

# Pressureless Euler with nonlocal interactions as a singular limit of degenerate Navier-Stokes system

José A. Carrillo<sup>1</sup>, Aneta Wróblewska-Kamińska<sup>2</sup>, Ewelina Zatorska<sup>3</sup>

1. Mathematical Institute, University of Oxford,  
Oxford OX2 6GG, United Kingdom.  
e-mail: carrillo@maths.ox.ac.uk

2. Institute of Mathematics, Polish Academy of Sciences,  
Śniadeckich 8, 00-656 Warszawa, Poland.  
e-mail: awrob@impan.pl

3. Department of Mathematics, University College London,  
Gower Street, London WC1E 6BT, United Kingdom.  
e-mail: e.zatorska@ucl.ac.uk

**Abstract.** We show that weak solutions of degenerate Navier-Stokes equations converge to the strong solutions of the pressureless Euler system with linear drag term, Newtonian repulsion and quadratic confinement. The proof is based on the relative entropy method using the artificial velocity formulation for the one-dimensional Navier-Stokes system.

**Keywords:** compressible Navier-Stokes equations, pressureless Euler equations, nonlocal attraction-repulsion, relative entropy.

**AMS Subject Classification:** 35Q35, 76N17, 92D25

## 1 Introduction

Hydrodynamic models of collective behaviour provide a macroscopic description of large groups of interacting individuals. They can be derived from the particle models via BBGKY hierarchies or mean-field limits [11, 18] and integration of the moments of the Vlasov-type kinetic equation seen at the intermediate mesoscopic level. We are particularly interested in the pressureless Euler system with nonlocal interactions modelled by potential  $W$  and a linear damping term

$$\begin{cases} \partial_t \bar{\rho} + \partial_x(\bar{\rho} \bar{u}) = 0, \\ \partial_t(\bar{\rho} \bar{u}) + \partial_x(\bar{\rho} \bar{u}^2) = -\bar{\rho} \bar{u} - (\partial W * \bar{\rho}) \bar{\rho}, \\ W(x) = -|x| + \frac{|x|^2}{2}, \end{cases} \quad (1)$$

for  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ . This system is a macroscopic model for individuals whose short-range repulsion is described by the Newtonian potential  $K(x) = -|x|$ , the long-range attraction is described by the quadratic confinement  $L(x) = \frac{1}{2}x^2$ , and constant alignment is described by the linear drag term  $\bar{\rho} \bar{u}$ . System (1) was recently considered in [9] where threshold conditions

for the existence of global-in-time solutions emanating from smooth initial data (ref. Theorem 3.1) were derived. It was also proven that the long-time asymptotic profile of the density is a step function determined by the total mass, the first moment of the initial density, and the initial momentum (ref. Theorem 4.1).

In this paper we compare the classical solution of (1) to a weak solution of the corresponding Navier-Stokes type system

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) = 0, \\ \partial_t(\varrho u) + \partial_x(\varrho u^2) - \varepsilon \partial_x(\mu(\varrho) \partial_x u) + \varepsilon \partial_x p(\varrho) = -\varrho u - \varrho \partial_x W * \varrho, \\ W = -|x| + \frac{|x|^2}{2}, \end{cases} \quad (2)$$

for  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ , where  $\varepsilon > 0$  denotes a small parameter that will tend to 0. The unknowns of system (2) are  $\varrho = \varrho(t, x)$  the density and  $u = u(t, x)$  the velocity,  $p(\varrho) = \varrho^\gamma$ ,  $\gamma > 1$ , denotes the barotropic pressure, and  $\mu(\varrho) = \gamma \varrho^\gamma$  denotes the density-dependent viscosity coefficient. Note that the forms of the pressure and the viscosity coefficient are related, we will explain the reason for it later on.

Our main result is the proof that when  $\varepsilon \rightarrow 0$ , weak solutions to system (2) converge to the strong solution of (1) as long as the latter exists. The basic idea is to use the momentum equation of (2) written in a modified form. Following [17] we introduce an artificial velocity  $v$ :

$$v = u + \frac{\varepsilon \gamma}{\gamma - 1} \partial_x \varrho^{\gamma-1}, \quad (3)$$

which, at least at the formal level, satisfies the equation

$$\partial_t(\varrho v) + \partial_x(\varrho v u) = -\varrho v - \varrho \partial_x W * \varrho. \quad (4)$$

Let us observe that (4) is very similar to the momentum equation of system (1). Moreover, for the limit system (1) the standard velocity  $\bar{u}$  and the artificial velocity  $\bar{v}$  are in fact the same, since  $\varepsilon = 0$  in this limit. Using this observation we will construct a relative entropy functional allowing to measure the distance between the weak solutions to the primitive system (2) and the strong solutions to the limit system (1).

We refer to system (2) as the *degenerate* Navier-Stokes system because, unlike for usual Navier-Stokes system [26], when the density vanishes the velocity vector field may not be defined. Therefore, the notion of weak solution must be adjusted to deal with this fact. The first attempt to study such system can be found in the work of Veigant and Kazhikhov [28] who considered the initial-value problem on a square for the density dependent viscosity coefficients satisfying additional growth conditions. We will be looking for weak solutions to system (2) that satisfy certain energy-entropy estimates. In this context, the first results devoted to weak solutions of the degenerate Navier-Stokes system in the multi-dimensional case are due to Bresch, Desjardins and coauthors [2–5]. In this series of papers they showed essentially the weak sequential stability of solutions to such systems in space dimension  $d \geq 1$ , and existence of weak solutions to various augmented variants of such system (involving singular pressure, or higher order friction terms). The proof of sequential stability of weak solutions without any of such terms is due to Mellet and Vasseur [24], who combined the entropy estimate, called the Bresch-Desjardins entropy, with additional estimate for the velocity. This extra estimate provides sufficient information to prove compactness in the convective term  $\varrho u \otimes u$ . However, construction of weak solutions satisfying all these extra entropy/energy estimates has been

a long time unsolved problem. Meanwhile, the one-dimensional variant of the problem was addressed by several different authors, and we refer to the work of Jiang, Xin and Zhang [19] as well as to another work of Mellet and Vasseur [25], where existence of global-in-time strong solutions was proven for various forms of degenerate viscosities. Construction of a weak solution in 1D satisfying all entropy/energy inequalities in the situation when the density may touch the vacuum was first proven by Li, Li and Xin [22] for the initial-boundary problem and then adapted by Jiu and Xin [20] to the whole space case. Very recently, a complete proof of existence in the multi-dimensional case was provided by Vasseur and Yu [27] for shallow water model and by Bresch, Vasseur and Yu [7] for the general case. Once again it turned out that the three estimates: classical energy estimate, Bresch-Desjardins entropy estimate and Mellet-Vasseur estimate of the velocity play an essential role in the proof. As far as we know, the existence of solutions to (2) with the nonlocal interaction forces is so far restricted to the Navier-Stokes-Poisson type systems [14, 23]. In this paper we show that the three major estimates necessary to repeat the construction from [27] or from [20] are true for the Navier-Stokes system with more general interaction terms, provided they are combined with the estimates of higher moments of the density. With these a-priori estimates at hand, the compactness arguments from [24], see also [17, 29, 30], allow to show the sequential stability of weak solutions to (2). This is shown in the Section 3 of the paper. Later on, in Section 4, we show that when  $\varepsilon \rightarrow 0$  the weak solutions to system (2) coincide with the strong solutions of system (1) as long as the latter exist. This is achieved thanks to the relative entropy inequality that relies on the reformulation of the momentum equation (4). Similar form of the entropy inequality was used by Haspot in [15] to prove the weak-strong uniqueness of the solutions to the degenerate Navier-Stokes equations in one dimension and in [6] for applications to various singular limits problems. It is also worth to mention a result of Brenier [1] who used the monotonic rearrangement operator to reduce the degenerate one-dimensional Navier-Stokes system (also the Navier-Stokes-Poisson system) on a torus to an elementary differential equation with noise. In this work the pressure and viscosity are linked to each other through a given smooth and strictly convex function, and the limit passage  $\varepsilon \rightarrow 0$  does not rely on the relative entropy functional.

## 2 Preliminaries and the main result

In this section we introduce the concept of weak solution to system (2), the strong solution to system (1), and formulate our main result.

### 2.1 Weak solutions to the Navier-Stokes system

We start from introducing the hypothesis on the initial data

$$(\varrho(t, \cdot), u(t, \cdot))|_{t=0} = (\varrho_0, u_0), \quad (5)$$

for which we assume that

$$\begin{aligned} \varrho_0 \geq 0, \quad \varrho_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad |x|^{\kappa+2} \varrho_0 \in L^1(\mathbb{R}), \quad (\varrho_0^{\gamma-\frac{1}{2}})_x \in L^2(\mathbb{R}), \\ \lim_{|x| \rightarrow \infty} \varrho_0(x) u_0(x) = 0, \quad \varrho_0 u_0^2 \in L^1(\mathbb{R}), \quad \varrho_0 |u_0|^{2+\kappa} \in L^1(\mathbb{R}), \end{aligned} \quad (6)$$

where  $0 < \kappa \leq \min\{2\gamma - 1, \frac{2}{\gamma}\}$ . The total initial mass and momentum are given s.t.

$$\begin{aligned} M_0 &:= \int_{\mathbb{R}} \varrho_0(x) \, dx \quad \text{and} \quad M_1 = \int_{\mathbb{R}} \varrho_0(x) u_0(x) \, dx, \\ 0 &< M_0 < \infty, \quad |M_1| < \infty. \end{aligned} \tag{7}$$

**Definition 1** For fixed  $\varepsilon > 0$  the pair of functions  $(\varrho, \sqrt{\varrho}u)$  is called a weak solution to the system (2) with initial data (5) satisfying (6) and (7) if:

$$\begin{aligned} \limsup_{|x| \rightarrow +\infty} |\varrho(t, x) u(t, x)| &= 0, \quad \text{for a.a. } t \in (0, T), \\ \varrho &\in C_w([0, T]; L^\gamma(\mathbb{R})), \quad \varrho \in L^\infty(0, T; L^\gamma(\mathbb{R}) \cap L^1(\mathbb{R})), \\ \sqrt{\varrho}u &\in L^\infty(0, T; L^2(\mathbb{R})), \\ \partial_x(\varrho^{\gamma-\frac{1}{2}}) &\in L^\infty(0, T; L^2(\mathbb{R})), \\ \varrho^{\gamma-\frac{1}{2}} &\in L^\infty(0, T; L^\infty(\mathbb{R})). \end{aligned} \tag{8}$$

Moreover, the following weak formulation of the continuity equation is satisfied

$$\int_{\mathbb{R}} \varrho \psi(t_2) \, dx - \int_{\mathbb{R}} \varrho \psi(t_1) \, dx = \int_{t_1}^{t_2} \int_{\mathbb{R}} (\varrho \partial_t \psi + \varrho u \partial_x \psi) \, dx \, dt \tag{9}$$

for any  $T \geq t_2 \geq t_1 \geq 0$  and any  $\psi \in C^1([t_1, t_2] \times \mathbb{R})$ , and denoting  $\varrho v = \varrho u + \varepsilon \partial_x \varrho^\gamma$  the following equality holds

$$\int_{\mathbb{R}} \varrho \psi(t_2) \, dx - \int_{\mathbb{R}} \varrho \psi(t_1) \, dx = \int_{t_1}^{t_2} \int_{\mathbb{R}} (\varrho \partial_t \psi + \varrho v \partial_x \psi - \varepsilon \partial_x \varrho^\gamma \partial_x \psi) \, dx \, dt. \tag{10}$$

Furthermore, the the weak formulation of the momentum equation

$$\begin{aligned} \int_{\mathbb{R}} \varrho_0 u_0 \psi(0) \, dx + \int_0^T \int_{\mathbb{R}} ((\varrho u) \partial_t \psi + (\varrho u^2 + \varepsilon \varrho^\gamma) \partial_x \psi) \, dx \, dt - \varepsilon \langle \mu(\varrho) \partial_x u, \partial_x \psi \rangle \\ = \int_0^T \int_{\mathbb{R}} \varrho u \psi \, dx \, dt + \int_0^T \int_{\mathbb{R}} \varrho (\partial_x W * \varrho) \psi \, dx \, dt, \end{aligned} \tag{11}$$

is satisfied for any  $\psi \in C_c^\infty([0, T] \times \mathbb{R})$ , where the diffusion term is defined as follows:

$$\begin{aligned} \langle \mu(\varrho) \partial_x u, \partial_x \psi \rangle \\ = -\gamma \int_0^T \int_{\mathbb{R}} \varrho^{\gamma-\frac{1}{2}} \sqrt{\varrho} u \partial_{xx} \psi \, dx \, dt - \frac{2\gamma}{2\gamma-1} \int_0^T \int_{\mathbb{R}} \partial_x \left( \varrho^{\gamma-\frac{1}{2}} \right) \sqrt{\varrho} u \partial_x \psi \, dx \, dt. \end{aligned} \tag{12}$$

**Remark 1** Let us emphasize that all terms in the above formulation are well defined. In particular, notice that

$$\int_0^T \int_{\mathbb{R}} ((\varrho u) \partial_t \psi + (\varrho u^2)) \, dx \, dt = \int_0^T \int_{\mathbb{R}} (\sqrt{\varrho}(\sqrt{\varrho}u) \partial_t \psi + ((\sqrt{\varrho}u)^2)) \, dx \, dt.$$

