NUMERICAL SOLUTIONS OF CAUCHY INTEGRAL EQUATIONS AND APPLICATIONS

by

JOSE A. CUMINATO
Wolfson College
Oxford

A Thesis submitted for the degree of Doctor of Philosophy

Hilary Term 1987

ABSTRACT

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This thesis investigates the polynomial collocation method for the numerical solution of Cauchy type integral equations and the use of those equations and the related numerical techniques to solve two practical problem in Acoustics and Aerodynamics.

Chapters I and II include the basic background material required for the development of the main body of the thesis.Chapter I discusses a number of practical problems which can be modelled as a singular integral equations. In Chapter II the theory of those equations is given in great detail.

In Chapter III the polynomial collocation method for singular integral equations with constant coefficients is presented. A particular set of collocation points, namely the zeros of the first kind Chebyshev polynomials, is shown to give uniform convergence of the numerical approximation for the cases of the index $\kappa = 0, 1$. The convergence rate for this method is also given. All these results were obtained under slightly stronger assumptions than the minimum required for the existence of an exact solution.

Chapter IV contains a generalization of the results in Chapter III to the case of variable coefficients.

In Chapter V an example of a practical problem which results in a singular integral equation and which is successfully solved by the collocation method is described in substantial detail. This problem consists of the interaction of a sound wave with an elastic plate freely suspended in a fluid. It can be modelled by a system of two coupled boundary value problems - the Helmholtz equation and the beam equation. The collocation method is then compared with asymptotic results and a quadrature method due to Miller.

In Chapter VI an efficient numerical method is developed for solving problems with discontinuous right-hand sides. Numerical comparison with other methods and possible extensions are also discussed.

ACKNOWLEDGEMENTS

I would like to thank my family for all the support they have given me throughout the years of preparation of this work.

I am also grateful to my supervisor, Prof. S. McKee, for the guidance and advice that he has given me over the past four years, to Dr. D. Butler for all the help with the problem in chapter V to the Brazilian funding agency CAPES, to the Instituto de Ciencias Matemáticas de São Carlos - Universidade de São Paulo - and to the Overseas Research Scheme for financially supporting my research during this period.

J. A. Cuminato

March 1987

CONTENTS

CHAPTER I

Examples of Singular Integral Equations Arising from Practical Problems in Mathematical Physics

1. Introduction

It is the purpose of this chapter to present a number of practical problems which give rise to singular integral equations (SIE). SIE's are an important tool in the solution of practical problems arising in Aerodynamics (Golberg [1979], Bland [1970]), Mechanics of Fracture (Erdogan & Gupta [1971a,1971b,1973], Boiko & Karpenko [1984]), Acoustics (Miller [1982], Wickham [1982]) and more generally in the solution of boundary value problems of elliptic type with mixed boundary conditions (Venturino [1984], Razali [1982,1984]).

The techniques used to reduce a problem in those areas to a SIE are usually: eigenfunction expansions, Fourier transforms and the boundary integral method. In the examples which follow we shall present applications of those techniques to problems in different areas.

2. Examples

EXAMPLE 1

Let $D = R^2 - [-1,1]$ where, here and elsewhere when this notation is used, $[-1,1]$ denotes the segment $\{-1 \le x \le 1\}$. The problem we are concerned with can be stated as:

Find the function $\phi(x,y)$, solution of

$$\nabla^2\phi + \alpha^2\phi = 0 \text{ in } D \tag{I.2.1}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, satisfying the boundary condition

$$\frac{\partial\phi}{\partial y} + \frac{\partial\phi_o}{\partial y} = 0 \text{ on } [-1,1] \tag{I.2.2}$$

the radiation condition

$$\sqrt{r}\left(\frac{\partial}{\partial r} - i\alpha\right)\phi \to 0 \text{ as } r^2 = x^2 + y^2 \to \infty,\ 0 \le \theta < 2\pi \tag{I.2.3}$$

and which is bounded everywhere in R^2 (see figure I.2.1).

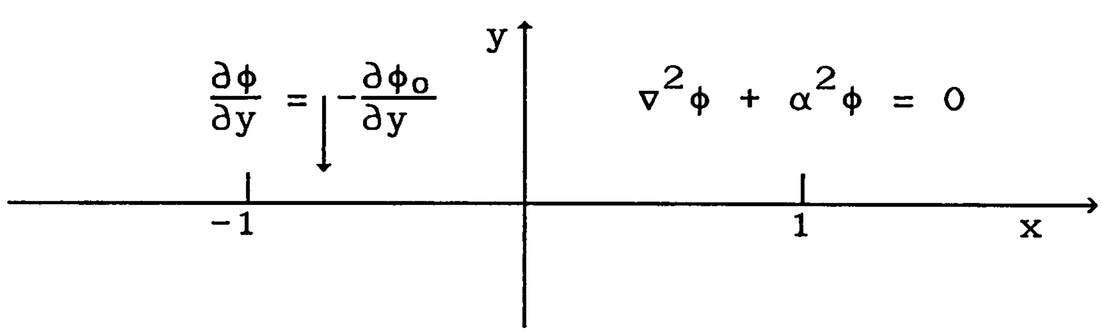

Figure I.2.1

The function ϕ represents the field scattered from a rigid barrier immersed in an ideal compressible fluid when it is illuminated by time harmonic waves of wave number α and velocity potential ϕ_o.

Using the Green's function for the Helmholtz equation it will be shown in chapter V that the function

$$\phi(x,y) = \int_{-1}^{1} G(x,y,s)\mu(s)ds \tag{I.2.4}$$

satisfies (I.2.1) and (I.2.3) where $\mu(s)$ is an arbitrary function and $G(x,y,s)$ is the Green's function of the Helmholtz equation defined by

$$G(x,y,s) = \left[\frac{i}{4}\frac{\partial}{\partial z}H_o^{(1)}(\alpha R)\right]_{z=0},$$

where $R^2 = (x-s)^2 + (y-z)^2$ and $H_o^{(1)}$ is the first kind Hankel function of order zero.

On imposing the boundary condition (I.2.2) on $\phi(x,y)$ we obtain, after some manipulation (see chapter V), a singular integral equation for $\mu'(s)$, namely

$$\frac{1}{\pi}⨍_{-1}^{1}\frac{\Psi(s)}{s-x}ds + \int_{-1}^{1}k(x,s)\Psi(s)ds = -\frac{\partial}{\partial y}\phi_o(x,0)$$

where $\Psi(s) = \frac{1}{2}\mu'(s)$ and

$$⨍_{-1}^{1}\frac{\Psi(s)}{s-x}ds \equiv \lim_{\varepsilon\to 0}\left[\int_{-1}^{x-\varepsilon}\frac{\Psi(s)}{s-x}ds + \int_{x+\varepsilon}^{1}\frac{\Psi(s)}{s-x}ds\right].$$

The integral on the left of this definition is called Cauchy principal value integral; further details will be given in chapter II.

Having found the function μ, equation (I.2.4) defines ϕ everywhere in R^2.

EXAMPLE 2

In this example we consider the problem of finding a complex function $\phi(z) = u(x,y) + iv(x,y)$ holomorphic in the upper half plane $y \geq 0$ and bounded at infinity, and constants $C_1, C_2, \ldots, C_p$

such that

$$v(x,0) = v_o(x) + C_j \text{ if } x \in [a_j,b_j],\ j = 1,\dots,p. \qquad (I.2.5)$$

$$u(x,0) = u_o(x) \text{ if } x \in L = R - \bigcup_{j=1}^{p}[a_j,b_j] \qquad (I.2.6)$$

where u_o and v_o are two given Hölder continuous functions and the intervals $[a_j,b_j]$ are all disjoints finite subsets of the real line ($-\infty < a_1 < b_1 < a_2 < \dots < b_p < \infty$).

The problem described above is usually called the Volterra problem in the literature (see for instance Razali & Thomas [1982]).

To solve this problem we derive an SIE for u(x,0) on the real line and then the function $\phi(z)$ can be found from (see Muskhelishvili [1953 p.112])

$$\phi(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{u(t,0)}{t - z}dt + iC \qquad (I.2.7)$$

where C is an arbitrary constant.

To derive an integral equation for u(x,0) we need to make use of the Plemelj-Sokhotski formulae (see chapter II equations (II.3.1)). Applying those formulae to (I.2.7) we obtain

$$\phi^+(x,0) = u(x,0) + \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{u(t,0)}{t - x}dt + iC.$$

In order that the real and imaginary parts of ϕ^+ satisfy condition (I.2.5) and (I.2.6) we must have

$$\frac{-1}{\pi}\int_{-\infty}^{\infty}\frac{u(t,0)}{t-x}dt + C = v_o(x) + C_j,\quad x \in [a_j,b_j],\quad j = 1,\ldots,p \qquad (I.2.8)$$

$$u(x,0) = u_o(x) \qquad x \in L \qquad (I.2.9)$$

If we then substitute (I.2.9) into (I.2.8) we obtain the SIE

$$\frac{-1}{\pi}\sum_{j=1}^{p}\int_{a_j}^{b_j}\frac{u(t,0)}{t-x}dt = v_o(x) + \tilde{C}_j + \frac{1}{\pi}\int_L \frac{u_o(t)}{t-x}dt. \qquad (I.2.10)$$

where $\tilde{C}_j = C_j - C$.

Notice that equation (I.2.10) is a SIE in several disjoint intervals and that the constants C_j as well as $u(x,0)$ are the unknowns.

As an application of the method described above let us consider the problem:

Solve Laplace's equation in the first quadrant of the unit circle with the boundary conditions given in figure I.2.2

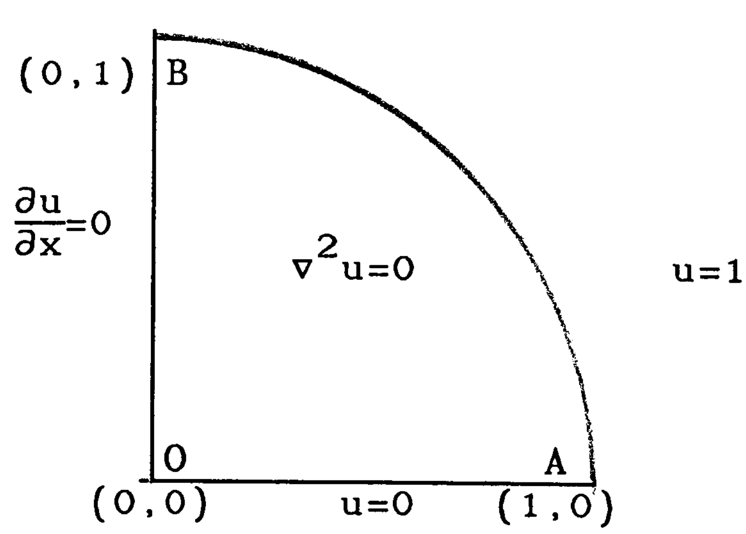

Figure I.2.2

Firstly, we conformally transform the region on figure I.2.2

into the upper half plane by the composition of the two conformal mappings

$$w_1(z) = \left(\frac{z^2 + 1}{z^2 - 1}\right)^2$$

$$w_2(z) = \frac{4w_1 - 3}{3 - 2w_1}.$$

The transformed space is

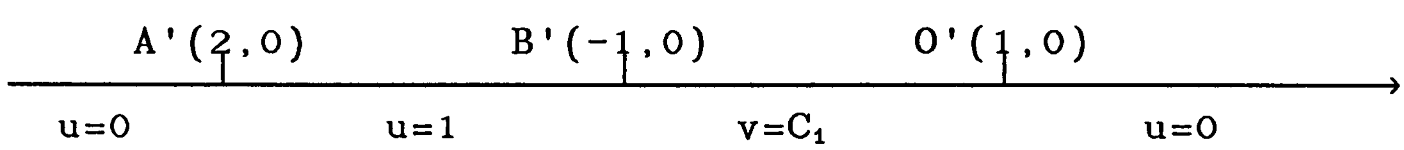

Figure I.2.3

where A', B' and O' are the images of A, B, and O of figure I.2.2 by the conformal mappings. The Neumann boundary condition $\frac{\partial u}{\partial x} = 0$ is transformed into a boundary condition involving v by making use of the Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

to give $v(0,y) = C_1$.

Equation (I.2.9) in this case takes the form

$$\frac{-1}{\pi}\int_{-1}^{1} \frac{u(t,0)}{t - x}dt = C_1 + \frac{1}{\pi}\log\left|\frac{1 + x}{2 + x}\right|.$$

<u>EXAMPLE 3</u>

Consider the problem of finding the steady state temperature distribution u in a semi-infinite strip, described by the mixed boundary value problem

$$\nabla^2 u = 0 \quad \text{in } D = \{(x,y)\ 0 < x < a,\ y > 0\} \qquad \text{(I.2.11)}$$

$$u(0,y) = \frac{\partial u(a,y)}{\partial x} = 0 \qquad \text{(I.2.12)}$$

$$\lim_{\substack{y\to 0\\ y>0}} \frac{\partial u(x,y)}{\partial y} = f(x), \quad x \in L = (b,c) \qquad \text{(I.2.13)}$$

$$u(x,0) = 0 \quad x \in L^c = (0,a) - (b,c) \qquad \text{(I.2.14)}$$

$$\lim_{y\to\infty} u(x,y) = 0 \quad \text{for all } x$$

where $0 < b < c < a$ are given real numbers.

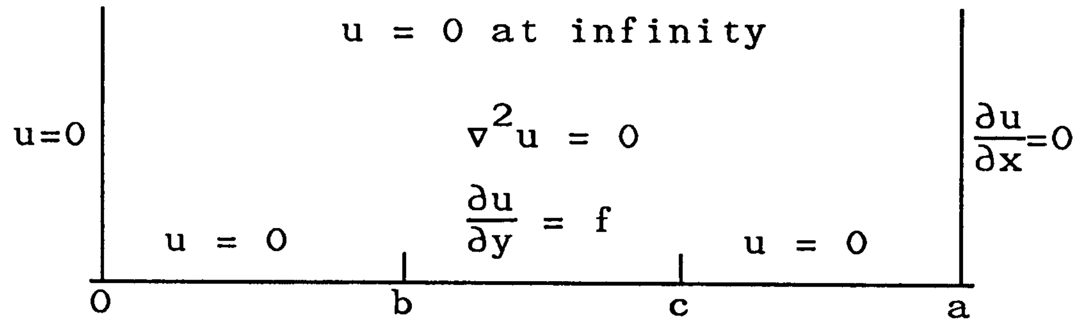

Figure I.2.4

Following Venturino [1984 pg.4] the solution is sought in the form

$$u(x,y) = \sum_{n=1}^{\infty} A_n \exp(-\lambda_n y)\sin(\lambda_n x) \quad \text{in } D \qquad \text{(I.2.15)}$$

where A_n are unknown constants and $\lambda_n = (2n-1)\frac{\pi}{2a}$.

Notice that (I.2.15) already satisfies (I.2.11), (I.2.12), and $u(x,\infty) = 0\ \forall x$. To find the constants A_n we use conditions (I.2.13) and (I.2.14)

$$\lim_{y\to 0^+} \sum_{n=1}^{\infty} \lambda_n A_n \exp(-\lambda_n y)\sin(\lambda_n x) = -f(x) \qquad x \in L \qquad (I.2.16)$$

$$\sum_{n=1}^{\infty} A_n \sin(\lambda_n x) = 0 \qquad x \in L^c. \qquad (I.2.17)$$

Equations (I.2.16) and (I.2.17) are called the dual series equations.

To obtain a SIE we let

$$g(x) = \frac{\partial u(x,0)}{\partial x} = \sum_{n=1}^{\infty} A_n \lambda_n \cos(\lambda_n x) \qquad 0 < x < a. \qquad (I.2.18)$$

Now noting that $u(x,0) = 0$ for $x \in L^c$ we deduce

$$\frac{\partial u(x,0)}{\partial x} = 0 \quad \text{for } x \in \mathrm{Int}(L^c),$$

where $\mathrm{Int}(L^c)$ is the interior of L^c.

Note also that $\{\cos(\lambda_n x)\}$ form an orthogonal system on $[0,a]$, hence

$$\lambda_n A_n = \frac{2}{a}\int_0^a g(t)\cos(\lambda_n t)dt = \frac{2}{a}\int_L g(t)\cos(\lambda_n t)dt. \qquad (I.2.19)$$

This formula tells us that the coefficients A_n can be determined in terms of the function $g(x)$.

From (I.2.16) and (I.2.19) we have

$$-f(x) = \lim_{y\to 0^+} \sum_{n=1}^{\infty} \frac{2}{a}\int_L g(t)\cos(\lambda_n t)dt \; \exp(-\lambda_n y)\sin(\lambda_n x)$$

$$= \frac{2}{a}\lim_{y\to 0^+} \int_L g(t) \sum_{n=1}^{\infty} \cos(\lambda_n t)\sin(\lambda_n x)\exp(-\lambda_n y)dt$$

$$= \frac{1}{a}\lim_{y\to 0^+}\int_L g(t)\sum_{n=1}^{\infty}(\sin\lambda_n(x-t) + \sin\lambda_n(x+t))\exp(-\lambda_n y)dt$$

which after lengthy algebra, (see Venturino [1984]) gives the following SIE for the function g(x)

$$\frac{1}{\pi}\int_L \frac{g(t)}{t-x}dt + \int_L k(x,t)g(t)dt = -f(x)$$

where the kernel k(x,t) is given by

$$k(x,t) = \frac{1}{2a}\Big[\cos\alpha(x-t)\cot\alpha(x-t) + \sin\alpha(x-t)$$
$$+ \cos\alpha(x+t)\cot\alpha(x+t) + \sin\alpha(x+t) - \frac{\pi}{x-t}\Big]$$

where $\alpha = \frac{\pi}{2a}$.

We observe that the argument above still holds true for the case in which the interval [0,a] is split into a number of subintervals each of which has the boundary condition

$$\frac{\partial u(x,0)}{\partial y} = f_i(x) \qquad x \in L_i = (b_i, c_i).$$

The final SIE would still be the same with a different definition for the domain of integration L.

EXAMPLE 4

In this example we consider the plane strain problem for two semi-infinite elastic plates bonded through an elastic layer of thickness h. Suppose the resulting medium has a crack of length 2a between the layer and one of the plates. Suppose also that the crack length is normalized to be 2 and the symmetric traction on the crack surface

$$\sigma_{yy}(x,0) = p_1(x) = p_1(-x)$$
$$\sigma_{xy}(x,0) = p_2(x) = -p_2(x) \qquad (I.2.20)$$

are the only external loads acting on the medium.

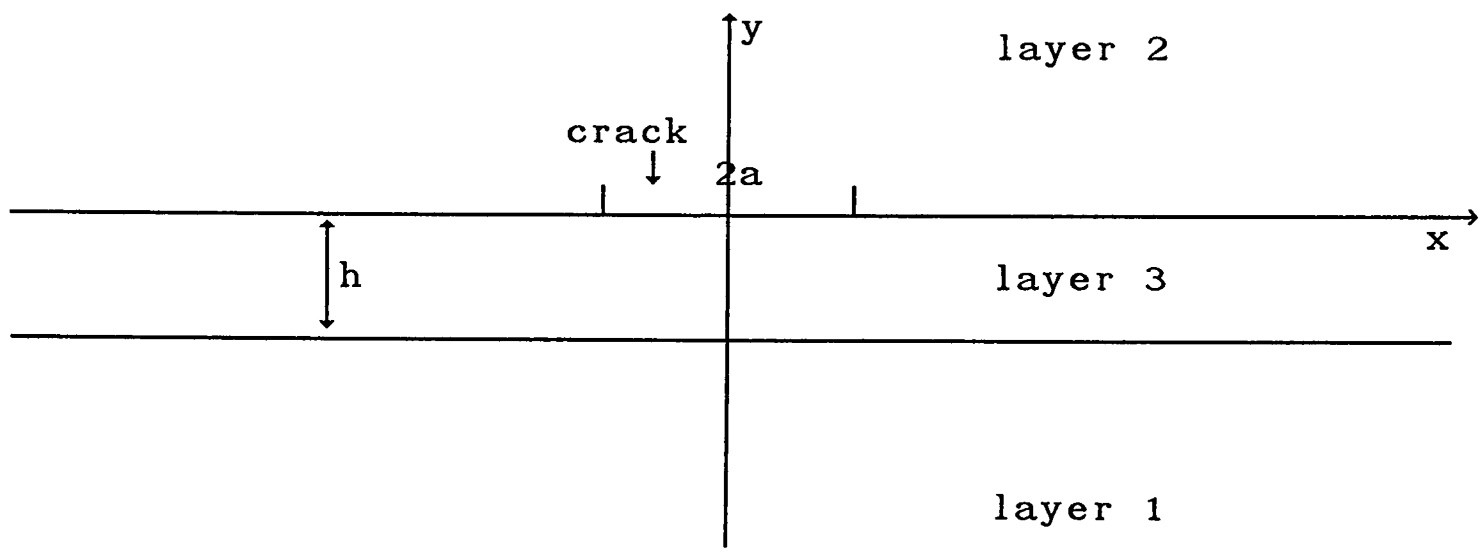

Figure I.2.5

Denoting the x and y components of the displacements in the i^{th} layer by u_i and v_i respectively, we have the equations

$$\mu_i \nabla^2 u_i + (\lambda_i+\mu_i)\frac{\partial}{\partial x}\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right) = 0$$
$$\mu_i \nabla^2 v_i + (\lambda_i+\mu_i)\frac{\partial}{\partial y}\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right) = 0 \qquad i=1,2,3 \qquad (I.2.21)$$

where μ_i is the shear modulus and $\lambda_i = \lambda_i(\mu_1,\mu_2,\mu_3;\kappa_1,\kappa_2,\kappa_3)$ with $\kappa_i = 3 - 4\nu_i$ for plane strain and $\kappa_i = (3 - \nu_i)/(1 + \nu_i)$ for generalized plane stress, ν_i being the Poisson's ratio.

Let u_i and v_i be sine and cosine Fourier transforms of the functions ϕ_i and ψ_i in the x variable, respectively, that is,

$$u_i(x,y) = \frac{2}{\pi}\int_0^\infty \phi_i(\alpha,y)\sin(\alpha x)d\alpha$$

$$v_i(x,y) = \frac{2}{\pi}\int_0^\infty \psi_i(\alpha,y)\cos(\alpha x)d\alpha.$$

Throughout this example we denote the transformed variable of x by α.

With this notation equations (I.2.21) are transformed into a pair of linear ordinary differential equations in the variable y for the functions ϕ_i and ψ_i which can be solved to give

$$u_i(x,y) = \frac{2}{\pi}\int_0^\infty \left[(A_{i1} + A_{i2}y)e^{-\alpha y} + (A_{i3} + A_{i4}y)e^{\alpha y}\right]\sin(\alpha x)d\alpha$$

$$v_i(x,y) = \frac{2}{\pi}\int_0^\infty \left\{\left[A_{i1} + \left(\frac{\kappa_i}{\alpha} + y\right)\right]e^{-\alpha y}\right.$$

$$\left. + \left[-A_{i3} + \left(\frac{\kappa_i}{\alpha} - y\right)\right]e^{\alpha y}\right\}\cos(\alpha x)d\alpha \tag{I.2.22}$$

where A_{ij} are functions of α which will be determined from the boundary and the continuity conditions. Notice that we have 12 of those functions in all.

Making use of Hooke's law and denoting the non-vanishing components of the stress vector at the interfaces and boundaries for the i^{th} layer by σ^i_{yy} and σ^i_{xy} we can write

$$\frac{1}{2\mu_i}\sigma^i_{yy} = \frac{2}{\pi}\int_0^\infty \left\{-\left[\alpha(A_{i1} + A_{i2}y) + 2(1 - \nu_i)A_{i2}\right]e^{-\alpha y}\right.$$

$$\left. + \left[-\alpha(A_{i3} + A_{i4}y) + 2(1 - \nu_i)A_{i4}\right]e^{-\alpha y}\right\}\cos(\alpha x)d\alpha$$

$$\frac{1}{2\mu_i}\sigma^i_{xy} = \frac{2}{\pi}\int_0^\infty\left\{-\left[\alpha(A_{i1} + A_{i2}y) + (1 - 2\nu_i)A_{i2}\right]e^{-\alpha y}\right.$$
$$\left. + \left[\alpha(A_{i3} + A_{i4}y) - (1 - 2\nu_i)A_{i4}\right]e^{-\alpha y}\right\}\sin(\alpha x)d\alpha \tag{I.2.23}$$

The boundary and continuity conditions are (see figure II.2.5)

1) $u_1(x,\infty) = u_2(x,\infty) = v_1(x,\infty) = v_2(x,\infty) = 0$

2) $u_1(x,-h) = u_3(x,-h), \quad v_1(x,-h) = v_3(x,-h)$

3) $\sigma^1_{yy} = \sigma^3_{yy}, \quad \sigma^1_{xy} = \sigma^3_{xy}, \quad \sigma^3_{yy} = \sigma^2_{yy}, \quad \sigma^3_{xy} = \sigma^2_{xy}.$

All of the above conditions are taken to be valid for $0 < x < \infty$. To obtain the two extra conditions needed to match the number of unknowns A_{ij} we assume that at $y = 0$ the bond between the two adjacent layers is perfect except for the dislocations at $x = t$ defined by

4) $\frac{\partial}{\partial x}(u_2-u_3) = f_1\delta(x-t), \quad \frac{\partial}{\partial x}(v_2-v_3) = f_2\delta(x-t)$

where $\delta(x)$ is the Dirac delta function.

Using conditions 1), 2), 3) and 4) it can be shown (see Erdogan & Gupta [1971a]) that a system of linear equations for the functions A_{ij} is determined which can be solved in the general form

$$A_{ij}(\alpha) = F^1_{ij}(\alpha)f_1\cos(\alpha t) + F^2_{ij}(\alpha)f_2\sin(\alpha t) \tag{I.2.24}$$

where the functions $F^k_{ij}(\alpha)$ depend on the material constants and on the thickness h of the layer.

Substituting (I.2.24) into (I.2.23) the two components of the stress in the layer 3 can be expressed as

$$\frac{1}{2\mu_3}\sigma^3_{yy} = h_{11}(x,y,t)f_1 + h_{12}(x,y,t)f_2$$

$$\frac{1}{2\mu_3}\sigma^3_{xy} = h_{21}(x,y,t)f_1 + h_{22}(x,y,t)f_2. \tag{I.2.25}$$

If instead of the boundary conditions 4) above we have the x-derivative of the relative displacements given as functions of t, say $f_1(t)$ and $f_2(t)$ along the interface $y = 0$, the stresses in layer 3 may be obtained by simply replacing f_i in (I.2.25) by $f_i(t)$ and integrating in t along the length of the crack.

Taking into account the known tractions on the crack surface given by equations (I.2.20), equation (I.2.25) can be written as:

$$\frac{1}{2\mu_3}p_1(x) = \lim_{y\to 0}\int_0^1 \left[h_{11}(x,y,t)f_1(t) + h_{12}(x,y,t)f_2(t)\right]dt$$

$$\frac{1}{2\mu_3}p_2(x) = \lim_{y\to 0}\int_0^1 \left[h_{21}(x,y,t)f_1(t) + h_{22}(x,y,t)f_2(t)\right]dt \tag{I.2.26}$$

where we recall

$$f_1(x) = \frac{\partial}{\partial x}(u_2^+ - u_3^-)$$

$$f_2(x) = \frac{\partial}{\partial x}(v_2^+ - v_3^-)$$

and the superscripts + and - refer to the limiting values of the displacements as y tends to zero from positive and negative sides, respectively.

For $y = 0$ and $|x| > 1$ we must have $u_2^+ - u_3^- = v_2^+ - v_3^- = 0$ for the continuity of the displacements; hence on integrating the above expressions, we obtain the conditions

$$\int_{-1}^{1} f_1(x)dx = \int_{-1}^{1} f_2(x)dx = 0. \tag{I.2.27}$$

From $u_2^+ - u_3^- = v_2^+ - v_3^- = 0$ for $|x| > 1$ we also deduce that $f_1(x) = f_2(x) = 0$ for $|x| > 1$.

Turning now to the definition of $h_{ij}(x,y,t)$ we see that

$$h_{ij}(x,y,t) = P_{ij}(x,y)\begin{cases}\cos(\alpha t) & \text{if } j=1\\ \sin(\alpha t) & \text{if } j=2\end{cases}.$$

As a consequence of this expression and the fact that f_1 and f_2 vanish for $|x| > 1$ equations (I.2.26) can be written as

$$\begin{aligned}\frac{1}{2\mu_3}p_1(x) &= \lim_{y\to 0}\left[P_{11}(x,y)\int_0^\infty \cos(\alpha t)f_1(t)dt\right.\\ &\qquad\left. + P_{12}(x,y)\int_0^\infty \sin(\alpha t)f_2(t)dt\right]\\ \frac{1}{2\mu_3}p_2(x) &= \lim_{y\to 0}\left[P_{21}(x,y)\int_0^\infty \cos(\alpha t)f_1(t)dt\right.\\ &\qquad\left. + P_{22}(x,y)\int_0^\infty \sin(\alpha t)f_2(t)dt\right]\end{aligned} \tag{I.2.28}$$

After considerable algebra the functions $P_{ij}(x,y)$ can be calculated, and the equations (I.2.28) take the form (see Erdogan & Gupta [1971b]).

$$\frac{-1 + \kappa_3}{\mu_3}p_1(x) = \lim_{y\to 0}\left[\frac{2}{\pi}\int_0^\infty \sum_{j=1}^{2} a_{1j}e^{\alpha y}C_1(\alpha)\cos(\alpha x)d\alpha\right.$$

$$+ \frac{2}{\pi}\int_0^\infty \sum_{j=1}^{2} H_{1j}(\alpha)C_j(\alpha)\cos(\alpha x)d\alpha\Bigg]$$

$$\frac{-1+\kappa_3}{\mu_3}p_2(x) = \lim_{y\to 0}\Bigg[\frac{2}{\pi}\int_0^\infty \sum_{j=1}^{2} a_{2j}e^{\alpha y}C_1(\alpha)\cos(\alpha x)d\alpha$$

$$+ \frac{2}{\pi}\int_0^\infty \sum_{j=1}^{2} H_{2j}(\alpha)C_j(\alpha)\cos(\alpha x)d\alpha\Bigg]$$

where

$$C_1(\alpha) = \int_0^\infty f_1(t)\cos(\alpha t)dt$$

$$C_2(t) = \int_0^\infty f_2(t)\sin(\alpha t)dt$$

and the constants a_{ij} and the functions $H_{ij}(\alpha)$, $i,j = 1,2$ are defined in Erdogan & Gupta [1971b pgs. 1091,1105].

From the symmetry properties we can deduce that $f_1(t)$ is even and $f_2(t)$ is odd, from this it can easily be shown that

$$\int_0^\infty H(\alpha)\cos(\alpha x)d\alpha\int_0^1 f_1(t)\cos(\alpha t)dt = \frac{1}{2}\int_{-1}^{1} f_1(t)dt\int_0^\infty H(\alpha)\cos\alpha(t-x)d\alpha \tag{I.2.29}$$

with a similar expression being valid for other combinations of sines and cosines.

Using these formulae, evaluating the infinite integrals, letting y tend to zero and dividing by $-a_{21}$ finally gives

$$\frac{1+\kappa_3}{a_{21}\mu_3}p_1(x) = \gamma f_1(x) + \frac{1}{\pi}\int_{-1}^{1}\frac{f_2(t)}{t-x}dt - \frac{1}{a_{21}\pi}\int_{-1}^{1}\sum_{j=1}^{2}k_{1j}(x,t)f_j(t)dt$$
(I.2.30)
$$\frac{1+\kappa_3}{a_{21}\mu_3}p_2(x) = -\gamma f_2(x) + \frac{1}{\pi}\int_{-1}^{1}\frac{f_1(t)}{t-x}dt - \frac{1}{a_{21}\pi}\int_{-1}^{1}\sum_{j=1}^{2}k_{2j}(x,t)f_j(t)dt$$

where

$$k_{11}(x,t) = \int_0^\infty H_{11}(\alpha)\cos\alpha(x-t)d\alpha$$

$$k_{12}(x,t) = \int_0^\infty H_{12}(\alpha)\sin\alpha(x-t)d\alpha$$

$$k_{21}(x,t) = \int_0^\infty H_{21}(\alpha)\sin\alpha(x-t)d\alpha$$

$$k_{22}(x,t) = \int_0^\infty H_{22}(\alpha)\cos\alpha(x-t)d\alpha$$

$$\gamma = \frac{a_{11}}{a_{12}}.$$

Equations (I.2.30) form a system of two SIE. To obtain a single SIE we write $\phi(x) = f_1(x) + if_2(x)$

EXAMPLE 5

In this example we consider the problem of predicting unsteady airloads on oscillating thin aerofoils in ventilated wind tunnels. The thin two-dimension aerofoil is subjected to small amplitude oscillations about the central plane of the tunnel, see figure I.2.6

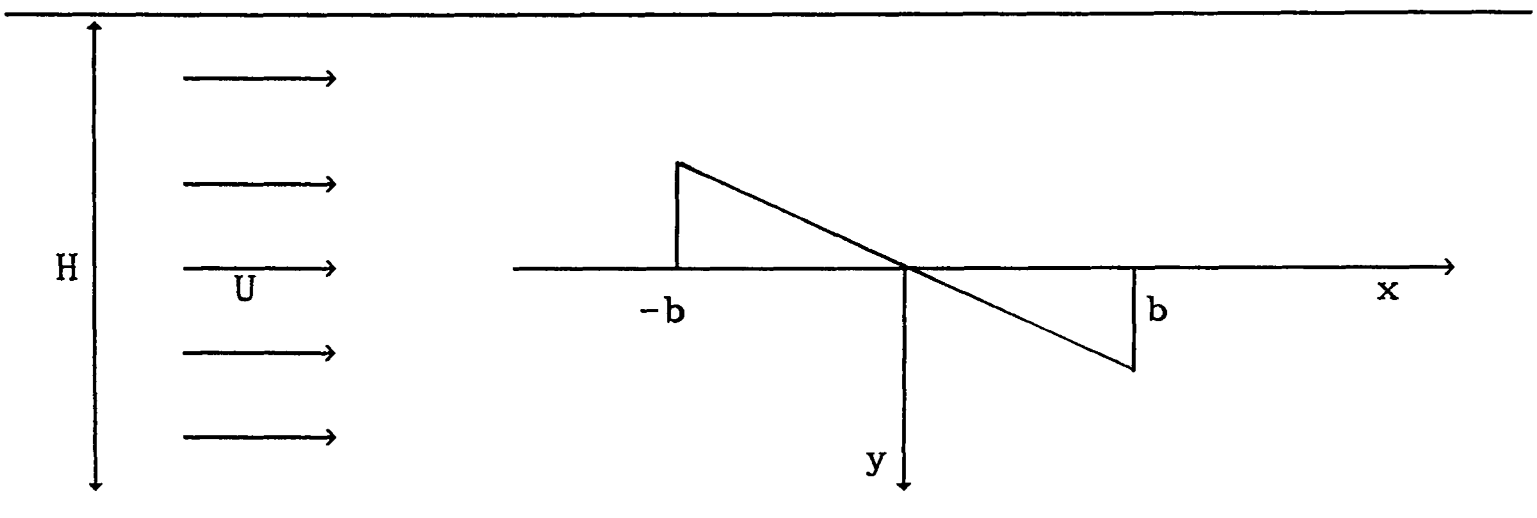

Figure I.2.6 Wind Tunnel

The non-dimensional equation, using the aerofoil semichord b as reference length and making the assumption for inviscid irrotational flow, is as follows

$$\beta^2\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} - \frac{2M^2b}{U}\frac{\partial^2\Psi}{\partial x\partial t} - \left(\frac{Mb}{U}\right)^2\frac{\partial^2\Psi}{\partial t^2} = 0$$

where $\Psi(x,t,t) = -(p(x,y,t) - p_o)/\rho_o$ is the disturbance pressure potential, $p(x,y,t)$ is the local static pressure, p_o and ρ_o are the undisturbed free stream static pressure and density, M is the free stream Mach number, $\beta = \sqrt{1 - M^2}$ and U is the free stream velocity.

For an aerofoil as in figure I.2.6 the pressure will be an odd function of y. Since it is continuous except at the aerofoil surface, we have

$$\Psi(x,0,t) = 0, \quad |x| > 1 \text{ and } t > 0.$$

On the aerofoil surface we have

$$\Psi(x,0^+,t) = \frac{U^2\Delta p(x,t)}{4q}, \quad |x| < 1$$

where $\frac{\Delta p(x,t)}{q}$ is the unknown lifting pressure and $q = \frac{1}{2}U^2\rho_o$. For the sake of simplicity of notation we shall write $\Delta p_q(x,t)$ in place of the quotient $\frac{\Delta p(x,t)}{q}$.

We need also to assume that the flow is smooth at the aerofoil trailing edge, that is

$$\lim_{x \to 1} \Delta p_q(x,t) = 0,$$

which is commonly known as the Kutta condition. At the tunnel wall we impose the boundary condition

$$\Psi(x,H/2,t) + c\frac{\partial\Psi(x,H/2,t)}{\partial y} = 0$$

where $c \in [0,\infty]$ is given.

