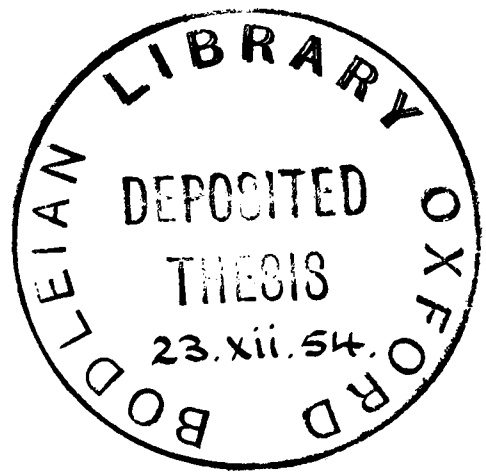
THE MATHEMATICAL FORM AND PHYSICAL CONTENT OF UNIFIED FIELD THEORIES DERIVED FROM A VARIATIONAL PRINCIPLE

by

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Preface

This thesis is concerned with the problem of constructing a unified field theory which will describe satisfactorily in the same formalism gravitation, electromagnetism and perhaps other force fields. Part I contains a brief historical survey of the subject in its first chapter. Very little space is given to the many unified field theories which have already been proposed, for to do so in detail would require a sizeable book. In the second chapter a new extension of the so-called affine theory is obtained. This extension, Schrödinger's affine theory, and Einstein's recent theories are discussed and each is shown not to be a fully satisfactory theory. The third chapter is concerned with certain aspects of static spherically symmetric solutions of Einstein's theory.

Part II is concerned with a new unified field theory. After deriving the theory and its interpretation, such topics as spherically symmetric static solutions of the field equations, the problem of motion for an n-particle system, and possible models in the theory for the elementary particles are discussed.

The decimal notation is used throughout in numbering equations, the first number denoting the chapter. Bracketed numbers in the text refer to equations, the first number being omitted when the equation is in the same chapter as that in which the reference is

made. Unbracketed decimals refer to paragraphs, the first number again giving the chapter. Other numbers, generally written as superscripts in the text refer to the list of references at the end of Part II. This list is by no means exhaustive.

I would like to thank Professor Pryce, my supervisor, for his valuable advice and continued interest in the work. Also I thank Dr. D. K. C. MacDonald for his help during the first eighteen months of my research before I turned to the work dealt with in this thesis. Grateful thanks are also due to the Department of Scientific and Industrial Research from whom I received a maintenance grant.

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The Mathematical Form and Physical Content of Unified Field Theories Derived from a Variational Principle.

Abstract

The Einstein theory of general relativity reduces as a first approximation to the old Newtonian theory and provides a satisfactory account of such small effects as the motion of the perihelion of Mercury and the gravitational deflection of light. The greatest triumph of the theory, however, is that the field equations themselves determine the motion of an n-particle system - as a first approximation the Newtonian attraction. In this way the separate concepts of matter, field and field - matter interaction are not distinct in general relativity.

Unified field theory seeks to extend general relativity so that other force fields such as electromagnetism are included in the formalism with gravitation. Such a theory, to be successful, must as a first approximation both reduce to the Newtonian - Maxwell field theories and imply the Newtonian - Lorents equations of motion. The exact form of, or higher approximation to, the field equations would be expected to give deviations from the Maxwell theory just as general relativity deviates from the Newtonian theory. Indeed the electromagnetic and gravitational fields may not be separable except as an approximation. It is reasonable to expect that a unified field theory with these characteristics would increase our understanding of matter and provide an improved

basis for a new quantum theory.

Taking the so-called affine standpoint one looks for a simple Lagrangian, constructed from the affine connection alone, from which a possible unified field theory can be derived by a variational principle. For a non-symmetric affine connection the simplest Lagrangian seems to be the determinant of a linear combination of the two tensors which result by contracting the curvature tensor. It is found that this Lagrangian gives rise to only two different sets of field equations, one set constituting Schrödinger's theory and the other a new theory. For both sets identity relations or conservation laws can be derived and these provide a powerful tool in suggesting possible interpretations of the theories. The next step is to see if the approximate form for small fields of either set of field equations contains a subset which could be identified as Maxwell's equations. While Schrödinger's theory is satisfactory on this point the new theory is not and so must be discarded. Schrödinger's theory contains a factor λ which must be identified as the cosmological constant. From observations of the solar system this constant must be very small. Consequently Schrödinger's theory cannot differ greatly from Einstein's 'weak' theory which may be obtained formally from Schrödinger's by putting λ equal to zero. Also since the conservation law for Schrödinger's theory contains a Lorentz term with λ as a multiplicative constant it looks as if the smallness of λ is effectively going to remove electromagnetic effects. This conclusion is supported by the fact

that both Einstein's 'weak' and 'strong' theories fail to give equations of motion of a particle system containing force terms other than the gravitational ones. It is difficult to see how Schrödinger's theory with such a small λ could give a significantly different result. All these arguments mean that the new affine theory, Schrödinger's theory, and Einstein's theories each fail to give a satisfactory unified field theory.

For the sake of completeness we derive the detail form for the conservation laws of the new theory. These are found to be of similar form as those found by Schrödinger for his theory. Also the solution of Einstein's 'strong' and 'weak' equations for a static spherically symmetric field are discussed. It is known that in the magnetic case (no electric field) whereas the weak equations have solutions corresponding to a magnetic pole with mass, the strong equations require the mass to be zero. An argument is given to show that in the general combined electric-magnetic case the weak equations may allow the presence of an independent mass parameter, whereas the strong equations, as is known, require the mass to be zero.

All this has been concerned with a non-symmetric affine connection. Part II of this thesis is concerned with a new theory which involves a symmetric affine connection and a non-symmetric metric tensor. Thus the affine approach, according to which the affine connection alone characterises the structure of space, is discarded. The theory developed has a superficial resemblance to a recent theory of Kursunoglu but it is far simpler and has a quite different interpretation.

The form of the conservation laws not only helps to suggest an interpretation of the theory but also plays a part in the derivation of the field equations themselves. The new theory does in first approximation contain a Maxwell-like subset. The suggested exact form for Maxwell's equations involves in a complicated way both the gravitational and electromagnetic field quantities. The field equations can be solved almost exactly for a static spherically symmetric field and the solutions may be interpreted as a stationary charged particle with mass. A structure parameter is involved and from its influence on the electrostatic field and charge distribution it may be interpreted as the radius of the particle. The gravitational mass of the particle contains a positive contribution which it is reasonable to suppose is the electrostatic energy. For large distances the electrostatic field falls off exponentially and the best models for the elementary particles will depend upon just how slow this fall off must be to fit observation. The theory also requires that in regions where the field is small the current vector must not vanish since it also acts as the electromagnetic potential. A finite mass for the photon is predicted, the precise value being inversely proportional to the effective range of the electromagnetic field. For ranges of the order of 1000 Km it is approximately 10^{-44} gms. Finally the field equations are found to require a slowly moving n-particle system to be subject to the usual Newtonian - Lorentz force terms.

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PART I

Introduction and General Theory

Chapter 1

Introductory Survey and Notation

1.1 Introduction: General Relativity

During the latter part of 1915 Einstein in a series of papers[I] published his general relativity theory of gravitation. Although this theory was the culmination of a gradual process of development it brought about a fundamental change in classical physics. Previously gravitation was one among many types of natural force which influenced the motion of material particles in an independently existing Euclidean space, but in the general theory of relativity gravitation is distinguished from other types of force in that it becomes merely a manifestation of the structure of space no longer necessarily Euclidean. Put rather differently it means that massive particles are accompanied by a certain structure of space which shows itself in the gravitational effects experienced by other particles. Gravitation has been in effect 'geometrised'. Although this is the orthodox view of the theory it should be noticed that several attempts have been made to view it simply as a potential theory in Euclidean space[II]. It is convenient to summarise briefly the main ideas which lie at the foundation of Einstein's theory. (There are of course many excellent reviews of the subject, e.g.

Whittaker's 'Aether and Electricity'[33]).

Any event can be assigned a descriptive label of four numbers, one for the time and the other three for the coordinates in 3-space. These labels are assumed to form a continuum in the sense that the motion of an idealised point particle is represented by a continuous curve in space time or 4-space. These statements assume the existence of point events, a concept closely related to that of a point particle, and of the means of determining their position to absolute precision in 4-space. These abstractions, which have no operational justification, are characteristic of classical physics and will not be discussed here. Rather we shall from the beginning take the old classical standpoint and ignore all quantum considerations. Taking this for granted we notice that physics is not interested in isolated point events but is concerned with the interrelation and correlation of many point events. It is clear that both the point events themselves and their interrelation, if this is real and not merely apparent, must be independent of the particular labelling system used to describe them - at least in some class of such systems. In pre-relativity physics it was assumed that the fundamental relationships were the instantaneous relative distance (in 3-space) between any two point events and the relative acceleration (defined as the second derivative with respect to the time of the relative space coordinates) of point particles as measured in Cartesian frames.

These two relationships are covariant to the class of coordinate transformations $\underline{x} \rightarrow \underline{x}'$ given by

$$\underline{x}' = A\,\underline{x} - \underline{v}t \qquad (1)$$
$$t' = t$$

where A is any unitary 3 x 3 matrix with constant components, $\underline{v}$ a velocity vector and $\underline{x}'$, $\underline{x}$ with t', t the space and time coordinates respectively in the two coordinate systems. From this standpoint it was reasonable to require all the laws of physics to be likewise covariant to these Galilean transformations.

The observed properties of light, namely the independence of its finite velocity on the uniform relative motion of source and observer, made these basic assumptions of pre-relativity physics untenable operationally. Instead the fundamental relation between point events must be the interval as defined in space time by the differential relation

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta \qquad (2)$$

where $x^\alpha \equiv (x^a, ct)$ and $\eta_{\alpha\beta} \equiv (-\delta_{ab}, 1)$. Special relativity requires that all physical laws should be covariant to the group of transformations which leave this interval invariant i.e. the Lorentz group. Nothing need be said about the invariance of acceleration since the Lorentz transformation only involves the coordinates linearly. Of course the Lorentz group reduces to the Galilean group for the limiting case of an infinite velocity for light.

The paths of light rays are given by the null geodesics

$$ds = 0 \qquad 3$$

while the path of a particle under the action of no external force by the geodesics

$$\delta\left(\int ds\right) = 0 \qquad 4$$

i.e. straight lines in 4-space.

Special Relativity resembles the old physics in that only relatively unaccelerated frames are included in its transformation group. Just as in the old physics, however, we can certainly derive the form the laws of nature take in an accelerated frame. If we change to a coordinate system $\bar{x}^\alpha$ related to the system x^α by a more general transformation than the Lorentz one the interval (2) will take the form

$$ds^2 = g_{\alpha\beta}\, d\bar{x}^\alpha d\bar{x}^\beta$$

where

$$g_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial \bar{x}^\alpha} \cdot \frac{\partial x^\nu}{\partial \bar{x}^\beta} = \eta_{\mu\nu}\, \partial^\mu_\alpha\, \partial^\nu_\beta \qquad 5$$

the second form for $g_{\alpha\beta}$ uses a convenient notation for the partial differential coefficients. The path of a particle under the action of no forces is given in the $\bar{x}^\alpha$ coordinates by (4) with the ds given by (5). Clearly in the $\bar{x}^\alpha$ frame the path will no longer be rectalinear so that the transformation may be said to cause the appearance of fictitious forces - fictitious because a coordinate transformation will remove them entirely - but the paths are independent of any properties of the particular moving particle.

Suppose we generalise the interval (5) to the case when $g_{\alpha\beta}$ cannot be reduced to $\eta_{\alpha\beta}$ everywhere by a suitable coordinate transformation. Equation (4) will then give paths which cannot be described as rectalinear in any frame and we can say that the body moving in this way is under the action of a force. Again these paths will not depend on the particular particle concerned and the fact that a coordinate frame always exists for any symmetric tensor $g_{\alpha\beta}$ such that[22 §4]

$$g_{\alpha\beta} = \eta_{\alpha\beta} \quad , \quad g_{\alpha\beta,\gamma} = 0$$

along a given line in 4-space means that at least in the locality of the moving frame we can transform away the force. The question arises as to whether any force exists in nature, having this characteristic, which could be described by the generalisation (5) of the metric (2) together with the equations of motion (4). If such a force does exist then the description outlined would have the advantage of being covariant with respect to all transformations of coordinates (restricted only to be 1 - 1), a much wider group than the Lorentz transformations.

The Eötvös[24] experiment shows that gravitation is of this character. To great precision it shows that gravitational and inertial mass are the same so that in a freely falling lift the gravitational field of the earth, for instance, is effectively transformed away. Since this is true only in the vicinity of the lift it does suggest that the gravitational field could be described in a covariant way by the 10 components of the metric tensor $g_{\alpha\beta}$

instead of a single scalar potential.

Between 1913 - 14 Einstein made this suggestion and justified the greatly increased complication of gravitational theory by showing that a scalar potential led in certain cases to inconsistencies[36]. The old scalar potential was determined in terms of the matter distribution by Poisson's equation and the next step was to discover similar field equations for the ten components $g_{\alpha\beta}$. This final stage in building general relativity theory is most easily taken by seeking for a natural analogue to Poisson's equation. Thus we expect a tensor equation involving $g_{\alpha\beta}$ and its derivatives up to the second order on the left, with some generalisation of mass density on the right. Now it can be shown (Eddington, 30 § 36) that any tensor which contains only $g_{\mu\nu}$ with its first and second derivatives must be a function of $g_{\mu\nu}$ and $B^{\iota}{}_{\kappa\lambda\mu}$, the curvature tensor constructed from the second Christoffel symbols of $g_{\mu\nu}$ i.e.

$$B^{\iota}{}_{\kappa\lambda\mu} = -\Gamma^{\iota}_{\kappa\lambda,\mu} + \Gamma^{\iota}_{\kappa\mu,\lambda} + \Gamma^{\iota}_{\alpha\lambda}\Gamma^{\alpha}_{\kappa\mu} - \Gamma^{\iota}_{\alpha\mu}\Gamma^{\alpha}_{\kappa\lambda}, \qquad \Gamma^{\iota}_{\kappa\lambda} = \left\{ {\iota \atop \kappa\lambda} \right\} \qquad 6$$

Also the natural generalisation of the mass density is a stress energy momentum tensor T_{ik} with the property that

$$T^{ik}{}_{\|k} = 0 \qquad 7$$

where indices are raised by the inverse metric tensor g^{ik} in the usual way, and where the $\|$ refers to covariant differentiation based on the Christoffel symbol of $g_{\alpha\beta}$. Based upon these

considerations the simplest possible analogue to Poisson's equation is

$$R_{\mu\kappa} - \tfrac{1}{2} g_{\mu\kappa} R = -\kappa T_{\mu\kappa} \qquad 8$$

where

$$R_{\mu\kappa} = B^{\sigma}_{\mu\kappa\sigma} \quad , \qquad R = g^{\alpha\beta} R_{\alpha\beta}$$

Bianchi's identities for $B^{\cdot}_{\kappa\lambda\mu}$ are sufficient to show that the covariant divergence of the left of (8) vanishes as required. These equations (8) with the equations of motion (4) and (5) for a particle in the field constitute the Einstein theory of gravitation as first proposed.

The disadvantage of these field equations lies in the fact that the $T_{\mu\kappa}$ tensor involved is of unknown form. However, making a reasonable assumption for $T_{\mu\kappa}$ the theory can be shown to reduce as a first approximation to the Newtonian theory. For a purely gravitational field in empty space we can take $T_{\mu\kappa}$ to be zero everywhere and the equations then take the more definite form

$$R_{\mu\kappa} = 0 \qquad 9$$

We would expect that a system of mass-point particles would be described by this equation and indeed it can be shown (see later) that if they do describe a system with mass then there must be point singularities in the field. The case of a spherically symmetric static field represents the especially simple particle described by

the well known Schwarzschild solution.

On the scale of the solar system the sun may be regarded to a good approximation as a Schwarzschild particle. Using the geodesic postulate (3), (4) and (5) it is then possible to apply the model to the problem of the deflection of light by the sun and to the motion of the perihelion of Mercury. The agreement with observation on these two issues constitute the strongest empirical evidence in favour of the theory. Another prediction of the Einstein theory, namely the gravitational reddening of photons, is by no means well established observationally(cf 38).

To the mind of the author, however, the great triumph of the theory does not lie in its simple covariant form or on its ability to account quantitatively for the motion of the perihelion of Mercury and the gravitational deflection of light but rather in the fact that the field equations themselves imply the motion of particles in the field. At first, and in the above development of the theory, the geodesic postulate for the equations of motion of a particle in the field was thought an indispensible addition to the field equations. The position then resembles that of the old Newtonian theory where the equations determining the field in terms of the matter distribution are incomplete without another equation giving the influence of the field on the motion of matter. Now in the Newtonian theory the field equations are linear so that given

any two solutions a third can be found by taking their sum in any proportion. One has only to look at the complicated dependence of $R_{\mu\kappa}$ upon $g_{\mu\kappa}$ to see that no similar property holds for the Einstein theory. Indeed it is apparent that the field equations do themselves limit the possible motion and distribution of matter. However, it was only from 1938 onwards that Einstein, Infeld and Hoffmann in a series of papers[20,21] were able to show that the field equations (9) applied to a slowly moving n-particle system themselves implied equations of motion for the particles. To a first approximation the motion is in accord with the Newtonian inverse square law of gravitation with a constant mass for each particle. By an entirely different method Infeld and Schild showed in 1949[22] that the geodesic postulate (4) and (5) for the motion of a test particle could also be deduced from the field equation (9). Thus General Relativity has welded together the concepts of field and matter in the sense that only as a first approximation is it valid to speak of matter acting as a source to a field which then influences the motion of the particles.

1.2 The identity relations

As Einstein[21] has already pointed out the reason why the field equations do themselves imply the equations of motion lies not so much in their non-linearity as in the existence of certain identity relations or conservation laws which they satisfy. It is convenient

to write down here for comparison with later similar results the various forms these conservation laws take. The first form is directly equivalent to the vanishing of the covariant divergence of the left hand side of (8):-

$$\mathfrak{T}^{\alpha}_{\beta\|\alpha} = 0 \qquad \text{where} \qquad \mathfrak{T}^{\alpha}_{\beta} = \sqrt{-g}\left(g^{\sigma\alpha}R_{\sigma\beta} - \tfrac{1}{2}\delta^{\alpha}_{\beta}g^{\mu\nu}R_{\mu\nu}\right) \qquad \text{(a)}$$

The second is the simple divergence form for these same equations,

$$\left(\mathfrak{T}^{\alpha}_{\beta} + t^{\alpha}_{\beta}\right)_{,\alpha} = 0 \qquad \text{(b)}$$

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where

$$t^{\alpha}_{\beta} = -\tfrac{1}{2}\left(\delta^{\alpha}_{\beta}\Lambda + \Lambda^{\alpha}_{\mu\nu}g^{\mu\nu}{}_{,\beta}\right)$$

The third set gives a divergence form for $\mathfrak{T}^{\alpha}_{\beta} + t^{\alpha}_{\beta}$,

$$\mathfrak{T}^{\alpha}_{\beta} + t^{\alpha}_{\beta} = -\tfrac{1}{2}\left(\Lambda^{\sigma}_{\alpha\rho}g^{\beta\rho} + \Lambda^{\sigma}_{\rho\alpha}g^{\rho\beta} - \delta^{\beta}_{\alpha}\Lambda^{\sigma}_{\mu\nu}g^{\mu\nu}\right)_{,\sigma} \qquad \text{(c)}$$

Here (b) and (c) use the notation

$$\Lambda = g^{\mu\nu}\left(\left\{{\alpha \atop \mu\beta}\right\}\left\{{\beta \atop \alpha\nu}\right\} - \left\{{\beta \atop \alpha\beta}\right\}\left\{{\alpha \atop \mu\nu}\right\}\right) \quad , \qquad \Lambda^{\alpha}_{\mu\nu} = \left\{{\alpha \atop \mu\nu}\right\} - \delta^{\alpha}_{\nu}\left\{{\sigma \atop \mu\sigma}\right\} \qquad \text{(d)}$$

t^{α}_{β} , clearly not a tensor, is known as the pseudo - stress - energy - momentum - tensor and we shall call $\mathfrak{T}^{\alpha}_{\beta} + t^{\alpha}_{\beta}$ the stress - energy - momentum array for the field. It is not difficult to show that the mass of a finite system is given by

$$m = -\frac{1}{8\pi}\int\left(\mathfrak{T}^{4}_{4} + t^{4}_{4}\right)d\sigma \qquad \text{(a)}$$

while for a quasi-static system an alternative useful form is valid

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i.e.

$$m = -\frac{1}{8\pi}\int\left(\mathfrak{T}^{4}_{4} - \mathfrak{T}^{1}_{1} - \mathfrak{T}^{2}_{2} - \mathfrak{T}^{3}_{3}\right)d\sigma \qquad \text{(b)}$$

(for these formulae see Tolman 34 §§ 91, 92 and for a much simpler method for deriving (11 b) Papapetrou 35). From (11 b) and the definition (10 a) of $\mathfrak{T}^{\prime}_{\beta}$ we can deduce that a static system with mass described by the field equations (9) must contain singularities in the $g_{\mu\beta}$ field. This justifies a remark made earlier on this matter.

We can conclude this brief review of general relativity by saying that Einstein has succeeded in constructing a very successful covariant theory of gravitation. However, gravitation is but one type of natural force known to us and it is reasonable to seek to describe in the same or similar formalism other natural force fields such as electromagnetism. Such a successful extension of the theory is required to contain as a first approximation not only the Newtonian theory of gravitation but also the classical theory of electromagnetism. Before we discuss attempts to construct such a theory it is convenient to consider the equations of classical electromagnetism and possible covariant forms of them.

1.3 The Maxwell Lorentz Equations

Maxwell's equations and the Lorentz equation for the mechanical force may be divided into the three sets

(i) $\quad \operatorname{curl} H - \frac{1}{c}\dot{D} = I \qquad \operatorname{div} D = \rho$

(ii) $\quad \operatorname{curl} E + \frac{1}{c}\dot{B} = 0 \qquad \operatorname{div} B = 0$

(iii) $\quad F = \rho E + I \wedge H$

where D , E and ρ are measured in e.s.u. and B , H and I

in e.m.u. except that the units involved in the current - charge vector (I, ρ) have been modified so as to contain the factor 4π . Using the coordinate system $x^\alpha = (x^r, ct)$ there are two obvious ways of writing these equations in covariant form

	A	B
(i)	$f^{\iota\kappa}{}_{,\kappa} = 1^\iota$	$\frac{1}{2}\varepsilon^{\sigma\iota\kappa\lambda} f_{\kappa\lambda,\iota} = 1^\sigma = \frac{1}{6}\varepsilon^{\sigma\iota\kappa\lambda} I_{\iota\kappa\lambda}$
(ii)	$\varphi_{\iota\kappa,\lambda} + \text{cyclic} = 0$	$\Phi^{\iota\kappa}{}_{,\kappa} = 0$
(iii)	$\mathcal{F}_\alpha = \varphi_{\alpha\sigma} 1^\sigma$	$\mathcal{F}_\alpha = -\frac{1}{2} I_{\alpha\mu\nu} \Phi^{\mu\nu}$

where 13

$$f^{\iota\kappa} = \frac{1}{2}\varepsilon^{\iota\kappa\lambda\mu} f_{\lambda\mu} \quad \text{and} \quad \varphi_{\iota\kappa} = \frac{1}{2}\varepsilon_{\iota\kappa\lambda\mu} \Phi^{\lambda\mu}$$

$$= \begin{pmatrix} 0 & H_3 & -H_2 & -D_1 \\ -H_3 & 0 & H_1 & -D_2 \\ H_2 & -H_1 & 0 & -D_3 \\ D_1 & D_2 & D_3 & 0 \end{pmatrix} \qquad = \begin{pmatrix} 0 & B_3 & -B_2 & +E_1 \\ -B_3 & 0 & B_1 & E_2 \\ B_2 & -B_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix}$$

and also $1^\alpha = (I, \rho)$

These two formulations are not independent the second being known as the dual of the first. In formulation B both $\varepsilon^{\iota\kappa\lambda\mu}$ and $\varepsilon_{\iota\kappa\lambda\mu}$ have the same components in all frames (namely +1 or -1 according as $\iota\kappa\lambda\mu$ is an even or odd permutation of 1 2 3 4 and zero otherwise) but whereas $\varepsilon^{\iota\kappa\lambda\mu}$ is a contravariant tensor of weight unity, $\varepsilon_{\iota\kappa\lambda\mu}$ is a covariant tensor of weight minus unity. Admittedly it is perhaps a little arbitrary to write the current vector as a density 1^α but it seems reasonable. Another note about notation concerns the use of dots below three indices, this denoting the sum

over cyclical permutations of those indices. Some authors write this as $\varphi_{[\iota\kappa,\lambda]}$. As we have written these covariant forms for the Maxwell - Lorentz equations there is no connection between the tensor density $f^{\iota\kappa}$ and the tensor $\varphi_{\iota\kappa}$. In special relativity the distinction between densities and ordinary tensors disappears and the relation is

$$f^{\iota\kappa} = \eta^{\iota\alpha}\eta^{\kappa\beta}\varphi_{\alpha\beta} \qquad 14$$

where $\eta^{\iota\kappa}$ is the Minkowski matrix, or equivalently just $H = B$, $E = D$. In analogy with this it was suggested long ago that the general covariant relation should be

$$f^{\iota\kappa} = \sqrt{-g}\, g^{\iota\alpha} g^{\kappa\beta} \varphi_{\alpha\beta} \qquad 15$$

This would mean that B and D differ from H and E respectively in regions where a gravitational field is present. Again in the special relativity case it is convenient to define an energy momentum tensor $t_{\iota\kappa}$ by

$$t_{\iota\kappa} = \varphi_{\iota\cdot}^{\ \alpha}\varphi_{\alpha\kappa} - \tfrac{1}{4}\gamma_{\iota\kappa}\varphi^{\alpha\beta}\varphi_{\beta\alpha} \qquad \text{(a)}$$

so that 16

$$\mathfrak{f}_\alpha = \varphi_{\alpha\sigma}\, 1^\sigma = t_{\alpha\cdot,\sigma}^{\ \ \sigma} \qquad \text{(b)}$$

One might suppose that the appropriate general covariant form for $t_{\iota\kappa}$ was

$$\mathcal{E}_\kappa^\iota = f^{\iota\alpha}\varphi_{\alpha\kappa} - \tfrac{1}{4}\delta_\kappa^\iota f^{\alpha\beta}\varphi_{\beta\alpha} = \sqrt{-g}\, E_\kappa^\iota \qquad \text{(a)}$$

because then 17

$$\mathfrak{f}_\alpha = \varphi_{\alpha\sigma}\, 1^\sigma = \mathcal{E}_{\alpha\|\sigma}^{\ \ \sigma} \qquad \text{(b)}$$

as is easy to verify using (15). It is also interesting to note that the $(^4_4)$ component of $\mathcal{E}'_4$ is given according to (13) by

$$\mathcal{E}^4_4 = \tfrac{1}{2}(B.H + D.E) \qquad 18$$

i.e. the usual form for the energy of an electromagnetic field in a medium whose dielectric constant and magnetic permeability are not necessarily unity.

The suggested covariant expressions (15) and (17) must be regarded as tentative only. We do not necessarily expect the exact form of the Maxwell - Lorentz equations to be given by (13) with the relation (15) but we require at least that every proposed exact form should reduce as a first approximation for small fields to the special relativity equations given by (13) with (14). The alternative expressions (13) have been written down only to help us in constructing and interpreting possible field theories aimed at providing a unified description of the fields of gravitation and electromagnetism.

1.4 Notation

To avoid any possible confusion it is best now, before going further, to outline the notation and basic geometrical ideas involved in the theory. Some of these have been used already but it was more convenient not to launch out at the start on a discussion of these topics.

Transfer and covariant differentiation

In constructing a tensor calculus some definition of parallel transfer is essential and the simplest definition is provided by the three index symbol or affine connection $\Gamma^{\iota}_{\kappa\lambda}$. When this symbol is not symmetric in its two lower indices there exist three different expressions for the parallel transfer $A^{\iota} + \delta A^{\iota}$ of the contravariant vector A^{ι} from the point x^{κ} to the neighbouring point $x^{\kappa} + \delta x^{\kappa}$ denoted by,

$$\delta A^{\iota} = -\Gamma^{\iota}_{\sigma\rho} A^{\sigma} \delta x^{\rho} \quad , \quad \delta A^{\dot{+}} = -\Gamma^{\iota}_{\sigma\rho} A^{\sigma} \delta x^{\rho} \quad , \quad \delta A^{\dot{-}} = -\Gamma^{\iota}_{\rho\sigma} A^{\sigma} \delta x^{\rho} \qquad (19)$$

Only the ' $\circ$ ' transfer satisfies the parallelogram rule of displacements i.e. given two displacements $\delta^{(1)}x^{\iota}$ and $\delta^{(2)}x^{\iota}$ at the point x^{ι} the transfer of one displacement along the other is commutative only for the ' $\circ$ ' transfer. Indeed one can make out a good case from this fact that the three index symbol $\Gamma^{\iota}_{\kappa\lambda}$ should be symmetric in (κ, λ) (cf. Eddington 30 §91), but we shall consider in Part I the more general case. The condition that the parallel transfer $A^{\iota} + \delta A^{\iota}$ should be a tensor at $x^{\iota} + \delta x^{\iota}$ for any one of the expressions (19) requires that $\Gamma^{\iota}_{\kappa\lambda}$ should transform on charge of coordinate frame from x^{ι} to $\bar{x}^{\iota}$ as

$$\bar{\Gamma}^{\iota}_{\kappa\lambda} = \delta^{\bar{\iota}}_{a} \, \delta^{b}_{\bar{\kappa}} \, \delta^{c}_{\bar{\lambda}} \, \Gamma^{a}_{bc} + \delta^{\bar{\iota}}_{\alpha} \, \delta^{\alpha}_{\bar{\kappa}\bar{\lambda}} \qquad (20)$$

where

$$\delta^{\bar{\iota}}_{a} = \frac{\partial \bar{x}^{\iota}}{\partial x^{a}} \quad , \quad \delta^{\alpha}_{\bar{\kappa}\bar{\lambda}} = \frac{\partial^{2} x^{\alpha}}{\partial \bar{x}^{\kappa} \partial \bar{x}^{\lambda}} \quad \text{etc.}$$

Corresponding to the three expressions for parallel transfer (19)

there are three types of covariant differentiation, namely

$$A^{\overset{\circ}{\underline{\ }}}_{;k} = A^{\overset{\circ}{\underline{\ }}}_{,k} + \Gamma^{\overset{\circ}{\underline{\ }}}_{\underline{\sigma k}} A^{\sigma}, \quad A^{\overset{+}{\underline{\ }}}_{;k} = A^{\overset{+}{\underline{\ }}}_{,k} + \Gamma^{\overset{+}{\underline{\ }}}_{\sigma k} A^{\sigma}, \quad A^{\overset{-}{\underline{\ }}}_{;k} = A^{\overset{-}{\underline{\ }}}_{,k} + \Gamma^{\overset{-}{\underline{\ }}}_{k\sigma} A^{\sigma} \qquad 21$$

For scalar quantities the operations of covariant and ordinary differentiation coincide. Using this with the demand that the covariant derivative of the scalar product of a contravariant and covariant vector shall comply with the product rule

$$(A_\sigma B^\sigma)_{;\iota} = (A_\sigma B^\sigma)_{,\iota} = A_{\sigma;\iota} B^\sigma + A_\sigma B^\sigma{}_{;\iota} = A_{\sigma,\iota} B^\sigma + A_\sigma B^\sigma{}_{,\iota}$$

for each type of covariant derivative, it follows that the types of covariant derivative of a covariant vector corresponding to those of a contravariant vector are

$$A_{\overset{\circ}{\underline{\ }};k} = A_{\iota,k} - \Gamma^{\sigma}_{\underline{\iota k}} A_\sigma, \quad A_{\overset{+}{\underline{\ }};k} = A_{\iota,k} - \Gamma^{\sigma}_{\iota k} A_\sigma, \quad A_{\overset{-}{\underline{\ }};k} = A_{\iota,k} - \Gamma^{\sigma}_{k\iota} A_\sigma \qquad 22$$

It necessarily follows that only indices of like sign character can be contracted within a covariant derivative.

Turning now to densities, which transform according to

$$\overline{\mathfrak{A}} = |\partial \bar{x}|^n \mathfrak{A} \qquad 23$$

where n is the weight of the density, we need expressions only for the covariant derivative of a density of weight unity. Again there are in the general case three types

$$\mathfrak{A}_{\overset{\circ}{\underline{\ }};\iota} = \mathfrak{A}_{,\iota} - \Gamma^{\sigma}_{\underline{\sigma\iota}} \mathfrak{A}, \quad \mathfrak{A}_{\overset{+}{\underline{\ }};\iota} = \mathfrak{A}_{,\iota} - \Gamma^{\sigma}_{\sigma\iota} \mathfrak{A}, \quad \mathfrak{A}_{\overset{-}{\underline{\ }};\iota} = \mathfrak{A}_{,\iota} - \Gamma^{\sigma}_{\iota\sigma} \mathfrak{A} \qquad 24$$

For a given tensor density (of unit weight understood henceforth) there are many covariant derivatives, each contravariant and covariant index following one of the patterns (21) and (22) respectively with in addition a term like one of those in (24) for the density 'index'.

(the usual convention of using a gothic letter for a density is followed: if the letter is gothic then there is a density index). The particular character of each index under covariant differentiation is as before denoted by a $\circ$, $+$ or $-$ sign below the index e.g. for a tensor density

$$\mathfrak{g}^{ik}_{\ \ \ ;\lambda} = \mathfrak{g}^{ik}_{\ \ \ ,\lambda} + \Gamma^{i}_{\sigma\lambda}\,\mathfrak{g}^{\sigma k} + \Gamma^{k}_{\lambda\sigma}\,\mathfrak{g}^{i\sigma} - \Gamma^{\sigma}_{\sigma\lambda}\,\mathfrak{g}^{ik} \qquad 25$$

In the following, when the sign character of an index is not explicitly written the $+$ one will be implied.

