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Essays on Credit and Default under the Presence of Collateral



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Abstract

This thesis is a collection of three essays, which study a range of topics related to leverage, credit and default under the presence of collateral. The essay on belief disagreement and leverage proposes a novel approach in general equilibrium frameworks to assess how beliefs affect equilibrium and contract origination. Binding collateral constraints raise asset prices above the fundamental value. The model establishes that changes in market average belief rather than belief disagreement drives the asset price movement. The essay on mortgages and Minsky effect presents a micro-founded framework to study the macroeconomic effect of mortgage default, and shows that the endogenous leverage of the mortgage contract together with the bank capital determine the comovement of prices, leverage and credit extension. Default is more severe after good states than after bad states, due to high leverage and risk-taking. Minsky effect amplifies the financial instability. The essay on systemic risk measurement presents a calibration method for equilibrium models of banking systems, identifies channels that co-determine the distress dependence structures in the banking system, and the results suggest that the "Minsky effect", in addition to the widely accepted feedback channels such as the interbank lending and common exposure, serves as a systemic risk amplification mechanism.

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Contents

Figures **viii**

Tables **x**

1 Introduction **1**

1.1 Leverage, collateral and belief 2

1.2 Mortgages and the credit cycles 6

1.3 Systemic risk measurement and Minsky effect 10

2 Belief Disagreement, Collateral, and Endogenous Leverage **14**

2.1 Introduction 15

2.1.1 Related literature 20

2.2 The Model 23

2.2.1 The Optimization Problem 25

2.2.1.1 The pessimistic traders 25

2.2.1.2 The optimistic traders 26

2.2.2 The Equilibrium 27

2.2.2.1 Market Clearing Conditions 27

2.2.2.2 Contract Determination 28
2.2.3 Equilibrium Characterization 34
2.2.4 Numerical Examples 36
2.2.5 Belief heterogeneity 39
2.3 Extensions 44
2.3.1 Risk-free asset 44
2.3.1.1 The optimization problem 44
2.3.1.2 The equilibrium 46
2.3.1.3 Numerical example 47
2.3.2 Endogenous Trading Volume 50
2.3.2.1 The optimization problem 51
2.3.2.2 The equilibrium 52
2.3.2.3 Numerical example 53
2.4 Conclusion 56
2.A Non-convexity 57
2.B Proof of Theorem 1 58
2.C No-lending benchmark for the collateralized equilibrium 61
2.D Collateral equilibrium with risk-free asset 62
2.E Collateral equilibrium with endogenous trading volume 65

3 Mortgages, Financial Intermediation and Optimism 69

3.1 Introduction 70
3.1.1 Related literature 75
3.2 A mortgage model 78

3.2.1 The optimization problem 80
3.2.2 The equilibrium 84
3.2.3 Numerical Example 89
3.3 Credit cycle 92
3.3.1 The Economy 94
3.3.2 The Optimization Problem 98
3.3.3 Numerical Example 106
3.4 The Minsky effect 109
3.4.1 Beliefs 110
3.4.2 Comparative Statics 111
3.5 Conclusion 115
3.A First order conditions: Two-period mortgage model 117
3.B Benchmark economy 117
3.C Araujo's collateral equilibrium 119
3.D Definition of equilibrium, credit cycle model in Section 3.3 121
3.E Bayesian updating beliefs 124

4 An equilibrium framework for systemic risk: How economic theory meets statistics 126

4.1 Introduction 127
4.1.1 Related literature 137
4.2 The model 139
4.2.1 The economy 140
4.2.2 The banks' optimization problem 143

4.2.3 Central bank and regulator . . . 147
4.2.4 Private agent sector . . . 148
4.2.5 GDP . . . 148
4.2.6 Market clearing conditions . . . 149
4.3 CIMDO . . . 150
4.3.1 Computation method . . . 151
4.3.2 Dependence structure . . . 153
4.3.3 Expected shortfalls in profits . . . 154
4.4 Model assessment . . . 156
4.4.1 Equilibrium model revisited . . . 159
4.4.1.1 Default and distress dependence . . . 159
4.4.1.2 Incorporating the Minsky effect . . . 161
4.4.2 Data description . . . 163
4.4.3 Calibrating the risk-related deep parameters . . . 164
4.4.4 Model prediction without Minsky effect . . . 167
4.4.5 The Minsky effect . . . 170
4.4.6 Robustness . . . 174
4.5 Conclusion . . . 175
4.A First order conditions of the equilibrium model . . . 176
4.B Proof of Proposition 5 . . . 177
4.C Data . . . 179