Because $\Delta p_q(x,t)$ is unknown, to render the problem well-posed we need an additional piece of information. We shall assume that the vertical velocity $W(x,y,t)$ at the aerofoil surface is known. The function W relates to the potential Ψ through

$$W(x,y,t) = \frac{1}{U}\int_{-\infty}^{x} \frac{\partial\Psi(\lambda,y,t - b\frac{x-\lambda}{U})}{\partial y} d\lambda.$$

Hence the final set of equations and boundary conditions is:

$$\beta^2\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} - \frac{2M^2b}{U}\frac{\partial^2\Psi}{\partial x\partial t} - \left(\frac{Mb}{U}\right)^2\frac{\partial^2\Psi}{\partial t^2} = 0 \qquad \text{(I.2.31)}$$

$$\Psi(x,0,t) = \begin{cases} \frac{U^2}{4}\Delta p_q(x,t) & |x| < 1 \\ 0 & |x| > 1 \end{cases} \qquad \text{(I.2.32)}$$

$$\Psi(x,H/2,t) + c\frac{\partial\Psi(x,H/2,t)}{\partial y} = 0 \tag{I.2.33}$$

$$\lim_{x\to 1} \Delta p_q(x,t) = 0 \tag{I.2.34}$$

$$W(x,y,t) = \frac{1}{U}\int_{-\infty}^{x} \frac{\partial\Psi(\lambda,y,t - b\frac{x-\lambda}{U})}{\partial y} d\lambda. \tag{I.2.35}$$

Assuming that the time dependence is given by the exponential factor $e^{i\kappa t}$ where $\kappa = \omega - ig$, ω being the oscillation frequency we can write

$$\Psi(x,y,t) = \varphi(x,y)e^{i\kappa t}$$

and equations (I.2.31) through to (I.2.34) become

$$\beta^2\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} - 2iM^2\kappa \frac{\partial\varphi}{\partial x} - M^2\kappa^2\varphi = 0 \tag{I.2.36}$$

$$\varphi(x,0) = \begin{cases} \frac{U^2}{4} \Delta p_q(x) & |x| < 1 \\ 0 & |x| > 1 \end{cases} \tag{I.2.37}$$

$$\varphi(x,H/2) + c\frac{\partial\varphi(x,H/2)}{\partial y} = 0 \tag{I.2.38}$$

$$\lim_{x\to 1} \Delta p_q(x) = 0. \tag{I.2.39}$$

We recall that $\Delta p_q(x)$ is not known.

We solve equations (I.2.36)-(I.2.38) by taking Fourier transforms in the x-variable, that is,

$$\hat{\varphi}(s,y) = \int_{-\infty}^{\infty} \varphi(x,y)e^{-isx}dx$$

$$\varphi(x,y) = \int_{-\infty}^{\infty} \hat{\varphi}(s,y)e^{isx}ds$$

to obtain the following boundary value problem for ordinary differential equations

$$\hat{\varphi}'' - \alpha^2\hat{\varphi} = 0$$

$$\hat{\varphi}(s,0) = T(s) \qquad (I.2.40)$$

$$\hat{\varphi}(s,H/2) + c\hat{\varphi}'(s,H/2) = 0$$

where ' means differentiation with respect to the y-variable,

$$\alpha^2(s) = \beta^2 s^2 - 2M^2\kappa s - M^2\kappa^2$$

and

$$T(s) = \frac{U^2}{4}\int_{-1}^{1} \Delta p_q(x)e^{-isx}dx.$$

The solution of (I.2.40) is

$$\hat{\varphi}(s,y) = T(s)g(s)$$

where

$$g(s) = \frac{\sinh\alpha(H/2-y) + c\alpha\cosh\alpha(H/2-y)}{\sinh(\alpha H/2) + c\alpha\cosh(\alpha H/2)}.$$

Making use of the inversion formula yields

$$\varphi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} T(s)g(s)e^{ixs}ds. \qquad (I.2.41)$$

Substituting (I.2.41) into (I.2.35) and writing $W(x,y,t) = w(x,y)e^{i\kappa t}$ we obtain the equation

$$w(x,y) = \frac{1}{\pi}\int_{-\infty}^{x} e^{-i\kappa(x-\lambda)} \frac{\partial}{\partial y}\left[\int_{-\infty}^{\infty} T(s)g(s)e^{i\lambda s}ds\right]d\lambda.$$

The uniform convergence of all the integrals involved in the analysis above has been established by Bland [1970], so that interchanging the order of integration and differentiation is permissable and after extensive algebra we obtain

$$w(x,y) = \int_{-1}^{1} \Delta p_q(\xi)K(M,\kappa,x-\xi,y,H,c)d\xi \qquad (I.2.42)$$

where

$$K(M,\kappa,x-\xi,y,H,c) = \frac{iU}{8\pi}\int_{-\infty}^{\infty} \frac{\alpha e^{is(x-\xi)}}{(s+\kappa)} \frac{(\cosh\alpha(H/2-y) + c\alpha \sinh\alpha(H/2-y))}{(\sinh(\alpha H/2) + c\alpha \cosh(\alpha H/2))}ds$$

and equation (I.2.42) is to be solved for $y = 0$.

According to Bland [1970] the final SIE can be written in the form

$$\int_{-1}^{1} \hat{K}(x-\xi)\Delta p_q(\xi)d\xi = w(x,0) \qquad (I.2.43)$$

where

$$\hat{K}(x-\xi) = \frac{\beta}{4\pi(x-\xi)} - \frac{i\kappa}{4\pi\beta}\log|x-\xi| - K_b(x-\xi)$$

with K_b a smooth function defined in Bland [1970 pg.838].

EXAMPLE 6

A more interesting problem than example 5 is the case when a flap is hinged at the trailing edge, say at the position $x = x_1$.

We observe that if $h(x,t)$ is the profile of the aerofoil then the vertical velocity $W(x,y,t)$ at $y = 0$ can be given in terms of $h(x,t)$ by the formula

$$W(x,0,t) = \left[U\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right]h(x,t).$$

Hence on assuming that

$$h(x,t) = e^{i\kappa t}h(x)$$

we obtain

$$w(x,0) = Uh'(x) + i\kappa h(x).$$

For the case of a single flap at $x = x_1$,

$$h(x) = \begin{cases} 0 & -1 < x < x_1 \\ x - x_1, & x_1 < x < 1 \end{cases}$$

and in this case

$$w(x,0) = \begin{cases} 0 & -1 < x < x_1 \\ U + i\kappa(x-x_1), & x_1 < x < 1 \end{cases}.$$

The final equation will be (I.2.43) with the right-hand side having a jump discontinuity at the point $x = x_1$.

CHAPTER II

A Survey of the Theory of Singular Integral Equations

1. Introduction

The main recent developments in the theory of SIE are almost entirely due to Russian mathematicians. The classical books on the subject are Muskhelishvili [1953] and Gakhov [1966]. The classical approach used by those authors to solve a SIE is to make use of the so called Plemelj-Sokhotski formulae to reduce it to a Hilbert boundary value problem (HBVP) and then develop a theory for the solution of the later. Indeed Muskhelishvili shows that these two problems are completely equivalent. It is customary to present the solution of the HBVP for a smooth closed contour in the complex plane and for Hölder continuous data and then extend those results to the case of data with jump discontinuities. The case of open arcs is then solved by extending the arcs to a closed curve and conveniently redefining the data on the extension.

In this chapter we shall not follow the approach outlined above due to it being too lengthy. We chose instead to present the solution of the HBVP for open arcs directly and then derive the solution of the SIE from it.

We shall also present some results on the uniform convergence of the polynomial interpolation of smooth data.

2. Definitions and Basic Theorems

Definition II.2.1

We shall call a *smooth contour* an open or closed line in the complex plane which is non-intersecting, has continuously varying tangent and has no cusps. We denote such a countour by L.

Definition II.2.2

Let L be a smooth closed contour orientated in a direction which we shall call positive direction. That part of the plane which lies on the left will always be denoted D^+ and that on the right D^-. In the case when L is an open arc the complex plane with a cut along L will be called the *deleted complex plane*.

Definition II.2.3

Let L be a smooth contour and $\varphi(x)$ be a function of position on L. $\varphi(x)$ is said to satisfy the Hölder condition on L if for arbitrary points x_1 and x_2 on L

$$|\varphi(x_1) - \varphi(x_2)| \leq A|x_1 - x_2|^{\mu}$$

where A is a positive constant and $0 < \mu \leq 1$.
We will use the notation $\varphi \in H^{\mu}(L)$.

Definition II.2.4

Let L be an open arc and $\varphi(t)$ be a function of position on L satisfying the Hölder condition on every closed part of L not containing the ends, and such that near any of the ends c it is of the form $\varphi(t) = \dfrac{\varphi^*(t)}{(t-c)^{\alpha}}$ with $\varphi^*(t) \in H^{\mu}(L)$, then $\varphi(t)$ will be said to be in H^*.

Definition II.2.5

Let L be a smooth contour and $\Psi(x,t)$ be a function of two variables on L. $\Psi(x,t)$ is said to satisfy the Hölder condition on L if for any pair of points (x_1,t_1) and (x_2,t_2)

$$|\Psi(x_1,t_1) - \Psi(x_2,t_2)| < C\Big[|x_1 - x_2|^{\mu} + |t_1 - t_2|^{\mu}\Big].$$

where C is a constant and $0 < \mu \leq 1$.

We shall use the same notation as for functions of one variable, i.e. $\Psi \in H^{\mu}(L)$.

Definition II.2.6

Let L be a smooth contour and $\varphi \in H^{\mu}(L)$. We define and denote the *Cauchy principal value integral* of φ at a point x of L, by the expression

$$\fint_L \frac{\varphi(t)}{t - x}dt \equiv \lim_{\delta\to 0^+} \int_{L-\Omega} \frac{\varphi(t)}{t - x}dt$$

where $L-\Omega$ is the contour obtained by subtracting from L the intersection of L with a circle centred at the point x and radius δ.

Notation Convention

Throught this thesis we shall use the notation $P_n(x)$ to denote a polynomial of degree n, with the convention that for $n < 0$ $P_n(x) \equiv 0$. For a polynomial of unespecified degree we write simply $P(x)$.

We shall use the letter z to denote a generic complex number in the set $\mathbb{C}\backslash L$ and the letters x,t,s to denote a point in the countour L.

Lemma II.2.1

Let L be a smooth contour and $\varphi \in H^{\mu}(L)$. Then the function

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t - z}dt \tag{II.2.1}$$

is continuous on L from the left and from the right except at those ends where $\varphi(x) \neq 0$.

Remark:

The proof of lemma II.2.1 can be found in Muskhelishvili [1953 pg. 38]. This proof assumes that L is a closed contour but as the author points out on page 41, it can easily be extended to open arcs. In the case of open arcs we need only assume that $\varphi \in H^*$ on L and the continuity will hold except at those ends where $\varphi(x) \neq 0$.

Lemma II.2.2

Let $L = \overrightarrow{ab}$ be a smooth open arc and φ be a function given on L satisfying one of the following properties:

1) $\varphi \in H^{\mu}(L)$

2) $\varphi \in H^{\mu}$ everywhere on L except at a point $c \in L$ where it has a jump discontinuity, ie. $\varphi(c-0) \neq \varphi(c+0)$

3) $\varphi(x) = \dfrac{\varphi^*(x)}{(x - c)^{\gamma}}$, $\gamma = \alpha + i\beta$, $0 \leq \alpha < 1$, c is assumed to be one of the ends a or b and $\varphi^* \in H^{\mu}(L)$.

Let

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t - z}dt.$$

$$\Psi(x) = \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t - x}dt.$$

Then

a) if φ satisfies 1) $\Psi \in H^{\mu}(L)$ except at those ends c where $\varphi(c) \neq 0$ and in that case we have

$$\Phi(z) = \mp\frac{\varphi(c)}{2\pi i}\log(z-c) + \Phi_0(z)$$

$$\Psi(x) = \mp\frac{\varphi(c)}{2\pi i}\log(x-c) + \Psi_0(x)$$

where the upper sign goes with $c = a$ and the lower with $c = b$ and Φ_0 and Ψ_0 are bounded near c.

b) if φ satisfies 2)

$$\Phi(z) = \frac{\varphi(c-0) - \varphi(c+0)}{2\pi i}\log(z-c) + \Phi_0(z)$$

$$\Psi(x) = \frac{\varphi(c-0) - \varphi(c+0)}{2\pi i}\log(x-c) + \Psi_0(x)$$

where Φ_0 and Ψ_0 are bounded in the vicinity of c.

c) if φ satisfies 3) and $\gamma = \alpha + i\beta \neq 0$ then

$$\Phi(z) = \pm\frac{e^{\pm\gamma\pi i}}{2i\sin\gamma\pi}\,\frac{\varphi^*(c)}{(z - c)^{\gamma}} + \Phi_0(z)$$

$$\Psi(x) = \pm\frac{\cot(\gamma\pi)}{2i}\,\frac{\varphi^*(c)}{(x - c)^{\gamma}} + \Psi_0(x)$$

where the signs Φ_0 and Ψ_0 are as in a).

The proof of this result can be found in Muskhelishvili [1953 §29, §30, §31, §32 and §33].

3. The Plemelj-Sokhotski and Poincaré-Bertrand Formulae

Theorem II.3.1 (Plemelj-Sokhotski formulae)

Let $\varphi \in H^{\mu}(L)$, and for each $z \notin L$ and each $x \in L$, x not an end point define

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t - z}dt$$

$$\Phi^+(x) = \lim_{z\to x} \Phi(z), \quad z \in D^+$$

$$\Phi^-(x) = \lim_{z\to x} \Phi(z), \quad z \in D^-$$

Then

$$\Phi^+(x) = \frac{1}{2}\varphi(x) + \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t - x}dt$$

$$\Phi^-(x) = \frac{-1}{2}\varphi(x) + \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t - x}dt. \tag{II.3.1}$$

Proof: We assume that L is closed.

It is well known that

$$\int_L \frac{1}{t - z}dt = \begin{cases} 2\pi i & z \in D^+ \\ 0 & z \in D^- . \\ \pi i & z \in L \end{cases} \tag{II.3.2}$$

where the orientation of the contour L is such that D^+ forms the finite part of the domain.

Let $x \in L$, then

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t - z}dt = \frac{1}{2\pi i}\int_L \frac{\varphi(t) - \varphi(x)}{t - z}dt + \frac{\varphi(x)}{2\pi i}\int_L \frac{dt}{t - z}.$$

Using lemma II.2.1 and equation (II.3.2) ,we obtain

$$\Phi^+(x) = \varphi(x) + \frac{1}{2\pi i}\int_L \frac{\varphi(t) - \varphi(x)}{t - x}dt$$

$$= \varphi(x) + \frac{1}{2\pi i}\left[\int_L \frac{\varphi(t)}{t - x}dt - \varphi(x)\int_L \frac{dt}{t - x}\right]$$

$$= \frac{1}{2}\varphi(x) + \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t - x}dt.$$

The formula for $\Phi^-(x)$ is proved in the same way.■

Adding and subtracting formulae (II.3.1) we obtain an equivalent form of the Plemelj-Sokhotski formulae

$$\Phi^+(x) + \Phi^-(x) = \frac{1}{\pi i}\int_L \frac{\varphi(t)}{t - x}dt \qquad \text{(II.3.3)}$$

$$\Phi^+(x) - \Phi^-(x) = \varphi(x). \qquad \text{(II.3.4)}$$

Remark:

In the case when L is an open arc theorem II.3.1 can be proved under the weaker assumption that $\varphi \in H^*$. The proof follows from the remark following lemma II.2.1.

Theorem II.3.2 (Poincaré-Bertrand formula)

Let $\Psi(x,t)$ be Hölder continuous in both variables. Then, if x is a fixed point of L not coinciding with one of its ends,

$$\frac{1}{\pi i}\int_L \frac{dt}{t - x}\,\frac{1}{\pi i}\int_L \frac{\Psi(t,s)}{s - t}ds = \Psi(x,x) + \frac{1}{\pi i}\int_L ds\frac{1}{\pi i}\int_L \frac{\Psi(t,s)}{(t-x)(s-t)}dt.$$

(II.3.5)

The proof of (II.3.5) is fairly technical and can be found in

Muskhelishvili [1953 pg 56].

We consider in more detail the case in which $L = [-1,1]$, where we use the notation

$$\mathfrak{F}(\varphi,x) = \frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(t)}{t-x}dt. \tag{II.3.6}$$

As before the validity of theorem II.3.2 for an open arc can be established under the weaker assumption that the function $\Psi(x,t)$ is in H^* with respect to both variables.

Assume the functions f and g are in H^*, by taking $\Psi(x,t) = f(x)g(t)$ in theorem II.3.2, using the notation (II.3.6) and the identity

$$\frac{1}{(t-x)(s-t)} = \frac{1}{(s-x)}\left[\frac{1}{t-x} + \frac{1}{s-t}\right]$$

we obtain

$$\mathfrak{F}(f\mathfrak{F}(g,y),x) + \mathfrak{F}(g\mathfrak{F}(f,y),x) = \mathfrak{F}(f,x)\mathfrak{F}(g,x) - fg \tag{II.3.7}$$

4. Inversion Formula and Parseval's Identity

Theorem II.4.1 (inversion formula)

Let $\varphi \in H^{\mu}(L)$, L be a closed contour and for all $x \in L$ define

$$\hat{\varphi}(x) = \frac{1}{\pi i}\int_{L}\frac{\varphi(t)}{t-x}dt. \tag{II.4.1}$$

Then

$$\varphi(x) = \frac{1}{\pi i}\int_{L}\frac{\hat{\varphi}(t)}{t-x}dt.$$

Proof: On multiplyng (II.4.1) by $\frac{1}{\pi i(x-t)}$ and integrating with respect to x we obtain, after using (II.3.5)

$$\frac{1}{\pi i}\int_L \frac{\hat{\varphi}(x)}{x - t}dx = \frac{1}{\pi i}\int_L \frac{dx}{x - t}\, \frac{1}{\pi i}\int_L \frac{\varphi(s)}{s - x}ds$$

$$= \varphi(t) + \frac{1}{\pi i}\int_L \varphi(s)ds\, \frac{1}{\pi i}\int_L \frac{dx}{(x-t)(s-x)}.$$

The identity

$$\frac{1}{(x-t)(s-x)} = \frac{1}{t - s}\left[\frac{1}{x - s} - \frac{1}{x - t}\right]$$

allows us to deduce that

$$\int_L \frac{dx}{(x-t)(s-x)} = 0. \blacksquare$$

For the case in which L is the real line, we have the well known Hilbert reciprocity formulae

$$H(\varphi,x) \equiv \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varphi(t)}{t - x}dt$$

$$\varphi(x) = \frac{-1}{\pi}\int_{-\infty}^{\infty} \frac{H(\varphi,t)}{t - x}dt. \qquad \text{(II.4.2)}$$

Formulae (II.4.2) are valid for a more general class of functions, the class $L_p(-\infty,\infty)$ of functions whose p^{th} power is Lebesgue integrable in the real line; for more details see Titchmarsh [1937 pg.132].

<u>Theorem II.4.2</u> (Parseval's identity)

Let the functions $\varphi_1(x)$ and $\varphi_2(x)$ be in the classes $L_{p_1}(-\infty,\infty)$

and $L_{p_2}(-\infty,\infty)$ respectively, with $p_1 > 1$ and $p_2 > 1$. Then if

$$\frac{1}{p_1} + \frac{1}{p_2} = 1$$

we have

$$\int_{-\infty}^{\infty} \varphi_1(t)\varphi_2(t)dt = \int_{-\infty}^{\infty} H(\varphi_1,t)H(\varphi_2,t)dt. \quad (II.4.3)$$

The proof of this result can be found in the book by Titchmarsh [1937 pg. 138].

Notice that from (II.4.2) we deduce

$$H(H(\varphi,y),x) = -\varphi(x)$$

and then by taking $\varphi_2(t) = H(\Psi,t)$ and $\varphi_1(t) = \varphi(t)$ in (II.4.3) we can write

$$\int_{-\infty}^{\infty} \Big[\varphi(t)H(\Psi,t) + \Psi(t)H(\varphi,t)\Big]dt = 0 \qquad (II.4.4)$$

where φ and Ψ are functions in the classes L_{p_1} and L_{p_2} respectively.

For functions defined in the interval $[-1,1]$ we can still apply formula (II.4.4) by redefining the given function to be zero outside $[-1,1]$, and in this case using the notation of (II.3.6) formula (II.4.4) can be written as

$$\int_{-1}^{1} \Big[\varphi(t)\mathcal{F}(\Psi,t) + \Psi(t)\mathcal{F}(\varphi,t)\Big]dt = 0 \qquad (II.4.5)$$

5. The Hilbert Boundary Value Problem for Open Arcs

Let L_1, L_2, ..., L_p be p smooth non-intersecting open arcs with defined positive directions. We shall denote by L the union of those arcs and we write

$$L = L_1 + L_2 + \ldots + L_p.$$

The ends of the arc L_j will be denoted by a_j and b_j and are such that the positive direction of L_j is from a_j to b_j.

The plane cut along L will be denoted by S.

<u>Definition II.5.1</u>

A function $\Phi(z)$ will be called *sectionally holomorphic* with line of discontinuity L if it is holomorphic in S except possibly at the point at infinity, continuous on L from the left and from the right with the possible exception of the ends c and in a neighbourhood of c it satisfies

$$|\Phi(z)| \leq \frac{\text{const}}{|z - c|^{\alpha}}, \quad 0 \leq \alpha < 1. \qquad \text{(II.5.1)}$$

Moreover if for $|z|$ large

$$\Phi(z) = A_k z^k + A_{k-1} z^{k-1} + \quad \ldots \qquad \text{(II.5.2)}$$

then $\Phi(z)$ is said to have degree k at infinity.

<u>Definition II.5.2</u>

Let $L = L_1 + L_2 + \ldots + L_p$ with L_j a smooth open arc. Let also $G(x)$ and $g(x)$ be Hölder continuous functions on L, $G(x) \neq 0$ on L. The *Hilbert boundary value problem* consists of finding a sectionally holomorphic function $\Phi(z)$, having finite degree at infinity for which one of the following boundary conditions hold.

$$\Phi^+(x) = G(x)\Phi^-(x) \quad \text{on } L \quad \text{(Homogeneous problem)} \qquad \text{(II.5.3)}$$

$$\Phi^+(x) = G(x)\Phi^-(x) + g(x) \quad \text{on } L \quad \text{(Inhomogeneous problem)}, \qquad \text{(II.5.4)}$$

The solution of the homogeneous problem will be considered first.

Let

$$\Gamma(z) = \frac{1}{2\pi i}\int_L \frac{\log(G(t))}{t - z}dt$$

where log(G(t)) is taken to be any branch of the multi-valued function which varies continuously on each of the arcs L_j.

We now show that $e^{\Gamma(z)}$ satisfies (II.5.3).

Applying the Plemelj-Sokhotski formula (II.3.4) to the function $\Gamma(z)$ we obtain

$$\Gamma^+(x) - \Gamma^-(x) = \log(G(x))$$

so that

$$e^{\Gamma^+(x)}e^{-\Gamma^-(x)} = G(x)$$

or

$$e^{\Gamma^+(x)} = G(x)e^{\Gamma^-(x)} \tag{II.5.5}$$

which is in the form (II.5.3). However, the function $e^{\Gamma(z)}$ may not have satisfied all the requirements of the Hilbert problem. Notice that in the statement of the Hilbert problem (HP) we required $\Phi(z)$ to be sectionally holomorphic, and condition (II.5.1) may in fact not have been satisfied by $e^{\Gamma(z)}$.

For j = 1, 2,...,p define $c_{2j-1} = a_j$ and $c_{2j} = b_j$ where a_j and b_j are the ends of the arc L_j. For a generic end c_k, k = 1, 2,...,2p we then have from lemma II.2.2

$$\Gamma(z) = \frac{1}{2\pi i}\left[\int_L \frac{\log(G(c_k))}{t - z}dt + \int_L \frac{\log((G(t)) - \log(G(c_k))}{t - z}dt\right]$$

$$= \mp\frac{\log((G(c_k))}{2\pi i}\log(z-c_k) + \Gamma_o(z)$$

where $\Gamma_o(z)$ is bounded near and at c_k, the upper sign goes with $c_k = a_j$ and the lower with $c_k = b_j$. We observe that the above expression for $\Gamma(z)$ is only valid in the vicinity of the end c_k. Hence

$$e^{\Gamma(z)} = (z-c_k)^{\alpha_k+i\beta_k}e^{\Gamma_o(z)} = (z-c_k)^{\alpha_k+i\beta_k}\Omega(z)$$

where

$$\alpha_k + i\beta_k = \mp\frac{\log((G(c_k))}{2\pi i} \text{ and } \Omega(z) = e^{\Gamma_o(z)}.$$

Now select integers λ_k with the property

$$-1 < \alpha_k + \lambda_k < 1 \tag{II.5.6}$$

and write

$$\prod(z) = (z-c_1)^{\lambda_1}(z-c_2)^{\lambda_2} \ldots (z-c_{2p})^{\lambda_{2p}}$$

then the function

$$X(z) = \prod(z)e^{\Gamma(z)} \tag{II.5.7}$$

satisfies all the conditions of the problem. In fact

$$X^+(x) = \prod(x)e^{\Gamma^+(x)}$$

$$X^-(x) = \prod(x)e^{\Gamma^-(x)}$$

and from (II.5.5) we see that X(z) satisfies the boundary condition (II.5.3). From (II.5.6) we see that X(z) is sectionally holomophic and so it solves the Hilbert boundary value problem.

The solution X(z) however is not completely determined by the conditions (II.5.6). It is uniquely defined only for those ends in which α_k is an integer where we have $\lambda_k = -\alpha_k$. Those ends will be called special ends. Notice that if α_k is an integer then $G(c_k)$ is real and positive.

For all the other ends, which we shall call non-special, the integers λ_k may take one of two consecutive values and they can be chosen in such a way that either $\alpha_k + \lambda_k > 0$ or $\alpha_k + \lambda_k < 0$. To fix the choice of the λ_k one more condition must be added, this can be done by introducing the concept of a class of solutions.

Definition II.5.3

Let $c_1, c_2, \ldots, c_m$ be all the non-special ends. The solution $\Phi(z)$ is said to be in the class $h(c_1, c_2, \ldots, c_q)$ where $q \leq m$, if it is bounded at the ends $c_1, c_2, \ldots, c_q$.

The class of solutions unbounded at all non-special ends will be denoted by h_o and the one bounded at all non-special ends by h_m.

Notice that by requiring that the solution $\Phi(z)$ be in a certain class we fix the choice of λ_k in (II.5.6) in such a way that at non-special ends, where the solution of that class are bounded, $\alpha_k + \lambda_k > 0$ while $\alpha_k + \lambda_k < 0$ at all the remaining non-special ends.

Definition II.5.4

The solution $X(z)$ obtained by this choice of the λ_k or any other solution of the form const$\times X(z)$, will be called the *fundamental solution* of the given class.

Definition II.5.5

The integer

$$\kappa = -\sum_{j=1}^{2p} \lambda_j \tag{II.5.8}$$

will be called the *index* of the class of solutions $h(c_1, c_2,\dots,c_q)$.

Different choices of the logarithm will produce values of α_k differing by an integer. Further, because of the different signs in the definition of α_k, for each of the different ends this integer will enter in the summation in (II.5.8) twice with oposite signs, giving a zero contribuition to the final result. Therefore the value of the index given by (II.5.8) does not depend on the choice of the branch of the logarithm function.

It is obvious from (II.5.7) and the definition of $\Gamma(z)$ that $X(z)$ has degree $-\kappa$ at infinity and that

$$\lim_{z\to\infty} z^{\kappa} X(z) = 1.$$

<u>Definition II.5.6</u>

The function

$$Z(x) = \prod(x)e^{\Gamma(x)} \quad , \; x \in L \tag{II.5.9}$$

where

$$\Gamma(x) = \frac{1}{2\pi i}\int_L \frac{\log(G(t))}{t - x}dt$$

will be called the *fundamental function* of the class $h(c_1, c_2,\dots,c_q)$.

This definition of the fundamental function differs slightly from that of Muskhelishvili [1953].

Lemma II.5.1

$$X^{+}(x) = \sqrt{G(x)}Z(x) \tag{II.5.10}$$

$$X^{-}(x) = \frac{Z(x)}{\sqrt{G(x)}} \tag{II.5.11}$$

$$Z(x) = \sqrt{X^{+}(x)X^{-}(x)} \tag{II.5.12}$$

$$Z(x) = w_0(x)(x-c_1)^{\gamma_1}(x-c_2)^{\gamma_2} \ldots (x-c_{2p})^{\gamma_{2p}} \in H^* \tag{II.5.13}$$

where $\gamma_k = \alpha_k + \lambda_k + i\beta_k$, $w_0(x)$ is Hölder continuous and the branch of the square root in (II.5.10-11) is fixed by $\sqrt{G(x)} = \exp(\frac{1}{2}\log(G(x))$.

Proof: Formulae (II.5.10) and (II.5.11) follow directly from the Plemelj-Sokhotski formulae. (II.5.12) can be derived by multiplying (II.5.10) and (II.5.11) together. Finally to prove (II.5.13) we recall that

$$\Gamma(x) = \frac{1}{2\pi i}\int_L \frac{\log(G(t))}{t - x}dt \tag{II.5.14}$$

an then in the vicinity of an end c_k $\Gamma(x)$ can be written in the form

$$\Gamma(x) = \frac{1}{2\pi i}\left[\int_L \frac{\log(G(c_k))}{t - x}dt + \int_L \frac{\log((G(t)) - \log(G(c_k))}{t - x}dt\right]$$

$$= \mp\frac{\log((G(c_k))}{2\pi i}\log(x-c_k) + \Gamma_0(x)$$

where $\Gamma_0(x)$ is Hölder continuous, this follows from part a) of lemma II.2.2 applied to the last integral. Repeating this argument for each of the ends and using (II.5.9) we obtain

$$Z(x) = \prod(x)e^{\Gamma(x)} = w_o(x)(x-c_1)^{\gamma_1}(x-c_2)^{\gamma_2} \ldots (x-c_{2p})^{\gamma_{2p}}.$$

The fact that $Z(x) \in H^*$ on L follows from the definition of λ_k.∎

We can now show:

<u>Theorem II.5.1</u>

A sectionally holomorphic function $\Phi(z)$ with finite degree at infinity is a solution of the homogeneous HBVP (II.5.3) if and only if

$$\Phi(z) = P(z)X(z)$$

where $P(z)$ is a polynomial.

Proof: Assume first that Φ is a solution of (II.5.3), that is

$$\Phi^+(x) = G(x)\Phi^-(x) \text{ on } L$$

but $X(z)$ is also a solution of that problem, so that

$$X^+(x) = G(x)X^-(x) \text{ on } L.$$

Hence

$$\frac{\Phi^+(x)}{X^+(x)} = \frac{\Phi^-(x)}{X^-(x)} \text{ on } L.$$

This last equation indicates that the function $\Psi(z) = \Phi(z)/X(z)$ is holomorphic in the entire plane with the possible exception of the end points of the arcs L_j and the point at infinity. At the end points it can become infinite with degree less than 1. Since the degree of Ψ at infinity is finite it is a polynomial $P(z)$ and we have

$$\Phi(z) = P(z)X(z).$$

The converse follows directly from applying the Plemelj-Sokhotski formulae to $\Phi(z)$ and taking into account the properties of $X(z)$.■

A number of important consequences follows from this theorem

<u>Corollary II.5.1</u>

1) Every solution of (II.5.3) is bounded near a special end.

2) If a solution of (II.5.3) is bounded at a non-special end then it vanishes there.

Proof: 1) This follows immediately from the fact that $X(z)$ is bounded near a special end.

2) If c is a non-special end determining the class of $X(z)$ then $X(z)$ vanishes at c by construction. If, however, c is not in the class determining $X(z)$ then in order that the solution be bounded at this end we need $P(c) = 0$, that is $(z-c)$ is a factor of $P(z)$; hence $\Phi(c) = P(c)X(c) = 0$ because the factor $(z-c)$ appearing in $X(z)$ has an exponent with real part > -1.■

We turn now to the solution of the inhomogeneous problem (II.5.4). As for the homogeneous problem we shall seek a solution belonging to a given class $h(c_1, c_2, \dots, c_q)$. Let $X(z)$ be the fundamental solution of the homogeneous problem in that class; from

$$X^+(x) = G(x)X^-(x)$$

we obtain

$$G(x) = \frac{X^+(x)}{X^-(x)}.$$

Substituting this last result into (II.5.4) we obtain the equation

$$\frac{\Phi^+(x)}{X^+(x)} = \frac{\Phi^-(x)}{X^-(x)} + \frac{g(x)}{X^+(x)}. \tag{II.5.15}$$

Since near the ends c_1, $c_2, \ldots, c_q$ where $X(z)$ becomes zero, we have

$$\left|\frac{\Phi(z)}{X(z)}\right| < \frac{\text{const}}{|z - c_j|^\alpha}, \quad \alpha < 1, \tag{II.5.16}$$

$\Phi(z)/X(z)$ is sectionally holomorphic and has finite degree at infinity.

From this we can see that in order to solve the inhomogeneous problem (II.5.4) we need only solve the problem: Let $h(x)$ be a function of position on L such that $h \in H^*$. Find a sectionally holomophic function $\Psi(z)$ with degree κ at infinity and which satisfies the boundary condition

$$\Psi^+(x) = \Psi^-(x) + h(x) \text{ on } L. \tag{II.5.17}$$

The solutions of this problem are of the form

$$\Psi(z) = \frac{1}{2\pi i}\int_L \frac{h(t)}{t - z}dt + P_\kappa(z) \tag{II.5.18}$$

where $P_\kappa(z)$ is a polynomial of degree κ.

The fact that (II.5.18) satisfies (II.5.17) follows immediately from the Plemelj-Sokhotski formulae (II.3.4).

To prove that all the solutions of (II.5.17) are of the form (II.5.18) we show that any two solutions $\Psi_1(z)$ and $\Psi_2(z)$ differ by a polynomial of degree κ. Let $R(z) = \Psi_1(z) - \Psi_2(z)$, from

(II.5.17) we see that $R^+(x) = R^-(x)$ on L. This implies that R(z) is a holomorphic function on the entire plane except at the ends c_j where it has singularities of order less than 1, and hence it is necessarily bounded near those points, so that R(z) is holomorphic in the entire plane and has degree κ at infinity. Therefore R(z) is a polynomial of degree κ.

Having found a solution of problem (II.5.17) in the form (II.5.18) we easily see that the general solution of problem (II.5.4) is given by

$$\Phi(z) = \frac{X(z)}{2\pi i}\int_L \frac{g(t)}{X^+(t)(t-z)}dt + X(z)P(z) \qquad \text{(II.5.19)}$$

where P(z) is an arbitrary polynomial.

Noting that the function X(z) has degree $-\kappa$ at infinity we can prove the following theorem

Theorem II.5.2

1) For $\kappa \geq 0$ the solutions of the inhomogeneous problem of a given class, vanishing at infinity are given by

$$\Phi(z) = \frac{X(z)}{2\pi i}\int_L \frac{g(t)}{X^+(t)(t-z)}dt + X(z)P_{\kappa-1}(z). \qquad \text{(II.5.20)}$$

2) For $\kappa < 0$ the solution of a given class vanishing at infinity exist if and only if the conditions

$$\int_L \frac{t^j g(t)}{X^+(t)}dt = 0, \quad j = 0, 1, \ldots, -\kappa-1 \qquad \text{(II.5.21)}$$

hold and in this case the solution is given by (II.5.20) with

$P_{\kappa-1} \equiv 0$.

Proof: The proof of 1) is obvious from (II.5.19).

The proof of 2) is as follows:

For large z we can write

$$\int_L \frac{g(t)}{X^+(t)(t-z)}dt = \frac{-1}{z}\int_L \frac{g(t)}{X^+(t)(1-\frac{t}{z})}dt = \frac{-1}{z}\int_L \frac{g(t)}{X^+(t)}\left[1 + \frac{t}{z} + \frac{t^2}{z^2} + \ldots\right]dt$$

$$= -\sum_{j=0}^{\infty} \frac{1}{z^{j+1}}\int_L \frac{t^j g(t)}{X^+(t)}dt.$$

Substituting this result in (II.5.20) and recalling that the degree of X(z) at infinity is $-\kappa$ we see that $\Phi(z)$ has degree -1 at infinity if and only if (II.5.21) holds.∎

6. Solution of SIE for Open Arcs

Let $L = L_1 + L_2 + \ldots + L_p$ be the union of p disconnected smooth open arcs L_j with ends a_j and b_j and with the positive direction running from a_j to b_j.

The equation we are concerned with in this section has the form:

$$a(x)\phi(x) + \frac{b(x)}{\pi}\int_L \frac{\phi(t)}{t - x}dt + \int_L m(x,t)\phi(t)dt = f(x) \qquad \text{(II.6.1)}$$

where the functions a,b,m and f are real valued functions which satisfy the Hölder condition on L.

We define the operators

$$A\phi = a(x)\phi(x) + \frac{b(x)}{\pi}\int_L \frac{\phi(t)}{t - x}dt \tag{II.6.2}$$

$$M\phi = \int_L m(x,t)\phi(t)dt. \tag{II.6.3}$$

The equation

$$A\phi = f \tag{II.6.4}$$

is called the dominant equation and

$$A\phi + M\phi = f \tag{II.6.5}$$

the complete equation.

