t u = -\nabla \cdot F(u) + (cu + \lambda)\, \partial_t W, \tag{4.1}$$

or

$$\partial_t u = -\nabla \cdot F(u) + \nabla \cdot (\mathbf{A}(u) \cdot \nabla u) + (cu + \lambda)\, \partial_t W, \tag{4.2}$$

in the linear multiplicative noies case, with $c, \lambda \in \mathbb{R}$, and $c \neq 0$, we have the following calculation for the "spatial average":

Since $\partial_t u = \partial_t(u + \lambda/c)$ integrating through leaves one with,

$$\frac{1}{c}\partial_t \int u + \frac{\lambda}{c}\, dx = \int u + \frac{\lambda}{c}\, dx\; \partial_t W,$$

Writing $\int u + \lambda/c \, dx$ as $X^{(1)}$, we have,

$$dX^{(1)} = cX^{(1)} \, dW,$$

which can be solved – the solution being,

$$X_t^{(1)} = X_0^{(1)} \exp(cW - c^2 t/2).$$

We then can make two observations. First, the sign of $X^{(1)}$ does not change. Secondly, by the law of the iterated logarithm, the Brownian motion satisfies [60] (Theorem 2.9.23, and first proven in [65]),

$$\limsup_{t \to \infty} \frac{|W(t)|}{\sqrt{2t \log \log(t)}} = 1 \quad a.s.,$$

which implies that almost surely, the spatial average of $X^{(1)} \to 0$ as tends $t \to \infty$. Of course, the strong law of large numbers also already implies such a limit.

Remark. In fact for the above calculation, we could have considered the even more general form,

$$\partial_t u + \mathrm{div} A(u, \nabla u) = u \, \partial_t W.$$

We further have the theorem:

Theorem 18. *Let F and $\mathbf{A}$ be Hölder continuous with polynomial growth, and let the Hölder index γ of $\mathbf{A}$ satisfy $2\gamma > 1$. Let $\sigma(u) = cu + \lambda$. Let p be a positive even integer. Then a kinetic solution to,*

$$\partial_t u = -\nabla \cdot F(u) + \nabla \cdot (\mathbf{A}(u) \cdot \nabla u) + \sigma(u) \partial_t W$$

exhibits the following long-time decay – almost surely,

$$\int \left| u + \frac{\lambda}{c} \right|^p \, dx \to 0.$$

The first-order equation is a special case of this theorem.

Proof. Specialising to the case of the two equations (4.1, 4.2), because we have a handle on the sign on the defect measures arising from the kinetic (and parabolic) dissipative effects, for any *even* moment, multiplying the equation through by $p(u-\lambda/c)^{p-1}$, we have

$$
\begin{aligned}
d\int\left(u+\frac{\lambda}{c}\right)^p\,dx =&cp\int\left(u+\frac{\lambda}{c}\right)^p\,dx\,dW+\frac{c^2}{2}p(p-1)\int\left(u+\frac{\lambda}{c}\right)^p\,dx\,dt\\
&+\iint p(\xi+\lambda/c)^{p-1}\partial_\xi m_u(d\xi,dx,t)\,dt\\
=&cp\int\left(u+\frac{\lambda}{c}\right)^p\,dx\,dW+\frac{c^2}{2}p(p-1)\int\left(u+\frac{\lambda}{c}\right)^p\,dx\,dt\\
&-\iint p(p-1)(\xi+\lambda/c)^{p-2}(m_u+n_u)(d\xi,dx,t)\,dt,
\end{aligned}
$$

where m_u is the non-negative kinetic measure that captures dissipative effects from shocks, and n_u is the non-negative parabolic defect measure that captures parabolic dissipation. This is effectively the geometric Brownian motion.

We shall use $D\geq 0$ to denote the non-negative quantity

$$
D=\iint p(p-1)(\xi+\lambda/c)^{p-2}(m_u+n_u)(d\xi,dx,t).
$$

Calling $X^{(p)}=\int(u+\lambda/c)^p\,dx$, we have

$$
\begin{aligned}
dX^{(p)} =&cpX^{(p)}dW+\frac{c^2}{2}p(p-1)X^{(p)}\,dt-D\,dt\\
d\log(X^{(p)})\leq&-\frac{c^2}{2}p\,dt+cp\,dW.
\end{aligned}
$$

This leads to

$$
X_t^{(p)}\leq X_0^{(p)}\exp(p(cW-c^2t/2)). \tag{4.3}
$$

And so we see that by integrating away the spatial variable, we transformed an SPDE into an SDE.

For $u_0\in L^2$, we have that, $u_0+\lambda/c\in L^2$ also, and

$$
\begin{aligned}
&⨍\left(\left(u+\frac{\lambda}{c}\right)-⨍ u+\frac{\lambda}{c}\,dy\right)^2\,dx+\left(⨍ u+\frac{\lambda}{c}\,dx\right)^2\\
=&⨍\left(u+\frac{\lambda}{c}\right)^2\,dx
\end{aligned}
$$

$$=\frac{1}{\mu(\mathbb{T}^d)}X_t^{(2)} \to 0,$$

the last limit holding again by the law of iterated logarithms.

□

This shows that both the spatial average and the spatial fluctuations (of $u + \lambda/c$) tend to zero. Any initial condition $u_0 \in L^1$ can be approximated by an L^2-valued random variable v_0 in the L^1 norm, and the L^1 contraction formula suggests that

$$\mathbb{E}(\|u(t) - v(t)\|_{L^1}) \leq \mathbb{E}(\|u_0 - v_0\|_{L^1}).$$

This shows that for $u_0 \in L^1$ with $v_0 \in L^2$ so that $\mathbb{E}(\|u_0 - v_0\|_{L^1}) \leq \varepsilon$, one has

$$\begin{aligned} \mathbb{E}(\|u(t) + \lambda/c\|_{L^1}) &\leq \mathbb{E}(\|u(t) - v(t)\|_{L^1} + \|v(t) + \lambda/c\|_{L^1}) \\ &\leq \varepsilon + \mu^{1/2}(\mathbb{T}^d)\mathbb{E}(\|v(t) + \lambda/c\|_{L^2}) \to \varepsilon, \end{aligned} \tag{4.4}$$

as $t \to \infty$, and ε is arbitrarily small.

From (4.3) we can derive a rate for the decay – again, this is a small variation of the geometric Brownian motion. Since p is even, we have that $X_0^{(p)} > 0$. By Jensen's inequality on the convex function $\exp(\cdot)$, and the fact that martingales have zero expectation, we have

$$0 \leq \mathbb{E}(X_t^{(p)})) \leq \mathbb{E}(X_0^{(p)} \exp(cW_t)) \exp(-c^2pt/2)). \tag{4.5}$$

Now we know the distribution of W, and in fact this decay provides a much better description of long-time behaviour than do invariant measures. Nevertheless, we prove the uniqueness of invariant measures to this and similar equations with different noises in §4.3.3 below.

Remark. Allowing c and λ to be random variables, we still have the formula

$$X_t^{(p)} \leq X_0^{(p)} \exp(p(cW - c^2t/2)),$$

and still, almost surely, $X^{(p)} \to 0$. But the rate depends on the distributions of c and λ.

4.1.1 Estimate on the relative sizes of $m_u + n_u$ and p_u

This decay allows us to derive an estimate on the size of $m_u+n_u-p_u$, which is the sum of the kinetic and parabolic defect measures, less the Itô correction. In [30], the authors derived a total variation bound as adapted in Lemma 14 above, but it did not take into account the cancellation effects in the difference between the non-negative measures $m_u + n_u$, the kinetic and parabolic defect measures, and p_u, the Itô correction. We can provide a better estimate here in the case of a multiplicative linear noise.

By a construction similar to 2.22, let $\tilde{\eta}_\rho$ be a smooth, convex approximation to the absolute value function on $\mathbb{R}$, such that $\tilde{\eta}_\rho(r) = |r|$ outside the ball $B_\rho(r)$. Convexity necessarily means that $\tilde{\eta}_\rho(r) \geq |r|$. We also have the inequality that $\tilde{\eta}_\rho(r) \leq |r| + \rho$. Therefore

$$\mathbb{E}(\|\tilde{\eta}_\rho(u)\|_{L^1}) \leq \mathbb{E}(\|u\|_{L^1}) + \rho\mu(\mathbb{T}^d).$$

Keeping in mind the L^1 bound above, we also have,

$$\mathbb{E}(\|\tilde{\eta}_\rho(u(t))\|_{L^1}) = \mathbb{E}(\|\tilde{\eta}_\rho(u_0)\|_{L^1}) + \mathbb{E}\left(\frac{1}{2}\int_0^t \int \tilde{\eta}_\rho''(u)\sigma^2(u)\ dx\, ds\right) + \text{non-positive dissipation.}$$

Now using $\|u\|_{L^1} \leq \|u - \lambda/c\|_{L^1} + \lambda/c\mu(\mathbb{T}^d)$, we have,

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{2}\int_0^t \int \tilde{\eta}_\rho''(u)\sigma^2(u)\ dx\, ds\right) - \mathbb{E}\left(\int_0^t \iint \tilde{\eta}_\rho''(\xi)(m_u+n_u)(d\xi, dx, ds)\right) \\
\leq &\mathbb{E}(\|u+\lambda/c\|_{L^1}) + (\lambda/c+\rho)\mu(\mathbb{T}^d) - \mathbb{E}(\|\tilde{\eta}_\rho(u_0)\|_{L^1}).
\end{aligned}$$

Now the first summand on the last line decays to an arbitrarily small quantity in enough time. So for sufficiently small ρ, and if $\lambda/c \leq 0$ (that is, the root of $\sigma(u) = cu+\lambda$ is non-positive), we see that the kinetic dissipation and parabolic defect measures eventually overcome the Itô correction.

4.2 Sublinear Unbounded Noises

Let v_1 be a solution to the equation with noise $\tau_1(v_1) = c_1 v_1 + \lambda_1$; and let v_2 be a solution to the equation with noise $\tau_2(v_2) = c_2 v_2 + \lambda_2$, such that $c_1/\lambda_1 = c_2/\lambda_2 = -R$ – that is, τ_1 and τ_2 have the same root.

The triangle inequality implies

$$\|v_1 - v_2\|_{L^1} \leq \|v_1 - R\|_{L^1} + \|v_2 - R\|_{L^1}. \tag{4.6}$$

From (4.4), we see that $\mathbb{E}(\|v_1 - v_2\|_{L^1})$ becomes arbitrarily small asymptotically in time. This is of course much better than can be expected from the simple continuous dependence result of the Second Chapter.

This motivates us to explore scenarios for which the noise $\sigma(u)$ is almost linear, perhaps between two linear noises with the same root. We know that this approximately linear noise cannot have more than one root if we are to see decay, and where there are multiple roots, we expect that with an initial condition taking values between two roots, we revert to a situation much like noises of the form $\sigma(u) = u(1-u)$, which was studied intensively [20].

Let us first consider almost linear noises, writing

$$\partial_t u = -\nabla \cdot F(u) + \nabla \cdot (\mathbf{A}(u) \cdot \nabla u) + \sigma(u)\partial_t W,$$

We think of $\sigma(u)$ as a perturbation of $cu + \lambda$, and use the previously established name $R = -\lambda/c$. By almost linear, we mean the conditions (4.7) and (4.8) given below, which we give in the context in which they arise.

Carrying out the calculations as before, we have

$$\begin{aligned}
&d\left(\int (u-R)^p \, dx\right) \\
=&p\int (u-R)^{p-1}\sigma(u)\, dxdW + \frac{1}{2}p(p-1)\int (u-R)^{p-2}\sigma^2(u)\, dx\, dt \\
&- p(p-1)\iint (\xi - R)^{p-2}(m_u + n_u)(d\xi, dx)\, dt \\
=&cp\int (u-R)^p \, dxdW + \frac{c^2}{2}p(p-1)\int (u-R)^p \, dx\, dt \\
&+ cp\int (u-R)^{p-1}\left(\frac{\sigma(u)}{c} - (u-R)\right) dxdW \\
&+ \frac{c^2}{2}p(p-1)\int (u-R)^{p-2}\left(\frac{\sigma^2(u)}{c^2} - (u-R)^2\right) dx\, dt \\
&- p(p-1)\iint (\xi - R)^{p-2}(m_u + n_u)(d\xi, dx)\, dt.
\end{aligned}$$

Suppose both that

$$|\sigma(u)/c| \leq |u - R|, \tag{4.7}$$

and,

$$|\sigma(u)/c - (u - R)| \leq (1 - \varepsilon)|u - R|. \tag{4.8}$$

These two conditions respectively ensure that for p even,

$$\frac{c^2}{2} p(p-1) \int (u - R)^{p-2} \left(\frac{\sigma^2(u)}{c^2} - (u - R)^2 \right) \, dx \leq 0,$$

and,

$$\left| \frac{1}{X^{(p)}} \int (u - R)^{p-1} \left(\frac{\sigma(u)}{c} - (u - R) \right) \, dx \right| < 1 - \varepsilon,$$

where again we have used the old notation,

$$X^{(p)} = \int (u - R)^p \, dx.$$

The two conditions jointly ensure that,

$$\operatorname{sgn}(\sigma(u)/c) = \operatorname{sgn}(u - R).$$

We also point out the much more important fact that by the two conditions, σ is bounded between $cu + \lambda$ and $c_1 u - c_1 R$, where $\operatorname{sgn}(c_1) = \operatorname{sgn}(c)$.

The noise we condsider is the following:

We shall now prove the theorem:

Theorem 19. *Let F and $\mathbf{A}$ be Hölder continuous with polynomial growth in their arguments, and let the Hölder index γ of $\mathbf{A}$ satisfy $2\gamma > 1$ Suppose there are real numbers λ, c with $|c| > 0$ for which $\sigma : \mathcal{R} \to \mathcal{R}$ is Lipschitz, and satisfies (4.7) and (4.8) with fixed, even p, and*

$$\varepsilon = \left(1 - \frac{1}{2\sqrt{p}} \right).$$

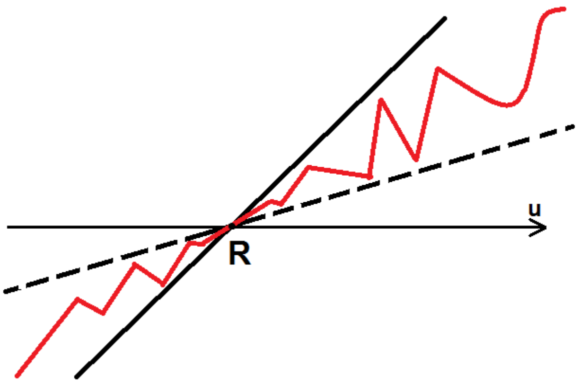

Figure 4.1: red line: admissible noise, solid black line: $cu + \lambda$, broken black line: $\varepsilon(cu + \lambda)$, line with arrowhead: horizontal axis.