5 Conclusions **183**

Bibliography **186**

Figures

2.1 Simulation of Baseline Model 41

2.2 Comovement of asset prices and leverage 43

2.3 Simulation with risk-free asset 49

2.4 Simulation with endogenous trading volume 55

3.1 House Price Index 71

3.2 Residential real estate loans, percentage change per year 72

3.3 Volatility S&P 500 (VIX) 72

3.4 Credit Cycle 93

3.5 The Economy 95

3.6 Time Structure 97

4.1 Simplified structure of banking system 130

4.2 Interbank channel of distress dependence 131

4.3 Common asset exposure channel of distress dependence 131

4.4 Systemic Risk Amplification / Minsky effect 132

4.5 Input and output variables in the equilibrium model 136

4.6 Input and output variables in CIMDO 136

4.7 Calibrating the Minsky effect 136

4.8 The Economy 141

4.9 The Time Structure 142

4.10 CIMDO-density, adjustment factor 153

4.11 Model Assessment 158

4.12 Expected Shortfalls in Profits, June 2007 170

4.13 Leverage 174

Tables

2.1 List of Variables 25

2.2 Exogenous variables 37

2.3 Endogenous Variables 37

2.4 Exogenous variables 47

2.5 Exogenous variables 54

3.1 List of Variables 81

3.2 Simulation, two-period mortgage model 90

3.3 List of Variables and Notations 99

3.4 Exogenous variables 107

3.5 Endogenous Variables, Credit Cycles 108

3.6 Beliefs 111

3.7 Endogenous variables, the Minsky effect 113

3.8 Bank's learning effects on financial contracts 113

3.9 Endogenous variables, the Minsky effect 115

3.10 All agents' learning effects on financial contracts 115

3.11 Comparison, two-period mortgage model 121

4.1 Commercial bank's balance sheet 142

4.2 Variables in the model 144

4.3 Variables for first-step model calibration 165

4.4 Variables for the second period model prediction without Minsky effect 169

4.5 Banks' expected shortfalls in profits in June 2007, measured by CIMDO 170

4.6 Variables for the second period with Minsky effect 172

4.7 Balance Sheet data of the UK banks 179

4.8 Values of variables in the first-step model calibration 180

4.9 Values of variables in th second period, without Minsky effect 181

4.10 Values of variables in the second period, with Minsky effect 182

Chapter 1

Introduction

This thesis is a collection of three papers, which study a range of topics related to leverage, credit and default under the presence of collateral. My research intends to extend the understanding of the trade-off between leverage and interest rate, how beliefs affect leverage and credit, as well as the systemic risk of the financial system.

There are two key concepts in the thesis. The first is collateralized loans. Classic economic theories usually assume that agents always repay their debt in full, without further requirements. When agents lack the ability to commit, either a punishment for default should be introduced as in Dubey et al. (2005), or collateral has to be pledged. The global financial crisis highlighted the importance and influence of collateral, thus my thesis focuses on models of loans with collateral requirement. The value of collateral is usually higher than the loan amount when borrowing occurs, and the percentage difference is referred to as the haircut or the margin. The contract leverage can be defined as the inverse of the margin. Given the maturity of the loans, the two most important characters are the required rate of return and collateral. The thesis proposes

a novel method to jointly determine the two variables under different model settings. The second key concept in the thesis is belief disagreement. Borrowers and lenders may have different beliefs on the future value of the collateral, and such difference creates frictions on the loan market. I do not explain why beliefs are different, but only look at how belief disagreement affects the prices and leverage. Moreover, one of the chapters models belief updating, and shows how the cyclicality of optimism contributes to the credit cycles and systemic risk.

1.1 Leverage, collateral and belief

The chapter, "Belief disagreement, collateral and endogenous leverage", provides a general equilibrium model with incomplete markets, collateral and default to assess how beliefs affect equilibrium and contract origination. Due to the lack of ability to commit repayment, collateral is required for loans. A collateralized loan is characterized by the leverage and required rate of return, and the question is to determine which contract is actively traded. I introduce the following mechanism to determine the equilibrium properties of the loans. The market offers a menu of collateralized debt contract with different leverage and required rate of return. Then borrowers choose one contract from the menu, knowing that different leverage level correspond to different required rate of return. Meanwhile, the lenders take the choice as given. Such a simple tractable mechanism determines uniquely the equilibrium leverage and rate of return for the loan as well as the underlying asset prices. When borrowers become more optimistic, they demand more assets and loans, thus raising the prices, whereas when lenders become more optimistic, they tend to extend more loans, since