With this basic definition at hand, following Dafermos [13] and Haspot [15], we introduce the relative entropy (energy) functional

$$\begin{aligned} \mathcal{E}(\varrho, v | \tilde{\varrho}, \tilde{v})(t) &:= \\ \int_{\mathbb{R}} \left( \frac{\varrho(v - \tilde{v})^2}{2} + \frac{1}{2}(\varrho - \tilde{\varrho})W * (\varrho - \tilde{\varrho}) + \varepsilon (H(\varrho) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - H(\tilde{\varrho})) \right) &dx, \end{aligned} \tag{13}$$

where  $v$  is given by (3),  $H(s) = \frac{1}{\gamma}s^\gamma$ , and  $\tilde{\varrho}(t, x), \tilde{v}(t, x)$  are smooth functions defined on  $[0, T] \times \mathbb{R}$ , such that

$$\tilde{\varrho} > 0, \text{ on } [0, T] \times \mathbb{R}, \quad \tilde{\varrho}(t, x)\tilde{v}(t, x) \rightarrow 0 \text{ as } x \rightarrow \pm\infty. \quad (14)$$

To simplify presentation, we immediately assume that  $\tilde{\varrho}(t, x), \tilde{v}(t, x)$  satisfy the limit system (1), so in fact we take

$$\tilde{\varrho}(t, x) = \bar{\varrho}(t, x), \quad \tilde{v}(t, x) = \bar{u}(t, x).$$

Note that since  $H(s)$  is a strictly convex function we have

$$\varepsilon (H(\varrho) - H'(\bar{\varrho})(\varrho - \bar{\varrho}) - H(\bar{\varrho})) \geq \varepsilon H''(\xi)R^2 = \varepsilon(\gamma - 1)\xi^{\gamma-2}R^2,$$

for some  $\xi$  strictly between  $\varrho$  and  $\bar{\varrho}$ . Therefore, this term gives nonnegative contribution in (13). Concerning the non-local terms in the relative entropy functional (13), the same form was obtained in [10] where the weak-strong uniqueness of solutions to the compressible Euler system with nonlocal terms in multi-dimensional setting was studied, and in [8] where a viscous approximation of measure-valued solutions was considered. Because the potential  $W$  is symmetric, the nonlocal terms give positive contrigution in (13). Moreover, the Newtonian repulsion term  $K(x) = -|x|$  implies in particular that the  $W^{-1,2}$  distance between the solutions  $\varrho$  and  $\bar{\varrho}$  is controlled.

Following some tedious calculations presented in Section 4, we will derive the relative entropy inequality at the level of sufficiently smooth approximation of system (2):

$$\begin{aligned} [\mathcal{E}(\varrho, v | \bar{\varrho}, \bar{u})]_{t=0}^{t=\tau} + \left( \frac{\varepsilon\gamma}{\gamma-1} \right)^2 \int_0^\tau \int_{\mathbb{R}} \varrho (\partial_x \varrho^{\gamma-1})^2 dx dt + 2\varepsilon \int_0^\tau \int_{\mathbb{R}} \varrho^{\gamma+1} dx dt \\ \leq \int_0^\tau \mathcal{R}(\varrho, v, \bar{\varrho}, \bar{u}) dt, \end{aligned} \quad (15)$$

for almost all  $\tau \in (0, T)$ , with the reminder term given by

$$\begin{aligned} \mathcal{R}(\varrho, v, \bar{\varrho}, \bar{u}) = - \int_{\mathbb{R}} \varrho(v - \bar{u})(u - \bar{u})(\partial_x \bar{v} + 1) dx - \int_{\mathbb{R}} \varrho(u - \bar{u})\varepsilon \frac{\gamma}{\gamma-1} \partial_x \bar{\varrho}^{\gamma-1} dx \\ + \int_{\mathbb{R}} (\varrho - \bar{\varrho})\bar{v} \partial_x W * (\varrho - \bar{\varrho}) dx + 2\varepsilon \int_{\mathbb{R}} \varrho^\gamma \bar{\varrho} dx - \gamma \int_{\mathbb{R}} \partial_x \bar{v} F dx, \end{aligned} \quad (16)$$

and

$$F = F(\varrho, \bar{\varrho}) = \varepsilon (H(\varrho) - H'(\bar{\varrho})(\varrho - \bar{\varrho}) - H(\bar{\varrho})).$$

**Remark 2** A weak solution to (2) that satisfies inequality similar to (15) but for general functions as in (14) is sometimes referred to as a suitable weak solution. It is not clear, however, whether this inequality could be justified for the weak solutions. Therefore, we derive it at the level of approximation, estimate the reminder, and pass to the limit with the parameter of the approximation in the resulting formula.

## 2.2 Strong solutions to the Euler system

As mentioned in the introduction, the target system is pressureless compressible Euler-type of system with nonlocal interactions (1). The critical thresholds for the initial data for existence of global in time strong solutions were determined in [9]. Following their approach we consider system (1) supplemented by the initial values for the density and the velocity

$$(\bar{\varrho}(t, \cdot), \bar{u}(t, \cdot))|_{t=0} = (\bar{\varrho}_0, \bar{u}_0) \in C^2(\Omega_0) \times C^3(\Omega_0), \quad (17)$$

s.t.  $\bar{\varrho}_0 > 0$  on  $\Omega_0$ , where  $\Omega_0 := \Omega(0)$  is either an open interval  $(a_0, b_0)$  (in which case  $\bar{\varrho}(t, x)$  is extended by 0 outside  $\Omega(t)$ ) or the whole line  $\Omega_0 = \mathbb{R}$ . In addition, we assume boundedness of total initial mass and momentum (we assume that both are the same as in the previous system, see (7), for relative entropy inequality)

$$M_0 = \int_{\Omega_0} \bar{\varrho}_0(x) dx \quad \text{and} \quad M_1 = \int_{\Omega_0} \bar{\varrho}_0(x) \bar{u}_0(x) dx, \quad (18)$$

as well as boundedness of the first moment of the density and integrability of the initial momentum

$$\int_{\mathbb{R}} |x| \bar{\varrho}_0(x) dx < \infty \quad \text{and} \quad \int_{\mathbb{R}} \bar{\varrho}_0(x) |\bar{u}_0(x)| dx < \infty. \quad (19)$$

**Definition 2** We say that a pair of functions  $(\bar{\varrho}, \bar{u})$  is a classical local-in-time solution to (1) with the initial data (17), if:

- There exists time  $T > 0$  such that  $\bar{\varrho}(t, x) > 0$ , and  $\bar{\varrho}, \bar{u}$  are  $C^1$  and  $C^2$  respectively in the set  $\{(t, x) \in [0, T) \times \Omega(t)\}$  and  $\bar{\varrho}$  and  $\bar{u}$  satisfy the equations (1) pointwisely in  $\{(t, x) \in [0, T) \times \Omega(t)\}$  with initial data (17).
- The characteristics  $\eta(t, x)$  associated to  $\bar{u}$  defined by

$$\frac{d\eta(t, x)}{dt} = \bar{u}(t, \eta(t, x)) \quad \text{with} \quad \eta(0, x) = x \in \Omega_0 \quad (20)$$

are diffeomorphisms for all  $t \in [0, T)$  with  $\Omega(t) = \eta(t, \Omega_0)$ .

For the sake of completeness let us recall Theorem 3.1 from [9] stating the global existence or finite-time blow up of classical solutions to (1) depending on the initial data  $\partial_x \bar{u}_0$ ,  $\bar{\varrho}_0$  and  $M_0$ .

**Theorem 1 (Carrillo, Choi, Zatorska, 2016)** Assume that  $(\varrho, u)$  is a classical solution to the system (1) with initial data (17), then:

**Case A:** If  $1 - 4M_0 > 0$ , the solution blows up in finite time if and only if there exists a  $x^* \in \Omega_0$  such that

$$\partial_x \bar{u}_0(x^*) < 0, \quad M_0 - 2\bar{\varrho}_0(x^*) < \lambda_1 \partial_x \bar{u}_0(x^*),$$

and

$$2\bar{\varrho}_0(x^*) \leq (\lambda_1 \partial_x \bar{u}_0(x^*) - M_0 + 2\bar{\varrho}_0(x^*))^{-\lambda_2/\sqrt{\Xi}} (\lambda_2 \partial_x \bar{u}_0(x^*) - M_0 + 2\bar{\varrho}_0(x^*))^{\lambda_1/\sqrt{\Xi}},$$

where

$$\lambda_1 := \frac{-1 + \sqrt{1 - 4M_0}}{2}, \quad \lambda_2 := \frac{-1 - \sqrt{1 - 4M_0}}{2},$$

$$\Xi := 1 - 4M_0.$$

**Case B:** If  $1 - 4M_0 = 0$ , the solution blows up in finite time if and only if there exists a  $x^* \in \Omega_0$  such that

$$\partial_x \bar{u}_0(x^*) < \min \left\{ 0, 4\bar{\varrho}_0(x^*) - \frac{1}{2} \right\},$$

and

$$\ln \left( \frac{8\bar{\varrho}_0(x^*)}{8\bar{\varrho}_0(x^*) - 2\partial_x \bar{u}_0(x^*) - 1} \right) \leq \frac{2\partial_x \bar{u}_0(x^*)}{8\bar{\varrho}_0(x^*) - 2\partial_x \bar{u}_0(x^*) - 1}.$$

**Case C:** If  $1 - 4M_0 < 0$ , the solution blows up in finite time if and only if there exists a  $x^* \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$  where

$$\begin{aligned}\mathcal{S}_1 &:= \left\{ x \in \Omega_0 : \partial_x C_5(x) < 0, \partial_x C_6(x) > 0, \frac{2\bar{\varrho}_0(x)}{M_0} - C_7(x) \exp\left(\frac{C_8(x)}{\sqrt{\square}}\right) \leq 0 \right\}, \\ \mathcal{S}_2 &:= \left\{ x \in \Omega_0 : \partial_x C_5(x) > 0, \partial_x C_6(x) < 0, \frac{2\bar{\varrho}_0(x)}{M_0} - C_7(x) \exp\left(\frac{C_8(x) - \pi}{\sqrt{\square}}\right) \leq 0 \right\}, \\ \mathcal{S}_3 &:= \left\{ x \in \Omega_0 : \partial_x C_5(x) < 0, \partial_x C_6(x) < 0, \frac{2\bar{\varrho}_0(x)}{M_0} - C_7(x) \exp\left(\frac{C_8(x) - \pi}{\sqrt{\square}}\right) \leq 0 \right\}, \\ \mathcal{S}_4 &:= \left\{ x \in \Omega_0 : \partial_x C_5(x) > 0, \partial_x C_6(x) > 0, \frac{2\bar{\varrho}_0(x)}{M_0} - C_7(x) \exp\left(\frac{C_8(x) - 2\pi}{\sqrt{\square}}\right) \leq 0 \right\}\end{aligned}\tag{21}$$

where

$$\begin{aligned}\square &:= -\Xi, \\ \partial_x C_5(x) &= \partial_x \bar{u}_0(x), \quad \partial_x C_6(x) = \frac{2}{\sqrt{\square}} \left( -\frac{1}{2} \partial_x \bar{u}_0(x) - M_0(x) + 2\bar{\varrho}_0(x) \right), \\ C_7(x) &:= \left( \frac{2\sqrt{\square}}{1 + \square} \right) \sqrt{\frac{1 + \square}{\square} (\partial_x \bar{u}_0(x))^2 + \frac{4}{\square} (2\bar{\varrho}_0(x) - M_0) (2\bar{\varrho}_0(x) - M_0 - \partial_x \bar{u}_0(x))},\end{aligned}$$

and

$$C_8(x) := \arctan \left( \frac{\sqrt{\square} \partial_x \bar{u}_0(x)}{4\bar{\varrho}_0(x) - 2M_0 - \partial_x \bar{u}_0(x)} \right).$$

Moreover, for all cases, if there is no finite-time blow-up, then the classical solution  $(\bar{\varrho}, \bar{u})$  exists globally in time.

The proof of this theorem was based on derivation of an explicit formula for the solution  $u = u(t, \eta(t, x))$  in the Lagrangian coordinates. This allowed, for instance, to determine the exact form of  $\eta(t, x)$  from (20), and from there the exact form of the density on the characteristics, using the continuity equation

$$\bar{\varrho}(t, \eta(t, x)) \partial_x \eta(t, x) = \bar{\varrho}_0(x), \quad x \in \Omega_0.\tag{22}$$

For the purposes of this paper it is important to observe the following estimates of the classical solutions to (1) for large  $x$ .

**Corollary 1** *Let  $\Omega_0 = \mathbb{R}$  and let*

$$|\bar{u}_0(x)| \leq C(1 + |x|), \quad \partial_x \bar{u}_0, \quad \partial_x^2 \bar{u}_0 \in L^\infty(\mathbb{R}),\tag{23a}$$

$$\lim_{|x| \rightarrow \infty} \eta_x(t, x) \geq c > 0,\tag{23b}$$

for any  $t \in (0, \infty)$ .

Then the solution  $\bar{u}$  to (1) has the following properties:

$$|\bar{u}(t, x)| \leq c_1 + c_2 |x| \quad \text{for } t \in (0, T), \quad x \in \mathbb{R},\tag{24}$$

$$|\partial_x \bar{u}(t, x)| \leq c_3 \quad \text{for } t \in (0, T), \quad x \in \mathbb{R},\tag{25}$$

where  $c_i$ ,  $i = 1, 2, 3$  are positive constants depending on initial data and  $T$ .