Then a kinetic solution to

$$\partial_t u = -\nabla \cdot F(u) + \nabla \cdot (\mathbf{A}(u) \cdot \nabla u) + \sigma(u)\partial_t W$$

exhibits the following long-time decay

$$\int |u - R|^p \, dx \to 0 \qquad a.s..$$

Proof. Seeking again an expression for $d\log(X^{(p)})$ leads us to

$$\begin{aligned}
d\log(X^{(p)}) \leq& cp\, dW + \frac{c^2}{2}p(p-1)\, dt \\
&+ cp\frac{1}{X^{(p)}}\int (u-R)^{p-1}\left(\frac{\sigma(u)}{c} - (u-R)\right)\, dx\, dW \\
&+ \frac{c^2 p(p-1)}{2X^{(p)}}\int (u-R)^{p-2}\left(\frac{\sigma^2(u)}{c^2} - (u-R)^2\right)\, dx\, dt \\
&- \frac{c^2p^2}{2}\frac{1}{(X^{(p)})^2}\left(X^{(p)} + \int (u-R)^{p-1}\left(\frac{\sigma(u)}{c} - (u-R)\right)\, dx\right)^2 dt \\
=& cp\, dW - \frac{c^2}{2}p\, dt \qquad\qquad (4.9) \\
&+ cp\frac{1}{X^{(p)}}\int (u-R)^{p-1}\left(\frac{\sigma(u)}{c} - (u-R)\right)\, dx\, dW \\
&+ \frac{c^2 p(p-1)}{2X^{(p)}}\int (u-R)^{p-1}\left(\frac{\sigma(u)}{c} - (u-R)\right)\frac{\sigma(u)/c + (u-R)}{(u-R)}\, dx\, dt
\end{aligned}$$

$$
\begin{aligned}
&- \frac{c^2p^2}{2}\frac{2}{X^{(p)}}\int (u-R)^{p-1}\left(\frac{\sigma(u)}{c}-(u-R)\right)\,dx\,dt \\
&- \frac{c^2p^2}{2}\frac{1}{(X^{(p))})^2}\left(\int (u-R)^{p-1}\left(\frac{\sigma(u)}{c}-(u-R)\right)\,dx\right)^2 dt.
\end{aligned}
$$

By (4.7) and (4.8), the quantity below is bounded and signed, respectively,

$$
1+\varepsilon \leq \frac{\sigma(u)/c+(u-R)}{(u-R)} < \infty.
$$

The calculation above compels us to investigate the quantity

$$
\begin{aligned}
&\frac{c^2p(p-1)}{2X^{(p)}}\int (u-R)^{p-1}\left(\frac{\sigma(u)}{c}-(u-R)\right)\frac{\sigma(u)/c+(u-R)}{(u-R)}\,dx \\
&- \frac{c^2p^2}{2}2\frac{1}{X^{(p)}}\int (u-R)^{p-1}\left(\frac{\sigma(u)}{c}-(u-R)\right)\,dx \\
&- \frac{c^2p^2}{2}\frac{1}{(X^{(p))})^2}\left(\int (u-R)^{p-1}\left(\frac{\sigma(u)}{c}-(u-R)\right)\,dx\right)^2 \\
\leq& \frac{c^2p(p-1)(1+\varepsilon)}{2X^{(p)}}\int (u-R)^{p-1}\left(\frac{\sigma(u)}{c}-(u-R)\right)\,dx \\
&- \frac{c^2p^2}{2}2\frac{1}{X^{(p)}}\int (u-R)^{p-1}\left(\frac{\sigma(u)}{c}-(u-R)\right)\,dx \\
&- \frac{c^2p^2}{2}\frac{1}{(X^{(p))})^2}\left(\int (u-R)^{p-1}\left(\frac{\sigma(u)}{c}-(u-R)\right)\,dx\right)^2.
\end{aligned}
$$

Writing

$$
Y = \frac{1}{X^{(p))}}\left(\int (u-R)^{p-1}\left(\frac{\sigma(u)}{c}-(u-R)\right)\,dx\right),
$$

we seek to optimize,

$$
\frac{c^2p^2}{2}\frac{(p-1)(1+\varepsilon)}{p}Y - \frac{c^2p^2}{2}(2Y) - \frac{c^2p^2}{2}Y^2
$$

over $Y \in [-1,0]$, and use the the variable ε of which we have control to bound the optimized expression. This expression is optimized at,

$$
Y = \frac{(1+\varepsilon)(p-1)}{2p} - 1,
$$

where the expression has the value,

$$\frac{c^2p^2}{2}\left(\frac{(1+\varepsilon)(p-1)}{2p}-1\right)^2.$$

In view of the drift term in (4.9), we require that

$$\frac{c^2p^2}{2}\left(\frac{(1+\varepsilon)(p-1)}{2p}-1\right)^2 \leq \frac{c^2p}{2}(1-\delta)$$

for some $\delta > 0$.

By choosing,

$$\varepsilon = \left(1-\frac{1}{2\sqrt{p}}\right),$$

we have the bound above.

This gives us

$$\begin{aligned} d\log(X^{(p)}) \leq & cpdW - \frac{c^2p}{2}\delta\ dt \\ & + cp\frac{1}{X^{(p)}}\int (u-R)^{p-1}\left(\frac{\sigma(u)}{c}-(u-R)\right)dW. \end{aligned} \tag{4.10}$$

Since

$$0 \geq Y = \frac{1}{X^{(p)}}\int (u-R)^{p-1}\left(\frac{\sigma(u)}{c}-(u-R)\right)\ dx \geq -(1-\varepsilon)$$

is bounded, integrating (4.10) again allows us to conclude that almost surely,

$$X_t^{(p)} \to 0.$$

Of course, this hinges on the pathwise property of $\int_0^t Y\ dW$. Whilst it may be some work to prove an interated logarithm theorem for the stochastic integral, we can see that, by the Burkhölder-Davis-Gundy inequality,

$$\mathbb{E}\left(\left|\int_0^t Y\ dW\right|\right) \leq \mathbb{E}^{1/2}\left(\int_0^t |Y|^2\ dt\right) \leq \|Y\|_{L^\infty_{\Omega,t}}\sqrt{t}.$$

So suppose there exists a set $A \subseteq \Omega$ with $\mathbb{P}(A) > 0$ on which the limit superior

$$\limsup_{t\to\infty} \frac{\left|\int_0^t Y_s \, dW_s\right|}{\|Y\|_{L^\infty_{\Omega,t}} t^{1/2+0}}$$

is non-zero. Then the Burkhölder-Davis-Gundy inequality would be violated. $\square$

We now make a few remarks. The Lipschitzness of σ is required for the existence and uniqueness of solutions to the stochastic hyperbolic-parabolic equation. This can be relaxed to a Hölder type continuity (see Chapter 2 or [28]), but since σ lies between two straight lines, at the root, σ must be strictly Lipschitz.

As mentioned previously, the first-order equation is a special case of this theorem. Also, Theorem 18 is a direct corollary of this theorem. In the cases of the piecewise linear noises

$$\sigma(u) = \begin{cases} c_1 u + \lambda_1 & u \le R \\ c_2 u + \lambda_2 & u > R \end{cases},$$

where c_1 and c_2 do not differ too much, is also a corollary of this theorem. Additionally, this theorem frees the root from the constraint of being at the meeting of the two linear pieces.

In particular, we have the corollary:

Corollary 20. *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be given by*

$$\sigma(u) = \begin{cases} c_1 u + \lambda_1 & u \le M \\ c_2 u + \lambda_2 & u > M \end{cases},$$

with

$$M = -\frac{(\lambda_1 - \lambda_2)}{(c_1 - c_2)}.$$

Let R be the single root of σ. We require that

$$|c_1| \ge |c_2| \leftrightarrow R \le M.$$

Suppose finally that

$$\frac{1}{(2+\varepsilon)p} \leq \frac{c_1}{c_2} \leq (2+\varepsilon)p.$$

Then, almost surely,

$$\int |u-R|^p \, dx \to 0.$$

We can derive an expression for the rate at which $\mathbb{E}(X_t^{(p)})$ tends to zero. From (4.10), it holds that

$$\mathbb{E}(X_t^{(p)}) = O\left[\exp\left(-\frac{c^2 p\delta}{2}t + O(\sqrt{2t\log\log(t)})\right)\right].$$

4.2.1 Piecewise multiplicative noise

The equation 4.6, as well as the corollary following the previous proof also inspires us to consider noises that are made of two linear segments, meeting at the point $u = R$. This choice of having the meeting point of $\tau_1(u) = c_1 u + \lambda_1$ and $\tau_2(u) = c_2 u + \lambda_2$ at their common root is so that we do not have multiple zeros, which introduces other phenomena, as an initial state at one root will never flow away from that root. However, as we show here, our techniques are unable to show a similar kind of decay for noises of the form

$$\sigma(u) = \begin{cases} c_1 u + \lambda_1 & u \leq R \\ c_2 u + \lambda_2 & u > R \end{cases}.$$

where $R = -\lambda_1/c_1 = -\lambda_2/c_2$, and $\mathrm{sgn}(c_1) \neq \mathrm{sgn}(c_2)$.

Demonstration. Again, by the Itô formula, we have

$$\begin{aligned}\partial_t(u-R)^p =& p(u-R)^{p-1}\nabla\cdot(F(u) - \mathbf{A}(u)\cdot\nabla u) \\ &+ p(u-R)^{p-1}(c_1(u-R)\chi_{\{u<R\}} + c_2(u-R)\chi_{\{u>R\}})\partial_t W \\ &+ \frac{p(p-1)}{2}(u-R)^{p-2}(c_1(u-R)\chi_{\{u<R\}} + c_2(u-R)\chi_{\{u>R\}})^2 \\ &- \int p(p-1)(\xi-R)^{p-2}(m_u+n_u)\, d\xi.\end{aligned}$$

Owing to the orthogonality (or mutual singularity) of $\chi_{\{u<R\}}$ and $\chi_{\{u>R\}}$, one can write

$$(c_1(u-R)\chi_{\{u<R\}} + c_2(u-R)\chi_{\{u>R\}})^2 = c_1^2(u-R)\chi_{\{u<R\}} + c_2^2(u-R)\chi_{\{u>R\}},$$

and,

$$\begin{aligned}
&(u-R)^{p-1}(c_1(u-R)\chi_{\{u<R\}} + c_2(u-R)\chi_{\{u>R\}}) \\
=&(c_1(u-R)^p\chi_{\{u<R\}} + c_2(u-R)^p\chi_{\{u>R\}}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\partial_t \int (u-R)^p \, dx = &\int pc_1(u-R)^p\chi_{\{u<R\}} + pc_2(u-R)^p\chi_{\{u>R\}} \, dx\partial_t W \\
&+ \frac{p(p-1)}{2}\int c_1^2(u-R)^p\chi_{\{u<R\}} + c_2^2(u-R)^p\chi_{\{u>R\}} \, dx \\
&- \int p(p-1)(\xi-R)^{p-2}(m_u+n_u)(d\xi, dx).
\end{aligned} \tag{4.11}$$

Let,

$$X_1^{(p)} = \int_{\{u<R\}} (u-R)^p \, dx, \qquad X_2^{(p)} = \int_{\{u>R\}} (u-R)^p \, dx,$$

and denote by D the non-negative quantity,

$$D = \int p(p-1)(\xi-R)^{p-2}(m_u+n_u)(d\xi, dx).$$

Supressing the superscript (p), we can write the expression (4.11) as

$$d(X_1+X_2) = p(c_1X_1+c_2X_2)\, dW + \frac{p(p-1)}{2}(c_1^2X_1+c_2^2X_2)\, dt - Ddt.$$

Where p is even, both X_1 and X_2 are non-negative, so $0 \le X_1, X_2 \le X_1+X_2$, and $X_1/X^{(p)}$ and $X_2/X^{(p)}$ are signed and bounded.

$$\begin{aligned}
d\log(X^{(p)}) \le &p\frac{c_1X_1+c_2X_2}{X^{(p)}}dW + \frac{1}{2}p(p-1)\frac{c_1^2X_1+c_2^2X_2}{X^{(p)}}dt \\
&- \frac{1}{2}p^2\left(\frac{c_1X_1+c_2X_2}{X^{(p)}}\right)^2 dt.
\end{aligned}$$

We require the following sufficient condition for decay:

$$(p-1)\left(c_1^2\frac{X_1}{X}+c_2^2\frac{X_2}{X}\right) < p\left(\frac{c_1X_1}{X}+\frac{c_2X_2}{X}\right)^2.$$

Since we have no a-priori bound on $X_1/X^{(p)}$ and $X_2/X^{(p)}$ except that they are individually non-negative and sum to unity, we know only that it is sufficient that

$$(p-1)\left(c_1^2a+c_2^2(1-a)\right)-p\left(c_1a+c_2(1-a)\right)^2<0$$

for any $a\in[0,1]$.

We find that the expression above is maximized at,

$$a=\frac{1}{2}-\frac{c_1+c_2}{2p(c_1-c_2)}.$$

Putting this back in the expression, we arrive at the condition,

$$\begin{aligned}0>&(p-1)\left(\frac{c_1^2+c_2^2}{2}-(c_1^2-c_2^2)\frac{(c_1+c_2)}{2p(c_1-c_2)}\right)\\&-p\left(\frac{c_1+c_2}{2}-(c_1-c_2)\frac{(c_1+c_2)}{2p(c_1-c_2)}\right)^2\\=&(p-1)\left(\frac{(c_1^2+c_2^2)}{2}-\frac{(c_1+c_2)^2}{2p}\right)-\frac{(p-1)^2}{4p}(c_1+c_2)^2,\end{aligned}$$

which would be impossible if c_1 and c_2 were of opposite signs, even if we take $p=2$.

□

4.3 Uniqueness of Invariant Measures

Before we explain the framework from which uniqueness is very easily derived, we lay some theoretical groundwork.