We first consider the solution of the dominant equation. Writing

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\phi(t)}{t - z}dt \tag{II.6.6}$$

from formulae (II.3.3-4) we have that

$$\Phi^+(x) + \Phi^-(x) = \frac{1}{\pi i}\int_L \frac{\phi(t)}{t - x}dt \tag{II.6.7}$$

$$\Phi^+(x) - \Phi^-(x) = \phi(x). \tag{II.6.8}$$

Substituting those formulae into (II.6.4) we obtain

$$a(x)(\Phi^+(x) - \Phi^-(x)) + ib(x)(\Phi^+(x) + \Phi^-(x)) = f(x)$$

or

$$\Phi^+(x) = \frac{a(x) - ib(x)}{a(x) + ib(x)}\Phi^-(x) + \frac{f(x)}{a(x) + ib(x)}. \tag{II.6.9}$$

From quation (II.6.6) we deduce that $\Phi(z)$ has degree -1 at infinity, so that equation (II.6.9) and the condition that $\Phi(z)$ have degree -1 at infinity form an inhomogeneous Hilbert boundary value problem with

$$G(x) = \frac{a(x) - ib(x)}{a(x) + ib(x)}$$

if we assume that $a(x) - ib(x) \neq 0$ and $a(x) + ib(x) \neq 0$ or equivalently

$$r(x) \equiv \sqrt{a^2(x) + b^2(x)} \neq 0. \qquad \text{(II.6.10)}$$

The function $G(x)$ is Hölder continuous on L and $G(x) \neq 0$ $\forall x \in L$.

The equivalence of the SIE with a HBVP implies that the solutions of (II.6.4) will be divided into classes, as for the HBVP, and the general solution of the class $h(c_1, c_2, \ldots, c_q)$ will be given by formulae (II.5.20), ie.

$$\Phi(z) = \frac{X(z)}{2\pi i}\int_L \frac{f(t)}{(a(t) + ib(t))X^+(t)(t-z)}dt + X(z)P_{\kappa-1}(z) \qquad \text{(II.6.11)}$$

together with formulae (II.6.8), where $X(z)$ is the solution of the homogeneous problem of the same class.

From (II.5.10) and (II.6.10) we have

$$r(x)Z(x) = \frac{r(x)X^+(x)}{\sqrt{G(x)}} = X^+(x)(a(x) + ib(x)).$$

The following theorem naturally results

Theorem II.6.1

The solution ϕ of the dominant equation (II.6.4) is given by

$$\phi(x) = A^I f(x) - \frac{2ib(x)Z(x)}{r(x)} P_{\kappa-1}(x), \qquad \text{if } \kappa \geq 0$$

$$\phi(x) = A^I f(x) \qquad \text{if } \kappa < 0 \tag{II.6.12}$$

where

$$A^I f(x) = \frac{a(x)f(x)}{r^2(x)} - \frac{b(x)Z(x)}{r(x)\pi} \int_L \frac{f(t)}{r(t)Z(t)(t-x)} dt$$

and for $\kappa < 0$ the solution exists if and only if the conditions

$$\int_L \frac{t^j f(t)}{r(t)Z(t)} dt = 0, \quad j = 0, 1, \ldots, -\kappa-1 \tag{II.6.13}$$

hold.

Proof: From (II.6.8) we have

$$\phi(x) = \Phi^+(x) - \Phi^-(x) = X^+(x)\Psi^+(x) + X^+(x)P_{\kappa-1}(x)$$

$$- X^-(x)\Psi^-(x) - X^-(x)P_{\kappa-1}(x) \tag{II.6.14}$$

where

$$\Psi(z) = \frac{1}{2\pi i} \int_L \frac{f(t)}{r(t)Z(t)(t-z)} dt.$$

From the Plemelj-Sokhotski formulae (II.3.1) we write

$$\Psi^+(x) = \frac{1}{2} \frac{f(x)}{Z(x)r(x)} + \frac{1}{2\pi i} \int_L \frac{f(t)}{r(t)Z(t)(t-x)} dt$$

$$\Psi^-(x) = \frac{-1}{2} \frac{f(x)}{Z(x)r(x)} + \frac{1}{2\pi i} \int_L \frac{f(t)}{r(t)Z(t)(t-x)} dt.$$

Substituting these expressions into (II.6.14) we obtain

$$\phi(x) = \frac{1}{2}\,\frac{f(x)}{Z(x)r(x)}\left[X^+(x) + X^-(x)\right] + \frac{1}{2\pi i}\int_L \frac{f(t)}{r(t)Z(t)(t-x)}dt\left[X^+(x)\right.$$

$$\left. - X^-(x)\right] + P_{\kappa-1}(x)\left[X^+(x) - X^-(x)\right].$$

From (II.5.10) and (II.5.11) we have

$$X^+(x) + X^-(x) = \sqrt{G(x)}Z(x) + \frac{Z(x)}{\sqrt{G(x)}}$$

$$= \left[\sqrt{\frac{a(x) - ib(x)}{a(x) + ib(x)}} + \sqrt{\frac{a(x) + ib(x)}{a(x) - ib(x)}}\right]Z(x)$$

$$= \frac{2a(x)Z(x)}{r(x)}.$$

Similarly

$$X^+(x) + X^-(x) = \frac{-2ib(x)Z(x)}{r(x)}.$$

Hence

$$\phi(x) = \frac{a(x)f(x)}{r^2(x)} - \frac{b(x)Z(x)}{r(x)\pi}\int_L \frac{f(t)}{r(t)Z(t)(t-x)}dt - \frac{2ib(x)Z(x)}{r(x)}P_{\kappa-1}(x).$$

For $\kappa < 0$ condition (II.6.13) follows immediately from (II.5.21).■

Equations (II.6.11-12) have the important consequences

$$\begin{aligned} &1)\ A^IAf = f + \frac{b(x)Z(x)}{r(x)}P_{\kappa-1}(x) && \text{if } \kappa \geq 0 \\ &\quad A^IAf = f && \text{if } \kappa < 0. \end{aligned} \qquad \text{(II.6.15)}$$

for all Hölder continuous functions f.

2) For $\kappa \geq 0$ the kernel of the operator A has dimension κ and when $\kappa \geq 1$ is spanned by the set of functions

$$\frac{b(x)Z(x)}{r(x)}\left\{1,\ x,\ x^2,\ \ldots,\ x^{\kappa-1}\right\}. \qquad \text{(II.6.16)}$$

3) From lemma II.2.2 part 3 it also follows that the solution $\phi \in H^*$, moreover the solution ϕ is in the class $h(c_1, c_2, \ldots, c_q)$ of solutions of the Hilbert problem.

7. Solution of the Complete Equation

Consider now the solution of the complete equation (II.6.5) in the class of functions $h(c_1, c_2, \ldots, c_q)$. From

$$A\phi + M\phi = f$$

we obtain

$$A\phi = f - M\phi \qquad \text{(II.7.1)}$$

where $f - M\phi$ is Hölder continuous if we insist that $f(x)$ and $m(x,t)$ are. Assuming that the right-hand side of (II.7.1) is known, theorem II.6.1 allows us to write the solution of (II.7.1) in the form

$$\phi(x) + \int_L N(x,t)\phi(t)dt = f_1(x) \qquad \text{(II.7.2)}$$

where

$$N(x,t) = \frac{a(x)m(x,t)}{r^2(x)} - \frac{b(x)Z(x)}{r(x)\pi}\int_L \frac{m(s,t)}{Z(s)r(s)(s-x)}dt \qquad \text{(II.7.3)}$$

$$f_1(x) = \frac{a(x)f(x)}{r^2(x)} - \frac{b(x)Z(x)}{r(x)\pi}\int_L \frac{f(t)}{r(t)Z(t)(t-x)}dt$$

$$- \frac{b(x)Z(x)}{r(x)}P_{\kappa-1}(x). \qquad \text{(II.7.4)}$$

In operator notation

$$\phi(x) + A^I M\phi(x) = A^I f + \frac{b(x)Z(x)}{r(x)} P_{\kappa-1}(x)$$

where

$$A^I M\phi = \int_L N(x,t)\phi(t)dt$$

$$f_1(x) = A^I f + \frac{b(x)Z(x)}{r(x)} P_{\kappa-1}(x).$$

We now show that equation (II.7.2) can be transformed into a Fredholm integral equation. From (II.5.13) we know that

$$Z(x) = w_o(x)(x-c_1)^{\gamma_1}(x-c_2)^{\gamma_2} \ldots (x-c_{2p})^{\gamma_{2p}}$$

where the ends c_j have been split into 3 different groups with the properties

$$0 < \mathrm{Re}(\gamma_k) < 1, \; k = 1,2, \ldots, q$$

$$-1 < \mathrm{Re}(\gamma_k) < 0, \; k = q+1, \ldots, \ell$$

$$\mathrm{Re}(\gamma_k) = 0, \; k = \ell+1, \ldots, 2p.$$

It is readly seen from (II.7.3) and (II.7.4) that $N(x,t)$ and $f_1(x)$ behave at the ends $c_{q+1}, \ldots, c_\ell$ as $Z(x)$, and hence have infinities of order $\mathrm{Re}(\gamma_k)$ there. However by a change of the dependent variable ϕ and the independent variable t in (II.7.2) we can transform (II.7.2) into a Fredholm integral equation. We shall now demonstrate this.

Let $T(x) = (x-c_{q+1})^{\gamma_{q+1}} \ldots (x-c_m)^{\gamma_\ell}$ and write $\phi(x) = T(x)\phi_o(x)$ then (II.7.2) can be written as

$$\phi_o(x) + \int_L \frac{N(x,t)T(t)}{T(x)}\phi_o(t)dt = \frac{f_1(x)}{T(x)}. \tag{II.7.5}$$

The kernel in (II.7.5) is now free of singularities in the x-variable but singularities in the t-variable have appeared. To remove those we change the independent variable by writing

$$t_1 = \int_{a_k}^{t} T(s)ds, \quad x_1 = \int_{a_k}^{x} T(s)ds \text{ on } L_k, \; k = 1,2, \ldots, p.$$

Thus (II.7.5) can be written as the Fredholm equation

$$\phi_o(x_1) + \int_{L'} N_1(x_1,t_1)\phi_o(t_1)dt_1 = f_1(x_1)$$

where L' is the transformed arc of L and N_1 and f_1 are now bounded functions of the new independent variables. This allows us to infer that equation (II.7.2) has a unique solution for any right-hand side as long as -1 is not an eigenvalue of the kernel N_1.

8. Uniform Convergence of Polynomial Interpolation

We start this section by proving the fundamental result due to Kalandiya [1975 pg. 105].

Lemma II.8.1

Let $f:[-1,1] \to R$ be a function such that, for each

$n \geq 2$, there exists a polynomial $p_n(x)$ of degree n, with

$$\max_{x\in[-1,1]} |s_n(x)| \equiv \max_{x\in[-1,1]} |f(x) - p_n(x)| \leq \frac{c}{n^p}$$

where $p > 0$, and c is an arbitrary constant.

Then

$$|s_n(x) - s_n(y)| \leq \frac{A(\xi)}{n^{p-2\xi}}|x-y|^{\xi}, \quad 0 < 2\xi < p, \qquad \text{(II.8.1)}$$

for all $x,y \in [-1,1]$, where $A(\xi)$ is a constant which depends of ξ only.

Proof: For each fixed value of n, let

$$U_k(x) = p_{i(k)}(x) - p_{i(k-1)}(x), \quad k = 1,2, \ \ldots \qquad \text{(II.8.2)}$$

where $i(k) = 2^k n$.

From the assumptions we may deduce

$$s_n(x) \equiv f(x) - p_n(x) = \sum_{k=1}^{\infty} U_k(x). \qquad \text{(II.8.3)}$$

An estimate for $U_k(x)$ is given by

$$\begin{aligned} |U_k(x)| &= |p_{i(k)}(x) - p_{i(k-1)}(x)| \\ &\leq |f(x) - p_{i(k)}(x)| + |p_{i(k-1)}(x) - f(x)| \\ &\leq \frac{c}{(2^k n)^p} + \frac{c}{(2^{k-1}n)^p} = \frac{c_k^*}{(2^{k-1}n)^{2\xi}} \end{aligned} \qquad \text{(II.8.4)}$$

where $c_k^* = \dfrac{(1+2^{-p})c}{(2^{k-1}n)^{p-2\xi}}$ and ξ is such that $0 < 2\xi < p$.

Notice that the c_k^*'s are monotonically decreasing, more specifically,

$$c_k^* = 2^{-(p-2\xi)}c_{k-1}^* \tag{II.8.5}$$

or

$$c_k^* = (2^{p-2\xi})^{1-k}c_1^* \leq c_1^* \tag{II.8.6}$$

because $p-2\xi > 0$.

Given an arbitrary positive number h, $0 \leq h \leq 2$, first suppose $4n^2h < 1$, and choose an integer m such that

$$2^{-2(m+1)}n^{-2} < h < 2^{-2m}n^{-2} \tag{II.8.7}$$

Notice that this choice is always possible due to $4n^2h < 1$.

Having chosen m, we split the summation in (II.8.3)

$$s_n(x) = F_1(x) + F_2(x)$$

where

$$F_1(x) = \sum_{k=1}^{m} U_k(x) \ , \quad F_2(x) = \sum_{k=m+1}^{\infty} U_k(x).$$

In the sequel we are going to use the notation

$$\Delta_h f(x) = f(x+h) - f(x).$$

Notice that to prove the lemma we need to show that

$|\Delta_h s_n(x)| \leq \frac{c}{n^{p-2\xi}} h^{\xi}$, $\forall x \in [-1,1]$, and $\forall h$ such that $x+h \in [-1,1]$.

Let us start by finding a bound for $|\Delta_h F_1(x)|$.

First recall that if $q_n(x)$ is a polynomial of degree n in $[-1,1]$, then

$$|q_n'(x)| \le n^2 \max_{t\in[-1,1]} |q_n(t)|, \quad \forall x \in [-1,1],$$

see Cheney [1966 pg. 91].

Using the mean value theorem, the above result and the bounds (II.8.4),(II.8.5) and (II.8.6) allow us to write

$$|\Delta_h F_1(x)| \le \sum_{k=1}^{m} |U_k(x+h) - U_k(x)| \le \sum_{k=1}^{m} h|U_k'(x+\theta_k h)|$$

$$\le h\sum_{k=1}^{m} (2^k n)^2 \max_t |U_k(t)| \le 2^{2\xi} h \sum_{k=1}^{m} (2^k n)^{2(1-\xi)} c_k^*$$

$$\le 2^{2\xi} h c_1^* n^{2(1-\xi)} \sum_{k=1}^{m} 2^{2k(1-\xi)}$$

$$\le 2^{2\xi} h c_1^* (2^m n)^{2(1-\xi)} \sum_{k=1}^{m} 2^{-2(1-\xi)(m-k)}$$

$$\le 2^{2\xi} c_1^* h (2^m n)^{2(1-\xi)} \sum_{k=0}^{m-1} 2^{-2(1-\xi)k}. \qquad \text{(II.8.8)}$$

Also from (II.8.7)

$$(2^m n)^{2(1-\xi)} \le h^{\xi-1}, \qquad \text{(II.8.9)}$$

and

$$\sum_{k=0}^{m-1} 2^{-2(1-\xi)k} \le 2^{2(1-\xi)} \left[2^{2(1-\xi)} - 1\right]^{-1}. \qquad \text{(II.8.10)}$$

Using (II.8.9) and (II.8.10), (II.8.8) can be written as

$$|\Delta_h F_1(x)| \le 2^{2\xi} c_1^* h^{\xi}\, 2^{2(1-\xi)} \left[2^{2(1-\xi)} - 1\right]^{-1}$$

$$= 4(1+2^{-p}) \left[2^{2(1-\xi)} - 1\right]^{-1} \frac{c h^{\xi}}{n^{p-2\xi}} \qquad \text{(II.8.11)}$$

where use has been made of the definiton of c_1^*.

To bound $|\Delta_h F_2(x)|$ we require

$$|\Delta_h F_2(x)| \leq 2 \sum_{k=m+1}^{\infty} \underset{x\in[-1,1]}{\operatorname{Max}} |U_k(x)|. \tag{II.8.12}$$

Now by using (II.8.5), (II.8.4) and making the change of variables $k = r + m + 1$ in the summation, we obtain

$$|\Delta_h F_2(x)| \leq 2c^{*}_{m+1} \sum_{k=m+1}^{\infty} (2^{k-1}n)^{-2\xi} = 2^{2\xi+1} c^{*}_{m+1} (2^{m+1}n)^{-2\xi} \sum_{r=0}^{\infty} 2^{-2\xi r}.$$

Using (II.8.7) and the definition of c^{*}_{m+1} we have

$$|\Delta_h F_2(x)| \leq 2^{4\xi+1}(1+2^{-p})\left[2^{2\xi}-1\right]^{-1} \frac{ch^{\xi}}{n^{p-2\xi}}. \tag{II.8.13}$$

Thus we have shown that, for any h, such that $4n^2h < 1$,

$$|\Delta_h s_n(x)| = |\Delta_h F_1(x) + \Delta_h F_2(x)| \leq |\Delta_h F_1(x)| + |\Delta_h F_2(x)|$$

$$\leq \left[\frac{4(1+2^{-p})}{2^{2(1-\xi)}-1} + \frac{2^{4\xi+1}(1+2^{-p})}{2^{2\xi}-1}\right] \frac{ch^{\xi}}{n^{p-2\xi}} \tag{II.8.14}$$

from (II.8.11) and (II.8.13).

To complete the proof we shall show that (II.8.14) also holds for h such that $4n^2h \geq 1$.

$$h^{-\xi}|\Delta_h s_n(x)| \leq (4n^2)^{\xi} \underset{x\in[-1,1]}{\operatorname{Max}} |\Delta_h s_n(x)| \leq 2(4n^2)^{\xi} \underset{x\in[-1,1]}{\operatorname{Max}} |s_n(x)|$$

$$\leq 2(4n^2)^{\xi} cn^{-p} = 2^{2\xi+1} \frac{c}{n^{p-2\xi}}$$

that is,

$$|\Delta_h s_n(x)| \le 2^{2\xi+1} \frac{ch^\xi}{n^{p-2\xi}} \qquad \text{(II.8.15)}$$

which is of the same form as (II.8.14).

Finally we can write

$$|\Delta_h s_n(x)| \le \frac{A(\xi)h^\xi}{n^{p-2\xi}}$$

where

$$A(\xi) = c \text{ Max } \left\{2^{2\xi+1}, \frac{4(1+2^{-p})}{2^{2(1-\xi)}-1} + \frac{2^{4\xi+1}(1+2^{-p})}{2^\xi-1}\right\}. \blacksquare$$

The next lemma gives the rate of convergence of the interpolating polynomial to a smooth function.

Lemma II.8.2 (Rivlin [1969])

Let $f \in C^p[-1,1]$ with $f^{(p)} \in H^\mu[-1,1]$ and let the points $x_i \in [-1,1]$ be the zeros of the $(n+1)^{th}$ Chebyshev polynomial of the first kind, i.e.

$$x_i^{(n)} = \cos\left[\frac{(2i+1)\pi}{2(n+1)}\right], \ i = 0,1,\dots n. \qquad \text{(II.8.16)}$$

Let also $p_n(x)$ be the interpolating polynomial to $f(x)$ on the points x_i, then

$$\operatorname*{Max}_{x\in[-1,1]} |f(x) - p_n(x)| \le (5 + \frac{2}{\pi}\log(n))\frac{c}{n^{p+\mu}} \qquad \text{(II.8.17)}$$

Proof: Let $p_n^*(x)$ be the best polynomial approximation of degree n in the uniform norm to the function f, ie.

$$E_n(f) \equiv \underset{x\in[-1,1]}{\text{Max}} |f(x) - p_n^*(x)| \le \underset{x\in[-1,1]}{\text{Max}} |f(x) - q_n(x)|$$

for any polynomial q_n of degree n, then

$$\underset{x\in[-1,1]}{\text{Max}} |f(x) - p_n(x)| \le \underset{x\in[-1,1]}{\text{Max}} |f(x) - p_n^*(x)|$$

$$+ \underset{x\in[-1,1]}{\text{Max}} |p_n^*(x) - p_n(x)|. \qquad \text{(II.8.18)}$$

Notice now that

$$p_n^*(x) = \sum_{j=0}^{n} L_j(x) p_n^*(x_j)$$

$$p_n(x) = \sum_{j=0}^{n} L_j(x) f(x_j)$$

where $L_j(x)$ are the Lagrange polynomials defined by

$$L_j(x) = \frac{(x - x_1)(x-x_2) \;\ldots\; (x-x_n)}{x-x_j}.$$

From (II.8.18) and the above expressions for $p_n^*(x)$ and $p_n(x)$ we obtain

$$\underset{x\in[-1,1]}{\text{Max}} |f(x) - p_n(x)| \le E_n(f)$$

$$+ \sum_{j=0}^{n} \underset{x\in[-1,1]}{\text{Max}} |L_j(x)|\,|f(x_j) - p_n^*(x_j)|$$

$$\le \left[1 + \sum_{j=0}^{n} \underset{x\in[-1,1]}{\text{Max}} |L_j(x)|\right] E_n(f)$$

$$= (1 + \Omega_n) E_n(f). \qquad \text{(II.8.19)}$$

The constants Ω_n are called the Lebesgue constants and can be shown (see Rivlin [1969 pgs 88-93]) to satisfy

$$\Omega_n \leq 4 + \frac{2}{\pi}\log(n).$$

As for $E_n(f)$ we have from Jackson's theorem (see Cheney [1966 pg 147]) that

$$E_n(f) \leq \frac{\text{const}}{(n+1)(n)\ \ldots\ (n-p+2)} W_p\left[\frac{1}{n-p}\right]$$

where

$$W_p\left[\frac{1}{n-p}\right] = \sup_{|x_1-x_2| \leq \frac{1}{n-p}}\left\{|f^{(p)}(x_1) - f^{(p)}(x_2)|\right\}.$$

Using the Hölder condition of $f^{(p)}$ and taking $n > p$ we obtain from (II.8.19) the required result (II.8.17).■

Remark:

The notation introduced in (II.8.16) will be used throughout this thesis to denote the zeros of first kind Chebyshev polynomial of degree n+1. When no confusion arises we shall drop the superscript n.

We now prove a similar result for the function of two variables $k(x,t)$. First let us introduce the polynomial which interpolates $k(x,t)$ in the x-variable as

$$k_n(x,t) = \sum_{j=0}^{n} \phi_j(x)\psi_j(t) \tag{II.8.20}$$

where the sequence of polynomials $\{\phi_j\}$ satisfies the Haar conditions on the interval $[-1,1]$ and the functions ψ_j are defined by

$$\sum_{j=0}^{n} \phi_j(x_i)\psi_j(t) = k(x_i,t), \quad i = 0,1,\ldots,n.$$

Notice that the Ψ_j are well defined because $\det\{\phi_j(x_i)\} \neq 0$ and so we can always solve for Ψ_j (see Cheney [1966 pgs 74-77].

Lemma II.8.3

Let $k(x,t) \in C^q[-1,1]$ in the x-variable, with $\frac{\partial^q k(x,t)}{\partial x^q} \in H^\sigma[-1,1]$ uniformly in the t variable, that is,

$$\left|\frac{\partial^q k(x_1,t)}{\partial x^q} - \frac{\partial k(x_2,t)}{\partial x^q}\right| \leq M|x_1 - x_2|^\sigma \tag{II.8.21}$$

where the constants M and σ are independent of t.

Then the function $k_n(x,t)$ defined by (II.8.20) satisfies

$$\underset{(x,t)\in[-1,1]}{\mathrm{Max}} |k(x,t) - k_n(x,t)| \leq \frac{c\log(n)}{n^{q+\sigma}}, \text{ as } n \to \infty. \tag{II.8.22}$$

Proof: As in lemma II.8.2 above, for $t \in [-1,1]$ and $n > q$ we have that

$$\underset{x\in[-1,1]}{\mathrm{Max}} |k(x,t) - k_n(x,t)| \leq (5 + \frac{2}{\pi}\log(n))\frac{c}{n^q}W_q\left[\frac{1}{n - q}\right] \tag{II.8.23}$$

where

$$W_q\left[\frac{1}{n - q}\right] = \underset{|x_1-x_2| \leq \frac{1}{n - q}}{\mathrm{Sup}}\left\{\left|\frac{\partial^q k(x_1,t)}{\partial x^q} - \frac{\partial k(x_2,t)}{\partial x^q}\right|\right\}.$$

From (II.8.21) we have

$$W_q\left[\frac{1}{n - q}\right] \leq M|x_1 - x_2|^\sigma \leq \frac{1}{(n - q)^\sigma}.$$

Substituting this result into (II.8.23) and recalling that M is independent of t we obtain (II.8.22).■

CHAPTER III

Uniform Convergence of a Collocation Method for the Numerical Solution of SIE with Constant Coefficients

1. Introduction

In several recent papers various authors have studied the numerical solution of singular integral equations with constant coefficients. Among the methods used to solve such singular integral equations are the direct weighted Galerkin method see Erdogan [1969], and the polynomial collocation method see Golberg [1984]. L_2 convergence of the direct weighted Galerkin method has been established by Dzhishkariani [1980] and Golberg [1983] and uniform convergence by Ioakimidis [1981,1983], Linz [1977], Venturino [1985].

The L_2 convergence of polynomial collocation method has been established in a series of papers by Golberg [1984,1979] and Fromme [1981]. Uniform convergence of collocation methods have been considered by Miel [1986] and Junghanns [1984]. It is the main purpose of the work presented in this chapter to demonstrate the uniform convergence of the collocation method and give its convergence rates.

In this chapter we adapt the general theory developed in chapter II to the particular case we are interested in here. We introduce some important properties of the Jacobi orthogonal polynomials, describe the polynomial collocation method we are going to work with, prove the uniform convergence of this method and in the last section we present some numerical experiments which produce results in good agreement with the theoretical predictions.

All the results presented in this chapter will be generalized in chapter IV for SIE with variable coefficients. The reason for presenting the constant coefficient case separately is that this class of equations has an importance of its own; notice that all the examples in chapter I are of SIE with constant coefficients. Also, the results concerning the uniform convergence of the collocation method for SIE with constant coefficients were first proved and later generalized to this wider class.

2. The Integral Equation

The singular integral equation to be considered here has the form

$$a\phi(x) + \frac{b}{\pi}\int_{-1}^{1}\frac{\phi(t)}{t-x}dt + \int_{-1}^{1}k(x,t)\phi(t)dt = f(x) \quad -1<x<1. \qquad \text{(III.2.1)}$$

The kernel $k(x,t)$ and the right-hand side $f(x)$ are assumed to be Hölder continuous and $a^2 + b^2 = 1$.

From the results in chapter II we have that

$$G(x) = \frac{a - ib}{a + ib}$$

which is constant and then

$$\Gamma(x) = \frac{1}{2\pi i}\log\left[\frac{a - ib}{a + ib}\right]\int_{-1}^{1}\frac{dt}{t - x} = \frac{1}{2\pi i}\log\left[\frac{a - ib}{a + ib}\right]\log\left[\frac{1 - x}{1 + x}\right].$$

Hence

$$e^{\Gamma(x)} = (1-x)^{\gamma}(1+x)^{-\gamma}$$

where

$$\gamma = \frac{1}{2\pi i}\log\left[\frac{a - ib}{a + ib}\right].$$

According to the theory in chapter II we have to find integers M and N such that

$$\begin{aligned} -1 < \gamma + M < 1 \\ -1 < -\gamma + N < 1; \end{aligned} \qquad \text{(III.2.2)}$$

thus

$$Z(x) = (1-x)^{M}(1+x)^{N}(1-x)^{\gamma}(1+x)^{-\gamma} = (1-x)^{\alpha}(1+x)^{\beta}$$

where

$$\alpha = \gamma + M$$

$$\beta = -\gamma + N$$

and γ is defined as above.

The index is then given by (II.5.8):

$$\kappa = -(M+N) = -(\alpha+\beta).$$

The conditions (III.2.2) on M and N restrict the value of the index to -1, 0, 1. In the case $\kappa = 1$ the solution is in the class h_0 and is unbounded at both ends +1 and -1. It can be seen from (II.6.11) that in this case (III.2.1) has infinitely many solutions. To obtain uniqueness the integral equation must be supplemented by a side condition usually of the form

$$\int_{-1}^{1} \phi(x)dx = \text{const.} \qquad \text{(III.2.3)}$$

For $\kappa = 0$, (III.2.1) possesses a unique solution.

For $\kappa = -1$, the solution is in the class h_m and is bounded at both ends. From (II.6.13) the solution must satisfy the consistency condition,

$$\int_{-1}^{1} \frac{1}{Z(x)}\left[a\phi(x) + \frac{b}{\pi}\int_{-1}^{1} \frac{\phi(t)}{t - x}dt\right]dx = 0$$

if it is to exist.

It will be shown in chapter IV that the solution $\phi(x)$ can be decomposed as $\phi(x) = Z(x)g(x)$ with $g(x)$ Hölder continuous. As the function $Z(x)$ will play the role of a weight function from now on we shall write $w(x)$ in place of $Z(x)$.

In terms of the new unknown $g(x)$ (III.2.1) can be written as

$$aw(x)g(x) + \frac{b}{\pi}\int_{-1}^{1} \frac{w(t)g(t)}{t - x}dt + \int_{-1}^{1} k(x,t)w(t)g(t)dt = f(x) \tag{III.2.4}$$

or in operator form

$$Hg + Kg = f \tag{III.2.5}$$

where

$$Hg = aw(x)g(x) + \frac{b}{\pi}\int_{-1}^{1} \frac{w(t)g(t)}{t - x}dt$$

and

$$Kg = \int_{-1}^{1} k(x,t)w(t)g(t).$$

Defining the operator H^I by

$$H^{I}f = aw^{*}(x)f(x) - \frac{b}{\pi}\int_{-1}^{1}\frac{w*(t)f(t)}{t-x}dt \qquad \text{(III.2.6)}$$

where $w^{*}(x) = \frac{1}{w(x)}$, it will also be shown in chapter IV that

$$\begin{aligned} H^{I}Hg &= g - g_0, \text{ if } \kappa = 1 \\ H^{I}Hg &= g, \quad \text{ if } \kappa \leq 0 \end{aligned} \qquad \text{(III.2.7)}$$

for all Hölder-continuous g, with $g_0 \in Ker(H)$ a constant.

Pre-multiplying equation (III.2.7) by H^{I} we obtain

$$\begin{aligned} g + H^{I}Kg &= H^{I}f + g_0 \quad \text{if } \kappa = 1 \\ g + H^{I}Kg &= H^{I}f \quad \text{if } \kappa \leq 0. \end{aligned} \qquad \text{(III.2.8)}$$

On the assumption that the function k(x,t) is Hölder continuous, it will be shown in chapter IV that $H^{I}K$ is a Fredholm operator, so that (III.2.8) are Fredholm equations of the second kind, and have a unique solution , for each fixed g_0, provided -1 is not an eigenvalue of $H^{I}K$, a condition which will be assumed throughout this chapter.

Notice that equations (III.2.8) can be written in the compact form

$$g + H^{I}Kg = H^{I}f + g_0\delta_{1,\kappa} \qquad \text{(III.2.9)}$$

where δ_{ij} is the Kronecker's symbol, given by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

This form of the integral equation will be extensively used in all the analysis which follows.

Notice also that from (III.2.9) and the fact that $(I + H^I K)$ is invertible we can rewrite g in the form

$$g = (I + H^I K)^{-1}(H^I f + g_o \delta_{1\kappa}).$$

In the case $\kappa = 1$, multiplying the above expression by $w(x)$, integrating over $[-1,1]$ and making use of (III.2.3) gives

$$\text{const} = \int_{-1}^{1} w(x)(I + H^I K)^{-1} H^I f(x)dx + \int_{-1}^{1} (I + H^I K)^{-1} g_o(x)dx.$$

As g_o is a constant we can write

$$\text{const} = \int_{-1}^{1} w(x)(I + H^I K)^{-1} H^I f(x)dx + g_o \int_{-1}^{1} (I + H^I K)^{-1} 1(x)dx.$$

Therefore, in order that g_o can be uniquely determined from condition (III.2.3) we need

$$\int_{-1}^{1} (I + H^I K)^{-1} 1(x)dx \neq 0 \qquad \text{(III.2.10)}$$

where 1 denotes the constant function $f(x) \equiv 1$.

Condition (III.2.10) will be of crucial importance in the proof of the results in section 5 and we shall assume it holds throughout.

3. Numerical Methods for the Solution of SIE

Before presenting the collocation method for the general problem we introduce a simple illustrative example and discuss briefly the various methods available for its numerical solution. Consider the SIE

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi(x)}{t-x}dt + \int_{-1}^{1}k(x,t)\phi(t)dt = f(x), \quad -1<x<1 \qquad \text{(III.3.1)}$$

where k and f are Hölder continuous functions.

This equation is simply (III.2.1) with $a \equiv 0$ and $b \equiv 1$. This gives

$$\gamma = \frac{1}{2\pi i}\log(-1)$$

the value of which is dependent on the choice of the branch of the logarithm. If we choose $\log(-1) = \pi i$ then $\gamma = \frac{1}{2}$ and if we want the solution to be unbounded at both ends we must have

$$-1 < \alpha = \tfrac{1}{2} + M < 0$$

$$-1 < \beta = -\tfrac{1}{2} + N < 0$$

giving $M = -1$ and $N = 0$. Therefore if $\alpha = \beta = -\frac{1}{2}$ then the index κ is 1 and the function $w(x)$ defined in section 2 is given by

$$w(x) = \frac{1}{\sqrt{1-x^2}}.$$

As the index is 1 then according to (III.2.3) we need a side condition for uniqueness, and we take this to be

$$\int_{-1}^{1} \phi(x)dx = 0. \qquad \text{(III.3.2)}$$

It can easily be seen that formulae (III.4.1) and (III.4.2) of the next section are reduced to the two remarkable equations

$$\frac{1}{\pi}\int_{-1}^{1} \frac{T_j(t)}{\sqrt{1 - t^2}(t - x)}dt = \begin{cases} 0 & j = 0 \\ U_{j-1}(x) & j \geq 1 \end{cases} \qquad \text{(III.3.2)}$$

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1 - t^2}U_j(t)}{t - x}dt = -T_{j+1}(x) \qquad \text{(III.3.3)}$$

where T_j and U_j are the Chebyshev polynomials of first and second kind respectively, defined by

$$T_j(x) = \cos(j\arccos(x))$$

$$U_j(x) = \frac{\sin((j+1)\arccos(x))}{\sin(\arccos(x))}.$$

From (III.2.4) the integral equation (III.3.1) can be written in the form

$$\frac{1}{\pi}\int_{-1}^{1} \frac{g(t)}{\sqrt{1 - t^2}(t - x)}dt + \int_{-1}^{1} \frac{k(x,t)}{\sqrt{1 - t^2}}g(t)\, dt = f(x) \qquad \text{(III.3.4)}$$

where

$$\phi(x) = \frac{g(x)}{\sqrt{1 - x^2}}.$$

The main difficulty in solving equation (III.3.4) lies in approximating the first integral, due to the strong singularity at $t = x$.

If, however we approximate $g(x)$ by $g_n(x)$ in the form

$$g_n(x) = a_o T_o(x) + a_1 T_1(x) + \dots + a_n T_n(x)$$

we see, on substituting the expression for g_n into equation (III.3.4), by virtue of formulae (III.3.2) that the singular integrals are evaluated exactly. The following equation is obtained by the procedure described above

$$\sum_{j=1}^{n} a_j U_{j-1}(x) + \sum_{j=0}^{n} a_j \int_{-1}^{1} \frac{k(x,t)T_j(t)}{\sqrt{1-t^2}} dt = f(x). \qquad \text{(III.3.5)}$$

Equation (III.3.5) is the basis of all current global methods.

The Galerkin method consists of multiplying (III.3.5) by $\sqrt{1-x^2}U_i(x)$, $i = 0, 1,\dots,n-1$ and integrating over $[-1,1]$ to obtain a linear system of equations for the unknowns a_o, $a_1,\dots,a_n$. This system of equations is represented in the form

$$A\mathfrak{a} = b \qquad \text{(III.3.6)}$$

where the matrix A is given by

$$A_{ij} = \int_{-1}^{1} \sqrt{1-x^2}U_i(x) \int_{-1}^{1} \frac{k(x,t)T_j(t)}{\sqrt{1-t^2}} dt + \begin{cases} \frac{\pi}{2} & \text{if } i = j-1 \\ 0 & \text{if } i \neq j-1 \end{cases}$$

$\mathfrak{a} = (a_o, a_1,\dots,a_n)$ and b is a vector with components

$$b_i = \int_{-1}^{1} f(x)\sqrt{1-x^2}U_i(x)dx.$$

It is seen that (III.3.6) is a system of n equations in n+1 unknowns and condition (II.3.2) on ϕ must be used to obtain the extra equation needed.