1.5 Metric and Interval

Given a metric tensor g_{ik} (any second rank covariant tensor will do) we can define an invariant interval ds between two neighbouring points by the differential relation

$$ds^2 = g_{\alpha\beta}\,dx^\alpha\,dx^\beta \qquad 26$$

Clearly any skew part of $g_{\alpha\beta}$ is irrelevant. But the symmetric part of an affine connection, $\Gamma^{i}_{k\lambda}$, defines along each geodesic of the connection another invariant interval ' $d\sigma$ ', namely that ' $d\sigma$ ' for which the geodesics take the simple form

$$\frac{d^2x^\alpha}{d\sigma^2} + \Gamma^{\alpha}_{\mu\nu}\,\frac{dx^\mu}{d\sigma}\,\frac{dx^\nu}{d\sigma} = 0 \qquad 27$$

It is interesting to note the condition that these two measures of interval ds and $d\sigma$ should be compatible in the domain of their coexistence i.e. along the geodesics. Schrödinger[7] has shown that the necessary and sufficient condition is

$$\Gamma^{\cdot}_{\underline{k\lambda}} = \left\{{\cdot \atop k\lambda}\right\}_h + h^{\cdot\sigma} T_{\sigma k\lambda} \qquad \text{(a)}$$

where $T_{\sigma k\lambda}$ is any tensor satisfying 28

$$T_{\underset{\cdot\cdot}{\sigma k\lambda}} = 0 \qquad T_{\sigma k\lambda} = T_{\sigma\lambda k} \qquad \text{(b)}$$

and $\left\{{\cdot \atop k\lambda}\right\}$ is the Christoffel symbol of $h_{\cdot k} = g_{\underline{\cdot k}}$

1.6 Integrability and the Curvature Tensor

An operation of parallel transfer is said to be integrable if its repeated application to a vector around a finite closed circuit leaves that vector unchanged. Thus the + transfer is integrable if

$$\oint \Gamma^{\cdot}_{\sigma\rho} A^{\sigma} dx^{\rho} = 0$$

where $\oint$ means integration around a closed circuit. It is well known that the necessary and sufficient condition for this integral to vanish is that the + curvature tensor should vanish i.e.

$$\underset{+}{\beta}{}^{\cdot}_{\kappa\lambda\mu} = 0$$

where $$\underset{+}{\beta}{}^{\cdot}_{\kappa\lambda\mu} = - \Gamma^{\cdot}_{\kappa\lambda,\mu} + \Gamma^{\cdot}_{\kappa\mu,\lambda} + \Gamma^{\cdot}_{\alpha\lambda}\Gamma^{\alpha}_{\kappa\mu} - \Gamma^{\cdot}_{\alpha\mu}\Gamma^{\alpha}_{\kappa\lambda}$$ 29

The same condition must also hold if the + transfer of a covariant vector is integrable. The condition that o or - transfer should be integrable is given by the vanishing of the corresponding curvature tensor.

An interesting case arises when the zero transfer is integrable for then it can be shown that a class of coordinate frames exist in which the symmetric part of the affine connection vanishes everywhere (see Schrödinger 32 p.47). In this class, each member of which is

related to every other by a linear coordinate transformation, the affine connection reduces to a tensor skew in its covariant indices. The case of an integrable symmetric affine connection characterised by

$$\Gamma^{\cdot}_{\underset{\nu}{\kappa\lambda}} = 0 \qquad \beta^{\cdot}_{\kappa\lambda\mu} = 0 \tag{30}$$

is especially simple for then a class of coordinate frames exist in which the affine connection vanishes everywhere. There is then only one covariant derivative and this coincides with the ordinary derivative.

For future reference we write down the two tensors which arise from $\overset{\cdot}{\underset{+}{\beta}}{}_{\kappa\lambda\mu}$ by contraction.

$$R_{\kappa\lambda} = \overset{\sigma}{\underset{+}{\beta}}{}_{\kappa\lambda\sigma} = -\Gamma^{\sigma}_{\kappa\lambda,\sigma} + \Gamma^{\sigma}_{\kappa\sigma,\lambda} + \Gamma^{\sigma}_{\alpha\lambda}\Gamma^{\alpha}_{\kappa\sigma} - \Gamma^{\sigma}_{\alpha\sigma}\Gamma^{\alpha}_{\kappa\lambda}$$

$$S_{\lambda\mu} = \overset{\sigma}{\underset{+}{\beta}}{}_{\sigma\lambda\mu} = -\Gamma^{\sigma}_{\sigma\lambda,\mu} + \Gamma^{\sigma}_{\sigma\mu,\lambda} \tag{31}$$

Finally we note that when the affine connection is symmetric there is only one transfer, one covariant derivative and one curvature tensor. Furthermore in this case the curvature tensor has in effect only one contraction for then

$$R_{\kappa\lambda} = -\tfrac{1}{2} S_{\kappa\lambda} \tag{32}$$

1.7 The Lagrangian derivation of Einstein's gravitation theory

Given a symmetric metric tensor and a symmetric affine connection the most simple scalar density we can construct from them both is

$$\mathfrak{L} = g^{\cdot\kappa} R_{\cdot\kappa} \qquad \text{where} \qquad g^{\cdot\kappa} = \sqrt{-g}\, g^{\kappa\cdot}, \quad g^{\cdot\sigma} g_{\kappa\sigma} = \delta^{\cdot}_{\kappa}$$

Consider the invariant

$$I = \int \mathcal{L}\, d\tau \qquad d\tau = dx^1 dx^2 dx^3 dx^4$$

and apply the principle that the variation of I vanishes for all independent variations which vanish on the boundary of integration of g_{ik} and $\Gamma^{i}_{k\lambda}$. Using Palatini's formula for δR_{ik} this gives for the variation of $\Gamma^{i}_{k\lambda}$

$$0 = \int g^{ik}\, \delta R_{ik}\, d\tau = \int g^{ik} \left(- \delta\Gamma^{\sigma}_{ik;\sigma} + \delta\Gamma^{\sigma}_{i\sigma;k} \right) d\tau$$

$$= \int \left(g^{ik}{}_{;\sigma} - \delta^{k}_{\sigma}\, g^{ip}{}_{;p} \right) \delta\Gamma^{\sigma}_{ik}\, d\tau$$

The last step follows by adding the covariant divergence of a vector density to the integral - since for any

$$\mathfrak{A}^{\alpha}_{;\alpha} = \mathfrak{A}^{\alpha}_{,\alpha}$$

this divergence term is equivalent to a surface integral which vanishes because $\delta\Gamma^{i}_{k\lambda}$ vanishes on the boundary. Since $\delta\Gamma^{i}_{k\lambda}$ is arbitrary within the region of integration it is then a simple matter to deduce that

$$g^{ik}{}_{;\sigma} = 0 \qquad \text{or} \qquad g_{ik;\sigma} = 0$$

This simply means that $\Gamma^{i}_{k\lambda}$ is the Christoffel symbol of g_{ik}.

Finally the variation of g_{ik} requires

$$R_{ik} = 0$$

where R_{ik} is constructed from the Christoffel symbol of g_{ik} according to (31). But this is just Einstein's field equation (9) for a mass - point - particle system.

1.8 The General Relativity - Maxwell theory

The simplest and most obvious way to bring electromagnetism into the formalism of general relativity is to substitute into the field equation (8) the stress-energy-momentum-tensor (17 a) of the electromagnetic field and the adjoin the Maxwell equations (13) with the relation (15). Gathering this together the field equations are

$$R_{ik} - \tfrac{1}{2} g_{ik} R = -8\pi E_{ik} \qquad \text{a } 33$$

Since the covariant divergence of the left hand side vanishes identically this equation is by (17 b) only valid in regions unoccupied by charge. In addition there are the Maxwell type equations

$$f^{ik}{}_{,k} = 0 \quad , \quad f^{ik} = \sqrt{-g}\; g^{i\alpha} g^{k\beta} \varphi_{\alpha\beta} \quad , \quad \varphi_{ik,\lambda} = 0 \qquad \text{b } 33$$

The solution of these equations for a spherically symmetric static field is (see Eddington 30 § 78)

$$g_{ik} = -\gamma^{-1} dr^2 - r^2(\sin^2\theta\, d\varphi^2 + d\theta^2) + \gamma dt^2$$

where

$$\gamma = 1 - \frac{2m}{r} + \frac{4\pi\varepsilon^2}{r^2} \qquad 34$$

and

$$\varphi_{14} = E_r = \frac{\varepsilon}{r^2}$$

This solution may be identified as a point mass particle with charge. Furthermore the Einstein - Infeld - Hoffmann method developed for the purely gravitational theory can be used to derive from these equations the Newtonian - Lorentz equations of motion.

In spite of these results the theory is regarded as unsatisfactory on three counts

(i) The electromagnetic and gravitational fields enter the equations on entirely different levels.

(ii) No new insight into the constitution of matter is provided by the equations.

(iii) The Lagrangian from which the field equations follow by a variation principle i.e.

$$\mathcal{L} = g^{ik} R_{ik} - 4\pi f^{ik} \varphi_{ik} \qquad 35$$

is unnatural and odd in composition (Eddington 30 §79).

1.9 Unified Field Theory

Many attempts have been made to construct a truly unified field theory embracing both gravitation and electromagnetism. The success of any proposed theory is to be judged on the following criteria

(i) The equation must as a first approximation for small fields reduce to the field equations of Newton and Maxwell for the gravitational and electromagnetic components. It is not expected possible to separate the two fields except in the small field approximation.

(ii) The field equations themselves must imply equations of motion which to a first approximation are in accord with the Newton - Lorents ideas.

(iii) The field equations must have solutions which correspond to known physical objects. A theory which permits, for instance, the existence of magnetic poles with or without mass is untenable. (iv) The fields of electromagnetism and gravitation must enter the field equations on a common basis not always perhaps being capable of separation. The very structure of space must contain both electromagnetism and gravitation.

So far no theory has been discovered which is known to be satisfactory on all these points. We will not attempt to discuss the many theories proposed over the years but will be content with an outline of the main developments. All the theories have involved an extension of the geometrical basis of general relativity. Weyl modified the requirement that the interval ds of (5) should be invariant[12]: Eddington laid stress on the role played in both the geometry and tensor calculus by the affine connection[30]: Einstein developed a theory based only on a symmetrical affine connection[5]: Schrödinger continued the so-called affine method by considering in a number of papers a non-symmetrical affine connection[6-9]: Einstein developed the so-called metric-affine approach by using both a non-symmetrical metric tensor and affine connection.[1-4,37] A Hermitian postulate was used to reduce the possibilities and the Bianchi identities played an occasional important part in constructing the field equations[3]: Kursunoğlu modified further Einstein's work

by laying greater stress on the form of the Bianchi identities in constructing the field equations. Of all these theories the last is most promising. Unfortunately, however, the equations are so complicated that only their approximate form for small fields can at present be considered in detail.[10] A theory very similar to it superficially but of a far simpler character and with a fundamentally different interpretation is developed in Part II of this thesis.

Apart from this main line of more or less continuous progress there were many other theories, in particular the 5 -dimensional theory of Kaluza[13] with its developments and more recently the six dimensional theory of Podolanski[15]. In its early development the Kaluza theory seemed promising but later work showed that it was equivalent to the General Relativity - Maxwell theory. Attempts to generalise the Kaluza theory particularly by Einstein and Mayer[14] have not proved successful. The Podolanski theory is really an extension of the Kaluza theory to six dimensions and contains on that account two skew fields. No doubt it was first thought that one of these would describe electromagnetism while the other would be a short range field capable of describing inter-nuclear forces. Unfortunately the theory as so far developed contains the fields symmetrically without any indication of such a difference of character. On the whole it seems best to continue to work in four dimensions.

In the second chapter we will consider an extension of the affine theory and we continue to discuss non-symmetric affine connection theories in the remainder of Part I. In Part II we consider a new theory based on a symmetric affine connection and a non-symmetric metric tensor.

Chapter 2

Affine Theory

2.1 The Variation Principle

The aim of unified field theory is to develop a geometry whose structure shall be capable of describing such properties of matter as gravitation and electromagnetism in one formalism. Because of its successful account of gravitation it is reasonable to expect that some generalisation of general relativity will provide the unified field theory sought. In fact we look for a modification of Riemanian Geometry assuming as before the existence of a 4-space continuum and the means of labelling events in that continuum to absolute precision. Of course it may be that electromagnetism is essentially quantal in nature in which case such a classical approach as we undertake is doomed to failure from the start. Only future developments can resolve the issue.

In the so-called affine approach to the problem, with which we are concerned in this chapter, the structure of space is assumed to be entirely characterised by the affine connection $\Gamma^{\cdot}_{\kappa\lambda}$. From this point of view the existence of a metric tensor is superfluous and is not assumed at the start. As we have seen, when $\Gamma^{\cdot}_{\kappa\lambda}$ satisfies equations (1.30) i.e. is integrable and symmetric there exist a class of coordinate frames in which $\Gamma^{\cdot}_{\kappa\lambda}$ vanishes everywhere. The structure of space is then especially simple and it is

reasonable to suppose that it corresponds to a space devoid of matter. It is also reasonable on this basis to suppose that some generalisation of equations (1.30) will describe space containing matter.

The most powerful and simple way of deriving a complete set of compatible field equations is to use some variational principle. Adopting this method the problem is then to find a suitable scalar density constructed from the tensors involved in (1.30) which can play the role of Lagrangian. The only guide here is the one of simplicity. We note that a scalar density is most easily obtained from a second rank covariant tensor by taking the square root of its determinant, and also that two such tensors are provided by contracting $B^{\iota}_{\kappa\lambda\mu}$, namely $R_{\kappa\lambda}$ and $S_{\lambda\mu}$ given by (1.31). On this basis the most simple class of Lagrangian is

$$\mathfrak{L} = A\sqrt{-|R_{ik}|} + B\sqrt{-|R_{ik}+\alpha S_{ik}|} + C\sqrt{-|S_{ik}|} \qquad 1$$

where A, B, C and α are constants. The field equations will follow from the variational principle

$$\delta I = 0 \quad \text{where} \quad I = \int_V \mathfrak{L}\, d\tau \qquad d\tau = dx^1 dx^2 dx^3 dx^4 \qquad 2$$

for all variations of $\Gamma^{\iota}_{\kappa\lambda}$ which vanish on the boundary of the region V of integration. Since $S_{\lambda\mu}$ is, by (1.31) the curl of $\Gamma^{\alpha}_{\alpha\lambda}$ (not a vector) the last term in (1) can easily be seen to make no contribution to the variation and so may be omitted. The first term in (1) has already been used alone by Schrödinger[8a] for the

derivation of field equations similar to those of Einstein[2-4,27] except for an additional cosmological term and is, furthermore, contained as a special case in the second term. From these considerations it appears that the second term in (1) will by itself provide the most general Lagrangian of the simple type under consideration. We write it in the equivalent form

$$\mathfrak{L} = \frac{1}{\lambda}\sqrt{-|R_{ik}+\alpha S_{ik}|}$$

or

$$\mathfrak{L} = 2\lambda\sqrt{-g}\,, \quad g = |g_{ik}| \qquad \lambda g_{ik} = R_{ik} + \alpha S_{ik} \tag{3}$$

2.2 General Formulae from the Variational Principle

In this section we work out in detail the variation principle (2) applied to the Lagrangian (3). It is convenient to write

$$C_{ik} = R_{ik} + \alpha S_{ik} = \lambda g_{ik}$$

so that

$$\frac{\partial \mathfrak{L}}{\partial C_{ik}} = g^{ik}$$

where

$$g^{ik} = \sqrt{-g}\, g^{ik}\,, \quad g = |g_{ik}|\,, \qquad g^{i\sigma} g_{k\sigma} = \delta^i_k = g^{\sigma i} g_{\sigma k} \tag{4}$$

Using this notation the variation principle now reduces to

$$0 = \int_V \frac{\partial \mathfrak{L}}{\partial C_{ik}}\, \delta C_{ik}\, d\tau = \int_V g^{ik}\left(\delta R_{ik} + \alpha\, \delta S_{ik}\right) d\tau \tag{5}$$

where δR_{ik} and δS_{ik} depend on the variation of $\Gamma^i_{k\lambda}$ which is limited only by the condition that $\delta\Gamma^i_{k\lambda}$ vanishes on the boundary of V. For convenience we consider the two terms involved in the integrand separately.

(1) First take

$$\delta \mathfrak{L}_1 = g^{ik}\,\delta R_{ik} = g^{k\lambda}\,\delta^\mu_i\, B^i_{k\lambda\mu}$$

By a straightforward calculation using (1.29) Palatini's formula may

be generalised to,

$$\delta B^{\cdot}_{\kappa\lambda\mu} = -\left(\delta\Gamma^{\cdot}_{\kappa\overset{+}{-}}\right)_{;\mu} + \left(\delta\Gamma^{\cdot}_{\mu\overset{+}{-}}\right)_{;\lambda}$$

Hence we can write

$$\delta K_1 = -g^{\kappa\lambda}\delta^{\mu}_{\cdot}\left(\delta\Gamma^{\cdot}_{\kappa\overset{+}{-}}\right)_{;\mu} + g^{\kappa\lambda}\delta^{\mu}_{\cdot}\left(\delta\Gamma^{\cdot}_{\mu\overset{+}{-}}\right)_{;\lambda}$$

$$= \left(g^{\kappa\lambda}_{\overset{+}{-};\cdot} - g^{\kappa\sigma}_{\overset{+}{-};\sigma}\delta^{\lambda}_{\cdot}\right)\delta\Gamma^{\cdot}_{\kappa\lambda} - \left(g^{\kappa\lambda}\delta\Gamma^{\mu}_{\kappa\lambda}\right)_{;\mu} + \left(g^{\kappa\lambda}\delta\Gamma^{\sigma}_{\kappa\sigma}\right)_{;\lambda} \qquad 6$$

where we have used the product rule for differentiation remembering that only indices of like sign character can be contracted within the operation of differentiation. Now it is easy to show that

$$\mathfrak{A}^{\mu}_{\overset{+}{-};\mu} = \mathfrak{A}^{\mu}_{;\mu} + 2\Gamma_{\sigma}\mathfrak{A}^{\sigma} \quad , \quad \mathfrak{B}^{\lambda}_{\overset{+}{-};\lambda} = \mathfrak{B}^{\lambda}_{;\lambda}$$

where

$$\Gamma_{\sigma} = \Gamma^{\alpha}_{\sigma\alpha}$$

Using the result in (6) we get

$$\delta K_1 = \left(g^{\kappa\lambda}_{\overset{+}{-};\cdot} - g^{\kappa\sigma}_{\overset{+}{-};\sigma}\delta^{\lambda}_{\cdot} - 2\Gamma_{\cdot}g^{\kappa\lambda}\right)\delta\Gamma^{\cdot}_{\kappa\lambda} - \left(g^{\kappa\lambda}\delta\Gamma^{\mu}_{\kappa\lambda}\right)_{;\mu} + \left(g^{\kappa\lambda}\delta\Gamma^{\sigma}_{\kappa\sigma}\right)_{;\lambda} \qquad 7$$

Where this is inserted into the integral (5), the last two terms transformed by Green's theorem into a surface integral over the boundary of V give a vanishing contribution since $\delta\Gamma^{\cdot}_{\kappa\lambda}$ vanishes on the boundary. We can therefore omit these last two terms from δK_1. The first set of bracketed terms may be simplified by the substitution

$$\dot{\Gamma}^{\cdot}_{\kappa\lambda} = \Gamma^{\cdot}_{\kappa\lambda} + \tfrac{1}{3}\delta^{\cdot}_{\kappa}\Gamma_{\lambda}$$

from which

$$\dot{\Gamma}_{\kappa} = \dot{\Gamma}^{\sigma}_{\kappa\sigma} \qquad 8$$

Denoting the covariant derivative with respect to this dotted connection by a single bar the expression for δK_1 becomes

$$\delta k_1 = \left(g^{\kappa\lambda}_{+-|\iota} - \delta^\lambda_\iota g^{\kappa\sigma}_{+-|\sigma}\right)\delta\Gamma^\iota_{\kappa\lambda} \qquad 9$$

The fact that the substitution (8) does simplify in this way the curly bracket terms in (7) means that they only depend on 60 of the 64 components of $\Gamma^\iota_{\kappa\lambda}$. There is no obvious reason why this should be so. As far as is known to the author the simplification afforded by the use of the dotted connection defined by (8) was first discovered by Schrödinger.[8a]

(ii) Now take

$$\delta k_2 = g^{\iota\kappa}\delta S_{\iota\kappa}$$

Using the definition (1.31) of $S_{\iota\kappa}$ this may immediately be rewritten as

$$\delta k_2 = g^{\iota\kappa}\left(-\delta\Gamma^\alpha_{\alpha\iota,\kappa} + \delta\Gamma^\alpha_{\alpha\kappa,\iota}\right)$$

$$= 2\delta^\kappa_\iota\, g^{\lambda\sigma}_{+-|\sigma}\,\delta\Gamma^\iota_{\kappa\lambda} + \text{simple divergence terms} \qquad 10$$

As in (7) the simple divergence terms make no contribution to the integral (5) and so may be omitted. From the results (9) and (10) the condition (5) may now be written as,

$$0 = \int_V \left(g^{\kappa\lambda}_{+-|\iota} - \delta^\lambda_\iota g^{\kappa\sigma}_{+-|\sigma} + 2\alpha\delta^\kappa_\iota g^{\lambda\sigma}_{+-|\sigma}\right)\delta\Gamma^\iota_{\kappa\lambda}\, d\tau \qquad 11$$

This then is the detail form of the variation principle applied to the Lagrangian (3). With the definition in (3) for $g_{\iota\kappa}$ it forms the basis of the various sets of field equations which will be derived and discussed in a later paragraph. It is convenient to write down here the forms taken by $R_{\iota\kappa}$ and $S_{\iota\kappa}$ in terms of the dotted

connection, given by (8), and Γ_λ . They are

$$R_{ik} = \dot{R}_{ik} - \tfrac{2}{3}(\Gamma_{i,k} - \Gamma_{k,i})$$

$$S_{ik} = \dot{S}_{ik} + \tfrac{2}{3}(\Gamma_{i,k} - \Gamma_{k,i}) \qquad 12$$

where $\dot{R}_{ik}$ and $\dot{S}_{ik}$ are the same functions of the dot affinity as R_{ik} and S_{ik} are of $\Gamma^i_{k\lambda}$.

2.3 The Conservation Laws or Identity Relations

It is convenient at this point to derive certain identity relations satisfied by R_{ik} , S_{ik} and g_{ik} in virtue of the variation principle (5). Consider the invariant integral defined by

$$I = \int_V g^{ik}(R_{ik} + \alpha S_{ik})\, d\tau$$

where g^{ik} is given in terms of $\Gamma^i_{k\lambda}$ by (3) and (4). For any variation of $\Gamma^i_{k\lambda}$ which vanishes on the boundary of integration it follows, since (5) is satisfied, that

$$\delta I = \int_V (R_{ik} + \alpha S_{ik})\, \delta g^{ik}\, d\tau = \int_V C_{ik}\, \delta g^{ik}\, d\tau \qquad 13$$

We now specialise the variation to be that due to the coordinate variation

$$\bar{x}^i = x^i + \xi^i \qquad 14$$

where ξ^i , a function of x^α , vanishes on the boundary of V and is so small elsewhere that only terms linear in ξ^i need be retained. Since I is an invariant, δI must vanish for this coordinate variation. Denoting a variation of g^{ik} consequent upon

(14) by $\delta^+ g^{ik}$ it is easy to show from the way g^{ik} behaves under a coordinate transformation i.e.

$$\bar{g}^{ik}(\bar{x}) = \left|\frac{\partial x}{\partial \bar{x}}\right| \frac{\partial \bar{x}^i}{\partial x^\mu} \frac{\partial \bar{x}^k}{\partial x^\nu} g^{\mu\nu}(x)$$

that to the first order in ξ^i, $\delta^+ g^{ik}$ is given by

$$\delta^+ g^{ik} = \xi^i_{,\mu} g^{\mu k} + \xi^k_{,\mu} g^{i\mu} - (g^{ik}\xi^\mu)_{,\mu}$$

Altogether for the variation (14), (13) becomes

$$0 = \int_V C_{ik}\left(\xi^i_{,\mu} g^{\mu k} + \xi^k_{,\mu} g^{i\mu} - [g^{ik}\xi^\mu]_{,\mu}\right) d\tau$$

Using the fact that ξ^i vanishes on the boundary of V this can be transformed in a straightforward manner to give

$$-\int_V \xi^\alpha \{(C_{\alpha\sigma} g^{\beta\sigma} + C_{\sigma\alpha} g^{\sigma\beta})_{,\beta} - g^{\sigma\beta} C_{\sigma\beta,\alpha}\} d\tau = 0$$

Hence since ξ^i is arbitrary throughout the region V we must have

$$(C_{\alpha\sigma} g^{\beta\sigma} + C_{\sigma\alpha} g^{\sigma\beta})_{,\beta} - g^{\sigma\beta} C_{\sigma\beta,\alpha} = 0 \qquad 16$$

These are the identity relations between $R_{\alpha\beta}$, $S_{\alpha\beta}$, and $g_{\alpha\beta}$ which hold when the variation equation (5) is satisfied.

<u>A simpler form for the identities</u>

As written in (16) the identities are not in a very useful or surveyable form. Written in terms of the skew and symmetric parts of $C_{\alpha\beta}$ and $g^{\alpha\beta}$ (16) may be written as

$$(g^{\beta\sigma} C_{\underline{\alpha\sigma}} - \tfrac{1}{2}\delta^\beta_\alpha g^{\underline{\mu\nu}} C_{\underline{\mu\nu}})_{,\beta} + g^{\underline{\mu\nu}}{}_{,\alpha} C_{\underline{\mu\nu}} + 2 g^{\underline{\sigma\beta}}{}_{,\beta} C_{\underset{\vee}{\sigma\alpha}} + g^{\underline{\sigma\beta}} C_{\underset{\vee}{\sigma\alpha},\beta} = 0 \qquad 17$$

Let us first look at the terms dependent on the symmetric components. Define a symmetric tensor b_{ik} by the relation

$$b^{ik} = g^{\underline{ik}} \qquad (i)$$

where as usual

$$b^{ik} = \sqrt{-b}\, b^{ik} , \quad b^{\sigma k} b_{\sigma i} = \delta^k_i , \quad b = |b^{ik}| \qquad \text{(ii)}$$

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It is easy to prove that

$$b^{\mu\nu}{}_{,\alpha} = \sqrt{-b}\left(b^{\mu\nu}{}_{,\alpha} - \tfrac{1}{2} b^{\mu\nu} b_{\rho\sigma} b^{\rho\sigma}{}_{,\alpha}\right)$$

and so

$$R_{\mu\nu}\, g^{\mu\nu}{}_{,\alpha} = \sqrt{-b}\left(R_{\mu\nu} - \tfrac{1}{2} b_{\mu\nu}\, b^{\rho\sigma} R_{\rho\sigma}\right) b^{\mu\nu}{}_{,\alpha} = \mathfrak{T}_{\mu\nu}\, b^{\mu\nu}{}_{,\alpha} \text{ (say)}$$

In this notation, noting that from (4) $C_{\mu\nu} = R_{\mu\nu}$, the terms in (17) dependent on the symmetric components take the form

$$2\left(\mathfrak{T}^{\beta}_{\alpha,\beta} + \tfrac{1}{2}\mathfrak{T}_{\mu\nu} b^{\mu\nu}{}_{,\alpha}\right) = 2\,\mathfrak{T}^{\beta}_{\alpha\|\beta}$$

19

where $\|$ denotes differentiation with respect to the Christoffel symbol of $b_{\alpha\beta}$, and where the indices of $\mathfrak{T}_{\mu\nu}$ have been raised by $b^{\mu\nu}$.

Turning now to the terms in (17) dependent on the skew components we note that from (4), since ς_μ in the curl of $\Gamma^\sigma_{\sigma\alpha}$,

$$C_{\underset{\vee}{\sigma\alpha},\beta} = R_{\underset{\vee}{\sigma\alpha},\beta}$$

Finally using this result and (19) equation (17) becomes

$$\mathfrak{T}^{\beta}_{\alpha\|\beta} = C_{\alpha\sigma}\, g^{\sigma\nu}{}_{,\beta} + \tfrac{1}{2} g^{\mu\nu} R_{\underset{\vee}{\mu\nu},\alpha} \qquad 20$$

where

$$\mathfrak{T}^{\beta}_{\alpha} = b^{\beta\sigma} R_{\alpha\sigma} - \tfrac{1}{2}\delta^{\beta}_{\alpha}\, b^{\mu\nu} R_{\mu\nu}$$

This is the form of the identities (16) which will be found useful in the following paragraphs.

2.4 The Affine Field Equations

Let us now return to equation (11), the detail form of the variation principle (5). For a non-symmetric Γ^{i}_{kl} this can only be satisfied if throughout V

$$g^{kl}_{\pm}{}_{|i} - \delta^{l}_{i} g^{k\sigma}_{\pm}{}_{|\sigma} + 2\alpha \delta^{k}_{i} g^{l\sigma}_{\vee}{}_{,\sigma} = 0 \qquad 21$$

We now show the unexpected result that α can only effectively take two values. First contract the indices (k, i),

$$g^{ml}_{\pm}{}_{|m} - g^{lm}_{\pm}{}_{|m} = -8\alpha g^{l\sigma}_{\vee}{}_{,\sigma} \qquad 22$$

When the first two terms are written in full one gets

$$g^{ml}_{\pm}{}_{|m} - g^{lm}_{\pm}{}_{|m} = -2 g^{l\sigma}_{\vee}{}_{,\sigma} \qquad 23$$

Equations (22) and (23) require, therefore, that either

$$\alpha = \tfrac{1}{4} \qquad \text{or} \qquad g^{l\sigma}_{\vee}{}_{,\sigma} = 0 \qquad 24$$

Now contract the indices (l, i) in (21):-

$$g^{km}_{\pm}{}_{|m} = \tfrac{2}{3}\alpha\, g^{k\sigma}_{\vee}{}_{,\sigma} \qquad 25$$

Case 1, $\alpha \neq \frac{1}{4}$ From the last two results (21) is in this case

$$g^{kl}_{\pm}{}_{|i} - \tfrac{1}{6}\delta^{l}_{i}\, g^{k\sigma}_{\vee}{}_{,\sigma} + \tfrac{1}{2}\delta^{k}_{i}\, g^{l\sigma}_{\vee}{}_{,\sigma} = 0 \qquad 26$$

To this we must add the definition (3) for g_{ik} which by (12) takes the form

$$\dot{R}_{ik} + \dot{S}_{ik} = \lambda g_{ik} \qquad 27$$

We notice at once that both of these equations (26) and (27) are functions of the dotted affinity alone. By adding the equation

$$\dot{\Gamma}_{\cdot} = 0 \qquad 28$$

there would then be no need to use the distinguishing | and · in (26) and (27) respectively. But in fact it is even unnecessary to add this equation for it is implied already in (26). To show this subtract the two possible contractions of (26) giving

$$g^{\underset{\cdot}{\lambda}\underset{\cdot}{m}}{}_{|m} - g^{\underset{\cdot}{m}\underset{\cdot}{\lambda}}{}_{|m} - 2g^{\underset{\cdot}{\lambda}\underset{\cdot}{\sigma}}{}_{,\sigma} = 0$$

Writing out the bar differentiations in full this reduces to

$$\dot{\Gamma}_{\sigma} g^{\sigma\lambda} = 0$$

Since in general we require $|g^{\mu\nu}| \neq 0$ it follows that (28) must hold.

From all this and using (3) and (20) the Lagrangian, field equations and conservation law are in the case of $\alpha = \frac{1}{4}$;

$$\mathfrak{L} = \frac{2}{\lambda}\sqrt{-|R_{\cdot\kappa} + \tfrac{1}{4}S_{\cdot\kappa}|} \qquad (i)$$

$$g^{\underset{\cdot}{\kappa}\underset{\cdot}{\lambda}}{}_{;\iota} - \tfrac{1}{6}\delta^{\lambda}_{\iota}\, g^{\underset{\cdot}{\kappa}\underset{\cdot}{\sigma}}{}_{,\sigma} + \tfrac{1}{2}\delta^{\kappa}_{\iota}\, g^{\underset{\cdot}{\lambda}\underset{\cdot}{\sigma}}{}_{,\sigma} = 0 \qquad (ii)$$

29

$$R_{\cdot\kappa} + \tfrac{1}{4}S_{\cdot\kappa} = \lambda g_{\cdot\kappa} \qquad (iii)$$

$$\mathfrak{T}^{\beta}_{\alpha\|\beta} = \left(R_{\underset{\cdot}{\alpha}\underset{\cdot}{\sigma}} + \tfrac{1}{4}S_{\alpha\sigma}\right) g^{\underset{\cdot}{\sigma}\underset{\cdot}{\nu}}{}_{,\nu} + \tfrac{1}{2} g^{\underset{\cdot}{\mu}\underset{\cdot}{\nu}}\left(R_{\underset{\cdot}{\mu}\underset{\cdot}{\nu},\alpha}\right) \qquad (iv)$$

We have also shown that (ii) necessarily implies

$$\Gamma_{\cdot} = 0$$

Interpretation and discussion

Using (iii) the conservation law (iv) may be written

$$\mathfrak{T}^{\beta}_{\alpha\|\beta} = \lambda\, g_{\alpha\underset{\cdot}{\sigma}}\, g^{\underset{\cdot}{\sigma}\underset{\cdot}{\nu}}{}_{,\nu} + \tfrac{1}{2}\lambda\, g^{\underset{\cdot}{\mu}\underset{\cdot}{\nu}}\, g_{\underset{\cdot}{\mu}\underset{\cdot}{\nu},\alpha} \qquad 30$$

When the skew fields vanish the right hand side vanishes and we are left with the general relativity conservation law (1.10 a). This and the fact that the $\|$ differentiation is based on the Christoffel symbol of $b_{\mu\nu}$ suggest that $b_{\mu\nu}$, related to $g_{\mu\nu}$ by (18), should be interpreted as the gravitational tensor. Comparison of the right hand side term of (30) with the expressions given in (1.13) shows that both terms look like a Lorentz force. The interpretation for each case would be,

(i) $g_{\mu\sigma}$ the electromagnetic field tensor, $g^{\mu\nu}{}_{,\nu}$ the current vector.

(ii) $g^{\mu\nu}$ the electromagnetic field tensor, $g_{\mu\nu,\lambda}$ the current vector.

Apart from this ambiguity there is the further disadvantage that neither of the skew tensors $g_{\mu\nu}$ or $g^{\mu\nu}$ is a curl. If this feature is found to persist in the small field approximation, to be considered later, then it would be most unlikely that either of the interpretations (i) and (ii) of the skew fields was tenable.

