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On the group rings of abelian minimax groups, II: The singular case

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Abstract

The main results of [D. Segal, On the group rings of abelian minimax groups, J. Algebra 237 (2001) 64–94], and of related papers by J.E. Roseblade and C.J.B. Brookes, are extended to group rings kG where the characteristic of the field k occurs in the spectrum of the abelian minimax group G.

Keywords: Abelian group rings; Abelian groups of finite rank

In the paper [S1] I established some basic results about ideals in and modules for the group rings mentioned in the title. Most of those results require a hypothesis of *non-singularity*. This seemed natural enough in the context, but in trying to apply the theory to problems about soluble groups one inevitably comes up against modules that may be singular, and something more is required. The first aim of this note is to show how most of the main results of [S1] can be adapted to the singular situation. The second aim is to extend to the singular case the main theorem of [S2, §2]: this concerns the group ring of a group with operators and establishes a kind of enhanced Jacobson property, originally introduced by Roseblade in the context of polycyclic groups (Theorem E of [R1]). In order to do this, we also have to extend a theorem about 'control' of prime ideals, due to Roseblade and Brookes ([R2, Theorem D]; [B, Theorem A]).

This paper thus closes a gap left in [S1,S2] and [B]. The need to fill this gap arose in my work with Pyber on groups with 'polynomial index growth' [PS]; the machinery developed here enables us to settle a 20-year old conjecture about the structure of such groups.

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1. Ideals

An abelian group G is *minimax* if G contains a finitely generated subgroup H such that G/H satisfies the minimal condition. We can choose H so that G/H is a direct sum of finitely many quasi-cyclic groups C_{q^∞}, and the set of primes q that occur is the *spectrum* $\mathrm{spec}(G)$ of G. A G-module M is *non-singular* if M has no q-torsion for $q \in \mathrm{spec}(G)$. An ideal I in a group ring kG is called *regular* if the module kG/I is non-singular.

We fix a prime p and a finitely generated $\mathbb{F}_p$-algebra k, and study the group ring kG where G is as above. We also choose and fix a subgroup N of G such that

$$p \notin \mathrm{spec}(N),$$
$$G/N \cong (C_{p^\infty})^d;$$

if $p \notin \mathrm{spec}(G)$ then $N = G$—this case presents nothing new as all kG-modules are then non-singular. Otherwise, all non-zero kG-modules are singular, while in any case all kN-modules are non-singular.

Definition. For any subset S of kG,

$$\sqrt{S} = \{r \in kG \mid r^n \in S \text{ for some } n \in \mathbb{N}\},$$
$$S_0 = S \cap kN.$$

Note that if S is an ideal of kG then $\sqrt{S}$ is again an ideal, the radical of S.

Lemma 1.

(i) *For each $r \in kG$ there exists $n \in \mathbb{N}$ such that $r^{p^n} \in kN$.*
(ii) *Let $S \subseteq kG$ and suppose that $s \in S \Rightarrow s^p \in S$. Then*

$$S \subseteq \sqrt{S} = \sqrt{S_0}.$$

Proof. (i) holds because G/N is a p-group and k has characteristic p. (ii) follows easily. □

Let $\mathcal{P}_G$ denote the set of prime ideals of kG and $\mathcal{P}_N$ denote the set of prime ideals of kN. For an ideal I of kG (respectively kN) we put

$$I^\dagger = (I + 1) \cap G$$
$$\big(\text{respectively } (I + 1) \cap N\big).$$

Proposition 1. *There are mutually inverse bijections*

$$\mathcal{P}_N \xrightarrow{\Phi} \mathcal{P}_G \xrightarrow{\Psi} \mathcal{P}_N$$

with

$$\Phi(P)=\sqrt{P}\quad (P\in\mathcal{P}_N),$$
$$\Psi(Q)=Q_0\quad (Q\in\mathcal{P}_G).$$

Proof. Certainly Ψ maps $\mathcal{P}_G$ into $\mathcal{P}_N$, and does so surjectively since kG is integral over kN. Let $P\in\mathcal{P}_N$, with $P=\Psi(Q)$ where $Q\in\mathcal{P}_G$. Lemma 1 shows that

$$\Phi(P)=\sqrt{P}=\sqrt{Q_0}=\sqrt{Q}=Q;$$

it follows that $\Psi\Phi(P)=\Psi(Q)=P$ and $\Phi\Psi(Q)=\Phi(P)=Q$. $\square$

Corollary 1. *Let $Q\in\mathcal{P}_G$. Then Q is maximal in kG if and only if Q_0 is maximal in kN. If either kG/Q or kN/Q_0 is algebraic over $\mathbb{F}_p$ then $kG/Q\cong kN/Q_0$.*

Proof. The first claim is clear since the maps Φ and Ψ are inclusion-preserving. Suppose that kN/Q_0 is algebraic over $\mathbb{F}_p$ (this certainly holds if kG/Q is algebraic), and write $\bar{\ }: kG\to kG/Q$ for the quotient map. Then $\overline{kG}$ is an algebraic extension of the locally finite field $\overline{kN}\cong kN/Q_0$, so $\overline{kG}$ is a locally finite field. It follows that $\overline{G}$ is a p'-group, and hence that $\overline{G}=\overline{N}$. Thus

$$\overline{kG}=k[\overline{G}]=k[\overline{N}]=\overline{kN}$$

and the result follows. $\square$

Recall that an abelian minimax group A is said to be *reduced* if the torsion subgroup of A is finite, or equivalently if A is residually finite. Let us say that A is *p'-reduced* if the p'-part of A is finite; this holds iff the finite residual of A is a p-group.

Note (i). If $Q\in\mathcal{P}_G$ then $G/Q^\dagger$ is reduced if and only if it is p'-reduced, because the integral domain kG/Q has no units of p-power order.

Note (ii). Any quotient group of N is reduced if and only if it is p'-reduced, because N has no infinite p-torsion sections.

We can now generalise one of the main results of [S1]. Writing

$$L \mathrel{\underset{\max f}{\lhd}} kG$$

for 'L is a maximal ideal of finite index in kG,' we have

Theorem 1. *Let $Q\in\mathcal{P}_G$ and suppose that $G/Q^\dagger$ is reduced. Then*

$$Q=\bigcap\Big\{L \;\Big|\; Q\subseteq L \mathrel{\underset{\max f}{\lhd}} kG\Big\}.$$

Proof. Write $P=Q_0$. Let $a\in kG\setminus Q$. For some n we have $a^{p^n}\in kN$, and certainly $a^{p^n}\notin P$. Now P is a regular prime ideal of kN and $N/P^\dagger$, a subgroup of $G/Q^\dagger$, is reduced; according to

Theorem 4.2 of [S1] there exists $T \lhd_{\max f} kN$ with $P \subseteq T$ and $a^{p^n} \notin T$. The preceding corollary shows that then $L = \sqrt{T} \lhd_{\max f} kG$. Now $Q = \sqrt{P} \subseteq L$; and $a \notin L$, else $a^{p^n} \in L \cap kN = L_0 = T$. The result follows. □

2. Modules

We turn now to some module theory. G, N and k remain as in the preceding section. A G-module M is said to be *qrf* ('quasi-residually finite') if $G/\mathrm{C}_G(a)$ is p'-reduced for every $a \in M$. (Note (ii) above shows that this is consistent with the definition given in [S1] when applied to non-singular modules.)