*Proof.* Let us denote  $\bar{w}(t, x) := \bar{u}(t, \eta(t, x))$ . Following [9, Section 2] we have

$$\partial_{tt}\bar{w} + \partial_t\bar{w} + M_0\bar{w} = M_1e^{-t}, \quad t > 0, \quad \bar{w}_0 = \bar{u}_0 \quad (26)$$

and the initial data  $\partial_t\bar{w}(t, x)|_{t=0} = \bar{w}'_0(x)$  are given through

$$\bar{w}'_0(x) = -\bar{w}_0(x) - (x+1)M_0 + \int_{\mathbb{R}} y\bar{\varrho}_0(y)dy + 2 \int_{-\infty}^x \bar{\varrho}_0(y)dy \quad \text{for } x \in \mathbb{R}. \quad (27)$$

Moreover, we can compute

$$\eta(t, x) = x + \int_0^t \bar{w}(s, x) ds \quad \text{and} \quad \partial_x\eta(t, x) = 1 + \int_0^t \partial_x\bar{w}(s, x) ds. \quad (28)$$

Depending on the size of the initial mass  $M_0$ , as long as the solution exists, it satisfies:

- **Case A** ( $1 > 4M_0$ ):

$$\bar{w}(t, x) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} + \frac{M_1}{M_0}e^{-t}, \quad (29)$$

- **Case B** ( $1 = 4M_0$ ):

$$\bar{w}(t, x) = C_3e^{-t/2} + C_4te^{-t/2} + \frac{M_1}{M_0}e^{-t}, \quad (30)$$

- **Case C** ( $1 < 4M_0$ ):

$$\bar{w}(t, x) = C_5e^{-t/2} \cos\left(\frac{\sqrt{4M_0-1}}{2}t\right) + C_6e^{-t/2} \sin\left(\frac{\sqrt{4M_0-1}}{2}t\right) + \frac{M_1}{M_0}e^{-t}, \quad (31)$$

where  $\lambda_1, \lambda_2$ , and  $C_i, i = 1, \dots, 6$  are given by

$$\lambda_1 := \frac{-1 + \sqrt{1-4M_0}}{2}, \quad \lambda_2 := \frac{-1 - \sqrt{1-4M_0}}{2}, \quad (32a)$$

$$C_1 := \frac{1}{\lambda_2 - \lambda_1} \left( \lambda_2 \bar{w}_0 - \bar{w}'_0 + \lambda_1 \frac{M_1}{M_0} \right), \quad C_2 := \frac{1}{\lambda_2 - \lambda_1} \left( -\lambda_1 \bar{w}_0 + \bar{w}'_0 - \lambda_2 \frac{M_1}{M_0} \right), \quad (32b)$$

$$C_3 := \bar{w}_0 - \frac{M_1}{M_0}, \quad C_4 := \frac{\bar{w}_0}{2} + \bar{w}'_0 + \frac{M_1}{2M_0}, \quad (32c)$$

$$C_5 := \bar{w}_0 - \frac{M_1}{M_0}, \quad \text{and} \quad C_6 = \frac{2}{\sqrt{4M_0-1}} \left( \bar{w}'_0 + \frac{\bar{w}_0}{2} + \frac{M_1}{2M_0} \right). \quad (32d)$$

For abbreviation, we set

$$\Xi := 1 - 4M_0 \quad \text{and} \quad \square := -\Xi.$$

Note that due to this explicit formula for the solution, assumption (23b) is in fact an assumption for the initial condition  $\bar{u}_0, \bar{u}'_0$ . Indeed, since in all the cases A, B, C recalled above, the solution  $\bar{w}$  is expressed as combination of functions that are integrable in time ( $e^{-ct}$ ,  $te^{-ct}$ ,  $e^{-c_1 t} \cos(c_2 t)$  for some positive constants  $c, c_1, c_2$ ), the assumption (23b) is met for example in the case when the  $L^\infty$  norms of  $\partial_x \bar{w}_0$  and  $\partial_x \bar{w}'_0$  are sufficiently small.

From these representations we also immediately check that  $\bar{w}$  and  $\partial_x \bar{w}$  satisfy:

- **Case A**

$$\begin{aligned} |\bar{w}(t, x)| &\leq C_{A,1} e^{\max(\lambda_1, \lambda_2)t} (|x| + 1) \quad \text{as } |x| \rightarrow \infty, \\ |\partial_x \bar{w}(t, x)| &\leq C_{A,2} e^{\max(\lambda_1, \lambda_2)t} \quad \text{as } |x| \rightarrow \infty, \end{aligned} \quad (33)$$

• **Case B**

$$\begin{aligned} |\bar{w}(t, x)| &\leq C_{B,1}(1+t)e^{-\frac{t}{2}}(|x|+1) \quad \text{as } |x| \rightarrow \infty, \\ |\partial_x \bar{w}(t, x)| &\leq C_{B,2}e^{-\frac{t}{2}} \quad \text{as } |x| \rightarrow \infty, \end{aligned} \tag{34}$$

• **Case C**

$$\begin{aligned} |\bar{w}(t, x)| &\leq C_{C,1}e^{-\frac{t}{2}}(|x|+1) \quad \text{as } |x| \rightarrow \infty, \\ |\partial_x \bar{w}(t, x)| &\leq C_{C,2}e^{-\frac{t}{2}} \quad \text{as } |x| \rightarrow \infty. \end{aligned} \tag{35}$$

One can summarise that for any fixed  $T$  we have that

$$|\bar{w}(t, x)| \leq c|x| \quad \text{as } |x| \rightarrow \infty.$$

In order to come back to Eulerian coordinates we notice that

$$|\bar{u}(t, \eta(t, x))| \leq c|x| \quad \text{as } |x| \rightarrow \infty$$

and consequently

$$|\bar{u}(t, y)| \leq c|\eta^{-1}(t, y)| \quad \text{with } y = \eta(t, x) \quad \text{as } |x| \rightarrow \infty \tag{36}$$

Note that  $\partial_x \eta|_{t=0} = 1$ , moreover from the second equality in (28) and formulas for  $\partial_x \bar{w}$  in all three cases we verify that for classical solutions  $\lim_{t \rightarrow \infty} \partial_x \eta(t, x) = 1$ . The positive bound from below for  $\partial_x \eta$  for  $|x| \rightarrow \infty$  follows also from (28) together with the assumption (23b). Since for classical solutions  $\partial_x \eta$  is a continuous function, its minimal value is bounded away from 0. The bound from above follows directly from the assumption (23a), hence we have

$$0 < \underline{c} \leq \partial_x \eta \leq \bar{c} < \infty. \tag{37}$$

Consequently, the growth of  $\eta^{-1}$  is not faster than linear. Next writing

$$\partial_y \bar{u}(t, y) = \partial_x \bar{w}(t, \eta^{-1}(t, y)) \partial_y (\eta^{-1})(t, y),$$

since by (33), (34), (35), we in fact have that  $|\partial_y \bar{w}| \leq C$  for any fixed time, we immediately find that

$$|\bar{u}(t, y) - \bar{u}(t, 0)| \leq c|y|,$$

where  $\bar{u}(t, 0) = \bar{w}(t, \eta^{-1}(t, 0))$ . □

Additionally to what has been said above let us observe that  $\bar{\varrho}$  behaves similarly to  $\bar{\varrho}_0$ . Indeed, one can write

$$\bar{\varrho}(t, \eta(t, x)) \partial_x \eta(t, x) = \bar{\varrho}_0 \tag{38}$$

and consequently

$$\underline{c} \leq \left| \frac{\bar{\varrho}(t, \eta(t, x))}{\bar{\varrho}_0(x)} \right| \leq \bar{c}.$$

In particular, if  $\bar{\varrho}_0$  is positive on  $\mathbb{R}$ , then  $\bar{\varrho}$  remains positive on  $\mathbb{R}$  for all  $t \in (0, T)$ .

### 2.3 The main result

The main theorem of this paper consists of two parts. We first prove that the weak solution to (2) is sequentially stable for  $\varepsilon > 0$  fixed. This means that having a smooth approximation of system (2) satisfying all energy-entropy bounds, there exists a subsequence whose limit satisfies the Definition 1 together with the relative entropy inequality. Secondly, we prove the convergence of the weak solutions to (2), as  $\varepsilon \rightarrow 0$ , to the strong solution of (1), emanating from the same initial data, as long as the latter exist.

**Theorem 2** *Assume that  $\gamma \in (1, 3/2]$ .*

*(i) Let  $(\varrho_n, u_n)$  be a sequence of weak solutions to (2) in the sense of Definition 1 satisfying energy inequalities (48), (51), and (68), where  $v$  is defined by (3). Let the initial conditions*

$$\varrho_n|_{t=0} = \varrho_0, \quad u_n|_{t=0} = u_0 \quad (39)$$

*satisfy assumptions (6) and (7). Then, up to a subsequence  $(\varrho_n, \sqrt{\varrho_n}u_n)$  converges strongly to  $(\varrho_\varepsilon, \sqrt{\varrho_\varepsilon}u_\varepsilon)$  a weak solution of (2) satisfying energy inequalities (48), (51), and (68). More precisely*

$$\begin{aligned} \varrho_n &\rightarrow \varrho_\varepsilon \quad \text{strongly in } C([0, T] \times \mathbb{R}_{loc}), \\ \partial_x \left( \varrho_n^{\gamma-\frac{1}{2}} \right) &\rightarrow \partial_x \left( \varrho_\varepsilon^{\gamma-\frac{1}{2}} \right) \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \\ \sqrt{\varrho_n}u_n &\rightarrow \sqrt{\varrho_\varepsilon}u_\varepsilon, \quad \text{strongly in } L^2(0, T; L^2(\mathbb{R})), \\ \varrho_n v_n &\rightarrow \varrho_\varepsilon v_\varepsilon, \quad \text{strongly in } L^2(0, T; L^2(\mathbb{R})), \\ \gamma \varrho_n^\gamma \partial_x u_n &\rightarrow \Lambda_\varepsilon \quad \text{weakly in } L^2(0, T; L^2_{loc}(\mathbb{R})), \end{aligned} \quad (40)$$

where  $\Lambda_\varepsilon$  satisfies

$$\int_0^T \int_{\mathbb{R}} \Lambda_\varepsilon \phi \, dx \, dt = -\gamma \int_0^T \int_{\mathbb{R}} \varrho_\varepsilon^{\gamma-\frac{1}{2}} \sqrt{\varrho_\varepsilon} u_\varepsilon \partial_x \phi \, dx \, dt - \frac{2\gamma}{2\gamma-1} \int_0^T \int_{\mathbb{R}} \partial_x (\varrho_\varepsilon^{\gamma-\frac{1}{2}}) \sqrt{\varrho_\varepsilon} u_\varepsilon \phi \, dx \, dt \quad (41)$$

for any  $\phi \in C_c^\infty((0, T) \times \mathbb{R})$ .

*(ii) Let  $(\varrho_n, u_n)$  satisfy in addition the entropy inequality (15)-(16), with  $(\bar{\varrho}, \bar{u})$  being the strong solution to (1) on the time interval  $(0, T)$  in the sense specified in Definition 2. Let the initial data  $(\bar{\varrho}_0, \bar{u}_0) = (\varrho_0, u_0)$  satisfy (6) and (7) together with*

$$(\varrho_0^{\gamma-1})_x \in L^2(\mathbb{R}). \quad (42)$$

*Then the limiting weak solution obtained in (40) converges to the strong solution of (1) in the following sense*

$$\operatorname{ess\,sup}_{t \in (0, T)} \mathcal{E}(\varrho_\varepsilon, v_\varepsilon | \bar{\varrho}, \bar{u})(t) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

**Remark 3** *Note that the initial conditions for the primitive as well as the limit problem are the same. This assumption could be relaxed, but we do not focus on this aspect here. Note, in particular, that assumptions (6) and (7) imply (19) that is necessary for existence of strong solutions to the limit system.*

**Remark 4** *Thanks to (6) and (7), via (37) and (38) we can verify that  $\bar{\varrho}\bar{u} \rightarrow 0$  and  $x^{3/2}\bar{\varrho} \rightarrow 0$  for  $x \rightarrow \infty$  and that  $\bar{\varrho} \in L^{\gamma+1}((0, T) \times \mathbb{R})$ . Thanks to (42) we also get that  $\bar{\varrho} \in L^{\gamma+1}((0, T) \times \mathbb{R})$ . These properties of the limit solution are needed in order to close the relative entropy estimate.*

The rest of this paper is devoted to the proof of this theorem. The proof of existence of approximation  $(\varrho_n, u_n)$  could be obtained following [20] or [27] and is left for the future study.

### 3 A-priori estimates for the Navier-Stokes system

The purpose of this section is to provide various essential a-priori estimates for the approximate solutions  $(\varrho_n, u_n)$  to the primitive system (2). We will show that these a-priori estimates provide uniform with respect to  $n$  bounds that allow to deduce the weak sequential stability of solutions. Similar reasoning was performed before in [17, 24].

In what follows we drop the subindex  $n$  when no confusion can arise, and we assume that the assumptions of Theorem 2 are satisfied. In particular,  $(\varrho, u)$  are sufficiently smooth functions satisfying equations of system (2) pointwise, s.t.  $\varrho \geq 0$ , and  $\lim_{|x| \rightarrow \infty} \varrho(t, x)u(t, x) \rightarrow 0$ . We clearly do not expect that  $u(t, x)$  decays at infinity itself, but we a-priori assume that  $\partial_x u$  is bounded at infinity, in accordance with what is known for the limiting system, see (25).

#### 3.1 Conservation of mass and momentum

It is straightforward to deduce that  $\varrho$  is nonnegative. Integrating the continuity equation with respect to space variable, and using decay of  $\varrho(t, x)u(t, x)$  at infinity we obtain that

$$\frac{d}{dt} \int_{\mathbb{R}} \varrho \, dx = 0, \quad (43)$$

and so the total mass is conserved, in particular

$$\|\varrho_n\|_{L^\infty(0, T; L^1(\mathbb{R}))} \leq C, \quad (44)$$

uniformly w.r.t.  $n$ .