Case 2: $\alpha \neq \frac{1}{4}$ From (24) and (25) it follows that in this case

$$g^{km}_{+-}{}_{|m} = 0 = g^{k\sigma}_{+-}{}_{,\sigma} \qquad 31\text{ a}$$

so that (21) becomes simply

$$g^{kl}_{+-}{}_{|l} = 0 \qquad 31\text{ b}$$

or equivalently, as is easy to show using the relations (4)

$$g_{+-}^{kl}{}_{|l} = 0 \qquad 31\text{ c}$$

Writing this equation out in full and multiplying by g^{kl} one gets the relation

$$\Gamma^{\sigma}_{\sigma l} = (\log \sqrt{-g})_{,l}$$

Consequently it follows that $\dot{S}_{\mu\kappa}$, the curl of this quantity, vanishes identically. Then using (12) it is seen that $g_{\mu\kappa}$ is given by

$$\lambda g_{\mu\kappa} = R_{\mu\kappa} + \alpha S_{\mu\kappa} = \dot{R}_{\mu\kappa} - \tfrac{2}{3}(1-4\alpha)(\Gamma_{\mu\kappa} - \Gamma_{\kappa\mu}) \qquad 32$$

Now the field equations in this case are (31 a); (31 b) or (31 c) and (32). Inspection of these equations shows that a non zero α merely multiplies by a factor the vector Γ_ι which only occurs in the second set of field equations (32). We conclude that it makes no difference whatever to put $\alpha = 0$. This justifies the statement made above that α in (21) can in effect only take two values, namely zero and $\frac{1}{4}$.

But the Lagrangian when $\alpha = 0$ reduces to that used by Schrödinger[8a] and so we get his set of field equations. Changing slightly the notation in (31) and (32) these equations are

$$g_{\underset{\sim}{\mu\kappa};\iota} = 0 \qquad (i)$$

$$\Gamma_\iota = 0 \quad \text{or} \quad g^{\underset{\sim}{\sigma}}_{\underset{\sim}{\sigma},\sigma} = 0 \qquad (ii)$$

$$R_{\underset{\sim}{\mu\kappa}} = \lambda\, g_{\underset{\sim}{\mu\kappa}} \qquad (iii) \qquad 33$$

$$R_{\underset{\sim}{\mu\kappa},\iota} = \lambda\, g_{\underset{\sim}{\mu\kappa},\iota} \qquad (iv)$$

The conservation law in this case is from (20) using (ii) and (iv):-

$$\Sigma^{\beta}_{\alpha\|\beta} = \tfrac{1}{2}\, g^{\underset{\sim}{\mu\nu}} R_{\underset{\sim}{\mu\nu},\alpha} = \tfrac{1}{2}\lambda\, g^{\underset{\sim}{\mu\nu}} g_{\underset{\sim}{\mu\nu},\alpha} \qquad 34$$

From what has already been said there is one obvious interpretation of this theory; it is

$h_{\mu\nu}$ is the gravitation tensor

$g^{\mu\nu}$ is the electromagnetic field tensor density or $\mathfrak{F}^{\mu\nu}$ in (1.13)

$g_{\mu\nu\rho}$ is the current vector or $f_{\mu\nu\rho}$ in (1.13) β

Indeed looking at (1.13) β it is clear that the right hand side of (34) is then apart from the λ factor simply the Lorentz force term $\mathfrak{F}_\lambda$ while the exact form of Maxwell's equations would be

$$g^{\mu\nu}{}_{,\sigma} = 0$$

$$i^\sigma = \tfrac{1}{2}\varepsilon^{\sigma\mu\kappa\lambda} g_{\mu\kappa,\lambda} = \tfrac{1}{6}\varepsilon^{\sigma\mu\kappa\lambda} g_{\mu\kappa\lambda}$$

35

All this shows how powerful the form of the conservation law is in suggesting a possible interpretation of the theory. Whether the theory and its interpretation is a successful unified field theory is a different question. Indeed we can see one possible flaw already. Equations (33) reduce, as they should, to the equations of general relativity with a cosmological term when the skew fields vanish. It is known from observations concerning the motion of the planets that this cosmological constant must be very small - the largest admissable value being about 10^{-44} cms. (Bondi [29] p.96). Unfortunately this constant also occurs as a factor in the Lorentz force term appearing on the right in (34), and it looks as if its small value will effectively make the influence of the term negligible. This fault in the theory is seen to hinge on (33 iii). If we could adjust

this equation in such a way that when the skew fields vanish λ does not remain as a cosmological term then there would no longer be any necessity for λ to be so small. This is in effect what Kursunoğlu[10] has achieved in his field theory for in it (33 iii) is replaced by

$$R_{\mu\nu} = \lambda(g_{\mu\nu} - b_{\mu\nu}) \qquad 36$$

When the skew fields vanish $b_{\mu\nu}$ is identical with $g_{\mu\nu}$ and the right hand side vanishes. The field equation then reduces to (1.9) of general relativity without a cosmological constant. The Kursunoğlu theory retains the other equations in (33) unchanged. We will not discuss here Kursunoğlu's theory any further. The field theory to be developed in Part II resembles it superficially but has the advantage of being far more simple in character.

Einstein's weak field equations[1-4] correspond to (33) with a negligibly small λ while his strong set[37] have in addition instead of (33 iv) the equation

$$R_{\mu\nu} = 0 \qquad 37$$

Because of the above comments about λ we would expect both these theories of Einstein to be in a worse plight than that of Schrödinger's. Our argument is supported by the fact that it has not been possible to derive from either of the Einstein Unified field theories the Lorentz - Newtonian equations of motion. Applying the Einstein - Infeld - Hoffmann method (and this is valid for as

in the gravitational case one can show that a field with mass must have point singularities[25]) no electromagnetic Lorentz force terms appear, the equations of motion being just the gravitational ones of general relativity[23a,b]. Bearing in mind the small value that λ has in (33) it seems clear that similar investigations for Schrödinger's unified field theory would give the same unsatisfactory result.

2.5 The weak field approximation

In this section we derive the approximate form for small fields of the field theory developed under Case 1 above, the main object being to see whether any of the skew fields in that theory reduces to a curl form. The Einstein - Schrödinger theories will follow from the first case by adding the conditions

$$g^{\mu\nu}{}_{,\sigma} = 0 \quad , \qquad \Gamma_i = 0 \tag{38}$$

Make the substitution

$$g_{ik} = \eta_{ik} + \gamma_{ik} \quad , \qquad \gamma_{ik} = a_{ik} \qquad \gamma_{[ik]} = f_{ik}$$

It is sufficient for our purposes to work to the first order in γ_{ik}. The work is quite straightforward. (29 ii) is first solved for Γ^{i}_{kl} correct to the first order in γ_{ik} and then this solution is inserted into (29 iii) to give a second order differential equation for γ_{ik}. One obtains for the skew and symmetric parts respectively,

$$-\eta^{\alpha\beta} f_{ik,\alpha\beta} + \tfrac{1}{2}(j_{i,k} - j_{k,i}) = 2\lambda f_{ik} \tag{i}$$

where $j_\alpha = -\eta^{\mu\nu} f_{\alpha\mu,\nu}$ 39

and $$\eta^{\alpha\beta}\left[a_{\alpha\beta,ik} - a_{\beta i,\alpha k} - a_{\beta k,\alpha i} - a_{ik,\alpha\beta}\right] = 2\lambda(\eta_{ik} + a_{ik}). \tag{ii}$$

Both the skew fields $g_{\mu\nu}$ and $g^{\mu\nu}$ reduce in this approximation to $f_{\mu\nu}$ and it is immediately seen that in general they are not of curl form. This makes it extremely improbable that the field theory (29) can account for electromagnetism despite the fact that to this linear approximation the right hand side of the conservation law (30) is

$$(f_{\mu\sigma} f^{\sigma\nu} - \tfrac{1}{4} \delta^{\nu}_{\mu} f^{\rho\sigma} f_{\sigma\rho})_{,\nu}$$

i.e. of the form of the divergence of the electromagnetic-stress-energy-momentum tensor.

The first order approximation to Schrödinger's theory (33) is for the skew field,

$$-\eta^{\alpha\beta} (f_{\mu\nu,\lambda})_{,\alpha\beta} = 2\lambda f_{\mu\nu,\lambda} \qquad (1)$$

and 40

$$-\eta^{\alpha\beta} f_{\mu\alpha,\beta} = 0 \qquad (11)$$

while the symmetric field has to satisfy (39 ii). If the interpretation of the theory set out above is tenable these equations must be in accord with (1.13)B for the special relativity case (1.14). It is clear that (40 ii) is in fact the second equation in (1.13)B and that

$$f_{\mu\nu} = \begin{pmatrix} 0 & E_3 & -E_2 & -H_1 \\ -E_3 & 0 & E_1 & -H_2 \\ E_2 & -E_1 & 0 & -H_3 \\ H_1 & H_2 & H_3 & 0 \end{pmatrix}$$

41

Also (40 i) may be written as

$$-\eta^{\alpha\beta} z_{k,\alpha\beta} = 2\lambda z_k \qquad 42$$

which is a Yukawa-like equation for the current four-vector z_k . This is not at variance with observation as may be seen by considering the electrostatic case. It is simpler to use (42) with the set (1.13) A to obtain for this case the solutions

$$z_4 = -2\lambda \frac{\rho}{r} e^{-\sqrt{2\lambda}\, r}$$

$$E_m = \frac{\varepsilon x^m}{r^3} + \frac{\rho x^m}{r^3}(1+\sqrt{2\lambda}\, r) e^{-\sqrt{2\lambda} r} \qquad 43$$

where ε and ρ are constants and m takes the values 1, 2, 3. Now E_m is small and so for $r < (2\lambda)^{-\frac{1}{2}}$ z_4 is much smaller still. However, Schrödinger's theory does imply that the charge density need not be zero everywhere outside a small region.

Einstein's 'weak' equations may be obtained formally by replacing λ by zero (of course they cannot be derived in this way). The first order approximation thus has instead of (40 i) or (42) the equation

$$\eta^{\alpha\beta} z_{k,\alpha\beta} = 0 \qquad 44$$

For the electrostatic case this has the general solution for a field with spherical symmetry

$$z_k = \frac{\rho}{r}$$

In Schrödinger's theory the cosmological constant is all important and in Einstein's theory it is possible that the boundary conditions for f_{ik} could overcome the unsatisfactory appearance of both this particular solution and the general equation (44). Because (44)

is so much weaker than Maxwell's equations Einstein was led to 'strengthen' his field equations by the condition (37). In this way (40 i) (with $\lambda = 0$) is replaced by

$$\eta^{\alpha\beta} f_{\nu,\alpha\beta} = 0$$

However it seems that Einstein has now discarded these strong field equations[31]. In connection with the 'weak' field equations it should be noted that Schrödinger has recently shown that the electric four-vector defined as in (1.13 B) is in general different from zero throughout the field[28].

Conclusion

In conclusion we may say from §§ 2.4 and 2.5 that Schrödinger's theory given by (33) seems promising at first but the smallness of the factor λ undermines it. The equations of motion determined for the Einstein form of the theory ($\lambda = 0$) fail to show any Lorentz force term and it is hardly likely that Schrödinger's theory with the very small value of λ required would give a significantly different result. All this seems to indicate that neither the affine theory nor the Einstein theories provide a successful unified field theory. As has already been mentioned, Kursunoğlu's theory seems to be the only promising non-symmetric affine one.

2.6 Alternative expressions for the conservation law

For completeness we derive alternative forms for the conservation law (16) or (20) valid for the new affine theory (29). The method followed is that used by Schrödinger[9] in deriving similar expressions for his affine theory (33).

The g-Γ relation for the affine theory (29) is

$$g^{kl}_{+-;\iota} - \tfrac{1}{6}\delta^{l}_{\iota}\, g^{kr}_{+-,\sigma} + \tfrac{1}{2}\delta^{k}_{\iota}\, g^{lr}_{+-,\sigma} = 0 \qquad 45$$

and follows from the variation equation

$$0 = {}_{V}\!\int g^{ik}\left(\delta R_{ik} + \tfrac{1}{4}\delta S_{ik}\right) d\tau \qquad 46$$

where δR_{ik} and δS_{ik} are due to the variation of Γ^{i}_{kl} which apart from vanishing on the boundary of V is arbitrary. The identities which we derive here are valid provided the g-Γ relation (45) is satisfied.

Consider the invariant I defined by

$$I = {}_{V}\!\int g^{ik}\left(R_{ik} + \tfrac{1}{4}S_{ik}\right) d\tau \qquad 47$$

For a variation of the entities involved in the integrand limited only by the condition that they should vanish on the boundary of integration we see that provided (45) or (46) is satisfied

$$\begin{aligned} \delta I &= {}_{V}\!\int \left(R_{ik} + \tfrac{1}{4}S_{ik}\right) \delta g^{ik}\, d\tau \\ &= {}_{V}\!\int C_{ik}\, \delta g^{ik}\, d\tau \\ &= {}_{V}\!\int \mathfrak{C}_{ik}\, \delta g^{ik}\, d\tau \end{aligned} \qquad (i)$$

48

where $$\mathfrak{C}_{ik} = \sqrt{-g}\left(C_{ik} - \tfrac{1}{2} g_{ik}\, g^{\mu\nu} C_{\mu\nu}\right) \qquad (ii)$$

This result will be of use later.

Consider now in detail the integrand of (47). Using (1.31) for R_{ik} and S_{ik}, written explicitly in terms of Γ^{i}_{kl} it is

$$\mathfrak{L}_1 = \left(-g^{ik}\Gamma^{\alpha}_{ik} + g^{i\alpha}\Gamma^{\sigma}_{i\sigma} - \tfrac{1}{2}g^{i\alpha}\Gamma^{\sigma}_{\sigma i}\right)_{,\alpha} + \mathfrak{L}_2 + \Lambda$$

where $$\Lambda = g^{ik}\Lambda_{ik} = g^{ik}\left(\Gamma^{\alpha}_{i\beta}\Gamma^{\beta}_{\alpha k} - \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{ik}\right) \qquad 49$$

and $$\mathfrak{L}_2 = g^{ik}{}_{,\alpha}\Gamma^{\alpha}_{ik} - g^{ik}{}_{,k}\Gamma^{\alpha}_{i\alpha} + \tfrac{1}{2}g^{i\alpha}{}_{,\alpha}\Gamma^{\sigma}_{\sigma i}$$

This expression for $\mathfrak{L}_2$ may be transformed by using (46) to obtain the derivatives of g^{ik} in terms of Γ^{i}_{kl}, g^{kl} and Γ^{k}, defined by

$$\Gamma^{k} = g^{k\sigma}{}_{,\sigma} \qquad 50$$

In this way it is straightforward to show that

$$\mathfrak{L}_2 = -2\Lambda \qquad 51$$

Thus we have shown that the integrand in (47) is on account of (46) equivalent to

$$\mathfrak{L}_1 = -\Lambda + \left(-g^{ik}\Gamma^{\alpha}_{ik} + g^{i\alpha}\Gamma^{\sigma}_{i\sigma} - \tfrac{1}{2}g^{i\alpha}\Gamma^{\sigma}_{\sigma i}\right)_{,\alpha} \qquad 52$$

Forgetting temporarily the dependence of g_{ik} upon R_{ik} and S_{ik}, we see that the g-Γ relation (46) follows from the condition that $\delta I = 0$ for an arbitrary variation, restricted only to vanish on the boundary of V, of Γ^{i}_{kl}. Clearly for such a variation the divergence term in (52) will make no contribution. This suggests that we consider instead of the invariant I, the non-invariant J defined by

$$J = \int_V \Lambda\, d\tau \qquad 53$$

This integral, with Λ defined by (49), will indeed yield the identities we require.

We will need in the following an expression for .

From (51) we see that

$$-2\delta\Lambda = g^{ik}{}_{,\alpha}\,\delta\Gamma^{\alpha}_{ik} - g^{ik}{}_{,\kappa}\,\delta\Gamma^{\alpha}_{i\alpha} + \tfrac{1}{2} g^{\alpha\beta}{}_{,\alpha}\,\delta\Gamma^{\sigma}_{\sigma\beta} + \Gamma^{\alpha}_{ik}\,\delta(g^{ik}{}_{,\alpha}) - \Gamma^{\alpha}_{i\alpha}\,\delta(g^{ik}{}_{,\kappa}) + \tfrac{1}{2}\Gamma^{\sigma}_{\sigma\beta}\,\delta g^{\alpha\beta}$$

Again using (46) for the derivatives of g^{ik} we get for

$$g^{ik}{}_{,\alpha}\,\delta\Gamma^{\alpha}_{ik} - g^{ik}{}_{,\kappa}\,\delta\Gamma^{\alpha}_{i\alpha} = -g^{ik}\delta\Lambda_{ik} - \tfrac{1}{2}g^{\alpha\beta}\delta\Gamma^{\sigma}_{\sigma\beta} = -\delta\Lambda + \Lambda_{ik}\delta g^{ik} - \tfrac{1}{2}g^{\alpha\beta}\delta\Gamma^{\sigma}_{\sigma\beta}$$

Hence the previous expression becomes

$$-\delta\Lambda = \Lambda_{ik}\,\delta g^{ik} + \Lambda^{\alpha}_{ik}\,\delta(g^{ik}{}_{,\alpha}) \qquad (i)$$

54

where

$$\Lambda^{\alpha}_{\epsilon\eta} = \Gamma^{\alpha}_{\epsilon k} - \delta^{\alpha}_{\kappa}\Gamma^{\sigma}_{i\sigma} + \tfrac{1}{4}\left(\delta^{\alpha}_{\epsilon}\Gamma^{\sigma}_{\sigma\kappa} - \delta^{\alpha}_{\kappa}\Gamma^{\sigma}_{\sigma\epsilon}\right) \qquad (ii)$$

This is the required expression for $\delta\Lambda$

Consider the effect of a variation dependent upon the co-

ordinate variation

$$\bar{x}^{i} = x^{i} + \xi^{i}$$

55

where ξ^{i}, a function of x^{α}, is everywhere small enough for its square and higher powers to be neglected. Denoting this variation by δ^{+} we need the following theorem (see Part II, p 27),

$$\delta^{+}J = \int_{V} \delta^{+}\Lambda\, d\tau + \int_{V} (\Lambda\xi^{\alpha})_{,\alpha}\, d\tau$$

This result is valid for any 4-space integral. Hence from (54 i), after rearranging, we get

$$\delta^{+}J = -\int_{V} (\Lambda_{ik} - \Lambda^{\alpha}_{ik,\alpha})\,\delta^{+}g^{ik}\,d\tau + \int_{V}(-\Lambda^{\alpha}_{ik}\delta^{+}g^{ik} + \Lambda\xi^{\alpha})_{,\alpha}\,d\tau$$

But it is easy to see from the definitions (49) and (54 ii) that

$$\Lambda_{ik} - \Lambda^{\alpha}_{ik,\alpha} = R_{ik} + \tfrac{1}{4}S_{ik}$$

Hence using (48)

$$\delta^+ J = -\int_V \mathcal{C}_{ik}\,\delta^+ g^{ik}\,d\tau + \int_V \left(-\Lambda^\alpha_{ik}\,\delta^+ g^{ik} + \Lambda \xi^\alpha\right)_{,\alpha}\,d\tau \qquad 56$$

To evaluate this expression we need the following formulae for $\delta^+ g^{ik}$ and $\delta^+ g_{ik}$ which are simply established by applying (55) to the transformation law of g^{ik} and g_{ik} respectively.

$$\delta^+ g^{ik} = g^{\sigma k}\,\xi^i_{,\sigma} + g^{i\sigma}\,\xi^k_{,\sigma} - g^{ik}_{,\sigma}\,\xi^\sigma$$

$$\delta^+ g_{ik} = -g_{\sigma k}\,\xi^\sigma_{,i} - g_{i\sigma}\,\xi^\sigma_{,k} - \left(g_{ik}\,\xi^\sigma\right)_{,\sigma}$$

Using these expressions and rearranging we can write

$$\mathcal{C}_{ik}\,\delta^+ g^{ik} = -2\left(\mathcal{C}^\beta_{\alpha,\beta} + \tfrac{1}{2}\,\mathcal{C}_{\sigma\rho}\,g^{\sigma\rho}{}_{,\alpha}\right)\xi^\alpha + 2\left(\mathcal{C}^\beta_\alpha\,\xi^\alpha\right)_{,\beta} \qquad 57$$

where

$$2\,\mathcal{C}^\beta_\alpha = g^{\beta\rho}\,\mathcal{C}_{\alpha\rho} + g^{\rho\beta}\,\mathcal{C}_{\rho\alpha}$$

and

$$\Lambda^\alpha_{ik}\,\delta^+ g^{ik} = -\Lambda^\alpha_{ik}\,g^{ik}{}_{,\sigma}\,\xi^\sigma + \left(\Lambda^\alpha_{\sigma\rho}\,g^{\beta\rho} + \Lambda^\alpha_{\rho\sigma}\,g^{\rho\beta} - \delta^\beta_\sigma\,\Lambda^\alpha_{ik}\,g^{ik}\right)\xi^\sigma_{,\beta}$$

Using these results in (56) we finally get for $\delta^+ J$,

$$\delta^+ J = 2\int_V \left(\mathcal{C}^\beta_{\alpha,\beta} + \tfrac{1}{2}\,\mathcal{C}_{\sigma\rho}\,g^{\sigma\rho}{}_{,\alpha}\right)\xi^\alpha\,d\tau \qquad 58$$

$$+ \int_V \left\{\left(-2\,\mathcal{C}^\alpha_\sigma + \Lambda\,\delta^\alpha_\sigma + \Lambda^\alpha_{ik}\,g^{ik}{}_{,\sigma}\right)\xi^\sigma - \left(\Lambda^\alpha_{\sigma\rho}\,g^{\beta\rho} + \Lambda^\alpha_{\rho\sigma}\,g^{\rho\beta} - \delta^\beta_\sigma\,\Lambda^\alpha_{ik}\,g^{ik}\right)\xi^\sigma_{,\beta}\right\}d\tau$$

Now consider the following cases.

1. ξ^i vanishes on the boundary of V. It follows from (56) that

$$\delta^+ J = \int_V \mathcal{C}_{ik}\,\delta^+ g^{ik}\,d\tau = \delta^+ I$$

But I is an invariant so that $\delta^+ I$, due to a coordinate variation, must vanish. Hence $\delta^+ J$ must also vanish and we see from (58) that this requires, since ξ^i is arbitrary throughout V

$$\mathcal{C}^\beta_{\alpha,\beta} + \tfrac{1}{2}\,\mathcal{C}_{\sigma\rho}\,g^{\sigma\rho}{}_{,\alpha} = 0. \qquad 59$$

2. ξ^i is a constant everywhere. For this case the affine connection transforms like a tensor for the transformation (55). Thus Λ behaves like a scalar density and J like an invariant so that $\delta^* J$ must be zero. Using (59) we see that (58) requires

$$(\mathfrak{C}^\alpha_\beta + t^\alpha_\beta)_{,\alpha} = 0$$

where

$$t^\alpha_\beta = -\tfrac{1}{2}(\Lambda^\alpha_{ik} g^{ik}{}_{,\beta} + \Lambda \delta^\alpha_\beta) \qquad 60$$

3. $\xi^i_{,\alpha}$ is a constant i.e. the transformation (55) is linear. Γ^i_{kl} behaves like a tensor for linear transformations and so, as in 2. , $\delta^* J$ must vanish in this case also. Taking account of (59) and (60), (58) requires in this case

$$\mathfrak{C}^\beta_\sigma + t^\beta_\sigma = -\tfrac{1}{2}(\Lambda^\alpha_{\sigma\rho} g^{\beta\rho} + \Lambda^\alpha_{\rho\sigma} g^{\rho\beta} - \delta^\beta_\sigma \Lambda^a_{ik} g^{ik})_{,\alpha} \qquad 61$$

These $4 + 4 + 16 = 24$ relations, in conjunction with the definitions (49), (48 ii), (57) and (54 ii) for Λ and Λ_{ik}, C_{ik}, $\mathfrak{C}^\beta_\sigma$ and Λ^α_{ik} respectively, are the identities we wished to establish. They have the same form as the corresponding expressions of general relativity† (1.10) and of Schrödinger's theory[9]. The identity (60) is an alternative simple divergence expression for (59) itself equivalent to the conservation law (20) or (16). The remaining identity (61) gives the stress-energy-momentum array $\mathfrak{C}^\beta_\sigma + t^\beta_\sigma$ as a simple divergence.

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† See (1.10) page 10

Chapter 3

Solutions of Einstein's 'Weak' Unified Field Theory

3.1 Introduction

In this theory proposed by Einstein and Strauss in 1946 the field equations are[4]

$$g_{(ik),\lambda} = 0 \qquad \text{(i)}$$

$$g^{(ik}_{\ \ ,\sigma} = 0 \qquad \text{or} \qquad \Gamma_\lambda = 0 \qquad \text{(ii)}$$

$$R_{[ik]} = 0 \qquad \text{(iii)}$$

$$R_{(ik),\lambda} = 0 \qquad \text{(iv)}$$

where $R_{(ik)}$ and $R_{[ik]}$ are the skew and symmetric parts of the tensor R_{ik} given by

$$R_{ik} = -\Gamma^{\alpha}_{ik,\alpha} + \Gamma^{\alpha}_{i\alpha,k} + \Gamma^{\alpha}_{i\beta}\Gamma^{\beta}_{\alpha k} - \Gamma^{\beta}_{i\alpha}\Gamma^{\alpha}_{\beta k}$$

$$= -\Gamma^{\alpha}_{(ik),\alpha} + \tfrac{1}{2}\left(\Gamma^{\alpha}_{i\alpha,k} + \Gamma^{\alpha}_{\alpha k,i}\right) + \Gamma^{\alpha}_{(i\beta)}\Gamma^{\beta}_{\alpha k} - \Gamma^{\beta}_{[i\alpha]}\Gamma^{\alpha}_{\beta k}$$

The latter is the form usually quoted for R_{ik} but it is the same as the first since (i) and (ii) require

$$\Gamma^{\alpha}_{[\alpha i]} \equiv -\Gamma_i = 0 \quad , \qquad S_{ik} \equiv -\Gamma^{\alpha}_{[\alpha i],k} + \Gamma^{\alpha}_{\alpha k,i} = 0$$

The non-symmetric tensor g_{ik} corresponding to a static field with spherical symmetry takes in polar coordinates the form[16]

$$g_{ik} = \begin{pmatrix} -\alpha & 0 & 0 & \omega \\ 0 & -3 & f\sin\theta & 0 \\ 0 & -f\sin\theta & -3\sin^2\theta & 0 \\ -\omega & 0 & 0 & \gamma \end{pmatrix}$$

where α , β , γ , f and ω are functions of r only. We will only be interested here in fields of this type. The physical interpretation of f and ω is best seen by writing out given by (3) in Cartesian coordinates

$$g_{\mu\nu} = \begin{pmatrix} 0 & f/r^2\,\frac{x}{r} & -\frac{f}{r^2}\frac{y}{r} & \omega\frac{z}{r} \\ & 0 & \frac{f}{r^2}\frac{x}{r} & \omega\frac{y}{r} \\ & & 0 & \omega\frac{z}{r} \\ & & & 0 \end{pmatrix} \qquad 4$$

Thus according to the interpretation of the theory (2.41) , $-\omega$ is the radial component of the magnetic field and f/r^2 is the radial component of the electric field. In the light of this identification the following boundary conditions are imposed on all solutions of the equations

$$\alpha \to 1 \ , \ \gamma \to 1, \ \beta \to r^2 \ , \quad f/r^2 \to 0 \ , \ \omega \to 0 \quad \text{as } r \to \infty \qquad 5$$

We note again that the strong form of Einstein's equations consist of (1) with (iv) replaced by

$$R_{\mu\nu} = 0 \qquad 6$$

3.2 Solutions already known

Wyman[17], Papapetrou[16] and Bonner[18,19] have investigated the spherically symmetric solutions of both the 'strong' and 'weak' forms of Einstein's unified field theory. The weak set (1) have only been solved for the electric case ($\omega = 0$) and the magnetic case

($f = 0$), and not for the case of a combined electric and magnetic field (both $\omega \neq 0$, $f \neq 0$). In the magnetic case the solutions correspond to a magnetic pole with mass, the pole strength and mass being independent parameters. In the electric case all the solutions correspond to continuous charge distributions but the asymptotic form of the field at large distances is that of a point charge in classical theory. Again the mass of the system is an independent parameter. Several of the solutions are singular for an infinity of values of the radial coordinate and in these the charge density is not of constant sign. There are, however, two solutions for which the charge density is of constant sign throughout the field and which have no singularities outside a definite region surrounding the particle ($r > 2m$, m = mass). The corresponding solutions of the 'strong' equations differ only in the magnetic case where the magnetic pole must now have zero mass. This is considered to be in favour of the 'strong' equations[26] although as far as the author can see a theory which permits the existence of magnetic poles at all with or without mass is not wholly satisfactory.

In addition to these solutions Bonnor has succeeded in solving the 'strong' equations for the general case of an electric and a magnetic field combined. The general solutions correspond to a system without mass, having a charge distribution of variable sign, and with a point magnetic pole at the origin. The solutions have an

infinity of singular surfaces but at large distances correspond to a point charge and a point magnetic pole without mass.

There are two features which we notice concerning the solutions. The first is that when the solution contains an electric field the charge is distributed continuously throughout the field. This is in agreement with the findings of Schrödinger[28] mentioned in the previous chapter - namely that for the 'weak' field theory the current four-vector is in general non-zero throughout the field. The second feature is that in the magnetic case the weak equations permit the system to have mass whereas the strong equations do not.

We note also that the solution of the 'strong' equations for a combined electric and magnetic field has no associated mass. Unfortunately we have not succeeded in solving the weak equations for this general combined field case but it is possible to show that the solutions are likely to have associated mass unlike those of the 'strong' equations. The argument is outlined in this chapter.

3.3 Preliminary Calculations

The first task is to solve (1 i) for $\Gamma'_{\kappa\lambda}$ when the metric tensor is given by the general spherically symmetric time independent form (3). This has been done by Bonnor[19] and there is no need to write down here the resulting 30 non zero components of $\Gamma'_{\kappa\lambda}$. Equation (1 ii) then requires that either

$$\omega = 0 \qquad \text{or} \qquad A = \tfrac{1}{2}\left[\log\left(1 - \frac{\alpha r}{\omega^2}\right)\right]' \tag{1}$$

where

$$A = \tfrac{1}{2}\left[\log\left(f^2+\beta^2\right)\right]' \tag{11}$$ 7

the dash denoting differentiation with respect to r. The appropriate condition is the second one since we are only interested in the $\omega \neq 0$, $f \neq 0$ case.

The next step is to calculate R_{ik} given by (2) in terms of α, β, γ, f and ω by means of the values found for the components of $\Gamma^i_{k\lambda}$. Again we quote Bonnor's results for the non-zero components of R_{ik}. They are

$$R_{11} = A' + \tfrac{1}{2}(A^2+B^2) - \frac{\alpha'}{2\alpha}A - \Gamma^4_{14}\left(\frac{\alpha'}{2\alpha} - \Gamma^4_{14}\right) + \Gamma^4_{14,1}$$

$$-R_{22} = \left(\frac{fB-\beta A}{2\alpha}\right)' + \frac{fB-\beta A}{2\alpha}\cdot\tfrac{1}{2}\left[\log\left(\omega^2-\alpha\gamma\right)\right]' + B\,\frac{fA+\beta B}{2\alpha} + 1$$

$$R_{33} = \sin^2\theta\, R_{22}$$

$$-R_{44} = -\Gamma^1_{44,1} + \frac{\omega^2}{4\alpha^2}\left[\log\left(1-\frac{\alpha\gamma}{\omega^2}\right)\right]' + \frac{\omega^2}{2\alpha^2}\left(A^2+B^2\right) + \Gamma^1_{44}\left(\frac{\alpha'}{2\alpha} - \Gamma^4_{14} + A\right)$$ 8

$$-R_{23}\operatorname{cosec}\theta = R_{32}\operatorname{cosec}\theta$$

$$= \left(\frac{fA+\beta B}{2\alpha}\right)' + \frac{fA+\beta B}{2\alpha}\cdot\left(\frac{\alpha'}{2\alpha} + \Gamma^4_{14}\right) - B\,\frac{f\beta-\beta A}{2\alpha}$$

$$-R_{14} = R_{41} = \Gamma^1_{14,1} + \frac{\omega}{2\alpha}\left(A^2+B^2\right) + A\,\Gamma^1_{14}$$

It should be noted that these expressions differ from Bonnor's by a -1 factor since our definition (2) for R_{ik} differs from Bonnor's by the same multiplicative factor. As we might expect R_{ik} is, like g_{ik}, spherically symmetric. In these expressions A is given by (7 i) and

$$B = \frac{f\beta' - \beta f'}{f^2+\beta^2}$$

while

$$\Gamma^1_{44} = \frac{4ww'\alpha\gamma - 2w^2\alpha'\gamma - (w^2+\alpha\gamma)\alpha\gamma'}{2\alpha^2(w^2-\alpha\gamma)} = \frac{\gamma'}{2\alpha} + \frac{2w^2}{\alpha^2}A$$

$$\Gamma^4_{14} = \frac{2ww'\alpha - w^2\alpha' - \alpha^2\gamma'}{2\alpha(w^2-\alpha\gamma)} = \frac{\gamma'}{2\gamma} + \frac{w^2}{\alpha\gamma}A \qquad 9$$

$$\Gamma^1_{14} = \frac{2\alpha\gamma w' - w(\alpha'\gamma+\alpha\gamma')}{2\alpha(w^2-\alpha\gamma)} = \frac{w}{\alpha}A$$

The second forms for Γ^1_{44} and Γ^4_{14} were not given by Bonnor but are easily established. In each case the first term is independent of the skew field while the second term vanishes when the skew field vanishes ($w = 0 = f$). In fact it is possible to split up all Bonnor's expressions for the components of Γ^i_{kl} in this way.

