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MATHEMATICS OF OPERATIONS RESEARCH

Strategic Bonding Curves in Automated Market Makers*

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Abstract. The bonding curves of decentralised exchanges (DEXs) define, with mathematical formulae, the relationship between liquidity supply and prices. Most DEXs use the bonding curves of constant function markets (CFMs) to clear the supply and demand of liquidity. At present, liquidity providers (LPs) operate at a loss in CFMs, on average. We generalise CFMs and introduce decentralised liquidity pools (DLPs) which provide LPs with the tools to design dynamic bonding curves according to strategic preferences. In DLPs, impact functions encode how orders affect prices and quote functions determine the price of liquidity. To illustrate the strategic flexibility of DLPs, we develop models for bonding curves when prices form across multiple venues, within the DLP, or in a competing venue. In fragmented markets, the DLP estimates the fundamental price from the trading flow to adjust the bonding curve and reduce arbitrage losses. Our models may be used as hooks in Uniswap v4 when the DLP's impact functions are the constant product function.

Key words: Blockchain Economics, Decentralised Finance, Automated Market Making, Stochastic AMMs, Uniswap v4
MSC2000 subject classification: Primary: 91G15, 93E20; secondary: 91B24, 93E35
OR/MS subject classification: Primary: Financial institutions: markets; secondary: Dynamic programming/optimal control: applications

1. Introduction

Matching buyers with sellers is a basic function of financial markets. In traditional electronic markets, the aggregation of participants' pending orders in limit order books (LOBs) or the aggregation of quotes from market makers in over-the-counter (OTC) markets determine supply curves that map quantities to prices, and the interactions between market participants determine the effect of orders on prices. In decentralised exchanges (DEXs), the supply of liquidity is aggregated into liquidity pools, and liquidity takers trade directly with the pool. In contrast to traditional markets, the mechanics of DEXs are defined in smart contracts, which are hard-coded programs running on a blockchain. These programs use mathematical formulae to determine execution prices and the price impact of orders, where both are a function of the supply of liquidity.

* An earlier version of this work was previously circulated under the title "Automated Market Makers Designs Beyond Constant Functions"; see [34].

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These functions, known as *bonding curves*, are key in DEXs because they underlie the mechanics of most trading venues and they determine how markets clear.

Currently, most DEXs operate as constant function markets (CFMs) where bonding curves are fixed trading functions. A key drawback of CFMs is that LPs are not fairly compensated due to exposure to predictable losses; see [29]. To mitigate these losses, liquidity takers (LTs) are charged a fee that is shared among the LPs who are counterparty to the trade. However, empirical evidence shows that despite collecting fees, LPs lose money in CFMs, on average.[1] Theoretical and experimental work attribute the losses of LPs to the design of CFMs: (i) LPs do not fully participate in price discovery because of their limited strategic behaviour, where price discovery is solely due to liquidity taking trades,[2] and (ii) the slippage incurred by LTs is determined solely by the aggregate liquidity reserves in the pool.

In this paper, we propose the decentralised liquidity pool (DLP). The DLP is a new design that overcomes the limitations of CFMs so DEXs may compete as viable trading venues. The bonding curve of the DLP is defined by (i) *impact functions* that determine how liquidity taking orders impact the marginal price of the pool, and (ii) *quote functions* that set the price of liquidity at the bid and at the ask around the marginal price. The DLP accommodates any set of impact and quote functions that preclude roundtrip arbitrages by LTs. Our design of DLP includes CFMs as a particular case; that is, there exist impact functions and quotes that replicate the marginal price dynamics and the execution costs implied by the trading function and the fee rate of a CFM.

In DLPs, LPs possess the tools to build bonding curves. More precisely, LPs set or modify the impact and quote functions to adjust buy and sell execution prices strategically. In this work, we show that DLPs offer a broad strategic flexibility to LPs. Specifically, we propose models of liquidity demand and price formation to derive dynamic bonding curves in the DLP. In our models, the wealth dynamics of LPs result from the liquidity taking trading flow which is sensitive to the price of liquidity in the DLP, and the bonding curve maximises a risk-reward objective for LPs. The price of liquidity is determined by the distance between the execution prices implied by the bonding curve and the fundamental price. In practice, the fundamental price is not directly observed so it is infeasible to encode it within a DEX's (hard-coded) smart contract. On the other hand, using an oracle price is problematic due to manipulation; see [23] and [24].

We propose three models of liquidity demand and price formation and derive corresponding strategic bonding curves when liquidity takers are sensitive to the price of liquidity. In the first model, prices form in multiple exchanges including the DLP. When markets are fragmented, the DLP employs the liquidity-taking flow to estimate the difference between the marginal price and the (unobserved) fundamental price. In this model, when the buying or selling pressure increases significantly, the probability of under- or over-estimating the (unobserved) fundamental price increases, so the DLP's bonding curve adjusts the price of liquidity at the bid and at the ask accordingly. In the second model, we assume that price discrepancies between the marginal and fundamental prices seldom appear because the trading activity is high or because the DLP is the sole trading venue for the pair of assets – such as many crypto-assets exclusively traded on DEXs. In this case, the DLP's bonding curve adjusts the price of liquidity according to the magnitude of the price impact of liquidity taking trades. Finally, the third model assumes price formation in an alternative trading venue, so the fundamental price of the asset is observed. In this model, the DLP's bonding curve adjusts the price of liquidity according to the difference between the marginal and fundamental

[1] See [49], [65], and [30] for instance.

[2] The strategic behaviour of LPs in CFMs with concentrated liquidity (CL) is focused on increasing fee revenue. Moreover, CL increases the predictable losses of LPs; see [30] and [57].

(oracle) prices, minimising the predictable losses of LPs. At present, the third model is best suited for DEXs, because crypto-currency prices mainly form in centralised venues such as Binance.

If the impact functions of the DLP are those of the trading function in a CFM, e.g., the constant product trading function of Uniswap, then the quote functions derived in our models may be used as a dynamic fee rate. In particular, our models may be used as a *hook* in Uniswap v4, i.e., as a smart contract which defines a dynamic fee rate. In this case, the impact functions are those of Uniswap and they employ the liquidity supply level in the pool; see [1, 70].

Numerous works study the microstructure of AMMs and liquidity provision. Early literature that describes the mechanics of AMMs is in [9], [6], and [8], where the authors show that the convexity of the bonding curve is key to describe the costs for both LTs and LPs in CFMs. [26] study the microstructure of AMMs theoretically and empirically. [61] describes competition between CFMs and LOBs. [56] shows that higher fees in AMMs increase liquidity supply, thereby decreasing the overall trading costs for LTs, and increasing trading volumes on blockchains; [19] and [7] also study the impact of the CFMs bonding curves on the returns of LPs in AMMs. [18] introduce an axiomatic framework for AMMs and define measures of price impact of trades. [68, 69] introduce LVR as a measure of the arbitrage gains of LTs in CFMs. [29] derives the losses of LPs in CFMs and in concentrated liquidity markets.[3] [72] studies the design of AMMs and the different types of cost when trading in these venues. [57] studies the risks and benefits of liquidity supply in decentralised concentrated liquidity markets. [66] explores the potential for AMMs to supply liquidity in equity markets. Finally, [27] studies the influence of gas fees on the informativeness of trades in AMMs and show, empirically, that high-fee trades contain more private information. Furthermore, recent work explores strategic liquidity provision to compensate for arbitrage losses in AMMs. [71] and [30] study strategic liquidity provision in CFMs with concentrated liquidity, [63] studies strategic liquidity provision in different types of AMMs, and [52] studies the hedging of the impermanent losses of LPs. For liquidity takers in AMMs, [31], [28], and [60] propose trading strategies in AMMs.

The literature that studies the design of AMMs mainly explores optimal dynamic fees in CFMs. [50] studies optimal fees in geometric markets, [74] studies variable fees in constant product markets (CPMs), [54] studies an AMM with a dynamic trading function that incorporates the beliefs of LPs about future asset prices, and [12] finds the fee structure that maximises order flow processed by the venue. Furthermore, [40] derives an upper bound for fee revenue to make liquidity provision profitable in CFMs, [43] frames optimal AMM design as an optimal transport problem, and [10] as a stochastic game between the venue and a representative LP.

Our models of liquidity demand in the DLP build on the literature for market making in OTC and LOB markets [11, 53, 58].[4] We combine a CFM's proportional fees and the execution costs from consuming its limited liquidity into the quote functions, which capture the effective cost of walking the bid or ask side of an equivalent LOB. In decentralised exchanges, LPs cannot choose dynamically the mid-price around which to provide liquidity; marginal prices are aligned to fundamentals only through arbitrage. In a DLP, LPs can use impact functions to replicate part of the strategic behaviour of market makers in traditional electronic exchanges. In these markets, market makers adjust prices and liquidity using prices observed in competing venues, client order flow, or private information.

Finally, the literature studies crypto-assets and blockchain economics.[5] [42] and [33] study the adoption of digital platforms for trading cryptocurrencies. [15] studies the drivers of cryptocurrency

[3] See [46] for a thorough description of the mechanics of concentrated liquidity markets.

[4] These frameworks have been extended in several directions; see [32, 38, 45, 47].

[5] For a more comprehensive review of this literature, see [16] and references therein.

prices and risks. [41] studies the economic implications of *layer-2* scaling solutions for AMMs and other smart contracts. [5] studies consensus in blockchain protocols and propose a model to produce only valid blocks in equilibrium. Finally, [25] study the implications of the observability of pending transactions in blockchain memory pools.

The remainder of this paper proceeds as follows. Section 2 introduces the mechanics of DLPs. We formalize the DLP bonding curve, derive conditions on the impact and quote functions to prevent roundtrip arbitrage, show that CFMs are a special case of DLPs, and discuss their practical implementation on blockchains. Section 3 illustrates how LPs can use DLPs to provide liquidity strategically. Specifically, Section 3.1 proposes a strategic bonding curve model that uses trading flow to estimate the fundamental price; Section 3.2 studies the case where the fundamental price is determined by an oracle; Section 3.3 develops a model for the setting in which the asset is traded exclusively in the DLP; and Section 3.4 presents numerical experiments demonstrating that the DLP design and our proposed models can outperform Uniswap v3.

2. The decentralised liquidity pool This section introduces decentralised liquidity pools (DLPs) as a new design for DEXs. The DLP is a liquidity pool that provides a venue to trade a pair of assets X and Y. The reference asset is X and Z is the pool's marginal price. The marginal price is the price in terms of X for an infinitesimal quantity of asset Y and is akin to the midprice in an LOB.

DLPs employ deterministic impact functions η^b and η^a of pool reserves to determine marginal price innovations, and they employ deterministic quote functions δ^b and δ^a of pool reserves and other variables, to determine execution prices. The impact and quote functions define the bonding curves that determine the mechanics of price and liquidity. More precisely, they define price impact and execution prices of liquidity taking trades as a function of liquidity supply. In contrast to CFMs, the bonding curves of DLPs can be tailored for strategic liquidity provision.

Below, Section 2.1 describes the mechanics of price and liquidity in DLPs, provides conditions on the impact functions to preclude roundtrip arbitrages, and formalises the DLP's bonding curves. Section 2.2 shows that CFMs are a special case of DLPs, and Section 2.3 studies the wealth of LPs in the DLP.

2.1. Formalisation of DLPs

Price and liquidity LTs interact with the DLP to buy and sell asset Y over a trading window $[0,T]$, where $T>0$ is fixed and known, and during which all LPs do not deposit nor withdraws any assets. At the start of the trading window, LPs deposit initial liquidity consisting of reserves x_0 and y_0 of both assets, and let Z_0 be the initial marginal price. LPs also set the impact and quote functions of the DLP. In practice, the impact and quote functions can be dynamically set by LPs as smart contract code via a blockchain transaction, much like Uniswap v4 hooks in the Ethereum decentralised finance ecosystem; see Section 2.5 for further details. Our definitions and models presented below can be extended to accommodate random trading windows $[0,\tau]$, where τ is a stopping time corresponding to events where an LP deposits or withdraws liquidity, or when the quote and impact functions are adjusted.[6]

Next, we specify how the pool reserves and the marginal prices are updated in the DLP.[7] Let $(x_t)_{t\in[0,T]}$ and $(y_t)_{t\in[0,T]}$ denote the reserves of both assets in the DLP, and let $(Z_t)_{t\in[0,T]}$ denote the

[6] See Section 2.5 for implementation and practical details for DLPs running on a blockchain.

[7] In the remainder of this paper, we work on a filtered probability space $(\Omega,\mathcal{F},\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$ that supports the processes we introduce and satisfies the usual conditions.

marginal price. In the DLP, LTs buy or sell any quantity that is a multiplier of the minimal unit $\zeta > 0$ of asset Y at any time $t \in [0,T]$, provided the reserve constraint $y_t \in \{\underline{y}, \underline{y}+\zeta, \ldots, \overline{y}-\zeta, \overline{y}\} = \mathcal{Y}$ is satisfied. Here, $\underline{y}$, $\overline{y}$, and y_0 are integer multiples of ζ satisfying $\underline{y} < y_0 < \overline{y}$. In practice, these constraints are not restrictive because $\underline{y}$ may be arbitrarily low (or negative), and $\overline{y}$ may be set arbitrarily large. We do not impose specific conditions on the reserve levels in asset X; however, it is implied that the reserves in asset X must be sufficient to accommodate liquidity taking trades and price changes resulting from adjustments between $\underline{y}$ to $\overline{y}$ in the reserves of asset Y. If reserves in one of the assets are depleted, then the DLP provides liquidity on only one side of the price.

To keep the exposition general, we introduce an adapted process $(I_t)_{t\in[0,T]}$ which takes values in some set $\mathcal{I}$ and represents any additional observable information, beyond the level of the reserves y, that the DLP may employ.[8] The DLP sets deterministic quote functions $\delta^b(\cdot,\cdot)$ and $\delta^a(\cdot,\cdot)$ which define the dynamic shifts $\delta^b_t = \delta^b(y_t, I_t)$ and $\delta^a_t = \delta^a(y_t, I_t)$. These shifts define the quoted bid price $Z_{t^-} - \delta^b_t$ and ask price $Z_{t^-} + \delta^a_t$ at which the pool would execute a buy and a sell order (respectively) for the minimal unit ζ of asset Y.[9] In our models below, the additional information process I includes time of day, marginal prices, oracle prices, and the (filtered) probability of under or over-valuation of the fundamental price.

Let $\left(N^b_t\right)_{t\in[0,T]}$ and $\left(N^a_t\right)_{t\in[0,T]}$ be Poisson processes that count the number of sell and buy orders of size ζ filled by the DLP, respectively, throughout the trading period $[0,T]$. As trading evolves, the reserves in assets X and Y in the DLP follow the dynamics

$$\mathrm{d}y_t = \zeta\,\mathrm{d}N^b_t - \zeta\,\mathrm{d}N^a_t \quad \text{and} \quad \mathrm{d}x_t = -\zeta\left(Z_{t^-} - \delta^b_t\right)\mathrm{d}N^b_t + \zeta\left(Z_{t^-} + \delta^a_t\right)\mathrm{d}N^a_t\,.$$

When the DLP receives a buy (resp. sell) order of quantity ζ, it is executed at the price $Z + \delta^a$ (resp. $Z - \delta^b$), after which the marginal price Z in the pool is updated, and liquidity is offered at a bid and an ask around the new marginal price. In practice, any liquidity-taking order with quantity greater than ζ is subdivided in multiple trades, each of size ζ. The updates in the marginal price Z are given by $\eta^a(\cdot,\cdot)$ and $\eta^b(\cdot,\cdot)$, which are non-negative deterministic impact functions of the pool reserves. These functions define the dynamic effect $\eta^b_t = \eta^b(y_{t^-}, I_t)$ and $\eta^a_t = \eta^a(y_{t^-}, I_t)$ of liquidity taking trades on the marginal price. Specifically, the impact functions determine the pool's marginal price response to incoming trades as a function of the reserves y in asset Y as follows:

$$\mathrm{d}Z_t = -\eta^b_t\,\mathrm{d}N^b_t + \eta^a_t\,\mathrm{d}N^a_t\,. \tag{1}$$

Thus, throughout a trading period, the DLP is characterised by initial reserves $y_0 \in \mathcal{Y}$, initial marginal price Z_0, unit ζ, trading horizon T, impact functions $\{\eta^b, \eta^a\}$, and quote functions $\{\delta^b, \delta^a\}$.

Impact functions Consider a DLP with impact functions $\eta^a(\cdot,\cdot)$ and $\eta^b(\cdot,\cdot)$ which receives a (roundtrip) sequence of liquidity taking trades $\{\epsilon_1, \epsilon_2\}$, where $\epsilon_1 = -\epsilon_2 = \pm 1$. General non-negative deterministic impact functions do not guarantee that the marginal price $Z + \eta^a(y, I) - \eta^b(y-\zeta, I)$ or $Z - \eta^b(y, I) + \eta^a(y+\zeta, I)$ at the end of the sequence is Z. Thus, we restrict the impact functions η^a and η^b to a set of functions which guarantees that the marginal price returns to its starting value for any roundtrip sequence of trades. To simplify notation, let $y_0 = \underline{y}$, $y_1 = \underline{y} + \zeta$, $\ldots$, and $y_N = \overline{y} = \underline{y} + N\,\zeta$. The following proposition provides necessary and sufficient conditions for roundtrip trades not to change the marginal price, and is the first step to design general no-arbitrage conditions for the DLP. A proof is in Appendix D.1.

[8] The set $\mathcal{I}$ will be specified below for each one of our strategic bonding curve models.

[9] t^- denotes the time just before t.

PROPOSITION 1. *The marginal price Z only takes the ordered finitely many values $\mathcal{Z} = \{z_0, \dots, z_N\}$, with the property that $Z_0 \in \mathcal{Z}$ and for $i \in \{0, \dots, N-1\}$*

$$z_{i+1} - \eta^b(y_{N-i}, I) = z_i \qquad \text{and} \qquad z_i + \eta^a(y_{N-i} + \zeta, I) = z_{i+1}\,,$$

if and only if η^a and η^b are such that, for $i \in \{0, \dots, N-1\}$,

$$\eta^b(y_i, I) = \eta^a(y_i + \zeta, I)\,. \tag{2}$$

The proposition above gives a necessary and sufficient condition for the marginal price Z to live within a fixed grid with lattice-style dynamics, that is, the marginal price can only move from z_i to z_{i-1} or to z_{i+1} provided it stays in the grid. In particular, this implies that there is a one-to-one mapping between the level of reserves y and the marginal price Z.

Simple examples of impact functions that satisfy (2) are constant impact functions $\eta^a(y, I) = \eta^b(y, I) = \eta \in \mathbb{R}^+$, and impact functions of the form $\{\eta^a(y, I) = f(y)\,,\ \eta^b(y, I) = f(y + \zeta)\}$ for any function f. In practice, while it is not strictly required, impact functions are generally expected to decrease as the level of reserves increases.[10] Finally, Theorem 1 below provides the impact functions that replicate the marginal price dynamics in a CFM with level function φ. These functions satisfy condition (2).

