Supersymmetric Geometries in String Theory

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Abstract

Supersymmetric vacua of string theory endow the internal space with a special geometric structure. We refer to these structures collectively as supersymmetric geometries. The objective of this thesis is to study two classes of supersymmetric geometries and their associated quantum field theories arising from string compactifications.

In the first part of this thesis, we study M-theory compactifications on G_2-manifolds. We focus on the gauge sector of such compactifications by studying the Higgs bundle, characterizing a local ALE-fibration over a supersymmetric three-cycle M_3, obtained from a partially twisted 7d super Yang-Mills theory on M_3. We derive the BPS equations and find the massless spectrum for both abelian and non-abelian gauge groups in 4d. The mathematical tool that allows us to determine the spectrum is Morse theory, and more generally Morse-Bott theory. We make contact with twisted connected sum (TCS) G_2-manifolds, which form the largest class of examples of compact G_2-manifolds. M-theory on TCS G_2-manifolds is known to result in a non-chiral 4d spectrum. We determine the Higgs bundle for this class of G_2-manifolds and provide a prescription for how to engineer singular transitions to models with chiral matter.

In the second part, we consider GK geometries that arise in AdS compactifications of IIB string theory and M-theory and play a key role in the geometric dual to c-extremization. We provide a mathematically oriented exposition of GK geometries in general dimension. We study the extension of the geometric dual of c-extremization for 2d $(0,2)$ superconformal field theories (SCFTs) that have an AdS_3 dual realized in Type IIB with varying axio-dilaton, i.e. F-theory. M/F-duality implies that such AdS_3 solutions can be mapped to AdS_2 solutions in M-theory, which are holographically dual to superconformal quantum mechanics (SCQM), obtained by dimensional reduction of the 2d SCFTs. We analyze the corresponding map between holographic c-extremization in F-theory and $\mathcal{I}$-extremization in M-theory, where in general the latter receives corrections relative to the F-theory result.

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Statement of Originality

Chapter II of the thesis is based on the article (written in collaboration):

A. P. Braun, S. Cizel, M. Hübner and S. Schäfer-Nameki, *Higgs bundles for M-theory on* G_2*-manifolds*, *JHEP* **03** (2019) 199, [1812.06072].

Chapter III is partially based on the article (written in collaboration):

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and unpublished work done in collaboration with Prof. James Sparks.

Parts of the work presented in this thesis will appear in forthcoming DPhil theses by coauthors listed above.

CONTENTS

CHAPTER

I

INTRODUCTION

The past four decades have been witness to a remarkable interaction between geometry and string theory. The first superstring revolution coincided with the ideas of quantum field theory making a forceful appearance in geometry through the groundbreaking work of Donaldson and Witten [1–4]. Despite the mathematical difficulties with a rigorous definition of quantum field theories, these developments started to shape the ubiquitous influence of physical insights in geometry and mathematics in general. With the advent of string theory, physicists uncovered a singularly effective method of constructing supersymmetric quantum field theories in various dimensions, which in turn relies on special classes of geometric objects whose properties are encoded in the physics of the resulting theory. This facilitated an ongoing, though occasionally pugnacious, dialogue between physics and geometry wherein advances in the former could have dramatic implications for the latter and vice-versa. This connection provides the backdrop for the work described in this thesis. We study multiple classes of geometries that arise as compactification spaces in superstring theory as well as the interplay between their geometric properties and the associated classes of quantum field theories.

The starting point of constructing vacuum solutions of superstring theory and M-theory, dating back to the seminal work [5], is to consider a spacetime of the form

$$\mathbb{M}^{1,d} = \mathfrak{M} \times X \, , \tag{1.1}$$

where $\mathfrak{M}$ is a maximally symmetric Lorentzian space with constant curvature, X is a compact internal manifold of Euclidean signature, and $d = 9$ for superstring theories and $d = 10$ for M-theory. It is a well-known result in Lorentzian geometry that, up to covers and quotients, maximally symmetric manifolds of signature $(1, n)$ are [6]: flat Minkowski space $\mathbb{R}^{1,n}$, de Sitter space dS_{n+1}, and anti-de Sitter space AdS_{n+1}. While these have all been discussed extensively in the context of string theory compactifications, spacetimes with a de Sitter factor cannot give rise to a supersymmetric vacuum solution of string theory [7]. In this thesis, we focus on the remaining two classes of compactifications, involving the AdS and Minkowski spaces. Both have resulted in a vast number of supersymmetric vacuum solutions of string and M-theory.

A key ingredient that places strong restrictions on the geometry of the backgrounds that can appear in (1.1) is supersymmetry. The basic compactification set-up considers the low energy limit of string theory — the corresponding supergravity theory — on a product background. The field content of supergravity in particular includes fermions, which are sections of the spinor bundle over the space $\mathbb{M}^{1,d}$. The product ansatz implies that the spin bundle (as well as all the associated vector bundles) decomposes as the product of the spin bundles over the factors in (1.1). This *a priori* implies that X is necessarily a spin manifold. A dramatically more restrictive condition arises from requiring that a certain fraction of supersymmetry of the starting supergravity theory is preserved in the compactified theory. By standard Kaluza-Klein analysis, this imposes a differential constraint in the form of a generalized Killing spinor equation on the internal manifold X. Schematically, the equation is of the form

$$\nabla_{\mathfrak{M},\mathcal{F}} \epsilon = 0 \, , \tag{1.2}$$

where $\nabla_{\mathfrak{M},\mathcal{F}}$ is a connection defined on a spinor bundle on X that depends both on the space $\mathfrak{M}$ and potentially also on fluxes that are turned on, which we collectively denote by $\mathcal{F}$. For further details of the general set-up see the review [8]. In general, this equation provides a restriction on the metrics on X that can admit solutions to the Killing spinor equation (see for instance [9]). Frequently it also leads to strong topological restrictions on the space X itself. In this thesis we refer to manifolds that admit solutions to (1.2) as *supersymmetric geometries*, owing to their string theory interpretation as being the vacuum solutions that preserve supersymmetry. It is worth underscoring the strong dependence of the Killing spinor equations on both the maximally symmetric space $\mathfrak{M}$ (in particular, its dimension) and the flux configuration of the particular supergravity set-up. This implies that studying different compactifications gives rise to diverse classes of supersymmetric geometries, each with its particular geometric flavor.

The simplest and most studied compactifications arise when $\mathbb{M} = \mathbb{R}^{1,n}$ and the background fluxes are all set to zero. The equation (1.2) in this case reduces to the parallel spinor equation

$$\nabla \epsilon = 0\,, \tag{1.3}$$

where ∇ is the Levi-Civita spin connection. A straightforward consequence of this is that the internal manifold is necessarily Ricci flat. Moreover, the existence of solutions to parallel spinors has deep connections to the geometry of manifolds with special and exceptional holonomy that were classified by Berger [10]. We summarize Berger's list and a selection of relevant geometric facts in table 1[1]. It is immediately clear from the table that only manifolds with holonomy $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, G_2 and $\mathrm{Spin}(7)$ belong to the class of supersymmetric geometries in this particular context. In particular, Calabi-Yau manifolds (that is, manifolds with $\mathrm{SU}(n)$-holonomy) have occupied a central role in string theory compactifications from the very beginning. This is largely because, amongst manifolds with special and exceptional holonomy, both the existence and geometry of Calabi-Yau manifolds are best understood. This has led to monumental advances in the understanding of supersymmetric quantum field theories in various dimensions by geometrically engineering families of examples from string theory on Calabi-Yau manifolds. Conversely, string theory has generated highly non-trivial insights in the geometry of Calabi-Yau manifolds, ranging

[1] The original Berger's result includes $\mathrm{Spin}(9)$ as a possible holonomy group as well. It was shown later [11, 12] that manifolds with holonomy $\mathrm{Spin}(9)$ are necessarily locally symmetric.

Hol(g)	dim(M)	Properties	Parallel spinors	
SO(n)	n	—	0	Generic
U(n)	$2n$	Kähler	0	Special
SU(n)	$2n$	Ricci-flat Kähler	2	
Sp(1)Sp(n)	$4n$	Einstein	0	
Sp(n)	$4n$	Hyperkähler	$n+1$	
G_2	7	Ricci-flat	1	Exceptional
Spin(7)	8	Ricci-flat	1	

Table 1: Overview of the possible holonomy groups according to Berger's classification. The table is a combination of theorems 3.4.1 and 3.6.1 in [16]. The compact manifold M is assumed to be simply connected, irreducible and not locally symmetric.

from the celebrated mirror symmetry conjectures [13,14] to string theory calculation of the Donaldson-Thomas invariants [15].

In chapter II of this thesis, we study compactifications of M-theory on G_2-manifolds, where the current understanding of both physics and mathematics is significantly more limited. These compactifications are of particular physical interest since they give rise to 4d quantum field theories with minimal $\mathcal{N} = 1$ supersymmetry, which play an important role in constructing 4d super Yang-Mills theories as well as the minimally supersymmetric extension of the Standard Model. The phenomenological motivation for M-theory on G_2-manifolds has certainly waned in recent years in view of the absence of experimental evidence for supersymmetry. However, as string theory on Calabi-Yau manifolds shows, understanding the physics of compactifications can provide valuable constructions of new QFTs and can shed light on the geometry of the compact internal space. The latter is especially important in the case of G_2-manifolds since their geometry still evades a systematic understanding. The general question of sufficient conditions for the existence of a G_2-holonomy metric on a given seven-fold seems to be far beyond reach. In particular, the supply of examples of compact G_2-manifolds is very limited; the known examples arise from intricate constructions (we review a selection of them in section 2.3). A shared feature of all the constructions is that they all build metrics by combining lower dimensional structures and deforming them to obtain a genuine G_2-metric. Such deformations inevitably lie near the boundary of the moduli space of G_2-metrics on a given manifold and therefore the global structure of the moduli space is still very mysterious.[2]

The second class of compactifications, with $\mathfrak{M} = \mathrm{AdS}_{n+1}$, presents a distinct set of geometric characteristics. The interest in AdS compactifications is not directly related to phenomenology; rather the impetus for their study comes from the celebrated *AdS/CFT correspondence*, which relates quantum gravity theories in $n+1$ dimensions, that arise from compactifications on $\mathrm{AdS}_{n+1} \times X$, with conformal field theories in one dimension less. This provides yet another interface between the geometry of X and conformal field theories that gave rise to a spectrum of breakthrough results on both sides [18–20]. The key examples of supersymmetric geometries in this context are the Sasaki-Einstein manifolds that feature in the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence. The supersymmetry equation (1.2) takes the form of the classical Killing spinor equation

$$\nabla_Y \epsilon = \pm \frac{1}{2} Y \cdot \epsilon \,, \tag{1.4}$$

where Y is a vector field on X and ∇ is the spin connection. Manifolds admitting real Killing spinors admit a similar classification as manifolds with parallel spinors [21, 22]. However, two notable features are particularly distinctive for AdS supersymmetric geometries more generally. First, the metric cone $\mathrm{C}(X)$ frequently admits a natural geometric structure that is induced by the Killing spinor equations. This is not coincidental; compactifications on $\mathrm{AdS}_{n+1} \times X$ are in fact formally equivalent to compactifications on $\mathbb{R}^{1,n-1} \times \mathrm{C}(X)$ [8]. Additionally, for odd-dimensional X, the supersymmetry requirements commonly entail an existence of a 1d foliation on X with the transverse space again endowed with a special geometric structure. These properties are most clearly exemplified when X is Sasaki-Einstein. As is well known, X has a transverse Kähler-Einstein structure, and the cone $\mathrm{C}(X)$ is necessarily Calabi-Yau.

Chapter III of this thesis is focused on geometries that arise when compactifying IIB supergravity on backgrounds with AdS_3 factor and M-theory on background with AdS_2 factor, and are collectively known as GK geometries. In many ways, their geometry bears a cursory resemblance to Sasaki-Einstein manifolds. On any GK-manifold X there exists

[2] For instance, only very recently it was shown that G_2-moduli space can be disconnected [17].

a 1d foliation with a transverse Kähler structure as well as a natural complex structure on $C(X)$ with vanishing first Chern class. There are two fundamental differences, however. The transverse Kähler metric can never be Kähler-Einstein and the complex cone $C(X)$ has no distinguished Kähler structure. GK geometry is therefore quite different from Sasaki-Einstein geometry and many of its properties are still unexplored. On the CFT side, GK geometries are dual to 2d $\mathcal{N} = (0, 2)$ superconformal field theories for type IIB and 1d $\mathcal{N} = 2$ super quantum mechanics for M-theory solutions. Like the corresponding GK geometries, the understanding of these two families is constrained to special cases. This is especially true for 1d theories and we use a combination of physical and geometric arguments to study the relation between 2d and 1d theories.

The objective of this thesis is to contribute to these exciting topics at the nexus of geometry and physics. As we have alluded to in this introduction, the two parts of the thesis are independent of each other, with chapter II dedicated to the study of G_2-manifolds and their geometry through the lens of M-theory compactifications, and chapter III focusing on GK geometries both from a purely mathematical perspective and by leveraging physical arguments such as M/F-duality. With this, we attempt to shed some light on the GK-geometries themselves as well as their field theory duals.

CHAPTER

II

G_2-MANIFOLDS

1 Introduction

The central focus of this chapter is on M-theory compactifications on G_2-manifolds. Their importance lies in the fact that in this framework, one is able to generate minimally supersymmetric theories in 4d, which have, compared to other string theoretic constructions, the distinguishing feature of being purely geometric; one can obtain $\mathcal{N} = 1$ supersymmetry without any additional fluxes or bundles. Despite significant research interest, the understanding of M-theory on G_2-manifolds is still fundamentally lacking. This is in no small part due to the intractability of G_2-manifolds and the consequent lack of a complete understanding of their geometry. As is well known, the physically interesting examples of G_2-manifold have to admit singularities, which yield both gauge (codimension 4) and chiral matter (codimension 7) degrees of freedom in 4d [23–29]. There have been recent advances in the direction of producing compact G_2-manifolds with conical singularities [30–35], but it still remains an open problem.

The largest known class of examples of compact G_2-manifolds was given using a construction called *twisted connected sum* (TCS) [17, 36–38]. Starting from non-compact Calabi-Yau building blocks, it gives rise to millions of new compact G_2-manifolds with a detailed understanding of their topology in terms of the building blocks. The physics of M-theory and string theory on TCS G_2-manifolds has been investigated in [39–48]. The key property common to all TCS manifolds, which is a direct consequence of this particular construction, is that singularities will occur (if at all) in codimension 4 and 6, but not 7. From the standard geometric engineering dictionary for G_2-manifolds it then follows that the resulting models in 4d will not have chiral matter. A natural question is then whether the TCS manifolds can be deformed in a way that yields a chiral spectrum in 4d. The present chapter will provide a setting that gives some answers to this question and explores how such transitions would be characterized in TCS geometries, by providing a local model description in terms of a Higgs bundle. To achieve this, we first refine and extend the local model framework of [49], to incorporate the local limit of TCS G_2-manifolds, and then determine the type of deformations that are required.

The approach of using local Higgs bundle models and their spectral covers in F-theory [50–60] has proven very successful in model building, and more importantly as a precursor to the study of compact F-theory models. The Higgs bundles characterize the gauge sector of a compactification in terms of the local geometry in the vicinity of an ADE-singularity. Local examples of G_2-manifolds with isolated conical singularities have been studied in [26, 28, 29]. Here we will take a slightly different approach, starting much like in F-theory with the statement that a local geometry that realizes in M-theory an ADE gauge group in 4d, will necessarily have a description in terms of an ALE-fibration over a compact associative (supersymmetric) cycle M_3

$$\mathbb{C}^2/\Gamma_{ADE} \to M_3 \,. \tag{1.1}$$

This approach was first studied in [49], however much of the details of their paper remained somewhat ad hoc and more importantly, does not e.g. include the case of TCS G_2-manifolds as we shall explain. We provide both a first principle derivation and an analysis of the solutions of the Higgs bundle associated to this model, which, in particular, lends itself to generalizations.

As M-theory compactified on an ALE space gives a 7d super Yang-Mills (SYM) theory with ADE gauge group, the effective 4d $\mathcal{N} = 1$ theory of an ALE-fibration can be found by studying a topologically twisted 7d SYM-theory on a three-manifold M_3. The BPS equations then determine the field configurations along M_3 that ensure that $\mathcal{N} = 1$ supersymmetry is preserved in 4d. They are given in terms of a Higgs bundle specified by an adjoint valued one-form Higgs field ϕ and a gauge connection W along M_3. We focus entirely on diagonalizable Higgs fields, which implies that the connection W furthermore has to be flat. The diagonalizability implies that we can equivalently describe the Higgs bundle in terms of its eigenvalues or spectral data.

The BPS equations imply that $\mathrm{d}\phi = \mathrm{d}^\dagger\phi = 0$ and so $\phi = \mathrm{d}f$, where f is a harmonic function. This in turn implies that f is constant as long as we require M_3 to be compact and f to be regular. To obtain interesting solutions we introduce 'sources' or equivalently singularities for f, $\Delta f = \rho$. Alternatively, we may excise the loci where sources are located and study the corresponding f-twisted Laplace equation on the resulting three-manifold with boundary $\mathcal{M}_3$.

In general, the solutions to this zero-mode counting are difficult to determine. However, if we assume a fully factored spectral cover, the problem of finding the zero mode spectrum and interactions maps to Morse-Bott cohomology on $\mathcal{M}_3$. In this case, the resulting 4d gauge theory has U(1)-gauge symmetries, which are determined by the number of factors of the spectral cover. The zero modes can then be computed in terms of relative cohomology of $\mathcal{M}_3$ with respect to its boundary. The Higgs bundle spectral cover provides a construction of the three-cycles in the ALE-fibration, and determines the matter fields and couplings in 4d. There is an alternative description — again in the case of fully factored spectral covers — in terms of supersymmetric quantum mechanics (SQM), whose grounds states can be computed using Morse (more generally Morse-Bott) theory as in Witten's classic work [4]. This characterization in terms of SQM identifies matter and couplings in terms of gradient flow trees in $\mathcal{M}_3$.

This setup in particular allows modeling the local geometry of M-theory compactifications on TCS G_2-manifolds, which have an ALE-fibration over $\mathbb{S}^3$ (e.g. as in [44]).

Moreover, it will allow us — in the framework of the local Higgs bundle description of the geometry — to make a concrete proposal for the types of deformations and transitions that the geometry needs to undergo. Although we necessarily lose the global description of the geometry offered in terms of a twisted connected sum[3] we may nevertheless track what happens to our model in the language of the local geometry, which may be useful in modifying/improving the TCS construction.

Outline

This chapter proceeds as follows. In sections 2 and 3 we lay the groundwork by outlining the fundamentals of G_2-geometry and the M-theory physics. In particular, we discuss a selection of known examples of G_2-manifolds and the dictionary between the singularities in G_2-manifolds and the corresponding M-theory physics. The next section 4 starts with a careful derivation of the partially topologically twisted 7d Super-Yang-Mills (SYM) theory on M_3, which in turn determines the BPS equations. We then discuss solutions in terms of Higgs bundles, which characterize the local geometry, and discuss the spectrum of gauge and bulk matter. The spectral cover approach for these Higgs bundles is set up in section 5 and localized matter is studied in section 6. A description of abelian Higgs field backgrounds in terms of supersymmetric quantum mechanics and its connection with Morse and more generally Morse-Bott theory is given in section 7. This setup is then applied to the study of matter couplings in section 8. Finally, in section 9 we apply this framework to describe the local models for TCS G_2-manifolds and study the deformations of the associated local models. A summary of results useful for model building applications together with some concrete models is given in section 10. We conclude with a summary of recent developments and future research directions in section 11.

2 G_2-geometry

The mathematical interest in manifolds with holonomy group G_2 can be traced back to the classification of possible holonomy groups of a simply connected, irreducible Riemanninan

[3] Studying such transitions in a compact setting seems to go beyond the current tools available in geometry, as it can no longer be a TCS. However, see also the recent paper by Chen [30].

manifold that is not locally symmetric [10]. The simple Lie group G_2 occurs alongside Spin(7) as one of the two *exceptional* holonomy groups, with the term exceptional reflecting the fact that it only occurs in dimension 7. The complete Berger's list of special and exceptional holonomy groups is summarized in table 1. While Berger showed that G_2-holonomy is possible, it took more than three decades before Bryant [61] found first local examples of G_2-holonomy metrics in $\mathbb{R}^7$. First complete and non-compact examples were proved to exist by Bryant and Salamon [62] shortly thereafter. Compact examples remained elusive for a further decade until Joyce [63, 64] produced the first compact G_2-manifolds. Since then, a range of both compact and non-compact examples have been constructed (see [32, Section 6.2] for a recent overview of known constructions).

2.1 G_2-structures

We begin our overview of G_2 geometry by introducing the notion of a G_2-structure. The results we present in this section are covered in more detail in many standard references on G_2-geometry such as [16, 32, 65, 66].

Consider the differential form $\Phi_0 \in \Omega^3(\mathbb{R}^7)$ given by

$$\Phi_0 = e^{123} - e^{167} - e^{527} - e^{563} - e^{415} - e^{426} - e^{437}, \tag{2.1}$$

where e^i are an orthonormal coframe on $\mathbb{R}^7$ and we used the shorthand $e^{ijk} = e^i \wedge e^j \wedge e^k$. The group G_2 can be defined as the stabilizer subgroup $\mathrm{Stab}(\Phi_0) \subset \mathrm{GL}(7, \mathbb{R})$ under the usual action of $\mathrm{GL}(7, \mathbb{R})$ on forms. This form is referred to as the *associative* 3-form.[4] The group G_2 is a compact 14 dimensional subgroup of SO(7) and can in particular be defined also as the stabilizer of the dual 4-form.

$$\Psi_0 = *\Phi_0 = e^{4567} - e^{4523} - e^{4163} - e^{4127} - e^{2637} - e^{1537} - e^{1526}, \tag{2.2}$$

which is referred to as the *coassociative* 4-form. The Hodge star is taken with respect to

[4] We remark that the choice of the associative form is not quite unique due to a choice of orientation and we follow the conventions in [32], which are also the conventions of [31]. Note however that a different choice is used in e.g. [16, 37]. In [67] there is a summary of different conventions across mathematical and physical literature.

the standard Euclidean metric on $\mathbb{R}^7$. Moreover, G_2 is a 2-connected, simple Lie group and the crucial property of Φ_0 is that its orbit under the $GL(7,\mathbb{R})$ action is open in $\Omega^3(\mathbb{R}^7)$. Note that this is only the case because $\dim GL(7,\mathbb{R}) = \dim G_2 + \dim \Omega^3(\mathbb{R}^7)$. Forms with open orbits can exist only in a handful of dimensions and this coincidence is in fact the source of special geometric features of G_2-manifolds [68].

Let now M be an oriented 7 dimensional manifold and denote by $\Omega^3_+(M)$ the set of all 3-forms Φ for which at every point $m \in M$ there exists an oriented isomorphism

$$f : \mathbb{R}^7 \to T_m M \tag{2.3}$$

such that $f^*\Phi = \Phi_0$. That is, Φ can be pointwise identified with the model form Φ_0. Elements of $\Omega^3_+(M)$ are often called *positive forms* for reasons that will become apparent in the next paragraph. A choice of a positive 3-form Φ on M endows it with a G_2-structure that is a reduction of the structure group of the frame bundle to G_2. Conversely, if M admits a G_2-structure, the corresponding positive form can be defined locally as in (2.1), which patches together to a global positive form. Note that G_2-structures (equivalently positive forms) on M exist if and only if M is orientable and spin.

As G_2 is a subgroup of $SO(7)$ the G_2-structure defined by a positive form Φ induces a metric g and a volume form on M. To see this consider the symmetric bilinear form defined at a given point as

$$B(v,w) = \frac{1}{6}(v \lrcorner \Phi) \wedge (w \lrcorner \Phi) \wedge \Phi. \tag{2.4}$$

Therefore, B is a map $B : T_m M \otimes T_m M \to \Omega^7(T^*_m M)$. B is in fact positive definite, so we can use it to define the metric and volume form via the relation

$$B_{ij} = g_{ij} \otimes \text{vol}. \tag{2.5}$$

In terms of a local coframe in which Φ looks like the model form (2.1) there is an explicit expression for the metric in terms of Φ. The metric is given by [69]

$$g_{ij} = (\det B)^{-\frac{1}{9}} B_{ij}, \tag{2.6}$$

where

$$B_{ij} = \frac{1}{3!4!} \Phi_{ikl}\Phi_{jmn}\Phi_{opr}\epsilon^{klmnopr}\,, \tag{2.7}$$

and $\epsilon^{klmnopr}$ is a totally anti-symmetric tensor with $\epsilon^{1234567} = 1$. With this metric one can define the usual operators such as the Hodge star and the Levi-Civita connection ∇. The Hodge dual form $\Psi = *\Phi$ also plays an important role and is in local coframe exactly of the form (2.2). Crucially, both $*$ and ∇ implicitly depend on Φ through the equations above.

2.2 Manifolds with Holonomy G_2

The language of G_2-structures provides the most convenient framework to study manifolds with $\mathrm{Hol}(M) \subseteq G_2$. The holonomy group acting on a tangent space at a point m is defined by parallel transport of vectors around loops based at m. Clearly, if $\mathrm{Hol}(M) \subseteq G_2$, there exists a positive form Φ obtained simply by defining $\Phi_m = \Phi_0$ and then parallel transporting it to other points on M. By construction this form has the property that $\nabla\Phi = 0$. The converse statement is also true. Remarkably, for a 7-fold M with a G_2-structure Φ the following are equivalent [70]:

$$\mathrm{Hol}(M) \subseteq G_2 \quad \Longleftrightarrow \quad \nabla\Phi = 0 \quad \Longleftrightarrow \quad \mathrm{d}\Phi = 0 \text{ and } \mathrm{d}\Psi = 0. \tag{2.8}$$

The G_2-structure satisfying $\mathrm{d}\Phi = \mathrm{d}\Psi = 0$ is called *torsion-free*. This provides a very useful criterion for establishing if the holonomy of M is contained in G_2. Furthermore, by a lemma of Joyce [63], the holonomy of M admitting a torsion-free G_2-structure is precisely equal to G_2 if and only if $\pi_1(M)$ is finite. We take a G_2-manifold to mean a manifold with $\mathrm{Hol}(M) = G_2$.

We consider some fundamental topological facts about G_2-manifolds. Since their fundamental group is finite, $b_1(M)$ vanishes and, by Poincaré duality, the only independent

Betti numbers are hence b_2 and b_3. Moreover, G_2 acts on differential forms, which split into irreducible representations. For dimensional reasons, the relevant representations are $\mathbf{1}, \mathbf{7}, \mathbf{14}$ and $\mathbf{27}$. The splitting descends to cohomology and we get

$$\begin{aligned} H^2(M,\mathbb{R}) &= H^2_{\mathbf{7}}(M,\mathbb{R}) \oplus H^2_{\mathbf{14}}(M,\mathbb{R})\,, \\ H^3(M,\mathbb{R}) &= H^3_{\mathbf{1}}(M,\mathbb{R}) \oplus H^3_{\mathbf{7}}(M,\mathbb{R}) \oplus H^3_{\mathbf{27}}(M,\mathbb{R})\,, \end{aligned} \tag{2.9}$$

where we wrote out only the two independent cohomology groups. This splitting in fact holds for any manifold with a G_2-structure. For a G_2-manifold we have the additional constraint

$$H^k_{\mathbf{7}}(M,\mathbb{R}) = 0\,, \tag{2.10}$$

for all $k = 0, \ldots, 7$ and so, for a G_2-manifold,

$$\begin{aligned} H^2(M,\mathbb{R}) &= H^2_{\mathbf{14}}(M,\mathbb{R})\,, \\ H^3(M,\mathbb{R}) &= H^3_{\mathbf{1}}(M,\mathbb{R}) \oplus H^3_{\mathbf{27}}(M,\mathbb{R})\,. \end{aligned} \tag{2.11}$$

Moreover, the group $H^3_{\mathbf{1}}(M,\mathbb{R})$ is one dimensional and is spanned by the cohomology class of the harmonic G_2-form Φ.

G_2-manifolds have two distinguished classes of submanifolds that are central to the later parts of this chapter. Their definition requires a momentary digression to the theory of *calibrated geometry* [71]. A closed k-form η on a manifold M is called a *calibration* if for every point $m \in M$ the following inequality holds

$$\eta|_{V_k} \leq \mathrm{vol}(V_k), \tag{2.12}$$

for all oriented k-planes V_k in T_mM. A submanifold Q of dimension k is called a calibrated submanifold if *equality* holds at all points of Q. In other words, η restricts to a volume form on Q. Calibrated submanifolds are homologically volume minimizing.

In the context of G_2-structures, both Φ and Ψ are calibrations in the sense defined above and, correspondingly, they define two classes of calibrated submanifolds:

1. *associative* submanifolds, which are calibrated by Φ and hence are three-dimensional,

2. *coassociative* submanifolds, which are four-dimensional and calibrated by Ψ.

One can equivalently define a coassociative submanifold by requiring the restriction of Φ to vanish along the submanifold.

The model examples of these two classes of calibrated submanifolds can be found by considering the decomposition $\mathbb{R}^7 = \mathbb{R}^3 \oplus \mathbb{R}^4$. Examining the forms Φ_0 and Ψ_0 it is immediately clear that

$$\{(x_1, x_2, x_3, 0, 0, 0, 0)\} \subset \mathbb{R}^7 \tag{2.13}$$

is calibrated by Φ_0 and is hence associative and

$$\{(0, 0, 0, x_4, x_5, x_6, x_7)\} \subset \mathbb{R}^7 \tag{2.14}$$

is a coassociative submanifold.

2.3 Examples of G_2-manifolds

Having introduced the basic vocabulary of G_2-manifolds, the next natural step is to consider examples. This is where the challenges of G_2 geometry quickly become obvious. In stark contrast to the world of Calabi-Yau manifolds, where there is a plentiful supply of both compact and non-compact examples, the known examples of G_2-manifolds are extremely scarce and all known constructions are based on intricate *ad hoc* methods. Despite significant progress over the past decade [31, 33, 37, 72–75], generating examples remains a central problem for a systematic understanding of G_2-manifolds.

In this section we review three constructions of G_2-manifolds: Bryant-Salamon, Joyce-Karigiannis gluing, and the twisted connected sum construction. In passing, we also describe Joyce orbifolds, which have play an important part both in geometry and physics. While this is by no means an exhaustive list, each of them highlights facets of G_2 geometry that will feature prominently in the remainder of this chapter.

Bryant-Salamon Manifolds

In their landmark paper [62] Bryant and Salamon constructed the first examples of complete metrics with holonomy G_2 on three non-compact manifolds:

1. $\mathbf{S}(\mathbb{S}^3)$ — the spin bundle over the three-sphere,
2. $\Omega^2_-(\mathbb{S}^4)$ — the bundle of anti-self-dual forms over the four-sphere,
3. $\Omega^2_-(\mathbb{CP}^2)$ — the bundle of anti-self-dual forms over the complex projective plane.

We refer to these as the Bryant-Salamon manifolds and they were further studied in [76,77]. All three are *asymptotically conical* and admit a *cohomogeneity one* Lie group action. Due to recent work of Karigiannis and Lotay [31], they are also the first known explicit examples of G_2-manifolds that admit *coassociative fibrations*. This means that if M is any of the three manifolds above, there exists a three-dimensional space B and a family of coassociative submanifolds N_b parametrized by B such that

$$\begin{array}{ccc} N_b & \hookrightarrow & M \\ & & \downarrow \\ & & B \end{array} \tag{2.15}$$

Note that to make this rigorously true, one must allow the fibration to have singular and, in the case of $\Omega^2_-(\mathbb{CP}^2)$, even intersecting fibers. We are mainly interested in $M = \mathbf{S}(\mathbb{S}^3)\,, \Omega^2_-(\mathbb{S}^4)$ so we will gloss over these technicalities and refer the reader to [31, Definition 1.2] and the surrounding discussion for more details.

$\mathbf{S}(\mathbb{S}^3)$

The simplest Bryant-Salamon manifold is $M = \mathbf{S}(\mathbb{S}^3)$, which is topologically just a trivial fibration $\mathbf{S}(\mathbb{S}^3) = \mathbb{S}^3 \times \mathbb{R}^4$. Consider a metric $g_{\mathbb{S}^3}$ with a constant scalar curvature and take an orthonormal coframe b_1, b_2, b_3 on $\mathbb{S}^3$ in which this metric takes the form

$$g_{\mathbb{S}^3} = b_1^2 + b_2^2 + b_3^2\,. \tag{2.16}$$

Additionally, let $\zeta_0, \zeta_1, \zeta_2, \zeta_3$ be the vertical coframe[5] and

$$\omega_1 = \zeta_0 \wedge \zeta_1 - \zeta_2 \wedge \zeta_3\,, \quad \omega_2 = \zeta_0 \wedge \zeta_2 - \zeta_3 \wedge \zeta_1\,, \quad \omega_3 = \zeta_0 \wedge \zeta_3 - \zeta_1 \wedge \zeta_2\,, \tag{2.17}$$

[5] Note that we follow [31] and use the letter b to denote forms defined on the base of the vector bundle and ζ to denote forms on the fibers.

the basis of the anti-self-dual forms on the fibers. The fibral coframe ζ_i naturally has an implicit dependence on b_i (see [31, eq. (3.9)]), but it is not essential for our exposition. The Bryant-Salamon metric is given by

$$\begin{aligned} g &= 3(c+r^2)^{\frac{2}{3}} g_{\mathbb{S}^3} + 4(c+r^2)^{-\frac{1}{3}} \left(\zeta_0^2 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2\right) \\ \Phi &= 3\sqrt{3}(c+r^2)\mathrm{vol}_{\mathbb{S}^3} + 4\sqrt{3}\left(b_1 \wedge \omega_1 + b_2 \wedge \omega_2 + b_3 \wedge \omega_3\right) \end{aligned} \tag{2.18}$$

Here, r is the distance to the zero section in the fibers. The free parameter c controls the size of the distinguished $\mathbb{S}^3$ that sits inside $\mathbf{S}(\mathbb{S}^3)$ as the zero section. It is clear from the explicit expression for Φ that the zero section is an associative submanifold and that the $\mathbb{R}^4$ fibers are coassociative (Φ manifestly vanishes on the fibers). In the limit $c \to 0$ the zero section collapses to a point and M converges to the G_2-cone $M = \mathbb{R}^+ \times \mathbb{S}^3 \times \mathbb{S}^3$. One can also consider a different limit, the so-called *flat limit*, where the scalar curvature of the base $\mathbb{S}^3$ goes to zero. In this case, this is equivalent to taking the limit where the volume of $\mathbb{S}^3$ goes to infinity. One can show that the G_2-structure converges to the flat structure on $\mathbb{R}^7$, viewed as a trivial coassociative fibration of $\mathbb{R}^4$ over $\mathbb{R}^3$.

$\Omega^2_-(\mathbb{S}^4)$ and $\Omega^2_-(\mathbb{CP}^2)$

The remaining two examples arise from a construction of torsion-free G_2-structures on the bundle of anti-self-dual forms on a self-dual Einstein 4-manifold N. If N has positive scalar curvature (so N is either $\mathbb{S}^4$ or $\mathbb{CP}^2$) the resulting G_2-holonomy metric is smooth and complete.

Starting with a local orthonormal coframe b_1, b_2, b_3, b_4 on N, we take the basis of the anti-self-dual forms to be

$$\omega_1 = b_0 \wedge b_1 - b_2 \wedge b_3\,, \quad \omega_2 = b_0 \wedge b_2 - b_3 \wedge b_1\,, \quad \omega_3 = b_0 \wedge b_3 - b_1 \wedge b_2\,. \tag{2.19}$$

Using the natural connection on $\Omega^2_-(N)$, induced by the Einstein metric, we can define vertical forms $\zeta_1, \zeta_2, \zeta_3$ on the fibers. In this coframe, the torsion-free G_2-structure and metric are given by

$$
\begin{aligned}
g &= (c+r^2)^{-\frac{1}{2}}\left(\zeta_1^2+\zeta_2^2+\zeta_3^2\right)+2(c+r^2)^{\frac{1}{2}}(b_1^2+b_2^2+b_3^2+b_4^2)\,,\\
\Phi &= (c+r^2)^{-\frac{3}{4}}\zeta_1\wedge\zeta_2\wedge\zeta_3+2(c+r^2)^{\frac{1}{4}}\left(\zeta_1\wedge\omega_1+\zeta_2\wedge\omega_2+\zeta_3\wedge\omega_3\right)\,.
\end{aligned}
\tag{2.20}
$$

As before the coordinate r measures the distance to the zero section in the $\mathbb{R}^3$ fiber. Additionally, the parameter c again controls the size of the zero section, which is now a coassociative submanifold. The G$_2$-cone obtained by taking $c\to 0$ is $\mathbb{R}^+\times\mathbb{CP}^3$ for $\Omega^2_-(\mathbb{S}^4)$ and $\mathbb{R}^+\times(\mathrm{SU}(3)/\mathrm{T}^2)$ for $\Omega^2_-(\mathbb{CP}^2)$.

In the case of $\mathbf{S}(\mathbb{S}^3)$ the coassociative fibration was very easy to identify, since it presents as a fibration of a four-dimensional space over a three-dimensional space. This is certainly not the case here, which makes identifying the coassociative fibers non-trivial. We focus on the example of $\Omega^2_-(\mathbb{S}^4)$, as $\Omega^2_-(\mathbb{CP}^2)$ presents some additional technical difficulties that are not consequential for our purposes. Using a detailed analysis of the G$_2$-structure the authors of [31] were able to provide an explicit description of the coassociative fibration structure. The generic fiber is topologically the Eguchi-Hanson space i.e. the total space of $\mathcal{O}(-2)$, fibered over $\mathbb{R}^3$.[6] In every smooth fiber there is a sphere $\mathbb{S}^2$, with its volume determined by the induced metric. Over a circle $\mathbb{S}^1\subset\mathbb{R}^3$ the two-sphere in the fiber collapses to zero volume. These are the singular fibers, all of which are diffeomorphic to $\mathbb{C}^2/\mathbb{Z}_2$.

Another interesting feature of the coassociative fibration is its flat limit. The flat limit of the G$_2$-structure is again the standard structure on $\mathbb{R}^7$, but the coassociative fibration is given by the product of the Lefschetz fibration $\pi:\mathbb{C}^3\to\mathbb{C}$, given by

$$
\pi(z_1,z_2,z_3)=z_1^2+z_2^2+z_3^2\,,\tag{2.22}
$$

with the real line. The singularities are now located along the points $(x,0)\in\mathbb{R}\oplus\mathbb{C}$ in the base. This is a simple local model for the singular behavior of Higgs bundles we consider

[6]The caveat is that the induced metric is not the Eguchi-Hanson hyperkähler metric but it does admit a slightly weaker structure called *hypersymplectic* structure [78,79], given by a triplet of two-forms ω_i, that satisfy

$$
\mathrm{d}\omega_i=0\qquad\text{and}\qquad\omega_i\wedge\omega_j=2Q_{ij}\nu\,,\tag{2.21}
$$

where ν is an arbitrary volume form and Q_{ij} is a symmetric positive definite matrix of functions.

later in this chapter. In particular, we discuss the properties of the Lefschetz fibration in more detail in section 4.2.

Joyce Orbifolds

Tough they are not smooth manifolds, Joyce orbifolds play an important role in G$_2$ geometry, since they are the starting point of Joyce's original construction of the first compact examples of G$_2$-manifolds [63, 64]. In contrast to their smooth counterparts, orbifolds are somewhat easier to construct by taking quotients of the torus $\mathbb{T}^7$, which admits the natural flat G$_2$-structure inherited from $\mathbb{R}^7$. The simplest example of a non-trivial Joyce orbifold is given by $\mathbb{T}^7/\mathbb{Z}_2^3$, where $\mathbb{Z}_2^3$ is generated by

$$\begin{aligned}
\alpha(x_1, \dots, x_7) &= (-x_1, -x_2, x_3, -x_4, x_5, x_6, -x_7) \ , \\
\beta(x_1, \dots, x_7) &= \left(\tfrac{1}{2} - x_1, x_2, -x_3, -x_4, x_5, -x_6, x_7\right) \ , \\
\gamma(x_1, \dots, x_7) &= \left(x_1, \tfrac{1}{2} - x_2, -x_3, \tfrac{1}{2} - x_4, -x_5, x_6, x_7\right) \ .
\end{aligned} \tag{2.23}$$

The G$_2$-structure is invariant under this action and so it descends to the quotient. The fixed points of the generators are 12 disjoint copies of $\mathbb{T}^3$, which becomes the singular locus of the orbifold quotient. Near the singular locus, the orbifold looks like $\mathbb{T}^3 \times \mathbb{C}^2/\mathbb{Z}_2$. Orbifolds with this property are required for the construction of smooth G$_2$-manifolds we review in the following subsection.

Especially in physics one is also frequently interested orbifolds where the normal space to the singular locus is a general ADE singularity. Compact orbifolds of this type are much harder to construct, but non-compact examples are easy to write down. Consider the space $\mathbb{C}^2 \times \mathbb{T}^3$, where z_1, z_2 are complex coordinates on $\mathbb{C}^2$ and x_1, x_2, x_3 are coordinates on $\mathbb{T}^3$. The maps

$$\begin{aligned}
\alpha(z_1, z_2, x_1, x_2, x_3) &= \left(e^{2\pi i/n} z_1, e^{-2\pi i/n} z_2, x_1, x_2, x_3\right) \ , \\
\beta(z_1, z_2, x_1, x_2, x_3) &= \left(-z_1, z_2, \tfrac{1}{2} - x_1, -x_2, \tfrac{1}{2} + x_3\right) \ , \\
\gamma(z_1, z_2, x_1, x_2, x_3) &= \left(-\overline{z_1}, -\overline{z_2}, -x_1, \tfrac{1}{2} + x_2, -x_3\right) \ ,
\end{aligned} \tag{2.24}$$

generate an action of $\mathbb{Z}_2^2 \ltimes \mathbb{Z}_n$ on $\mathbb{C}^2 \times \mathbb{T}^3$. The singular set is now $\mathbb{T}^3/\mathbb{Z}_2^2$, with the normal space given by $\mathbb{C}^2/\mathbb{Z}_n$.

Joyce-Karigiannis Manifolds

The Joyce-Karigiannis construction [74] is the most recent method of constructing compact examples of G_2-holonomy manifolds. The fundamental idea of the construction is to obtain smooth G_2-manifold by resolving the singularities of a G_2-orbifold. One can for instance apply the construction to the compact orbifold described in (2.23). The resolution of singularities is achieved by gluing a family of Eguchi-Hanson spaces in place of the singular locus. The key piece of data entering the construction is a *harmonic one-form on the singular locus* that parametrizes the family of Eguchi-Hanson spaces.

The starting point of the construction is a smooth connected G_2-manifold M together with an involution ι that preserves the three-form

$$\iota^* \Phi = \Phi \,. \tag{2.25}$$

If the fixed point set of ι is not empty then

$$L = \{ p \in M \mid \iota(p) = p \} \tag{2.26}$$

is an associative submanifold of M. Taking the quotient by the involution one obtains a G_2-orbifold $M/\langle \iota \rangle$ with the singular locus L. The key point is that the local neighborhood of the singularities is diffeomorphic to $\mathbb{R}^3 \times \mathbb{C}^2/\mathbb{Z}_2$. We now resolve the orbifold singularities by consistently resolving every $\mathbb{C}^2/\mathbb{Z}_2$ fiber.

According to the Kronheimer's classification of ALE spaces [80] the space of complex structures on the resolution $\widetilde{\mathbb{C}^2/\mathbb{Z}_2}$ is parametrized by $\mathbb{R}^3$. In other words, a family of resolutions $\mathbb{R}^3 \times \widetilde{\mathbb{C}^2/\mathbb{Z}_2}$ can be described by a one-form ϕ on $\mathbb{R}^3$ such that, at every point $x \in \mathbb{R}^3$, ϕ_x determines the complex structure on the fiber $\widetilde{\mathbb{C}^2/\mathbb{Z}_2}$ over x. To be able to carry out this resolution globally on L we need to require the one-form ϕ to be a global one-form on L. Joyce and Karigiannis show that to construct an appropriate G_2-structure on the resolved manifold $\widetilde{M}$, ϕ needs to be a nowhere vanishing, harmonic one-form on L. If such ϕ exists, then $\widetilde{M}$ admits a G_2-holonomy metric, provided $M/\langle \iota \rangle$ has a finite fundamental group.

The requirement that ϕ has no zeros and is harmonic is very restrictive and such forms are difficult to generate in practice. Generically, a harmonic form will have isolated zeros. From the perspective of physics this is in fact the more interesting scenario as isolated zeros could translate to conical singularities in the 'resolved' $\widetilde{M}$. In fact, the harmonic one-form ϕ that describes the ALE-fibration is exactly the Higgs field we consider in the core part of this chapter. By analyzing the gauge theories arising from M-theory we argue that Higgs fields with isolated zeros give rise to G$_2$-metrics with conical singularities.

Twisted Connected Sum Manifolds

In this section we describe the twisted connected sum construction, which is the most successful method so far in producing compact examples of G$_2$-holonomy manifolds. The method is originally due to Kovalev [36]. It later and was extended by Corti, Haskins, Nordström and Pacini [37, 38], that also gave a plethora of building blocks that allowed millions of new compact G$_2$-manifolds to be constructed. Since then, further extensions were given in [17,44]. We review the main geometric ideas behind the original construction [36].

The basic ingredient for the twisted connected sum construction is a pair of algebraic threefolds $Z_\pm$, which admit a K3 fibration

$$\begin{array}{ccc} S_\pm & \hookrightarrow & Z_\pm \\ & & \downarrow{\scriptstyle \pi_\pm} \\ & & \mathbb{CP}^1_\pm\,, \end{array} \tag{2.27}$$

where $S_\pm$ denotes the generic K3 fiber. The manifolds $Z_\pm$ have to satisfy

$$c_1(Z_\pm) = [S_\pm] \tag{2.28}$$

i.e. the first Chern class of $Z_\pm$ must be equal to the class of a generic K3 fiber. The manifolds $Z_\pm$ satisfying some further conditions are then called the building blocks. The precise conditions for the building blocks are not immediately relevant for our purposes and we refer to [37, Definition 3.5] for details. Excising a generic fiber from $Z_\pm$ one obtains a pair of asymptotically cylindrical Calabi-Yau threefolds $X_\pm = Z_\pm \setminus S_\pm$, fibered over a punctured Riemann sphere

$$\begin{array}{ccc} S_\pm & \hookrightarrow & X_\pm \\ & & \big\downarrow \pi_\pm \\ & & \mathbb{CP}^1_\pm \setminus \{\infty_\pm\} = \mathbb{R}^+ \times \mathbb{S}^1_\pm \,. \end{array} \tag{2.29}$$

The manifolds $X_\pm$ are asymptotic to the metric cylinder $V_\pm = \mathbb{R}^+ \times \mathbb{S}^1_\pm \times S_\pm$, which can be viewed as a trivial K3 fibration over $\mathbb{C}$. Let us denote the hyperkähler triplet on $S_\pm$ by $(\omega_{1\pm}, \omega_{2\pm}, \omega_{3\pm})$, the first among which we take to be the Kähler form on $S_\pm$ and the remaining two combine in the holomorphic volume form

$$\Omega_\pm = \omega_{2\pm} + i\omega_{3\pm}. \tag{2.30}$$

We now consider $\mathbb{S}^1_{e\pm} \times X_\pm$ equipped with the product metric. To obtain the compact manifold M, we perform a connected sum construction by identifying the asymptotic limits $\mathbb{S}^1_{e\pm} \times V_\pm$ while exchanging the internal and external circles. Explicitly, we identify

$$\begin{array}{ccc} \mathbb{S}^1_+ & & \mathbb{S}^1_- \\ & \searrow \!\!\!\!\!\! \nearrow & \\ \mathbb{S}^1_{e+} & & \mathbb{S}^1_{e-} \,. \end{array} \tag{2.31}$$

The reason for this twist is that the naively identifying the external circles would result in a manifold with an infinite fundamental group and hence the holonomy group would be properly contained in G_2 by Joyce's lemma. To glue together the metrics, consider the torsion free G_2 structure on the asymptotic limits $\mathbb{S}^1_{e\pm} \times \mathbb{R}^+ \times \mathbb{S}^1_\pm \times S_\pm$ given by

$$\Phi_\pm = \mathrm{d}\theta_\pm \wedge \mathrm{d}t \wedge \mathrm{d}\vartheta_\pm + \mathrm{d}\theta_\pm \wedge \omega_{1\pm} + \mathrm{d}\vartheta_\pm \wedge \omega_{2\pm} + \mathrm{d}t \wedge \omega_{3\pm}. \tag{2.32}$$

Here $\theta_\pm$ and $\vartheta_\pm$ are coordinates on $\mathbb{S}^1_{e\pm}$ and $\mathbb{S}^1_\pm$ respectively. The twisting described above identifies $\theta_\pm$ with $\vartheta_\mp$. In order for this map to be an isometry of the G_2-structures, i.e. pull back Φ_- to Φ_+, the following matching must be imposed on the hyperkähler triplets of $S_\pm$

$$\begin{aligned} \omega_{1+} &= \omega_{2-} \\ \omega_{2+} &= \omega_{1-} \\ \omega_{3+} &= -\omega_{3-}, \end{aligned} \tag{2.33}$$

This matching condition is known as the Donaldson matching or hyperkähler rotation. The gluing argument now proceeds by fixing a large parameter $T \in \mathbb{R}^+$, which control the length of the neck regions of the manifolds $X_\pm$. Sufficiently far along the neck, we can interpolate between G_2-structure on $X_\pm \times \mathbb{S}^1_{e\pm}$ and the asymptotic G_2-structure $\Phi_\pm$. Applying this procedure and gluing the asymptotic structures as described above introduces a G_2-structure on the connected sum M. The interpolation process has introduced torsion, which exponentially decreases as the parameter T is increased. The analytic results of [36, 37] then imply that for sufficiently large T there exists a torsion-free G_2-structure.

Finally, note that TCS G_2-manifolds are clearly fibered by K3 surfaces. The base of the fibration can also be easily identified. After interpolating the G_2-structure and cutting procedure described above the base becomes $\Delta_\pm \times \mathbb{S}^1_{e\pm}$, where Δ is a bounded disc in $\mathbb{C}$. The twisting of the base identifies

$$\begin{array}{ccc} \partial\Delta_+ & & \partial\Delta_- \\ & \times & \\ \mathbb{S}^1_{e+} & & \mathbb{S}^1_{e-}. \end{array} \tag{2.34}$$

This amounts to gluing two solid tori, while exchanging the boundary cycle with the external $\mathbb{S}^1_e$, which gives the base to be $\mathbb{S}^3$. Note that since the torsion-free G_2-metric is obtained using perturbation of the glued G_2-structure it is not known if the K3 fibers are in fact calibrated for the so obtained G_2-metric. Conjecturally, this is the case at least near an appropriate adiabatic limit [78, 81].

The Donaldson matching is the most restrictive condition present in the construction, since finding appropriate building blocks $Z_\pm$ for which there exist a Donaldson matching between generic fibers is highly non-trivial. However, with the results of [37] (as well as [38]) a sizable amount of matching pairs have been found.