<u>Some useful formulae</u>. One easily sees that

$$2A\frac{w^2}{\alpha\gamma} = \frac{w^2}{\alpha\gamma}\left[\log\left(1-\frac{\alpha\gamma}{w^2}\right)\right]' = \left[\log\left(\frac{w^2}{\alpha\gamma}-1\right)\right]'$$

or alternatively

$$2A\frac{w^2}{\alpha\gamma} = \left[\log\left(1-\frac{\alpha\gamma}{w^2}\right)\right]' - \left[\log\frac{\alpha\gamma}{w^2}\right]' = 2A - \frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma} + \frac{2w'}{w} \qquad 10$$

Hence it follows from (9) that

$$\Gamma^1_{44} = \frac{\gamma}{\alpha}\left\{2A + \frac{2w'}{w} - \frac{\alpha'}{\alpha} - \frac{\gamma'}{2\gamma}\right\}$$

and $\qquad 11$

$$\Gamma^4_{14} = A + \frac{w'}{w} - \frac{\alpha'}{2\alpha}$$

3.4 The Field Equations

Noting from (8) the form of the eight non zero elements of R_{ik} we see that the ten field equations (1 iii) reduce to three, namely

$$R_{11} = 0 \qquad R_{22} = 0 \qquad R_{44} = 0 \qquad 12$$

while the four field equations (1 iv) reduce to one, namely

$$R_{23,1} = 0 \qquad \text{or} \qquad R_{23} = c\sin\Theta \tag{13}$$

where c is an arbitrary constant. The 'strong' set of equations solved by Bonnor involve (6) instead of (1 iv) and so require in addition to (12) and (13),

$$c = 0 \qquad R_{14} = 0 \tag{14}$$

Using (8) and (11) we can write the first equation in (12) as

$$A' + \tfrac{1}{2}(A^2+B^2) - \tfrac{\alpha'}{2\alpha}A - \left(A + \tfrac{\omega'}{\omega} - \tfrac{\alpha'}{2\alpha}\right)\left(\tfrac{\alpha'}{\alpha} - \tfrac{\omega'}{\omega} - A\right) + \left(A + \tfrac{\omega'}{\omega} - \tfrac{\alpha'}{2\alpha}\right)' = 0$$

i.e.

$$\left(2A + \tfrac{\omega'}{\omega} - \tfrac{\alpha'}{2\alpha}\right)' + \left(\tfrac{\omega'}{\omega} - \tfrac{\alpha'}{\alpha}\right)\left(2A + \tfrac{\omega'}{\omega} - \tfrac{\alpha'}{2\alpha}\right) + \tfrac{1}{2}(3A^2+B^2) = 0$$

or finally

$$\left[\tfrac{\omega}{\alpha}\left(2A + \tfrac{\omega'}{\omega} - \tfrac{\alpha'}{\alpha}\right)\right]' + \tfrac{1}{2}\tfrac{\omega^2}{\alpha}(3A^2+B^2) = 0 \tag{15}$$

Now again use (8), (11) with (7 ii) to get for the R_{44} equation of (12)

$$\left[\tfrac{\gamma}{\alpha}\left(2A + \tfrac{2\omega'}{\omega} - \tfrac{\alpha'}{\alpha} - \tfrac{\gamma'}{2\gamma}\right)\right]' + \tfrac{\omega^2}{2\alpha^2 c}(3A^2+B^2) + \tfrac{\gamma}{\alpha}\left(2A + \tfrac{2\omega'}{\omega} - \tfrac{\alpha'}{\alpha} - \tfrac{\gamma'}{2\gamma}\right)\left(\tfrac{\alpha'}{\alpha} - \tfrac{\omega'}{\omega}\right)$$

i.e.

$$\left[\tfrac{\gamma}{\omega}\left(2A + \tfrac{2\omega'}{\omega} - \tfrac{\alpha'}{2\alpha} - \tfrac{\gamma'}{2\gamma}\right)\right]' + \tfrac{\omega}{2\alpha}(3A^2+B^2) = 0 \tag{16}$$

From these fairly symmetrical forms for the R_{11} and R_{44} equations we proceed to get a first integral. Subtract (15) from (16) and use the identity (10) in the form

$$\tfrac{\gamma}{\omega}\left(2A + \tfrac{2\omega'}{\omega} - \tfrac{\alpha'}{2\alpha} - \tfrac{\gamma'}{\gamma}\right) = 2A\tfrac{\omega}{\alpha}$$

to give after integration

$$\tfrac{\gamma'}{2\omega} - \tfrac{\omega'}{\alpha} - \tfrac{\alpha'\omega}{2\alpha^2} = -\tfrac{1}{2}b_1 \tag{17}$$

where b_1 is an arbitrary constant. This may be written as

$$\left(\frac{\omega^2}{\alpha}-\gamma\right)' = \omega t_1 \qquad 18$$

or writing $\omega = x'$ we can integrate at once to give

$$\frac{\omega^2}{\alpha}-\gamma = x t_1 + t_2 \qquad 19$$

where t_1 is another arbitrary constant.

Now from (9) we see that

$$\Gamma^4_{14} = \frac{\alpha\omega}{\omega^2-\alpha\gamma}\left(\frac{\omega'}{\alpha} - \frac{\alpha'\omega}{2\alpha^2} - \frac{\gamma'}{2\omega}\right)$$

and so using (17) and (19)

$$\Gamma^4_{14} = \tfrac{1}{2}\frac{t_1 x'}{t_1 x + t_2} \qquad 20$$

3.5 The Mass of the System

Bonner, using the 'strong' equations (14) in addition to (13), finds that (19 eq^ 2.19) $\Gamma^4_{14} = 0$

From (20) we see that this can only be so if $t_1 = 0$. We will now show that this condition effectively eliminates the possibility of the system having mass.

On account of the boundary condition (5) the asymptotic expansions of α , γ and ω in terms of $\frac{1}{r}$ may be assumed to take the form,

$$\alpha = 1 + \frac{a_1}{r} + \frac{a_2}{r^2} + \cdots \quad , \quad \gamma = 1 + \frac{c_1}{r} + \frac{c_2}{r^2} + \cdots \quad , \quad \omega = \frac{\omega_1}{r} + \frac{\omega_2}{r^2} + \cdots$$

Now when t_1 is zero (19) may be written as

$$\omega^2 - \alpha\gamma = \alpha t_2$$

It follows immediately from the boundary conditions that $t_2 = -1$ and then using the asymptotic expansion for α , γ and ω one gets

$$-1 - \frac{a_1 + c_1}{r} + O\left(\frac{1}{r^2}\right) = -1 - \frac{a_1}{r} + O\left(\frac{1}{r^2}\right)$$

From this it follows immediately that c_1 must be zero. Since we identify the coefficient of $\frac{1}{r}$ in the asymptotic form of γ as the mass of the system this means that the mass must be zero.

It is not difficult to see by a similar procedure that when $t_1 \neq 0$ no such restriction on the mass of the system is involved in (17) or equivalently (19).

Conclusion

In conclusion we can say that whereas the combined electric - magnetic field solution of the 'strong' field equations contains no mass parameter, there is a good possibility that the corresponding solution of the 'weak' field equations would contain a mass parameter. Unfortunately it has not so far proved possible to obtain these general solutions of the weak equations.

PART II

A Unified Field Theory with a Symmetric Affine Connection

Chapter 1

The Field Equations and their Form in First Approximation

Introduction

This second part is concerned with a new unified field theory based upon a symmetric affine connection. In constructing the theory we do not adopt the so-called affine approach but assume from the start the existence of a metric tensor $g_{\mu\nu}$, not necessarily symmetric, in addition to the affine connection.

The restriction that the affine connection $\Gamma^{\lambda}_{\mu\nu}$ be symmetric in its two lower indices (μ, ν) is seen not to be unreasonable if we examine again the need for the existence of an affine connection at all. To construct a tensor calculus covariant to general and arbitrary coordinate transformations some means of parallel transfer is essential and the simplest means is provided by the three index symbol $\Gamma^{\lambda}_{\mu\nu}$. This symbol is not, of course, a tensor but if it is not symmetric its skew part certainly is a tensor. Further, such a general symbol provides not only one but no less than three different definitions of parallel transfer, namely those based on $\overset{r}{\Delta}{}^{\lambda}_{\mu\nu}$ given by

$$\overset{1}{\Delta}{}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\underline{\mu\nu}} \quad , \quad \overset{2}{\Delta}{}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} \quad , \quad \overset{3}{\Delta}{}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$$

As we saw in the introduction to Part I this implies three different definitions of covariant differentiation and three different curvature

tensors. This multiplicity can only be controlled by superposing some additional hypothesis such as the requirement that the theory should be Hermitian in the sense defined and used by Einstein[3]. Altogether it would seem simpler to try and construct a theory based on a symmetric affine connection. The additional skew tensor and the ambiguity it causes then disappears.

Whereas the addition of a skew component to a symmetric affine connection effects the parallel transfer based upon it, the addition of a skew tensor to a symmetric metric tensor does not affect the metric at all. Since the aim of the theory is to construct an extension of general relativity that will incorporate electromagnetism we certainly expect a skew second rank tensor to occur somewhere in the theory. The aim, however, is also to obtain a natural unification of the fields of gravitation and electromagnetism. For such a unified field theory there would seem to be no simple or more natural place to expect the appearance of the skew tensor than in company with the tensor which plays the role of gravitational potential in general relativity, i.e., the metric tensor.

Assuming then the existence of a symmetric affine connection Γ^{i}_{kl} and a non symmetric metric tensor g_{ik} the first task is to construct a satisfactory set of field equations which will determine

these 56 quantities. This we do by first fixing the metric - affine relation - a relation comprised of 40 equations and itself derived from a variational principle. At once an invariant integral comes to light from which certain identity relations can be deduced in virtue of this $g-\Gamma$ relation. These identity relations or conservation laws play an important part not only in suggesting the remaining 16 field equations but also in suggesting an interpretation of the theory. Only when the complete set of field equations have been obtained in this way can we find the Lagrangian from which they all spring. The existence of the Lagrangian reassures us of the compatibility of the set.

One of the ways of deducing Einstein's gravitational theory consists in first fixing the affine connection as the second Christoffel symbols of the metric and then using the Bianchi identities for the curvature tensor constructed from these symbols to suggest the remaining 10 field equations. Our method of deriving a unified field theory is not disimilar to this. Furthermore, it closely resembles the method adopted by Kursunoğlu in a recent unified field theory suggested by him[10]. Kursunoğlu emplys an affine connection and metric which are both non-symmetric and uses the Hermitian argument mentioned above. Although our Lagrangian (1.26) looks very similar to Kursunoğlu's the field equations are much different and have the advantage of being fundamentally simpler involving as they do a symmetric affine connection. In particular

his g-Γ relation, the usual one employed by Einstein

$$g_{\mu\nu;\lambda} = 0$$

cannot be solved for $\Gamma^{\lambda}_{\mu\nu}$ exactly whereas ours, namely (1.7) or (1.8) can. The interpretation of the two theories is also fundamentally different.

In this first chapter a complete set of field equations with an accompanying interpretation is derived. Then their form and that of the Lagrangian is examined for a weak field approximation, solutions being obtained in this approximation for a static spatially spherically symmetric field. At the end of the chapter we demonstrate the non-trivial result that the exact field equations are symmetric to the change of sign of the electromagnetic field. In the succeeding chapters we study in more detail the conservation laws for the theory, obtain more exact static spatially spherically symmetric solutions of the equations, determine to a first approximation the motion of an n-particle system, and finally in the last chapter discuss the theory and its significance in the light of these results.

A few sections of the work in the first chapter have already been done in Part I. They are included here to give continuity and to make this second part self contained.

1.1 The $g-\Gamma$ relation

One can construct only one scalar density from a metric tensor g_{ik} and the curvature tensor B^{i}_{klm} of a symmetric affine connection taken together. It is

$$\mathfrak{L}_1 = g^{ik} R_{ik}$$

where $g^{ik} = \sqrt{g}\, g^{ik}$, $g^{il} g_{kl} = \delta^{i}_{k} = g^{il} g_{lk}$, $g = |g_{ik}|$ 1

and $$R_{ik} = -\Gamma^{\alpha}_{ik,\alpha} + \Gamma^{\alpha}_{i\alpha,k} + \Gamma^{\alpha}_{i\beta}\Gamma^{\beta}_{\alpha k} - \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{ik}$$

In developing the following theory based on a symmetric affine connection we make the natural assumption that the field equations are based on a variational principle applied to a Lagrangian $\mathfrak{L}$ which contains the affine connection only through $\mathfrak{L}_1$. In fact we shall suppose

$$\mathfrak{L} = \mathfrak{L}_1 + \mathfrak{L}_2 \qquad 2$$

where $\mathfrak{L}_2$ is independent of Γ^{i}_{kl}. Moreover the field equations will follow from

$$\delta I = 0 \qquad I = \int_V \mathfrak{L}\, d\tau \qquad d\tau = dx^1 dx^2 dx^3 dx^4 \qquad 3$$

where δ denotes an arbitrary variation that vanishes on the boundary of the region of integration V. In this variation the various entities of which $\mathfrak{L}$ is composed are of course to be varied independently.

The variation of Γ^i_{kl}

From the preceding equations this variation requires

$$0 = \int_V g^{ik}\,\delta R_{ik}\,d\tau \qquad 4$$

This is simply dealt with by using Palatini's formula for

$$\delta R_{ik} = -(\delta\Gamma^\alpha_{ik})_{;\alpha} + (\delta\Gamma^\alpha_{i\alpha})_{;k}$$

~~By partial integration~~ (4) may then be transformed into

$$0 = \int_V \left\{ (g^{ik}_{\ ;\alpha} - \delta^k_\alpha\, g^{i\sigma}_{\ ;\sigma})\,\delta\Gamma^\alpha_{ik} + (g^{ik}\,\delta\Gamma^\alpha_{i\alpha})_{;k} - (g^{ik}\,\delta\Gamma^\alpha_{ik})_{;\alpha} \right\} d\tau \qquad 5$$

But for any vector density $\mathfrak{A}^\alpha$

$$\mathfrak{A}^\alpha_{;\alpha} = \mathfrak{A}^\alpha_{,\alpha}$$

and so the last two terms in the integrand of (5) are simple divergences. By Green's theorem they contribute only through a surface integral and this vanishes because $\delta\Gamma^i_{kl}$ is restricted to vanish on the boundary. Consequently (5) becomes

$$0 = \int_V (g^{ik}_{\ ;\alpha} - \delta^k_\alpha\, g^{i\sigma}_{\ ;\sigma})\,\delta\Gamma^\alpha_{ik}\,d\tau \qquad 5'$$

or since $\delta\Gamma^i_{kl}$ is symmetric in (k,l), and $\delta\Gamma^i_{kl}$ is arbitrary throughout the region V

$$g^{ik}_{\ ;\alpha} - \tfrac{1}{2}\delta^i_\alpha\, g^{k\sigma}_{\ ;\sigma} - \tfrac{1}{2}\delta^k_\alpha\, g^{i\sigma}_{\ ;\sigma} = 0 \qquad 6$$

This may be simplified by contracting (k,α)

$$g^{im}_{\ ;m} - \tfrac{5}{2}\, g^{i\sigma}_{\ ;\sigma} = 0 \quad \text{or} \quad -\tfrac{3}{2}\, g^{i\sigma}_{\ ;\sigma} = g^{i\sigma}_{\ ;\sigma} = g^{i\sigma}_{\ ,\sigma}$$

Thus (6) may be rewritten

$$g^{ik}_{;\alpha} + \tfrac{1}{3}\delta^i_\alpha g^{k\sigma}_{\nu,\sigma} + \tfrac{1}{3}\delta^k_\alpha g^{i\sigma}_{\nu,\sigma} = 0$$

or $b^{ik}_{;\alpha} + \tfrac{1}{3}\delta^i_\alpha 1^k + \tfrac{1}{3}\delta^k_\alpha 1^i$ where $b^{ik} = g^{ik}$, $1^\alpha = g^{\sigma\sigma}_{\nu,\sigma}$ 7

Defining b and b_{ik} by the usual relations

$$b = |b^{ik}|, \quad b^{ik} = \frac{1}{\sqrt{-b}} b^{ik}, \quad b_{ik} b^{il} = \delta^i_k \qquad 7'$$

it is a straightforward matter to transform (7) into

$$b_{ik;l} + \tfrac{1}{3} b_{ik} z_l - \tfrac{1}{3} b_{kl} z_i - \tfrac{1}{3} b_{li} z_k = 0 \qquad 8$$

where

$$z_\alpha = b_{\alpha\beta} z^\beta \qquad z^\beta = \frac{1}{\sqrt{-b}} 1^\beta \qquad 8'$$

The solution of (8) for Γ^i_{kl} in terms of b_{ik} and z_α is well known i.e.

$$\Gamma^i_{kl} = \{^i_{kl}\}_b + \tfrac{1}{6}\delta^i_k z_l + \tfrac{1}{6}\delta^i_l z_k - \tfrac{1}{2} b_{kl} z_i \qquad 9$$

where $\{^i_{kl}\}_b$ is the Christoffel symbol of the second kind derived from b_{ik}.

The curvature tensor R_{ik} when written in terms of Γ^i_{kl} given by (9) takes the form

$$R_{ik} = G_{ik} + \tfrac{1}{6} z_i z_k + \tfrac{1}{6}(z_{i,k} - z_{k,i}) \qquad 10$$

where G_{ik} is the curvature tensor constructed from $\{^i_{kl}\}_b$.

1.2 The Conservation Law

The only assumption made so far about the complete Lagrangian $\mathcal{L}$ is that it takes the form (2), $\mathcal{L}_1$ being given by (1) whereas $\mathcal{L}_2$ is some unknown function of $g_{\mu\nu}$. Evidently the variation of $g_{\mu\nu}$ in (3) will give

$$R_{\mu\nu} = -\frac{\delta \mathcal{L}_2}{\delta g^{\mu\nu}} \qquad 11$$

To suggest a choice of $\mathcal{L}_2$ we turn to the identity relations or 'conservation law' which $R_{\mu\nu}$ based on the affine connection (9) must satisfy. These relations may be derived from the well-known Bianchi identities but it is easier to proceed as follows. Consider the invariant $\mathcal{J}$ defined by

$$\mathcal{J} = \int_V g^{\mu\nu} R_{\mu\nu}\, d\tau$$

where V is a large region in 4-space. By definition $\mathcal{J}$ must be invariant under any coordinate transformation. Consider, therefore, the infinitesimal coordinate transformation

$$\bar{x}^\mu = x^\mu + \xi^\mu \qquad 12$$

where ξ^μ, a function of x^α, is always so small that only terms of the first degree in ξ^μ need be retained. Apart from this and the further condition that it vanishes on the boundary of V, ξ^μ is quite arbitrary. Denoting the operation of the coordinate variation by δ^+ we evidently have

$$0 = \delta^+ \mathcal{J} = \int_V g^{\mu\nu}\, \delta^+ R_{\mu\nu}\, d\tau + \int_V R_{\mu\nu}\, \delta^+ g^{\mu\nu}\, d\tau \qquad 13$$

where $\delta^+ g^{\mu\nu}$ and $\delta^+ \Gamma^{\lambda}_{\mu\nu}$ (from which $\delta^+ R_{\mu\nu}$ is calculated) are the variations of $g^{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ consequent upon (12).

In the search for the complete Lagrangian $\mathfrak{L}$, the part dependent on $\Gamma^{\lambda}_{\mu\nu}$ i.e. $\mathfrak{L}_1$ is fixed. This has been seen to involve (9) or more fundamentally (4) in which δ denotes an arbitrary variation of $\Gamma^{\lambda}_{\mu\nu}$ restricted only to vanish on the boundary. It follows that the first integral in (13), being dependent on the far less general coordinate variation $\delta^+ \Gamma^{\lambda}_{\mu\nu}$ must itself vanish. Thus (13) reduces to

$$0 = \int_V R_{\mu\nu}\, \delta^+ g^{\mu\nu}\, d\tau \qquad 14$$

Before proceeding to simplify this condition it is convenient to split the integrand into the parts dependent on the symmetrical and skew parts of $R_{\mu\nu}$ respectively

$$0 = \int_V R_{(\mu\nu)}\, \delta b^{\mu\nu}\, d\tau + \int_V R_{[\mu\nu]}\, \delta^+ g^{\mu\nu}\, d\tau \qquad 15$$

In the first use the relation $\delta b^{\mu\nu} = \sqrt{-b}\,(\delta b^{\mu\nu} - \tfrac{1}{2} b^{\mu\nu} b_{\rho\sigma}\, \delta b^{\rho\sigma})$

so that

$$\int_V R_{[\mu\nu]}\, \delta^+ b^{\mu\nu}\, d\tau = \int_V \Sigma_{\mu\nu}\, \delta^+ b^{\mu\nu}\, d\tau \qquad 16$$

where $\Sigma_{\mu\nu} = \sqrt{-b}\,(R_{[\mu\nu]} - \tfrac{1}{2} b_{\mu\nu}\, b^{\rho\sigma} R_{[\rho\sigma]})$

From (12) it follows that $\delta^+ b^{\mu\nu} = \xi^{\nu}_{,\rho}\, b^{\rho\mu} + \xi^{\mu}_{,\rho}\, b^{\rho\nu} - b^{\mu\nu}_{\ \ ,\rho}\, \xi^{\rho}$

Hence

$$\int_V \Sigma_{\mu\nu}\, \delta^+ b^{\mu\nu}\, d\tau = -\int_V \left\{ (\Sigma_{\mu\sigma} b^{\rho\sigma} + \Sigma_{\sigma\mu} b^{\sigma\rho})_{,\rho} + \Sigma_{\alpha\beta} b^{\alpha\beta}_{\ \ ,\mu} \right\} \xi^{\mu}\, d\tau$$

Here the integrand has been transformed by adding simple divergence terms which by Green's theorem are equivalent to surface integrals that vanish due to ξ^i vanishing on the boundary of V.
Raising the indices of $\mathfrak{T}_{ik}$ with b_{ik} it is a simple matter to transform the integrand again so that finally

$$\int_V \mathfrak{T}_{ik}\, \delta^+ b^{ik}\, d\tau = -\int 2\,\mathfrak{T}^{\mu}_{i\|\mu}\,\xi^i\, d\tau \qquad 17$$

where $\|$ denotes covariant differentiation based upon the second Christoffel symbol of b_{ik}.
For the second integral in (15) use

$$\delta^+ g^{\mu\nu} = \xi^{\mu}_{,\rho}\, g^{\rho\nu} + \xi^{\nu}_{,\rho}\, g^{\mu\rho} - (\xi^{\rho} g^{\mu\nu})_{,\rho}$$

Making the substitution and adding simple divergence terms we get

$$R^{ik}_{\mu\nu}\,\delta^+ g^{\mu\nu} = \{-2(R^{i\sigma}_{\mu\nu} g^{\mu\nu})_{,\rho} + R^{\sigma\rho,i}_{\mu\nu}\, g^{\mu\nu}\}\xi^k \quad + \text{simple divergence terms}$$

$$= -2\,1^{\sigma} R^{i}_{\sigma} + g^{\mu\nu} R^{\sigma\rho,i}_{\mu\nu} \quad + \text{simple divergence terms}$$

But from (1)

$$R^{ik}_{\mu\nu} = -\tfrac{1}{2} S^{ik} = -\tfrac{1}{2}(-\Gamma^{\alpha}_{\alpha i,k} + \Gamma^{\alpha}_{\alpha k,i})$$

so that

$$R^{\sigma\rho,i}_{\mu\nu} = 0$$

Altogether from these results with (17) we get for the condition (15)

$$\int_V (\mathfrak{T}^{\sigma}_{i\|\sigma} + 1^{\sigma} R^{}_{\sigma i})\,\xi^i\, d\tau = 0$$

But ξ^i is arbitrary everywhere within V. Hence in V we must have

$$\mathfrak{T}^{\sigma}_{i\|\sigma} = R^{}_{i\sigma}\, 1^{\sigma} \qquad 18$$

where from (16)

$$\mathfrak{T}_i^k = \sqrt{-b}\,(R_{il}\,b^{kl} - \tfrac{1}{2}\delta_i^k\, b^{\mu\nu}R_{\mu\nu}) = g^{\underline{ik}}R_{il} - \tfrac{1}{2}\delta_i^k\, g^{\underline{\mu\nu}}R_{\mu\nu}$$

The 4 equations (18) are the required identities or conservation law satisfied by R_{ik} and g_{ik} when the $g-\Gamma$ relation (9) holds.

1.3 Interpretation and Complete Field Equation

From equation (10)

$$R_{\underline{ik}} = G_{ik} + \tfrac{1}{6}z_i z_k$$

$$R_{\underline{ik}} = \tfrac{1}{6}(z_{i,k} - z_{k,i}) \qquad 19$$

Thus from (18) using (7') and (8') we have

$$\mathfrak{T}_i^k = \sqrt{-b}\,(G_{il}\,b^{kl} - \tfrac{1}{2}\delta_i^k\, b^{\mu\nu}G_{\mu\nu}) + \tfrac{1}{6}\sqrt{-b}\,(z^k z_i - \tfrac{1}{2}\delta_i^k\, z^\alpha z_\alpha) \qquad 20$$

These expressions and the conservation law (18) suggest at once the following interpretation

$\mathfrak{T}_i^k$ is the matter energy-momentum tensor

1^σ is the electric 4-current vector density

$R_{\underline{\mu\sigma}}$ is the electromagnetic field tensor

b_{ik} is the potential of the gravitational field

} $R_{\underline{\mu\sigma}}1^\sigma$ is the Lorentz field-current interaction.

Remembering that $z_\alpha = \frac{1}{\sqrt{-b}}\, b_{\alpha\sigma}1^\sigma$, it is seen that according to this interpretation the current vector is intimately related to the vector potential. Indeed it will be shown later that apart from

a numerical factor the vector potential and 4-current are as a first approximation for small fields identical. The object of the remainder of the present work will be to develop a possible complete set of field equations and to determine whether the above interpretation is tenable or not.

The Complete Field Equations.

Consider the invariant $\mathcal{J}' = 2\int \sqrt{-g}\, d\tau$ and apply the variation (12) where ξ^ι is restricted in the same way as before. Then

$$0 = \delta^+ \mathcal{J}' = \int g_{\mu\nu}\, \delta^+ g^{\mu\nu}\, d\tau \qquad 21$$

This closely resembles (14) and indeed (18) with $R_{\mu\nu}$ replaced by $g_{\mu\nu}$ follows from (21) just as (18) does from (14).

Thus we obtain

$$\left(g^{\underline{\kappa\lambda}}\, g_{\underline{\mu\lambda}} - \tfrac{1}{2}\delta^\kappa_\mu\, g^{\underline{\nu\sigma}}\, g_{\underline{\nu\sigma}}\right)_{\|\kappa} = g_{\underline{\mu\sigma}}\, I^\sigma \qquad 22\,a$$

and it is easy to verify the identity by direct calculation. For comparison we write out (18) again

$$\left(g^{\underline{\kappa\lambda}}\, R_{\underline{\mu\lambda}} - \tfrac{1}{2}\delta^\kappa_\mu\, g^{\underline{\nu\sigma}}\, R_{\underline{\nu\sigma}}\right)_{\|\kappa} = R_{\underline{\mu\sigma}}\, I^\sigma \qquad 22\,b$$

Further, bearing in mind that the $\|$ differentiation is based on $\left\{{\iota \atop \kappa\lambda}\right\}_b$ note that

$$\left(g^{\underline{\kappa\lambda}}\, b_{\underline{\mu\lambda}} - \tfrac{1}{2}\delta^\kappa_\mu\, g^{\underline{\nu\sigma}}\, b_{\underline{\nu\sigma}}\right)_{\|\kappa} = \left(\sqrt{-b}\right)_{\|\mu} = 0 \qquad 22\,c$$

The form of (22 a - c) suggests that the missing field equations

foreshadowed in (11) should be obtained by equating their right and left hand sides, i.e.

$$G_{ik} + \tfrac{1}{6} z_i z_k = R_{ik} = r g_{ik} + s b_{ik} \qquad \text{(a)}$$

(r, s numerical constants) 23.

$$\tfrac{1}{6}(z_{,k} - z_{k,i}) = R_{[ik]} = r g_{[ik]} \qquad \text{(b)}$$

Returning to equation (11) it is seen that the additional part of the Lagrangian L independent of Γ^i_{kl} must be

$$L_2 = -2r\sqrt{-g} - 2s\sqrt{-b}$$

As yet no relation is suggested for the relative values of the numerical constants r and s. However, it is necessary that the field equations reduce for a symmetric g_{ik} tensor to the equations of General Relativity, i.e.

$$G_{ik} = 0 \qquad \text{or} \qquad G_{ik} = \lambda g_{ik} \qquad 24$$

where observations show that λ is extremely small ($\sim 3/R^2$ where R is the radius of the universe: thus $\lambda \sim 10^{-50}$ cm^{-2} at most)* Now when g_{ik} is symmetric (23) reduces to

$$G_{ik} = (r+s)\, g_{ik}$$

---

* Observations for the solar system require 10^{-44} cm^{-1} at most. See Bondi, H. 'Cosmology', C.U.P. (1952) p.96.

There are two possibilities: $r+s$ is extremely small $\sim\lambda$; r and s both positive; or r,s of opposite signs, their difference being $\sim\lambda$. The influence of the right hand sides of (23) would appear to be negligible when r and s are both as small as the cosmological constant λ - indeed (23) would then be but a trivial modified form of the equation (24) of gravitation. In the following we shall find many indications that while r and s must be small, they must also be large compared with λ . Ignoring possible cosmological terms which surely should only be apparent in large regions of the universe write

$$r = -s = -p^2$$

where p is real. The negative sign will later be shown to be necessary. Thus we adopt as the complete set of field equations

$$R_{ik} = -p^2(g_{ik} - b_{ik}) \quad \text{or} \quad G_{ik} + \tfrac{1}{2}g_{ik}\lambda = -p^2(g_{ik} - b_{ik}) \qquad \text{(a)}$$

$$\lambda_{,k} - 2\kappa_{,i} = -6p^2 g_{ik} \qquad \text{(b)} \qquad 25$$

in the notation of (7'), (8') and (10). Note that the Lagrangian form which (25) follow by the variational principle is

$$L = g^{ik}R_{ik} + 2p^2(\sqrt{-g} - \sqrt{-b}) \qquad 26$$

where R_{ik} is the Einstein-tensor (1) of a symmetric affinity Γ'_{kl} .

1.4 An approximate form for the field equations (25)

Put $g_{ik} = \eta_{ik} + \gamma_{ik}$ $\qquad \gamma_{ik} = a_{ik}$ $\qquad \gamma_{ik} = f_{ik}$

where η_{ik} **is the Minkowski matrix**

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$$\eta_{ik} = \begin{pmatrix} -1 & & \\ & -1 & 0 \\ & 0 & -1 \\ & & & 1 \end{pmatrix}$$

Assuming that γ_{ik} **is small we work to the following approximation in** a_{ik} , f_{ik}

(i) a_{ik} **are so small that all second degree terms in** a_{ik} **may be neglected**

(ii) f_{ik} **are so small that all terms above the second degree in** f_{ik} **as well as all mixed** $a-f$ **terms may be neglected.**

The first step consists in finding expressions for g , $\sqrt{-g}$, g^{ik} **and** g^{ik} **correct to the second order in** γ_{ik} .

For g **one obtains**

$$g = |\eta_{ik} + \gamma_{ik}| = -1 - \eta^{\alpha\beta}\gamma_{\alpha\beta} - \tfrac{1}{2}(\eta^{\alpha\beta}\gamma_{\alpha\beta})^2 + \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\delta}\gamma_{\alpha\gamma}\gamma_{\beta\delta} \qquad \text{(a)}$$

and from this

$$\sqrt{-g} = 1 + \tfrac{1}{2}\eta^{\alpha\beta}\gamma_{\alpha\beta} + \tfrac{1}{4}\left[-\eta^{\alpha\beta}\eta^{\gamma\delta}\gamma_{\alpha\gamma}\gamma_{\beta\delta} + \tfrac{1}{2}(\eta^{\alpha\beta}\gamma_{\alpha\beta})^2\right] \qquad \text{(b)}$$

writing

$$g^{ik} = \eta^{ik} + \gamma^{ik}$$

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and using the definition of g^{ik} **in terms of** g_{ik}

i.e.,

$$g^{i\alpha} g_{k\alpha} = \delta^i_k = g^{\alpha i} g_{\alpha k}$$

one gets for g^{ik} ,

$$g^{ik} = \eta^{ik} - \eta^{i\alpha}\gamma_{\beta\alpha}\eta^{\beta k} + \eta^{i\alpha}\eta_{\beta\alpha}\eta^{\beta\varepsilon}\gamma_{\delta\varepsilon}\eta^{\delta k} \qquad \text{(c)}$$

Remembering that $g^{ik} = \sqrt{-g}\, g^{ik}$ one gets from (b) and (c)

$$g^{ik} = \eta^{ik} + [-\eta^{il}\gamma_{\beta l}\eta^{\beta k} + \tfrac{1}{2}\eta^{ik}\eta^{\alpha\beta}\gamma_{\alpha\beta}] + \{[\eta^{il}\gamma_{\beta l}\eta^{\beta c}\gamma_{\varepsilon c}\eta^{\varepsilon k} - \tfrac{1}{2}\eta^{il}\gamma_{\beta l}\eta^{\beta k}(\eta^{\sigma\rho}\gamma_{\sigma\rho})] - \tfrac{1}{4}\eta^{ik}[\eta^{\alpha\beta}\eta^{\gamma\delta}\gamma_{\alpha\gamma}\gamma_{\beta\delta} - \tfrac{1}{2}(\eta^{\alpha\beta}\gamma_{\alpha\beta})^2]\}$$

It will not cause confusion now to use the Minkowski matrix η_{ik} and η^{ik} to lower and raise indices. The expression for g^{ik} can then be written rather more simply as

$$g^{ik} = \eta^{ik} + [-\gamma^{ik} + \tfrac{1}{2}\eta^{ik}\eta^{\alpha\beta}\gamma_{\alpha\beta}] + \{[\gamma^{i}{}_{\varepsilon}\gamma^{\varepsilon k} - \tfrac{1}{2}\eta^{ik}\eta^{\sigma\rho}\gamma_{\sigma\rho}] - \tfrac{1}{4}\eta^{ik}[\gamma^{\alpha\beta}\gamma_{\beta\alpha} - \tfrac{1}{2}(\eta^{\alpha\beta}\gamma_{\alpha\beta})^2]\} \qquad 29$$

From this the symmetric part is

$$g^{ik} = \eta^{ik} + [-a^{ik} + \tfrac{1}{2}\eta^{ik}a] + \{[a^{i}{}_{c}a^{ck} + f^{i}{}_{c}f^{ck} - \tfrac{1}{2}a^{ik}a] - \tfrac{1}{4}\eta^{ik}[a^{\alpha\beta}a_{\alpha\beta} + f^{\alpha\beta}f_{\beta\alpha} - \tfrac{1}{2}a^2]\} \qquad 30$$

where a_{ik} and f_{ik} are introduced according to (27) and $a = \eta^{\rho\sigma}a_{\rho\sigma}$. Now according to (7) this is just b^{ik}. To obtain the components of b_{ik} in terms of those of g_{ik} notice that if

$$b_{ik} = \eta_{ik} + d_{ik} \qquad (d_{ik}\ \text{symmetric}) \qquad 31$$

then correct to quadratic terms in d_{ik} one obtains by analogy with (30)

$$b^{ik} = \eta^{ik} + [-d^{ik} + \tfrac{1}{2}\eta^{ik}d] + \{(d^{i}{}_{c}d^{ck} - \tfrac{1}{2}d^{ik}d) - \tfrac{1}{4}\eta^{ik}(d^{\alpha\beta}d_{\alpha\beta} - \tfrac{1}{2}d^2)\} \qquad 32$$

Comparing this with g^{ik} one sees that correct to quadratic terms in γ_{ik} (i.e. in a_{ik} and f_{ik})

$$d_{ik} = a_{ik} - t_{ik} \qquad \text{or} \qquad b_{ik} = \eta_{ik} + a_{ik} - t_{ik} \qquad 33$$

where $$t_{ik} = f_{i}{}^{\alpha}f_{\alpha k} - \tfrac{1}{4}\eta_{ik}f^{\alpha\beta}f_{\beta\alpha}$$

We emphasise that this result is correct to quadratic terms in both a_{ik} and f_{ik} because we shall need to use this fact in another connection later.