Similarly, integrating the momentum equation, we check that since  $\partial_x W = -\operatorname{sgn}(x) + x$  is antisymmetric, the nonlocal term integrates to zero, and hence we have

$$\frac{d}{dt} \int_{\mathbb{R}} \varrho u \, dx = - \int_{\mathbb{R}} \varrho u \, dx. \quad (45)$$

Integrating with respect to time, we find

$$\int_{\mathbb{R}} \varrho u(t) \, dx = e^{-t} \int_{\mathbb{R}} \varrho_0 u_0 \, dx = e^{-t} M_1, \quad (46)$$

and so the total momentum is bounded if the initial momentum is.

#### 3.2 The basic energy estimate

We recall the classical energy inequality that is derived by multiplying the momentum equation of (2) by  $u$  and integration by parts. For the nonlocal term, this leads to

$$- \int_{\mathbb{R}} \varrho u \partial_x W * \varrho \, dx = \int_{\mathbb{R}} \partial_x (\varrho u) W * \varrho \, dx = - \int_{\mathbb{R}} \partial_t \varrho W * \varrho \, dx = - \frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} \varrho W * \varrho \, dx, \quad (47)$$

where we used the continuity equation to get the second equality. Similarly, for the pressure term we have that

$$\varepsilon \int_{\mathbb{R}} \partial_x \varrho^\gamma u \, dx = \varepsilon \frac{1}{\gamma - 1} \frac{d}{dt} \int_{\mathbb{R}} \varrho^\gamma \, dx.$$

Consequently, the energy inequality reads

$$\frac{d}{dt} \int_{\mathbb{R}} \left( \frac{1}{2} \varrho u^2 + \frac{\varepsilon}{\gamma-1} \varrho^\gamma + \frac{1}{2} \varrho W * \varrho \right) dx + \varepsilon \gamma \int_{\mathbb{R}} \varrho^\gamma (\partial_x u)^2 dx + \int_{\mathbb{R}} \varrho u^2 dx \leq 0. \quad (48)$$

Note that  $W = -|x| + x^2 \geq -C$ , so the l.h.s. can be made positive, by adding to  $W$  a constant, and by using the fact that the mass is conserved.

Integrating (48) over time interval  $[0, T]$  we obtain the following estimates

$$\begin{aligned} \|\sqrt{\varrho_n} u_n\|_{L^\infty(0,T;L^2(\mathbb{R}))} &\leq C, \\ \|\varrho_n\|_{L^\infty(0,T;L^\gamma(\mathbb{R}))} &\leq C, \\ \|\varrho_n^{\gamma/2} \partial_x u_n\|_{L^2(0,T;L^2(\mathbb{R}))} &\leq C \end{aligned} \quad (49)$$

are uniform w.r.t.  $n$ . Smoothness of  $\varrho_n$  and  $u_n$  for  $n$  fixed together with the bound from above implies also that  $\lim_{|x| \rightarrow 0} \sqrt{\varrho_n(t, x)} u_n(t, x) = 0$ , which will become important later on.

### 3.3 Bresch-Desjardins entropy estimate

Testing (4) by solution  $v$ , and using the continuity equation we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} \frac{\varrho v^2}{2} dx + \int_{\mathbb{R}} \varrho v^2 dx + \int_{\mathbb{R}} \varrho \left( u + \varepsilon \frac{\gamma}{\gamma-1} \partial_x \varrho^{\gamma-1} \right) \partial_x W * \varrho dx = 0,$$

where in the last term we used the definition of  $v$  (3). Proceeding as in (47) we therefore get

$$\frac{d}{dt} \int_{\mathbb{R}} \left( \frac{1}{2} \varrho v^2 + \frac{1}{2} \varrho W * \varrho \right) dx + \int_{\mathbb{R}} \varrho v^2 dx + \varepsilon \int_{\mathbb{R}} \partial_x \varrho^\gamma \partial_x W * \varrho dx = 0.$$

Let us now focus on the last term on l.h.s., integrating by parts we get

$$\varepsilon \int_{\mathbb{R}} \partial_x \varrho^\gamma \partial_x W * \varrho dx = -\varepsilon \int_{\mathbb{R}} \varrho^\gamma \partial_x^2 W * \varrho dx, \quad (50)$$

where we used the fact that for  $n$  fixed  $\varrho$  is smooth and integrable, therefore  $\varrho_n(t, x) \rightarrow 0$  when  $|x| \rightarrow +\infty$ . Due to definition of  $W$  we have  $\partial_x^2 W = -2\delta + 1$ , and so

$$-\varepsilon \int_{\mathbb{R}} \varrho^\gamma \partial_x^2 W * \varrho dx = 2\varepsilon \int_{\mathbb{R}} \varrho^{\gamma+1} dx - \varepsilon \int_{\mathbb{R}} \varrho dx \int_{\mathbb{R}} \varrho^\gamma dx = 2\varepsilon \int_{\mathbb{R}} \varrho^{\gamma+1} dx - \varepsilon M_0 \int_{\mathbb{R}} \varrho^\gamma dx,$$

which gives rise to the mathematical entropy inequality

$$\frac{d}{dt} \int_{\mathbb{R}} \left( \frac{1}{2} \varrho v^2 + \frac{1}{2} \varrho W * \varrho dx \right) dx + \int_{\mathbb{R}} \varrho v^2 dx + 2\varepsilon \int_{\mathbb{R}} \varrho^{\gamma+1} dx \leq \varepsilon M_0 \int_{\mathbb{R}} \varrho^\gamma dx. \quad (51)$$

After integration over time we have

$$\int_{\mathbb{R}} \left( \frac{\varrho v^2}{2} + \frac{1}{2} \varrho W * \varrho(t) \right) dx + \int_0^T \int_{\mathbb{R}} (\varrho v^2 + 2\varepsilon \varrho^{\gamma+1}) dx dt \leq C + \varepsilon CT. \quad (52)$$

From this estimate it follows in particular by (3) that uniformly w.r.t.  $n$  we have

$$\varepsilon \|\partial_x (\varrho_n^{\gamma-\frac{1}{2}})\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C, \quad (53)$$

for  $\varepsilon$  small enough and so, by the Sobolev imbedding, since  $\varrho \in L^1 \cap L^\gamma(\mathbb{R})$ ,

$$\varepsilon \|\varrho_n^{\gamma-\frac{1}{2}}\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C. \quad (54)$$

### 3.4 Higher moments estimates

Let us first check the behaviour of the center of mass of the density. Multiplying continuity equation by  $x$  and integrating by parts we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} x \varrho dx = \int_{\mathbb{R}} \varrho u dx. \quad (55)$$

Indeed, this can be justified using as a test function for the continuity equation  $x\Psi_k$  where  $\Psi_k \in C_c^\infty(\mathbb{R})$ ,  $\Psi_k = 1$  for  $x \in [-k+1, k-1]$ ,  $\Psi_k = 0$  for  $x \in (-\infty, -2k-1) \cup (2k+1, \infty)$ ,  $|\partial_x \Psi_k| \leq \frac{1}{k}$ . Then

$$\frac{d}{dt} \int_{\mathbb{R}} x \varrho \Psi_k dx = - \int_{\mathbb{R}} \partial_x(\varrho u) x \Psi_k dx = \int_{\mathbb{R}} \varrho u \Psi_k dx + \int_{\mathbb{R}} \varrho u x \partial_x \Psi_k dx.$$

Since for a.a.  $t \in (0, \infty)$ ,  $\varrho u = \sqrt{\varrho} \sqrt{\varrho} u \in L^1(\mathbb{R})$  (notice  $\sqrt{\varrho} \in L^2(\mathbb{R})$  and  $\sqrt{\varrho} u \in L^2(\mathbb{R})$ ), we may pass with  $k \rightarrow \infty$  and obtain (55).

Next, using the momentum estimate (46), we obtain that

$$M_2(t) := \int_{\mathbb{R}} x \varrho(t) dx = \int_{\mathbb{R}} x \varrho_0 dx + (1 - e^{-t}) M_1. \quad (56)$$

With this at hand, we go one level higher and we estimate the second moment of the density. Using again the continuity equation we have

$$\frac{d}{dt} \int_{\mathbb{R}} x^2 \varrho dx = 2 \int_{\mathbb{R}} x \varrho u dx. \quad (57)$$

This again can be justified using appropriate test function in the continuity equation. We now take  $\Phi_k \in C_c^\infty(\mathbb{R})$ , such that  $\Phi_k = 1$  for  $x \in [-k^2+1, k^2-1]$ ,  $\Phi_k = 0$  for  $x \in (-\infty, -k^2-k-1) \cup (k^2+k+1, \infty)$ ,  $|\partial_x \Phi_k| \leq \frac{1}{k^2}$ . Then integrating by parts

$$\frac{d}{dt} \int_{\mathbb{R}} x^2 \varrho \Phi_k dx = - \int_{\mathbb{R}} \partial_x(\varrho u) x^2 \Phi_k dx = \int_{\mathbb{R}} x \varrho u \Phi_k dx + \int_{\mathbb{R}} \varrho u x^2 \partial_x \Phi_k dx.$$

Since we already know that  $x \varrho u \in L^1(\mathbb{R})$ , we may pass with  $k \rightarrow \infty$  and obtain (57).

The r.h.s. of (57) can be evaluated using the momentum equation, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} x \varrho u dx \\ &= \int_{\mathbb{R}} \varrho u^2 dx - \varepsilon \gamma \int_{\mathbb{R}} \varrho^\gamma \partial_x u dx + \varepsilon \int_{\mathbb{R}} \varrho^\gamma dx - \int_{\mathbb{R}} x \varrho u dx - \int_{\mathbb{R}} x \varrho \partial_x W * \varrho dx, \end{aligned} \quad (58)$$

hence we get

$$\int_{\mathbb{R}} x(\varrho u)(t) dx = e^{-t} \int_{\mathbb{R}} x \varrho_0 u_0 dx + \int_0^t e^{s-t} f(s) ds,$$

where

$$f(s) = \int_{\mathbb{R}} (\varrho u^2 - \varepsilon \gamma \varrho^\gamma \partial_x u + \varepsilon \varrho^\gamma - x \varrho \partial_x W * \varrho)(s) dx.$$

From the energy estimate we can check that  $f \in L^2((0, T))$ . Clearly,  $\|\varrho u^2\|_{L^\infty(0, T; L^1(\Omega))} \leq C$ ,  $\sqrt{\varepsilon}\|\varrho^{\frac{\gamma}{2}}\|_{L^\infty(0, T; L^2(\Omega))} \leq C$ ,  $\sqrt{\varepsilon}\|\varrho^{\frac{\gamma}{2}}\partial_x u\|_{L^2(0, T; L^2(\Omega))} \leq C$ . The only issue is the last term, for which we can write

$$\begin{aligned} - \int_{\mathbb{R}} x \varrho \partial_x W * \varrho \, dx &= - \int_{\mathbb{R}} \int_{\mathbb{R}} x \varrho(x) \partial W(x-y) \varrho(y) \, dx \, dy \\ &= - \int_{\mathbb{R}} \int_{\mathbb{R}} (x-y) \varrho(x) \partial W(x-y) \varrho(y) \, dx \, dy - \int_{\mathbb{R}} \int_{\mathbb{R}} y \varrho(x) \partial W(x-y) \varrho(y) \, dx \, dy \\ &= - \int_{\mathbb{R}} \int_{\mathbb{R}} (x-y) \varrho(x) \partial W(x-y) \varrho(y) \, dx \, dy - \int_{\mathbb{R}} \int_{\mathbb{R}} x \varrho(y) \partial W(y-x) \varrho(x) \, dx \, dy \\ &= - \int_{\mathbb{R}} \int_{\mathbb{R}} (x-y) \varrho(x) \partial W(x-y) \varrho(y) \, dx \, dy + \int_{\mathbb{R}} \int_{\mathbb{R}} x \varrho(y) \partial W(x-y) \varrho(x) \, dx \, dy. \end{aligned}$$

Thus, it follows that

$$- \int_{\mathbb{R}} x \varrho \partial_x W * \varrho \, dx = - \frac{1}{2} \int_{\mathbb{R}} \varrho(x \partial_x W) * \varrho \, dx. \quad (59)$$

Now note that  $|x \partial_x W| \leq C(W+1)$ , then we have thanks to the energy estimate (48) and conservation of mass that the term (59) is bounded uniformly in time. Consequently we deduce that

$$\int_{\mathbb{R}} x(\varrho u)(t) \, dx \leq C \left( T, \int_{\mathbb{R}} x \varrho_0 u_0 \, dx \right),$$

and so, by (57) we infer that

$$\int_{\mathbb{R}} x^2 \varrho(t) \, dx \leq c \left( T, \int_{\mathbb{R}} x^2 \varrho_0 \, dx, \int_{\mathbb{R}} x \varrho_0 u_0 \, dx \right). \quad (60)$$

Finally, we want to estimate higher order moments. Due to continuity equation for any  $\kappa > 0$  we have

$$\frac{d}{dt} \int_{\mathbb{R}} x^2 |x|^\kappa \varrho \, dx = (2 + \kappa) \int_{\mathbb{R}} x |x|^\kappa \varrho u \, dx. \quad (61)$$

We cannot proceed as previously using the momentum equation. However, applying the Young inequality to the r.h.s of (61) we can write

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} |x|^{\kappa+2} \varrho \, dx &\leq C \int_{\mathbb{R}} |x|^{\kappa+1} \varrho |u| \, dx \\ &\leq C \int_{\mathbb{R}} (|x|^{\kappa+2} \varrho)^{\frac{\kappa+1}{\kappa+2}} \varrho^{\frac{1}{\kappa+2}} |u| \, dx \\ &\leq C_1(\kappa) \int_{\mathbb{R}} |x|^{\kappa+2} \varrho \, dx + C_2(\kappa) \int_{\mathbb{R}} \varrho |u|^{2+\kappa} \, dx. \end{aligned} \quad (62)$$

And that will be used later in forthcoming section (see (68)).