they believe that the collateralized assets are less likely to lose value. Therefore, asset price is high when everyone is optimistic, and low when everyone is pessimistic. The effect on leverage varies under different model settings. The setup of the model is akin to Simsek (2013), who claims that optimism is asymmetrically disciplined in collateral equilibrium, and that belief disagreement moves the asset prices and leverage in the same direction. Also, the type of disagreement matters more than the disagreement itself. Unlike his model, I assume that investors are risk-averse and consume in both periods. Instead of using an arbitrage argument to pin down the leverage of collateralized loans, I propose an endogenous contract selection mechanism to determine leverage in equilibrium under different model settings.

General equilibrium models with incomplete markets (GEI) have accommodated default and bankruptcy, while preserving perfect competition, rational expectations, and the general existence of equilibrium. An earlier version of Dubey et al. (2005) is among the first to extend the standard GEI model to allow for default and punishment. When agents default, they suffer from a utility loss. The market chooses the promises, default penalties, and quantity constraints of actively traded assets. Zame (1993) argues that default plays an important positive role in the economy when markets are incomplete. It promotes efficiency by expanding possibilities for contingent contracting, which opening new markets does not achieve. Chatterjee et al. (2007) provides a theory for unsecured consumer loans, where competitive financial intermediaries offer a menu of loan sizes and interest rates, and the default option of the consumers resembles a bankruptcy filing under Chapter 7 of the U.S. Bankruptcy Code.

The roles of collateral have long been studied. Besanko and Thakor (1987) discuss the economic function of collateral and how the credit market structure affects its usefulness. In a monopolistic market structure, collateral is not used unless it's sufficiently valuable to the bank to make the loan riskless; while in a perfectly competitive market, banks can sort borrowers into risk classes by designing credit contracts with inversely related interest rates and collateral requirement. Bester (1987) states that under imperfect information, collateralization and credit rationing are alternative means to solve adverse selection and moral hazard problems. Lenders may use collateral as a self-selection and incentive mechanism.

Geanakoplos (1997) introduces collateral into the GEI framework. Instead of assuming that agents automatically keep all their promises, collateral must be pledged to back the loans. Default leads to foreclosure, without further utility loss. Geanakoplos (2003) adds heterogeneous beliefs, and shows that an increase in belief disagreements can considerably increases margins and decrease asset prices. Geanakoplos (2010) extends the model into multi-period settings, and brings about the concept of the leverage cycle. Market leverage is determined by the marginal buyers. Over-leveraged agents, who are the most optimistic, are removed from the market when a bad event occurs. Thus overall optimism and leverage are pro-cyclical. Araujo et al. (2002) endogenize the collateral level by allowing each consumer to choose the optimal durable bundle and the respective loan, knowing that the interest spread will depend on the collateral choice. Araujo et al. (2012) examine the effects of default and collateral on risk sharing, and demonstrate that the scarcity of collateralizable durable goods leads to large welfare losses. Regulation of collateral requirements never lead to Pareto improvement,

although it often benefits a large group of agents. My thesis adopts the definition of a collateralized loan contract in Geanakoplos and Zame (2014), who prove existence of the collateral equilibrium under general conditions. Under a binomial time structure with financial assets serving as collateral, Fostel and Geanakoplos (2015) prove that, any equilibrium is equivalent to another one with no default, and that equilibrium loan-to-value (LTV) ratio for noncontingent debt contract is determined by the worst case return of the underlying asset. My thesis deviates from their two assumptions. Chapter Two deviates from the assumption of *binomial time structure* by considering continuous payoff states. Chapter Three deviates from the assumption of *financial asset* by considering real assets that provides direct utility to the asset holder. My results highlight the effect of default in equilibrium.

The role of collateral and default has long been studied in the literature of contract theory. Stiglitz and Weiss (1981) show that increasing interest rates or increasing collateral requirements could increase the riskiness of the bank's loan portfolio, thus decreasing the bank's profits. Diamond (1984) provides some general analysis of the effects of diversification on resolving incentive problems, and shows that debt contracts with costly bankruptcy is optimal. Bester (1985) show that no credit rationing will occur in equilibrium if banks compete by choosing collateral requirements and the rate of interest simultaneously to screen investors' riskiness, thus using different contracts as a self-selection mechanism becomes possible. Chan and Kanatas (1985) show that collateral improves lenders' estimates of their expected returns when lenders and borrowers have different information. Besanko and Thakor (1987) show that banks can sort borrowers into risk classes by designing credit contracts with inversely re-

lated interest rates and collateral requirements. Boot et al. (1991) show that private information may either accentuate or retard the positive relationship between collateral requirements and borrower risk encountered with just moral hazard, thus higher collateral may be posted by either safer or riskier borrowers, making secured lending widespread despite deadweight cost associated with collateral.