From (29) it also follows that within the approximations (i) and (ii) s^k and z_k defined by (7) and (8') are given by

$$s^k = f^{k\sigma}{}_{,\sigma} \qquad \text{and} \qquad z_k = \gamma_{k\alpha} z^\alpha \tag{34}$$

After these preliminaries the field equation (25) correct to the approximations (i) and (ii) above can be written down at once. The expression for G_{ik} calculated from the Christoffel symbols of b_{ik} is of well known form and need not be derived here in detail. It is convenient, as is usual with similar expressions involving G_{ik}, not to write down (25 a) as it stands but to subtract $\frac{1}{2}\gamma_{ik} \times$ its trace so that G_{ik} occurs in the combination $G_{ik} - \frac{1}{2}\gamma_{ik}\gamma^{\mu\nu}G_{\mu\nu}$. Thus (25 a) becomes

(with $\beta_{ik} = \alpha_{ik} - \frac{1}{2}\gamma_{ik}\alpha$)

$$\begin{aligned}(G_{ik} - \tfrac{1}{2}\gamma_{ik}G) \equiv \tfrac{1}{2}\eta^{\alpha\beta}\left(-\beta_{\beta i,\alpha k} - \beta_{\beta k,\alpha i} + \beta_{ik,\alpha\beta} + \gamma_{ik}\eta^{\rho\sigma}\beta_{\beta\rho,\alpha\sigma}\right) \\ = -\tfrac{1}{2}\left(z_{i,k} - \tfrac{1}{2}\gamma_{ik}z^\alpha{}_{,\alpha}\right) - \rho^2 t_{ik}\end{aligned} \tag{a}$$

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and (25 b) becomes $6R_{ik} = z_{i,k} - z_{k,i} = \eta^{\alpha\beta}(f_{i\alpha,\beta k} - f_{k\alpha,\beta i}) = -6\rho^2 f_{ik}$ (b)

Before proceeding it is interesting to look more closely at these equations. The interpretation suggested by the conservation law (18) is that R_{ik} is the electromagnetic field tensor in natural

units. (35 b) tells us that it is the curl of the vector i_α which from (34) is, correct to the approximations (i) and (ii), the current vector. Thus the current vector must to this approximation also play the role of vector potential. Apart from this unusual feature the remainder of (35) is highly satisfactory. According to the Einstein theory of gravitation the left hand side of (35 a) should be equated to the energy momentum tensor of the system. By (33) the right hand side contains the electromagnetic energy momentum tensor t_{ik} (in natural units) and the terms involving i_α. These latter terms closely resemble the matter energy momentum tensor,

$$T^{ik} = m \frac{dx^i}{ds} \cdot \frac{dx^k}{ds} = m u^i u^k$$

Indeed where the current vector i^α due to the motion of point charges we should have

$$i^\alpha = \varepsilon u^\alpha$$

at the point occupied by the charge.

Thus we have seen that to the approximation considered the field equations closely resemble the equations of general relativity with introduced matter and electromagnetic energy momentum tensors.

It is interesting to see the form taken to the approximation (i) and (ii) by the Lagrangian (26) upon which the present theory is based, i.e.

$$R = g^{ik} R_{ik} + 2\rho^2 (\sqrt{-g} - \sqrt{-b})$$

According to the field equation (25)

$$g^{ik} R_{ik} = -\rho^2 g^{ik}(g_{ik} - b_{ik}) = -4\rho^2(\sqrt{-g} - \sqrt{-b})$$

where we have used (7) coupled with the fact that b_{ik} is symmetric. Hence

$$R = -2\rho^2(\sqrt{-g} - \sqrt{-b})$$

But to the approximation (1) and (11) (28 b) gives

$$\sqrt{-g} = 1 + \tfrac{1}{2}\gamma^{\alpha\beta}a_{\alpha\beta} - \tfrac{1}{4}f^{\alpha\beta}f_{\beta\alpha} \qquad \text{(a)}$$

and similarly the same approximation 36

$$\sqrt{-b} = 1 + \tfrac{1}{2}\eta^{\alpha\beta}a_{\alpha\beta} = 1 + \tfrac{1}{2}\eta^{\alpha\beta}a_{\alpha\beta} \qquad \text{(b)}$$

using (33). Hence

$$R = \tfrac{1}{2}\rho^2 f^{\alpha\beta}f_{\beta\alpha} = \rho^2(E^2 - H^2) \qquad 38$$

according to our interpretation. This is, within an irrelevant constant factor, the Lagrangian of Maxwell's electrodynamics.

1.5 A static solution of the approximate equation (33)

We first solve (35 b) for the current - potential vector z_α, which in the static case will have only one non-vanishing component z_0. Operating on (35 b) with $\eta^{kl}\partial_l$ we get in virtue of (34)

$$\eta^{kl} z_{\alpha,kl} = -\rho^2 z_\alpha$$

In the static case this reduces to

$$z_{0,ss} = \rho^2 z_0 = \nu^2 z_0 \qquad \text{(say)} \qquad (s=1,2,3 \text{ only})$$

The solution of this equation which tends to zero for r increasing is

$$z_0 = -\rho^2 \frac{c}{r} e^{-\nu r}$$

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Returning to (35 b) it follows from this that

$$f_{os} = -\frac{\varepsilon x^s}{r^3}(1+\nu r)\, e^{-\nu r} \qquad 40$$

If ρ_0 is to represent the charge density, and f_{os} the electric field vector $-E_s$ it is necessary to suppose that ν or $\sqrt{6\rho^2}$ is very small. Accordingly it will be sufficient to solve (35 a) correct to the first order in ρ^2. To this approximation the ρ_α terms in (35) can be neglected since by (39) they involve ρ^4. Furthermore since t_{ik} already has ρ^2 as coefficient we may neglect the ν terms in (40) i.e. we may take

$$f_{os} = -\frac{\varepsilon x^s}{r^3} \qquad 41$$

From (33) it is now easy to calculate the components of t_{ik}. In fact for the non-vanishing components

$$t_{mn} = -f_{om} f_{on} + \tfrac{1}{2}\delta_{mn} f_{or} f_{or} \qquad (m,n,r = 1,2,3 \text{ only}) \qquad 42$$

$$t_{oo} = \tfrac{1}{2} f_{or} f_{or}$$

The fact that the trace of t_{ik}, $\eta^{\mu\nu} t_{\mu\nu}$, vanishes serves as a check to these values. Using (41) these components become

$$t_{mn} = \left[-\frac{x^m x^n}{r^6} + \tfrac{1}{2}\delta_{mn}\frac{1}{r^4}\right]\rho^2\varepsilon^2$$

$$t_{oo} = \tfrac{1}{2}\frac{1}{r^4}\rho^2\varepsilon^2 \qquad 43$$

The left hand side of (35 a) is too complicated as it stands but it simplifies by imposing the usual coordinate conditions

$$\eta^{\mu\nu}\beta_{\lambda\mu,\nu} = 0$$

into

$$\tfrac{1}{2}\eta^{\mu\nu}\beta_{ik,\mu\nu} = -\tfrac{1}{2}\beta_{ik,ss} \qquad 44$$

The last step follows because only static fields are to be considered here.

Consequently (35 a) becomes with the use of (43) and ignoring as explained the τ_α terms:-

$$\beta_{mn,55} = 2\rho^2\varepsilon^2\left[-\frac{x^m x^n}{r^6} + \tfrac{1}{2}\delta_{mn}\frac{1}{r^4}\right]$$

$$\beta_{on,55} = 0$$

$$\beta_{oo,55} = \rho^2\varepsilon^2\frac{1}{r^4}$$

The general solution of these equations is

$$\beta_{mn} = \frac{\rho_{mn}}{r} + \tfrac{1}{2}\frac{x^m x^n}{r^4}\rho^2\varepsilon^2$$

$$\beta_{on} = \frac{\rho_n}{r} \qquad 45$$

$$\beta_{oo} = -\frac{4m}{r} + \tfrac{1}{2}\frac{\rho^2\varepsilon^2}{r^2}$$

where ρ_{mn}, ρ_n and m are some constants. These solutions must satisfy the coordinate conditions (44) which in the static case are

$$\beta_{mn,n} = 0 \qquad \beta_{on,n} = 0$$

A simple calculation shows that these are only satisfied if $\rho_{mn} = 0 = \rho_n$. Collecting together the results (39), (40) and (45) we see the static solution correct to the first order in of the approximate field equation (35) is given by:-

Charge density: $$\tau_0 = -6\rho^2\frac{\varepsilon}{r}e^{-\nu r} \qquad (a)$$

Electric field: $$f_{s0} = E_s = \frac{\varepsilon x^s}{r^3}(1+\nu r)e^{-\nu r} \qquad (b)$$ 46

Gravitational potential: $$g_{mn} = -1 + \tfrac{1}{2}\frac{x^m x^n}{r^4}\rho^2\varepsilon^2 - \frac{2m}{r}\delta_{mn} \qquad (c)$$

$$g_{oo} = 1 - \frac{2m}{r} + \tfrac{1}{2}\rho^2\frac{\varepsilon^2}{r^2} \qquad (d)$$

[The gravitational potential $g_{ik} = \eta_{ik} + a_{ik}$, $a_{ik} = \beta_{ik} - \tfrac{1}{2}\eta_{ik}\eta^{\rho\sigma}\beta_{\rho\sigma}$, $\eta^{\rho\sigma}\beta_{\rho\sigma} = -\frac{4m}{r}$]

Just as in general relativity,† whose equations (35) so closely resemble, the charge contributes to the gravitational potential only through

† Compare (46) with (1.34) page 21, Pt I

a $\frac{1}{r^2}$ term. Later when more exact solutions of the true field equations (25) are found it will be shown that the electromagnetic field does contribute directly to the gravitational mass m .

The solution (46) for small fields displays two reasons why ρ^2 must be very small. From (34) it follows that within the approximations made ι. plays the role of charge density according to the interpretation of the theory. This can only be in agreement with experiment if ρ^2 is very small. Furthermore from the exponential factors in (a) and (b) $\frac{1}{\sqrt{6}}\rho^{-1}$ must be of the same order as the range of the electromagnetic field - again ρ^2 must be small.

1.6 Charge symmetry

It has just been noted that the charge ε contributes to the gravitational potential b_{ik} only through $\frac{1}{r^2}$ terms but in addition this contribution is only dependent on ε through its square. Thus according to the approximate solutions (46) there is no gravitational difference between a positive or negative charge. This result is not however peculiar to these approximate solutions for small fields of (25) but is generally true for all solutions. To show this we first need to examine the complicated relationship (7) and (7') between b_{ik} and g_{ik} . Consider

$$g_{ik} = h_{ik} + f_{ik} \quad \text{where} \quad g_{\underline{ik}} = h_{ik} \,, \; g_{\underline{\underline{ik}}} = f_{ik}. \qquad 47$$

From this it follows that

$$g = |g_{\mu\nu}| = h + \tfrac{1}{2} h\, h^{\mu\lambda} h^{\nu\rho} f_{\mu\nu} f_{\lambda\rho} + \mathfrak{J}^2 \,, \quad \mathfrak{J} = \tfrac{1}{8} \varepsilon^{\mu\nu\lambda\rho} f_{\mu\nu} f_{\lambda\rho} \qquad 48$$

where h is the determinant of $h_{\mu\nu}$ and $h^{\mu\lambda}$ is the inverse matrix of $h_{\mu\nu}$. In this section $h_{\mu\nu}$ and $h^{\mu\nu}$ will be used for raising and lowering indices. Now $b_{\mu\nu}$ was defined by (7) according to the relation

$$b^{\mu\nu} = g^{(\mu\nu)} \qquad 7$$

coupled with (7'). $g^{(\mu\nu)}$ may readily be calculated in terms of $h_{\mu\nu}$ and $f_{\mu\nu}$ by noting that

$$dg = g\, g^{\mu\nu} dg_{\mu\nu} = -\sqrt{g}\, g^{\mu\nu} dg_{\mu\nu} = -\sqrt{g}\,(g^{(\mu\nu)} dh_{\mu\nu} + g^{[\mu\nu]} df_{\mu\nu})$$

whence

$$g^{(\mu\nu)} = -\frac{1}{\sqrt{g}} \frac{\partial g}{\partial h_{\mu\nu}} \,, \quad g^{[\mu\nu]} = -\frac{1}{\sqrt{g}} \frac{\partial g}{\partial f_{\mu\nu}}$$

Using (48) it is now straightforward to obtain

$$g^{(\mu\nu)} = -\frac{h}{\sqrt{g}} \left[h^{\mu\nu}(1 + f^{ab} f_{ba}) - f^{\mu a} f^{a\nu} \right]$$

$$g^{[\mu\nu]} = -\frac{h}{\sqrt{g}} \left[f^{\mu\nu} + \tfrac{1}{2} h\, \mathfrak{J}\, \varepsilon^{\mu\nu\lambda\rho} f_{\lambda\rho} \right] \qquad 49$$

These results for $g^{(\mu\nu)}$ and $g^{[\mu\nu]}$ were first published by Schrödinger[8b]. From (49) and (48) it is evident that $g^{(\mu\nu)}$ and g are even in $f_{\mu\nu}$ whereas $g^{[\mu\nu]}$ is odd. From (7) and (7') it follows that $h_{\mu\nu}$ is also even in $f_{\mu\nu}$. Now the field equations (25) may be written

$$G_{\mu\nu}(b) + \tfrac{1}{6} z_\mu z_\nu = -\rho^2 (h_{\mu\nu} - b_{\mu\nu})$$

$$z_{\mu,\nu} - z_{\nu,\mu} = -6\rho^2 f_{\mu\nu}$$

where $z_\mu = b_{\mu\sigma} \frac{1}{\sqrt{b}}\, g^{[\sigma\nu]}{}_{,\nu}$

Inspection shows that for any given solution $h_{\mu\nu}$, $f_{\mu\nu}$ there corresponds the solution $h_{\mu\nu}$, $-f_{\mu\nu}$. Thus according to the

theory and its interpretation there is complete symmetry between positive and negative charges. In particular $b_{\mu\nu}$, if it contains terms depending on the charge ε must always do so through the square of the charge.

1.7 Maxwell's equations and the mass of the photon

The tentative interpretation of the theory identifies i^α with the current 4 - vector and $f_{\mu\nu} = g_{\mu\nu}$ with the electromagnetic field tensor. The exact form of Maxwell's equations would then consist of the second field equation (25 b)

$$-\nu^2 f_{\mu\nu} = \imath_{\nu,\mu} - \imath_{\mu,\nu} \qquad \nu^2 = 6\rho^2 \tag{a}$$

50

and

$$i^\alpha = g^{\mu\nu}{}_{,\sigma} \tag{b}$$

The definition of the 4 - vector current. These equations are meaningless without the definition (7) - (7') of the quantities involved, i.e., briefly

$$\imath_\alpha = b_{\alpha\beta}\imath^\beta \,, \qquad \imath^\beta = \frac{1}{\sqrt{-b}}\, g^{\beta\sigma}{}_{,\sigma} \,, \qquad b^{\mu\nu} = g^{\mu\nu} \tag{51}$$

In this way the electromagnetic and gravitational fields are inextricably linked. To the approximation discussed in 1.4 the pair (50) reduce to

$$-\nu^2 f_{\mu\nu} = \imath_{\nu,\mu} - \imath_{\mu,\nu}$$

$$i^\mu = \eta^{\mu\sigma}\imath_\sigma \,, \qquad \imath_\sigma = \eta^{\mu\nu} f_{\sigma\mu,\nu} \tag{52}$$

Thus in regions where the field is weak the current 4 - vector is also the potential vector of the electromagnetic field. We know that ν must be small and this ensures that the current vector can

be much smaller than the field $f_{\mu\nu}$ itself small. It is seen that the vector z_α satisfies

$$\eta^{\mu\nu} z_{\alpha,\mu\nu} = -\nu^2 z_\alpha$$

This is a Yukawa like equation and implies that a photon should have a finite mass. Possible values of ν will be discussed in the last chapter and this will give an idea of the mass involved for photons. The mass is formally

$$m_{ph} = \frac{h\nu}{c}$$

Chapter 2

The Conservation Laws and Mass

2.1 A more detailed form for the conservation laws

The conservation identities or laws in the form (18) have already played an important part in suggesting the form of the field equations. In this section alternative useful forms of the same conservation law will be derived. The results are closely similar to those known for general relativity and for both the Einstein and Schrödinger unified field theories [9]. The method of calculation is, however, somewhat different.

Consider the invariant I defined by

$$I = \int_V \mathcal{L}\, d\tau \qquad (a) \qquad\qquad \mathcal{L} = g^{\mu\nu} R_{\mu\nu} \qquad (b) \qquad 1$$

where $R_{\mu\nu}$ is given by (1.1) and where $\Gamma^{\iota}_{\kappa\lambda}$ is given in terms of $g_{\mu\nu}$ by (1.7-9). The identities (18) were obtained by applying to an infinitesimal coordinate variation which vanished on the boundary of the 4 - volume of integration V. Consider now the effect of a variation which does not necessarily vanish on the boundary - i.e. take

$$\bar{x}^{\iota} = x^{\iota} + \xi^{\iota}$$

ξ^{ι} always infinitesimal 2

Denoting this variation and the consequent variation of tensors by δ^{+} it follows since I is an invariant that

$$0 = \delta^{+} I = \delta^{+} \int_V \mathcal{L}\, d\tau \qquad 3$$

The variation will affect the limits of integration and the following lemma is needed.

Lemma For the variation (2) and with the $\mathfrak{L}$ of (1a) any function of x^α

$$\delta^+ I = \int_V \left[\delta^+ \mathfrak{L} + (\mathfrak{L}\xi^\alpha)_{,\alpha}\right] d\tau \qquad 4$$

Clearly $$\delta^+ I = \int_{\bar V} \bar{\mathfrak{L}}(\bar x)\, d\bar\tau - \int_V \mathfrak{L}(x) d\tau \quad \text{where} \quad \bar{\mathfrak{L}}(x) - \mathfrak{L}(x) = \delta^+ \mathfrak{L}(x) \qquad 5$$

By changing the variables of integration according to (2) the first integral becomes

$$\int_V \bar{\mathfrak{L}}(x^i + \xi^i)\, |\partial \bar x_\nu|\, d\tau = \int_V (\bar{\mathfrak{L}}(x) + \xi^\alpha \bar{\mathfrak{L}}_{,\alpha})(1 + \xi^\nu_{,\alpha})\, d\tau$$

$$= \int_V \left\{ \bar{\mathfrak{L}} + (\mathfrak{L}\xi^\alpha)_{,\alpha} \right\} d\tau$$

to the first order in ξ^i. Using this in (5) the result (4) follows at once. This simple lemma was published by Schrödinger but it seemed convenient to establish it here in detail.

Now use the lemma for the particular $\mathfrak{L}$, a scalar density, given by (1 b) Then by (3)

$$0 = \delta^+ I = \int_V \left\{ R_{ik}\, \delta^+ g^{ik} + g^{ik} \delta^+ R_{ik} + (g^{ik} R_{ik} \xi^\alpha)_{,\alpha} \right\} d\tau \qquad 6$$

It is convenient to consider the first two terms in the integrand separately.

(a) Since $$\delta g^{ik} = \sqrt{-g}\left(\delta g^{ik} - \tfrac{1}{2} g^{ik} g_{\mu\nu} \delta g^{\mu\nu}\right)$$ it follows that

$$R_{ik}\, \delta^+ g^{ik} = \mathfrak{T}_{ik}\, \delta^+ g^{ik} \qquad \text{where} \quad \mathfrak{T}_{ik} = \sqrt{-g}\left(R_{ik} - \tfrac{1}{2} g_{ik} g^{\mu\nu} R_{\mu\nu}\right) \qquad 7$$

From the tensor character of g^{ik} , $\delta^+ g^{ik}$ due to the variation (2) can easily be calculated and is, correct to the first order in ξ^i ,

$$\delta^+ g^{ik} = \xi^i_{;m}\, g^{mk} + \xi^k_{;m}\, g^{im} - g^{ik}_{;m}\, \xi^m$$

Using this result the first term in the integrand takes the form -

$$R_{ik}\, \delta^+ g^{ik} = \left(\Sigma_{ik}\, g^{mk} + \Sigma_{ki}\, g^{km}\right)\xi^i_{;m} + \Sigma_{ik}\, g^{ik}_{;m}\, \xi^m$$

$$= 2\left(\Sigma^m_\alpha\, \xi^\alpha\right)_{;m} - \left(2\Sigma^m_{\alpha,m} + \Sigma_{\mu\nu}\, g^{\mu\nu}{}_{;\alpha}\right)\xi^\alpha \qquad 7$$

where $\Sigma^m_\alpha = \frac{1}{2}\left(\Sigma_{\alpha k}\, g^{mk} + \Sigma_{k\alpha}\, g^{km}\right)$

(b) Using Palatini's formula for δR_{ik} in terms of $\delta\Gamma^i_{kl}$ -

$$\delta R_{ik} = -\left(\delta\Gamma^\alpha_{ik}\right)_{;\alpha} + \left(\delta\Gamma^\alpha_{i\alpha}\right)_{;k}$$

the second term can be written

$$g^{ik}\, \delta^+ R_{ik} = -\left[\, g^{ik}\left(\delta^+\Gamma^\alpha_{ik} - \delta^\alpha_k\, \delta^+\Gamma^\sigma_{i\sigma}\right)\right]_{;\alpha} + \left(g^{ik}{}_{;\alpha} - \delta^k_\alpha\, g^{im}{}_{;m}\right)\delta\Gamma^\alpha_{ik}$$

Now the $g-\Gamma$ relation (16) or (1.8) is equivalent to (1.5') for any variation of Γ^i_{kl} . Hence the last term here vanishes. For the first bracket note that for any vector density

$$\mathfrak{U}^\alpha_{;\alpha} = \mathfrak{U}^\alpha_{,\alpha}$$

Using therefore the notation $\Lambda^\alpha_{ik} = \Gamma^\alpha_{ik} - \delta^\alpha_k\, \Gamma^\sigma_{i\sigma}$ 8

one gets $g^{ik}\, \delta^+ R_{ik} = -\left(g^{ik}\, \delta^+ \Lambda^\alpha_{ik}\right)_{,\alpha}$ 9

<u>An explicit expression for $\delta^+\Lambda^\alpha_{ik}$</u>

It is first necessary to calculate $\delta^+\Gamma^\alpha_{ik}$, and to do this recall that Γ^α_{ik} transforms according to

$$\bar\Gamma^\mu_{ik}(\bar x) = \partial^{\bar\mu}_a\, \partial^b_{\bar i}\, \partial^c_{\bar k}\, \Gamma^a_{bc}(x) + \partial^{\bar\mu}_\sigma\, \partial^\sigma_{\bar i\bar k} \qquad \text{where } \partial^{\bar\mu}_\sigma = \frac{\partial \bar x^\mu}{\partial x^\sigma} \text{ etc.}$$

For the variation (2) and to the first order in

$$\bar{\Gamma}^{\mu}_{\iota\kappa}(\bar{x}) = \Gamma^{\mu}_{\iota\kappa}(x) + \xi^{\mu}_{,a}\Gamma^{a}_{\iota\kappa} - \xi^{b}_{,\iota}\Gamma^{\mu}_{b\kappa} - \xi^{c}_{,\kappa}\Gamma^{\mu}_{\iota c} - \xi^{\mu}_{,\iota\kappa}$$

and so $$\delta^{+}\Gamma^{\mu}_{\iota\kappa} = \xi^{\alpha}_{,\beta}\{\delta^{\mu}_{\alpha}\Gamma^{\beta}_{\iota\kappa} - \delta^{\beta}_{\iota}\Gamma^{\mu}_{\alpha\kappa} - \delta^{\beta}_{\kappa}\Gamma^{\mu}_{\iota\alpha}\} - \xi^{\mu}_{,\iota\kappa} - \Gamma^{\mu}_{\iota\kappa,\alpha}\xi^{\alpha}$$

From this $$\delta^{+}\Gamma^{\sigma}_{\iota\sigma} = \xi^{\alpha}_{,\beta}(-\delta^{\beta}_{\iota}\Gamma^{\sigma}_{\alpha\sigma}) - \xi^{\sigma}_{,\iota\sigma} - \Gamma^{\sigma}_{\iota\sigma,\alpha}\xi^{\alpha}$$

These last two results then give

$$\delta^{+}\Lambda^{\mu}_{\iota\kappa} = \xi^{\alpha}_{,\beta}\{\delta^{\mu}_{\alpha}\Gamma^{\beta}_{\iota\kappa} - \delta^{\beta}_{\iota}\Gamma^{\mu}_{\alpha\kappa} - \delta^{\beta}_{\kappa}\Gamma^{\mu}_{\iota\alpha} + \delta^{\beta}_{\iota}\delta^{\mu}_{\kappa}\Gamma^{\sigma}_{\alpha\sigma}\} - \xi^{\mu}_{,\iota\kappa} + \delta^{\mu}_{\kappa}\xi^{\sigma}_{,\iota\sigma} - \Lambda^{\mu}_{\iota\kappa,\alpha}\xi^{\alpha}$$

$$= \xi^{\alpha}_{,\beta}\{\delta^{\mu}_{\alpha}\Lambda^{\beta}_{\iota\kappa} - \delta^{\beta}_{\iota}\Lambda^{\mu}_{\alpha\kappa} - \delta^{\beta}_{\kappa}\Lambda^{\mu}_{\iota\alpha}\} - \Lambda^{\mu}_{\iota\kappa,\alpha}\xi^{\alpha} - (\xi^{\mu}_{,\iota\kappa} - \delta^{\mu}_{\kappa}\xi^{\sigma}_{,\iota\sigma}) \qquad (10)$$

and this is the required expression for $\delta^{+}\Lambda^{\mu}_{\iota\kappa}$ in terms of $\Lambda^{\alpha}_{\iota\kappa}$ defined by (8).

Using the results (7'), (9) and (10), (6) can now be written:-

$$0 = \delta^{+}I = \int_{V}\{-2(\Sigma^{\mu}_{a,\mu} + \tfrac{1}{2}\Sigma_{\mu\nu}g^{\mu\nu}{}_{,a})\xi^{a} + [(2\Sigma^{\mu}_{\alpha} + \delta^{\mu}_{\alpha}\Lambda + g^{\iota\kappa}\Lambda^{\mu}_{\iota\kappa,\alpha})\xi^{\alpha}$$

$$+ (g^{\beta\kappa}\Lambda^{\mu}_{\alpha\kappa} + g^{\mu\beta}\Lambda^{\mu}_{\kappa\alpha} - \delta^{\mu}_{\alpha}g^{\iota\kappa}\Lambda^{\beta}_{\iota\kappa})\xi^{\alpha}_{,\beta} + \xi^{\alpha}_{,\iota\beta}(g^{\iota\beta}\delta^{\mu}_{\alpha} - g^{\iota\mu}\delta^{\beta}_{\alpha})]_{,\mu}\}d\tau \qquad (11)$$

Consider the following cases -

<u>Case (i)</u> $\xi^{\iota} = 0$ on the boundary of V. The divergence terms, i.e. $[\cdots]_{,\mu}$ in (11) when transformed into a surface integral by Green's theorem now give a vanishing contribution. Since ξ^{ι} is arbitrary within the region V,

$$\Sigma^{\mu}_{a,\mu} + \tfrac{1}{2}\Sigma_{\mu\nu}g^{\mu\nu}{}_{,a} = 0 \qquad (12)$$

This is equivalent to the conservation law (1.18) although it should be noted that $\mathfrak{T}^{\mu}_{\nu}$ is not the same in the two equations. As written in (1.18) its tensor character is obvious.

Case (ii) ξ^{μ} = constant everywhere. In this case, and because (12) holds, (11) reduces to

$$\int_{V} \left(2\mathfrak{T}^{\mu}_{\alpha} + \delta^{\mu}_{\alpha}\mathfrak{L} + g^{\nu\kappa}\Lambda^{\mu}_{\nu\kappa,\alpha}\right)_{,\mu} \xi^{\alpha} d\tau = 0$$

or as ξ^{α} is an arbitrary constant and V is an arbitrary region

$$\left[\mathfrak{T}^{\mu}_{\alpha} + \tfrac{1}{2}(\delta^{\mu}_{\alpha}\mathfrak{L} + g^{\nu\kappa}\Lambda^{\mu}_{\nu\kappa,\alpha})\right]_{,\mu} = 0$$

or

$$(\mathfrak{T}^{\mu}_{\alpha} + \tau^{\mu}_{\alpha})_{,\mu} = 0 \qquad 13$$

where

$$\tau^{\mu}_{\alpha} = \tfrac{1}{2}(\delta^{\mu}_{\alpha}\mathfrak{L} + g^{\nu\kappa}\Lambda^{\mu}_{\nu\kappa,\alpha})$$

Case (iii) $\xi^{\alpha}_{,\beta}$ = constant everywhere. In this case because of (12) and (13), (11) reduces to

$$0 = \int \left\{ \xi^{\alpha}_{,\beta} \left[\left(2\mathfrak{T}^{\beta}_{\alpha} + \delta^{\beta}_{\alpha}\mathfrak{L} + g^{\nu\kappa}\Lambda^{\beta}_{\nu\kappa,\alpha}\right) + \left(g^{\beta\kappa}\Lambda^{\mu}_{\alpha\kappa} + g^{\kappa\beta}\Lambda^{\mu}_{\kappa\alpha} - \delta^{\mu}_{\alpha} g^{\nu\kappa}\Lambda^{\beta}_{\nu\kappa}\right)_{,\mu} \right] \right\} d\tau$$

But $\xi^{\alpha}_{,\beta}$ are arbitrary constants and V is an arbitrary region so that we must have

$$\mathfrak{T}^{\beta}_{\alpha} + \tau^{\beta}_{\alpha} = -\tfrac{1}{2}\left(g^{\beta\kappa}\Lambda^{\mu}_{\alpha\kappa} + g^{\kappa\beta}\Lambda^{\mu}_{\kappa\alpha} - \delta^{\mu}_{\alpha} g^{\nu\kappa}\Lambda^{\beta}_{\nu\kappa}\right)_{,\mu} \qquad 14$$

It is possible to proceed and obtain two futher identities from (11) but as neither of them is useful we shall not do so here. The identities (12), (13) and (14) together constitute 24 equations and they should reduce when the skew fields vanish to the corresponding well-known expressions of general relativity. While for (12) this is obviously true, (13) and (14) are written in unusual form

and need closer investigation. Indeed it is clear from (13) that T^{μ}_{α} does not reduce to the conventional pseudo energy-momentum-stress tensor of general relativity.† However, (13) and (14) can be transformed into conventional shape by using the relations:-

$$\mathcal{L} = \Lambda - g^{\iota\kappa}\Lambda^{\beta}_{\iota\kappa,\beta} \quad , \text{ where} \qquad \Lambda = g^{\iota\kappa}\left(\Gamma^{\beta}_{\iota\alpha}\Gamma^{\alpha}_{\beta\kappa} - \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\iota\kappa}\right) \quad (i)$$

15

and $$g^{\iota\kappa}{}_{,\beta}\,\Lambda^{\beta}_{\iota\kappa} = -2\Lambda \qquad (ii)$$

The first relation is self evident from the definitions of $\mathcal{L}$, $\Lambda^{\beta}_{\iota\kappa}$ and Λ while the second is established very simply as follows;

$$\begin{aligned} g^{\iota\kappa}{}_{,\beta}\,\Lambda^{\beta}_{\iota\kappa} &= g^{\iota\kappa}{}_{,\beta}\left(\Gamma^{\beta}_{\iota\kappa} - \delta^{\beta}_{\kappa}\Gamma^{\sigma}_{\iota\sigma}\right) \\ &= g^{\iota\kappa}{}_{,\beta}\left(\Gamma^{\beta}_{\iota\kappa} - \delta^{\beta}_{\kappa}\Gamma^{\sigma}_{\iota\sigma}\right) - g^{\iota\beta}{}_{,\beta}\,\Gamma^{\sigma}_{\iota\sigma} \end{aligned}$$

where we have used the symmetry of $\Gamma^{\beta}_{\iota\kappa}$. Now substitute for $g^{\iota\kappa}{}_{,\beta}$ from the $g-\Gamma$ relation 1.7 to give

$$\left(-\Gamma^{\iota}_{\sigma\beta}\,g^{\sigma\kappa} - \Gamma^{\kappa}_{\sigma\beta}\,g^{\iota\sigma} - \tfrac{1}{2}\delta^{\iota}_{\beta}\,1^{\kappa} - \tfrac{1}{3}\delta^{\kappa}_{\beta}\,1^{\iota}\right)\left(\Gamma^{\beta}_{\iota\kappa} - \delta^{\beta}_{\kappa}\Gamma^{\sigma}_{\iota\sigma}\right) - 1^{\iota}\Gamma^{\sigma}_{\iota\sigma}$$

$$= -2\Lambda$$

and this is the result quoted above.