### 3.5 Mellet-Vasseur velocity estimate

The final estimate is the improved estimate of the velocity a'la Mellet and Vasseur [24]. It is obtained by testing the momentum equation of the system (2) by  $u|u|^\kappa$  for  $0 < \kappa \leq$

$\min\{2\gamma - 1, \frac{2}{\gamma}\}$ , and by testing the continuity equation by  $\frac{|u|^{2+\kappa}}{2+\kappa}$ . Summing up the obtained expressions we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \frac{\varrho |u|^{2+\kappa}}{2+\kappa} dx + \varepsilon \gamma (\kappa + 1) \int_{\mathbb{R}} \varrho^\gamma |u|^\kappa |\partial_x u|^2 dx + \int_{\mathbb{R}} \varrho |u|^{2+\kappa} dx \\ = -\varepsilon \int_{\mathbb{R}} \partial_x \varrho^\gamma u |u|^\kappa dx - \int_{\mathbb{R}} \varrho u |u|^\kappa \partial_x W * \varrho dx = I_1 + I_2. \end{aligned} \quad (63)$$

Due to decay of  $\varrho$  and  $\varrho u^2$  at infinity the lack of boundary term coming from integration by parts in the viscosity term is justified provided  $\kappa \leq 2\gamma - 1$ . Under the same restriction  $I_1$  can be integrated by parts and estimated as follow

$$\begin{aligned} I_1 &= -\varepsilon \int_{\mathbb{R}} \partial_x \varrho^\gamma u |u|^\kappa dx = \varepsilon (\kappa + 1) \int_{\mathbb{R}} \varrho^\gamma |u|^\kappa \partial_x u dx \\ &\leq \frac{\varepsilon (\kappa + 1)}{2} \int_{\mathbb{R}} \varrho^\gamma |u|^\kappa |\partial_x u|^2 dx + \frac{\varepsilon (\kappa + 1)}{2} \int_{\mathbb{R}} \varrho^\gamma |u|^\kappa dx. \end{aligned}$$

The first term can be absorbed by the l.h.s. of (63), while to control the second term, we use Young's inequality to write for  $\kappa \leq 2$

$$\begin{aligned} \frac{\varepsilon (\kappa + 1)}{2} \int_{\mathbb{R}} \varrho^\gamma |u|^\kappa dx &= \frac{\varepsilon (\kappa + 1)}{2} \int_{\mathbb{R}} \varrho^{\gamma - \frac{\kappa}{2}} \varrho^{\frac{\kappa}{2}} |u|^\kappa dx \\ &\leq C(\kappa) \int_{\mathbb{R}} \varrho u^2 dx + \varepsilon C(\kappa) \int_{\mathbb{R}} \varrho^{\frac{2\gamma - \kappa}{2 - \kappa}} dx \end{aligned}$$

for  $\varepsilon$  small enough and so, the first term can be controlled by the energy estimate (48), while the second one is bounded by the  $L^1$  norm of  $\varrho^{\gamma+1}$  from (52) provided  $\kappa \leq \frac{2}{\gamma}$ . For the estimate of  $I_2$ , we use the explicit form of  $\partial_x W * \varrho$ , i.e.

$$\partial_x W * \varrho = M_0 - 2 \int_{-\infty}^x \varrho(t, y) dy + x M_0 - \int_{\mathbb{R}} x \varrho dx. \quad (64)$$

This means that

$$\begin{aligned} I_2 &\leq \left| - \int_{\mathbb{R}} \varrho u |u|^\kappa \partial_x W * \varrho dx \right| \\ &\leq 3M_0 \int_{\mathbb{R}} \varrho |u|^{\kappa+1} dx + M_0 \int_{\mathbb{R}} |x| \varrho |u|^{\kappa+1} dx + \int_{\mathbb{R}} |x| \varrho dx \int_{\mathbb{R}} \varrho |u|^{\kappa+1} dx. \end{aligned} \quad (65)$$

Using the estimates of the second moment of  $\varrho$  (60) and of the total mass we find that

$$\int_{\mathbb{R}} |x| \varrho(t) dx \leq c(M_0, m_0, T, \int_{\mathbb{R}} x^2 \varrho_0 dx),$$

and therefore

$$\begin{aligned} I_2 &\leq C \int_{\mathbb{R}} |x| \varrho |u|^{\kappa+1} dx + C \int_{\mathbb{R}} \varrho |u|^{\kappa+1} dx \\ &= C \int_{\mathbb{R}} (|x|^{2+\kappa} \varrho)^{\frac{1}{2+\kappa}} \left( \varrho^{\frac{1}{2+\kappa}} |u| \right)^{\kappa+1} dx + C \int_{\mathbb{R}} \varrho^{\frac{1}{2+\kappa}} \left( \varrho^{\frac{1}{2+\kappa}} |u| \right)^{\kappa+1} dx. \end{aligned}$$

Now, we use the Young inequality again with  $p = \kappa + 2$ ,  $p' = \frac{\kappa+2}{\kappa+1}$ , similarly as in (62), for both terms at the same time, so that we get

$$I_2 \leq C(\kappa) \left( \int_{\mathbb{R}} |x|^{2+\kappa} \varrho dx + \int_{\mathbb{R}} \varrho dx + \int_{\mathbb{R}} \varrho |u|^{\kappa+2} dx \right). \quad (66)$$

Summarizing, the formula (63) might be now rewritten as:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \frac{\varrho |u|^{2+\kappa}}{2+\kappa} dx + \frac{\varepsilon(\kappa+1)(2\gamma-1)}{2} \int_{\mathbb{R}} \varrho^\gamma |u|^\kappa |\partial_x u|^2 dx + \int_{\mathbb{R}} \varrho |u|^{2+\kappa} dx \\ \leq C(\kappa) \left( 1 + \int_{\mathbb{R}} |x|^{2+\kappa} \varrho dx + \int_{\mathbb{R}} \varrho |u|^{\kappa+2} dx \right). \end{aligned} \quad (67)$$

In the above  $C(\kappa)$  depends also on  $M_0, m_0, T, \int_{\Omega} x^2 \varrho_0 dx$ . In order to deduce some useful estimates, we now add it to formula (62), to get

$$\begin{aligned} \frac{d}{dt} \left( \int_{\mathbb{R}} \frac{\varrho |u|^{2+\kappa}}{2+\kappa} dx + \int_{\mathbb{R}} |x|^{2+\kappa} \varrho dx \right) + \frac{\varepsilon(\kappa+1)(2\gamma-1)}{2} \int_{\mathbb{R}} \varrho^\gamma |u|^\kappa |\partial_x u|^2 dx \\ \leq C(\kappa) \left( 1 + \int_{\mathbb{R}} |x|^{2+\kappa} \varrho dx + \int_{\mathbb{R}} \varrho |u|^{\kappa+2} dx \right) \end{aligned} \quad (68)$$

for  $\kappa$  small enough, and we conclude by Gronwall's inequality that if  $T < \infty$ ,  $\varrho_0 |u_0|^{2+\kappa} \in L^1(\mathbb{R})$ ,  $|x|^{2+\kappa} \varrho_0 \in L^1(\mathbb{R})$  then

$$\begin{aligned} \|\varrho_n |u_n|^{2+\kappa}\|_{L^\infty(0,T;L^1(\mathbb{R}))} &\leq C, \\ \| |x|^{2+\kappa} \varrho_n \|_{L^\infty(0,T;L^1(\mathbb{R}))} &\leq C, \end{aligned} \quad (69)$$

where  $C$  depends on:  $\kappa, M_0, m_0, T, \int_{\Omega} x^2 \varrho_0 dx, \varrho_0 |u_0|^{2+\kappa}, |x|^{2+\kappa} \varrho_0$ , but is uniform w.r.t.  $n$ . We summarize the findings of this section in the following lemma.

**Lemma 1** *Let  $(\varrho_n, u_n)$  be a sequence of sufficiently smooth approximate solutions to system (2) with the initial data (5) satisfying (6) and (7). Then, uniformly in  $n$  we have*

$$\begin{aligned} \|\varrho_n\|_{L^\infty(0,T;L^1(\mathbb{R}))} &\leq C, \\ \|\sqrt{\varrho_n} u_n\|_{L^\infty(0,T;L^2(\mathbb{R}))} &\leq C, \\ \|\varrho_n\|_{L^\infty(0,T;L^\gamma(\mathbb{R}))} &\leq C, \\ \|\varrho_n^{\gamma/2} \partial_x u_n\|_{L^2(0,T;L^2(\mathbb{R}))} &\leq C, \\ \varepsilon \|\partial_x (\varrho_n^{\gamma-\frac{1}{2}})\|_{L^\infty(0,T;L^2(\mathbb{R}))} &\leq C, \\ \varepsilon \|\varrho_n^{\gamma-\frac{1}{2}}\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} &\leq C, \\ \|\varrho_n |u_n|^{2+\kappa}\|_{L^\infty(0,T;L^1(\mathbb{R}))} &\leq C, \\ \| |x|^{2+\kappa} \varrho_n \|_{L^\infty(0,T;L^1(\mathbb{R}))} &\leq C \end{aligned} \quad (70)$$

for  $\kappa$  small enough.

## 4 Relative entropy

In this section we first derive the relative entropy inequality and then the relative entropy estimate. We again assume that  $(\varrho_n, u_n)$  are sufficiently smooth functions satisfying equations of system (2) pointwise, s.t.  $\varrho \geq 0$ , and  $\lim_{|x| \rightarrow \infty} \varrho(t, x) u(t, x) \rightarrow 0$ .

Recall that due to (3), at the level of sufficiently regular solutions, the system (2) is equivalent to the following one:

$$\begin{cases} \partial_t \varrho + \partial_x (\varrho u) = 0, \\ \partial_t (\varrho v) + \partial_x (\varrho u v) = -\varrho v - \varrho \partial_x W * \varrho. \end{cases} \quad (71)$$

Setting  $\bar{v} = \bar{u}$  in the Euler system (1) we obtain the following equations:

$$\begin{cases} \partial_t \bar{\varrho} + \partial_x(\bar{\varrho} \bar{u}) = 0, \\ \partial_t(\bar{\varrho} \bar{v}) + \partial_x(\bar{\varrho} \bar{u} \bar{v}) = -\bar{\varrho} \bar{v} - \bar{\varrho} \partial_x W * \bar{\varrho}. \end{cases} \quad (72)$$

Note that in both cases  $W = -|x| + \frac{|x|^2}{2}$ .

#### 4.1 Derivation of the relative entropy inequality

We will use the equations (71) and (72) to construct the relative entropy functional similar to the one obtained in [15], modulo the nonlocal attraction/repulsion term and linear damping term. Note also that in [15] it was assumed that  $\bar{v}$  is a bounded function which is not the case here. Therefore, all integrations by parts and estimates must take this fact into account. In this part of the proof it is important that both systems (71) and (72) are considered on the whole domain  $\mathbb{R}$ . Adapting our arguments to the free boundary problem would require some tedious extensions of the solution to the limit problem preserving the Sobolev norms, and we leave this for future research.

For brevity of formulas, we introduce the notation:

$$V = v - \bar{v}, \quad U = u - \bar{u}, \quad R = \varrho - \bar{\varrho},$$

and so, subtracting equations of system (72) from equations of system (71), respectively, we obtain:

$$\begin{cases} \partial_t R + \partial_x(\varrho U) + \partial_x(R \bar{u}) = 0, \\ (\varrho \partial_t + \varrho u \partial_x) V + \varrho V = -R(\partial_t \bar{v} + \bar{u} \partial_x \bar{v}) - R \bar{v} - \varrho U \partial_x \bar{v} - \varrho \partial_x W * \varrho + \bar{\varrho} \partial_x W * \bar{\varrho}. \end{cases} \quad (73)$$

Next, we reduce the right hand side of the new momentum equation using the fact that in the momentum equation of (72) all terms are multiplied by  $\bar{\varrho}$ . Note that the classical solution to the limit system has positive density in whole domain provided this was the case for the initial condition. Therefore, by the continuity equation of (72), the momentum equation may be reduced to

$$\partial_t \bar{v} + \bar{u} \partial_x \bar{v} = -\bar{v} - \partial_x W * \bar{\varrho}.$$

So, inserting it to the momentum equation of (73), we get

$$\begin{cases} \partial_t R + \partial_x(\varrho U) + \partial_x(R \bar{u}) = 0, \\ (\varrho \partial_t + \varrho u \partial_x) V + \varrho V = -\varrho U \partial_x \bar{v} - \varrho \partial_x W * \varrho + \varrho \partial_x W * \bar{\varrho}. \end{cases} \quad (74)$$