3 M-theory Compactifications on G_2-manifolds

So far our discussion revolved exclusively around geometry and the particulars of G_2-manifolds. In this section we remedy this and connect the geometry to physics by discussing the role of G_2 geometry in M-theory compactifications. Since its proposal in [82], M-theory has been a received significant research attention. In addition to its obvious appeal as a theory unifying all five superstring theories in 10d it also provides a powerful method of engineering supersymmetric field theories in various dimensions by compactifying M-theory on different classes of manifolds. The most studied among them are compactifications on G_2-manifolds since they give rise to 4d $\mathcal{N} = 1$ field theories that are especially interesting for model building and phenomenological purposes.

3.1 Rudiments of 11d Supergravity

M-theory was originally proposed as a putative 11d strong coupling limit of IIA string theory [82]. Since then, considerable effort went into elucidating the foundations of M-theory, however a fundamental formulation remains elusive. We start our discussion by considering the low energy effective theory of M-theory: *11d supergravity.*[7]

Eleven dimensional supergravity is supersymmetric theory on a Lorentzian manifold $\mathbb{M}^{1,10}$. The field content of the theory consists of a three-form C_3, the metric g and corresponding gravitino. Denoting $G_4 = \mathrm{d}C_3$, the bosonic part of the action takes the form

$$S = \int \mathrm{d}^{11}x \,\sqrt{g}R - \frac{1}{2}G_4 \wedge *G_4 - \frac{1}{6}C_3 \wedge G_4 \wedge G_4 \,. \tag{3.1}$$

The standard ansatz for dimensional reduction to 4d Minkowski space is to assume that the Lorentzian structure $\mathrm{SO}(1,10)$ reduces to $\mathrm{SO}(1,3) \times \mathrm{SO}(7)$, in other words $\mathbb{M}^{1,10}$ is a Riemannian product

$$\mathbb{M}^{1,10} = \mathbb{R}^{1,3} \times M, \tag{3.2}$$

where M is a compact seven-fold. In the simplest setting, where we assume $G = 0$, the requirement that $\mathcal{N} = 1$ supersymmetry is preserved in the 4d theory is equivalent to the

[7] This material is standard and is covered in great detail in the literature e.g. [24, 83–86].

condition that there exists a spinor ϵ on M that satisfies

$$\nabla\epsilon = 0\,. \tag{3.3}$$

This necessitates $\mathrm{Hol}(M) = \mathrm{G}_2$. Any further reduction of the holonomy group would entail more preserved supersymmetry in the 4d theory. Smooth G_2-manifolds are therefore supersymmetric vacuum solutions of M-theory with precisely $\mathcal{N} = 1$ supersymmetry in 4d.

It is readily apparent that compactifications on smooth G_2-manifolds do not give rise to interesting physical models in 4d. Using a standard Kaluza-Klein analysis of the 11d fields [87] one can derive that the massless spectrum of the 4d $\mathcal{N} = 1$ theory consists of

1. a gravity multiplet,

2. $b_2(M)$ *abelian* vector multiplets,

3. $b_3(M)$ *neutral* chiral multiplets.

From the perspective of building a physically realistic low energy 4d theory there are two glaring issues. First, the gauge group in 4d is necessarily abelian and second, there is no massless fermionic matter charged under the gauge group, both of which are foundational components of the Standard Model physics. It is therefore manifest that M-theory on smooth G_2-manifolds is unsuitable as a starting point for engineering a minimally supersymmetric standard model or any extension thereof.

M-theory on G_2-manifolds becomes much more interesting if the manifold is singular and therefore the supergravity approximation is no longer valid. Crucially, M must admit singularities of two types: ADE singularities in codimension 4 and conical singularities in codimension 7. These have a direct translation in to the properties of the 4d theory; the dictionary is as follows

$$\begin{aligned} \text{non-abelian gauge fields} \quad &\leftrightarrow \quad \text{ADE singularities in codimension 4} \\ \text{chiral fermions} \quad &\leftrightarrow \quad \text{codimension 7 conical singularities.} \end{aligned}$$

We now examine both entries in the dictionary in more detail.

3.2 Non-abelian Gauge Fields

To understand the physics leading to this dictionary we follow [85] and consider a simple set-up of M-theory on the singular space with the simplest ADE singularity $\mathbb{R}^{1,6} \times \mathbb{C}^2/\mathbb{Z}_2$. We start by considering M-theory on the smooth ALE-space $\mathbb{R}^{1,6} \times \widetilde{\mathbb{C}^2/\mathbb{Z}_2}$. Recall that $\widetilde{\mathbb{C}^2/\mathbb{Z}_2} \cong \mathcal{O}(-2)$ is the cotangent space of $\mathbb{CP}^1$ and, in particular, the zero section is a rigid holomorphic two-sphere σ in $\mathcal{O}(-2)$. This cycle generates the second homology group and there exists a dual compactly supported harmonic two-form α. Recall also that the moduli space of complex structures on $\mathcal{O}(-2)$ is parametrized by a single vector $\phi \in \mathbb{R}^3$.

M-theory on ALE-space gives rise to 7d gauge theory coupled to supergravity, with the field content consisting of the gravity multiplet and vector multiplets. In what follows we are only interested in the gauge sector. Writing the C_3 as

$$C_3 = A \wedge \alpha \,, \tag{3.4}$$

it clearly gives rise to a U(1) gauge field in 7d, so the 7d theory is an abelian gauge theory as expected. Together with the three scalars $\phi = (\phi_1, \phi_2, \phi_3)$, the gauge field A forms a single vector multiplet in 7d. Now consider an M2-brane wrapped on the cycle σ, where the rigid two sphere σ is holomorphic and is hence calibrated with respect to the ALE metric on $\widetilde{\mathbb{C}^2/\mathbb{Z}_2}$. Wrapping an M2-brane on a calibrated cycle gives rise to a BPS state, which is a particle charged under the U(1) from the 7d perspective. Its mass is proportional to the volume of σ and in turn

$$\mathrm{Vol}(\sigma) \propto |\phi| \,. \tag{3.5}$$

Wrapping an M2-brane with an opposite orientation, we likewise get a gauge boson with a negative U(1) charge compared to previously. When σ collapses to zero volume, the gauge bosons become massless; this is the usual setting for gauge symmetry enhancement.

As the notation suggests, the Higgs field ϕ in the 7d gauge theory gets identified with the complex structure modulus of the ALE-fiber in M-theory. In fact, from the gauge theory data of the 7d Higgs field, one can construct a corresponding ALE-fibration in M-theory. We discuss this correspondence 4.2. This shows that we can engineer a SU(2) gauge theory from M-theory on $\mathbb{R}^{1,6} \times \mathbb{C}^2/\mathbb{Z}_2$. We can immediately generalize the above argument to any ADE group Γ_{ADE} by considering M-theory on the singular space $\mathbb{C}^2/\Gamma_{\text{ADE}}$. The rigid two-cycles in the resolved ALE-space $\widetilde{\mathbb{C}^2/\Gamma_{\text{ADE}}}$ fill out the corresponding ADE Dynkin diagram. Shrinking them to a point results in the gauge symmetry enhancing from $\mathrm{U}(1)^n$ to the corresponding simply laced Lie group.

To derive the low energy effective field theory of M-theory four dimensions, we fiber the ADE singularities over a compact M_3. In other words, we consider M-theory on a non-compact G_2-manifold, where M is a fibration

$$\begin{array}{ccc} \mathbb{C}^2/\Gamma_{\text{ADE}} & \hookrightarrow & M \\ & & \downarrow \\ & & M_3\,, \end{array} \tag{3.6}$$

and first reduce along the ALE-fiber and then along M_3. The above arguments show that the 7d theory is well-approximated by a maximally supersymmetric 7d super Yang-Mills theory with the gauge group being the ADE dual of Γ_{ADE}. This approximation is valid in the adiabatic limit, where the fibration varies slowly over M_3, in other words, when the fibration is approximated by the product ALE-fibration $M_3 \times \mathbb{C}^2/\Gamma_{\text{ADE}}$. However, in this context, the reduction of the 7d theory requires an additional subtlety. On a generic three-manifold there can be no parallel spinors, so a reduction on such manifold breaks all the supersymmetry. In order to preserve $\mathcal{N} = 1$ supersymmetry in the effective 4d theory dimensionally reduce the 7d SYM with a *partial topological twist*. We provide a detailed discussion of the dimensional reduction in section 4.1.

3.3 Charged Chiral Fermions

We now switch gears and address the question of how to engineer chiral fermions in M-theory. Chiral fermions appear at a different type of singularity to the one discussed above — codimension 7 conical singularity. In the context of M-theory examples of conical G_2-manifolds were initially studied in [26] and were shown to yield chiral fermions by utilizing the duality to IIA theory. We review the Witten's anomaly inflow argument [27] that uses anomaly cancellation to argue for the presence of chiral fermions at conical singularities.

Consider a G_2-cone M. This means that $M = \mathbb{R}^+ \times M'$ with the metric

$$g_M = \mathrm{d}r^2 + r^2 g_N \,. \tag{3.7}$$

We can think of the 6d manifold M' as the boundary of M at infinity. The Chern-Simons interaction in the M-theory action is

$$S_{CS} = \frac{1}{6} \int_{\mathbb{R}^{1,3} \times M} C_3 \wedge G_4 \wedge G_4 \,. \tag{3.8}$$

If we change C_3 by a gauge transformation

$$C_3 \longrightarrow C_3 + \mathrm{d}\epsilon_2 \,, \tag{3.9}$$

the overall action changes by

$$\delta S \sim \int_{\mathbb{R}^{1,3} \times M} \mathrm{d} \left(\epsilon_2 \wedge G_4 \wedge G_4 \right) \,. \tag{3.10}$$

Note however, that supergravity is not defined at the singularity, so we excise the singular point and integrate by parts

$$\delta S \sim \int_{\mathbb{R}^{1,3} \times M'} \epsilon_2 \wedge G_4 \wedge G_4 \,. \tag{3.11}$$

Decomposing ϵ_2 and C_3 using the Kaluza-Klein ansatz we write

$$\begin{aligned} \epsilon_2 &= \sum_i \epsilon^i \beta_i \,, \\ C_3 &= \sum_i \beta_i \wedge A_i \,. \end{aligned} \tag{3.12}$$

Here, β^i are harmonic two-forms on M, so the index i runs from 1 to $b_2(M)$. Plugging this back in the δS, we see that the variation can be expressed as

$$\delta S \sim \int_{\mathbb{R}^{1,3}} \epsilon^i F^j \wedge F^k \int_{M'} \beta_i \wedge \beta_j \wedge \beta_k \,, \tag{3.13}$$

where $F^j = \mathrm{d}A^j$. Notice that the integral

$$\int_{M'} \beta_i \wedge \beta_j \wedge \beta_k \tag{3.14}$$

is purely topological. If it is non-zero, there is a 4d interaction indicative of an anomaly in the 4d theory. Assuming that the 4d theory is consistent, there must be a phenomenon supported at the singular point of M, that cancels the anomaly. This leads to the conjecture that there exist massless chiral fermions, charged under U(1) symmetries generated by A^i, localized at the singular point.

This is still not completely satisfactory since we ultimately want to generate massless chiral spectrum, charged under a non-abelian gauge group. However, this argument can be extended by incorporating more general ADE singularities into the mix. To do this, we start with a G_2-manifold with a family of ADE singularities along a 3d locus L and, at a single point in L, the singularity worsens to a conical singularity. An example of such a manifold is the cone $M = \mathbb{R}^+ \times \mathbb{WCP}_{n,n,1,1}$, where $\mathbb{WCP}_{n,n,1,1}$ is the weighted projective space with weights $(n, n, 1, 1)$. Its key feature is that it admits a $\mathbb{CP}^1$ family of $\mathbb{Z}_n$ singularities, given by the points, where the first two coordinates vanish. In M this translates to the singular locus L being the cone $\mathbb{R}^+ \times \mathbb{CP}^1$. At the apex of the cone, where $r = 0$, the singularity degenerates further. Indeed, one can think of the $r \to 0$ limit as shrinking the $\mathbb{CP}^1$ to zero, which in fact results in the $\mathbb{Z}_n$ singularity degenerating to a $\mathbb{Z}_{n+1}$ singularity.[8] This is a key example that geometrically encodes the gauge symmetry enhancement from SU(n) to SU$(n+1)$ at the apex. Using generalizations of this one can also construct weighted projective spaces that give rise to non-abelian gauge enhancements [28].

The anomaly cancellation argument can be generalized to the context of SU$(n) \times$ U(1) symmetry enhancing to SU$(n+1)$ [27,85]. In this context one can write down an interaction term

$$S = \int_{\mathbb{R}^{1,3} \times L} K \wedge \mathrm{Tr}\left[F \wedge F \wedge F\right] . \tag{3.15}$$

Here, F is the field strength of SU(n) gauge field we will denote by A, whereas K is the

[8] We provide a slightly different perspective that elucidates this further in section 4.2.

field strength of the remaining U(1) gauge field. Performing a gauge transformation

$$A \longrightarrow A + D_A\lambda \tag{3.16}$$

the interaction term transforms as

$$\delta S \sim \int_{\mathbb{R}^{1,3}\times L} K \wedge \mathrm{d}\,\mathrm{Tr}\left[\lambda F \wedge F\right] . \tag{3.17}$$

For a normal U(1)-bundle, the two-form K is exact so the variation vanishes. However, of the $\mathbb{Z}_n$ singularity degenerates to a $\mathbb{Z}_{n+1}$ singularity at the apex p, the derivative of K picks up a source term supported at p

$$\mathrm{d}K = 2\pi q \delta_p \,. \tag{3.18}$$

The variation of the interaction term is then

$$\delta S \sim -q \int_{\mathbb{R}^{1,3}} \mathrm{Tr}\,\lambda F \wedge F \,. \tag{3.19}$$

This again indicates an anomaly. This time, to cancel the anomaly, there must be chiral fermions charged under the SU(n) gauge group.

We have therefore established the two key pieces of the puzzle, that allow us to engineer non-abelian gauge theories with charged chiral matter: codimension 4 ADE singularities and codimension 7 conical singularities, that *lie on the codimension 4 singular locus.* While there are plenty of examples of non-compact local G_2-manifolds that have the required singularities, these have to be embedded in a compact G_2-manifold to yield a *bona fide* vacuum solution of M-theory. However, as we have alluded to in section 2.3, finding examples of compact singular G_2-manifolds with codimension 7 singularities remains an open problem.

4 The Gauge Theory Sector of M-theory on G$_2$-manifolds

In this section we study gauge theories obtained from M-theory compactifications on ALE-fibrations (3.6) in more detail. A useful way to characterize the gauge sector is to think in terms of the 7d SYM-theory obtained from M-theory on the ALE-fiber: the gauge bosons in the Cartan subalgebra of the gauge group arise from dimensional reduction of the M-theory three-form C_3 on the two-forms in the ALE-fiber, and the remaining non-abelian gauge bosons arise from wrapped M2-branes. In an adiabatic approximation the 4d effective action can be obtained by dimensionally reducing this 7d SYM-theory on the three-cycle M_3, with a partial topological twist. In this section we carry out this reduction and determine the spectrum of gauge and matter fields, which are determined by solutions of BPS equations along M_3 (see (4.18)). The solutions are given in terms of a Higgs bundle over M_3, that is specified by a one-form Higgs field ϕ and an internal gauge field W.

4.1 Partial Topological Twist and BPS Equations

We start with 7d SYM with ADE gauge group $\widetilde{G}$. This theory can be obtained by dimensional reduction of the maximally supersymmetric 10d SYM on $\mathbb{R}^{1,6} \times T^3$. Our conventions are such that the 10d gauge multiplet consists of a (hermitian) gauge field A and a Majorana-Weyl spinor λ both valued in the adjoint representation of an ADE group $\widetilde{G}$. The Lorentz group, and thereby the vector multiplet, reduce as follows

$$\begin{array}{rrcl} & \mathrm{SO}(1,9)_L & \to & \mathrm{SO}(1,6)_L \times \mathrm{SO}(3)_R \\ A: & \mathbf{10} & \to & (\mathbf{7},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \equiv (A_M, \phi_i) \\ \lambda: & \mathbf{16} & \to & (\mathbf{8},\mathbf{2}) \equiv (\lambda_{\alpha\hat{\alpha}}) \,, \end{array} \tag{4.1}$$

where the 10d vector indices are split into $M = 0, \ldots, 6$ and $i = 1, 2, 3$ and the spinor indices decompose as $\alpha = 1, \ldots, 8$ and $\hat{\alpha} = 1, 2$, where we denote the R-symmetry indices with a hat. The 10d Majorana-condition descends to a 7d symplectic Majorana-condition[9].

[9] We refer to appendix A.2 for our conventions with regard to spinors and supersymmetry.

Denoting the gauge coupling in 7d by g_7 the action becomes

$$
\begin{aligned}
S_{7\mathrm{d}} =& \frac{1}{g_7^2}\int \mathrm{d}^7x \left[-\frac{1}{4}\mathrm{Tr}\left(F_{MN}F^{MN}\right) - \frac{1}{2}\mathrm{Tr}\left(D_M\phi_i D^M\phi^i\right) + \frac{1}{4}\mathrm{Tr}\left([\phi_i,\phi_j][\phi^i,\phi^j]\right)\right] \\
&+ \frac{1}{g_7^2}\int \mathrm{d}^7x \left[+\frac{i}{2}\mathrm{Tr}\left(\bar\lambda^{\alpha\hat\alpha}(\hat\gamma^M)_\alpha{}^\beta D_M\lambda_{\beta\hat\alpha}\right) - \frac{i}{2}\mathrm{Tr}\left(\bar\lambda^{\alpha\hat\alpha}(\sigma^i)_{\hat\alpha}{}^{\hat\beta}[\phi_i,\lambda_{\alpha\hat\beta}]\right)\right] ,
\end{aligned}
\tag{4.2}
$$

where $D_M = \partial_M - i[A_M,\cdot\,]$ and F is the field strength associated to A. The supersymmetry variations are

$$
\begin{aligned}
\delta A_M &= +\frac{i}{2}\bar\epsilon^{\alpha\hat\alpha}(\hat\gamma_M)_\alpha{}^\beta\lambda_{\beta\hat\alpha} \\
\delta\phi_i &= +\frac{1}{2}\bar\epsilon^{\alpha\hat\alpha}(\sigma_i)_{\hat\alpha}{}^{\hat\beta}\lambda_{\alpha\hat\beta} \\
\delta\lambda_{\alpha\hat\alpha} &= -\frac{1}{4}F_{MN}(\hat\gamma^{MN})_\alpha{}^\beta\epsilon_{\beta\hat\alpha} + \frac{i}{2}D_M\phi_i(\hat\gamma^M)_\alpha{}^\beta(\sigma^i)_{\hat\alpha}{}^{\hat\beta}\epsilon_{\beta\hat\beta} - \frac{1}{4}[\phi_i,\phi_j]\epsilon^{ij}{}_k(\sigma^k)_{\hat\alpha}{}^{\hat\beta}\epsilon_{\alpha\hat\beta}\,,
\end{aligned}
\tag{4.3}
$$

where $\hat\gamma$ denotes the 7d gamma matrices.

This 7d SYM theory is the starting point for the analysis of gauge degrees of freedom in a local G$_2$-holonomy compactification of M-theory. For a given ALE-fiber, the singularity determines the 7d gauge group $\widetilde{G}$. We now reduce this theory further on an associative three-cycle M_3. Since this will be generically curved with holonomy group SO(3), the 4d theory will in turn only retain supersymmetry if we partially topologically twist the local Lorentz group SO(3)$_M$ with the R-symmetry SU(2)$_R$. Upon compactification on M_3 the local Lorentz symmetry is broken to

$$
\begin{array}{rcccl}
\mathrm{SO}(1,6)_L\times\mathrm{SU}(2)_R & & \rightarrow & & \mathrm{SO}(1,3)_L\times\mathrm{SO}(3)_M\times\mathrm{SU}(2)_R \\
A: & (\mathbf{7},\mathbf{1}) & \rightarrow & & (\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1}) \equiv (A_\mu, W_{\underline{i}}) \\
\phi: & (\mathbf{1},\mathbf{3}) & \rightarrow & & (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3}) \equiv (\phi_i) \\
\epsilon,\ \lambda: & (\mathbf{8},\mathbf{2}) & \rightarrow & & (\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{2})\oplus(\mathbf{1},\mathbf{2};\mathbf{2},\mathbf{2}) \equiv (\lambda_{\alpha\underline{\alpha}\hat\alpha}, \bar\lambda_{\dot\alpha\underline{\alpha}\hat\alpha})\,,
\end{array}
\tag{4.4}
$$

where the vector indices split as $\mu = 0,\ldots,3$ and $\underline{i},\hat\imath = 1,2,3$ and the spinor indices are $\alpha,\dot\alpha,\underline{\alpha},\hat\alpha = 1,2$. The fermions $\lambda_{\alpha\underline{\alpha}\hat\alpha}$ satisfy a Majorana-condition as described in the appendix in (A.14).

The supersymmetry parameter ϵ transforms non-trivially under SO(3)$_M$, so that to preserve supersymmetry in 4d, we redefine the local Lorentz group SO(3)$_M$ by an R-symmetry transformation[10]

$$\mathrm{SU}(2)_{\mathrm{twist}} = \mathrm{diag}(\mathrm{SO}(3)_M, \mathrm{SU}(2)_R)\,, \tag{4.5}$$

with generators $(\Sigma_M)_i + (\Sigma_R)_i$, where Σ denotes the generators of the respective algebras. The field content and supersymmetry parameters transform under the partially twisted Lorentz group as follows

$$\begin{array}{rccl} \mathrm{SO}(1,3)_L \times \mathrm{SU}(2)_M \times \mathrm{SU}(2)_R & & \rightarrow & \mathrm{SO}(1,3)_L \times \mathrm{SU}(2)_{\mathrm{twist}} \\ A: & (\mathbf{2},\mathbf{2};\mathbf{1};\mathbf{1}) & \rightarrow & (\mathbf{2},\mathbf{2};\mathbf{1}) \equiv (A_\mu) \\ W: & (\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1}) & \rightarrow & (\mathbf{1},\mathbf{1};\mathbf{3}) \equiv (W_i) \\ \phi: & (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3}) & \rightarrow & (\mathbf{1},\mathbf{1};\mathbf{3}) \equiv (\phi_i) \\ \epsilon,\ \lambda: & (\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{2}) & \rightarrow & (\mathbf{2},\mathbf{1};\mathbf{1}) \oplus (\mathbf{2},\mathbf{1};\mathbf{3}) \equiv (\chi_\alpha, \psi_{i\alpha}) \\ \bar\epsilon,\ \bar\lambda: & (\mathbf{1},\mathbf{2};\mathbf{2},\mathbf{2}) & \rightarrow & (\mathbf{1},\mathbf{2};\mathbf{1}) \oplus (\mathbf{1},\mathbf{2};\mathbf{3}) \equiv (\bar\chi^{\dot\alpha}, \bar\psi_i{}^{\dot\alpha})\,. \end{array} \tag{4.6}$$

It follows that there are four real supercharges, as required for 4d $\mathcal{N}=1$ supersymmetry,

$$\epsilon_\alpha = (\mathbf{2},\mathbf{1};\mathbf{1})\,, \qquad \bar\epsilon_{\dot\alpha} = (\mathbf{1},\mathbf{2};\mathbf{1})\,. \tag{4.7}$$

After the twist the fermions χ and ψ transform as singlets and triplets of the twisted Lorentz group and are identified with 0- and 1-forms on M_3 valued in $\mathrm{ad}(P)$, i.e.

$$\begin{aligned} \chi &\in \Omega^0(M_3, \mathrm{ad}(P)) \otimes \mathbb{C} \\ \psi &\in \Omega^1(M_3, \mathrm{ad}(P)) \otimes \mathbb{C}\,, \end{aligned} \tag{4.8}$$

where P is a $\widetilde{G}$-principal bundle. We denote the field strengths associated to the gauge fields A_μ and the Wilson lines W_i by $F_{\mu\nu}$ and $(F_W)_{ij}$, respectively, and their associated covariant derivatives as D_μ and D_i. The latter can be combined with the scalars ϕ_i, which

[10] We will be slightly casual here and in the following, in that the twist involves the Lie algebras, rather than the groups.

both transform as a **3** of $\mathrm{SU}(2)_{\text{twist}}$, into a complex 1-form

$$\varphi_i = \phi_i + iW_i\,, \qquad \bar{\varphi}_i = \phi_i - iW_i\,, \qquad \mathcal{D}_i = \partial_i + [\varphi_i, \cdot\,]\,, \qquad \bar{\mathcal{D}}_i = \partial_i - [\bar{\varphi}_i, \cdot\,]\,. \tag{4.9}$$

Note that $\varphi, \bar{\varphi}$ and $\mathcal{D}, \bar{\mathcal{D}}$ are related by conjugation in the gauge algebra. We further introduce

$$(\mathcal{F}_\varphi)_{ij} = [\mathcal{D}_i, \mathcal{D}_j]\,, \qquad (\mathcal{F}_\varphi)_{\mu i} = [D_\mu, \mathcal{D}_i]\,, \qquad (\mathcal{F}_{\bar{\varphi}})_{\mu i} = [D_\mu, \bar{\mathcal{D}}_i]\,, \tag{4.10}$$

and its conjugate $\mathcal{F}_{\bar{\varphi}} = \mathcal{F}_\varphi^\dagger$. We assume that the 4d gauge fields A_μ are independent of the internal coordinates along M_3, so that the latter two expressions become standard space-time derivatives of the complex scalars $\varphi, \bar{\varphi}$

$$(\mathcal{F}_\varphi)_{\mu i} = D_\mu \varphi_i\,, \qquad (\mathcal{F}_{\bar{\varphi}})_{\mu i} = D_\mu \bar{\varphi}_i\,. \tag{4.11}$$

Define the interaction term

$$I_{\varphi,\bar{\varphi}} \equiv 2D_i\phi^i = \partial_i\varphi^i + \partial_i\bar{\varphi}^i + [\varphi_i, \bar{\varphi}^i]\,. \tag{4.12}$$

The partially twisted 7d SYM action is then

$$\begin{aligned}
S_{F,\,\text{twist}} &= \frac{1}{g_7^2}\int \mathrm{d}^7x \Big\{ \mathrm{Tr}\Big[-i\bar{\chi}\bar{\sigma}^\mu D_\mu\chi - i\bar{\psi}^i\bar{\sigma}^\mu D_\mu\psi_i + \sqrt{2}i\bar{\chi}\mathcal{D}^i\bar{\psi}_i - \sqrt{2}i\chi\bar{\mathcal{D}}^i\psi_i \\
&\qquad\qquad + \frac{i}{\sqrt{2}}\epsilon^{ijk}\bar{\psi}_i\bar{\mathcal{D}}_j\bar{\psi}_k - \frac{i}{\sqrt{2}}\epsilon^{ijk}\psi_i\mathcal{D}_j\psi_k\Big]\Big\} \\
S_{B,\,\text{twist}} &= \frac{1}{g_7^2}\int \mathrm{d}^7x \Big\{ -\frac{1}{4}\,\mathrm{Tr}\Big[F_{\mu\nu}F^{\mu\nu}\Big] - \mathrm{Tr}\Big[(\mathcal{F}_\varphi)_{\mu i}(\mathcal{F}_{\bar{\varphi}})^{\mu i}\Big] - \mathrm{Tr}\Big[(\mathcal{F}_\varphi)_{ij}(\mathcal{F}_{\bar{\varphi}})^{ij}\Big] \\
&\qquad\qquad - \frac{1}{2}\,\mathrm{Tr}\Big[I^2_{\varphi,\bar{\varphi}}\Big]\Big\}\,.
\end{aligned} \tag{4.13}$$

The supersymmetry variations for the bosonic fields are

$$
\begin{aligned}
\delta A_\mu &= i\epsilon\sigma_\mu\bar{\chi}\,, & \bar{\delta}A_\mu &= -i\bar{\epsilon}\bar{\sigma}_\mu\chi \\
\delta W_i &= -\frac{i}{\sqrt{2}}\epsilon\psi_i\,, & \bar{\delta}W_i &= \frac{i}{\sqrt{2}}\bar{\epsilon}\bar{\psi}_i \\
\delta\phi_i &= \frac{1}{\sqrt{2}}\epsilon\psi_i\,, & \bar{\delta}\phi_i &= \frac{1}{\sqrt{2}}\bar{\epsilon}\bar{\psi}_i \\
\delta\varphi_i &= \sqrt{2}\epsilon\psi_i\,, & \bar{\delta}\varphi_i &= 0 \\
\delta\bar{\varphi}_i &= 0\,, & \bar{\delta}\bar{\varphi}_i &= \sqrt{2}\bar{\epsilon}\bar{\psi}_i\,,
\end{aligned}
\tag{4.14}
$$

and for the fermionic ones we find

$$
\begin{aligned}
\delta\chi &= F_{\mu\nu}\sigma^{\mu\nu}\epsilon + iI_{\varphi,\bar{\varphi}}\,\epsilon\,, & \bar{\delta}\chi &= 0 \\
\delta\bar{\chi} &= 0\,, & \bar{\delta}\bar{\chi} &= F_{\mu\nu}\bar{\epsilon}\bar{\sigma}^{\mu\nu} - iI_{\varphi,\bar{\varphi}}\,\bar{\epsilon} \\
\delta\psi^k &= i(\mathcal{F}_{\bar{\varphi}})_{ij}\epsilon^{ijk}\epsilon\,, & \bar{\delta}\psi^k &= \sqrt{2}i(\mathcal{F}_{\varphi})_\mu{}^k\sigma^\mu\bar{\epsilon} \\
\delta\bar{\psi}^k &= \sqrt{2}i(\mathcal{F}_{\bar{\varphi}})_\mu{}^k\bar{\sigma}^\mu\epsilon\,, & \bar{\delta}\bar{\psi}^k &= -i(\mathcal{F}_{\varphi})_{ij}\epsilon^{ijk}\bar{\epsilon}\,.
\end{aligned}
\tag{4.15}
$$

To obtain a 4d supersymmetric theory upon twisted dimensional reduction, the field configuration along M_3 needs to preserve supersymmetry. We further require the background to enjoy 4d Poincaré-invariance and therefore require it to be independent of the coordinates along $\mathbb{R}^{1,3}$

$$
(\mathcal{F}_{\varphi})_{\mu i} = 0\,, \qquad (\mathcal{F}_{\bar{\varphi}})_{\mu i} = 0\,. \tag{4.16}
$$

The BPS equations are then obtained by setting $\langle\delta\lambda\rangle = 0$ and are

$$
I_{\varphi,\bar{\varphi}} = \partial_i\varphi^i + \partial_i\bar{\varphi}^i + [\varphi_i, \bar{\varphi}^i] = 0\,, \qquad (\mathcal{F}_{\varphi})_{ij} = 0\,, \qquad (\mathcal{F}_{\bar{\varphi}})_{ij} = 0\,, \tag{4.17}
$$

where the first equation is obtained by setting the real and imaginary parts of $\delta\chi$ to zero separately. 4d Poincaré invariance requires $\langle F_{\mu\nu}\rangle = 0$. Rewriting (4.17) with respect to the notation in (4.6) the BPS equations become the F- and D-term equations

$$\boxed{\begin{aligned} 0 &= F_W - i[\phi, \phi] \\ 0 &= D_W \phi \\ 0 &= D_W^\dagger \phi \,. \end{aligned}} \tag{4.18}$$

Background values for the Higgs field ϕ and gauge field W along M_3 that solve these equations will determine the effective field theory in 4d[11]. In components the BPS equations are

$$\begin{aligned} 0 &= \partial_i W_j - \partial_j W_i + i[W_i, W_j] - i[\phi_i, \phi_j] \\ 0 &= \partial_i \phi_j + i[W_i, \phi_j] - \partial_j \phi_i - i[W_j, \phi_i] \\ 0 &= g^{ij} \left(\partial_i \phi_j + i[W_i, \phi_j]\right) \,. \end{aligned} \tag{4.19}$$

Depending on the topology of M_3 there are various solutions to these equations. The simplest set of solutions are obtained for commuting Higgs fields

$$[\phi, \phi] = 0 \,, \qquad F_W = 0 \,. \tag{4.20}$$

We will generally assume this to be the case. The remaining equations are $D_W \phi = D_W^\dagger \phi = 0$. If M_3 is a compact three-manifold without boundaries and ϕ is regular, there are two cases to consider:

$$\begin{aligned} \pi_1(M_3) = 0 \quad &\Rightarrow \quad W = 0 \,, \quad \mathrm{d}\phi = \mathrm{d}^\dagger \phi = 0 \quad \Rightarrow \quad \phi = 0 \\ \pi_1(M_3) \neq 0 \quad &\Rightarrow \quad D_W \phi = D_W^\dagger \phi = 0 \,. \end{aligned} \tag{4.21}$$

In the first case ϕ has to be a harmonic 1-form and thus must be trivial, in the second case it can be non-trivial.

We will be interested in simply-connected three-manifolds (i.e. a homology three-spheres) in the following. To nevertheless have non-trivial solutions we relax the assumption that ϕ is regular, which can be achieved by including sources into the D-term equations.

[11]Note that we have chosen Hermitian representatives for the gauge algebra. Transitioning to anti-Hermitian representatives we recover the results of [49].

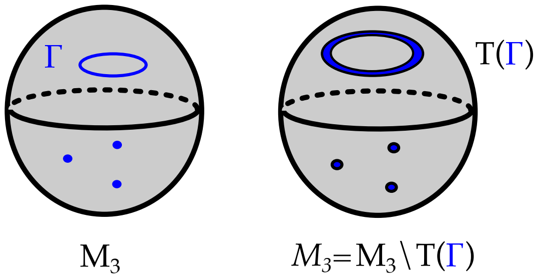

Figure 1: On the left-hand side the three-cycle M_3 is shown, with the charge distribution ρ that is located along Γ. On the right-hand side, a tubular neighborhood $T(\Gamma)$ is excised and the resulting manifold is $\mathcal{M}_3$.

Writing $\phi = \mathrm{d}f$, the function f is then required to satisfy Poisson's equation

$$\phi = \mathrm{d}f \,, \qquad \Delta f = \rho \,, \tag{4.22}$$

where ρ models the sources supported on a closed subset Γ. This maps the solution of the BPS equations to an electrostatics problem with the identification

$$\begin{aligned} f &= \text{electrostatic potential} \\ \rho &= \text{charge density, supported on } \Gamma \subset M_3 \,. \end{aligned} \tag{4.23}$$

Alternatively this system can be described by excising a tubular neighborhood $T(\Gamma)$ of the charge support Γ, and studying the problem of finding solutions on $\mathcal{M}_3 = M_3 \setminus T(\Gamma)$ – see figure 1. In this case ϕ needs to be regular, with suitable boundary conditions along $\partial\mathcal{M}_3$. In summary, we are going to consider the following setup

$$\boxed{\phi \text{ regular} \,, \quad \phi = \mathrm{d}f \,, \quad \Delta f = 0 \,, \quad \partial\mathcal{M}_3 = T(\Gamma) \neq \emptyset} \tag{4.24}$$

which will be used in section 6 to determine physically interesting solutions to the BPS equations including localized matter. Localized matter is characterized by the vanishing of ϕ. When f is Morse (i.e. it has no degenerate critical points) these are isolated points and we will discuss this setup in section 6. By relaxing the constraint of f only having isolated critical points this can be generalized to situations where f is a Morse-Bott function and higher-dimensional matter loci can be included as well. We will discuss this in section 7.4

and apply it to TCS G_2-manifolds in section 9.

In the remainder of this section, we assume that ϕ is non-trivial and regular, but make no further assumptions on the details of the loci $\phi = 0$.

4.2 Higgs Bundles

Before studying the low energy effective theory, let us briefly recall the relation between the Higgs bundle and the local ALE-fibration. The BPS equations (in the absence of sources) (4.18) are in fact precisely the odd dimensional analogue of the Hitchin equations for the Higgs field ϕ giving rise to the data of a Higgs bundle. In the case $[\phi, \phi] = 0$ there is an elegant geometric description of the Higgs field ϕ in terms of an ALE-fibration over M_3, which we now summarize [49]. This construction is analogous to the one in F-theory, where the Higgs field specifies the unfolding (a complex structure deformation) of the ALE singularity and is closely connected to the compact Calabi-Yau underlying the F-theory compactification [51, 55, 60]. Recently this was developed also for Spin(7) manifolds [88]. In our case the Higgs field describes the deformations of the full hyperkähler structure of an ALE-fiber.

Recall that ϕ is an adjoint valued 1-form $\Omega^1(\mathcal{M}_3)$ or a section of $T^*(\mathcal{M}_3)$, and we take it to be non-trivial along the commutant $G_\perp$ of the 4d gauge group G in

$$\widetilde{G} \;\rightarrow\; G_\perp \times G\,. \tag{4.25}$$

The Higgs field is

$$\phi \in \Gamma(T^*(\mathcal{M}_3) \otimes \mathrm{Ad}(G_\perp))\,, \tag{4.26}$$

i.e. ϕ lives in a local geometry in the vicinity of $\mathcal{M}_3$ which is the total space of the cotangent bundle $T^*(\mathcal{M}_3)$. This is a local Calabi-Yau threefold. Since $[\phi, \phi] = 0$, we can diagonalize the Higgs field to obtain n 1-forms ϕ_j, where n is the rank of the Lie algebra $\mathfrak{g}_\perp$ of $G_\perp$. To locally recover the ALE-fibration over $\mathcal{M}_3$ associated to this Higgs field, we use the Kronheimer construction [28, 80]. Every ALE-space is of the form $\mathbb{C}^2/\Gamma_{\mathrm{ADE}}$, where Γ_{ADE} is a finite subgroup of SU(2), which are classified by the corresponding ADE Dynkin diagrams. The second homology of the resolution of singularities of $\mathbb{C}^2/\Gamma_{\mathrm{ADE}}$ is isomorphic to $\mathfrak{g}$ and we can think of the components ϕ_j as measuring the periods of the hyperkähler structure forms. More explicitly, over a local patch of M_3 we can write the fibration as $\mathbb{R}^3 \times \mathbb{C}^2/\Gamma_{\rm ADE}$. We choose a basis σ_j of $H_2(\widetilde{\mathbb{C}^2/\Gamma_{\rm ADE}}, \mathbb{Z})$ and fix a hyperkähler triple $(\omega_I, \omega_J, \omega_K)$. The 1-form ϕ_j can be written as

$$\phi_j = \phi_{j,I} \mathrm{d}x^1 + \phi_{j,J} \mathrm{d}x^2 + \phi_{j,K} \mathrm{d}x^3 \,, \tag{4.27}$$

where we identify

$$\phi_{j,I} = \int_{\sigma_j} \omega_I, \qquad \phi_{j,J} = \int_{\sigma_j} \omega_J, \qquad \phi_{j,K} = \int_{\sigma_j} \omega_K \,. \tag{4.28}$$

This uniquely defines the hyperkähler structure on each fiber. Observe that the Higgs field has an SO(3) symmetry arising from the SO(3) acting on ω_I, ω_J and ω_K.

In geometric terms we can describe our situation as follows. For simplicity, assume that we have a $G_\perp = \mathrm{U}(1)$-valued Higgs field ϕ. We are considering a local model for a G_2-manifold with ADE-singularities located along an associative submanifold $\mathcal{M}_3$, which physically means that gauge degrees of freedom are localized along $\mathcal{M}_3$ and the gauge group is given by the ADE type of the singularity. Consider the gauge group $\widetilde{G}$, which by turning on a non-trivial background vev for ϕ generically higgses to $\widetilde{G} \ \rightarrow \ G \times \mathrm{U}(1)$. This means that the ALE-fiber over a generic point of $\mathcal{M}_3$ will have the singularity corresponding to G via the ADE correspondence and there will be a two-cycle in the U(1) direction with non-zero volume, given by ϕ. Over the points where $\phi = 0$, the two-cycle collapses and the ALE singularity worsens; equivalently the gauge group enhances from G to $\widetilde{G}$. We elaborate this point in section 8.

We can in fact make the local geometry of the gauge enhancement fairly explicit. For the moment let us restrict our attention to the case where $\widetilde{G} = \mathrm{SU}(2)$ which corresponds to a $\mathbb{C}^2/\mathbb{Z}_2$ singularity over $\mathcal{M}_3$. Giving a non-trivial background vev for ϕ corresponds to deforming the generic fiber to a smooth Eguchi-Hanson space. More precisely, consider the generator σ of $H_2(\widetilde{\mathbb{C}^2/\mathbb{Z}_2}, \mathbb{Z})$ of the resolved geometry. Recall that σ is topologically a two-sphere. From (4.27) and (4.28) we see that at a generic point $x \in \mathcal{M}_3$ (which for this purpose is approxated locally by $\mathbb{R}^3$) we have

$$\mathrm{Vol}(\sigma) = |\phi(x)|\,, \tag{4.29}$$

by which we mean the volume of σ in the Eguchi-Hanson space above x. Consider now a neighborhood of a non-degenerate zero of ϕ, which we can assume to be at $0 \in \mathbb{R}^3$. We can locally write $\phi = \mathrm{d}f$, where

$$f(x_1, x_2, x_3) = f(0) + \frac{1}{2}\sum_{i=1}^{3} \pm x_i^2\,. \tag{4.30}$$

The signs depend on the eigenvalue of the Hessian at 0. The Higgs field ϕ now has an isolated zero at the origin. The explicit local description of the ALE-fibration is given by

$$X = \left\{ (z_1, z_2, z_3),\ (x_1, x_2, x_3) \;\middle|\; z_1^2 + z_2^2 + z_3^2 = \sum_{i=1}^{3} x_i^2 \right\} \subset \mathbb{C}^3 \times \mathbb{R}^3\,. \tag{4.31}$$

Viewing X as a fibration over $\mathbb{R}^3$ all of the fibers are smooth apart from the fiber over $(0,0,0)$ i.e. the zero of ϕ. Moreover, X is a cone in $\mathbb{C}^3 \times \mathbb{R}^3$ with the apex at the origin. The link of the cone can be found by intersecting X with the unit sphere in $\mathbb{C}^3 \times \mathbb{R}^3$ and is in fact $\mathbb{CP}^3$ realized as the twistor bundle over $\mathbb{S}^4$. The approximate G_2-metric on X is given by

$$\Phi = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}x_1 \wedge \omega_I + \mathrm{d}x_2 \wedge \omega_J + \mathrm{d}x_3 \wedge \omega_K + \phi \wedge \eta\,, \tag{4.32}$$

where η is the 2-form dual to the two-cycle σ.

This can be generalized to arbitrary ALE-fibrations. The local geometry is of the form $\mathbb{C}^2/\Gamma_G \times \mathbb{R}^3$, with a $\mathbb{C}^2/\Gamma_{\widetilde{G}}$ fiber over the origin. We again work with $\widetilde{G} = \mathrm{SU}(n+1)$. For the deformations of other ADE singularities see [89]. The topology in a neighborhood of an isolated zero is

$$X = \left\{ (z_1, z_2, z_3),\ (x_1, x_2, x_3) \;\middle|\; z_1^2 + z_2^2 + z_3^n \left(z_3 - \sum_{i=1}^{3} x_i^2 \right) = 0 \right\} \subset \mathbb{C}^3 \times \mathbb{R}^3\,. \tag{4.33}$$

This describes a family of $\mathrm{SU}(n)$ singularities, with enhancement to $\mathrm{SU}(n+1)$ at the origin (note that we again write $\phi = \mathrm{d}f$ as above). There are also explicit deformations for other ADE groups. Topologically X is now a cone over the weighted projective space $\mathbb{WCP}^3_{n,n,1,1}$ with coordinates (y_1, y_2, y_3, y_4). Notice that this is precisely the geometry where we expect chiral fermions In the link, there is a family of SU(n) singularities along a $\mathbb{S}^2$ given by $y_3 = y_4 = 0$. In the ambient space, the location of the singularities is a cone $\mathbb{R}^+ \times \mathbb{S}^2 = \mathbb{R}^3$, which is identified in our context with a local patch of the base $\mathbb{R}^3$ of X. As before, the apex of the cone is where the cycle σ collapses to zero volume.

This therefore establishes a key piece of the dictionary between properties of ϕ and the ambient G_2-geometry. The isolated zeroes of ϕ give rise to conical singularities of the ALE-fibered G_2-manifold. As we show in section 7, this fits together nicely with the physics side as zeroes of ϕ which occur at codimension 7 are precisely the loci where chiral fermions are localized.

4.3 Massless Spectrum

Given a solution to the BPS equations (4.18) with regular Higgs field we can ask what the spectrum of the 4d gauge theory is. The equations of motion of the fermions follow from (4.13) to be

$$\begin{aligned} 0 &= \bar{\sigma}^\mu D_\mu \chi - \sqrt{2}\mathcal{D}_i \bar{\psi}^i \,, \\ 0 &= \bar{\sigma}^\mu D_\mu \psi^i + \sqrt{2}\mathcal{D}^i \bar{\chi} - \sqrt{2}\epsilon^{ijk}\bar{\mathcal{D}}_j \bar{\psi}_k \,, \end{aligned} \tag{4.34}$$

which are equivalent to the decoupled equations

$$\begin{aligned} 0 &= D_\mu D^\mu \chi + 2\mathcal{D}_i \bar{\mathcal{D}}^i \chi \,, \\ 0 &= D_\mu D^\mu \psi_i + 2[\mathcal{D}_i, \bar{\mathcal{D}}_j]\psi^j + 2\bar{\mathcal{D}}_j \mathcal{D}^j \psi_i \,. \end{aligned} \tag{4.35}$$

So far we have not imposed $[\phi, \phi] = 0$. Define the twisted exterior derivative and Laplace operator

$$\mathcal{D} = \mathrm{d} + [\varphi \wedge \cdot\,] \,, \qquad \bar{\mathcal{D}} = \mathrm{d} - [\bar{\varphi} \wedge \cdot\,] \,, \qquad \Delta = \mathcal{D}^\dagger \mathcal{D} + \mathcal{D}\mathcal{D}^\dagger \,, \qquad \bar{\Delta} = \bar{\mathcal{D}}^\dagger \bar{\mathcal{D}} + \bar{\mathcal{D}}\bar{\mathcal{D}}^\dagger \,, \tag{4.36}$$

where the adjoint is taken with respect to the Hermitian inner product

$$\begin{aligned} \langle\,\cdot\,,\,\cdot\,\rangle : \Omega^p(M_3,\mathrm{ad}(P)) \times \Omega^p(M_3,\mathrm{ad}(P)) \;&\to\; \mathbb{C}\,, \\ (\alpha,\beta) \;&\to\; \langle\alpha,\beta\rangle = \int_{M_3} \mathrm{Tr}\left(\bar{\alpha}\wedge *\beta\right)\,. \end{aligned} \tag{4.37}$$

Acting on functions $g \in \Omega^0(M_3, \mathrm{ad}\, P)$ and written in coordinates, e.g. the operator $\bar{\Delta}$ becomes

$$\bar{\Delta} g = \bar{\mathcal{D}}^\dagger \bar{\mathcal{D}} g = \bar{\mathcal{D}}^\dagger(\bar{\mathcal{D}}_m g\, \mathrm{d}x^m) = \mathcal{D}^m \bar{\mathcal{D}}_m g\,, \tag{4.38}$$

where we pick up a conjugation due to the inner product. We find that (4.35) may be rewritten as

$$\begin{aligned} 0 &= D_\mu D^\mu \chi + 2\bar{\Delta}\chi\,, \\ 0 &= D_\mu D^\mu \psi + 2\Delta\psi\,, \end{aligned} \tag{4.39}$$

where by (4.8), χ and ψ are 0- and 1-forms, respectively. Massless modes are therefore described by the kernels of the Laplacians $\Delta, \bar{\Delta}$ or equivalently by closed and co-closed forms with respect to the operators in (4.36)

$$\begin{aligned} \bar{\mathcal{D}}\chi &= 0\,, \qquad \bar{\mathcal{D}}^\dagger\chi = 0\,, \\ \mathcal{D}\psi &= 0\,, \qquad \mathcal{D}^\dagger\psi = 0\,. \end{aligned} \tag{4.40}$$

By the BPS equations the co-boundary operators $\mathcal{D}, \bar{\mathcal{D}}$ and their adjoints close $\mathcal{D}^2 = \bar{\mathcal{D}}^2 = 0$ and $(\mathcal{D}^\dagger)^2 = (\bar{\mathcal{D}}^\dagger)^2 = 0$, and via the Hodge correspondence for elliptic complexes we can describe the zero-modes equivalently as cohomology groups. The non-vanishing background value of ϕ or W oriented along a subgroup $G_\perp$ of $\widetilde{G}$ breaks the gauge group to its commutant $G \subset \widetilde{G}$. The adjoint fermions ψ, χ will decompose accordingly to give matter valued in irreducible representation. In this higgsed theory the fermions are sections of the associated gauge bundles, E. The action of $\mathcal{D}$ restricts to each of these subbundles allowing us to make the identification

$$\begin{aligned} \chi_\alpha &\in H^0_{\mathcal{D}}(M_3, E)\,, \qquad \bar{\chi}_{\dot{\alpha}} \in H^0_{\mathcal{D}}(M_3, E)\,, \\ \psi_\alpha &\in H^1_{\mathcal{D}}(M_3, E)\,, \qquad \bar{\psi}_{\dot{\alpha}} \in H^1_{\mathcal{D}}(M_3, E)\,. \end{aligned} \tag{4.41}$$

We next rewrite these cohomology groups with respect to the same co-boundary operator by dualizing $H^0_{\bar{\mathcal{D}}}, H^1_{\bar{\mathcal{D}}}$ with the Hodge star. Note that by (4.37) we have $\mathcal{D}^\dagger = *\bar{\mathcal{D}}*$ and $\bar{\mathcal{D}}^\dagger = *\mathcal{D}*$ so that taking $\chi_\alpha \in H^0_{\bar{\mathcal{D}}}(M_3, E)$ for example we find that $*\chi_\alpha$ is annihilated by the operators $\mathcal{D}, \mathcal{D}^\dagger$

$$\mathcal{D}^\dagger(*\chi_\alpha) = *\bar{\mathcal{D}}\chi_\alpha = 0\,, \qquad \mathcal{D}*\chi_\alpha = *\bar{\mathcal{D}}^\dagger\chi = 0\,. \tag{4.42}$$

This precisely states that $*\chi_\alpha \in H^3_{\mathcal{D}}(M_3, E)$, i.e. we have mapped from $\bar{\mathcal{D}}$-cohomology to $\mathcal{D}$-cohomology using the Hodge star. The same observations hold true for $\bar{\psi}_\alpha$. The Hodge star relates

$$H^0_{\bar{\mathcal{D}}}(M_3, E) \cong H^3_{\mathcal{D}}(M_3, E)\,, \qquad H^1_{\bar{\mathcal{D}}}(M_3, E) \cong H^2_{\mathcal{D}}(M_3, E)\,. \tag{4.43}$$

This allows us to make the following identifications

$$\boxed{\begin{aligned} &\chi_\alpha \in H^3_{\mathcal{D}}(M_3, E)\,, \qquad \bar{\chi}_{\dot{\alpha}} \in H^0_{\mathcal{D}}(M_3, E)\,,\\ &\psi_\alpha \in H^1_{\mathcal{D}}(M_3, E)\,, \qquad \bar{\psi}_{\dot{\alpha}} \in H^2_{\mathcal{D}}(M_3, E)\,,\end{aligned}} \tag{4.44}$$

where now all cohomologies are with respect to $\mathcal{D}$ and forms of all degrees are employed. Note that the $\mathbb{Z}_2$-grading of the exterior algebra aligns with the 4d chirality of the fermionic zero-modes. The Hodge star depends on the metric of M_3 which itself is induced from the metric of G_2-holonomy of the ambient 7d manifold.