4.3.1 Aside: The dual Lipschitz norm

Before we continue on to proving the continuous dependence estimates that we need, let us pause to consider the dual Lipschitz norm on $\mathfrak{M}_1(\mathfrak{X})$, where $(\mathfrak{X},d)$ is a metric space. It is shown in [35] that on the space of *probability* measures, the following topologies are equivalent. On the space of Radon measures, they are distinct.

The topologies are characterized by measures $\{\nu_n\} \subseteq \mathfrak{M}_1(\mathfrak{X})$ converging to $\nu \in \mathfrak{M}_1(\mathfrak{X})$ after the following modes:

(i) weak* convergence:

$$\forall \varphi \in C(\mathfrak{X}), \qquad \int \varphi(\nu_n - \nu) \to 0;$$

(ii) dual Lipschitz convergence:

$$\forall \varphi \in \mathrm{Lip}_b(\mathfrak{X}), \text{with } \mathrm{Lip}(\varphi) = 1 \qquad \int \varphi(\nu_n - \nu) \to 0.$$

(iii) For a set $A \subseteq \mathfrak{X}$, set

$$A^\varepsilon = \{y \in \mathfrak{X} : (\exists x \in A : d(x, y) < \varepsilon)\}$$

the Lévy-Prohorov convergence is:

$$\inf\{\varepsilon > 0 : \nu_n(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ for every Borel } A \subseteq \mathfrak{X}\} \to 0$$

Theorem 21. *A sequence $(\nu_n) \subseteq \mathfrak{M}_1(\mathfrak{X})$ converges to ν_n in one of the above topologies if and only if it converges in any of the other two.*

The equivalence of the three preceding topologies shows that the dual Lipschitz norm defined by

$$\|\nu\|_{\mathrm{Lip}^*} = \sup_{f \in \mathrm{Lip}_b(X), \mathrm{Lip}(f)=1} \left| \int f \nu_n \right|,$$

does indeed metrize the subspace of *probability* measures in the weak* topology. The following proof is found in [35].

Proof. Since bounded Lipschitz functions with unit Lipschitz constant are continuous, if $\lambda_n \to \lambda$ in the weak* topology, they do so in the dual Lipschitz topology.

Next suppose that $\lambda_n \to \lambda$ in the dual Lipschitz topology.

For any Borel set A, set

$$f(x) := \max(0, 1 - \varepsilon^{-1} d(x, A)),$$

so that $\mathrm{Lip}(f) \leq 1 + \varepsilon^{-1}$. From the definition of the dual Lipschitz norm, we have,

$$\lambda_n(A) \leq \int f\lambda_n \leq \int f\lambda + (1+\varepsilon^{-1})\|\lambda_n - \lambda\|_{\mathrm{Lip}^*} \leq \lambda(A^\varepsilon) + (1+\varepsilon^{-1})\|\lambda_n - \lambda\|_{\mathrm{Lip}^*}.$$

Therefore,

$$\inf\{r > 0 : \nu_n(A) \leq \nu(A^r) + r \text{ for every Borel } A \subseteq X\} \leq \max(\varepsilon, (1+\varepsilon^{-1})\|\nu_n - \nu\|_{\mathrm{Lip}^*}.$$

Now choosing n large enough such that $\|\nu_n - \nu\|_{\mathrm{Lip}^*} \leq \varepsilon^2$, we have that

$$\inf\{r > 0 : \nu_n(A) \leq \nu(A^r) + r \text{ for every Borel } A \subseteq X\} \leq \varepsilon + \varepsilon^2.$$

This shows that convergence in the dual Lipschitz topology implies convergence in the Lévy-Prohorov topology.

Next suppose that $\nu_n \to \nu$ in the Lévy-Prohorov topology.

Let A be a set of continuity of ν (that is, $\nu(\partial A) = 0$, where $\partial A = \bar{A} \backslash \mathrm{int}(A)$ is the topological boundary of A in the metric topology). Set also $\varepsilon > 0$. Then for a small enough δ,

$$\nu(A^\delta \backslash A) \leq \varepsilon \quad \text{and} \quad \nu((A^c)^\delta \backslash A^c) \leq \varepsilon.$$

By assumption, for a large enough n,

$$\nu_n(A) \leq \nu(A^\delta) + \delta \leq \nu(A) + 2\varepsilon,$$

and,

$$\nu_n(A^c) \leq \nu((A^c)^\delta) + \delta \leq \nu(A^c) + 2\varepsilon.$$

Hence for any A,

$$|\nu_n(A) - \nu(A)| \leq 2\varepsilon,$$

this is convergence in total variation, which implies convergence in the weak* topology as continuous functions can be approximated in $L^1(\nu_n)$ by indicator functions of Borel sets. □

Remark. Since we required bounds for both $\nu_n(A)$ and $\nu_n(A^c)$, this proof cannot hold for general Borel measures.

In our context, $\mathfrak{X}$ is the space L^1.

4.3.2 Framework of uniqueness of invariant measure in infinite dimensions

Here we use a framework suggested by §3.1 of [73], which makes clear the theoretical underpinnings of the uniqueness results. The uniqueness of an invariant measure depends generally on two properties [73] (§3.1) – stability, and recurrence.

As remarked in [48], recurrence boils down to the idea that a path through solution space does not move away forever, or that any collection of initial data eventually mix reasonably homogeneously. This is a reflection of the fact that the collection of invariant measures is convex, with ergodic measures as extreme points. This convexity implies that if there are more than one invariant measure, there must be at least two ergodic measures. Since two ergodic measures are either the same or singular with respect to one another, the support of one measure cannot flow into the support of the other. Heuristically, then, if each point of positive measure set is recurrent, then there cannot be another invariant measure. Therefore it is not surprising that as all initial conditions flow within an arbitrary ball of the constant function taking the value of the root, there is only one invariant measure, but it is a matter of completeness to show it.

> **Stability**: Let $P(s, t, u, E)$ be the probability transition function that starts at time s and at point u,such that u_t is inside the set E at time t, i.e.,
>
> $$P(0, t, u, E) = (\mathscr{P}_t^* \theta_u)(E),$$
>
> or more generally,
>
> $$(\mathscr{P}_t^* \Upsilon)(E) = \int_X P(0, t, v, E)\ \Upsilon(dv).$$
>
> There exists a sequence $\theta_k \to 0$, a sequence of positive times (T_k), and a sequence of closed subsets $B_k \subseteq \mathfrak{X}$, such that, for any two starting points $u, v \in B_k$,
>
> $$\sup_{t \geq T_k} \|P(0, t, u, \cdot) - P(0, t, v, \cdot)\|_{\text{Lip}^*} \leq \theta_k,$$
>
> where $\|\cdot\|_{\text{Lip}^*}$ is the dual Lipschitz norm.
>
> **Recurrence**: With the same sequence of B_k as in the condition above, let u_t and v_t be the Feller Markov processes with transition function P as

above, starting at u and v, respectively. Let $\mathcal{T}(B_k)$ be the hitting time,

$$\mathcal{T}(B_k) = \min\{t \geq 0 : u_t \in B_k,\ v_t \in B_k\}.$$

Recurrence is the requirement that,

$$\mathbb{P}(\{\mathcal{T}(B_k) < \infty\} | u_0 = u,\ v_0 = v) = 1.$$

We have the following general theorem:

Theorem 22 ([73] (Theorem 3.1.3)). *For an invariant measure ν satisfying the conditions of recurrence and stability, for any other probability measure Υ on $\mathfrak{X}$, we have,*

$$\|\mathscr{P}_t^* \Upsilon - \nu\|_{\mathrm{Lip}^*} \to 0$$

as $t \to 0$, and μ is unique as an invariant measure.

The proof is included below for completeness. It is as found in Theorem 3.1.3 of [73]. Still, the intuition is clear: If the balls B_m exist with associated $\theta_m \to 0$, and increase to fill all space, stability alone will suffice – stability already is the theorem. If such a sequence of balls does not exist, and the sequence does not increase to fill all space, yet no matter how small a ball B_m gets, still the process enters it in finite time, and within the safety of B_m, there is stability – up to θ_m.

Proof. For any $k \geq 1$, and $t \geq T_k$, set,

$$s(k,t) = \min(\mathcal{T}(B_k), t), \qquad p(u, v, m, t) = \mathbb{P}(\{\mathcal{T}(B_k) + T_k > t\} | u_0 = u,\ v_0 = v).$$

Where $\underline{P}(0, t, (u, v), E_1 \times E_2)$ is the transition probability on $\mathfrak{X} \times \mathfrak{X}$ having $P(0, t, u, E)$ and $P(0, t, v, E)$ as marginals is the transition probability for the coupled, or extended Markov process, we can use the strong Markov property,

$$\underline{P}(0, t, (u, v), E) = \mathbb{E}_{(u,v)}(\underline{P}(\mathcal{T}, t, (u_{\mathcal{T}}, v_{\mathcal{T}}), E)),$$

to calculate thus:

$$P(0, t, u, E) = \underline{P}(0, t, (u, v), E \times \mathfrak{X})$$

$$
\begin{aligned}
&=\mathbb{E}_{(u,v)}(\underline{P}(\mathcal{T}, t, (u_{\mathcal{T}}, v_{\mathcal{T}}), E \times \mathfrak{X})) \\
&=\mathbb{E}_{(u,v)}(P(\mathcal{T}, t, u_{\mathcal{T}}, E));
\end{aligned}
$$

and similarly,

$$
P(0, t, v, E) = \mathbb{E}_{(u,v)}(P(\mathcal{T}, t, v_{\mathcal{T}}, E)).
$$

Therefore,

$$
\begin{aligned}
\|P(0,t,u,\cdot)-&P(0,t,v,\cdot)\|_{\mathrm{Lip}^*} \\
&\leq \mathbb{E}_{(u,v)}(\|P(0, t-s(k,t), u_{s(k,t)}, \cdot) - P(0, t-s(k,t), v_{s(k,t)}, \cdot)\|_{\mathrm{Lip}^*}).
\end{aligned}
$$

Since $\mathcal{T}(B_k) + T_k \leq t$ implies $s(k,t) = \mathcal{T}(B_k)$ and $t - s(k,t) \geq T_k$, by the stability condition,

$$
\|P(0, t-s(k,t), u, \cdot) - P(0, t-s(k,t), v, \cdot)\|_{\mathrm{Lip}^*} \leq \theta_k,
$$

so,

$$
\|P(0,t,u,\cdot) - P(0,t,v,\cdot)\|_{\mathrm{Lip}^*} \leq \theta_k + p(u,v,m,t). \tag{4.12}
$$

By the recurrence condition,

$$
\mathbb{P}(\mathcal{T}(B_k) < \infty | u_0 = u,\ v_0 = v) = 1.
$$

One can therefore first fix m, and then take t to be suffienciently large, so that $p(u,v,m,t)$ becomes sufficiently small.

Therefore,

$$
\|P(0,t,u,\cdot) - P(0,t,v,\cdot)\|_{\mathrm{Lip}^*}
$$

is arbitrarily small for sufficiently large t and $\to 0$ as $t \to \infty$.

From this follows that for any bounded Lipschitz function $f \in \mathrm{Lip}_b(\mathfrak{X})$, we have,

$$
\begin{aligned}
|\langle f, \mathscr{P}_t^* \Upsilon - \nu\rangle| =& |\langle f, \mathscr{P}_t^*(\Upsilon - \nu)\rangle| \\
=& \left| \iint_{\mathfrak{X}\times\mathfrak{X}} (\mathscr{P}_t f(u) - \mathscr{P}_t f(v))\ \Upsilon(du)\nu(dv) \right|
\end{aligned}
$$

$$\leq \iint_{\mathfrak{X}\times\mathfrak{X}} |\mathscr{P}_t f(u) - \mathscr{P}_t f(v)| \; \Upsilon(du)\nu(dv),$$

which tends to 0 from (4.12) above, via the Lebesgue dominated convergence theorem. □

4.3.3 Uniqueness of invariant measures: Proof

We use the framework of Kuksin and Shirikyan, noting that from L^1-contraction, we have the Feller property.

Stability:

First we consider the issue of stability:

Let $u, v \in B_k$. Using the definition of the dual Lipschitz norm and the L^1-contraction inequality,

$$\begin{aligned}
&\|P^m(0,t,u,\cdot) - P^m(0,t,v,\cdot)\|_{\mathrm{Lip}^*} \\
=& \sup_{f:\mathrm{Lip}(f)=1} \left| \int_{\mathfrak{X}} f\left((\mathscr{P}_t^m)^*\delta_u - (\mathscr{P}_t^m)^*\delta_v\right)\right| \\
=& \sup_{f:\mathrm{Lip}(f)=1} |\mathscr{P}_t^m f(u) - \mathscr{P}_t^m f(v)| \\
\leq& \|u(t) - v(t)\|_{L^1} \\
\leq& \|u - v\|_{L^1} \\
\leq& 2 \cdot \mathrm{radius}(B_k).
\end{aligned}$$

Therefore choosing (B_k) such that the radii of the balls tend to 0 gives us a collection of balls, as well as corresponding (θ_k), (T_k), which satisfy the stability condition. Naturally we shall choose these balls to be centred about the constant function $w(x) = R$.

Recurrence:

Since $X^{(2)} \to 0$ for each of our processes it follows that $X^{(2)}$ is arbitarily small in finite time. Therefore any solution u enters an arbitrarily small L^2 ball around $w(x) = R$ in finite time.

From (4.4), we can change these balls to L^1 balls, writing,

$$\begin{aligned}
&\mathbb{E}(\|u(t) - R\|_{L^1}) \leq \mathbb{E}(\|u(t) - v(t)\|_{L^1} + \|v(t) - R\|_{L^1}) \\
\leq& \varepsilon + \mu^{1/2}(\mathbb{T}^d)\mathbb{E}(\|v(t) - R\|_{L^2}) \to \varepsilon,
\end{aligned}$$

where v_0 is an L^2 approximation to u_0 with $\mathbb{E}(\|v_0 - u_0\|_{L^1}) \leq \varepsilon$. By L^1 contraction, this means $\mathbb{E}(\|u(t) - v(t)\|_{L^1}) \leq \mathbb{E}(\|v_0 - u_0\|_{L^1}) \leq \varepsilon$.

In order to free ourselves from the expectation, we employ a Borel-Cantelli argument in the spirit of one that Debussche and Vovelle used in [30].

Let T_l be a sequence of increasing deterministic times.

Let

$$A_l = \{u \notin B_k; t \in [0, T_l]\}.$$

Here B_k are balls of radii θ_k centred at the constant function $R \in L^1$.