Our task now is to derive an analogue of the classical energy estimate for the system (74). We start from multiplying the second equation of (74) by  $V$  and integrating over the whole space to get

$$\frac{d}{dt} \int_{\mathbb{R}} \frac{\varrho V^2}{2} = - \int_{\mathbb{R}} \varrho V^2 dx - \int_{\mathbb{R}} \varrho V U \partial_x \bar{v} dx - \int_{\mathbb{R}} \varrho (\partial_x W * (\varrho - \bar{\varrho})) V dx, \quad (75)$$

where we also used the continuity equation of (71) tested by  $\frac{1}{2} V^2$  to obtain the energy term on the l.h.s of (75). Although the first term on the r.h.s. of (75) has already a right sign we

further transform it using  $V = U + \varepsilon \frac{\gamma}{\gamma-1} \partial_x \varrho^{\gamma-1}$ :

$$\begin{aligned}
\int_{\mathbb{R}} \varrho V^2 \, dx &= \int_{\mathbb{R}} \varrho V U \, dx + \int_{\mathbb{R}} \varrho V \varepsilon \frac{\gamma}{\gamma-1} \partial_x \varrho^{\gamma-1} \, dx \\
&= \int_{\mathbb{R}} \varrho V U \, dx + \int_{\mathbb{R}} \varrho U \varepsilon \frac{\gamma}{\gamma-1} \partial_x \varrho^{\gamma-1} \, dx + \int_{\mathbb{R}} \varrho \left( \frac{\varepsilon \gamma}{\gamma-1} \right)^2 (\partial_x \varrho^{\gamma-1})^2 \, dx \\
&= \int_{\mathbb{R}} \varrho V U \, dx + \int_{\mathbb{R}} \varrho U \varepsilon \frac{\gamma}{\gamma-1} \partial_x \bar{\varrho}^{\gamma-1} \, dx \\
&\quad + \int_{\mathbb{R}} \varrho U \varepsilon \frac{\gamma}{\gamma-1} \partial_x (\varrho^{\gamma-1} - \bar{\varrho}^{\gamma-1}) \, dx + \int_{\mathbb{R}} \varrho \left( \frac{\varepsilon \gamma}{\gamma-1} \right)^2 (\partial_x \varrho^{\gamma-1})^2 \, dx.
\end{aligned} \tag{76}$$

Inserting this back to (75) and rearranging the terms we get

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} \frac{\varrho V^2}{2} \, dx &+ \int_{\mathbb{R}} \varrho U \varepsilon \frac{\gamma}{\gamma-1} \partial_x (\varrho^{\gamma-1} - \bar{\varrho}^{\gamma-1}) \, dx + \int_{\mathbb{R}} \varrho \left( \frac{\varepsilon \gamma}{\gamma-1} \right)^2 (\partial_x \varrho^{\gamma-1})^2 \, dx \\
&= - \int_{\mathbb{R}} \varrho V U (\partial_x \bar{v} + 1) \, dx - \int_{\mathbb{R}} \varrho U \varepsilon \frac{\gamma}{\gamma-1} \partial_x \bar{\varrho}^{\gamma-1} \, dx - \int_{\mathbb{R}} \varrho (\partial_x W * R) V \, dx.
\end{aligned} \tag{77}$$

In the next step we will complement the kinetic part of the relative entropy appearing on the l.h.s. of (77) by the potential part that is linked to the non-local forces. To this purpose we evaluate

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} (\varrho - \bar{\varrho}) W * (\varrho - \bar{\varrho}) \, dx &= \int_{\mathbb{R}} (\varrho - \bar{\varrho}) W * (\partial_t \varrho - \partial_t \bar{\varrho}) \, dx \\
&= - \int_{\mathbb{R}} (\varrho - \bar{\varrho}) W * \partial_x (\varrho u - \bar{\varrho} \bar{u}) \, dx = \int_{\mathbb{R}} (\varrho u - \bar{\varrho} \bar{u}) \partial_x W * (\varrho - \bar{\varrho}) \, dx.
\end{aligned} \tag{78}$$

Note that above we only used the fact that  $W$  is symmetric, and not the integration by parts. Substituting  $u = v - \varepsilon \frac{\gamma}{\gamma-1} \partial_x \varrho^{\gamma-1}$ ,  $R = \varrho - \bar{\varrho}$ , and  $\bar{u} = \bar{v}$  we obtain

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} R W * R \, dx &= \int_{\mathbb{R}} (\varrho v - \varepsilon \varrho \frac{\gamma}{\gamma-1} \partial_x \varrho^{\gamma-1} - \bar{\varrho} \bar{v}) \partial_x W * R \, dx \\
&= \int_{\mathbb{R}} R \bar{v} \partial_x W * R \, dx + \int_{\mathbb{R}} \varrho (v - \bar{v}) \partial_x W * R \, dx - \varepsilon \int_{\mathbb{R}} \partial_x \varrho^\gamma \partial_x W * R \, dx \\
&= \int_{\mathbb{R}} R \bar{v} \partial_x W * R \, dx + \int_{\mathbb{R}} \varrho V \partial_x W * R \, dx - \varepsilon \int_{\mathbb{R}} \partial_x \varrho^\gamma \partial_x W * R \, dx.
\end{aligned} \tag{79}$$

Summing up (77) and (79) we obtain

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} \frac{\varrho V^2}{2} \, dx &+ \frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} R W * R \, dx \\
&+ \int_{\mathbb{R}} \varrho U \varepsilon \frac{\gamma}{\gamma-1} \partial_x (\varrho^{\gamma-1} - \bar{\varrho}^{\gamma-1}) \, dx + \left( \frac{\varepsilon \gamma}{\gamma-1} \right)^2 \int_{\mathbb{R}} \varrho (\partial_x \varrho^{\gamma-1})^2 \, dx \\
&= - \int_{\mathbb{R}} \varrho V U (\partial_x \bar{v} + 1) \, dx - \int_{\mathbb{R}} \varrho U \varepsilon \frac{\gamma}{\gamma-1} \partial_x \bar{\varrho}^{\gamma-1} \, dx \\
&+ \int_{\mathbb{R}} R \bar{v} \partial_x W * R \, dx - \varepsilon \int_{\mathbb{R}} \partial_x \varrho^\gamma \partial_x W * R \, dx.
\end{aligned} \tag{80}$$

To deal with the last term on the r.h.s. we will separate  $W(x)$  into two parts  $K(x) = -|x|$ , and  $L(x) = \frac{x^2}{2}$ . Note that  $\partial_x L * (\varrho - \bar{\varrho}) = 0$ , indeed we have

$$\begin{aligned}\partial_x L * (\varrho - \bar{\varrho}) &= \int_{\mathbb{R}} (x - y)(\varrho(t, y) - \bar{\varrho}(t, y)) dy \\ &= x \int_{\mathbb{R}} (\varrho(t, y) - \bar{\varrho}(t, y)) dy - \int_{\mathbb{R}} (x\varrho - x\bar{\varrho}) dx \\ &= - \int_{\mathbb{R}} (x\varrho - x\bar{\varrho}) dx = 0.\end{aligned}\tag{81}$$

The two last equalities follow from the fact the total masses for both  $\varrho$  and  $\bar{\varrho}$  as well as their first moments are the same. To see the equality between the moments it is enough to integrate the momentum equation for the target system (1), and to check that it implies the same formula (56) as for the  $\varepsilon$ -dependent system, and so  $\int_{\mathbb{R}} x\bar{\varrho} dx = \int_{\mathbb{R}} x\varrho dx$ .

Using (81) we immediately simplify the last term in (80) as follows

$$\begin{aligned}-\varepsilon \int_{\mathbb{R}} \partial_x \varrho^\gamma \partial_x W * R dx &= -\varepsilon \int_{\mathbb{R}} \partial_x \varrho^\gamma \partial_x K * R dx = \varepsilon \int_{\mathbb{R}} \varrho^\gamma \partial_{xx} K * R dx \\ &= -2\varepsilon \int_{\mathbb{R}} \varrho^\gamma (\varrho - \bar{\varrho}) dx = -2\varepsilon \int_{\mathbb{R}} \varrho^{\gamma+1} dx + 2\varepsilon \int_{\mathbb{R}} \varrho^\gamma \bar{\varrho} dx,\end{aligned}\tag{82}$$

where we used the explicit form of  $K$  to compute  $\partial_{xx} K(x) = -2\delta_0(x)$ , and the fact that  $\varrho^\gamma(t, x) \rightarrow 0$  as  $|x| \rightarrow +\infty$  and  $\partial_x K * R$  is bounded, so the boundary term in the integration by parts disappears. Note that the first term on the r.h.s. of (82) has a good sign and so it can be moved to the l.h.s. of (80).

Let us now focus on the the third term on the l.h.s. of (80) and show that it has the right meaning of the distance. Indeed, integrating by parts and using the continuity equation from (73), we get

$$\begin{aligned}\int_{\mathbb{R}} \varrho U \varepsilon \frac{\gamma}{\gamma-1} \partial_x (\varrho^{\gamma-1} - \bar{\varrho}^{\gamma-1}) dx &= - \int_{\mathbb{R}} \partial_x (\varrho U) \varepsilon \frac{\gamma}{\gamma-1} (\varrho^{\gamma-1} - \bar{\varrho}^{\gamma-1}) dx \\ &= \int_{\mathbb{R}} (\partial_t R + \partial_x (R\bar{u})) \varepsilon \frac{\gamma}{\gamma-1} (\varrho^{\gamma-1} - \bar{\varrho}^{\gamma-1}) dx.\end{aligned}\tag{83}$$

Note that the boundary terms  $[\varrho U (\varrho^{\gamma-1} - \bar{\varrho}^{\gamma-1})]_{x \rightarrow -\infty}^{x \rightarrow +\infty}$  disappears because  $\varrho U \rightarrow 0$  for  $|x| \rightarrow +\infty$ , and  $[\varrho \bar{u} (\varrho^{\gamma-1} - \bar{\varrho}^{\gamma-1})]_{x \rightarrow -\infty}^{x \rightarrow +\infty}$  disappears because for  $|x| \rightarrow +\infty$ , we know that  $\bar{u}$  grows at most linearly (24), while  $x\varrho$  tends to 0 due to (60), and assumption that  $\varrho$  is smooth, and  $(\varrho^{\gamma-1} - \bar{\varrho}^{\gamma-1})$  remains bounded. Now we define our free energy  $F(R, \bar{\varrho})$  as follows

$$F(R, \bar{\varrho}) = \frac{\varepsilon}{\gamma} (R + \bar{\varrho})^\gamma - \varepsilon \bar{\varrho}^{\gamma-1} R - \frac{\varepsilon}{\gamma} \bar{\varrho}^\gamma.\tag{84}$$

Note that because  $R + \bar{\varrho} = \varrho$ , we have that

$$\frac{\partial F}{\partial R} = \varepsilon ((R + \bar{\varrho})^{\gamma-1} - \bar{\varrho}^{\gamma-1}) = \varepsilon (\varrho^{\gamma-1} - \bar{\varrho}^{\gamma-1}),\tag{85}$$

later on we will also need

$$\frac{\partial F}{\partial \bar{\varrho}} = \varepsilon ((R + \bar{\varrho})^{\gamma-1} - (\gamma-1)R\bar{\varrho}^{\gamma-2} - \bar{\varrho}^{\gamma-1}).\tag{86}$$

Using (85) in (83) we get that

$$\begin{aligned}
& \int_{\mathbb{R}} \varrho U \varepsilon \frac{\gamma}{\gamma-1} \partial_x (\varrho^{\gamma-1} - \bar{\varrho}^{\gamma-1}) \, dx = \int_{\mathbb{R}} (\partial_t R + \partial_x (R\bar{u})) \frac{\gamma}{\gamma-1} \frac{\partial F}{\partial R} \, dx \\
& = \frac{\gamma}{\gamma-1} \int_{\mathbb{R}} \partial_t R \frac{\partial F}{\partial R} \, dx + \frac{\gamma}{\gamma-1} \int_{\mathbb{R}} \bar{u} \partial_x R \frac{\partial F}{\partial R} \, dx + \frac{\gamma}{\gamma-1} \int_{\mathbb{R}} \partial_x \bar{u} R \frac{\partial F}{\partial R} \, dx \\
& = \frac{\gamma}{\gamma-1} \left( \int_{\mathbb{R}} \partial_t F \, dx - \int_{\mathbb{R}} \partial_t \bar{\varrho} \frac{\partial F}{\partial \bar{\varrho}} \, dx + \int_{\mathbb{R}} \bar{u} \partial_x F \, dx - \int_{\mathbb{R}} \bar{u} \partial_x \bar{\varrho} \frac{\partial F}{\partial \bar{\varrho}} \, dx + \int_{\mathbb{R}} \partial_x \bar{u} R \frac{\partial F}{\partial R} \, dx \right), \tag{87}
\end{aligned}$$

where to pass to the last line we used the fact that  $\partial_s F(R, \bar{\varrho}) = \partial_s R \frac{\partial F}{\partial R} + \partial_s \bar{\varrho} \frac{\partial F}{\partial \bar{\varrho}}$ , for  $s = t, x$ .