DLP bonding curve Bonding curves are a central concept in DEXs and are the underlying mechanism for automated market makers. They define, with mathematical formulae, the relationship between the (finite) liquidity supply in a pair of assets and the price impact and execution prices of liquidity taking trades. More precisely, the bonding curve maps changes in the reserves of asset Y to changes in the reserves in asset X in the pool following liquidity taking trades. The bonding curve of a CFM is defined by the level function (see Section 2.2). The level functions of CFMs are popular due to their minimal storage and computational requirements. Below, we formalise the concept of bonding curve for the DLP.

Fix impact functions η^b and η^a and fix the additional information I. Let $y \in \mathcal{Y}$ and $Z = Z(y) \in \mathcal{Z}$ be the current reserves in asset Y in the DLP and the current marginal price, respectively. Let $\delta^b(\cdot, I)$ and $\delta^a(\cdot, I)$ be the DLP's quote functions. We define the DLP's bonding curve $\varphi : \mathcal{Y} \mapsto \mathbb{R}$ as follows. First, fix $\varphi(y) = x$ as the reserves in asset X. For each $y_i \in \mathcal{Y}$, with $i \in \{1, \dots, N\}$, we write

$$\varphi(y_i) = \varphi(y_{i-1}) + \zeta\, \delta^b(y_{i-1}, I) - \zeta\, z_{N-i+1}\,, \tag{3}$$

and

$$\varphi(y_i) = \varphi(y_{i+1}) + \zeta\, \delta^a(y_{i+1}, I) + \zeta\, z_{N-i-1}\,, \tag{4}$$

where z_{N-i} is the DLP's marginal price when the reserves are y_i.

More precisely, when the reserves in the DLP are $y_{i-1} = y_i - \zeta$ and the marginal price z_{N-i+1}, a sell order of size ζ moves the pool's reserves from $y_{i-1} = y_i - \zeta$ to y_i (and moves the marginal price from z_{N-i+1} to z_{N-i}). By definition of the unitary bid price, the cash received by the LT is

$$\zeta\Big(\underbrace{z_{N-i+1}}_{\text{price before the trade}} - \underbrace{\delta^b(y_i - \zeta, I)}_{\text{bid quote before the trade}} \Big) = \varphi(\text{old state}) - \varphi(\text{new state}) = \varphi(y_i - \zeta) - \varphi(y_i)\,,$$

which establishes (3). The equality in (4) follows by a symmetry argument.

[10] The impact functions for DLPs offering liquidity for pairs of stablecoins that replicate the same currency may be set to zero.

Finally, if the current DLP reserves are y_i and the marginal price z_{N-i}, then for a sell liquidity taking trade of quantity $\Delta y = n\,\zeta$, the execution price is

$$\frac{\varphi(y_i+\Delta y)-\varphi\left(y_i\right)}{\Delta y}=\frac{\zeta}{\Delta y}\sum_{j=1}^{n}\left(\delta^b\left(y_{i+j},I\right)-z_{N-i-j}\right),$$

and for a buy liquidity taking trader of quantity $\Delta y = n\,\zeta$, the execution price is

$$\frac{\varphi(y_i-\Delta y)-\varphi\left(y_i\right)}{\Delta y}=\frac{\zeta}{\Delta y}\sum_{i=1}^{n}\left(\delta^a\left(y_{i-j},I\right)+z_{N-i+j}\right).$$

Thus, the bonding curve φ maps liquidity supply to execution prices by determining the changes in the reserves of asset X and Y following liquidity taking trades. The function φ changes whenever the additional information I changes or when the quote and impact functions change. For CFMs, one employs $\mathcal{I}=\emptyset$, the impact functions (6), and the quote functions (7) defined below to obtain the CFM bonding curve, or more simply, the *level function*. Next, we show that CFMs are a particular class of DLPs.

2.2. CFMs are a particular case of DLP In CFMs, the bonding curve is determined by a trading function; see [28]. Here, we show that CFMs are a particular case of the DLP for appropriate choices of the impact functions $\{\eta^b,\eta^a\}$ and quote functions $\{\delta^b,\delta^a\}$.

Consider a CFM that makes liquidity in the pair of assets X and Y throughout the trading period $[0,T]$. The trading function $f:\mathbb{R}_+\times\mathbb{R}_+\mapsto\mathbb{R}_+$ of the CFM is twice differentiable and increasing in both arguments. Let m be the depth of the CFM pool and let $\left(x_t^{\mathrm{CFM}}\right)_{t\in[0,T]}$ and $\left(y_t^{\mathrm{CFM}}\right)_{t\in[0,T]}$ be the processes describing the amount of asset X and of asset Y, respectively, in the CFM pool, where x_0^{CFM} and y_0^{CFM} are known. Recall that the liquidity taking trading condition $f(x_t,y_t)=m^2$ in the CFM must hold for all $t\in[0,T]$, and there exists an appropriate *level* function $\varphi:\mathbb{R}_+\mapsto\mathbb{R}_+$ such that $x_t=\varphi(y_t)$ for all $t\in[0,T]$.

By Proposition 1 in [30], the level function φ is convex and the marginal price in the CFM pool at time t^- is $-\varphi'\left(y_{t^-}^{\mathrm{CFM}}\right)$. In the absence of fees, when processing a trade for the minimum trade size ζ, the dynamics of the amounts of asset X and asset Y and the marginal price Z^{CFM} in the CFM pool are given by

$$\begin{aligned}
\mathrm{d}y_t^{\mathrm{CFM}}&=\zeta\,\mathrm{d}N_t^b-\zeta\,\mathrm{d}N_t^a,\\
\mathrm{d}x_t^{\mathrm{CFM}}&=\left(\varphi\left(y_{t^-}^{\mathrm{CFM}}+\zeta\right)-\varphi\left(y_{t^-}^{\mathrm{CFM}}\right)\right)\mathrm{d}N_t^b+\left(\varphi\left(y_{t^-}^{\mathrm{CFM}}-\zeta\right)-\varphi\left(y_{t^-}^{\mathrm{CFM}}\right)\right)\mathrm{d}N_t^a,\\
\mathrm{d}Z_t^{\mathrm{CFM}}&=\left(-\varphi'\left(y_{t^-}^{\mathrm{CFM}}+\zeta\right)+\varphi'\left(y_{t^-}^{\mathrm{CFM}}\right)\right)\mathrm{d}N_t^b+\left(-\varphi'\left(y_{t^-}^{\mathrm{CFM}}-\zeta\right)+\varphi'\left(y_{t^-}^{\mathrm{CFM}}\right)\right)\mathrm{d}N_t^a.
\end{aligned}\tag{5}$$

When the CFM charges a proportional fee $\pi\in[0,1)$, the dynamics of x^{CFM} become

$$\begin{aligned}
\mathrm{d}x_t^{\mathrm{CFM}}=&\left(\varphi\left(y_{t^-}^{\mathrm{CFM}}+\zeta\right)-\varphi\left(y_{t^-}^{\mathrm{CFM}}\right)+\pi\,\zeta\left(-\varphi'(y_{t^-})\right)\right)\mathrm{d}N_t^b\\
&+\left(\varphi\left(y_{t^-}^{\mathrm{CFM}}-\zeta\right)-\varphi\left(y_{t^-}^{\mathrm{CFM}}\right)+\pi\,\zeta\left(-\varphi'(y_{t^-})\right)\right)\mathrm{d}N_t^a.
\end{aligned}$$

The next theorem shows that there are impact functions and quotes in the DLP that replicate the marginal price dynamics and the execution costs in a CFM.

THEOREM 1. *Let φ be the level function of a CFM. Assume the DLP chooses the impact functions*

$$\eta^a(y) = \varphi'(y) - \varphi'(y-\zeta)\,, \qquad \eta^b(y) = -\varphi'(y) + \varphi'(y+\zeta)\,, \tag{6}$$

and chooses the quote functions

$$\begin{aligned}\delta^a\left(y_{t^-}\right) &= \frac{\varphi(y_{t^-}-\zeta)-\varphi(y_{t^-})}{\zeta} + \varphi'(y_{t^-}) + \pi\left(-\varphi'(y_{t^-})\right),\\ \delta^b\left(y_{t^-}\right) &= \frac{\varphi(y_{t^-}+\zeta)-\varphi(y_{t^-})}{\zeta} - \varphi'(y_{t^-}) + \pi\left(-\varphi'(y_{t^-})\right).\end{aligned} \tag{7}$$

Then, the marginal price dynamics, reserve dynamics, and execution costs in the DLP are the same as those in the CFM with level function φ.

For a proof, see D.2.

2.3. Wealth of LPs in the DLP The bonding curve defines execution prices for any quantity of asset Y for LTs. Thus, it also determines the wealth of LPs. Here, we show that the dynamics of the wealth of LPs in the DLP are mathematically tractable, so they can be more easily optimised.

We write the dynamics of the pool's value in terms of asset X for any pair of counting processes $\left(N^b_t\right)_{t\in[0,T]}$ and $\left(N^a_t\right)_{t\in[0,T]}$, which count the number of sell and buy orders filled by the DLP, respectively. Consider a trading period $[0,T]$ in the DLP. At time $t=0$, LPs deposit and commit initial liquidity (x_0, y_0) in assets X and Y, respectively, and they set the impact and quote functions. In the DLP, LPs can only withdraw or deposit additional liquidity at the end of the trading period $[0,T]$, so the value of the DLP's reserves is self-financing. The dynamics of the value of the DLP's liquidity reserves $(\alpha_t)_{t\in[0,T]} = (x_t + y_t\, Z_t)_{t\in[0,T]}$ in terms of asset X are

$$\mathrm{d}\left(x_t + y_t\, Z_t\right) = y_t\,\mathrm{d}Z_t + \zeta\,\left(\delta^b_t - \eta^b_t\right)\,\mathrm{d}N^b_t + \zeta\,\left(\delta^a_t - \eta^a_t\right)\,\mathrm{d}N^a_t\,. \tag{8}$$

Assume Z represents the fundamental price. The dynamics (8) reveal that providing liquidity in the DLP introduces an additional term compared with the dynamics of holding the reserves y_t outside the pool, that is

$$\mathrm{d}\left(x_t + y_t\, Z_t\right) - y_t\,\mathrm{d}Z_t = \zeta\,\left(\delta^b_t - \eta^b_t\right)\,\mathrm{d}N^b_t + \zeta\,\left(\delta^a_t - \eta^a_t\right)\,\mathrm{d}N^a_t\,. \tag{9}$$

This component represents the revenue δ^b_t or δ^a_t following sell or buy trades, respectively, and the loss in value $-\eta^b_t$ or $-\eta^a_t$ due to adverse selection for providing liquidity if the change in the marginal price corresponds to a change in the value of the fundamental price.

The case of CFMs To replicate a CFM with level function φ, the impact functions of the DLP are in (6) and the price of liquidity is in (7). Assume that the proportional transaction fee π in (7) is zero. The dynamics in (9) become

$$\begin{aligned}\zeta\,\left(\delta^b_t - \eta^b_t\right)\,\mathrm{d}N^b_t + \zeta\,\left(\delta^a_t - \eta^a_t\right)\mathrm{d}N^a_t = {}& \left(\varphi(y_{t^-}+\zeta) - \varphi(y_{t^-}) - \zeta\,\varphi'\left(y_{t^-}+\zeta\right)\right)\,\mathrm{d}N^b_t\\ &+ \left(\varphi(y_{t^-}-\zeta) - \varphi(y_{t^-}) + \zeta\,\varphi'\left(y_{t^-}-\zeta\right)\right)\,\mathrm{d}N^a_t\,.\end{aligned} \tag{10}$$

The work in [48], [7], and [30] shows that a no-arbitrage condition implies that the level function φ must be convex. Thus

$$\begin{cases}\varphi(y_{t^-}+\zeta) - \varphi(y_{t^-}) - \zeta\,\varphi'\left(y_{t^-}+\zeta\right) & \leq 0\,,\\ \varphi(y_{t^-}-\zeta) - \varphi(y_{t^-}) + \zeta\,\varphi'\left(y_{t^-}-\zeta\right) & \leq 0\,,\end{cases} \tag{11}$$

so the terms (11) in (10) are always nonpositive, so liquidity provision in CFMs comes at a cost.

The next section derives conditions for no-arbitrage in the DLP.

2.4. No-arbitrage in the DLP Consider a DLP with reserves $y_0 \in \mathcal{Y}$ in asset Y, marginal price Z_0, minimal buy/sell quantity ζ, and impact functions $\eta^a(\cdot)$ and $\eta^b(\cdot)$. The DLP trades with only one LT who starts with zero cash. Here, arbitrage is any (roundtrip) sequence of trades $\{\epsilon_1, \dots, \epsilon_{\mathfrak{m}}\}$, where $\epsilon_k = \pm 1$ (buy/sell) and $y_k \in \mathcal{Y}$, for $k \in \{1, \dots, \mathfrak{m}\}$ and $\sum_{k=1}^{\mathfrak{m}} \epsilon_k = 0$, such that the value of the terminal cash of the LT is positive. In AMMs that run on blockchains, LTs use atomic transactions to implement arbitrage sequences; see [76].[11] Thus, all the trades ϵ_k of the roundtrip sequence are executed in an orderly manner and simultaneously (hence in the same block and with the same timestamp), and we drop the time variable t.

For any liquidity provision strategy $(\delta^b, \delta^a) = (\mathfrak{d}^b(y, I), \mathfrak{d}^a(y, I))$, where $\mathfrak{d}^b$ and $\mathfrak{d}^a$ are functions of the pool reserves y and other information I, there are choices of impact functions $\eta^a(\cdot)$ and $\eta^b(\cdot)$ that make arbitrage possible. Let $\mathfrak{m} = 2$ and assume that the LT does either the roundtrip trade (i) buy-then-sell ($\epsilon_1 = 1$ and $\epsilon_1 = -1$) or (ii) sell-then-buy ($\epsilon_1 = -1$ and $\epsilon_1 = 1$).[12] Use the dynamics in (2.1) and (1) to obtain the profit and loss (P&L) of the LT after the roundtrip trade as

$$\begin{aligned} &\text{(i)} \qquad \text{P\&L} = \zeta \left(\eta^a (y_0, I) - \mathfrak{d}^a(y_0, I) - \mathfrak{d}^b(y_0 - \zeta, I) \right) \\ &\text{or (ii)} \qquad \text{P\&L} = \zeta \left(\eta^b (y_0, I) - \mathfrak{d}^b(y_0, I) - \mathfrak{d}^a(y_0 + \zeta, I) \right) . \end{aligned} \tag{12}$$

If the DLP sets impact functions $\eta^a(\cdot, \cdot)$ and $\eta^b(\cdot, \cdot)$ such that either of the above expressions is positive, then, arbitrageurs can execute roundtrip trades with risk-free profits, which would drive LPs out of business. Clearly, the profits in (12) are negative if the bid quote after a buy trade

$$Z_0 + \eta^a(y_0, I) - \mathfrak{d}^b(y_0 - \zeta, I)$$

is less than the ask quote before the trade

$$Z_0 + \mathfrak{d}^a(y_0, I)$$

because it guarantees

$$\eta^a(y_0) \leq \mathfrak{d}^b(y_0 - \zeta, I) + \mathfrak{d}^a(y_0, I) , \tag{13}$$

and conversely for a sell trade.

The no-arbitrage condition (13) is for roundtrip sequences of length $\mathfrak{m} = 2$. The next result provides a general no-arbitrage condition for the DLP for roundtrip sequences of any length.

THEOREM 2. *Let $\eta^a(\cdot, \cdot)$ and $\eta^b(\cdot, \cdot)$ satisfy* (2) *for $i \in \{0, \dots, N-1\}$. For any liquidity provision strategy of the form $(\delta^b, \delta^a) = (\mathfrak{d}^b(y, I), \mathfrak{d}^a(y, I))$, for all $i \in \{0, \dots, N-1\}$ and all $I \in \mathcal{I}$, there is no roundtrip sequence of trades that a liquidity taker can execute to arbitrage the DLP if*

$$\eta^a(y_{i+1}, I) \leq \mathfrak{d}^a (y_{i+1}, I) + \mathfrak{d}^b (y_{i+1} - \zeta, I) \quad \text{and} \quad \eta^b(y_i, I) \leq \mathfrak{d}^b (y_i, I) + \mathfrak{d}^a (y_i + \zeta, I) , \tag{14}$$

or equivalently

$$\eta^a(y_{i+1}, I) \leq \mathfrak{d}^a (y_{i+1}, I) + \mathfrak{d}^b (y_i, I) \quad \text{and} \quad \eta^b(y_i, I) \leq \mathfrak{d}^b (y_i, I) + \mathfrak{d}^a (y_{i+1}, I) .$$

[11] Transactions in the Ethereum blockchain, where the majority of AMMs run, are atomic. A transaction is an ordered set of instructions to different protocols. A blockchain instruction is *atomic* because each instruction has to be valid for the whole transaction to be executed. In Ethereum, all the instructions of a transaction are executed *simultaneously*.

[12] Our results below hold for any $m \geq 2$ as shown in the proof of Theorem 2.

For a proof, see D.3. The conditions of Theorem 2 are standard in a trading venue that is competitive for LPs. For general liquidity provision strategies, condition (14) implies that in any state of the pool, the bid quote after a buy is lower than the ask quote before the trade is executed, and the ask quote after a sell is higher than the bid quote before the trade is executed.

Below, we illustrate how LPs can provide liquidity strategically. We propose models to construct dynamic bonding curves in the DLP. More precisely, for fixed impact functions, we derive the price of liquidity in the form of deterministic quote functions of the reserves y and other variables detailed in the following sections.[13]

2.5. Discussion Before presenting specific models for liquidity demand and price formation and derive strategic DLP bonding curves, we first discuss the DLP design in a general framework. In particular, we compare DLPs to CFMs and highlight their strategic advantages, contrast DLPs with dynamic fee models, examine the implications of DLP design for welfare and market fragmentation, and discuss how DLPs can be implemented in current blockchains.

CFMs and DLPs. In current CFMs, the effect of liquidity-taking orders on the pool's marginal rate and execution costs is set by the convexity of the trading function [28]. This convexity is proportional to the aggregate reserves of all LPs: trading is cheaper for LTs in large pools. An LP in a CFM who believes liquidity is too cheap—say, relative to price volatility—cannot provide liquidity that reflects this view, because the price of liquidity is determined collectively by the aggregate reserves of all LPs, not by any single provider. In a DLP, by contrast, each LP can set quotes to reflect such strategic views.[14]

More generally, in any liquidity provision framework where a representative LP provides liquidity to maximise an arbitrary performance criterion J, if the impact functions in the DLP match those of a CFM in (6), then the performance $J(\delta^{\mathrm{CFM}})$ in the CFM using the convexity costs δ^{CFM} in (7), is sub-optimal compared to that in a DLP if the LP chooses optimal quote functions $\delta^{\mathrm{DLP}} = \sup_{\delta}\{J(\delta)\}$ (see Section 3 for examples).