Since M_3 is associative and so calibrated with respect to Φ_{ijk} we equivalently could have used the G_2 3-form Φ_{ijk} to dualize since it restricts to a volume form of M_3. Contracting elements of $H^0_{\bar{\mathcal{D}}}$ and $H^1_{\bar{\mathcal{D}}}$ with the 3-form Φ_{ijk} is then exactly the same as taking their Hodge dual.

4.4 Bulk Matter

The first type of matter we will discuss arises from a background Higgs bundle, where $\langle\phi\rangle = 0$, which solves the BPS equations, but $W \neq 0$ with $F_W = 0$. This will be referred to as bulk matter, as the modes will not be localized. We will see that for $\pi_1(\mathcal{M}_3) = 0$ there is no chiral index for this matter type. It may be interesting to extend this to non-trivial π_1 setups, which we relegate to future work, and also has been discussed in earlier works from a different point of view (see e.g. [90]).

Turning on a flat gauge field along a subgroup $G_\perp \subset \widetilde{G}$ the gauge group $\widetilde{G}$ is Higgsed to the commutant G of $G_\perp$ in $\widetilde{G}$ and the adjoint representation of $\widetilde{G}$ decomposes as

$$\begin{aligned} \widetilde{G} &\rightarrow G \times G_\perp \\ \mathrm{Ad}(\widetilde{G}) &\rightarrow (\mathrm{Ad}(G)\otimes \mathbf{1}) \oplus (\mathbf{1}\otimes \mathrm{Ad}(G_\perp)) \oplus \bigoplus_n \mathbf{R}_n \otimes \mathbf{S}_n \,. \end{aligned} \tag{4.45}$$

For the fields of the theory this decomposition is lifted to the bundle level, where $\mathrm{Ad}(P)$ decomposes into the vector bundles $\mathcal{R}_n \otimes \mathcal{S}_n$ in the representations $\mathbf{R}_n$ and $\mathbf{S}_n$ of G and $G_\perp$, respectively. The chiral and conjugate-chiral zero modes transforming in $\mathbf{R}_n$ are then counted by the cohomology groups

$$\text{Chiral}: \quad \begin{matrix} (\chi_{\mathbf{R}_n})_\alpha \in H^0_{\mathcal{D}}(M_3, \mathcal{S}_n) \\ (\psi_{\mathbf{R}_n})_\alpha \in H^1_{\mathcal{D}}(M_3, \mathcal{S}_n) \end{matrix} \qquad \text{Conjugate-chiral}: \quad \begin{matrix} (\bar\chi_{\mathbf{R}_n})_{\dot\alpha} \in H^0_{\bar{\mathcal{D}}}(M_3, \mathcal{S}_n) \\ (\bar\psi_{\mathbf{R}_n})_{\dot\alpha} \in H^1_{\bar{\mathcal{D}}}(M_3, \mathcal{S}_n)\,. \end{matrix} \tag{4.46}$$

Their CPT-conjugate zero modes in $\overline{\mathbf{R}}_n$ are obtained by Hermitian conjugation in the gauge algebra or equivalently from (4.41) with $E = \bar{\mathcal{S}}$. In order to rewrite these cohomology groups with respect to the same boundary operator $\mathcal{D}$ we again dualize $H^0_{\bar{\mathcal{D}}}, H^1_{\bar{\mathcal{D}}}$ using the Hodge star and obtain

$$\text{Chiral}: \quad \begin{matrix} (\chi_{\mathbf{R}_n})_\alpha \in H^3_{\mathcal{D}}(M_3, \mathcal{S}_n) \\ (\psi_{\mathbf{R}_n})_\alpha \in H^1_{\mathcal{D}}(M_3, \mathcal{S}_n) \end{matrix} \qquad \text{Conjugate-chiral}: \quad \begin{matrix} (\bar\chi_{\mathbf{R}_n})_{\dot\alpha} \in H^0_{\mathcal{D}}(M_3, \mathcal{S}_n) \\ (\bar\psi_{\mathbf{R}_n})_{\dot\alpha} \in H^2_{\mathcal{D}}(M_3, \mathcal{S}_n)\,. \end{matrix} \tag{4.47}$$

These cohomology groups completely determine the chiral and conjugate-chiral spectrum in 4d transforming in $\mathbf{R}_n$ of the remnant gauge symmetry G

$$\begin{aligned} \text{Chiral fermion zero-modes}: &\quad H^3_{\mathcal{D}}(M_3, \mathcal{S}_n) \oplus H^1_{\mathcal{D}}(M_3, \mathcal{S}_n)\,, \\ \text{Conjugate-chiral fermion zero-modes}: &\quad H^0_{\mathcal{D}}(M_3, \mathcal{S}_n) \oplus H^2_{\mathcal{D}}(M_3, \mathcal{S}_n)\,. \end{aligned} \tag{4.48}$$

The chiral index of the representation $\mathbf{R}_n$ is

$$\chi(M_3, \mathbf{R}_n, \mathcal{D}) = \sum_{i=0}^{3} (-1)^i \dim_{\mathbb{C}} H^i_{\mathcal{D}}(M_3, \mathcal{S}_n) \,, \tag{4.49}$$

which is nothing other than the Euler characteristic of the $\mathcal{D}$-complex. In the case of trivial fundamental group $\pi_1(M_3)$, there is no flat bundle to break the gauge group, and $\dim H^i_{\mathcal{D}}(M_3, \mathcal{S}_n) = b^i(M_3, \mathcal{D})$ reduce to the Betti numbers of the de Rham complex on M_3. The chiral index is then given by the usual Euler characteristic, which vanishes for odd dimensional closed manifolds

$$\pi_1(M_3) = 0 : \qquad \chi(M_3, \mathbf{R}_n, \mathcal{D}) = 0 \,. \tag{4.50}$$

This concludes our discussion of 'bulk' matter. In the following we will focus our attention on localized matter modes, which arise from non-trivial ϕ background values. Since these are best characterized in terms of spectral covers we will first develop the framework for that. We will briefly discuss interactions between bulk matter and localized matter fields later on.

5 Spectral Covers

5.1 Spectral Cover for the Higgs Field

For the case when a higher rank Higgs bundle is turned on but the Higgs field commutes, it is useful to describe the solution to the BPS equations in terms of the spectral data of the Higgs field. This framework is of course very familiar from F-theory spectral covers, see e.g. [55,56,58,60], and for the Lagrangians in Calabi-Yau threefolds and the associated G_2-manifolds with pointlike singularities was touched upon in [49]. Here we will prepare the setup to also account for more general Higgs field configurations, with the goal to apply it to the TCS-manifolds.

Recall that ϕ is a section of $\Omega^1(\mathcal{M}_3) \otimes \mathrm{Ad}(G_\perp)$. For concreteness let $G_\perp = \mathrm{SU}(n)$. For a commuting Higgs field we can choose to diagonalize it and study the resulting spectral

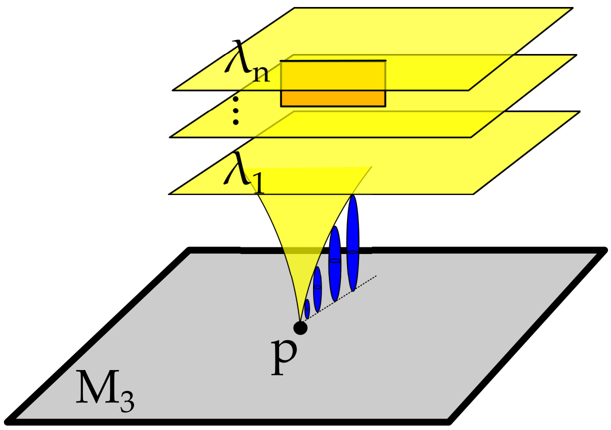

Figure 2: Higgs bundle spectral cover over $\mathcal{M}_3$. Each sheet is labeled by an eigenvalue λ_i of $G_\perp$. Each λ_i is a one-form and their vanishing thus implies that they intersect the 'zero section', i.e. the locus of the ADE singularity of type G present in every ALE-fiber, over points p_k on M_3. Those points p_i are precisely the loci where matter is localized. In the generic case of non-factored spectral covers, the sheets are furthermore connected by branch-cuts (orange). As we will see later on: In case the point p is connected by a flow line in M_3 to another critical point, there is a corresponding associative three-cycle which is built by fibering the collapsing $\mathbb{S}^2$ (blue) over the flow line. The resulting contribution to the superpotential gives a mass term for the localized states.

cover

$$\mathcal{C}: \qquad 0 = \det(\phi - s) = \sum_{i=0}^{n} b_{n-i} s^i = b_0 \prod_{i=1}^{n} (s - \lambda_i)\,, \tag{5.1}$$

where b_i are symmetric polynomials in the eigenvalues of ϕ and for SU(n), $b_1 = 0$. The eigenvalues λ_i are one-forms which give rise to an n-sheeted cover of M_3 and

$$b_i \in S^i(T^*\mathcal{M}_3)\,, \tag{5.2}$$

where b_0 is the zero-section. A cartoon of this is given in figure 2. Using the spectral cover the associated ALE-fibration is simply

$$y^2 = x^2 + \mathcal{C}(s)\,. \tag{5.3}$$

Each λ_i parametrizes the volumes of a corresponding two-sphere in the $G_\perp$-ALE-fiber. The gauge symmetry $\widetilde{G}$ is generically higgsed to G, except at the loci

$$b_n = b_0 \prod_{i=1} \lambda_i = 0\,, \tag{5.4}$$

when the gauge symmetry enhances. Since λ_i is a one-form, this condition implies that this happens generically over points in $\mathcal{M}_3$, though we will encounter other situations as well. The relation between the eigenvalues λ_i and coefficients in the ALE-fibration b_i is not linear, and generically the sheets of the spectral cover will be connected by branch-cuts. This effect implies in particular that the U(1)-symmetries associated to the Higgs bundle are not actually present in the low energy effective theory.

The classic example for spectral cover models starts with an $E_8 \to \mathrm{SU}(5) \times \mathrm{SU}(5)_\perp$. The spectral cover is five-sheeted and $\lambda_i = 0$ characterizes the location of **10** matter representations (we refer the reader to the F-theory literature where these models have been studied in depth [55, 56, 58, 60]). We will construct an example of this kind in detail at the end of this thesis.

5.2 U(1) Symmetries

Generically the sheets of the cover are connected by branch-cuts and therefore, although locally it may appear otherwise, the independent gauge group is G, the commutant of $G_\perp = \mathrm{SU}(n)$ in $\widetilde{G}$. If however the coefficients of the spectral cover are tuned such that it globally factors over $\mathcal{M}_3$

$$\mathcal{C}(s) = \prod_{k=1}^{N+1} \mathcal{C}^{(k)}(s)\,, \tag{5.5}$$

then this corresponds to N independent U(1) factors in the 4d effective theory [58]. The possibilities of factorization depend on the monodromy group that acts on the spectral cover, which for SU(n) covers is S_n. If the group acts transitively on the n sheets then there is no additional U(1) symmetry. If it has $N+1$ orbits then there are N globally defined two-forms, which define U(1) symmetries. To see this, we consider the difference between the factored cover components $\mathcal{C}^{(k)} - \mathcal{C}^{(l)}$. Fibered over $\mathcal{M}_3$, the associated two-cycles define a non-trivial five-cycle in both the local model and in the compact G_2 manifold J. The Poincaré dual two-form to this five-cycle then gives a U(1) gauge boson in the Kaluza-Klein reduction of the three-form C_3 in compactification of M-theory. This can be also be seen concretely in the context of TCS G_2, see section 9.

Perhaps the only important difference to the F-theory spectral cover story is that here it will be paramount that the spectral cover factors, in order to determine the spectrum of the 4d theory. Although the general twisted cohomology will continue to compute the zero mode spectrum, we do not have a computational tool to determine these cohomologies, unless the spectral cover factors completely.

6 Localised Matter

We will now study the more interesting and richer class of matter fields, localized on points or one-dimensional loci of $\mathcal{M}_3$. So far in section 4.4 we considered only flat gauge fields on along $\mathcal{M}_3$, which corresponds to bulk matter. Turning on vevs for the Higgs fields ϕ will enlarge the possible matter structure and will allow us to engineer spectra with non-trivial chiral index. The simplest case is an abelian Higgs field configuration

$$
\begin{aligned}
\widetilde{G} &\rightarrow G \times \mathrm{U}(1)_\perp \\
\mathrm{Ad}(\widetilde{G}) &\rightarrow \mathrm{Ad}(G) \oplus \mathrm{Ad}(\mathrm{U}(1)) \oplus \mathbf{R}_{+q} \oplus \overline{\mathbf{R}}_{-q} \,,
\end{aligned}
\tag{6.1}
$$

where G is the 4d gauge group and $\mathrm{U}(1)_\perp$ the commutant, along which the Higgs fields is turned on. The expectation is that since ϕ is in the $\mathbf{3}$ of $\mathrm{SO}(3)_{\text{twist}}$, the condition for local gauge enhancement to G occurs at codimension 3 in the base M_3, i.e. codimension 7 in the G_2-manifold J. This is also suggested by the earlier spectral cover discussion. We will discuss this case of codimension 7 localized matter first. In less generic situations, such as the twisted connected sums, however, enhancement occurs at codimension 6 loci.

The solution to the BPS equations (4.24) on M_3 will be constructed by excising a tubular neighborhood $T(\Gamma)$ of a graph Γ, with boundary conditions, which we will discuss in detail. The central question is how the zero modes in $\mathbf{R}_{+q}$ and $\overline{\mathbf{R}}_{-q}$ are counted. In this section we provide the cohomological answer to this question, which applies to both codimension 6 and 7 gauge enhancements. In the next section we will provide specific solutions to the BPS-equations, to which the general analysis in this section can be applied, thereby computing the zero mode spectrum.

6.1 Zero Modes from Relative Cohomology

We now turn on a background value for the Higgs field ϕ, which to begin with is U(1)-valued. As explained in section 4, we now set out to solve the D-term equation (4.18) for $\phi = \mathrm{d}f$ with sources, i.e. the Poisson equation

$$\Delta f = \rho\,, \tag{6.2}$$

where the charge density ρ satisfies charge conservation

$$\int_{M_3} \rho = 0\,. \tag{6.3}$$

We take ρ to be localized on links Γ_i in M_3 of definite signs of the charges, $\Gamma_\pm$,

$$\Gamma = \Gamma_+ \cup \Gamma_-\,. \tag{6.4}$$

Both the Higgs field ϕ and f diverge along Γ. We again excise a tubular neighborhood as in section 4. The boundary $\partial\mathcal{M}_3$ splits into connected components Σ_i, which correspond to the connected components of the underlying links Γ_i and correspondingly the boundary splits as

$$\partial\mathcal{M}_3 = \bigcup_i \Sigma_i = \Sigma_+ \cup \Sigma_-\,. \tag{6.5}$$

The normal derivatives of f, which are computed with respect to the outward pointing unit normal vector fields, have to be positive (resp. negative) restricted to Σ_+ (resp. Σ_-).

The zero modes of the fields in the representation $\mathbf{R}_q$ and $\overline{\mathbf{R}}_{-q}$ in the presence of a background Higgs vev $\phi = \mathrm{d}f$ are obtained from the twisted Laplacian

$$\Delta_f = \mathcal{D}\mathcal{D}^\dagger + \mathcal{D}^\dagger\mathcal{D} = \left(\mathrm{d}^\dagger\mathrm{d} + \mathrm{d}\mathrm{d}^\dagger\right) + q^2|\mathrm{d}f|^2 + q\sum_{i,j=1}^{3} (H_f)_{ij}\left[(a^i)^\dagger, a^j\right]\,, \tag{6.6}$$

where

$$\mathcal{D} = \mathrm{d} + q\mathrm{d}f\wedge\,, \qquad \mathcal{D}^\dagger = \mathrm{d}^\dagger + q\,\iota_{\mathrm{grad}f}\,, \tag{6.7}$$

and H_f is the Hessian of f. Furthermore we defined the raising/lowering operators

$$(a^i)^\dagger = \mathrm{d}x^i \wedge \,, \qquad a^i = \iota_{\partial_i} \,. \tag{6.8}$$

Note that $\mathcal{D}^\dagger$ is not necessarily adjoint to $\mathcal{D}$ on manifolds with boundary Σ as

$$\langle \mathcal{D}\alpha, \beta \rangle - \langle \alpha, \mathcal{D}^\dagger \beta \rangle = \int_\Sigma \bar{\alpha} \wedge *\beta \,. \tag{6.9}$$

Requiring appropriate boundary conditions fixes this problem. Consider a form α split into its tangential and normal component to the boundary

$$\alpha = \alpha_t + \alpha_n \,. \tag{6.10}$$

The tangential part α_t is defined as the pullback of α to the boundary and the normal part as $\alpha_n = \alpha - \alpha_t$. The boundary contribution is sensitive only to the tangential components i.e.

$$\int_\Sigma \bar{\alpha} \wedge *\beta = \int_\Sigma \bar{\alpha}_t \wedge *\beta_n = \sum_i \int_{\Sigma_i} \bar{\alpha}_t \wedge *\beta_n \,, \tag{6.11}$$

where we have used the fact that $(*\alpha)_t = *\alpha_n$. The two types of boundary conditions are

$$\begin{aligned} \text{Dirichlet}: \qquad & \alpha_t|_{\Sigma_i} = 0 \\ \text{Neumann}: \qquad & *\alpha_n|_{\Sigma_i} = 0 \,, \end{aligned} \tag{6.12}$$

which can be imposed on every boundary component Σ_i independently. Choosing one of the above boundary conditions for every Σ_i amounts to restricting the domains of the operators $\mathcal{D}$ and $\mathcal{D}^\dagger$ to an appropriate subspace of forms. Within the restricted domains, the operators then become adjoints to each other. Moreover, by restricting the domain of Δ_f to make it self-adjoint, we can identify the zero-modes of Δ_f with the elements of cohomology groups $H^*_{\mathcal{D}}(\mathcal{M}_3)$ using Hodge theory. We supply more details on the boundary conditions in appendix B.

A natural choice is to split the boundary conditions according to whether the normal derivative $\partial_n f$ is inward or outward pointing at a particular component of the boundary.

This is the unique choice of boundary conditions, which precludes localization of zero-modes on the boundary Σ. The relevance of this choice will become clear in section 7.4. Extending the set-up in [91] we first restrict the domains of d and $\mathrm{d}^\dagger$ to

$$\begin{aligned} D(\mathrm{d}) &:= \left\{\alpha \in \Omega^p(\mathcal{M}_3) \,\middle|\, \alpha_t|_{\Sigma_-} = 0 \text{ (Dirichlet)}\right\} \\ D(\mathrm{d}^\dagger) &:= \left\{\alpha \in \Omega^p(\mathcal{M}_3) \,\middle|\, {*\alpha_n}|_{\Sigma_+} = 0 \text{ (Neumann)}\right\} , \end{aligned} \tag{6.13}$$

i.e. we are imposing Neumann conditions on the positive boundary and Dirichlet conditions on the negative. Moreover, we define the domains of $\mathcal{D}$ and $\mathcal{D}^\dagger$ to be $D(\mathcal{D}) = D(\mathrm{d})$ and $D(\mathcal{D}^\dagger) = D(\mathrm{d}^\dagger)$. The corresponding boundary conditions on the metric Laplace operator are given as

$$D^{\text{matter}}(\Delta) = \left\{\alpha \in \Omega^p(\mathcal{M}_3) \,\middle|\, \alpha_t|_{\Sigma_-} = (\mathrm{d}^\dagger\alpha)_t|_{\Sigma_-} = 0 \quad \text{and} \quad *\,\alpha_n|_{\Sigma_+} = *(\mathrm{d}\alpha)_n|_{\Sigma_+} = 0\right\} , \tag{6.14}$$

where we set again $D^{\text{matter}}(\Delta_f) = D^{\text{matter}}(\Delta)$. Note that the d-complex and $\mathcal{D}$-complex are isomorphic (cf. appendix B), so they have isomorphic cohomology groups. In this case, the d-complex is restricted to forms which vanish on Σ_-. This computes the relative cohomology of the pair $(\mathcal{M}_3, \Sigma_-)$ [92] so we get

$$H^p_{\mathcal{D}}(\mathcal{M}_3) = H^p(\mathcal{M}_3, \Sigma_-)\,. \tag{6.15}$$

The sign of the U(1)-charge q is important. Changing it amounts to changing the sign of f, which inverts the signs of normal derivatives and consequently exchanges the boundary conditions imposed on the positive and negative boundaries, and we obtain the cohomology groups with respect to the positive boundary. In terms of the operators defined in section 4.3 changing the sign of q corresponds to computing the cohomology with respect to the operator $\bar{\mathcal{D}}$, which is isomorphic to the $\mathcal{D}$-cohomology but this time with respect to the conjugate representation $\overline{\mathbf{R}}$.

Returning to the analysis of the spectrum above, we have seen that it is computed by the relative cohomology with respect to the negatively charged boundary components. Clearly, $H^0(\mathcal{M}_3, \Sigma_-)$ vanishes since any constant function which vanishes on the boundary is identically zero. Moreover, by Lefschetz duality $H^3(\mathcal{M}_3, \Sigma_-)$ also vanishes. Therefore, the discussion from section 4.3 shows that the chiral fermions are counted by $H^1(\mathcal{M}_3, \Sigma_-)$, while the conjugate-chiral fermions are counted by $H^2(\mathcal{M}_3, \Sigma_-)$

$$\begin{aligned} \text{chiral}: &\qquad H^1(\mathcal{M}_3, \Sigma_-) \\ \text{conjugate-chiral}: &\qquad H^2(\mathcal{M}_3, \Sigma_-)\,. \end{aligned} \tag{6.16}$$

The net amount of chiral matter transforming in the representation $\mathbf{R}$ is therefore given by the relative Euler characteristic

$$\chi(\mathcal{M}_3, \Sigma_-) = b^2(\mathcal{M}_3, \Sigma_-) - b^1(\mathcal{M}_3, \Sigma_-), \tag{6.17}$$

where $b^1(\mathcal{M}_3, \Sigma_-)$ and $b^2(\mathcal{M}_3, \Sigma_-)$ are the dimensions of the respective cohomology groups. The Hodge star induces the isomorphism $H^*(\mathcal{M}_3, \Sigma_-) = H^{3-*}(\mathcal{M}_3, \Sigma_+)$, so that

$$\chi(\mathcal{M}_3, \Sigma_-) = -\chi(\mathcal{M}_3, \Sigma_+)\,. \tag{6.18}$$

We have seen that for an M_3 without boundary there is a 4d vector multiplet in the spectrum. Once we introduce sources along Γ and excise a tubular neighborhood around them, we need to check that the vector multiplets remain in the spectrum. Since these adjoint fields are uncharged under the U(1), the associated forms cannot have any tangential boundary conditions, and we impose purely normal boundary conditions. In this case the domain of the relevant Laplace operator becomes

$$D^{\text{gauge}}(\Delta) := \{\alpha \in \Omega^p(\mathcal{M}_3) \mid *\alpha_n|_\Sigma = *(\mathrm{d}\alpha)_n|_\Sigma = 0\}\,. \tag{6.19}$$

The kernel is then isomorphic to the de Rham cohomology groups [91] and we obtain the required zero modes for the vector multiplets in 4d.

6.2 Higher Rank Higgs bundles

Next we generalize to higher rank Higgs bundles in $G_\perp$. We still assume that $[\phi, \phi] = 0$. If we cannot diagonalize the Higgs bundle globally, (i.e. in the spectral cover language the spectral cover does not fully factor) then we still have a local description in terms of the Higgs field along the CSA, but not globally:

$$\text{globally on } M_3: \qquad \phi = \mathrm{d}f\,, \qquad \Delta f = \rho\,, \qquad \int_{M_3} \rho = 0\,, \tag{6.20}$$

$$\text{locally on } M_3: \qquad \phi = \mathfrak{t}^i \mathrm{d}f_i\,, \qquad \rho = \mathfrak{t}^i \rho_i \qquad \Delta f_i = \rho_i\,, \tag{6.21}$$

i.e. we can only diagonalize locally in a patch U of M_3. Here $f_i, \rho_i : U \to \mathbb{R}$ are functions, $n = \mathrm{rk}\, G_\perp$ and $\mathfrak{t}^i$ the generators of the CSA. Locally this background breaks the gauge symmetry into

$$\begin{aligned} \widetilde{G} \quad &\to \quad G \times \mathrm{U}(1)^n\,, \\ \mathrm{Ad}\,\widetilde{G} \quad &\to \quad \mathrm{Ad}\,G \oplus \mathrm{Ad}(\mathrm{U}(1)^n) \oplus \bigoplus_{Q=(q_1,\ldots,q_n)} \mathbf{R}_Q\,, \end{aligned} \tag{6.22}$$

where $Q = (q_1, \ldots, q_n)$ denotes a vector of U(1)-charges. If the spectral cover has $N+1$ irreducible components (as in (5.5)), N of these n U(1) factors descend to the gauge group of the 4d effective theory. The operator $\mathcal{D}$ defined in (4.36) acts on $\mathbf{R}_Q$ by

$$\begin{aligned} \mathcal{D}|_{\mathbf{R}_Q} &= \mathcal{D}_Q = \mathrm{d} + (q_1 \mathrm{d}f_1 + \cdots + q_n \mathrm{d}f_n) \wedge\,, \\ \mathcal{D}|^\dagger_{\mathbf{R}_Q} &= \mathcal{D}^\dagger_Q = \mathrm{d}^\dagger + \iota_{\mathrm{grad}(q_1 f_1 + \cdots + q_n f_n)}\,. \end{aligned} \tag{6.23}$$

Let us introduce

$$f_Q = q_1 f_1 + \cdots + q_n f_n\,. \tag{6.24}$$

The zero-modes are counted by (4.44) where $E = \mathrm{Ad}\, G_\perp$. If the spectral cover does not factor, i.e. the sheets mix under monodromy, the cohomologies of the operator $\mathcal{D}$ cannot be rewritten in terms of e.g. de Rham cohomologies. For the case of rank 1 Higgs bundles the isomorphism given between the corresponding complexes was given by conjugation with e^{qf} (this is explained in greater detail in appendix B). This required a globally defined function f whose role for fully reducible Higgs bundles is played by f_Q as we will explain in the next section. This isomorphism cannot be adapted in a straightforward manner to general Higgs bundles.

	$\mathrm{Ad}\, G$	$\mathrm{Ad}\, \mathrm{U}(1)^n$	$\mathbf{R}_Q$	$\overline{\mathbf{R}}_{-Q}$
Vector multiplets	1	1	0	0
Chiral multiplets	$b^1(\mathcal{M}_3)$	$b^1(\mathcal{M}_3)$	$b^1(\mathcal{M}_3, \Sigma_Q^-)$	$b^2(\mathcal{M}_3, \Sigma_Q^-)$

Table 2: The 4d $\mathcal{N} = 1$ matter content for a background given by a $\mathrm{U}(1)^n$ valued Higgs bundle whose spectral cover is fully factored. Here Σ_Q^- denotes the negative boundary of $\mathcal{M}_3$ with respect to the function f_Q. Note $b^1(\mathcal{M}_3) = b^2(\mathcal{M}_3)$ and $b^1(\mathcal{M}_3, \Sigma_Q^{\mp}) = b^2(\mathcal{M}_3, \Sigma_Q^{\pm})$.

Restricting $\mathcal{D}$ to $\mathrm{Ad}\, G$ or $\mathrm{Ad}(\mathrm{U}(1)^n)$, it is reduced to the exterior derivative

$$\mathcal{D}|_{\mathrm{Ad}\, G} = \mathcal{D}|_{\mathrm{U}(1)} = \mathrm{d}\,, \qquad \mathcal{D}|^{\dagger}_{\mathrm{Ad}\, G} = \mathcal{D}|^{\dagger}_{\mathrm{U}(1)} = \mathrm{d}^{\dagger}\,. \tag{6.25}$$

Vector and chiral multiplets transforming in these representations are thus simply counted by the zeroth and first Betti numbers of $\mathcal{M}_3$, respectively.

However, if the Higgs bundle diagonalizes globally, i.e. if we have rank $G_{\perp}$ many $\mathrm{U}(1)$ symmetries, then a simple generalization of the rank one case applies. The zero modes are counted with respect to

$$\mathcal{D} = \mathrm{d} + \mathrm{d}f_Q \wedge\,, \tag{6.26}$$

where f_Q is globally well-defined and a function. As a consequence the results of section 6.1 carry over upon making the replacement $qf \to f_Q$. $\mathcal{M}_3$ is obtained by excising the singularities of all the f_i and the boundary decomposes again into positive and negative parts

$$\Sigma = \Sigma_Q^+ \cup \Sigma_Q^-\,, \tag{6.27}$$

depending on whether $f_Q \to \pm\infty$ when approaching the excised charge. By (6.24) the charge vector can flip the sign of a boundary as seen by the individual functions f_i used to define f_Q, i.e. for differently charged representation $\mathbf{R}_Q$ each zero mode counting requires an alternate decomposition of the boundary. We therefore find the fermionic zero-mode

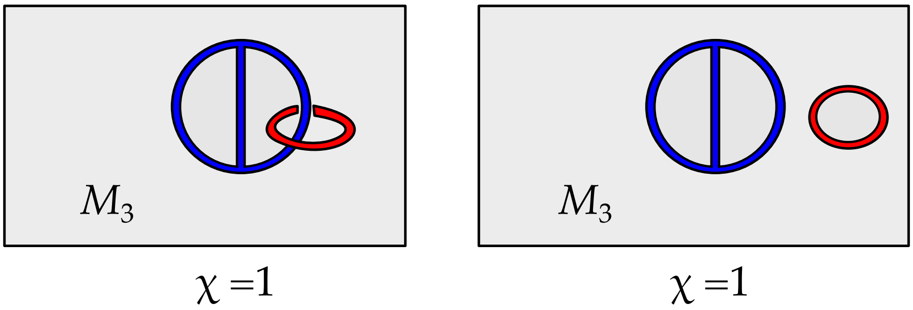

Figure 3: Examples of charged graphs in $\mathcal{M}_3$. Positive, negative charges are colored red, blue respectively. Both charge distribution give rise to the same chiral index but a different number of zero-modes.

spectrum in the representation $\mathbf{R}_Q$ to be enumerated by the relative Betti numbers

$$\begin{aligned} b^1(\mathcal{M}_3, \Sigma_Q^-) &= \text{chiral zero-modes in } \mathbf{R}_Q \,, \\ b^2(\mathcal{M}_3, \Sigma_Q^-) &= \text{conjugate-chiral zero-modes in } \mathbf{R}_Q \,. \end{aligned} \tag{6.28}$$

This parallels the identification of cohomologies as in (6.15). Each of these fermionic zero-modes contributes to a chiral multiplet upon reduction to 4d by supersymmetry. The CPT conjugate of the fermionic zero-modes enumerated by $b^2(\mathcal{M}_3, \Sigma_Q^-)$ will be of positive chirality in 4d and contribute to a chiral multiplet valued in $\overline{\mathbf{R}}_{-Q}$.

For the representations uncharged under any of the factors of U(1) we have $\mathcal{D} = d$ and their boundary conditions on $\mathcal{M}_3$ are chosen purely normal as in (6.19), and they are counted by de Rham cohomology. The complete 4d spectrum is summarized in table 2.

6.3 Example 1: Wires in $\mathbb{S}^3$

We now turn to describing concrete charge configurations — these configurations were studied in [49] and we revisit them here. Let $M_3 = \mathbb{S}^3$ and embed charges in $\mathbb{S}^3$ which are localized on a graph Γ. The positively and negatively charged components of the graph are disjoint $\Gamma = \Gamma_+ \cup \Gamma_-$. We denote by n_+, n_- the number of components and by ℓ_+, ℓ_- the number of loops of Γ_+, Γ_- respectively. The total charge on $\Gamma_\pm$ is again constrained to vanish. Excising tubular neighborhoods of $\Gamma_\pm$ we obtain $\mathcal{M}_3$ with associated boundaries $\Sigma_\pm$. By (6.15) the number of non perturbative chiral and conjugate-chiral zero-modes are then given by the Betti numbers $b^i(\mathcal{M}_3, \Sigma_-)$ for $i = 1, 2$ respectively. The top and bottom cohomologies vanish as discussed in section 6.1. The first and second cohomology are

$$\begin{aligned} b^1(\mathcal{M}_3, \Sigma_-) &= \ell_+ + n_- - r - 1 \\ b^2(\mathcal{M}_3, \Sigma_-) &= \ell_- + n_+ - r - 1\,, \end{aligned} \tag{6.29}$$

where r counts the number of negative loops which are independent in homology when embedded in $M_3 \setminus \Gamma_+$. The chiral index is then computed to be

$$\chi(\mathcal{M}_3, \mathbf{R}_q) = (n_+ - \ell_+) - (n_- - \ell_-)\,. \tag{6.30}$$

It solely depends on the charge configuration Γ and is independent of the number r. A chiral spectrum is therefore easily generated. Multiple charged graphs will give rise to the same spectrum. Another point to note here is that a non-trivial chiral index will only arise if for some sign of the charge, the number of loops and components is different, i.e. the charge distribution is not localized solely on a disjoint union of circles. This will later on give hints as to how to deform the Higgs bundles for TCS G$_2$-manifolds.

7 BPS-Configurations, Super-QM and Morse-Bott Theory

In the discussion above we were not interested in any particular features of the harmonic function f on $\mathcal{M}_3$ and the computation of the spectrum is in fact valid for any such f. In this section we first specialize to the case where the Higgs field $\phi = \mathrm{d}f$ has isolated, non-degenerate zeros, which is the same as requiring f to be a Morse function – this is the case already studied in [49]. We show that massless chiral matter is localized at the zeros of the Higgs field. We then generalize this to the case where f can have critical loci of dimension one, in which case it is Morse-Bott. The latter will be essential for the TCS geometries. The main tool here is reformulating the problem of finding the kernel of the Laplacian Δ_f in terms of a supersymmetric quantum mechanics and Morse theory. This approach is useful as it lends itself to the generalized Morse-Bott setup that we are interested in.

7.1 Matter, Morse and (Super-Quantum-) Mechanics

Let us consider again the abelian case where $\phi = \mathrm{d}f$ with f harmonic in the decomposition (6.1), which counts the fermionic zero-modes transforming in the representation $\mathbf{R}_q$, that are in the kernel of

$$\Delta_f = \mathcal{D}\mathcal{D}^\dagger + \mathcal{D}^\dagger\mathcal{D} = \left(\mathrm{d}^\dagger\mathrm{d} + \mathrm{d}\mathrm{d}^\dagger\right) + q^2|\mathrm{d}f|^2 + q\{\mathrm{d}, \iota_{\mathrm{grad}\,f}\} + q\{\mathrm{d}^\dagger, \mathrm{d}f\wedge\}\,. \tag{7.1}$$

The twisted Laplacian Δ_f can be interpreted as the Hamiltonian of a supersymmetric quantum mechanics (SQM) with the target space $\mathcal{M}_3$ where the supercharges are given by the operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ [4]. In section 4.3 we have shown that (due to the partial topological twist) the state space is identified with the space of differential forms on $\mathcal{M}_3$. However, since $\mathcal{M}_3$ is now a manifold with boundary, we have to restrict the state space to forms satisfying the boundary conditions given in (6.13), which we denote by $\mathcal{H} = \oplus_i \Omega^i_b(\mathcal{M}_3)$. The subscript b indicates that the forms satisfy the boundary conditions. The function f now plays the role of a superpotential and the kernel of Δ_f characterizing the true zero modes in the reduction to 4d is now enumerating the supersymmetric ground states of the SQM [4]. In summary:

4d Effective Theory	SQM
Matter fields	State Space
$\mathcal{D}$, $\mathcal{D}^\dagger$	Supercharges
Δ_f	Hamiltonian
Higgs field $\phi = \mathrm{d}f$	$f =$ Superpotential
Matter zero modes	Ground states

As in Witten's analysis, we can now use perturbation theory to compute the zero mode spectrum. To compute the perturbative kernel of Δ_f, rescale $f \mapsto tf$. In terms of the electrostatics problem (6.2), this amounts to rescaling the charges globally by a factor of t, which does not alter the overall ground state count. The term $q^2|\mathrm{d}f|^2$ in (7.1) scales quadratically in t. Hence, for large t, the solutions of the equation $\Delta_{tf}\psi = 0$ are localized at the points where $\mathrm{d}f = 0$ i.e. the zeros of the Higgs field ϕ.

In this discussion we focus on harmonic functions f which are Morse. The local physics will then be given by a supersymmetric harmonic oscillator. Before continuing with the computation we recall the definition of a Morse function. A smooth function $f : \mathcal{M}_3 \to \mathbb{R}$ is called Morse if its set of critical points

$$N = \{p \in \mathcal{M}_3 : \mathrm{d}f(p) = 0\} \tag{7.2}$$

is discrete and all points $p \in N$ are non-degenerate. A critical point $p \in N$ is called non-degenerate if its Hessian $H_f(p)$ is non-degenerate as a bilinear map. In this case $p \in N$ is assigned a number $\mu(p)$ called the Morse index given by the number of negative eigenvalues of $H(p)$

$$p \in N : \qquad \mu(p) = |\{c = \text{eigenvalue of } H_f(p);\ c < 0\}|\ . \tag{7.3}$$

In the case of manifolds with boundary, we further assume that there are no critical points of f on $\partial\mathcal{M}_3$. Note that this is true in our case, since the normal derivative of f at the boundary is non-zero (see section 6.1). For more details on Morse theory we refer the reader to [14, 93].

We can choose a normal coordinate system in which f and the metric g on $\mathcal{M}_3$ take the form

$$\begin{aligned} f(x) &= f(0) + \frac{1}{2}\sum_{i=1}^{3} c_i (x^i)^2 + \mathcal{O}((x^i)^3)\,, \\ g_{ij}(x) &= \delta_{ij} + \mathcal{O}((x^i)^2)\,, \end{aligned} \tag{7.4}$$

where we assumed that $p = 0$ and c_i are the eigenvalues of the Hessian, which due to the harmonicity of f sum to zero. This means that only points with Morse index 1 and 2 can occur. Expanded in these coordinates Δ_{tf} reduces to the Hamiltonian of a supersymmetric harmonic oscillator with

$$\Delta_{tf} = \sum_{i=1}^{3}\left(-\frac{\partial^2}{\partial(x^i)^2} + q^2t^2c_i^2(x^i)^2 + qtc_i[\mathrm{d}x^i, \iota_{\partial/\partial x^i}]\right) + \mathcal{O}((x^i)^3)\,. \tag{7.5}$$

Solving for the ground states of the harmonic oscillator locally, near a critical point p of

Morse index $\mu(p)$, we find a unique solution given by a differential form of degree $\mu(p)$. The zero modes of ψ, which are identified with 1-forms in (4.44), localize at critical points of Morse index 1. For c_i with signature $(-,+,+)$, the solution to leading order is

$$\mu(p) = 1: \quad \psi = \psi_{(p,q)} \exp\left(-qt\sum_{i=1}^{3}|c_i|(x^i)^2\right) \mathrm{d}x^1 \,. \tag{7.6}$$

In other words the form part is oriented along the negative eigenspaces of the Hessian of the function f. Here we have decomposed the 7d spinor ψ into a Weyl-spinor $\psi_{(p,q)}$ carrying the anti-commuting, gauge and 4d spinor structure and its internal profile along $\mathcal{M}_3$. The index (p,q) indicates the point p, where the corresponding perturbative ground state localizes and q keeps track of the charge of $\mathbf{R}_q$. The boundary conditions we described in section 6.1 are exactly such that the solutions of (7.6) collected from all critical points of f of Morse index 1 span the complete perturbative kernel of Δ_f at degree 1 [91].

If p has Morse index 2, the ground state localized near p is of degree 2 and letting c_i have signature $(-,-,+)$, the solution is

$$\mu(p) = 2: \quad \bar{\psi} = \bar{\psi}_{(p,q)} \exp\left(-qt\sum_{i=1}^{3}|c_i|(x^i)^2\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \,. \tag{7.7}$$

Likewise the fermions in $\overline{\mathbf{R}}_{-q}$ are obviously counted by replacing f with $-f$.

7.2 Exact Spectrum from SQM

The perturbative calculation in the previous section does not necessarily give the exact spectrum of the full theory. On the SQM side this is due to the fact that quantum mechanical instanton corrections can cause perturbative ground states to acquire a mass and be lifted in the full theory [4, 14]. We now complete the dictionary between the 4d effective theory of 7d SYM and SQM by showing that masses of perturbative zero modes in the 4d theory arise precisely from instanton corrections on the SQM side.

We start our analysis by splitting the complexified Higgs field (4.9) and splitting $\varphi = \varphi_0 + \delta\varphi$ into its background $\varphi_0 = t\mathrm{d}f$ and fluctuations $\delta\varphi$. The 7d fields are expanded in terms of a basis of perturbative ground states of the twisted Laplacian as

$$\begin{aligned} \psi(x,y) &= \psi_{(a,q)}(x)\psi^{(a,q)}(y)\,, \\ \varphi(x,y) &= t\mathrm{d}f(y) + \delta\varphi(x,y) = t\mathrm{d}f(y) + \delta\varphi_{(a,q)}(x)\delta\varphi^{(a,q)}(y)\,, \end{aligned} \tag{7.8}$$

where $(x,y) \in \mathbb{R}^{1,3} \times \mathcal{M}_3$. Here, the sum runs over the charged representations, $\mathbf{R}_q$ and $\overline{\mathbf{R}}_{-q}$, and all critical points p_a of Morse index 1 with respect to the relevant Morse function, f and $-f$ respectively. The fermionic field $\psi_{(a,q)}(x)$ carries the anti-commuting, gauge and 4d spinor structure while $\psi^{(a,q)}(y)$ is a 1-form on $\mathcal{M}_3$ annihilated by the twisted Laplacian in perturbation theory. In leading order in t these are (7.6) or the CPT conjugate of (7.7). The decompositions for $\delta\varphi$ are of analogous structure.

A mass term in 4d descends from the 7d interaction

$$\mathrm{Tr}\,[\psi \wedge \mathcal{D}\psi] = \mathrm{Tr}\,[\psi \wedge (\mathrm{d}\psi + [\varphi\wedge, \psi])]\ , \tag{7.9}$$

which for an abelian Higgs background yields the mass matrix

$$M^{ab} = \int_{\mathcal{M}_3} \psi^{(a,-q)} \wedge (\mathrm{d} + tq\mathrm{d}f\wedge)\psi^{(b,q)} = \int_{\mathcal{M}_3} \bar{\psi}^{(a,q)} \wedge *(\mathrm{d} + tq\mathrm{d}f\wedge)\psi^{(b,q)}\,. \tag{7.10}$$

This precisely computes the instanton corrections between the perturbative ground states in SQM theory and is simply the matrix element

$$M^{ab} = \langle \psi^{(a,q)} | \mathcal{D}\psi^{(b,q)} \rangle\ . \tag{7.11}$$

Let us briefly summarize the classic results on these instanton corrections, see [4, 14] for a detailed treatment. The (Euclidean) action of the SQM with target space $\mathcal{M}_3$ is given by a standard sigma-model action

$$\begin{aligned} S_{\mathrm{SQM}} = \int_{\mathbb{R}} ds \Bigg(&\frac{1}{2} g_{ij} \frac{d\gamma^i}{ds}\frac{d\gamma^j}{ds} + \frac{q^2t^2}{2} g^{ij}\partial_i f \partial_j f \\ &+ g_{ij}\bar{\eta}^i D_s \eta^j + qt D_i\partial_j f \bar{\eta}^i\bar{\eta}^j + \frac{1}{2}R_{ijkl}\eta^i\bar{\eta}^j\eta^k\bar{\eta}^l \Bigg)\ , \end{aligned} \tag{7.12}$$

where g_{ij} is the metric on $\mathcal{M}_3$, D the covariant derivative and R_{ijkl} the curvature tensor.

Canonically quantizing this action, one gets the SQM we have described in the previous section [4]. The matrix element (7.11) now has the following path integral expression

$$\begin{aligned}\langle\psi^{(a,q)}|\mathcal{D}\psi^{(b,q)}\rangle &= \frac{1}{qf(p_a)-qf(p_b)+O(1/t)}\lim_{T\to\infty}\langle\psi^{(a,q)}|e^{T\Delta_{tf}}[\mathcal{D},f]e^{-T\Delta_{tf}}\psi^{(b,q)}\rangle \\ &= \frac{1}{qf(p_a)-qf(p_b)+O(1/t)}\int_{\substack{\gamma(+\infty)=p_a\\ \gamma(-\infty)=p_b}} D\gamma D\eta D\bar{\eta}\,[\mathcal{D},f]e^{-S_{\mathrm{SQM}}}\,,\end{aligned} \tag{7.13}$$

which is valid to leading order in $1/t$. The path integral is taken over the space of all trajectories γ connecting the critical point p_b to p_a, where $\mu(p_b)=1$ and $\mu(p_a)=2$. The integrand $[\mathcal{D},f]$ is $\mathcal{D}$-exact and hence the path integral receives contributions only from fixed points of the fermionic variations generated by the corresponding supercharge $\mathcal{D}$. Such fixed points are given by trajectories γ

$$\frac{\mathrm{d}\gamma^i}{\mathrm{d}s} = tqg^{ij}\partial_j f\,, \tag{7.14}$$

which is the gradient flow equation. With this the mass matrix is evaluated in [14] to leading order in $1/t$ as

$$M^{ab} = \sum_\gamma n_\gamma e^{-tq(f(p_a)-f(p_b))}\,. \tag{7.15}$$

Here the sum runs over all ascending gradient flow lines γ starting at p_b and ending at p_a. The contribution from a flow line γ is weighted by a sign $n_\gamma=\pm1$, which arises from a choice of orientation on the moduli space of gradient trajectories. The precise derivation from the SQM context is intricate and is given in [94, Appendix F]. The main takeaway is that perturbative ground states form a complex, where the coboundary operator is given by

$$\mathcal{D}\psi^{(b,q)} = \sum_a M^{ab}\bar{\psi}^{(a,q)}\,. \tag{7.16}$$

This is exactly the Morse-Witten complex for the Morse function f. Massless states are counted by the cohomology of this complex and can be found by diagonalising M^{ab}. Recall from section 6.1 that f is a solution of an electrostatics problem and satisfies $\partial_n f<0$ (resp. $\partial_n f>0$) on Σ_- (resp. Σ_-). The Morse-Witten complex therefore recovers the relative cohomology of a pair $(\mathcal{M}_3,\Sigma_-)$ [95]. In 4d these give rise to $b^1(\mathcal{M}_3,\Sigma_-)$ chiral multiplets

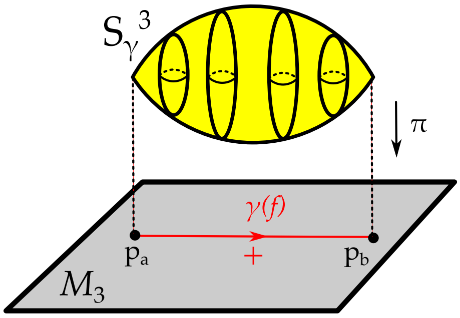

Figure 4: The supersymmetric three-cycle responsible for mass terms. The two critical points p_a and p_b of the function Morse f in the base of the ALE-fibration $\mathcal{M}_3$ are connected by a gradient flow line $\gamma(f)$. Above each point along this path there is a two-spheres in the ALE-fiber. Traversing along a gradient flow line of f a 3-sphere $\mathbb{S}^3_\gamma$ is traced out.

valued in $\mathbf{R}_q$ and $b^2(\mathcal{M}_3, \Sigma_-)$ chiral multiplets valued in $\overline{\mathbf{R}}_{-q}$.

It is possible that the boundary operator of the Morse-Witten complex is trivial. This is equivalent to a vanishing of the mass matrix $M^{ab} = 0$, i.e. all perturbative ground states are true ground states. In this case the Morse function f is called perfect. This is precisely the case when f has $b^i(\mathcal{M}_3, \Sigma_-)$ critical points of Morse index i, for $i = 1, 2$.

We can consider these mass terms also in the M-theory picture. In section 4.2 we have interpreted the Higgs field $\phi = \mathrm{d}f$ as measuring the periods of the vanishing cycle in an ALE-fibration, with respect to a reference hyperkähler structure. For an abelian Higgs field there is exactly one such vanishing cycle which is of finite volume through-out $\mathcal{M}_3$ and collapses precisely at the critical points of f. As this vanishing cycle is a two-sphere, paths connecting two critical points lifts to a 3-sphere in the ALE geometry. This 3-sphere is of minimal volume whenever it projects to a gradient flow line in $\mathcal{M}_3$. This is depicted in figure 4.

M2-instanton wrapped on such a three-sphere $\mathbb{S}^3_\gamma$ reduces to SQM [96, 97]. The stationary points of the M2-brane action, which correspond to associative three-cycles, hence become a fibration of the vanishing cycle of the ALE-fiber over the gradient trajectories $\gamma(f)$ determined by the Morse function f. These associatives then give a non-perturbative correction to the superpotential [96, 97] which is of the form

$$\Delta W = n_\gamma \exp\left(i \int_{\mathbb{S}^3_\gamma} (C + i\Phi) \right) . \tag{7.17}$$

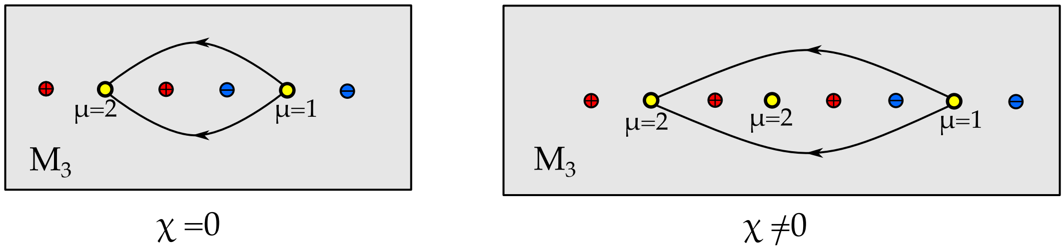

Figure 5: Examples of point like charge configurations in $\mathcal{M}_3$. Depicted are positive (red) and negative (blue) charges, critical points (yellow) and flow lines starting and ending at critical points. The critical points have Morse index μ. The contributions of the flow lines cancels and for generic set-ups each critical point will give rise to a ground state of positive, negative chirality if $\mu = 1, 2$ respectively. The LHS thus has an equal number of chiral and conjugate-chiral ground states, the chiral index vanishes. For the same reasons the chiral index does not vanish on the RHS.

In particular, the coefficients originating from a one-loop determinant in the M2-brane action are the same as the those computed in the supersymmetric quantum mechanics and hence give the same coefficients $n_\gamma = \pm 1$ as those appearing in the Morse theory analysis. In the case of several flow lines connecting the same critical loci p_a, p_b, the corresponding associatives are homologous and there can hence be cancellations among the different contributions depending on the relative orientation.