Using these relations (15 i, ii) it is simplest not to transform (13) and (14) separately but to go back to the following terms which occur within the square brackets of (11):-

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†See PtI, (4.10) page 10

$$\delta^{\mu}_{\alpha}\,\mathcal{L}\,\xi^{\alpha} = (\Lambda - g^{ik}\Lambda^{\beta}_{ik,\beta})\xi^{\mu}$$

$$+ g^{ik}\Lambda^{\mu}_{ik,\alpha}\xi^{\alpha} = (g^{ik}\Lambda^{\mu}_{ik}\xi^{\beta})_{,\beta} - g^{ik}{}_{,\beta}\Lambda^{\mu}_{ik}\xi^{\beta} - g^{ik}\Lambda^{\mu}_{ik}\xi^{\alpha}_{,\alpha}$$

$$- g^{ik}\Lambda^{\beta}_{ik}\xi^{\mu}_{,\beta} = -(g^{ik}\Lambda^{\beta}_{ik}\xi^{\mu})_{,\beta} + g^{ik}{}_{,\beta}\Lambda^{\beta}_{ik}\xi^{\mu} + g^{ik}\Lambda^{\beta}_{ik,\beta}\xi^{\mu}$$

The second term in the first line cancels with the last in the third: the first term in each of the last two lines taken together are skew in the indices μ , β and so vanish under the combined x^{β} differentiation and the x^{μ} differentiation outside the square bracket in (11). Using (15 ii) for the second term of the last line one gets for the original three terms on the left,

$$-\Lambda\delta^{\mu}_{\alpha}\xi^{\alpha} - g^{ik}{}_{,\beta}\Lambda^{\mu}_{ik}\xi^{\beta} - g^{ik}\Lambda^{\mu}_{ik}\delta^{\beta}_{\alpha}\xi^{\alpha}_{,\beta}$$

Inserting this into (11) the square bracket terms become

$$[(2\mathfrak{T}^{\mu}_{\alpha} - \delta^{\mu}_{\alpha}\Lambda - g^{ik}{}_{,\alpha}\Lambda^{\mu}_{ik})\xi^{\alpha} + (g^{\beta k}\Lambda^{\mu}_{\alpha k} + g^{k\beta}\Lambda^{\mu}_{k\alpha} - g^{ik}\Lambda^{\mu}_{ik}\delta^{\beta}_{\alpha})\xi^{\alpha}_{,\beta}]$$

Following through the arguments of cases (ii) and (iii) for this modified form in (11) the identities corresponding to (13) and (14) are seen to be

$$(\mathfrak{T}^{\mu}_{\alpha} + t^{\mu}_{\alpha})_{,\mu} = 0 \qquad 13'$$

and
$$\mathfrak{T}^{\beta}_{\alpha} + t^{\beta}_{\alpha} = -\tfrac{1}{2}(g^{\beta k}\Lambda^{\mu}_{\alpha k} + g^{k\beta}\Lambda^{\mu}_{k\alpha} - \delta^{\beta}_{\alpha}\,g^{ik}\Lambda^{\mu}_{ik})_{,\mu} \qquad 14'$$

where
$$t^{\beta}_{\alpha} = -[\Lambda\delta^{\beta}_{\alpha} + g^{ik}{}_{,\alpha}\Lambda^{\beta}_{ik}] \qquad 13'\,a$$

These are conventional in form and reduce at once to the well known equations of general relativity when the skew field g^{ik} vanishes.

2.2 The Mass and Energy of a finite static system

With the help of these conservation identities and of the weak field solutions (1.46) of the field equations we can now obtain useful relations for the mass and energy of a spherically symmetric (in 3-space) finite static system. From (14'), integrating over a volume V in 3-space and using Green's theorem we get

$$\int_V (T^4_4 + t^4_4)\,dw = -\tfrac{1}{2}\int_V (g^{44}\Lambda^\mu_{44} + g^{4\nu}\Lambda^\mu_{4\nu} - g^{\sigma\nu}\Lambda^\mu_{\sigma\nu})_{,\mu}\,dw$$

$$= -\tfrac{1}{2}\int_S (g^{44}\Lambda^m_{44} + g^{4\nu}\Lambda^m_{4\nu} - g^{\sigma\nu}\Lambda^m_{\sigma\nu})\,n_m\,dS$$

where m takes the values 1,2,3 only and n_m is the outward normal defined in the Euclidian sense to the surface S bounding the three dimensional volume V. The time derivative on the right of the first line vanishes because the system is static. Provided the region V is taken sufficiently large we can use the weak field solutions (1.46) to evaluate the surface integral. When the skew field vanishes (16) reduces to the general relativity formula and the solutions (1.46) assume the form of the Schwarzschild solution. The presence of the skew field, i.e. $\epsilon \neq$ in (1.46), will add to the integrand of the surface integral various terms in r all of which either contain the exponential factor $e^{-\nu r}$ or are multiples of $\frac{1}{r^3}$. Hence in the limit for large r the

surface integral in (16) will simply reduce to the general relativity value of $-8\pi m$ where m is the parameter occurring in (1.46) and the Schwarzschild solution identified as the mass of the system. Thus we have shown that for a static, finite and spherically symmetric system

$$m = -\frac{1}{8\pi}\int (\mathfrak{T}_4^4 + t_4^4)\, d\omega \qquad 17$$

This result and (13') enables us to indentify the array $-\frac{1}{8\pi}(\mathfrak{T}_\beta^\alpha + t_\beta^\alpha)$ as the stress - energy - momentum of the field.

An Equivalent Expression for the mass in terms of τ_β^α

Instead of (16) consider the equation which follows from (14) for the same system i.e.

$$\int (\mathfrak{T}_4^4 + \tau_4^4)\, d\omega = -\frac{1}{2}\int \left\{ (g^{44}\Lambda^\mu_{4k} + g^{k4}\Lambda^\mu_{k4})_{,\mu} - (g^{ik}\Lambda^4_{ik})_{,4} \right\} d\omega$$
$$= -\frac{1}{2}\int (g^{44}\Lambda^m_{4k} + g^{k4}\Lambda^m_{k4})\, n_m\, dS \qquad 18$$

By the same arguments as were used in the preceding section we need only consider the contribution of the Schwarzschild line element to the surface integral. From the definition (8) for Λ^α_{ik} we have

$$\Lambda^\alpha_{ik} \longrightarrow \left\{{\alpha \atop ik}\right\} - \delta^\alpha_k \left\{{\sigma \atop i\sigma}\right\}$$

for large distances, where the curly brackets denote as usual the Christoffel symbols of the second kind derived from the line element. Since only the limit of the surface integral for large r is required it is sufficient to know these Christoffel symbols correct

to $\frac{1}{r^2}$ in r. The only contributing components are easily found to be (the indices are not summed)

$$\begin{Bmatrix} a \\ a\,a \end{Bmatrix} = -\frac{m}{r^2}\frac{x^a}{r} \qquad a \neq 4$$

$$\begin{Bmatrix} b \\ a\,a \end{Bmatrix} = \frac{m}{r^2}\frac{x^b}{r}$$

$$\begin{Bmatrix} a \\ b\,a \end{Bmatrix} = \pm\frac{m}{r^2}\frac{x^b}{r} \qquad \begin{cases} a = 4 \\ a \neq 4 \end{cases} \qquad b \neq 4$$

To give $\frac{1}{r^2}$ terms in the integrand it is sufficient to take

$$g^{45} = 0 \qquad g^{44} = 1 \qquad s = 1,2,3$$

With these values the surface integral becomes

$$-\tfrac{1}{2}\int 2g^{44}\Lambda^m_{44}\,n_m\,ds = -\int \begin{Bmatrix} m \\ 44 \end{Bmatrix}\frac{x^m}{r}\,r^2 d\omega = -4\pi m$$

Thus corresponding to (17) we have shown that for a finite, static and spherically symmetric system

$$m = -\frac{1}{4\pi}\int_V (T^4_4 + \tau^4_4)\,dv \qquad (19)$$

From this result and (13) we can deduce that $-\frac{1}{4\pi}(T^\beta_\alpha + \tau^\beta_\alpha)$ can also be considered as the stress - energy - momentum array for the field.

2.3 Some useful formulae

It is convenient to derive here some useful formulae. First from the definitions (7) and (7') for T^β_α we get,

$$T^\beta_\alpha = \tfrac{1}{2}\left(g^{\beta\mu}R_{\alpha\mu} + g^{\mu\beta}R_{\mu\alpha}\right) - \tfrac{1}{2}\delta^\beta_\alpha\, g^{\mu\nu}R_{\mu\nu}$$

Then using the field equations (1.25) for $R_{\mu\nu}$ one gets

$$R = g^{\mu\nu} R_{\mu\nu} = -\rho^2 (g^{\mu\nu} g_{\mu\nu} - g^{\mu\nu} b_{\mu\nu})$$
$$= -4\rho^2 (\sqrt{-g} - \sqrt{-b}) \qquad 20$$

and $$g^{\beta\mu} R_{\alpha\mu} + g^{\mu\beta} R_{\mu\alpha} = -\rho^2 (2\delta^\beta_\alpha \sqrt{-g} - g^{\beta\mu} b_{\alpha\mu} - g^{\mu\beta} b_{\mu\alpha})$$

or as is symmetric, $$= -2\rho^2 \delta^\beta_\alpha (\sqrt{-g} - \sqrt{-b})$$

Hence altogether $$\mathfrak{T}^\beta_\alpha = \rho^2 \delta^\beta_\alpha (\sqrt{-g} - \sqrt{-b}) = -\tfrac{1}{4} \delta^\beta_\alpha R \qquad 21$$

A further result of especial use in proving the next theorem concerns the behaviour of $\mathfrak{T}^\beta_\alpha + t^\beta_\alpha$ at large distances from a finite system. For weak fields - i.e. at large distances from a finite system - (21) takes the form found in (1.32[8]) and so

$$\mathfrak{T}^\beta_\alpha = -\tfrac{1}{8} \rho^2 \delta^\beta_\alpha f^{\mu\nu} f_{\nu\mu}$$

The weak field solution for $f_{\mu\nu}$ given in (1.46) then shows that

$$\mathfrak{T}^\beta_\alpha \sim \frac{1}{r^4} e^{-2\nu r}$$

at large distances. Now consider t^β_α which by (13' a) is given by

$$2 t^\beta_\alpha = -[\Lambda \delta^\beta_\alpha + g^{\mu\nu}{}_{\alpha} \Lambda^\beta_{\mu\nu}]$$

As already observed this reduces to the corresponding expression of general relativity when the skew field vanishes. Indeed when the skew or dextric field vanishes the whole theory reduces to the purely gravitational theory of general relativity. In this case

$$g^{\mu\nu} = 0 \qquad b_{\mu\nu} = \eta_{\mu\nu} + \tfrac{1}{r}\text{terms} \quad \text{for large } r$$

Since $\Lambda^\beta_{\mu\nu}$ is linear and Λ is of the second degree in the Christoffel

symbols of $g_{\mu\nu}$ (again only in the gravitational case) it follows that

$$t^{\beta}_{\alpha} \sim \frac{1}{r^4} \qquad \text{for large } r$$

From these observations we can see the behaviour of $\mathfrak{T}^{\beta}_{\alpha} + t^{\beta}_{\alpha}$ at large distances when the skew field does not vanish. The weak field solution (1.46) shows that it will differ from the gravitational value only by terms involving the exponential factor $e^{-\nu r}$. Consequently

$$t^{\beta}_{\alpha} \sim \frac{1}{r^4} \text{ or } r^n e^{-\nu r} \qquad (n \text{ finite})$$

and so

$$\mathfrak{T}^{\beta}_{\alpha} + t^{\beta}_{\alpha} \sim \frac{1}{r^4}, r^n e^{-\nu r} \qquad \text{when } r \text{ is large.}$$

We use this result to establish an important theorem.

2.4 The Laue Theorem.*

Consider the conservation law (13') in the form

$$\Sigma^{\alpha}_{\beta,\alpha} = 0 \qquad \Sigma^{\alpha}_{\beta} = \mathfrak{T}^{\alpha}_{\beta} + t^{\alpha}_{\beta}$$

It is evident from this that

$$(x^{\gamma}\,\Sigma^{\alpha}_{\beta})_{,\alpha} = \Sigma^{\gamma}_{\beta}$$

and so

$$\int_V \Sigma^{\gamma}_{\beta}\,dv = \int_V (x^{\gamma}\Sigma^{\alpha}_{\beta})_{,\alpha}\,dv + \int_V (x^{\gamma}\Sigma^{4}_{\beta})_{,4}\,dv$$

the integration being over a region in 3-space. Provided $\gamma \neq 4$ the last integral vanishes for a static system and so

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* The method used here closely follows the method developed by Papapetrou for the General Relativity - Maxwell theory in P.R.I.A., A51, 191, 1947.(35)

$$\int_V \Sigma^c_\beta \, dv = \int_V (x^c \Sigma^a_\beta)_{,a} \, dv \qquad a,c = 1,2,3$$

But it has been shown that Σ^α_β at large distances contains terms behaving like $\frac{1}{r^4}$ while the remainder contain the factor $e^{-\nu r}$. Transforming the divergence into a surface integral by Green's theorem it immediately follows that in the limit for large r

$$\int_V \Sigma^c_\beta \, dv = 0 \qquad (23)$$

This result is valid for a finite static system where the integration is taken over the whole of 3-space. It is the so-called Laue theorem.

2.5 Another form for the Mass of a static system

The expression (1) for the mass of a finite static system can be written in a form not involving the pseudo tensor t^β_α by the use of the preceding theorem. First observe that from (23), the Laue theorem,

$$\int_V (\mathfrak{T}^4_4 + t^4_4) \, dv = 2 \int_V (\mathfrak{T}^4_4 + t^4_4) \, dv - \int_V (\mathfrak{T}^\alpha_\alpha + t^\alpha_\alpha) \, dv \qquad (24)$$

Also from the definition (13' a) for t^β_α and using the identity (15 ii)

$$t^\alpha_\alpha = -2\Lambda - \tfrac{1}{2} g^{\mu\nu}{}_{,\beta} \Lambda^\beta_{,\mu\nu}$$

$$= -\Lambda$$

and

$$2 t^4_4 = -\Lambda$$

From these two results we have

$$2t^4_4 - t^\alpha_\alpha = 0$$

This result inserted in (24) then gives

$$\int_V (\mathfrak{T}^4_4 + \tau^4_4)\, dv = \int_V (2\mathfrak{T}^4_4 - \mathfrak{T}^\alpha_\alpha)\, dv \qquad 25$$

The range of validity of this expression is the same as that of the Laue theorem. Hence the mass of a finite system given by (17) is

$$m = -\frac{1}{8\pi}\int_V (\mathfrak{T}^4_4 + t^4_4)\, dv$$

$$= -\frac{1}{8\pi}\int_V (\mathfrak{T}^4_4 - \mathfrak{T}^1_1 - \mathfrak{T}^2_2 - \mathfrak{T}^3_3)\, dv \qquad 26$$

Like our other formulae this expression is of the same form as the similar one in general relativity to which of course it reduces when the skew fields vanish. In making these comparisons of (17) and (26) with general relativity it should be noted that the usual definition of $\mathfrak{T}^\beta_\alpha$ and t^β_α are the same as ours multiplied by a minus sign.

Using (21) for $\mathfrak{T}^\beta_\alpha$ (26) becomes

$$m = \frac{1}{4\pi}\rho^2 \int_V (\sqrt{-g} - \sqrt{-\delta})\, dv \qquad 27$$

The other volume integral expression for the mass i.e. (19) can be seen to give the same result by noting that for a static system (13) gives

$$\tau^4_4 = \tfrac{1}{2}\mathcal{L} + \tfrac{1}{2} g^{ik} \Lambda^4_{ik,4} = \tfrac{1}{2}\mathcal{L}$$

Using this, and the formula (21), (19) then reduces to

$$m = -\frac{1}{4\pi}\int_V (\mathfrak{T}^4_4 + \tfrac{1}{2}\mathcal{L})\, dv$$

$$= \frac{1}{4\pi}\rho^2 \int_V (\sqrt{-g} - \sqrt{-\delta})\, dv \qquad 28$$

This entirely independent evaluation for the mass of the system obtained directly from (19) serves as a useful check to the arguments linking the result (27) with (17).

According to (1.36) the integrand in this expression for the mass takes at large distances for the static spherically symmetric case the form-

$$-\tfrac{1}{4} f^{\alpha\beta} f_{\beta\alpha} = -\tfrac{1}{2} \mathcal{E}^2$$

where $\underline{\mathcal{E}}$ is the electrostatic field vector. Hence in regions where the field is small the contribution to the mass is

$$\Delta m = -\frac{1}{8\pi} \rho^2 \int_{\Delta V} \mathcal{E}^2 d\sigma \qquad (29)$$

Assuming that the correct sign has been chosen for ρ^2 this means that the electrostatic energy at regions where the field is small contributes negative mass to the system. This odd result will be discussed later.

Chapter 3

Solutions of the Field Equations for Static Spatially Spherically Symmetric Fields

3.1 Introduction

It is convenient first to write down the field equations (25) with the definitions (7) and (7')

$$R_{\mu\nu} \equiv G_{\mu\nu} + \tfrac{1}{2} z_\mu z_\nu = -\rho^2(g_{\mu\nu} - b_{\mu\nu})$$
$$6R_{\mu\nu} \equiv z_{\mu;\nu} - z_{\nu;\mu} = -6\rho^2 g_{\mu\nu} \qquad 1$$

where $G_{\mu\nu}$ is the Einstein tensor calculated from the Christoffel symbols of $b_{\mu\nu}$ itself defined in terms of $g_{\mu\nu}$ by

$$b^{\mu\nu} = g^{\mu\nu}, \quad b = |b^{\mu\nu}|, \quad b^{\mu\nu} = \frac{1}{\sqrt{-b}} b^{\mu\nu}, \quad b_{\mu\sigma} b^{\nu\sigma} = \delta^\nu_\mu \qquad 2$$

The vector z_α is defined by

$$z_\alpha = \frac{1}{\sqrt{-b}} b_{\alpha\beta} g^{\beta\sigma}_{\ \ ;\sigma}$$

It has been found that this theory has as its conservation laws or identity relations the 4 equations (18):-

$$\left(b^{\alpha\mu} R_{\beta\mu} - \tfrac{1}{2}\delta^\alpha_\beta b^{\mu\nu} R_{\mu\nu}\right)_{\|\alpha} = R_{\beta\mu} g^{\mu\nu}_{\ \ ;\nu} = -6\rho^2 g_{\beta\mu} g^{\mu\nu}_{\ \ ;\nu} \qquad 3$$

where the $\|$ differentiation is based on the Christoffel symbol of $b_{\mu\nu}$. All this suggested the interpretation

(i) $T^\alpha_\beta = b^{\alpha\mu} R_{\beta\mu} - \tfrac{1}{2}\delta^\alpha_\beta b^{\mu\nu} R_{\mu\nu}$ is the matter energy momentum tensor

(ii) $g^{\mu\nu}_{\ \ ;\nu}$ is the electric current vector density

(iii) $g_{\mu\nu}$ or $R_{\mu\nu}$ is the electromagnetic field tensor

} Lorentz force $\sim g_{\beta\mu} g^{\mu\nu}_{\ \ ;\nu}$

(iv) $b_{\mu\nu}$ is the gravitational tensor potential.

* See page 43, footnote

All this is very tentative. For instance, no evidence has been found which definitely shows that b_{ik} and not g_{ik} plays the role of gravitational potential.

After this brief summary of the proposed unified field theory the present purpose is to attempt to solve the field equations for a static spatially spherically symmetric field. In addition the boundary condition will be imposed - namely that both b_{ik} and g_{ik} approach the flat space metric at large distances. This requires in cartesians

$$b_{ik}, g_{ik} \to \eta_{ik} \qquad g_{ij} \to 0 \tag{4}$$

As is well known, and as will be shown the boundary condition when expressed in polar coordinates leads to an ambiguity, at least for a spherically symmetric field.

3.2 Preliminary calculations

Papapetrou[16] has shown that the general form for a second order tensor spherically symmetric in space and static in time is, in polar coordinates $x^\mu = (r, \theta, \varphi, ct)$

$$g_{\mu\nu} = \begin{pmatrix} -\alpha & 0 & 0 & \omega \\ 0 & -\beta & f\sin\theta & 0 \\ 0 & -f\sin\theta & -\beta\sin^2\theta & 0 \\ -\omega & 0 & 0 & \gamma \end{pmatrix} \tag{5}$$

Here $(g_{23}, g_{31}, g_{12}) = (H_1 H_2 H_3)$, the magnetic field components: $(g_{14}\, g_{24}\, g_{34}) = (E_1 E_2 E_3)$, the electric field components.

where α , β , γ , f and ω are all functions of r only. The boundary condition requires that $g_{\mu\nu}$ take the diagonal form of flat space for large r i.e.

$$g_{\mu\nu} \to (-1, -r^2, -r^2\sin^2\theta, 1) \qquad 6$$

Hence the components of $g_{\mu\nu}$ must satisfy the conditions

$$\alpha \to 1, \quad \gamma \to 1, \quad \beta \to r^2, \quad f \to 0, \quad \omega \to 0 \quad \text{as} \quad r \to \infty \qquad 7$$

The transformation to Cartesian coordinates will involve the symmetric and skew parts of $g_{\mu\nu}$ independently (both parts must be tensors). The skew part of (5) becomes in Cartesians

$$g_{[\mu\nu]} = \begin{pmatrix} 0 & \frac{f}{r^2}\frac{x}{r} & -\frac{f}{r^2}\frac{y}{r} & \omega\frac{z}{r} \\ & 0 & \frac{f}{r^2}\frac{z}{r} & \omega\frac{y}{r} \\ & & 0 & \omega\frac{x}{r} \\ & & & 0 \end{pmatrix}$$

The boundary condition applied to this requires

$$\frac{f}{r^2} \to 0 \qquad \omega \to 0 \qquad \text{as} \quad r \to \infty \qquad 8$$

These two different conditions for f given by (7) and (8) is the ambiguity mentioned above. In fact it will not effect the following since the present theory will be shown to involve $f = 0$ everywhere.

From (5)

$$g = |g_{\mu\nu}| = -(\alpha\gamma - \omega^2)(f^2 + \beta^2)\sin^2\theta, \quad \sqrt{-g} = \sin\theta\sqrt{(\alpha\gamma - \omega^2)(f^2 + \beta^2)} \qquad 9$$

and then defining g^{ik} by the relation

$$g_{k\sigma}\, g^{i\sigma} = \sqrt{-g}\, \delta^i_k = g_{\sigma k}\, g^{\sigma i}$$

it follows that

$$g^{ik} = \begin{pmatrix} -\gamma \sin\theta\, \Delta & 0 & 0 & -\omega \sin\theta\, \Delta \\ 0 & -\beta \sin\theta \Delta^{-1} & f\Delta^{-1} & 0 \\ 0 & -f\Delta^{-1} & -\beta \operatorname{cosec}\theta \Delta^{-1} & 0 \\ \omega \sin\theta\, \Delta & 0 & 0 & \alpha \sin\theta \Delta \end{pmatrix} \qquad (1)$$

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where $\Delta = \sqrt{\dfrac{f^2+\beta^2}{\alpha\gamma - \omega^2}}$ (11)

Consequently the symmetric part of g^{ik} has only its diagonal components non-zero. Thus according to (2) b^{ik} is given by

$$b^{ik} = g^{(ik)} = \{ -\gamma \sin\theta\, \Delta,\ -\beta \sin\theta\, \Delta^{-1},\ -\beta \operatorname{cosec}\theta\, \Delta^{-1},\ \alpha \sin\theta \Delta \}$$

and taking determinants

$$b = |b^{ik}| = -\alpha\gamma\beta^2 \sin^2\theta \qquad \text{and} \qquad \sqrt{-b} = \sqrt{\alpha\gamma}\ \beta \sin\theta$$

Then once more according to the definition (2)

$$b^{ik} = \frac{1}{\sqrt{-b}}\, b^{ik} = \{ -\frac{1}{\alpha D},\ -\frac{1}{\beta} D,\ -\frac{1}{\beta} D \operatorname{cosec}^2\theta,\ \frac{1}{\gamma D} \}$$

where

$$D = \frac{\beta}{\sqrt{\alpha\gamma}}\, \Delta^{-1} = \sqrt{\frac{\beta^2(\alpha\gamma - \omega^2)}{\alpha\gamma\,(f^2+\beta^2)}} \qquad 11$$

From this the components of b_{ik}, the inverse matrix, may be written down at once.

$$b_{ik} = \{ -\alpha D,\ -\beta D^{-1},\ -\beta D^{-1} \sin^2\theta,\ \gamma D \} \qquad 12$$

The second field equation 1 (ii)

So far the field equations themselves have not been used and the results (5) - (12) hold for any theory involving the two tensors

$b_{\mu\kappa}$ and $g_{\mu\kappa}$ related according to (2). Before the field equation (1 ii) can be considered z_μ must be calculated in terms of the components of $g_{\mu\kappa}$. To do this first note that from (10 i)

$$g^{\kappa\sigma}_{\nu,\sigma} = \{0, 0, 0, (\omega\Delta)' \sin\theta\} \qquad 13$$

where a dash here and in the following denotes differentiation with respect to the coordinate r . Hence from (2) using (12)

$$z_\mu = \frac{1}{\sqrt{-b}} b_{\mu\kappa} g^{\kappa\sigma}_{\nu,\sigma} = \left\{0, 0, 0, \frac{D}{\beta}\sqrt{\frac{\gamma}{\alpha}}(\omega\Delta)'\right\} \qquad 14$$

Since z_4 is the only non-zero component of z_μ , and since this component is a function only of the coordinate r , it follows from (2) that $g_{\mu\kappa}$ has only one non-zero component, namely g_{14} given by

$$z_{4,1} = 6\rho^2 g_{\mu\nu}$$

Inserting the components for $g_{\mu\kappa}$ given by (5) and the result (14) for z_μ this gives

$$f = 0 \qquad (1)$$

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$$\left\{\frac{D}{\beta}\sqrt{\frac{\gamma}{\alpha}}(\omega\Delta)'\right\}' = 6\rho^2\omega r \qquad (11)$$

Thus the field equation requires f to be zero everywhere and the ambiguity, mentioned above, between the boundary condition (7) and (8) is avoided. Physically the vanishing of f means, according to the interpretation, that a spherically symmetric magnetic field cannot exist - i.e. magnetic poles are inconsistent with the theory, at least in the static case.

A more convenient choice for the unknown functions

The field equations (1) involve the curvature tensor constructed from the Christoffel symbol of b_{ik} . From (12), or merely the requirement that it should be a static spatially spherically symmetric tensor, b_{ik} takes the form

$$b_{ik} = \{ -\alpha_0, -\beta_0, -\beta_0 \sin^2\theta, \gamma_0 \} \qquad 16$$

It is in fact far more convenient to work in terms of these components of b_{ik} than in the components (12) constructed from the functions α, β, γ and ω. They are all functions of r only, and $\alpha_0, \beta_0, \gamma_0$ are clearly related to $\alpha, \beta, \gamma, \omega$ by

$$\alpha_0 = \alpha D \qquad (i)$$

$$\beta_0 = \beta D^{-1} \qquad (ii) \qquad 17$$

$$\gamma_0 = \gamma D \qquad (iii)$$

The whole scheme takes symmetrical shape if ω_0 is defined by

$$\omega_0 = \omega D \qquad (iv)$$

for then by (11) [note $f = 0$ from (15)]

$$D = \sqrt{1 - \frac{\omega^2}{\beta\gamma}} = \sqrt{1 - \frac{\omega_0^2}{\alpha_0\beta_0}} = D_0 \qquad 18$$

From the boundary conditions (7) for α, γ and ω it follows at once that

$$D \to 1 \qquad \text{as} \qquad r \to \infty$$

and so $\alpha_0, \beta_0, \gamma_0$ and ω_0 must each satisfy the same boundary conditions as their counterparts α, β, γ and ω. Δ defined by (10 ii) may also be expressed in terms of the new functions,

$$\Delta = \frac{\beta}{\sqrt{\alpha\gamma}} D^{-1} = \frac{\beta_0}{\sqrt{\alpha_0\gamma_0}} D_0 \qquad 19$$

A further useful result then follows for τ_4 given by (14)

$$\tau_4 = \frac{D}{\beta}\sqrt{\frac{\gamma}{\alpha}}\,(\omega\Delta)' = \frac{1}{\alpha_0}\left(\frac{\beta_0}{\sqrt{\alpha_0\gamma_0}}\right)^{-1}\left(\omega_0\frac{\beta_0}{\sqrt{\alpha_0\gamma_0}}\right)' \qquad 20$$

The field equations (1) in terms of α_0 , β_0 , γ_0 and ω_0 .

Using the preceding results the field equations (1) may now be written

$$G_{11} = -\rho^2\alpha_0\,(1-D^{-1}) \qquad (i)$$

$$G_{22} = -\rho^2\beta_0\,(1-D) \qquad (ii) \qquad 21$$

$$G_{44} + \tfrac{1}{6}\tau_4\tau_4 = \rho^2\gamma_0\,(1-D^{-1}) \qquad (iii)$$

$$\tau_{4,1} = 6\rho^2\omega_0 D^{-1} \qquad (iv)$$

where D and τ_4 are given in terms of α_0 β_0 γ_0 ω_0 by (18) and (20) respectively, and where $G_{\mu\nu}$ is the curvature tensor calculated from (16). The equation involving G_{33} has been omitted since apart from a $\sin^2\theta$ factor it is the G_{22} equation.

The Christoffel symbol and $G_{\mu\nu}$ constructed from α_0 β_0 γ_0

The Christoffel symbols of $\Gamma_{\mu\nu}$ and the curvature tensor constructed from them assume the form well known in general relativity. The Christoffel symbol of the second kind has only nine non-zero components, namely -

$$\left\{\begin{matrix}1\\11\end{matrix}\right\} = \frac{\alpha_0'}{2\alpha_0} \qquad \left\{\begin{matrix}2\\12\end{matrix}\right\} = \frac{\beta_0'}{2\beta_0} \qquad \left\{\begin{matrix}3\\13\end{matrix}\right\} = \frac{\beta_0'}{2\beta_0} \qquad \left\{\begin{matrix}4\\14\end{matrix}\right\} = \frac{\gamma_0'}{2\gamma_0}$$

$$\left\{\begin{matrix}1\\22\end{matrix}\right\} = -\frac{\beta_0'}{2\alpha_0} \qquad \left\{\begin{matrix}3\\23\end{matrix}\right\} = \cot\theta$$

$$\left\{\begin{matrix}1\\33\end{matrix}\right\} = -\frac{\beta_0'}{2\alpha_0}\sin^2\theta \qquad \left\{\begin{matrix}2\\33\end{matrix}\right\} = -\sin\theta\cos\theta$$

$$\left\{\begin{matrix}1\\44\end{matrix}\right\} = \frac{\gamma_0'}{2\gamma_0}$$

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The curvature tensor from $\left\{\begin{matrix}\alpha\\ik\end{matrix}\right\}$ is according to (1.1)

$$G_{ik} = -\left\{\begin{matrix}\alpha\\ik\end{matrix}\right\}_{,\alpha} + \left\{\begin{matrix}\alpha\\i\alpha\end{matrix}\right\}_{,k} + \left\{\begin{matrix}\alpha\\i\beta\end{matrix}\right\}\left\{\begin{matrix}\beta\\\alpha k\end{matrix}\right\} - \left\{\begin{matrix}\beta\\\alpha\beta\end{matrix}\right\}\left\{\begin{matrix}\alpha\\ik\end{matrix}\right\} \qquad 23$$

Using (22) the components of G_{ik} are calculated in a straightforward manner. Only the diagonal elements are non-zero, and they are

$$G_{11} = \tfrac{1}{2}\left(\frac{\gamma_0'}{\gamma_0}\right)' + \left(\frac{\beta_0'}{\beta_0}\right)' + \tfrac{1}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2 + \tfrac{1}{4}\left(\frac{\gamma_0'}{\gamma_0}\right)^2 - \frac{\alpha_0'}{2\alpha_0}\left(\frac{\beta_0'}{\beta_0} + \frac{\gamma_0'}{\gamma_0}\right)$$

$$G_{22} = \tfrac{1}{2}\left(\frac{\beta_0'}{\alpha_0}\right)' - 1 + \frac{1}{4}\frac{\beta_0'}{\alpha_0}\left(\frac{\alpha_0'}{\alpha_0} + \frac{\gamma_0'}{\gamma_0}\right) \qquad 24$$

$$G_{33} = \sin^2\theta\, G_{22}$$

$$G_{44} = -\tfrac{1}{2}\left(\frac{\gamma_0'}{\alpha_0}\right)' - \frac{\gamma_0'}{4\alpha_0}\left(\frac{\alpha_0'}{\alpha_0} + \frac{2\beta_0'}{\beta_0} - \frac{\gamma_0'}{\gamma_0}\right)$$

In gravitational theory β_0 is generally replaced by its asymptotic form r^2. There is, of course, no need for this and in the following β_0 will occur as an arbitrary function in the solutions for α_0, γ_0 and ω_0. Now make the substitution

$$\alpha_0 = e^{\mu}, \qquad \beta_0 = e^{\rho}, \qquad \gamma_0 = e^{\nu} \qquad 25$$

(24) then becomes,

$$G_{11} = \tfrac{1}{2}\nu'' + \tfrac{1}{4}\nu'(\nu'-\mu') + \rho'' + \tfrac{1}{2}\rho'(\rho'-\mu')$$

$$G_{22} = e^{\rho-\mu}\left[\tfrac{1}{2}\rho'' + \tfrac{1}{4}\rho'(2\rho'-\mu'+\nu')\right] - 1$$

$$G_{33} = e^{\nu-\mu}\left[-\tfrac{1}{2}\nu'' - \tfrac{1}{4}\nu'(\nu'-\mu'+2\rho')\right] \qquad 26$$

3.3 The solution of the field equations

With the help of (26) the first three field equations of (21) may be rewritten

$$\tfrac{1}{2}\nu'' + \tfrac{1}{4}\nu'(\nu'-\mu') + \rho'' + \tfrac{1}{2}\rho'(\rho'-\mu') = -\rho^2 e^{\mu}(1-\mathbb{D}^{-1}) \qquad (1)$$

$$\tfrac{1}{2}\rho'' + \tfrac{1}{4}\rho'(2\rho'-\mu'+\nu') - e^{\mu-\rho} = -\rho^2 e^{\mu}(1-\mathbb{D}) \qquad (11) \qquad 27$$

$$-\tfrac{1}{2}\nu'' - \tfrac{1}{4}\nu'(\nu'-\mu'+2\rho') + \tfrac{1}{6}z_4 z_4 e^{\mu-\nu} = \rho^2 e^{\mu}(1-\mathbb{D}^{-1}) \qquad (111)$$

In addition there is the equation for z_4 ,

$$z_{4,1} = 6\rho^2 \omega \qquad 28$$

or writing $\omega = \varphi'$, $\qquad z_4 = 6\rho^2\varphi$

when the arbitrary constant is absorbed into φ . According to the interpretation of the theory z_4 plays the role of electrostatic potential and is also closely linked with the current four-vector $g^{\sigma}{}_{,\sigma}$. Indeed as has been seen in the weak field approximations z_4 closely approaches the charge density for large distances. Evidently z_4 - and hence φ - must approach zero as r becomes large. Furthermore it has been found from the discussion on the weak field solution that ρ^2 must be very small if the theory and its interpretation is to be in agreement with observation. Consequently it is reasonable to expect z_4 to be everywhere small $O(\rho^2)$ except perhaps in a very small region surrounding the origin. The smallness of the ρ^2 factor will nevertheless allow ω , the radial component of the electric field, to be quite large (say ~ 1)

So far it has not proved possible to obtain exact solutions of (27) and (28). The difficulty hinges on the γ_4^2 term in (iii) which from the preceding is $O(\rho^4)$. However, progress is possible if one works to the first degree only in ρ^2. In this way the γ_4^2 term in (iii) can be neglected. Furthermore since ω occurs in (27) only through the D terms which already have a ρ^2 factor, it will be sufficient to solve (28) correct to $O(1)$ in ρ^2. For the moment this approximation procedure will be adopted without further discussion.