To illustrate the added strategic value of DLPs, consider the following example. An LP provides liquidity for a pair of stablecoins whose fundamental exchange rate is fixed at $Z_t = Z$. The cost of liquidity is the difference between the trade price and the fundamental price Z. For a sell order, the cost is $Z - Z^b_t$, where Z^b_t is the bid price. For a buy order, the cost is $Z^a_t - Z$, where Z^a_t is the ask price. We assume that the flow of buy and sell orders in the market is sensitive to this cost of liquidity. To illustrate, assume the parametric form for sell orders sensitivity is $c\,\exp\left(-\kappa\left(Z - Z^b_t\right)\right)$, and that of buy orders is $c\,\exp\left(-\kappa\left(Z^a_t - Z\right)\right)$, where $c > 0$ is a base intensity and $\kappa > 0$ measures sensitivity to liquidity costs. Under these assumptions, if the LP's objective is to maximize the expected terminal value of reserves $x_T + y_T\,Z$, it can be shown that the optimal bid and ask prices

$$Z^b_t = Z - \frac{1}{\kappa} \quad \text{and} \quad Z^a_t = Z + \frac{1}{\kappa}, \tag{15}$$

are symmetric.

[13] Future work will explore the joint optimisation of the impact and quote functions.

[14] In practice, in platforms such as Uniswap v4, LPs can create pools with specific dynamic fee models. Optimal routing of trading flow among LP positions in a DLP is outside the scope of this paper. Optimal routing in the context of blockchains is studied in the literature; see [39, 44]. For instance, Uniswap v4 routes LT trades across multiple pools for the same asset pair with different fee tiers and *hooks*; see [75].

Let y_t^{CFM} be the reserves in a CFM and let Z_t^{CFM} be the marginal price. If $Z_t^{\mathrm{CFM}} = Z$, then in most cases it is impossible for an LP to set a symmetric price for liquidity at the bid and the ask in a CFM, because

$$\delta^a(y_t) - \delta^b(y_t) = \frac{\varphi(y_t - \zeta) - \varphi(y_t + \zeta)}{\zeta} + 2\,\varphi'(y_t) \neq 0. \tag{16}$$

In addition, the impact functions (6) in a CFM are never zero, so the marginal price

$$Z_t^{\mathrm{CFM}} = -\varphi'(y_t^{\mathrm{CFM}})$$

is dynamic. To keep the cost of liquidity in a CFM equal to the optimal (15), the LP must adjust the reserve level y_t^{CFM} so that the CFM quotes in (7) satisfy

$$\begin{aligned}
\delta^a(y_t^{\mathrm{CFM}}) &= \frac{1}{\kappa} + \left(Z_t^{\mathrm{CFM}} - Z\right) = \frac{\varphi(y_t^{\mathrm{CFM}} - \zeta) - \varphi(y_t^{\mathrm{CFM}})}{\zeta} + \varphi'(y_t^{\mathrm{CFM}}) + \pi\left(-\varphi'(y_t^{\mathrm{CFM}})\right), \\
\delta^b(y_t^{\mathrm{CFM}}) &= \frac{1}{\kappa} - \left(Z_t^{\mathrm{CFM}} - Z\right) = \frac{\varphi(y_t^{\mathrm{CFM}} + \zeta) - \varphi(y_t^{\mathrm{CFM}})}{\zeta} - \varphi'(y_t^{\mathrm{CFM}}) + \pi\left(-\varphi'(y_t^{\mathrm{CFM}})\right).
\end{aligned} \tag{17}$$

These conditions imply that, after a buy order, the price of liquidity at the ask must be set below that at the bid to maintain the optimal cost of liquidity. This requires *asymmetric quotes*. Standard CFMs cannot implement such asymmetry because the relation between bid- and ask-side liquidity costs is fixed by the shape of the bonding curve; see (16). In CFMs with concentrated liquidity, approximating this behavior would require non-trivial parameter tuning and operational complexity [29, 30]. Furthermore, because the optimal quotes depend on y_t^{CFM}, frequent reserve adjustments would be necessary and increase gas costs. In contrast, in a DLP, the LP can set $\eta^a(\cdot) = \eta^b(\cdot) = 0$ and post constant quotes $1/\kappa$. This ensures the DLP offers the optimal price of liquidity (15) with minimal gas usage for any desired reserve level.

DLPs and dynamic fees. Multiple works have extended classical CFMs to incorporate dynamic fees that adjust to market conditions; see [30, 50, 54]. In these models, an optimal fee π is computed to compensate LPs for adverse selection losses. Some features of DLPs discussed above—such as dynamic and potentially asymmetric prices of liquidity—can be replicated in a dynamic proportional fee model.[15] For instance, in the stablecoin example above, the optimal spread could be achieved by setting an appropriately asymmetric fee π in (17).

The key difference between a DLP and a dynamic fee model within a CFM is that in a DLP the bonding curve is jointly determined by both the quote functions and the impact functions. This allows LPs to influence changes in the marginal price. In contrast, in CFMs, marginal prices are aligned to fundamentals only through the activity of arbitrageurs and informed traders, even when LPs know fundamentals differ.[16]

To illustrate this difference, consider the example of a decentralised exchange for assets X and Y. Suppose LPs use a process S as an oracle for the price of Y in terms of X. Similar to above, assume liquidity takers are sensitive to the bid and ask prices relative to the fundamental price, and let the sensitivity of sell orders be $c\,\exp\left(-\kappa\,(S_t - Z_t^b)\right)$, and that of buy orders be $c\,\exp\left(-\kappa\,(Z_t^a - S_t)\right)$,

[15] **The optimal quote models we propose below are new to the literature and tailored to decentralised markets.**

[16] A representative LP in a CFM with concentrated liquidity can influence price changes by withdrawing liquidity at the current marginal price and reallocating it within a different price range. This, however, requires frequent on-chain operations and incurs substantial rebalancing costs.

where $c > 0$ is a base intensity and $\kappa > 0$ measures sensitivity to liquidity costs. If the LP maximises the expected terminal value of holdings, the optimal bid and ask are:

$$Z_t^b = S_t - \frac{1}{\kappa}, \quad Z_t^a = S_t + \frac{1}{\kappa}.$$

In a CFM, these optimal quotes can be implemented using dynamic fees:

$$\pi^a = 1 - \frac{S_t - \frac{1}{\kappa}}{Z_t^{\mathrm{CFM}}} - \frac{\varphi(y_t^{\mathrm{CFM}} - \zeta) - \varphi(y_t^{\mathrm{CFM}})}{\zeta\, Z_t^{\mathrm{CFM}}}, \quad \pi^b = -1 + \frac{S_t + \frac{1}{\kappa}}{Z_t^{\mathrm{CFM}}} - \frac{\varphi(y_t^{\mathrm{CFM}}) - \varphi(y_t^{\mathrm{CFM}} + \zeta)}{\zeta\, Z_t^{\mathrm{CFM}}}.$$

However, in this case, the marginal price at any time $T > t$ satisfies $\mathbb{E}[Z_T^{\mathrm{CFM}}] = Z_t \neq S_T$, meaning that the marginal price does not move on average and, in particular, does not track S_t. This can lead to inefficiency: the reference price in the CFM may drift far from the fundamental price, even when an oracle provides it. Economists often measure market efficiency by the gap between clearing prices and fundamental values [22, 51, 59]. In a DLP, LPs can set impact functions to track the oracle S_t whenever an order arrives and maintain constant quotes $1/\kappa$ between oracle updates.

More generally, LPs in a DLP have greater strategic flexibility because they can influence both prices and liquidity. In particular, when marginal price changes are driven partly or fully by: (i) other prices in exogenous markets, (ii) order flow in the blockchain, or (iii) known changes in fundamentals, LPs can encode strategic behaviour in the impact functions. This mirrors common practice in traditional centralised markets, where market makers use client order flow and price signals to adjust the mid-price around which they provide liquidity; see [55, 67].

DLPs in blockchains. In traditional markets, high barriers to entry for market making mean that wealth transfers from uninformed to informed traders occur primarily through market makers. In DEXs, where anyone can become an LP with minimal effort, these transfers often flow from both uninformed traders and uninformed LPs to informed traders, making arbitrage relatively more profitable. The DLP framework addresses this by enabling transparent, strategically designed liquidity provision that less-informed LPs can join, which reduces their exposure to adverse selection.

In the context of Ethereum blockchains, the impact and quote functions are implemented as smart contracts that customise the DLP and encode designated actions. LPs may change the impact and quote functions with blockchain transactions that redefine the code of these functions. The DLP design accommodates any set of impact and quote functions. In blockchains, these elements must be computationally efficient to limit gas fees, and any computationally demanding operations can be conducted off-chain. Moreover, on modern layer-2 blockchains, the DLP can be implemented with lower gas fees, even when computational complexity is significant.

Similar to the broader DeFi ecosystem on Ethereum, LPs can choose the strategic bonding curve of an existing DLP to deposit their liquidity. Historical performance data for each DLP is public on the blockchain, so less informed LPs can easily evaluate and compare different pools. This transparency means that, even if the DLP design leads to greater market fragmentation, less informed LPs can still participate by allocating funds to historically well-performing DLPs, thereby sharing risks. In turn, this changes the strategic advantage of individual skills: the benefits of superior strategy design do not arise from exclusive access to private trading methods, but from making the DLP's strategy available publicly and charging LPs protocol fees for access.

LPs can also create new DLPs which better align with their strategic preferences, akin to Uniswap v4, but with a broader range of strategic flexibility. In this work, we do not address issues related to the governance of a particular DLP once it is created. However, the DLP may issue so-called

governance tokens to LPs who decide, by vote, on changes to the DLP's impact and quote functions; see [64] for more details.

DLPs offer an ecosystem where LPs can act strategically. Akin to the coexistence, for one pair of assets X and Y, of (i) CPM pools of varying fee tiers and liquidity depths in Uniswap, (ii) liquidity pools from a large number of other protocols that run different designs on blockchains, and (iii) centralised venues, we expect that several DLPs with different impact and quote functions that optimise different utility functions may coexist for a single pair of assets. It is beyond the scope of this paper to study the outcome of competition between different DLPs that run different strategies; see [17, 20, 62, 73] for works on competition among liquidity providers in different markets.

Finally, the models that we propose in this work optimise specific performance criteria along with specific assumptions to derive the strategic bonding curves. As discussed in Section 2.1, LPs that have different preferences and different assumptions over the liquidity taking trading flow must derive different strategic bonding curves.

3. Examples of strategic bonding curves Section 3.1 derives quote functions when prices form in various trading venue including the DLP, Section 3.2 assumes price formation in the DLP, and Section 3.3 considers oracle prices.

3.1. Strategic bonding curve when markets are fragmented

The model In this section, we derive quote functions when the trading flow faced by the DLP is sensitive to the price of liquidity. For simplicity, we consider impact functions that depend on the level of the reserves only. The DLP maximises the expected wealth of LPs while penalising deviations from a target marginal price, or equivalently, a reserve level $\tilde{y} \in \mathcal{Y}$. Fix a pair of impact functions η^b and η^a that satisfy condition (2), so by Proposition 1, the marginal price of the DLP takes the ordered values $\mathcal{Z} = \{Z_0, \dots, Z_N\}$, which correspond, respectively, to the reserve levels $\{\overline{y}, \dots, \underline{y}\}$ in asset Y in the DLP.

The DLP proposes a bid quote $Z^b_t = Z_t - \delta^b_t$ and an ask quote $Z^a_t = Z_t + \delta^a_t$ at which LTs can sell or buy the asset, respectively. In our model, there is a baseline of LTs that trade in the pool who are sensitive to the price of liquidity $Z_t - \delta^b_t$ at the bid, and to the price of liquidity $Z_t + \delta^a_t$ at the ask. The DLP is continuously updating the price of liquidity δ^b_t and δ^a_t at the bid and the ask, respectively, in (2.1) to optimise a performance criterion that we specify below. For $t \in [0, T]$, we define the set $\mathcal{A}_t$ of admissible strategies

$$\mathcal{A}_t = \left\{(\delta^b_s, \delta^a_s)_{s\geq t},\ \mathbb{R}^2\text{-valued},\ \mathbb{F}\text{-predictable, square-integrable, bounded from below by } \underline{\delta}\right\}, \tag{18}$$

where $\underline{\delta} \in \mathbb{R}$ is given and write $\mathcal{A} := \mathcal{A}_0$.[17]

In our model, the intensity of the liquidity taking trading flow depends on whether the marginal price in the DLP is above or below the fundamental price of the asset. However, the fundamental price is unknown and is not privately observed by the DLP. Thus, we design a model where the DLP employs (filters) the liquidity taking trading flow to assign probabilities to the magnitude of the discrepancy between the fundamental price and the marginal price, and to tailor the price of liquidity accordingly. Specifically, if it is more likely that the fundamental price is below (resp. above) the marginal price, then the liquidity at the bid (resp. ask) is more expensive.

[17] In the models of this section, we do not provide a rigorous characterization of the underlying probability space, nor do we specify the change of measure involved in the weak formulation of the control problems we solve. Below, we denote the controlled measure by $\mathbb{P}$.

We introduce a process g, which is not adapted to the filtration $\mathbb{F}$, that describes the gap between the marginal price in the DLP and the fundamental price. We model g as a continuous-time Markov Chain (CTMC) which describes the probability of the gap transitioning between states, or regimes. The regimes take values in a finite state space $\mathcal{S} = \{1, \dots, M\}$. The transition rate matrix of the CTMC, which is known to the DLP, is given by

$$\Pi = \left(\pi_{ij}\right)_{i,j\in\{1,\dots,M\}^2}, \tag{19}$$

where $\pi_{ii} = -\sum_{i\neq j}\pi_{ij}$, and for all $i \neq j$, $\pi_{ij} \geq 0$. For every $i, j \in \{1, \dots, M\}^2$, the value of π_{ij} in (19) is the instantaneous intensity of the CTMC transitioning from state i to state j.[18]

We assume that the true intensity of liquidity taking order arrivals depends on the difference between the unobserved fundamental price and the marginal price in the pool. We introduce $\boldsymbol{c}^i = \{c^{i,1}, \dots, c^{i,M}\}$, for $i \in \{a, b\}$, where $c^{a,j}$ and $c^{b,j}$ are constants for all j. The vectors $\boldsymbol{c}^b$ and $\boldsymbol{c}^a$ describe the baseline intensity of order arrivals at the bid and at the ask in each regime $j \in \{1, \dots, M\}$. In particular, regimes where the marginal price in the DLP underestimates (resp. overestimates) the fundamental price correspond to increased buying (resp. selling) trading activity from liquidity takers. We also define $\tilde{\boldsymbol{c}}^i = \{c^{i,1}, \dots, c^{i,M-1}\}$ for $i \in \{a, b\}$ to simplify notation in the mathematical analysis below.

The intensity of the arrival of liquidity taking orders in the DLP follows

$$\lambda_t^b\left(\delta_t^b, g_t\right) = \sum_{j=1}^{M} \underbrace{c^{b,j}\, e^{-\kappa\,\delta_t^b}}_{\lambda_t^{b,j}(\delta_t^b)}\, \mathbb{1}^b\left(y_{t^-}\right)\, \mathbb{1}\left(g_t = j\right),$$

$$\lambda_t^a\left(\delta_t^a, g_t\right) = \sum_{j=1}^{M} \underbrace{c^{a,j}\, e^{-\kappa\,\delta_t^a}}_{\lambda_t^{a,j}(\delta_t^a)}\, \mathbb{1}^a\left(y_{t^-}\right)\, \mathbb{1}\left(g_t = j\right),$$

where

$$\mathbb{1}^b(y) = \mathbb{1}_{\{y+\zeta\leq\overline{y}\}} \quad \text{and} \quad \mathbb{1}^a(y) = \mathbb{1}_{\{y-\zeta\geq\underline{y}\}} \tag{20}$$

indicate that the DLP stops trading in a given direction if it would breach the reserve limits $\underline{y}, \overline{y}$; in practice, the magnitude of $\underline{y}$ and $\overline{y}$ may be as large as needed. The positive constant κ models the sensitivity of LTs to the price of liquidity. In particular, the value of κ increases when competing trading venues offer cheaper liquidity.

The DLP does not observe the regime g directly, i.e., it does not observe the fundamental price. However, the DLP observes the trading flow of liquidity takers, and uses this information to compute (filter) the probability Ψ_t^j that regime g is in state j. We denote the vector of filtered regime probabilities by $\boldsymbol{\Psi}_t = \{\Psi_t^1, \dots, \Psi_t^M\}$. The DLP computes the probability Ψ_t^j that the CTMC is in state j at time t given all the available information $\mathcal{F}_t$ and we write

$$\Psi_t^j = \mathbb{E}\left[\mathbb{1}(g_t = j) \mid \mathcal{F}_t\right] = \mathbb{P}\left[g_t = j \mid \mathcal{F}_t\right].$$

The following result, which is proved in [37], provides the dynamics of the filtered regime probability from observing the liquidity taking flow in the DLP.[19]

[18] The diagonal entries of the transition rate matrix are negative and are interpreted as a rate of *no transition*. To obtain the steady state probabilities p for the CTMC one solves the matrix equation $a^\intercal\,\Pi = 0$ subject to $\sum_i a_i = 1$.

[19] The integrability condition of Theorem 3.1 in [37] is satisfied.

PROPOSITION 2. *The filter* Ψ_t^j *satisfies* $\Psi_t^j = \frac{\Gamma_t^j}{\sum_{i=1}^M \Gamma_t^i}$, *where the* score *process* Γ^j, *for* $j \in \{1, \dots, M\}$, *follows the dynamics*

$$\frac{\mathrm{d}\Gamma_t^j}{\Gamma_t^j} = \left(\lambda_{t^-}^{a,j}\left(\delta_t^a\right) - 1\right)\left(\mathrm{d}N_t^a - \mathrm{d}t\right) + \left(\lambda_{t^-}^{b,j}\left(\delta_t^b\right) - 1\right)\left(\mathrm{d}N_t^b - \mathrm{d}t\right) + \frac{\sum_{i=1}^M \Gamma_{t^-}^i}{\Gamma_{t^-}^j}\,\pi_{ji}\,\mathrm{d}t\,.$$

To derive the quote functions and corresponding bonding curve, we consider a liquidity provision model in which buying and selling pressure is determined by the weighted average of baseline pressures across regimes, with weights assigned based on the filtered probability of each regime. More precisely, given the (observed) score process $\mathbf{\Gamma}$, the DLP considers the projected stochastic intensities $\hat{\lambda}^b$ and $\hat{\lambda}^a$ given by

$$\begin{cases} \hat{\lambda}_t^b\left(\delta_t^b, \mathbf{\Gamma}_t\right) & = e^{-\kappa\,\delta_t^b}\,\mathbb{1}^b\left(y_{t^-}\right)\,\mathbf{c}^b \cdot \mathbf{\Psi}_t = e^{-\kappa\,\delta_t^b}\,\mathbb{1}^b\left(y_{t^-}\right)\,\sum_{j=1}^M c^{b,j}\,\Psi_t^j\,, \\ \hat{\lambda}_t^a\left(\delta_t^a, \mathbf{\Gamma}_t\right) & = e^{-\kappa\,\delta_t^a}\,\mathbb{1}^a\left(y_{t^-}\right)\,\mathbf{c}^a \cdot \mathbf{\Psi}_t = e^{-\kappa\,\delta_t^a}\,\mathbb{1}^a\left(y_{t^-}\right)\,\sum_{j=1}^M c^{a,j}\,\Psi_t^j\,, \end{cases} \tag{21}$$

where the operator $\cdot$ denotes the scalar product and

$$\sum_{j=1}^M c^{a,b,j}\,\frac{\Gamma_t^j}{\sum_i \Gamma_t^i} = \sum_{j=1}^M c^{a,b,j}\,\Psi_t^j = \sum_{j=1}^M c^{a,b,j}\,\mathbb{E}\left[\mathbb{1}(g_t = j) \mid \mathcal{F}_t\right]$$

is the expected trading intensity given the information up to time t.