My thesis models the belief heterogeneity and disagreement as different subjective probability density functions on the payoff states. Fostel and Geanakoplos (2008) consider belief heterogeneity as different subjective probabilities in a binomial structure and show how leverage cycles can cause contagion, flight to collateral and issuance rationing. Shleifer and Vishny (2010) use a different term, "investor sentiment", and assume that it affects the price of securitized debt directly. Banerjee and Kremer (2010) show that when investors have infrequent but major disagreements there are positive autocorrelation in volume and positive correlation between volume and volatility. My modelling of belief disagreement is close to Simsek (2013). However, in the numerical examples of Chapter Two, I choose simplified linear belief functions, and compute equilibria with different value of slopes to resemble different level of optimism/pessimism.

1.2 Mortgages and the credit cycles

The global financial crisis of 2007-2009 is considered the worst since the Great Depression in the 1930s. During the crisis, the burst of the U.S. housing bubble caused the values of securities tied to U.S. real estate pricing to plummet, damaging the financial

institutions globally. The chapter, "Mortgages, Financial Intermediation and Optimism[1]", introduces general equilibrium models with collateral and default to study the mortgage market, the credit cycles and the Minsky effect. As documented by Fostel and Geanakoplos (2016), the leverage, prices and investment in the US housing market moved together during the crisis. This implies some interesting relationship among those economic variables. When demand in housing market starts to drop, house price declines, depressing the demand for mortgage loans. But why does this lead to an unanticipated crisis? We know that due to the low credit quality in subprime mortgages, defaults occurred inevitably when the house price plummeted. Due to financial innovation, mainly securitization, the crisis in the mortgage market quickly spread into the whole financial sector. The anxiety in the market crystallized, persisting and amplifying the crisis. This chapter aims at offering an coherent model to include house price, leverage, default, optimism, and financial intermediaries.

There is a wide range of literature on borrowing constraints and credit cycles. Shleifer and Vishny (1992) identify the fire-sale mechanism in which a firm in financial distress sells assets when its industry peers are likely to be experiencing problems. Holmstrom and Tirole (1997) show that when firms and financial intermediaries are capital constrained, capital tightening hits poorly capitalized firms the hardest, but that interest rate effects and the intensity of monitoring will depend on relative changes in the various components of capital. Kiyotaki and Moore (1997) construct a dynamic economy in which durable assets are both factors of production and collateral for loans, and show that the dynamic interaction between credit limits and asset prices is a pow-

[1]This is a joint work with Prof. Dimitrios Tsomocos

erful transmission mechanism by which the effects of shocks persist, amplify and spill over to other sectors. Bernanke et al. (1999) develop a DSGE model which clarifies the role of credit market frictions in business fluctuations. The framework exhibits a financial accelerator, where endogenous developments in credit markets work to amplify and propagate shocks to the macroeconomy. Brunnermeier and Pedersen (2009) link an asset's market liquidity and traders' funding liquidity, and show that under certain conditions, margins are destabilizing and market liquidity and funding liquidity are mutually reinforcing, leading to liquidity spirals. Their model explains why market liquidity can suddenly dry up, has commonality across securities, is related to volatility, and is subject to flight to quality and co-moves with the market. Adrian and Shin (2010) document evidence that there is a strongly positive relationship between changes in leverage and changes in balance sheet size. Financial intermediaries adjust their balance sheets actively, and doing so in such a way that leverage is high during booms and low during busts. So leverage is procyclical, and this affects volatility and particularly the price of risk.