The operator

$$Hg = \frac{1}{\pi}\int_{-1}^{1}\frac{g(t)}{\sqrt{1-t^2}(t-x)}dt$$

can be shown to be a bounded invertible operator from the space $L_{2w}[-1,1] \rightarrow L_{2w*}[-1,1]$ where

$$L_{2w} = \left\{f \text{ such that } \int_{-1}^{1} w(x)f^2(x)dx < \infty\right\}$$

and $w^*(x) = \frac{1}{w(x)}$ (see Golberg [1984]).
Having the above result in mind it is seen that the choice of the space spanned by the Chebyshev polynomials of the second kind and weight $w^*(x)$ as the test space for the Galerkin method in (III.3.5) is a natural one. Indeed, the convergence of the approximation g_n to the true solution g(x) in the weighted norm, that is, $\|g(x) - g_n(x)\|_{L_{2w}} \rightarrow 0$ as $n \rightarrow \infty$, can be proved, see Ioakimidis [1981,1982], Linz [1977], Elliott [1983], Dzhishkariani [1980] and Golberg [1983a,1983b] for instance.

The polynomial collocation method, on the other hand, consists of choosing n distinct points $y_1, y_2, \ldots, y_n$ on [-1,1] and determining the coefficients $a_o, a_1, \ldots, a_n$ such that

$$\sum_{j=1}^{n} a_j U_{j-1}(y_i) + \sum_{j=0}^{n}\int_{-1}^{1}\frac{k(y_i,t)T_j(t)}{\sqrt{1-x^2}}dt = f(y_i) \quad i = 1,\ldots,n.$$

A common choice for the collocation points is at the zeros of $U_n(x)$ given by the formula

$$y_i = \cos(\frac{i\pi}{n+1}), \quad i = 1,\dots,n.$$

Making use of the Erdös-Turan theorem, see Szegö [1959] it is possible to prove convergence in the weighted spaces for this particular choice of the collocation points, see Golberg [1979,1981,1984] and Dzhishkariani [1981].

Other choices of the collocation points are, of course, possible and we shall present a case for the choice of the zeros of T_n as the collocation points in the next section of this chapter.

A different approach is taken by the authors Elliott [1982a,1984] and Srivastav [1983a,1983b] who consider quadrature methods and collocation to solve equation (III.3.4).

Another technique for solving SIE is the spline approximation of g(x) and then the use of either Galerkin or collocation. The main problem with this approach is that, generally the Cauchy integrals involved are very hard to evaluate exactly and too costly to evaluate numerically. However, there has been considerable interest in those methods, and substantial progress has been made recently by such authors as, Gerasoulis [1980,1986], Gerasoulis & Srivastav [1981], Srivastav & Jen [1981] and Miller & Keer [1985]. The L_2 convergence of the Galerkin method with splines has only been proved recently for the case of the index $\kappa = 0$, see Thomas [1981] and Elschner [1984].

4. The Collocation Method

Before introducing the polynomial collocation method we present some standard properties of the Jacobi polynomials $P_i^{(\alpha,\beta)}(x)$ of degree i, where α and β have been defined in section 2.

Let $\phi_i = P_i^{(\alpha,\beta)}(x)$ and $\phi_i^* = P_i^{(-\alpha,-\beta)}$, so that $\{\phi_i\}$ and $\{\phi_i^*\}$ form orthogonal sequences with respect to the weights w(x) and $w^*(x)$ respectively. We have the following relationships between the Jacobi polynomials, see Krenk [1975 pg. 226].

Lemma III.4.1

$$aw(x)\phi_i(x) + \frac{b}{\pi}\int_{-1}^{1}\frac{w(t)\phi_i(t)}{t-x}dt = \frac{-b2^{-\kappa}}{\sin\pi\alpha}\phi^*_{i-\kappa}(x) \qquad (III.4.1)$$

$$aw^*(x)\phi_i^*(x) - \frac{b}{\pi}\int_{-1}^{1}\frac{w^*(t)\phi_i^*(t)}{t-x}dt = \frac{-b2^{\kappa}}{\sin\pi\alpha}\phi_{i+\kappa}(x) \qquad (III.4.2)$$

where $\phi_i \equiv \phi_i^* \equiv 0$ for $i < 0$, and κ is the index of (III.2.1).

Proof: From Szegö [1959 pg. 75] we have the formula

$$-\tfrac{1}{2}(z-1)^{-\alpha}(1+z)^{-\beta}\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}\frac{\phi(t)}{t-z}dt = 2^{n+\alpha+\beta}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)}\times$$

$$\times(z-1)^{-n-\alpha-1}(z+1)^{-\beta}F\left[n+\alpha+1,n+1,2n+\alpha+\beta+2,\frac{2}{1-z}\right] \qquad (III.4.3)$$

where $\Gamma(x)$ is the gamma function and F(a,b,c,z) is the hypergeometric function defined, for example in Abramowitz & Stegun [1964 pgs. 255,556]. From Abramowitz & Stegun

[1964 pg. 559], we have

$$F\left[n+\alpha+1, n+1, 2n+\alpha+\beta+2, \frac{2}{1-z}\right]$$

$$= \frac{\Gamma(2n+\alpha+\beta+2)}{\Gamma(n+1)\Gamma(n+\beta+1)}\Gamma(-\alpha)\left[\frac{-2}{1-z}\right]^{-n-\alpha-1}F\left[n+\alpha+1, -(n+\beta), \alpha+1, \frac{1-z}{2}\right]$$

$$+ \frac{\Gamma(2n+\alpha+\beta+2)\Gamma(\alpha)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}\left[\frac{-2}{1-z}\right]^{-n-1}F\left[n+1, -n-\alpha-\beta, 1-\alpha, \frac{1-z}{2}\right]. \qquad \text{(III.4.4)}$$

Multiplying (III.4.3) by $-2(z-1)^{\alpha}(1+z)^{\beta}$ and making use of (III.4.4) allows us to write

$$\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}\frac{\phi_n(t)}{t-z}dt$$

$$= -\frac{2^{\beta}\Gamma(-\alpha)\Gamma(n+\alpha+1)}{\Gamma(n+1)}(z-1)^{\alpha}F\left[n+\alpha+1, -(n+\beta), \alpha+1, \frac{1-z}{2}\right]$$

$$- 2^{\alpha+\beta}\frac{\Gamma(\alpha)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}F\left[n+1, -n-\alpha-\beta, 1-\alpha, \frac{1-z}{2}\right]. \qquad \text{(III.4.5)}$$

Using relations 15.3.5 of Abramowitz & Stegun [1964 pg. 559] we can deduce

$$F\left[n+\alpha+1, -(n+\beta), \alpha+1, \frac{1-z}{2}\right] = \left[\frac{1+z}{2}\right]^{n+\beta}F\left[-(n+\beta), -n, \alpha+1, \frac{z-1}{z+1}\right]$$

(III.4.6)

$$F\left[n+1, -(n+\alpha+\beta), 1-\alpha, \frac{1-z}{2}\right] = \left[\frac{1+z}{2}\right]^{n+\beta}F\left[-(n+\alpha+\beta), -(n+\alpha), 1-\alpha, \frac{z-1}{z+1}\right]$$

By substituting (III.4.6) into (III.4.5) and by making use of formulae in Abramowitz & Stegun [1964 pg. 779], (III.4.5) gives

$$\Psi(x+i0^{+}) = \frac{\pi e^{\alpha\pi i}}{\sin\pi\alpha}(1-x)^{\alpha}(1+x)^{\beta}\phi_n(x)$$

$$- 2^{\alpha+\beta} \frac{\Gamma(\alpha)\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+1)} F(n+1,-n-\alpha-\beta,1-\alpha,\frac{1-x}{2}\Big] \qquad (III.4.7)$$

where

$$\Psi(z) = \int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta}\frac{\phi_n(t)}{t - z}dt$$

and use has been made of the well-known relation $\Gamma(\alpha)\Gamma(\alpha-1) = \pi\text{cosec}(\alpha\pi)$.

From (III.4.3) and formulae 15.3.4 of Abramowitz & Stegun [1964 pg 559] we have

$$\Psi(z) = 2^{n+\alpha+\beta} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)}(z-1)^{-n-1}F\left[n+1,n+\alpha+1,2n+\alpha+\beta+2,\frac{2}{1-z}\right]$$

$$= 2^{n+\alpha+\beta} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)}(z-1)^{-n-1}F\left[n+1,n+\beta+1,2n+\alpha+\beta+2,\frac{2}{1+z}\right].$$

(III.4.8)

Writing $\Psi(x+i0^+) = u(x) + iv(x)$ and because $0 < \frac{2}{1 - x} < 1$ for $x < -1$, and $0 < \frac{2}{1 + x} < 1$ for $x > 1$ it follows from formulae (III.4.8) that $v(x) \equiv 0$ for $x \notin [-1,1]$.

It also follows from the definition of $\Psi(z)$ that $\Psi(z) = \frac{1}{|z|}$ for large z and from Szegö [1959 pgs. 76-77], that $\Psi(z) \sim (z-1)^{\alpha}(z+1)^{\beta}$ for $z \to \pm 1$. So that the real and imaginary parts of $\Psi(x+i0^+)$ belong to the class L_p, $p \geq 1$.

From Hilbert's reciprocity formulae (II.4.2) we obtain

$$\frac{1}{\pi}\int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta}\frac{\phi_n(t)}{t - x}dt$$

$$= \cot(\alpha\pi)w(x)\phi_n(x) - 2^{\alpha+\beta}\frac{\Gamma(\alpha)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} F\left[n+1, -n-\alpha-\beta, 1-\alpha, \frac{1-x}{2}\right]. \tag{III.4.9}$$

By making use of (III.4.6), from the formulae on Abramowitz & Stegun [1964 pg. 779] and recalling that $\alpha+\beta = -\kappa$, (III.4.9) becomes

$$\frac{1}{\pi}\int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta}\frac{\phi_n(t)}{t-x}dt = \cot(\alpha\pi)w(x)\phi_n(x) - \frac{2^{\kappa}}{\sin(\pi\alpha)}\phi^{*}_{n-\kappa}(x). \tag{III.4.10}$$

It remains to show that $\cot(\alpha\pi) = -a/b$. Writing $a - ib = \rho\exp(i\theta)$, then

$$\alpha = \frac{1}{2\pi}\arg(\exp(2i\theta)) + M = \frac{\theta}{\pi} + M.$$

From the definition of θ we have

$$\tan^{-1}(-b/a) = \theta = (\alpha-M)\pi$$

which gives the final result as

$$\cot(\alpha\pi) = -a/b. \tag{III.4.11}$$

Substituting (III.4.11) into (III.4.10) we obtain (III.4.1).

To demonstrate (III.4.2) we observe that by changing the sign of α in (III.4.11) we change the sign of either a or b, say b, and from $\kappa = -(\alpha + \beta)$, we change the sign of κ, so that by rewriting equation (III.4.1) with α, β replaced by $-\alpha$, $-\beta$ respectively we obtain (III.4.2).

Finally, it still remains to show that for $i = 0$, and

$\kappa = 1$, (III.4.1) is reduced to

$$aw(x)\phi_o(x) + \frac{b}{\pi}\int_{-1}^{1}\frac{w(t)\phi_o(t)}{t-x}dt = 0.$$

This is a direct consequence of the special case discussed in Szegö [1959 pg. 75].■

Consider now approximating $g(x)$ by $g_n(x)$, where

$$g_n(x) = \sum_{j=o}^{n} a_j\phi_j(x)$$

and a_j, $j = 0,1,\ldots,n$, are unknown constants. In order to determine these we form the residual

$$r_n(x) = (Hg_n + Kg_n - f)(x)$$

$$= aw(x)g_n(x) + \frac{b}{\pi}\int_{-1}^{1}\frac{w(t)g_n(t)}{t-x}dt + \int_{-1}^{1} w(t)k(x,t)g_n(t)dt - f(x).$$

By making use of (III.4.1) we have

$$r_n(x) = \sum_{j=0}^{n} a_j\left[\frac{-b2^{-\kappa}}{\sin\pi\alpha}\phi^*_{j-\kappa}(x) + \int_{-1}^{1} w(t)k(x,t)\phi_j(t)dt\right] - f(x). \qquad \text{(III.4.12)}$$

If we let $r_n(x_m) = 0$, $m = 0,1,\ldots,n-\kappa$, where $x_m \in [-1,1]$ are the zeros of the first kind Chebyshev polynomials of degree $n - \kappa + 1$, then (III.4.12) gives rise to a system of linear equations. In what follows we shall restrict attention to the cases $\kappa = 0$ and $\kappa = 1$. For $\kappa = 1$, this system will have n equations in the n+1 unknowns $a_0, a_1, \ldots, a_n$ so that it must be augmented by another equation, which is given by letting the approximate solution $g_n(x)$ satisfy (III.2.3). For $\kappa = 0$ we have n+1 equations in n+1 unknowns and we shall show that the system can be solved.

Writing $\ell = n - \kappa$, define the projection operator P_ℓ by

$$P_\ell : C[-1,1] \to C[-1,1]$$

$$P_\ell f = \sum_{m=0}^{\ell} f(x_m) L_m(x),$$

where $\{x_m\} = \{x_m^{(\ell)}\}$ are the zeros of the first kind Chebyshev polynomial of degree ℓ and $L_m(x)$ are the Lagrange polynomials satisfying $L_m(x_k) = \delta_{mk}$.

Lemma III.4.2

$$r_n(x_m) = 0, \; m = 0,1,\ldots,\ell, \text{ if and only if } P_\ell r_n \equiv 0.$$

Proof: Assume first that $r_n(x_m) = 0$. From the definition of P_ℓ we deduce

$$P_\ell r_n(x) = \sum_{m=0}^{\ell} r_n(x_m) L_m(x) \equiv 0.$$

Conversely, from $P_\ell r_n \equiv 0$, we can say that $P_\ell r_n(x_k) = 0$ for all k, but

$$P_\ell r_n(x_k) = \sum_{m=0}^{\ell} r_n(x_m) L_m(x_k) = r_n(x_k).$$

Hence $r_n(x_k) = 0$, $k=1,\ldots,\ell$. ∎

Taking into account the result of lemma III.4.2 the system of equations $r_n(x_m) = 0$ can be rewritten as

$$P_\ell H g_n + P_\ell K g_n = P_\ell f. \tag{III.4.13}$$

Notice from (III.4.1) that Hg_n is a polynomial of degree at most $n - \kappa$, so that

$$P_\ell H g_n = H g_n$$

and then (III.4.13) becomes

$$H g_n + P_\ell K g_n = P_\ell f. \tag{III.4.14}$$

<u>Lemma III.4.3</u>

Let $k(x,t)$ be a Hölder continuous function and $\forall t$ let $k_n(x,t)$ be the interpolating polynomial to the function $k(x,t)$ in the x-variable on the set of points $y_0, y_1, \ldots, y_n$. Let also the operators K and K_n be defined by

$$Kf(x) = \int_{-1}^{1} k(x,t) f(t) dt$$

$$K_n f(x) = \int_{-1}^{1} k_n(x,t) f(t) dt \tag{III.4.15}$$

then

$$P_n K = K_n. \tag{III.4.16}$$

Proof: From (II.8.20) we have that

$$k_n(x,t) = \sum_{j=0}^{n} \phi_j(x) \psi_j(t) \tag{III.4.17}$$

where $\{\phi_j\}$ is a basis for the set of polynomials of degree n satisfying the Haar condition and the functions ψ_j are so chosen that $k_n(y_i, t) = k(y_i, t)$, for $i = 0, 1, \ldots, n$.

Substituting (III.4.17) into (III.4.15) we obtain for $x = y_i$ and arbitrary function f

$$K_n f(y_i) = \int_{-1}^{1} k(y_i, t) f(t) dt = Kf(y_i). \qquad \text{(III.4.18)}$$

From (III.4.15) and (III.4.17) $K_n f$ is a polynomial of degree n and from (III.4.18) it interpolates the function $Kf(x)$. As $P_n Kf(x)$ is by definition the interpolating polynomial of $Kf(x)$, from the uniqueness of the interpolating polynomial they must coincide; hence

$$P_n Kf = K_n f.$$

As f is arbitrary we finally obtain

$$P_n K = K_n. \blacksquare$$

Before stating and proving the main result in this chapter we need the following lemma on the boundedness of the inverse of a linear operator.

<u>Lemma III.4.4</u>

Let X and Y be Banach spaces, and $T, S: X \to Y$ bounded linear operators such that

1) S^{-1} exists as a bounded linear operator from Y onto X

2) $\|T - S\| \, \|S^{-1}\| < 1$.

Then T^{-1} exists as a bounded linear operator from Y onto X and

$$\|T^{-1}\| \leq \frac{\|S^{-1}\|}{1 - \|S^{-1}\| \, \|T - S\|}. \qquad \text{(III.4.19)}$$

For the proof of this lemma see Atkinson [1976 pg.15].

5. Convergence

First of all we shall restrict attention to the case $\kappa = 0$ and we prove the following theorem.

Theorem III.5.1

Let $f \in C^p[-1,1]$ with $f^{(p)} \in H^{\mu}[-1,1]$ $p \geq 0$, and let $k(x,t)$ be a Hölder continuous function such that, $k(x,t) \in C^q[-1,1]$ with respect to the variable x and $\frac{\partial^q k(x,t)}{\partial x^q} \in H^{\sigma}[-1,1]$, $q \geq 0$, uniformly in the variable t. Let also $g(x)$ be the solution of the following singular integral equation

$$Hg + Kg = f. \qquad \text{(III.5.1)}$$

Assume that the index κ of (III.5.1) is zero and

$$\Gamma_2 \equiv \tfrac{1}{2}\text{Min}\{p+\mu, q+\sigma\} > \text{Max}\{0,\alpha,\beta\} \equiv \Gamma_1. \qquad \text{(III.5.2)}$$

Then for sufficiently large n the equation

$$Hg_n + P_n K g_n = P_n f \qquad \text{(III.5.3)}$$

has a unique solution and

$$\underset{x\in[-1,1]}{\mathrm{Max}} |g(x) - g_n(x)| \leq cn^{-r} \qquad \text{(III.5.4)}$$

where $r = \mathrm{Min}\{p+\mu, q+\sigma\} - 2\xi$, $\xi \in (\Gamma_1, \Gamma_2)$.

Proof: As the index is zero, the inverse of H exists and $H^{-1} = H^I$, so that equations (III.5.1) and (III.5.3) are equivalent to the equations:

$$g + H^I Kg = H^I f \qquad \text{(III.5.5)}$$

$$g_n + H^I K_n g_n = H^I f_n. \qquad \text{(III.5.6)}$$

Subtracting (III.5.6) from (III.5.5) yields

$$g-g_n + H^I K_n(g-g_n) = H^I(K_n-K)g + H^I(f-f_n) \qquad \text{(III.5.7)}$$

The operator $H^I K_n$ is a Fredholm integral operator, as we have already observed in section 2.

For all continuous functions h, with $\|h\|_\infty = 1$ we have, using (III.2.6),

$$\|H^I(K-K_n)h\|_\infty = \underset{x\in[-1,1]}{\mathrm{Max}} \left| aw^*(x)h_n(x) - \frac{b}{\pi}\int_{-1}^{1} \frac{w^*(t)h_n(t)}{t-x}dt \right| \qquad \text{(III.5.8)}$$

where

$$h_n(x) = \int_{-1}^{1} w(t)\Big[k(x,t) - k_n(x,t)\Big]h(t)dt.$$

Summing and subtracting $h_n(x)$ in the singular integral in

(III.5.8) we obtain

$$\|H^I(K-K_n)h\|_\infty = \max_{x\in[-1,1]} \left| aw^*(x)h_n(x) - \frac{b}{\pi}h_n(x)\int_{-1}^{1}\frac{w^*(t)}{t-x}dt \right.$$

$$\left. - \frac{b}{\pi}\int_{-1}^{1}\frac{w^*(t)}{t-x}\Big[h_n(t)-h_n(x)\Big]dt \right|. \qquad \text{(III.5.9)}$$

From (III.4.2) with i = 0 we have

$$aw^*(x)\phi_0^*(x) - \frac{b}{\pi}\int_{-1}^{1}\frac{w^*(t)\phi_o^*(t)}{t-x}dt = \frac{-b2^\kappa}{\sin\pi\alpha}\phi_\kappa(x).$$

But $\phi_0^*(x)$ is a constant (because it is a polynomial of degree 0), so that

$$aw^*(x) - \frac{b}{\pi}\int_{-1}^{1}\frac{w^*(t)}{t-x}dt = \frac{-b2^\kappa}{\sin\pi\alpha\ \ \phi_o^*(x)}\phi_\kappa(x). \qquad \text{(III.5.10)}$$

Also from lemmas II.8.3 and II.8.1, respectively, we have

$$\max_{x\in[-1,1]} |h_n(x)| \le \frac{c\ \log(n)}{n^{q+\sigma}} \qquad \text{(III.5.11)}$$

and

$$\left|\int_{-1}^{1}\frac{w^*(t)}{t-x}\Big[h_n(t) - h_n(x)\Big]dt\right| \le \frac{c\ \log(n)}{n^{q+\sigma-2\xi}} \qquad \text{(III.5.12)}$$

where $\xi \in (\Gamma_1,\Gamma_2)$ and use has been made of (III.5.2).

From (III.5.10) and (III.5.11) we have

$$\max_{x\in[-1,1]} \left| aw^*(x)h_n(x) - \frac{b}{\pi}h_n(x)\int_{-1}^{1}\frac{w^*(t)}{t-x}dt \right|$$

$$= \max_{x\in[-1,1]} |h_n(x)| \left| aw^*(x) - \frac{b}{\pi}\int_{-1}^{1} \frac{w^*(t)}{t-x}dt \right|$$

$$\leq \max_{x\in[-1,1]} |h_n(x)| \max_{x\in[-1,1]} \left| \frac{-b2^{\kappa}}{\sin\pi\alpha \;\phi_o^*(x)}\phi_\kappa(x) \right|$$

$$\leq \frac{c \log(n)}{n^{q+\sigma}} \qquad \text{(III.5.13)}$$

because $\dfrac{-b2^{\kappa}}{\sin\pi\alpha \;\phi_o^*(x)}\phi_\kappa(x)$ is bounded independently of n.

Substituting (III.5.13) and (III.5.12) into (III.5.9) we obtain

$$\|H^I(K - K_n)h\|_\infty \leq \frac{c \log(n)}{n^{q+\sigma}} + \frac{c' \log(n)}{n^{q+\sigma-2\xi}}$$

where c' is a constant independent of n.

Hence we have shown that given $\xi' \in (\Gamma_1, \Gamma_2)$, there exists an n_o such that for any $n \geq n_o$ we have

$$\|H^I(K-K_n)\|_\infty \leq \frac{c}{n^{q+\sigma-2\xi'}}. \qquad \text{(III.5.14)}$$

One important consequence of (III.5.14) is that $H^IK_n \to H^IK$, uniformly, as $n \to \infty$. Lemma III.4.4 applied with $T = I + H^IK_n$ and $S = I + H^IK$ implies that the operator $(I + H^IK_n)^{-1}$ exists and is uniformly bounded for n suficiently large; more precisely

$$\|(I + H^IK_n)^{-1}\|_\infty \leq \frac{\|(I + H^IK)^{-1}\|_\infty}{1 - \|(I + H^IK)^{-1}\|_\infty \|H^I(K - K_n)\|_\infty}$$

for n sufficiently large provided -1 is not an eigenvalue of H^IK.

This proves the existence of a unique solution for equation (III.5.6) and hence for equation (III.5.3).

Premultiplying equation (III.5.7) by $(I + H^I K_n)^{-1}$ we obtain the equation

$$g - g_n = (I + H^I K_n)^{-1} G_n \tag{III.5.15}$$

where

$$G_n = H^I (K_n - K) g + H^I (f - f_n).$$

From (III.5.15) on taking norm we obtain

$$\|g - g_n\|_\infty \le c \|G_n\|_\infty. \tag{III.5.16}$$

A bound will now be found for $\|G_n\|_\infty$. The operator $H^I(K-K_n)$ has already been shown to be bounded in norm (see (III.5.14)) and in a similar manner a bound can be constructed for $\|H^I(f-f_n)\|_\infty$, namely,

$$\|H^I(f-f_n)\|_\infty \le \frac{c}{n^{p+\mu-2\xi''}} \quad \xi'' \in (\Gamma_1, \Gamma_2) \tag{III.5.17}$$

And so, from (III.5.14), (III.5.17), and (III.5.16) we obtain the required result (III.5.4).■

We shall now discuss, from a practical point of view only, the case $\kappa = 1$, assuming that a solution of the discrete equation (III.5.3) exists.

Equation (III.5.7) has now an additional term on the right-hand side and can be expressed in the form

$$g - g_n + H^I K_n (g - g_n) = H^I (K_n - K) g + H^I (f - f_n) + t_n \tag{III.5.18}$$

where t_n is a constant (with respect to x) chosen such that

$$\int_{-1}^{1} w(x)(g - g_n)(x)dx = 0. \tag{III.5.19}$$

The existence of the operator $(I + H^I K_n)^{-1}$ can be proved as in theorem III.5.1 above, so that

$$g - g_n = (I + H^I K_n)^{-1}(G_n + t_n) \tag{III.5.20}$$

where G_n is as in (III.5.15) above.

Clearly the bounds (III.5.14) and (III.5.17) still hold, so that to obtain an error bound similar to that of theorem III.5.1 we only need to show that the sequence (t_n) satisfies

$$t_n = O(\|G_n\|_\infty) \text{ as } n \to \infty.$$

From (III.5.20) it can be seen that G_n can also be written in the form

$$G_n(x) = (I + H^I K_n)(g - g_n)(x) - t_n.$$

As already pointed out in chapter II $H^I K_n$ is a Fredholm integral operator and $g-g_n$ is Hölder continuous. Therefore G_n is continuous on $[-1,1]$ and hence bounded (as a function of x).

On multiplying (III.5.20) by w(x), integrating over $[-1,1]$ and using (III.5.19) we obtain

$$t_n = \frac{\int_{-1}^{1} w(x)(I + H^I K_n)^{-1} G_n(x)dx}{\int_{-1}^{1} w(x)(I + H^I K_n)^{-1} 1(x)dx} \tag{III.5.21}$$

or on taking modulus

$$|t_n| \leq \frac{A\|(I + H^I K_n)^{-1}\|_\infty \|G_n\|_\infty}{A_n} \tag{III.5.22}$$

where

$$A_n = \left|\int_{-1}^{1} w(x)(I + H^I K_n)^{-1} 1(x)dx\right| \text{ and } A = \int_{-1}^{1} w(x)dx.$$

Since $(I + H^I K_n)^{-1} \to (I + H^I K)^{-1}$ as $n \to \infty$ (III.5.22) yields

$$t_n = O(\|G_n\|_\infty) \quad \text{as } n \to \infty$$

as required.

To end the argument we need only take norm in (III.5.20) to obtain

$$\|g - g_n\|_\infty \leq c\|G_n\|_\infty$$

and use the bounds (III.5.14) and (III.5.17).

Notice that equation (III.5.21) implicitly assumes that the denominator does not vanish for n large. This follows from (III.2.10) by invoking the convergence of the operator $(I + H^I K_n)^{-1}$ to $(I + H^I K)^{-1}$ as $n \to \infty$. This proof of (III.5.22) is due to Linz [1977]. In that paper Linz demonstrates the validity of (III.5.22) for the particular class of equations with $a = 0$ and $b = 1$ and for $\kappa = 1$.

6. Numerical Examples

(i) Consider the equation

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\phi(t)}{t - x}dt = \frac{2}{\pi}\left[1 + \frac{x^2}{\sqrt{1 - x^2}}\log\left|\frac{\sqrt{1 - x^2} - x + 1}{\sqrt{1 - x^2} + x - 1}\right|\right] = f(x)$$

which has the exact solution

$$\phi(x) = \frac{x|x|}{\sqrt{1 - x^2}}$$

if the additional condition $\int_{-1}^{1} \phi(x)dx = 0$ is imposed.

This example is used to illustrate the rate of convergence of the method described previously. It is seen that the exact solution can be written in the form

$$\phi(x) = \frac{g(x)}{\sqrt{1 - x^2}}$$

where $g'(x)$ is Lipschitz-continuous. By considering a series expansion of the function $\log\left|\frac{1 + x}{1 - x}\right|$, $x^2 < 1$, it is possible to show that $f'(x)$ exists for $x \in [-1,1]$ and is Lipschitz continuous. Therefore, using the notation of theorem III.5.1 we have $p = 1$ and $\mu = 1$, and the discussion following this theorem predicts uniform convergence of order $r = 2 - \xi$ for $\xi > 0$. Table III.6.1 below shows an approximation to the order of convergence r, calculated from

$$r = -\log\left(\frac{e_{2n}}{e_n}\right)/\log(2)$$

where

$$e_n = \max_{x\in[-1,1]} |g(x) - g_n(x)|.$$

In tables III.6.1 and III.6.2 below the numbers under the heading "Chebyshev first kind" were obtained by applying the collocation method described in section 4 to the relevant equation using the zeros of the first kind Chebyshev polynomial of degree n as the collocation points. The numbers under the heading "Chebyshev second kind" were obtained similarly using the zeros of the second kind Chebyshev polynomial.

	Chebyshev first kind		Chebyshev second kind	
n	$\|g - g_n\|_\infty$	r	$\|g - g_n\|_\infty$	r
2	0.2669		0.1930	
		2.322		2.26
4	5.33E-02		4.02E-02	
		2.081		1.946
8	1.26E-02		1.04E-02	
		2.034		1.916
16	3.07E-03		2.76E-03	
		2.004		1.943
32	7.66E-04		7.19E-04	

Table III.6.1

(ii) We also consider the equation:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi(t)}{t - x}dt + \int_{-1}^{1}\frac{(t^2 - x^2)^2}{t^2 + x^2}\phi(t)dt = f(x)$$

with f(x) as in example (i).

The exact solution of this equation is the same as that of problem (i), because $K\phi = 0$. However, the numbers

$$b_{ij} = \int_{-1}^{1}\frac{k(x_i, t)\phi_j(t)}{\sqrt{1 - t^2}}dt,$$

with x_i the collocation points and $\phi_j(t)$ the Chebyshev polynomials of first kind, do not vanish; indeed, the b_{ij}'s are not necessarily even small. The numerical results for this example are shown on table III.6.2 below and they behave as predicted by theorem III.5.1.

The numbers b_{ij} were calculated by the Gauss quadrature formula

$$\int_{-1}^{1} \frac{h(t)}{\sqrt{1 - t^2}} dt = \frac{\pi}{N+1} \sum_{j=0}^{N} h(x_j^{(N)})$$

with N = 35 and with $x_j^{(N)}$ defined as in (II.8.16).

	Chebyshev first kind		Chebyshev second kind	
n	$\|g - g_n\|_\infty$	r	$\|g - g_n\|_\infty$	r
2	0.2978		0.2142	
		2.369		2.37
4	5.76E-02		4.14E-02	
		2.121		1.969
8	1.33E-02		1.05E-02	
		2.072		1.924
16	3.16E-03		2.78E-03	
		2.009		1.943
32	7.85E-04		7.25E-04	

Table III.6.2

CHAPTER IV

Uniform Convergence of a Collocation Method for the Numerical Solution of Singular Integral Equations with Variable Coefficients

1. Introduction

The numerical solution of SIE with variable coefficients has not been as widely studied as SIE with constant coefficients. The difficulty in treating the variable coefficients case, as far as the Galerkin method is concerned, lies in the fact that, unlike the constant coefficients case, the solution of a SIE using the Galerkin method is not equivalent to the solution of the regularized Fredholm equation. The consequences of this are far reaching: the generalization of the well-understood theory of Fredholm equations to the solution of SIE is not possible.

It was not until Elliott [1982] introduced a class of orthogonal polynomials associated with SIE that L_2 convergence of the Galerkin method could be proved, see Elliott [1983]. Elliott also described and demonstrated the uniform convergence of a quadrature and collocation method which he called the Classical Collocation Method for SIE with variable coefficients, see Elliott [1981,1982a,1982b,1984] also Dow & Elliott [1979].

As we have already pointed out our main goal in this chapter is to generalize the proof of convergence of the method described in chapter III; in doing so the main difficulty lies in proving a parallel result to that of lemma III.4.1

We shall also adapt the theory of SIE developed previously to the special case we are concerned with here, present a series of results related to orthogonal polynomials associated with the SIE and prove some outstanding results from chapter III. Finally we shall present numerical examples along the same line as those of chapter III.

2. The Integral Equation

In this chapter we are concerned with the integral equation

$$a(x)\phi(x) + \frac{b(x)}{\pi}\int_{-1}^{1} \frac{\phi(t)}{t - x}dt + \int_{-1}^{1} k(x,t)\phi(t)dt = f(x), \quad -1 < x < 1. \tag{IV.2.1}$$

where the functions $a(x)$, $b(x)$ the kernel $k(x,t)$ and the right-hand side $f(x)$ are assumed to be Hölder-continuous and real valued.

We shall assume that the function

$$r(x) = \sqrt{(a(x))^2 + (b(x))^2} > 0 \quad \forall x \in [-1,1].$$

From section 6 of chapter II we have that

$$G(x) = \frac{a(x) - ib(x)}{a(x) + ib(x)}$$

so that

$$\frac{1}{2\pi i}\log(G(x)) = \frac{i}{2\pi i}\arg[G(x)] = \frac{1}{2\pi}\Big[\arg[a(x) - ib(x)] - \arg[a(x) + ib(x)]\Big] = \frac{-1}{\pi}\arg[a(x) + ib(x)]. \tag{IV.2.2}$$

According to the theory in chapter II, before we can solve the integral equation (IV.2.1) we have to specify a class of functions in which the solution of the SIE is to lie. Throughout this chapter we shall work with the class h_o, that is we are seeking a solution which is unbounded at both ends x = -1 and x = 1.

Definition IV.2.1

Let θ be a real continuous function defined on [-1,1] such that θ(x) takes one of the values of the multi-valued function

$$\frac{1}{\pi}\arg[a(x) + ib(x)]$$

and such that

$$-1 < \theta(-1) \leq 0. \qquad \text{(IV.2.3)}$$

From this definition we can easily see that θ(x) must have the form

$$\theta(x) = \frac{1}{\pi}\arctan\left[\frac{b(x)}{a(x)}\right] + N(x) \qquad \text{(IV.2.4)}$$

where N(x) is a function which takes only integer values and may have discontinuities at the zeros of a(x)/b(x), and

$$\frac{-\pi}{2} < \arctan(x) \leq \frac{\pi}{2}.$$

Condition (IV.2.3) determines the branch of θ uniquely at x = -1. From (IV.2.2) and (IV.2.4) we see that

$$\frac{1}{2\pi i}\log(G(x)) = -\theta(x). \qquad \text{(IV.2.5)}$$

Therefore the function Γ(z) can be defined by

$$\Gamma(z) = -\!\!\!\!\!\!\int_{-1}^{1} \frac{\theta(t)}{t - z} dt$$

and as a consequence of (IV.2.3) the condition (II.5.6) is already satisfied at the end $x = -1$; so from (II.5.7)

$$\begin{aligned} X(z) &= (1 - z)^{-\kappa} e^{\Gamma(z)} \\ Z(x) &= (1 - x)^{-\kappa} \exp\left[-\!\!\!\!\!\!\int_{-1}^{1} \frac{\theta(t)}{t - x} dt\right] \end{aligned} \tag{IV.2.6}$$

where κ is the index of (IV.2.1) with respect to the class h_0.

The reason we can write $X(z)$ as above is that from (IV.2.3) we have $\lambda_1 = 0$ in (II.5.6) and therefore the definition of the index in (II.5.8) gives $-\kappa = \lambda_2$.

As a direct consequence of (IV.2.6) we see that

Remarks

1) $Z(x) = (1 - x)^{-(\kappa+\theta(x))}(1 + x)^{\theta(x)} \exp\left(-\int_{-1}^{1} \frac{\theta(t) - \theta(x)}{t - x} dt\right)$,

   and from the conditions on $\theta(-1)$ and $\theta(1)$ we can deduce that $Z(x)$ and $\frac{1}{Z(x)}$ are integrable on $[-1,1]$;
2) From lemma II.5.1 we also have that $Z(x) = (1-x)^{\alpha}(1+x)^{\beta} w_0(x)$, with $-1 < \alpha,\beta < 1$ and w_0 Hölder continuous;
3) $X(z)$ is analytic in the deleted complex plane;
4) $X(z) = (-1)^{-\kappa} z^{-\kappa}$ + lower terms as $z \to \infty$;
5) $X(z)$ has no zeros in the deleted complex plane;
6) $X^{\pm} \in H^{*}$.

The fact that the solution $\phi(x)$ can be decomposed as $\phi(x) = w(x)g(x)$ with $w(x) \in H^*$ and non-negative and $g(x)$ Hölder continuous, has already been used in chapter III and is an important result for the theory of SIE as well as for its numerical solution. We proceed now with a series of results aimed at the demonstration of this fact.