For any subset S of M we write

$$S^* = \mathrm{ann}_{kG}(S), \qquad S_0^* = \mathrm{ann}_{kN}(S).$$

Lemma 2. *Let M be a kG-module, let $Q \in \mathcal{P}_G$ and put $P = Q_0$.*

(i) *M is qrf if and only if M is qrf as an N-module.*
(ii) *The following are equivalent*:
   (a) *$G/Q^\dagger$ is reduced,*
   (b) *$N/P^\dagger$ is reduced,*
   (c) *kG/Q is qrf,*
   (d) *kN/P is qrf as N-module.*

Proof. (i) Let $a \in M$, write $T_G/\mathrm{C}_G(a)$ for the torsion subgroup of $G/\mathrm{C}_G(a)$ and $T_N/\mathrm{C}_N(a)$ for the torsion subgroup of $N/\mathrm{C}_N(a)$. We have an exact sequence

$$1 \to T_N/\mathrm{C}_N(a) \to T_G/\mathrm{C}_G(a) \to G/N\mathrm{C}_G(a).$$

As the last term is a p-group and N has no infinite p-sections, it follows that $T_N/\mathrm{C}_N(a)$ is finite if and only if the p'-part of $T_G/\mathrm{C}_G(a)$ is finite.

(ii) Taking $M = kG/Q$ we have $\mathrm{C}_G(a) = Q^\dagger$ and $\mathrm{C}_N(a) = P^\dagger$ for every $a \in M \setminus \{0\}$, so (ii) follows from (i) (and Note (i) above). □

Lemma 3. *Let M be a qrf kG-module, let $0 \neq a \in M$ and put $J = \sqrt{a^*}$. Then*

(i) *$(J_0)^n \subseteq a_0^*$ for some $n \in \mathbb{N}$;*
(ii) *if J is prime then $J_0 = b_0^*$ for some $b \in a.kN \setminus \{0\}$.*

Proof. Put $I = a_0^*$. Then $kN/I \cong a.kN$ is non-singular and qrf as N-module. By [S1, Corollary 6.6], there exist $P_1, \ldots, P_m \in \mathcal{P}_N$ and $n_i \in \mathbb{N}$ such that

$$\prod_{i=1}^{m} P_i^{n_i} \subseteq I \subseteq \bigcap_{i=1}^{m} P_i,$$

and for each i there exists $x_i \in kN$ such that $P_i = (ax_i)_0^*$.

Now $J = \sqrt{I}$ by Lemma 1, so $J_0 = (\sqrt{I})_0 \subseteq (\sqrt{P_i})_0 = P_i$ for each i. Hence $(J_0)^n \subseteq I$ where $n = n_1 + \cdots + n_m$. If J is prime then $J_0 = P_i$ for some i and we have (ii) with $b = ax_i$; also $b \neq 0$ since $1 \notin J_0$. □

In [S1] I defined the set of *associated primes* $\mathcal{P}(M)$ of a module M as the set of prime annihilators of non-zero elements of M. We now modify this definition.

Definition. Let M be a kG-module.

$$\mathcal{P}(\!(M)\!) = \bigl\{Q \in \mathcal{P}_G \bigm| Q = \sqrt{a^*} \text{ for some } a \in M \setminus \{0\}\bigr\},$$
$$\mathcal{P}_0(M) = \bigl\{P \in \mathcal{P}_N \bigm| P = a_0^* \text{ for some } a \in M \setminus \{0\}\bigr\}.$$

Thus $\mathcal{P}_0(M)$ is the set of associated primes of the kN-module M as defined before.

Definition. Let M be a kG-module, $\mathcal{X} \subseteq \mathcal{P}_N$ and $\mathcal{Y} \subseteq \mathcal{P}_G$.

$$M(\!(\mathcal{Y})\!) = \bigl\{a \in M \bigm| \sqrt{a^*} \supseteq Q_1 \cdots Q_m \text{ for some } Q_1, \ldots, Q_m \in \mathcal{Y}\bigr\},$$
$$M(\mathcal{X}) = \{a \in M \mid aP_1 \cdots P_m = 0 \text{ for some } P_1, \ldots, P_m \in \mathcal{X}\}$$

(the Q_i and the P_i are not necessarily distinct). Again, $M(\mathcal{X})$ is the set defined in [S1] for M considered as a kN-module.

Proposition 2. *Let M be a qrf kG-module, $\mathcal{Y} \subseteq \mathcal{P}_G$ and $\mathcal{X} = \Psi(\mathcal{Y}) \subseteq \mathcal{P}_N$.* Then

$$\mathcal{P}(\!(M)\!) = \Phi\bigl(\mathcal{P}_0(M)\bigr), \qquad \mathcal{P}_0(M) = \Psi\bigl(\mathcal{P}(\!(M)\!)\bigr)$$

and

$$M(\!(\mathcal{Y})\!) = M(\mathcal{X}).$$

Proof. The first line follows from Proposition 1, Lemma 1(ii) and Lemma 3(ii).

To prove the final claim, let $a \in M(\mathcal{X})$. Then $aP_1 \cdots P_m = 0$ with $P_1, \ldots, P_m \in \mathcal{X}$, and then $\sqrt{P_1} \cdots \sqrt{P_m} \subseteq \sqrt{P_1 \cdots P_m} \subseteq \sqrt{a^*}$ so $a \in M(\!(\Phi(\mathcal{X}))\!) = M(\!(\mathcal{Y})\!)$.

Now suppose $a \in M(\!(\mathcal{Y})\!)$, so $Q_1 \cdots Q_m \subseteq \sqrt{a^*}$ for some $Q_1, \ldots, Q_m \in \mathcal{Y}$. By Lemma 3(i) there exists $n \in \mathbb{N}$ such that $(Q_1 \cdots Q_m \cap kN)^n \subseteq a_0^*$, and putting $P_i = \Psi(Q_i) \in \mathcal{X}$ we have $P_1^n \cdots P_m^n \subseteq a_0^*$. Thus $a \in M(\mathcal{X})$. □

Proposition 3. *Let I be a proper ideal of kG such that kG/I is qrf. Then I has finitely many minimal primes $Q_1, \ldots, Q_m$, each of the groups $G/Q_i^\dagger$ is reduced, each of the modules kG/Q_i is qrf, and $\sqrt{I} = Q_1 \cap \cdots \cap Q_m$.*

Proof. The hypothesis implies that kN/I_0 is qrf as N-module. According to [S1, Corollary 6.6], the ideal I_0 has finitely many minimal primes $P_1, \ldots, P_m$ in kN, and they satisfy

$$\prod_{i=1}^{m} P_i^{n_i} \subseteq I_0 \subseteq \bigcap_{i=1}^{m} P_i$$

for some $n_i \in \mathbb{N}$. Put $Q_i = \sqrt{P_i}$ for each i. Then

$$Q_1 \cdots Q_m \subseteq \sqrt{I_0} \subseteq \bigcap_{i=1}^{m} Q_i$$

and as $\sqrt{I_0} = \sqrt{I}$ we have $I \subseteq \sqrt{I} \subseteq \bigcap_{i=1}^{m} Q_i$. Conversely, if $w \in \bigcap_{i=1}^{m} Q_i$ then $w^n \in I_0$ for some n, so $w \in \sqrt{I}$. Thus $\sqrt{I} = \bigcap_{i=1}^{m} Q_i$. It follows that every minimal prime of I is one of the Q_i. If $Q_i \supseteq L \supseteq I$ with $L \in \mathcal{P}_G$ then $P_i \supseteq L_0 \supseteq I_0$, so $L_0 = P_i$ and consequently $L = \Phi(L_0) = Q_i$; thus each Q_i is a minimal prime of I.