The mortgage model in the thesis is closely related with Goodhart et al. (2010), which incorporates heterogeneous banks and households, as well as collateral backed mortgages. My model highlights the importance of collateral constraints. Other mortgage models include Campbell and Cocco (2003) who study the household choice between a fixed-rate and an adjustable-rate mortgage. Iacoviello (2005) develops a monetary business cycle model with nominal loans and collateral constraints tied to housing values. Forlati and Lambertini (2011) develop a DSGE model with housing, risky mortgages and endogenous default. Various explanations have been offered for

the housing bubble and the mortgage crisis. Lack of incentive to monitor due to securitization, which leads to the loosening of lending standard and weakening of credit quality is one of the most popular explanations, such as in Demyanyk and Van Hemert (2009), Duca et al. (2010), Keys et al. (2010), Purnanandam (2010), DellAriccia et al. (2012), etc. Others argue that the housing bubble is rather rational, and based on expectation about housing price, without the finance insiders deceiving the uninformed mortgage borrowers and investors (Foote et al. (2012), Case et al. (2012)). Greenlaw et al. (2008), Mian and Sufi (2009, 2011) provide evidence on the bank credit and leverage. Iacoviello and Neri (2010) suggest that housing demand and housing technological shock also explain the volatility in the housing investment and prices.

Bank capital is always an important issue in financial stability. A series of work by Diamond and Rajan on bank capital, liquidity, financial fragility and bank banking crisis (Diamond and Rajan (2000, 2001, 2005, 2009, 2011)) suggest that optimal bank capital structure trades off effects on liquidity creation, costs of bank distress, and the ability to force borrower repayment. Fragility commits banks to creating liquidity, enabling depositors to withdraw when needed, while buffering borrowers from depositors' liquidity needs. Stabilization policies, such as capital requirements, narrow banking, and suspension of convertibility, may reduce liquidity creation. However, the fact that banks make hard-to-collect relationship loans and can influence the production decisions of their borrowers may lead to bank failures. The bank failures can shrink the common pool of liquidity, creating, or exacerbating aggregate liquidity shortages, which may lead to contagion or total meltdown of the system. Kashyap and Stein (2000) show that the impact of monetary policy on lending is stronger for banks with

lower ratios of securities to assets, and the pattern is largely attributable to the smaller banks. Allen et al. (2011) suggest that, in perfectly competitive markets, banks can find it optimal to use costly capital rather than the interest rate on the loan to commit to monitoring, since it allows higher borrower surplus.

1.3 Systemic risk measurement and Minsky effect

The chapter, "An equilibrium framework for systemic risk: How economic theory meets statistics[2]", provides a novel approach combining theoretical and statistical methods to measure the systemic risk, and to identify different channels that contribute to the distress dependence structure in the banking system, especially the Minsky effect. Minsky (1977, 1992) propose the Financial Instability Hypothesis, which states that, over a prolonged prosperity, optimism changes, and agents tend to invest more in the riskier assets. This behavior is the source of financial instability in the economy. Bhattacharya et al. (2015) provide a formalized model, by considering investors updating beliefs following the Bayes' rule. Whenever a good state realizes, investors become more optimistic. After several good realizations, the investors become so optimistic that they invest in assets which are considered too risky were it not for the belief updating. This chapter modifies and extends the model by Goodhart, Sunirand, and Tsomocos (2005) into a multi-period setting, incorporates a systemic risk amplification mechanism (the Minsky effect), and offers a calibration method using Segoviano (2006)'s CIMDO approach to measure the deep parameters related with the banks' risk-taking, altitude towards default, and the Minsky effect.

[2]This chapter is a joint work with Dr. Raphael Espinoza and Dr. Miguel Segoviano

The Consistent Information Multivariate Density Optimizing (CIMDO) by Segoviano (2006), is a non-parametric method to construct a consistent multivariate density from a very limited amount of information. Based on the Kullback (1959) cross-entropy approach, CIMDO infers the unknown multivariate distribution of asset values in a system of firms containing different risk quality. The recovered distribution is closest to the prior distribution but is also consistent with the empirically observed probabilities of default. Segoviano and Espinoza (2016) apply CIMDO to estimate several indicators of systemic risk in a network of institutions representing the global banking and shadow banking system. In this chapter, we adopt the estimates for expected equity growth rate to measure the conditional profit/loss in different states. Both CIMDO and the equilibrium model are able to estimate the conditional expected profit/loss, thus provide a valid metric to compare the two approaches, and enable us to identify the implied channels that contributes to the distress dependence structure of the banking system.