First add (i) and (iii) to give,

$$\rho'' + \tfrac{1}{2}\rho'(\rho' - \mu' - \nu') = 0 \qquad 29$$

or integrating, $\rho + 2\log\rho' = \mu + \nu + \text{const.}$

and then using (25), $\alpha_0\gamma_0 = e^{\mu+\nu} = c(\rho')^2 e^{\rho} = c\dfrac{(\beta_0')^2}{\beta_0}$

The constant c can now be determined by the boundary conditions (7) - they give $c = \frac{1}{4}$. Hence

$$e^{\mu+\nu} = \alpha_0\gamma_0 = \frac{1}{4\beta_0}(\beta_0')^2 = \frac{1}{4}(\rho')^2 e^{\rho} \qquad 30$$

This expression reduces, of course, when $\beta_0 = r^2$ to $\alpha_0\gamma_0 = 1$, but the more general case (30) is more useful. Now use (29) to eliminate ν' from (27 ii) to give an equation in ρ, μ and β only. The result when multiplied by $e^{\rho-\mu}$ is

$$e^{\rho-\mu}\left[\rho' + \tfrac{1}{4}\rho'(3\rho' - 2\mu')\right] = 1 - \rho^2 e^{\rho}(1 - D)$$

When this is further multiplied by $\rho' e^{\frac{1}{2}\rho}$,

$$\left[\tfrac{1}{2}(\rho')^2 e^{\frac{3}{2}\rho-\mu}\right]' = \rho' e^{\frac{1}{2}\rho} - \rho^2 e^{\frac{3}{2}\rho}(1-D)\rho'$$

and integrating

$$e^{-\mu} = \frac{4}{(\rho')^2} e^{-\rho} - \rho^2 \frac{2e^{-\frac{3}{2}\rho}}{(\rho')^2} \int^{\rho} e^{\frac{3}{2}\rho}(1-D)\,d\rho$$

Regarding D as a function of β_0 this may be written in terms of β_0 by using (25) to give

$$\frac{1}{\alpha_0} = e^{-\mu} = \frac{4\beta_0}{(\beta_0')^2}\left\{1 - \rho^2 \frac{1}{2\beta_0^{\frac{1}{2}}}\int^{\beta_0} \beta_0^{\frac{1}{2}}(1-D)\,d\beta_0\right\} \qquad 31$$

In all this an arbitrary constant is contained in the lower limit of integration. From (30) the expression for γ_0 or e^{ν} can be written at once as

$$\gamma_0 = e^{\nu} = \frac{\beta_0')^2}{4\beta_0} e^{-\mu} = 1 - \rho^2 \frac{1}{2\beta_0^{\frac{1}{2}}}\int^{\beta_0} \beta_0^{\frac{1}{2}}(1-D)\,d\beta_0 \qquad 32$$

Both these expressions, (31) for $\frac{1}{\alpha_0}$ and (32) for γ_0 , contain D which according to (18) is a function of β_0 and ω_0 . It remains therefore to determine ω_0 as a function of β_0 correct to $O(1)$ in ρ^2 from the remaining field equation (26) before α_0 and γ_0 are given explicitly as functions of β_0 correct to $O(\rho^2)$. To the approximation required (2') is

$$z_{4,1} = 0 \quad \text{or} \quad z_4 = l \quad \text{(constant)}$$

i.e. using (20) for z_4 ,

$$\frac{1}{\alpha_0}\left(\frac{\beta_0}{\sqrt{\alpha_0\gamma_0}}\right)^{-1}\left(\omega_0 \frac{\beta_0}{\sqrt{\alpha_0\gamma_0}}\right)' = l \qquad 33$$

Either by applying the boundary conditions direct to ζ_4 or to the last expression in α_0, β_0 and γ_0 it is easy to see that ℓ must be zero. In fact the asymptotic form of (33) is

$$\frac{1}{r^2}(\omega_0 r^2)' = \ell \qquad \text{or} \qquad \omega_0 = \tfrac{1}{3}\ell r + \frac{\text{const}}{r^2}$$

Clearly in order that ω_0 may itself satisfy the boundary condition $\omega_0 \to 0$, as $r \to \infty$, ℓ must be zero. Hence (33) reduces to

$$\omega_0 = k^2 \frac{\sqrt{\alpha_0\gamma_0}}{\beta_0}$$

from which it follows that

$$\mathcal{D} = \sqrt{1 - \frac{\omega_0^2}{\alpha_0\gamma_0}} = \sqrt{1 - k^4/\beta_0^2} \qquad 34$$

where k^2 is an arbitrary constant, not necessarily positive.

Finally using (30) for $\alpha_0\gamma_0$ the expression for ω is

$$\omega = \frac{\omega_0}{\mathcal{D}} = \frac{\beta_0'}{2\beta_0'^2} \frac{k^2}{(\beta_0^2 - k^4)^{\frac{1}{2}}} \qquad 35$$

Equations (31) and (32) with (34) for $\mathcal{D}$ give the required expressions for α_0 and γ_0 as functions of β_0 correct to $O(\rho^4)$. However not all the field equations (27) have been fully used in their derivation - the second one has, but the first and last entered only through their sum. Thus it is necessary to show that the solutions do indeed satisfy one or other of these latter equations taken alone. It is simplest to consider the last equation (27 iii) which to the approximation taken is

$$-\tfrac{1}{2}\nu'' - \tfrac{1}{4}\nu'(\nu' - \mu' + 2\rho') = \rho^2 e^{\mu}(1 - \mathcal{D}^{-1})$$

This becomes after multiplication by $e^{\frac{1}{2}(\nu-\mu)+\rho}$

$$-\tfrac{1}{2}\left[\nu' e^{\frac{1}{2}(\nu-\mu)+\rho}\right]' = p^2 e^{\frac{1}{2}(\mu+\nu)+\rho}(1-\mathfrak{D}^{-1})$$

Now use (30) for $e^{\frac{1}{2}(\mu+\nu)}$ on the right,

$$\left[\nu' e^{\frac{1}{2}(\nu-\mu)+\rho}\right]' = -p^2 \rho' e^{\frac{3}{2}\rho}(1-\mathfrak{D}^{-1})$$

Introducing β_0 instead of e^{ρ}, integrating and then multiplying by $e^{\frac{1}{2}(\nu+\mu)}$ as given by (30) one gets

$$\nu' e^{\nu} = -p^2 \frac{\beta_0'}{2\beta_0^{\frac{3}{2}}} \int^{\beta_0} \beta_0^{\frac{1}{2}}(1-\mathfrak{D}^{-1})\, d\beta_0 \qquad 36$$

Taking that part of the integrand involving $\mathfrak{D}^{-1}$ and transforming by partial integration the following results:-

$$\int^{\beta_0} \frac{\beta^{\frac{1}{2}}}{(1-k^4/\beta^2)^{\frac{1}{2}}}\, d\beta = \int^{\beta_0} \frac{\beta_0 \beta_0^{\frac{1}{2}}}{(\beta_0^2-k^4)^{\frac{1}{2}}}\, d\beta_0$$

$$= \beta_0^{\frac{1}{2}}(\beta_0^2-k^4) - \tfrac{1}{2}\int^{\beta_0} \beta_0^{-\frac{1}{2}}(\beta_0^2-k^4)^{\frac{1}{2}}\, d\beta_0$$

$$= \beta_0^{\frac{3}{2}}\mathfrak{D} - \tfrac{1}{2}\int^{\beta_0} \beta_0^{\frac{1}{2}}\mathfrak{D}\, d\beta_0$$

Using this result (36) may be written

$$\nu' e^{\nu} = -p^2 \frac{\beta_0'}{2\beta_0^{\frac{3}{2}}} \int^{\beta_0} \tfrac{1}{2}\beta_0^{\frac{1}{2}}\mathfrak{D}\, d\beta_0 + \tfrac{1}{2}\beta_0' p^2 \mathfrak{D} - \tfrac{1}{3}p^2\beta_0'$$

$$= p^2 \frac{\beta_0'}{4\beta_0^{\frac{3}{2}}} \int^{\beta_0} \beta_0^{\frac{1}{2}}(1-\mathfrak{D})\, d\beta_0 - p^2 \cdot \tfrac{1}{2}\beta_0'(1-\mathfrak{D})$$

This is immediately seen to be in complete agreement with (32) and so (31) and (32) with (34) are the required solutions of the complete set of field equations.

The solutions (31), (32), (34) and (35) can now be written together as,

$$\gamma_0 = 1 - p^2 \frac{1}{2\beta_0^{\frac{1}{2}}} \int^{\beta_0} \left[x^{\frac{1}{2}} - x^{-\frac{1}{2}} (x^2 - k^4)^{\frac{1}{2}} \right] dx \qquad (i)$$

$$\frac{1}{\alpha_0} = \frac{4\beta_0}{(\beta_0')^2} \gamma_0 \qquad (ii)$$

$$\omega = \frac{\beta_0'}{2\beta_0^{\frac{1}{2}}} \frac{k^2}{(\beta_0^2 - k^4)^{\frac{1}{2}}} \qquad (iii)$$

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Also from the relation (17), using (34) for D

$$\gamma = \left(1 - \frac{k^4}{\beta_0^2}\right)^{-\frac{1}{2}} \gamma_0 \qquad (iv)$$

$$\frac{1}{\alpha} = \left(1 - \frac{k^4}{\beta_0^2}\right)^{\frac{1}{2}} \frac{1}{\alpha_0} \qquad (v)$$

$$\beta = (\beta_0^2 - k^4)^{\frac{1}{2}} \qquad (vi)$$

In this way solutions of the field equations have been found correct to terms in p^2 expressed as functions of β_0 . It is interesting and important to write down the same solutions as functions of β given by (vi) instead of β_0 . This is important since as yet there is no criterion to decide which of $g_{\mu\nu}(\alpha, \beta, \delta)$ and $h_{\mu\nu}(\alpha_0 \beta_0 \gamma_0)$ is the more fundamental tensor although the conservation law (18) does seem to indicate that it is $h_{\mu\nu}$ which plays the role of gravitational potential. In particular we wish to know which of the two possible choices for the functions β and β_0 , namely

$$\beta = r^2 \qquad \beta_0 = (r^4 + k^4)^{\frac{1}{2}} \qquad (a)$$

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$$\beta = (r^4 - k^4)^{\frac{1}{2}} \qquad \beta_0 = r^2 \qquad (b)$$

is most convenient. To express the solutions (37) in terms of

first note that

$$\beta_0 = (\beta^2 + k^4)^{\frac{1}{2}}, \qquad D = \left(1 - \frac{k^4}{\beta_0^2}\right)^{\frac{1}{2}} = \left(1 + \frac{k^4}{\beta^2}\right)^{-\frac{1}{2}}, \qquad \beta_0' = \frac{\beta\beta'}{(\beta^2 + k^4)^{\frac{1}{2}}} \tag{39}$$

Using these relations it is easy to see that (37) becomes-

$$\gamma_0 = 1 - p^2 \cdot \frac{1}{2\beta^{\frac{1}{2}}} \left(1 + \frac{k^4}{\beta^2}\right)^{-\frac{1}{4}} \int^{(\beta^2+k^4)^{\frac{1}{2}}} [x^{\frac{1}{2}} - x^{-\frac{1}{2}}(x^2 - k^4)^{\frac{1}{2}}]\,dx$$

$$\frac{1}{\alpha_0} = \frac{4\beta}{(\beta')^2}\left(1 + \frac{k^4}{\beta^2}\right)^{\frac{3}{2}}\gamma_0$$

$$\omega = \frac{k^2\beta'}{2(\beta^2 + k^4)^{\frac{3}{4}}} \tag{40}$$

$$\gamma = \left(1 + \frac{k^4}{\beta^2}\right)^{\frac{1}{2}}\gamma_0$$

$$\frac{1}{\alpha} = \frac{4\beta}{(\beta')^2}\left(1 + \frac{k^4}{\beta^2}\right)\gamma_0$$

From (37) and (40) it can now be seen that the choice (38 b), $\beta_0 = r^2$, leads to a singularity of the solutions when $r = k$ whereas no singularity arises from the choice (38 a) i.e. $\beta = r^2$. At this point it should again be remembered that the field equations were solved on the basis that p^2 was so small that its square could be neglected. This basis arose because p^{-1} was found to be of the same order of magnitude as the range of the electromagnetic field. Now the neglect of terms containing p^4 and higher powers will only seriously be in error when the field is either very large or small. The latter case is covered by the weak field solutions (1.46). As we have seen the choice $\beta_0 = r^2$ introduces a singularity into the solutions, limiting their range of validity to $r > k$. The avoidance of this singularity by the choice $\beta = r^2$, on the other hand, leads to no such limitations. On these grounds

we make the choice $\beta = r^2$, i.e. (38 a), and write the solutions as functions of r from (40):-

$$\gamma_0 = 1 - p^2 \frac{|k|^3}{r} \left(1 + \frac{k^4}{r^4}\right)^{-\frac{1}{4}} \overline{J}(r) \qquad \text{(i)}$$

$$\frac{1}{\alpha_0} = \left(1 + \frac{k^4}{r^4}\right)^{\frac{3}{2}} \gamma_0 \qquad \text{(ii)}$$

$$w = \frac{k^2 r}{(r^4 + k^4)^{\frac{3}{4}}} \qquad \text{(iii)}$$

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$$\gamma = \left(1 + \frac{k^4}{r^4}\right)^{\frac{1}{2}} \gamma_0 \qquad \text{(iv)}$$

$$\frac{1}{\alpha} = \left(1 + \frac{k^4}{r^4}\right) \gamma_0 \qquad \text{(v)}$$

where

$$\overline{J}(r) = \tfrac{1}{2} \int^{\left(\frac{r^4}{k^4}+1\right)^{\frac{1}{2}}} \left[y^{\frac{1}{2}} - y^{-\frac{1}{2}} (y^2-1)^{\frac{1}{2}}\right] dy = \int^{\frac{r}{|k|}} \frac{(z^4+1)^{\frac{1}{2}} - z^2}{(z^4+1)^{\frac{3}{4}}} z^3 dz \qquad \text{(vi)}$$

$|k|$ occurs in (i) and (vi) because we shall find it necessary to allow k to be imaginary (not complex) and γ_0 is clearly seen in (39) to be dependent only upon $|k|$.

A full discussion of these solutions and their significance is delayed until after the next chapter in which the problem of motion of an n-particle system will be studied.

Chapter 4

The Equations of Motion of an n-particle System

4.1 The Method

In this chapter we shall derive the equations of motion of a charged particle from the field equations (1.25). The method used is a simple extension of that developed in a number of papers by Einstein, Infeld and Hoffmann[20,21] for the theory of general relativity. When possible we shall try to adhere to the notation used in the last paper of the series, namely that of Einstein and Infeld (1949).

Consider the field equations (1.25) not in their original form but with $\frac{1}{2}\eta_{\mu\kappa} \times$ their trace subtracted. These field equations will involve the field quantities $b_{\mu\kappa}$ and $f_{\mu\kappa}$ in a complicated manner but at least we know that the terms linear in $b_{\mu\kappa}$ and $f_{\mu\kappa}$ are given by the left hand sides of the equations (35 a, b). Separating these linear terms from the rest and using the notation,

$$b_{\mu\kappa} = \eta_{\mu\kappa} + \beta_{\mu\kappa} \quad , \qquad \bar{\beta}_{\mu\kappa} = \beta_{\mu\kappa} - \tfrac{1}{2}\eta_{\mu\kappa}\eta^{\rho\sigma}\beta_{\rho\sigma}$$

the field equations may be written as,

$$\tfrac{1}{2}\eta^{\alpha\beta}\left[-\bar{\beta}_{\beta\mu,\alpha\nu} - \bar{\beta}_{\beta\nu,\alpha\mu} + \bar{\beta}_{\mu\nu,\alpha\beta} + \eta_{\mu\nu}\eta^{\rho\sigma}\bar{\beta}_{\beta\rho,\alpha\sigma}\right] \qquad \text{(a)}$$

$$= -\tfrac{1}{6}\left(\chi_\mu\chi_\nu - \tfrac{1}{4}\eta_{\mu\nu}\chi^\alpha\chi_\alpha\right) - p^2\bar{T}_{\mu\nu} - \Lambda'_{\mu\nu} \qquad 1$$

$$\eta^{\alpha\beta}\left(f_{\mu\alpha,\nu\beta} - f_{\nu\alpha,\mu\beta}\right) = -6p^2 f_{\mu\nu} + F_{\mu\nu} \qquad \text{(b)}$$

where $T_{\mu\nu} = t_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}t_{\rho\sigma}$, $t_{\mu\nu} = g_{\underline{\mu\nu}} - b_{\mu\nu}$

$\Lambda'_{\mu\nu}$ are the non-linear terms involving $\bar{\beta}_{\mu\nu}$ alone.

$F_{\mu\nu}$ are non linear terms involving both $\bar{\beta}_{\mu\nu}$ and $f_{\mu\nu}$.

Roughly speaking the method consists in adopting an approximation procedure for solving these equations in successive stages. For an n -particle system restrictions on the motion of the particles are found to arise. Since in the procedure the ' o ' index will be distinguished from the others we shall need (1 a) written as the three equations

$$\bar{\Phi}_{oo} + 2\Lambda''_{oo} = 0 \qquad (1)$$

$$\bar{\Phi}_{om} + 2\Lambda''_{om} = 0 \qquad (11)$$

$$\bar{\Phi}_{mn} + 2\Lambda''_{mn} = 0 \qquad (111)$$

where it is convenient to split the terms as follows:

$$\bar{\Phi}_{oo} = -\bar{\beta}_{oo|ss}$$

$$\bar{\Phi}_{om} = -\bar{\beta}_{om|ss} + \bar{\beta}_{os|sm}$$

$$\bar{\Phi}_{mn} = -\bar{\beta}_{mn|ss} + \bar{\beta}_{ms|ns} + \bar{\beta}_{ns|sm} - \delta_{mn}\bar{\beta}_{rs|rs}$$

and

$$2\Lambda''_{oo} = \bar{\beta}_{sr|sr} + 2\Lambda'_{oo} + \tfrac{1}{3}\bar{I}_{oo} + 2\rho^2\bar{T}_{oo}$$

$$2\Lambda''_{om} = \bar{\beta}_{ms|so} - \bar{\beta}_{oo|mo} + 2\Lambda'_{om} + \tfrac{1}{3}\bar{I}_{om} + 2\rho^2\bar{T}_{om}$$

$$2\Lambda''_{mn} = -\bar{\beta}_{om|on} - \bar{\beta}_{on|om} + 2\delta_{mn}\bar{\beta}_{os|os} + \bar{\beta}_{mn|oo} - \delta_{mn}\bar{\beta}_{oo|oo} + 2\Lambda'_{mn} + \tfrac{1}{3}\bar{I}_{mn} + 2\rho^2\bar{T}_{mn}$$

or

$$2\Lambda''_{\mu\nu} = 2\Lambda_{\mu\nu} + \tfrac{1}{3}\bar{I}_{\mu\nu} + 2\rho^2\bar{T}_{\mu\nu}$$

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where $\Lambda''_{\mu\nu}$ is the quantity corresponding to $\Lambda_{\mu\nu}$ in general relativity theory. $\Lambda''_{\mu\nu}$ reduces to $\Lambda_{\mu\nu}$ when the skew field, and therefore $I_{\mu\nu}$, $T_{\mu\nu}$, vanish. Here we are again using Latin letters for the index values 1, 2, 3 only. The bar has been used instead of the comma to denote differentiation solely for future convenience.

With the terms divided in this way the field equations (1 a) are

$$\Phi_{\mu\nu} + 2\Lambda''_{\mu\nu} = 0 \qquad 4$$

From the structure of $\Phi_{\mu k}$ it follows that

$$\Phi_{\mu k|k} = 0 \qquad \int_S \Phi_{\mu k} n_k \, dS = 0 \qquad 5$$

where n_k is the outward normal, defined as in Euclidean geometry, to the closed surface S in 3 space. The surface S is here quite an arbitrary one. Because of the field equations (4) $\Lambda''_{\mu k}$ must satisfy similar conditions to (5) i.e.

$$\Lambda''_{\mu k|k} = 0 \qquad \int_S \Lambda''_{\mu k} n_k \, dS = 0 \qquad 6$$

It is found that the condition for the vanishing of the surface integral is identically satisfied unless the surface surrounds a singularity of the field. In that case, and for the purely gravitational theory, Einstein and Infeld found that the vanishing of the surface integral determined the equations of motion of the singularity or particle. By Green's theorem, the first condition in (6) means that the value of the surface integral is independent of the particular surface surrounding the singularity.

4.2 The approximation procedure

The field equations, and (6), are of course far too difficult to solve exactly for a general system. Instead we require an approximation procedure similar to that used by Einstein and Infeld. Write

$$g_{ik} = \eta_{ik} + \gamma_{ik} \quad , \quad g^{ik} = a^{ik} \quad \gamma^{ik} = f^{ik} \qquad 7$$

and consider $\bar{\gamma}_{\mu\nu}$ defined in terms of $\gamma_{\mu\nu}$ by

$$\bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\gamma_{\rho\sigma} \qquad 8$$

We suppose that $\bar{\gamma}_{\mu\nu}$ can be expanded in terms of a parameter λ according to:-

$$\bar{\gamma}_{00} = \lambda^2 \underset{2}{\bar{\gamma}}_{00} + \lambda^4 \underset{4}{\bar{\gamma}}_{00} + \cdots \qquad 9$$

$$\bar{\gamma}_{0m} = \lambda^2 \underset{2}{\bar{\gamma}}_{0m} + \lambda^3 \underset{3}{\bar{\gamma}}_{0m} + \lambda^4 \underset{4}{\bar{\gamma}}_{0m} + \cdots$$ odd powers symmetric, even skew.

$$\bar{\gamma}_{mn} = \lambda^3 \underset{3}{\bar{\gamma}}_{mn} + \lambda^4 \underset{4}{\bar{\gamma}}_{mn} + \cdots$$ even powers symmetric, odd skew.

This parameter λ is related to $\frac{v}{c}$, the ratio of the velocity of the particles in the field to the velocity of light. The expansion chosen here is in agreement for the symmetrical components with the expansion used by Einstein and Infeld for the gravitational theory. The expansion for the skew components has been chosen in the light of the interpretation of the present theory whereby

$f_{mn} = \underset{1}{\bar{\gamma}}_{mn}$ constitutes the magnetic field

and $f_{om} = \overline{\gamma}_{om}$ constitutes the electric field vector

For slowly moving particles it is reasonable to require

$$f_{mn} \sim \lambda f_{on}$$

The expansion for f_{on} is begun at λ^2 in order to bring in electromagnetic effects at the same stage of the approximation that Newton's law arises in the gravitational case.

The procedure is confined to the case of slowly moving particles and slowly changing fields. Differentiation with respect to x° will always increase the order of a term by the factor λ. This will be written explicitly as:-

$$\gamma_{\alpha\beta|0} = \lambda \gamma_{\alpha\beta,0} \qquad 10$$

This convention was the reason behind the use of the bar instead of the comma for differentiation in (3). The bar derivation with respect to x^a will not increase the order of a term so that

$$\gamma_{\alpha\beta|a} = \gamma_{\alpha\beta,a} \qquad 11$$

Having adopted the expansion (9) for $\overline{\gamma_{\mu\nu}}$ we wish to know the consequent expansions for $\beta_{\mu\nu}$ and $\overline{\beta_{\mu\nu}}$ where

$$b_{\mu\kappa} = \eta_{\mu\kappa} + \beta_{\mu\kappa} \quad ; \quad \overline{\beta_{\mu\kappa}} = \beta_{\mu\kappa} - \tfrac{1}{2}\eta_{\mu\kappa}\gamma^{\rho\sigma}\beta_{\rho\sigma} \qquad 12$$

and where $b_{\mu\kappa}$ is defined in terms of $g_{\mu\kappa}$ by the relation (1.7) or

$$b^{\mu\kappa} = g^{\mu\kappa} \qquad 13$$

To do this it is convenient to use the expansion for $\gamma_{\mu\nu}$ rather than $\overline{\gamma_{\mu\nu}}$. From (8) and (9) it follows that the expansion for $\gamma_{\mu\nu}$ is of the form:-

$$\gamma_{oo} = \lambda^2 \gamma_{2oo} + \lambda^4 \gamma_{4oo} + \cdots$$

$$\gamma_{om} = \lambda^2 \gamma_{2om} + \lambda^3 \gamma_{3om} + \lambda^4 \gamma_{4om} + \cdots \quad \text{even skew, odd symmetrical}$$

$$\gamma_{mn} = \lambda^2 \gamma_{2mn} + \lambda^3 \gamma_{3mn} + \lambda^4 \gamma_{4mn} + \cdots \quad \text{odd skew, even symmetrical}$$

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The relation (13) is a complicated one but as we shall only need β_{ik} and $\overline{\beta_{ik}}$ correct to λ^4 it is sufficient to consider the approximate form of this relation accurate to terms involving the squares of γ_{ik} and β_{ik} . This approximation has already been worked out in (1.30) and (1.31') and is, after lowering the indices with η_{ik} :-

$$-\beta_{ik} + \tfrac{1}{2}\eta_{ik}\beta + (\beta_{i\mu}\eta^{\mu\nu}\beta_{\nu k} - \tfrac{1}{2}\beta_{ik}\beta) - \tfrac{1}{4}\eta_{ik}\left(\eta^{\alpha\beta}\eta^{\mu\nu}\beta_{\alpha\mu}\beta_{\nu\beta} - \tfrac{1}{2}\beta^2\right)$$

$$= -\underset{\sim}{\gamma}_{ik} + \tfrac{1}{2}\eta_{ik}\gamma + \left(\underset{\sim}{\gamma}_{i\mu}\eta^{\mu\nu}\underset{\sim}{\gamma}_{\nu k} - \tfrac{1}{2}\underset{\sim}{\gamma}_{ik}\gamma\right) - \tfrac{1}{4}\eta_{ik}\left(\eta^{\alpha\beta}\eta^{\mu\nu}\underset{\sim}{\gamma}_{\alpha\mu}\underset{\sim}{\gamma}_{\nu\beta} - \tfrac{1}{2}\gamma^2\right)$$
$$+ \left(\underset{\sim}{\gamma}_{i\mu}\eta^{\mu\nu}\underset{\sim}{\gamma}_{\nu k} - \tfrac{1}{4}\eta_{ik}\eta^{\alpha\beta}\eta^{\mu\nu}\underset{\sim}{\gamma}_{\alpha\mu}\underset{\sim}{\gamma}_{\nu\beta}\right)$$

where $\beta = \eta^{ik}\beta_{ik}$, $\gamma = \eta^{ik}\gamma_{ik}$

Multiplying this relation by η^{ik} gives

$$\beta = \gamma$$

Hence the relation may be simplified to

$$-\beta_{ik} + \left(\beta_{i\mu}\eta^{\mu\nu}\beta_{\nu k} - \tfrac{1}{2}\beta_{ik}\beta\right) - \tfrac{1}{4}\eta_{ik}\eta^{\alpha\beta}\eta^{\mu\nu}\beta_{\alpha\mu}\beta_{\nu\beta}$$

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$$= -\underset{\sim}{\gamma}_{ik} + \left(\underset{\sim}{\gamma}_{i\mu}\eta^{\mu\nu}\underset{\sim}{\gamma}_{\nu k} - \tfrac{1}{2}\gamma_{ik}\gamma\right) - \tfrac{1}{4}\eta_{ik}\eta^{\alpha\beta}\eta^{\mu\nu}\underset{\sim}{\gamma}_{\alpha\mu}\underset{\sim}{\gamma}_{\nu\beta} + \left(\underset{\sim}{\gamma}_{i\mu}\eta^{\mu\nu}\underset{\sim}{\gamma}_{\nu k} - \tfrac{1}{2}\eta_{ik}\eta^{\alpha\beta}\eta^{\mu\nu}\underset{\sim}{\gamma}_{\alpha\mu}\underset{\sim}{\gamma}_{\nu\beta}\right)$$

Clearly the expansion of β_{ik} must also begin with λ^2. Inserting (14) into (15) and equating the coefficients of λ^2 and λ^3 one gets,

$$\beta_{2ik} = \underset{\sim}{\gamma}_{2ik} \quad \text{or} \quad \beta_{2oo} = \gamma_{2oo}\,,\ \beta_{2om} = 0\,,\ \beta_{2mn} = \gamma_{2mn}$$

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$$\beta_{3ik} = \underset{\sim}{\gamma}_{3ik} \quad \text{or} \quad \beta_{3oo} = 0\,,\ \beta_{3om} = \gamma_{3om}\,,\ \beta_{3mn} = 0$$

Using these results and again inserting the expansion for $\gamma_{\mu\nu}$ into (15) one gets by equating the coefficients of λ^4 ,

$$-\beta_{4ik} = -\gamma_{4ik} + \left(\tfrac{1}{2}\eta^{\mu\nu}\gamma_{2\nu k} - \tfrac{1}{4}\eta_{ik}\eta^{\alpha\beta}\eta^{\mu\nu}\gamma_{2\alpha\mu}\gamma_{2\nu\beta}\right)$$

or

$$\beta_{400} = \gamma_{400} + \tfrac{1}{2}\gamma_{20s}\gamma_{2s0}$$

$$\beta_{40m} = 0$$

$$\beta_{4mn} = \gamma_{4mn} - \left(\gamma_{2m0}\gamma_{2on} - \tfrac{1}{2}\delta_{mn}\gamma_{2os}\gamma_{2s0}\right) = \gamma_{4mn} - t_{4mn} \qquad 17$$

From the results (16) and (17) we now know the expansion of β_{ik} correct to terms in λ^4 . If we note that $\eta^{\rho\sigma}\beta_{\rho\sigma} = \eta^{\rho\sigma}\gamma_{\rho\sigma}$ and take into account the relationship between the expansions (9) and (14) for $\bar{\gamma}_{\mu\nu}$ and $\gamma_{\mu\nu}$ respectively it is an easy matter to write down the expansion for $\bar{\beta}_{\mu\nu}$ defined in terms of $\beta_{\mu\nu}$ by (12). We get

$$\bar{\beta}_{00} = \lambda^2\bar{\gamma}_{200} + \lambda^4\left(\bar{\gamma}_{400} + \tfrac{1}{2}\bar{\gamma}_{20s}\bar{\gamma}_{2s0}\right) + \cdots$$

$$\bar{\beta}_{0m} = \lambda^3\bar{\gamma}_{20s} + \cdots \qquad 18$$

$$\bar{\beta}_{mn} = \lambda^4\left(\bar{\gamma}_{4mn} - t_{4mn}\right) + \cdots$$

In the following it is convenient, however, to work in terms of $\bar{\beta}_{\mu\nu}$ rather than $\bar{\gamma}_{\mu\nu}$. For this reason we conclude by writing down the expansions (9) and (18) for $\bar{\gamma}_{\mu\nu}$ and $\bar{\beta}_{\mu\nu}$ in the equivalent form

$$\bar{\beta}_{00} = \lambda^2\bar{\beta}_{200} + \lambda^4\bar{\beta}_{400} + \cdots \qquad \bar{\gamma}_{00} = \lambda^2\bar{\beta}_{200} + \lambda^4\left(\bar{\beta}_{400} - \tfrac{1}{2}f_{20s}f_{2s0}\right) + \cdots$$

$$\bar{\beta}_{0m} = \lambda^3\bar{\beta}_{30m} + \cdots \qquad \bar{\gamma}_{0m} = \lambda^2 f_{20m} + \lambda^3\bar{\beta}_{30m} + \lambda^4 f_{40m} + \cdots \qquad 19$$

$$\bar{\beta}_{mn} = \lambda^4\bar{\beta}_{4mn} + \cdots \qquad \bar{\gamma}_{mn} = \lambda^3 f_{3mn} + \lambda^4\left(\bar{\beta}_{4mn} + t_{4mn}\right) + \cdots$$

where as before

$$t_{4mn} = f_{2m0}f_{2on} - \tfrac{1}{2}\delta_{mn}f_{2os}f_{2s0} \qquad 20$$

In addition to this we need the first terms in the expansion of ψ . It is clear from the relation (1.34), which is correct to first order terms in $\gamma_{\mu\nu}$, that

$$\text{where} \quad \begin{cases} {}^2_3m = {}^2_1f_{mo,0} - {}^2_3f_{ms,s} \\ {}^2_2o = -{}^2_1f_{os,s} \end{cases} \qquad 21$$

4.3 The Field Equations and the Equations of Motion.

We go back to the field equations as expressed in (1), (2) with (3) and introduce the power series expansions for $\overline{\beta}_{\mu\nu}$, $\overline{\gamma}_{\mu\nu}$ and ${}^2_\mu$. Apart from the skew equation (1 b) and the terms $I_{\mu\nu}$ and $T_{\mu\nu}$ involved in $\Lambda''_{\mu\nu}$ these are the same as those in gravitational theory used by Einstein and Infeld. These authors (1949) exhaustively justified the application of the approximation procedure to the field equations and our procedure in the unified field theory under discussion can be justified on similar lines. We shall not go into details here but go straight to the coefficients of the lowest powers of λ involved in the field equations. In this way it will be possible to deduce the Coulomb force law from the equations at the stage in the procedure named the Newtonian approximation in the gravitational case.

Turning to (2 i), λ^2 is seen to be the lowest power of contained. The vanishing of its coefficient gives (there is no confusion if the bars are henceforth omitted)

$$\beta_{2\,oo,ss} = 0 \qquad 22$$

This, of course, occurs also in the gravitational case and the solution to be considered is

$$\beta_{2\,oo} = 2\varphi \qquad \varphi = \sum_{s=1}^{n} \left(-2\,{}^s_2m\,{}^s\psi\right)$$

$${}^s\psi = \left[(x^r - {}^s_5\xi^r)(x^r - {}^s_5\xi^r)\right]^{-\frac{1}{2}} = ({}^sr)^{-1} \qquad 23$$

and corresponds to an n-particle system each associated with a number $\overset{s}{m}$ (at most a function of time). The conditions (6) are obviously empty in this case.