The DLP sets the bid and ask quotes $Z - \delta^b$ and $Z + \delta^a$ to optimise the expected value of the DLP's reserves in terms of asset X, while controlling exposure to asset Y by keeping the reserves close to some level $\hat{y} \in \mathcal{Y}$ in units of asset Y, or equivalently by keeping the marginal price close to some level $\hat{Z} \in \mathcal{Z}$; see Section 2.1.

Performance criterion The DLP's performance criterion from implementing a bonding curve implied by the quotes $\delta = \left(\delta^b, \delta^a\right) \in \mathcal{A}$ is a function $w^\delta\colon [0,T] \times \mathbb{R} \times \mathcal{Y} \times \mathcal{Z} \times \mathbb{R}_{++}^M \to \mathbb{R}$, which is given by

$$w^\delta(t, x, y, z, \boldsymbol{\gamma}) = \mathbb{E}_{t,x,y,z,\boldsymbol{\gamma}}\left[x_T + a\,y_T - \alpha\,(y_T - \hat{y})^2 - \phi\int_t^T (y_s - \hat{y})^2\,\mathrm{d}s\right], \tag{22}$$

where $\mathbb{E}_{t,x,y,z,\boldsymbol{\gamma}}$ is the expectation operator conditional on the values of the controlled (using the quotes δ) processes $(x_s)_{s\in[t,T]}$, $(y_s)_{s\in[t,T]}$, $(Z_s)_{s\in[t,T]}$, and $(\mathbf{\Gamma}_s)_{s\in[t,T]}$ at time t.

In the performance criterion, we make the simplifying assumption that the DLP's horizon T is fixed. While LPs can technically withdraw liquidity at any time, our assumption is motivated by the discrete-time nature of blockchain markets. In practice, state changes (including liquidity withdrawals) become effective only upon block inclusion. Hence, an LP who decides to adjust or exit a position must wait at least until the next block is produced. On Ethereum, this implies a minimum commitment over a block interval of 12 seconds. More generally, liquidity providers may face uncertainty about their exit time or may optimally choose when to withdraw liquidity. Appendix C extends the model to allow for (i) a random exit time, and (ii) an endogenously chosen exit time. We expect the main economic mechanisms of the model to remain unchanged in these extensions.

The performance criterion (22) maximises the level of reserves in the DLP, where the reserves in asset Y are scaled according to the coefficient $a > 0$. For instance, the scaling factor can be set to

the initial marginal price, or it can be adjusted to a desired level that reflects the DLP's preference for one of the two assets. Furthermore, the constants $\alpha, \phi \in \mathbb{R}^+$ determine the terminal and running penalty, respectively, on reserve deviations from some level $\hat{y} \in \mathbb{R}$. The DLP's performance criterion incorporates a target reserve level $\hat{y}$, which, by Proposition 1, is associated with a target marginal price on the discrete price grid. Deviations from $\hat{y}$ can therefore be interpreted as deviations from a reference price that captures the DLP's beliefs about the fundamental value of the asset. More generally, allowing $\hat{y}$ to be stochastic makes it possible to encode deviations from an external benchmark, such as a price oracle. Moreover, when the target coincides with the initial reserve level, the criterion mechanically limits reserve variation. Finally, by choosing $\hat{y}$ away from the current pool state, the DLP can strategically skew quotes that reflect price beliefs. The penalty parameters can be set to zero if no such adjustment is desired.

As described below, the model derived in this section employs the trading flow of liquidity takers to estimate the difference between the DLP's marginal price and the unobserved fundamental price, and to adjust the price of liquidity accordingly. Thus, the DLP attracts trading flow that adjusts the marginal price towards the fundamental price. B solves the problem of this section when the terminal reserves in asset Y are valued at the terminal marginal price Z_T, i.e., when it is assumed that the terminal marginal price Z_T is close to the fundamental price.[20]

The optimal quote functions The value function w of the DLP is

$$w(t,x,y,z,\boldsymbol{\gamma}) = \sup_{\delta \in \mathcal{A}_t} w^{\delta}(t,x,y,z,\boldsymbol{\gamma})\,. \tag{23}$$

Proposition 3 shows that the performance criterion (22) is finite and that the associated value function (23) is well defined.

PROPOSITION 3. *There is $C \in \mathbb{R}$ such that for all $(\delta_s)_{s\in[t,T]} \in \mathcal{A}_t$, the performance criterion of the LP satisfies $w^{\delta}(t,x,y,z,\boldsymbol{\gamma}) \leq C < \infty$, so the value function w in* (23) *is well defined.*

For a proof, see D.4.

Define $\boldsymbol{\psi} = \{\psi^1, \ldots, \psi^M\}$ and $\tilde{\boldsymbol{\psi}} = \{\psi^1, \ldots, \psi^{M-1}\}$.

The next result, whose proof is in D.5, provides the strategic bonding curve in feedback form when the DLP filters the difference between the marginal price it offers and the fundamental price from the liquidity taking flow. It is beyond the scope of this work to characterise the bonding curve formally, however, we provide numerical examples to study the behaviour of the strategic bonding curve in different market environments.

PROPOSITION 4. *Let z be the marginal price corresponding to the reserves level y. The quote functions are*

$$\begin{aligned}
\delta^{\star,b}(y,\tilde{\boldsymbol{\psi}}) =& \frac{1}{\kappa} - \frac{\theta\left(t, y+\zeta, \frac{\tilde{\boldsymbol{\psi}}\odot\tilde{\mathbf{c}}^b}{\boldsymbol{\psi}\cdot\mathbf{c}^b}\right) - \theta(t,y,\tilde{\boldsymbol{\psi}})}{\zeta} + \frac{1}{\zeta}\left(\frac{\tilde{\boldsymbol{\psi}}\odot\tilde{\mathbf{c}}^b}{\boldsymbol{\psi}\cdot\mathbf{c}^b}\cdot\nabla_{\tilde{\boldsymbol{\psi}}}\theta - \tilde{\boldsymbol{\psi}}\cdot\nabla_{\tilde{\boldsymbol{\psi}}}v + (a-z)\,\zeta\right) \\
\delta^{\star,a}(y,\tilde{\boldsymbol{\psi}}) =& \frac{1}{\kappa} - \frac{\theta\left(t, y-\zeta, \frac{\tilde{\boldsymbol{\psi}}\odot\tilde{\mathbf{c}}^a}{\boldsymbol{\psi}\cdot\mathbf{c}^a}\right) - \theta(t,y,\tilde{\boldsymbol{\psi}})}{\zeta} + \frac{1}{\zeta}\left(\frac{\tilde{\boldsymbol{\psi}}\odot\tilde{\mathbf{c}}^a}{\boldsymbol{\psi}\cdot\mathbf{c}^a}\cdot\nabla_{\tilde{\boldsymbol{\psi}}}\theta - \tilde{\boldsymbol{\psi}}\cdot\nabla_{\tilde{\boldsymbol{\psi}}}v + (z-a)\,\zeta\right),
\end{aligned} \tag{24}$$

[20] When $\phi = \alpha = 0$ and $a = Z_T$ in (22), [19] show that if one assumes a complete market and takes expectations in the risk-neutral measure, then there exist profitable arbitrages when the performance criterion is greater than the cost of trading.

where θ solves the PDE

$$0 = \partial_t\theta - \phi\,(y-\hat{y})^2 + \sum_{j=1}^{M-1}\left(2\psi^j + \psi\cdot\pi_{j,:}\right)\partial_{\psi^j}\theta - \tilde{\psi}\cdot\nabla_{\tilde{\psi}}\theta\sum_{j=1}^{M}\left(2\psi^j + \psi\cdot\pi_{j,:}\right) \tag{25}$$
$$+\frac{\zeta(\psi\cdot\mathbf{c}^b)}{\kappa\,e}\exp\left(\frac{\kappa}{\zeta}\left(\theta\Big(t,y+\zeta,\frac{\tilde{\psi}\odot\tilde{\mathbf{c}}^b}{\psi\cdot\mathbf{c}^b}\Big)-\theta(t,y,\tilde{\boldsymbol{\psi}})-\Big(\frac{\tilde{\psi}\odot\tilde{\mathbf{c}}^b}{\psi\cdot\mathbf{c}^b}-\tilde{\psi}\Big)\cdot\nabla_{\tilde{\psi}}\theta+(a-z)\;\zeta\right)\right)$$
$$+\frac{\zeta\,(\psi\cdot\mathbf{c}^a)}{\kappa\,e}\exp\left(\frac{\kappa}{\zeta}\left(\theta\Big(t,y-\zeta,\frac{\tilde{\psi}\odot\tilde{\mathbf{c}}^a}{\psi\cdot\mathbf{c}^a}\Big)-\theta(t,y,\tilde{\boldsymbol{\psi}})-\Big(\frac{\tilde{\psi}\odot\tilde{\mathbf{c}}^a}{\psi\cdot\mathbf{c}^a}-\tilde{\psi}\Big)\cdot\nabla_{\tilde{\psi}}\theta+(z-a)\;\zeta\right)\right),$$

with terminal condition

$$\theta\left(T,y,\tilde{\boldsymbol{\psi}}\right) = -\alpha\,y^2\,,$$

where $\odot$ denotes the Hadamard (element-wise) product.

Discussion Our result is valid for any pair of non-negative deterministic impact functions η^b and η^a . The first two terms in the quote functions (24) are similar to those in the classical models of [11] and [35]. The term $1/\kappa$ optimises the instantaneous expected profit from a roundtrip trade—the expected profit from a roundtrip trade is $2\,\zeta\,(\delta^b\,\hat{\lambda}^b(\delta^b)+\delta^a\,\hat{\lambda}^a(\delta^a))$ which is maximal for $\delta^b = 1/\kappa$ and $\delta^a = 1/\kappa$, where $\hat{\lambda}^b$ and $\hat{\lambda}^a$ are in (21). The second term in (24) is an adjustment to the the $1/\kappa$ strategy, i.e., an additional or reduced wealth, that the DLP demands to compensate for the change in her value function as a function of the current reserves y and the position of the fundamental price. The last term in (24) is an adjustment to the price of liquidity stemming from (i) the intensity of the trading flow at the bid and the ask as a consequence of the difference between the marginal price of the pool and the fundamental price, and (ii) changes in the value of the reserves as a consequence of changes in the marginal price.

The price of liquidity can become negative to attract the liquidity taking trading flow in a specific direction, as required by the DLP's strategic preferences. This aspect of the DLP's bonding curve is analogous to market making models in traditional electronic exchanges. In our context, negative quotes indicate that potential losses from providing liquidity at a negative spread are acceptable if they mitigate greater risks such as holding excessive reserves or observing extreme deviations from the fundamental price.

Figures 1 and 2 show a numerical example with two regimes, i.e., $M = 2$. In the first regime, the liquidity taking trading flow indicates that the fundamental price is below the marginal price of the DLP. In this regime, the selling pressure at the bid is five times the buying pressure at the ask and we set $c^{1,b} = 50$ and $c^{1,a} = 10$. In the second regime, the fundamental price is estimated to be above the marginal price, so the buying pressure is five times the selling pressure and we set $c^{1,b} = 10$ and $c^{1,a} = 50$.

Figure 1 shows the price at the bid and at the ask for the minimal quantity ζ as a function of the likelihood of being in the first regime ψ^1, and Figure 2 shows the bonding curves for various values of Ψ^1. The bonding curves of Figure 2 map the reserves in asset Y to the reserves in asset X, for fixed values of the regime probabilities $\boldsymbol{\Psi}$, following liquidity taking trades of any quantity (akin to the level curves of CFMs). Figure 1 shows that the ask (resp. bid) price increases (resp. decreases) with the estimated likelihood of underestimating (resp. overestimating) the fundamental price, because the DLP anticipates trading flow that will align the marginal price with the fundamental price. Figure 2 shows that these effects become more significant when the traded quantity increases.

A key advantage of our model is that it uses information in the liquidity taking flow to tailor the price of liquidity autonomously according to the position of the fundamental price. In particular,

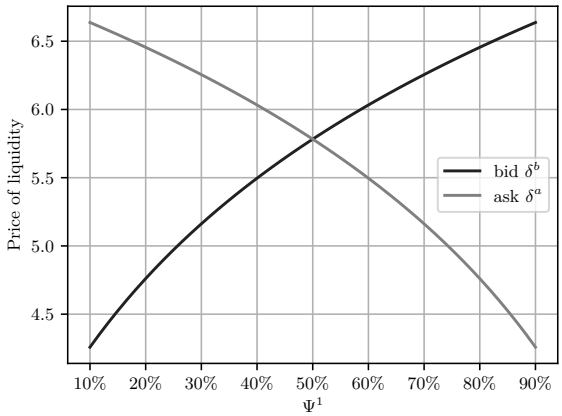


FIGURE 1. Price of liquidity at the bid and the ask, for a quantity ζ, for different values of Ψ^1, and when $\phi = 0.1$ and $y = \hat{y}$. Other model parameters are in Table 3.

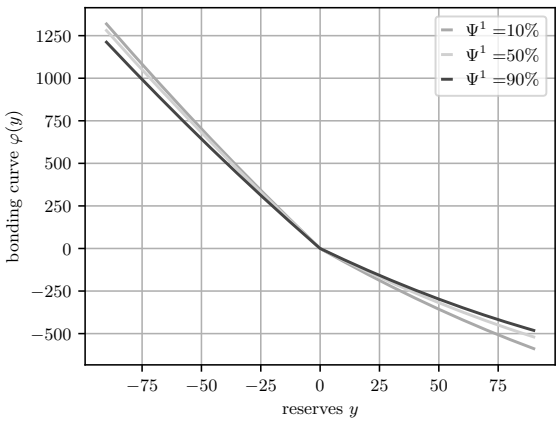


FIGURE 2. Bonding curves (geometric representation of execution prices as a function of traded quantity) for different values of Ψ when $\phi = 0.1$. Other model parameters are in Table 3.

it is designed to make the inherently passive liquidity provision in DEXs strategic. DLPs that run the bonding curve (24) may be implemented on blockchains provided the necessary calculations to solve the PDE (25) are performed off-chain.

3.2. Strategic bonding curve when price formation is in the DLP This section proposes a model for the strategic bonding curve when the DLP is the sole trading venue for the pair of assets X and Y and trading activity is significant. In this case, we assume that the marginal price in the DLP is efficient and sufficiently close to the fundamental price. Similar to Section 3.1, we derive the quote functions for a fixed pair of impact functions η^b and η^a when the arrival intensity of liquidity taking orders decays exponentially as a function of the shift from the marginal price quoted at the bid and the ask. The stochastic intensity of order arrivals $\lambda_t^b\left(\delta_t^b\right)$ and $\lambda_t^a\left(\delta_t^b\right)$ are

$$\lambda_t^b\left(\delta_t^b\right) = c^b\, e^{-\kappa\,\delta_t^b}\, \mathbb{1}^b\left(y_{t^-}\right) \quad \text{and} \quad \lambda_t^a\left(\delta_t^a\right) = c^a\, e^{-\kappa\,\delta_t^a}\, \mathbb{1}^a\left(y_{t^-}\right),$$

where c^a and c^b are two non-negative constants that capture the baseline selling and buying intensity, respectively, and where $\mathbb{1}^b(y)$ and $\mathbb{1}^a(y)$ are in (20). The positive constant κ models how sensitive the LT is to the price of liquidity, which here is defined relative to the marginal price.

The DLP sets the bid and ask quotes $Z - \delta^b$ and $Z + \delta^a$ to optimise expected final wealth, while controlling exposure to asset Y. The performance criterion from implementing the strategy $\delta = \left(\delta^b, \delta^a\right) \in \mathcal{A}$ is

$$w^\delta(t,x,y,z) = \mathbb{E}_{t,x,y,z}\left[x_T + y_T\, Z_T - \alpha\,(y_T - \hat{y})^2 - \phi \int_t^T (y_s - \hat{y})^2\,\mathrm{d}s\right], \tag{26}$$

The following results provides the quote functions. A proof is in D.6.

THEOREM 3. *Let $\underline{N} = \underline{y}/\zeta$, $\overline{N} = \overline{y}/\zeta$, and $N = \overline{N} - \underline{N} + 1$. The quote functions are*

$$\begin{aligned}
\delta^{b\star}(t, y_{t^-}) &= \frac{1}{\kappa} - \frac{\theta\left(t, y_{t^-} + \zeta\right) - \theta\left(t, y_{t^-}\right)}{\zeta} - \frac{\left(y_{t^-} + \zeta\right)\eta^b\left(y_{t^-}\right)}{\zeta}, \\
\delta^{a\star}(t, y_{t^-}) &= \frac{1}{\kappa} - \frac{\theta\left(t, y_{t^-} - \zeta\right) - \theta\left(t, y_{t^-}\right)}{\zeta} + \frac{\left(y_{t^-} - \zeta\right)\eta^a\left(y_{t^-}\right)}{\zeta},
\end{aligned} \tag{27}$$

where $\theta(t,y) = \frac{\zeta}{\kappa}\log u(t,y)$, $u(t,m\,\zeta) = \mathbf{U}(T-t)_m$, *and*

$$\mathbf{U}(t) = \exp(\mathbf{K}\,t)\;\mathbf{U}(0)\,,\quad \mathbf{U}(0)_m = e^{-\alpha\,\frac{\kappa}{\zeta}\,(\zeta\,m-\hat{y})^2}\,,\ m\in[\underline{N},\bar{N}]\cap\mathbb{Z},\quad t\in[0,T]\,, \tag{28}$$

where the matrix $\mathbf{K}\in\mathbb{R}^{N\times N}$ *is defined as*

$$\mathbf{K}_{mn} = \begin{cases} c^a\,e^{-1}\,e^{\kappa\,(m-1)\,\eta^a(m\,\zeta)} & \text{if } n = m-1 \text{ and } m > \underline{N}\,,\\ -\kappa\,\phi\,(m\,\zeta-\hat{y})^2\,/\,\zeta & \text{if } n = m\,,\\ c^b\,e^{-1}\,e^{-\kappa\,(m+1)\,\eta^b(m\,\zeta)} & \text{if } n = m+1 \text{ and } m < \overline{N}\,, \end{cases} \tag{29}$$

for $m,n\in\{\underline{N},\underline{N}+1,\dots,\overline{N}\}$.