By [S1, Corollary 6.6] again, for each P_i there exists $x_i \in kN$ such that $P_i = \mathrm{ann}_{kN}((x_i + I_0)/I_0)$. Since kN/I_0 is qrf, it follows that $N/P_i^{\dagger}$ is reduced. With Lemma 2(ii) this shows that $G/Q_i^{\dagger}$ is reduced and that kG/Q_i is qrf as G-module. □

Using Proposition 2, we can translate most of the structural results from Sections 6 and 7 of [S1] to the present context. Specifically, the following results remain valid for kG-modules, with the hypothesis 'non-singular' omitted, provided we everywhere replace terms of the form $M(\mathcal{X})$ by $M((\mathcal{X}))$ and $\mathcal{P}(M)$ by $\mathcal{P}((M))$:

- Lemma 6.5, Proposition 6.7, Lemma 7.1, Corollary 7.2

and all of Proposition 7.3 except for the final claim. To paraphrase the introduction of [S1], we may deduce that

- every qrf kG-module has a natural finite filtration in which each factor is unmixed;
- every unmixed qrf kG-module has a natural decomposition as a subdirect sum of primary modules.

A module M is *unmixed* if $\mathcal{P}((M))$ has no inclusions (all chains have length one). M is *primary* if $\mathcal{P}((M))$ is a singleton.

In the singular case, primary modules are not so easy to handle, but they do behave nicely under an additional hypothesis.

Definition. The kG-module M is said to *locally radical* (l.r.) if $a^* = \sqrt{a^*}$ for each $a \in M$.

Proposition 4. *Let M be a qrf kG-module and suppose that M is l.r. If $\mathcal{P}((M)) = \{Q\}$ is a singleton then M is Q-*prime*, i.e. $MQ = 0$ and M is torsion-free as a module for kG/Q.*

Proof. Put $P = Q_0$, so $\mathcal{P}_0(M) = \{P\}$. Let $0 \neq a \in M$. By [S1, Lemmas 6.3 and 6.5(iv)], we have

$$P \supseteq a_0^* \supseteq P^n$$

for some n. Hence

$$Q = \sqrt{P} = \sqrt{P^n} = \sqrt{a_0^*} = \sqrt{a^*} = a^*. \qquad \square$$

Certain modules are easily seen to be locally radical:

Lemma 4. *Every residually finite-simple kG-module is locally radical.*

Proof. Let M be a residually finite-simple kG-module. Let $a \in M$ and suppose that $ax^n = 0$ ($x \in kG$, $n \in \mathbb{N}$). If $ax \neq 0$ then $ax \notin ML$ for some maximal ideal L of finite index p^f in kG. Now $x^{p^{mf}} \equiv x \pmod L$ for every m so taking $p^{mf} \geqslant n$ we obtain

$$ax \in ax^{p^{mf}} + aL \subseteq ML,$$

a contradiction. Thus $ax = 0$ so $x \in a^*$. □

When applying the structure theory as in [S1], one is led to consider certain quotients as in the following lemma; it is important to know that good properties of a module are preserved under this process:

Lemma 5. *Let M be a qrf kG-module and suppose that M is l.r. Let $\mathcal{Y} \subseteq \mathcal{P}((M))$ and put $V = M((\mathcal{Y}))$. Then M/V is again qrf and l.r.*

Proof. Write $\bar{\ }: M \to M/V$ for the quotient map. Lemma 6.5(i) of [S1], with Lemma 2(i) above, shows that $\overline{M}$ is qrf. Now let $a \in M \setminus V$. Then

$$a^* = \sqrt{a^*} = Q_1 \cap \cdots \cap Q_s$$

for some $Q_1, \ldots, Q_s \in \mathcal{P}_G$; let us label these so that $Q_1, \ldots, Q_t$ contain no member of $\mathcal{Y}$ while $Q_j \supseteq Y_j \in \mathcal{Y}$ for $j = t+1, \ldots, s$. I claim that

$$\bar{a}^* = Q_1 \cap \cdots \cap Q_t;$$

this will complete the proof as $Q_1 \cap \cdots \cap Q_t$ is a radical ideal.

To establish the claim, let $r \in Q_1 \cap \cdots \cap Q_t$. Then $rY_{t+1} \cdots Y_s \subseteq \bigcap_1^s Q_j = a^*$ so $Y_{t+1} \cdots Y_s \subseteq (ar)^*$, $ar \in M((\mathcal{Y})) = V$ and $r \in \bar{a}^*$. Conversely, if $r \in \bar{a}^*$ then $ar \in V = M((\mathcal{Y}))$ so

$$(ar)^* = \sqrt{(ar)^*} \supseteq Z_1 \cdots Z_n$$

for some $Z_1, \ldots, Z_n \in \mathcal{Y}$, and then $rZ_1 \cdots Z_n \subseteq a^* = Q_1 \cap \cdots \cap Q_s$. Since Q_i contains no Z_j if $i \leqslant t$ it follows that $r \in Q_1 \cap \cdots \cap Q_t$. □

These results are applied in the following way. Starting with a kG-module M that is residually finite-simple, we form the natural filtration (M_j) as given in [S1, Proposition 6.7]. Each factor M_{i-1}/M_i is unmixed, and locally radical. Such a factor decomposes in turn as a subdirect sum of primary modules, as in [S1, Lemma 7.1(ii)]. Each of these primary modules is again l.r., and hence *prime*. Thus certain problems about M may be reduced to the case of prime modules. This strategy is employed in [PS].

3. Group rings with operators, 1: Control

From now on, we take k to be a *finite field*, of size p^e. Most of the results extend to the case where k is locally finite (algebraic over a finite prime field)—this is left to the reader. To begin with, G denotes an arbitrary abelian group.

Let P be an ideal of kG. A subgroup H of G is said to *control* P if $P = (P \cap kH)kG$. The intersection $\mathcal{C}(P)$ of all such subgroups H is called the *controller* of P; it is an elementary fact that $\mathcal{C}(P)$ itself controls P, hence is the unique minimal subgroup of G that controls P (see [P, Lemma 8.1.1]).

Let Γ be a group acting by automorphisms on G. Then Γ acts on kG, and if Γ fixes P then it fixes $\mathcal{C}(P)$. Under certain circumstances, the induced action of Γ on $\mathcal{C}(P)$ is surprisingly restricted. In Theorem D of [R2], Roseblade proved that if G is finitely generated, k is any field, and P is a Γ-invariant faithful prime ideal of kG then $|\Gamma/\mathrm{C}_\Gamma(\mathcal{C}(P))|$ is finite.

This was extended by Brookes to the case where G is torsion-free of finite rank [B, Theorem A]. However, there is a gap in the proof of this latter result: to make it valid one has to assume that $\mathrm{char}(k) \notin \mathrm{spec}(G)$, i.e. that the prime ideal P is *regular*. A counterexample with kG/P singular is given in [S1, Example 1.9]. Here G is the additive group of $\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$ (considered as a multiplicative group), and P is generated by the elements

$$\big(p^n, 0, 0\big) + \big(0, p^n, 0\big) - \big(0, 0, p^n\big) \quad (n \in \mathbb{Z}),$$

in the group algebra $\mathbb{F}_p G$. It is shown in [S1] that P is faithful and (implicitly) that $\mathcal{C}(P) = G$; on the other hand, it is clear that the automorphism $x \mapsto x^p$ fixes P and has infinite order.

Fortunately, this kind of behaviour is essentially the worst that can happen.

Definition. Let $H \neq 1$ be a torsion-free abelian group of finite rank and Γ a group acting on H. Then (H, Γ) is a *Brookes pair* (for the prime p) if there exist a subgroup Γ_0 of finite index in Γ and characters

$$\chi_1, \dots, \chi_r : \Gamma_0 \to \langle p \rangle < \mathbb{Q}^*$$

such that the quotient

$$H/(H_{\chi_1} \times \cdots \times H_{\chi_r})$$

is periodic, where for a character χ we write

$$H_\chi = \big\{ g \in H \bigm| g^\gamma = g^{\chi(\gamma)},\ \forall \gamma \in \Gamma_0 \big\}.$$

This amounts to saying that Γ_0 acts diagonally on the rational completion $H^{\mathbb{Q}}$ of H and all eigenvalues are powers of p.