7.3 Example 2: $n_+ + n_-$ Point Charges in $\mathbb{S}^3$

We apply the analysis of section 7.1 and 7.2 to point charges on the three-sphere. Example configurations are shown in figure 5. Let $M_3 = \mathbb{S}^3$ and $\widetilde{G} = \mathrm{SU}(n+1)$. Consider $n_\pm$ positive/negative point charges with the total charge vanishing. The function $f : M_3 \to \mathbb{R}$ is the electrostatic potential generated by these charges. This function gives rise to a singular abelian Higgs field background on $\mathbb{S}^3$ via $\phi = \mathrm{d}f$ which breaks

$$\mathrm{Ad}\,\mathrm{SU}(n+1) \to \mathrm{Ad}\,\mathrm{SU}(n) \oplus \mathrm{Ad}\,\mathrm{U}(1) \oplus \mathbf{n}_q \oplus \overline{\mathbf{n}}_{-q}\,, \tag{7.18}$$

Perturbative ground states localize at the critical points of the harmonic function f. Let n_μ be the number of points with Morse index μ, then there are n_1 chiral fermions ψ and n_2 conjugate-chiral fermions $\bar{\psi}$ transforming in $\mathbf{n}_q$. The harmonicity of f forbids points of Morse index 0 or 3 as these are minima or maxima respectively. The chiral index as defined (4.49) is given by the difference

$$\chi\left(\mathbb{S}^3, \mathbf{n}_q\right) = n_2 - n_1\,, \tag{7.19}$$

as perturbative ground states are lifted by M2-brane corrections in pairs leaving the difference of ground states of positive and negative chirality unchanged.

Next smear out the charges to small balls so that the singularities of f are removed without altering f away from the support of the charge distribution. In this case $\operatorname{grad} f$ becomes a smooth vector field on M_3 and the Poincaré-Hopf theorem can be applied. We denote the critical points of f by x_i, then the topological index $I(x_i, f)$ of $\operatorname{grad} f$ at x_i is determined by the topological index of the map

$$\frac{\operatorname{grad} f}{|\operatorname{grad} f|}: \quad \mathbb{S}^2_{x_i} \to \mathbb{S}^2\,, \tag{7.20}$$

where $\mathbb{S}^2_{x_i}$ is a small ball containing the critical point x_i. The Poincaré-Hopf theorem asserts that the sum of all indices is the Euler characteristic of $M_3 = \mathbb{S}^3$

$$\sum_i I(x_i, f) = \chi\left(\mathbb{S}^3\right) = 0\,. \tag{7.21}$$

Note that $I(x_i, f) = (-1)^{\mu(x_i)}$ for all critical points x_i and that each charge contributes one maximum or minimum upon smearing it out, whereby (7.21) simplifies to

$$0 = n_- - n_1 + n_2 - n_+\,. \tag{7.22}$$

Combining this result with (7.19) we find the chiral index to be determined solely by the composition of the initial charge configuration

$$\chi\left(\mathbb{S}^3, \mathbf{n}_q\right) = n_+ - n_-\,. \tag{7.23}$$

We thus find a rather simple criterion to determine whether the true ground state spectrum of the theory is chiral or not:

$$n_+ \neq n_- \quad \leftrightarrow \quad \text{chiral spectrum}\,. \tag{7.24}$$

Two examples are shown in figure 5. This result is of course recovered from the more general charge distributions discussed in section 6.3 upon setting the number of loops l_+ and l_- to zero. In particular for generic placements of the $n_+ + n_-$ charges one has

$$n_1 = n_- - 1\,, \qquad n_2 = n_+ - 1\,. \tag{7.25}$$

Each critical point thus constitutes a true ground state and we recover (6.29). This is made explicit in figure 5. If flow lines between critical points exist, they always do so in pairs with $n_\gamma = \pm 1$. Hence the corresponding ground states are not lifted.

7.4 Generalized Critical Loci and Morse-Bott Theory

The setup studied in [49] and in the last section assumes that the critical loci of the function f are isolated points. Although this is the generic situation, it will be important to relax this assumption and consider the generalized setup in which the critical locus of f can be one-dimensional, which happens for the recent TCS constructions of G_2-manifolds. Functions f with critical loci of dimension greater than zero whose Hessian at its critical closed submanifold is non-degenerate in the normal direction are called Morse-Bott functions. An example is given in figure 6. For further background on this see [14,98].

The starting point is once more an abelian Higgs field $\phi = \mathrm{d}f$ as in section 7.1 where now f is taken to be a harmonic Morse-Bott function. We are again interested in the fermionic zero modes transforming in the representation $\mathbf{R}_q$ which are in the kernel of the twisted Laplacian (7.1). As before, rescaling $f \to tf$ these localize on the critical loci of f and we can solve for the zero mode solutions locally. However, f now has higher dimensional critical loci and our previous analysis needs to be adapted. We begin by analyzing the critical loci of f.

The local analysis of the perturbative ground states is now the same as in section 7.1, although some extra care is required to keep track of the critical loci of different dimensions.

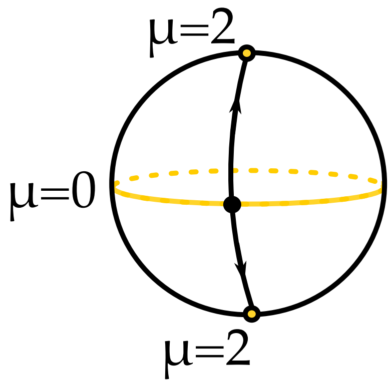

Figure 6: $\mathbb{S}^2$ with the Morse-Bott function given by $f(x, y, z) = z^2$. The critical locus is colored in yellow and consists of two critical points of index 2 (north ant south pole) and a critical circle of index 0 (the equatorial circle). The gradient curves are depicted in black. Note that $\mathcal{M}(N_0, N_2) = \mathbb{S}^1 \coprod \mathbb{S}^1$. These two circles parametrize the gradient trajectories in the upper and lower hemisphere.

The critical locus of f splits into connected components all of which are compact closed submanifolds of $\mathcal{M}_3$. Let N denote a single connected component. The normal bundle νN splits into the positive and negative eigenspace of the Hessian H_f of f

$$\nu N = \nu_+ N \oplus \nu_- N \tag{7.26}$$

and the Morse index of N is defined as the rank of $\nu_- N$. In our context the Morse-Bott function f is also harmonic. This precludes critical submanifolds of dimension 2 since harmonicity of f implies that $\operatorname{Tr} H_f = 0$, which would mean that H_f is degenerate in the normal direction, which is not possible since f is Morse-Bott by assumption. For harmonic Morse-Bott functions on a three-manifold, N can thus only be a point or a circle. Moreover, if $N = \mathbb{S}^1$, it can only have index 1. This is again due to the requirement that $\operatorname{Tr} H_f$ vanishes everywhere. The case where N is a point has been analyzed in section 7.1.

If $N = \mathbb{S}^1$ we can proceed analogously. As N has index 1, f is locally of the form

$$f(x) = f(0) - \frac{c}{2}\left((x^1)^2 - (x^2)^2\right) + O((x^i)^3)\,, \tag{7.27}$$

in a suitable normal coordinate chart centered at a point $p \in N$. In this coordinate system x^3 is the coordinate tangential to N and the Hessian H_f is diagonalized with the eigenvalues c and $-c$. In these coordinates the twisted Laplacian (7.1) now takes the form

$$
\begin{aligned}
\Delta_{tf} &= (\Delta_{tf})_\perp + (\Delta_{tf})_\parallel + \mathcal{O}((x^i)^3)\,, \\
(\Delta_{tf})_\perp &= \sum_{i=1}^{2}\left(-\frac{\partial^2}{\partial(x^i)^2} + q^2t^2c^2(x^i)^2\right) - qtc[\mathrm{d}x^1, \iota_{\partial/\partial x^1}] + qtc[\mathrm{d}x^2, \iota_{\partial/\partial x^2}]\,, \\
(\Delta_{tf})_\parallel &= -\frac{\partial^2}{\partial(x^3)^2}\,.
\end{aligned}
\tag{7.28}
$$

The analysis of perturbative ground states thus splits into normal and tangential parts relative to N. In the normal direction we get a single 1-form solution $\psi_\perp$ given by

$$
\psi_\perp = \psi_{(n,q)} \exp\left(-qtc\left((x^1)^2 + (x^2)^2\right)\right) \mathrm{d}x^1\,. \tag{7.29}
$$

Here we have split $\psi_\perp$ into a 4d Weyl spinor $\psi_{(n,q)}$ carrying the anti-commuting, gauge and spinor structure and its internal profile normal to N. In principle $\psi_\perp$ is defined only locally on N. However, observe that $\psi_\perp$ is a volume form on the fiber of $\nu_- N$. Hence, assuming that the negative eigenbundle $\nu_- N$ is orientable, the local solutions can be patched together to a global form on N. Since f is constant on N the tangential equation reduces to a Laplace equation on $\mathbb{S}^1$. Let the coordinate on the circle be θ. Then we obtain two solutions

$$
\psi_1 = \psi_\perp\,, \qquad \psi_2 = \mathrm{d}\theta \wedge \psi_\perp\,. \tag{7.30}
$$

For every circle N contributing to the perturbative spectrum we therefore obtain a pair of states consisting of a 1- and 2-form. From (4.44) we know that the degree of the ground state correlates with the 4d chirality of fermions, i.e. the state described by a 1,2-form has positive, negative chirality upon a reduction to 4d. These fermionic states again contribute to chiral multiplets in 4d.

As in the case of Morse functions, perturbative zero modes for $\chi, \bar{\chi}$ transforming in $\mathbf{R}_q$ are absent as f is harmonic. To conclude we again remark that the analysis above extends to fermionic ground states transforming in $\overline{\mathbf{R}}_{-q}$ by replacing f with $-f$. The function $-f$ now exhibits the same critical loci. A critical point of Morse index μ with respect to f has a Morse index of $\mu - 3$ with respect to $-f$, however critical circles exhibit an unchanged Morse index of 1 with respect to both f and $-f$. The modes localizing on the critical circles of $-f$ transforming in $\overline{\mathbf{R}}_{-q}$ are CPT conjugate to the solutions found in (7.30). As a consequence we find the localized perturbative ground states on every critical circle contributing to the perturbative spectrum to assemble to two chiral multiplets transforming in $\mathbf{R}_q$ and $\overline{\mathbf{R}}_{-q}$.

7.5 Generalized Critical Loci and SQM

We now turn to the computation of the exact spectrum from the perturbative solutions in the Morse-Bott case, where the critical loci of f consist of points and circles. While it is possible to compute the SQM instanton correction in much greater generality [14, 98], the applications for TCS local models allow us to consider only the set-up with this restriction. The instanton calculation in this case effectively reduces to the one considered in section 7.2.

To find the exact spectrum, we again want to compute the matrix element (7.15) between perturbative zero modes localized at critical submanifolds we use the analogous SQM computation. Let N_m denote the disjoint union of critical submanifolds of Morse-Bott index m (recall that this is the dimension of the negative eigenspace of the Hessian matrix). In our case, m can take the values 1 or 2. For $m = 2$, all of the components of N_2 must be points, whereas N_1 can contain points as well as circles.

Recall that among the ground states localized at critical circles there are chiral multiplets transforming in the representation $\mathbf{R}_q$ and $\overline{\mathbf{R}}_{-q}$. As already discussed in section 7.4, this is because perturbative ground states are of the form

$$\psi = \alpha \wedge \psi_{\perp}, \tag{7.31}$$

with $\deg(\psi_{\perp}) = 1$ and α a harmonic form on N_1. When N_1 is a circle, α can be a function or a one-form. Consider again the matrix element

$$M^{ab} = \int_{\mathcal{M}_3} \bar{\psi}^{(a,q)} \wedge *(\mathrm{d} + tq\mathrm{d}f\wedge)\psi^{(b,q)} \,. \tag{7.32}$$

Here we again use the indices a and b to enumerate all the perturbative ground states of total degree 2 and 1 respectively. However, note that for Morse-Bott functions the index is no longer in one-to-one correspondence with critical loci since there are two perturbative ground states localized at each critical $\mathbb{S}^1 \subset N_1$. For the following we will require the assumption that there are no ascending gradient flow lines between connected components in N_1.[12]

To compute M^{ab} we need to consider three cases. First, both $\bar{\psi}^{(a,q)}$ and $\psi^{(b,q)}$ may be localized at points in which case the discussion of section 7.2 applies verbatim. We now turn to the second possibility, where the ground states are both localized at the same circle critical circle $\mathbb{S}^1 \subset N_1$. The matrix element is then given by the integral

$$\int_{\mathcal{M}_3} \mathrm{d}\theta \wedge \psi_\perp \wedge *(\mathrm{d} + tq\mathrm{d}f\wedge)\psi_\perp \,, \tag{7.33}$$

where we have used the explicit expression of for such ground states given in (7.30). Using the expression for $\psi_\perp$ in (7.29) one can see that $d\theta \wedge *(\mathrm{d}f \wedge \psi_\perp) = 0$ and also $\mathrm{d}\theta \wedge \psi_\perp \wedge *\mathrm{d}\psi_\perp = 0$. This implies that the matrix element M^{ab} is zero, if $\bar{\psi}^{(a,q)}$ and $\psi^{(b,q)}$ are both localized at the same circle.

The third possibility is that $\bar{\psi}^{(a,q)}$ is localized at a point p_a in N_2 and $\psi^{(b,q)}$ is localized at a circle $\mathbb{S}^1_b \in N_1$. To keep track of all of the gradient curves between critical loci of f, we introduce the moduli space of gradient trajectories between N_m and N_n Here, the quotient is taken with respect to the remaining reparametrization invariance of the gradient flow: $\gamma(t) \mapsto \gamma'(t) = \gamma(t + \delta t)$. The moduli space $\mathcal{M}(N_m, N_n)$ is a smooth manifold, and it follows from simple dimensional analysis that its dimension is $m - n - 1$. An illustrative example is given by $\mathbb{S}^2$ with the Morse-Bott function $f(x, y, z) = z^2$, see figure 6.

For our purposes, the only relevant case is $m = 1$ and $n = 2$ in which case the moduli space is a finite set of points. This means that there are finitely many gradient trajectories connecting N_1 and N_2 and there are finitely many ascending gradient flow lines connecting $\mathbb{S}^1_b$ and p_a. We can now continue with the computation. In terms of the SQM path integral

[12] In this case f is said to be weakly self-indexing. This assumption can be avoided at a cost of making the exposition much more technical [98].

we have the expression

$$M^{ab} = \langle \psi^{(a,q)} | \mathcal{D} \psi^{(b,q)} \rangle = \frac{1}{qf(p_a) - qf(p_b) + O(1/t)} \int_{\substack{\gamma(+\infty)=p_a \\ \gamma(-\infty)\in\mathbb{S}^1_b}} D\gamma D\eta D\bar{\eta}\, [\mathcal{D}, f] e^{-S_{\mathrm{SQM}}}\,, \tag{7.34}$$

where p_b is an arbitrary point in $\mathbb{S}^1_b$ (note that f is constant along $\mathbb{S}^1_b$). This is nearly the same expression as in (7.13), with the only difference being that we integrate over all curves with $\gamma(-\infty) \in \mathbb{S}^1_b$. However, the same localization argument as before applies. As we have seen above, the number of gradient trajectories is still finite and the result of the path integral computation has exactly the same form as for points, i.e. (7.15). The expression for the operator $\mathcal{D}$ also remains unchanged

$$\mathcal{D}\psi^{(b,q)} = \sum_a M^{ab} \bar{\psi}^{(a,q)}\,. \tag{7.35}$$

The exact spectrum is given as the cohomology of $\mathcal{D}$, which acts on the following complex

$$C^1 = \Omega^0(N_1)\,, \qquad C^2 = \Omega^1(N_1) \oplus \Omega^0(N_2)\,. \tag{7.36}$$

This complex is a convenient way to arrange all the perturbative ground states of degree p in C^p. It is a specific instance of a Morse-Bott complex for f, which can be defined for f with critical loci of arbitrary dimension [98]. If f is a solution to the electrostatics problem in section 6.1, the Morse-Bott cohomology again recovers the relative cohomology of a pair $(\mathcal{M}_3, \Sigma_-)$.

7.6 Chiral Index from Spectral Covers

We close this section by introducing yet another picture for counting the perturbative zero modes, namely using the spectral cover introduced in section 5. For certain configurations it is possible to read off the exact spectrum using the spectral cover, this was already observed for the U(1) case in [49].

For simplicity let us begin by recalling the statement for the rank 1 Higgsing in (7.18) where $\widetilde{G} = \mathrm{SU}(n+1)$. There we turned on a single abelian Higgs background parametrized by the Morse function f via $\phi = \mathrm{d}f$. The spectral cover $\mathcal{C}$ in this case is simply the graph of ϕ. The intersection number of $\mathcal{C}$ with the zero section $b_0 = 0$ (i.e. $\mathcal{M}_3$) at a critical point p is denoted by n_p. This can be identified with the degree of the vector field $\mathrm{grad}\, f$ at the critical point p. In a coordinate system where the Hessian H_f is diagonal it follows immediately that the degree is determined by the Morse index $\mu(p)$ of f at p as $n_p = (-1)^{\mu(p)}$. We can therefore recast the counting of perturbative ground states as

$$\begin{aligned} |(\mathcal{C} \cap \mathcal{M}_3)_-| &= \text{perturbative zero modes in } \mathbf{R}_q \\ |(\mathcal{C} \cap \mathcal{M}_3)_+| &= \text{perturbative zero modes in } \overline{\mathbf{R}}_{-q}\,, \end{aligned} \tag{7.37}$$

where $(\mathcal{C} \cap \mathcal{M}_3)_\pm$ counts the number of critical points p with $n_p = \pm 1$. The chiral index is thus simply given by the signed count of all points of intersection

$$\chi(\mathcal{M}_3, \mathbf{R}_q) = \mathcal{C} \cap \mathcal{M}_3 = \sum_{p \in \mathcal{M}_3 \,:\, \mathrm{d}f(p)=0} n_p = (\mathcal{C} \cap \mathcal{M}_3)_+ - (\mathcal{C} \cap \mathcal{M}_3)_- \,. \tag{7.38}$$

The above carries over straightforwardly to higher rank Higgs bundles if their corresponding spectral cover factors completely. We start from the set-up in which we have broken the gauge symmetry to $G \times \mathrm{U}(1)^n$ by turning on sources for the Higgs field along the CSA of $\widetilde{G}$ as in section 6.2. The representation $\mathrm{Ad}\,\widetilde{G}$ decomposes into irreducible representation $\mathbf{R}_Q$ of $G \times \mathrm{U}(1)^n$ where Q denotes a vector of U(1) charges. Generically the representation $\mathrm{Ad}\,\widetilde{G}$ decomposes into irreducible representation of $G \times G_\perp$ with the weights λ_i of the representation of $G_\perp$ determining the different spectral covers. Due to the special choice of background the representations of $G_\perp$ have decomposed into representations of $\mathrm{U}(1)^n$ and to construct the spectral cover we must group the representations $\mathbf{R}_Q$ according to this decomposition. This grouping depends on $\widetilde{G}$ but the weights will always be determined by the corresponding effective Morse functions as $\lambda_i = \mathrm{d}f_{Q_i}$ where $i = 1, \ldots, N$. The effective Morse function f_{Q_i} was defined in (6.24) and N denotes the rank of the spectral cover. A spectral cover is thus the union of graphs of multiple $\mathrm{d}f_{Q_i}$ and an N-fold covering of $\mathcal{M}_3$. The matter loci are as before the critical points of f_{Q_i}, i.e. the intersection of the spectral cover with the zero section. This is just $b_0 = 0$ in the language of section 5.

To compute the perturbative spectrum we thus just need to count the intersections of the different sheets with their signs as in the rank 1 case above. Let $\mathcal{C}_i \subset \mathcal{C}$ denote the sheet of a spectral cover $\mathcal{C}$ with $\text{Graph}(\mathrm{d}f_{Q_i}) = \mathcal{C}_i$ then

$$\begin{aligned}|(\mathcal{C}_i \cap \mathcal{M}_3)_-| &= \text{perturbative zero modes in } \mathbf{R}_{Q_i} \\ |(\mathcal{C}_i \cap \mathcal{M}_3)_+| &= \text{perturbative zero modes in } \overline{\mathbf{R}}_{-Q_i} \,,\end{aligned} \tag{7.39}$$

where the notation is as in (7.37). Similarly we compute the chiral index to

$$\chi(\mathcal{M}_3, \mathbf{R}_{Q_i}) = \mathcal{C}_i \cap \mathcal{M}_3 = (\mathcal{C}_i \cap \mathcal{M}_3)_+ - (\mathcal{C}_i \cap \mathcal{M}_3)_- \,. \tag{7.40}$$

Perturbative zero modes transforming a representation $\mathbf{R}_{Q_i}$ which is not associated by $\lambda_i = \mathrm{d}f_{Q_i}$ to a sheet of this spectral cover are enumerated by the intersection of the different sheets

$$\begin{aligned}|(\mathcal{C}_i \cap \mathcal{C}_j)_-| &= \text{perturbative zero modes in } \mathbf{R}_{Q_i - Q_j} \\ |(\mathcal{C}_i \cap \mathcal{C}_j)_+| &= \text{perturbative zero modes in } \overline{\mathbf{R}}_{-(Q_i - Q_j)} \,.\end{aligned} \tag{7.41}$$

The chiral index again given by the difference

$$\chi(\mathcal{M}_3, \mathbf{R}_{Q_i - Q_j}) = \mathcal{C}_i \cap \mathcal{C}_j = (\mathcal{C}_i \cap \mathcal{C}_j)_+ - (\mathcal{C}_i \cap \mathcal{C}_j)_- \,. \tag{7.42}$$

This is pictorially most clear in the case of A_n singularities. In this case the ALE-fiber is given by a circle fibration over $\mathbb{R}^3$ and the eigenvalues λ_i, which are characterized by the sheets of the spectral cover, correspond to the points at which the circle collapses. A vanishing sphere is stretched between any pair of these points and collapses whenever they come together, i.e. when the sheets intersect. This enhances the spectrum and constitutes an additional ground state.

8 Yukawa Couplings and Higher-Point Interactions

In this section we discuss the interactions of localized matter. It will be useful to consider the case of a fully factored spectral cover, in which case we can compute the zero-modes. In M-theory interactions between localized matter fields come from M2-instantons wrapped

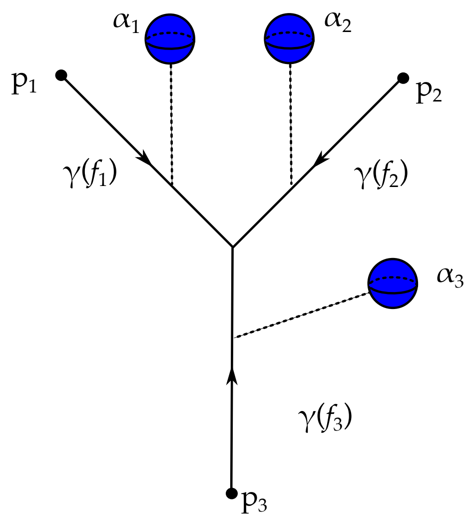

Figure 7: Gradient flow tree for Yukawa couplings. The picture shows three critical points p_i of the functions f_i of Morse index $\mu_{f_i}(p_i) = 1$. The gradient flow lines $\gamma(f_i)$ of the f_i are marked by arrows. Every f_i controls the size of a 2-cycle α_i which has the topology of a two-sphere and collapses over the points p_i. The three three-chains formed by fibering the two-spheres α_i over the segments $\gamma(f_i)$ can be joined at their meeting point as $\alpha_1+\alpha_2+\alpha_3 = 0$, and the resulting three-cycle is an associative.

on calibrated 3-spheres of the local ALE-fibration. This is simply a generalization of the results of section 7.2, where we interpreted non-perturbative mass terms as arising from M2-instantons wrapping three-cycles which connect two critical points over a gradient flow line. For higher point interactions these three-cycles project to gradient flow trees on $\mathcal{M}_3$ and studying the moduli space of these constrains the corresponding interactions in 4d. Corrections to these couplings are obtained from integrating out states with masses induced by M2-instantons as discussed in section 7.2.

Consider again the background as in section 6.2

$$\text{globally on } M_3: \qquad \langle \phi \rangle = \text{diag}(\lambda_1, \cdots, \lambda_n) = \sum_{i=1}^{n} \mathrm{t}^i \mathrm{d} f_i \,, \quad \Delta f_i = \rho_i \,, \quad \int_{M_3} \rho_i = 0 \,. \tag{8.1}$$

The matter content is summarized in table 2.

8.1 Yukawa Couplings

For Yukawa couplings we need a rank $n = 2$ Higgs bundle (or higher). There are two Morse functions f_1 and f_2 and the combination $f_Q = q_1 f_1 + q_2 f_2$. From the effective field theory we obtain this coupling by 4d effective action in perturbative zero modes

$$Y^{abc}_{pqr} = \int_{\mathcal{M}_3} \psi^{(a,p)} \wedge \varphi^{(b,q)} \wedge \psi^{(c,r)}\,, \qquad Q_p + Q_q + Q_r = 0\,, \tag{8.2}$$

where (a,p) refers to the internal profile of the perturbative zero mode localized at the critical point p_a transforming in $\mathbf{R}_{Q_p}$. The Yukawa couplings arise from M2-instantons wrapping associative three-cycles. To characterize the three-cycles consider the Morse functions

$$\begin{aligned} Q_1 &= (1,0)\,, & Q_2 &= (0,-1)\,, & Q_3 &= (-1,1) \\ f_{Q_1} &= f_1\,, & f_{Q_2} &= -f_2\,, & f_{Q_3} &= -f_1 + f_2 = f_3\,, \end{aligned} \tag{8.3}$$

which describe an SU(3) ALE-fibration over the base $\mathcal{M}_3$. Each of the functions f_i controls the volume of a corresponding two-sphere α_i in the ALE-fiber, which satisfy

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \tag{8.4}$$

in the homology of every fiber. Recall that α_i shrinks to zero volume precisely over the points p_i where $\mathrm{d}f_i = 0$. To every gradient trajectory $\gamma(f_i)$ starting at a point p_i we can associate a 3-chain, which is given by tracing out the corresponding α_i in the ALE-fibration. Given three sufficiently generic Morse functions f_i, there will be finitely many gradient flow trees connecting the three critical points p_i (see figure 7). Adding the associated three-chains produces a three-cycle, the boundary of which is given by $\sum_i \alpha_i$ in the ALE-fiber. We may produce a closed three-cycle with the topology of a three-sphere by adding a three-cycle β such that $\partial\beta = \alpha_1 + \alpha_2 + \alpha_3$. Moreover, this $\mathbb{S}^3$ is in fact associative, since it projects to the tree of gradient trajectories and hence minimizes the volume among all the three-cycles which project down to trees connecting p_1, p_2 and p_3. Wrapping an M2-brane on such a cycle gives rise to Yukawa couplings between modes localized at the critical points of f_1, f_2 and f_3. Consequently, the overlap integral (8.2) vanishes if there exists no trivalent gradient flow tree connecting the critical points[13].

[13]Massless chiral multiplets are found when expanding the 7d action in true zero modes. These are in general linear combinations of the localized perturbative profiles used in (8.2). The relevant linear

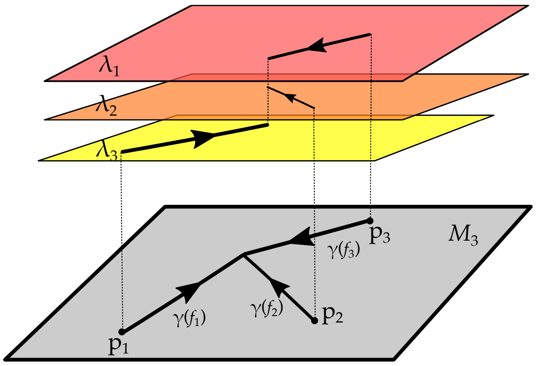

Figure 8: Construction of three-cycle that gives rise to the Yukawa couplings in the spectral Cover picture. The critical loci p_i correspond to the loci where two of the weights λ_j are equal, i.e. the corresponding sheets of the spectral cover meet. The uplift of the gradient flow lines $\gamma(f_i)$ sweeps out the associative three-cycle $\mathbb{S}^3$ that can then be wrapped by an M2-instanton. This gives rise to the coupling between the three matter states localized at $\lambda_i = 0$. The combined flow lines give rise to the gradient flow tree $\gamma(f_1, f_2, f_3)$.

Similarly, in the spectral cover description, the Yukawa coupling is modeled in terms of a three-sheeted cover, whoch is determined by the graph of $\mathrm{d}f_i$. The segments of the gradient flow trees determined by the function f_i thus lift to paths on the corresponding sheets; see figure 8. The paths connect the points where two sheets pairwise intersect. One can think of the 2-cycles α_j in the ALE-fibration as being stretched between the sheets and the corresponding cycle collapses precisely at points where two sheets meet.

The strength of these interactions is governed by the choice of functions f_i. The three-sphere giving rise to the Yukawa coupling is a supersymmetric rigid homology sphere within the G_2-manifold and its contribution to the superpotential is again given by (7.17). The sign $n_\gamma = \pm 1$ arises in the same manner and is given by an orientation on the moduli space of gradient flow trees. As the Higgs field ϕ_i and the gauge field W_i are identified with the periods of the supergravity 3-form C and associative 3-form Φ the integral is evaluated as

$$\int_{\mathbb{S}^3_\gamma} (C + i\Phi) = \sum_{j=1}^{3} \int_{\gamma(f_j)} \int_{\alpha_j} (C + i\Phi) = \sum_{j=1}^{3} \int_{\gamma(f_j)} (W_j + i\phi_j) = i \sum_{j=1}^{3} \int_{\gamma(f_j)} t\mathrm{d}f_{Q_j} \, , \quad (8.5)$$

Here, we have used that we can gauge the background for the gauge field W_i to zero. Evaluating the final integrals and using that the homological relation between the α implies

combinations are determined by the Morse-Witten complex. The overlap integral determining the Yukawa couplings between the massless modes are thereby linear combinations of (8.2).

$\sum_i^3 f_i = 0$ we find

$$\Delta W = n_\gamma \exp\left(-\sum_{i=1}^{3} t f_{Q_i}(p_i)\right) . \tag{8.6}$$

8.2 Associatives and Gradient Flow Trees

The generation of Yukawa couplings and mass terms from associative three-cycles which project to flow trees on M_3 has a natural generalization [99], which in the effective theory realizes higher point couplings.

We consider a setup in which $G_\perp = S[\mathrm{U}(1)^k]$, so that the Higgs background is described by k smooth Morse functions f_i. As the associated two-spheres α_i in the ALE-fiber sum to zero in homology, the same must be true of the functions f_i. Choosing a critical point p_i of each f_i with Morse index $\mu(p_i)$, one can define the moduli space of gradient flow trees

$$\mathcal{M}(M; f_1, \ldots, f_k\, ; p_1, \ldots, p_k)\, , \tag{8.7}$$

as the set of gradient flow trees with external vertices $p_1, \ldots, p_k$ such that the lines emanating from p_i are ascending gradient flow lines of f_i. These form the external edges of the gradient flow tree. Of course we also allow for internal vertices and edges. The flow of these is governed by the associated integral linear combinations of the f_i, which are in turn determined by a charge conservation constraint. This moduli space $\mathcal{M}$ has dimension

$$\dim \mathcal{M}(M; f_1, \ldots, f_k\, ; p_1, \ldots, p_k) = k - \sum_{i=1}^{k} \mu(p_i)\, , \tag{8.8}$$

and there are thus finitely many gradient flow trees connecting k points of Morse index 1. An example of a gradient flow tree for the case of $k = 5$ is shown in figure 9.

As before, we can construct a three-cycle by fibering the two-sphere α associated with the Morse function f over each segment $\gamma(f)$. This both guarantees that we end up with an associative, and also that α collapses at the end-points of the flow tree. Furthermore, the fact that we have a tree in M_3 implies that the resulting associative three-cycle has the topology of a three-sphere, so that it contributes to the effective superpotential. Using

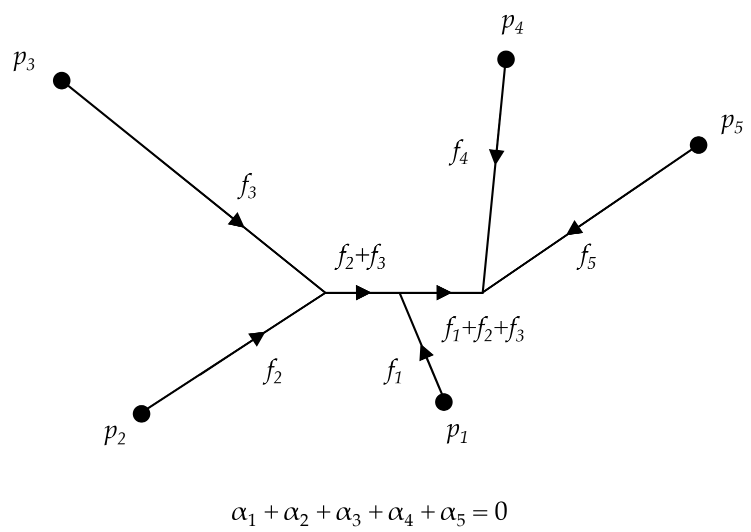

Figure 9: A gradient flow tree with 5 external vertices of Morse index 1.

the same manipulations as in (8.5), we can compute the volume of such a 3-sphere γ as

$$\mathrm{Vol}\,\gamma(f_1,\ldots,f_k;p_1,\ldots,p_k)=\sum_{i=1}^{k}f_i(p_i)\,, \tag{8.9}$$

so that the resulting contribution to the superpotential is

$$\Delta W=\frac{1}{M_\phi^{k-3}}\sum_\gamma n_\gamma e^{-\sum_{i=1}^{k}f_i(p_{a_i})}\,. \tag{8.10}$$

The scale M_ϕ is set by the vev of ϕ (see our discussion of this in section 10.1). Note that there can in general exist several flow trees connecting matter localized at the same loci p_i, which can cancel out.

The modes participating in the Yukawa (and higher) couplings are not just the massless states, but in fact all perturbative ground states of the SQM. Below the mass scale

$$M_{\mathrm{Inst}}\sim M_\phi e^{-tV} \tag{8.11}$$

induced by associates over flow lines between two points, we may integrate out the corresponding massive fields, thereby generating higher-dimensional operators in the effective field theory. As $M_{\mathrm{Inst}}\ll M_\phi$ these corrections are dominant compared to the couplings induced between the same fields by associatives. An example is shown in figure 10.

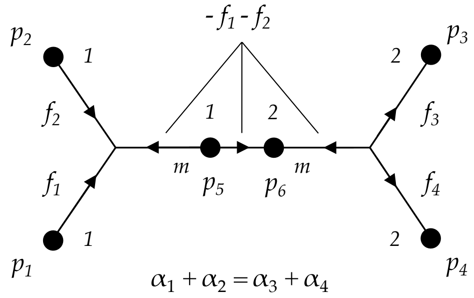

Figure 10: The figure shows how perturbative ground states participate in 4-point interaction between states localized at p_1, p_2, p_3, p_4. The gradient flow tree consists of two trivalent trees connected by a gradient flow line between p_5 and p_6. We have indicated the relevant Morse functions and Morse indices in the picture. The states localized at p_5 and p_6 are lifted by instanton corrections and develop a mass $m \sim M_{\rm Inst}$. In the 4d effective field theory this gives rise to a 4-point interaction.

For non-generic choices of the charge distribution the moduli space of gradient flow lines may increase and (8.8) is no longer valid. In this case the moduli space of gradient flow lines is not discrete but of dimension 1 and isomorphic to a circle. In the ALE geometry this corresponds to a continuous family of associative submanifolds. In [97] it is shown that the contribution of such a family $\mathcal{C}$ of associatives is proportional to $\chi(\mathcal{C})$.

In the more generic set-up of unfactored spectral cover of rank n the Higgs background can only be diagonalized locally as in (6.20). The source terms ρ in the BPS equations are now oriented arbitrarily along $G_\perp$ breaking the gauge symmetry to its commutant G in $\widetilde{G}$. Assuming we can diagonalizing the Higgs field in a tubular neighborhood T of the singularity as

$$U(x)\phi(x)U^{-1}(x) = \mathrm{diag}(\lambda_1(x), \ldots, \lambda_n(x)) \quad x \in T \supset \mathrm{supp}\,(\rho)\,, \tag{8.12}$$

we can impose boundary conditions on our field content as in section 6.2 and proceed with a local analysis. Chiral multiplets still localize at the vanishing points of the Higgs field and the boundary conditions again preclude perturbative zero modes from localizing at the boundary. Due to the mixing of the sheets of the spectral cover the background is no longer determined by a set of globally defined functions and we can not relate the cohomologies of $\mathcal{D}$ counting the zero modes to de Rham cohomologies. The local geometric picture however persists, all interactions are determined by three-cycles of the ALE geometry as in the previous sections with strengths determined by their volumes as in (8.6).

Finally, let us briefly comment on the case in which the critical loci are circles, i.e. we are allowing f_Q to be Morse-Bott. Perturbing the set-up slightly we return to the case of Morse theory. The ground states of (7.31) now decompose into multiple perturbative ground states

$$\alpha \wedge \psi_\perp \quad \to \quad \frac{1}{\sqrt{n}}\eta_i \quad i = 1, \ldots, n\,, \tag{8.13}$$

where we have assumed that the circle decomposes into $2n$ critical point of which n have Morse index 1 and n have a Morse index of 2. The forms η_i are 1, 2-forms depending on whether α is a 0, 1-form and localize at these critical points of Morse index 1, 2 respectively. We are further assuming that the states $\alpha \wedge \psi_\perp$ and η_i are of unit norm. After this perturbation, the previous analysis applies. The true ground state corresponding to $\alpha \wedge \psi_\perp$ is

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\eta_i\,. \tag{8.14}$$

Finally, let us apply these observations to the TCS constructions. In [46] a chain of string dualities was used to argue for the existence of infinitely many associatives on a class of TCS G$_2$-manifolds, and this result was recovered in an orbifold limit in [48]. These associatives are furthermore in one-to-one correspondence with elements of the lattice $E_8 \oplus E_8$. The local limits of these models must be such that $\tilde{G} = G_\perp = E_8 \times E_8$, and the associatives argued for in [46, 48] must correspond to flow trees in these local models. In fact, the description of the associatives in terms of string junctions in [46, 48] is already deceptively close to our description in terms of flow trees. It would certainly be interesting to flesh out this correspondence in detail.

9 Higgs Bundles and Twisted Connected Sum G$_2$-manifolds

In this section we consider local models associated with twisted connected sum (TCS) G$_2$-manifolds, which form the largest known class of examples of compact G$_2$-manifolds [36,37]. We reviewed the details of the construction in section 2.3. In a nutshell, the power of the TCS construction is that it shows how compact G$_2$-manifolds can be glued from simpler

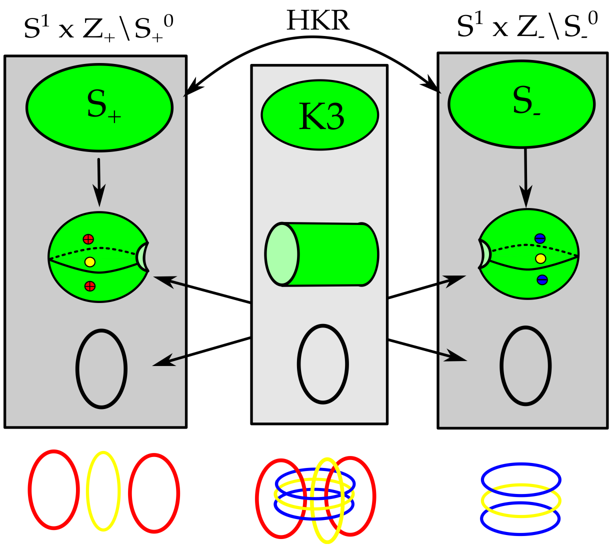

Figure 11: TCS construction of G_2-manifolds. Top: Building blocks that are aCyl Calabi-Yau and hyper-Kähler rotation (HKR) in asymptotic cyclindrical region. Bottom: Higgs bundle data. The critical loci of the Morse-Bott function f (in yellow) and the charge configuration ρ (in red and blue) corresponding to the local limit of a TCS G_2-manifold. The figure on the top shows the decomposition of $\mathbb{S}^3$ into $\mathbb{C}_\pm \times \mathbb{S}^1_\pm$ and the figure on the bottom shows the location of the same critical loci and charges in a patch $\mathbb{R}^3$ of $\mathbb{S}^3$. Every circle in $X_+ \times \mathbb{S}^1_+$ has linking number one with any of the circles in $X_+ \times \mathbb{S}^1_+$. Note that charge conservation requires that not all loops carry identical charges in this example.

building blocks, which can in turn be constructed using algebraic geometry. Although this makes finding examples relatively straight-forward, TCS G_2-manifolds appear to be a rather special class within the set of all G_2-manifolds [44]. Our analysis of local models for G_2-manifolds allows us to move away (at least in local models) from the TCS description and explore how to connect TCS G_2-manifolds to G_2-manifolds giving rise to chiral spectra.

9.1 TCS G_2-Manifolds

Recall that the basic ingredient for the twisted connected sum construction is a pair of algebraic threefolds $Z_\pm$, which each admit a K3 fibration over $\mathbb{CP}^1$ with a generic fiber $S_\pm$. Excising a generic fiber $S^0_\pm$ from $Z_\pm$ one obtains a pair of non-compact threefolds $X_\pm = Z_\pm \backslash S^0_\pm$, fibered over a punctured Riemann sphere, which are asymptotically cylindrical (aCyl) Calabi-Yau threefolds. Away from a compact submanifold, the $X_\pm$ have the topology of the cylinder $\mathbb{R}^+ \times \mathbb{S}^1_{b,\pm} \times S^0_\pm$ and the Ricci-flat metrics on $X_\pm$ asymptote to the Ricci-flat product metric on this cylinder. The situation is sketched in figure 11.

To form a compact G_2-manifold, the aCyl Calabi-Yaus $X_\pm$ are then multiplied by an extra circle $\mathbb{S}^1_{e,\pm}$ and glued together along a their cylindrical regions. The diffeomorphism used for the gluing exchanges the 'internal' circles $\mathbb{S}^1_{b,\pm}$ with the 'external' circles $\mathbb{S}^1_{e,\mp}$ and identifies the K3 surfaces $S^0_\pm$ by a diffeomorphism which induces a hyperkähler rotation, or Donaldson matching,

$$\begin{aligned} \mathrm{Re}(\Omega^{2,0}_\pm) &= \omega_\mp \\ \mathrm{Im}(\Omega^{2,0}_+) &= -\,\mathrm{Im}(\Omega^{2,0}_+)\,. \end{aligned} \tag{9.1}$$

Here, $\Omega^{2,0}_\pm$ and $\omega_\pm$ are the holomorphic $(2,0)$ forms and Kähler forms on $S^0_\pm$ which are induced by the complex structures on $X_\pm$. The compact topological manifold J resulting from this gluing then admits a metric with holonomy G_2, which is close to the Ricci-flat metrics on $X_\pm \times \mathbb{S}^1_{e,\pm}$. More precisely, there exists a limit, which we will call 'Kovalev limit', in which the cylindrical region becomes arbitrarily long and in this limit the Ricci flat G_2-holonomy metric approaches the Calabi-Yau metrics on $X_\pm$. In compactifications of M-theory, modes localized only on $X_+ \times \mathbb{S}^1_{e,+}$ or $X_- \times \mathbb{S}^1_{e,-}$ give rise to subsectors with enhanced $\mathcal{N} = 2$ supersymmetry in four dimensions. These subsectors are coupled such that they mutually only preserve $\mathcal{N} = 1$ supersymmetry, and we may think of the length of the cylindrical region as the inverse of their coupling [42, 45].

As both $X_\pm \times \mathbb{S}^1_{e,\pm}$ are fibered by K3 surfaces and the gluing acts separately on the fiber and base, J is (topologically) fibered by K3 surfaces as well. The base of this fibration is a three-sphere $\mathbb{S}^3$ glued together from two solid tori. We can hence think of the local models associated with TCS G_2-manifolds as describing an ALE space which is cut out from the K3 fiber over a base space M_3 which is $\mathbb{S}^3$. To engineer non-abelian gauge groups, every ALE-fiber of the local geometry and hence every K3 fiber of the associated compact G_2-manifold must be singular. It is straightforward to construct acyl Calabi-Yau threefolds in which every K3 fiber has a singularity of ADE type and the work of [44, 45] suggests that gluing such singular three-folds indeed results in a singular G_2-manifold.

Let us consider this in more detail. Denote the image of

$$\rho_\pm : H^2(Z_\pm, \mathbb{Z}) \to H^2(S^0_\pm, \mathbb{Z}) \tag{9.2}$$

by $N_\pm$. The Donaldson matching implies an idenfication of $H^2(S^0_+, \mathbb{Z}) \simeq H^2(S^0_-, \mathbb{Z})$. Using this map, every element of

$$\mathfrak{g} = N_+ \cap N_- \, , \tag{9.3}$$

gives rise to an associated harmonic two-form on J: the Poincaré dual cycle to such a form is algebraic for both S_+ and S_-, so that its fibration over the whole base $\mathbb{S}^3$ of J is trivial and it sweeps out a five-cycle, which is Poincaré dual to a two-form on J. The number of independent such two-forms on J is simply given by the rank of $\mathfrak{g}$ [37]. In compactifications of M-theory on J, there are hence $|\mathfrak{g}|$ massless U(1) vectors from the Kaluza-Klein reduction of the three-form C_3[14].

The hyper-Kähler structure on $S^0_\pm$ is forced by the Donaldson matching to be such that the integral of both $\Omega^{2,0}_\pm$ and $\omega_\pm$ vanishes for every cycle contained in $\mathfrak{g}$. This means that whenever there is a root, i.e. a lattice vector of length -2, contained in $\mathfrak{g}$, the K3 fibers $S^0_\pm$ are singular. As $\mathfrak{g}$ sits inside of the polarizing lattices[15] of the algebraic families $X_\pm$, this implies that every single K3 fiber has a singularity. The type of singularity can be read of by finding the sublattice $\mathfrak{g}_{\text{root}} \subset \mathfrak{g}$ generated by the roots of $\mathfrak{g}$. This sublattice must be a (sum of) ADE root lattice(s) and its type determines the corresponding singularity and the resulting simply-laced[16] non-abelian gauge group upon compactification of M-theory.

The matter loci in these models arise as the degeneration loci of the singular K3-fibration i.e. where the singularity worsens. This happens over points in $\mathbb{CP}^1_\pm$, each of which gets multiplied with a circle in the TCS construction. This implies that in M-theory compactification on a TCS manifold J, matter is localized along circles. This is true at least in the Kovalev (stretched neck) limit in which the metric on the J is well approximated by the metrics on each of the building blocks, which can be thought of as being contained

[14]There are in general further massless U(1) vectors associated with classes in the kernel of $\rho_\pm$ [37], which associated with the irreducible components of reducible fibers of the K3 fibrations on $Z_\pm$.

[15]The polarizing lattice of a family of K3 surfaces is the sublattice of $H^2(K3, \mathbb{Z})$ which is orthogonal to $\Omega^{2,0}$ for all members of the family.

[16]While this data is sufficient to find the singularities associated with simply-laced gauge groups, it is slightly more tricky to find non-simply laced gauge groups. Their emergence in TCS G_2-manifolds parallels their emergence in F-theory [100] in that the exceptional divisors of resolutions of ADE singularities of $S_\pm$ may globally become a single divisor in $X_\pm$ [44]. In terms of lattice data, this can be expressed by saying that a cycle of self-intersection $n < -2$ contained in $\mathfrak{g}$ can force an ADE singularity in every fiber if it is a linear combination of -2 curves in S_+ or S_- which are all in the polarizing lattices of the families S_+ and S_-. The difference between the polarizing lattices and $N_\pm$ determines the 'folding' of the ADE Dynkin diagram.

inside J (more precisely, the products $X_{\pm} \times \mathbb{S}^1_{\pm}$ are in J).

9.2 Higgs Bundles of TCS G_2-manifolds

We start by considering the local models of the two building blocks individually. As the discussion is the same for both sides, we will drop the $\pm$ subscripts. The first step is to replace the K3 fibration with a local ALE model. The precise details of this local limit depends on the ADE group corresponding to the type of ADE singularity, and are well known in the literature [101, 102]. Besides an ADE singularity, every ALE-fiber contains a number of compact cycles, the volumes of which vary over the base. Such cycles may collapse over points in the base $\mathbb{C}$. At these loci the singularity present in the generic fiber is enhanced and matter is localized. Let

$$\sigma \in H_2(S, \mathbb{Z}) \tag{9.4}$$

be such cycle which vanishes at (some) of the corresponding points in the base $\mathbb{C}$. Let us denote the hyperkähler triple by $\Theta = (\omega_I, \omega_J, \omega_K)$. In terms of (9.1) the hyperkähler structure is simply

$$\begin{aligned} \omega &= \omega_I \\ \Omega^{2,0} &= \omega_J + i\omega_K. \end{aligned} \tag{9.5}$$

After taking the local limit and integrating over σ we get a meromorphic function ϕ on $\mathbb{C}$:

$$\phi = \int_\sigma \Theta, \tag{9.6}$$

with zeros precisely where σ shrinks to zero volume. The poles and zeros of ϕ are located away from ∞. Moreover, we can identify ϕ with the meromorphic (1,0)-form as $\phi\, dz$. Since the base $\mathbb{C}$ is contractible $\phi = \mathrm{d}f$, where f is now a Morse function with critical points of index 1 and singular loci corresponding to the poles of ϕ. After taking a product with the circle we trivially get a Morse-Bott function.

If (unit) charges are placed at points $a_i \in \mathbb{C}$, the function ϕ will be of the form

$$\phi(z) = \sum_{i=1}^{n} \frac{1}{z - a_i} \,. \tag{9.7}$$

Therefore ϕ can have at most $n-1$ critical loci, which is generically the case. If we impose charge conservation on each side, there can be at most $n-2$ critical loci of ϕ.

With this information let us now consider how the Higgs bundle for TCS manifolds. After gluing $\mathbb{C}_+ \times \mathbb{S}^1_{e,+}$ with $\mathbb{C}_- \times \mathbb{S}^1_{e,-}$, the base manifold is $M_3 = \mathbb{S}^3$. In the Kovalev limit, the critical locus of the harmonic Morse-Bott function f consists of a disjoint union of m circles of Morse index 1. As before, we may engineer such an f by an appropriate configuration of charges Γ on $\mathbb{S}^3$. On $\mathbb{C}_\pm \times \mathbb{S}^1_\pm$, these charges will simply be given by a collection of points on $\mathbb{C}_\pm$ times the circle $\mathbb{S}^1_\pm$.