Choose ε to be less than half the radius of the given ball we seek to have the solutions enter, that is, $\varepsilon \leq \theta_k/2$, and let v_0, $v(t)$ be as before. We show that solutions do enter B_k in finite time thus:

$$\begin{aligned}
&\mathbb{P}(\{u \notin B_k; t \in (T_l, T_{l+1}]\}|\mathscr{F}_{T_l}) \\
\leq&\mathbb{P}\left(\left\{\fint_{T_l}^{T_{l+1}} \|u(t) - R\|_{L^1}\, dt \geq \theta_k\right\}\middle|\mathscr{F}_{T_l}\right) \\
\leq&\theta_k^{-1}\mathbb{E}\left(\fint_{T_l}^{T_{l+1}} \|u(t) - R\|_{L^1}\, dt\middle|\mathscr{F}_{T_l}\right) \\
\leq&\theta_k^{-1}\mathbb{E}\left(\fint_{T_l}^{T_{l+1}} \|u(t) - v(t)\|_{L^1}\, dt\middle|\mathscr{F}_{T_l}\right) + \theta^{-1}\mathbb{E}\left(\fint_{T_l}^{T_{l+1}} \|v(t) - R\|_{L^1}\, dt\right) \\
\leq&\varepsilon\theta_k^{-1} + \mu^{1/2}(\mathbb{T}^d)\theta_k^{-1}\mathbb{E}\left(\fint_{T_l}^{T_{l+1}} \|v(t) - R\|_{L^2}\, dt\middle|\mathscr{F}_{T_l}\right).
\end{aligned}$$

Multiplying through by χ_{A_l} and taking (uncondition) expectation on both sides, one arrives at

$$\mathbb{P}(A_{l+1}) \leq \frac{1}{2}\mathbb{P}(A_l) + \mu^{1/2}(\mathbb{T}^d)\theta_k^{-1}\mathbb{E}\left(\fint_{T_l}^{T_{l+1}} \|v(t) - R\|_{L^2}\, dt\right).$$

Since $\mathbb{E}(\|v - R\|_{L^2})$ decays, we can choose $T_{l+1} - T_l$ to be long enough that

$$\mu^{1/2}(\mathbb{T}^d)\theta_k^{-1}\mathbb{E}\left(\fint_{T_l}^{T_{l+1}} \|v(t) - R\|_{L^2}\, dt\right) \leq 2^{-l}.$$

This in turn shows that $\mathbb{P}(A_l) \leq l2^{-l}$, so that almost surely,

$$l_0 = \inf\{l \geq 0 : \inf_{s\in[T_l, T_{l+1}]} \|u(s) - R\|_{L^1} \leq \theta_k\}$$

is finite. It follows that the first entry time into B_k is bounded by T_{l_0+1}, and hence almost surely finite as well.

Both stability and recurrence together show that there can be no more than one invariant measure, and this invariant measure is the Dirac delta supported at $w(x) = R$.

4.4 Further Remarks on Multiplicative Noises

Where the noise has more than a root it may be that non-trivial asymptotic behaviour emerges. A hueristic reason can be gleaned from the L^1 contraction inequality.

Suppose R_1 and R_2 are two roots of $\sigma(u)$. Let $u_0 \in L^1$, not necessarily taking values between R_1 and R_2. Then we see that, both,

$$\begin{aligned}\mathbb{E}(\|u(t) - R_1\|_{L^1}) \leq \mathbb{E}(\|u_0 - R_1\|_{L^1}), \text{ and} \\ \mathbb{E}(\|u(t) - R_2\|_{L^1}) \leq \mathbb{E}(\|u_0 - R_2\|_{L^1}).\end{aligned}$$

So that in expectation, $u(t)$ remains in the intersection between the spheres of radii $\|u_0 - R_1\|_{L^1}$ and $\|u_0 - R_2\|_{L^1}$ centred at R_1 and R_2, respectively. Apart from this condition there does not seem to be any a-priori reason that the solution map cannot explore this intersection of spheres densely. This is a question into which I intend to look further.

4.4.1 Compactness results for bounded Lipschitz noises

In this section we show that modifying the techniques of Debussche and Vovelle, that is, using velocity averaging techniques, it is possible to show that the solution operator of

$$\partial_t u = -\nabla \cdot F(u) + \nabla \cdot (\mathbf{A}(u) \cdot \nabla u) + \sigma(u)\, \partial_t W \tag{4.13}$$

smoothens for a bounded multiplicative noise σ, and F and $\mathbf{A}$ satisfying appropriate nonlinearity conditions as dictated by the requirements of velocity averaging.

Let us suppose that $|\sigma|$ is uniformly bounded by $\Sigma > 0$.

Again, let

$$
\begin{aligned}
u^0 &= \int \mathcal{S}(t)\chi_u(\xi, x, 0)\, d\xi, \\
u^\flat &= \int \int_0^t \mathcal{S}(s)(B_\gamma \chi_u + \theta\chi_u)(\xi, x, t-s)\, ds\, d\xi, \\
\langle M_1(t), \varphi\rangle &= -\int_0^t \int_{\mathbb{T}^d} \int \partial_\xi(\mathcal{S}^*(t-s)\varphi)\, d(m_u + n_u - p_u)(\xi, x, t), \\
\langle M_2(t), \varphi\rangle &= \int_{\mathbb{T}^d} (\mathcal{S}^*(t-s)\varphi)(x, u(x,s))\sigma(u(x,s))\, dW_s\, dx.
\end{aligned}
$$

The analyses of u_0 and $u^\flat$ coincide exactly with their analysis in the previous chapter, except that there is no way to bound the zeroth mode. Suppose as in (3.6) that

$$
\sup_{\substack{n\in\mathbb{R}^n \\ \tau\in\mathbb{R}}} \int \frac{(4\pi^2\mathbf{A}(\xi) : n\otimes n + \gamma|n|^{2\alpha} + \theta)}{(2\pi F'(\xi)\cdot n + \tau)^2 + (4\pi^2\mathbf{A}(\xi) : n\otimes n + \gamma|n|^{2\alpha} + \theta)^2}\, d\xi = O(|n|^{-2\alpha-b})
$$

That is we have shown that

$$
\mathbb{E}\left(\int_0^T |u^0 + u^\flat|_{\mathcal{X}}^2\, dt\right) \le C\left(\|u_0\|_{L^1} + \mathbb{E}\left(\|u(t)\|_{L^1([0,T];L^1(\mathbb{T}^d))}\right)\right), \tag{4.14}
$$

where C depends only on the dimension, d, and $\mathcal{X}$ is a compact subspace of equicontinuous functions, and $|\cdot|_{\mathcal{X}}$ its associated seminorm.

Now we turn to the analysis of M_1 and M_2.

Analysis of M_1

Writing out the integral we have

$$
\begin{aligned}
\langle(-\Delta)^{\beta/2}M_1(t), \varphi\rangle &= -\int_0^t \int_{\mathbb{T}^d} \int (-\Delta)^{\beta/2}\partial_\xi(\mathcal{S}^*(t-s)\varphi)\, d(m_u + n_u - p_u)(\xi, x, t) \\
&= -\int_0^t \int_{\mathbb{T}^d} \int F''(\xi)\cdot\nabla(-\Delta)^{\beta/2}(\mathcal{S}^*(t-s)\varphi)\, d(m_u + n_u - p_u)(\xi, x, t) \\
&\quad -\int_0^t \int_{\mathbb{T}^d} \int \mathbf{A}'(\xi) : \nabla^2(-\Delta)^{\beta/2}(\mathcal{S}^*(t-s)\varphi)\, d(m_u + n_u - p_u)(\xi, x, t).
\end{aligned}
$$

Using a kernel estimate as before,

$$
\begin{aligned}
\|(-\Delta)^{\beta/2}\nabla(\mathcal{S}^*(t-s)\varphi)\|_{L^\infty_x} \le & C(\gamma(t-s))^{-(\beta+1)/(2\alpha)-d/(2p'\alpha)}e^{-\theta(t-s)}\|\varphi\|_{L^{p'}_x}, \\
\|(-\Delta)^{\beta/2}\nabla^2(\mathcal{S}^*(t-s)\varphi)\|_{L^\infty_x} \le & C(\gamma(t-s))^{-(\beta+2)/(2\alpha)-d/(2p'\alpha)}e^{-\theta(t-s)}\|\varphi\|_{L^{p'}_x}.
\end{aligned}
$$

Therefore we have in turn

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T \langle (-\Delta)^{\beta/2} M_1(t), \varphi(t)\rangle \, dt\right) \\
\leq & C\left(\sup_{s\in[0,T]} \frac{1}{\gamma}\int_s^T (\gamma(t-s))^{1-(\beta+1)/(2\alpha)-d/(2p'\alpha)} e^{-\theta(t-s)} \, dt\right) \\
&\cdot \|\varphi\|_{L^\infty_{\omega,t}L^{p'}_x} \mathbb{E}\left(\int_0^T \int_{\mathbb{T}^d}\int |F''(\xi)| d|M_1|(\xi,x,s)\right) \\
&+ C\left(\sup_{s\in[0,T]} \frac{1}{\gamma}\int_s^T (\gamma(t-s))^{1-(\beta+2)/(2\alpha)-d/(2p'\alpha)} e^{-\theta(t-s)} \, dt\right) \\
&\cdot \|\varphi\|_{L^\infty_{\omega,t}L^{p'}_x} \mathbb{E}\left(\int_0^T \int_{\mathbb{T}^d}\int \sup_{i,j} |\mathbf{A}'(\xi)| d|M_1|(\xi,x,s)\right).
\end{aligned}$$

Where $|F''(r)| \leq 1+|r|^{\kappa_1}$ and $|\mathbf{A}'(r)| \leq 1+|r|^{\kappa_2}$, with $\kappa_1,\kappa_2 \leq 1$, using the total variation bound of Lemma 14 on $|M_1|$ ($\sigma(u)$ in place of $\sigma(x)$ remains L^∞-bounded) and L^1-contraction estimate on u, we have a bound for the left-hand side and this yields the familiar inclusion

$$M_1 \in L^1_{\omega,t}W^{\beta,p}_x \subseteq L^1_{\omega,t}W^{\beta,1}_x,$$

via the bound

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T \langle (-\Delta)^{\beta/2} M_1(t), \varphi(t)\rangle \, dt\right) \\
\leq & C\left(\frac{\gamma}{\theta}\right)^{-(\beta+1)/(2\alpha)-d/(2p'\alpha)} \theta^{-2}\Gamma\left(2-\frac{\beta+1}{2\alpha}-\frac{d}{2p'\alpha}\right) \\
&\cdot \|\varphi\|_{L^\infty_{\omega,t}L^{p'}_x} \mathbb{E}\left(\int_0^T 1+\|u(t)\|_{L^1_x}\Sigma^2_m \, dt + 1 + \|u_0\|^3_{L^3_x}\right) \\
&+ C\left(\frac{\gamma}{\theta}\right)^{-(\beta+2)/(2\alpha)-d/(2p'\alpha)} \theta^{-2}\Gamma\left(2-\frac{\beta+2}{2\alpha}-\frac{d}{2p'\alpha}\right) \\
&\cdot \|\varphi\|_{L^\infty_{\omega,t}L^{p'}_x} \mathbb{E}\left(\int_0^T 1+\|u(t)\|_{L^1_x}\Sigma^2_m \, dt + \mu(\mathbb{T}^d) + \|u_0\|^3_{L^3_x}\right) \\
\leq & C_{M_1}(\alpha,\beta,\gamma,\theta)\|\varphi\|_{L^\infty_{\omega,t}L^{p'}_x} \mathbb{E}\left(1+T+\int_0^T \|u(t)\|_{L^1_x}\Sigma^2_m \, dt + \mu(\mathbb{T}^d) + \|u_0\|^3_{L^3_x}\right). \quad (4.15)
\end{aligned}$$

Analysis of M_2

We again use the inequality of Burkhölder, Davis, and Gundy:

$$\mathbb{E}\left(\sup_{s\le t}\langle(-\Delta)^{-\beta/2}M_2(s),\varphi\rangle^2\right)$$
$$\le\mathbb{E}\Bigg(\int_0^t\left(\int\int_{\mathbb{T}^d}e^{-\theta(t-s)}e^{(B_\gamma+\mathbf{A}(\xi):\nabla\otimes\nabla)(t-s)}((-\Delta)^{-\beta/2}\varphi)(x+F'(\xi)(t-s))\right.$$
$$\left.\left.\sigma(\xi)\delta(\xi-u(x,s))\,dx\,d\xi\right)^2\,ds\right)$$
$$\le\mathbb{E}\left(\int_0^t\left(\int\int_{\mathbb{T}^d}(e^{-\theta(t-s)}(-\Delta)^{-\beta/2}e^{(B_\gamma+\mathbf{A}(\xi):\nabla\otimes\nabla)(t-s)}\varphi(x+F'(\xi)(t-s)))^2\Xi(\xi-u(x,s))\,dx\,d\xi\right)\right.$$
$$\left.\cdot\left(\int\int_{\mathbb{T}^d}\sigma^2(\xi)\delta(\xi-u(x,s))\,dx\,d\xi\right)\,ds\right).$$

We apply the kernel estimate to $\varphi(x+F'(\xi)(t-s)))^2$, which, as $\varphi\in L^\infty$, is also $\in L^\infty_x$. Writing $\varphi(x-F'(\xi)(t-s))=:\phi(\xi,x,s)\in L^\infty_{\xi,x,s}$, we have

$$\begin{aligned}&\left\|(-\Delta)^{-\beta/2}e^{(B_\gamma+\mathbf{A}(\xi):\nabla\otimes\nabla)(t-s)}\phi(\xi,x,s)\right\|_{L^\infty_x}\\ \le&C(\gamma(t-s))^{-\beta/(2\alpha)}\|\phi(\xi,x,s)\|_{L^\infty_x}=C(\gamma(t-s))^{-\beta/(2\alpha)}\|\varphi\|_{L^\infty}.\end{aligned}$$

Whereupon, we also have

$$\begin{aligned}&\mathbb{E}\left(\sup_{s\le t}\langle(-\Delta)^{-\beta/2}M_2(s),\varphi\rangle^2\right)\\ \le&C\mathbb{E}\left(\left(\int_0^t e^{-2\Xi(t-s)}(\gamma(t-s))^{-\beta/\alpha}\|\varphi\|^2_{L^\infty}\right)\cdot\left(\int_{\mathbb{T}^d}\sigma^2(u)\,dx\right)\,ds\right)\\ &\le C'\mathbb{E}\left(\left(\int_0^t e^{-2\Xi(t-s)}(\gamma(t-s))^{-\beta/\alpha}\|\varphi\|^2_{L^\infty}\right)\cdot\left(\Sigma^2\mu(\mathbb{T}^d)\right)\,ds\right)\\ \le&C'\|\varphi\|^2_{L^\infty}\left(\frac{\gamma}{\theta}\right)^{-\beta/\alpha}\theta^{-1}\Gamma\left(1-\frac{\beta}{\alpha}\right)\cdot\left(\Sigma^2\mu(\mathbb{T}^d)\right)\\ =&C_{M_2}(\alpha,\beta,\gamma,\theta)\cdot\left(\Sigma^2\mu(\mathbb{T}^d)\right)\|\varphi\|^2_{L^\infty}.\end{aligned}\tag{4.16}$$

A judiciously chosen $\beta>0$, as well as γ and θ (which have both heretofore been unconstrained except by positivity), ensures that the left-hand side of this inequality remains bounded.