Theorem. (C.J.B. Brookes, personal communication) *Let G be a torsion-free abelian group of finite rank, Γ a group acting on G, and $P \neq 0$ a faithful Γ-invariant prime ideal of kG where k is a field of characteristic p. Then $(\mathcal{C}(P), \Gamma)$ is a Brookes pair.*

I am most indebted to Chris Brookes for drawing my attention to the gap, for acquainting me with this correct version, and for patiently explaining the main ideas of the proof.

Here I will sketch the proof of a slightly sharper version of a special case, sufficient for present applications. The Brookes pair (H, Γ) as above will be called *strict* if $\Gamma = \Gamma_0$, $\mathrm{spec}(H) \subseteq \{p\}$ and $H/(H_{\chi_1} \times \cdots \times H_{\chi_r})$ is a finite p'-group.

Henceforth, G again denotes an *abelian minimax group*.

Theorem 2. *Let G be a torsion-free abelian minimax group, Γ a group acting on G, and $P \neq 0$ a faithful Γ-invariant prime ideal of kG where k is a finite field of characteristic p. Then G has a Γ-invariant subgroup H such that*

(i) $P = (P \cap kH)kG$ *and*
(ii) (H, Γ_0) *is a strict Brookes pair for some subgroup Γ_0 of finite index in Γ.*

The proof depends on

Lemma 6. *Let Q be a faithful prime ideal of kG. Then G has a subgroup C such that G/C is a divisible p'-group,* $\mathrm{spec}(C) \subseteq \{p\}$, *and*

$$Q = (Q \cap kC)kG.$$

Proof. G has a finitely generated subgroup H such that G/H is a divisible torsion group, and then $G/H = A/H \times B/H$ with A/H a p-group and B/H a p'-group. Since $p \notin \mathrm{spec}(B)$ we have

$$Q \cap kB = (Q \cap kH_1)kB$$

for some finitely generated subgroup H_1 of B, and we may choose $H_1 \geqslant H$ [S1, Corollary 1.2]. Put $C = AH_1$ and write $Q_C = Q \cap kC$.

Now let $w \in Q$. Write

$$w = \sum r_i b_i$$

with each $r_i \in kC$ and $b_1, b_2, \ldots$ in distinct cosets of C in G. By Lemma 1 there exists $n \in \mathbb{N}$ such that

$$\sum r_i^{p^n} b_i^{p^n} = w^{p^n} \in Q \cap kB \subseteq Q_C kG.$$

Since G/C is a p'-group the elements $b_i^{p^n}$ $(i = 1, 2, \ldots)$ lie in distinct cosets of C. Therefore $r_i^{p^n} \in Q_C$ for each i and so $r_i \in Q_C$ for each i. Hence $w \in Q_C kG$. The result follows. □

Proof of Theorem 2. Put $H = \mathcal{C}(P)$. Writing H as an additive group, let V be the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes H$. Then V is a Γ-module. In [BG], Bieri and Groves present a proof of Roseblade's Theorem D based on the properties of a certain geometric invariant; they remark that the same argument applies more generally to the case where H (denoted B in [BG]) is a group of finite rank; what it shows is that Γ has a subgroup Γ_0 which acts diagonally on V (the finiteness of k in the present case ensures that the subgroup denoted A in [BG] is trivial). (The stronger claim made in [BG] that the corollary to Theorem B also extends to the finite-rank case is not quite correct, as shown by the example above.)

Now Lemma 6 shows that in our situation, $\mathrm{spec}(H) \subseteq \{p\}$. If $\gamma \in \Gamma_0$ and $h \in H$ is an eigenvector for γ then

$$h \cdot \mathbb{Z}\langle\gamma\rangle \subseteq \mathbb{Q}h \cap H.$$

As $\mathbb{Q}h \cap H$ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}[\frac{1}{p}]$ it follows that the eigenvalues of γ must be powers of p.

Straightforward linear algebra now shows that $V = \bigoplus_{i=1}^{r} V_{\chi_i}$ where $\chi_1, \ldots, \chi_r$ are distinct homomorphisms $\Gamma_0 \to \langle p\rangle$. Writing $H_i = H_{\chi_i}$, it remains to show that $\widetilde{H} = H/\bigoplus H_i$ is a finite p'-group. Now $\widetilde{H}$ is a torsion group since H spans V. As $\mathrm{spec}(\widetilde{H}) \subseteq \{p\}$ it follows that the p'-part of $\widetilde{H}$ is finite, so it will suffice to show that $\widetilde{H}$ has trivial p-part. Now the group H_i is p-divisible for each non-trivial character χ_i; say this holds for $i > 1$. If

$$h = p^{-1}(h_1 + \cdots + h_r) \in H$$

with $h_i \in H_i$ for each i then $p^{-1}h_i \in H_i$ for $i = 2, \ldots, r$ and

$$p^{-1}h_1 = h - p^{-1}(h_2 + \cdots + h_r) \in H \cap V_{\chi_1} = H_1.$$

Thus $h \in \bigoplus H_i$. This shows that $\widetilde{H}$ has no elements of order p. □

4. Group rings with operators, 2: Orbital primes

Throughout, G will denote an abelian minimax group, Γ a group acting on G, and k a field of size p^e. For convenience I will call a prime ideal Q of kG *good* if $G/Q^\dagger$ is reduced. Thus, according to Theorem 1, a prime ideal is good if and only if it is an intersection of maximal ideals of finite index.

We begin with some technical lemmas.

Lemma 7. *Let $C \leqslant G$ such that G/C is a p'-group. Let P be a prime ideal of kC, and suppose that $G/P^\dagger$ is reduced. Then PkG has finitely many minimal primes $Q_1, \ldots, Q_m$,*

$$PkG = Q_1 \cap \cdots \cap Q_m, \tag{1}$$

and each Q_i is a good prime of kG and satisfies $Q_i \cap kC = P$.

Proof. If $0 \neq v \in kG/PkG$ then $\mathrm{C}_G(v)/\mathrm{C}_C(v)$ is finite, because $\mathrm{C}_G(v)$ permutes the support of v by multiplication (considering kG/PkG as a direct sum of copies of kC/P indexed by the cosets of C in G). Also $\mathrm{C}_C(v) = P^\dagger$ because P is prime. Therefore $G/\mathrm{C}_G(v)$ is reduced. Thus kG/PkG is qrf as kG-module.

An argument used in the proof of Lemma 6 shows that $PkG = \sqrt{PkG}$. Now Proposition 3 shows that PkG has finitely many minimal primes $Q_1, \ldots, Q_m$, each of which is good, and that (1) holds.

If $m = 1$ then $Q_1 = PkG$ clearly intersects kC in P. Suppose that $m \geqslant 2$, and let $w \in Q_i \cap kC$. Then w annihilates some non-zero element of kG/PkG, which is torsion-free as a module for kC/P, so $w \in P$. Thus $Q_i \cap kC = P$. □

Lemma 8. *Let $C \leqslant G$ such that G/C is a p'-group. Let P be a prime ideal of kC, and suppose that $PkG \leqslant L$ for some $L \lhd_{\max f} kG$. Then the set of good prime ideals of kG containing PkG has finitely many minimal members $N_1, \ldots, N_m$, and for each i we have $N_i \cap kC = P$.*

Proof. Assume without loss of generality that $P^\dagger = 1$. Then C has no p-torsion. Let D be the divisible part of $\tau(G)$. I claim that $D \cap C = 1$. To see this, suppose that $x \in D \cap C$ has order $q > 1$. Then $p \nmid q$ so $X^q - 1 = (X - 1)f(X)$ in the polynomial ring $\mathbb{F}_p[X]$, where $f(1) \neq 0$. Since $x^q - 1 \in P$ and $x - 1 \notin P$ we have $f(x) \in P$. On the other hand, since $G/L^\dagger$ is finite we have $L^\dagger \geqslant D$, so $x - 1 \in L$. As $L \supseteq P$ it follows that

$$f(1) \equiv f(x) \equiv 0 \mod L$$

whence $f(1) \in L \cap \mathbb{F}_p = 0$, a contradiction. This establishes the claim.