From the above discussion we only need the simple observation that matter loci in TCS G_2-manifold, at least in the Kovalev limit, are circles. Suppose that there are m matter circles and no points. Using the results of section 7.5 we see that the Morse-Bott complex is

$$C^1 = \Omega^0(\mathbb{S}^1)^m \,, \qquad C^2 = \Omega^1(\mathbb{S}^1)^m \,, \tag{9.8}$$

and the cohomology gives just

$$H^1(\mathcal{M}_3, \Sigma_-) \cong \mathbb{R}^m \,, \qquad H^2(\mathcal{M}_3, \Sigma_-) \cong \mathbb{R}^m \,. \tag{9.9}$$

We find that every perturbative ground state constitutes a true ground state, the Morse-Bott function f is thus perfect. As each circle gives rise to a pair of chiral and conjugate-chiral zero modes upon Kaluza-Klein reduction, the spectrum associated to this Higgs field configuration $\phi = \mathrm{d}f$ is non-chiral

$$\chi(\mathcal{M}_3, \mathbf{R}_q) = 0 \,. \tag{9.10}$$

We can use this result to derive constraints on the function f. By the above results the relative Euler characteristic $\chi(\mathcal{M}_3, \Sigma_-) = 0$ vanishes and by Lefschetz duality we find that this implies $\chi(\mathcal{M}_3, \Sigma_+) = 0$. We obtain the topological constraint

$$\chi(\mathcal{M}_3) = \chi(\Sigma_+) = \chi(\Sigma_-) = 0\,. \tag{9.11}$$

There has been a recent attempt to modify the TCS construction to yield singular G_2-manifolds with codimension 6 singularities by Chen [30]. Instead of smooth bulding blocks Chen takes the Z_+ building block to be a threefold with isolated nodal singularities, which means that the non-compact aCyl G_2-manifold $X_+ \times \mathbb{S}^1_{e+}$ has singularities in codimension 6. However, the standard TCS gluing argument does not work in this case; rather it is conjectured [30] that if circles of nodal singularities are replaced by pairs of isolated conical singularities it is possible to glue to a G_2-manifold with conical singularities using a modified version of the connected sum construction. In terms of the local model, the collapse of circles into points corresponds to deforming the Morse-Bott function to a generic Morse function, where the same collapse of critical circles to critical points occurs (recall that critical points correspond precisely to isolated singularities of the total space of the G_2-manifold). However, even if this conjecture is true, such G_2-manifold will still give rise to a non-chiral spectrum by the arguments above.

Finally, let us discuss the spectral covers for a TCS G_2-manifold which is given to us in terms of building blocks $X_\pm$ and a gluing map

$$\gamma: \qquad S^0_+ \rightarrow S^0_-\,, \tag{9.12}$$

where $S_\pm$ have ADE singularities over $U_\pm \subset \mathbb{C}_\pm$. This gluing of the K3 fibers in the TCS geometry also implies a consistent gluing map for the ALE-fibrations associated with the local model. In general, to be able to glue two given ALE-fibrations together, the two ALE-spaces need to be of the same type, $\tilde{G}_+ = \tilde{G}_-$, and furthermore the periods of the ALE-fibers must satisfy a matching condition. By Torelli theorem for ALE-spaces [103], the structure of an ALE-space is completely determined by the periods of the hyperkähler structure forms over the 2-cycles in the root lattice of the algebra of $\widetilde{G}$. Explicitly, the

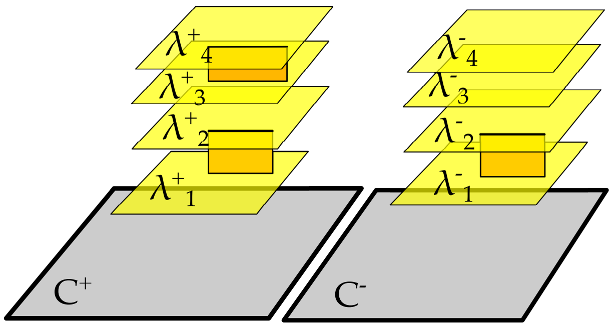

Figure 12: Each of the building blocks $X_\pm$ defines a spectral cover over $\mathbb{C}_\pm$, and these are then glued to a spectral cover over $\mathbb{S}^3$. In the example shown here, the cover on $\mathbb{C}_+$ factors into two components and the cover on $\mathbb{C}_-$ factors into three components. These covers are glued such that the resulting spectral cover over $\mathbb{S}^3$ has two components with two sheets each. Hence the resulting model has $G_\perp = S[\mathrm{U}(2) \times \mathrm{U}(2)]$ and there is a single unbroken $\mathrm{U}(1)$.

matching condition is

$$\int_{\sigma_j} \omega_{I,+} = \int_{\sigma_j} \omega_{J,-}\,, \qquad \int_{\sigma_j} \omega_{J,+} = \int_{\sigma_j} \omega_{J,-}\,, \qquad \int_{\sigma_j} \omega_{K,+} = -\int_{\sigma_j} \omega_{K,-}\,, \tag{9.13}$$

where σ_j are the 2-cycles generating the root lattice. Note that this implies that the non-abelian part of the group G, i.e. the type if ADE singularity, must be the same on both sides.

Each $X_\pm$ furthermore has a local model, which is a Higgs bundle $\phi_{(\pm)}$ over $\mathbb{C}_\pm$, and a corresponding spectral cover $\mathcal{C}_{(\pm)}$. The asymptotic values the Higgs fields $\phi_{(\pm),0}$ are similarly related by

$$\phi_{(+),0} = \gamma^* \phi_{(-),0}\,, \tag{9.14}$$

which induces a gluing of the spectral covers.

Let us explain the origin of U(1) gauge symmetries in glued spectral covers. Each cover $\mathcal{C}_{(\pm)}$ can have a factorization structure, which defines two-forms (five-cycles) and locally U(1) symmetries. This can be detected by the restriction of the map (9.2) to the ALE-fibrations over $\mathbb{C}_\pm$. Factorization of the spectral cover $\mathcal{C}$ over $\mathbb{S}^3$ after gluing $\mathcal{C}_{(\pm)}$ can likewise be detected by (9.3), and only those two-forms in the image that lie in the intersection will globally give rise to a two-form and thereby a U(1) symmetry. An example is shown in figure 12, where $\mathcal{C}_{(\pm)} \to \mathbb{C}_\pm$ each is a four-sheeted cover. However

$\mathcal{C}_{(\pm)}$ is factored into two (three), and thus locally gives one (two) U(1) symmetries. The gluing is such that the spectral cover $\mathcal{C} \to \mathbb{S}^3$ has only two factors, and thus only gives rise to a single U(1) symmetry. The scale at which the other U(1) is broken is set by the size of the neck region of the TCS-construction.

Besides an analysis via Higgs bundles, the matter spectrum of M-theory on TCS G_2-manifolds can also be found using a purely geometric reasoning. The geometry in the vicinity of each matter locus is that of a Calabi-Yau threefold times a circle. The local Calabi-Yau geometry is that of a fibration of an ADE singularity over $\mathbb{C}$ with a further degeneration at a point. Using the usual dictionary between singularities and gauge theory for M-theory or type IIA on Calabi-Yau threefolds, the Cartan generators and weight vectors can be identified with exceptional divisors and curves in the resolved Calabi-Yau geometry [60,89,104,105]. Our analysis of Higgs bundles now implies that the multiplicities must be such that each matter locus gives rise to a single vector-like pair of representations. Furthermore, we may determine the U(1) charges by simply integrating the two-forms in $\mathfrak{g}$ which give rise to the U(1)s over the exceptional curves of the resolution associated with the matter.

9.3 Deformation of TCS Higgs Bundles

Given the local model for TCS G_2-manifolds we now consider the behavior of the physics under deformations of the Morse-Bott function. We have seen above that circular critical loci arise in the non-generic $\mathbb{S}^1$-invariant distributions of charges in $\mathbb{S}^3$ which are present in the Kovalev limit. The natural question is what happens if this invariance is broken by a slight deformation. The strategy we will use to describe deformations is to exploit the construction of Morse(-Bott) functions in terms of charge distributions. For every charge distribution ρ, there is an associated Morse-Bott function, which in turn lifts to an ALE-fibration, our local model of a G_2-manifold. For every deformation of the charge distribution there is hence an associated deformation of the local model. Note that this deformation might be trivial: contrary to the number of deformations of Higgs bundles or the deformations of G_2-manifolds, which are finite in number, there are infinitely many deformations of any given charge configuration.

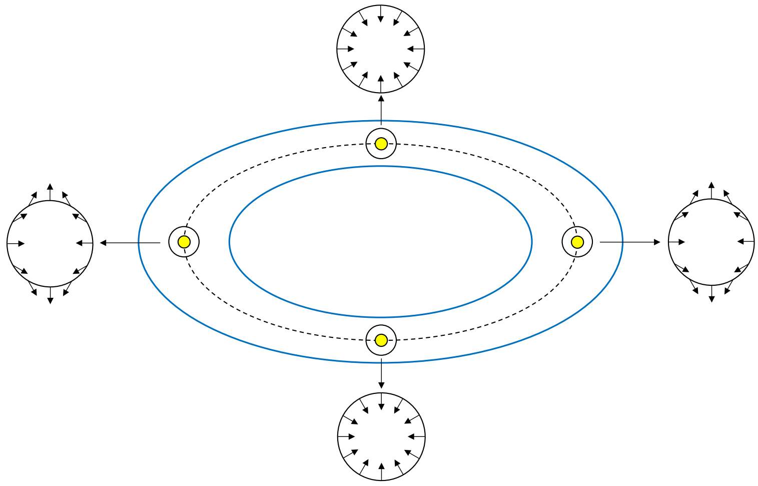

Figure 13: Two homogeneously negatively charged coplanar circles which have been stretched to ellipses. On the ellipse which is equidistant to the two charge loci, the electric field only vanishes at the four points marked as black dots. To find the Morse index, consider the small circles (coplanar to the ellipses) around these points and the restriction of the electric field to these circles. The Morse index of the points near the vertices of the ellipse is 2 and while those near the co-vertices have a Morse index of 1. As before, the critical loci are shown in yellow, and the charges in blue.

A configuration of charges which produces the Morse-Bott function associated with a TCS G$_2$-manifold in the Kovalev limit must of course be finely tuned, as a generic configuration of charges will always result in critical loci of dimension zero. Let us discuss this in a simple example — see figure 13: consider a charge distribution of two equally charged coplanar and cocentric circles in $\mathbb{R}^3$. This setup has rotational symmetry and correspondingly the critical locus is another coplanar and concentric circle. A generic deformation will destroy the rotational symmetry and lead to critical points instead of a circle. Consider e.g. deforming the charges to ellipses while preserving coplanarity. This collapses the critical locus to two points of Morse index 1 near the vertices of the ellipses and two points of index 2 near the co-vertices.

More generally, the function f will become Morse with isolated critical points for a generic deformation. However, since the topology of $\Sigma_\pm$ does not change, we still have

$$\chi(\mathcal{M}_3, \Sigma_-) = 0\,. \tag{9.15}$$

Physically this means that any deformation of f will give rise to chiral spectrum if under the deformation the topology of Σ_- remains unchanged. Denoting the number of points with Morse index i by m_i, the Morse inequalities for manifolds with boundary imply

$$\chi(\mathcal{M}_3, \Sigma_-) = m_2 - m_1 = 0\,. \tag{9.16}$$

Equally, every deformation of the local model of a TCS G$_2$-manifold that has an associated charge distribution which consists of a number of circles will satisfy $n_\pm = l_\pm$, so that the resulting spectrum is seen to be non-chiral.

9.4 Chirality and Singular Transitions

It is not at all surprising that TCS G$_2$-manifolds do not give rise to chiral spectra and that small deformations do not change the chiral index. However the result we have found already has fairly interesting geometrical implications: for a generic small deformation of a TCS G$_2$-manifold, the loci at which matter is localized are no longer one-dimensional but become point-like. This of course implies that the product structure of $X_\pm \times \mathbb{S}^1_{e,\pm}$ must be broken and the periods of the hyper-Kähler triplet on the K3 fiber must have a non-trivial dependence along $\mathbb{S}^1_{e,\pm}$. Although such small deformations will not yield G$_2$-manifolds giving rise to chiral spectra, the crucial ingredient, which are point-like singularities, is already present for small deformations of TCS G$_2$-manifolds.

Engineering the ALE-fibration from a Morse function which in turn is determined by a configuration of charges allows us to make the key observation for how to deform TCS G$_2$-manifolds to situation with chiral spectra: we need to make a transition after which $n_\pm = l_\pm$ no longer holds. The simplest way to do so is to bring two circles of equal charge together and then deform them to an object with $l = 2$. For TCS G$_2$-manifolds, there are essentially two different ways to achieve this.

The first option is to take e.g. a positive charge on $\mathbb{C}_+$ and another positive charge on $\mathbb{C}_-$, bring them together, and fuse them as shown in figure 14. As now $l_+ - n_+ = 1$ while $l_- - n_- = 0$, the resulting spectrum must be chiral. In a generic situation in which f is Morse, i.e. f only has isolated critical points, the critical locus of f hence consists of an o number of points now. As we started from a non-chiral configuration with an even

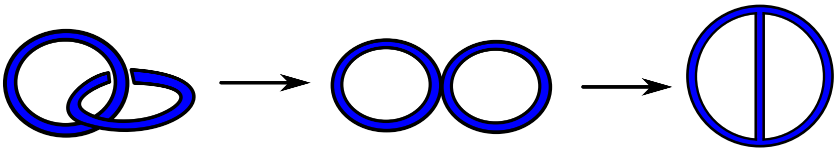

Figure 14: A transition of the charge configuration which results in a transition between a non-chiral and a chiral spectrum. Starting from a TCS configuration of charges deforming the configuration to one that results in a chiral spectrum.

number of critical points, this implies that some of the critical points must have fused. As the circles of positive charge we have fused originated from different ends of the TCS G_2-manifold we started from, the critical points which have fused must likewise originate from different ends. Geometrically, these critical points are nothing but degeneration loci of the K3 fibration of the G_2-manifold, so that we have effectively taken specific singular fibers of the K3 fibration into what used to be cylindrical region of the TCS and collided them. As expected from our earlier statement about the absence of chiral spectra in TCS G_2-manifolds, this signifies a definite departure from the TCS set-up, where the K3-fibration must be constant in the cylinder region.

In fact, the type of transition we have just sketched can also be anticipated from the heterotic duals of TCS G_2-manifolds, which are given by compactifications on the Schoen Calabi-Yau threefold with different vector bundles [44]. Such models always have non-chiral spectra and a singular transition connecting the Schoen Calabi-Yau threefold to a different Calabi-Yau threefold (together with appropriate vector bundles) is needed to find a chiral spectrum. The Schoen Calabi-Yau threefold can be described as a fiber product of two dP_9s, and it allows singular transitions in which a singular fiber of one dP_9 is collided with a singular fiber of the other dP_9. As discussed in [44, 46], the duality to a TCS G_2-manifold implies that the singular fibers of these two dP_9s are separated into disjoint regions of the common $\mathbb{CP}^1$ base. A collison between singular fibers from both ends translates to a collision of singular K3 fibers coming from the two separate ends X_+ and X_- of the dual TCS G_2-manifold.

The second option is to change the charge configuration corresponding to a TCS G_2-manifold by colliding two circles of equal charge which are both located in the same building block. The picture of such a deformation will be similar to the one in figure 14, however initially the charged circles will be unlinked. Again, it is clear that this signals a departure from a TCS G$_2$-manifold (and must result in a singular transition on the heterotic side as well): after the transition e.g. $X_+ \times \mathbb{S}^1_{e,+}$ must become a non-compact G$_2$-manifold without the structure of a product.

10 Higgs Bundles for G$_2$s: A User's Manual

We will now give a user-friendly summary of how to build local Higgs bundle models for G$_2$-manifolds, stripping off most of the mathematical baggage and condensing it to the essentials, which might be useful for the practitioners in the field.

10.1 Scales

Let us briefly discuss the mass scales in the problem, and specify what scale separation gives rise to the decoupling of gravity. For this purpose consider the gauge theory on the associative three-cycle M_3 on which the gauge degrees of freedom are localized. The compactification geometry determines the size of the cycle M_3 and the volume of the G$_2$-manifold J

$$\mathrm{Vol}(M_3) \sim R^3_{\mathrm{M}_3}\,, \qquad \mathrm{Vol}(J) \sim R^7_{\mathrm{G}_2}\,, \tag{10.1}$$

and the characteristic size $R_\perp$ of the directions transverse to M_3 in J is hence

$$R^4_\perp \sim R^7_{\mathrm{G}_2}/R^3_{\mathrm{M}_3}\,. \tag{10.2}$$

In terms of these length scales, the Plank masses M_4 in 4D and M_7 in 7D are given by

$$M_4^2 \sim M_{11d}^9 R^7_{\mathrm{G}_2}\,, \qquad M_7^5 \sim M_{11d}^9 R^4_\perp\,. \tag{10.3}$$

The scale M_ϕ at which $\tilde{G}$ is broken to G is set by the an appropriate average of the Higgs background. As the volumes of the compact cycles in the ALE-fiber over M_3 are set by $\langle\phi\rangle$, the local limit corresponds to the limit in which we can decouple gravity from the gauge theory degrees of freedom:

$$M_\phi \ll M_7\,. \tag{10.4}$$

Approximating the effective physics by the gauge degrees of freedom is only valid if the two scales M_ϕ and M_7 can be decoupled and permit the limit $M_7 \to \infty$ while keeping M_ϕ constant. Keeping ϕ fixed, this limit is equivalent to shrinking M_3 inside of J. Finally, the coupling of the 4D gauge theory G is given in terms of

$$\frac{1}{\alpha_{\mathrm{GUT}}} \sim M_{11d}^3 R_{M_3}^3 \,. \tag{10.5}$$

10.2 Matter Content and Interactions

Here we summarize the construction of some simple backgrounds and the resulting matter content and interactions.

1. Choose the rank n of the Higgs bundle, which is equal to the number of Cartan generators $\mathfrak{t}^i$ along which a Higgs field background $\mathrm{d}f_i$ has been turned on. Each of these n abelian directions is sourced by a charge distribution ρ_i of different type which determines the background function $f_i : M_3 \to \mathbb{R}$ completely, and must integrate to zero on M_3. The Higgs bundle is therefore given by

$$i = 1, \ldots, n \,: \qquad \phi = \mathfrak{t}^i \mathrm{d}f_i \,, \qquad \rho = \mathfrak{t}^i \rho_i \,, \qquad \Delta f_i = \rho_i \,, \qquad \int_{M_3} \rho_i = 0 \,. \tag{10.6}$$

2. This background breaks the gauge symmetry $\widetilde{G} \to G \times \mathrm{U}(1)^n$ and determines the count of representations in $\oplus_Q \mathbf{R}_Q$, where $Q = (q_1, \ldots, q_n)$. To count the zero modes in $\mathbf{R}_Q$ we need the effective charge distribution and its corresponding potential

$$\rho_Q = \sum_{i=1}^n q_i \rho_i \,, \qquad f_Q = \sum_{i=1}^n q_i f_i \,. \tag{10.7}$$

At every point in M_3 where $\mathrm{d}f_Q = 0$, there is a localized chiral multiplet transforming in $\mathbf{R}_Q$. For every flow line governed by f_Q between a pair of such points, there is a mass term for the associated chiral multiplets. Given a charge configuration ρ_Q, the resulting massless spectrum can be described in terms of the numbers $n_\pm^Q$ of positively and negatively charged component, and the total number $\ell_\pm^Q$ of loops.

The massless spectrum is counted by

$$\begin{aligned} \#\,\text{chiral multiplets in } \mathbf{R}_Q &= \ell_+^Q + n_-^Q - r^Q - 1\,, \\ \#\,\text{chiral multiplets in } \overline{\mathbf{R}}_{-Q} &= \ell_-^Q + n_+^Q - r^Q - 1\,, \\ \chi_Q &= (l_-^Q - l_+^Q) + (n_+^Q - n_-^Q)\,. \end{aligned} \tag{10.8}$$

Here r^Q denotes the number of negatively charged loops which are independent in homology when embedded into $\mathcal{M}_3 \setminus \rho_Q^+$.

3. Interactions between three 4d chiral fields localized at points p_s transforming in $\mathbf{R}_{Q_s}$ can only arise if $Q_1 + Q_2 + Q_3 = 0$ and if there exists a trivalent gradient flow tree between them. In general, there can be several such flow trees and cancellations between them can occur. Furthermore, there are mass terms for some of the localized zero modes and one needs to integrate out those massive fields to find the Yukawa couplings between the massless fields.

10.3 Retro-Model-Building 1: Top Yukawa

We close this section with two retro-inspired model building applications. First we will consider the top Yukawa coupling in an $\mathcal{N} = 1$ SU(5) toy model. We take $M_3 = \mathbb{S}^3$ and $\widetilde{G} = E_6$, i.e. two abelian directions parametrized by f_a and f_b are turned on. The corresponding decompositions read

$$\begin{aligned} E_6 &\rightarrow \mathrm{SU}(5) \times \mathrm{U}(1)_a \times \mathrm{U}(1)_b\,, \\ \mathbf{78} &\rightarrow \mathbf{1}_{0,0} \oplus \mathbf{1}_{0,0} \oplus \mathbf{1}_{-5,-3} \oplus \mathbf{1}_{5,3} \oplus \mathbf{24}_{0,0} \\ &\quad \oplus \mathbf{5}_{-3,3} \oplus \overline{\mathbf{5}}_{3,-3} \oplus \mathbf{10}_{-1,-3} \oplus \overline{\mathbf{10}}_{1,3} \oplus \mathbf{10}_{4,0} \oplus \overline{\mathbf{10}}_{-4,0}\,. \end{aligned} \tag{10.9}$$

The effective Morse functions are

$$\begin{aligned} \mathbf{5}_{-3,3}: &\qquad f_5 = -3f_a + 3f_b\,, \\ \mathbf{10}_{-1,-3}: &\qquad f_{10}^{(1)} = -f_a - 3f_b\,, \\ \mathbf{10}_{4,0}: &\qquad f_{10}^{(2)} = 4f_a\,. \end{aligned} \tag{10.10}$$

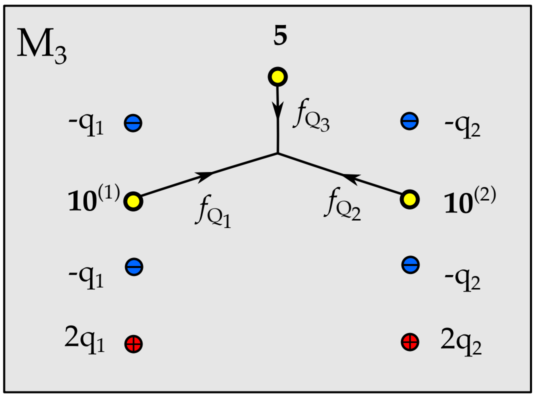

Figure 15: The point charge distributions ρ_1 and ρ_2 with two types of U(1) charges (1) and (2). Negative charges (blue) and positive charges (red) add to zero for both configurations and they each give rise to critical points (yellow) at which matter transforming in $\mathbf{10}^{(1)}$ and $\mathbf{10}^{(2)}$ is localized. The charge distribution $\rho_3 = -\rho_1 - \rho_2$ gives rise to a third critical point (yellow) at which matter transforming in $\mathbf{5}$ resides. A flow tree between these 3 critical points gives rise to a Yukawa coupling of type $\mathbf{10}^{(1)}\,\mathbf{10}^{(2)}\,\mathbf{5}$.

and they are determined by two independent charge distributions ρ_a and ρ_b. Using the linear combinations $\rho_1 = -\rho_a - 3\rho_b$ and $\rho_2 = 4\rho_a$ we can write the charge vectors Q_s as

$$Q_1 = Q_{f_{10}^{(1)}} = (1,0)\,, \qquad Q_2 = Q_{f_{10}^{(2)}} = (0,1)\,. \qquad Q_3 = Q_{f_5} = (-1,-1)\,. \tag{10.11}$$

In our model, we distribute $n_\pm^{(1)} + 1$ and $n_\pm^{(2)} + 1$ negative and positive point charges of types 1 and 2 throughout M_3. This yields generically $n_\pm^{(1)}$ and $n_\pm^{(2)}$ critical points of the functions f_{Q_1} and f_{Q_2} as seen in example 7.3. Of these $n_-^{(1)}, n_-^{(2)}$ have Morse index 1, and $n_+^{(1)}, n_+^{(2)}$ have Morse index 2 with respect to f_{Q_1} and f_{Q_2}.

An example is shown in figure 15. Here we have embedded 3 point charges of type 1 and 2 in M_3 such that their total charge vanishes. The charge configurations of type 1 and 2 exhibit a single critical point each (yellow), while their linear combination has a critical point (yellow) in the patch of M_3 depicted. Their interaction is determined by the gradient flow tree connecting the critical points and is of type $\mathbf{10}^{(1)}\,\mathbf{10}^{(2)}\,\mathbf{5}$. Denoting the critical points at which the $\mathbf{10}^{(1,2)}$ matter localizes by p_1, p_2 and the critical point at which the $\mathbf{5}$ matter localizes by p_3 we have to leading order in $1/t$

$$\lambda_{(\mathbf{10},\mathbf{10},\mathbf{5})} \sim e^{-t(f_{Q_1}(p_1)+f_{Q_2}(p_2)+f_{Q_3}(p_3))}\,, \qquad \mathrm{d}f_{Q_i}(p_i) = 0\,. \tag{10.12}$$

So far the analysis has only concentrated on a local patch in $\mathcal{M}_3$ and in principle there may exist further gradient flow trees connecting the points p_i in a complete model, and therefore further contributions to the $\mathbf{10}^{(1)}\,\mathbf{10}^{(2)}\,\mathbf{5}$ coupling.

10.4 Retro-Model-Building 2: And SU(5) GUT

Finally, we provide a full SU(5) GUT type model with all matter and Yukawa couplings. The goal is to construct three generations of chiral matter in $\mathbf{10}$ and $\overline{\mathbf{5}}$, as well as a pair of Higgs fields in the $\mathbf{5}_H$ and $\overline{\mathbf{5}}_H$, and the top and bottom Yukawa couplings

$$W_{\text{Yuk}} = \lambda_{\text{top}}\,\mathbf{10}\,\mathbf{10}\,\mathbf{5}_H + \lambda_{\text{bottom}}\,\overline{\mathbf{5}}\,\overline{\mathbf{5}}_H\,\mathbf{10}\,. \tag{10.13}$$

The top Yukawa was already engineered in the local $\mathrm{G_2}$ spectral cover in section 10.3. To break the GUT group the standard model gauge group can e.g. be achieved by turning on discrete Wilson lines. We will not discuss this here, but it would be interesting to incorporate this into the Higgs bundle framework, see also [90] for a discussion.

There are numerous ways in generating the chiral matter content of an SU(5) GUT model in the Higgs bundle setup. We will choose one that is minimal and requires only point charges. Furthermore to incorporate the superpotential couplings we will consider a Higgs field with $\widetilde{G} = E_8$ and $\langle\phi\rangle$ taking values in $G_\perp = S[\mathrm{U}(1)^5]$.

The Higgsing that we consider is a special case of

$$\begin{aligned} E_8 \quad &\rightarrow \quad \mathrm{SU}(5)_{\text{GUT}} \times \mathrm{SU}(5)_\perp \\ \mathbf{248} \rightarrow \quad &(\mathbf{24},\mathbf{1}) \oplus (\mathbf{1},\mathbf{24}) \oplus (\mathbf{10},\mathbf{5}) \oplus (\overline{\mathbf{5}},\mathbf{10}) \oplus (\overline{\mathbf{10}},\overline{\mathbf{5}}) \oplus (\mathbf{5},\overline{\mathbf{10}})\,. \end{aligned} \tag{10.14}$$

in which

$$E_8 \quad \rightarrow \quad \mathrm{SU}(5)_{\text{GUT}} \times \mathrm{U}(1)^4\,. \tag{10.15}$$

The way to parametrize the charges is in terms of the embedding of a $\mathrm{U}(1)^4$ into the Cartan subalgebra (CSA) of SU(5), where we have five generators $\mathfrak{t}_i$, which satisfy

$$\sum_{i=1}^{5} \mathfrak{t}_i = 0\,. \tag{10.16}$$

Any $U(1)_\alpha$ can be parametrized now as a linear combination

$$\mathfrak{t}_\alpha = \sum m_\alpha^i \mathfrak{t}_i \,. \tag{10.17}$$

The charges under the CSA generators $(\mathfrak{t}_1, \cdots, \mathfrak{t}_5)$ of the matter fields of the $SU(5)_{GUT}$ are

Matter Field	U(1)-Charges under CSA	Spectral Cover
$\mathbf{10}_i$	$(\delta_{i,n})_{n=1,\cdots,5}$	$\lambda_i = 0$
$\overline{\mathbf{5}}_{ij}\,,\ i > j$	$(\delta_{i,n} + \delta_{j,n})_{n=1,\cdots,5}$	$\lambda_i + \lambda_j = 0$
$\mathbf{1}_{ij}\,,\ i > j$	$(\delta_{i,n} - \delta_{j,n})_{n=1,\cdots,5}\,,\ i > j$	$\lambda_i - \lambda_j = 0$

(10.18)

We also include the loci where these matter fields are localized in $\mathcal{M}_3$ in terms of the λ_i, $i = 1, \cdots, 5$, with $\sum_i \lambda_i = 0$, which are the weights of the fundamental representation of the $SU(5)_\perp$. For each λ_i, we define a corresponding Morse function f_i with

$$\lambda_i = \mathrm{d} f_i \,. \tag{10.19}$$

Let us assign the matter points as follows

GUT Multiplet	SC realization	Locus
$\mathbf{10}_M^{(1)}$	$\mathbf{10}_1$	$\lambda_1 = 0$
$\mathbf{10}_M^{(2)}$	$\mathbf{10}_2$	$\lambda_2 = 0$
$\mathbf{10}_M^{(2)}$	$\mathbf{10}_3$	$\lambda_3 = 0$
$\mathbf{5}_H$	$\mathbf{5}_{23}$	$-(\lambda_2 + \lambda_3) = 0$
$\mathbf{5}_{\text{ex}}$	$\mathbf{5}_{45}$	$-(\lambda_4 + \lambda_5) = 0$
$\overline{\mathbf{5}}_M^{(1)}$	$\mathbf{5}_{24}$	$\lambda_2 + \lambda_4 = 0$
$\overline{\mathbf{5}}_M^{(2)}$	$\mathbf{5}_{15}$	$\lambda_1 + \lambda_5 = 0$
$\overline{\mathbf{5}}_M^{(3)}$	$\mathbf{5}_{35}$	$\lambda_3 + \lambda_5 = 0$

(10.20)

The top Yukawa coupling takes the form $\mathbf{10}_M^{(2)} \times \mathbf{10}_M^{(3)} \times \mathbf{5}_H$ as all other combinations are forbidden by the U(1) symmetries, i.e. there is no bottom Yukawa coupling, this must be generated beyond the set-up. There is one additional multiplet valued in **5** beyond the required matter for the GUT model. This example has one extra matter multiplet, and it would be interesting to see whether different charge configurations give rise to exactly the GUT spectrum.

To realize this spectrum via electrostatic charge distributions we translate the above into the language of section 7 and its higher rank generalizations. Note from (10.15) that 4 factors of U(1) have been broken off, this gives us 4 types of charge with which to build the model. Due to this special abelian background the decomposition in (10.14) decomposes further and the fundamental weights of $\mathrm{SU}(5)_\perp$ are now associated with fundamental charge

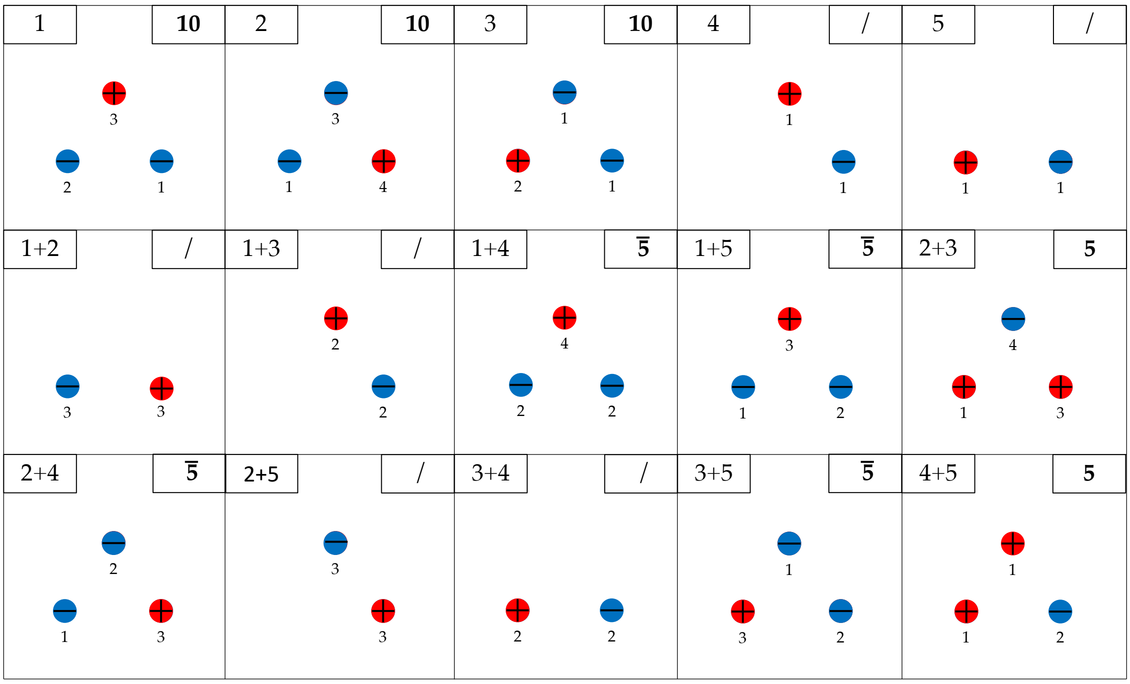

Figure 16: A charge configuration leading to a chiral spectrum with 3 multiplets transforming in **10**, 4 in $\overline{\mathbf{5}}$ and 2 in **5**. There are further multiplets transforming in **1** which are not depicted. Each box shows the same 3 points in M_3. The first row shows 5 fundamental charge distributions with the units of charge denoted as subscripts. The total charge of each box vanishes and also adding the first row of pictures yields a vanishing charge distribution. This reflects (10.21). The bottom two rows show superpositions of the fundamental charge distributions as noted in the top left corner. Each box contributes a single chiral multiplet, denoted in the top right corner of each box, if it depicts 3 charged points. This realizes the matter content as in (10.20).

vectors Q_i^F such that (10.19) now becomes

$$\lambda_i = \mathrm{d}f_i = \mathrm{d}f_{Q_i^F} \,, \qquad \sum_{i=1}^{5} Q_i^F = 0 \,. \tag{10.21}$$

Upon a redefinition of U(1) generators we may take these charge vectors to be

$$Q_5^F = (-1,-1,-1,-1) \,, \qquad Q_i^F = \delta_{ik} \quad k = 1,\ldots,4 \,. \tag{10.22}$$

Placing n_i^- negative and n_i^+ positive point charges of type $i = 1,\ldots,4$ we obtain by (7.25)

$$\begin{aligned} &n_i^- - 1 \text{ chiral multiplets transforming in } \mathbf{10}_{Q_i^F} \,, \\ &n_i^+ - 1 \text{ chiral multiplets transforming in } \overline{\mathbf{10}}_{-Q_i^F} \,, \end{aligned} \tag{10.23}$$

generically. The number of chiral multiplets valued in $\bar{\mathbf{5}}_M$ and $\mathbf{5}_H$ is fixed by these choices and computed by taking the relevant linear combinations of the charges as listed in (10.20).

The simplest charge configuration possible is obtained by collecting the four types of different charge at three distinct points in M_3 where $i = 1, 2, 3$. The only constraint on each charge configuration is that it must be of vanishing total charge. A fifth charge configuration is generated via the last relation in (10.21). We depict a possible distribution of point charges in figure 16. The charge distribution of this example yields a chiral spectrum with $3, 4, 2$ chiral multiplets transforming in the representations $\mathbf{10}, \bar{\mathbf{5}}, \mathbf{5}$ respectively. These multiplets reside at the critical points of the relevant combinations of the charge. The set-up allows for a single top Yukawa coupling. Clearly there is a lot of room to extend these models and improve them and it would be interesting to see the full extent of the phenomenological implications of this framework.

11 Conclusions and Outlook

The main result of this chapter is a study of the gauge sector of M-theory compactifications on G_2-holonomy manifolds to 4d $\mathcal{N} = 1$ supersymmetric gauge theories. The structure that governs this theory is a Higgs bundle on an associative three-cycle M_3, i.e. a gauge field W and a one-form Higgs field ϕ on M_3, satisfying the BPS equations (4.18). We have focused exclusively on the case of $W = 0$ and ϕ Higgs field, and have given a detailed description on how to engineer and analyze backgrounds satisfying the BPS equations for abelian Higgs fields. Furthermore, we have shown how to apply this formalism to the case of TCS G_2-manifolds. Although these are not interesting for phenomenological applications, we have qualitatively shown under which conditions singular transitions of such compactifications can give rise to chiral 4d spectra and thus bring us somewhat closer to the main open question in this field, i.e. the construction of compact G_2-manifolds with codimension 7 singularities. Since the publication of these results they have been extended in several directions [106–109], but there is a number of avenues deserving of further research.

In connection with G_2-geometry, it would be interesting to understand how the data of a singular abelian Higgs field arises as a formal limit of the G_2-structure data. This is related to the ongoing adiabatic limit program of Donaldson [110] but is likely to be a somewhat different limit (closer to what [31] call the flat limit). In particular, the geometric meaning of the source terms we consider is still unclear and it would be interesting to elucidate it further. This would shed light on what are the constraints on the local models arising from the geometry, in particular, what are the possible source terms that can arise.

To understand the physical spectrum of the 4d effective field theory one needs to be able to compute the cohomology groups of the twisted $\mathcal{D}$ operators in (4.44). Here again very little is known about the general case where W is a non-trivial connection. It would be of great interest to extend the understanding and provide ways of computing the twisted cohomology groups in this case.

Appendices

A Conventions

A.1 Glossary

Label	Meaning
M_3	Associative three-cycle
$\widetilde{G}$	Unhiggsed gauge group
G	Gauge group in 4d (arising from Higgsing from $\widetilde{G} \to G \times G_\perp$)
ϕ	One-form Higgs field in $\Omega^1(\mathcal{M}_3) \otimes \mathrm{Ad}(G_\perp)$
f	Morse-Bott function or electrostatic potential with $\phi = \mathrm{d}f$
ρ	Charge distribution on M_3 supported on Γ
Q	Vector of U(1) charges
f_Q	Charge weighted sum of Morse-Bott functions
Γ	Subspace of M_3 where electrostatic charge distribution is localized
$\mathcal{M}_3$	$M_3 \setminus T(\Gamma)$, where $T(\Gamma)$ is a tubular neighborhood of Γ.
Σ	$\partial \mathcal{M}_3$
$\gamma(f_1, \cdots, f_n)$	Gradient flow tree specified by Morse-Bott functions f_i

A.2 Spinors

The Clifford-algebras in $4, 7, 10$ dimension are denoted by

$$4d \leftrightarrow \gamma\,, \qquad 7d \leftrightarrow \hat{\gamma}\,, \qquad 10d \leftrightarrow \Gamma\,. \tag{A.1}$$

We realize the gamma matrices as

$$
\begin{aligned}
\Gamma^0 &= \sigma^1 \otimes \hat{\gamma}^0 \otimes \sigma^0 = \sigma^1 \otimes \gamma^0 \otimes \sigma^0 \otimes \sigma^0 \\
\Gamma^1 &= \sigma^1 \otimes \hat{\gamma}^1 \otimes \sigma^0 = \sigma^1 \otimes \gamma^1 \otimes \sigma^0 \otimes \sigma^0 \\
\Gamma^2 &= \sigma^1 \otimes \hat{\gamma}^2 \otimes \sigma^0 = \sigma^1 \otimes \gamma^2 \otimes \sigma^0 \otimes \sigma^0 \\
\Gamma^3 &= \sigma^1 \otimes \hat{\gamma}^3 \otimes \sigma^0 = \sigma^1 \otimes \gamma^3 \otimes \sigma^0 \otimes \sigma^0 \\
\Gamma^4 &= \sigma^1 \otimes \hat{\gamma}^4 \otimes \sigma^0 = \sigma^1 \otimes \gamma_5 \otimes \sigma^1 \otimes \sigma^0 \\
\Gamma^5 &= \sigma^1 \otimes \hat{\gamma}^5 \otimes \sigma^0 = \sigma^1 \otimes \gamma_5 \otimes \sigma^2 \otimes \sigma^0 \\
\Gamma^6 &= \sigma^1 \otimes \hat{\gamma}^6 \otimes \sigma^0 = \sigma^1 \otimes \gamma_5 \otimes \sigma^3 \otimes \sigma^0 \\
\Gamma^7 &= \sigma^2 \otimes I_8 \otimes \sigma^1 = \sigma^2 \otimes I_4 \otimes \sigma^0 \otimes \sigma^1 \\
\Gamma^8 &= \sigma^2 \otimes I_8 \otimes \sigma^2 = \sigma^2 \otimes I_4 \otimes \sigma^0 \otimes \sigma^2 \\
\Gamma^9 &= \sigma^2 \otimes I_8 \otimes \sigma^3 = \sigma^2 \otimes I_4 \otimes \sigma^0 \otimes \sigma^3 \,.
\end{aligned}
\tag{A.2}
$$

Here $\sigma^0 = \mathrm{Id}_2$ and the 4d gamma matrices are

$$
\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} , \qquad (\sigma^\mu) = (-\sigma^0, \sigma^i)\,, \qquad (\bar{\sigma}^\mu) = (-\sigma^0, -\sigma^i)\,, \tag{A.3}
$$

where 4d signature is $\mathbb{R}^{3,1}$. The chirality, B-matrices, charge conjugation matrices and Lorentz-generators will be denoted by:

$$
\gamma_5\,, \Gamma_c\,, \qquad B_4\,, B_{10}\,, \qquad C_4\,, C_{10}\,, \qquad \Sigma_4\,, \Sigma_7\,, \Sigma_{10}\,, \tag{A.4}
$$

respectively for $4d, 7d, 10d$ gamma matrices.

$$
\begin{aligned}
\gamma_5 &= i\gamma^0 \cdots \gamma^3\,, & \Gamma_c &= -\Gamma^0 \cdots \Gamma^9 \\
B_4 &= \gamma_5 \gamma^0 \gamma^1 \gamma^3\,, & B_{10} &= -\Gamma^0\Gamma^1\Gamma^3\Gamma^5\Gamma^7\Gamma^9\,, & \text{(A.5)} \\
C_4 &= B_4 \gamma^0\,, & C_{10} &= -B_{10}\Gamma^0\,, & \text{(A.6)}
\end{aligned}
$$

and

$$
\Sigma_4^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]\,, \qquad \Sigma_7^{\mu\nu} = -\frac{i}{4}[\hat{\gamma}^\mu, \hat{\gamma}^\nu]\,, \qquad \Sigma_{10}^{\mu\nu} = -\frac{i}{4}[\Gamma^\mu, \Gamma^\nu]\,. \tag{A.7}
$$

As one of the defining properties of the B-matrices is $B^*B = 1$ there strictly speaking does not exist a matrix B_7. However the definition of the B-matrices as product of all imaginary gamma matrices can be extended to odd dimensions. We thus define B_7 and its corresponding charge conjugation matrix C_7 by

$$B_7 = \hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^3\hat{\gamma}^5 \,, \qquad C_7 = B_7\hat{\gamma}^0 \,. \tag{A.8}$$

The three B-matrices fit together as

$$B_{10} = \sigma^0 \otimes B_7 \otimes i\sigma_2 = \sigma^0 \otimes B_4 \otimes (-\sigma^2) \otimes i\sigma_2 \,. \tag{A.9}$$

We collect the relations satisfied by the above matrices in table A.2. Finally we list anti-symmetric combinations needed to specify the Lorentz generators $\Sigma_{10}, \Sigma_7, \Sigma_4$

$$\Gamma^{\mu\nu} = \sigma^0 \otimes \gamma^{\mu\nu} \otimes I_4 \,, \qquad \Gamma^{\mu\underline{k}} = \sigma^0 \otimes \gamma^\mu\gamma_5 \otimes \sigma^{\underline{k}} \otimes \sigma^0 \,, \tag{A.10}$$

$$\Gamma^{\underline{kl}} = I_8 \otimes \sigma^{\underline{kl}} \otimes \sigma^0 \,, \qquad \Gamma^{\mu\hat{\imath}} = i\sigma^3 \otimes \gamma^\mu \otimes \sigma^0 \otimes \sigma^{\hat{\imath}} \,, \tag{A.11}$$

$$\Gamma^{\hat{\imath}\hat{\jmath}} = I_{16} \otimes \sigma^{\hat{\imath}\hat{\jmath}} \,, \qquad \Gamma^{\underline{k}\hat{\imath}} = i\sigma^3 \otimes \gamma_5 \otimes \sigma^{\underline{k}} \otimes \sigma^{\hat{\imath}} \,, \tag{A.12}$$

where indices run as $\mu = 0, \ldots, 3$ and $\underline{k}, \underline{l} = 1, 2, 3$ and $\hat{\imath}, \hat{\jmath} = 1, 2, 3$.

The 10d Majorana-condition $B_{10}\lambda = \lambda^*$ leads to a symplectic Majorana-constraint on the 7d spinors and a Majorana-constraints on the 4d spinors. We trace through the decomposition of the spinor representation as detailed in (4.1), (4.4) and (4.6) and make these constraints explicit.

By (A.9) the constraint inherited by the 7d spinors $\lambda_{\alpha\hat{\alpha}}$ is

$$\lambda_{\alpha\hat{\alpha}} = (i\sigma^2)_{\hat{\alpha}}{}^{\hat{\beta}}\lambda^*_{\alpha\hat{\beta}} \,, \tag{A.13}$$

which is a symplectic Majorana-condition, [111]. Here the indices run as $\alpha = 1, \ldots, 8$ and $\hat{\alpha} = 1, 2$. The 7d spinors satisfy no further constraints.

We next turn to the spinors $\lambda_{\alpha\underline{\alpha}\hat{\alpha}}$ in 4d of the untwisted symmetry group. By (A.9)

	Chirality	B-matrix	Charge Conj.
4d	$\gamma_5\gamma_5 = 1$	$B_4^* B_4 = 1$	$C_4\gamma C_4^{-1} = +\gamma^T$
	$\{\gamma_5, \gamma\} = 0$	$B_4\gamma B_4^{-1} = +\gamma^*$	$C_4\Sigma_4 C_4^{-1} = -\Sigma_4^T$
	$[\gamma_5, \Sigma_4] = 0$	$B_4\gamma_5 B_4^{-1} = -\gamma_5^*$	$C_4 C_4 = -1$ (†)
	$\gamma_5 = \gamma_5^*$ (†)	$B_4\Sigma_4 B_4^{-1} = -\Sigma_4^*$	$C_4 = C_4^*$ (†)
	$\gamma_5 = \gamma_5^T$ (†)	$B_4 = B_4^T$ (†)	$C_4 = -C_4^T$ (†)
7d	does not exist	$B_7^* B_7 = -1$	$C_7\Sigma_7 C_7^{-1} = -\Sigma_7^T$
		$B_7\hat{\gamma} B_7^{-1} = +\hat{\gamma}^*$	$C_7 C_7 = -1$ (†)
		$B_7\Sigma_7 B_7^{-1} = -\Sigma_7^*$	$C_7 = -C_7^*$ (†)
		$B_7 = -B_7^T$ (†)	$C_7 = C_7^T$ (†)
10d	$\Gamma_c\Gamma_c = 1$	$B_{10}^* B_{10} = 1$	$C_{10}\Gamma C_{10}^{-1} = -\Gamma^T$
	$\{\gamma_5, \Gamma\} = 0$	$B_{10}\Gamma B_{10}^{-1} = +\Gamma^*$	$C_{10}\Sigma_{10} C_{10}^{-1} = -\Sigma_{10}^T$
	$[\Gamma_c, \Sigma_{10}] = 0$	$B_{10}\Gamma_c B_{10}^{-1} = +\Gamma_c^*$	$C_{10} C_{10} = 1$ (†)
	$\Gamma_c = \Gamma_c^*$ (†)	$B_{10}\Sigma_{10} B_{10}^{-1} = -\Sigma_{10}^*$	$C_{10} = -C_{10}^*$ (†)
	$\Gamma_c = \Gamma_c^T$ (†)	$B_{10} = B_{10}^T$ (†)	$C_{10} = -C_{10}^T$ (†)

Table A.1: List of matrix relations. The unmarked relations are fundamental and necessary to the definition of the chiral, B and charge conjugation matrices. The daggered relations are a consequence of the explicit realization of the gamma matrices. The B_7 is defined in analogy to the B_4, B_{10} matrices but cannot be used to implement a Majorana condition as $B_7^* B_7 = -1$. In odd dimensions there exists no notion of chirality as the representation of the Clifford-algbra is already irreducible. We suppress space-time indices.

these are required to satisfy the Majorana-constraint

$$\lambda_{\alpha\underline{\alpha}\hat{\alpha}} = (-\sigma^2)_{\underline{\alpha}}{}^{\underline{\beta}} (i\sigma^2)_{\hat{\alpha}}{}^{\hat{\beta}} \lambda^*_{\alpha\underline{\beta}\hat{\beta}} \,, \tag{A.14}$$

where the indices run as $\alpha, \underline{\alpha}, \hat{\alpha} = 1, 2$. There are no further constraints on the spinors.

After performing the twist we find the Dirac spinors (λ_0, λ_i), which carry twisted indices, to be constrained as

$$iB_4\lambda_0 = \lambda_0^* \,, \qquad iB_4\lambda_i = -\lambda_i^* \,, \tag{A.15}$$

with $i = 1, 2, 3$. Decomposing these Dirac spinors into Weyl spinors $\lambda_0 = (i\chi_\alpha, i\bar{\xi}^{\dot{\alpha}})$ and $\lambda_i = (\psi_{i\alpha}, \bar{\zeta}_i{}^{\dot{\alpha}})$ the conditions are rewritten explicitly with the charge conjugation matrix as

$$(i\bar{\sigma}^2)^{\dot{\alpha}\alpha}\chi_\alpha = -\left(\bar{\xi}^{\dot{\alpha}}\right)^* \,, \qquad (i\bar{\sigma}^2)^{\dot{\alpha}\alpha}\psi_{i\alpha} = -\left(\bar{\zeta}_i{}^{\dot{\alpha}}\right)^* \,, \tag{A.16}$$

which are, due to the introduction of a factor of i, nothing but two canonical Majorana-conditions

$$\chi_\alpha = \xi_\alpha \,, \qquad \psi_{i\alpha} = \zeta_{i\alpha} \,. \tag{A.17}$$

Using this we can rewrite the 4d dimensionally reduced action of 10d SYM interms of 1+3 unconstrained Weyl spinors χ, ψ_i in 4d. The resulting action is given in (4.13).

B Boundary Conditions

In this appendix we provide some details underpinning the computation of the $\mathcal{D}$-cohomology groups in section 6.1. The argument was originally introduced by Witten [4] for closed manifolds. The extension to the manifolds with boundary includes the additional subtlety of choosing the appropriate boundary conditions, which allow us to apply the Hodge theory arguments on manifolds with boundary.