Lemma IV.2.1 (Dow & Elliott [1979])

Let Φ be analytic in the deleted complex plane and such that $\Phi^{\pm} \in H^*$. Suppose there exists a polynomial $P(x)$ such that

$$\lim_{z\to\infty} (\Phi(z) - P(z)) = 0.$$

Then

$$\Phi(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\Phi^+(t) - \Phi^-(t)}{t - z}dt + P(z), \quad z \notin [-1,1] \qquad \text{(IV.2.7)}$$

and

$$\frac{1}{\pi i}\int_{-1}^{1} \frac{\Phi^+(t) - \Phi^-(t)}{t - x}dt = \Phi^+(x) + \Phi^-(x) - 2P(x), \quad x \in (-1,1). \qquad \text{(IV.2.8)}$$

Proof: As $\Phi^{\pm} \in H^*$, the function

$$\varphi(x) = \Phi^+(x) - \Phi^-(x)$$

also satisfies $\varphi \in H^*$ and from (II.5.18) we have

$$\Phi(z) - P(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\varphi(t)}{t - z}dt = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\Phi^+(t) - \Phi^-(t)}{t - z}dt.$$

From the Plemlj-Sokhotski formulae (II.3.1) we obtain

$$\Phi^{+}(x) - P(x) = \frac{1}{2}\varphi(x) + \frac{1}{2\pi i}\int_{-1}^{1}\frac{\varphi(t)}{t - x}dt \qquad (IV.2.9)$$

$$\Phi^{-}(x) - P(x) = \frac{-1}{2}\varphi(x) + \frac{1}{2\pi i}\int_{-1}^{1}\frac{\varphi(t)}{t - x}dt. \qquad (IV.2.10)$$

Adding (IV.2.9) and (IV.2.10) we obtain

$$\frac{1}{\pi i}\int_{-1}^{1}\frac{\Phi^{+}(t) - \Phi^{-}(t)}{t - z}dt = \Phi^{+}(x) + \Phi^{-}(x) - 2P(x) \qquad x \in (-1,1).\blacksquare$$

Theorem IV.2.1

The limiting values of the function $X(z)$ satisfies

$$X^{\pm}(x) = \{a(x) \mp ib(x)\}\frac{Z(x)}{r(x)}. \qquad (IV.2.11)$$

Proof: Let us define

$$\Phi(z) = \int_{-1}^{1}\frac{\theta(t)}{t - z}dt \quad z \notin [-1,1].$$

From the Plemelj-Sokhotski formulae (II.3.1) we have

$$\Phi^{+}(x) = \frac{1}{2}\theta(x) + \frac{1}{2\pi i}\int_{-1}^{1}\frac{\theta(t)}{t - x}dt$$

$$\Phi^{-}(x) = \frac{-1}{2}\theta(x) + \frac{1}{2\pi i}\int_{-1}^{1}\frac{\theta(t)}{t - x}dt$$

so that

$$(1 - x)^{-\kappa}\exp(-2\pi i\Phi^{\pm}(x)) = Z(x)\exp(\mp\pi i\theta(x))$$

or

$$X^{\pm}(x) = Z(x)\exp(\mp\pi i\theta(x)).$$

To complete the proof we need to show that

$$\exp(\mp\pi i\theta(x)) = \frac{a(x) \mp ib(x)}{r(x)}.$$

From the definition of $\theta(x)$ we see that

$$\theta(x) = \frac{1}{\pi}\arg(a(x) + ib(x)) + N(x) = \frac{-i}{\pi}\log\left[\frac{a(x) + ib(x)}{r(x)}\right] + N(x)$$

and hence the required result follows.■

<u>Definition IV.2.2</u>

Let f be a complex function defined in the deleted complex plane and suppose that

$$f(z) = \sum_{j=-\infty}^{n} f_j z^j \text{ for large } z$$

where n is any integer. The *principal part* of f at a point $x \in (-1,1)$ is defined and denoted by

$$pp(f,x) = \sum_{j=0}^{n} f_j x^j.$$

Note that if $n < 0$ then $pp(f,x) \equiv 0$.

Notice from (IV.2.6) that

$$X(z) = (-1)^{-\kappa}z^{-\kappa} + \text{lower order terms}$$

and

$$X^{-1}(z) = (-1)^{-\kappa}z^{\kappa} + \text{lower order terms}$$

as $z \to \infty$, so that $pp(X,z)$ is a polynomial of degree $-\kappa$ for $\kappa < 0$ and $pp(X^{-1},z)$ is a polynomial of degree κ for $\kappa \geq 0$.

Theorem IV.2.2 (Dow & Elliott [1979])

Let Q and R be functions such that:

a) QX - R is analytic in the deleted complex plane and zero at infinity;

b) $Q^+(x) = Q^-(x)$, $R^+(x) = R^-(x)$ $x \in (-1,1)$ and $\frac{aQZ}{r} - R$ and $\frac{bQz}{r}$ are in H^*.

Then for $x \in (-1,1)$ we have

$$\frac{1}{\pi}\int_{-1}^{1} \frac{b(t)Q(t)Z(t)}{r(t)(t-x)}dt = \frac{-a(x)Q(x)Z(x)}{r(x)} + R(x). \qquad \text{(IV.2.12)}$$

Proof: Define $\Phi = QX - R$, then from (IV.2.11) we have

$$\Phi^{\pm}(x) = \frac{Q(x)Z(x)}{r(x)}\{a(x) \mp ib(x)\} - R(x) \quad x \in (-1,1).$$

Therefore $\Phi^{\pm} \in H^*$. Since Φ is analytic in the deleted complex plane and vanishes at infinity, lemma IV.2.1 with $P(z) \equiv 0$ gives the required result.∎

Similarly we can show

Theorem IV.2.3

Let S and T be functions such that:

a) $SX^{-1} - T$ is analytic in the deleted complex plane and zero at infinity;

b) $S^+(x) = S^-(x)$, $T^+(x) = T^-(x)$ $x \in (-1,1)$ and $\frac{aS}{rZ} - T$ and $\frac{bS}{rZ}$ are in H^*.

Then for $x \in (-1,1)$ we have

$$\frac{1}{\pi}\int_{-1}^{1}\frac{b(t)S(t)}{r(t)Z(t)(t-x)}dt = \frac{a(x)S(x)}{r(x)Z(x)} - T(x). \qquad (IV.2.13)$$

As an important consequence of theorem IV.2.2 and IV.2.3 we can prove

Theorem IV.2.4

Let P be any polynomial.

Then

$$\frac{1}{\pi}\int_{-1}^{1}\frac{b(t)P(t)Z(t)}{r(t)(t-x)}dt = \frac{-a(x)P(x)Z(x)}{r(x)} + pp(PX,x) \qquad (IV.2.14)$$

and

$$\frac{1}{\pi}\int_{-1}^{1}\frac{b(t)P(t)}{r(t)Z(t)(t-x)}dt = \frac{a(x)P(x)}{r(x)Z(x)} - pp(PX^{-1},x). \qquad (IV.2.15)$$

Proof: Take Q = P and R = pp(PX,x) in theorem IV.2.3, then Q and R satisfy conditions a) and b) of that theorem so that (IV.2.14) is obtained from (IV.2.12). (IV.2.15) is proved similarly.∎

Corollary IV.2.1

$$\frac{1}{\pi}\int_{-1}^{1}\frac{b(t)Z(t)}{r(t)(t-x)}dt = \frac{-a(x)Z(x)}{r(x)} + pp(X,x) \qquad (IV.2.16)$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{b(t)}{r(t)Z(t)(t-x)}dt = \frac{a(x)}{r(x)Z(x)} - pp(X^{-1},x) \qquad (IV.2.17)$$

Proof: The proof follows immediately from theorem IV.2.4 by taking $P \equiv 1$ in that theorem.■

Theorem IV.2.5

Let $\phi(x)$ be the solution of the dominant equation

$$a(x)\phi(x) + \frac{b(x)}{\pi}\int_{-1}^{1}\frac{\phi(t)}{t-x}dt = f(x)$$

where a, b and f are Hölder continuous. Let also X(z) be the fundamental solution of the associated HBVP of the class h_o. Then, for $-1 < x < 1$ $\phi(x)$ can be expressed in the following forms

$$\phi(x) = \frac{Z(x)}{r(x)}\left\{f(x)pp(X^{-1},x) + \frac{1}{\pi}\int_{-1}^{1}\frac{f(x)b(t) - f(t)b(x)}{r(t)Z(t)(t-x)}dt + b(x)P_{\kappa-1}(x)\right\} \quad (IV.2.18)$$

$$\phi(x) = \frac{Z(x)}{r(x)}\left\{f(x)\left(pp(X^{-1},x) + \frac{1}{\pi}\int_{-1}^{1}\frac{b(t) - b(x)}{r(t)Z(t)(t-x)}dt\right) + b(x)\left[-\frac{1}{\pi}\int_{-1}^{1}\frac{f(t) - f(x)}{r(t)Z(t)(t-x)}dt + P_{\kappa-1}(x)\right]\right\}. \quad (IV.2.19)$$

Proof: From theorem II.6.1 we have

$$\phi(x) = \frac{a(x)f(x)}{r^2(x)} - \frac{b(x)Z(x)}{r(x)}\left[\frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{r(t)Z(t)(t-x)}dt + b(x)P_{\kappa-1}(x)\right]$$

$$= \frac{a(x)f(x)}{r^2(x)} - \frac{Z(x)}{r(x)}\left[\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)b(t) + f(t)b(x) - f(x)b(t)}{r(t)Z(t)(t - x)}dt\right.$$

$$\left. + b(x)P_{\kappa-1}(x)\right].$$

Substituting expression (IV.2.17) into the above expression for $\phi(x)$ we obtain

$$\phi(x) = \frac{a(x)f(x)}{r^2(x)} - \frac{Z(x)}{r(x)}\left[f(x)\left(\frac{a(x)}{r(x)Z(x)} - pp(X^{-1},x)\right)\right.$$

$$\left. + \frac{1}{\pi}\int_{-1}^{1}\frac{f(t)b(x) - f(x)b(t)}{r(t)Z(t)(t - x)}dt + b(x)P_{\kappa-1}(x)\right]$$

$$= \frac{Z(x)}{r(x)}\left[f(x)pp(X^{-1},x) + \frac{1}{\pi}\int_{-1}^{1}\frac{f(x)b(t) - f(t)b(x)}{r(t)Z(t)(t - x)}dt\right.$$

$$\left. + b(x)P_{\kappa-1}(x)\right].$$

The last identity proves (IV.2.18).

To obtain (IV.2.19) we substitute the identity $f(x)b(t) - f(t)b(x) = f(x)(b(t) - b(x)) - b(x)(f(t) - f(x))$ into (IV.2.18).∎

As we are seeking a solution in the class h_0 the function $\frac{1}{Z(x)}$ is bounded everywhere in $[-1,1]$ and vanishes at the ends so from the results of Gakhov [1966 page 41] (see also Dow & Elliott [1979]) it follows that the functions

$$\frac{1}{\pi}\int_{-1}^{1}\frac{b(t) - b(x)}{r(t)Z(t)(t - x)}dt$$

and

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(t) - f(x)}{r(t)Z(t)(t - x)}dt$$

satisfy the Hölder condition if f(x) and b(x) do. These facts and (IV.2.19) together imply

$$\phi(x) = \frac{Z(x)}{r(x)}g(x) \tag{IV.2.20}$$

where g(x) is Hölder continuous.

Expression (IV.2.20) holds for the solution of the dominant equation. To prove it is also true for the solution of the complete equation we use the technique employed in (II.7.1), that is, we take the term involving the compact operator to the right-hand side and treat it as known. It can then be seen that the function $K\phi(x)$ satisfies the Hölder condition and so does the right-hand side: this allows us to conclude that the solution of the complete equation also satisfies (IV.2.20).

Rewriting the integral equation (IV.2.1) in terms of the unknown g(x) we obtain

$$\frac{Z(x)}{r(x)}a(x)g(x) + \frac{b(x)}{\pi}\int_{-1}^{1}\frac{Z(t)g(t)}{r(t)(t - x)}dt + \int_{-1}^{1}\frac{Z(t)k(x,t)g(t)}{r(t)}dt = f(x). \tag{IV.2.21}$$

In operator form (IV.2.21) becomes

$$Hg + Kg = f \tag{IV.2.22}$$

where

$$Hg = \frac{Z(x)}{r(x)}a(x)g(x) + \frac{b(x)}{\pi}\int_{-1}^{1}\frac{Z(t)g(t)}{r(t)(t - x)}dt$$

$$= w_1(x)a(x)g(x) + \frac{b(x)}{\pi}\int_{-1}^{1} \frac{w_1(t)g(t)}{t-x}dt$$

$$Kg = \int_{-1}^{1} w_1(t)k(x,t)g(t)dt$$

$$w_1(x) = \frac{Z(x)}{r(x)} .$$

We define the operator

$$H^I g = \frac{a(x)g(x)}{r(x)Z(x)} - \frac{b(x)}{\pi}\int_{-1}^{1} \frac{g(t)}{r(t)Z(t)(t-x)}dt$$

$$= a(x)w_2(x)g(x) - \frac{b(x)}{\pi}\int_{-1}^{1} \frac{w_2(t)g(t)}{t-x}dt \qquad \text{(IV.2.23)}$$

where

$$w_2(x) = \frac{1}{r(x)Z(x)}.$$

For functions ϕ of the form $\phi(x) = w_1(x)g(x)$ with g Hölder continuous we have from (II.6.2) that the operator A satisfies $A\phi = Hg$ and for a general function h in the domain of A^I we have from (II.6.12) that $A^I h = w_1 H^I h$, so from (II.6.15) it follows that

1) $H^I Hf = f + bP_{\kappa-1}$ for $\kappa > 0$;

(IV.2.24)

2) $H^I Hf = f$ for $\kappa \leq 0$

for any f Hölder continuous, where $P_{\kappa-1}(x)$ is a polynomial of degree $\kappa-1$;

3) The kernel of H, ker(H), has dimension κ and is spanned by the set of functions

$$b(x)\left\{1, x, x^2, \ldots, x^{\kappa-1}\right\}. \qquad \text{(IV.2.25)}$$

This follows directly from the relation $A\phi = Hg$ and (II.6.16) Notice also that $w_1(x)$ and $w_2(x)$ are non-negative integrable functions on $[-1,1]$.

On pre-multiplying equation (IV.2.22) by H^I we obtain

$$\begin{aligned} g + H^I Kg &= H^I f + bP_{\kappa-1} && \text{if } \kappa > 0 \\ g + H^I Kg &= H^I f && \text{if } \kappa \leq 0. \end{aligned} \tag{IV.2.26}$$

For the sake of simplicity of notation we rewrite equations (IV.2.26) in the form

$$g + H^I Kg = H^I f + bP_{\kappa-1} \tag{IV.2.27}$$

where $P_j \equiv 0$ for $j < 0$.

For $\kappa > 0$, we see from (IV.2.25) that the solution of (IV.2.22) is non-unique. To have uniqueness a further set of κ side conditions must be imposed on the solution. For example, for $\kappa = 1$ a common choice is

$$\int_{-1}^{1} \phi(x)dx = \text{const}$$

For $\kappa < 0$ the function f and the solution ϕ must satisfy consistency conditions in order that (IV.2.22) have a solution. The consistency conditions, which will be assumed to be satisfied, are of the form

$$\int_{-1}^{1} w_2(t)t^{j-1}(f(t) - Kg(t))dt = 0, \quad j = 1,2,\ldots,-\kappa.$$

To close this section we shall now show that the operator $H^I K$ is a Fredholm operator (compact operator) in the space of continuous functions, that is we shall show that, for every continuous function g(x) we have

$$H^I Kg = \int_{-1}^{1} h(x,t)g(t)dt \tag{IV.2.28}$$

with h(x,t) weakly singular.

Let v(x) = Kg, then v is Hölder continuous because of the assumptions we made on k(x,t), therefore $H^I v = H^I Kg$ is well defined. From (IV.2.23) it follows that

$$H^I v = a(x)w_2(x)v(x) - \frac{b(x)}{\pi}\int_{-1}^{1} \frac{w_2(t)v(t)}{t-x}dt$$

$$= a(x)w_2(x)v(x) - \frac{1}{\pi}\int_{-1}^{1} \frac{w_2(t)(b(t)v(x) + b(x)v(t) - b(t)v(x))}{t-x}dt$$

$$= v(x)pp(X^{-1},x) + \frac{1}{\pi}\int_{-1}^{1} \frac{w_2(t)(b(t)v(x) - b(x)v(t))}{t-x}dt$$

$$= v(x)pp(X^{-1},x) + \frac{1}{\pi}\int_{-1}^{1} \frac{w_2(t)v(x)(b(t) - b(x))}{t-x}dt$$

$$- \frac{1}{\pi}\int_{-1}^{1} \frac{w_2(t)b(x)(v(t) - v(x))}{t-x}dt. \tag{IV.2.29}$$

Now

$$\frac{1}{\pi}\int_{-1}^{1} \frac{w_2(t)b(x)(v(t) - v(x))}{t-x}dt$$

$$= \frac{b(x)}{\pi}\int_{-1}^{1} \frac{w_2(t)}{t-x}\left\{\int_{-1}^{1} (k(t,s) - k(x,s))g(s)ds\right\}dt$$

$$= \frac{1}{\pi}\int_{-1}^{1} g(s)\left\{\int_{-1}^{1} \frac{b(x)w_2(t)(k(t,s) - k(x,s))}{t - x}dt\right\}ds$$

by interchanging the order of integration.

From this expression and (IV.2.29) it follows that (IV.2.28) is true with h(x,t) given by

$$h(x,t) = k(x,t)\left[pp(X^{-1},x) + \frac{1}{\pi}\int_{-1}^{1} \frac{w_2(s)(b(s) - b(x))}{s - x}ds\right]$$

$$+ \frac{1}{\pi}\int_{-1}^{1} \frac{b(x)w_2(s)(k(s,t) - k(x,t))}{s - x}ds.$$

We have shown that (IV.2.27) is a Fredholm integral equation: as a consequence it has a unique solution provided that -1 is not an eigenvalue of H^IK, a condition which will be assumed throughout. The process of reduction of (IV.2.22) to (IV.2.27) is called regularization.

EXAMPLE

Consider equation (IV.2.1) where

$$a(x) = (-1)^{\kappa}\text{sign}(\kappa-1)\sqrt{1 - x^2}\; U_{|\kappa-1|-1}(x)$$

$$b(x) = (-1)^{\kappa-1}T_{|\kappa-1|}(x)$$

and T_j and U_j denote the Chebyshev polynomials of degree j, of first and second kind respectively and κ is an arbitrary integer.

It can be shown see Elliott [1982a] that the index of the resulting SIE with respect to the class of unbounded integrable functions at both ends is κ, and it is easily seen that

$$r^2 = \sqrt{a^2 + b^2} = 1.$$

It may be shown that Z(x) is given simply by

$$Z(x) = \frac{2^{\kappa-1}}{\sqrt{1 - x^2}}.$$

Therefore $w_1(x) = Z(x)$ and $w_2(x) = 2^{1-\kappa}\sqrt{1 - x^2}$.

3. Orthogonal Polynomials Associated with SIE

The results we are going to present in this section require a further assumption on the coefficient b(x) of the SIE (IV.2.1). It will be assumed that b(x) is a polynomial of degree m. Though this is not the most general assumption under which these results can be proved, they can be easily generalized to the case in which b(x) satisfies the following condition:

Suppose a Hölder-continuous function q defined on [-1,1] can be found such that:

i) $q(x) \geq 0$ on [-1,1];

ii) $B(x) = q(x)b(x)$ is a polynomial of degree m and such that $B(x) = 0$ implies $x \in [-1,1]$;

iii) if $B(x_0) = 0$ then $b(x_0) = 0$.

By writing $g(x) = \frac{q(x)r(x)\phi(x)}{Z(x)}$ and transforming equation (IV.2.1) into

$$\frac{a(x)Z(x)}{r(x)}g(x) + \frac{B(x)}{\pi}\int_{-1}^{1} \frac{Z(t)g(t)}{q(t)r(t)(t-x)}dt$$

$$+ \int_{-1}^{1} \frac{q(x)k(x,t)Z(t)g(t)}{q(t)r(t)}dt = q(x)f(x), \quad -1<x<1 \qquad \text{(IV.3.1)}$$

the results of the previous section can easily be adapted to cope with equation (IV.3.1). We shall omit the details here.

We have already mentioned that the functions $w_1(x)$ and $w_2(x)$, defined in the previous section, are nonnegative on $[-1,1]$ and integrable. It is a standard result, see Szegö [1959], that there exist sequences of polynomials $\{p_n\}$ and $\{q_n\}$, with strictly positive leading coefficients, such that

$$\int_{-1}^{1} w_1(t)p_i(t)p_j(t)dt = h_i\delta_{ij}, \quad i,j = 0,1, \ldots .$$

$$\int_{-1}^{1} w_2(t)q_i(t)q_j(t)dt = h_i^*\delta_{ij}, \quad i,j = 0,1, \ldots .$$

where h_i and h_i^* are strictly positive constants.

The main objective of this section is to obtain a pair of relations involving the polynomials p_n and q_n similar to the one we proved in lemma III.4.1 of chapter III. We start with the following definition

<u>Definition IV.3.1</u>

For $n \geq \text{Max}\{0,-\kappa\}$ we define the polynomials u_n by

$$u_n(z) = pp((-1)^\kappa X p_{n+\kappa}, z) \text{ for all } z$$

where p_n are the orthogonal polynomials with respect to the weight w_1. We have already remarked that X is of order $-\kappa$ at infinity with coefficient $(-1)^\kappa$, so that $(-1)^\kappa X p_{n+\kappa}$ is of order n at infinity with a strictly positive leading coefficient.

Theorem IV.3.1

For $n \geq \text{Max}\{0, m-\kappa\}$ and $x \in (-1,1)$

$$(-1)^{\kappa} u_n(x) = a(x) w_1(x) p_s(x) + \frac{b(x)}{\pi} \int_{-1}^{1} \frac{w_1(t) p_s(t)}{t-x} dt \qquad (IV.3.2)$$

where m = degree of b and $s = n + \kappa$.

Proof: From (IV.2.14) with $P(x) = p_{n+\kappa}(x)$ and the definition of u_n we have

$$(-1)^{\kappa} u_n = pp(Xp_s, x)$$

$$= a(x) w_1(x) p_s(x) + \frac{1}{\pi} \int_{-1}^{1} \frac{b(t) w_1(x) p_s(t)}{t-x} dt. \qquad (IV.3.3)$$

To complete the proof we need to show that

$$\frac{1}{\pi} \int_{-1}^{1} \frac{b(t) w_1(x) p_s(t)}{t-x} dt = \frac{b(x)}{\pi} \int_{-1}^{1} \frac{w_1(t) p_s(t)}{t-x} dt. \qquad (IV.3.4)$$

Indeed

$$\frac{1}{\pi} \int_{-1}^{1} \frac{b(t) w_1(x) p_s(t)}{t-x} dt - \frac{b(x)}{\pi} \int_{-1}^{1} \frac{w_1(t) p_s(t)}{t-x} dt$$

$$= \frac{1}{\pi} \int_{-1}^{1} w_1(x) \left[\frac{b(t) - b(x)}{t-x}\right] p_s(t) dt.$$

Now, as b(x) is a polynomial of degree m the term in brackets in the integral above is a polynomial of degree m - 1 in t. As the polynomials p_n form a orthogonal sequence with respect to the weight $w_1(x) = \frac{Z(x)}{r(x)}$ (IV.3.4) is satisfied and so is (IV.3.2).∎

We can also obtain an expression equivalent to (IV.3.2) for the polynomials p_n in terms of u_n, namely

<u>Theorem IV.3.2</u>

For $n \geq \mathrm{Max}\{\kappa,m\}$ and $x \in (-1,1)$

$$(-1)^{\kappa}p_n(x) = a(x)w_2(x)u_s(x) - \frac{b(x)}{\pi}\int_{-1}^{1}\frac{w_2(t)u_s(t)}{t-x}dt \qquad \text{(IV.3.5)}$$

where $s = n - \kappa$.

Proof: In order to simplify the notation we shall make use of the notation introduced in (II.3.6) to represent the singular integral.

From (IV.3.2) with n replaced by $n - \kappa$ we obtain

$$(-1)^{\kappa}\ a(x)u_s(x)w_2(x) = \frac{a^2(x)}{r^2(x)}p_n(x) + b(x)a(x)w_2(x)\mathcal{F}(w_1p_n,x)$$

and

$$(-1)^{\kappa}b(x)\mathcal{F}(w_2u_s,x) = b(x)\mathcal{F}(\frac{a}{r^2}p_n,x) + b(x)\mathcal{F}(bw_2\mathcal{F}(w_1p_n,t),x).$$

Hence

$$(-1)^{\kappa}\left[a(x)u_s(x)w_2(x) - \frac{b(x)}{\pi}\int_{-1}^{1}\frac{w_2(t)u_s(t)}{t-x}dt\right]$$

$$= \frac{a^2(x)}{r^2(x)}p_n(x) + b(x)a(x)w_2(x)\mathcal{F}(w_1p_n,x)$$

$$- b(x)\mathcal{F}(\frac{a}{r^2}p_n,x) - b(x)\mathcal{F}(bw_2\mathcal{F}(w_1p_n,t),x). \qquad \text{(IV.3.6)}$$

From (IV.2.15) with $P \equiv 1$ we have

$$\mathcal{F}(bw_2, x) = -a(x)w_2(x) - pp(X^{-1}, x). \tag{IV.3.7}$$

Using Poincaré-Bertrand formula (II.3.5), the last term on the right-hand side of (IV.3.6) can be written as

$$\mathcal{F}(bw_2\mathcal{F}(w_1p_n)) = \frac{1}{\pi}\int_{-1}^{1} w_1(s)p_n(s)\left[\frac{1}{\pi}\int_{-1}^{1} \frac{b(t)w_2(t)}{(t-x)(s-t)}dt\right]ds$$

$$- b(x)w_2(x)w_1(x)p_n(x).$$

Making use of the identity $\frac{1}{(t-x)(s-t)} = \frac{1}{s-x}\left[\frac{1}{t-x} + \frac{1}{s-t}\right]$ and (IV.2.15) we obtain

$$\mathcal{F}(bw_2\mathcal{F}(w_1p_n)) = -\mathcal{F}(w_1p_n(aw_2 - pp(X^{-1})))$$

$$+ (aw_2 - pp(X^{-1}))\mathcal{F}(w_1p_n) - bw_2w_1p_n$$

$$= -\mathcal{F}(w_2w_1p_na) + \mathcal{F}(pp(X^{-1})w_1p_n) + aw_2\mathcal{F}(w_1p_n)$$

$$- \mathcal{F}(w_1p_n)pp(X^{-1}) - bw_1w_2p_n. \tag{IV.3.8}$$

Now

$$\mathcal{F}(w_1p_npp(X^{-1})) - pp(X^{-1})\mathcal{F}(w_1p_n)$$

$$= \frac{1}{\pi}\int_{-1}^{1} w_1(t)\left[\frac{pp(X^{-1},t) - pp(X^{-1},x)}{t - x}\right]p_n(t)dt. \tag{IV.3.9}$$

As $pp(X^{-1},z)$ is a polynomial of degree κ, the term in brackets in the above integral is a polynomial of degree $\kappa - 1$ in the variable t. As $n \geq \kappa$ and $\{p_n\}$ forms a sequence of orthogonal polynomials with respect to the weight w_1, the expression in

(IV.3.9) vanishes. Hence

$$\mathcal{F}(bw_2\mathcal{F}(w_1p_n)) = -\mathcal{F}(w_1w_2p_na) + aw_2\mathcal{F}(w_1p_n) - bw_1w_2p_n.$$

Substituting $w_1(x) = \frac{Z(x)}{r(x)}$ and $w_2(x) = \frac{1}{Z(x)r(x)}$ we have

$$\mathcal{F}(bw_2\mathcal{F}(w_1p_n)) = -\mathcal{F}(\frac{ap_n}{r^2}) + aw_2\mathcal{F}(w_1p_n) - \frac{bp_n}{r^2}. \qquad \text{(IV.3.10)}$$

Substituting (IV.3.10) into (IV.3.6) and recalling that $r^2 = a^2 + b^2$ we obtain (IV.3.5).■

Now we prove that the polynomials u_n given by definition IV.3.1 are in fact orthogonal on $[-1,1]$ with respect to the weight w_2.

<u>Theorem IV.3.3</u>

Let $\{p_n\}$ be a sequence of polynomials such that

$$\int_{-1}^{1} w_1(t)p_i(t)p_j(t)dt = h_i\delta_{ij}, \quad i,j = 0,1, \ldots$$

and let the polynomials u_n be as in definition IV.3.1
Then, for $i,j \geq \text{Max}\{0, m-\kappa\}$

$$\int_{-1}^{1} w_2(t)u_i(t)u_j(t)dt = h_i^*\delta_{ij}$$

where h_i and h_i^* are strictly positive constants.

Proof: Setting $s = j + \kappa$, from (IV.3.2) we have that

$$\int_{-1}^{1} w_2(t)u_i(t)u_j(t)dt$$

$$= (-1)^{\kappa}\int_{-1}^{1} w_2(t)u_i(t)\left[a(t)w_1(t)p_s(t) + b(t)\mathcal{F}(w_1p_s, t)\right]dt$$

$$= (-1)^{\kappa}\int_{-1}^{1} p_s(t)\left[\frac{a(t)w_1(t)u_i(t)}{Z(t)r(t)} - w_1(t)\mathcal{F}(bw_2u_i, t)\right]dt$$

$$= (-1)^{\kappa}\int_{-1}^{1} w_1(t)p_s(t)\left[\frac{a(t)u_i(t)}{z(t)r(t)} - \mathcal{F}(bw_2u_i, t)\right]dt$$

$$= (-1)^{\kappa}\int_{-1}^{1} w_1(t)p_s(t)pp(u_iX^{-1}, t)dt$$

where use has been made of (II.4.5), the definition of w_2 and (IV.2.15) respectively.

Now $pp(u_iX^{-1})$ is a polynomial of degree $i+\kappa$ and the polynomials $p_n(x)$ are orthogonal with respect to $w_1(x)$; hence

$$\int_{-1}^{1} w_2(t)u_i(t)u_j(t)dt = 0 \text{ for } i < j.$$

In the case $i > j$ we just exchange the roles of i and j in the above proof and we obtain the same final result for $i > j$. Therefore the polynomials u_n form a orthogonal sequence with respect to the weight w_2.■

The combined force of theorems (IV.3.1), (IV.3.2) and (IV.3.3) allows us to prove the following result:

Theorem IV.3.4

Let $\{p_n\}$ and $\{q_n\}$ be sequences of orthogonal polynomials on $[-1,1]$ with respect to weights w_1 and w_2 respectively.
Then

$$Hp_n = (-1)^{-\kappa} q_{n-\kappa} \quad \text{for } n \geq \mathrm{Max}\{m,\kappa\} \tag{IV.3.11}$$

$$H^I q_n = (-1)^{\kappa} p_{n+\kappa} \quad \text{for } n \geq \mathrm{Max}\{m,-\kappa\} \tag{IV.3.12}$$

where m is the degree of the polynomial b.

We shall also require the following property of the operator H.

Theorem IV.3.5

Let s_j be any polynomial of degree j.
Then

Hs_j is a polynomial of degree $\mathrm{Max}\{j-\kappa, m-1\}$

$H^I s_j$ is a polynomial of degree $\mathrm{Max}\{j+\kappa, m-1\}$.

Proof: Let s_j be a polynomial of degree j, from the definition of the operator H, see (IV.2.22), we obtain

$$Hs_j = a(x)w_1(x)s_j(x) + \frac{b(x)}{\pi}\int_{-1}^{1} \frac{w_1(t)s_j(t)}{t-x}dt =$$

$$= pp(s_j X, x) - \frac{1}{\pi}\int_{-1}^{1} w_1(t)\left[\frac{b(t)-b(x)}{t-x}\right]s_j(t)dt$$

where the last equality is due to (IV.2.14).

As $b(z)$ is a polynomial of degree m the term
$\frac{1}{\pi}\int_{-1}^{1} w_1(t)\left[\frac{b(t) - b(x)}{t - x}\right]s_j(t)dt$ is a polynomial of degree $m - 1$ in x. In fact let $b(z) = b_o + b_1 z + \ldots + b_m z^m$ then

$$\frac{b(t) - b(x)}{t - x} = \frac{b_1 t + \ldots + b_m t^m - b_1 x - \ldots - b_m t^m}{t - x}$$

$$= b_1 + b_2(t+x) + \ldots + b_m(t^{m-1} + t^{m-2}x + \ldots\ tx^{m-2} + x^{m-1})$$

$$= A_1(x) + A_2(x)t + \ldots + A_{m-1}(x)t^{m-1}$$

where each $A_i(x)$ is a polynomial of degree less than or equal to m-1. Hence Hs_j is a polynomial of degree $\mathrm{Max}\{j-\kappa, m-1\}$.

The same proof applies to $H^I s_j$ using expression (IV.2.15).■

<u>Lemma IV.3.1</u>

$$\max_{x\in[-1,1]} |H^I f(x)| \le M < \infty.$$

where $f(x) \equiv 1$.

Proof: Since $f(x)$ is a polynomial of degree 0, lemma IV.3.5 implies that $H^I f(x)$ is a polynomial of degree $\mathrm{Max}\{\kappa, m-1\}$, and therefore uniformly bounded.■

<u>4. The Collocation Method and its Uniform Convergence</u>

We observe that the collocation method we are going to use in this section is essentially the same as that introduced in chapter III, so that we shall not discuss it again in detail.

As before consider approximating $g(x)$ by $g_n(x)$, where

$$g_n(x) = \sum_{j=0}^{n} a_j p_j(x) \qquad \text{(IV.4.1)}$$

$p_j(x)$ are the orthogonal polynomials with respect to the weight $w_1(x)$ and a_j, $j = 0,1,\ldots,n$, are unknown constants. In order to determine these we form the residual

$$r_n(x) = (Hg_n + Kg_n - f)(x) = \sum_{j=0}^{n} a_j[Hp_j(x) + Kp_j(x)] - f(x).$$

Making use of theorem IV.3.4 we can write, for $n > s$

$$r_n(x) = \sum_{j=0}^{s} a_j Hp_j(x) + \sum_{j=s+1}^{n} a_j(-1)^{\kappa} q_{j-\kappa}(x)$$
$$+ \sum_{j=0}^{n} a_j Kp_j(x) - f(x) \qquad \text{(IV.4.2)}$$

where $s = \text{Max}\{m,\kappa\} - 1$.

Remark: According to theorem IV.3.5 the degree of the polynomial in the first summation of (IV.4.2) is an integer independent of n and less than $\text{Max}\{s-\kappa, m-1\}$.

By setting $r_n(x_m) = 0$, $m = 0,1,\ldots,n-\kappa$, where $x_m \in [-1,1]$ are the zeros of the first kind Chebyshev polynomial of degree $n-\kappa+1$, (IV.4.2) gives rise to a system of linear equations. As in chapter III we shall restrict attention to the case $\kappa \geq 0$.

For $\kappa > 0$, this system has $n - \kappa + 1$ equations in the $n+1$ unknowns $a_0, a_1, \ldots, a_n$ so that it must be augmented by another κ equations, which are given by asking the approximate solution $g_n(x)$ to satisfy κ prescribed side conditions. For $\kappa = 0$ the number of equations matches that of the unknowns and we shall show the system can be solved.

As before we rewrite the system of equations $r_n(x_m) = 0$, $m = 0,1,\dots,\ell$, where $\ell = n - \kappa$ as

$$P_\ell r_n \equiv 0, \tag{IV.4.3}$$

where P_ℓ is the interpolating projector defined previously.

Recalling the definition of r_n, (IV.4.3) can be written as

$$P_\ell H g_n + P_\ell K g_n = P_\ell f. \tag{IV.4.4}$$

and from (IV.3.11) and the above remark Hg_n is a polynomial of degree $n-\kappa$, for large n. So that

$$P_\ell H g_n = H g_n. \tag{IV.4.5}$$

Hence, our final set of linear equations can be expressed in operator form as

$$H g_n + P_\ell K g_n = P_\ell f. \tag{IV.4.6}$$

For the case of the index $\kappa = 0$ we can prove the following theorem

<u>Theorem IV.4.1</u>

Let $f \in C^p[-1,1]$ with $f^{(p)} \in H^\mu[-1,1]$ $p \geq 0$, and let $k(x,t)$ be a Hölder continuous function such that, $k(x,t) \in C^q[-1,1]$ with respect to the variable x and $\dfrac{\partial^q k(x,t)}{\partial x^q} \in H^\sigma[-1,1]$, $q \geq 0$, uniformly with respect to the variable t (see (II.8.21)). Let also $g(x)$ be the solution of the SIE

$$Hg + Kg = f. \tag{IV.4.7}$$

Assume that the index κ of (IV.4.7) relatively to the class of functions h_o is zero and

$$\Gamma_2 \equiv \tfrac{1}{2}\text{Min}\{p+\mu, q+\sigma\} > \text{Max}\{0, \alpha, \beta\} \equiv \Gamma_1.$$

Then for sufficiently large n the solution of

$$Hg_n + K_n g_n = f_n. \tag{IV.4.8}$$

exists and

$$\max_{x\in[-1,1]} |g(x) - g_n(x)| \leq cn^{-\nu} \tag{IV.4.9}$$

where $\nu = \text{Min}\{p+\mu, q+\sigma\} - 2\xi$, $\xi \in (\Gamma_1, \Gamma_2)$.
Note that as we are seeking a solution in the class h_o we have $\Gamma_1 \equiv 0$.

Proof: Subtracting (IV.4.8) from (IV.4.7) gives

$$H(g-g_n) + K_n(g-g_n) = (K_n-K)g + f - f_n \tag{IV.4.10}$$

where, for simplicity of notation, we write $K_n = P_n K$.