Since CD/C is divisible it is a direct factor of G/C, so there exists $E \leqslant G$ such that $E \geqslant C$ and $G = E \times D$. Then E is reduced, so the preceding lemma shows that PkE has finitely many minimal primes $Q_1, \ldots, Q_m$,

$$PkE = Q_1 \cap \cdots \cap Q_m,$$

and each Q_i is a good prime of kE and satisfies $Q_i \cap kC = P$.

The inclusion $E \to G$ induces an isomorphism

$$\theta : kE \to kG/\mathfrak{d}$$

where $\mathfrak{d} = (D - 1)kG$, and we put

$$\theta(Q_i) = N_i/\mathfrak{d}$$

for each i. Then each N_i is a good prime ideal of kG and

$$N_1 \cap \cdots \cap N_m = PkG + \mathfrak{d}.$$

Also for each i we have

$$N_i \cap kC = N_i \cap (kC + \mathfrak{d}) \cap kC = (P + \mathfrak{d}) \cap kC = P.$$

As kG is integral over kC, this implies that N_i is minimal among the prime ideals of kG containing P. On the other hand, since D is divisible, every ideal of finite index in kG contains $\mathfrak{d}$. Therefore every good prime containing PkG contains $PkG + \mathfrak{d}$ and hence one of the N_i. It follows that $N_1, \ldots, N_m$ are precisely the minimal good primes containing PkG. □

Lemma 9. *Suppose that* $\mathrm{spec}(G) \subseteq \{p\}$. *Let Q be a prime ideal of kG and let $a \in kG \setminus Q$. Then the set of prime ideals of kG containing $Q + akG$ has finitely many minimal members $P_1, \ldots, P_m$, and for each i we have* $\mathrm{ht}(P_i/Q) = 1$.

Note that when $\mathrm{spec}(G) \subseteq \{p\}$ every prime ideal Q is good, since $G/Q^\dagger$ has no p-torsion.

Proof. There exists a finitely generated subgroup H of G such that G/H is a divisible p-group and $a \in kH$. Put $Q_0 = Q \cap kH$. Now $Q_0 + akH$ has finitely many minimal primes $M_1, \ldots, M_m$ in the finitely generated algebra kH, and $\mathrm{ht}(M_i/Q_0) = 1$ for each i by Krull's Principal Ideal Theorem. Put $P_i = \sqrt{M_i}$ for $i = 1, \ldots, m$. The result now follows from Proposition 1. □

Lemma 10. *Suppose that G is torsion-free and that $H \leqslant G$ satisfies* $\mathrm{spec}(H) \subseteq \{p\}$. *Then there exists $C \leqslant G$ such that $H \leqslant C$,* $\mathrm{spec}(C) \subseteq \{p\}$ *and G/C is a p'-group.*

Proof. Write G additively and identify G with $1 \otimes G \leqslant \mathbb{Q} \otimes G = V$. Choose a vector space complement W for $\mathbb{Q}H$ in V and let W_0 be the $\mathbb{Z}$-span of a basis for W. It is easy to see that

$$C = \left(\mathbb{Z}\left[\frac{1}{p}\right]H + \mathbb{Z}\left[\frac{1}{p}\right]W_0\right) \cap G$$

has the required properties. □

We now come to the main result of this section. An ideal M of kG is said to be *Γ-orbital* if its orbit under Γ is finite, that is, if $|\Gamma : \mathrm{N}_\Gamma(M)| < \infty$. Since kG has only finitely many ideals of a given finite index, each of these is orbital; so Theorem 1 implies that every good prime ideal is an intersection of orbital primes. In certain circumstances this can be refined:

Proposition 5. *Assume that G is torsion-free, and let $Q \neq 0$ be a good faithful prime ideal of infinite index in kG. If Q is Γ-orbital then*

$$Q = \bigcap \mathcal{G}$$

where $\mathcal{G}$ is the set of all good Γ-orbital prime ideals M of kG with $M > Q$ and $\mathrm{ht}(M/Q) = 1$.

Proof. Replacing Γ by a suitable subgroup of finite index, we may assume that Q is Γ-invariant. Then Theorem 2 shows that $Q = (Q \cap kH)kG$ where H is a subgroup of G such that (H, Γ_0) is a strict Brookes pair for some subgroup Γ_0 of finite index in Γ. In particular, $\mathrm{spec}(H) \subseteq \{p\}$, so by the preceding lemma there exists $C \geqslant H$ with $\mathrm{spec}(C) \subseteq \{p\}$ such that G/C is a p'-group. Writing $Q_0 = Q \cap kC$ we then have $Q = Q_0 kG$.

Fix $L \lhd_{\max f} kG$ with $Q \leqslant L$. We shall see that L contains some member of $\mathcal{G}$. As Q is an intersection of ideals like L, this will suffice to establish the proposition.

I will call a subset S of kG *sound* if $S \subseteq L^\gamma$ for some $\gamma \in \Gamma$.

Since (H, Γ_0) is a strict Brookes pair there exist $c \in H \setminus \{1\}$ and a character $\chi : \Gamma_0 \to \langle p \rangle$ such that $c^\gamma = c^{\chi(\gamma)}$ for all $\gamma \in \Gamma_0$. Since $L \cap k\langle c \rangle$ has finite index in $k\langle c \rangle$ we may choose an element a with

$$0 \neq a \in L \cap k\langle c \rangle.$$

Since Q_0 is faithful (while every non-zero ideal of $k\langle c \rangle$ has finite index), we have $Q_0 \cap k\langle c \rangle = 0$; so $a \notin Q_0$. According to Lemma 9, there are finitely many prime ideals $P_1, \ldots, P_t$ of kC minimal over $Q_0 + akC$, and $\mathrm{ht}(P_i/Q_0) = 1$ for each i. Let us label these so that $P_1, \ldots, P_s$ are sound while no P_j is sound for $s < j \leqslant t$. Note that $s \geqslant 1$ since at least one of the P_i is contained in $L \cap kC$.

Let $i \in \{1, \ldots, s\}$. According to Lemma 8, the set of good prime ideals of kG containing $P_i kG$ has finitely many minimal members; let $\mathcal{N}_i$ denote the set of those minimal ones that are sound. Then $\mathcal{N}_i$ is non-empty, and each $N \in \mathcal{N}_i$ satisfies $N \cap kC = P_i$.

Now let $\Gamma_1 = \chi^{-1}\langle p^e \rangle$ (where $p^e = |k|$). I claim that if $\gamma \in \Gamma_1$, $i \leqslant s$ and $N \in \mathcal{N}_i$ then $N^\gamma \in \mathcal{N}_j$ for some $j \leqslant s$.