Following [91] we consider the case where we have a single boundary component. We first focus on defining the appropriate domains of the standard operators d and $\mathrm{d}^\dagger$ and then show that the cohomology groups are invariant under the deformation by f. We define d_1 and $\mathrm{d}_1^\dagger$ as operators with the domains given by

$$\begin{aligned} D(\mathrm{d}_1) &= \Omega^p(\mathcal{M}_3), \\ D(\mathrm{d}_1^\dagger) &= \{\alpha \in \Omega^p(\mathcal{M}_3) \mid *\alpha_n|_\Sigma = 0\}. \end{aligned} \tag{B.1}$$

This is the Neumann boundary condition. To obtain a self-adjoint Laplacian for these boundary conditions we have to further restrict the domain of the Laplace operator. We define Δ_1 to be the standard metric Laplacian, with the domain

$$D(\Delta_1) = \{\alpha \in \Omega^p(M) \mid *\alpha_n|_\Sigma = *(\mathrm{d}\alpha)_n|_\Sigma = 0\}. \tag{B.2}$$

Note that d_1 is simply the standard de Rham differential, so the cohomology of the resulting complex is the (absolute) de Rham cohomology $H^*(\mathcal{M}_3)$. For the Dirichlet boundary conditions we define the operators d_2 and $\mathrm{d}_2^\dagger$, with domains

$$\begin{aligned} D(\mathrm{d}_2) &= \{\alpha \in \Omega^p(\mathcal{M}_3) \mid \alpha_t|_\Sigma = 0\}, \\ D(\mathrm{d}_2^\dagger) &= \Omega^p(\mathcal{M}_3), \end{aligned} \tag{B.3}$$

and Δ_2, with the domain

$$D(\Delta_2) = \left\{\alpha \in \Omega^p(\mathcal{M}_3) \,\middle|\, \alpha_t|_\Sigma = (\mathrm{d}^\dagger\alpha)_t|_\Sigma = 0\right\}. \tag{B.4}$$

We use the indices to formally distinguish between the two operators, based on their different domains. With these domains, the two Laplace operators are self-adjoint and we have the Hodge decompositions [91, 112]

$$\begin{aligned} \Omega^p(\mathcal{M}_3) &= R(\mathrm{d}_i^{p-1}) \oplus R((\mathrm{d}_i^\dagger)^{p+1}) \oplus N(\Delta_i^p), \\ N(\mathrm{d}_i^p) &= R(\mathrm{d}_i^{p-1}) \oplus N(\Delta_i^p), \\ N((\mathrm{d}_i^\dagger)^{p-1}) &= R((\mathrm{d}_i^\dagger)^p) \oplus N(\Delta_i^p). \end{aligned} \tag{B.5}$$

Here, N and R denote the nullspace and range respectively. Superscripts denote the degree of forms on which the operators act. Observe that this implies

$$N(\Delta_i^p) = N(\mathrm{d}_i^p)/R(\mathrm{d}_i^{p-1}). \tag{B.6}$$

The quotient on the right side is precisely the p-th cohomology group of the operator d_i. As already noted above, the cohomology of operator d_1 is the de Rham cohomology $H^p(\mathcal{M}_3)$ by definition. To identify the cohomology groups of the operator d_2, observe that $\alpha_t|_\Gamma = 0$ is the same as saying $\alpha|_\Sigma = 0$. This implies that the cohomology of d_2 is isomorphic to the relative cohomology $H^p(\mathcal{M}_3, \Sigma)$ [92].

Let us now consider how the deformation of the complex by a smooth function f relates to the above. Recall that $\mathcal{D}$ is in fact the deformed de Rham differential $\mathcal{D} = e^{-qf}\mathrm{d}e^{qf}$. Denote $\mathcal{D}_i = e^{-qf}\mathrm{d}_i e^{qf}$ and $\mathcal{D}_i^\dagger = e^{qf}\mathrm{d}_i^\dagger e^{-qf}$ i.e. we impose the same boundary conditions on $\mathcal{D}_i$ (resp $\mathcal{D}_i^\dagger$) as we did on d_i (resp. $\mathrm{d}_i^\dagger$). Let $\Delta_{f,i} = \mathcal{D}_i\mathcal{D}_i^\dagger + \mathcal{D}_i^\dagger\mathcal{D}_i$ denote the twisted Laplacian from (6.6). We set $D(\Delta_{f,i}) = D(\Delta_i)$. We have the same Hodge decomposition for the operators $\mathcal{D}_i, \mathcal{D}_i^\dagger, \Delta_{f,i}$ as in (B.5) as well as the identification

$$H^p_{\mathcal{D}_i}(\mathcal{M}_3) = N(\Delta^p_{f,i}). \tag{B.7}$$

Finally, the map $\alpha \mapsto e^{-qf}\alpha$ induces an isomorphism between the d_i-complex and $\mathcal{D}_i$-complex (multiplication by smooth functions preserves the boundary conditions). This in turn induces isomorphisms of the cohomology groups. Combining the arguments above, we see that

$$N(\Delta^p_{f,i}) = N(\Delta^p_i), \tag{B.8}$$

so the number of ground states of $\Delta_{f,i}$ is independent of f. Moreover,

$$\begin{aligned} H^p_{\mathcal{D}_1}(\mathcal{M}_3) &= H^p(\mathcal{M}_3) \\ H^p_{\mathcal{D}_2}(\mathcal{M}_3) &= H^p(\mathcal{M}_3, \Sigma). \end{aligned} \tag{B.9}$$

This provides the simplest example where the cohomology of $\mathcal{D}$ is computed directly in terms of the cohomology of the underlying manifold with boundary. However, the case with single boundary component is not interesting in our context, as it violates the charge conservation condition we obtain from the electrostatics problem. However, the above discussion clearly outlines the general structure, which persists in the case of mixed boundary conditions, which we consider in 6.1.

CHAPTER

III

GK GEOMETRIES

1 Introduction

In this chapter, we shift our attention to compactifications of string theory on spaces with an AdS factor. One of the key motivations for considering this set-up is the AdS/CFT correspondence, which provides a remarkable equivalence between the quantum gravity theory, derived by compactifying the low energy limit of string theories (supergravity theories) in dimensions 10 and 11 on spaces of the form $\mathrm{AdS}_{d+1} \times Y$, and a conformal field theory on a d-dimensional space isomorphic to the conformal boundary of the AdS_{d+1}. The physical properties of supergravity compactifications are to a large extent determined by the geometry of the internal space Y, which is itself constrained to be a solution of partial differential equations arising from supersymmetry and the supergravity equations of motion. From this perspective, the AdS/CFT correspondence predicts a dictionary between the geometric properties of Y and the physics of d-dimensional CFT, which one can hope to leverage to turn intractable problems on one side, into tractable problems on the other. Mapping out this dictionary in various dimensions continues to be an immensely fruitful area of research in the AdS/CFT correspondence [113–117].

Perhaps the most striking interaction between geometry and field theory in the context of AdS/CFT arises from the study of the geometric duals of *extremization principles* for superconformal field theories with a $U(1)_R$ symmetry. For such theories, there is a unique superconformal R-symmetry whose conserved current is in the same multiplet as the stress-energy tensor. SCFTs are generally strongly coupled and have no perturbative description and are hence studied indirectly, as the IR limit of some weakly coupled UV theory via the RG flow. However, the UV descriptions do not usually present with a distinguished R-symmetry. In fact, given a fixed R-symmetry generator R_0, any combination of the form

$$R = R_0 + \sum_{i=1}^{n} s_i F_i \,, \tag{1.1}$$

where s_i are real parameters and F_i generate the global abelian symmetries, is also an R-symmetry. The challenge then is to identify the superconformal R-symmetry amongst all the R-symmetries present in the UV theory. Selecting the exact R-symmetry of the superconformal fixed point (provided one exists) amounts to determining the correct parameters s_i, given some initial choice for R_0.

Remarkably, there exist general methods of determining the exact R-symmetry based on extremizing certain functions of (1.1) over all global abelian symmetries. These extremization principles are somewhat confusingly named with a different letter depending on the dimension; for 2d, 3d and 4d theories they are c-extremization, F-maximization and a-maximization respectively [118–122]. As the naming conventions suggest, the exact function depends on the dimension. The functions are particularly simple in even dimensions where they are given by the c and a central charges in 2d and 4d respectively.[17] The central charges determine the conformal anomaly of any CFT and, crucially, are determined by a simple expression in terms of the 't Hooft anomaly of the exact R-symmetry for SCFTs. Both the c-extremization and the a-maximization arguments then proceed by considering a corresponding expression for a trial R-symmetry (1.1) called the *trial* central charge. The fundamental observation is then that the exact R-symmetry locally extremizes the

[17] There are some caveats. The IR theory can have accidental global symmetries that are not visible in the UV but can nevertheless mix with the R-symmetry. For the extremization argument to be valid, one must start with the correct global abelian symmetry group.

trial central charge over the space parametrized by s_i. Using this one can easily determine the exact R-symmetry and the central charge using only the UV description. The corresponding argument in odd dimensions is somewhat more subtle since there is no conformal anomaly so the corresponding function is harder to identify. For 3d $\mathcal{N}=2$ SCFTs this was done in [119, 120], where the corresponding F function was shown to be the negative logarithm of the partition function of the SCFT placed on the three-sphere.

The prototypical example of an extremization principle and its geometric dual is the a-maximization for 4d $\mathcal{N}=1$ SCFTs, introduced by Intriligator and Wecht [122]. On the gravity side, these field theories are dual to solutions of IIB supergravity on spaces of the form $\mathrm{AdS}_5 \times Y_5$, where Y_5 is a *Sasaki-Einstein* manifold. Sparks, Martelli, and Yau [123, 124] showed that the geometric dual a-maximization is *Sasakian volume minimization* and provided a detailed correspondence between the geometry and field theory. Together with related works [125–131] this led to a drastic improvement in the understanding of the 4d field theories as well as a plethora of new results in Sasakian geometry.

The problem motivating the work presented in this chapter is formulating a geometric dual of c-extremization for 2d $\mathcal{N}=(0,2)$ SCFTs. The central charge is given by

$$c = 3\,\mathrm{Tr}\,\gamma^3 R^2\,,^{18} \tag{1.2}$$

which was shown to be extremized by the exact superconformal R-symmetry in [118, 132]. The corresponding supergravity solutions again arise from IIB supergravity but this time on spaces $\mathrm{AdS}_3 \times Y_7$, with a with D3-branes. The geometric requirements imposed on Y_7 by supersymmetry and equations of motion were first identified in [133]. While Sasaki-Einstein manifolds were studied in the mathematical literature prior to their appearance in the AdS/CFT correspondence, the geometric structure in this context has not been previously known. Exactly the same structure, albeit in two dimensions higher, arises when considering M-theory (11d supergravity) on the space $\mathrm{AdS}_2 \times Y_9$ with M2-branes [134]. The dual field theory in this case is 1d $\mathcal{N}=2$ superconformal quantum mechanics. This structure was later generalized to an arbitrary odd dimension by Gauntlett and Kim [135] and the geometric structure has since acquired the name *GK geometry*.

[18] The γ^3 is the 2d chirality matrix and that trace is taken over the Weyl fermions.

GK-manifolds generated substantial interest in the physics literature and a number of different families of solutions have been found [133–146]. Many of their properties parallel those of Sasaki-Einstein geometry. In particular, the metric cone over a GK-manifold admits a natural complex structure with a vanishing first Chern class. In sharp contrast to the Sasaki-Einstein case though, this cone does not have a natural Kähler structure. Despite the interesting geometric properties and mathematical novelty, GK geometry has so far flown completely under the radar of the mathematical community. In hopes of providing an entry point to the mathematically inclined reader, we provide a detailed overview of the geometric structure in section 3, extracting the underlying mathematics from the physics vernacular and provide some novel observations on the geometry of GK-manifolds.

A geometric dual of c-extremization was very recently proposed in [147] and has since been explored and extended in subsequent work [148–150]. Analogously to the case of a-maximization, there exists a natural geometric functional that depends on the choice of a Killing vector (geometric counterpart to R-symmetry) and additional geometric parameters. When extremized over the parameter space this functional calculates the c central charge of the dual SCFT. In fact, the extremization problem of [147] is not restricted to any particular dimension. This is especially important since it also provides a geometric extremization problem the M-theory on $\mathrm{AdS}_2 \times Y_9$, where no general field theory extremization principle for the dual 1d SCQMs is known. A 1d extremization principle, called $\mathcal{I}$-extremization, has been formulated only in a narrow class of theories that arise as topologically twisted compactifications of a 3d $\mathcal{N} = 2$ theory on a Riemann surface [151]. The topologically twisted index of the 3d theory is expected to yield the 1d partition function [152–154] after extremization. The corresponding geometric extremization problem is indeed dual to the $\mathcal{I}$-extremization [155, 156].

Our discussion of c-extremization and its geometric dual has so far been anchored in the pure IIB set-up, where the axio-dilaton is constant. A natural generalization of this is to consider the F-theory extension, where the axio-dilaton varies holomorphically and can be encoded as the complex structure of an elliptic fibration over the compact internal space. A class of AdS_3 solutions of F-theory was obtained in [157, 158] and the dual field theories are obtained by wrapping D3-branes on curves, above which the axio-dilaton varies. From the point of view of the 4d $\mathcal{N} = 4$ Super-Yang-Mills (SYM) theory on the D3-branes, this corresponds to a varying complexified coupling, and the 2d SCFT is obtained by a duality-twist [159–162]. Generalizations of these F-theory solutions were obtained in [163, 164] and the dual field theories were studied in [165, 166].

In the central part of this chapter, we generalize the geometric dual of c-extremization to include a varying axio-dilaton, which provides a powerful tool for identifying F-theory AdS_3 supergravity solutions that can arise from configurations of D3-branes and 7-branes. Furthermore, from this point of view the duality with M-theory AdS_2 backgrounds not only provides a description where the elliptic fibration associated with the varying axio-dilaton is physically manifest, it also implies that in this specific context holographic $\mathcal{I}$- and c-extremization are dual to each other.

More precisely, we will find that in general the two quantities obtained by extremization in M/F-theory only agree up to leading order in an expansion in terms of the volume of the elliptic fiber. Namely, we find

$$\log Z_{1\mathrm{d}} = \frac{1}{4G_2} = \frac{\Delta\phi}{12} c_{\mathrm{sugra}} + \mathcal{O}(k_0) = \sqrt{\frac{2}{3}} \pi N_0^{1/2} \cdot c_{\mathrm{sugra}}^{1/2} + \mathcal{O}(k_0) \,, \tag{1.3}$$

where k_0 is the volume of the elliptic fiber. Here on the left-hand side $Z_{1\mathrm{d}}$ is the partition function of the 1d SCQM, which via holography is related to the 2d Newton constant G_2 of the dual M-theory AdS_2 solution. On the right-hand side c_{sugra} is the leading order 2d central charge of the F-theory AdS_3 solution, $N_0 \in \mathbb{N}$ is a certain quantized flux number, while $\Delta\phi$ is the size of the circle upon which the 2d SCFT is compactified. The correction terms in (1.3) are $\mathcal{O}(k_0)$. In M-theory, this fiber volume is a physical quantity, whereas in F-theory the elliptic fiber is an auxiliary geometric structure, where only the complex structure has a physical meaning, and the volume is strictly taken to zero. The correction terms then generically arise because the M-theory backgrounds include the full backreaction of the 7-branes on the F-theory side, which in particular break the circle isometry on which we T-dualize.

Let us conclude by making some comments on the physical interpretation of (1.3). The left-hand side is the logarithm of the 1d partition function of the SCQM on a circle. On the other hand, the first expression involving c_{sugra} is precisely the *Casimir energy* of the 2d $(0,2)$ theory placed on a torus, as one might have expected on general grounds. The final expression on the right-hand side of (1.3) is proportional to $c_{\text{sugra}}^{1/2}$, with a proportionality constant that is a fixed number. In particular, this shows that to leading order in k_0 the two extremization principles are dual to each other. We shall see explicit examples, where the $\mathcal{O}(k_0)$ correction terms are either zero or non-zero.

Outline

The chapter is organized as follows: we start by recounting the details of AdS backgrounds that give rise to GK geometries in section 2. Section 3 is of a more mathematical flavor and focuses the geometry of GK-manifolds and the geometric extremization problem. In section 4 we present the details of the supersymmetric F-theory AdS_3 geometries and generalize the method of holographic c-extremization to accommodate a varying axio-dilaton. In section 5 we review the M/F-duality for the supersymmetric $\text{AdS}_2/\text{AdS}_3$ geometries and specialize holographic $\mathcal{I}$-extremization to the case where the compactification space contains a non-trivial elliptic fibration. We then determine the map between $\mathcal{I}$- and c-extremization in section 6. In section 7 we consider a large class of toric examples and apply $\mathcal{I}/c$-extremization to a novel set of M/F-theory setups, and rederive a known class of solutions, the elliptic surface universal twist solutions, using this new framework. For these theories the M- and F-theory computations agree without any corrections. Finally, section 8 contains an analysis of a related known class of M/F-theory solutions, the elliptic three-fold universal twist solutions, where the M-theory result for $1/G_2$ receives corrections compared to the F-theory computation. We conclude in section 9.

2 AdS Backgrounds

The class of AdS_3 solutions of IIB supergravity we consider is given by the following ansatz for the 10d metric and the Ramond-Ramond self-dual five-form F_5:

$$
\begin{aligned}
\mathrm{d}s^2_{10} &= L_{10}^2 e^{-B_{10}/2}(\mathrm{d}s^2(\text{AdS}_3) + \mathrm{d}s^2(Y_7))\,, \\
F_5 &= -L_{10}(\text{vol}_{\text{AdS}_3} \wedge F + *_7 F)\,.
\end{aligned}
\tag{2.1}
$$

All the remaining IIB fields are set to zero. In particular, for the purposes of this initial exposition, we only consider a constant axio-dilaton. We will discuss the extension to the case of varying axio-dilaton later in this chapter.

On the $D = 11$ supergravity side we consider the AdS_2 solutions where the metric and the four-form G are given by:

$$\begin{aligned} \mathrm{d}s^2_{11} &= L^2_{11} e^{-2B_{11}/3}(\mathrm{d}s^2(\mathrm{AdS}_2) + \mathrm{d}s^2(Y_9))\,, \\ G_4 &= L^3_{11}\mathrm{vol}_{\mathrm{AdS}_2} \wedge F\,. \end{aligned} \tag{2.2}$$

In both expressions, the AdS space is endowed with the unit radius metric and vol is the corresponding volume form. The parameter L is the overall dimensionful length scale and the function B appearing in the conformal factor in the metric is a smooth function defined on the compact internal manifold. Finally, F is a two-form on the internal space. Since both the IIB and 11D geometries have similar properties we will unify the treatment by switching to Y_{2n+1} in the following. While the physically interesting cases are $n = 3$ and $n = 4$, this also anticipates the generalization to arbitrary dimension first formulated in [135] which is the subject of the next section.

A key observation is that both IIB and $D = 11$ equations of motion are equivalent to the equations arising form an action defined on the internal space

$$S = \int_{Y_{2n+1}} e^{(1-n)B}\left(R_{2n+1} - \frac{2n}{(n-2)^2} + \frac{n(2n-3)}{2}(\mathrm{d}B)^2 + \frac{1}{4}e^{2B}F^2\right)\mathrm{vol}_{2n+1}\,. \tag{2.3}$$

The dynamical fields are the metric $\mathrm{d}s^2_{2n+1}$, function B and the local one-form potential A, where $F = \mathrm{d}A$. Furthermore, R_{2n+1} is the Ricci scalar of $\mathrm{d}s^2_{2n+1}$, vol_{2n+1} is the corresponding volume form, and the terms $(\mathrm{d}B)^2$ and F^2 are the square norms of the respective 2-forms i.e. $F^2 = F_{ab}F^{ab}$.

The requirement to preserve supersymmetry gives rise to a Killing spinor equation on the compact internal space. We call the manifolds Y_{2n+1} *supersymmetric* if there exists a solution to the Killing spinor equation. The existence of such a Killing spinor places strong restrictions on the geometry of Y_{2n+1} [135]. It implies that Y_{2n+1} admits a unit norm Killing vector field ξ that in turn defines a transverse foliation $\mathcal{F}_\xi$. Writing the

Killing vector in local coordinates as

$$\xi = \frac{1}{c}\partial_z\,, \tag{2.4}$$

where $c = \frac{(n-2)}{2}$, the metric splits as

$$\mathrm{d}s^2_{2n+1} = c^2(\mathrm{d}z + P)^2 + e^B \mathrm{d}s^2\,. \tag{2.5}$$

Further, the transverse metric $\mathrm{d}s^2$ is in fact Kähler. The one-form P is the local one-form potential of the transverse Ricci form i.e. $\mathrm{d}P = \rho$, where ρ is the Ricci form. The transverse Kähler metric determines the function B and the two-form F via the equations

$$\begin{aligned} e^B &= \frac{c^2}{2}R\,, \\ F &= -\frac{1}{c}J + c\,\mathrm{d}\left(e^{-B}(\mathrm{d}z + P)\right)\,. \end{aligned} \tag{2.6}$$

Here R is the Ricci scalar and J the Kähler form of the transverse metric. The above properties are direct consequences of the existence of a Killing spinor and they are also sufficient, in that every Y_{2n+1} with these properties admits a spinor that solves the supersymmetry equation. For supersymmetric geometries, the equations of motion of (2.3) reduce to

$$\mathrm{d}\left[e^{(3-n)B} *_{2n+1} F\right] = 0\,. \tag{2.7}$$

This equation is in fact equivalent to the closure of F_5 for $n = 3$ and closure of $*_{11}G_4$ for $n = 4$.

To obtain consistent string theory backgrounds from (2.1) and (2.2) we need to impose flux quantization conditions. Additionally, these conditions play a crucial role in defining the extremization problem we will turn to in the following sections. Starting with the IIB backgrounds, the relevant flux quantization condition is a Dirac quantization of the form F_5 over five-cycles in Y_7. More precisely, given a basis S_α of $H_5(Y_7, \mathbb{Z})$, the condition is

$$\frac{1}{(2\pi l_s)^4 g_s}\int_{S_\alpha} F_5 = N_\alpha \in \mathbb{Z}\,. \tag{2.8}$$

The parameters l_s and g_s in the equation are the string length and the constant string coupling respectively. Notice that the integral in (2.8) is not well defined since it can depend on the representative of S_α. However, the type IIB equations of motion imply the closure of F_5, which in turn ensures that the flux integrals depend only on the homology class.

Flux quantization in 11d presents some additional subtleties. As in [167] we assume from the outset that the first Pontryagin class $p_1(Y_9)$ is divisible by four. The corresponding flux quantization condition is

$$\frac{1}{(2\pi l_p)^6}\int_{S_\alpha} *_{11}G = N_\alpha \in \mathbb{Z}\,, \tag{2.9}$$

where the homology classes S_α are elements of any basis of the free part of $H_7(Y_9,\mathbb{Z})$.

We conclude by stating two key equations that relate the 3d (respectively 2d) Newton constant to the volume of Y_7 (respectively Y_9). Starting with the former, one can dimensionally reduce the IIB supergravity action to 3d to obtain [151, Appendix B]

$$\frac{1}{G_3} = \frac{L_{10}^7}{G_{10}}\int_{Y_7} e^{-2B_{10}}\mathrm{vol}_7\,. \tag{2.10}$$

Here, G_{10} is the ten-dimensional Newton constant

$$G_{10} = \frac{(2\pi)^7 g_s^2 l_s^2}{16\pi}\,. \tag{2.11}$$

For a given supergravity solution with properly quantized fluxes we can therefore calculate the central charge of the holographically dual 2d $(0,2)$ SCFT by using the Brown-Henneaux formula [168]

$$c_{\mathrm{sugra}} = \frac{2L_{10}}{2G_3}\,. \tag{2.12}$$

Similarly, we can calculate the 2d Newton constant for the $\mathrm{AdS}_2 \times Y_9$ solutions. It is given by

$$\frac{1}{G_2} = \frac{L_{11}^9}{G_{11}}\int_{Y_9} e^{-3B_{11}}\mathrm{vol}_9\,, \tag{2.13}$$

where

$$G_{11} = \frac{(2\pi)^8 l_p^9}{16\pi} \,. \tag{2.14}$$

In general, the precise meaning of $\frac{1}{G_2}$ on the 1d SQM side has not yet been established, however, the expectation is that $\frac{1}{4G_2}$ equals the negative logarithm of the partition function of the dual supersymmetric quantum mechanics [147].

3 GK Geometry

The physical considerations in the previous section have motivated interest in a new class of odd-dimensional manifolds that we will refer to as GK-manifolds, named after Gauntlett and Kim who have initiated the study in [135], which was then extended in [147–149, 155]. In this section, we approach the geometries we have discussed before from a distinctly more mathematical perspective and analyze their properties in detail.

3.1 Supersymmetric Manifolds

We start by considering *supersymmetric manifolds* as defined in the previous section. We assume throughout that Y_{2n+1} is a smooth closed Riemannian manifold of dimension $2n+1$, where $n \geq 2$. Recall that, by definition, Y_{2n+1} admits a unit norm Killing vector ξ and a transversely conformally Kähler foliation $\mathcal{F}_\xi$. We denote the metric dual one-form of ξ by η so that

$$\eta = c(\mathrm{d}z + P) \,, \tag{3.1}$$

and $\mathrm{d}\eta = c\rho$. The Killing vector is in this context also called the *R-symmetry vector.*

The existence of the transverse conformally Kähler structure has some immediate consequences. To study them we introduce the basic cohomology of $\mathcal{F}_\xi$, which, by definition, is the cohomology of the complex of ξ-basic forms

$$\Omega^p_B(\mathcal{F}_\xi) = \{\alpha \in \Omega^p(Y_{2n+1}) \ \mid \ \xi \lrcorner \alpha = 0 \,, \quad \mathcal{L}_\xi \alpha = 0\} \,, \tag{3.2}$$

with the differential being the restriction of the standard exterior derivative. There is a natural map at the level of cohomology

$$H_B^*(\mathcal{F}_\xi) \longrightarrow H^*(Y_{2n+1}, \mathbb{R}) \,, \tag{3.3}$$

since the representatives of the basic cohomology classes are simply the closed basic forms on Y_{2n+1}. Notice that the map is not injective. The transverse Ricci form on a supersymmetric manifold satisfies $\rho = \frac{1}{c}\mathrm{d}\eta$, so $[\rho]$ is zero in $H^2(Y_{2n+1}, \mathbb{R})$, while it is non-zero in $H_B^2(\mathcal{F}_\xi)$, provided that the foliation is not integrable.

An important technical observation is that the foliation $\mathcal{F}_\xi$ is transversely orientable, which means that

$$H_B^{2n}(\mathcal{F}_\xi) \cong \mathbb{R} \,. \tag{3.4}$$

To show this, we use the Gysin long exact sequence for the foliation, the relevant part of which is [22, Equation (7.2.1)]

$$\cdots \longrightarrow H_B^{2n+1}(\mathcal{F}_\xi) \longrightarrow H^{2n+1}(Y, \mathbb{R}) \longrightarrow H_B^{2n}(\mathcal{F}_\xi) \longrightarrow H_B^{2n+2}(\mathcal{F}_\xi) \longrightarrow \cdots \tag{3.5}$$

Since $\mathcal{F}_\xi$ is a Riemannian foliation, the groups $H_B^p(\mathcal{F}_\xi)$ vanish for $p > 2n$. Hence, we get the required isomorphism. The importance of the transverse orientability lies in the fact that it allows one to leverage Hodge theoretic arguments to establish results familiar from the compact setting [169]. In particular, the transverse Kähler structure implies a Dolbeault type splitting of the basic cohomology groups

$$H_B^p(\mathcal{F}_\xi) = \bigoplus_{i+j=p} H_B^{i,j}(\mathcal{F}_\xi) \,. \tag{3.6}$$

We return to the Gysin sequence to note an additional observation. The beginning is given by

$$0 \longrightarrow H_B^1(\mathcal{F}_\xi) \longrightarrow H^1(Y_{2n+1}, \mathbb{R}) \longrightarrow H_B^0(\mathcal{F}_\xi) \cong \mathbb{R} \xrightarrow{\delta} H_B^2(\mathcal{F}_\xi) \longrightarrow H^2(Y_{2n+1}, \mathbb{R}) \longrightarrow \cdots \tag{3.7}$$

The map δ is given by wedging with $\mathrm{d}\eta$. If $[\mathrm{d}\eta]$ is a non-trivial class in $H_B^2(\mathcal{F}_\xi)$, then δ is injective and, in particular,

$$H^1_B(\mathcal{F}_\xi) \cong H^1(Y_{2n+1}, \mathbb{R}) \,. \tag{3.8}$$

The geometric picture is most straightforward for the case of a *quasi-regular* Killing vector. By definition this means that the orbits of ξ close and Y_{2n+1} is the total space of the circle fibration

$$\begin{array}{ccc} \mathbb{S}^1 & \hookrightarrow & Y_{2n+1} \\ & & \downarrow \\ & & \mathcal{M}_{2n} \end{array} \,, \tag{3.9}$$

where the transverse Kähler space $\mathcal{M}_{2n}$ is a compact Kähler orbifold. This means that the R-symmetry vector generates a global locally free action on Y_{2n+1}. If the U(1) action is free, ξ is said to be *regular*. For a quasi-regular Killing vector there is an isomorphism

$$H^*_B(\mathcal{F}_\xi) \cong H^*(\mathcal{M}_{2n}) \,, \tag{3.10}$$

that is, the basic cohomology recovers the standard orbifold cohomology groups of the transverse Kähler orbifold. If the orbits of ξ do not close, we say that ξ is *irregular*. In this case, there is no globally defined transverse space.

3.2 Complex Cone

Another essential feature of supersymmetric geometries that was established in [135] is that there exists a natural complex structure on the metric cone $\mathrm{C}(Y_{2n+1})$ over Y_{2n+1}. Recall that the metric cone is the product $\mathrm{C}(Y_{2n+1}) = \mathbb{R}^+ \times Y_{2n+1}$ endowed with the metric

$$\mathrm{d}s^2_{2n+2} = \mathrm{d}r^2 + r^2 \mathrm{d}s^2_{2n+1} \,. \tag{3.11}$$

Define a $\mathrm{SU}(n+1)$ structure on $\mathrm{C}(Y_{2n+1})$ by

$$\begin{aligned} \mathcal{J} &= -cr\mathrm{d}r \wedge \eta + r^2 e^B J \\ \Omega_{(n+1,0)} &= e^{iz}\left(e^{B/2} r\right)^n (\mathrm{d}r - irc\eta) \wedge \Omega \,. \end{aligned} \tag{3.12}$$

Here, Ω is the locally defined holomorphic volume form, that is, a $(n,0)$-form on the transverse Kähler space. Moreover, it satisfies

$$d\Omega = iP \wedge \Omega \,, \tag{3.13}$$

where $dP = \rho$. Notice that, despite Ω being only locally defined, the holomorphic volume form on the cone is a global form. This is due to the fact that (3.1) implies that the local coordinate z is a local section of the anti-canonical bundle of the transverse Kähler space an hence the product $e^{iz}\Omega$ extends to a global form. The SU$(n+1)$ structure in fact gives rise to an integrable almost complex structure on C(Y_{2n+1}). Indeed, the conformally rescaled form

$$\Psi = e^{-nB/2} r^{\frac{n(n-1)}{n-2}} \Omega_{(n+1,0)} \tag{3.14}$$

is closed, so it defines an integrable SL$(n, \mathbb{C})$ structure. In addition, $d\Psi = 0$ implies that the first Chern class of the cone C(Y_{2n+1}) vanishes. This is reminiscent of Calabi-Yau cones that arise in Sasaki-Einstein geometry, however, the fundamental difference here is that the non-degenerate two-form $\mathcal{J}$ is neither closed nor conformally closed, which means that there is no natural symplectic form on C(Y_{2n+1}). For certain explicit solutions it can be shown that there is in fact no Kähler structure compatible with the natural complex structure on the cone [147].

The R-symmetry vector field ξ can be naturally extended to the cone by letting it be independent of the radial coordinate. We conflate the vector field on Y_{2n+1} and its extension by using ξ to denote both. The Lie derivative of Ψ is given by

$$\mathcal{L}_\xi \Psi = \frac{i}{c} \Psi \,, \tag{3.15}$$

which implies that ξ is holomorphic with respect to the complex structure $\mathcal{I}$ defined by Ψ.

For the purposes of formulating the extremization problem in subsection 3.5 we consider the case where C(Y_{2n+1}) admits a holomorphic U$(1)^s$ action. By choosing a basis of real holomorphic vector fields ∂_{ϕ_i} such that

$$\mathcal{L}_{\partial_{\phi_1}} \Psi = \Psi \,, \qquad \text{and} \qquad \mathcal{L}_{\partial_{\phi_i}} \Psi = 0 \quad \text{for } i = 2, \ldots, s \,, \tag{3.16}$$

we can express a general R-symmetry vector field in terms if this basis as

$$\xi = \sum_{i=1}^{s} b_i \partial_{\phi_i} = \xi_0 + \sum_{i=2}^{s} c_i \partial_{\phi_i} \,, \tag{3.17}$$

where $b_i, c_i \in \mathbb{R}$. The vector ξ_0 can be interpreted as a fixed initial choice of an R-symmetry vector and, in particular, Ψ has a charge $1/c$ under ξ_0. Additionally, the choice of basis fixes the coefficient b_1 to be equal to $1/c$.

3.3 GK-manifolds

Let Y_{2n+1} be a supersymmetric manifold. The equation (2.7) is equivalent to the following PDE on the *transverse Kähler metric*

$$\Box R = \frac{1}{2} R^2 - R_{ij} R^{ij} \,. \tag{3.18}$$

The $\Box$ operator is the metric Laplacian and R_{ij} is the Ricci tensor. This is a fourth order partial differential equation on the transverse metric. This equation is commonly referred to as the *master equation*. A *GK-manifold* is a supersymmetric manifold that solves (3.18).

We can immediately derive a simple topological constraint that must be satisfied by solutions of (3.18). Using the fact that

$$\eta \wedge \rho^2 \wedge \frac{J^{n-2}}{(n-2)!} = \left(\frac{1}{2} R^2 - R_{ij} R^{ij} \right) \eta \wedge \frac{J^n}{n!} \,, \tag{3.19}$$

and integrating the equation over Y we obtain

$$\int_{Y_{2n+1}} \eta \wedge \rho^2 \wedge \frac{J^{n-2}}{(n-2)!} = 0 \,. \tag{3.20}$$

Another foundational property of GK-manifolds is that they are extremal points of a natural geometric functional, given by the action (2.3), which for supersymmetric geometries simplifies to

$$S = \int_{Y_{2n+1}} \eta \wedge \rho \wedge \frac{J^{n-1}}{(n-1)!} \,. \tag{3.21}$$

This remarkably elegant functional, called *the supersymmetric action*, is central to the extremization problem we discuss later in this section.

Notice that the action manifestly depends on the R-symmetry vector ξ and the transverse Kähler metric. However, the action remains unchanged if we vary the metric within a fixed Kähler class. To show this, we take $J' = J + \mathrm{d}\alpha$, where $\alpha \in \Omega^1_B(\mathcal{F}_\xi)$. The respective Ricci forms likewise differ by an exact basic two-form i.e. $\rho' = \rho + \mathrm{d}\beta$. The difference of the integrands is

$$\eta \wedge \rho' \wedge \frac{(J')^{n-1}}{(n-1)!} - \eta \wedge \rho \wedge \frac{J^{n-1}}{(n-1)!} = \eta \wedge \mathrm{d}\alpha \wedge \Theta + \eta \wedge \mathrm{d}\beta \wedge \Psi \,, \tag{3.22}$$

where Θ, Ψ are closed, basic $(2n-2)$-forms. Integrating by parts we obtain

$$\int_{Y_{2n+1}} \eta \wedge \mathrm{d}\alpha \wedge \Theta = \int_{Y_{2n+1}} \mathrm{d}\eta \wedge \alpha \wedge \Theta = 0 \,. \tag{3.23}$$

The final equality follows from the fact that $\mathrm{d}\eta = c\rho$, which implies that the integrand has zero contraction with ξ. Applying the same argument to the second term on the right-hand side of (3.22) we reach the conclusion that the supersymmetric action depends on the transverse Kähler metric only via its basic Kähler class.

3.4 Quasi-regular GK-manifolds

The problem of finding solutions for (3.18) is formidable, especially in view of the fact that, for an irregular R-symmetry vector ξ, the transverse Kähler space itself implicitly depends on ξ. In this section we restrict our attention to quasi-regular GK-manifolds, where the master equation is a *bona fide* Kähler problem.

We start by analyzing the constraint equation (3.20) in this setting. We can integrate out the R-symmetry dependance to get

$$\int_{\mathcal{M}_{2n}} \rho^2 \wedge J^{n-2} = 0 \,. \tag{3.24}$$

This can be rephrased as the intersection identity

$$c_1(\mathcal{M}_{2n})^2 \cdot [J]^{n-2} = 0 \,, \tag{3.25}$$

where $c_1(\mathcal{M}_{2n})$ is the orbifold first Chern class of $\mathcal{M}_{2n}$. We view the Kähler orbifold as fixed, which means that the above equation is manifestly a restriction on the Kähler class, provided that $n > 2$. In the edge case, where $n = 2$, the Kähler class dependance drops out and the constraint is purely topological

$$c_1(\mathcal{M}_4)^2 = 0\,. \tag{3.26}$$

In combination with the assumption on the scalar curvature of $\mathcal{M}_4$ this places significant restrictions on Kähler surfaces that can admit solutions to the master equation. Yau [170] proved that a smooth compact Kähler surface $\mathcal{M}_4$ admits a Kähler metric of positive scalar curvature precisely if $\mathcal{M}_4$ is either a *rational surface* or a *ruled surface*.[19] Restricting our attention to *minimal*[20] Kähler surfaces, the only candidates that satisfy (3.26) are ruled surfaces over an elliptic curve.

We now switch gears and explore the consequences of the topological constraints for a general $n \geq 3$. First, note that, while orbifolds with vanishing first Chern class (Calabi-Yau orbifolds) trivially satisfy the constraint (and indeed solve the master equation) we are excluding them since we require the scalar curvature to be positive. It is quickly apparent that $\mathbb{CP}^n$ does not admit any solutions to the master equation for any $n \geq 2$. Indeed, the Kähler class is $[J] = kH$ and $c_1(\mathbb{CP}^n) = (n+1)H$, where H is the hyperplane class and k is a positive real number, so

$$c_1(\mathcal{M}_{2n})^2 \cdot [J]^{n-2} = (n+k+1) \neq 0\,. \tag{3.27}$$

More generally, on a smooth Fano manifold, $c_1(\mathcal{M}_{2n})$ is ample and this implies that

$$c_1(\mathcal{M}_{2n})^2 \cdot [J]^{n-2} > 0\,, \tag{3.28}$$

since the intersection product of ample divisors is positive [171, Example 1.2.5].

[19] A rational surface is a compact complex surface birational to $\mathbb{CP}^2$. Minimal rational surfaces are $\mathbb{CP}^2$ and the Hirzebruch surfaces $\mathbb{F}_n$ for $n = 0$ and $n \geq 2$. While there are slightly different definitions of ruled surfaces in the literature we take a ruled surface of genus $g \geq 1$ to be a compact complex surface isomorphic to a fiber bundle with fiber $\mathbb{CP}^1$ over a Riemannian surface of genus $g \geq 1$.

[20] A minimal surface is a surface without any -1-curves.

The supersymmetric action also reduces to a functional on the transverse Kähler orbifold. Integrating over the circle direction we obtain

$$S = \frac{(2\pi)^2 c I_{\mathcal{M}_{2n}}}{\mathfrak{m}} \int_{\mathcal{M}_{2n}} c_1(\mathcal{M}_{2n}) \wedge \frac{J^{n-1}}{(n-1)!} \,, \tag{3.29}$$

where $I_{\mathcal{M}_{2n}}$ is the Fano index of $\mathcal{M}_{2n}$ and $\mathfrak{m}$ is an integer that divides $I_{\mathcal{M}_{2n}}$. The reason for the numerical factor involving the Fano index is that, in general, the z coordinate can have period $\frac{2\pi I_{\mathcal{M}_{2n}}}{\mathfrak{m}}$. This is the case when Y_{2n+1} is the total space of the circle bundle associated to the (orbifold) line bundle $K^{\mathfrak{m}/I_{\mathcal{M}_{2n}}}$ over $\mathcal{M}_{2n}$.

Simple examples of GK-manifolds can be constructed by considering the product

$$\mathcal{M}_{2n} = \Sigma \times M \,, \tag{3.30}$$

where Σ is a Riemannian surface endowed with a constant scalar curvature metric and M is a compact manifold. In addition, we assume that $n \geq 3$. Dimensionally reducing the master equation (3.18) we obtain

$$\Box_M R_M = \frac{1}{2} R_M^2 + R_\Sigma R_M - (R_M)_{ij} (R_M)^{ij} \,. \tag{3.31}$$

The scalar curvature of Σ can be expressed using the Gauss-Bonnet theorem

$$R_\Sigma = \frac{2\pi\chi(\Sigma)}{\mathrm{Vol}(\Sigma)} \,, \tag{3.32}$$

where the volume of Σ is an *a priori* free parameter. Let now M be Kähler-Einstein with the Einstein constant λ. The scalar curvature is then constant, so the master equation reduces to the constraint

$$R_\Sigma = -n\lambda \,. \tag{3.33}$$

The scalar curvature of $\mathcal{M}_{2n}$ is

$$R = R_\Sigma + R_M = (n-2)\lambda \,. \tag{3.34}$$

Notice that, since we require the scalar curvature R to be positive, this entails that the Kähler-Einstein constant needs to be positive as well. In turn, this forces the Riemann surface Σ to be hyperbolic. To construct an example, one therefore can start with a fixed Kähler-Einstein manifold of dimension $2n-2$ with positive λ and take the product with a hyperbolic Riemann surface Σ with the volume

$$\mathrm{Vol}(\Sigma) = -\frac{2\pi\chi(\Sigma)}{n\lambda}\,. \tag{3.35}$$

Solutions of this form were first considered in [137] are known in the literature as the *universal twist solutions.*

3.5 Extremization Problem

We now turn to the formulation of a geometric dual of c-extremization, originally devised in [147]. This geometric extremization problem is in many ways analogous to the volume minimization problem in Sasaki-Einstein geometry [124] where minimizing the volume functional over the space of Sasaki structures gives rise to a well-defined convex optimization problem with a unique minimum. The corresponding construction in GK geometry is more subtle. The main difficulty resides in defining an appropriate space over which the supersymmetric action is extremized. This is the crux of the extremization problem we review in this section.

Recall from section 2 that the putative supergravity solutions need to satisfy flux quantization conditions to give rise to consistent string theory backgrounds. However, the conditions (2.8) and (2.9) are well-defined only for GK-manifolds and not for supersymmetric geometries. To extend them to the class of supersymmetric geometries we need to study the topology of Y_{2n+1} in more detail and impose some additional restrictions.

We start by requiring that the natural injective map

$$H^2_B(\mathcal{F}_\xi)/[\rho] \longrightarrow H^2(Y_{2n+1},\mathbb{R})\,, \tag{3.36}$$

is also surjective, so that

$$H^2_B(\mathcal{F}_\xi)/[\rho] \cong H^2(Y_{2n+1},\mathbb{R})\,. \tag{3.37}$$

This is not a particularly restrictive assumption. Notice that (3.8) together with the Gysin sequence implies that $H^1(Y_{2n+1},\mathbb{R}) = 0$ is sufficient for the above isomorphism to hold, so it is true for simply-connected Y_{2n+1}. For quasi-regular supersymmetric manifolds, the isomorphism is equivalent to requiring that every homology class in $H_{2n-1}(Y_{2n+1},\mathbb{R})$ admits a representative tangent to ξ.

By choosing tangent representatives of homology classes $S_\alpha \in H_{2n-1}(Y_{2n+1},\mathbb{R})$ we can rewrite the flux quantization conditions (2.8) and (2.9) as

$$\int_{S_\alpha} \eta \wedge \rho \wedge \frac{J^{n-2}}{(n-2)!} = \nu_n N_\alpha \,, \tag{3.38}$$

where N_α are integers and ν_n is a real number. While the conditions are physically meaningful only in dimensions 3 and 4, we can readily generalize them to an arbitrary n in this form. Note that the reason for including ν_n is again due to the physical origins of the problem, as ν_3 and ν_4 have a definite meaning in terms of the fundamental constants

$$\nu_3 = \frac{2(2\pi l_s)^4 g_s}{L_{10}^4} \qquad \text{and} \qquad \nu_4 = \frac{(2\pi l_p)^6}{L_{11}^6} \,. \tag{3.39}$$

However, the integral in (3.38) is not *a priori* well defined. To demonstrate this we restrict our attention to the quasi-regular supersymmetric geometries. The topological constraint (3.37) implies that S_α are in fact circle (orbi)bundles over transverse cycles $C_\alpha \in H_{2n-2}(\mathcal{M}_{2n},\mathbb{R})$. Consider $(2n-1)$-cycles S_α and S'_α and let C_A and C'_A be the corresponding transverse $(2n-2)$-cycles. Suppose now that

$$C_A - C'_A = \lambda \mathrm{P} \,, \tag{3.40}$$

where P is the Poincaré dual of $[\rho]$. This means that the cycles S_α and S'_α are homologous. However, calculating the integrals in (3.38) we obtain

$$\int_{S_\alpha} \eta \wedge \rho \wedge J^{n-2} - \int_{S'_\alpha} \eta \wedge \rho \wedge J^{n-2} = \lambda \int_{Y_{2n+1}} \eta \wedge \rho^2 \wedge J^{n-2} \,. \tag{3.41}$$

For (3.38) to be well defined we must therefore require that

$$\int_{Y_{2n+1}} \eta \wedge \rho^2 \wedge J^{n-2} = 0\,. \tag{3.42}$$

Notice that this is precisely the integrated version of the master equation we first encountered in (3.20). With this condition in place, the flux integrals depend only on the Killing vector ξ, the basic class $[J]$, and the homology classes S_α.

We can now formulate the general extremization problem for supersymmetric geometries. We start with a fixed choice of a complex cone $\mathrm{C}(Y_{2n+1}) = \mathbb{R}^+ \times Y_{2n+1}$, with a holomorphic volume form Ψ and a holomorphic $\mathrm{U}(1)^s$ action. Fix a choice of an R-symmetry vector ξ_0 and parametrize the space of R-symmetry vector fields as in (3.17). This gives rise to the foliation $\mathcal{F}_{\xi_0}$ and we may choose a basic transverse Kähler class $[J] \in H^{1,1}_B(\mathcal{F}_{\xi_0})$. Finally, we impose the flux quantization conditions (3.38) and, as a prerequisite for this, the topological conditions (3.37) and (3.20). We refer to Y_{2n+1} that satisfies all these conditions as an *off-shell* supersymmetric manifold. As we noted in section 3.3 the supersymmetric action is a function of the Killing vector ξ and the transverse Kähler class parameters. The topological conditions and flux quantization give constraints on the parameter space and we extremize the action over the remaining free parameters. GK-manifolds are necessarily extremal points of the supersymmetric action S so we can calculate the value S for a GK-manifold by using the extremization argument above. Note that the extremal point is a *bona fide* GK-manifold only if it is a solution of the master equation. This is completely analogous to the Sasakian geometry, where the minimum of the Sasakian volume is Sasaki-Einstein only if the transverse Kähler space is Kähler-Einstein.

The fundamental physical motivation to formulate the extremization principle is to provide a geometric dual of c-extremization for 2d $\mathcal{N} = (0, 2)$ superconformal field theories. This is now straightforward. Using the formula for the central charge (2.12) we can define the *trial* central charge as

$$c_{\text{trial}} = \frac{12(2\pi)^2}{\nu_3^2} S\,, \tag{3.43}$$

where S is the supersymmetric action with $n = 3$. Extremizing S we have

$$c_{\text{sugra}} = \frac{12(2\pi)^2}{\nu_3^2} S \Big|_{\text{on-shell}} . \tag{3.44}$$

Similarly, we can formulate a geometric extremization principle for 1d $\mathcal{N} = 2$ super quantum mechanics when $n = 4$. In this case, we can write down the *trial entropy*

$$\mathscr{S} = \frac{8\pi^2}{\nu_4^{3/2}} S . \tag{3.45}$$

On-shell this computes the 2d Newton's constant via (2.13)

$$\frac{1}{4G_2} = \mathscr{S}|_{\text{on-shell}} . \tag{3.46}$$

3.6 Toric Fibrations

Having established the general geometric extremization problem a natural next step is to consider a setting where the formulas can be explicitly calculated so that one is able to carry out the complete extremization argument. In Sasakian geometry one can rely on using the methods of toric Kähler geometry that has been extensively studied in the mathematics literature [172–174]. Applying this for toric Kähler cones, the authors of [123] were able to provide completely explicit expressions for the toric Sasakian volume in terms of the toric data. The analogous problem in the present context is more complicated. The most obvious problem is that, as noted in 3.2, the complex cone does not admit a natural Kähler structure, so the cones are not toric in the sense of [174]. To get around this Gauntlett, Martelli and Sparks [155] considered fibrations of the form

$$\begin{array}{ccc} X_{2r+1} & \hookrightarrow & Y_{2r+2k+1} \\ & & \downarrow \\ & & B_{2k} \end{array} , \tag{3.47}$$

such that $\mathrm{C}(X_{2r+1})$ is a Gorenstein toric Kähler cone, and B_{2k} is a compact Kähler manifold. For such toric fibrations all the data in the extremization problem can be expressed in terms of the toric volume of the Sasakian fiber X_5 and its derivatives. Since fibrations of this form are central to section 7 we provide a brief outline of the relevant geometry. We focus only on the case where $r = 2$ (i.e. on X_5), which is the case of interest for us.

Toric fiber

We begin by reviewing the geometry of the toric fiber. Consider a Sasaki five-fold X_5 with a Gorenstein toric Kähler cone $\mathrm{C}(X_5)$. By definition there exists an isometric holomorphic $\mathrm{U}(1)^3$ action on $\mathrm{C}(X_5)$ and we denote the generators by ∂_{φ_i}, for $i = 1, 2, 3$. In this basis, an R-symmetry vector can be expressed as

$$\xi = \sum_{i=1}^{3} b_i \partial_{\varphi_i} \,, \tag{3.48}$$

so the vector $\vec{b} = (b_1, b_2, b_3)$ parametrizes the choice of the R-symmetry vector. As shown in [174], the cone $\mathrm{C}(X_5)$ is a torus fibration over a three-dimensional polyhedral cone $\mathcal{C}$. We denote the normals to the facets of $\mathcal{C}$ by $\vec{v}_a$ for $a = 1, \ldots, d$. A fixed choice of $\mathrm{C}(X_5)$ and ξ the space transverse to ξ is Kähler. We wish to study the *master volume functional*

$$\mathcal{V} = \int_{X_5} \eta \wedge \frac{\omega^2}{2} \,, \tag{3.49}$$

as a function of ξ and the transverse Kähler class of ω. Importantly, here η is the contact form on the Sasakian X_5. To provide an explicit parametrization of the transverse Kähler classes we write

$$[\omega] = -2\pi \sum_{i=1}^{d} \lambda_a c_a \,, \tag{3.50}$$

where $\lambda_a \in \mathbb{R}$ and $c_a \in H^2_B(\mathcal{F}_\xi)$[21] are obtained as follows. On general grounds [174] there exist d toric divisors (corresponding to the edges of $\mathcal{C}$) in $\mathrm{C}(X_5)$ that restrict to integral classes $S_a \in H_3(X_5, \mathbb{Z})$. We take c_a to a basic representative of the cohomology class Poincaré dual to S_a. The transverse first Chern class has a particularly simple expression in terms of c_a

$$\frac{[\rho]}{2\pi} = \sum_{a=1}^{d} c_a \,. \tag{3.51}$$

[21] Notice that in the present context $\mathcal{F}_\xi$ refers to the transverse foliation on X_5.