General equilibrium provides useful framework for research on financial stability. Tsomocos (2003) extends the canonical GEI model with money and default to allow for competitive banking and financial instability. Capital requirements are introduced for the banking sector to access the short and medium term macroeconomic consequences of the Basel II accord. Goodhart et al. (2004, 2006a) illuminate problems relating to individual bank behavior, possible contagious inter-relationships among banks, and the appropriate design of prudential requirements and incentives to limit excessive risk-taking. The model includes heterogeneous agents, endogenous default, multiple commodity, and credit and deposit markets, and indicates how monetary

policy may affect financial fragility, thus highlighting the trade-off between financial stability and economic efficiency. Bhattacharya et al. (2007) prove that sequential contagion is caused via the interbank market channel even when banks specialize in their portfolio composition and their returns are uncorrelated. Limited participation emerges naturally whenever banks incorporate a relative performance criterion in their maximization problem. Goodhart et al. (2012, 2013) consider a financial system including both banks and shadow banks, and assess the effects and interactions of different policy options such as limits on loan to value ratios, capital requirements for banks, liquidity coverage ratios for banks, dynamic loan loss provisioning for banks, and margin requirements on repurchase agreements used by shadow banks.

Literature provides different measures of systemic risk. Lehar (2005) measures the risk of a regulator's portfolio by modelling the individual liabilities that the regulator has to each bank as contingent claims on the bank's asset. Billio et al. (2012) propose econometric measures of connectedness based on principal-components analysis and Granger-causality networks, and such measures can identify and quantify financial crisis periods. Huang et al. (2009, 2012) measure the systemic risk a banking sector as a hypothetical distress insurance premium, which is the insurance cost to protect against distressed losses. The measure indicates market perceived risk, which reflects expected default risk of individual banks, risk premia as well as correlated defaults. Adrian and Brunnermeier (2016) propose a measure of systemic risk, ΔCoVaR, defined as the change in the value at risk of the financial system conditional on an institution being under distress relative to its median state. Reversing the conditioning illustrates how much a particular institutions risk increases given that the whole financial

system is in distress. Acharya et al. (2017) measure each financial institution's contribution of systemic risk as its systemic expected shortfall, which is the propensity to be undercapitalized when the system as a whole is undercapitalized.

Chapter 2

Belief Disagreement, Collateral, and Endogenous Leverage

This chapter proposes a novel approach in general equilibrium frameworks to assess how beliefs affect equilibrium and contract origination. The market offers a menu of collateralized debt contracts with different leverage and interest rates. Borrowers select a contract from the menu, whereas lenders take the choice as given. Binding collateral constraints raise asset prices above the fundamental value. The model establishes that changes in market *average* belief rather than *belief disagreement* drives the asset price movement. When a risk-free asset is absent, leverage comoves with asset prices. Extensions of the model considers different market structures. When a risk-free asset exists, the leverage decreases with belief disagreement. When the trading volume of the underlying asset is endogenous, the leverage increases with belief disagreement.

2.1 Introduction

Leverage was at the centre of the global financial crisis in 2007. When a trader borrows to purchase securities, or a household contracts mortgage loans to purchase houses, collateral has to be pledged. The *margin* or *haircut*, is the percentage difference between the value of the collateral and the loan amount, while the *leverage* is the inverse of the margin. When borrowers become more optimistic, they borrow more, thus boosting asset prices; whereas when lenders become more optimistic, they extend credit more easily, raising the price as well. However, the way beliefs affect leverage is not that straightforward.

Within a general equilibrium model with incomplete markets, this paper shows that beliefs affect the relationship between leverage and the underlying asset price in a nontrivial manner. Due to the lack of ability to commit repayment, collateral is required for loans. In the paper a collateralized loan is characterized by the leverage and required rate of return. Obviously there can be different loan contracts, and the question is which contract is actively traded. I introduce the following mechanism to determine the loan characters in equilibrium. The market offers a menu of collateralized debt contract with different leverage and required rate of return. Then borrowers choose one contract from the menu, knowing that different leverage level correspond to different required rate of return. Meanwhile, the lenders take the choice as given. Such a simple tractable mechanism uniquely determines the equilibrium leverage and rate of return for the loan as well as the underlying asset prices. When borrowers become more optimistic, they demand more assets and loans, raising the prices, whereas when

lenders become more optimistic, they tend to extend more loans, since they believe that the collateralized assets are less likely to lose value. Therefore, asset price is high when everyone is optimistic, and low when everyone is pessimistic. The effect of beliefs on leverage varies through different channels. In the baseline model with inelastic asset supply and no risk-free asset, the leverage co-moves with the asset price in most cases.

The baseline model establishes that it is the market *average* belief rather than the *belief disagreement* that drives the cyclicality and the co-movement of asset price and market leverage. However, under certain conditions, asset prices and leverage can move in opposite directions. In addition, when collateral constraints are binding, the underlying asset prices are above their fundamental value, and this difference captures the collateral value. Due to the collateral constraints, the collateral value is more volatile than the fundamental asset value, contributing to the asset price volatility.