Note first that if $\chi(\gamma^{-1}) = p^{ne}$ where $n \geqslant 0$ then $a^{\gamma^{-1}} = a^{p^{ne}} \in N$, while if $\chi(\gamma^{-1}) = p^{-ne}$ then $(a^{\gamma^{-1}})^{p^{ne}} = a \in N$, so in either case $a^{\gamma^{-1}} \in N$. Thus $N^\gamma \cap kC$ contains $Q_0 + akC$, hence contains P_j for some $j \leqslant s$. Then $P_j kG \leqslant N^\gamma$, so there exists $M \in \mathcal{N}_j$ with $M \leqslant N^\gamma$. Taking γ^{-1} in place of γ in the above argument we see that $M^{\gamma^{-1}} \cap kC$ contains P_l for some $l \leqslant s$. Then

$$P_l \leqslant M^{\gamma^{-1}} \cap kC \leqslant N \cap kC = P_i$$

so in fact $l = i$ and $M^{\gamma^{-1}} \cap kC = N \cap kC$. As kG is integral over kC it follows that $M^{\gamma^{-1}} = N$. Thus

$$N^\gamma = M \in \mathcal{N}_j.$$

Thus Γ_1 permutes the finite set $\bigcup_{i=1}^{s} \mathcal{N}_i$. As Γ_1 has finite index in Γ it follows that all members of this set are Γ-orbital.

Now let $N \in \mathcal{N}_1$. Then $N \cap kC = P_1$ and $N \geqslant P_1 kG \geqslant Q_0 kG = Q$. As kG is integral over kC and $\mathrm{ht}(P_1/Q_0) = 1$ it follows that $\mathrm{ht}(N/Q) = 1$. Thus $N \in \mathcal{G}$, and then $N^\gamma \in \mathcal{G}$ for each $\gamma \in \Gamma$. As N is sound, we also have $N^\gamma \leqslant L$ for some $\gamma \in \Gamma$, and the result follows. □

5. Roseblade's 'Theorem E' revisited

Here we extend Theorem 2.1 of [S2], generalising Theorem E of [R1]. As above k will denote a finite field of size p^e, G an abelian minimax group and Γ a group acting on G. For a prime ideal Q of kG and $\lambda \in kG$ we write

$$\mathcal{L}(\Gamma, Q, \lambda)_{kG} = \left\{ L \mathrel{\underset{\max f}{\lhd}} kG \;\middle|\; Q \leqslant L,\ \lambda^\gamma \notin L,\ \forall \gamma \in \Gamma \right\}.$$

I will usually omit the subscript kG when the context is clear.

When the action of Γ is trivial, Theorem 1 implies that $\bigcap \mathcal{L}(\Gamma, Q, \lambda) = Q$ for each $\lambda \notin Q$. We generalise this to

Theorem 3. *Let G be an abelian minimax group and Γ a virtually soluble group acting on G. Let Q be a Γ-invariant prime ideal of kG such that $G/Q^\dagger$ is reduced and let $\lambda \in kG \setminus Q$. Then*

$$\bigcap \mathcal{L}(\Gamma, Q, \lambda) = Q.$$

We will need some simple observations.

Lemma 11. *Suppose that $H < G$ are abelian groups and that G/H is torsion-free. If P_0 is a prime ideal of kH then $P_0 kG$ is a prime ideal of kG.*

Proof. Every finitely generated subgroup of G is contained in one of the form $H \times X$ with X free abelian. It follows that any two elements of kG/P_0kG lie in a subring isomorphic to a group algebra like $(kH/P_0)X$, an integral domain. □

Lemma 12. *Let G be an abelian minimax group and $\mathcal{X}$ an infinite set of maximal ideals of finite index in kG. Then $\mathcal{X}$ contains an infinite subset $\mathcal{Y}$ such that $\bigcap \mathcal{Y}$ is a prime ideal of kG.*

Proof. Put $I = \bigcap \mathcal{X}$. Then kG/I is residually finite as a kG-module and $I = \sqrt{I}$, so Proposition 3 shows that $I = Q_1 \cap \cdots \cap Q_m$ where $Q_1, \ldots, Q_m$ are the minimal primes of I. If $m = 1$ take $\mathcal{Y} = \mathcal{X}$. Suppose that $m \geqslant 2$. For each i let

$$\mathcal{X}(i) = \{L \in \mathcal{X} \mid Q_i \subseteq L\}.$$

Then $\mathcal{X} = \bigcup_{i=1}^{m} \mathcal{X}(i)$ so $\mathcal{X}(i)$ is infinite for at least one value of i, say $i = 1$. Now there exists $y \in (Q_2 \cap \cdots \cap Q_m) \setminus Q_1$. If $a \in \bigcap \mathcal{X}(1)$ then $ya \in \bigcap_{i=1}^{m} (\bigcap \mathcal{X}(i)) = \bigcap \mathcal{X} = I \subseteq Q_1$, so $a \in Q_1$. It follows that $\bigcap \mathcal{X}(1) = Q_1$. □

Now let G, Γ and Q be as in Theorem 3.

Remark 1. Let $\Delta \leqslant_f \Gamma$, let Z be a transversal to $\Delta \backslash \Gamma$ and let $\lambda \in kG$. Put $\mu = \prod_{z \in Z} \lambda^z$. Then $\lambda \notin Q$ if and only if $\mu \notin Q$, and

$$\mathcal{L}(\Gamma, Q, \lambda) = \mathcal{L}(\Delta, Q, \mu).$$

This is clear.

Remark 2. To prove the theorem it will suffice to establish that $\mathcal{L}(\Gamma, Q, \lambda)$ is non-empty for each $\lambda \in kG \setminus Q$.

Indeed, if $\lambda' \in \bigcap \mathcal{L}(\Gamma, Q, \lambda) \setminus Q$ then $\mathcal{L}(\Gamma, Q, \lambda\lambda') = \emptyset$.

Remark 3. Let $H \leqslant_f G$ and $\Delta \leqslant_f \Gamma$ with $H = H^{\Delta}$. Put $Q_0 = Q \cap kH$. If $\mathcal{L}(\Delta, Q_0, \lambda)_{kH} \neq \emptyset$ for each $\lambda \in kH \setminus Q_0$ then $\mathcal{L}(\Gamma, Q, \lambda)_{kG} \neq \emptyset$ for each $\lambda \in kG \setminus Q$.

To see this, let $\lambda \in kG \setminus Q$ and define μ as in Remark 1. Since kG is integral over kH and Q is prime, μ satisfies an equation

$$\mu^n + c_1\mu^{n-1} + \cdots + c_{n-1}\mu + \nu = 0$$

with each $c_j \in kH$ and $\nu \in kH \setminus Q_0$. Let $L_0 \in \mathcal{L}(\Delta, Q_0, \nu)_{kH}$. Then $L_1/Q = (L_0 + Q)/Q$ is a maximal ideal of $(kH + Q)/Q$, and as kG/Q is finite over $(kH + Q)/Q$ there exists a maximal ideal L/Q of finite index in kG/Q such that $L_1 = L \cap (kH + Q)$. Then $L \cap kH = L_0$, so if $\mu^{\delta} \in L$ where $\delta \in \Delta$ then $\nu^{\delta} \in L_0$, contradicting the choice of L_0. It follows that $L \in \mathcal{L}(\Delta, Q, \mu) = \mathcal{L}(\Gamma, Q, \lambda)$.

Proof of Theorem 3. We may clearly assume that Q is faithful. Then G is reduced, and replacing G by a subgroup of finite index, as we may in view of Remarks 3 and 2, we may suppose that G is torsion-free.

First we consider the case where $Q = 0$, and argue by induction on the torsion-free rank $\mathrm{r}_0(G)$ of G.

Subcase 1: where G is a *plinth* for Γ; that is, G is rationally irreducible for every subgroup of finite index in Γ.

This case is addressed in [S2, p. 403, 'Proof of theorem 2.1: Conclusion']. What is shown there is the following: there exist a subgroup Γ_1 of finite index in Γ and an infinite cyclic subgroup $\langle x\rangle$ of Γ_1 such that G is a plinth for $\langle x\rangle$, and if $\lambda \in kG$ is such that $\mathcal{L}(\Gamma_1, 0, \lambda) = \emptyset$ then the set

$$\mathcal{X}_\lambda = \left\{ L \underset{\max f}{\lhd} kG \;\middle|\; \lambda \in L = L^x \right\}$$

is infinite.