Discussion Similar to Section 3.1, the first two terms in the strategy (27) are those in the classical model of [11]. Here, the last term in (27) is an impact component that adjusts the bonding curve to the impact of LT activity on the marginal pool rate Z. In particular, the adjustment in the price of liquidity is interpreted as an additional wealth that LTs pay for adverse selection losses. When the DLP fills a buy order, the reserves in asset Y decrease but the price increases, and when the DLP fills a sell order, the reserves in asset Y increase but the price decreases. Thus, the price of liquidity at the bid and the ask compensates for the loss in value stemming for losses to LTs due to price impact.

The quote functions $\{\delta^{b\star},\delta^{a\star}\}$ can become negative as a result of reserve constraints and impact functions. Without the constraints $\delta^{b\star},\delta^{a\star}\geq 0$, the bonding curve is in closed-form so the AMM can rapidly update the price of liquidity for different levels of the marginal price and the reserves in asset Y before providing liquidity. In practice, the bonding curve derived in this section can be implemented for pairs of assets that are solely traded on the blockchain, or for pairs of stablecoins whose relative prices are stable.

3.3. Strategic bonding curve when price formation is exogenous This section considers the case when the DLP is not the primary trading venue for the pair of assets X and Y, so prices form in an exogenous market. We denote by $(S_t)_{t\in[0,T]}$ the fundamental price of asset Y in terms of asset X in the alternative trading venue with dynamics

$$\mathrm{d}S_t = \sigma\,\mathrm{d}W_t\,,$$

where $(W_t)_{t\in[0,T]}$ is a standard Brownian motion. The exchange rate S is an *oracle* rate which may be observed in a centralised venue or as the aggregate price of different trading venues.

The DLP proposes a bid quote $Z_t^b = Z_t - \delta_t^b$ and an ask quote $Z_t^a = Z_t + \delta_t^a$ at which LTs can sell or buy the asset, respectively. There is a baseline of LTs that trade in the pool who are sensitive to the price of liquidity $S_t - Z_t + \delta_t^b$ at the bid, and the price of liquidity $Z_t + \delta_t^a - S_t$ and the ask. Specifically, let $\lambda_t^b\left(\delta_t^b\right)$ and $\lambda_t^a\left(\delta_t^b\right)$ denote the stochastic intensity of order arrivals and write

$$\lambda_t^b\left(\delta_t^b\right) = c^b\,e^{-\kappa\left(S_t-Z_t+\delta_t^b\right)}\,\mathbb{1}^b\left(y_{t^-}\right)\quad\text{and}\quad \lambda_t^a\left(\delta_t^a\right) = c^a\,e^{-\kappa\left(Z_t-S_t+\delta_t^a\right)}\,\mathbb{1}^a\left(y_{t^-}\right)\,,$$

where c^a and c^b are two non-negative constants that capture the baseline selling and buying intensity, respectively, and where $\mathbb{1}^b(y)$ and $\mathbb{1}^a(y)$ are in (20).

For $t\in[0,T]$, we define the set $\mathcal{A}_t$ of admissible shifts is in (18). The DLP sets the bid and ask quotes $Z-\delta^b$ and $Z+\delta^a$ to optimise expected final wealth, while penalising deviations of the reserves y from some level $\hat{y}\in\mathcal{Y}$. Recall that reserve levels in $\mathcal{Y}$ correspond to marginal price levels

in $\mathcal{Z}$; see Proposition 1. The performance criterion corresponding to the strategy $\delta = (\delta^b, \delta^a) \in \mathcal{A}$ is the function $w^\delta \colon [0,T] \times \mathbb{R} \times \mathcal{Y} \times \mathcal{Z} \times \mathbb{R} \to \mathbb{R}$ given by

$$w^\delta(t,x,y,z,S) = \mathbb{E}_{t,x,y,z,S}\left[x_T + y_T\, S_T - \alpha\,(y_T - \hat{y})^2 - \phi \int_t^T (y_s - \hat{y})^2 \,\mathrm{d}s \right].$$

Below, Theorem 4 provides the quote functions and corresponding strategic bonding curve with an oracle price. We omit the proofs because the proofs follow the arguments in the results of the previous sections.

THEOREM 4. *The optimal quote functions are*

$$\begin{aligned} \delta^{b\star}(t, y_{t^-}, Z_{t^-}, S) &= \frac{1}{\kappa} - \frac{\theta\left(t, y_{t^-} + \zeta, Z_{t^-} - \eta^b(y_{t^-})\right) - \theta(t, y_{t^-}, Z_{t^-})}{\zeta} + (Z_{t^-} - S_{t^-})\,, \\ \delta^{a\star}(t, y_{t^-}, Z_{t^-}, S) &= \frac{1}{\kappa} - \frac{\theta\left(t, y_{t^-} - \zeta, Z_{t^-} + \eta^a(y_{t^-})\right) - \theta(t, y_{t^-}, Z_{t^-})}{\zeta} + (S_{t^-} - Z_{t^-})\,, \end{aligned} \tag{30}$$

where $\theta(t,y) = \frac{\zeta}{\kappa} \log u(t,y)$, $u(t, m\,\zeta) = \mathbf{U}(T-t)_m$, *and*

$$\mathbf{U}(t) = \exp(\mathbf{K}\,t)\;\mathbf{U}(0)\,, \quad \mathbf{U}(0)_m = e^{-\alpha\,\frac{\kappa}{\zeta}\,(\zeta\, m - \hat{y})^2}\,, \;\; m \in [\underline{N}, \bar{N}] \cap \mathbb{Z}, \quad t \in [0,T]\,,$$

where the matrix $\mathbf{K} \in \mathbb{R}^{N\times N}$ *is defined as, for* $m, n \in \{\underline{N}, \underline{N}+1, \ldots, \overline{N}\}$,

$$\mathbf{K}_{mn} = \begin{cases} c^a\, e^{-1} & \text{if } n = m-1 \text{ and } m > \underline{N}\,, \\ -\kappa\,\phi\,(y_m - \hat{y})^2 \,/\, \zeta & \text{if } n = m\,, \\ c^b\, e^{-1} & \text{if } n = m+1 \text{ and } m < \overline{N}\,. \end{cases}$$

Discussion In contrast to the two previous models, the last term in the quote functions (30) is an adjustment to the quotes stemming from the difference between the marginal price of the pool and the fundamental price. In particular, when the marginal price underestimates (resp. overestimates) the fundamental price, the pool anticipates trading flow that will align the marginal price, so the price of liquidity at the ask increases (resp. decreases) and the price of liquidity at the bid decreases (resp. increases); see Figure 3. Specifically, the price of liquidity at the ask (resp. bid) increases when the oracle price is above (resp. below) the marginal price. Moreover, Figure 4 shows how higher values of the penalty parameter ϕ increases the price of liquidity at the bid and the ask, resulting in increased convexity in the bonding curve of the DLP.

In the next result and specific to this model, we derive the no-arbitrage condition of Theorem 2 for the impact functions of an DLP that implements the dynamic quotes (30). In particular, we show that for the liquidity provision strategy (30), condition (31) reduces to ensuring that the jump in the marginal price after a trade is less than the baseline half bid-ask spread $1/\kappa$.

PROPOSITION 5. *For the liquidity provision strategy in* (30)*, the no-arbitrage condition* (14) *simplifies to, for all* $i \in \{1, \ldots, N\}$,

$$\eta^a(y_i) \le \frac{1}{\kappa} \quad \text{and} \quad \eta^b(y_i) \le \frac{1}{\kappa}\,. \tag{31}$$

For a proof, see D.7.

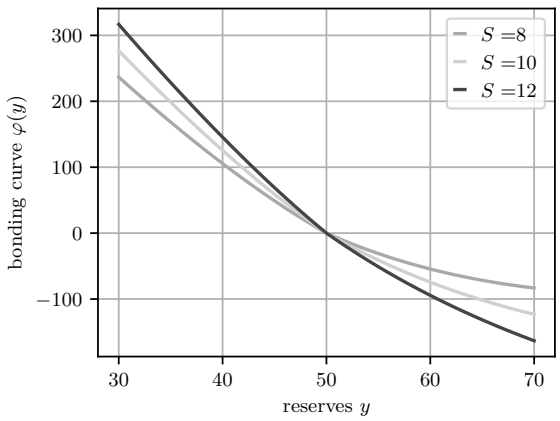


FIGURE 3. Bonding curves for multiple values of the oracle S when the marginal price is $Z = 10$.

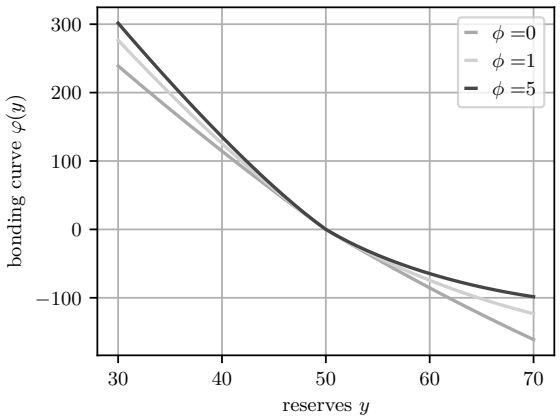


FIGURE 4. Bonding curves for multiple values of the penalty parameter ϕ when the marginal price is $Z = 10$.

3.4. Numerical experiments At present, price formation for most pairs of cryptocurrencies is in the traditional price-time priority LOBs of Binance. In this section, we use transaction data for the currency pair ETH/USDC from Binance and from the Uniswap v3 pool ETH/USDC 0.05% to illustrate the performance of the model in Section 3.2. Uniswap v3 is the most liquid CFM in the Ethereum blockchain. More precisely, it is a constant product market (CPM) which implements concentrated liquidity; see [2] and [31]. Table 1 shows descriptive statistics for both sets of transaction data.

	ETH/USDC 0.05%		Binance
	LT	LP	
Number of transactions	216,739	42,022	12,341,854
Average transaction size	\$ 109,037	\$ 2,765,499	\$ 1,735
Gross USD volume	$\approx$ \$ 185.57 $\times 10^9$	$\approx$ \$ 116.2 $\times 10^9$	$\approx$ \$ 21.42 $\times 10^9$
Average trading frequency	18.27 seconds	12.3 minutes	2.56 seconds
Median LP holding time	86 minutes		n.a.
Average pool depth	19,788,327 $\sqrt{\text{ETH} \cdot \text{USDC}}$		n.a.

TABLE 1. LT and LP activity statistics in the Uniswap v3 pool ETH/USDC 0.05% and in Binance between 5 May 2021 (Uniswap inception) and 30 April 2022; see [46] for more details.

Impact functions. In the numerical examples below, we use the impact functions

$$\eta^b(y) = \frac{\zeta}{\frac{1}{2}y + \zeta} L\,, \quad \eta^a(y) = \frac{\zeta}{\frac{1}{2}y - \zeta} L\,, \tag{32}$$

where $L > 0$ is an *impact parameter*. We choose $L < 1/\kappa$ so that $\eta^a(\cdot)$ and $\eta^b(\cdot)$ satisfy (2), and by Theorem 2, the DLP does not admit arbitrage. The optimal quotes (27) become

$$\delta^{b\star}(t,y) = \frac{1}{\kappa} - \frac{\theta(t,y+\zeta) - \theta(t,y)}{\zeta} - L\,, \qquad \delta^{a\star}(t,y) = \frac{1}{\kappa} - \frac{\theta(t,y-\zeta) - \theta(t,y)}{\zeta} + L\,, \tag{33}$$

and the matrix $\mathbf{K}$ in (29) becomes

$$\mathbf{K}_{mn} = \begin{cases} c\, e^{-1}\, e^{\kappa L}, & n = m-1,\ m > \underline{N}, \\ -\kappa\, \phi\, (m\,\zeta - \hat{y})^2/\zeta, & n = m, \\ c\, e^{-1}\, e^{-\kappa L}, & n = m+1,\ m < \overline{N}. \end{cases}$$

The impact functions (32) are inversely proportional to the reserves y; the price impact of trades decreases with market depth as in traditional markets. For small trades, one can approximate $\eta^b(y) \approx \eta^a(y) \approx 2\,\zeta\, L/y$, and the bid-ask spread as $\delta^{b\star}(t,y) + \delta^{a\star}(t,y) \approx 2/\kappa$. The parameter L scales how changes in the reserves affect the marginal price Z. An LP can choose L based on their assessment of the proportion of informed (toxic) versus uninformed (noise) order flow: high L yields faster price adjustments, which helps mitigate losses when toxic flow dominates, while low L leads to slower adjustments, which is preferable when noise flow is prevalent.

To compute the optimal quotes in (33), the DLP solves for the value function (28), which involves matrix exponentiation. In practice, the dimension of $\mathbf{K}$ is large, making direct exponentiation inefficient. Two features make the computation tractable: (i) $\mathbf{K}$ is sparse, and (ii) we only require $\exp(\mathbf{K}\,t)$ acting on $\mathbf{U}(0)$; see (38). We use the algorithm in [4], which applies the scaling-and-squaring method with a Padé approximation, optimised for sparse matrices. Moreover, for known impact functions η^b and η^a, optimal quotes can be precomputed off-chain for parameter tuples $(T, \zeta, y, \kappa, c, \phi, \alpha)$. In practice, this shifts the computational cost of the LP position in the DLP to a storage requirement.

3.4. Numerical experiments Consider an example where a single representative LP provides liquidity in the DLP and only arbitrageurs trade with it to align the marginal price with the fundamental price on Binance. This setting is realistic when a more liquid venue exists where price discovery takes place. In such cases, LPs lose money on average when providing liquidity to informed traders.[21]

An LP who expects trading flow to be primarily driven by informed LTs should increase the price of liquidity in the pool. In particular, differences between initial and terminal prices when providing passive liquidity generate adverse selection losses. To mitigate these losses, the LP can raise the penalty parameter ϕ, which increases the price of liquidity whenever reserves deviate from the initial level, or equivalently, whenever the DLP's marginal price departs from its initial value. However, because a higher price of liquidity reduces liquidity-taking activity, if part of the flow is noise and therefore profitable, the LP must balance both effects.

We compare the performance of LPs using the strategy (27) in the DLP with the historical performance of LPs in Uniswap v3 in two scenarios; scenario I where only arbitrageurs trade in the DLP to align its rate Z with the Binance exchange rate; scenario II where where the DLP additionally receives noise-trader flow, accounting for half of the trading activity observed in Uniswap v3. We perform multiple in-sample parameter estimations and out-of-sample liquidity provision simulations. The procedure is as follows: (i) fix a one-hour in-sample window and a 30-minute out-of-sample window, (ii) estimate parameters from the in-sample window, (iii) compute the LP's out-of-sample performance, and (iv) shift both windows forward by 30 minutes and repeat. During the out-of-sample period, if the LP's reserves reach $\underline{y}$ or $\overline{y}$, the position is liquidated, the accrued PnL recorded, and a new position is initiated.

[21] In this example, providing liquidity against purely informed traders is not economically rational; we use this setup only for illustrative purposes.

In each run, the LP starts with target reserves $y_0 = \hat{y} = 500$ ETH.[22] The trade size ζ is the average in-sample traded size in Binance; see Table 1. The LP estimates a fill ratio exponent $\kappa = 0.8$.[23] For the buy/sell pressure parameter c, we use the in-sample average order arrival intensity in Binance. The reserves limits are $\overline{y} = 1000$ ETH and $\underline{y} = 0$ ETH, and the terminal penalty parameter is $\alpha = 10^{-3}\,\text{USDC}\cdot\text{ETH}^{-2}$. Finally, to capture the LP's strategic behavior, we fix the rate impact parameter $L = 1$ and the penalty parameter $\phi = 10^{-1}\,\text{USDC}\cdot\text{ETH}^{-2}$ in scenario I, and $L = 0.5$ and $\phi = 10^{-3}\,\text{USDC}\cdot\text{ETH}^{-2}$ in scenario II.

Table 2 reports the average performance of LPs in the DLP for scenarios I and II, alongside the historical performance of LPs in the Uniswap v3 ETH/USDC 0.05% pool. LPs in the DLP outperform Uniswap LPs in both scenarios. Furthermore, when noise traders are present (scenario II), providing liquidity in the DLP is more profitable than simply holding the assets.

	Average	Standard deviation
DLP (scenario I)	−0.004%	0.719%
DLP (scenario II)	0.998%	1.688%
Buy and Hold	0.001%	0.741%
Uniswap v3	−1.485%	7.812%

TABLE 2. Average and standard deviation of 30-minutes performance of LPs in the DLP for both simulation scenarios, LPs in Uniswap, and buy-and-hold. The historical performance on Uniswap v3 LPs is in [30].

4. Conclusion In this work, we showed that CFMs are not an optimal design for decentralised venues and that it is possible to build AMMs that are viable alternative trading venues because they provide LPs with the tools to construct bonding curves that align with their strategic preferences. We introduced the DLP; an AMM design that captures CFMs and that can be implemented in a blockchain as a smart contract. We proposed models to build strategic bonding curves in DLPs where liquidity provision is based on beliefs over the trading flow of liquidity takers. There is a number of ways this work can be extended. First, we derive the strategic bonding curves for fixed impact functions, and one may develop models to optimise the impact and quote functions jointly. Second, one can consider impact functions that are driven by additional exogenous information. Third, in practice,one can incorporate the significant delays of blockchains and the pre-trade transparency in the memory pools of blockchains into strategic bonding curve models to improve trading performance of LPs; see [26] and [36]. Finally, our models can be extended to multiple related pools or multiple currency pairs within the same pool using multivariate models, see [14].

Appendix A: Model parameters

TABLE 3. Model parameters

Parameter	$\{\eta^a,\eta^b\}$	Z_0	S_0	y_0	$\hat{y}$	$\{\underline{y},\overline{y}\}$	κ	T	α	ϕ	$\{c^a,c^b\}$	ζ
Value	$\{0.1,0.1\}$	10	10	50	50	$\{10,100\}$	0.5	1	0	1	$\{10,10\}$	1

[22] Penalising deviations from the initial price reduces the expected impermanent loss.

[23] The average bid-ask spread in the DLP is $2.5.

Appendix B: Strategic bonding curve when reserves are valued at the marginal price Here, we consider the model of Section 3.1 but where the performance criterion is

$$w^{\delta}(t,x,y,z,\boldsymbol{\gamma}) = \mathbb{E}_{t,x,y,z,\boldsymbol{\gamma}}\left[x_T + y_T\, Z_T - \alpha\,(y_T-\hat{y})^2 - \phi\int_t^T (y_s-\hat{y})^2\,\mathrm{d}s\right],$$

and the value function w of the DLP is

$$w(t,x,y,z,\boldsymbol{\gamma}) = \sup_{\delta\in\mathcal{A}_t} w^{\delta}(t,x,y,z,\boldsymbol{\gamma})\,. \tag{34}$$

The following result provides the optimal quote functions in this setup. The proof is omitted and follows the same arguments as in the proof of Proposition 4.