Next, we eliminate the second and fourth terms in above using the continuity equation for  $\bar{\varrho}$ , from which it follows that:

$$-\partial_t \bar{\varrho} - \bar{u} \partial_x \bar{\varrho} = \bar{\varrho} \partial_x \bar{u}, \tag{88}$$

therefore, integrating the third term on the r.h.s. of (87) by parts we find that

$$\begin{aligned}
& \int_{\mathbb{R}} \varrho U \varepsilon \frac{\gamma}{\gamma-1} \partial_x (\varrho^{\gamma-1} - \bar{\varrho}^{\gamma-1}) \, dx \\
& = \frac{d}{dt} \int_{\mathbb{R}} \frac{\gamma}{\gamma-1} F \, dx + \frac{\gamma}{\gamma-1} \int_{\mathbb{R}} \partial_x \bar{u} \left( -F + \bar{\varrho} \frac{\partial F}{\partial \bar{\varrho}} + R \frac{\partial F}{\partial R} \right) \, dx. \tag{89}
\end{aligned}$$

Indeed, let us notice that the integration by parts is justified here, similarly as for estimates for higher order moments, by Corollary 1 and the fact that we keep center of mass and second moment for both densities bounded due to assumptions. Finally, using (85) and (86), we find that

$$\bar{\varrho} \frac{\partial F}{\partial \bar{\varrho}} + R \frac{\partial F}{\partial R} = \gamma F.$$

After this simplification (80) can be rewritten in its final form as

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}} \left( \frac{\varrho V^2}{2} + \frac{1}{2} R W * R + \frac{\gamma}{\gamma-1} F \right) dx \\
& + \left( \frac{\varepsilon \gamma}{\gamma-1} \right)^2 \int_{\mathbb{R}} \varrho (\partial_x \varrho^{\gamma-1})^2 \, dx + 2\varepsilon \int_{\mathbb{R}} \varrho^{\gamma+1} \, dx \\
& \leq - \int_{\mathbb{R}} \varrho V U (\partial_x \bar{v} + 1) \, dx - \int_{\mathbb{R}} \varrho U \varepsilon \frac{\gamma}{\gamma-1} \partial_x \bar{\varrho}^{\gamma-1} \, dx \\
& + \int_{\mathbb{R}} R \bar{v} \partial_x W * R \, dx + 2\varepsilon \int_{\mathbb{R}} \varrho^\gamma \bar{\varrho} \, dx - \gamma \int_{\mathbb{R}} \partial_x \bar{u} F \, dx. \tag{90}
\end{aligned}$$

Note that we have

$$\begin{aligned}
& \int_{\mathbb{R}} \left( \frac{\varrho V^2}{2} + \frac{1}{2} R W * R + \frac{\gamma}{\gamma-1} F \right) dx \\
& = \int_{\mathbb{R}} \left( \frac{\varrho(v - \bar{v})^2}{2} + \frac{1}{2} (\varrho - \bar{\varrho}) W * (\varrho - \bar{\varrho}) + \varepsilon (H(\varrho) - H'(\bar{\varrho})(\varrho - \bar{\varrho}) - H(\bar{\varrho})) \right) dx \tag{91} \\
& = \mathcal{E}(\varrho, v | \bar{\varrho}, \bar{u})(t)
\end{aligned}$$

for  $H(s) = \frac{1}{\gamma} s^\gamma$ , as specified in (15), and therefore, after integration of (90) over  $(0, \tau)$ , we obtain (15) with the reminder as in (16).

## 4.2 Estimate of the reminder

In what follows we estimate the terms on the r.h.s. of the relative entropy inequality (90), i.e.

$$\begin{aligned}\mathcal{R}(\varrho, v, \bar{\varrho}, \bar{u}) &= - \int_{\mathbb{R}} \varrho V U (\partial_x \bar{v} + 1) \, dx - \int_{\mathbb{R}} \varrho U \varepsilon \frac{\gamma}{\gamma - 1} \partial_x \bar{\varrho}^{\gamma-1} \, dx \\ &\quad + \int_{\mathbb{R}} R \bar{v} \partial_x W * R \, dx + 2\varepsilon \int_{\mathbb{R}} \varrho^\gamma \bar{\varrho} \, dx - \gamma \int_{\mathbb{R}} \partial_x \bar{u} F \, dx \\ &= I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned}\tag{92}$$

To this purpose we use the general a-priori estimates derived in Section 3 as well as the l.h.s. of (90).

*Estimate of  $I_1$ .* For the first term, we use the fact that

$$U = u - \bar{u} = v - \varepsilon \frac{\gamma}{\gamma - 1} \partial_x \varrho^{\gamma-1} - \bar{v} = V - \varepsilon \frac{\gamma}{\gamma - 1} \partial_x \varrho^{\gamma-1},$$

therefore

$$\begin{aligned}I_1 &= - \int_{\mathbb{R}} \varrho V U (\partial_x \bar{v} + 1) \, dx \\ &= - \int_{\mathbb{R}} \varrho V^2 (\partial_x \bar{v} + 1) \, dx + \int_{\mathbb{R}} \varrho V \varepsilon \frac{\gamma}{\gamma - 1} \partial_x \varrho^{\gamma-1} (\partial_x \bar{v} + 1) \, dx \\ &\leq - \int_{\mathbb{R}} \varrho V^2 (\partial_x \bar{v} + 1) \, dx + \frac{1}{4} \int_{\mathbb{R}} \left( \frac{\varepsilon \gamma}{\gamma - 1} \right)^2 \varrho (\partial_x \varrho^{\gamma-1})^2 \, dx + \int_{\mathbb{R}} \varrho V^2 (\partial_x \bar{v} + 1)^2 \, dx \\ &\leq \left( \|(\partial_x \bar{v} + 1)\|_{L^\infty((0,T) \times \mathbb{R})}^2 + 1 \right) \int_{\mathbb{R}} \varrho V^2 \, dx + \frac{1}{4} \int_{\mathbb{R}} \left( \frac{\varepsilon \gamma}{\gamma - 1} \right)^2 \varrho (\partial_x \varrho^{\gamma-1})^2 \, dx.\end{aligned}\tag{93}$$

The first term can be controlled by the Gronwall inequality, while the second one can be absorbed by the l.h.s. of (90).

*Estimate of  $I_2$ .* Using the same decomposition of  $U$  as in case of  $I_1$  we obtain

$$\begin{aligned}I_2 &= - \int_{\mathbb{R}} \varrho U \frac{\varepsilon \gamma}{\gamma - 1} \partial_x \bar{\varrho}^{\gamma-1} \, dx \\ &= - \int_{\mathbb{R}} \varrho V \frac{\varepsilon \gamma}{\gamma - 1} \partial_x \bar{\varrho}^{\gamma-1} \, dx + \int_{\mathbb{R}} \varrho \left( \frac{\varepsilon \gamma}{\gamma - 1} \right)^2 (\partial_x \bar{\varrho}^{\gamma-1}) (\partial_x \varrho^{\gamma-1}) \, dx \\ &\leq \left( \frac{\varepsilon \gamma}{\gamma - 1} \right)^2 \int_{\mathbb{R}} (\partial_x \bar{\varrho}^{\gamma-1})^2 \varrho \, dx + \frac{1}{4} \int_{\mathbb{R}} \varrho V^2 \, dx \\ &\quad + \frac{1}{4} \int_{\mathbb{R}} \left( \frac{\varepsilon \gamma}{\gamma - 1} \right)^2 \varrho (\partial_x \varrho^{\gamma-1})^2 \, dx + \left( \frac{\varepsilon \gamma}{\gamma - 1} \right)^2 \int_{\mathbb{R}} (\partial_x \bar{\varrho}^{\gamma-1})^2 \varrho \, dx \\ &= \frac{1}{4} \int_{\mathbb{R}} \varrho V^2 \, dx + \frac{1}{4} \int_{\mathbb{R}} \left( \frac{\varepsilon \gamma}{\gamma - 1} \right)^2 \varrho (\partial_x \varrho^{\gamma-1})^2 \, dx + 2 \left( \frac{\varepsilon \gamma}{\gamma - 1} \right)^2 \int_{\mathbb{R}} (\partial_x \bar{\varrho}^{\gamma-1})^2 \varrho \, dx.\end{aligned}\tag{94}$$

Again, the first two terms will be handled by the Gronwall inequality and by moving to the l.h.s. of (90), respectively. The last term on the r.h.s. of (94) is a small constant due to the presence of  $\varepsilon$ , provided that  $\partial_x \bar{\varrho}^{\gamma-1} \in L^2(0, T; L^2(\mathbb{R}))$ . Indeed, from (54), we may deduce that  $\varepsilon^2 \|\varrho\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C \varepsilon^{\frac{4(\gamma-1)}{2\gamma-1}}$ .

*Estimate of  $I_3$ .* To deal with this term we again separate  $W(x)$  into two parts  $K(x) = -|x|$ , and  $L(x) = \frac{x^2}{2}$

$$I_3 = \int_{\mathbb{R}} (\varrho - \bar{\varrho}) \bar{v} \partial_x K * (\varrho - \bar{\varrho}) dx + \int_{\mathbb{R}} (\varrho - \bar{\varrho}) \bar{v} \partial_x L * (\varrho - \bar{\varrho}) dx. \quad (95)$$

The part of the potential coming from the Newtonian repulsion is good because we can write

$$\partial_{xx} K * (\varrho - \bar{\varrho}) = -2(\varrho - \bar{\varrho}) \quad (96)$$

and so

$$\begin{aligned} \int_{\mathbb{R}} (\varrho - \bar{\varrho}) \bar{v} \partial_x K * (\varrho - \bar{\varrho}) dx &= -\frac{1}{2} \int_{\mathbb{R}} \bar{v} \partial_{xx} K * (\varrho - \bar{\varrho}) \partial_x K * (\varrho - \bar{\varrho}) dx \\ &= -\frac{1}{4} \int_{\mathbb{R}} \bar{v} \partial_x |\partial_x K * (\varrho - \bar{\varrho})|^2 dx \\ &= \frac{1}{4} \int_{\mathbb{R}} \partial_x \bar{v} |\partial_x K * (\varrho - \bar{\varrho})|^2 dx \leq \frac{1}{4} \|\partial_x \bar{v}\|_{\infty} \int_{\mathbb{R}} |\partial_x K * (\varrho - \bar{\varrho})|^2 dx. \end{aligned} \quad (97)$$

Note that  $\lim_{|x| \rightarrow \infty} \partial_x K * (\varrho - \bar{\varrho}) = 0$  but  $\bar{v}$  may diverge at infinity, so the lack of boundary terms in above requires an extra justification. Note however that

$$\partial_x K * (\varrho - \bar{\varrho}) = - \int_{\mathbb{R}} \text{sgn}(x - y) (\varrho(y) - \bar{\varrho}(y)) dy = -2 \int_{-\infty}^x (\varrho(y) - \bar{\varrho}(y)) dy, \quad (98)$$

and therefore

$$\begin{aligned} \lim_{x \rightarrow +\infty} \bar{v} |\partial_x K * (\varrho - \bar{\varrho})|^2 &\leq \lim_{x \rightarrow +\infty} |\sqrt{|\bar{v}|} \partial_x K * (\varrho - \bar{\varrho})|^2 \\ &\leq \lim_{x \rightarrow +\infty} \left| 2\sqrt{x} \int_{-\infty}^x (\varrho(y) - \bar{\varrho}(y)) dy \right|^2 = 4 \left( \lim_{x \rightarrow +\infty} \frac{\int_{-\infty}^x (\varrho(y) - \bar{\varrho}(y)) dy}{x^{-1/2}} \right)^2 \\ &= 16 \left( \lim_{x \rightarrow +\infty} \frac{\varrho(x) - \bar{\varrho}(x)}{x^{-3/2}} \right)^2 = 0, \end{aligned} \quad (99)$$

where we used d'Hôpital rule and (60), subsequently, in the two last equalities. Coming back to (97) the term on the r.h.s. can be now controlled by the l.h.s. of (90), indeed we have

$$\begin{aligned} \int_{\mathbb{R}} (\varrho - \bar{\varrho}) K * (\varrho - \bar{\varrho}) dx &= -\frac{1}{2} \int_{\mathbb{R}} \partial_{xx} K * (\varrho - \bar{\varrho}) K * (\varrho - \bar{\varrho}) dx \\ &= \frac{1}{2} \int_{\mathbb{R}} |\partial_x K * (\varrho - \bar{\varrho})|^2 dx, \end{aligned} \quad (100)$$

where again we dropped the boundary term due to the fact that  $\lim_{|x| \rightarrow \infty} \partial_x K * (\varrho - \bar{\varrho}) = 0$ . Therefore, the Newtonian repulsion part of potential  $W$  can be controlled by the Gronwall inequality. For the quadratic confinement  $L(x) = \frac{x^2}{2}$  in the integral in (95), we have by (81) that

$$\int_{\mathbb{R}} (\varrho - \bar{\varrho}) \bar{v} (\partial_x L * (\varrho - \bar{\varrho})) dx = 0. \quad (101)$$

*Estimate of  $I_4$ .* The fourth term on the r.h.s. of (90) can be immediately absorbed by the

last term on the l.h.s., we have

$$I_4 = 2\varepsilon \int_{\mathbb{R}} \varrho^\gamma \bar{\varrho} \, dx \leq \varepsilon^{\frac{1}{\gamma+1}} \varepsilon \int_{\mathbb{R}} \varrho^{\gamma+1} \, dx + \varepsilon^{\frac{1}{\gamma+1}} \int_{\mathbb{R}} \bar{\varrho}^{\gamma+1} \, dx, \quad (102)$$

and so, for sufficiently small  $\varepsilon$  the first term is absorbed by  $\varepsilon \int_{\mathbb{R}} \varrho^{\gamma+1} \, dx$  on the l.h.s. of (90), while the second term is arbitrary small provided that  $\bar{\varrho} \in L^{\gamma+1}((0, T) \times \mathbb{R})$ .

*Estimate of  $I_5$ .* Estimate of the last term is almost straightforward as we have

$$I_5 = -\gamma \int_{\mathbb{R}} \partial_x \bar{u} F \, dx \leq \gamma \|\partial_x \bar{v}\|_{L^\infty((0, T) \times \mathbb{R})} \int_{\mathbb{R}} F \, dx \quad (103)$$

that can be treated by the Gronwall inequality.