This implies $M_2\in L^2_\omega L^\infty_t W^{\beta,1}_x\subseteq L^1_\omega L^\infty_t W^{\beta,1}_x$.

A degree of compactness

Again, let $\mathfrak{Z}$ be the compact space of equicontinuous functions. Now either $\mathcal{X} \subseteq W^{\beta,1}(\mathbb{T}^d)$ or the reverse inclusion holds. In either case, we end up with $B_R \subseteq L^1_x$ being mapped into a compact set.

From (4.14), we have

$$\mathbb{E}\left(\int_0^T |u^0 + u^\flat|^2_{\mathcal{X}}\, dt\right) \leq C\left(\|u_0\|_{L^1} + \mathbb{E}\left(\|u(t)\|_{L^1([0,T];L^1(\mathbb{T}^d))}\right)\right),$$

where C depends only on the dimension, d.

So

$$\mathbb{E}\left(\int_0^T |u^0 + u^\flat|_{\mathcal{X}}\, dt\right) \leq CT + \frac{1}{4}\left(\|u_0\|_{L^1} + \mathbb{E}\left(\|u(t)\|_{L^1([0,T];L^1(\mathbb{T}^d))}\right)\right).$$

From (4.15), and (4.16), we have

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T \|M_1 + M_2\|_{W^{\beta,1}_x}\, dt\right) \\
\leq &C_{M_1}(\alpha,\beta,\gamma,\theta,p)\left(\Sigma^2\|u\|_{L^1([0,T];L^1(\mathbb{T}^d))} + \Sigma^2 T + \mu(\mathbb{T}^d) + \|u_0\|^3_{L^3(\mathbb{T}^d)}\right) \\
&+ C_{M_2}(\alpha,\beta,\gamma,\theta)\Sigma^2\mu(\mathbb{T}^d)T
\end{aligned}$$

By taking $|\cdot|_{\mathfrak{Z}}$ to be the seminorm of the bigger space of the two, we have

$$\begin{aligned}
\fint_0^T \mathbb{E}\left(|u|_{\mathfrak{Z}}\right)\, dt \leq &C + \frac{1}{4T}\|u_0\|_{L^1} + C_{M_1}(\Sigma^2 + \frac{\mu(\mathbb{T}^d)}{T} + \frac{1}{T}\|u_0\|^3_{L^3_x}) + C_{M_2}\Sigma^2\mu(\mathbb{T}^d) \\
&+ \left(\frac{1}{4} + C_{M_1}\Sigma^2\right)\fint_0^T \|u(t)\|_{L^1}\, dt.
\end{aligned}$$

Since $\mathfrak{Z}$ is compact in L^1, this shows that the solution map associated with the equation (4.13) does indeed map solutions into a compact subspace given the appropriate nonlinearity conditions. However, as there is no a-priori control on $\|u(t)\|_{L^1}$, one cannot give a bound to show that the Krylov-Bogoliubov hypothesis of tightness is satisfied. Of course, if the zero function were a solution – that is, if $\sigma(0) = 0$, then the L^1 contraction property would provide a bound on $\|u(t)\|_{L^1}$, but in that case the existence of an invariant measure is trivial.

Chapter 5

Invariant Measures for a Simple System of Equations

Here [1] we consider the compressible Navier-Stokes system,

$$\begin{aligned} \partial_t \varrho &= -\,\partial_x(\varrho u) \\ \partial_t(\varrho u) &= -\,\partial_x(\varrho u^2 + p(\varrho)) + \mu \partial^2_{xx} u + \sigma(\varrho, u, x)\partial_t W_t, \end{aligned} \tag{5.1}$$

with initial data $\varrho(0) = \varrho_0$, $u(0) = u_0$, and with $x \in \mathbb{R}$ and $t \geq 0$.

Here $p(\varrho) = \varrho^\gamma$ is the adiabatic relation between the pressure, p, and the density, ϱ. Here for simplicity we are going to require σ to be bounded in all its arguments and compactly supported in x. We use only that $\sigma \in L^2$ in x, and is bounded in all its other arguments.

We are seeking an invariant measure.

The strategy is to use Hoff's discrete difference scheme in [51] to show existence of solutions and use Kanel"s classical energy method argument [59] to prove some asymptotic bounds relating to higher regularity norms in both u and $v = 1/\varrho$, in the Lagrangian coordinates. Since we are using a discrete scheme, Kanel"s argument requires some modification, in particular, a new auxiliary function to show that v is bounded above from infinity and below from nought in finite time.

The background to this work is roughly as follows, though the literature is immense. Kanel' [59] showed existence and regularity of solutions to the system (5.1) in Lagrangian coordinates on $\mathbb{R}$. Kazhikov and Shelukhin [63] made the next landmark

[1] *post-viva addendum: This chapter is incorrect – see Coti-Zelati, Glatt-Holtz, Trivisa 2018.*

step and proved a similar result for the system of three equations,

$$\begin{aligned} \partial_t v =& \partial_y u \\ \partial_t u =& \partial_y \left(-P(v) + \frac{\varkappa}{v} \partial_y u \right) \\ \partial_t \left(e + \frac{u^2}{2} \right) =& \partial_y \left(\varkappa v^{-1} \partial_y \boldsymbol{\theta} \right) + \partial_y \left[u \left(-P(v) + \frac{\varkappa}{v} \partial_y u \right) \right], \end{aligned} \tag{5.2}$$

on a bounded domain, where $\boldsymbol{\theta}$ is "temperature", and $e = c_V \boldsymbol{\theta}$ is the "specific energy". Kanel' and Kazhikov Shelukhin both considered, essentially, the specific volume on a particle path,

$$\begin{aligned} \dot{X}(t) =& u(X(t), t) \\ X(0) =& x_0, \end{aligned}$$

showing that the logarithm of the specific volume remains bounded. This was expanded upon in Hoff's papers in [50, 51], where Hoff employed the same argument as Kazhikov and Shelukhin on a discrete collection of difference equations — and further explained in greater clarity in [52]. The question of the boundedness of the density, or specific volume, away from nought and infinity is fundamental to the smoothness of solutions, as shall later be explained. The corresponding physical phenomenon for a blowup of the specific volume is a vacuum, which leads to extreme effects such as cavitation. Infinite density is also a physically troubling prospect.

Jiang [58] studied asymptotic behaviour on the half-line for problems with large data, finding time-independent bounds above and below for density. Li and Liang [76] improved on this in a very recent contribution to the subject by showing that temperature is also bounded pointwise from zero and infinity.

5.1 Existence of Solutions

5.1.1 Lagrangian Coordinates

In Lagrangian coordinates, these equations become

$$\partial_t v = \partial_y u$$

$$\partial_t u = -\,\partial_y P(v) + \partial_y(\varepsilon(v)\partial_y u) + \sigma \partial_t W_t, \tag{5.3}$$

where $P(v) = p(1/v) = v^{-\gamma}$, and $\varepsilon(v) = \mu/v$.

We shall continue to call the first equation "the continuity equation", and the second equation "the momentum equation". Of course, for precision, the solution u in this set of equations should properly be labelled u_L, with $u_L(y,t) = u(x,t)$, but for simplicity and legibility, we leave the distinction implicit.

In Lagrangian coordinates, the variables of interest to us are (y,t), where $y = x(0)$, and (x,t) are the Eulerian coordinates, for which fluid particles take the positions.

$$\begin{aligned} \dot{x} =& u(x(t),t) \qquad t \le s \\ x(s) =& y. \end{aligned}$$

It is well known that the Lagrangian coordinates satisfy the transformation

$$y = \int_{x_0}^{x} \varrho(s,t)\, ds,$$

as can also be seen from the continuity equation

$$\partial_t \varrho + \partial_x(\varrho u) = 0,$$

by postulating

$$\varrho = \partial_x y, \qquad \varrho u = -\partial_t y.$$

Therefore changing $\partial_t + u\partial_x$ to ∂_t and ∂_x to $\varrho\partial_y$ changes an equation from the Eulerian to the Lagrangian perspective. It is important that the Jacobian of the change-of-coordinates be non-singular.

We see from the above that it is of primary importance that ϱ remains bounded away from zero – that vacuums do not spontaneously arise. This is of paramount importance to the entire problem of well-posedness. Physically, where vacuums arise, we would expect cavitation, shocks, and classical well-posedness to break-down, so that the mathematical model fails adequately to describe the phenomenon at the vacuum/cavitation limit which had done so up to that limit.

The theorem we shall to prove in the subsequent sections will be the following:

Theorem 23. *Assume that the initial data* (v_0, u_0) *satisfy* $0 < \underline{v}(0) < v_0 < \bar{v}(0)$, *and* $v_0 - v' \in L^2(\mathbb{R}) \cap BV(\mathbb{R})$ *for some fixed* $v' > 0$, *and* $u_0 \in L^2(\mathbb{R})$. *Assume that* $\sigma = \sigma(u, v, x)$ *in (5.3), is compactly supported in the last argument, and bounded in all three. Then the Cauchy problem (5.3) has a weak solution almost surely for every time* $t \geq 0$, *satisfying*

$$0 \leq \underline{v}(t) < v(x,t,\omega) < \bar{v}(t)\omega - a.s. \tag{5.4}$$

$$v(\cdot,t) - v', u(\cdot,t) \in L^2_x\omega - a.s., and \tag{5.5}$$

$$\epsilon^{1/2}(v)u_x \in L^2_{t,x}\omega - a.s.. \tag{5.6}$$

Furthermore, an invariant measure exists for the process $Y_t = (v(t) - v', u(t))$.

We mean by an almost sure (or pathwise) weak solution the notion that Hoff used, being that of Kazhikov and Shelukin's [63], that (v, u) almost surely satisfy

$$\begin{aligned}
&v, v^{-1} \in L^\infty([0,T]; \dot{H}^1(\mathbb{R})),\\
&v(\cdot,t) - v' \in L^\infty([0,T]; L^2(\mathbb{R})), and\\
&u \in L^2([0,T]; H^1(\mathbb{R})) \cap H^1([0,T], L^2(\mathbb{R})),
\end{aligned}$$

and satisfy the equations almost everywhere in $\mathbb{R} \times [0,T]$ for any $T < \infty$.

5.1.2 Bounds for Specific Volume

In this section we derive the four bounds (5.4) - (5.6). In the next section we conclude the proof of the theorem.

Therefore first we derive a bound for the density, or more accurately, the specific volume $v = 1/\varrho$.

Following Hoff in[50] and [51], we introduce the semidiscrete difference scheme, with $h > 0$, and $x_k = kh$, with $k \in \mathbb{Z}$. We approximate u at integral multiples of h and at the midpoint between integeral multiples we approximate v, writing therefore $u_k(0)$ for $u(x_k, 0)$ and $v_{k+1/2}(0)$ for $v(x_{k+1/2}, 0)$.

We denote the forward difference by Δ, so that, e.g.,

$$(\Delta u)_{k+1/2} = \frac{1}{h}(u_{k+1} - u_k),$$

and

$$(\Delta v)_k = \frac{1}{h}(v_{k+1/2} - v_{k-1/2}).$$

The approximate solutions $u_k(t)$ and $v_{k+1/2}(t)$ are then computed via the system of SDEs:

$$\begin{aligned} \partial_t v_{k+1/2} =& (\Delta u)_{k+1/2}(t) \\ \partial_t u_k =& \Delta(-P + \epsilon(v)\Delta u)_k + \sigma_k \partial_t W_t, \end{aligned} \tag{5.7}$$

where $P(v) = p(1/v) = v^{-\gamma}$.

First we prove the usual, *a priori*, energy bound. For convenience we follow Hoff and define

$$\psi(v) := \int_{v'}^{v} P(v') - P(r)\, dr. \tag{5.8}$$

Since P is a decreasing function of its argument, the integral ψ is non-negative.

We enforce the boundary conditions that $(u_k, v_{k+1/2}) \to (0, v')$ as $|k| \to \infty$, where v' is a constant.

Multiplying the second equation in (5.7) by $u_k h$, and using the Itô formula, we have

$$\frac{1}{2}\partial_t u_k^2 h = -\Delta(P(v) - P(v'))_k u_k h + \Delta(\epsilon \Delta u)_k u_k h + \frac{1}{2}\sigma_k^2 h + \sigma_k u_k h \partial_t W_t,$$

noting that $(\Delta P)_k = \Delta(P - c)_k$ for any constant c.

Next we sum the indices k from $-N$ to N and integrate in time from 0 to T, summing-by-parts,

$$\begin{aligned} &\frac{1}{2}\sum_{-N}^{N} u_k^2 h \Big|_0^T + \sum_{-N}^{N} \psi(v_{k+1/2}) h \Big|_0^T + \int_0^T \sum_{-N}^{N} \epsilon(v_{k+1/2})(\Delta u)_{k+1/2}^2 h\, ds \\ &\quad = \int_0^T \big((P(v') - P_{N+1/2})u_N - (P(v') - P(v_{-N-1/2}))u_{-N}\big)\, ds \\ &\quad\quad + \int_0^T \big(u_N \epsilon(v_{N+1/2})(\partial u)_{N+1/2} - u_{-N}\epsilon(v_{-N-1/2})(\partial u)_{-N-1/2}\big)\, ds \\ &\quad\quad + \frac{1}{2}\int_0^T \sum_{-N}^{N} \sigma_k^2 h\, ds + \int_0^T \sum_{-N}^{N} \sigma_k u_k h\, dW_t \end{aligned}$$

Enforcing the boundary conditions, taking $N \to \infty$, we can write

$$\begin{aligned}\frac{1}{2}\sum u_k^2(T)h + \sum \psi(v_{k+1/2})(T)h + &\int_0^T \sum \epsilon(v_{k+1/2})(\Delta u)_k^2 h\, dt \\ \leq &\frac{1}{2}\sum u_k^2(0)h + \sum \psi(v_{k+1/2}(0))h \\ &+ \frac{1}{2}\int_0^T \sum \sigma_k^2 h\, dt + \int_0^T \sum u_k \sigma_k h\, dW_t,\end{aligned} \tag{5.9}$$

where C is independent of T and h, depending only on initial data $\|u_0\|_{L^2}$ and $\|\psi(v_0)\|_{L^1}$.