PROPOSITION 6. *If there exists a unique solution $\theta:[0,T]\times\mathcal{Y}\times[0,1]^{M-1}$ to the PDE*

$$\begin{aligned}
0 = {}& \partial_t\theta - \phi\,(y-\hat{y})^2 + \sum_{j=1}^{M-1}\left(2\psi^j + \psi\cdot\pi_{j,:}\right)\partial_{\psi^j}\theta - \tilde{\psi}\cdot\nabla_{\tilde{\psi}}\theta\sum_{j=1}^{M}\left(2\psi^j+\psi\cdot\pi_{j,:}\right)\\
&+\frac{\zeta(\psi\cdot\mathbf{c}^b)}{\kappa\, e}\exp\left(\frac{\kappa}{\zeta}\left(\theta\Big(t,y+\zeta,\frac{\tilde{\psi}\odot\tilde{\mathbf{c}}^b}{\psi\cdot\mathbf{c}^b}\Big)-\theta\Big(t,y,\tilde{\boldsymbol{\psi}}\Big)-\Big(\frac{\tilde{\psi}\odot\tilde{\mathbf{c}}^b}{\psi\cdot\mathbf{c}^b}-\tilde{\psi}\Big)\cdot\nabla_{\tilde{\psi}}\theta-(y+\zeta)\eta^b(y)\right)\right)\\
&+\frac{\zeta(\psi\cdot\mathbf{c}^a)}{\kappa\, e}\exp\left(\frac{\kappa}{\zeta}\left(\theta\Big(t,y-\zeta,\frac{\tilde{\psi}\odot\tilde{\mathbf{c}}^a}{\psi\cdot\mathbf{c}^a}\Big)-\theta\Big(t,y,\tilde{\boldsymbol{\psi}}\Big)-\Big(\frac{\tilde{\psi}\odot\tilde{\mathbf{c}}^a}{\psi\cdot\mathbf{c}^a}-\tilde{\psi}\Big)\cdot\nabla_{\tilde{\psi}}\theta+(y-\zeta)\eta^a(y)\right)\right).
\end{aligned}$$

with terminal condition $\theta\left(T,y,\tilde{\boldsymbol{\psi}}\right)=-\alpha\,y^2$, then the function $\omega\,(t,x,y,z,\boldsymbol{\gamma}) = x+y\,Z+\theta\left(t,y,\tilde{\boldsymbol{\psi}}\right)$ solves the HJB satisfied by the value function (34)*, and the optimal quote functions are*

$$\begin{aligned}
\delta^{\star,b}\left(y,\tilde{\boldsymbol{\psi}}\right) &= \frac{1}{\kappa} - \frac{\theta\Big(t,y+\zeta,\frac{\tilde{\psi}\odot\tilde{\mathbf{c}}^b}{\psi\cdot\mathbf{c}^b}\Big)-\theta(t,y,\tilde{\boldsymbol{\psi}})}{\zeta} + \frac{1}{\zeta}\left(\frac{\tilde{\psi}\odot\tilde{\mathbf{c}}^b}{\psi\cdot\mathbf{c}^b}\cdot\nabla_{\tilde{\psi}}\theta - \tilde{\psi}\cdot\nabla_{\tilde{\psi}}v - (y+\zeta)\,\eta^b(y)\right)\\
\delta^{\star,a}\left(y,\tilde{\boldsymbol{\psi}}\right) &= \frac{1}{\kappa} - \frac{\theta\Big(t,y-\zeta,\frac{\tilde{\psi}\odot\tilde{\mathbf{c}}^a}{\psi\cdot\mathbf{c}^a}\Big)-\theta(t,y,\tilde{\boldsymbol{\psi}})}{\zeta} + \frac{1}{\zeta}\left(\frac{\tilde{\psi}\odot\tilde{\mathbf{c}}^a}{\psi\cdot\mathbf{c}^a}\cdot\nabla_{\tilde{\psi}}\theta - \tilde{\psi}\cdot\nabla_{\tilde{\psi}}v + (y-\zeta)\,\eta^a(y)\right).
\end{aligned}$$

Appendix C: DLP bonding curve when the horizon is random. Two extensions of the DLP problems studied above are (i) random exit time (for instance, exponentially distributed), and (ii) exit time chosen optimally by the DLP.

We first consider case (i) and argue that a substantial degree of tractability can be preserved. To illustrate this point, consider the model with endogenous price formation studied in Section 3.2, and let the exit time τ be exponentially distributed with parameter $\underline{\lambda}$.

First, we augment the filtration of the controller (DLP) with

$$\mathcal{H}_t = \sigma(H_u : u\le t)\,,$$

where $H_u = \mathbb{1}(\tau \le u)$. The performance criterion with random horizon is defined as in (26), with the fixed terminal time T replaced by $T \wedge \tau$, and the associated value function is defined analogously.

The Hamilton–Jacobi–Bellman equation characterizing the value function is similar to (41), with the addition of a killing term

$$\underline{\lambda}\left(\omega(t,x,y,z) - x - yz + \alpha(y-\hat{y})^2\right).$$

Next, use the ansatz $\omega(t,x,y,z) = x + yz + \theta(t,y)$ and proceed as in the baseline to obtain a partial differential equation for θ. At this stage, if $\eta^{a,b}$ are linear, a linear approximation of the resulting exponential terms would lead to a tractable PDE. Otherwise, one can employ numerical methods to obtain the optimal bonding curve as in Section 3.1. We leave the details to the interested reader and refer to [45] for closely related arguments.

Next, we turn to case (ii) when the exit time is controlled by the DLP. This extension leads to quasi-variational inequalities in the characterization of the value function and requires a more delicate analytical treatment. Related optimal stopping problems for liquidity providers are in [3, 13]. For instance, [13] derives the solution to the optimal stopping problem when the controls $\delta_t^{a,b}$ correspond to Uniswap v2 (including fees). In that setting, the exit region depends jointly on price dislocations, time, and the state of the liquidity pool.

Appendix D: Proofs

D.1. Proof of Proposition 1 Assume $Z_0 \in \mathcal{Z}$ and that for $i \in \{0,\dots,N-1\}$,

$$z_{i+1} - \eta^b(y_{N-i}) = z_i \qquad \text{and} \qquad z_i + \eta^a(y_{N-i} + \zeta) = z_{i+1}\,,$$

Then for any $i \in \{0,\dots,N-1\}$, the price following two consecutive and opposite trades of size ζ is

$$z_i - \eta^b(y_{N-i-1}) + \eta^a(y_{N-i-1} + \zeta),$$

or

$$z_i + \eta^b(y_{N-i}) - \eta^b(y_{N-i} + \zeta).$$

These prices are equal to z_i for all i if and only if

$$\eta^b(y_i) = \eta^a(y_i + \zeta)\,.$$

D.2. Proof of Theorem 1 Let $\left(x_t^{\mathrm{CFM}}\right)_{t\in[0,T]}$, $\left(y_t^{\mathrm{CFM}}\right)_{t\in[0,T]}$, and $\left(Z_t^{\mathrm{CFM}}\right)_{t\in[0,T]}$ be the processes describing the amount of asset X, amount of asset Y, and marginal price respectively, in a CFM pool. From the trading condition of CFMs we know that

$$f\left(x_{t^-}^{\mathrm{CFM}}-\zeta^x, y_{t^-}^{\mathrm{CFM}}+\zeta\right)=f\left(x_{t^-}^{\mathrm{CFM}}, y_{t^-}^{\mathrm{CFM}}\right)=m^2\,,$$
$$\text{and}\quad f\left(x_{t^-}^{\mathrm{CFM}}+\zeta^x, y_{t^-}^{\mathrm{CFM}}-\zeta\right)=f\left(x_{t^-}^{\mathrm{CFM}}, y_{t^-}^{\mathrm{CFM}}\right)=m^2\,,$$

for a sell order of size ζ and a buy order of size ζ of asset Y, respectively, where m is the fixed depth of the CFM pool, and $\zeta^x>0$ is in terms of asset X.

Thus, for a sell order of size ζ, $x_{t^-}^{\mathrm{CFM}}-\zeta^x=\varphi\left(y_{t^-}^{\mathrm{CFM}}+\zeta\right)$ and $x_{t^-}^{\mathrm{CFM}}=\varphi\left(y_{t^-}^{\mathrm{CFM}}\right)$ hold, so the (bid) execution price is

$$\frac{\zeta^x}{\zeta}=\frac{\varphi\left(y_{t^-}^{\mathrm{CFM}}\right)-\varphi\left(y_{t^-}^{\mathrm{CFM}}+\zeta\right)}{\zeta}\,.$$

Thus, the execution cost in the CFM pool is

$$\frac{\varphi\left(y_{t^-}^{\mathrm{CFM}}+\zeta\right)-\varphi\left(y_{t^-}^{\mathrm{CFM}}\right)}{\zeta}-\varphi'\left(y_{t^-}^{\mathrm{CFM}}\right)\,.$$

Similarly, the execution cost at the ask is

$$\frac{\varphi\left(y_{t^-}^{\mathrm{CFM}}\right)-\varphi\left(y_{t^-}^{\mathrm{CFM}}+\zeta\right)}{\zeta}+\varphi'\left(y_{t^-}^{\mathrm{CFM}}\right)\,,$$

so the execution costs in the CFM and the DLPs are the same when (7) holds.

Next, following a sell trade of size ζ in the CFM pool, the innovation in the marginal price is $-\varphi'(y_{t^-}^{\mathrm{CFM}}+\zeta)+\varphi'\left(y_{t^-}^{\mathrm{CFM}}\right)$. Similarly, following a buy trade of size ζ in the CFM pool, the innovation in the marginal price is $-\varphi'(y_{t^-}^{\mathrm{CFM}}-\zeta)+\varphi'\left(y_{t^-}^{\mathrm{CFM}}\right)$. Thus, the marginal price dynamics in the CFM and the DLPs are the same when (6) holds and $Z_0^{\mathrm{CFM}}=Z_0$. Similarly, compare (2.1) and (5) for the choice of $\delta^{a,b}$ above, to see that if $y_0^{\mathrm{CFM}}=y_0$ and $x_0^{\mathrm{CFM}}=x_0$ then $y_t^{\mathrm{CFM}}=y_t$ and $x_t^{\mathrm{CFM}}=x_t$ for all $t\in[0,T]$. Q.E.D.

D.3. Proof of Theorem 2 Let $y_0\in\mathcal{Y}$ be the initial reserves of the LP and without loss of generality assume that $\underline{y}<y_0=y_j<\bar{y}$ for $j\in\{1,\dots,N-1\}$.[24] A roundtrip sequence of trades contains an even number of individual trades, that is, the total number of trades used by the LT is $\mathfrak{m}=2m$ for some $m\in\mathbb{N}$. We proceed by induction over $m\in\mathbb{N}$; our proof is similar to the proof of Lemma 3 in [21]. For $m=1$ the LT either executes a (i) buy-then-sell or (ii) sell-then-buy strategy.

[24] If y_0 is at either of the boundaries we just not consider an initial buy (resp. an initial sell) from the LT when y_0 is $\underline{y}$ (resp. $\bar{y}$).

The P&L of the LT is given by (12) for general liquidity provision strategies, which is negative if (14) is satisfied. Recall that $\eta^a(\cdot)$ and $\eta^b(\cdot)$ satisfy (2) so we have that (12) becomes

$$\begin{aligned} \text{case (i)} \qquad & \text{P\&L} = -\frac{2\zeta}{\kappa} + 2\zeta\,\eta^a(y_0)\,, \\ \text{case (ii)} \qquad & \text{P\&L} = -\frac{2\zeta}{\kappa} + 2\zeta\,\eta^b(y_0)\,. \end{aligned} \tag{35}$$

It follows that (31) implies that the P&L in both cases of (35) is negative.

Our goal is to show that if there is no roundtrip arbitrage in a sequence of $\mathfrak{m} = 2\,(m-1)$ trades, then there is no roundtrip arbitrage in a sequence of $\mathfrak{m} = 2\,m$ trades. Let $\{\epsilon_1, \ldots, \epsilon_{\mathfrak{m}}\}$ be a roundtrip sequence of trades, i.e., $\epsilon_k \in \{1, -1\}$ and $\sum_{k=1}^{\mathfrak{m}} \epsilon_k = 0$, such that $\mathfrak{m} = 2\,m$.

Define the inventory of the LT as $(Q_k)_{k\in\{0,1,\ldots,n\}}$ by setting $Q_0 = 0$ and $Q_k = Q_{k-1} + \epsilon_k$ for $k = 1, \ldots, n$. Consider the case $\epsilon_1 = 1$ (the case $\epsilon_1 = -1$ is analogous) and observe that if an element of the series $(Q_k)_{k\in\{1,\ldots,n\}}$ is negative, then $\exists\, m \in \{2, \ldots, n-1\}$ such that $Q_m = 0$ and $Q_{m+1} = -1$. Define the series $(R_k)_{k\in\{0,\ldots,n-m\}}$ by $R_0 = 0$ and $R_k = R_{k-1} + \epsilon_{k+m}$ for $k \in \{1, \ldots, n-m\}$. Then the inductive hypothesis that both sequences of roundtrip trades are not profitable holds because $m < n$ and $n - m < n$. Thus, it suffices to restrict our attention to the case where all elements of the series $(Q_k)_{k\in\{1,\ldots,n\}}$ are non-negative. It follows that $Q_{\mathfrak{m}-1} = 1$ and the sequence of trades $\{\epsilon_2, \ldots, \epsilon_{\mathfrak{m}-1}\}$ form a roundtrip sequence of $\mathfrak{m} - 2$ trades. By the inductive hypothesis it follows that this sequence of $\mathfrak{m} - 2$ trades is not profitable. Last, observe that the P&L associated with $\epsilon_1 = +1$ and $\epsilon_{\mathfrak{m}} = -1$ is given by

$$\text{P\&L} = \zeta\left(\eta^a\left(y_j\right) - \delta^a_{j,-} - \delta^b_{j,+}\right),$$

for general liquidity provision strategies, which is negative as a consequence of (14).

By induction, it follows that there is not a roundtrip sequence of trades of any length that would be profitable for the LT. Q.E.D.

D.4. Proof of Proposition 3 To prove that (22) is finite we observe that for $\delta \in \mathcal{A}_t$ it follows that $|y_s| \le y^*$ for $s \in [t, T]$ where $y^* = \max\{|\underline{y}|, |\bar{y}|\}$. Next, define $\eta^* = \max\{\eta^b(\underline{y}), \ldots, \eta^b(\bar{y}), \eta^a(\underline{y}), \ldots, \eta^a(\bar{y})\}$, and

$$\eta^b(y) = \max_{\Psi\in[0,1]^M} \eta^b(y, \Psi)\,,$$

for $y \in \mathcal{Y}$. We also let $c^* = \max\{c^{a,1}, c^{b,1}, \ldots, c^{a,M}, c^{b,M}\}$, then, a simple calculation yields the bound

$$\mathbb{E}[|Z_t|] \leq |Z_0| + 2\,T\,\eta^*\,c^*,$$

for $t \in [0,T]$, which we use to show that

$$\begin{aligned}
&\left|\mathbb{E}_{t,x,y,z}\left[x_T + y_T\,Z_T - \alpha\,(y_T - \hat{y})^2 - \phi\int_t^T (y_s - \hat{y})^2\,\mathrm{d}s\right]\right| \\
&\quad \leq |x_0| + 2\,\left(T\,\zeta\,\left(|Z_0| + 2\,T\,\eta^*\,c^*\right)\,c^*\right) + \zeta\,c^*\,\mathbb{E}\left[\int_t^T |\delta_s^a| + |\delta_s^b|\mathrm{d}s\right] \\
&\qquad + y^*\,\left(|Z_0| + 2\,T\,\eta^*\,c^*\right) + \alpha\,\left(y^* + |\hat{y}|\right)^2 + \phi\,T\,\left(y^* + |\hat{y}|\right)^2 < \infty
\end{aligned}$$

because $\delta \in \mathcal{A}$. Next, use the inequality $\delta^{a,b}\,e^{-\kappa\,\delta^{a,b}} \leq \frac{1}{\kappa}\,e^{-1}$, to show that $w^\delta(t,x,y,z)$ is bounded from above:

$$\begin{aligned}
&\left|\mathbb{E}_{t,x,y,z}\left[x_T + y_T\,Z_T - \alpha\,(y_T - \hat{y})^2 - \phi\int_t^T (y_s - \hat{y})^2\,\mathrm{d}s\right]\right| \\
&\quad \leq |x_0| + 2\,\left(T\,\zeta\,\left(|Z_0| + 2\,T\,\eta^*\,c^*\right)\,c^* + T\,\zeta\,c^*\,\frac{1}{\kappa}e^{-1}\right) \\
&\qquad + y^*\,\left(|Z_0| + 2\,T\,\eta^*\,c^*\right) + \alpha\,\left(y^* + \hat{y}\right)^2 + \phi\,T\,\left(y^* + \hat{y}\right)^2 < \infty.
\end{aligned}$$

Thus, there is $C \in \mathbb{R}$ such that for all $\delta \in \mathcal{A}_t$, $w^\delta(t,x,y,z) \leq C < \infty$ which implies that the supremum over $\mathcal{A}_t$ is finite. Thus, the value function $w\colon\ [0,T] \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R} \to \mathbb{R}$

$$w(t,x,y,z) = \sup_{\delta \in \mathcal{A}_t} w^\delta(t,x,y,z)$$

is well defined. Q.E.D.