Crucially, the parametrization (3.50) has some redundancy; amongst the d parameters only $d-2$ of them are independent.

The master volume (3.49) can be expressed explicitly in terms of the toric data, the vector $\vec{b}$ and the Kähler parameters. The expression was derived in [148] to be

$$\mathcal{V}(\vec{b},\{\lambda_a\},\{\vec{v}_a\}) = \frac{(2\pi)^3}{2}\sum_{a=1}^{d}\lambda_a \frac{\lambda_{a-1}[\vec{v}_a,\vec{v}_{a+1},\vec{b}] - \lambda_a[\vec{v}_{a-1},\vec{v}_{a+1},\vec{b}] + \lambda_{a+1}[\vec{v}_{a-1},\vec{v}_a,\vec{b}]}{[\vec{v}_{a-1},\vec{v}_a,\vec{b}][\vec{v}_a,\vec{v}_{a+1},\vec{b}]}\,. \tag{3.52}$$

Here $[\cdot,\cdot,\cdot]$ denotes a 3×3 determinant, and we cyclically order $\vec{v}_0 = \vec{v}_d$, $\vec{v}_{d+1} = \vec{v}_1$, with similar identifications for the λ_a. Note that two of the Kähler class parameters are redundant, so that $\mathcal{V}$ is effectively only a function of $d-2$ of the d Kähler class parameters λ_a and moreover it is quadratic in the Kähler parameters λ_a. We can rewrite it as

$$\mathcal{V} = (2\pi)^2\sum_{a,b=1}^{d}\frac{1}{2!}I_{ab}\lambda_a\lambda_b\,, \tag{3.53}$$

where I_{ab} are reminiscent of the intersection numbers of the toric divisors

$$I_{ab} = \int_{X_5}\eta\wedge c_a\wedge c_b = \frac{1}{(2\pi)^2}\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial\lambda_b}\,. \tag{3.54}$$

In general, these are not precisely the transverse intersection numbers since they manifestly depend on the Reeb vector ξ. They are however independent of the transverse Kähler parameters. With these in hand we can calculate integrals over other cohomology classes that will play a role in what follows. Using the expression for the first Chern class (3.51) we have

$$\int_{X_5}\eta\wedge\rho\wedge\omega = -(2\pi)^2\sum_{b=1}^{d}I_{ab}\lambda_b = -\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a}\,. \tag{3.55}$$

Finally, for the integrals over toric divisors S_a we have

$$\int_{S_a}\eta\wedge\omega = -2\pi\sum_{b=1}^{d}I_{ab}\lambda_b = -\frac{1}{2\pi}\frac{\partial\mathcal{V}}{\partial\lambda_a}\,. \tag{3.56}$$

Fibration

We now fiber X_5 over the Kähler base B_{2k}. The cases relevant to us in section 7 are $k = 1, 2$, but the argument does not depend on dimension. We can therefore leave the k to be arbitrary for the moment. Recall that there is a holomorphic $\mathrm{U}(1)^3$ action on $\mathrm{C}(X_5)$. Geometrically, we wish to fiber X_5 over the base B_{2k} by choosing three $\mathrm{U}(1)$ bundles on B_{2k} and gluing the fibers according using the transition functions of the bundles. This gives rise to Y_{2k+5} and additionally exhibits the cone $\mathrm{C}(Y_{2k+5})$ as a fibration of $\mathrm{C}(X_5)$ over B_{2k}.

To implement this explicitly we choose three $\mathrm{U}(1)$ gauge fields A_i with the corresponding curvature forms $F_i = \mathrm{d}A_i$. With this we can consider the twisted equivalents of all the forms on the toric fiber. The contact one-form gets replaced by

$$\eta \to \eta_{\text{twisted}} = 2\sum_{d=1}^{3} w_i(\mathrm{d}\varphi_i + A_i)\,, \tag{3.57}$$

where

$$w_i = \frac{1}{2}\partial_{\phi_i} \lrcorner \eta\,. \tag{3.58}$$

Analogously, we can write down the twisted Kähler form

$$\omega_{\text{twisted}} = \sum_{i=1}^{3} \mathrm{d}x_i \wedge (\mathrm{d}\phi_i + A_i) + \sum_{i=1}^{3} x_i F_i\,, \tag{3.59}$$

where the coordinates x_i are global Hamiltonian functions on X_5.[22] The transverse Kähler form can be expressed as

$$J = \omega_{\text{twisted}} + J_{B_{2k}} + \text{basic exact}. \tag{3.60}$$

The basic exact term is irrelevant, since all the quantities of interest depend only on the cohomology classes of the relevant forms.

In order to complete the set-up we need a holomorphic volume form on the cone $\mathrm{C}(Y_{2k+5})$. Notice that we started with the assumption that $\mathrm{C}(X_5)$ is Gorenstein, which

[22] Note that these exist since $b_1(X_5) = 0$ for all toric Sasaki manifolds [22].

means that it admits a global section $\Psi_{(3,0)}$ of the canonical bundle. We can choose a basis of the $\mathrm{U}(1)^3$ action such that

$$\mathcal{L}_{\partial_{\varphi_1}} \Psi_{(3,0)} = i\Psi_{(3,0)} \,, \tag{3.61}$$

and $\Psi_{(3,0)}$ is invariant under ∂_{φ_2} and ∂_{φ_3}. This means that the form $\Psi_{(3,0)}$ transforms as the section of the line bundle on the base B_{2k} with first Chern class equal to $\left[\frac{F_1}{2\pi}\right]$. To construct a holomorphic volume form on $\mathrm{C}(Y_{2k+5})$ we wedge $\Psi_{(3,0)}$ with a (local) section of the canonical bundle $K_{B_{2k}}$. The product gives rise to a nowhere vanishing holomorphic volume form provided

$$\left[\frac{F_1}{2\pi}\right] = c_1(B_{2k}) \,. \tag{3.62}$$

This fixes one of the three U(1) bundles we have chosen at the beginning. The choice of the remaining two remains unconstrained. We can now write down the toric formulas for the expressions appearing in the extremization problem explicitly. We only state the final formulas for $k = 1$ and $k = 2$, which we need in section 7. For more general results and additional details we refer the reader to [148, 149].

When the base is a Riemann surface Σ (i.e. $k = 1$), the formulas are particularly compact. The $\mathrm{U}(1)^3$ bundle is determined by a vector of three Chern numbers $\vec{n} = (n_1, n_2, n_3)$, where $n_1 = c_1(\Sigma) = 2 - 2g$. We take the Kähler form on Σ to be $J_\Sigma = A\mathrm{vol}_\Sigma$, where A is a free positive parameter. Then the supersymmetric action is

$$S = -A \sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a} - 2\pi b_1 \sum_{i=1}^{3} n_i \frac{\partial \mathcal{V}}{\partial b_i} \,. \tag{3.63}$$

The topological constraint equation can be expressed as

$$A \sum_{a,b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial \lambda_b} - 2\pi n_1 \sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a} + 2\pi b_1 \sum_{a=1}^{d} \sum_{i=1}^{3} n_i \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial b_i} = 0 \,. \tag{3.64}$$

Finally, there are two classes of five-cycles to consider for flux quantization. First is the fiber X_5 over a fixed point in Σ. The corresponding flux quantization condition is

$$\nu_3 N = -\sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a} \,. \tag{3.65}$$

The second class consists of toric divisors S_a fibered over the base Σ. In this case we denote the flux numbers with M_a and they are given by

$$\nu_3 M_a = \frac{A}{2\pi}\sum_{b=1}^{d}\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial\lambda_b} + b_1\sum_{i=1}^{3} n_i \frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial b_i}\,. \tag{3.66}$$

We now move to the case where X_5 is fibered over B_4. The supersymmetric action now takes the form

$$S = -\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a}\mathrm{vol}(B_4) - b_1\sum_{i=1}^{3}\frac{\partial\mathcal{V}}{\partial b_i}\int_{B_4} F_i\wedge J_{B_4} + \frac{b_1}{4}\sum_{a=1}^{3}\sum_{i,j=1}^{4}\lambda_a\frac{\partial^2\mathcal{V}}{\partial b_j\partial v_a^i}\int_{B_4}\frac{1}{2}F_i\wedge F_j\,, \tag{3.67}$$

and the topological constraint is

$$\begin{aligned}&\sum_{a,b=1}^{d}\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial\lambda_b}\mathrm{vol}(B_4) + b_1\sum_{i=1}^{3}\sum_{a=1}^{d}\frac{\partial\mathcal{V}^2}{\partial\lambda_a\partial b_i}\int_{B_4}F_i\wedge J_{B_4} - \sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a}\int_{B_4}F_1\wedge J_{B_4}\\ &\quad + b_1^2\sum_{i,j=1}^{4}\frac{\partial^{\mathcal{V}}}{\partial b_i\partial b_j}\int_{B_4}\frac{1}{2}F_i\wedge F_j = 0\,.\end{aligned} \tag{3.68}$$

We now quantize fluxes over seven-cycles, which again fall into two classes. One can fiber X_5 over the two-cycle C_α in B_4 or fiber the toric divisors S_a over the Kähler surface B_4. The flux quantization condition for the former reads

$$\nu_4 N_\alpha = -\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a}\int_{C_\alpha}J_{B_4} - b_1\sum_{i=1}^{3}\frac{\partial\mathcal{V}}{\partial b_i}\int_{C_\alpha}F_i\,. \tag{3.69}$$

For the latter we have

$$\begin{aligned}\nu_4 M_a = {}& \frac{1}{2\pi}\sum_{b=1}^{d}\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial\lambda_b}\mathrm{vol}(B_4) + \frac{b_1}{2\pi}\sum_{i=1}^{3}\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial b_i}\int_{B_4}F_i\wedge J_{B_4}\\ &- \frac{b_1}{2\pi}\sum_{i,j=1}^{4}\lambda_a\frac{\partial^2\mathcal{V}}{\partial b_j\partial v_a^i}\int_{B_4}\frac{1}{2}F_i\wedge F_j\,.\end{aligned} \tag{3.70}$$

This collection of formulas gives the complete description of the extremization problem in terms of the toric data and specifically the derivatives of the master volume. Recall that, for supersymmetric geometries, b_1 is fixed. In particular, $b_1 = 2$ for $k = 1$ and $b_1 = 1$ for $k = 2$. Importantly, in the above formulas, one must first take the partial derivative with respect to b_1 and only then fix it to the relevant constant. The extremization procedure now amounts to explicitly solving the topological constraint and the flux quantization condition for the Kähler parameters and R-symmetry vector, and subsequently extremizing the supersymmetric action with respect to the remaining free parameters. We provide concrete examples of this procedure in section 7.

4 Holographic c-Extremization in F-Theory

Until this point we have been focused almost entirely on the geometry of GK-manifolds. In this section we switch gears and extend the geometric dual of c-extremization to encompass AdS_3 geometries in *F-theory*, i.e. Type IIB supergravity with a holomorphically varying axio-dilaton τ, which are holographically dual to 2d $\mathcal{N} = (0, 2)$ SCFTs. To begin with we review the class of geometries [158] that generalize the pure type IIB AdS_3 backgrounds we considered in section 2. We then extend the holographic c-extremization in Type IIB (*cf.* section 3.5) to these F-theory geometries.

4.1 AdS_3 Backgrounds

We consider holographic duals to 2d $(0, 2)$ SCFTs realized in Type IIB with five-form flux and varying axio-dilaton [158]. The geometry underlying the solutions is $\mathrm{AdS}_3 \times Y_7$, and is supported by RR five-form flux

$$
\begin{aligned}
\mathrm{d}s^2_{10} &= L_{10}^2\, \mathrm{e}^{-B_{10}/2} \left[\mathrm{d}s^2(\mathrm{AdS}_3) + \mathrm{d}s^2(Y_7)\right] \,, \\
F_5 &= -L_{10}^4 \left(\mathrm{vol}_{\mathrm{AdS}_3} \wedge F + *_7 F\right) \,.
\end{aligned}
\tag{4.1}
$$

Note that this is formally the same as the IIB ansatz (2.1), however the axio-dilaton field is now allowed to vary over the internal space Y_7. The brane configuration corresponding to these geometries consists of N D3-branes on $\mathbb{R}^{1,1} \times C$, where C are curves in $\mathcal{M}_6$, above which the axio-dilaton varies. The auxiliary elliptic fiber degenerates over the loci that are subspaces wrapped by the 7-branes, which in the present case have world-volume $\mathcal{W}_8 = \mathrm{AdS}_3 \times \widetilde{S}$, where $\widetilde{S}$ are five-cycles in Y_7.

The supersymmetry equations of Type IIB get modified when the axio-dilaton is varying. The $SL(2,\mathbb{Z})$ self-duality of Type IIB induces a so-called $\mathrm{U}(1)_D$ symmetry, which acts on the fermions and supercharges by

$$\mathrm{U}(1)_D: \qquad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}): \qquad \mathrm{e}^{\mathrm{i}\alpha_\gamma} = \frac{|c\tau+d|}{c\tau+d}\,. \tag{4.2}$$

The action on the fermions with half-integral charge extends the $SL(2,\mathbb{Z})$ by a $\mathbb{Z}_2$ to the metaplectic group [175]. The duality U(1)-symmetry $\mathrm{U}(1)_D$ can be gauged, and then defines a line bundle $\mathcal{L}$, with connection

$$Q = -\frac{1}{2\tau_2}\mathrm{d}\tau_1\,, \tag{4.3}$$

where $\tau = \tau_1 + \mathrm{i}\tau_2$. Furthermore, it is convenient to define the one-form

$$\mathcal{P} = \frac{\mathrm{i}}{2\tau_2}\mathrm{d}\tau\,. \tag{4.4}$$

Supersymmetry implies that τ is preserved by ξ (i.e. $\mathcal{L}_\xi\tau = 0$) and that it varies holomorphically over the transverse Kähler space. The bundle $\mathcal{L}$ is then transversely holomorphic with the curvature given by

$$\mathrm{i}\,\mathrm{d}\mathcal{P} = \mathrm{d}Q = -\mathrm{i}\mathcal{P}\wedge\bar{\mathcal{P}}\,. \tag{4.5}$$

Next we consider how the geometry of Y_7 itself is constrained by supersymmetry. Let η be the one-form dual to ξ. Choosing a local coordinate z so that $\xi = 2\partial_z$, the local expression for η is given by $\eta = \frac{1}{2}(\mathrm{d}z + P)$.[23] The derivative of the local one-form P then satisfies

$$\mathrm{d}P = \rho_6 - \mathrm{i}\mathcal{P}\wedge\bar{\mathcal{P}}\,, \tag{4.6}$$

where ρ_6 is the transverse Ricci form. Finally, there is a relation between the scalar curvature R_6 of the transverse Kähler space and the warp factor B_{10}

$$\mathrm{e}^{B_{10}} = \frac{1}{8}\left(R_6 - 2|\mathcal{P}|^2\right)\,. \tag{4.7}$$

[23] Note that we are following the conventions in [147], which are different from the conventions in [158]. The naming differences are particularly subtle when it comes to the connection one-forms. The reader should be aware that $P_{\text{here}} = -\rho_{\text{there}}$, $\mathcal{P}_{\text{here}} = P_{\text{there}}$ and $(\rho_6)_{\text{here}} = (\mathfrak{R}_6)_{\text{there}}$.

Before proceeding let us summarize all the expressions for 10d fields after having imposed the supersymmetry equations:

$$
\begin{aligned}
\mathrm{d}s^2_{10} &= L^2_{10}\, \mathrm{e}^{-B_{10}/2} \left[\mathrm{d}s^2\left(\mathrm{AdS}_3\right) + \frac{1}{4}\left(\mathrm{d}z + P\right)^2 + \mathrm{e}^{B_{10}} \mathrm{d}s^2\left(\mathcal{M}_6\right) \right] , \\
F_5 &= -L^4_{10}\left(\mathrm{vol}_{\mathrm{AdS}_3} \wedge F + *_7 F\right) , \\
F &= -2J_6 + \frac{1}{2}\mathrm{d}\left[e^{-B_{10}}\left(\mathrm{d}z + P\right)\right] , \\
\mathrm{d}P &= \rho_6 - \mathrm{i}\mathcal{P} \wedge \bar{\mathcal{P}} , \\
\mathrm{e}^{B_{10}} &= \frac{1}{8}\left(R_6 - 2|\mathcal{P}|^2\right) .
\end{aligned}
\tag{4.8}
$$

Here J_6 is the Kähler form on $\mathcal{M}_6$. Notice that all the 10d fields are completely determined by the transverse Kähler metric together with the line bundle $\mathcal{L}$. We refer to Y_7 satisfying the supersymmetry equations, and therefore having all the properties outlined above, as a *supersymmetric geometry*. For constant axio-dilaton $\mathcal{P} = 0$, and the above reduce to the Type IIB equations in [133].

All the above results hold off-shell, by which we mean that we merely impose supersymmetry, without imposing the equations of motion. For supersymmetric geometries the equations of motion reduce to a PDE on the transverse Kähler space involving the metric and the connection on the line bundle $\mathcal{L}$. This is referred to as the *master equation* in [158], and is given by

$$
\Box_6(R_6 - 2|\mathcal{P}|^2) = \frac{1}{2}R_6^2 - (R_6)_{\mu\nu}(R_6)^{\mu\nu} + 2|\mathcal{P}|^2 R_6 - 4(R_6)_{\mu\nu}\mathcal{P}^\mu \bar{\mathcal{P}}^\nu . \tag{4.9}
$$

Geometries satisfying this equation will be called *on-shell*, and are solutions of the Type IIB supergravity equations with varying axio-dilaton, provided that the five-form flux is appropriately quantized. We will return to the flux quantization conditions in a later section.

The F-theory perspective amounts to giving the varying axio-dilaton a geometric interpretation in terms of an auxiliary elliptic fibration

$$\begin{array}{ccc} \mathbb{E}_\tau & \hookrightarrow & \mathcal{M}_8^\tau \\ & & \downarrow \\ & & \mathcal{M}_6 \end{array} \,. \tag{4.10}$$

The total space $\mathcal{M}_8^\tau$ is Kähler but not Calabi-Yau[24]. Locally, away from the singular fibers, the metric on the total space is

$$\mathrm{d}s^2(\mathcal{M}_8^\tau) = \frac{1}{\tau_2}\left[(\mathrm{d}\psi + \tau_1 \mathrm{d}\phi)^2 + \tau_2^2 \mathrm{d}\phi^2\right] + \mathrm{d}s^2(\mathcal{M}_6)\,. \tag{4.11}$$

The master equation can then be interpreted as a curvature condition on the total space $\mathcal{M}_8^\tau$. Taking this view, the master equation is

$$\Box_8 R_8 = \frac{1}{2}R_8^2 - (R_8)_{\mu\nu}(R_8)^{\mu\nu}\,, \tag{4.12}$$

which is precisely the form of the equation for constant axio-dilaton, just in two dimensions higher.

4.2 Supersymmetric Action

The Type IIB supergravity equations including varying τ are [158]

$$R_{\mu\nu} = 2\mathcal{P}_{(\mu}\bar{\mathcal{P}}_{\nu)} + \frac{1}{96}(F_5)_{\mu\sigma_1\cdots\sigma_4}(F_5)_\nu{}^{\sigma_1\cdots\sigma_4}\,, \qquad \mathrm{d} * F_5 = 0\,, \tag{4.13}$$

where $\mu,\nu = 0,1,...,9$. Writing out the components of the Einstein equations along the internal space Y_7 we obtain

$$\begin{aligned} 0 = R_{7ab} - 2\mathcal{P}_{(a}\bar{\mathcal{P}}_{b)} + \frac{1}{2}\nabla_a B_{10}\nabla_b B_{10} + 2\nabla_{ab}B_{10} + \frac{1}{4}\nabla^2 B_{10} g_{7ab} - \frac{1}{2}\left(\mathrm{d}B_{10}\right)^2 g_{7ab} \\ + \frac{1}{2}\mathrm{e}^{2B_{10}}F_{ac}F_b{}^d - \frac{1}{8}\mathrm{e}^{2B_{10}}F^2 g_{7ab}\,, \end{aligned} \tag{4.14}$$

[24] We will denote spaces which enjoy an elliptic fibration with a superscript τ, indicating the complex structure of the elliptic fiber.

where $a, b = 1, 2, ..., 7$. This arises by extremizing the following action functional

$$S_F = \int_{Y_7} \mathrm{e}^{-2B_{10}} \left[R_7 - 2|\mathcal{P}|^2 - 6 + \frac{9}{2} \left(\mathrm{d}B_{10}\right)^2 + \frac{1}{4}\mathrm{e}^{2B_{10}} F^2 \right] \mathrm{vol}_{Y_7} \,, \tag{4.15}$$

with respect to the 7d metric, and generalizes the action functional for constant τ in [135]. Varying the other fields in this action gives rise to the remaining Type IIB equations of motion.

We now specialize to the case where Y_7 is supersymmetric. Using the notation introduced in the previous subsection, the metric on Y_7 can be written as

$$\mathrm{d}s^2 \left(Y_7\right) = \eta^2 + \mathrm{e}^{B_{10}} \mathrm{d}s^2 \left(\mathcal{M}_6\right) \,, \tag{4.16}$$

where $\mathrm{d}s^2 \left(\mathcal{M}_6\right)$ is the transverse Kähler metric. Writing out the Ricci scalar we obtain (up to total derivatives)

$$R_7 = \mathrm{e}^{-B_{10}} R_6 - 5\mathrm{e}^{-B_{10}} \left(\mathrm{d}B_{10}\right)^2 - \frac{1}{16}\mathrm{e}^{-2B_{10}} (\mathrm{d}P)^2 \,. \tag{4.17}$$

Furthermore, the flux term in the action is

$$\frac{1}{4}\mathrm{e}^{2B_{10}} F^2 = 6 - \frac{1}{2}\mathrm{e}^{-B_{10}} \left(R_6 - 2|\mathcal{P}|^2\right) + \frac{1}{16}\mathrm{e}^{-2B_{10}} (\mathrm{d}P)^2 + \frac{1}{2}\mathrm{e}^{-B_{10}} \left(\mathrm{d}B_{10}\right)^2 \,. \tag{4.18}$$

Combining these, we find that the action evaluated on supersymmetric geometries is given by

$$\begin{aligned} S_F =& \frac{1}{2} \int_{Y_7} \mathrm{e}^{-3B_{10}} \left(R_6 - 2|\mathcal{P}|^2\right) \mathrm{vol}_{Y_7} \\ =& \int_{Y_7} \eta \wedge \left(\rho_6 - \mathrm{i}\mathcal{P} \wedge \bar{\mathcal{P}}\right) \wedge \frac{J_6^2}{2} \,. \end{aligned} \tag{4.19}$$

We may rewrite this in a slightly nicer way as follows. Notice that $\mathrm{i}\mathcal{P} \wedge \bar{\mathcal{P}}$ is the curvature of the connection (4.3) and hence is a representative of $2\pi c_1(\mathcal{L})$. The action only depends on the cohomology class and not the particular representative, so we can rewrite it in terms of $c_1(\mathcal{L})$ as

$$S_F = \int_{Y_7} \eta \wedge \left(\rho_6 - 2\pi c_1(\mathcal{L})\right) \wedge \frac{J_6^2}{2} \,. \tag{4.20}$$

For fixed R-symmetry vector ξ this function depends only on the transverse Kähler class of the Kähler form J_6, and here also the first Chern class of the line bundle $\mathcal{L}$. We can again write down the trial central charge using the Brown-Henneaux formula (2.12)

$$c_{\text{trial}} = \frac{12(2\pi)^2}{\nu_3^2} S_F \,, \tag{4.21}$$

An F-theory solution necessarily extremizes c_{trial} over the class of off-shell geometries, and the central charge of the holographic dual SCFT is determined by

$$c_{\text{sugra}} = \frac{12(2\pi)^2}{\nu_3^2} S_F \Big|_{\text{on-shell}} \,. \tag{4.22}$$

4.3 Flux Quantization

We now describe the flux quantization conditions for the F-theory extremization problem. These amount precisely the same considerations as in 3.5 with the only difference in replacing the ρ_6 with $\rho_6 - 2\pi c_1(\mathcal{L})$. For the sake of clarity we provide a synopsis of the flux quantization conditions specialized to the F-theory context.

The general flux integrals are the same as in (2.8) but the restriction of the five-form flux to Y_7 is now given by [158]

$$F_5|_{Y_7} = \frac{L_{10}^4}{4} \left[(\mathrm{d}z + P) \wedge \left(\rho_6 - \mathrm{i}\mathcal{P} \wedge \bar{\mathcal{P}}\right) \wedge J_6 + \frac{1}{2} *_6 \mathrm{d}\left(R_6 - 2|\mathcal{P}|^2\right) \right] . \tag{4.23}$$

If we impose the topological restriction

$$H^2(Y_7, \mathbb{R}) \cong H^2_B(\mathcal{F}_\xi) / \left[\rho_6 - 2\pi c_1(\mathcal{L})\right] \,, \tag{4.24}$$

the flux quantization conditions can also be expressed as

$$\nu_3 N_\alpha^F = \int_{S_\alpha} \eta \wedge (\rho_6 - 2\pi c_1(\mathcal{L})) \wedge J_6 \,, \tag{4.25}$$

where S_α are again representatives tangent to ξ. As in (3.5) this still does not guarantee that the integrals in (4.25) are independent of the choice of S_α.

The second topological condition we have to impose is

$$\int_{Y_7} \eta \wedge (\rho_6 - 2\pi c_1(\mathcal{L}))^2 \wedge J_6 = 0\,. \tag{4.26}$$

As in section 3.5 the two topological conditions ensure that the flux quantization conditions are well defined for supersymmetric geometries. Furthermore, (4.26) is just the integrated version of the F-theory master equation (4.9), which follows by writing the latter as

$$\Box_6 \left(R_6 - 2|\mathcal{P}|^2\right) = (J_6 \wedge J_6) \lrcorner \left[\left(\rho_6 - \mathrm{i}\mathcal{P} \wedge \bar{\mathcal{P}}\right) \wedge \left(\rho_6 - \mathrm{i}\mathcal{P} \wedge \bar{\mathcal{P}}\right)\right]\,. \tag{4.27}$$

Integrating this equation over Y_7, the left hand side vanishes using Stokes' theorem. Using the identity

$$[(J_6 \wedge J_6) \lrcorner (a \wedge a)] \frac{J_6^3}{3!} = 2a^2 \wedge J_6\,, \tag{4.28}$$

which holds for any (1,1)-form, we obtain precisely (4.26).

4.4 The Complex Cone and the Geometric Extremization Problem

Having set up the abstract extremization problem, we now turn to the question of how to parametrize the class of off-shell supersymmetric geometries over which we extremize the action, by constructing a complex cone associated to Y_7 that allows us to parametrize the space of R-symmetry vectors on Y_7.

Consider the cone $\mathrm{C}(Y_7)$ with metric

$$\mathrm{d}s^2(\mathrm{C}(Y_7)) = \mathrm{d}r^2 + r^2 \mathrm{d}s^2(Y_7)\,, \tag{4.29}$$

where $r \in \mathbb{R}_{>0}$. As for the constant axio-dilaton case, we can consider the natural, locally defined $(4,0)$-form on the cone that is given by

$$\Omega_{(4,0)} = \mathrm{e}^{\mathrm{i}z} \mathrm{e}^{3B_{10}/2} r^3 \left[\mathrm{d}r - \frac{\mathrm{i}r}{2}(\mathrm{d}z + P)\right] \wedge \Omega_6\,. \tag{4.30}$$

However, this form does not extend to a global form unless the duality bundle $\mathcal{L}$ is trivial, i.e. the axio-dilaton is constant. To see this note that (4.6) implies that $\mathrm{e}^{\mathrm{i}z}$ transforms as a local section[25] of $K^{-1}_{\mathcal{M}_6} \otimes \mathcal{L}^{-1}$, whereas Ω_6 is a local section of $K_{\mathcal{M}_6}$. The object $\Omega_{(4,0)}$ therefore transforms as a local section of $\mathcal{L}^{-1}$. Since $\mathcal{L}$ admits a global holomorphic section its dual does not, unless $\mathcal{L}$ is trivial. In particular, $\Omega_{(4,0)}$ is not globally defined as a form, when the axio-dilaton varies.

To circumvent this issue we use the auxiliary elliptic fibration introduced in (4.10), where the complex structure of the elliptic fiber encodes the axio-dilaton. Moreover, we assume that this fibration has a holomorphic section $\sigma : \mathcal{M}_6 \to \mathcal{M}_8^\tau$. Since τ is preserved by the Killing vector we can construct an elliptic fibration[26] over Y_7 by letting the elliptic fiber be constant along the orbits of ξ. This gives a 9d space, which we denote by Y_9^τ, endowed with the metric

$$\mathrm{d}s^2(Y_9^\tau) = \mathrm{d}s^2(Y_7) + \mathrm{e}^{B_{10}}\mathrm{d}s^2(\mathbb{E}_\tau) = \eta^2 + \mathrm{e}^{B_{10}}\mathrm{d}s^2\left(\mathcal{M}_8^\tau\right) . \tag{4.31}$$

One can think of Y_9^τ as an elliptic fibration over Y_7, with the elliptic fibers being invariant along the Killing vector direction $\xi = 2\partial_z$. The differential forms pull back from Y_7 to Y_9^τ, and as usual we conflate the forms with their lifts to avoid notational clutter. We can now define the cone over Y_9^τ as

$$\mathrm{d}s^2(\mathrm{C}(Y_9^\tau)) = \mathrm{d}r^2 + r^2\mathrm{d}s^2(Y_9^\tau) . \tag{4.32}$$

This cone admits a natural SU(5) structure, with the $(5,0)$-form locally given by

$$\Omega_{(5,0)} = \mathrm{e}^{\mathrm{i}z}\mathrm{e}^{2B_{10}}r^4\left[\mathrm{d}r - \frac{\mathrm{i}r}{2}\left(\mathrm{d}z + P\right)\right] \wedge \Omega_8 . \tag{4.33}$$

The fundamental two-form is exactly the same as in [147] and is not relevant for our purposes. The local holomorphic volume form on $\mathcal{M}_8^\tau$ is

$$\Omega_8 = \mathcal{P} \wedge \Omega_6 , \tag{4.34}$$

[25] We are suppressing the pullbacks in the notation for various bundles.

[26] This is a fibration in the sense of algebraic geometry, i.e. with a generic fiber being a smooth elliptic curve.

which satisfies

$$\mathrm{d}\Omega_8 = \mathrm{i}P \wedge \Omega_8 \,. \tag{4.35}$$

The local holomorphic volume form $\Omega_{(5,0)}$ now does extend to a global form, as Ω_8 is a section of $K_{\mathcal{M}_6} \otimes \mathcal{L}$ and the extra $\mathcal{L}$ now cancels with the $\mathcal{L}^{-1}$. In addition, by using (4.35) we can show that the holomorphic volume form is conformally closed

$$\mathrm{d}\Psi = 0 \,, \qquad \Psi \equiv \mathrm{e}^{-2B_{10}} r^{-7} \Omega_{(5,0)} \,, \tag{4.36}$$

i.e. $\mathrm{C}(Y_9^\tau)$ has vanishing first Chern class. We find that Ψ is charged under the R-symmetry vector field

$$\mathcal{L}_\xi \Psi = 2\mathrm{i}\Psi \,. \tag{4.37}$$

This implies that ξ is a holomorphic vector field, which is paired with the radial vector field under the complex structure $\mathcal{I}(\xi) = -r\partial_r$.

We are now in precisely the same set-up as in the section 3.5. Suppose now that $\mathrm{C}(Y_9^\tau)$ admits a holomorphic $\mathrm{U}(1)^s$ action, generated by a set of holomorphic vector fields ∂_{φ_i}, $i = 1, 2, ..., s$. We parametrize the general R-symmetry vector in terms of these holomorphic vector fields

$$\xi = \sum_{i=1}^{s} b_i \partial_{\varphi_i} \,, \tag{4.38}$$

and choose a basis where Ψ has charge 1 under ∂_{φ_1} and charge 0 under the remaining generators. This fixes $b_1 = 2$, and leaves the remaining b_i, $i = 2, 3, ..., s$ as free variables to be extremized over in S_F. We shall exemplify the extremization procedure in section 7.

5 M/F-Duality and Holographic $\mathcal{I}$-Extremization

The axio-dilaton in F-theory can at times be somewhat obscure, as it is not part of the geometry of the spacetime. To clarify the role of the elliptic fibration, it is often useful to consider a dual M-theory background. For AdS_3 solutions, this could either be a dual AdS_3 or AdS_2 solution of M-theory. In the current framework we will dualize to the latter, which in the field theory corresponds to the circle-reduction to a 1d SCQM. The associated geometric extremization principle, holographic $\mathcal{I}$-extremization, was studied in [147]. In this section we will apply this formalism to the class of geometries that are dual to the F-theory backgrounds and study the extremization principle.

5.1 M/F-Duality for AdS-Geometries

To begin with, we will briefly summarize M/F-duality, applied to the F-theory AdS_3 geometries discussed in section 4, which are mapped to AdS_2 geometries in M-theory.

Any M-theory geometry with an elliptic fibration can be dualized to obtain a corresponding F-theory geometry with varying axio-dilaton, by first reducing to Type IIA along one cycle of the elliptic fibration and subsequently T-dualizing along the second cycle. This approach is valid away from singular fibers, where locally the geometry of the elliptic fiber is

$$\mathrm{d}s^2\left(\mathbb{E}_\tau\right) = \frac{L_{11}^2}{\tau_2}\left((\mathrm{d}x+\tau_1\mathrm{d}y)^2+\tau_2^2\mathrm{d}y^2\right) . \tag{5.1}$$

We have introduced an overall M-theory length scale L_{11} and the periodic coordinates $x \sim x + 2\pi\Delta x$ and $y \sim y + 2\pi\Delta y$, where we set $\Delta x = \Delta y$. The M-theory background can be dimensionally reduced on a circle to yield a Type IIA background. Specifically, the two metrics are related as [176]

$$\mathrm{d}s_{11}^2 = L_{11}^2\left(\frac{l_s}{l_p}\right)^4 \mathrm{e}^{4\phi_{\mathrm{IIA}}/3}(\mathrm{d}x+C_1)^2 + \left(\frac{l_p}{l_s}\right)^2 \mathrm{e}^{-2\phi_{\mathrm{IIA}}/3}\mathrm{d}s_{\mathrm{IIA}}^2\,, \tag{5.2}$$

where l_s and l_p are the string and 11d Planck lengths, and $\mathrm{d}s_{\mathrm{IIA}}^2$, $\mathrm{e}^{\phi_{\mathrm{IIA}}}$ and C_1 are respectively the metric, the fluctuating dilaton and the RR one-form potential of Type IIA. Comparison with (5.1) allows us to immediately identify

$$C_1 = \tau_1\mathrm{d}y\,, \qquad \mathrm{e}^{4\phi_{\mathrm{IIA}}/3} = \left(\frac{l_p}{l_s}\right)^4\frac{1}{\tau_2}\,, \qquad \mathrm{d}s_{\mathrm{IIA}}^2 = L_{11}^2\left(\frac{l_s}{l_p}\right)^2 \mathrm{e}^{2\phi_{\mathrm{IIA}}/3}\tau_2\mathrm{d}y^2 + \mathrm{d}s_9^2\,, \tag{5.3}$$

where $\mathrm{d}s_9^2$ is the metric on the 9d space of the Type IIA geometry orthogonal to the y circle. Dimensionally reducing the M-theory action to that of Type IIA (here it is sufficient to consider the Ricci scalar term) fixes the period of the circle to be

$$L_{11}\Delta x = \frac{l_p^3}{l_s^2}\,, \tag{5.4}$$

where we have used $16\pi G_{11} = (2\pi)^8 l_p^9$ and $16\pi G_{10} = (2\pi)^7 l_s^8$ for the 11d and 10d Newton constants, respectively. Hence, we can express the volume of the elliptic fiber in terms of fundamental length scales as

$$\mathrm{vol}\,(\mathbb{E}_\tau) = (2\pi\Delta x)^2 = \frac{(2\pi)^2 l_p^6}{L_{11}^2 l_s^4}\,. \tag{5.5}$$

Carrying out T-duality along the y circle results in

$$R_{\mathrm{IIB}} = \frac{l_s^2}{L_{11}\Delta y} = \frac{l_s^4}{l_p^3}\,, \qquad C_0 = (C_1)_y = \tau_1\,, \qquad \mathrm{e}^{\phi_{\mathrm{IIB}}} = \frac{l_s}{\frac{l_s}{l_p} L_{11}\Delta y\, \mathrm{e}^{\phi_{\mathrm{IIA}}/3}\sqrt{\tau_2}}\mathrm{e}^{\phi_{\mathrm{IIA}}} = \frac{1}{\tau_2}\,. \tag{5.6}$$

This then identifies $\tau = \tau_1 + \mathrm{i}\tau_2 = C_0 + \mathrm{i}\,\mathrm{e}^{-\phi_{\mathrm{IIB}}}$.

Applied to the AdS$_3$ F-theory geometries of section 4, the key observation is that we dualize along the AdS direction by first writing AdS$_3$ as a circle fibration over AdS$_2$ [177]

$$\mathrm{d}s^2\,(\mathrm{AdS}_3) = \frac{1}{4}\left(-r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + (\mathrm{d}\phi + a_1)^2\right) = \frac{1}{4}\mathrm{d}s^2\,(\mathrm{AdS}_2) + \frac{1}{4}(\mathrm{d}\phi + a_1)^2\,, \tag{5.7}$$

where $\phi \sim \phi + \Delta\phi$ is the circle coordinate and $a_1 = r\mathrm{d}t$ so that $\mathrm{d}a_1 = \mathrm{vol}_{\mathrm{AdS}_2}$. The F-theory metric can then be written as

$$\mathrm{d}s_{10}^2 = L_{11}^2 \mathrm{e}^{-B_{11}/2}\left[\mathrm{d}s^2\,(\mathrm{AdS}_2) + (\mathrm{d}\phi + a_1)^2 + (\mathrm{d}z + P)^2 + \mathrm{e}^{B_{11}}\mathrm{d}s^2\,(\mathcal{M}_6)\right]\,, \tag{5.8}$$

where we have taken the M/F-theory length scales and warp factors to be related by

$$L_{10} = \sqrt{2}L_{11}\,, \qquad \mathrm{e}^{B_{10}} = \frac{1}{4}\mathrm{e}^{B_{11}}\,. \tag{5.9}$$

T-duality along the ϕ direction results in

$$\begin{aligned}
\mathrm{d}s^2_{\mathrm{IIA}} &= L_{11}^2\sqrt{\tau_2}\,\mathrm{e}^{B_{11}/2}\mathrm{d}\phi^2 + L_{11}^2\frac{\mathrm{e}^{-B_{11}/2}}{\sqrt{\tau_2}}\left[\mathrm{d}s^2\left(\mathrm{AdS}_2\right) + (\mathrm{d}z+P)^2 + \mathrm{e}^{B_{11}}\mathrm{d}s^2\left(\mathcal{M}_6\right)\right]\,, \\
\mathrm{e}^{-2\phi_{\mathrm{IIA}}} &= \frac{l_p^6}{l_s^6}\tau_2^{3/2}\mathrm{e}^{-B_{11}/2}\,, \\
H &= L_{11}^2\mathrm{d}\phi\wedge\mathrm{vol}_{\mathrm{AdS}_2}\,, \\
F_2 &= L_{11}\mathrm{d}\tau_1\wedge\mathrm{d}\phi\,, \\
F_4 &= \frac{1}{2}L_{11}^3\mathrm{vol}_{\mathrm{AdS}_2}\wedge F\,.
\end{aligned} \tag{5.10}$$

Finally, we uplift to M-theory using the metric in (5.2). We find that the M-theory geometries dual to the AdS_3 F-theory geometries in (4.8) are

$$\begin{aligned}
\mathrm{d}s^2_{11} &= L_{11}^2\mathrm{e}^{-2B_{11}/3}\left[\mathrm{d}s^2\left(\mathrm{AdS}_2\right) + (\mathrm{d}z+P)^2 + \mathrm{e}^{B_{11}}\mathrm{d}s^2\left(\mathcal{M}_8^\tau\right)\right]\,, \\
G_4 &= L_{11}^3\mathrm{vol}_{\mathrm{AdS}_2}\wedge\left[-J_8 + \mathrm{d}\left(\mathrm{e}^{-B_{11}}\left(\mathrm{d}z+P\right)\right)\right]\,, \\
\mathrm{d}P &= \rho_8\,, \\
\mathrm{e}^{B_{11}} &= \frac{1}{2}R_8\,,
\end{aligned} \tag{5.11}$$

where J_8, ρ_8 and R_8 denote the Kähler form, Ricci form, and Ricci scalar of the Kähler four-fold $\mathcal{M}_8^\tau$. This is exactly the space introduced in (4.10) with metric (4.11), which in M-theory forms part of the physical spacetime. Its Ricci form and scalar are related to the corresponding $\mathcal{M}_6$ quantities as

$$\rho_8 = \rho_6 - \mathrm{i}\mathcal{P}\wedge\bar{\mathcal{P}}\,, \qquad R_8 = R_6 - 2|\mathcal{P}|^2\,. \tag{5.12}$$

Notice that the duality determines the period of the ϕ circle in terms of fundamental length scales to be

$$\frac{L_{11}\Delta\phi}{2\pi} = \frac{l_s^4}{l_p^3}\,. \tag{5.13}$$

5.2 M-Theory Supersymmetric Action for Elliptic Fibrations

Finally, we specialize the M-theory geometries to those with F-theory duals, i.e. $\mathcal{M}_8^\tau$ is an elliptic fibration over a base $\mathcal{M}_6$ with a section $\sigma : \mathcal{M}_6 \to \mathcal{M}_8^\tau$. We are interested in determining how the flux quantization conditions (2.9) and supersymmetric action (3.21) depend on data of the base $\mathcal{M}_6$. For this purpose we will here focus on on-shell solutions, which allows us to assume a choice of a regular Killing vector. This in turn ensures that the transverse Kähler space $\mathcal{M}_8^\tau$ is a smooth manifold. We will return to the extremization problem in the subsequent sections.

The Shioda-Tate-Wazir theorem for elliptically fibered Kähler manifolds [178] asserts[27] that we can decompose the (cohomology class of the) Kähler form on $\mathcal{M}_8^\tau$ as

$$J_8 = k_0\omega_0 + \sum_\alpha k_\alpha\omega_\alpha + \sum_i k_i\omega_i \equiv \sum_I k_I\omega_I\,. \tag{5.14}$$

This decomposition corresponds to three divisor classes, which generate the Picard group of $\mathcal{M}_8^\tau$. These are: the divisor corresponding to the section σ with its dual (1,1)-form ω_0, the pullback divisors C_α with dual forms denoted by ω_α, and finally the resolution divisors (also referred to as Cartan divisors) D_i with dual forms ω_i. For a more thorough discussion see [157]. Note that we do not require the Kähler parameters k_I to be integers; rather they are real numbers, which will ultimately be determined by the flux integers. Moreover, the Killing vector is assumed to be regular, implying a smooth $\mathcal{M}_8^\tau$. We assume for simplicity that the elliptic fibration is a smooth Weierstrass model and thus only has Kodaira type I_1 fibers and no resolution divisors.

With the expansion (5.14) the supersymmetric action (3.21) becomes

$$S_M = \sum_{IJK} \frac{k_I k_J k_K}{3!} \int_{Y_9^\tau} \eta\wedge\rho_8\wedge\omega_I\wedge\omega_J\wedge\omega_K\,. \tag{5.15}$$

The integral in S_M can be pushed down to an intersection on the base using adjunction

$$c_1(\mathcal{M}_8^\tau) = c_1(\mathcal{M}_6) - c_1(\mathcal{L})\,. \tag{5.16}$$

Furthermore, since the Killing vector is regular, we can integrate out the circle direction,

[27] For this to be true we need to impose some topological restrictions, namely $h^{1,0}(\mathcal{M}_8^\tau) = h^{2,0}(\mathcal{M}_8^\tau) = 0$. Also we assume for simplicity that there are no extra sections, i.e. the Mordell Weil group is trivial. From now on we assume this to hold.

which we take to have period $2\pi\ell$, and write the supersymmetric action as

$$S_M = (2\pi)^2\ell \sum_{IJK} \frac{k_I k_J k_K}{3!} \int_{\mathcal{M}_8^\tau} (c_1(\mathcal{M}_6) - c_1(\mathcal{L})) \wedge \omega_I \wedge \omega_J \wedge \omega_K \,. \tag{5.17}$$

We define the intersection numbers

$$C_{IJK} \equiv (c_1(\mathcal{M}_6) - c_1(\mathcal{L})) \cdot C_I \cdot C_J \cdot C_K = \int_{\mathcal{M}_8^\tau} (c_1(\mathcal{M}_6) - c_1(\mathcal{L})) \wedge \omega_I \wedge \omega_J \wedge \omega_K \,. \tag{5.18}$$

Using the intersection identity

$$\sigma \cdot_{\mathcal{M}_8^\tau} (\sigma + c_1(\mathcal{L})) = 0 \,, \tag{5.19}$$

a short computation shows that

$$\begin{aligned}
C_{000} &= (c_1(\mathcal{M}_6) - c_1(\mathcal{L})) \cdot c_1(\mathcal{L}) \cdot c_1(\mathcal{L}) \,, \\
C_{00\alpha} &= -\,(c_1(\mathcal{M}_6) - c_1(\mathcal{L})) \cdot c_1(\mathcal{L}) \cdot C_\alpha \,, \\
C_{0\alpha\beta} &= (c_1(\mathcal{M}_6) - c_1(\mathcal{L})) \cdot C_\alpha \cdot C_\beta \,, \\
C_{\alpha\beta\gamma} &= 0 \,,
\end{aligned} \tag{5.20}$$

which are manifestly intersection numbers on the base $\mathcal{M}_6$. Then the supersymmetric action specialized to elliptic fibrations is given in terms of intersection numbers on the base as

$$S_M = (2\pi)^2\ell \sum_{IJK} \frac{k_I k_J k_K}{3!} C_{IJK} \,. \tag{5.21}$$

The flux quantization conditions (2.9) specialized to elliptic fibrations become

$$\nu_4 N_I^M = (2\pi)^2\ell \sum_{JK} \frac{k_J k_K}{2} C_{IJK} \,. \tag{5.22}$$

Finally, observe that the Kähler parameter k_0 of the elliptic fiber is exactly the volume of a (non-singular) fiber

$$\mathrm{vol}(\mathbb{E}_\tau) = \int_{\mathbb{E}_\tau} J_8 = k_0 \int_{\mathbb{E}_\tau} \omega_0 = k_0 \,. \tag{5.23}$$

From the discussion of the M/F-duality, specifically using (5.5), we find that k_0 is expressed in terms of fundamental lengths as

$$k_0 = \frac{(2\pi)^2 l_p^6}{L_{11}^2 l_s^4} \,. \tag{5.24}$$

6 $\mathcal{I}/c$-Extremization

We will now compare the extremization procedures in M/F-theory. We will first provide the map between the two geometric extremization procedures, and then discuss the dual field theory.

6.1 Geometry

What we have argued so far is that an F-theory AdS_3 geometry is characterized by the complex geometry of the internal space $\mathcal{M}_6$ and the axio-dilaton profile. They are conveniently thought of here in terms of the complex cone $C(Y_9^\tau)$, which is a $\mathbb{C}^*$ fibration over an elliptically fibered base $\mathcal{M}_8^\tau$. An on-shell solution is ultimately determined by imposing a topological constraint, as well as a choice of quantized flux numbers $N_\alpha^F \in \mathbb{Z}$, where $\alpha = 1, \dots, \dim H_5(Y_7, \mathbb{R})$, as these fix the Kähler class parameters of the internal space geometry. Such a solution is then dual to a 2d $(0,2)$ SCFT living on the conformal boundary of AdS_3, for example as written in the usual Poincaré slicing. The holographic central charge c_{sugra} of this theory is computed using equation (4.22).

Associated to any such F-theory solution is a different global form of AdS_3, which is a circle bundle over AdS_2, as in (5.7). Topologically the circle fibration is trivial, with the fiber coordinate ϕ having period $\Delta\phi$, which *a priori* is arbitrary. Since the size of the ϕ circle in the AdS_3 is bounded it becomes part of the internal space, and the remaining conformal boundary is 1-dimensional. This implies that the associated solutions have an interpretation as holographic duals to 1d SCQM.

T-dualizing along this circle and uplifting to M-theory, the circle becomes part of the internal space of the M-theory geometry and, together with the circle introduced in the uplift from Type IIA to M-theory, it makes up the elliptic fiber $\mathbb{E}_\tau$ with volume k_0. The M-theory AdS_2 geometries obtained in this way are determined by the complex geometry of the internal space $\mathcal{M}_8^\tau$. An analogous cone construction $C(Y_9^\tau)$ exists for the M-theory geometries [147], which provides a parametrization of the R-symmetry vector. Finding on-shell M-theory solutions amounts to imposing a topological constraint and a choice of flux numbers $N_I^M \in \mathbb{Z}$, where $I = 1, \ldots, \dim H_7(Y_9^\tau, \mathbb{R})$, which fix the Kähler class parameters of the internal complex geometry. The effective AdS_2 Newton constant is then computed as in (3.46).

The two supergravity duals each contain a set of parameters that are mapped to each other through the duality. On either side, the flux quantization conditions come with a dimensionless combination of length scales characteristic of each theory, namely ν_3 in F-theory and ν_4 in M-theory. Furthermore, on the F-theory side we have the circle length $\Delta\phi$ as an *a priori* free parameter, and on the M-theory side we have the fiber volume k_0. These parameters are given in terms of fundamental length scales as

$$\text{F-theory/IIB}: \quad \begin{cases} \nu_3 = \dfrac{2(2\pi l_s)^4}{L_{10}^4} \\ \dfrac{\Delta\phi}{2\pi} = \dfrac{\sqrt{2} l_s^4}{L_{10} l_p^3} \end{cases} \qquad \text{M-theory}: \quad \begin{cases} \nu_4 = \dfrac{(2\pi l_p)^6}{L_{11}^6} \\ k_0 = \dfrac{(2\pi)^2 l_p^6}{L_{11}^2 l_s^4} \end{cases} \tag{6.1}$$

With $L_{11} = L_{10}/\sqrt{2}$, we find the following relation

$$\Delta\phi = \frac{\sqrt{\nu_4}}{k_0} \,. \tag{6.2}$$

As T-duality inverts the radius of the circle, $\Delta\phi$ is indeed expected to be inversely related to the volume of the elliptic fiber. Given such an M-theory geometry, we can trace through the duality in the other direction by taking the F-theory limit, corresponding to shrinking the elliptic fiber to zero size, $k_0 \to 0$. This in turn takes $\Delta\phi \to \infty$, decompactifying the ϕ circle.

Any solution to the topological constraint together with some configuration of flux numbers makes for a perfectly consistent and physical M-theory solution. However, in this thesis we are not interested in a generic M-theory solution; rather, we wish to find the ones *with F-theory duals*, and the map that takes us from one to the other.