Pre-multiplying (IV.4.10) by H^I, using (IV.2.27) and setting $e_n = g - g_n$ we obtain

$$e_n + H^I K_n e_n = H^I(K_n-K)g + H^I(f-f_n). \tag{IV.4.11}$$

As we have already pointed out the operator $H^I K_n$ is a Fredholm integral operator, see (IV.2.28). Therefore (IV.4.11) is a second kind Fredholm integral equation for the error e_n.

We are now going to show that the operator $(I + H^I K_n)$ is invertible and its inverse is bounded on $C[-1,1]$, for all sufficiently large n, on the assumption that $(I + H^I K)$ is invertible and its inverse is bounded.

Let h be a continuous function with $\|h\|_\infty = 1$ then

$$\|H^I(K-K_n)h\|_\infty$$

$$= \max_{x\in[-1,1]} \left| a(x)w_2(x)h_n(x) - \frac{b(x)}{\pi}\int_{-1}^{1} \frac{w_2(t)h_n(t)}{t-x}dt \right| \qquad \text{(IV.4.12)}$$

where

$$h_n(x) = \int_{-1}^{1} w_1(t)\left[k(x,t) - k_n(x,t)\right]h(t)dt.$$

Summing and subtracting $h_n(x)$ in the singular integral in (IV.4.12) we obtain

$$\|H^I(K-K_n)h\|_\infty = \max_{x\in[-1,1]} \left| a(x)w_2(x)h_n(x) - \frac{b(x)h_n(x)}{\pi}\int_{-1}^{1} \frac{w_2(t)}{t-x}dt \right.$$

$$\left. + \frac{b(x)}{\pi}\int_{-1}^{1} \frac{w_2(t)}{t-x}\left[h_n(t) - h_n(x)\right]dt \right|.$$

Note now that

$$h_n(x)H^I 1(x) = h_n(x)\left[a(x)w_2(x) - \frac{b(x)}{\pi}\int_{-1}^{1} \frac{w_2(t)}{t-x}dt\right]$$

and so lemma IV.3.1 implies that

$$\max_{x\in[-1,1]} |h_n(x)H^I 1(x)| \le \max_{x\in[-1,1]} |h_n(x)|M.$$

So

$$\|H^{I}(K-K_n)h\|_\infty \le \underset{x\in[-1,1]}{\mathrm{Max}} |h_n(x)|M +$$

$$+ \underset{x\in[-1,1]}{\mathrm{Max}} \left| \frac{b(x)}{\pi} \int_{-1}^{1} \frac{w_2(t)}{t-x}\Big[h_n(x) - h_n(t)\Big]dt \right|. \qquad (\mathrm{IV}.4.13)$$

Now

$$\underset{x\in[-1,1]}{\mathrm{Max}} |h_n(x)| \le \frac{2c\log(n)}{n^{q+\sigma}} \qquad (\mathrm{IV}.4.14)$$

by lemma II.8.3 and writing

$$\omega(x) = \int_{-1}^{1} w_1(t)k(x,t)h(t)dt$$

$$\omega_n(x) = \int_{-1}^{1} w_1(t)k_n(x,t)h(t)dt$$

$\omega_n(x)$ is the interpolating polynomial to $\omega(x)$ on the set of points $\{x_0, x_1, \ldots, x_n\}$. From lemma II.8.3 again

$$\|h_n\|_\infty = \|\omega - \omega_n\|_\infty \le \frac{2c\log(n)}{n^{q+\sigma}}$$

and then lemma II.8.1 can be applied to give

$$|h_n(x) - h_n(t)| \le \frac{c|x-t|^{\xi}}{n^{q+\sigma-2\xi}}, \quad 0 < 2\xi < q+\sigma. \quad (\mathrm{IV}.4.15)$$

Substituting (IV.4.15) into (IV.4.13) and using (IV.4.14) and the fact that $\Gamma_2 > \Gamma_1$ we obtain

$$\|H^{I}(K-K_n)h\|_\infty \le \frac{c_1\log(n)}{n^{q+\sigma}} + \frac{c_2\log(n)}{n^{q+\sigma-2\xi}}$$

$$\le \frac{c}{n^{q+\sigma-2\xi'}}, \quad \xi'\in(\Gamma_1,\Gamma_2) \qquad (\mathrm{IV}.4.16)$$

As the bound in (IV.4.16) is independent of h we have

$$\|H^I(K-K_n)\|_\infty \le \frac{c}{n^{q+\sigma-2\xi'}}, \quad \xi'\in(\Gamma_1,\Gamma_2) \qquad \text{(IV.4.17)}$$

Hence, $H^IK_n \longmapsto H^IK$ as $n \longmapsto \infty$, in the uniform norm; therefore lemma III.4.4 applied with $T = I + H^IK_n$ and $S = I + H^IK$ implies that the operator $(I + H^IK_n)^{-1}$ exists and is uniformly bounded, provided $(I + H^IK)^{-1}$ exists. More precisely

$$\|(I + H^IK_n)^{-1}\|_\infty \le \frac{\|(I + H^IK)^{-1}\|_\infty}{1 - \|(I + H^IK)^{-1}\|_\infty \|H^I(K - K_n)\|_\infty}$$

for n sufficiently large. This implies the existence of a unique solution g_n of the equation $g_n + H^IK_ng_n = H^If_n$ which is equivalent to (IV.4.8) because for $\kappa = 0$ $H^{-1} \equiv H^I$.

From (IV.4.11) we can write

$$e_n = (I + H^IK_n)^{-1}G_n \qquad \text{(IV.4.18)}$$

where

$$G_n = H^I(K-K_n)g + H^I(f-f_n)$$

From the uniform boundedness of $(I + H^IK_n)^{-1}$ and (IV.4.18) we obtain

$$\|e_n\|_\infty \le c\|G_n\|_\infty .$$

To complete the proof of the theorem we need to find a bound for $\|G_n\|_\infty$.

Notice that

$$\|G_n\|_\infty \le \|H^I(K-K_n)g\|_\infty + \|H^I(f-f_n)\|_\infty \le$$

$$\le \|H^I(K-K_n)\|_\infty \|g\|_\infty + \|H^I(f-f_n)\|_\infty . \qquad \text{(IV.4.19)}$$

Following the same steps as for the bound on $\|H^I(K-K_n)\|_\infty$, namely, steps (IV.4.13) through to (IV.4.17) we can easily show that

$$\|H^I(f-f_n)\|_\infty \le \frac{c}{n^{p+\mu-2\xi''}} \qquad \xi''\in(\Gamma_1,\Gamma_2) \qquad \text{(IV.4.20)}$$

And so, from (IV.4.19), (IV.4.17) and (IV.4.20) we have

$$\|G_n\|_\infty \le cn^{-\nu}$$

where

$$\nu = \mathrm{Min}\{p+\mu \; q+\sigma\} - \xi \qquad \xi\in(\Gamma_1,\Gamma_2)$$

Finally

$$\|g - g_n\|_\infty = \|e_n\|_\infty \le cn^{-\nu}$$

which is exactly as (IV.4.9).■

We proceed now with the analysis of the case $\kappa = 1$. We shall assume that the SIE (IV.2.1) is to be solved subject the condition

$$\int_{-1}^{1} \phi(x)dx = \text{const.}$$

This analysis will be performed under the assumption that the solution of the discrete equation (IV.4.8) exists subject to the side condition

$$\int_{-1}^{1} w_1(x)g_n(x)dx = \text{const.} \qquad \text{(IV.4.21)}$$

Equation (IV.4.11) takes the form

$$e_n + H^I K_n e_n = H^I(K_n - K)g + H^I(f - f_n) + t_n$$

where the numbers t_n are determined such that g_n satisfies (IV.4.21). Clearly the bounds (IV.4.17) and (IV.4.20) still hold true and this allows us to show that $(I + H^I K_n)^{-1}$ exists and is uniformly bounded. This then implies

$$e_n \leq c\|G_n\|_\infty + |t_n|$$

with G_n defined as in (IV.4.18). Proceeding as in section 5 of chapter III we can easily show that $|t_n| = O(\|G_n\|_\infty)$ as $n \to \infty$, and from this we obtain the same error bound as (IV.4.9).

5. Numerical Examples

(i) Consider the SIE with constant coefficients

$$a\phi(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{\phi(t)}{t - x}dt = f(x)$$

together with

$$\int_{-1}^{1} \phi(x)dx = L.$$

In order that the the integral equation and the side condition have a solution we must have the index of the equation to be $\kappa = 1$. We choose $a = -0.5497546$ such that

$$\alpha = \frac{1}{\pi}\arctan(\frac{1}{a}) = -0.34$$

and

$$\beta = -1 + \alpha = -0.66,$$

so that the weight function $w(x)$ is given by

$$w_1(x) = (1 - x)^{\alpha}(1 + x)^{\beta}.$$

Choosing f to be a polynomial of degre k say, then from (III.4.1) $\phi(x)$ will have the form

$$\phi(x) = w(x)g(x)$$

where $g(x)$ is a polynomial of degree $k+1$. Choosing also $L = 0$, and more specifically, $f(x) = 1 + x + x^2 + x^3 + x^4 + x^5$ then $g(x)$ is a polynomial of degree 6.

We approximate $g(x)$ by $g_n(x)$ where

$$g_n(x) = a_o P_o^{(\alpha,\beta)}(x) + \ldots + a_n P_n^{(\alpha,\beta)}(x)$$

and $P_j^{(\alpha,\beta)}(x)$ are the Jacobi polynomials, orthogonal in $[-1,1]$ with respect to the weight $w_1(x)$. The side condition on $\phi(x)$ is required to be satisfied by $w_1(x)g_n(x)$ implying $a_o = 0$.

Table IV.5.1 below shows the numerical approximations for the solution $g(x)$ at the points $x = -0.5$, 0.0, 0.75 produced by the collocation method described in the last section. As the solution is a polynomial of degree 6 we predict that the method should reproduce the exact solution for $n = 6, 7 \ldots$. From table

IV.5.1 we observe that this is actually the case.

n	g(-0.5)	g(0)	g(0.75)
2	1.31355004867	2.26730005983	4.15823757657
3	1.26735056218	2.34006309010	6.61631940292
4	1.27372529999	2.08603632898	7.18219044254
5	1.28544661243	2.08206919847	7.15509941658
6	1.27373104227	2.10802364491	7.06227028837
7	1.27373104227	2.10802364491	7.06227028837

Table IV.5.1

(ii) Consider the equation in the example of section 2, with the right-hand side given by

$$f(x,\kappa) = (-1)^{\kappa-1}\left\{\frac{2}{\pi}T_{|\kappa-1|}(x)\left[1 + \frac{x^2}{\sqrt{1-x^2}}\log\left|\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right|\right] + \right.$$

$$\left. - \operatorname{sign}(\kappa-1)\; x|x|U_{|\kappa-1|-1}(x)\right\}.$$

The exact solution of the resulting SIE is

$$\phi(x) = \frac{x|x|}{\sqrt{1-x^2}}.$$

As we have already mentioned the index of this equation with respect to the class of functions h_0 is κ. If we choose $\kappa = 0$, then the final equation is

$$\sqrt{1-x^2}\phi(x) + \frac{x}{\pi}\int_{-1}^{1}\frac{\phi(t)}{t-x}dt$$

$$= x|x| + \frac{2}{\pi}\left[1 + \frac{x^2}{\sqrt{1-x^2}}\log\left|\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right|\right].$$

Notice that $b(x) = x$, $m = \text{degree}(b) = 1$, $r^2 = a^2 + b^2 = 1$

$w_1(x) = \dfrac{1}{2\sqrt{1 - x^2}}$ and $w_2(x) = 2\sqrt{1 - x^2}$. As a result of this observation the orthogonal polynomials $p_n(x)$ and $q_n(x)$ can be chosen to be the Chebyshev polynomials of first and second kind, respectively, ie. $p_n(x) = T_n(x)$ and $q_n(x) = U_n(x)$. Notice also that formula (IV.3.11) is valid only for $n \geq \mathrm{Max}\{1,0\} = 1$, that is

$$HT_n = U_n \text{ for } n \geq 1$$

and we have to explicitly evaluate HT_o. From theorem IV.3.5 we know that HT_o is a polynomial of degree $\mathrm{Max}\{0,0\} = 0$, and so is a constant. According to the definition of H

$$HT_o = H1 = \frac{1}{2}.$$

Hence $HT_o = \frac{1}{2}U_o$.

Writing $\phi(x) = \dfrac{g(x)}{\sqrt{1 - x^2}}$ we obtain

$$g(x) + \frac{x}{\pi}\int_{-1}^{1} \frac{g(t)}{\sqrt{1 - t^2}(t - x)}dt = f(x,0).$$

Approximating g(x) by

$$g_n(x) = \sum_{j=0}^{n} a_j T_j(x)$$

gives

$$\frac{1}{2}a_o U_o + \sum_{j=1}^{n} a_j U_j(x) = f(x,0).$$

Collocation at the zeros of $T_{n+1}(x)$ gives a set of n+1 linear equations for the coefficients a_j. Table IV.5.2 below, as those of chapter III, shows an approximation for the rate of convergence ν of the collocation method calculated from

$$\nu = -\log\left(\frac{e_{2n}}{e_n}\right)/\log(2)$$

where $e_n = \max\limits_{x\in[-1,1]} |g(x) - g_n(x)|$.

The results are also shown for the more common choice of collocation points, namely, the zeros of n^{th} Chebyshev polynomial of second kind.

	Tchebyshev first kind		Tchebyshev second kind	
n	$\|g - g_n\|_\infty$	ν	$\|g - g_n\|_\infty$	ν
2	0.2567		0.1893	
		2.324		2.28
4	5.12E-02		3.89E-02	
		2.084		1.955
8	1.20E-02		1.00E-02	
		2.044		1.920
16	2.92E-03		2.65E-03	
		2.011		1.964
32	7.26E-04		6.80E-04	

Table IV.5.2

From table IV.5.2 it can be seen that the rate of convergence does approximate $\nu = 2$, as is predicted by theorem IV.4.1.

CHAPTER V

Sound Scattering from an Underwater Finite Flexible Plate.

1. Introduction

The work reported in this chapter is concerned with the modelling of the interaction of a sound wave with an elastic plate freely suspended in a fluid.

The mathematical model for this problem consists of a system of two coupled boundary value problems - The Helmholtz equation for the scattered wave pressure and the beam equation for the displacements. The coupling of these equations is in the boundary conditions.

Using the boundary integral method we reduce this problem to a singular integro-differential equation which is in turn transformed into a singular integral equation and solved by the collocation method described in the previous chapters.

In solving this problem, we are solely interested in what is called the *near field solution* as opposed to the more commom practice of solving for the *far field pressure*. That is, we are interested in the efects of the elasticity of the plate on the pressure wave on the plate rather than far from it. The reduction to a singular integral equation is specially suited for this task because the integral equation has the pressure on the plate as the unknown function.

The same problem for the far field solution has been solved before by Shenderov [1973], but this author does not derive a singular integral equation as we do here. He suggests an expansion method using the eigenfunctions of the beam equation as basis functions and the use of general finite difference schemes for the solution of the partial differential equations before any transformation. The use of any of these methods for the problem at hand is inappropriate because they would be far too expensive in computing time and in the case of the expansion method inaccurate for very high wave numbers.

We present, in order to compare numerical results, asymptotic solutions for large and small values of a characteristic parameter (plate-length/wavelength) and results of a numerical method designed by Miller [1982] in connection with a similar problem.

2. Nondimensional Fluid-structure Equations

The pressure underwater and the bending of the finite flexible plate are described by the equations

$$\nabla^2 p + (\omega/c)^2 p = 0 \qquad \text{in } R^2 - [-\ell,\ell] \tag{V.2.1a}$$

$$Bu^{(4)}(\xi) - \tau\omega^2 u(\xi) + p(\xi,0^+) - p(\xi,0^-) = 0, \; -\ell \le \xi \le \ell \tag{V.2.1b}$$

together with the boundary conditions

$$\rho\omega^2 u(\xi) = p_\eta(\xi,0^\pm), \; -\ell \le \xi \le \ell \tag{V.2.2a}$$

$$u''(\pm\ell) = u'''(\pm\ell) = 0 \tag{V.2.2b}$$

and $p(\xi,\eta) \sim e^{-(i\omega/c)(\xi\sin\theta + \eta\cos\theta)}$ as $r^2 = \xi^2 + \eta^2 \mapsto \infty$.
Here $R^2 - [-a,a]$ denotes the line segment $\{-1 \leq \xi \leq 1,\ \eta = 0\}$ and $p(\xi,\eta)e^{-i\omega t}$ denotes the fluid pressure and $u(\xi)e^{-i\omega t}$ the plate displacement in the η-direction (see figure V.2.1); that is, it is assumed that the fluid pressure and plate displacement are periodic in time with the same frequency ω as the incident wave.

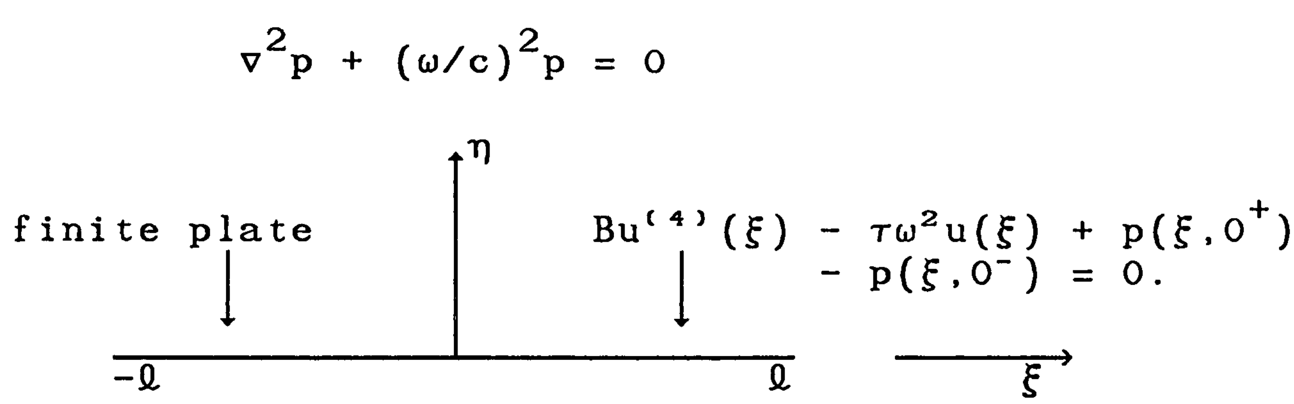

Figure V.2.1

In (V.2.1) and (V.2.2) the following notation has been used:

c = speed of sound in water,

ω = frequency of the incident wave,

θ = angle of incidence (of pressure wave),

ρ_p = density of the plate,

h = thickness of the plate,

$\tau = h\rho_p$ = mass/area,

$B = \dfrac{Eh^3}{12(1-\sigma^2)}$ = bending stiffness of the plate,

E = Young's modulus,

σ = Poisson's ratio,

2ℓ = width of the plate,

ρ = density of the water.

Subtracting the incident wave by writing

$$q(\xi,\eta) = p(\xi,\eta) - e^{-(i\omega/c)(\xi\sin\theta + \eta\cos\theta)}$$

implies that (V.2.1) and (V.2.2) hold with p replaced by q, with the exception of (V.2.2a) which becomes

$$q_\eta(\xi,0^\pm) = \rho\omega^2 u(\xi) + i(\omega/c)\cos\theta\ e^{-i(\omega/c)\sin(\theta)\xi}.$$

The equations and boundary conditions are nondimensionalised by the change of variables

$$x = \xi/\ell,\ y = \eta/\ell,\ \alpha = \ell\omega/c,\ S = \alpha\sin\theta,\ C = \alpha\cos\theta, \lambda = \ell\rho/\tau,$$

$$\beta^2 = B/(\tau\ell^2c^2),\ Q(x,y) = \frac{i}{C}q(\xi,\eta),\ U(x) = \frac{i\tau c^2}{\ell^2 C}u(\xi)$$

to give

$$\nabla^2 Q + \alpha^2 Q = 0 \text{ in } R^2-[-1,1] \tag{V.2.1a'}$$

$$\beta^2 U^{(4)}(x) - \alpha^2 U(x) + Q(x,0^+) - Q(x,0^-) = 0, \quad -1 \le x \le 1 \tag{V.2.1b'}$$

and

$$Q_y(x,0^\pm) = \lambda\alpha^2 U(x) - e^{-iSx}, \ -1 \le x \le 1 \tag{V.2.2a'}$$

$$U''(\pm 1) = U'''(\pm 1) = 0. \tag{V.2.2b'}$$

$$Q(x,y) \longmapsto 0 \text{ as } x^2 + y^2 \longmapsto \infty. \tag{V.2.2c'}$$

For physical reasons the nondimensional pressure Q(x,y) must be an odd function of y, ie Q(x,y) = -Q(x,-y) so that Q(x,0) = 0 for $|x| > 1$. Further (V.2.1b') becomes:

$$\beta^2 U^{(4)}(x) - \alpha^2 U(x) + 2Q(x,0^+) = 0.$$

3. General Outgoing Solution of the Helmholtz equation

Theorem V.3.1

Let μ be a real function satisfying

i) $\mu(\pm 1) = 0$

ii) $d\mu/ds$ exists and is Hölder continuous in $(-1,1)$.

Then the solution of the Helmholtz equation

$$\nabla^2\phi + k^2\phi = 0 \text{ in } R^2 - [-1,1] \qquad (V.3.1)$$

satisfying the conditions

a) $|\phi(x,y)| \leq M \quad \forall\ (x,y) \in R^2 - [-1,1]$

b) $\sqrt{r}\left(\frac{\partial}{\partial r} - ik\right)\phi \to 0$ as $r = \sqrt{x^2 + y^2} \to \infty$

c) $\phi(x,0^+) - \phi(x,0^-) = \mu(p)$

can be written in the form:

$$\phi(x,y) = \int_{-1}^{1} \mu(s)G(x,y,s)ds, \quad (x,y) \in R^2 - [-1,1] \qquad (V.3.2)$$

where

$$G(x,y,s) = \frac{i}{4}\left[\frac{\partial}{\partial t}H_0^{(1)}\left(k\sqrt{(x-s)^2 + (y-t)^2}\right)\right]_{t=0} \qquad (V.3.3)$$

and $H_o^{(1)}$ is the first kind Hankel function of order 0.

Proof: To prove this result we need to show that the function $\phi(x,y)$ defined by (V.3.2) satisfies the Helmholtz equation conditions a), b) and c). We start with equation (V.3.1).

We first observe that for fixed (s,t) the function $H_o^{(1)}(kR)$ with

$R(x,y) = \sqrt{(x - s)^2 + (y - t)^2}$ solves the Helmholtz equation for $(x,y) \neq (s,t)$ and satisfies the radiation condition b). For $(x,y) \in R^2 - [-1,1]$ and $s \in [-1,1]$ we have

$$(\nabla^2 + k^2)G(x,y,s) = (\nabla^2 + k^2)\frac{i}{4}\left[\frac{\partial}{\partial t}H_o^{(1)}\left(k\sqrt{(x - s)^2 + (y - t)^2}\right)\right]_{t=0}$$

$$= \frac{i}{4}\left[\frac{\partial}{\partial t}(\nabla^2 + k^2)H_o^{(1)}\left(k\sqrt{(x - s)^2 + (y - t)^2}\right)\right]_{t=0} = 0$$

from the observation above. From condition (ii) differentiation under the integral sign is permited in (V.3.2) and then ϕ satisfies (V.3.1). Writing $x - s = r\cos\theta$ and $y = r\sin\theta$ it can be easily shown that

$$G(x,y,s) = \frac{-i}{4}k\sin(\theta)H_o^{(1)\prime}(kr) = \frac{i}{4}k\sin(\theta)H_1^{(1)}(kr).$$

Thus

$$\sqrt{r}\left(\frac{\partial}{\partial r} - ik\right)G = \sqrt{r}\left(\frac{\partial}{\partial r} - ik\right)\frac{i}{4}k\sin(\theta)H_1^{(1)}(kr)$$

$$= \frac{i}{4}k^2\sin(\theta)\sqrt{r}\left(H_o^{(1)}(kr) - iH_1^{(1)}(kr)\right) + o(r^{-1/2})$$

from formulae (9.1.27) and (9.2.7) of Abramowitz & Stegun [1964]. Also from (9.2.7) of the same reference we deduce

$$H_0^{(1)}(kr) - iH_1^{(1)}(kr) = \sqrt{\frac{2}{\pi kr}}\left[e^{i(kr-\pi/4)} - e^{i(kr-3\pi/4)}\right] + o(r^{-1})$$

$$= \sqrt{\frac{2}{\pi kr}}e^{ikr}\left[e^{-i\pi/4} - e^{-i3\pi/4}\right] + o(r^{-1})$$

$$= o(r^{-1}).$$

This allows us to conclude that $\sqrt{r}\left(\frac{\partial}{\partial r} - ik\right)G = o(r^{-1/2})$ as $r \to \infty$, hence G satisfies the radiation condition b) and so does ϕ.

To prove part a) we notice that by performing the differentiation in the definition of G we obtain

$$G(x,y,s) = \frac{-i}{4}kH_0^{(1)}(kR)\frac{y}{R} = \frac{y}{2\pi R}2 + P(x,y,s)$$

where $R = \sqrt{(x - s)^2 + y^2}$ and P is a continuous function satisfying $P(x,y,s) \longrightarrow 0$ as $R \to \infty$, see Abramowitz & Stegun [1964, pg. 360]. We can then write

$$G(x,y,s) = \frac{1}{2\pi}\mathrm{Im}\left(\frac{1}{s - z}\right) + P(x,y,s)$$

where $z = x + iy$. Using this (V.3.2) can be rewritten in the form

$$\phi(x,y) = \frac{1}{2\pi}\mathrm{Im}\int_{-1}^{1}\frac{\mu(s)}{s - z}ds + \int_{-1}^{1}P(x,y,s)\mu(s)ds$$

and then property a) follows from i) and lemmas (II.2.1) and (II.2.2).

Finally the Plemelj-Sokhotski formula (II.3.4) applied to the function

$$\psi(z) = \frac{1}{2\pi}\int_{-1}^{1} \frac{\mu(s)}{s - z}ds$$

yields

$$\psi^{+}(x) - \psi^{-}(x) = i\mu(x)$$

and then from the continuity of $P(x,y,s)$

$$\phi^{+}(x) - \phi^{-}(x) = \mathrm{Im}(\psi^{+}(x) - \psi^{-}(x)) = \mu(x). \blacksquare$$

In order to find a solution to the Helmholtz equation when the boundary condition is

$$\frac{\partial\phi}{\partial y} + \frac{\partial\phi_o}{\partial y} = 0$$

where ϕ_o is a given function, we need to find a function μ such that the boundary condition:

$$-\frac{\partial\phi_o}{\partial y} = \frac{\partial}{\partial y}\int_{-1}^{1} \mu(s)G(x,y,s)ds, \quad (x,y) \in [-1,1]. \qquad \text{(V.3.4)}$$

Note that (V.3.4) is an integro-differential equation for the function μ.

4. Reduction to an Integro-differential Equation

To apply this theory to (V.2.1') and (V.2.2') we first note that equation (V.2.1a') and (V.2.2a') are of the desired form, if we assume for the moment that $U(x)$ is a given known function. Now, since Q is an odd function in y, condition c) can be written as

$$\mu(P) = 2Q(x,0^{+})$$

so that

$$Q(x,y) = \tfrac{1}{2}i\int_{-1}^{1} Q(s,0^+)\frac{\partial}{\partial t} H_o^{(1)}(\alpha R)\bigg|_{t=0} ds$$

where $R = \left[(x - s)^2 + (y - t)^2\right]^{\frac{1}{2}}$.

Thus, enforcing the boundary condition (V.2.2a') we have,

$$\lambda\alpha^2 U(x) - e^{-iSx} = \tfrac{1}{2}i\ \frac{\partial}{\partial y}\left[\int_{-1}^{1} Q(s,0^+)\frac{\partial}{\partial t}H_o^{(1)}(\alpha R)\bigg|_{t=0} ds\right]\bigg|_{y=0}.$$

On writing $f(x) = Q(x,0^+)$, since $\frac{\partial R}{\partial y} = -\frac{\partial R}{\partial t}$ we have

$$\lambda\alpha^2 U(x) - e^{-iSx} = -\tfrac{1}{2}i\ \frac{\partial}{\partial y}\left[\int_{-1}^{1} f(s)\frac{\partial}{\partial y}H_o^{(1)}(\alpha R)\bigg|_{t=0} ds\right]\bigg|_{y=0}$$

$$= -\tfrac{1}{2}i\ \frac{\partial}{\partial y}\left[\int_{-1}^{1} f(s)\frac{\partial}{\partial y}H_o^{(1)}(\alpha[(x-s)^2+y^2]^{\frac{1}{2}})ds\right]\bigg|_{y=0}$$

$$= -\tfrac{1}{2}i\ \frac{\partial^2}{\partial y^2}\left[\int_{-1}^{1} f(s)H_o^{(1)}(\alpha[(x-s)^2+y^2]^{\frac{1}{2}})\bigg|_{y=0} ds\right]$$

Or,

$$\lambda\alpha^2 U(x) - e^{-iSx} = \tfrac{1}{2}i\left(\frac{\partial^2}{\partial x^2} + \alpha^2\right)\int_{-1}^{1} f(s)H_o^{(1)}(\alpha|x-s|)ds \qquad (V.4.1)$$

using the differential identity $-\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x^2} + \alpha^2$ for functions satisfying (V.2.1a').

Determination of the Green's function for the boundary value problem consisting of (V.2.1b') and (V.2.2b') yields the solution

$$U(x) = 2\int_{-1}^{1} h(x,s)f(s)ds \qquad (V.4.2)$$

where

$$h(x,s) = \begin{cases} (h_1(x),h_2(x)) \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} h_1(-s) \\ h_2(-s) \end{bmatrix}, & x \geq s \\ h(s,x) & , x < s \end{cases}$$

with

$$h_1(x) = \sinh(\gamma(1-x)) + \sin(\gamma(1-x))$$

$$h_2(x) = \cosh(\gamma(1-x)) + \cos(\gamma(1-x))$$

and $\gamma = \sqrt{\alpha/\beta}$.

The constants are given by

$$a = -(\sinh(2\gamma) + \sin(2\gamma))/A$$

$$b = (\cosh(2\gamma) - \cos(2\gamma))/A$$

$$c = (-\sinh(2\gamma) + \sin(2\gamma))/A$$

where $A = 4\beta^2\gamma^3(1 - \cosh(2\gamma)\cos(2\gamma))$[1].

The unique solution is given by (V.4.2) provided γ is not an eigenvalue such that

$$\cosh(2\gamma)\cos(2\gamma) = 1.$$

Substituting for U(x) from (V.4.2) into (V.4.1) gives

$$-\tfrac{1}{2}i\left(\frac{\partial^2}{\partial x^2}+\alpha^2\right)\int_{-1}^{1} H_0^{(1)}(\alpha|x-s|)f(s)ds + 2\lambda\alpha^2\int_{-1}^{1} h(x,s)f(s)ds$$

$$= e^{-iSx} \qquad \text{(V.4.3)}$$

1

This calculation was performed by D. C. Handscomb and has been confirmed by the author using MACSYMA.

which is an integro-differential equation for $f(x)$.

The singularity in the integrand is such that the differential and integral operators cannot be interchanged. In order to achieve this it is first necessary to integrate by parts. Thus

$$\int_{-1}^{1} H_0^{(1)}(\alpha|x-s|)f(s)ds = -\int_{-1}^{1} K(x,s)f'(s)ds,$$

where

$$K(x,s) = \int_{-1}^{s} H_0^{(1)}(\alpha|x-w|)dw$$

and $f'(x)$ denotes the derivative of f. Use has been made of the fact that $f(\pm 1) = 0$. This is true because the pressure vanishes for $|x| > 1$ (see comment following formula (V.2.2c')) and been a continuous function must also vanish at the ends. Using this (V.4.3) becomes

$$\tfrac{1}{2}i\left(\frac{\partial^2}{\partial x^2} + \alpha^2\right)\int_{-1}^{1} K(x,s)f'(s)ds + 2\lambda\alpha^2\int_{-1}^{1} h(x,s)f(s)ds$$

$$= e^{-iSx}. \qquad \text{(V.4.4)}$$

This equation now admits the interchange of the differential and integral operators.

5. Derivation of a singular integrodifferential equation for f(x)

We start with the following result from Abramowitz & Stegun [1964 page 361]

$$\frac{d^2}{dz^2}H_0^{(1)}(z) = \tfrac{1}{2}\left[H_2^{(1)}(z) - H_0^{(1)}(z)\right] \tag{V.5.1}$$

$$\tfrac{1}{2}\left[H_2^{(1)}(z) + H_0^{(1)}(z)\right] = \frac{H_1^{(1)}(z)}{z} \tag{V.5.2}$$

where $H_n^{(1)}(z)$ denotes the first kind Hankel function.

From (V.5.1) and (V.5.2) we have

$$\left(\frac{\partial^2}{\partial x^2} + \alpha^2\right)H_0^{(1)}(\alpha|x-s|) = \frac{\alpha^2 H_1^{(1)}(\alpha|x-s|)}{\alpha|x-s|}$$

so that (V.4.4) becomes

$$\tfrac{1}{2}i\alpha^2\int_{-1}^{1} r(x,s)f'(s)ds + 2\lambda\alpha^2\int_{-1}^{1} h(x,s)f(s)ds = e^{-iSx} \tag{V.5.3}$$

where

$$r(x,s) = \int_{-1}^{s}\frac{H_1^{(1)}(\alpha|x-w|)}{\alpha|x-w|}dw$$

Now

$$\frac{H_1^{(1)}(\alpha|z|)}{\alpha|z|} = \frac{1}{\alpha|z|}\left[J_1(\alpha|z|) + iY_1(\alpha|z|)\right]$$

$$= \frac{1}{\alpha|z|}\left[\tfrac{1}{2}\alpha|z|\sum_{k=0}^{\infty}\frac{(-1)^k(\tfrac{1}{2}\alpha|z|)^{2k}}{k!(k+1)!}\right] + \frac{i}{\alpha|z|}\left[-\frac{2}{\pi\alpha|z|}\right.$$

$$+ \frac{2}{\pi}\log(\tfrac{1}{2}\alpha|z|)\tfrac{1}{2}\alpha|z|\sum_{k=0}^{\infty}\frac{(-1)^k(\tfrac{1}{2}\alpha|z|)^{2k}}{k!(k+1)!}$$

$$\left. - \tfrac{1}{2}\alpha|z|\sum_{k=0}^{\infty}(\psi(k+1) + \psi(k))\frac{(-1)^k(\tfrac{1}{2}\alpha|z|)^{2k}}{k!(k+1)!}\right]$$

where $\psi(x)$ is the digamma function (see Abramowitz & Stegun [1964 page 360]).

Thus

$$\frac{H_1^{(1)}(\alpha|z|)}{\alpha|z|} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{-2k-1}\alpha^{2k}z^{2k}}{k!(k+1)!} + i\left[-\frac{2}{\pi\alpha^2|z|^2}\right.$$

$$+ \frac{1}{\pi}\log(\tfrac{1}{2}\alpha|z|) + \frac{2}{\pi}\log(\tfrac{1}{2}\alpha|z|)\sum_{k=1}^{\infty}\frac{(-1)^k 2^{-2k-1}\alpha^{2k}z^{2k}}{k!(k+1)!}$$

$$\left. - \sum_{k=0}^{\infty}(\psi(k+1) + \psi(k))\frac{(-1)^k 2^{-2k-1}\alpha^{2k}z^{2k}}{k!(k+1)!}\right]$$

$$= R(z) - \frac{2i}{\pi\alpha^2 z^2} + \frac{i}{\pi}\log(\tfrac{1}{2}\alpha|z|) + I(z) \qquad (V.5.4)$$

where $R(z)$ is infinitely differentiable and $I(z)$ is continuous: in fact, $I(z)$ is Lipschitz continuous.

Thus

$$r(x,s) = \int_{-1}^{s}\frac{H_1^{(1)}(\alpha|x-w|)}{\alpha|x-w|}dw = \int_{-1}^{s}F(\alpha|x-w|)dw + \frac{2i}{\pi\alpha^2(s-x)}$$

$$+ \frac{i}{\pi}(s-x)\left[\log(\tfrac{1}{2}\alpha|x-s|) - 1\right] \qquad (V.5.5)$$

where $F(z) = R(z) + I(z)$.

From (V.5.3) and (V.5.5) we have the singular integrodifferential equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f'(s)}{s-x}ds + \frac{\alpha^2}{2\pi}\int_{-1}^{1}(s-x)\left[\log(\tfrac{1}{2}\alpha|x-s|) - 1\right]f'(s)ds$$

$$- \tfrac{1}{2}i\alpha^2\int_{-1}^{1}G(x,s)f'(s)ds - 2\lambda\alpha^2\int_{-1}^{1}h(x,s)f(s)ds = -e^{-iSx} \qquad (V.5.6)$$

where

$$G(x,s) = \int_{-1}^{s}F(\alpha|x-w|)dw.$$

6. Asymptotic solutions

From (V.4.1) and (V.2.1b') we have the set of equations

$$\lambda\alpha^2 U(x) - \tfrac{1}{2}i\left(\frac{\partial^2}{\partial x^2} + \alpha^2\right)\int_{-1}^{1} H_0^{(1)}(\alpha|x-s|)f(s)ds = e^{-iSx} \qquad \text{(V.6.1a)}$$

$$\beta^2 U^{(4)}(x) - \alpha^2 U(x) = -2f(x) \qquad \text{(V.6.1b)}$$

In this section we shall consider the two cases of high and low frequencies asymptotic behaviour of the solutions of (V.6.1).