Now suppose $\sigma_k = \sigma(u_k, v_{k+1/2}, kh)$, and that σ is compactly supported in its last argument, and bounded in all three. Then

$$\sum \sigma_k^2 h,\ \sum \sigma_k u_k h < \infty.$$

Now for $f(t)$ such that $\sup_{t\in[0,T]} |f(t)| < \infty$,

$$X_\cdot = \int_0^\cdot f(t)\, dW_t$$

has continuous-in-time paths (or is indistinguishable from a process having continuous-in-time paths), as

$$\begin{aligned}\mathbb{E}(|X_t - X_s|^{2\kappa}) =&\mathbb{E}\left(\left|\int_s^t f(\tau)\, d\beta_\tau\right|^{2\kappa}\right) \\ =&\mathbb{E}\left(\left|\int_s^t |f(\tau)|^2\, d\tau\right|^{\varkappa}\right) \leq \mathbb{E}\left(\sup_{\tau\in[0,T]} |f(t)|^{2\kappa}\right)|t-s|^{\varkappa},\end{aligned}$$

so that by the Kolmogorov continuity theorem, there is a version of X_t with sample paths that are almost surely in the class $C^{(\varkappa-1)/2\kappa}$ for every $\varkappa$. Also, where the stochastic process is indexed by a time variable in $\mathbb{R}^d$, different versions of the same process are indistinguishable from one another.

Alternatively, taking expectations, we have

$$\begin{aligned}&\mathbb{E}\left[\frac{1}{2}\sum u_k^2(T)h + \sum \psi(v_{k+1/2})(T)h + \int_0^T \sum \epsilon(v_{k+1/2})(\Delta u)_k^2 h\, dt\right] \\ \leq &C_0 + \mathbb{E}\left(\frac{1}{2}\int_0^T \sum \sigma_k^2 h\, dt\right),\end{aligned} \tag{5.10}$$

Hence, almost surely,

$$\frac{1}{2}\sum u_k^2(T)h + \sum \psi(v_{k+1/2})(T)h + \sum \int_0^T \epsilon(v_{k+1/2})(\Delta u)_k^2 \, dt \; h < \infty. \tag{5.11}$$

This boundedness implies that $u_k \in l^2$, and $(\Delta u)_{k+1/2} \in l^2$.

We also have the asymptotic bound

$$\begin{aligned} &\frac{1}{T}\mathbb{E}\left[\frac{1}{2}\sum u_k^2(T)h + \sum \psi(v_{k+1/2})(T)h + \int_0^T \sum \epsilon(v_{k+1/2})(\Delta u)_k^2 h \, dt\right] \\ \le &\frac{C_0}{T} + \frac{1}{T}\mathbb{E}\left(\frac{1}{2}\int_0^T \sum \sigma_k^2 h \, dt\right). \end{aligned} \tag{5.12}$$

Since σ_k is compactly supported in k and bounded in all its arguments,

$$\frac{1}{T}\mathbb{E}\left(\int_0^T \sum \sigma_k^2 h \, dt\right) = O(1),$$

as $T \to \infty$, independently of h.

Next, following Kanel' [59] (also found reproduced in [100]), we turn to an inequality bounding $(\Delta v)_k / v_{k+1/2}$.

Writing $E_{k+1/2} = \varkappa(\log(v_{k+1/2}) - \log(v'))$, we see that, as the continuity/mass equation does not have a stochastic forcing term, Itô formula coincides with the usual chain rule to give

$$\Delta(\epsilon(v)\Delta u)_k = \Delta(\partial_t E)_k.$$

Therefore,

$$\partial_t u_k + (\Delta P)_k - \Delta(\partial_t E)_k = \sigma_k \partial_t W_t.$$

Again, as the mass equation does not have a stochastic forcing term, we can multiply the momentum equation by $-\Delta E_k$, unpenalized by an additional correction term:

$$\begin{aligned} -(\Delta E)_k \partial_t u_k - (\Delta E)_k (\Delta P)_k + (\Delta E)_k \partial_t (\Delta E)_k &= -(\Delta E)_k - \sigma_k \partial_t W_t \\ -\int_0^T (\Delta E)_k \partial_t u_k - (\Delta E)_k (\Delta P)_k + \frac{1}{2}\partial_t (\Delta E)_k^2 \, dt &= -\int_0^T (\Delta E)_k \sigma_k dW_t. \end{aligned}$$

Integrating by parts and summing up $|k| \leq N$, we have

$$
\begin{aligned}
& -\int_0^T \sum_{|k|\leq N} (\Delta E)_k (\Delta P)_k h \; dt + \frac{1}{2} \sum_{|k|\leq N} (\Delta E)_k^2 h \Bigg|_0^T \\
&= \int_0^T \sum_{|k|\leq N} (\Delta E)_k \partial_t u_k h \; dt - \int_0^T \sum_{|k|\leq N} (\Delta E)_k \sigma_k h dW_t \\
&= \sum_{|k|\leq N} (\Delta E)_k u_k h \Bigg|_0^T - \int_0^T \sum_{|k|\leq N} \partial_t (\Delta E)_k u)_k h \; dt \\
&\quad - \int_0^T \sum_{|k|\leq N} (\Delta E)_k \sigma_k h \; dW_t \\
&= \sum_{|k|\leq N} (\Delta E)_k u_k h \Bigg|_0^T - \int_0^T \partial_t E_{k+1/2} u_k \Bigg|_{-N}^N dt \\
&\quad + \int_0^T \sum_{|k|\leq N} \partial_t E_{k+1/2} (\Delta u)_{k+1/2} h \; dt - \int_0^T \sum_{|k|\leq N} (\Delta E)_k \sigma_k h dW_t \\
&= \sum_{|k|\leq N} (\Delta E)_k u_k h \Bigg|_0^T - E_{k+1/2} u_k \Bigg|_{-N}^N \Bigg|_0^T \\
&\quad + \int_0^T E_{k+1/2} \partial_t u_k \Bigg|_{-N}^N dt \\
&\quad + \int_0^T \sum_{|k|\leq N} \epsilon(v_{k+1/2}) (\Delta u)_{k+1/2}^2 h \; dt \\
&\quad - \int_0^T \sum_{|k|\leq N} (\Delta E)_k \sigma_k h dW_t \\
&= \sum_{|k|\leq N} (\Delta E)_k (T) u_k(T) h - (E_{k+1/2}(T) u_k(T)) \Bigg|_{-N}^N \\
&\quad + \sum_{|k|\leq N} E_{k+1/2}(0) (\Delta u)_{k+1/2}(0) + \int_0^T E_{k+1/2} \partial_t u_k \Bigg|_{-N}^N dt \\
&\quad + \int_0^T \sum_{|k|\leq N} \epsilon(v_{k+1/2}) (\Delta u)_{k+1/2}^2 h \; dt
\end{aligned}
$$

$$-\int_0^T \sum_{|k|\le N} (\Delta E)_k \sigma_k h dW_t.$$

Via the bound

$$\sum_{|k|\le N} (\Delta E)_k(T) u_k(T) h \le \frac{1}{4} \sum_{|k|\le N} (\Delta E)_k^2(T) + \sum_{|k|\le N} u_k^2(T) h,$$

and using equation (5.9)

$$\frac{1}{2}\sum u_k^2(T)h + \sum \int_0^T \epsilon(v_{k+1/2})(\Delta u)_k^2 \; dt \; h$$
$$\le \frac{1}{2}\sum u_k^2(0)h + \sum \psi(v_{k+1/2})h\Big|_T^0 + \frac{1}{2}\int_0^T \sum \sigma_k^2 h \; dt + \int_0^T \sum u_k \sigma_k h \; dW_t,$$

we have

$$-\int_0^T \sum_{|k|\le N} (\Delta E)_k (\Delta P)_k h \; dt + \frac{1}{4}\sum_{|k|\le N} (\Delta E)_k^2(T) h + 2\sum_{|k|\le N} \psi(v_{k+1/2}(T))h$$
$$\le \frac{1}{2}\sum_{|k|\le N} (\Delta E)_k^2(0) h - (E_{k+1/2}(T) u_k(T))\Big|_{-N}^{N}$$
$$+ \sum_{|k|\le N} E_{k+1/2}(0)(\Delta u)_{k+1/2}(0) + \int_0^T E_{k+1/2} \partial_t u_k\Big|_{-N}^{N} \; dt$$
$$+ \sum_{|k|\le N} u_k^2(0) h + 2\sum_{|k|\le N} \psi(v_{k+1/2}(0))h + \int_0^T \sum_{|k|\le N} \sigma_k^2 h \; dt$$
$$+ 2\int_0^T \sum_{|k|\le N} u_k \sigma_k h \; dW_t - \int_0^T \sum_{|k|\le N} (\Delta E)_k \sigma_k h dW_t$$
$$+ 2\int_0^T (P(v') - P_{k+1/2}) u_k\Big|_{-N}^{N} \; ds + 2\int_0^T u_k \epsilon(v_{k+1/2})(\partial u)_{k+1/2}\Big|_{-N}^{N} \; ds.$$

First,

$$\int_0^T E_{k+1/2} \partial_t u_k\Big|_{-N}^{N} \; dt$$
$$= -\left(E_{k+1/2}(T) u_k(T) - E_{k+1/2}(0) u_k(0)\right)\Big|_{-N}^{N} - \int_0^T \partial_t E_{k+1/2} u_k\Big|_{-N}^{N} \; dt.$$

So, taking $N \to \infty$, and using the fact that $|u_k| \to 0$ as $|k| \to \infty$, we arrive at the inequality

$$\begin{aligned}
-\int_0^T \sum (\Delta E)_k (\Delta P)_k h \, dt + \frac{1}{4} \sum (\Delta E)_k^2(T) h + 2 \sum \psi(v_{k+1/2}(T)) h \\
\leq \frac{1}{2} \sum (\Delta E)_k^2(0) h + \sum E_{k+1/2}(0)(\Delta u)_{k+1/2}(0) \\
+ \sum u_k^2(0) h + 2 \sum \psi(v_{k+1/2}(0)) h + \int_0^T \sum \sigma_k^2 h \, dt \\
+ 2 \int_0^T \sum u_k \sigma_k h \, dW_t - \int_0^T \sum (\Delta E)_k \sigma_k h dW_t.
\end{aligned}$$

Now observe that, as $v \mapsto \log(v)$ in an increasing function and $v \mapsto v^{-\gamma}$ is a decreasing function:

$$(\Delta E)_k (\Delta P)_k = \frac{1}{h^2} (\log(v_{k+1/2}) - \log(v_{k-1/2}))(v_{k+1/2}^{-\gamma} - v_{k-1/2}^{-\gamma}) \leq 0,$$

so, coupled with the remark following (5.8) that $\psi(v) \geq 0$, each summand in the left-hand side of the preceding inequality is non-negative.

Next taking expectation and dividing by T, we attain the estimate

$$\frac{1}{T} \mathbb{E} \left(-\int_0^T \sum (\Delta E)_k (\Delta P)_k h \, dt + \frac{1}{4} \sum (\Delta E)_k^2(T) h + 2 \sum \psi(v_{k+1/2}(T)) h \right) \leq C_0, \tag{5.13}$$

as in (5.12).

Next, consider as in Kanel' [59] an auxiliary function. Kanel's function was

$$f(v) = \int_{v'}^{v} \epsilon(r) \sqrt{\psi}(r) \, dr = \int_{v'}^{v} \frac{\varkappa}{v} \sqrt{\psi(r)} \, dr.$$

This can be bounded thus:

$$\begin{aligned}
|f(v)| = \left| \int_{-\infty}^{v} \frac{\partial f}{\partial y} \, dy \right| = \left| \int \frac{\partial \log(v)}{\partial y} \sqrt{\psi(v)} \, dy \right| \\
\leq \left(\int_{-\infty}^{\infty} \frac{\partial \log(v)}{\partial y} \, dy \right)^{1/2} \left(\int_{-\infty}^{\infty} \psi(v) \, dy \right)^{1/2},
\end{aligned} \tag{5.14}$$

and $\partial_y \log(v)$ is the continuous analogue of ΔE, so that both factors are bounded. But as we are taking discrete differences here, we cannot use the same auxiliary function that Kanel' used. In particular the second equality in (5.14) above does not carry through easily, so we seek an auxiliary function for which that equality becomes a bound – an idea variously manifested and popular throughout analysis, such as the idea of weights that underpins Selberg's vast simplification of Brun's Sieve. We know that f satisfies

(i)

$$|f(v_{N+1/2})| = \left|\sum_{k\leq N}(\Delta f(v))_k h\right| \leq \sum_{k\leq N}|(\Delta f(v))_k|h,$$

and $|(\Delta f(v))_k|$ was in turn bounded by one of the summands on the left-hand side of (5.13); and

(ii)

$$|f(v)| \to \infty$$

as $v \to 0$ or $v \to \infty$,

from which we can conclude that $\log(v_N)$ is bounded.

To this end we define the auxiliary function

$$f(v) := \varkappa \log(v)\sqrt{\psi(v)}.$$

Recall that $\psi(v) \geq 0$.