Given such a $\lambda \neq 0$, Lemma 12 shows that some infinite subset of $\mathcal{X}_\lambda$ intersects in a prime ideal M, say, of kG. Then M is $\langle x\rangle$-invariant and kG/M is infinite. Thus $M^\dagger$ is an $\langle x\rangle$-invariant subgroup of infinite index in G, and as M is a good prime ideal $G/M^\dagger$ is reduced. Therefore $\mathrm{r}_0(M^\dagger) < \mathrm{r}_0(G)$, and as G is a plinth for $\langle x\rangle$ it follows that $M^\dagger = 1$. Now applying Theorem 2 we infer that $M = (M \cap kB)kG$ where $B = B^x \leqslant G$ and $(B, \langle x^n\rangle)$ is a strict Brookes pair for some $n > 0$. Thus B contains an $\langle x^n\rangle$-invariant subgroup C isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}[\frac{1}{p}]$, and as G is rationally irreducible for $\langle x^n\rangle$ the quotient G/C is periodic. Let $1 \neq b \in C$. Then $G/\langle b\rangle$ is periodic, so kG is integral over $k\langle b\rangle$, whence kG has Krull dimension 1. This is a contradiction since $0 < M < L$ for infinitely many maximal ideals L.

We conclude that if $\mathcal{L}(\Gamma_1, 0, \lambda) = \emptyset$ then $\lambda = 0$. The result follows by Remarks 1 and 2.

Subcase 2: where Γ has a subgroup Γ_1 of finite index and G has a Γ_1-invariant subgroup H with $0 < \mathrm{r}_0(H) < \mathrm{r}_0(G)$.

We may suppose that G/H is torsion-free, and (in view of Remark 1) that $H = H^\Gamma$. Let $0 \neq \lambda \in kG$. Then

$$\lambda = \lambda_1 + \sum_{1 \neq x \in X} \lambda_x x$$

where X is a transversal to $H\backslash G$ with $1 \in X$, each λ_x is in kH (almost all λ_x being 0); and replacing λ by λg for some $g \in G$ we may assume that $\lambda_1 \neq 0$. By inductive hypothesis, there exists

$$L_0 \in \mathcal{L}(\Gamma, 0, \lambda_1)_{kH}.$$

Put $K = L_0^\dagger$, a subgroup of finite index in H. By [S2, Lemma 2.3], there exist $\Delta \leqslant_f \Gamma$ and a complement Y/K to H/K in G/K such that Δ fixes Y and acts trivially on H/K. Let Z be a transversal to $\Delta\backslash\Gamma$ and put $\mu = \prod_{z \in Z} \lambda^z$.

I claim that $\mu \notin L_0 kG$. Indeed, since the latter is a prime ideal,

$$\begin{aligned} \mu \in L_0 kG \quad &\Longrightarrow \quad \lambda^z \in L_0 kG \quad \text{for some } z \in Z \\ &\Longrightarrow \quad \lambda_1^z \in L_0, \end{aligned}$$

contradicting the choice of L_0.

Now kG/L_0kG is Δ-isomorphic to the group algebra $F(Y/K)$ of Y/K over $F = kH/L_0$, where Δ acts trivially on F and fixes $Y/K \cong G/H$. Let $\bar{\mu}$ denote the image of μ in $F(Y/K)$. By inductive hypothesis again, we have

$$\mathcal{L}(\Delta, 0, \bar{\mu})_{F(Y/K)} \neq \emptyset.$$

Hence there exists a maximal ideal L of finite index in kG, with $L \geqslant L_0kG$, such that $\mu^\delta \notin L$ for all $\delta \in \Delta$. It follows that $L \in \mathcal{L}(\Gamma, 0, \lambda)_{kG}$. The result follows by Remark 2.

Now we turn to the case where $Q \neq 0$. In this case, Q has infinite index in kG: indeed, if kG/Q is finite then $G = 1$ since we are assuming that Q is faithful and G is torsion-free, whence $Q = 0$. We shall argue by induction on the Krull dimension $\mathrm{Dim}(kG/Q)$.

Let $\mathcal{G}$ be the set of all good Γ-orbital prime ideals M of kG with $M > Q$ and $\mathrm{ht}(M/Q) = 1$. Proposition 5 shows that

$$\bigcap \mathcal{G} = Q.$$

Note that, since Q is prime, this forces $\mathcal{G}$ to be an infinite set.

Now fix $\lambda \in kG \setminus Q$. Suppose that each member of $\mathcal{G}$ contains some Γ-conjugate of λ. Put

$$\mathcal{G}(\lambda) = \{M \in \mathcal{G} \mid \lambda \in M\}$$

and let $D = \bigcap \mathcal{G}(\lambda)$. Then $D = \sqrt{D}$ and kG/D is qrf, so $D = P_1 \cap \cdots \cap P_n$ for some prime ideals $P_1, \ldots, P_n$ by Proposition 3. If $M \in \mathcal{G}(\lambda)$ then for some i we have

$$M \geqslant P_i \geqslant Q + \lambda kG > Q,$$

and as $\mathrm{ht}(M/Q) = 1$ this implies that $M = P_i$. Thus each member of $\mathcal{G}$ is conjugate under Γ to one of the P_i. As $\mathcal{G}$ consists of Γ-orbital ideals it follows that $\mathcal{G}$ is finite, a contradiction.

Hence there exists $M \in \mathcal{G}$ such that $\lambda^\gamma \notin M$ for all $\gamma \in \Gamma$. Put $\Delta = \mathrm{N}_\Gamma(M)$, let Z be a transversal to $\Delta \backslash \Gamma$ and write $\mu = \prod_{z \in Z} \lambda^z$. Then $\mu \notin M$, so $\mathcal{L}(\Delta, M, \mu)$ is non-empty by inductive hypothesis. But it is evident that

$$\mathcal{L}(\Delta, M, \mu) \subseteq \mathcal{L}(\Gamma, Q, \lambda),$$

so $\mathcal{L}(\Gamma, Q, \lambda)$ is non-empty and the result follows by Remark 2.

This completes the proof. □

In applications it can be useful to know a little more. When $\mathcal{L}$ is a collection of ideals of finite index in kG let us write

$$\mathcal{L}(f) = \{L \in \mathcal{L} \mid |kG/L| = p^f\}.$$

I will say that $\mathcal{L}$ is *abundant* if the cardinalities $|\mathcal{L}(f)|$ are unbounded as $f \to \infty$.

Proposition 6. *Let G, Γ and Q be as above. Suppose that (G_0, Γ_0) is a strict Brookes pair for some $G_0 \leqslant_f G$ and $\Gamma_0 \leqslant_f \Gamma$, and that kG/Q is infinite. Then $\mathcal{L}(\Gamma, Q, \lambda)$ is abundant for each $\lambda \in kG \setminus Q$.*

This depends on the next lemma. For $p \nmid m \in \mathbb{N}$ let $f(m)$ denote the order of p modulo m, and $\phi(m)$ the Euler function.

Lemma 13.