Case I: $\alpha \gg 1$

The phase of the scattered wave on the plate is dominated by the phase of the driving force, i.e.,

$$f(x) = f_0(x)e^{-iSx} = f_0(x)e^{-i\alpha\sin(\theta)x}$$

with $f_0(x)$ a slowly varying function.

Since the integrand in (V.6.1a) has a singularity at the point $x = s$ we can write

$$\int_{-1}^{1} f_0(s)e^{-i\alpha\sin(\theta)s}\, H_0^{(1)}(\alpha|x-s|)ds$$

$$\simeq f_0(x)\int_{-1}^{1} e^{-i\alpha\sin(\theta)s}\, H_0^{(1)}(\alpha|x-s|)ds$$

$$= f_0(u/\alpha)\int_{-\alpha}^{\alpha} e^{-i\sin(\theta)v}\, H_0^{(1)}(|u-v|)\frac{dv}{\alpha}$$

$$\simeq (1/\alpha) f_o(u/\alpha) \int_{-\infty}^{\infty} e^{-i(u+w)\sin\theta} H_o^{(1)}(|w|)dw$$

$$= (1/\alpha) f_o(u/\alpha) e^{-iu\sin\theta} \int_0^{\infty} (e^{iw\sin\theta} + e^{-iw\sin\theta}) H_o^{(1)}(|w|)dw$$

$$= (2/\alpha) f_o(u/\alpha) e^{-iu\sin\theta} \int_0^{\infty} \cos(w\sin\theta) H_o^{(1)}(|w|)dw$$

$$= \frac{2f_o(u/\alpha)e^{-iu\sin(\theta)}}{\alpha|\cos\theta|}$$

see Gradshteyn & Ryzhik [1980 page 731].

Thus

$$\int_{-1}^{1} f_o(s) e^{-i\alpha\sin(\theta)s} H_o^{(1)}(\alpha|x-s|)ds$$

$$\simeq \frac{2f_o(x)e^{-i\alpha\sin(\theta)x}}{\alpha|\cos\theta|}. \qquad (V.6.2)$$

Now, for $\alpha >> 1$, $U(x)$ has a similar behaviour to $f(x)$, ie.

$$U(x) = U_o(x)e^{-i\alpha\sin(\theta)x}$$

with $U_o(x)$ a slowly varying function.

Neglecting the derivatives of $U_o(x)$, we have from (V.6.1b)

$$(\beta^2\alpha^2\sin^4\theta - 1)\alpha^2 U(x) = -2f(x)$$

and then from equations (V.6.1a) and (V.6.2) we obtain

$$\frac{-2\lambda f(x)}{((\alpha\beta)^2\sin^4\theta - 1)} - ½i(\frac{\partial^2}{\partial x^2} + \alpha^2)\frac{2f_o(x)e^{-i\alpha\sin(\theta)x}}{\alpha|\cos\theta|}$$

$$\simeq e^{-i\alpha\sin(\theta)x}. \tag{V.6.3}$$

Neglecting the derivatives of $f_o(x)$ (V.6.3) reads

$$\frac{-2\lambda f(x)}{((\alpha\beta)^2\sin^4\theta - 1)} - i\alpha(1 - \sin^2\theta)\frac{f_o(x)e^{-i\alpha\sin(\theta)x}}{|\cos\theta|} \simeq e^{-i\alpha\sin(\theta)x}$$

or

$$\frac{-2\lambda f_o(x)e^{-i\alpha\sin(\theta)x}}{((\alpha\beta)^2\sin^4\theta - 1)} - i\alpha|\cos\theta|\ f_o(x)e^{-i\alpha\sin(\theta)x} \simeq e^{-i\alpha\sin(\theta)x}$$

and so

$$f_o(x) \simeq \frac{(\alpha\beta)^2\sin^4(\theta) - 1}{-2\lambda - i\alpha|\cos\theta|(1 - (\alpha\beta)^2\sin^4\theta)}$$

$$= \frac{1 - (A\sin\theta)^4}{2\lambda - i\alpha|\cos\theta|(1 - (A\sin\theta)^4)} \tag{V.6.4}$$

with $A = \sqrt{\alpha\beta}$.

But

$$f(x) = Q(x,0^+) = \frac{iq(x,0^+)}{\alpha\cos\theta}$$

so that

$$q(x,0^+) \simeq \frac{-i\alpha\cos\theta\ (1 - (A\sin\theta)^4)e^{-i\alpha\sin(\theta)x}}{2\lambda - i\alpha|\cos\theta|(1 - (A\sin\theta)^4)} = \text{const}\times e^{-i\alpha\sin(\theta)x}. \tag{V.6.5}$$

The constant in (V.6.5) can be better represented in the form

$$\frac{-ib\cos\theta\ (1 - (A\sin\theta)^4)}{1 - ib|\cos\theta|(1 - (A\sin\theta)^4)}, \tag{V.6.6}$$

where

$$b = \frac{\alpha}{2\lambda} = \frac{\omega h \rho_p}{2\rho c}$$

Case II: $\alpha << 1$.

Let $f(x) = C\sqrt{1 - x^2}$ where the constant C is to be determined. (This assumption is made from consideration of the general form of the solution of equation (V.5.6)). We make the further approximation

$$H_o^{(1)}(\alpha|x-s|) \simeq 1 + i(\frac{2}{\pi}\log(\alpha|x-s|) - 0.0738). \qquad (V.6.7)$$

Note also that if we make the change of variables $x = \alpha z$ and write $\tilde{u}(z) = U(\alpha z) = U(x)$, equation (V.6.1b) reads

$$\beta^2\alpha^4\tilde{u}^{(4)}(z) - \alpha^2\tilde{u}(z) = -2f(\alpha z)$$

so that for α small the first term on the left hand side can be neglected compared with the other and we have

$$-\alpha^2 U(x) \simeq -2f(x) \qquad (V.6.8)$$

Substitution of (V.6.7) and (V.6.8) into (V.4.1) gives

$$2\lambda f(x) - \tfrac{1}{2}i(\frac{\partial^2}{\partial x^2} + \alpha^2)\int_{-1}^{1} C\sqrt{1 - s^2}(1 + i(\frac{2}{\pi}\log(\alpha|x-s|)-0.0738)ds$$

$$= e^{-i\alpha\sin(\theta)x} \qquad (V.6.9)$$

We must evaluate the integral

$$I(x) = \int_{-1}^{1} \sqrt{1 - s^2}(1 - 0.0738i)ds + \frac{2}{\pi}i \int_{-1}^{1} \sqrt{1 - s^2}\log(\alpha|x-s|)ds$$

$$= (1 - 0.0738i) \int_{-1}^{1} \sqrt{1 - s^2}ds + \frac{2}{\pi}i\log(\alpha) \int_{-1}^{1} \sqrt{1 - s^2}ds$$

$$+ \frac{2}{\pi}i \int_{-1}^{1} \sqrt{1 - s^2}\log|x-s|ds.$$

Let

$$I_1(x) = \int_{-1}^{1} \sqrt{1 - s^2}\log|x-s|ds$$

so that

$$I_1'(x) = \int_{-1}^{1} \sqrt{1 - s^2}\frac{ds}{x - s} = \pi x$$

and thus

$$I_1(x) = \frac{\pi}{2}x^2 + \text{const}.$$

Therefore, on evaluation

$$I(x) = (1 - 0.0738i)\frac{\pi}{2} + i\log(\alpha) + ix^2 + \text{const}.$$

Substituting this expression into (V.6.9) yields

$$2\lambda C\sqrt{1 - x^2} - \tfrac{1}{2}iC\left[2i + \alpha^2((1 - 0.0738i)\frac{\pi}{2} + \log(\alpha)+\tfrac{1}{2}ix^2+\text{const})\right]$$

$$= e^{-i\alpha\sin(\theta)x} = 1 - i\alpha\sin(\theta)x + \ldots$$

Neglecting terms of order α and greater gives

$$(2\lambda + 1)C = 1$$

or

$$C = \frac{1}{(2\lambda + 1)}.$$

But

$$f(x) = Q(x,0^+) = \frac{iq(x,0^+)}{\alpha\cos\theta}.$$

Hence an approximation to $q(x,0^+)$ for α small is given by

$$q(x,0^+) = \frac{-i\alpha\cos\theta}{2\lambda + 1}\sqrt{1 - x^2}. \qquad \text{(V.6.10)}$$

7. Numerical solution

By writing $v(x) = f'(x)$ equation (V.5.6) can be rewritten as

$$\frac{1}{\pi}\int_{-1}^{1}\frac{v(t)}{t - x}dt + \int_{-1}^{1}H(x,t)v(t)dt - 2\lambda\alpha^2\int_{-1}^{1}h(x,t)\left[\int_{-1}^{t}v(s)ds\right]dt$$

$$= -e^{-iSx} \qquad \text{(V.7.1)}$$

with the condition,

$$\int_{-1}^{1}v(x)dx = 0,$$

where

$$H(x,t) = ½\frac{\alpha^2}{\pi}(t-x)\left[\log(½\alpha|x-t|) - 1\right] - ½i\alpha^2G(x,t).$$

We assume Equation (V.7.1) has a unique solution. By a slight modification of the argument in section III.3 we may write v as:

$$v(x) = \frac{u(x)}{\sqrt{1 - x^2}}$$

where u(x) is Hölder continuous. This follows since the operator $T:C[0,1] \longmapsto C[0,1]$ defined by

$$Tf = \int_{-1}^{1} h(x,t)\left[\int_{-1}^{t} f(s)ds\right]dt$$

is a compact operator in C[-1,1].

According to the collocation method described in chapter III we approximate u(x) by a Chebyshev series

$$u_n(x) = a_oT_o(x) + a_1T_1(x) + \dots + a_nT_n(x) \qquad (V.7.2)$$

which in turn yields the approximation to f:

$$f_n(x) = \int_{-1}^{x} v_n(t)dt = \sum_{j=0}^{n} a_j \int_{-1}^{x} \frac{T_j(t)}{\sqrt{1 - t^2}}dt$$

$$= a_o(\pi - \arcos(x)) - \sum_{j=1}^{n} \frac{a_j}{j}\sin(j\arcos(x))$$

or on imposing the condition

$$\int_{-1}^{1} v_n(x)dx = 0$$

we obtain

$$f_n(x) = - \sum_{j=1}^{n} \frac{a_j}{j}\sin(j\arcos(x)). \qquad (V.7.3)$$

Let

$$r_n(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{u_n(t)}{(t - x)\sqrt{1 - t^2}}dt + \int_{-1}^{1} \frac{H(x,t)}{\sqrt{1 - t^2}}u_n(t)dt$$

$$- 2\lambda\alpha^2 \int_{-1}^{1} h(x,t) f_n(t)dt + e^{-iSx}$$

$$= \sum_{j=1}^{n} a_j \left[\frac{1}{\pi} \int_{1}^{1} \frac{T_j(t)}{(t - x)\sqrt{1 - t^2}} dt + \int_{-1}^{1} \frac{H(x,t)}{\sqrt{1 - t^2}} T_j(t)dt \right]$$

$$- 2\lambda\alpha^2 \sum_{j=1}^{n} \frac{a_j}{j} \int_{-1}^{1} h(x,t)\sin(j\arccos(t))dt + e^{-iSx} \qquad (V.7.4)$$

If we let $r_n(x_m) = 0$, $m = 1,2, \ldots, n$, $x_m \in [-1,1]$, then (V.7.4) gives rise to a system of linear equations for the complex unknowns a_j. The points x_m, as usual, are chosen to be

$$x_m = \cos\left(\frac{(2m-1)\pi}{2n}\right).$$

the zeros of the first kind Chebyshev polynomial of degree n.

The rest of this section will deal with the evaluation of the various integrals involved in (V.7.4). We first note that, since

$$\int_{-1}^{1} h(x,t)\sin(j\arccos(t))dt = \int_{-1}^{x} h(x,t)\sin(j\arccos(t))dt$$

$$+ \int_{x}^{1} h(t,x)\sin(j\arccos(t))dt,$$

and the function $h(x,t)$, being the Green's function from a boundary value problem, is a smooth function in each of the above intervals, the evaluation of the integrals by numerical methods should be straightforward. We make the change of variables $t = \cos(\theta)$ in each of those integrals and then calculate them by the composite trapezium rule. The number of subdivisions is taken to be equal to the degree of the aproximating polynomial $u_n(x)$, ie, n.

The integral

$$\int_{-1}^{1} H(x,t)\frac{\cos(j\arccos(t))}{\sqrt{1 - t^2}}dt$$

can be rewritten as

$$\frac{\alpha^2}{2\pi}\int_{-1}^{1}\frac{(t-x)[\ln(½\alpha|x-t|) - 1]\cos(j\arccos(t))}{\sqrt{1 - t^2}}dt$$

$$- ½i\alpha^2\int_{-1}^{1}\frac{G(x,t)\cos(j\arccos(t))}{\sqrt{1 - t^2}}dt. \qquad \text{(V.7.6)}$$

The integrals will be treated separately.

Let

$$J_j(x) = \frac{\alpha^2}{2\pi}\int_{-1}^{1}\frac{(t-x)[\ln(½\alpha|x-t|) - 1]\cos(j\arccos(t))}{\sqrt{1 - t^2}}dt. \qquad \text{(V.7.7)}$$

Then

$$J_j'(x) = -\frac{1}{\pi}\int_{-1}^{1}\frac{\ln(½\alpha|x-t|)\cos(j\arccos(t))}{\sqrt{1 - t^2}}dt$$

and

$$J_j''(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{\cos(j\arccos(t))}{\sqrt{1 - t^2}(t - x)}dt = U_{j-1}(x).$$

Using standard properties of the Chebyshev polynomials we obtain

$$J_j(x) = \begin{cases} ½x^2 + a_1x + b_1 & , j = 1 \\ \frac{1}{2j}\left[\frac{T_{j+1}(x)}{j + 1} - \frac{T_{j-1}(x)}{j - 1}\right] + a_jx + b_j, & j \neq 1 \end{cases} \qquad \text{(V.7.8)}$$

where a_j and b_j are determined such that

$$J_j(0) = \begin{cases} b_1 & , \ j = 1 \\ \dfrac{1}{2j}\left[\dfrac{\cos\frac{1}{2}(j+1)\pi}{j+1} - \dfrac{\cos\frac{1}{2}(j-1)\pi}{j-1}\right] + b_j, & j \neq 1 \end{cases}$$

$$J_j'(0) = \begin{cases} a_1 & j = 1 \\ \dfrac{\cos(\frac{1}{2}j\pi)}{j} + a_j & j \neq 1 \end{cases} .$$

In this way the evaluation of the first integral in (V.7.6) is reduced to the evaluation of the following integrals

$$J_j(0) = \frac{1}{\pi}\int_0^\pi \cos\theta \cos j\theta \log(\tfrac{1}{2}\alpha|\cos\theta|) - 1)d\theta \qquad \text{(V.7.9)}$$

$$J_j'(0) = -\frac{1}{\pi}\int_0^\pi \cos j\theta \log(\tfrac{1}{2}|\cos\theta|)d\theta. \qquad \text{(V.7.10)}$$

We can rewrite $J_j(0)$ and $J_j'(0)$ as

$$J_j(0) = \frac{1}{\pi}\left[\int_0^\pi \cos\theta \cos j\theta (\log(\tfrac{1}{2}\alpha)-1)d\theta + \int_0^\pi \cos\theta \cos j\theta \log|\cos\theta|d\theta\right]$$

$$J_j'(0) = -\frac{1}{\pi}\left[\int_0^\pi \cos j\theta \log(\tfrac{1}{2}\alpha)d\theta + \int_0^\pi \cos j\theta \log|\cos\theta|d\theta\right]$$

so that

$$J_j(0) = \begin{cases} \tfrac{1}{2}(\log(\tfrac{1}{2}\alpha) - 1) + \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos^2\theta \log|\cos\theta|d\theta, & j = 1 \\ \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos\theta \cos j\theta \log|\cos\theta|d\theta & j \neq 1 \end{cases} \quad \text{(V.7.11)}$$

$$J_j'(0) = -\frac{1}{\pi}\int_0^\pi \cos j\theta \log|\cos\theta|d\theta \quad \text{(V.7.12)}$$

We first note that (Gradshteyn & Ryzhik[1980 page 584]),

$$J_j'(0) = \begin{cases} -\log(2) & j = 0 \\ 0 & j \text{ odd} \\ \dfrac{(-1)^{k-1}}{j} & j = 2k \end{cases}$$

and that the integrals in (V.7.11) may be calculated from $J_j'(0)$ using the formula $\cos\theta \cos j\theta = \tfrac{1}{2}(\cos(j+1)\theta + \cos(j-1)\theta)$.

The second integral in (V.7.6) has still to be evaluated. From (V.5.6) we have

$$G(x,t) = \int_{-1}^{t} F(\alpha|x-s|)ds$$

$$= \int_{-1}^{t} \left[\frac{H_1^{(1)}(\alpha|x-s|)}{\alpha|x-s|} + \frac{2i}{\pi\alpha^2(x-s)^2} - \frac{i}{\pi}\log(\tfrac{1}{2}\alpha|x-s|)\right]ds. \quad \text{(V.7.13)}$$

Our strategy for evaluating these integrals is as follows: For each fixed x, we approximate G(x,t) by a cubic spline S(t) in the variable t, on a fixed set of knots, and then perform the integral

$$\int_{-1}^{1} \frac{S(t)\cos(j\arccos(t))}{\sqrt{1 - t^2}}dt \quad \text{(V.7.14)}$$

exactly. In order to calculate the cubic spline S(t) we need to be able to evaluate the function G(x,t) at any point $t \in [-1,1]$. We shall now describe how this is to be achieved.

We shall concentrate on the calculation of a function P(x) such that $P'(x) = \frac{H_1^{(1)}(x)}{x}$ for $x > 0$.

We observe that the following relation is valid for the Hankel function (see Abramowitz & Stegun [1964 page 361])

$$\frac{d}{dz}H_1^{(1)}(z) = H_o^{(1)}(z) - \frac{H_1{}^{(1)}(z)}{z}$$

so that the function

$$H_1^{(1)}(x) - \int_0^x H_o^{(1)}(t)dt \qquad (V.7.15)$$

has the desired property and is a natural choice.

From Abramowitz & Stegun [1964 pages 480-481] we have

$$\int_0^x H_o^{(1)}(t)dt = H_o^{(1)}(x) + \tfrac{1}{2}\pi x\left[\mathbf{H}_o(x)H_1^{(1)}(x) - \mathbf{H}_1(x)H_o^{(1)}(x)\right] \quad (V.7.16)$$

where $\mathbf{H}_n(x)$ is the Struve function defined, for example, in Abramowitz & Stegun [1964 page 496].

Formula (V.7.16) is valid for any value of x, but for large values of x, the evaluation of the Struve functions as well as the Hankel functions are time consuming. So for large values of x ($8 \leq x \leq \infty$) use

$$\int_0^x H_0^{(1)}(t)dt = 1 - x^{-\frac{1}{2}}e^{i(x-\pi/4)}\left[\sum_{k=0}^{7}(-1)^k(8/x)^{2k}(a_k(8/x) + ib_k) + \right.$$

$$\left. + \epsilon(x)\right], \qquad (V.7.17)$$

where $|\epsilon(x)| \leq 2x10^{-9}$ and the coefficients a_k and b_k are given in Abramowitz & Stegun [1964 page 481].

Formulae (V.7.16-17) and (V.7.15) allow us to calculate an approximation to the function P(x) for any given positive x.

For the sake of minimizing the length of the expressions which follow we shall make use of the notation

$$W(x) = \frac{2i}{\pi\alpha^2x^2} - \frac{i}{\pi}\log(\alpha x).$$

We observe that in (V.7.13) the integration does not distribute to each of the terms, because the integral of each of the first two terms does not exist on its own.

In order that we can make use of the formulae (V.7.15) and (V.7.16-17) to evaluate (V.7.13) we need to make a change of variable in (V.7.13).

$$\int_{-1}^{t}\left[\frac{H_1^{(1)}(\alpha|x-s|)}{\alpha|x-s|} + W(|x - s|)\right]ds$$

$$= \begin{cases} \displaystyle\int_{-1}^{t}\left[\frac{H_1^{(1)}(\alpha(x-s))}{\alpha(x-s)} + W(x-s)\right]ds & x\geq t \\ \displaystyle\int_{-1}^{x}\left[\frac{H_1^{(1)}(\alpha(x-s))}{\alpha(x-s)} + W(x-s)\right]ds + \int_{x}^{t}\left[\frac{H_1^{(1)}(\alpha(s-x))}{\alpha(s-x)} + W(s-x)\right]ds & x<t \end{cases}$$

$$= \begin{cases} -\displaystyle\int_{\alpha(x+1)}^{\alpha(x-t)} \left[\frac{H_1^{(1)}(\alpha u)}{\alpha u} + W(u)\right] du & x \geq t \\ -\displaystyle\int_{\alpha(x+1)}^{0} \left[\frac{H_1^{(1)}(\alpha u)}{\alpha u} + W(u)\right] du + \int_0^{\alpha(t-x)} \left[\frac{H_1^{(1)}(\alpha u)}{\alpha u} + W(u)\right] du & x < t \end{cases}$$

and finally

$$G(x,t) = \begin{cases} \dfrac{1}{\alpha}\left[\displaystyle\int_0^{\alpha(x+1)} \left[\frac{H_1^{(1)}(u)}{u} + W(u/\alpha)\right] du \right. \\ \dfrac{1}{\alpha}\left[\displaystyle\int_0^{\alpha(x+1)} \left[\frac{H_1^{(1)}(u)}{u} + W(u/\alpha)\right] du \right. \end{cases}$$

$$\begin{cases} \left. - \displaystyle\int_0^{\alpha(x-t)} \left[\frac{H_1^{(1)}(u)}{u} + W(u/\alpha)\right] du \right] & x \geq t \\ \left. + \displaystyle\int_0^{\alpha(t-x)} \left[\frac{H_1^{(1)}(u)}{u} + W(u/\alpha)\right] du \right] & x < t \end{cases} \tag{V.7.18}$$

Note that all the integrals in (V.7.18) are now in a form such that (V.7.15) may be used, so that they can be evaluated by using (V.7.15) in conjunction with (V.7.17).

8. Miller's Numerical Method

In this section we describe very briefly a method due to Miller, see Miller[1982 pg• 361]. This method is based on quadrature and solves equation (V.4.3). Miller writes equation (V.4.3) in the form

$$-\left(\frac{\partial^2}{\partial x^2} + \alpha^2\right) I_1(x;f) + I_2(x;f) = e^{-iSx} \tag{V.8.1}$$

where

$$I_1(x;f) = \int_{-1}^{1} \frac{i}{2}H_0^{(1)}(\alpha|x-s|)f(s)ds$$

$$I_2(x;f) = 2\lambda\alpha^2\int_{-1}^{1} h(x,s)f(s)ds.$$

By making use of the expression

$$\frac{i}{2}H_0^{(1)}(\alpha|x|) = \frac{-1}{\pi}\log(|x|) + V(x)$$

with $V(x)$ a function whose second derivative is integrable Miller shows that

$$\left(\frac{\partial^2}{\partial x^2} + \alpha^2\right)I_1(x;f)$$

$$= -\frac{\partial^2}{\partial x^2}L(x;f) - \frac{\alpha^2}{2}L(x;f) + \int_{-1}^{1} K_2(x-s)f(s)ds \qquad \text{(V.8.2)}$$

where

$$L(x;f) = \frac{1}{\pi}\int_{-1}^{1} \log|x-s|f(s)ds \qquad \text{(V.8.3)}$$

and K_2 is a function whose second derivative is integrable. Substituting (V.8.2) into (V.8.1) we obtain the integral equation

$$\frac{\partial^2}{\partial x^2}L(x;f) + \frac{\alpha^2}{2}L(x;f) + \int_{-1}^{1} K(x,s)f(s)ds = e^{-iSx} \qquad \text{(V.8.4)}$$

where $K(x,s) = 2\lambda\alpha^2 h(x,s) - K_2(x-s)$ is a smooth function.

Notice that if we interchange the differentiation with the integration in the first term on the left hand side of (V.8.4) we obtain a singularity in the integrand which is not integrable even in the Cauchy principal value sense.

It can easily be deduced from the work in the previous sections that $f(x) = \sqrt{1 - x^2}g(x)$ with g Hölder continuous, so that in order to solve equation (V.8.4) numerically by using quadrature formulae, we need to develop quadrature formulae to integrate each of the terms in (V.8.4). Miller developed such a quadrature formulae of the form

$$\int_{-1}^{1} w(x,y)g(y)dy \simeq \sum_{j=1}^{N-1} A_j^{(N)}(x)g(t_j) \tag{V.8.5}$$

for the cases when w(x,y) takes the functional form

a) $w(x,y) = \frac{1}{\pi}\sqrt{1 - y^2}\log|x-y|$

b) $w(x,y) = \frac{1}{\pi}\frac{\partial}{\partial x}\left\{\sqrt{1 - y^2}\log|x-y|\right\}$

c) $w(x,y) = \frac{1}{\pi}\frac{\partial^2}{\partial x^2}\left\{\sqrt{1 - y^2}\log|x-y|\right\}.$

In the particular case when $x = t_j = \cos(\frac{j\pi}{N})$, the weights $A_j^{(N)}$ can be given explicitly as:

a) $A_j^{(N)}(t_i) = -(1 - t_j^2)(\beta_{i+j} + \beta_{i-j})$

$$\beta_m = \frac{1}{N}\left\{\frac{\log(2)}{2} + \sum_{r=1}^{N\,*} \frac{1}{r}\cos\left(\frac{mr\pi}{N}\right)\right\}$$

in which the asterisk on the summation indicates that the last term is to be halved.

b) $$A_j^{(N)}(t_i) = \begin{cases} 0 & j-i \text{ even} \\ \dfrac{2(1 - t_j^2)}{N(t_i - t_j)} & j-i \text{ odd} \end{cases}$$

c) $A_j^{(N)}(t_i) = \begin{cases} \dfrac{-2(1 - t_j^2)}{N(1 - t_{j+i})(1 - t_{j-i})} & j-i \text{ odd} \\ 0 & j-i \text{ even, } j \neq i \\ \dfrac{N}{2} & j=i \end{cases}$

For the evaluation of the ordinary integrals we use the formula

$$\int_{-1}^{1} \sqrt{1 - y^2} F(y) dy \simeq \frac{\pi}{N} \sum_{j=1}^{N-1} (1 - x_j^2) F(x_j). \qquad (V.8.6)$$

Collocation at the points t_i $i = 1, 2, \ldots, N-1$ and the use of the quadrature formulae (V.8.5) in (V.8.4) produces a linear system of equations for the unknown values of the function g at the points t_i.

This quadrature method is much easier to program than the method we described earlier but it is not as accurate. The other problem with this method is that as far as I know no proof of conveergence of the quadrature formula in case c) is known.

As it will be illustrated by the examples in next section, in order that formula (V.8.6) give good accuracy we need a large number of quadrature points, specially for the case of the function K_2 which is oscillatory and slowly decaying.

9. Numerical Examples

We consider a square steel plate immersed in water and illuminated by a sound wave. The physical constants are

Speed of sound in water = c = 1500m/s

Density of steel = ρ_p = 7800 Kg/m^3

Thickness of the plate = h = 0.0508m

Young's modulus = E = 21.53 E+10N/m^2

Poisson's ratio = σ = 0.287

Width of the plate = 2ℓ = 1.2192m

Density of water = ρ = 1000Kg/m^3.

Figure V.9.1 and figure V.9.2 below show a comparison between the solutions produced by the collocation method (SIE), the method described in section 8 (MILLER) and the asymptotic solution (ASYMPT).

Figure V.9.1 shows those results for $\alpha << 1$, more precisely $\alpha \simeq 0.24$, or ω = 100 Hertz. The SIE and MILLER solutions are very close in this case because the function K_2 is easily approximated for α small. The difference between those solutions and the asymptotic solution is smaller than $\alpha^2 \simeq 0.04$ which is the error in the latter.

Here we used the degree of the approximating polynomial equal to 20 in SIE and 100 quadrature points in MILLER.

Figure V.9.2 shows the results for $\alpha >> 1$, $\alpha \simeq 32$ or ω = 12.5 kHz. The degree of the approximating polynomial in SIE was 40 and the number of quadrature points in MILLER 200. As anticipated, the SIE produces a more accurate solution (taking the asymptotic solution as exact) in much less computing time, about half. Neither of the solutions apperar to agree with the asymptotic solution for $\theta \simeq 90^o$. The reason for such a discrepancy is that the asymptotic solution is not valid for angles close to grazing incidence angles, see Shenderov [1973] for details.

Figure V.9.3 and V.9.4 show the numerical solution for the intermediate values of ω = 6kHz and ω = 8kHz, respectively, where no asymptotic solution is available. The fluctuations seen in the numerical solution, especially for angles greater than 90 degrees, are predictable from the physics of the problem. They are due to interference between waves diffracted from the edges of the plate. The response for angles less than 90 degrees also includes components due to the incident and reflected waves, which are much stronger than the others. For angles greater than 90 degrees, the transmitted wave will be present in addition to the diffracted ones but the incident and reflected waves will be absent.

Finally, figures V.9.5 and V.9.6 show the results of experiments obtained using the experimental apparatus pictured in figure V.9.7, for the same frequencies ω = 6kHz and ω = 8kHz, respectively.

The results of the calculations show qualitative agreement with the experiments. The pitch and amplitude of the fluctuations with angle of the pressure, measured at the centre of the elastic plate, show reasonable agreement between theory and experiment over the range of frequencies covered.

Examined in detail, however, for example whether there is a peak or a dip at 0^{o}, the agreement is less good. A possible explanation is due to the square shape of the experimental plate as opposed to the finite width, infinite length of the theoretical one. The top and bottom edges of the square plate introduce extra diffraction and the finite length will alter the mode structures of the response. Experiments have been done with a plate which may be considered as rigid for the range of frequencies used and good quantitative agreement with the theoretical solution is found for this plate. It would be instructive to have experimental results for a long, narrow plate, but this has not yet been done.

The model described here provides a way of investigating the change in behaviour which occurs with materials of different elasticity.

This investigation was suggested by the Plessey Co. PLC. Their support, including the provision of practical measurements made by the Company, is acknowledged.

CHAPTER VI

The Numerical Solution of Singular Integral Equations with Non-smooth Data

1. Introduction

Example 6 of chapter I is an instance of a practical problem which can be reduced to a SIE with non-smooth data. In applications in mechanics of fracture one often encounters the situation where the external load acting on the crack has different magnitude at certain points of the crack, with jump discontinuities between them. This, following the analysis of example 4 of chapter I, leads to a SIE with discontinuous right-hand side.

Throughout this discussion we shall concentrate on the numerical solution of a constant coefficient SIE having the Heaviside function as the right-hand side or forcing term. This function, being discontinuous, does not satisfy the Hölder condition and the analysis of convergence presented in chapter III does not therefore apply. It is possible, however, to demonstrate the convergence of the polynomial collocation method in a suitable weighted L_p space, see Golberg & Fromme [1979] and Golberg [1984]. The rate of convergence can be shown to be $n^{-½}$, in practice is observed to actually be n^{-1}, but this is still far too slow to be of any practical use.

Over the years, several techniques have been designed attempting to overcome this difficulty. A list and discussion of the more common can be found in Fromme & Golberg [1979 pg. 141]. In that paper the authors present a technique on extrapolation of the numerical approximation obtained from a Galerkin or collocation method. They justify the application of this technique on the empirical evidence that the relative error in the solution behaves like

$$\frac{(-1)^n}{n}a_1 + \frac{a_2}{n^2} + \ldots$$

where a_1, a_2,... are constants independent of n the degree of the approximating polynomial.

Although this is a substantial improvement on the numerical results of the original methods, it suffers from several drawbacks. The first and perhaps the most serious one is the assumption on the form of the asymptotic expansion of the error. Another is the fact that in order to have enough information for the extrapolation procedure we need to produce a table of the numerical solution for several values of n. This can be very costly in computer time.

Golberg himself (Golberg [1982 pg. 271]) recognised the weakness of the extrapolation procedure for solving SIE with non-smooth data and wrote: "Various convergence accelerating devices have been examined but do not seem computationally efficient".

The work reported here is an attempt to solve this problem efficiently by making use of more powerful quadrature formulae developed in recent years.

We start with a detailed description of those quadrature formulae then we go on to describe how these formulae can be used in the solution of a simple case of the SIE. Finally we give an introduction to the solution of the full problem.

2. The Quadrature Formulae

In this section we shall be concerned with the numerical evaluation of the integral

$$L_i(x) = \int_{x_i}^{x_{i+1}} \frac{w^*(t)}{t - x} dt \quad x \notin [x_i, x_{i+1}] \tag{VI.2.1}$$

where x_i, $x_{i+1} \in (-1,1)$, $x_i < x_{i+1}$ α, $\beta \in (-1,1)$ and

$$w^*(x) = (1-x)^{-\alpha}(1+x)^{-\beta}.$$

Theorem VI.2.1 (Gerasoulis [1986 pg. 896])

Let

$$p = \begin{cases} x_i & \text{if } x > x_{i+1} \\ x_{i+1} & \text{if } x < x_i \end{cases}. \tag{VI.2.2}$$

The integral $L_i(x)$ in (VI.2.1) can be evaluated from the steps a), b), and c) below.

a) If $0 \leq x_i < x_{i+1}$, then

$$L_i(x) = \sum_{k=0}^{\infty} B_k(x) L_{i,k}(x) \tag{VI.2.3}$$

where

$$B_k(x) = \frac{1}{p - x}, \; k = 0, 1, \ldots \qquad \text{if } \beta = 0. \tag{VI.2.4}$$

$$B_k(x) = \begin{cases} 0 & k = -1 \\ B_{k-1}(x) + b_k(x) & k = 0,\ 1,\dots \end{cases} \qquad \text{if } \beta \neq 0 \qquad \text{(VI.2.5)}$$

with

$$b_k(x) = \begin{cases} \dfrac{(1+p)^{-\beta}}{p - x} & k = 0 \\ -\dfrac{k-1+\beta}{k}\left(\dfrac{x - p}{1 + p}\right)b_{k-1}(x) & k = 1,\ 2,\dots \end{cases} \qquad \text{(VI.2.6)}$$

For the calculation of the $L_{i,k}(x)$ we have:

If $|x - p| \geq 1$, then

$$L_{i,k}(x) = (x - p)^{-k} I_{i,k}(x) \qquad \text{(VI.2.7)}$$

where, $I_{i,-1}(x) = 0$ and for $k = 1,\ 2,\dots$

$$I_{i,k}(x)$$
$$= C\left\{k(1-p)I_{i,k-1}(x) - \left[(1-x_{i+1})^{1-\alpha}(x_{i+1}-p)^k - (1-x_i)^{1-\alpha}(x_i-p)^k\right]\right\}$$

$$\text{(VI.2.8)}$$

where $C = \frac{1}{1+k-\alpha}$.

If $|x - p| < 1$ then the equivalent recursion

$$L_{i,k}(x) =$$
$$= C\left\{k\frac{1-p}{x-p}L_{i,k-1}(x) - \left[(1-x_{i+1})^{1-\alpha}\left(\frac{x_{i+1}-p}{x - p}\right)^k - (1-x_i)^{1-\alpha}\left(\frac{x_i-p}{x-p}\right)^k\right]\right\}$$

$$\text{(VI.2.9)}$$

should be used as follows:

(i) If $\left|\frac{1-p}{x-p}\right| < 1$ then $\left|\frac{k(1-p)}{(1+k-\alpha)(x-p)}\right| < 1$ for $k = 0,\ 1,\dots$ and the

$L_{i,k}(x)$ can be evaluated from (VI.2.9). In this case the recursion (VI.2.9) is numerically stable.

(ii) If $\left|\frac{1-p}{x-p}\right| > 1$ then there exists an integer $J \geq 1$ for which $\left|\frac{k(1-p)}{(1+k-\alpha)(x-p)}\right| > 1$ for all $k \geq J$ and (VI.2.9) will be numerically stable for $k \geq J$. In this case we choose a sufficiently large $N > J$, set $L_{i,N}(x) = 0$ and evaluate $L_{i,k}(x)$, $k = N-1,(-1),J$ by using (VI.2.9) backwards and $L_{i,k}(x)$, $k = 1$, $J-1$ by using (VI.2.9) forwards.

b) If $x_i < x_{i+1} \leq 0$ then

$$L_i(x) = \sum_{k=0}^{\infty} B_k(x) L_{i,k}(x)$$

where

$$B_k(x) = \frac{1}{p - x}, \quad k = 0, 1, \ldots \qquad \text{if } \alpha = 0.$$

$$B_k(x) = \begin{cases} 0 & k = -1 \\ B_{k-1}(x) + b_k(x) & k = 0, 1, \ldots \end{cases} \qquad \text{if } \alpha \neq 0$$

with

$$b_k(x) = \begin{cases} \dfrac{(1+p)^{-\alpha}}{p - x} & k = 0 \\ -\dfrac{k-1+\alpha}{k}\left(\dfrac{x-p}{1-p}\right) b_{k-1}(x) & k = 1, 2, \ldots \end{cases}$$

For the calculation of the $L_{i,k}(x)$ we have:

If $|x - p| \geq 1$, then

$$L_{i,k}(x) = (x - p)^{-k} I_{i,k}(x)$$

where, $I_{i,-1}(x) = 0$ and for $k = 1, 2, \ldots$

$I_{i,k}(x)$

$$= C\left\{-k(1+p)I_{i,k-1}(x) - \left[(1+x_{i+1})^{1-\beta}(x_{i+1}-p)^k - (1+x_i)^{1-\beta}(x_i-p)^k\right]\right\}$$

where $C = \frac{1}{1+k-\beta}$.