Working backwards, we also define

$$R_k := \sqrt{\psi(v_{k+1/2})} + \sqrt{\psi(v_{k-1/2})}.$$

Notice now

$$\begin{aligned}\frac{1}{\varkappa}(\Delta E)_k R_k =& \log(v_{k+1/2})\sqrt{\psi(v_{k+1/2})} - \log(v_{k-1/2})\sqrt{\psi(v_{k-1/2})} \\ &+ (\log(v_{k+1/2})\sqrt{\psi(v_{k-1/2}} - \log(v_{k-1/2})\sqrt{\psi(v_{k+1/2})},\end{aligned}$$

so

$$\begin{aligned}&\frac{1}{\varkappa}\sum(\Delta E)_k R_k h \\ =&\sum \Delta(\log(v)\sqrt{\psi(v)})_k h + \sum(\log(v_{k+3/2}) - \log(v_{k-1/2}))\sqrt{\psi(v_{k+1/2})}h\end{aligned}$$

$$
\begin{aligned}
=&\sum \Delta(\log(v)\sqrt{\psi(v)})_k h + \frac{1}{2}\sum[(\Delta \log(v))_{k+1} + (\Delta \log(v)_k)]\sqrt{\psi(v_{k+1/2})}h \\
=&\sum \Delta(\log(v)\sqrt{\psi(v)})_k h + \frac{1}{2}\sum((\Delta E_k) + (\Delta E)_{k+1})\sqrt{\psi(v_{k+1/2})}h.
\end{aligned}
$$

We bound $f(v_N)$ thus:

$$
\begin{aligned}
|f(v_N(T))| =& \left|\sum_{k\le N}(\Delta f(v))_k h\right| \\
\le& \sum |(\Delta E)_k||R_k|h + \frac{1}{2}\sum |(\Delta E)_k|\sqrt{\psi(v_{k+1/2})}h + \frac{1}{2}\sum |(\Delta E)_{k+1}|\sqrt{\psi(v_{k+1/2})}h \\
\le& 3\left(\sum(\Delta E)_k^2 h\right)^{1/2}\left(\sum \psi(v_{k+1/2})h\right)^{1/2} \\
\le& C(T) \\
<& \infty,
\end{aligned}
$$

recalling that $R_k = \sqrt{\psi(v_{k+1/2})} + \sqrt{\psi(v_{k-1/2})}$.

But also, $|f(v)| \to \infty$ as $v \to 0$ or $v \to \infty$. Therefore there is a constant $M(T) > 0$ such that

$$M(T)^{-1} \le v \le M(T).$$

5.2 Existence of Invariant Measure

Now, this bound on v implies that the SDEs (5.7) have strong solutions globally in time. Furthermore, we have the bounds independent of h that

$$\mathbb{E}\left(\int_0^T \sum \epsilon(v_{k+1/2})(\Delta u)_k^2 h\, dt + \frac{1}{4}\sum(\Delta E)_k^2(T)h\right) \le C_0 T.$$

This implies, as in [51], that

$$\mathbb{E}\left(\sum(\Delta v)_k^2 h\right) \le C_0 T,$$

Integrating in time, we have

$$\mathbb{E}\left(\int_0^t \sum(\Delta v)_k^2 h\, ds\right) \le C(1+T^2).$$

Therefore we have the temporal average bound:

$$\mathbb{E}\left[\left(\int_0^t \sum(\Delta v)_k^2 h\, ds\right)^{1/2}\right] \leq C(1+T).$$

All the bounds above are uniform in h.

This implies that the the solution map $S : (v_0 - v', u_0) \mapsto (v - v', u)$ is compact on $L^2 \times L^2$, by the Tychonoff theorem (for example, even though this is just a finite product). Since it is compact, it is also continuous. Therefore, composed with any bounded continuous $\varphi \in C_b(L^2)$, it holds that the associated process $\mathscr{P}_s$ which takes φ to $\mathscr{P}_s\varphi = \varphi \circ S$ – a composition of continuous functions – is Feller. Now we invoke the calculations in §3.1.3 and conclude that an invariant measure exists as a limit of

$$\nu_T = \frac{1}{T}\int_0^T \mathscr{P}_s^* \delta_{(v_0,u_0)}\, ds.$$

Chapter 6

Conclusion

6.1 Summary

In summary, in this dissertation,

(i) we gave a unified framework for the L^1 contraction, L^1 stability and BV/Nikolskii norm estimates;

(ii) we derived a Nikolskii semi-norm estimate for the spatial translation non-invariant stochastic anisotropic degenerate parabolic-hyperbolic equation, which was non-trivial and depended on the regularity of various coefficients, unlike in the translation invariant case, in which the BV estimate follows from the L^1 contraction;

(ii) we generalised the results of [30] concerning existence and uniqueness of invariant measures in the first order stochastic conservation law to the second order equations, with additive noise, using insights from [13, 14];

(iv) we derived long-time behaviour results for the second order stochastic scalar conservation law with a class of unbounded, not necessarily smooth, and multiplicative noises;

(v) we derived a simple result on the long-time behaviour of solutions to a system of equations.

The Overview (§1.2.1) gave a summary of the techniques and ideas behind each of these topics.

6.2 Looking Ahead

Here we discuss some further questions.

6.2.1 Multiplicative Noises

There are a few natural questions that follow from the studies cited. We restrict ourselves again to conservation laws (with stochastic force). The most natural is that of long-time behaviour of multiplicative noise. Noises of the form $\nabla \cdot A(u) \circ dW$ have been considered [41, 42], because in that case the dynamics remains contained in the zero-spatial-average subspace of $L^1(\mathbb{T}^d)$.

The well-posedness questions for equations with multiplicative noise is quite well understood from several different perspectives – the strong entropic stochastic solutions of Feng and Nualart [38], and of Chen, Ding, and Karlsen [10], the viscosity solution methods of Bauzet, Vallet, and Wittbold[3], and the kinetic approach of Debussche, Hofmanovà, and Vovelle [28, 29], which we have mentioned in the introductory chapter. Nevertheless, apart from compactness as shown in Chapter 4, the question of long-time behaviour is entirely open, because there is no effective way to control $\|u(t)\|_{L^1}$.

We remark on two aspects of the noise that can affect qualitative long-time behaviour:

(i) The question seems to be heavily dependent on the roots and growth of the noise σ – If the noise were degenerate (not cylindrical), and $\sigma(r) = 0$ for certain $r \in \mathbb{R}$, then $u(x) \equiv r$ is a fixed point of the evolution, and by L^1 contraction, it is possible to prove certain long-time behaviour results on solutions u as was done Chapter 4, which studied unbounded noise with one root. Both the growth of σ and how many roots it possesses affect the long time behaviour of solutions, as is evident also in studies of other equations like the KPP equation (discussed below). In the case that σ has no roots, there are no fixed points. It is possible that bounded noise with no roots exhibit other behaviours, such as the existence of a non-trivial invariant measure.

(ii) If the noise is $\sigma(u)dB = \sum_k g_k(u)dW^k$, where $B = W^k e_k$ is a cylindrical Wiener process, the behaviour is expected to be very different than if the noise is simply $\sigma(u)dW$.

6.2.2 Non-white noises

It would be interesting to explore the consequences of having more general Lévy noises in place of a white noise. Whilst most studies on stochastically driven nonlinear equations, such as the Burger's Equation [98, 93, 36, 49] append to the equation a Brownian noise term (and except in the case of [36], and thereafter, also a dissipation term), work has also appeared in which Lévy noises were considered [62, 73, 66].

It would also be of interest to explore the question of invariant measures for equations with noises arising from processes for which temporal increments are at least weakly correlated, as do arise from fractional Brownian motion. The goal would be to establish some correspondence between different types of noise and the corresponding invariant measures where they exist – a kind of continuous dependence of the sets of invariant measures on the noise, or the dynamics of the set of invariant measures (possibly null) as a (deterministic) map varies the statistics of the noise. To our knowledge, no such thing has been systematically done.

It is not immediately clear what it means to vary the statistics of the noise continuously. A starting point can be that of varying the Hurst parameter of noise arising from Brownian motion, and by and by in developing a topology for the statistics of (various classes of) noises.

6.2.3 Infinite Dimensional Dynamical Systems

As remarked in [49], infinite dimensional dynamical systems is a field still early in development compared to the finite dimensional theory. Properties of long-time behaviour are not limited to the existence and uniqueness of invariant measures or their negations. Invariant measures, whilst an indispensible tool in probing the asymptotic properties of dynamical systems, themselves have properties beyond existence and uniqueness that are of interest in the understanding of these systems. For example, it is well known that random attracting sets are intimately related to the supports of invariant measures [25, 24], and these random attracting sets have interesting properties of their own [26, 27] that give insight into statistical and asymptotic behaviour of solutions to infinite dimensional random dynamical systems. It would be interesting to study the finer properties of invariant measures to infinite dimensional random dynamical systems, and through them to learn about the long-time behaviour of such systems.

6.2.4 Systems of Equations

In the previous chapter we derived the existence of invariant measures for the one dimensional compressible Navier-Stokes system (*of equations* – not to be confused with dynamical "systems") with noise. A natural question to consider is the long-time behaviour of other common systems of equations. In systems of equations, such as for the isentropic Euler system, the kinetic formulation might not be "pure" – it contains instances of the solution u mixed with the kinetic equation [80]. At present it seems the methods of [30] are not directly applicable to this system, although [6] have developed a well-posedness theory for it and provided numerical evidence that we should expect the existence of an invariant measure. Smith [99] showed global well-posedness of weak solutions to the d-dimensional stochastic compressible Navier-Stokes equations. The incompressible fluid equations have seen comparably more effort [71, 72, 49] (and references in [83]). The intersection of infinite dimensional dynamical systems theory and the theories of systems of nonlinear equations, especially of compressible fluid equations, is a promising field as even the deterministic cases are rich with many phenomena such as solitons.

6.2.5 Malliavin Calculus

The Malliavin Calculus is important in some aspects of the study of stochastic PDEs. Malliavin originally invented his calculus to show that Hörmander's condition is sufficient for the existence of a smooth density for the solution of an SDE. In his book Stroock [101] used a technique suggestive of the Malliavin Calculus to show that linear degenerate parabolic equations have measure-valued solutions. The Calculus is also widely applied to questions in the field of mathematical finance [89].

As mentioned in §1.2.1 Karlsen and Storrøsten [61] used Malliavin Calculus in modifying the entropic framework of [3] to suggest an origin for the strong stochastic entropic property introduced [38]. They also discussed how there were different approaches to a well-posedness theory of stochastic first-order scalar conservation laws, mentioning that the kinetic framework of [28] sidestepped the noise-noise interaction that called for the strong stochastic entropic property. It would be interesting to study more generally the relationship between the kinetic formulation and the Malliavin Calculus in the setting of stochastic PDEs.

Bibliography

[1] L. Arnold. *Random Dynamical Systems*. Springer-Verlag Berlin Heidelberg, Germany, 1st edition, 1998.

[2] J. Ball and F. Murat. Remarks on Chacon's biting lemma. *Proc. A.M.S.*, 107(3):655 – 663, 1989.

[3] C. Bauzet, G. Vallet, and P. Wittbold. The Cauchy problem for conservation laws with a multiplicative stochastic perturbation. *J. Hyperbolic Diff. Eq.*, 9(4):661 – 709, 2012.

[4] P. Benilan and M. Crandall. The continuous dependence on φ of solutions to $u_t + \Delta\varphi = 0$. *Indiana Univ. Math. J.*, 30(2):161 – 177, 1980.

[5] B. Bergé and B. Saussereau. On the long-time behaviour of a class of parabolic spdes: Monotonicity methods and exchange of stability. *ESAIM: P & S*, 9:254 – 276, 2005.

[6] Berthelin and J. Vovelle. Stochastic isentropic Euler equations. *(preprint)*, pages 1 – 73, 2015.

[7] M. Bézard. Régularité L^p précisée des moyennes dans les équations de transport. *Bull. Soc. Math. France*, 122(1):29 – 76, 1994.

[8] N. Bogoliubov and N. Krylov. La théorie générale de la mesure dans son application à l'étude des systemes dynamiques de la mécanique non linéaire. *Annals of Math.*, 38(1):65 – 113, 1937.

[9] F. Bouchut and L. Desvillettes. Averaging lemmas without time Fourier transform and application to discretized kinetic equations. *Proc. Royal Soc. Edin. Sec. A*, 129:19 – 36, 1999.

[10] G.-Q. Chen, Q. Ding, and K. Karlsen. On nonlinear stochastic balance laws. *Arch. Rat. Mech. Anal.*, 204(3):707–743, 2012.

[11] G.-Q. Chen and H. Frid. Large time behaviour of entropy solutions in L^∞ for multidimensional conservation laws. *Advances in Nonlinear PDEs and Related Areas*, pages 28 – 44, 1998.

[12] G.-Q. Chen and P. Pang. Invariant measures for stochastic conservation laws. *Preprint in preparation*, 2017.

[13] G.-Q. Chen and B. Perthame. Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations. *Ann. l'I.H.P.*, 20(4):645–668, 2003.

[14] G.-Q. Chen and B. Perthame. Large-time behavior of periodic entropy solutions to anisotropic degenerate parabolic-hyperbolic equations. *Proc. A.M.S.*, 137(9):3003 – 3011, 2009.

[15] I. Chueshov and P. Vuillermot. On the large-time dynamics of a class of parabolic equations subjected homogeneous white noise: Stratonovitch's case. *C.R. Acad. Sci. Paris*, 323(1):29 – 33, 1996.

[16] I. Chueshov and P. Vuillermot. On the large-time dynamics of a class of random parabolic equations. *C.R. Acad. Sci. Paris*, 322(1):1181 – 1186, 1996.

[17] I. Chueshov and P. Vuillermot. Long-time behaviour of solutions to a class of quasilinear parabolic equations with random coefficients. *Ann. I.H.P. Anal. Non-Linéaires*, 15(2):191 – 232, 1998.

[18] I. Chueshov and P. Vuillermot. Long-time behaviour of solutions to a class of stochastic parabolic equations with homogeneous white noise: Stratonovitch's case. *Probab. theory Relat. Fields*, 112(2):149 – 202, 1998.

[19] I. Chueshov and P. Vuillermot. On the large-time dynamics of a class of parabolic equations subjected to homogeneous white noise: Itô's case. *C.R. Acad. Sci. Paris*, 326(1):1299 – 1304, 1998.

[20] I. Chueshov and P. Vuillermot. Long-time behaviour of solutions to a class of stochastic parabolic equations with homogeneous white noise: Itô's case. *Stochastic Analysis and Applications*, 18(4):581 – 615, 2000.

[21] B. Cockburn and G. Gripenberg. Continuous dependence on the nonlinearities of solutions of degenerate parabolic equations. *J. Diff. Eq.*, 151:231 – 251, 1999.

[22] H. Crauel. Markov measures for random dynamical systems. *Stochastics and Stochastic Reports*, 37(3):153 – 173, 1991.