Let us summarise, collecting the estimates for  $I_1 - I_5$ , we obtain

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}(\varrho, v | \bar{\varrho}, \bar{u})(t) + \int_{\mathbb{R}} \varrho \left( \frac{\varepsilon \gamma}{\gamma - 1} \right)^2 (\partial_x \varrho^{\gamma-1})^2 \, dx + 2\varepsilon \int_{\mathbb{R}} \varrho^{\gamma+1} \, dx \\ & \leq C(\|(\partial_x \bar{v} + 1)\|_{L^\infty((0, T) \times \mathbb{R})}^2) \int_{\mathbb{R}} \varrho V^2 \, dx + \frac{1}{4} \|\partial_x \bar{v}\|_{L^\infty(0, T \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x K * (\varrho - \bar{\varrho})|^2 \, dx \\ & \quad + \|\partial_x \bar{v}\|_{L^\infty((0, T) \times \mathbb{R})}^2 \gamma \int_{\mathbb{R}} F \, dx + \varepsilon C \\ & \leq C \mathcal{E}(\varrho, v | \bar{\varrho}, \bar{v})(t) + \varepsilon C, \end{aligned} \quad (104)$$

therefore applying the Gronwall inequality we obtain uniformly in  $n$  that

$$\begin{aligned} & \mathcal{E}(\varrho_n, v_n | \bar{\varrho}, \bar{u})(\tau) + \left( \frac{\varepsilon \gamma}{\gamma - 1} \right)^2 \int_0^\tau \int_{\mathbb{R}} \varrho_n (\partial_x \varrho_n^{\gamma-1})^2 \, dx \, dt + 2\varepsilon \int_0^\tau \int_{\mathbb{R}} \varrho_n^{\gamma+1} \, dx \, dt \\ & \leq C_1 (\mathcal{E}(\varrho_n, v_n | \bar{\varrho}, \bar{u})(0) + \varepsilon C_2). \end{aligned} \quad (105)$$

## 5 Conclusion of the proof of Theorem 2

With the uniform estimates from Section 3 it is relatively easy to repeat the arguments from [24] in order to prove strong convergence of a subsequence of  $(\varrho_n, \sqrt{\varrho_n} u_n)$  to  $(\varrho, \sqrt{\varrho} u)$  as specified in (40). The only modification which perhaps is worth explaining at this stage is the strong convergence of the density that we now have in much stronger sense, but also only for a restricted range of  $\gamma$ 's.

**Lemma 2** *Let  $\gamma \in (1, \frac{3}{2}]$ , then the sequence  $\varrho_n^{\gamma-\frac{1}{2}}$  satisfies*

$$\begin{aligned} & \varrho_n^{\gamma-\frac{1}{2}} \text{ is bounded in } L^\infty(0, T; H^1(\mathbb{R})); \\ & \partial_t \varrho_n^{\gamma-\frac{1}{2}} \text{ is bounded in } L^\infty(0, T; H^{-1}(\mathbb{R})). \end{aligned} \quad (106)$$

*As a consequence, up to a subsequence,  $\varrho_n^{\gamma-\frac{1}{2}}$  converges almost everywhere and strongly in  $C([0, T] \times B)$  for any compact subset  $B \subset \mathbb{R}$ . Moreover  $\varrho_n$  converges to  $\varrho$  in  $C([0, T] \times B)$ .*

*Proof.* The first estimate of (106) follows from (53) together with the conservation of mass (44). Next one can write

$$\partial_t \varrho_n^{\gamma-\frac{1}{2}} = -\partial_x (\varrho_n^{\gamma-\frac{1}{2}} u_n) - \left( \gamma - \frac{3}{2} \right) \varrho_n^{\gamma-\frac{1}{2}} \partial_x u_n,$$

which yields the second estimate due to (54). The Aubin's Lemma therefore provides the strong local convergence in the space of continuous functions. To prove the strong convergence of  $\varrho_n$  it is enough to apply the mean value theorem to the function  $f(s) = s^{2/(2\gamma-1)}$  together with the uniform (in  $n$ ) bound for the density (54). This is the only point of the proof when the assumption  $\gamma \leq 3/2$  becomes important.  $\square$

The rest of convergences in (40), i.e.

$$\begin{aligned} \partial_x \left( \varrho_n^{\gamma-\frac{1}{2}} \right) &\rightarrow \partial_x \left( \varrho_\varepsilon^{\gamma-\frac{1}{2}} \right) \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \\ \sqrt{\varrho_n} u_n &\rightarrow \sqrt{\varrho_\varepsilon} u_\varepsilon, \quad \text{strongly in } L^2(0, T; L^2(\mathbb{R})), \\ \varrho_n v_n &\rightarrow \varrho_\varepsilon v_\varepsilon, \quad \text{strongly in } L^2(0, T; L^2(\mathbb{R})), \\ \gamma \varrho_n^\gamma \partial_x u_n &\rightarrow \Lambda_\varepsilon \quad \text{weakly in } L^2(0, T; L_{loc}^2(\mathbb{R})), \end{aligned} \tag{107}$$

follows the same way as in [24] and [17].

Having proven (107), we can pass to the limit  $n \rightarrow \infty$  in all terms of system (2) to obtain a weak solution  $(\varrho_\varepsilon, \sqrt{\varrho_\varepsilon} u_\varepsilon)$  as specified in Definition 1. This is a very similar argument to the one performed in [17], modulo the passage to the limit in the nonlocal term. Here however, we can use uniform boundedness of higher moments of  $\varrho_n$ , see (69), together with strong convergence of  $\varrho_n$ . This concludes the proof of the first part of Theorem 2.

To prove the second part we realize that we can pass to the limit  $n \rightarrow \infty$  in (105). In this process we basically use the lower semicontinuity of convex functions that together with weak convergence of  $\partial_x \varrho_n^{\gamma-\frac{1}{2}}$  allows us to pass to the limit on the l.h.s. of (105) to obtain

$$\mathcal{E}(\varrho_\varepsilon, v_\varepsilon | \bar{\varrho}, \bar{u})(\tau) \leq C_1 (\mathcal{E}(\varrho_\varepsilon, v_\varepsilon | \bar{\varrho}, \bar{u})(0) + \varepsilon C_2). \tag{108}$$

Therefore if  $\varepsilon \rightarrow 0$ , and  $\mathcal{E}(\varrho_\varepsilon, v_\varepsilon | \bar{\varrho}, \bar{u})(0) = 0$ , then  $\mathcal{E}(\varrho_\varepsilon, v_\varepsilon | \bar{\varrho}, \bar{u})(\tau) \rightarrow 0$  for a.e.  $\tau \in (0, T)$ .  $\square$

**Remark 5** *The strong convergence  $v_\varepsilon \rightarrow \bar{u}$  does not follow from the relative entropy estimate, because although  $\varrho_\varepsilon > 0$ , we know that  $\varrho_\varepsilon(t, x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Actually, we do not know how to show a uniform bound from below on the densities of the form  $\varrho_n \geq \varphi(x) > 0$  with  $\varphi$  being a smooth enough decaying function at infinity. This estimate will allow us to relax the assumption  $\gamma \leq 3/2$  to deduce the strong convergence of  $\varrho_n$  to  $\varrho$  in Lemma 2 for all  $\gamma > 1$ . Another possible direction to investigate the sense of this convergence, would be to compare the solutions  $\varrho_\varepsilon, v_\varepsilon$  and  $\bar{\varrho}, \bar{u}$  on the support of the asymptotic profile of  $\bar{\varrho}$  for  $t \rightarrow \infty$  studied in [9], and it is postponed for future work.*

## Acknowledgements

JAC was partially supported by EPSRC grant number EP/P031587/1 and the Advanced Grant Nonlocal-CPD (Nonlocal PDEs for Complex Particle Dynamics: Phase Transitions, Patterns and Synchronization) of the European Research Council Executive Agency (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 883363). AWK is partially supported by a Newton Fellowship of the Royal Society and by the grant Iuventus Plus no. 0871/IP3/2016/74 of Ministry of Sciences and Higher Education RP. EZ was supported by the UCL Department of Mathematics Grant and grant Iuventus Plus no. 0888/IP3/2016/74 of Ministry of Sciences and Higher Education RP.

## References

- [1] Y. Brenier. Approximation of a simple Navier-Stokes model by monotonic rearrangement. *Discrete Contin. Dyn. Syst.*, 34(4):1285–1300, 2014.
- [2] D. Bresch and B. Desjardins. Existence of global weak solutions for a 2d viscous shallow water equations and convergence to the quasi-geostrophic model. *Commun. Math. Phys.*, 238(1):211–223, 2003.
- [3] D. Bresch, B. Desjardins, and D. Gérard-Varet. On compressible Navier-Stokes equations with density dependent viscosities in bounded domains. *J. Math. Pures Appl.* (9), 87(2):227–235, 2007.
- [4] D. Bresch, B. Desjardins, and C.-K. Lin. On some compressible fluid models: Korteweg, lubrication, and shallow water systems. *Comm. Partial Differential Equations*, 28(3-4):843–868, 2003.
- [5] D. Bresch, B. Desjardins, and E. Zatorska. Two-velocity hydrodynamics in fluid mechanics, part II: Existence of global  $\kappa$ -entropy solutions to the compressible Navier–Stokes systems with degenerate viscosities. *J. Math. Pures Appl.*, 104(4):801 – 836, 2015.
- [6] D. Bresch, P. Noble, and J.-P. Vila. Relative entropy for compressible Navier-Stokes equations with density dependent viscosities and various applications. *ESAIM: ProcS*, 58:40–57, 2017.
- [7] D. Bresch, A. F. Vasseur and C. Yu. Global Existence of Entropy-Weak Solutions to the Compressible Navier-Stokes Equations with Non-Linear Density Dependent Viscosities. *arXiv:1905.02701*, 2019.
- [8] J. Březina and V. Mácha. Inviscid limit for the compressible Euler system with non-local interactions. *arXiv:1611.07607*, 2016.
- [9] J. A. Carrillo, Y.-P. Choi, and E. Zatorska. On the pressureless damped Euler-Poisson equations with quadratic confinement: critical thresholds and large-time behavior. *Math. Models Methods Appl. Sci.*, 26(12):2311–2340, 2016.
- [10] J. A. Carrillo, E. Feireisl, P. Gwiazda, and A. Świerczewska-Gwiazda. Weak solutions for Euler systems with non-local interactions. *J. Lond. Math. Soc.*, 95(3):705–724, 2017.
- [11] J. A. Carrillo, M. Fornasier, G. Toscani, and F. Vecil. Particle, kinetic, and hydrodynamic models of swarming. *Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences*, Birkhauser Boston, 297–336, 2010.
- [12] J. A. Carrillo, A. Wróblewska-Kamińska, and E. Zatorska. On long-time asymptotics for viscous hydrodynamic models of collective behavior with damping and nonlocal interactions. To appear in *Math. Models Methods Appl. Sci.*, 2018.
- [13] C. M. Dafermos. The second law of thermodynamics and stability. *Arch. Ration. Mech. Anal.*, 70(2):167–179, 1979.
- [14] S. Ding, H. Wen, L. Yao, and C. Zhu. Global solutions to one-dimensional compressible Navier–Stokes–Poisson equations with density-dependent viscosity. *J. Math Phys.*, 50, 023101, 2009.

- [15] B. Haspot. Weak-strong uniqueness for compressible Navier-Stokes system with degenerate viscosity coefficient and vacuum in one dimension. *arXiv:1411.7679*, 2014.
- [16] B. Haspot. From the highly compressible Navier–Stokes equations to fast diffusion and porous media equations, existence of global weak solution for the quasi-solutions. *J. Math. Fluid Mech.*, 18(2):243–291, Jun 2016.
- [17] B. Haspot and E. Zatorska. From the highly compressible Navier-Stokes equations to the porous medium equation – rate of convergence. *Discrete Contin. Dyn. Syst.*, 36(6):3107–3123, 2016.
- [18] S.-Y. Ha and E. Tadmor. From particle to kinetic and hydrodynamic descriptions of flocking *Kinet. Relat. Models*, 1(3):415–435, 2008.
- [19] S. Jiang, Z. Xin, and P. Zhang. Global weak solutions to 1D compressible isentropic Navier-Stokes equations with density-dependent viscosity. *Methods Appl. Anal.*, 12(3):239–251, 2005.
- [20] Q. Jiu and Z. Xin. The Cauchy problem for 1D compressible flows with density-dependent viscosity coefficients. *Kinet. Relat. Models*, 1(2):313–330, 2008.
- [21] C. Lattanzio and A. E. Tzavaras. From gas dynamics with large friction to gradient flows describing diffusion theories. *Comm. Partial Differential Equations*, 42(2):261–290, 2017.
- [22] H.-L. Li, J. Li, and Z. Xin. Vanishing of vacuum states and blow-up phenomena of the compressible Navier-Stokes equations. *Comm. Math. Phys.*, 281(2):401–444, 2008.
- [23] H. Liu, H. Yuan, J. Qiao, and F. Li. Global existence of strong solutions of Navier–Stokes equations with non-Newtonian potential for one-dimensional isentropic compressible fluids. *Z. Angew. Math. Phys.*, 63(5):865–878, 2012.
- [24] A. Mellet and A. F. Vasseur. On the barotropic compressible Navier-Stokes equations. *Comm. Partial Differential Equations*, 32(1-3):431–452, 2007.
- [25] A. Mellet and A. F. Vasseur. Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations. *SIAM J. Math. Anal.*, 39(4):1344–1365, 2007/08.
- [26] P. B. Mucha. Compressible Navier-Stokes system in 1-D. *Math. Methods Appl. Sci.*, 24(9):607–622, 2001.
- [27] A. F. Vasseur and C. Yu. Existence of global weak solutions for 3d degenerate compressible Navier–Stokes equations. *Invent. Math.*, 206(3):935–974, 2016.
- [28] A. Veigant and A. Kazhikhov. On the existence of global solution to the two-dimensional Navier–Stokes equations for a compressible viscous flow. *Siberian Math. J.*, 36:1108–1141, 1995.
- [29] E. Zatorska. On the flow of chemically reacting gaseous mixture. *J. Differential Equations*, 253(12):3471–3500, 2012.
- [30] E. Zatorska. Mixtures: sequential stability of variational entropy weak solutions. *J. Math. Fluid Mech.* 17, No.3, 437–461, 2015.