The setup of the model is akin to Simsek (2013). However, unlike in his model, investors are risk-averse and consume in both periods. Instead of using an arbitrage argument to pin down the leverage of collateralized loans, I assume away the existence of the risk-free asset in the baseline model, and propose an endogenous contract selection mechanism to determine leverage in equilibrium[1]. Consider a two-period economy with a security and a continuous state space in the second period. The security provides no direct utility to the holder, and the payoff is the same regardless of the owner's identity. Two types of traders, with different beliefs on the distribution of the future

[1]I recently came across the work by Bailey et al. (2017) who use similar arguments to explain how house price beliefs affect the household mortgage leverage choice. However, I arrive at substantially different results and implications.

states, borrow from each other and invest in the security. The optimistic traders value the securities more, thus borrow from the pessimistic traders to purchase the securities, pledging which as collateral. If the realization of the collateral value is less than the debts face value, borrowers default and the only punishment is the foreclosure of collateral. There is no asymmetric information and all traders have rational expectations. Let there be a menu of collateralized debt contracts with different face value but the same collateral requirement in the market. Evidently, these contracts have different prices thus different interest rates. The borrowers are able to choose not only the quantity but also the type of the contracts, and therefore realize that contracts with different face values should have different prices and interest rates. The lenders take the menu of contracts as given, and the borrowers choose the optimal contract according to their preferences. Consequently, leverage, interest rates and asset prices are determined endogenously in equilibrium.

The main implication of the model is that beliefs affect asset prices and market leverage in a non-trivial way. If the beliefs of the borrowers and lenders change in the same direction, asset prices and market leverage co-move. However, in some extreme cases, if the two types of traders' beliefs move in opposite directions, it is also possible the asset prices and leverage to move in opposite directions. Mainstream literature emphasizes the effect of belief disagreement. Patton and Timmermann (2010) show that belief disagreement persists through time, and that differences in opinion move countercyclically, with heterogeneity being strongest during recessions. Buraschi et al. (2014) show that belief disagreement is positively related to the wedge between index and individual volatility risk premia, the different slope of the smile of index and

individual options, and the correlation risk premium. This paper, however, focuses on the mean rather than the variance of beliefs. I compute the equilibrium grid with different combinations of borrower/lender beliefs, *ceteris paribus*. In boom times, both types of traders become more optimistic, thus both asset prices and leverage increase, whereas in recessions, both types of traders become more pessimistic, hence both asset prices and leverage decrease. These results are consistent with the empirical evidences in Adrian and Shin (2010) that mark-to-market leverage and asset prices are strongly procylical, and in Gorton and Metrick (2012) that repo haircut index raised from zero in early 2007 to nearly 50% at the peak of the crisis in late 2008. Theoretical results in Simsek (2013) indicate that an increase in belief disagreements can increase or decrease asset prices and leverage together, whose directions depend on the concentration of belief disagreement changes. My simulation unifies their results, thus rendering Simsek (2013) a special case of the model, with additional testable implications. When the beliefs of the borrowers and lenders move in opposite direction in certain cases, it is possible for the asset price and leverage to move in opposite directions as well.

Whenever collateral constraints are binding, the asset price is above its fundamental value, and the difference represents the collateral value of the asset. In other words, the collateral value is the Lagrangian multiplier of the collateral constraint, which stays positive if the constraint binds. Intuitively, the asset provides dividend payoffs that are captured by the fundamental value, while it also allows the borrower to collateralize the asset holdings that generate further collateral value. These results are consistent with the theoretical work by Fostel and Geanakoplos (2008).

The equilibrium conditions of the model preserve the "asymmetric disciplining of optimism" property of Simsek (2013), which indicates that pessimistic beliefs are used to assess the probability of default, whereas optimistic beliefs are used to assess the value of the asset conditional on default not occurring. The optimistic beliefs, which emphasis on the upside states, may apply in the pre-crisis periods when Greenspan and Shiller argued that the "irrational exuberance had unduly escalated asset values". After a Minsky moment, the opposite could happen. The pessimistic beliefs apply during a crisis, when asset prices collapse and leverage plummets. The model generalizes Simsek (2013) who assumes risk neutral traders and the existence of an elastically supplied risk-free asset. Consequently, my argument with risk-averse traders, intertemporal consumption smoothing and endogenous contract origination is technically more demanding, however, much more closely related to the economic reality.