In (2 ii) the lowest order term is λ^3 and the requirement that its coefficient should vanish gives

$$\beta_{3\,0m,ss} - \beta_{3\,0s,sm} = -\beta_{2\,00,m_0^1} \qquad 24$$

This equation also occurs in the gravitational case. The right hand side is $\overset{''}{\Lambda}_{3\,0m}$ and satisfies the condition (6) for the solution (23) for $\beta_{2\,00}$ if and only if each $\overset{s}{m}$ is a constant independent of time (see Einstein and Infeld, 1949, p.220). Apart from this result it will not be necessary here to consider (24) in more detail.

As in the gravitational case we need now to apply the usual coordinate conditions

$$\beta_{3\,0s,s} - \beta_{2\,00,0^1} = 0$$

$$\beta_{4\,mr,r} = 0 \qquad 25$$

Many writers have discussed their significance and once more we will not go into details here. The lowest power of λ involved in (2 iii) is the fourth and applying the coordinate condition one gets the equation

$$\beta_{4\,mn,ss} = 2\Lambda_{4\,mn} + \tfrac{1}{3}\overline{\Gamma}_{4\,mn} + 2\rho^2\overline{\Gamma}_{4\,mn} \qquad 26$$

Here $\Lambda_{4\,mn}$ is independent of the skew field terms and is identical with the $\Lambda_{4\,mn}$ which occurs in the Einstein - Infeld treatment.

Based on the solution (23) and (24) they find that

$$\frac{1}{4\pi}\int^{s} 2\Lambda_{mn}\, n_n\, dS = 4m^{s}\left\{\ddot{\xi}^{s}_{m} + \tilde{\rho}^{s}_{,m}\right\}$$

$$\text{where}\quad \varphi^{s} = \sum_{r\neq s}\left(-\frac{m^{r}}{2}\,\psi^{r}\right) \qquad 27$$

$$\text{and}\quad \tilde{\varphi}^{s}_{,m} = \varphi^{s}_{,m} \quad \text{evaluated at} \quad x^{\sigma} = \xi^{s\sigma}$$

Here S is a surface surrounding the 's^{th}' particle, i.e. the particle at $\xi^{s\sigma}$, alone. Λ_{4mn} is also such that

$$\Lambda_{4\,mn,n} = 0 \qquad 26$$

It only remains now to consider the contribution of the additional $\overline{I}_{4mn}$ and $\overline{\Gamma}_{4mn}$ terms in (26) to the condition (6). From (1), (20) and (21)

$$\overline{I}_{4mn} = \tfrac{1}{2}\delta_{mn}\, f^{2}_{20}\, f^{2}_{20} \qquad f^{2}_{20} = -f^{2}_{0s,s} \qquad 29$$

and $$\overline{\Gamma}_{4mn} = t_{4mn} = f^{2}_{mo}\, f^{2}_{on} - \tfrac{1}{2}\delta_{mn}\, f^{2}_{os}\, f^{2}_{ro}$$

The coefficient of λ^{1} in the field equation (1 b) requires

$$-f^{2}_{0s,sn} = -6\rho^{2} f^{2}_{on} \quad \text{or} \quad f^{2}_{20,n} = -6\rho^{2} f^{2}_{on} \qquad 30$$

Taking the divergence of both sides gives

$$f^{2}_{20,ss} = 6\rho^{2} f^{2}_{20} \qquad 31$$

Hence a single pole solution of (30) is

$$f^{2}_{20} = -6\rho^{2}\chi \qquad \chi = \sum_{s=1}^{n}\left(\xi^{s}_{2}\,\omega^{s}\right)$$

$$f^{2}_{on} = \chi_{,n} \quad \text{where} \quad \omega^{s} = (r^{s})^{-1}\, e^{-\nu r^{s}}, \quad \nu^{2} = 6\rho^{2} \qquad 32$$

and from (31) $$\chi_{,ss} = 6\rho^{2}\chi$$

Here $\overset{s}{\varepsilon}$ is a parameter for each particle. It is assumed independent of time although unlike $\overset{s}{m}$ the field equations themselves do not seem to require this.

Now the first equation of conditions (6) requires that (remember (28))

$$\tfrac{1}{6}\ \overset{I}{4}_{mn,n} + \rho^2\ \overset{T}{4}_{mn,n} = 0 \qquad 38$$

But $$\overset{I}{4}_{mn,n} = \tfrac{1}{2}(\tfrac{2}{2}\circ\tfrac{2}{2}\circ)_{,m} = \tfrac{2}{2}\circ_{,m}\ \tfrac{2}{2}\circ = (6\rho^2)^2\ \chi\ \chi_{,m}$$

and $$\overset{T}{4}_{mn,n} = \tfrac{f}{2}_{m\circ}\ \tfrac{f}{2}_{\circ n,n} + \tfrac{f}{2}_{m\circ,n}\ \tfrac{f}{2}_{\circ n} - \tfrac{f}{2}_{\circ s}\ \tfrac{f}{2}_{s\circ,m}$$

$$= -\chi_{,m}\ \chi_{,nn} - \chi_{,mn}\ \chi_{,n} + \chi_{,s}\ \chi_{,sm} = -6\rho^2\ \chi_{,m}\chi$$

Thus (33) is identically satisfied. Now consider the surface integral taken around the sth particle alone,

$$2\int \left(\tfrac{1}{6}\ \overset{I}{4}_{mn} + \rho^2\ \overset{T}{4}_{mn}\right) n_n\, dS \qquad 34$$

On account of (33) its value does not depend upon the particular surface surrounding the particle. Taking the surface to be a sphere surrounding the particle we are therefore at liberty to consider the limiting case as the radius of the sphere, defined as $\overset{s}{r}$, approaches zero. We can write

$$n_a = \frac{x^a}{\overset{s}{r}}\ , \qquad dS = (\overset{s}{r})^2\, d\omega$$

It will only be necessary to consider terms behaving like $(\overset{s}{r})^{-2}$ near the particle.

Let us first consider the contribution of $\overset{I}{4}_{mn}$. It is clear from (32) and (29) that the only relevant term is,

$$\text{const.}\ (r^{\xi})^{-2}\ e^{-2\nu r^{\xi}}\ \delta_{mn}$$

and the surface integral will reduce within a constant factor to

$$\int^{\xi} n_m\, d\omega = 0$$

Hence T_{4mn} makes no contribution to the surface integral.

Consider now T_{4mn}. It is convenient to write χ as the sum of two parts i.e.

$$\chi = \overset{\xi}{\chi} + \overset{\xi}{\xi}_2\, \overset{\xi}{\omega} \quad , \quad \overset{\xi}{\chi} = \sum_{r+s} \overset{r}{\xi}_2\, \overset{r}{\omega}$$

and to note that

$$\overset{\xi}{\omega}_{,n} = -\frac{x^n}{(r^{\xi})^3}\,(1+\nu r^{\xi})\, e^{-\nu r^{\xi}}$$

Then inserting (32) into (29) the only terms which can contribute are seen to be

$$\left(-\overset{\xi}{\chi}_{,m}\,\overset{\xi}{\omega}_{,n} - \overset{\xi}{\chi}_{,n}\,\overset{\xi}{\omega}_{,m} + \delta_{mn}\,\overset{\xi}{\chi}_{,t}\,\overset{\xi}{\omega}_{,t}\right) + \left(-\overset{\xi}{\omega}_{,m}\,\overset{\xi}{\omega}_{,n} + \delta_{mn}\,\overset{\xi}{\omega}_{,t}\,\overset{\xi}{\omega}_{,t}\right)$$

The last group of terms will produce integrals of the type

$$\overset{\xi}{\int} n_m\, d\omega = 0$$

and so give a vanishing contribution to the surface integral. On the other hand the first group of terms produce

$$\begin{aligned}\int^{\xi} &\left(\overset{\xi}{\tilde{\chi}}_{,m}\,\overset{\xi}{\varepsilon}\,n_n + \overset{\xi}{\tilde{\chi}}_{,n}\,\overset{\xi}{\varepsilon}\,n_m - \delta_{mn}\,\overset{\xi}{\tilde{\chi}}_{,t}\,\overset{\xi}{\varepsilon}\,n_t\right) n_n\, d\omega \\ &= 4\pi\left(\overset{\xi}{\varepsilon}\,\overset{\xi}{\chi}_{,m} + \tfrac{1}{3}\,\overset{\xi}{\varepsilon}\,\overset{\xi}{\tilde{\chi}}_{,m} - \tfrac{1}{3}\,\overset{\xi}{\varepsilon}\,\overset{\xi}{\hat{\chi}}_{,m}\right) \\ &= 4\pi\,\overset{\xi}{\varepsilon}\,\overset{\xi}{\chi}_{,m}\end{aligned} \qquad 35$$

where $\overset{\xi}{\tilde{\chi}}_{,m}$ is $\overset{\xi}{\chi}_{,m}$ evaluated at $x^{\sigma} = \overset{\xi}{\xi}^{\sigma}$

Adding this result (remembering the $2\rho^2$ factor in (34)) to (27) one obtains for the integral condition (6)

$$\overset{\xi}{m}\left(\overset{\xi}{\ddot{\xi}}^{m} + \overset{\xi}{\tilde{\rho}}_{,m}\right) + \tfrac{1}{2}\rho^2\,\overset{\xi}{\varepsilon}\,\overset{\xi}{\tilde{\chi}}_{,m} = 0$$

i.e.

$$\overset{\xi}{\ddot{\xi}}^{m} + \overset{\xi}{\tilde{\rho}}_{,m} = \tfrac{1}{2}\rho^2\,\frac{\overset{\xi}{\varepsilon}}{\overset{\xi}{m}}\,\overset{\xi}{\tilde{f}}_{m} \qquad 36$$

This result means that in addition to the Newtonian gravitational force $\tilde{\varphi}_{,m}$ the charged particle is also subjected to a force due to the electrostatic field of all the <u>other</u> particles. Denoting the electrostatic field by $\tilde{\mathcal{E}}_m$ the additional force is

$$\tfrac{1}{2}\rho^2 \frac{\varepsilon}{m} \tilde{\mathcal{E}}_m \qquad 37$$

per unit mass, in complete accord with classical electrodynamics.

The method used above precludes the appearance of magnetic terms in the stage of the approximation procedure which yielded the equation of motion (36). To obtain magnetic effects it would be necessary either to go one stage further in the approximation or to alter the original expansion for $\gamma_{\mu\nu}$ in (14) so that the expansion for $f_{on}\ (\gamma_{0\nu})$ and $f_{mn}\ (\gamma_{m\nu})$ begin at λ and λ^2 instead of λ^2 and λ^3 respectively. Both these approaches, while simple in principle, would probably lead to great complication, for in both one would need the relation between $b_{\mu\nu}$ and $g_{\mu\nu}$ correct to cubic terms in $\beta_{\mu\nu}$ and $\gamma_{\mu\nu}$. However the mere requirement of Lorentz invariance would enable us to write (36) in the form

$$\ddot{\xi}^m + \tilde{\varphi}_{,m} = \tfrac{1}{2}\rho^2 \frac{\varepsilon}{m} \tilde{f}_{m\nu}\dot{\xi}^\nu \qquad 38$$

where $\xi^\nu = (\xi^n, x^0)$. The dot here as before denotes differentiation with respect to $x^0 = ct$. The right of (38) is then simply the Lorentz force due to the electromagnetic field of all the

particles other than the one at $\overset{s}{\xi}^{m}$.

The result (36) also gives the relationship between ε the measure of charge in the so called natural units and e the charge in c.g.s. units. From the gravitational terms we deduce in the usual way that

$$\overset{s}{m} = \kappa \overset{s}{m}_{o} \quad , \qquad \kappa = 1{\cdot}86 \times 10^{-27} \text{ cm gm}^{-1} \qquad 39$$

where $\overset{s}{m}_{o}$ is the mass of the sth particle in c.g.s. units. Remembering that f_{mo} implies a measure of charge we then deduce from the electrostatic term that

$$e^2 = \frac{\rho^2 c^2}{2\kappa}\varepsilon \qquad \text{or} \qquad e = \frac{\rho c}{\sqrt{2\kappa}}\varepsilon \qquad 40$$

Since ρ^{-1} has the dimensions of a length, and $\varepsilon = k^2$ is the square of a length this relation is easily seen to be dimensionally correct.

Incidentally the result (36) confirms the choice of sign before ρ^2 in the field equations. We were careful to choose the solutions (32) such that a positive $\overset{s}{\varepsilon}$ corresponded with a positive electric field $\overset{s}{E}_{m} = \overset{s}{f}_{mo}$ so that had the sign before been different (36) would have required the attraction of like charges. Note also that if on the other hand the equation of motion (36) had required a change in the sign of ρ^2 in the field equations then the exponential factors in (1.46) would be replaced by circular functions.

Chapter 5

Discussion and Conclusion

We have already noted at the end of chapter 4 that the equation of motion (4.36) confirms the sign before ρ^2 in the field equations. In addition these equations of motion also confirm the interpretation of the theory in the sense that if it is a successful theory (in other ways besides the equations of motion) then b_{ik} must be the gravitational potential and f_{ik} must be the electromagnetic field vector where $(f_{23}\ f_{31}\ f_{12})$ is the magnetic vector $(H_1\ H_2\ H_3)$ and $(f_{14}\ f_{24}\ f_{34})$ is the electric vector $(E_1\ E_2\ E_3)$. Our object in this chapter is to examine the other results of the preceding chapters to see how well the theory under discussion with this interpretation provides a satisfactory unified field theory of gravitation and electromagnetism.

5.1 Discussion of the Solutions (3.41)

(a) The Electromagnetic Field

Quite early in the solution of the field equations in chapter 3 it was found that f in (3.5) was zero. This means that for a static spherically symmetric field the magnetic component vanishes everywhere. The radial component of the electric field, however, is given by ε_r in 3.41 i.e.

$$\varepsilon_r = \frac{k^2 r}{(r^4 + k^4)^{3/4}} \qquad 1$$

For $r>>k$ this becomes

$$\varepsilon_r \sim \frac{k^2}{r^2} \qquad 2$$

i.e. the electric field due to a charge at the origin. Of course, k^2 is not the charge measured in c.g.s. units but in what we may call natural units and it has the dimensions of a length squared. By comparison with the weak field solution (1.46) it follows that

$$k^2 = \varepsilon \qquad 3$$

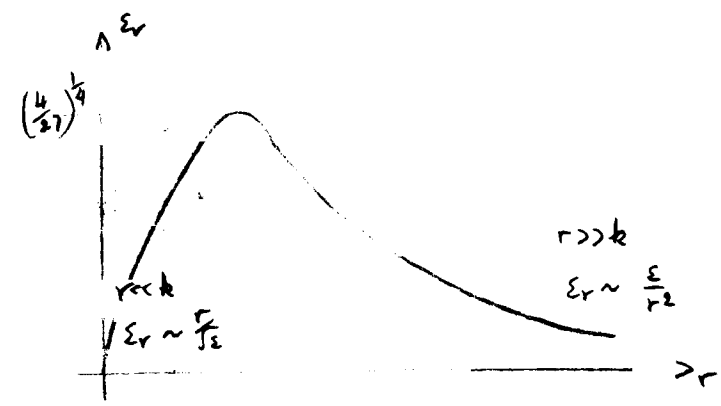
and then from (4.40) we can write the relation between the charge measured in c.g.s. units e and in natural units $\cdot\varepsilon$ or k^2 as

$$e = \frac{\rho c}{\sqrt{2\kappa}} \quad \varepsilon = \frac{\rho c}{\sqrt{2\kappa}} \quad k^2 \qquad 4$$

To describe both positive and negative charges k must take real or imaginary (not complex) values and this is the reason why $|k|$ occurs in (3.41). When $r \sim k$ the electric field ε_r given by (1) will differ greatly from the $\frac{1}{r^2}$ form and when $r<<k$, ε_r becomes a linear function of r,

$$\varepsilon_r \sim \pm \frac{1}{|k|} r \qquad 5$$

where the plus sign holds when k is real and the minus sign when k is imaginary. In other words ε_r has always the same sign as $\varepsilon = k^2$. The accompanying figure shows roughly the form of ε_r as a function of r for a positive charge ε. The maximum occurs



when $r_0 = \left(\frac{1}{2}\right)^{\frac{1}{4}} k = .84k$

and the corresponding value of $\mathcal{E}_r$ is about unity. The curve shows how the structure parameter k influences the electric field.

(b) The Charge distribution

First definition: Let us first consider the charge distribution defined in terms of $\mathcal{E}_r$ given by (1) according to Poisson's equation i.e. ρ the charge density given by

$$4\pi\rho = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\mathcal{E}_r) = \frac{3k^6}{(r^4+k^4)^{7/4}} \qquad 7$$

We get some idea of the dependence of ρ upon r from the three cases obtained from this

$$r << k \qquad \rho \sim \frac{3}{4\pi}\cdot\frac{1}{k}$$

$$r \sim k \qquad \rho \sim \frac{3}{4\pi}\cdot\frac{1}{k}$$

$$r >> k \qquad \rho \sim \frac{3k^6}{4\pi}\frac{1}{r^7}$$

A more precise picture of this dependence is gained by considering the total charge enclosed in a sphere of radius r . Denoting this by $\sigma(r)$

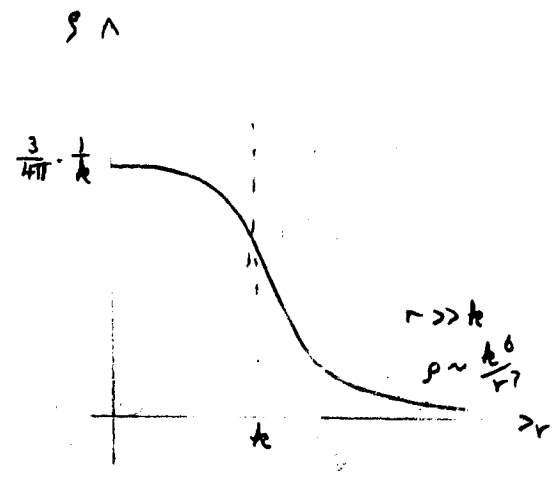
$$\sigma(r) = \int 4\pi\rho r^2 dr = r^2\mathcal{E}_r = \frac{k^2r^3}{(r^4+k^4)^{3/4}} \qquad 8$$

so that the total charge is $\lim_{r\to\infty}\sigma(r) = k^2$

and this is in agreement with the identification in the last section of k^2 as a measure of charge in natural units. The function rapidly approaches its asymptotic value as the radius increases through k . This is shown by the following brief table.

r =	k	$2k$	$10k$
$\sigma(r)$ =	$.6k^2$	$.956\,k^2$	$.99993\,k^2$

Thus practically all the charge is included in a sphere of radius $\sim 2k$ and both $\rho(r)$ the charge density and $\sigma(r)$ the charge within a sphere of radius r are regular smooth functions of r. For large r the charge density falls off as the inverse seventh power and for small r it is almost constant for $0 < r < k$. The accompanying figure gives a rough picture of the behaviour of ρ with r.



Once more we see the important part played by the form factor k. However, this definition of charge density is somewhat naive. It has certain obvious satisfactory features.

Second Definition: According to the tentative interpretation of the theory the charge - current density is

$$j^{\sigma}{}_{,\rho} = i^{\sigma} \qquad (9)$$

Since the charge density has nowhere occurred in our work so far it has not been established that (9) is preferable to the definition (7). For the spherically symmetric static case under discussion (9) is given by (3.13) i.e. only the fourth component is non zero

with the value

$$1^4 = (\omega\Delta)' \sin\theta \qquad 10$$

We can get an expression for this component correct to the first order in ρ^2 by first noting that from (3.14)

$$z_4 = \frac{\partial}{\beta}\sqrt{\frac{\gamma}{\alpha}}\,(\omega\Delta)'$$

and then from (28) to the first order in

$$z_4 = 6\rho^2 \int \omega\, dr$$

where ω is given by (3.41 iii). From these last two relations

$$1^4 = \sin\theta\ \frac{\beta}{D}\sqrt{\frac{\alpha}{\gamma}}\ 6\rho^2 \int \omega\, dr$$

and using (3.39) for D and (3.41) for α, β, γ, ω this becomes

$$1^4 = r^2 \sin\theta\ \frac{1}{\gamma_0}\ 6\rho^2 \left(1+\frac{k^4}{r^4}\right)^{-\frac{1}{4}} \int^r \frac{k^2 r\, dr}{(r^4+k^4)^{\frac{3}{4}}} \qquad 11$$

This gives 1^4 as an extremely complicated function of r. Since all the factors are finite for all r except possibly $\frac{1}{\gamma_0}$, it follows that 1^4 is everywhere $O(\rho^2)$ except in the vicinity of a zero of γ_0.

(c) <u>The Gravitational Potential</u>

Let us now consider the solution for γ_0 given by (3.41). The integral $J(r)$ contains an arbitrary constant in its lower bound and γ_0 is most conveniently written as,

$$\gamma_0 = 1 - \rho^2 \frac{|k|^3}{r}\left(1+\frac{k^4}{r^4}\right)^{-\frac{1}{4}} I(r) - \frac{2M}{r}\left(1+\frac{k^4}{r^4}\right)^{-\frac{1}{4}} \qquad 12$$

where

$$I(r) = \int_0^{\frac{r}{|k|}} \frac{(z^4+1)^{\frac{1}{2}} - z^2}{(z^4+1)^{\frac{3}{4}}}\, z^3\, dz = \frac{1}{2}\int_1^{\left(\frac{r^4}{k^4}+1\right)^{\frac{1}{2}}} \left[y^{\frac{1}{2}} - y^{-\frac{1}{2}}(y^2-1)^{\frac{1}{2}}\right] dy$$

From this it is easy to see that γ_0 is everywhere finite.

In particular when $r \ll k$

$$I(r) = \tfrac{1}{4} \frac{r^4}{|k|^4} \qquad \left(1 + \frac{k^4}{r^4}\right)^{-\frac{1}{4}} \sim \frac{r}{|k|}$$

and so $$\gamma_0 \sim 1 - \tfrac{1}{4} \rho^2 \frac{r^4}{|k|^2} - \frac{2M}{|k|} \qquad (13)$$

If we naively suppose that even at these distances γ_0 plays the role of gravitational potential (13) indicates that the force will be a repulsion $\sim \rho^2 \frac{r^3}{|k|^2}$

It is interesting to examine (12) when $r \gg k$. In this case the upper limits in the integrals defining $I(r)$ can be taken to infinity without sensibly affecting its value. The precise value of this integral is then most simply found from the second form, i.e.

$$I = \int_1^\infty \left[y^{\frac{1}{2}} - y^{-\frac{1}{2}}(y^2-1)^{\frac{1}{2}}\right] dy$$

Making the substitution $y = r^2$ this becomes

$$I = \int_1^\infty \left[r^2 - (r^4-1)^{\frac{1}{2}}\right] dr$$

Now by partial integration one obtains

$$\int (r^4-1)^{\frac{1}{2}} dr = \tfrac{1}{3} r (r^4-1)^{\frac{1}{2}} - \tfrac{2}{3} \int \frac{dr}{(r^4-1)^{\frac{1}{2}}}$$

and so

$$I = \tfrac{1}{3}\left[r^3 - r(r^4-1)^{\frac{1}{2}}\right]_1^\infty + \tfrac{2}{3} \int_1^\infty \frac{dr}{(r^4-1)^{\frac{1}{2}}}$$

$$= \tfrac{1}{3} + \tfrac{2}{3} \int_1^\infty \frac{dr}{(r^4-1)^{\frac{1}{2}}}$$

The value of the elliptic integral is found in tables and one finally gets

$$I = 1.1074$$

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With the value for I we conclude that for $r >> k$

$$\gamma_0 = 1 - \frac{2}{r}\{M + \tfrac{1}{2}\rho^2|k|^3 I\} \qquad 14$$

From (3.41) the other components of $g_{\mu\nu}$ for this case are given by $\gamma_0 = \frac{1}{\alpha_0} = \frac{1}{\alpha} = \gamma$

and $\beta = r^2$

These equations with (14) give the asymptotic form for large r of the components of $g_{\mu\nu}$ according to (3.41). They correspond to Schwarzschild's solution with a total mass m given by

$$m = M + \tfrac{1}{2}\rho^2|k|^3 I \qquad 15$$

All these results, like (2) for ε_r, hold when $r >> k$ ($r > 100k$ is in fact sufficient). From (15) we can deduce that when r is of this magnitude the gravitational mass is the sum of two components. One of them is due to the arbitrary constant M, while the other is directly dependent upon the presence of the electric field. When k vanishes the electric field given by (1) disappears and so does the latter component of the gravitational mass. Thus the electromagnetic field contributes to the gravitational mass for $r > 100k$ the positive quantity $= m_{em}$ given by,

$$m_{em} = \tfrac{1}{2}\rho^2|k|^3 I$$

16

5.2 Energy and Mass

Since for large r the symmetric part of g_{ik} given by the solution (3.41) reduces to the Schwarzschild line element with m given by (15) we can immediately deduce from (2.28) that

$$m = M + \tfrac{1}{2}\rho^2 |k|^3 I = \frac{1}{4\pi}\rho^2 \int_V (\sqrt{-g} - \sqrt{-b})\, dw \qquad 17$$

The integrand is interpreted as the energy density (in natural units) and it is interesting to examine the contributions made to it and its integral, the mass of the system, by the weak field solution (1.46) and the more exact solution (3.41).

The weak field contribution

From (2.29) we know that in the regions ΔV where the weak field solutions are valid the contribution Δm made to the mass of the whole system is given by

$$\Delta m = -\frac{1}{8\pi}\rho^2 \int_{\Delta V} \mathcal{E}^2\, dw \qquad 18$$

where $\underline{\mathcal{E}}$ is the electric field vector given by (2). We can express $\underline{E}$ the electric field vector measured in e.g.s. units in terms of $\underline{\mathcal{E}}$ given in natural units by using the relation (4), which is the relation between charge values measured in the two units. Hence

$$\underline{E} = \frac{\rho c}{\sqrt{2K}}\, \underline{\mathcal{E}} \qquad 19$$

Thus (18) may be written

$$\Delta m = k\,\Delta m_0 = -\frac{1}{8\pi}\frac{2K}{c^2}\int_{\Delta V} E^2 dv$$

or

$$c^2\,\Delta m_0 = -2\cdot\frac{1}{8\pi}\int_{\Delta V} E^2 dv \qquad 20$$

where m_0 is the mass in c.g.s. units given in terms of m by the usual relation (4.39). This result means that in regions where the field is small the contribution of the field to the mass of the system is <u>minus</u> twice the electrostatic energy.

<u>The contribution of the solutions (3.41)</u>

The integrand in (17) can be calculated quite easily in terms of the solution (3.41). From (3.9) and (3.12)

$$\sqrt{-g} - \sqrt{-b} = \beta \sin\theta\left[\sqrt{\alpha\gamma - \omega^2} - \sqrt{\alpha\gamma}\right] = \beta\sqrt{\alpha\gamma}\sin\theta\,(D-1)$$

where D is the usual expression (3.11) with $f = 0$. Using (3.39) for D and the solutions (3.41) for $\sqrt{\alpha\gamma}$ and β this becomes

$$\sqrt{-g} - \sqrt{-b} = (r^4 + k^4)^{-\frac{3}{4}}\left[r^2 - (r^4+k^4)^{\frac{1}{2}}\right] r^3 \sin\theta \qquad 21$$

If these expressions are valid in a region ΔV then their contribution Δm to the mass of the system is by (17) using spherical polar coordinates,

$$\Delta m = \frac{1}{4\pi}\rho^2 \int_{\Delta V} (\sqrt{-g} - \sqrt{-b})\, dr\, d\theta\, d\rho$$

Using (21) for the integrand and transforming by the substitution $r = kz$ we get,

$$\Delta m = -\rho^2 |k|^3 \int_{\Delta V} \frac{(z^4+1)^{\frac{1}{2}} - z^2}{(z^4+1)^{\frac{3}{4}}} z^3 dz = -\rho^2 |k|^3 [J]_{\Delta V} \qquad 22$$

where J is the integral which occurs in (3.41). Now the electromagnetic contribution to the mass of the system given by m_{em} in

(16) is

$$m_{em} = \tfrac{1}{2}\rho^2|k|^3 I = \tfrac{1}{2}\rho^2|k|^3[J]_V$$

where $[\ ]_V$ means that in the integral for $J(r)$, r is to be taken from zero to infinity - i.e. over its complete range. (22) may then be written in the form

$$\Delta m = -2[m_{em}]_{\Delta V} \qquad 23$$

If we regard m_{em} as being the energy of the electromagnetic field in natural units this result is in complete agreement with the form (20) valid for small fields.

It should be noted that the results (20) and (23) are based on different work. (20) depends on the units relation provided by the equations of motion in Chapter 4 and the expression for the mass (2.29). While (23) also depends on (29), it depends on the solutions (3.41) instead of the units relation. This comparative independence of (20) and (23) seems to minimise the possibility of an error in the -2 factor.

Nevertheless these results are rather puzzling. In any case, of course, the approximate solutions (3.41) were not expected to provide an integrand for (17) valid in all space which would give the mass of the whole system.

5.3 A charged particle with mass entirely electromagnetic in origin.

For such a particle M in (12) is zero. When $r >> k$ ($r > 100\,k$) the gravitational mass is given by

$$m = \tfrac{1}{2}\rho^2 |k|^3 I = \kappa m_0 \quad , \qquad \kappa = \frac{8\pi\gamma}{c^2} = 1.86 \times 10^{-27} \text{ cm gm}^{-1} \qquad 24$$

where m_0 is measured in c.g.s. units and γ is Newton's constant of gravitation. Recall also that the relationship between the charge e in c.g.s. units and k^2 is by (4)

$$e = \frac{\rho c}{\sqrt{2\kappa}}\, k^2 \qquad 25$$

Solving (24) and (25) for k and ρ we get

$$k = I \cdot \frac{e^2}{m_0 c^2} \quad , \quad \rho = e\, \frac{\sqrt{2\kappa}}{c I^2} \left(\frac{e^2}{m_0 c^2}\right)^{-2} \qquad 26$$

where I has the value (13).

The question arises as to whether the electron could be described by such a model. Using the well known values for e, m_0, c and κ (26) gives

$$k = 3.1 \times 10^{-13} \text{ cm.} \qquad \rho^{-1} = .985 \times 10^{8} \text{ cm.} \qquad 27$$

From what we know of the dependence of both the electrostatic field and charge distribution upon the structure constant k, the first result is satisfactory. According to the weak field solution (1.46) ρ enters the electric field through the exponential factor $e^{-\sqrt{6}\rho r}$. With the calculated value for ρ this gives an effective range for the electric field of

$$\frac{1}{\sqrt{6}}\,\rho^{-1} = \nu^{-1} \doteq 400\,\mathrm{Km}$$

This seems rather small. Schrödinger[27] in a paper discussing the permanent magnetic field of the earth reaches the conclusion that the range lies between $30 - 36 \times 10^8$ cm., or between eighty and ninety times our figure. Had we tried to fit the proton to this model k would have been about a thousand times smaller and the effective range no less than 10^6 smaller - an absurdly short range.

However it is not really certain that 400 Km is too low a figure. If it is not then the present model for an electron of a particle whose mass is entirely electromagnetic in origin would be satisfactory. Since ρ occurs in the field equations it would be fixed for all particles at the value given by (27). The positron could then be a similar negatively charged particle while a proton would be a positron with additional mass not electromagnetic in origin. The value of $|k|$ would be the same for all these particles since their charge is of the same magnitude. With ρ fixed the relation (25) gives k for particles of any other charge e.

If on the other hand the range 400 Km is definitely too small this would not mean that the present theory had failed. Rather it would mean that it was incorrect to ascribe all the mass of the electron to an electromagnetic origin. Reducing the electromagnetic

contribution to a tenth of the total mass would increase k by a factor 10, and the range ν^{-1} by a factor 100. In view of Schrödinger's work this might be said to give a reasonable value for the range but the structure parameter k (or effective radius) of all particles of the same numerical charge would be rather large ($\sim 10^{-12}$ cm).

A final decision upon the ability of the present theory to describe the fundamental particles, and upon the precise models applicable to these particles will depend upon the value of the lowest possible range for the electromagnetic field which is not inconsistent with observations. On the other hand it is quite reasonable to say that the value of 400 Km for the range is not in fact directly comparable with observation. As derived it applies to a spherically symmetric static field - i.e. to one isolated charged particle. At most one could say that it was also valid for a system of particles separated by distances of the order of 400 Km. Clearly a system of electrons separated in this way cannot be subject to observation.

In the last paragraph of Chapter 1 it was shown that the theory implied a finite mass for the photon. The expression (1.53) for the mass is

$$m_{ph} = \frac{k\nu}{c}$$

Some idea of the order of magnitude involved is gained by assuming

the range of the electromagnetic field to be about 1000 km., i.e. by putting $\nu = 10^{-8}$ cm^{-1}. One gets

$$m_{pl} \sim 10^{-44} \text{ gm.}$$

Conclusion.

Summing up we may say that the unified field theory given by the equations (1.25) with the definitions (1.7 - 1.8') has several interesting features. The equations of motion of an n-particle system as determined by the field equations themselves are in agreement with the usual Newtonian-Lorentz force terms. The solution (3.41) of the field equations for the case of static spherically symmetric field may be interpreted as the field of a stationary charged particle with mass. A magnetic pole cannot occur. The solution (3.41) contains a structure parameter k which also provides a measure of the charge. The manner in which the electrostatic field given by this solution and the corresponding charge distribution (according to one definition) depend on this parameter enables it to be identified as the effective radius of the particle. The gravitational mass of the particle contains for $r > 100\,k$ a positive contribution due to the electrostatic field which it is reasonable to suppose is the electrostatic energy. For large distances the electrostatic field falls off exponentially (according to the weak field solution (1.46)) and the best models for the elementary

particles will depend upon just how slow this fall off must be to fit observation. Also for small fields the theory requires that the current vector must not vanish since it also then plays the part of electromagnetic potential. A finite mass for the photon is predicted, the precise value being inversely proportional to the effective range of the electromagnetic field. For ranges of the order of 1000 km. it is approximately 10^{-44} gm.

Finally we note that the theory as developed so far is entirely classical and ignores quantum effects. Also the possibility of the theory providing a classical description of spin has not been investigated.

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