For this purpose, it is instructive to compare Kähler classes on the two sides of the duality, focusing on the k_0 dependence on the M-theory side, since the F-theory limit takes

k_0 to zero. In other words, we concentrate on the contributions coming from the volume of the elliptic fibration, which forms part of the physical data in M-theory, and ceases to have a physical interpretation in F-theory. Consider again the decomposition of the 8d Kähler form J_8 in (5.14)

$$J_8 = k_0\omega_0 + \sum_\alpha k_\alpha \omega_\alpha \,. \tag{6.3}$$

Recall that ω_α are pullbacks from the base $\mathcal{M}_6$ and together with ω_0 generate the second integral cohomology of $\mathcal{M}_8^\tau$. Once the M-theory topological constraint is imposed and the fluxes are properly quantized, the parameters k_α depend implicitly on the size of the elliptic fiber k_0. We will see this in examples in later sections. We denote the Kähler parameters of J_6 by k_α such that

$$J_6 = \sum_\alpha \mathsf{k}_\alpha \omega_\alpha \,. \tag{6.4}$$

The requirement for mapping a specific M-theory solution to its F-theory dual is that the Kähler class on $\mathcal{M}_8^\tau$ should match that of $\mathcal{M}_6$ in the F-theory limit, i.e.

$$J_6 = \lim_{k_0 \to 0} J_8 = \lim_{k_0 \to 0} \sum_\alpha k_\alpha \omega_\alpha \,. \tag{6.5}$$

In geometric terms we are collapsing the elliptic fiber, while keeping the volume of the total space bounded. The metric on $\mathcal{M}_8^\tau$, with Kähler form J_8, then under appropriate convergence conditions tends to a (singular) metric on $\mathcal{M}_6$, with Kähler form J_6. This implies that the 8d and 6d Kähler parameters are related by

$$k_\alpha = \mathsf{k}_\alpha + \mathcal{O}(k_0) \,. \tag{6.6}$$

This ansatz for the decomposition of the 8d Kähler form results in an M-theory topological constraint equation, which can be expanded order by order in k_0 to give

$$k_0 \int_{Y_7} \eta \wedge (c_1(\mathcal{M}_6) - c_1(\mathcal{L}))^2 \wedge J_6 + \mathcal{O}(k_0^2) = 0 \,. \tag{6.7}$$

The lowest order term is exactly the constraint equation for the F-theory geometries that was independently derived in section 4.3. Since this equation must be satisfied order by order in k_0, the F-theory constraint equation is thus built into its dual M-theory solution by imposing that J_8 satisfy (6.5). Requiring the higher order terms to vanish constrains the form of (6.6).

For every M-theory flux integer $N_\alpha^M \in \mathbb{Z}$ there exists an F-theory flux integer $N_\alpha^F \in \mathbb{Z}$, where the M-theory seven-cycle is exactly the corresponding F-theory five-cycle with the elliptic fibration. The requirement (6.5) ensures that in dual solutions these flux configurations match on the nose, i.e. we have $N_\alpha^M = N_\alpha^F \equiv N_\alpha \in \mathbb{Z}$. In a sense, this condition expresses the fact that every D3-brane is simply converted to an M2-brane.

When determining an on-shell F-theory solution, imposing the N_α flux quantization conditions and the topological constraint determine the complex geometry of Y_7 by fixing the R-symmetry vector and the Kähler parameters of $\mathcal{M}_6$. In M-theory there is an additional, distinguished flux integer $N_0^M \equiv N_0$, which has no F-theory analog, as it arises from the section σ of the elliptic fibration, i.e.

$$\nu_4 N_0 = \int_{Y_7} \eta \wedge \rho_8 \wedge \frac{J_8^2}{2} \,. \tag{6.8}$$

The distinguished flux integer does not map to any flux integer present in F-theory; rather, expanding in orders of k_0, we find that its leading contribution is determined by the Type IIB ϕ circle length and the central charge as

$$N_0 = \left(\frac{\Delta\phi}{2\pi}\right)^2 \frac{c_{\text{sugra}}}{24} + \mathcal{O}(k_0) \,. \tag{6.9}$$

This additional flux quantization condition is matched by the extra Kähler parameter k_0 in the compact space of the M-theory geometry. In practice, as we shall see in examples, fixing this additional flux number then fixes the period $\Delta\phi$. Hence, imposing the M-theory topological constraint and flux quantization conditions determines the decomposition of J_8 and the internal space geometry Y_9^τ.

The map from holographic $\mathcal{I}$-extremization in M-theory to c-extremization in F-theory is completed by considering the relation between the two actions. *Before* imposing the topological constraint or flux quantization, the dual supersymmetric actions are related as

$$S_M = 2k_0 S_F + \mathcal{O}(k_0^2)\,. \tag{6.10}$$

The factor of 2 comes from the relative rescaling of the Killing one-form η (see appendix A for a discussion of the relative normalizations in M- versus F-theory). The on-shell central charge of the 2d SCFT is then formally related to the AdS_2 Newton constant by

$$\frac{1}{G_2} = \frac{\Delta\phi}{3} c_{\text{sugra}} + \mathcal{O}(k_0)\,. \tag{6.11}$$

The reason this should only be read as a formal expression for the AdS_2 Newton constant is that the N_0 flux quantization condition has not been imposed, thus leaving in factors of k_0. Since a Kähler parameter of the internal space still appears explicitly in the equations, it cannot be understood as a physical quantity. From the above, we can thus conclude that *holographic $\mathcal{I}$-extremization in M-theory does not in general equal holographic c-extremization in F-theory.* In other words, extremizing $1/G_2$ does not necessarily correspond to finding an extremum for c_{sugra}.

This result generalizes the relation derived in [179], where the AdS_2 is considered as arising directly in Type IIB by writing AdS_3 as the total space of a circle fibration. The effective Newton constants are then related by dimensional reduction on this circle

$$G_3 = \frac{\Delta\phi}{2} G_2\,, \tag{6.12}$$

where the factor of 1/2 here arises as the length of the ϕ circle in the AdS_3 metric (5.7). Equation (6.11) takes into account corrections from the 7-branes and exactly reduces to the supergravity result in (6.12) when the elliptic fibration is trivial.

Interestingly, for many cases that we study later in this thesis, the $\mathcal{O}(k_0)$ terms are in fact absent in (6.11), even for a non-trivial elliptic fibration, so that (6.12) holds exactly. This is true for all the toric examples in section 7. As a proof of concept, we therefore also consider a known set of solutions, the universal twist solutions with elliptic three-fold factor, in section 8, which do have non-zero subleading terms.

6.2 Field Theory

Finally, we comment on the physical interpretation of (6.11) in terms of the holographically dual field theories. First recall how the two field theory duals are constructed. On the F-theory side, the dual field theories are realized on D3-branes along $\mathbb{R}^{1,1} \times C$, where C are curves in F-theory compactifications, above which the axio-dilaton profile is non-trivial. This induces a varying coupling τ of the 4d gauge theory on the D3-branes, and the 2d $(0,2)$ field theory along $\mathbb{R}^{1,1}$ acquires a dependence on the $\mathrm{U}(1)_D$ duality line bundle $\mathcal{L}$ [159–162, 165]. T-duality along a circle in the D3-brane world-volume gives rise to a configuration of D2-branes, which uplift in M-theory to M2-branes wrapped on the curves C, i.e. the M2-branes realize a 1d SCQM.

While AdS_2 holography is still very much under development, it is natural to identify minus the logarithm of the partition function of the 1d theory with the renormalized supergravity action. As shown in [147] we may thus identify

$$\log Z_{\mathrm{1d}} = \frac{1}{4G_2}\,. \tag{6.13}$$

If we consider the 1d SCQM as arising directly from a circle reduction of the 2d (0,2) SCFT, or, equivalently, from duality with M-theory on a trivially fibered torus, then we can use (6.12) and the standard Brown-Henneaux relation [168] to deduce that

$$\log Z_{\mathrm{1d}} = \frac{1}{4G_2} = \frac{\Delta\phi}{8G_3} = \frac{\Delta\phi}{12} c_{\mathrm{sugra}}\,. \tag{6.14}$$

Of course this is precisely equation (6.11), without the $\mathcal{O}(k_0)$ correction terms. The partition function on the left hand side of (6.13) is defined by putting the 1d SCQM on a circle. On the other hand, we have also effectively reduced from 2d to 1d on the ϕ circle. Physically one might then anticipate some relation between the 1d partition function and the 2d partition function, where the 2d $(0,2)$ theory is put on a torus T^2. We note that this is indeed precisely the case: putting a 2d CFT on a torus leads to a Casimir energy contribution to the partition function

$$Z_{T^2}(\text{Casimir}) = \exp\left(\frac{r_1}{r_2}\frac{c}{12}\right)\,, \tag{6.15}$$

where r_1, r_2 are the lengths of the circles in the T^2, and c is the central charge. We should then also recall that $\Delta\phi$ is dimensionless, but may be written as

$$\Delta\phi = \frac{2\pi R_{\text{IIB}}}{L_{11}}\,, \tag{6.16}$$

where R_{IIB} is the dimensionful Type IIB ϕ circle length and L_{11} is the overall dimensionful length scale in M-theory. The right hand side of (6.14) may then be identified with (the logarithm of) this Casimir contribution to the T^2 partition function. Recall here that in the M-theory solution $\Delta\phi$ depends on the additional M-theory flux number N_0, while the central charge c_{sugra} depends only on the F-theory data, which does not include N_0. In the above identification, the extra parameter N_0 determines, via $\Delta\phi$, the geometry of the T^2 on which the 2d $(0,2)$ SCFT is placed. Notice then that the 2d $(0,2)$ theory itself does not depend on the integer N_0, while the 1d SCQM that it reduces to does depend on N_0. It would be interesting to understand this in more detail, and in particular whether the integer N_0 has a simple 1d interpretation.

We will exemplify these general insights by considering several classes of solutions in the next two sections. In section 7 we study M/F-theory dual holographic setups, where the relation (6.11) holds precisely, without any $\mathcal{O}(k_0)$ corrections. We contrast this in section 8, where we study solutions where there are non-trivial corrections as predicted by (6.11). The key difference between these two sets of solutions is that in the former, the elliptic fiber is restricted to a complex curve, whereas in the latter the fibration is non-trivial over a complex surface, which results for instance in non-trivial terms of the type $c_1(\mathcal{L})^2$, which contribute the higher order terms in k_0.

7 Toric Fibrations over a Curve

In this section, we consider a class of toric geometries fibered over a complex curve or an elliptically fibered surface, where we can derive explicit formulas for the off-shell M/F-theory extremization problem. We show that, for these geometries, $\mathcal{I}$- and c-extremization are equivalent without any corrections in k_0, the volume of the elliptic fiber. Moreover, we apply the formalism to the cases referred to in the literature as the *universal* and *baryonic twists*.

7.1 F-Theory c-Extremization for Toric Fibrations

We first consider the fibration of X_5 over a Riemann surface Σ, and relate it to the toric fibration setup we considered in section 3.6. The fibration of X_5 over Σ can be parametrized as follows. The toric manifold is equipped with an isometric $\mathrm{U}(1)^3$ action, generated by a set of holomorphic vector fields $\partial_{\varphi_i}, i = 1, 2, 3$. We choose three line bundles $\mathcal{O}(n_i)$ on the Riemann surface so that, topologically, the compactification space is defined to be the total space of the associated bundle

$$Y_7 = \mathcal{O}(\vec{n}) \times_{\mathrm{U}(1)^3} X_5 \,. \tag{7.1}$$

For simplicity we shall assume that the axio-dilaton varies only over the Riemann surface Σ. That is, taking the F-theory perspective, the variation of the axio-dilaton is captured by an auxiliary elliptic fibration as in (4.10), where the total space that we will consider is

$$\begin{array}{ccccc} & & & & Y_9^\tau \\ & & & \swarrow & \big\downarrow{\scriptstyle p^*(\pi)} \\ \mathbb{E}_\tau & \hookrightarrow & B_4^\tau & & Y_7 \\ & & {\scriptstyle\sigma}\big\uparrow\big\downarrow{\scriptstyle\pi} & \swarrow{\scriptstyle p} & \\ & & \Sigma & & \end{array} \;. \tag{7.2}$$

Recall that the manifold Y_9^τ is obtained by pulling back the elliptic fibration π to Y_7 as in section 4.4. The existence of a global holomorphic $(5, 0)$-form on $\mathrm{C}(Y_9^\tau)$ places certain restrictions on $\vec{n}$. We may construct such a global $(5, 0)$-form by first noting that $\mathrm{C}(X_5)$ admits a global $(3, 0)$-form $\Omega_{(3,0)}$. The $(3, 0)$-form has an explicit $\mathrm{e}^{\mathrm{i}\varphi_1}$ dependence, since it has R-charge 2. On B_4^τ there is a local $(2,0)$-form $\Xi_{(2,0)}$, which is a local section of $K_{B_4^\tau}$. By adjunction we have

$$K_{B_4^\tau} = K_\Sigma \otimes \mathcal{L} \tag{7.3}$$

where $\mathcal{L}$ is the duality line bundle, whose connection depends on the variation of the axio-dilaton as introduced in section 4 and

$$\deg(K_{B_4^\tau}) = 2g - 2 + \deg \mathcal{L}\,. \tag{7.4}$$

The holomorphic volume form on $C(Y_9^\tau)$ is constructed as

$$\Omega_{(5,0)} = \Omega_{(3,0)} \wedge \Xi_{(2,0)}\,, \tag{7.5}$$

where $\Omega_{(3,0)}$ is twisted over Σ as in (7.1). Since $\mathrm{e}^{\mathrm{i}\varphi_1}$ is a section of $\mathcal{O}(n_1)$, we can ensure that $\Omega_{(5,0)}$ is a global non-vanishing form by taking

$$n_1 = 2 - 2g - \deg \mathcal{L}\,. \tag{7.6}$$

With this in place, the set of formulas (3.63), (3.64), (3.65) and (3.66) carry over verbatim to the present context. The difference between IIB and F-theory extremization problems in this toric setting lies entirely in the fact that n_1 develops a $\deg \mathcal{L}$ dependence in the F-theory case. This is due to the fact that the elliptic fibration is constrained to Σ and the higher powers of $c_1(\mathcal{L})$ vanish.

7.2 M-Theory $\mathcal{I}$-Extremization for Toric Fibrations

In this section, we establish the $\mathcal{I}$-extremization procedure dual to the c-extremization for fibered toric geometries set up in the previous section. In the context of M-theory, we are considering the physical compactification space

$$\begin{array}{ccccc} & & & & Y_9^\tau \\ & & & \swarrow & \\ \mathbb{E}_\tau & \hookrightarrow & B_4^\tau & & \\ & & \sigma \uparrow\downarrow \pi & & \\ & & \Sigma & & \end{array} \quad . \tag{7.7}$$

Here we view Y_9^τ as the space X_5 fibered over the elliptic surface B_4^τ. This is achieved in the same way as in (7.1), but the vector bundle $\mathcal{O}(\vec{n})$ is now pulled back from Σ to B_4^τ. The cone $\mathrm{C}(Y_9^\tau)$ also admits a non-vanishing holomorphic volume form precisely if

$$n_1 = 2 - 2g - \deg \mathcal{L} \, . \tag{7.8}$$

This geometric setup fits into the framework we considered in section 3.6. In what follows we specialize the formulas there to the elliptic surface case. We make the following ansatz for the Kähler form on B_4^τ

$$J_{B_4^\tau} = k_0 \omega_0 + \left(A + \frac{k_0 \deg \mathcal{L}}{2} \right) \mathrm{vol}_\Sigma \, . \tag{7.9}$$

In other words, we are assuming that the Kähler class on the elliptic surface is just a linear combination of the base and the fiber class. One can also derive similar formulas for a more general ansatz where Cartan divisors are added. The choice of the shift of A by $k_0 \deg \mathcal{L}/2$ is convenient in order to compare to the F-theory parameters at the end of this section. Using the ansatz the volume of the elliptic surface is

$$\mathrm{vol}(B_4^\tau) = \int_{B_4^\tau} \frac{J_{B_4^\tau}^2}{2} = -\frac{k_0^2 \deg \mathcal{L}}{2} + \left(A k_0 + \frac{k_0^2 \deg \mathcal{L}}{2} \right) = A k_0 \, , \tag{7.10}$$

where we used

$$\int_{B_4^\tau} \omega_0^2 = - \deg \mathcal{L} \, , \qquad \int_{B_4^\tau} \omega_0 \wedge \mathrm{vol}_\Sigma = 1 \, . \tag{7.11}$$

Furthermore, the curvature integrals specialize to

$$\int_{\Sigma} F_i = 2\pi n_i \,, \qquad \int_{\mathbb{E}_\tau} F_i = 0 \,, \tag{7.12}$$

and $F_i \wedge F_j = 0$ for dimensional reasons. With these results the M-theory constraint equation reduces to

$$A \sum_{a,b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial\lambda_a \partial\lambda_b} - 2\pi n_1 \sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a} + 2\pi b_1 \sum_{a=1}^{d}\sum_{i=1}^{3} n_i \frac{\partial^2 \mathcal{V}}{\partial\lambda_a \partial b_i} = 0 \,, \tag{7.13}$$

which is exactly the same as the F-theory constraint given in (3.64). The M-theory supersymmetric action becomes

$$S_M = -Ak_0 \sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a} - 2\pi k_0 b_1 \sum_{i=1}^{3} n_i \frac{\partial \mathcal{V}}{\partial b_i} \,. \tag{7.14}$$

Let us now focus on the flux quantization conditions. The seven-cycles fall into two classes, where the cycles in the first class are obtained by fibering X_5 over a two-cycle in the base, and the second class contains three-cycles in X_5 (associated with toric divisors on the cone) fibered over the entire base. The flux integer corresponding to fixing a point in Σ and quantizing over the cycle $X_5 \times \mathbb{E}_\tau$ is

$$\nu_4 N = -k_0 \sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a} \,. \tag{7.15}$$

The quantization conditions associated to fibrations of toric three-cycles over B_4^τ are

$$\nu_4 M_a = \frac{Ak_0}{2\pi} \sum_{b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial\lambda_a \partial\lambda_b} + k_0 b_1 \sum_{i=1}^{3} n_i \frac{\partial^2 \mathcal{V}}{\partial\lambda_a \partial b_i} \,. \tag{7.16}$$

There is one final cycle we need to consider, arising from the section of the elliptic fibration. Geometrically this is the space X_5 fibered over Σ and the corresponding flux number is given by

$$\nu_4 N_0 = -\left(A - \frac{k_0 \deg \mathcal{L}}{2}\right) \sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a} - 2\pi b_1 \sum_{i=1}^{3} n_i \frac{\partial \mathcal{V}}{\partial b_i} \,. \tag{7.17}$$

Notice that, combining the expressions for the flux numbers, the supersymmetric action can be rewritten as

$$S_M = k_0\nu_4\left(N_0 + \frac{1}{2}N\deg\mathcal{L}\right) . \tag{7.18}$$

This toric setup provides an instructive example of the M/F-theory relations we have described in section 6. In the ansatz for the Kähler class (7.9) we have explicitly included the k_0 corrections to the Kähler class on the base of the elliptic fibration. Indeed, the parameter A is precisely the F-theory Kähler parameter on Σ and the relation

$$\lim_{k_0\to 0} J_{B_4^\tau} = \lim_{k_0\to 0}\left(k_0\omega_0 + J_\Sigma + \frac{k_0\deg\mathcal{L}}{2}\mathrm{vol}_\Sigma\right) = J_\Sigma \tag{7.19}$$

holds. The way to derive the explicit form of the correction term is to start with a general ansatz for the Kähler form on B_4^τ and impose that the $\mathcal{O}(k_0^2)$ terms in the M-theory constraint equation cancel. In this way the constraint equation reduces just to the linear term, which is precisely the F-theory constraint equation. Moreover, the flux integers M_a and N also match on both sides if we take into account the relation $\nu_3 = \nu_4/2k_0$, as well as the fact that the master volume functions differ by a factor of 2. The detailed comparison of metrics and normalizations in M- versus F-theory is discussed in appendix A.

An interesting feature of these geometries is that, despite including the full backreaction of the 7-branes in the M-theory background, there are no k_0 corrections in the supersymmetric action, i.e. we find

$$S_M = 2k_0 S_F\,. \tag{7.20}$$

This implies that the resulting on-shell solutions will have

$$\frac{1}{4G_2} = \frac{\Delta\phi}{12}c_{\mathrm{sugra}} \tag{7.21}$$

on the nose, even though we are considering a non-trivial elliptic fibration. This is precisely the relation (6.14). We also see that this relation actually holds off-shell. The upshot of this discussion is that $\mathcal{I}$- and c-extremization are indeed equivalent for toric fibrations over a Riemann surface.

7.3 Universal Twist: Elliptic Surface

In this section, we focus on a known class of F-theory supergravity solutions found in [158], the so-called universal twist solution for elliptic surfaces. We apply the holographic $\mathcal{I}/c$-extremization developed in sections 4 and 5, which allows us to simultaneously re-derive the central charge of the 2d field theory dual to these F-theory solutions and determine $1/G_2$ of their M-theory duals, without ever explicitly solving the master equation.

The universal twist solutions are based on the ansatz

$$\begin{array}{ccc} S^1 & \hookrightarrow & Y_7 \\ & & \downarrow \\ & & \Sigma \times \mathcal{M}_4 \end{array} \tag{7.22}$$

which assumes that the transverse Kähler space $\mathcal{M}_6$ is a product of a complex curve and a Kähler surface. We are interested in the set of universal twist solutions where the elliptic fibration is non-trivial only over the complex curve, so that $\mathcal{M}_8^\tau$ contains an elliptic surface

$$\mathcal{M}_8^\tau = (\mathbb{E}_\tau \to \Sigma) \times \mathcal{M}_4 \,. \tag{7.23}$$

This corresponds to choosing the twist parameters n_i parallel to the R-symmetry vector, i.e. we take

$$n_i = \frac{n_1}{b_1} b_i \,, \tag{7.24}$$

which immediately implies

$$\sum_{a,b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial \lambda_b} = 8 b_1^2 \mathrm{Vol}(X_5) \,, \qquad b_1 \sum_{i=1}^{3} n_i \frac{\partial \mathcal{V}}{\partial b_i} = -n_1 \mathcal{V} \,. \tag{7.25}$$

The topological constraint, which must be imposed for either side of the duality, is

$$8A\, b_1^2\, \mathrm{Vol}(X_5) - 4\pi n_1 \sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a} = 0 \,. \tag{7.26}$$

The M/F-theory flux integers are given by

$$\begin{aligned} \nu_3 N &= -\sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a}\,, \\ \nu_3 M_a &= \frac{A}{2\pi} \sum_{b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial \lambda_b} - n_1 \frac{\partial \mathcal{V}}{\partial \lambda_a}\,. \end{aligned} \tag{7.27}$$

Here we have chosen to write the quantization conditions manifestly as F-theory equations.[28] The distinguished M-theory flux integer is N_0, which satisfies

$$\nu_4 N_0 = -\left(A - \frac{k_0 \deg \mathcal{L}}{2}\right) \sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a} + 2\pi n_1 \mathcal{V}\,. \tag{7.28}$$

The M/F-theory supersymmetric actions are

$$\begin{aligned} S_F &= A\,\nu_3 N + 2\pi n_1 \mathcal{V}\,, \\ S_M &= A\,\nu_4 N + 2\pi k_0 n_1 \mathcal{V}\,, \end{aligned} \tag{7.29}$$

so that again equation (7.20) holds, on the nose.

With these relations in place, we proceed to impose all common M/F-theory conditions (i.e. all but the N_0 flux quantization) and derive expressions for the supersymmetric actions that take these conditions into account. We start by rewriting the topological constraint in terms of the flux integer N as

$$2A\, b_1^2 \,\mathrm{Vol}(X_5) + \pi n_1 \nu_3 N = 0\,. \tag{7.30}$$

Solving this constraint for A and substituting into the supersymmetric actions yields

$$\begin{aligned} S_F &= -\frac{\pi n_1 \nu_3^2 N^2}{2 b_1^2 \mathrm{Vol}(X_5)} + 2\pi n_1 \mathcal{V}\,, \\ S_M &= -\frac{\pi n_1 \nu_4^2 N^2}{2 b_1^2 k_0 \mathrm{Vol}(X_5)} + 2\pi k_0 n_1 \mathcal{V}\,. \end{aligned} \tag{7.31}$$

[28] We use the convention that whenever the parameter ν_4/ν_3 appears in an equation, the volumes $\mathcal{V}$ and $\mathrm{Vol}(X_5)$ are implicitly understood to be functions of $\vec{b}_M/\vec{b}_F$ and the equation itself should be understood as an M/F-theory equation. To write an equation as it appears naturally in the dual description, one simply uses $\nu_3 = \nu_4/2k_0$ and the normalization conventions detailed in appendix A. Equations where neither parameter appears are invariant under $\vec{b}_M \leftrightarrow \vec{b}_F$.

We can impose quantization of the M_a by choosing $\lambda_a \equiv \lambda$. This ensures that the M_a are quantized as

$$M_a = -N\,. \tag{7.32}$$

This solution implies that the master volume is

$$\mathcal{V} = 4b_1^2\lambda^2 \mathrm{Vol}(X_5)\,, \tag{7.33}$$

and the flux quantization condition for N fixes λ to be

$$\lambda = -\frac{\nu_3 N}{8b_1^2 \mathrm{Vol}(X_5)}\,. \tag{7.34}$$

The supersymmetric actions can then be written as

$$\begin{aligned} S_F &= -\frac{3\pi n_1 \nu_3^2 N^2}{8b_1^2 \mathrm{Vol}(X_5)}\,, \\ S_M &= -\frac{3\pi n_1 \nu_4^2 N^2}{8k_0 b_1^2 \mathrm{Vol}(X_5)}\,. \end{aligned} \tag{7.35}$$

Since $\mathrm{Vol}(X_5)$ is extremized for a Reeb vector with $r_1 = 3$, we set the M/F-theory R-symmetry vector $\vec{b}_F = \frac{2}{3}\vec{r} = 2\vec{b}_M$. Let $\vec{r}_*$ denote the extremal Reeb vector, corresponding to a Sasaki-Einstein metric on X_5. We thus find the actions

$$\begin{aligned} S_F(\vec{r}_*) &= -\frac{\pi n_1 \nu_3^2 N^2}{36 \mathrm{Vol}(X_5)(\vec{r}_*)}\,, \\ S_M(\vec{r}_*) &= -\frac{\pi n_1 \nu_4^2 N^2}{72 k_0 \mathrm{Vol}(X_5)(\vec{r}_*)}\,. \end{aligned} \tag{7.36}$$

The 2d central charge is then given by (4.22) as

$$c_{\text{sugra}} = \frac{12(2\pi)^2}{\nu_3^2} S_F(\vec{r}_*) = -\frac{(2\pi)^3 n_1 N^2}{6 \mathrm{Vol}(X_5)(\vec{r}_*)}\,. \tag{7.37}$$

The AdS_2 Newton constant is given by (3.46) as

$$\frac{1}{G_2} = \frac{8(2\pi)^2}{\nu_4^{3/2}} S_M(\vec{r}_*) = -\frac{4\pi^3 n_1 \Delta\phi N^2}{9 \mathrm{Vol}(X_5)(\vec{r}_*)} = \Delta\phi \frac{c_{\text{sugra}}}{3}\,, \tag{7.38}$$

as expected. However, this expression for the Newton constant cannot yet be understood to reflect a physical quantity due to the presence of $\Delta\phi$, which is a parameter of the internal space. In order for this to constitute a genuine M-theory solution, we still need to impose the N_0 flux quantization condition

$$N_0 = -\frac{\pi n_1 \nu_4 N^2}{72 k_0^2 \mathrm{Vol}(X_5)(\vec{r}_*)} - \frac{1}{2} N \deg \mathcal{L} \,. \tag{7.39}$$

This condition fixes the period $\Delta\phi$ in terms of the distinguished flux number. We find

$$\Delta\phi = \pm \frac{6}{N} \sqrt{\frac{\mathrm{Vol}(X_5)(\vec{r}_*)\,(2N_0 + N \deg \mathcal{L})}{-\pi n_1}} \,. \tag{7.40}$$

The Newton constant of this genuine M-theory solution is then

$$\frac{1}{G_2} = \frac{8\pi^2 N}{3} \sqrt{\frac{-\pi n_1\,(2N_0 + N \deg \mathcal{L})}{\mathrm{Vol}(X_5)(\vec{r}_*)}} \,. \tag{7.41}$$

7.4 Baryonic Twist: $Y^{p,q}$

We now consider the so-called baryonic twist solutions [148,158]. For simplicity we present the computations for $X_5 = Y^{p,q}$. The $Y^{p,q}$ metrics first appeared in [125] and their toric data was derived in [127]. The $d = 4$ ordered inward pointing normal vectors are

$$v_1 = (1,0,0), \quad v_2 = (1,1,0), \quad v_3 = (1,p,p), \quad v_4 = (1, p-q-1, p-q) \,. \tag{7.42}$$

The $Y^{p,q}$ metrics have $p > q > 0$ and the polyhedral cone with vectors v_a, $a = 1, \ldots, 4$ is convex. We take the free twist parameters to be $n_2 = n_3 \equiv n$ for simplicity. As it turns out, the computational complexity of the problem is highly sensitive to the order in which the topological condition and flux quantization conditions are imposed, even though the resulting solution is clearly independent of this choice. We therefore include details of how the sets of equations are solved on each side of the duality.

We first discuss the F-theory side and proceed by using (7.42) to explicitly write down an expression for the master volume $\mathcal{V}$, which is a function of λ_a and b_i. With that, we derive expressions for the constraint, fluxes, and action in terms of $\mathcal{V}$ and set $b_1 = 2$. We use the flux quantization conditions for N and M_1 to solve for λ_4 and A, respectively, and solve the constraint equation for λ_1. Note that λ_2, λ_3 must necessarily drop out of any final result, since there are only two independent Kähler parameters. We rescale the fluxes and twist parameters as

$$M_a \equiv -n_1 m_a N, \qquad n \equiv -n_1 s\,, \tag{7.43}$$

and immediately rename $m_1 \equiv m$ for notational convenience. The remaining fluxes are

$$m_2 = m_4 = \frac{(1-m)p+s}{p+q}, \qquad m_3 = \frac{(m-1)(p-q)-2s}{p+q}\,. \tag{7.44}$$

We can then determine the trial central charge, which we do not quote here as the expression is extremely long. Extremizing with respect to b_2, b_3 gives the R-symmetry vector $\vec{b} = (2, b_2, b_2)$ with

$$b_2 = -2p\frac{p^3\left[2(m-1)^2q+(2m-3)s\right]-2p^2\left[(m-1)^2q^2+(2m-1)qs+2s^2\right]+pqs[(2m-3)q-2s]-2q^2s^2}{p^4(2m-1)+2p^3[(2m-1)q+s]+p^2\left[\left(4m^2-6m+3\right)q^2+4mqs+4s^2\right]+2pqs[(3-2m)q+2s]+4q^2s^2}\,, \tag{7.45}$$

and on-shell central charge

$$c_{\text{sugra}} = \frac{12N^2n_1p[(m-1)p-s]\left[p^3\left(2m^2-3m+1\right)+p^2\left(\left(-2m^2+3m-1\right)q-4ms+s\right)+pqs(2m-3)-2qs^2\right]}{p^4(2m-1)+2p^3[(2m-1)q+s]+p^2\left[\left(4m^2-6m+3\right)q^2+4mqs+4s^2\right]+2pqs[(3-2m)q+2s]+4q^2s^2}\,. \tag{7.46}$$

Note that, for a trivial line bundle with $\deg \mathcal{L} = 0$, we find that b_2 and c_{sugra} reduce to (6.6) and (6.7) in [148].

On the M-theory side we again start by explicitly writing down the master volume $\mathcal{V}$ as a function of λ_a and b_i using (7.42). We then derive expressions for the constraint, fluxes and action in which we set $b_1 = 1$. Noticing that the constraint equation does not depend on λ_1 and the flux quantization condition for M_1 does not depend on λ_3, we first solve the constraint equation for λ_3 and use the M_1 flux quantization condition to solve for λ_1. We then solve the N flux quantization condition for A. Note that λ_2, λ_4 then automatically drop out of subsequent results, since there are only two independent Kähler parameters. Having imposed the topological constraint and flux quantization for N and M_1, we reproduce the relations between the fluxes given in (7.44). The trial Newton constant can then be written down; however, the expression is not quoted here as it is very long. Extremizing with respect to b_2, b_3 gives the R-symmetry vector $\vec{b} = (1, b_2, b_2)$ with

$$b_2 = -p\frac{p^3\left[2(m-1)^2q+(2m-3)s\right]-2p^2\left[(m-1)^2q^2+(2m-1)qs+2s^2\right]+pqs[(2m-3)q-2s]-2q^2s^2}{p^4(2m-1)+2p^3[(2m-1)q+s]+p^2\left[\left(4m^2-6m+3\right)q^2+4mqs+4s^2\right]+2pqs[(3-2m)q+2s]+4q^2s^2}\,, \tag{7.47}$$

which is exactly half the corresponding R-symmetry component in F-theory, i.e. we have indeed found $\vec{b}_F = 2\vec{b}_M$. The preliminary Newton constant is

$$\frac{1}{G_2} = \frac{4\Delta\phi N^2 n_1 p(-mp+p+s)\left[p^3\left(2m^2-3m+1\right)+p^2\left(\left(-2m^2+3m-1\right)q-4ms+s\right)+pqs(2m-3)-2qs^2\right]}{p^4(2m-1)+2p^3((2m-1)q+s)+p^2\left(\left(4m^2-6m+3\right)q^2+4mqs+4s^2\right)+2pqs((3-2m)q+2s)+4q^2s^2} = \Delta\phi\frac{c_{\text{sugra}}}{3}\,. \tag{7.48}$$

In order for this to correspond to a genuine M-theory solution, we must still impose quantization of N_0. Solving the flux quantization condition for N_0 for k_0, the Newton constant in terms of M-theory fluxes is

$$\frac{1}{G_2} = 8\pi N\sqrt{\frac{-pn_1[2N_0+\deg\mathcal{L}N][(m-1)p-s]\left[p^3\left(2m^2-3m+1\right)+p^2\left(\left(-2m^2+3m-1\right)q-4ms+s\right)+pqs(2m-3)-2qs^2\right]}{p^4(2m-1)+2p^3((2m-1)q+s)+p^2\left(\left(4m^2-6m+3\right)q^2+4mqs+4s^2\right)+2pqs((3-2m)q+2s)+4q^2s^2}}\,. \tag{7.49}$$

This concludes the discussion of solutions where $\mathcal{I}$- and c-extremization agree exactly across M/F-theory duality.

8 Universal Twist Solutions: Elliptic Three-fold

We would like to demonstrate that generically $1/G_2$ and c_{sugra} do not match exactly, as in the examples in section 7, but rather $1/G_2$ includes higher order corrections in k_0 as argued for in (6.11). These are absent in the F-theory solution, where the volume of the elliptic fiber is strictly zero. To this end, we consider the (on-shell) universal twist elliptic three-fold solutions, which were determined in [158]. We will first give a brief summary of the known F-theory solutions and then provide the corresponding M-theory analysis, and a comparison of the two.

8.1 F-Theory

The universal twist solutions are based on the product ansatz

$$\begin{array}{ccc} S^1 & \hookrightarrow & Y_7 \\ & & \downarrow \\ & & \Sigma \times \mathcal{M}_4 \end{array} \quad , \tag{8.1}$$

where the transverse $\mathcal{M}_6$ factorizes as a product of a complex curve and a Kähler surface. To $\mathcal{M}_6$ we associate an auxiliary elliptic fibration $\mathcal{M}_8^\tau$, and assume that the fibration is non-trivial only over the $\mathcal{M}_4$ factor, so that the total space is given by

$$\mathcal{M}_8^\tau = \Sigma \times (\mathbb{E}_\tau \to \mathcal{M}_4) \ . \tag{8.2}$$

The metrics on Σ and $\mathcal{M}_4$ satisfy

$$\begin{aligned} \rho_4 + \mathrm{d}Q &= 6 J_{\mathcal{M}_4} \, , \\ \rho_\Sigma &= -3 J_\Sigma \, . \end{aligned} \tag{8.3}$$

Note that we assume that the Killing vector is regular throughout the following, and the period of the circle coordinate z is $2\pi\ell$. The volume of Σ is given by

$$\mathrm{Vol}(\Sigma) = \frac{2\pi}{3}(2g-2) \, , \tag{8.4}$$

which follows from the Gauss-Bonnet theorem. Moreover,

$$[J_{\mathcal{M}_4}] = \frac{\pi}{3}\left(c_1(\mathcal{M}_4) - c_1(\mathcal{L})\right) \ . \tag{8.5}$$

This equation implies that the volume of the circle-fibration over $\mathcal{M}_4$, denoted by $\mathcal{M}_5$, is

$$\mathrm{Vol}(\mathcal{M}_5) = \frac{\pi^3 \ell}{27} \int_{\mathcal{M}_4} \left(c_1(\mathcal{M}_4) - c_1(\mathcal{L})\right)^2 \ . \tag{8.6}$$

There are two classes of flux quantization conditions. The first corresponds to the cycle at fixed coordinates in Σ, which is a copy of $\mathcal{M}_5$. The flux integer is

$$\nu_3 N^F = 18\mathrm{Vol}(\mathcal{M}_5)\,. \tag{8.7}$$

The second class of flux quantization conditions is obtained as a U(1) fibration over the product of Σ with two-cycles C_α in $\mathcal{M}_4$. They are given by

$$\nu_3 M^F_\alpha = 3\pi\ell\mathrm{Vol}(\Sigma)C_\alpha\cdot[J_{\mathcal{M}_4}]\,. \tag{8.8}$$

Finally, the central charge of the 2d $(0,2)$ theory was computed in [158] to be

$$c_{\mathrm{sugra}} = \frac{2\pi^2\mathrm{Vol}(\Sigma)(N^F)^2}{\mathrm{Vol}(\mathcal{M}_5)}\,. \tag{8.9}$$

8.2 M-Theory

We start with the F-theory metric on $\mathcal{M}_6$ and construct the M-theory solution from it. We will specialize to the case where $\mathcal{M}_4 = \mathbb{CP}^2$. Consider the Kähler class ansatz

$$J_8 = k_0\omega_0 + xJ_\Sigma + yJ_{\mathcal{M}_4}. \tag{8.10}$$

The crucial point is that the form of the metrics on Σ and $\mathcal{M}_4$ are exactly the same as in the previous subsection, and we think of the F-theory solution as a 0-th order solution in a suitable expansion in the volume of the elliptic fiber. We have introduced x and y that parametrize the Kähler cone of $\mathcal{M}_6 = \Sigma\times\mathcal{M}_4$. Given this parametrization we now compute the Kähler class of the M-theory solution.

Note that the Kähler class on $\mathcal{M}_4$ is

$$[J_{\mathcal{M}_4}] = \frac{\pi}{3}(3-\deg\mathcal{L})[H]\,, \tag{8.11}$$

where $[H]$ is the hyperplane class of $\mathbb{CP}^2$. The line bundle associated to the elliptic fibration lives over $\mathcal{M}_4$ and in particular $c_1(\mathcal{L})^2 \neq 0$. From the M-theory constraint we derive the following equation

$$x = y - \frac{3k_0 \deg \mathcal{L}}{2\pi(3 - \deg \mathcal{L})}\,, \tag{8.12}$$

where

$$\deg \mathcal{L} = c_1(\mathcal{L}) \cdot [H]\,. \tag{8.13}$$

This allows us to eliminate the parameter x from the above ansatz. With this we can then compute the M-theory supersymmetric action

$$S_M = \frac{4\pi^2 \ell (g-1)}{3} k_0 \left[\pi^2 y^2 (3 - \deg \mathcal{L})^2 - 3\pi y (3 - \deg \mathcal{L}) \deg \mathcal{L} k_0 + 2 \deg \mathcal{L}^2 k_0^2\right]\,. \tag{8.14}$$

The remaining parameters, y and k_0 are fixed by flux quantization in M-theory

$$\begin{aligned}
\nu_4 N_0 &= \frac{4\pi^2 \ell (g-1)}{3} \left[\pi^2 y^2 (3 - \deg \mathcal{L})^2 - 4\pi y (3 - \deg \mathcal{L}) \deg \mathcal{L} k_0 + 3 \deg \mathcal{L}^2 k_0^2\right]\,, \\
\nu_4 N^M &= \frac{4\pi^3 \ell}{3} y k_0 (3 - \deg \mathcal{L})^2 + 2\pi^2 \ell (3 - \deg \mathcal{L}) \deg \mathcal{L} k_0^2\,, \\
\nu_4 M^M &= \frac{4\pi^2 \ell (g-1)}{3} \left[2\pi (3 - \deg \mathcal{L}) y k_0 + 3\left(1 - \frac{6}{(3 - \deg \mathcal{L})\pi}\right) \deg \mathcal{L} k_0^2\right]\,.
\end{aligned} \tag{8.15}$$

Imposing the flux quantization condition for N^M gives

$$y = \frac{3\nu_4 N^M}{4\pi^3 \ell (3 - \deg \mathcal{L})^2 k_0} - \frac{3 \deg \mathcal{L}}{2\pi (3 - \deg \mathcal{L})} k_0\,. \tag{8.16}$$

Let us briefly digress and examine this expression in more detail. Note that we can substitute N^F into the above to obtain

$$y = \frac{N^M}{N^F} - \frac{3 \deg \mathcal{L}}{2\pi (3 - \deg \mathcal{L})} k_0\,. \tag{8.17}$$

From this expression it is apparent that

$$\lim_{k_0 \to 0} J_8 = J_\Sigma + J_{\mathcal{M}_4} \tag{8.18}$$

holds if $N^M = N^F \equiv N$ are identified, as expected. Substituting (8.16) into the supersymmetric action results in

$$S_M = \frac{3(g-1)\nu_4^2 N^2}{4\pi^2\ell(3-\deg\mathcal{L})^2 k_0} - \frac{6\nu_4 N(g-1)\deg\mathcal{L}}{3-\deg\mathcal{L}} k_0 + \frac{35\pi^2\ell(g-1)}{3}(\deg\mathcal{L})^2 k_0^3 \,. \tag{8.19}$$

Using (3.46), we determine the leading contribution to the preliminary Newton constant to be

$$\frac{1}{G_2} = \frac{24(g-1)\sqrt{\nu_4}N^2}{\ell(3-\deg\mathcal{L})^2 k_0} + \mathcal{O}(k_0)\,. \tag{8.20}$$

Comparing this with the expression for c_{sugra} derived in the previous subsection, we indeed find

$$\frac{1}{G_2} = \frac{\Delta\phi}{3} c_{\text{sugra}} + \mathcal{O}(k_0)\,. \tag{8.21}$$

The distinguished flux number N_0 is given by

$$N_0 = \frac{3(g-1)\nu_4}{\pi^2\ell(3-\deg\mathcal{L})^2 k_0^2} N^2 - \frac{7(g-1)\deg\mathcal{L}}{3-\deg\mathcal{L}} N + 15\pi^2\ell(g-1)(\deg\mathcal{L})^2 \frac{k_0^2}{\nu_4}\,. \tag{8.22}$$

Substituting in the duality relations we find

$$N_0 = \left(\frac{\Delta\phi}{2\pi}\right)^2 \frac{c_{\text{sugra}}}{24} - \frac{7(g-1)\deg\mathcal{L}}{3-\deg\mathcal{L}} N + \frac{15\ell(g-1)(\deg\mathcal{L})^2}{4}\left(\frac{2\pi}{\Delta\phi}\right)^2 \,. \tag{8.23}$$

In particular, this is an expansion in $\Delta\phi$ where the quadratic term is proportional to c_{sugra}, again as expected. Finally, we record that

$$\begin{aligned}\frac{1}{G_2} = \frac{16\pi^2}{81\deg\mathcal{L}} &\left[16\pi(3-\deg\mathcal{L})N_0 + 3\text{Vol}(\Sigma)N\deg\mathcal{L} + \mathbf{V}\right] \\ &\times\sqrt{\frac{2(3-\deg\mathcal{L})N_0 - 21\text{Vol}(\Sigma)N\deg\mathcal{L} + 2\mathbf{V}}{15(3-\deg\mathcal{L})\text{Vol}(\mathcal{M}_5)\text{Vol}(\Sigma)}}\,,\end{aligned} \tag{8.24}$$

where

$$\mathbf{V} = \sqrt{4\pi^2(3-\deg\mathcal{L})^2 N_0^2 + 42\pi\text{Vol}(\Sigma)N_0 N(3-\deg\mathcal{L})\deg\mathcal{L} + 9\left[N\text{Vol}(\Sigma)\deg\mathcal{L}\right]^2}\,. \tag{8.25}$$

As a check on this expression we may formally expand it around the trivial torus fibration with $\deg \mathcal{L} = 0$

$$\frac{1}{G_2} = 8\pi^2 N \sqrt{\frac{N_0 \mathrm{Vol}(\Sigma)}{3\mathrm{Vol}(\mathcal{M}_5)}} - 5\pi N^2 \sqrt{\frac{\mathrm{Vol}(\Sigma)^3}{3N_0 \mathrm{Vol}(\mathcal{M}_5)}} \deg \mathcal{L} + \mathcal{O}(\deg \mathcal{L}^2)\,. \tag{8.26}$$

The first term should then match (7.41), and using the expression (8.4) for $\mathrm{Vol}(\Sigma)$ one can see that this is indeed the case.

9 Conclusions and Outlook

The geometry and physics related to GK geometry have attracted substantial attention recently, but many interesting questions are still open. Starting on the mathematical side, we presented an overview of the geometric features of GK geometry including novel observations on Kähler manifolds admitting the solutions to the master equation. Despite numerous classes of explicit solutions there is very little in the way of systematic understanding of the solutions of the master equation. As we argued in section 3.3 the only minimal complex surfaces that can admit solutions to the master equation are ruled surfaces over elliptic curves. In higher dimensions, the topological constraint equation is a restriction on the Kähler class. A natural question is, are there any further constraints on the solvability of the master equation or does a solution exist in every Kähler class satisfying the constraint.

Being a stationary point of the supersymmetric action is a global necessary condition for the transverse Kähler metric of Y_{2n+1} to solve the master equation. The extremization problem on the other hand requires the flux quantization conditions to be imposed, which are hard to understand from the mathematical perspective. A natural conjecture is that flux quantization is a way of fixing the transverse Kähler class. While this is true for all the explicit examples we are aware of it is not clear how this works in general because of the dependence of the flux integrals on the R-symmetry vector. This renders the extremization problem quite mysterious from the mathematical point of view, but there may exist an equivalent reformulation of the problem that could render it more mathematically tractable.

On the physics side, there has been much progress recently in tests of holography for 2d and 1d SCFTs. The class of theories we discussed in this thesis have the minimal amount of supersymmetry, whilst keeping a non-trivial R-symmetry. The U(1) R-symmetry in 2d and 1d can mix with global U(1) symmetries, and only after applying c- or $\mathcal{I}$-extremization is the true superconformal R-symmetry determined. The main paradigm in this thesis was to study this problem holographically in the context of Type IIB solutions, where the axio-dilaton has a non-trivial spacetime-dependent profile – i.e. F-theory. We showed that the c-extremization of 2d SCFTs obtained from wrapped D3-branes in F-theory compactifications define a geometric extremization problem in the holographically dual AdS_3 solutions. This allowed us to compute, using an off-shell approach, the central charge of the SCFTs from holography.

As a counterpoint to the 2d SCFTs, we discussed 1d SCQMs obtained by M2-branes wrapped on complex curves, and the dual holographic $\mathcal{I}$-extremization principle in M-theory. By M/F-duality, whereby an elliptic fiber in the M-theory geometry becomes the auxiliary elliptic fibration of F-theory, these two setups can be related. The F-theory result for the central charge is obtained by considering the limit in M-theory where the volume of the elliptic fiber is taken to zero ($k_0 \to 0$). As we showed, there are classes of SCFTs where the resulting identification (1.3) is true without any higher-order corrections in k_0 – these were discussed in section 7. In contrast, the class of solutions in section 8 showed that in general there can indeed be corrections to the F-theory expression of the central charge, in order for this to match the 1d partition function. Whenever both sides agree, the solution has an elliptic fibration that is non-trivial only over a complex curve. We observed that for elliptic fibrations over higher-dimensional base manifolds, there are generically correction terms, which arise from non-trivial higher intersection numbers on the base, e.g. from $c_1(\mathcal{L})^2$. This was exemplified in the elliptic three-folds of section 8.

Our analysis was largely focused on the geometric side of holography. Much is known about the wrapped D3-brane theories in F-theory, in terms of central charge computations. However, much less understood is the precise relation between the dimensional reduction of such 2d SCFTs with the 1d SCQM that arises from the dual M2-brane configuration and the associated 1d partition function. Related computations are known for higher (and non-chiral) supersymmetric theories, but for $(0, 2)$ this remains an exciting open problem.

Appendices

A Comparison of Normalizations in M/F-Theory

To streamline the notation in the main text, we implicitly always assume a particular normalization in M-theory and in F-theory. The purpose of this appendix is to explain these normalizations. We uniformly denote the compact spaces by Y_7 and Y_9^τ throughout the thesis, despite the fact that the metrics on these spaces differ depending on whether they appear in F- or M-theory: the normalizations of the Killing vectors ξ differ by a factor of 2

$$\text{F-theory/IIB}: \ \xi = 2\partial_z\,, \qquad \text{M-theory}: \ \xi = \partial_z\,. \tag{A.1}$$

Therefore, the b_i coefficients in the parametrization in (4.38) are related by $\vec{b}_F = 2\vec{b}_M$. The Killing one-form η is correspondingly normalized as

$$\text{F-theory/IIB}: \ \eta = \frac{1}{2}(\mathrm{d}z + P)\,, \qquad \text{M-theory}: \ \eta = \mathrm{d}z + P\,. \tag{A.2}$$

The metrics on the compact spaces are given by

$$\mathrm{d}s^2\left(Y_7\right) = \eta^2 + \mathrm{e}^B \mathrm{d}s^2\left(\mathcal{M}_6\right)\,, \qquad \mathrm{d}s^2(Y_9^\tau) = \eta^2 + \mathrm{e}^B \mathrm{d}s^2(\mathcal{M}_8^\tau)\,, \tag{A.3}$$

with the warp factor and normalization of η pertaining to F- or M-theory. Furthermore, we have also used the same symbol F for the closed two-form appearing in the fluxes F_5 in (4.1) and G_4 in (2.9) in F- and M-theory, respectively. It is given by

$$\text{F-theory/IIB}: F = -2J_6 + \mathrm{d}\left(e^{-B_{10}}\eta\right)\,, \qquad \text{M-theory}: F = -J_8 + \mathrm{d}\left(\mathrm{e}^{-B_{11}}\eta\right)\,, \tag{A.4}$$

with the warp factor, normalization of η and Kähler form pertaining to F- or M-theory. Finally, in the toric examples of section 7, the volumes $\mathcal{V}$ and $\mathrm{Vol}(X_5)$ implicitly depend on the R-symmetry vector $\vec{b}$ so that

$$\mathcal{V}(\vec{b}_M) = 2\mathcal{V}(\vec{b}_F)\,, \qquad \mathrm{Vol}(X_5)(\vec{b}_M) = 2^3\mathrm{Vol}(X_5)(\vec{b}_F)\,. \tag{A.5}$$

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