[23] H. Crauel. *Random Probability Measures on Polish Spaces.* Taylor and Francis, London, 1st edition, 2002.

[24] H. Crauel, A. Debussche, and F. Flandoli. Random attractors. *J. Dynam. Diff. Eq.*, 9(2):307 – 341, 1997.

[25] H. Crauel and F. Flandoli. Attractors for random dynamical systems. *Probab. Rel. Fields*, 100(3):365 – 393, 1994.

[26] A. Debussche. On the finite dimensionality of random attractors. *Stoch. Anal. Appl.*, 15(4):473 – 491, 1997.

[27] A. Debussche. Hausdorff dimension of a random invariant set. *J. Math. Pures Appl.*, 9(4):967 – 988, 1998.

[28] A. Debussche, M. Hofmanová, and J. Vovelle. Degenerate parabolic stochastic partial differential equations: Quasilinear case. *Ann. Prob.*, (to appear), 2014.

[29] A. Debussche and J. Vovelle. Scalar conservation laws with stochastic forcing. *J. Func. Anal.*, 259(4):1014–1042, 2010.

[30] A. Debussche and J. Vovelle. Invariant measure of scalar first-order conservation laws with stochastic forcing. *Prob. Theory Relat. Fields*, 163(3):575 – 611, 2015.

[31] R.J. DiPerna, P.-L. Lions, and Y. Meyer. l^p regularity of veolcity averages. *Ann. I.H.P. Ann. l'I.H.P. Anal. Non Linéaire*, 8(3-4):271 – 287, 1991.

[32] W. Doeblin. Éléments d'une théorie générale des chaînes simples constantes de Markoff. *Ann. Sci. École Normale Superieur*, 57:61 – 111, 1940.

[33] W. Doeblin and R. Fortet. Sur le chaînes à liaisons complète. *Bull. Soc. Math. France*, 65:132 – 148, 1937.

[34] J. Doob. Asymptotic property of Markoff transition probability. *Trans. Amer. Math. Soc.*, 64(3):393 – 421, 1948.

[35] R. Dudley. *Real Analysis and Probability.* Cambridge University Press, Cambridge, UK, 1st edition, 2002.

[36] W. E, K. Khanin, A Mazel, and Ya. Sinai. Invariant measures for Burgers equation with stochastic forcing. *Annals of Math.*, 151(3):877 – 960, 2000.

[37] Lawrence Craig Evans. *Weak Convergence Methods for Nonlinear Partial Differential Equations.* American Mathematical Society, USA, 1 edition, 1998.

[38] J. Feng and D.Nualart. Stochastic scalar conservation laws. *J. Func. Anal.*, 255(2):313–373, 2008.

[39] F. Flandoli. Stochastic flows and Lyapunov exponents for abstract stochastic PDEs of parabolic type. *in Lyapunov Exponents, L. Arnold, H. Crauel, J.-P. Eckamann eds., LNM 1486*, pages 196 – 205, 1991.

[40] F. Flandoli and B. Maslowski. Ergodicity of the 2-D Navier-Stokes equations under random perturbations. *Comm. Math. Phys.*, 172(1):119 – 141, 1995.

[41] B. Gess and P. Souganidis. Long-time behavior, invariant measures and regularizing effects for stochastic scalar conservation laws. *arXiv:1411.3939*, 2014.

[42] B. Gess and P. Souganidis. Stochastic non-isotropic degenerate parabolic-hyperbolic equations. *arXiv:1611.01303*, 2016.

[43] Y. Giga and T. Miyakawa. A kinetic construction of global solutions of first order quasilinear equations. *Duke Math. J.*, 50(2):505–515, 1989.

[44] F. Golse and P. Gérard. Averaging regularity results for pdes under transversality assumptions. *Comm. Pure Appl. Math.*, 45(1):1 – 26, 1992.

[45] F. Golse, P.-L. Lions, B. Perthame, and R. Sentis. Regularity of the moments of the solution of a transport equation. *J. func. Anal.*, 76(1):110 – 125, 1988.

[46] F. Golse, B. Perthame, and R. Sentis. Un résultat de compacité pour les équations de transport et application au calcul de la limite de la valeur propre principale dun

opérateur de transport. *C.R. Acad. Sci. Paris Séries I: Mathematique*, 301(7):341 – 344, 1985.

[47] F. Golse and L. St Raymond. Velocity averaging in l^1 for the transport equation. *C.R. Math. Acad. Sci. Paris*, 334(7):557–562, 2002.

[48] M. Hairer. Convergence of Markov processes. *http://www.hairer.org/notes/Convergence.pdf*, 2016.

[49] M. Hairer and J. Mattingly. Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing. *Annals of Math.*, 164:993 – 1032, 2006.

[50] D. Hoff. Construction of solutions for compressible isentropic Navier-Stokes equations in one space dimension with non-smooth initial data. *Proc. Roy. Soc. Edinburgh Sec. A*, 103:301 – 315, 1986.

[51] D. Hoff. Global existence for 1d, compressible isentropic Navier-Stokes equations with large initial data. *Transactions of the AMS*, 303(1):169 – 181, 1987.

[52] D. Hoff. Global solutions of the equations of one-dimensional, compressible flow with large data and forces, and with differing end states. *Z. agnew. Math. Phys.*, 49:774 – 785, 1998.

[53] M. Hofmanová. Degenerate parabolic stochastic partial differential equations: Quasilinear case. (to appear), 2013.

[54] H. Holden and N. Risebro. Conservation laws with a random source. *Applied Mathematics and Optimization*, 36(2):229 – 241, 1997.

[55] P.-E. Jabin and B. Perthame. Regularity in kinetic formulations via averaging lemmas. *ESAIM COCV*, 8:761 – 774, 2002.

[56] P.-E. Jabin and L. Vega. A real space method for averaging lemmas. *J. Maths. Pures Appl.*, 83(11):1307 – 1351, 2004.

[57] Y. Le Jan. Équilibre statistique pour les produits de difféomorphismes aléatoires indépendants. *Ann. Inst. Henri Poincaré Probab. Stat.*, 23(1):111 – 120, 1987.

[58] S. Jiang. Remarks on the asymptotic behaviour of solutions to the compressible Navier Stokes equations in the half-line. *Proc. Roy. Soc. Edinburgh Sec. A*, 132:627 – 638, 2002.

[59] Ya. Kanel'. On a model system of one dimensional gas motion. *Differential Equations*, 4(2):374 – 380, 1968.

[60] I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus.* Springer-Verlag New York, New York, 2nd edition, 1998.

[61] K. H. Karlsen and E. Storrøsten. On stochastic conservation laws and Malliavin calculus. *arXiv:1507.05518v2*, 2015.

[62] K.H. Karlsen and S. Ulusoy. Stability of entropy solutions for Lévy mixed hyperbolic-parabolic equations. *Electron. J. Diff. Equ.*, 2011(116):1 – 23, 2011.

[63] A.V. Kazhikov and V.V. Shelukhin. Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas. *PMM*, 41(2):282 – 291, 1977.

[64] R. Khas'minskii. Ergodic properties of recurrent diffusion processes and stabilization of the solutions to the Cauchy problem for parabolic equations. *Theory Probab. Appl.*, 5(2):179 – 196, 1960.

[65] A. Ya. Khinchin. Asymptotische gesetze der wahrscheinlichkeitsrechnung. *Ergebnisse der Mathematik*, 2(4), 1933.

[66] A. Løkka, B. Øksendal, and F. Proske. Stochastic partial differential equations driven by LÉVY space-time white noise. *Annals Appl. Probab.*, 14(3):1506 – 1528, 2004.

[67] S. Kruzhkov'. First order quasilinear equations with several space variables. *Math. USSR Sb.*, 10:217 – 273, 1970.

[68] S. Kruzhkov'. First order quasilinear equations in several independent variables. *Math. USSR Sb.*, 10:217 – 243, 1972.

[69] B. Øksendal, G. Våge, and H.Z. Zhao. Asymptotic properties of the solutions to stochastic kpp equations. *Proc. R. Soc. Edin. A*, 130:1363 – 1381, 2000.

[70] B. Øksendal, G. Våge, and H.Z. Zhao. Two properties of stochastic kpp equations: ergodicity and pathwise property. *Nonlinearity*, 14:639 – 662, 2001.

[71] S. Kuksin and A. Shirikyan. Stochastic dissipative PDEs and Gibbs measures. *Comm. Math. Phys.*, 221(2):351 – 366, 2000.

[72] S. Kuksin and A. Shirikyan. Ergodicity for the randomly forced 2D Navier-Stokes equations. *Math. Phys. Anal. Geom.*, 4(2):147 – 195, 2001.

[73] S. Kuksin and A. Shirikyan. *Mathematics of Two-Dimensional Turbulence*. Cambridge University Press, Cambridge, UK, 1st edition, 2012.

[74] F. Ledrappier. Positivity of the exponent for stationary sequences of matrices. *Lyapunov Exponents Proceedings (L. Arnold and V. Wihstutz (ed.))*, pages 56 – 73, 1986.

[75] C. De Lellis and M. Westdickenberg. On the optimality of velocity averaging lemmas. *Ann. l'I.H.P. Anal. Non Linéaire*, 20(6):1075 – 1085, 2003.

[76] J. Li and Z. Liang. Some uniform estimates and large-time behavior of solutions to one-dimensional compressible Navier-Stokes system in unbounded domains with large data. *Arch. Rat. Mech. Anal.*, 220(3):1195 – 1208, 2016.

[77] T. Lindvall. *Lectures on the Coupling Method*. John Wiley & Sons, New York, 1st edition, 1992.

[78] P.-L. Lions, B. Perthame, and P. Souganidis. Stochastic averaging lemmas for kinetic equations. *arXiv:1204.0317*, 2012.

[79] P.-L. Lions, B. Perthame, and E. Tadmor. A kinetic formulation of multidimensional scalar conservation laws and related equations. *J. A.M.S.*, 7(1):169–191, 1994.

[80] P.-L. Lions, B. Perthame, and E. Tadmor. Kinetic formulation of the isentropic gas dynamics and p-systems. *Comm. Math. Phys.*, 163:415 – 431, 1994.

[81] L. Lorenzi and M. Bertoldi. *Analytical Methods for Markov Semigroups*. Chapman and Hall/CRC, Boca Raton, Florida, 1st edition, July 2006.

[82] B. Maslowski and J. Seidler. Invariant measures for nonlinear SPDE's: uniqueness and stability. *Archivum Mathematicum*, 34(1):153 – 172, 1998.

[83] J.C. Mattingly. Recent progress for the stochastic Navier-Stokes equations. *Journeé ÉDP.*, (http://www.numdam.org/article/JEDP_2003____A11_0.pdf), 2003.

[84] H.P. McKean. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Comm. Pure Appl. Math.*, 28(3):323 – 331, 1975.

[85] F. Murat. Compacité par compensation. *Ann. Sc. Norm. Sup. Pisa*, 5(3):489 – 507, 1978.

[86] F. Murat. Compacité par compensation: condition nécessaire et suffisante de continuité faible sous une hypothèse de rang constant. *Ann. Sc. Norm. Sup. Pisa*, 8(1):69 – 102, 1981.

[87] H. Nakazawa. Stochastic Burgers' equation in the inviscid limit. *Advances in Applied Mathematics*, 3(1):18 – 42, 1982.

[88] E. Di Nezza, G Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *arXiv:1104.4345v3*, 2011.

[89] G. Di Nunno, B. Øksendal, and F. Proske. *Malliavin Calculus for Lévy Processes with Applications to Finance.* Springer Verlag Berlin Heidelberg, Berlin, 1st edition, 2009.

[90] R. J. Di Perna. Convergence of the viscosity method for isentropic gas dynamics. *Comm. Math. Phys.*, 91(1):1 – 30, 1983.

[91] B. Perthame. *Kinetic Formulation of Conservation Laws.* Oxford University Press, Oxford, UK, 1st edition, Dec 2002.

[92] S. Peszat and J. Zabczyk. Strong feller property and irreducibility for diffusions on hilbert spaces. *Annals of Prob.*, 23(1):157 – 172, 1995.

[93] G. Da Prato, A. Debussche, and R. Temam. Stochastic Burger's equation. *No.D.E.A.*, 1:389 – 402, 1994.

[94] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions.* Cambridge University Press, Cambridge, UK, reprint edition, 2008.

[95] Yu. Prohorov. Convergence of random processes and limit theorems in probability theory. *Theory Prob. Appl.*, 2(2):157 – 214, 1956.

[96] D. Serre. La capacité par compensation pour les systèmes nonlinéaires de deux équations à une dimension d'espace. *J. Math. Pures Appl.*, 65(2):423 – 468, 1987.

[97] J. Simon. Sobolev, Besov, and Nikolskii fractional spaces: Imbedding and comparisons for vector valued spaces on an interval. *Annali Math. pura ed applicata*, 52(4):117 – 148, 1990.

[98] Ya. Sinai. Two results concerning asymptotic behavior of solutions of the Burgers equation with force. *J. Stat. Phys.*, 64(1):1 – 12, 1991.

[99] S.A. Smith. Random perturbations of viscous, compressible fluids: Global existence of weak solutions. *arXiv:1504.00951v1*, 2015.

[100] J. Smoller. *Shock Waves and Reaction-Diffusion Equations.* Springer, USA, 2nd edition, 1983.

[101] Daniel W. Stroock. *Partial Differential Equations for Probabilists.* Cambridge University Press, Cambridge, UK, 1st edition, 2008.

[102] E. Tadmor and T. Tao. Velocity averaging, kinetic formulations and regularizing effects in quasi-linear pdes. *Comm. Pure Appl. Math.*, 60(10):1488 – 1521, 2007.

[103] L. Tartar. Compensated compactness and applications to partial differential equations. *Research Notes in Mathematics, Nonlinear Analysis and Mechanics: Heriot-Watt Symposium*, 4, 1979. , RJ Knops ed. Pitman Press, London.

[104] A. I. Vol'pert and S. I. Hudjaev (N. Koblitz tr.). Cauchy's problem for degenerate second order quasilinear parabolic equations. *Math. USSR Sbornik*, 7(3):365 – 387, 1969.

[105] R. De Vore and G. Petrova. The averaging lemma. *J. Math. Amer. Soc.*, 14(2):279 – 296, 2001.

[106] L. C. Young. *Lectures on the Calculus of Variations and Optimal Control Theory.* W.B. Saunders Company, Piladelphia, PA, 1st edition, 1969.