D.5. Proof of Proposition 4 The value function w satisfies the HJB equation

$$\begin{aligned}
0 = {} & \partial_t\omega - \phi\,(y - \hat{y})^2 + \sum_{j=1}^M \left(2\,\gamma^j + \sum_{k=1}^M \gamma^k\,\pi_{jk}\right)\partial_{\gamma_j}\omega \\
& + \sup_{\delta^b}\left\{\hat{\lambda}_t^b(\delta^b,\boldsymbol{\gamma})\left\{\omega\left(t, x - \zeta\,\left(z - \delta^b\right), y + \zeta, z - \eta^b(y), \boldsymbol{\gamma} + \boldsymbol{\Delta}^b\left(\delta^b\right)\right) - \omega\,(t,x,y,z,\boldsymbol{\gamma})\right\}\right. \\
& \qquad \left. - \sum_{j=1}^M \gamma^j\,\lambda^{b,j}\left(\delta^b\right)\,\partial_{\gamma^j}\omega\right\} \\
& + \sup_{\delta^a}\left\{\hat{\lambda}_t^a(\delta^a,\boldsymbol{\gamma})\left\{\omega\left(t, x + \zeta\,\left(z + \delta^a\right), y - \zeta, z + \eta^a(y), \boldsymbol{\gamma} + \boldsymbol{\Delta}^a\left(\delta^a\right)\right) - \omega\,(t,x,y,z,\boldsymbol{\gamma})\right\}\right.
\end{aligned}$$

$$- \sum_{j=1}^{M} \gamma^j \, \lambda^{a,j} \left(\delta^a\right) \, \partial_{\gamma^j} \omega \Bigg\},$$

with the terminal condition

$$\omega(T, x, y, z, \gamma) = x + y\, z - \alpha \left(y - \hat{y}\right)^2,$$

where

$$\begin{cases} \mathbf{\Delta}^b \left(\delta^b\right) = & \left(\gamma^1 \, \left(\lambda^{b,1} \left(\delta^b\right) - 1\right), \ldots, \gamma^M \, \left(\lambda^{b,M} \left(\delta^b\right) - 1\right)\right), \\ \mathbf{\Delta}^a \left(\delta^a\right) = & \left(\gamma^1 \, \left(\lambda^{a,1} \left(\delta^a\right) - 1\right), \ldots, \gamma^M \, \left(\lambda^{a,M} \left(\delta^a\right) - 1\right)\right). \end{cases}$$

Define $\boldsymbol{\psi} = \{\psi^1, \ldots, \psi^M\}$ and $\tilde{\boldsymbol{\psi}} = \{\psi^1, \ldots, \psi^{M-1}\}$. To further study the problem, consider the value function $w \colon [0,T] \times \mathbb{R} \times \boldsymbol{\mathcal{Y}} \times \mathbb{R} \times [0,1]^{M-1} \times \mathbb{R}_{++} \to \mathbb{R}$ by employing the change of variables

$$\omega\left(t, x, y, z, \gamma\right) = \omega\left(t, x, y, z, \tilde{\boldsymbol{\psi}}, \overline{\gamma}\right),$$

where

$$\begin{cases} \overline{\gamma} & = \sum\limits_{k=1}^{M} \gamma^k, \\ \psi^j & = \gamma^j \Big/ \sum\limits_{k=1}^{M} \gamma^k, \end{cases} \Longrightarrow \begin{cases} \gamma^j & = \psi^j \, \overline{\gamma}, \\ \gamma^M & = \left(1 - \sum_{k=1}^{M-1} \psi^k\right) \, \overline{\gamma}. \end{cases}$$

The change of variables yields the partial derivatives

$$\begin{cases} \partial_{\gamma^j} \omega & = \sum_{k=1}^{M-1} \partial_{\gamma^j} \psi^k \, \partial_{\psi^k} \omega + \partial_{\gamma^j} \overline{\gamma} \, \partial_{\overline{\gamma}} \omega \qquad \text{for } j \in \{1, \ldots, M-1\} \\ & = \frac{1}{\overline{\gamma}} \sum_{k=1}^{M-1} \left(\mathbf{1}_{\mathbf{j=k}} - \psi^k\right) \, \partial_{\psi^k} \omega + \partial_{\overline{\gamma}} \omega \\ & = -\frac{1}{\overline{\gamma}} \tilde{\psi} \cdot \nabla_{\tilde{\psi}} \omega + \frac{1}{\overline{\gamma}} \partial_{\psi^j} \omega + \partial_{\overline{\gamma}} \omega, \\ \partial_{\gamma^M} \omega & = \sum_{k=1}^{M-1} \partial_{\gamma^M} \psi^k \, \partial_{\psi^k} \omega + \partial_{\gamma^M} \overline{\gamma} \, \partial_{\overline{\gamma}} \omega \\ & = -\frac{1}{\overline{\gamma}} \sum_{k=1}^{M-1} \psi^k \, \partial_{\psi^k} \omega + \partial_{\overline{\gamma}} \omega \\ & = -\frac{1}{\overline{\gamma}} \tilde{\psi} \cdot \nabla_{\tilde{\psi}} \omega + \partial_{\overline{\gamma}} \omega. \end{cases}$$

Observe that

$$\begin{aligned} & \omega\left(t, x - \zeta \, \left(z - \delta^b\right), y + \zeta, z - \eta^b(y), \boldsymbol{\gamma} + \mathbf{\Delta}^b \left(\delta^b\right)\right) - \omega\left(t, x, y, z, \boldsymbol{\gamma}\right) \\ = & v\left(t, x - \zeta \, \left(z - \delta^b\right), y + \zeta, z - \eta^b(y), \frac{\tilde{\psi} \odot \tilde{\mathbf{c}}^b}{\psi \cdot \mathbf{c}^b}, \overline{\gamma} \, e^{-\kappa \, \delta^b} \, \left(\psi \cdot \mathbf{c}^b\right)\right) - v\left(t, x, y, z, \tilde{\boldsymbol{\psi}}, \overline{\gamma}\right), \end{aligned}$$

and similarly,

$$\begin{aligned}&\omega\left(t,x+\zeta\ \left(z+\delta^a\right),y-\zeta,z+\eta^a(y),\gamma+\boldsymbol{\Delta}^a\left(\delta^a\right)\right)-\omega\left(t,x,y,z,\gamma\right)\\=&v\left(t,x+\zeta\ \left(z+\delta^a\right),y-\zeta,z+\eta^a(y),\frac{\tilde{\boldsymbol{\psi}}\odot\tilde{\mathbf{c}}^a}{\boldsymbol{\psi}\cdot\mathbf{c}^a},\overline{\gamma}\,e^{-\kappa\,\delta^a}\ \left(\boldsymbol{\psi}\cdot\mathbf{c}^a\right)\right)-v\left(t,x,y,z,\tilde{\boldsymbol{\psi}},\overline{\gamma}\right)\,.\end{aligned}$$

Finally, one obtains

$$\begin{cases}\sum_{j=1}^M\gamma^j\,\lambda^{b,j}\left(\delta^b\right)\,\partial_{\gamma^j}\omega & =e^{-\kappa\,\delta^b}\left(\left[\tilde{\boldsymbol{\psi}}\odot\tilde{\mathbf{c}}^b-\left(\boldsymbol{\psi}\cdot\mathbf{c}^b\right)\tilde{\boldsymbol{\psi}}\right]\cdot\nabla_{\tilde{\boldsymbol{\psi}}}v+\boldsymbol{\psi}\cdot\mathbf{c}^b\,\overline{\gamma}\,\partial_{\overline{\gamma}}\omega\right)\\ \sum_{j=1}^M\gamma^j\,\lambda^{a,j}\left(\delta^a\right)\,\partial_{\gamma^j}\omega & =e^{-\kappa\,\delta^a}\left(\left[\tilde{\boldsymbol{\psi}}\odot\tilde{\mathbf{c}}^a-\left(\boldsymbol{\psi}\cdot\mathbf{c}^a\right)\tilde{\boldsymbol{\psi}}\right]\cdot\nabla_{\tilde{\boldsymbol{\psi}}}v+\boldsymbol{\psi}\cdot\mathbf{c}^a\,\overline{\gamma}\,\partial_{\overline{\gamma}}\omega\right)\,.\end{cases}$$

Next, we employ the ansatz

$$\omega\left(t,x,y,z,\tilde{\boldsymbol{\psi}},\overline{\gamma}\right)=v\left(t,x,y,z,\tilde{\boldsymbol{\psi}}\right),$$

and use the computations above to find that the function $v\colon\,[0,T]\times\mathbb{R}\times\mathcal{Y}\times\mathcal{Z}\times[0,1]^{M-1}\to\mathbb{R}$ solves the PDE

$$\begin{aligned}0=&\,\partial_t v-\phi\left(y-\hat{y}\right)^2\\&+\sup_{\delta^b}e^{-\kappa\,\delta^b}\left\{\left(\boldsymbol{\psi}\cdot\mathbf{c}^b\right)\,\left\{v\left(t,x-\zeta\ \left(z-\delta^b\right),y+\zeta,z-\eta^b(y),\frac{\tilde{\boldsymbol{\psi}}\odot\tilde{\mathbf{c}}^b}{\boldsymbol{\psi}\cdot\mathbf{c}^b}\right)-v\left(t,x,y,z,\tilde{\boldsymbol{\psi}}\right)\right\}\right.\\&\qquad\qquad\left.-\left(\tilde{\boldsymbol{\psi}}\odot\tilde{\mathbf{c}}^b-\left(\boldsymbol{\psi}\cdot\mathbf{c}^b\right)\tilde{\boldsymbol{\psi}}\right)\cdot\nabla_{\tilde{\boldsymbol{\psi}}}v\right\}\\&+\sup_{\delta^a}e^{-\kappa\,\delta^a}\left\{\left(\boldsymbol{\psi}\cdot\mathbf{c}^a\right)\,\left\{v\left(t,x+\zeta\ \left(z+\delta^a\right),y-\zeta,z+\eta^a(y),\frac{\tilde{\boldsymbol{\psi}}\odot\tilde{\mathbf{c}}^a}{\boldsymbol{\psi}\cdot\mathbf{c}^a}\right)-v\left(t,x,y,z,\tilde{\boldsymbol{\psi}}\right)\right\}\right.\\&\qquad\qquad\left.-\left(\tilde{\boldsymbol{\psi}}\odot\tilde{\mathbf{c}}^a-\left(\boldsymbol{\psi}\cdot\mathbf{c}^a\right)\tilde{\boldsymbol{\psi}}\right)\cdot\nabla_{\tilde{\boldsymbol{\psi}}}v\right\},\\&+\sum_{j=1}^{M-1}\left(2\,\boldsymbol{\psi}^j+\boldsymbol{\psi}\cdot\boldsymbol{\pi}_{j,:}\right)\partial_{\boldsymbol{\psi}^j}v-\tilde{\boldsymbol{\psi}}\cdot\nabla_{\tilde{\boldsymbol{\psi}}}v\sum_{j=1}^{M}\left(2\,\boldsymbol{\psi}^j+\boldsymbol{\psi}\cdot\boldsymbol{\pi}_{j,:}\right)\,,\end{aligned}$$

with terminal condition

$$v\left(T,x,y,z,\tilde{\boldsymbol{\psi}},\overline{\gamma}\right)=x+y\,z-\alpha\,y^2\,.$$

Next, use the ansatz

$$v\left(t,x,y,z,\tilde{\boldsymbol{\psi}}\right)=x+y\,Z+\theta\left(t,y,\tilde{\boldsymbol{\psi}}\right)\,,$$

to find that $\theta:[0,T]\times\mathcal{Y}\times[0,1]^{M-1}$ solves the PDE

$$0=\partial_t\theta-\phi\left(y-\hat{y}\right)^2+\sum_{j=1}^{M-1}\left(2\,\boldsymbol{\psi}^j+\boldsymbol{\psi}\cdot\boldsymbol{\pi}_{j,:}\right)\partial_{\boldsymbol{\psi}^j}\theta-\tilde{\boldsymbol{\psi}}\cdot\nabla_{\tilde{\boldsymbol{\psi}}}\theta\sum_{j=1}^{M}\left(2\,\boldsymbol{\psi}^j+\boldsymbol{\psi}\cdot\boldsymbol{\pi}_{j,:}\right)$$

$$
\begin{aligned}
&+\sup_{\delta^b} e^{-\kappa\,\delta^b}\left\{\left(\psi\cdot\mathbf{c}^b\right)\,\left\{\zeta\,\delta^b+(a-z)\;\zeta+\theta\left(t,y+\zeta,\frac{\tilde{\psi}\odot\tilde{\mathbf{c}}^b}{\psi\cdot\mathbf{c}^b}\right)-\theta\left(t,y,\tilde{\boldsymbol{\psi}}\right)+\tilde{\psi}\cdot\nabla_{\tilde{\psi}}\theta\right\}\right.\\
&\qquad\qquad\left.-\left(\tilde{\psi}\odot\tilde{\mathbf{c}}^b\right)\cdot\nabla_{\tilde{\psi}}\theta\right\}\\
&+\sup_{\delta^a} e^{-\kappa\,\delta^a}\left\{\left(\psi\cdot\mathbf{c}^a\right)\,\left\{\zeta\,\delta^a+(z-a)\;\zeta+\theta\left(t,y-\zeta,\frac{\tilde{\psi}\odot\tilde{\mathbf{c}}^a}{\psi\cdot\mathbf{c}^a}\right)-\theta\left(t,y,\boldsymbol{\psi}\right)+\tilde{\psi}\cdot\nabla_{\tilde{\psi}}\theta\right\}\right.\\
&\qquad\qquad\left.-\left(\tilde{\psi}\odot\tilde{\mathbf{c}}^a\right)\cdot\nabla_{\tilde{\psi}}\theta\right\},
\end{aligned}
$$

with terminal condition

$$\theta(T,y,\tilde{\boldsymbol{\psi}})=-\alpha\,y^2.$$

Solve the supremum terms above to obtain the strategic bonding curve in feedback form (24) and the PDE (25) for θ. In particular, if there exists a unique solution to the PDE (25), then the function

$$\omega\left(t,x,y,z,\boldsymbol{\gamma}\right)=x+a\,y+\theta\left(t,y,\tilde{\boldsymbol{\psi}}\right)$$

solves the HJB satisfied by the value function (23). Q.E.D.

D.6. Proof of Theorem 3 The value function $w\colon\,[0,T]\times\mathbb{R}\times\mathcal{Y}\times\mathcal{Z}\to\mathbb{R}$ of the LP is

$$w(t,x,y,z)=\sup_{\delta\in\mathcal{A}_t} w^{\delta}(t,x,y,z)\,. \tag{36}$$

The proof is divided into the following results. Proposition 7 shows that the performance criterion (26) is finite and that the associated value function (36) is well defined, Proposition 8 provides a candidate closed-form solution, and Theorem 5 is the verification theorem for the strategic bonding curve in the DLP when price formation is endogenous. For proofs, see D.6.1, D.6.2, and D.6.3.

PROPOSITION 7. *There is $C\in\mathbb{R}$ such that for all $(\delta_s)_{s\in[t,T]}\in\mathcal{A}_t$, the performance criterion of the LP satisfies $w^{\delta}(t,x,y,z)\leq C<\infty$, so the value function w in* (36) *is well defined.*

PROPOSITION 8 (**Candidate closed-form solution: DLP**). *Let $\underline{N}=\underline{y}/\zeta$, $\overline{N}=\overline{y}/\zeta$, and $N=\overline{N}-\underline{N}+1$. Define the matrix $\mathbf{K}\in\mathbb{R}^{N\times N}$ by*

$$
\mathbf{K}_{mn}=\begin{cases}
c^a\,e^{-1}\,e^{\kappa\,(m-1)\,\eta^a(m\,\zeta)} & \text{if } n=m-1 \text{ and } m>\underline{N},\\
-\kappa\,\phi\,(m\,\zeta-\hat{y})^2\,/\,\zeta & \text{if } n=m,\\
c^b\,e^{-1}\,e^{-\kappa\,(m+1)\,\eta^b(m\,\zeta)} & \text{if } n=m+1 \text{ and } m<\overline{N},
\end{cases}
\tag{37}
$$

for $m, n \in \{\underline{N}, \underline{N}+1, \dots, \overline{N}\}$. *Let* $\mathbf{U} \in C^1([0,T], \mathbb{R}^N)$ *be*

$$\mathbf{U}(t) = \exp(\mathbf{K}t)\ \mathbf{U}(0)\,, \quad \mathbf{U}(0)_m = e^{-\alpha\ \frac{\kappa}{\zeta}\ (\zeta m - \hat{y})^2}\ ,\ m \in [\underline{N}, \bar{N}] \cap \mathbb{Z}\,, \quad t \in [0,T]\,, \tag{38}$$

where $\exp(\cdot)$ *is matrix exponentiation. For* $m \in [\underline{N}, \bar{N}] \cap \mathbb{Z}$ *let* $u(t, m\,\zeta) = \mathbf{U}(T-t)_m$, *and define*

$$\theta(t,y) = \frac{\zeta}{\kappa}\ \log u(t,y)\,. \tag{39}$$

Then, the function $\omega\colon\ [0,T] \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R} \to \mathbb{R}$ *given by*

$$\omega(t,x,y,z) = x + y\,z + \theta(t,y) \tag{40}$$

solves the HJB equation associated with problem (36) *which is given by*

$$\begin{aligned} 0 = \partial_t \omega - \phi\,(y-\hat{y})^2 &+ \sup_{\delta^b} \lambda^b\left(\delta^b\right)\left\{\omega\left(t, x-\zeta\ \left(z-\delta^b\right), y+\zeta, z-\eta^b(y)\right) - \omega\,(t,x,y,z)\right\} \\ &+ \sup_{\delta^a} \lambda^a\,(\delta^a)\left\{\omega\,(t, x+\zeta\ (z+\delta^a), y-\zeta, z+\eta^a(y)) - \omega\,(t,x,y,z)\right\}, \end{aligned} \tag{41}$$

with the terminal condition

$$\omega(T,x,y,z) = x + y\,z - \alpha\,(y-\hat{y})^2\,.$$

THEOREM 5 **(Verification: DLP)**. *Let* ω *be defined as in Proposition 8. Then the function* ω *in* (40) *satisfies that for all* $(t,x,y,z) \in [0,T] \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R}$ *and* $\delta = (\delta_s)_{s\in[t,T]} \in \mathcal{A}_t$,

$$w^{\delta}(t,x,y,z) \le \omega\,(t,x,y,z)\ . \tag{42}$$

Moreover, equality is obtained in (42) *with the admissible optimal Markovian control* $\left(\delta_s^{\star}\right)_{s\in[t,T]} = \left(\delta_s^{b\star}, \delta_s^{a\star}\right)_{s\in[t,T]} \in \mathcal{A}_t$ *given by the feedback formulae*

$$\begin{aligned} \delta^{b\star}(t, y_{t^-}) &= \frac{1}{\kappa} - \frac{\theta\,(t, y_{t^-}+\zeta) - \theta\,(t, y_{t^-})}{\zeta} - \frac{(y_{t^-}+\zeta)\,\eta^b(y_{t^-})}{\zeta}\,, \\ \delta^{a\star}(t, y_{t^-}) &= \frac{1}{\kappa} - \frac{\theta\,(t, y_{t^-}-\zeta) - \theta\,(t, y_{t^-})}{\zeta} + \frac{(y_{t^-}-\zeta)\,\eta^a(y_{t^-})}{\zeta}\,, \end{aligned}$$

where θ *is in* (39). *In particular,* $\omega = w$ *on* $[0,T] \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R}$.