(i) *Let C be a cyclic group of order m where $p \nmid m$. If $e \mid f(m)$ then the group algebra kC has $\varphi(m)e/f(m)$ faithful maximal ideals of index $p^{f(m)}$.*
(ii) *If $m = p^{te} - 1$ then $f(m) = te$.*
(iii) *$\varphi(p^{te} - 1)/t \to \infty$ as $t \to \infty$.*

Proof. (i) Put $f = f(m)$ and $F = \mathbb{F}_{p^f}$. Then F^* is cyclic of order $p^f - 1$, so contains exactly $\varphi(m)$ elements of order m. Hence there are exactly $\varphi(m)$ monomorphisms $\theta : C \to F^*$. As $e \mid f$ the field F contains k, so each such θ extends to a k-algebra homomorphism $\bar{\theta} : kC \to F$. If $|\bar{\theta}(kC)| = p^{f_1}$ then $m \mid p^{f_1} - 1$, so in fact $f_1 = f$ and $\bar{\theta}$ is surjective. Thus $\ker \bar{\theta}$ is a faithful maximal ideal of index p^f in kC, and every such ideal arises this way. Now $\mathrm{Gal}(F/k)$ permutes the set Θ of all such $\bar{\theta}$ by composition, each orbit has length f/e, and members of Θ lie in the same orbit if and only if they have the same kernel.

(ii) is trivial, and (iii) is easy. □

Now to prove the proposition, we have to show that $\mathcal{L}(\Gamma, Q, \lambda)$ is not merely non-empty, but abundant. Remark 3 may again be used to replace G and Γ by suitable subgroups of finite index: in the notation of Remark 3, write $\mathcal{L}_H = \mathcal{L}(\Delta, Q_0, \nu)_{kH}$ and $\mathcal{L}_G = \mathcal{L}(\Gamma, Q, \lambda)_{kG}$, and suppose that $L_0 \in \mathcal{L}_H$ and L are as in the discussion of Remark 3. Then $L \in \mathcal{L}_G$ and

$$\left|\frac{kG}{L}\right| = \left|\frac{kH}{L_0}\right|^r$$

where $1 \leqslant r \leqslant |G : H| = m$, say. It follows that $|\mathcal{L}_H(f)| \leqslant \sum_{i=1}^{m} |\mathcal{L}_G(rf)|$, and so $\mathcal{L}_G$ is abundant if $\mathcal{L}_H$ is.

Replacing (G, Γ) by (G_0, Γ_0^e), we may assume that (G, Γ) is a strict Brookes pair, in particular that $\mathrm{spec}(G) \subseteq \{p\}$ and G is torsion-free, and that $\chi(\Gamma) \leqslant \langle p^e \rangle$ for each character χ with $G_\chi \neq 1$.

The inductive steps in the proof of Theorem 3 now carry over essentially verbatim; two core cases remain that need special treatment.

Case 1: where $Q = 0$ and G is a plinth for Γ. In this case G is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}[\frac{1}{p}]$ and Γ acts on G via a character $\chi : \Gamma \to \langle p^e \rangle$. If $\chi(\gamma) = p^{ne}$ with $n \geqslant 0$ then $a^\gamma = a^{p^{ne}}$ for each $a \in kG$, so every maximal ideal (indeed, every prime ideal) of kG is Γ-invariant. Now G has an infinite cyclic subgroup C such that G/C is a divisible p-group. If $0 \neq \lambda \in kG$ then $\lambda_1 = \lambda^{p^n} \in kC \setminus \{0\}$ for some $n \in \mathbb{N}$, and we have

$$\mathcal{L}(\Gamma, 0, \lambda) = \mathcal{L}(1, 0, \lambda_1).$$

In view of Corollary 1, it therefore suffices to show that the set of maximal ideals of kC not containing λ_1 is abundant. As λ_1 is contained in only finitely many maximal ideals of kC, this follows from Lemma 13.

Case 2: where $Q > 0$ and there exists a prime ideal $M > Q$ of kG such that $\mathrm{ht}(M/Q) = 1$ and kG/M is finite.

The principle is the same as in Case 1, but a little more justification is required. Since $\mathrm{spec}(G) \subseteq \{p\}$, there is a finitely generated subgroup N of G such that G/N is a divisible p-group. Write $Q_0 = Q \cap kN$ and $M_0 = M \cap kN$. Then $kN/Q_0 = R$ is a finitely generated k-algebra and M_0/Q_0 is a maximal ideal of height 1; it follows that R has Krull dimension and transcendence degree equal to one (cf. [E, §8.2]). Fix an element b of infinite order in $N \cap G_\chi$ where χ is as above. Since all non-zero ideals of $k\langle b\rangle$ have finite index, while Q is faithful, we have $Q \cap k\langle b\rangle = 0$. Hence

$$k\langle b\rangle \cong \frac{k\langle b\rangle + Q_0}{Q_0} = S \subseteq R = \frac{kN}{Q_0}.$$

Then R is a finitely generated k-algebra and is algebraic over S, so R contains a subring $T \supseteq S$ such that T is finitely generated as an S-module and $R \subseteq T[\mu^{-1}]$ for some $\mu \in S \setminus \{0\}$. Say T is generated by m elements as an S-module.

As in Case 1, it suffices to show that if $\lambda_1 \in kN \setminus Q_0$ then the set $\mathcal{L}(\Gamma, Q, \lambda_1)$ is abundant. Let $\bar{\lambda}$ denote the image of λ_1 in R. Then $0 \neq \bar{\lambda}\mu^t \in T$ for some $t \in \mathbb{N}$, and as T is integral over S there exists $\nu \in k\langle b\rangle$ such that $\bar{\nu} = \nu + Q_0$ satisfies

$$0 \neq \bar{\nu} \in \bar{\lambda}\mu^t T \cap S.$$

As above, Lemma 13 shows that the set $\mathcal{M}$ of maximal ideals of S not containing $\bar{\nu}\mu$ is abundant.

Let $L \in \mathcal{M}(f)$. Then $L = S \cap L_1$ where $L_1 \lhd_{\max f} T$ and $|T/L_1| = p^{rf}$ for some r with $1 \leqslant r \leqslant m$. As $\mu \notin L_1$, we have

$$L_1 R \cap T = L_1 \quad \text{and} \quad L_1 R + T = R;$$

it follows that $L_1 R = L_2$, say, is a maximal ideal of R and that $|R/L_2| = |T/L_1|$. Writing $L_2 = L_3/Q_0$ we see that L_3 is a maximal ideal of kN with

$$\left|\frac{kN}{L_3}\right| = \left|\frac{T}{L_1}\right| = p^{rf}.$$

Now Proposition 1 and Corollary 1 show that $\sqrt{L_3} = L_4$, say, is a maximal ideal of kG containing $Q = \sqrt{Q_0}$ and satisfies $|kG/L_4| = |kN/L_3| = p^{rf}$. Note that

$$\frac{(L_4 \cap k\langle b\rangle) + Q_0}{Q_0} = L_2 \cap S = L.$$

Suppose that $\lambda_1^\gamma \in L_4$ for some $\gamma \in \Gamma$. Then

$$\nu^\gamma \in \lambda_1^\gamma kG + Q \subseteq L_4.$$

If $\chi(\gamma) = p^{en}$ ($n \in \mathbb{N}$) then $\nu^\gamma = \nu^{p^{en}}$, while if $\chi(\gamma) = p^{-en}$ then $(\nu^\gamma)^{p^{en}} = \nu$. Hence in either case we have $\nu \in L_4 \cap k\langle b\rangle$, whence $\bar{\nu} \in L$, contradicting the choice of L.

Thus $L_4 \in \mathcal{L}(\Gamma, Q, \lambda_1)(rf)$ where $1 \leqslant r \leqslant m$. As L_4 uniquely determines L, this shows that

$$\left|\mathcal{M}(f)\right| \leqslant \sum_{i=1}^{m} \left|\mathcal{L}(\Gamma, Q, \lambda_1)(rf)\right|.$$

Since $\mathcal{M}$ is abundant it follows that $\mathcal{L}(\Gamma, Q, \lambda_1)$ is too, and the proof is complete.

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