If $|x - p| < 1$ then the equivalent recursion

$L_{i,k}(x) =$

$$= C\left\{-k\frac{1+p}{x-p}L_{i,k-1}(x) - \left[(1+x_{i+1})^{1-\beta}\left(\frac{x_{i+1}-p}{x - p}\right)^k - (1+x_i)^{1-\beta}\left(\frac{x_i-p}{x-p}\right)^k\right]\right\}$$

(VI.2.9a)

should again be used as follows:

(i) If $\left|\frac{1-p}{x-p}\right| < 1$ then $\left|\frac{k(1+p)}{(1+k-\beta)(x-p)}\right| < 1$ for $k = 0, 1, \ldots$ and the $L_{i,k}(x)$ can be evaluated from (VI.2.9a). In this case the recursion (VI.2.9a) is stable.

(ii) If $\left|\frac{1-p}{x-p}\right| > 1$ then there exists an integer $J \geq 1$ for which $\left|\frac{k(1+p)}{(1+k-\beta)(x-p)}\right| > 1$ for all $k \geq J$ and (VI.2.9a) will be numerically stable for $k \geq J$. In this case we choose a sufficiently large $N > J$, set $L_{i,N}(x) = 0$ and evaluate $L_{i,k}(x)$, $k = N-1,(-1),J$ by using (VI.2.9a) backwards and $L_{i,k}(x)$, $k = 1, J-1$ by using (VI.2.9a) forwards.

c) If $x_i < 0 < x_{i+1}$ the interval is split into the subintervals $[x_i,0]$ and $[0,x_{i+1}]$ and parts a) and b) are used for each of those intervals.

Proof: We present the proof of part a) only, since the proof of the other two are entirely similar.

The Taylor series expansion of $(1+t)^{-\beta}$ at a point $q \in [0,1]$ is given by

$$(1+t)^{-\beta} = (1+q)^{-\beta} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k+\beta)}{k!\Gamma(\beta)(1+q)^k}(t-q)^k. \qquad \text{(VI.2.10)}$$

From the definition of p and the fact that $x \notin [x_i, x_{i+1}]$ we have $\left|\frac{t-p}{x-p}\right| < 1$ for $x_i \le t \le x_{i+1}$.

Therefore,

$$(t-x)^{-1} = (p-x)^{-1}\left\{\frac{1}{1 - \frac{t - p}{x - p}}\right\} = (p-x)^{-1} \sum_{k=0}^{\infty} \left(\frac{t - p}{x - p}\right)^k. \qquad \text{(VI.2.11)}$$

From (VI.2.10) and (VI.2.11) it follows that

$$(t-x)^{-1}(1+t)^{-\beta} = \sum_{k=0}^{\infty} B_k(x)\left(\frac{t - p}{x - p}\right)^k \qquad \text{(IV.2.12)}$$

where

$$B_k(x) = (p-x)^{-1}(1+p)^{-\beta} \sum_{j=0}^{k} (-1)^j \frac{\Gamma(\beta+j)}{j!\Gamma(\beta)}\left(\frac{x - p}{1 + p}\right)^k. \qquad \text{(VI.2.13)}$$

Multiplying (VI.2.12) by $(1-t)^{-\alpha}$ and integrating over $[x_i, x_{i+1}]$ we obtain

$$L_i(x) = \sum_{k=0}^{\infty} B_k(x)L_{i,k}(x) \qquad \text{(VI.2.14)}$$

where

$$L_{i,k}(x) = \int_{x_i}^{x_{i+1}} (1-t)^{-\alpha}\left(\frac{t - p}{x - p}\right)^k dt.$$

We now prove that (VI.2.13) satisfies (VI.2.4-6.). It is easy to deduce from (VI.2.13) that for $\beta = 0$

$$B_k(x) = (p-x)^{-1}$$

and this proves (VI.2.4).

From (VI.2.13) we can also deduce that

$$B_k(x) = B_{k-1}(x) + \frac{(-1)^k\Gamma(\beta+k)}{(1+p)^\beta(p-x)\Gamma(\beta)k!}\left(\frac{x - p}{1 + p}\right)^k$$

and if we define

$$b_k(x) = \frac{(-1)^k\Gamma(\beta+k)}{(1+p)^\beta(p-x)\Gamma(\beta)k!}\left(\frac{x - p}{1 + p}\right)^k$$

then $b_k(x)$ satisfies the recurrence relation

$$b_k(x) = \begin{cases} \dfrac{(1 + p)^{-\beta}}{p - x} & k = 0 \\ -\dfrac{k-1+\beta}{k}\left(\dfrac{x - p}{1 + p}\right)b_{k-1}(x) & k = 1, 2, \ldots \end{cases}.$$

To prove the recurrence relations involving $L_{i,k}(x)$ we start from the formula

$$L_{i,k}(x) = \int_{x_i}^{x_{i+1}} (1-t)^{-\alpha}\left(\frac{t - p}{x - p}\right)^k dt = (x-p)^{-k} I_{i,k}(x),$$

where

$$I_{i,k}(x) = \int_{x_i}^{x_{i+1}} \frac{(t - p)^k}{(1 - t)^\alpha} dt. \qquad \text{(VI.2.15)}$$

Integrating (VI.2.15) by parts we obtain

$$I_{i,k}(x) = -(1-\alpha)^{-1}\left\{(t-p)^k(1-t)^{1-\alpha}\Big|_{x_i}^{x_{i+1}} - k\int_{x_i}^{x_{i+1}} (t-p)^{k-1}(1-t)^{1-\alpha}dt\right\}$$

$$= C\left\{k(1-p)I_{i,k-1}(x) - \left[(1-x_{i+1})^{1-\alpha}(x_{i+1}-p)^k - (1-x_i)^{1-\alpha}(x_i-p)^k\right]\right\}$$

(VI.2.16)

where $C = \frac{1}{1+k-\alpha}$.

Notice that for $p \in [0,1]$, $0 \leq \frac{k(1-p)}{1+k-\alpha} < 1$ and then the above recurrence relation is numerically stable.

Dividing equation (VI.2.16) by $(x - p)^k$ we obtain (VI.2.9).∎

The purpose of the separation of (VI.2.9) into the two cases (i) and (ii) is to achieve numerical stability of the recurrence relation.

In part (ii) a sufficiently large N must be chosen in order that we set $L_{i,N}(x) = 0$ and use (VI.2.9) backwards.

If we want to compute the $L_{i,k}(x)$ to within a given tolerance ε then we must have

$$|L_{i,N}(x)| = \left|\int_{x_i}^{x_{i+1}} (1-t)^{-\alpha}\left(\frac{t-p}{x-p}\right)^N dt\right| \leq \frac{|x_{i+1} - x_i|}{|x - p|}|I_{i,0}(x)| < \varepsilon$$

giving

$$N \geq \log(|\varepsilon/I_{i,0}(x)|)/\log(\lambda_2) \qquad \text{(VI.2.17)}$$

where

$$\lambda_2 = \frac{|x_{i+1} - x_i|}{|x - p|} = \frac{h_i}{|x - p|}.$$

We proceed now with an analysis of the error in the numerical approximation of $L_i(x)$ by truncating the series (VI.2.3) at the m^{th} term.

We first introduce some notation.

$$L_i^{(m)}(x) = \sum_{k=0}^{m} B_k(x)L_{i,k}(x) \qquad \text{(VI.2.18)}$$

$$R_i^{(m)}(x) = L_i(x) - L_i^{(m)}(x) \tag{VI.2.19}$$

Theorem VI.2.2 (Gerasoulis [1986 pg. 898])

If $0 \leq x_i < x_{i+1}$ then the remainder $R_i^{(m)}(x)$ satisfies

$$|R_i^{(m)}(x)| \leq C(x)|I_{i,0}(x)|\ \lambda_2^{m+1} \tag{VI.2.20}$$

for all $m \geq M$ = integer part$\left(\frac{1-\beta}{2}+1\right) \geq 1$

where

$$C(x) = \frac{|B_\infty(x)|}{1-\lambda_2} + \frac{|b_M(x)|\lambda_1^2}{(1-\lambda)(1-\lambda_2)} \tag{VI.2.21}$$

is independent of m and

$$B_\infty(x) = (1+x)^{-\beta}(p-x)^{-1}$$

$$\lambda_1 = (1+p)^{-1}|x-p| < 1$$

$$\lambda_2 = \frac{|x_{i+1} - x_i|}{|x-p|} = \frac{h_i}{|x-p|} < 1$$

$$\lambda = \lambda_1\lambda_2 < 1.$$

Proof: The choice of M and the fact that $\beta < 1$ gives $|j-\beta+1| < j$ for all $j \geq M$. From the recurrence relation (VI.2.6) we obtain

$$|b_j(x)| \leq \left|\frac{x-p}{1+p}\right| |b_{j-1}(x)|$$

or

$$|b_j(x)| \leq \lambda_1^{j-k}|b_k(x)|,\quad j = k,\ k+1, \ldots \tag{VI.2.22}$$

where $k \geq M$. From the expression

$$b_k(x) = \frac{(-1)^k \Gamma(\beta+k)}{(1+p)^\beta (p-x)\Gamma(\beta)k!}\left(\frac{x - p}{1 + p}\right)^k$$

and formula (VI.2.10) we obtain

$$\sum_{k=0}^{\infty} b_k(x) = (p-x)^{-1}(1+x)^{-\beta} = B_\infty(x).$$

From (VI.2.13) it follows that

$$B_k(x) = \sum_{j=0}^{k} b_j(x) = B_\infty(x) - \sum_{j=k+1}^{\infty} b_j(x)$$

so that, from (VI.2.22), we obtain

$$|B_k(x)| \le |B_\infty(x)| + \sum_{j=k+1}^{\infty} |b_j(x)| \le |B_\infty(x)| + |b_{k+1}(x)| \sum_{j=0}^{\infty} \lambda_1^j$$

$$= |B_\infty(x)| + \frac{|b_{k+1}(x)|}{1 - \lambda_1}. \tag{VI.2.23}$$

The function $f(t) = (1-t)^{-\alpha}\left(\frac{t - p}{x - p}\right)^k$ is of one sign for $t \in [x_i, x_{i+1}]$, so that

$$\left|\int_{x_i}^{x_{i+1}} f(t)dt\right| = \int_{x_i}^{x_{i+1}} |f(t)|dt.$$

Using this fact we deduce that

$$|L_{i,k}(x)| \le \lambda_2 |L_{i,k-1}(x)|$$

or

$$|L_{i,k}(x)| \le \lambda_2^{k-\nu} |L_{i,\nu}(x)|, \quad k = \nu,\ \nu+1, \ldots \quad \nu \ge 0. \tag{VI.2.24}$$

From (VI.2.19) we have

$$|R_i^{(m)}(x)| \leq \sum_{k=m+1}^{\infty} |B_k(x)|\ |L_{i,k}(x)|$$

$$\leq \sum_{k=m+1}^{\infty} \left[|B_\infty(x)| + \frac{|b_{k+1}(x)|}{1-\lambda_1}\right]\lambda_2^{k-m-1}|L_{i,m+1}(x)|$$

where use has been made of (VI.2.23) and (VI.2.24).

Using (VI.2.22) on the above inequality we obtain

$$|R_i^{(m)}(x)| \leq \sum_{k=m+1}^{\infty} \left[|B_\infty(x)| + \lambda_1^{k-m}\frac{|b_{m+1}(x)|}{1-\lambda_1}\right]\lambda_2^{k-m-1}|L_{i,m+1}(x)|$$

$$\leq |L_{i,m+1}(x)|\left\{|B_\infty(x)|\sum_{k=0}^{\infty}\lambda_2^k + \frac{|b_{m+1}(x)|}{1-\lambda_1}\lambda_1\sum_{k=0}^{\infty}(\lambda_1\lambda_2)^k\right\}$$

$$= |L_{i,m+1}(x)|\left\{\frac{|B_\infty(x)|}{1-\lambda_2} + \frac{|b_{m+1}(x)|\lambda_1}{(1-\lambda)(1-\lambda_1)}\right\}. \qquad \text{(VI.2.25)}$$

Using again (VI.2.22) and (VI.2.24) we can write

$$|L_{i,m+1}(x)| \leq \lambda_2^{m+1}|L_{i,0}(x)| = \lambda_2^{m+1}|I_{i,0}(x)|$$

$$|b_{m+1}(x)| \leq \lambda_1^{m+1-M}|b_m(x)| \leq \lambda_1|b_m(x)|.$$

Substituting these two expressions into (VI.2.25) we obtain the final result as in (VI.2.20) and (VI.2.21).■

Remarks:

1) A similar result can be proved for the case $x_i < x_{i+1} \leq 0$.

2) Given an $\epsilon > 0$ it is possible to find an m such that

$$|R_i^{(m)}(x)| \leq \epsilon.$$

Take any integer m satisfying

$$m > \log\left(\left|\frac{\epsilon}{C(x)I_{i,0}(x)}\right|\right)\Big/\log(\lambda_2). \qquad \text{(VI.2.26)}$$

Notice that all the quantities entering formula (VI.2.26) are known at a given point x.

3) As the point x approaches either end x_i or x_{i+1}, the value of m gets large, making evaluation of $L_i(x)$ for x close to any of those ends computationally costly.

3. The Integral Equation

The SIE which concerns us here has the form

$$a\phi(x) + \frac{b}{\pi}\int_{-1}^{1}\frac{\phi(t)}{t - x}dt + \int_{-1}^{1}k(x,t)\phi(t)dt = f(x) \qquad \text{(VI.3.1)}$$

where k(x,t) is an absolutely Riemann integrable function and f(x) is the Heaviside function defined by

$$f(x) = \begin{cases} 0 & -1 \leq x < x_0 \\ 1 & x_0 < x \leq 1 \end{cases} \qquad x_0 \in (-1,1).$$

The theory of SIE described in chapter II and used in chapters III and IV can be extended to include equation (VI.3.1), see Gakhov [1966 pg. 118 and 408] and Khvedelidze [1956]. Those authors investigate the Riemann problem on the assumption that G(x) (see II.5.3) is Hölder continuous but $f(x) \in L_p$. The results are the same as in the case of f being Hölder continuous except that the equations and solutions must be interpreted in the L_p context, that is, they exist almost everywhere.

The solution $\phi(x)$ of (VI.3.1) can still be decomposed as in chapter III into $\phi(x) = w(x)g(x)$ with the difference that now $g(x)$ is not Hölder continuous and in fact has a logarithmic singularity at the point x_o. The function $w(x)$ is given by

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}.$$

where α and β are defined as in section 2 of chapter III.
Using the operator notation as in chapter III we can write (VI.3.1) in the form

$$Hg + Kg = f. \qquad \text{(VI.3.2)}$$

Using the operator H^I we can tranform equation (VI.3.2) into the Fredholm equation

$$g + H^I Kg = H^I f + g_o \qquad \text{(VI.3.3)}$$

where, we recall, $Hg_o = 0$ and the operator H^I is defined by

$$H^I f = aw^*(x)f(x) - \frac{b}{\pi}\int_{-1}^{1} \frac{w^*(t)f(t)}{t - x}dt. \qquad \text{(VI.3.4)}$$

From lemma II.2.2 part c1 we have that $H^I f$ has a logarithmic singularity at the point x_o and so has g.

The strategy we propose for solving equation (VI.3.2) is very simple and consists of the following.

Suppose we can solve the equation

$$Hu = f \qquad \text{(VI.3.5)}$$

where f is the Heaviside function. Then the solution g of equation (VI.3.2) can be split into $g = u + v$ and substitution into (VI.3.2) yields

$$f = H(u+v) + K(u+v).$$

On using (VI.3.5) we obtain the following equation for v

$$Hv + Kv = -Ku. \tag{VI.3.6}$$

The important feature of equation (VI.3.6) is that its right-hand side is a lot smoother than that of the original equation. In fact if the kernel of the operator K satisfies the smoothness conditions of theorem III.5.1, as is the case for problems in mechanics of fracture, then $Ku \in C^q[-1,1]$. The collocation method discussed in chapter III can be applied to (VI.3.6) and its rate of convergence is given by that theorem. Even in the case of the kernel of example 6 of chapter I which is of the form $k(x,y) = \log|x-y|$ + smoother terms, bearing in mind that u has a logarithmic singularity at x_o the resulting function Ku is Hölder continuous.

Any numerical method used to solve (VI.3.6) will rely heavily on the evaluation of the right-hand side -Ku. Before we can obtain -Ku we need to evaluate the function u.

From (VI.3.5) and (VI.3.4) we obtain

$$u(x) = (H^I f)(x) = aw^*(x)f(x) - \frac{b}{\pi}\int_{x_o}^{1} \frac{w^*(t)}{t-x}dt. \tag{VI.3.7}$$

Therefore, the evaluation of u(x) reduces to the calculation of the singular integral in (VI.3.7). The numerical calculation of such integrals was described in section 2 above, apart from the fact that there we had $x \notin [x_o,1]$ and here we are going to need to calculate $u(x)$ for $x \in (x_o,1)$.

For $x \in (x_o,1)$, using (III.4.2) we can write

$$\frac{-b}{\pi}\int_{x_o}^{1}\frac{w^*(t)}{t-x}dt = \frac{-b}{\pi}\left[\int_{-1}^{1}\frac{w^*(t)}{t-x}dt - \int_{-1}^{x_o}\frac{w^*(t)}{t-x}dt\right]$$

$$= -aw^*(x) - \frac{b2^{\kappa}}{\sin\pi\alpha}P_{\kappa}^{(-\alpha,-\beta)}(x) + \frac{b}{\pi}\int_{-1}^{x_o}\frac{w^*(t)}{t-x}dt$$

where $\kappa = -(\alpha+\beta)$ and $P_{\kappa}^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree κ. From this, $u(x)$ can be evaluated from

$$u(x) = \begin{cases} \dfrac{-b}{\pi}\displaystyle\int_{x_o}^{1}\frac{w^*(t)}{t-x}dt & x\notin(x_o,1) \\ \dfrac{-b2^{\kappa}}{\sin\pi\alpha}P_{\kappa}^{(-\alpha,-\beta)}(x) + \dfrac{b}{\pi}\displaystyle\int_{-1}^{x_o}\frac{w^*(t)}{t-x}dt & x\in(x_o,1) \end{cases} \tag{VI.3.8}$$

where use has been made of the fact that $f(x) = 0$ for $x \in [-1,x_o)$ and $f(x) = 1$ for $x \in (x_o,1]$.

Notice that as $u(x)$ is singular at x_o this point has been excluded from the domain of u. Adding and subtracting $w^*(x)$ in the singular integral in (VI.3.7) we obtain

$$u(x) = aw^*(x)f(x) - \frac{b}{\pi}w^*(x)\log\left|\frac{1-x}{x_o-x}\right| - \frac{b}{\pi}\int_{x_o}^{1}\frac{w^*(t)-w^*(x)}{t-x}dt$$

so that u can be split into

$$u(x) = h(x) - \frac{b}{\pi}w^*(x)\log\left|\frac{1-x}{x_o-x}\right| \tag{VI.3.9}$$

with $h(x)$ Hölder continuous.

This splitting does not help with the evaluation of u

itself, but can be useful in the determination of Ku. Indeed

$$Ku = \int_{-1}^{1} k(x,t)w(t)u(t)dt$$
$$= \int_{-1}^{1} k(x,t)w(t)h(t)dt - \frac{b}{\pi}w^{*}(x)\int_{-1}^{1} k(x,t)w(t)\log\left|\frac{1-t}{x_o-t}\right|dt.$$

(VI.3.10)

This formula is computationally more efficient for the evaluation of Ku especially if the last integral can be calculated exactly or cheaply.

We now describe in detail how we intend to use the quadrature formulae of section 2 to calculate u(x) from (VI.3.8).

Let us take as an example the integral

$$r(x) = \int_{-1}^{x_o} \frac{w^{*}(t)}{t-x}dt \quad x \in (x_o,1) \tag{VI.3.11}$$

If x is not "close" to x_o then r(x) can be evaluated directly from the algorithm of theorem VI.2.1 by taking $x_i = -1$ and $x_{i+1} = x_o$. It remains to define what "close" means.

We say that x is close to x_o if for a given tolerance ϵ the value of m obtained from (VI.2.26) is larger than a preset value, say m_o. The reason for this is that we do not want to sum the series (VI.2.3) for a large number of terms. Table VI.3.1 below shows the number of terms m which would be necessary to make the remainder $|R^{(m)}(x)| \le \epsilon$, for the special case in which $x_o = 0$, $\alpha = -\beta = ½$, and $x \to 0^+$

ε \ x	0.5	0.1	0.001	0.00001	0.0000001
1.0E-05	33	153	19124	2371912	283241697
1.0E-10	61	273	30642	3523210	398370958

Table VI.3.1

In view of the numbers of table VI.3.1 it is necessary to have a better algorithm to calculate u(x) when x is close to x_o.

From the definition of λ_2 we have

$$\lambda_2 = \frac{|x_{i+1} - x_i|}{|x-p|} = \frac{h_i}{|x-p|} \qquad \text{(VI.3.12)}$$

which implies $\lambda_2 \to 1$ as $x \to x_{i+1}$ or $x \to x_i$ and then from (VI.2.26) we see that $m \to \infty$. By reducing the interval of integration we reduce the value of λ_2 and hence m, see figure VI.3.1.

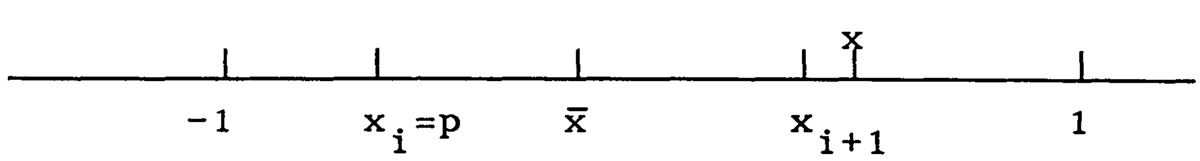

Figure VI.3.1

By introducing the point $\bar{x}$ in the interval $[x_i, x_{i+1}]$ we can write

$$L_i(x) = \int_{x_i}^{x_{i+1}} \frac{w^*(t)}{t-x}dt = \int_{x_i}^{\bar{x}} \frac{w^*(t)}{t-x}dt + \int_{\bar{x}}^{x_{i+1}} \frac{w^*(t)}{t-x}dt. \qquad \text{(VI.3.13)}$$

The value of λ_2 for the first integral is certainly smaller than that of (VI.3.12) and in this case this integral can be evaluated with fewer terms in the series (VI.3.2). By choosing $\bar{x}$ appropriately we can make the number of terms needed to obtain an error smaller than ϵ less than the preset value m_o.

For the second integral we repeat the splitting until x is not close to x_o, as defined above. Note that this will always happen in a finite (not necessarily small) number of steps. The final integral can then be calculated from

$$L_i(x) = \sum_{j=1}^{s} \int_{x_{i_j}}^{x_{i_{j+1}}} \frac{w^*(t)}{t - x} dt \qquad \text{(VI.3.14)}$$

where $x_i = x_{i_1} < \ldots < x_{i_s} = x_{i+1}$.

Each of the integrals in this sum will have been calculated with an error less than ϵ and using fewer than m_o terms in the series.

We implemented this technique by choosing $\bar{x} = \frac{1}{2}(x_{i+1}+x_i)$. For the case $x_o \leq 0$ the algorithm is as follows:

1) Input $x_i=-1$, $x_{i+1}=x_o$
2) Calculate m from (VI.2.26)
3) If m is less than m_o go to 6)
4) Set $x_i=x_i$ and $x_{i+1}= \frac{1}{2}(x_{i+1}+x_i)$
5) Repeat step 2)
6) Calculate $L_i(x)$ from theorem VI.2.1
7) Set $r(x) = r(x) + L_i(x)$
8) If $x_{i+1}=x_o$ stop

9) Set $x_i = x_{i+1}$, $x_{i+1} = x_o$ and repeat step 2).

We shall call this the *midpoint algorithm*.

In table VI.3.2 below we give the total number of terms used in the summation of the series (VI.3.2) to make the error smaller than ϵ, using this algorithm to cope with x close to x_o. Again we take $x_o = 0$, $\alpha = -\beta = ½$, with $m_o = 50$.

ϵ \ x	0.5	0.1	0.001	0.00001	0.0000001
1.0E-05	33	57	166	270	381
1.0E-10	54	116	329	548	766

Table VI.3.2

The use of $\bar{x}$ as the mid point represents a remarkable improvement on the algorithm of theorem VI.2.1, as the numbers in table VI.3.2 show. The assessment of those two algorithms by comparing solely the number of terms each take to achieve a prescribed accuracy is not very precise. In the case of the midpoint algorithm we have the additional overhead cost of the procedure that implements that algorithm. This cost is small compared with the overall cost and can be neglected.

One of the drawbacks of the midpoint algorithm is that the number s in (VI.3.14) can be very large, and although each term in the sum (VI.3.14) is calculated within an error ϵ the final result cannot be guaranteed to enjoy the same accuracy.

It is not possible, however, to completely overcome this problem by using any technique which splits the original integration interval. This is because we can only control the error in each term of the splitting and not in the final result. One possible way of alleviating this problem is by making as few splittings as possible. This could be achieved by choosing $\bar{x}$ in such a way that the interval $[x_i, \bar{x}]$ is the largest possible interval for which the integral is calculated with error less than ϵ and using m_o terms in the series, that is $\bar{x}$ is the solution of the equation

$$\lambda_2^{m_0+1} |I_{i,0}(x)| \left[\frac{|B_\infty(x)|}{1 - \lambda_2} + \frac{|b_M(x)|\lambda_1^2}{(1-\lambda)(1-\lambda_2)}\right] = \epsilon. \qquad \text{(VI.3.15)}$$

We implemented this technique but it did not seem to substantially improve on the mid point algorithm, basically because of the extra cost involved in solving the non-linear equation (VI.3.15).

As a final note we observe that the number of multiplication operations performed for the calculation of each term of the series is 10, so that, if N is the total number of terms in the series the cost of computing $L_i(x)$ is roughly 10N.

4. Numerical Examples

1) We start with the simple equation (see Golberg & Fromme [1979])

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\phi(t)}{t - x}dt = f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}. \qquad \text{(VI.4.1)}$$

The solution of this equation bounded at $x = 1$ and unbounded at $x = -1$ can be written in the form

$$\phi(x) = w(x)g(x) \tag{VI.4.2}$$

where

$$w(x) = \sqrt{\frac{1 - x}{1 + x}}.$$

Taking (VI.4.2) into (VI.4.1) we obtain the equation

$$\frac{1}{\pi}\fint_{-1}^{1} w(t)\frac{g(t)}{t - x}dt = f(x).$$

Using the inversion formula (VI.3.7) yields

$$g(x) = \frac{-1}{\pi}\fint_{-1}^{1} w^{*}(t)\frac{f(t)}{t - x}dt = \frac{-1}{\pi}\fint_{0}^{1} w^{*}(t)\frac{dt}{t - x}$$

so that we can calculate $g(x)$ for any $x \in (-1,1)$, $x \neq 0$, by making use of the algorithm described in section 3 and (VI.3.8).

The solution $g(x)$ is given in closed form by

$$g(x) = -\frac{1}{2} - \frac{1}{\pi}\sqrt{\frac{1 + x}{1 - x}}\log\left|\frac{\sqrt{1 + x} + \sqrt{1 - x}}{\sqrt{1 + x} - \sqrt{1 - x}}\right|.$$

In table VI.4.1 we show the relative error in the solution at the point x for several values of x. The total number of terms used to compute the approximation are also given. We used $\epsilon = 1.0E-10$.

x	-0.9	-0.1	-0.001	-0.00001	-.0000001
ERROR	-1.84E-14	-1.76E-12	-2.29E-12	-2.28E-12	-2.49E-12
N	40	116	329	549	775

x	0.9	0.1	0.001	0.00001	0.0000001
ERROR	-3.02E-13	-2.18E-12	-2.63E-12	-2.37E-12	-1.58E-11
N	38	113	325	546	772

Table VI.4.1

As we have already pointed out, g is singular at $x = 0$, so that it is not possible to have uniform convergence of the collocation method for equation (VI.4.1).

Briefly, the collocation method for this equation consists of approximating $g(x)$ by

$$g_n(x) = a_o P_o^{(\frac{1}{2},-\frac{1}{2})}(x) + \ldots + a_n P_n^{(\frac{1}{2},-\frac{1}{2})}(x)$$

to obtain the equation

$$\sum_{j=o}^{n} a_j \frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}}\, \frac{P_j^{(\frac{1}{2},-\frac{1}{2})}(t)}{t-x} dt = f(x) \qquad \text{(VI.4.3)}$$

which after using (III.4.1) can be written in the form

$$-\sum_{j=o}^{n} a_j P_j^{(-\frac{1}{2},\frac{1}{2})}(x) = f(x). \qquad \text{(VI.4.4)}$$

In order to calculate the coefficients a_j we require equation (VI.4.4) to be satisfied at a chosen set of collocation points $x_o, x_1, \ldots, x_n$. Golberg [1979] choses the collocation points at the zeros of the polynomial $P_{n+1}^{(-\frac{1}{2},\frac{1}{2})}(x)$. We used, in chapter III, the zeros of $T_{n+1}(x)$ the first kind Chebyshev polynomial of degree n+1. Any of these choices is known to give a convergent method in the weighted space L_{2w}, that is

$$\|g - g_n\|^2 = \int_{-1}^{1} \sqrt{\frac{1 - t}{1 + t}}(g(t) - g_n(t))^2 dt \longrightarrow 0 \text{ as } n \to \infty. \quad \text{(VI.4.5)}$$

Due to the difficulty in calculating (VI.4.5) we compare $\|g\|$ and $\|g_n\|$ instead. Since $|\|g\| - \|g_n\|| \leq \|g - g_n\|$ the convergence rates for g_n and $\|g_n\|$ should be comparable for n large.

Table VI.4.2 shows the error

$$\text{ERROR} = \frac{|\|g\| - \|g_\epsilon\||}{\|g\|}$$

for several values of ϵ, where g_ϵ is the approximation calculated by the algorithm of section 3 with ϵ as the error tolerance. We also give the total number N of terms required to achieve the given accuracy which is a guide for the total computing cost.

ϵ	ERROR	N
1.0E-03	2.74E-04	4212
1.0E-05	2.25E-07	13370
1.0E-10	2.26E-12	72034

Table VI.4.2

In table VI.4.3 we present the relative error as defined above for the numerical solution of problem (VI.4.1) by the collocation method. The approximations were calculated by collocating at the zeros of $P_{n+1}^{(-½,½)}(x)$ (GOLBERG) and by collocating at the zeros of $T_{n+1}(x)$ (CHEBYSHEV).

From column 3 we can see that the rate of convergence of the GOLBERG method is 1, whereas the rate of convergence of the CHEBYSHEV method is also globally 1 but for n odd the rate of convergence is in fact 2. We note that for n odd the point $x = 0$, which is a singular point for g, is not a collocation point; for n even $x = 0$ is a collocation point. This improved rate of convergence came as a nice surprise to us and backs up our claim that the collocation at the Chebyshev points is superior to the more common choice of the zeros of $P_{n+1}^{(-½,½)}(x)$, a fact that many authors appear to have failed to note.

Even though this is a remarkable improvement on the linear convergence of the polynomial collocation method, to achieve accuracies similar to those of table VI.4.2 a far too high value of n would have to be used.

Here we say that a sequence a_n of functions has rate of convergence υ to a limit a if $\|a_n - a\|_\infty = O(n^{-\upsilon})$

	GOLBERG		CHEBYSHEV		
n	ERROR	ERRORxN	ERROR	ERRORxN	ERRORxNxN
10	1.33E-02	0.133	4.73E-02	0.473	4.73
11	1.21E-02	0.133	8.92E-04	9.81E-03	0.107
12	1.13E-02	0.136	3.96E-02	0.476	5.71
13	1.04E-02	0.136	6.55E-04	8.51E-03	0.11
14	9.88E-03	0.138	3.41E-02	0.478	6.69
15	9.22E-03	0.138	5.01E-04	7.52E-03	0.112
16	8.75E-03	0.140	2.99E-02	0.479	7.67
17	8.23E-03	0.139	3.96E-04	6.73E-03	0.114
18	7.85E-03	0.140	2.67E-02	0.48	8.65
19	7.43E-03	0.140	3.20E-04	6.09E-03	0.114

Table VI.4.3

2) To back our claim that the method described in section 3 can be successfully applied to solve problem (VI.3.1) we solved the equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi(t)}{t-x}dt + \int_{-1}^{1}k(x,t)\phi(t)dt = f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}. \qquad \text{(VI.4.6)}$$

where $k(x,t) = \dfrac{xt}{1 + t^2 + x^2}$. The solution of equation (VI.4.6) in the class of functions bounded at $x = 1$, which is not known in closed form, can be written in the form $\phi(x) = \sqrt{\dfrac{1-x}{1+x}}g(x)$. Moreover, the function g can be split into $g = u + v$ where

$$u(x) = -1 + \frac{1}{\pi}\sqrt{\frac{1+x}{1-x}}\log\left|\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right|$$

and v satisfies (VI.3.6).

In table VI.4.4 we give the value of the approximation $v_n(x)$ produced by the collocation method used on equation (VI.3.6). We present two sets of results: One for the right-hand side $-Ku$ of (VI.3.6) calculated by the method of section 3 with $\epsilon = 1.0E-07$ and the other for $-Ku$ calculated using the exact function u given by (VI.4.6). The integrals involved in the calculation of Ku were approximated by the NAG routine D01AJF with error tolerance set to 1.0E-07. The integrals in the term Kv in (VI.3.6) were approximated by the Gauss formula

$$\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}f(x)dx \simeq \frac{4\pi}{2N+1}\sum_{k=1}^{N}\sin^2\left(\frac{k\pi}{2N+1}\right)f\left(\frac{2k\pi}{2N+1}\right)$$

with $N = 20$.

The CPU times shown in table VI.4.4 are measured in seconds and represent the cost of calculating the vector $((Ku)(x_1), (Ku)(x_2), \ldots, (Ku)(x_n))$.

All calculations were performed on a VAX 11/780 using double precision arithmetic.

	APPROXIMATE			
n	-0.95	0.0	0.5	CPU TIME
5	-1.199542283E-02	-0.3639580597	-0.4750228071	7.74
10	-1.185325724E-02	-0.3639496911	-0.4750120987	11.74
15	-1.185363252E-02	-0.3639605710	-0.4750260413	14.61
20	-1.185362738E-02	-0.3639605705	-0.4750260032	18.96
25	-1.185362741E-02	-0.3639605710	-0.4750260029	21.48
30	-1.185362742E-02	-0.3639605710	-0.4750260032	25.77

	EXACT			
n	-0.95	0.0	0.5	CPU TIME
5	-1.199542283E-02	-0.3639580597	-0.4750228071	3.61
10	-1.185325724E-02	-0.3639496910	-0.4750120987	3.80
15	-1.185363252E-02	-0.3639605710	-0.4750260413	3.86
20	-1.185362738E-02	-0.3639605705	-0.4750260032	3.92
25	-1.185362741E-02	-0.3639605710	-0.4750260028	4.14
30	-1.185362741E-02	-0.3639605709	-0.4750260031	4.56

Table VI.4.4

CHAPTER VII

Conclusion

The aim of this thesis is to discuss numerical methods for the solution of singular integral equations and to present applications of this technique to practical problems.

We describe a collocation method that uses the zeros of the $(n-\kappa)^{th}$ first kind Chebyshev polynomial as its collocation points. That fact implies that the collocation points are fixed and independent of the SIE. This is in contrast to the polynomial collocation method of Golberg [1984] who uses the zeros of the $(n-\kappa)^{th}$ orthogonal polynomial with respect to w_2 and which are dependent on the SIE.

The principal advantage of the collocation points we use is that only slightly stronger assumptions than those required by the theory of SIE to guarantee the existence of a solution are required for the proof of uniform convergence of the resulting method for the cases $\kappa = 0$,1. Uniform convergence is often achieved only on much stronger assumptions than the minimum required for the existence of a solution.

As we have already pointed out in chapter VI the collocation method described here appears to perform substantially better than the others in the literature when used to solve an SIE with discontinuous data.

We also describe how the boundary integral method can be used to reduce a problem in Acoustics to a singular integral equations and then how it can be successfully solved by the numerical techniques developed in earlier chapters. The nature of this problem required that it be solved accurately for a large span of a given parameter. This placed an extra burden on the numerical technique. Details of the strategies used to cope with these problems are described.

The results of the calculations show qualitative agreement with experiments. The model described provides a way of investigating the change in behaviour which occurs with materials of different elasticity.

Finally, an algorithm is presented for the numerical solution of SIE with discontinuous data. This algorithm is compared with existing extrapolation methods. The results are encouraging and show that this method is reliable and very fast.

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