D.6.1. Proof of Proposition 7 To prove that (26) is finite we observe that for $\delta \in \mathcal{A}_t$ it follows that $|y_s| \leq y^*$ for $s \in [t,T]$ where $y^* = \max\{|\underline{y}|, |\bar{y}|\}$. Next, define $\eta^* = \max\{\eta^b(\underline{y}), \ldots, \eta^b(\bar{y}), \eta^a(\underline{y}), \ldots, \eta^a(\bar{y})\}$, and $c^* = \max\{c^a, c^b\}$, then, a simple calculation yields the bound $\mathbb{E}[|Z_t|] \leq |Z_0| + 2\,T\,\eta^*\,c^*$, for $t \in [0,T]$, which we use to show that

$$\begin{aligned}
&\left|\mathbb{E}_{t,x,y,z}\left[x_T + y_T\,Z_T - \alpha\,(y_T - \hat{y})^2 - \phi\int_t^T (y_s - \hat{y})^2\,\mathrm{d}s\right]\right| \\
&\quad \leq |x_0| + 2\,\left(T\,\zeta\,\left(|Z_0| + 2\,T\,\eta^*\,c^*\right)\,c^*\right) + \zeta\,c^*\,\mathbb{E}\left[\int_t^T |\delta_s^a| + |\delta_s^b|\mathrm{d}s\right] \\
&\qquad + y^*\,\left(|Z_0| + 2\,T\,\eta^*\,c^*\right) + \alpha\,\left(y^* + |\hat{y}|\right)^2 + \phi\,T\,\left(y^* + |\hat{y}|\right)^2 < \infty
\end{aligned}$$

because $\delta \in \mathcal{A}$. Next, use the inequality

$$\delta^{a,b}\,e^{-\kappa\,\delta^{a,b}} \leq \frac{1}{\kappa}\,e^{-1}\,,$$

to show that $w^\delta(t,x,y,z)$ is bounded from above:

$$\begin{aligned}
&\left|\mathbb{E}_{t,x,y,z}\left[x_T + y_T\,Z_T - \alpha\,(y_T - \hat{y})^2 - \phi\int_t^T (y_s - \hat{y})^2\,\mathrm{d}s\right]\right| \\
&\quad \leq |x_0| + 2\,\left(T\,\zeta\,\left(|Z_0| + 2\,T\,\eta^*\,c^*\right)\,c^* + T\,\zeta\,c^*\,\frac{1}{\kappa}e^{-1}\right) \\
&\qquad + y^*\,\left(|Z_0| + 2\,T\,\eta^*\,c^*\right) + \alpha\,\left(y^* + \hat{y}\right)^2 + \phi\,T\,\left(y^* + \hat{y}\right)^2 < \infty.
\end{aligned}$$

Thus, there is $C \in \mathbb{R}$ such that for all $\delta \in \mathcal{A}_t$, $w^\delta(t,x,y,z) \leq C < \infty$ which implies that the supremum over $\mathcal{A}_t$ is finite. Thus, the value function $w\colon\ [0,T] \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R} \to \mathbb{R}$

$$w(t,x,y,z) = \sup_{\delta \in \mathcal{A}_t} w^\delta(t,x,y,z)$$

is well defined. Q.E.D.

D.6.2. Proof of Proposition 8 The function $\omega(t,x,y,z) = x + y\,z + \theta(t,y)$ solves the HJB in (41) if $\theta(t,y)$ solves

$$\begin{aligned}
0 = \partial_t\theta - \phi(y - \hat{y})^2 &+ \sup_{\delta^b} \lambda^b\left(\delta^b\right)\left\{\theta\left(t, y + \zeta\right) - \theta\left(t,y\right) + \zeta\,\delta^b - y\,\eta^b(y) - \zeta\,\eta^b(y)\right\}\mathbb{1}^b(y) \\
&+ \sup_{\delta^a} \lambda^a\left(\delta^a\right)\left\{\theta\left(t, y - \zeta\right) - \theta\left(t,y\right) + \zeta\delta^a + y\,\eta^a(y) - \zeta\,\eta^a(y)\right\}\mathbb{1}^a(y)
\end{aligned} \tag{43}$$

with the terminal condition $\theta(T,y) = -\alpha(y - \hat{y})^2$. Let

$$g^b(t,y) = \theta\left(t, y + \zeta\right) - \theta\left(t,y\right) - \left(y + \zeta\right)\eta^b(y)\,, \tag{44}$$

$$g^a(t,y) = \theta(t,y-\zeta) - \theta(t,y) + (y-\zeta)\,\eta^a(y)\,,$$

then, the supremum for the buy side in the HJB equation (43) satisfies

$$\begin{aligned}&\sup_{\delta^b} \lambda^b\left(\delta^b\right)\left(\theta(t,y+\zeta) - \theta(t,y) + \zeta\,\delta^b - y\,\eta^b(y) - \zeta\,\eta^b(y)\right)\\ &\qquad = \sup_{\delta^b} \lambda^b\left(\delta^b\right)\left(\zeta\,\delta^b + g^b(t,y)\right),\end{aligned}$$

where g^b is given by (44). The expression for maximising δ^a is similar. In both cases we maximise concave functions where the first order conditions are satisfied by

$$\delta^{b\star} = \frac{1}{\kappa} - \frac{g^b(t,y)}{\zeta}\,, \quad \text{and} \quad \delta^{a\star} = \frac{1}{\kappa} - \frac{g^a(t,y)}{\zeta}\,.$$

Next, use the expressions for the feedback controls to evaluate the supremum in (43), and obtain

$$\sup_{\delta^b} \lambda^b\left(\delta^b\right)\left(\zeta\,\delta^b + g^b(t,y)\right) = c^b\,\frac{\zeta}{\kappa}\,\exp\left(-1 + \frac{\kappa\, g^b(t,y)}{\zeta}\right).$$

A similar calculation for the ask side yields

$$\sup_{\delta^a} \lambda^a\left(\delta^a\right)\left(\zeta\,\delta^a + g^a(t,y)\right) = c^a\,\frac{\zeta}{\kappa}\,\exp\left(-1 + \frac{\kappa\, g^a(t,y)}{\zeta}\right).$$

The above implies that $\omega(t,x,y,z) = x + y\,z + \theta(t,y)$ solves the HJB in (41) if $\theta(t,y)$ solves the PDE

$$\begin{aligned}0 &= \partial_t\theta - \phi\,(y-\hat{y})^2 \\ &\quad + c^b\,\frac{\zeta}{\kappa}\,\exp\left(-1 + \frac{\kappa\, g^b(t,y)}{\zeta}\right)\mathbb{1}^b(y) + c^a\,\frac{\zeta}{\kappa}\,\exp\left(-1 + \frac{\kappa\, g^a(t,y)}{\zeta}\right)\mathbb{1}^a(y)\,,\end{aligned} \tag{45}$$

with terminal condition $\theta(T,y) = -\alpha(y-\hat{y})^2$. Recall that $\kappa > 0$, and $\theta(t,y)$ is given by (39). Then $g^b(t,y)$ and $g^a(t,y)$ become

$$\begin{aligned}g^b(t,y) &= \frac{\zeta}{\kappa}\,\log\left(\frac{u(t,y+\zeta)}{u(t,y)}\right) - (y+\zeta)\,\eta^b(y)\,,\\ g^a(t,y) &= \frac{\zeta}{\kappa}\,\log\left(\frac{u(t,y-\zeta)}{u(t,y)}\right) + (y-\zeta)\,\eta^a(y)\,.\end{aligned}$$

Thus, θ solves (45) with the terminal condition $\theta(T,y) = -\alpha(y-\hat{y})^2$ if u satisfies

$$\begin{aligned}0 = \frac{\zeta}{u(t,y)\,\kappa}\,\partial_t u(t,y) - \phi\,(y-\hat{y})^2 &+ c^b\,\frac{\zeta}{\kappa}\,e^{-1}\,\frac{u(t,y+\zeta)}{u(t,y)}\,\exp\left(-\frac{\kappa}{\zeta}(y+\zeta)\,\eta^b(y)\right)\mathbb{1}^b(y)\\ &+ c^a\,\frac{\zeta}{\kappa}\,e^{-1}\,\frac{u(t,y-\zeta)}{u(t,y)}\,\exp\left(\frac{\kappa}{\zeta}(y-\zeta)\,\eta^a(y)\right)\mathbb{1}^a(y)\,,\end{aligned} \tag{46}$$

with the terminal condition $u(T,y) = \exp(-\frac{\kappa\alpha}{\zeta}(y-\hat{y})^2)$. This ODE system is equivalent to

$$0 = \partial_t u(t,y) - \frac{\kappa\, u(t,y)}{\zeta}\phi\,(y-\hat{y})^2 + c^b\, e^{-1}\, u(t,y+\zeta)\, \exp\left(-\frac{\kappa}{\zeta}(y+\zeta)\,\eta^b(y)\right)\mathbb{1}^b(y) \\ + c^a\, e^{-1}\, u(t,y-\zeta)\, \exp\left(\frac{\kappa}{\zeta}(y-\zeta)\,\eta^a(y)\right)\mathbb{1}^a(y)\,.$$

Recall that $\underline{N} = \underline{y}/\zeta$ and $\overline{N} = \overline{y}/\zeta$. For $i \in (\underline{N}, \overline{N}) \cap \mathbb{Z}$ and $y = i\,\zeta$, we have that

$$0 = \partial_t u(t,y) - \frac{\kappa\, u(t,y)}{\zeta}\phi\,(y-\hat{y})^2 + c^b\, e^{-1}\, u(t,y+\zeta)\, e^{-\frac{\kappa}{\zeta}(y+\zeta)\,\eta^b(y)} \\ + c^a\, e^{-1}\, u(t,y-\zeta)\, e^{\frac{\kappa}{\zeta}(y-\zeta)\eta^a(y)}\,,$$

and when $i = \underline{N}$ or $i = \overline{N}$ we have that

$$0 = \partial_t u(t,\underline{N}\zeta) - \frac{\kappa\, u(t,\underline{N}\zeta)}{\zeta}\phi\,(\underline{N}\zeta-\hat{y})^2 + c^b\, e^{-1}\, u(t,(\underline{N}+1)\zeta)\, e^{-\frac{\kappa}{\zeta}(\underline{N}+1)\zeta\,\eta^b((\underline{N}+1)\zeta)}\,,$$
$$0 = \partial_t u(t,\overline{N}\zeta) - \frac{\kappa\, u(t,\overline{N}\zeta)}{\zeta}\phi\,(\overline{N}\,\zeta-\hat{y})^2 + c^a\, e^{-1}\, u\left(t,\left(\overline{N}-1\right)\zeta\right)\, e^{\frac{\kappa}{\zeta}(\overline{N}-1)\zeta\,\eta^a((\overline{N}-1)\zeta)}\,.$$

Recall that $N = \overline{N} - \underline{N} + 1$, and that the vector-valued function $\mathbf{U} : [0,T] \to \mathbb{R}^N$ is given by

$$\mathbf{U}(t) = \begin{pmatrix} u(T-t,\underline{N}\,\zeta) \\ u(T-t,(\underline{N}+1)\,\zeta) \\ \cdots \\ u(T-t,\overline{N}\,\zeta) \end{pmatrix},$$

and thus, u solves (46) with terminal condition $u(T,y) = e^{-\frac{\kappa\alpha}{\zeta}(y-\hat{y})^2}$ if $\mathbf{U}$ solves

$$0 = \partial_t \mathbf{U} - \mathbf{K}\,\mathbf{U}\,,\ \ t \in [0,T]\,,\ \ \mathbf{U}(0)\ \text{ given in (38)}\,,$$

where the matrix $\mathbf{K} \in \mathbb{R}^{N\times N}$ is given by (37). It is well-known that the unique solution to the previous equation is given by

$$\mathbf{U}(t) = \exp(\mathbf{K}\,t)\ \mathbf{U}(0)\,,$$

where $\exp(\,\cdot\,)$ is matrix exponentiation and this concludes the proof. Q.E.D.

D.6.3. Proof of Theorem 5 First, we show that the system is well defined and that the controls we propose are admissible. The optimal shifts $\delta_t^{a\star}, \delta_t^{b\star}$ in (27) are measurable functions of time and y_{t^-}. The function θ is C^1 in time and y_{t^-} can only take finitely many values in $\mathcal{Y}$. Thus, (i)

the function ω is also C^1 in time, (ii) the optimal shifts $\delta_t^{a\star}, \delta_t^{b\star}$ are bounded, square integrable, and thus admissible, and (iii) the stochastic intensity functions $\lambda_t^b = c^b\, e^{-\kappa\,\delta_t^{b\star}}$, $\lambda_t^a = c^a\, e^{-\kappa\,\delta_t^{a\star}}$ are $\mathbb{F}$-predictable and bounded. This implies that the jump processes $N^{a,b}$ with stochastic intensities $\lambda^{a,b}$ are well defined.

Let $t \in [0,T)$ and let $\left(\delta_s^b\right)_{s\in[t,T]}$ and $\left(\delta_s^a\right)_{s\in[t,T]}$ be two processes in $\mathcal{A}_t$. Use Itô's lemma to write

$$\begin{aligned}\omega(T,x_T,y_T,Z_T) = \omega\left(t,x,y,z\right) &+ \int_t^T \partial_s\omega\left(s,x_{s^-},y_{s^-},Z_{s^-}\right)\mathrm{d}s \\ &+ \int_t^T \left(\omega\left(s,x_{s^-}-\zeta\left(Z_{s^-}-\delta_s^b\right),y_{s^-}+\zeta,Z_{s^-}-\eta^b(y_{s^-})\right)-\omega\left(s,x_{s^-},y_{s^-},Z_{s^-}\right)\right)\mathrm{d}N_s^b \\ &+ \int_t^T \left(\omega\left(s,x_{s^-}+\zeta\left(Z_{s^-}+\delta_s^a\right),y_{s^-}-\zeta,Z_{s^-}+\eta^a(y_{s^-})\right)-\omega\left(s,x_{s^-},y_{s^-},Z_{s^-}\right)\right)\mathrm{d}N_s^a\,,\end{aligned} \tag{47}$$

where the processes $(x_s)_{s\in[t,T]}$, $(y_s)_{s\in[t,T]}$, and $(Z_s)_{s\in[t,T]}$ start at time t with initial values x, y, and z and we employ the controls $\left(\delta_s^b\right)_{s\in[t,T]}$ and $\left(\delta_s^a\right)_{s\in[t,T]}$.[25] Given that $\delta^{a\star}$ and $\delta^{b\star}$ attain the equality in (41), then, for any time $s \in [t,T]$ the following holds

$$\begin{aligned}\phi\left(y_{s^-}-\hat{y}\right)^2 \geq\ & \partial_s\omega\left(s,x_{s^-},y_{s^-},Z_{s^-}\right) \\ &+\lambda^b\left(\delta_s^b\right)\left\{\omega\left(s,x_{s^-}-\zeta\left(Z_{s^-}-\delta_s^b\right),y_{s^-}+\zeta,Z_{s^-}-\eta^b(y_{s^-})\right)-\omega\left(t,x_{s^-},y_{s^-},Z_{s^-}\right)\right\} \\ &+\lambda^a\left(\delta_s^a\right)\left\{\omega\left(s,x_{s^-}+\zeta\left(Z_{s^-}+\delta_s^a\right),y_{s^-}-\zeta,Z_{s^-}+\eta^a(y_{s^-})\right)-\omega\left(s,x_{s^-},y_{s^-},Z_{s^-}\right)\right\}.\end{aligned}$$

Next, take expectations in (47) and rearrange terms to obtain

$$\begin{aligned}\mathbb{E}\left[\omega(T,x_T,y_T,Z_T)\right] &= \mathbb{E}\left[x_T+y_T\,Z_T-\alpha\left(y_T-\hat{y}\right)^2\right] \\ &\leq \omega\left(t,x,y,z\right)+\mathbb{E}\left[\phi\int_t^T\left(y_{s^-}-\hat{y}\right)^2\mathrm{d}s\right].\end{aligned}$$

Thus, we have that $w^\delta(t,x,y,z) \leq \omega(t,x,y,z)$. Given that $\delta^{a,b}$ is arbitrary we have that $w(t,x,y,z) \leq \omega(t,x,y,z)$.

Furthermore, by Proposition 8, equality in (47) is obtained with $\delta^\star = \left(\delta^{a\star},\delta^{b\star}\right)$ which is an admissible control. Thus, $\omega(t,x,y,z) = w^{\delta^\star}(t,x,y,z) \leq w(t,x,y,z)$ which completes the proof. Q.E.D.

[25] For ease of notation, we drop the superscript in the controlled process notation $\left(x_s^{t,x,y,z,\delta}\right)_{s\in[t,T]}$, $\left(y_s^{t,y,\delta}\right)_{s\in[t,T]}$ and $\left(Z_s^{t,y,z,\delta}\right)_{s\in[t,T]}$.

D.7. Proof of Proposition 5 The proof follows the same arguments as those in the proof of Theorem 2 in D.3. In particular, we showed that, for general liquidity provision strategies, if there is no roundtrip arbitrage in a sequence of $\mathfrak{m} = 2\,(m-1)$ trades, then there is no roundtrip arbitrage in a sequence of $\mathfrak{m} = 2\,m$ trades.

For the strategy (27), the P&L of case (i) in (12) of a roundtrip associated with $\epsilon_1 = +1$ and $\epsilon_{\mathfrak{m}} = -1$ is given by

$$\begin{aligned}
\text{P\&L} &= \zeta\,\eta^a\left(y_j\right) - \frac{1}{\kappa} - \left(\zeta\,\left(z_{N-j} - S\right) + \theta\left(t, y_{j-1}, z_{N-j+1}, S\right) - \theta(t, y_j, z_{N-j}, S)\right) \\
&\quad - \frac{1}{\kappa} - \left(\zeta\,\left(S - z_{N-j+1}\right) + \theta\left(t, y_j, z_{N-j}, S\right) - \theta(t, y_{j-1}, z_{N-j+1}, S)\right) \\
&= \zeta\eta^a\left(y_j\right) - \frac{2\,\zeta}{\kappa} - \zeta\left(z_{N-j} - z_{N-j+1}\right) \\
&= 2\,\zeta\eta^a\left(y_j\right) - \frac{2\,\zeta}{\kappa}\,,
\end{aligned}$$

where y_j and z_{N-j} are the level of the reserves in asset Y in the DLP and the marginal price, respectively, at the start of the roundtrip sequence. Similarly, for case (ii) in (12), we obtain

$$\text{P\&L} = 2\,\zeta\eta^b\left(y_j\right) - \frac{2\,\zeta}{\kappa}\,.$$

The P&L for both cases is non-positive as a consequence of (31). Thus, the LT is not profitable with any roundtrip sequence of $\mathfrak{m}$ steps. Q.E.D.

Acknowledgments

We are grateful to Guillermo Angeris, Tarek Abou Zeid, Carol Alexander, Zachary Feinstein, Andre Rzym, and Lorenzo Schönleber, for insightful comments. We are also grateful to participants at the CBER Crafting the Cryptoeconomy Conference at Columbia University, the WBS Gillmore Centre Academic Conference on DeFi & Digital Currencies, the Harvard EconCS research group seminar, the Stevanovich Center microstructure conference at University of Chicago, the Oxford Mathematical Institute finance seminar, the Imperial finance & stochastics seminar, the Frontiers in DeFi seminar at Vienna, and the Stochastic Finance at Warwick seminar.

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