Extensions of the model considers the case with risk-free asset and elastic supply of the underlying asset, respectively. With the presence of a risk-free asset, the traders can save their initial-period endowments to the second period costlessly. The pessimistic traders face a portfolio problem of choosing between the collateralized debt and the risk-free asset, thus the shape of the contract menu is different from the previous case. The optimistic traders' trade-off between higher leverage and higher promised return changes accordingly, and the simulation results imply that more optimistic borrowers may demand lower leverage to take advantage of the lower repayment rate of the loans. This makes the leverage decreasing with belief disagreement. In a different case without risk-free asset but with elastic supply of the security, I assume that the pessimists initially own the security. Optimists purchase the security and

borrow from the optimists, pledging the security as collateral. In equilibrium, the pessimists keep part of their security endowment, and the equilibrium trading volume and the collateral amount are endogenous. Similarly, this assumption leads to a different contract menu as well, and the leverage in this case increases with belief disagreement.

2.1.1 Related literature

General equilibrium models with incomplete markets (GEI) have accommodated default and bankruptcy, while preserving perfect competition, rational expectations, and the general existence of equilibrium. An earlier version of Dubey et al. (2005) is among the first to extend the standard GEI model to allow for default and punishment. When agents default, they suffer from a utility loss. The market chooses the promises, default penalties, and quantity constraints of actively traded assets. Zame (1993) argues that default plays an important positive role in the economy when markets are incomplete. It promotes efficiency by expanding possibilities for contingent contracting, which opening new markets does not achieve. Chatterjee et al. (2007) provides a theory for unsecured consumer loans, where competitive financial intermediaries offer a menu of loan sizes and interest rates, and the default option of the consumers resembles a bankruptcy filing under Chapter 7 of the U.S. Bankruptcy Code.

Geanakoplos (1997) introduces collateral into the GEI framework. Instead of assuming that agents automatically keep all their promises or assuming non-pecuniary default penalties, collateral must be pledged to back the loans. Default leads to foreclosure, without further utility loss. Geanakoplos (2003) adds heterogeneous beliefs, and shows that an increase in belief disagreements can considerably increases margins

and decrease asset prices. Geanakoplos (2010) extends the model into multi-period settings, and brings about the concept of the leverage cycle. Market leverage is determined by the marginal buyers. Over-leveraged agents, who are the most optimistic, are removed from the market when a bad event occurs. Thus overall optimism and leverage are pro-cyclical. Araujo et al. (2002) endogenize the collateral level by allowing each consumer to choose the optimal durable bundle and the respective loan, knowing that the interest spread will depend on the collateral choice. Araujo et al. (2012) examine the effects of default and collateral on risk sharing, and demonstrate that the scarcity of collateralizable durable goods leads to large welfare losses. Regulation of collateral requirements never lead to Pareto improvement, although it often benefits a large group of agents. In this chapter I adopt the definition of a collateralized loan contract in Geanakoplos and Zame (2014), who prove existence of the collateral equilibrium under general conditions. Under a binomial time structure with financial assets serving as collateral, Fostel and Geanakoplos (2015) prove that, any equilibrium is equivalent to another one with no default, and that equilibrium loan-to-value (LTV) ratio for noncontingent debt contract is determined by the worst case return of the underlying asset. My thesis deviates from their two assumptions, and highlights the effect of default in equilibrium.

This chapter models the belief heterogeneity and disagreement as different subjective probability density functions on the payoff states. Fostel and Geanakoplos (2008) consider belief heterogeneity as different subjective probabilities in a binomial structure and show how leverage cycles can cause contagion, flight to collateral and issuance rationing. Shleifer and Vishny (2010) use a different term, "investor sentiment", and

assume that it affects the price of securitized debt directly. Banerjee and Kremer (2010) show that when investors have infrequent but major disagreements, there are positive autocorrelation in volume and positive correlation between volume and volatility. My modelling of belief disagreement is close to Simsek (2013). However, in the numerical examples, I choose simplified linear belief functions, and compute equilibria with different value of slopes to resemble different level of optimism/pessimism.

My paper is closed related to the large literature on borrowing constraints, credit and liquidity cycles, such as Shleifer and Vishny (1992), Holmstrom and Tirole (1997), Bernanke et al. (1999). Kiyotaki and Moore (1997) emphasize the feedback from the fall in collateral prices to a fall in borrowing capacity. Durable assets play as factors of production and collateral for loans. Gromb and Vayanos (2002) consider the case where specific collateral requirements limit liquidity supply, and show that the market outcome can fail to be allocatively efficient. Brunnermeier and Pedersen (2009) relate an asset's market liquidity and traders' funding liquidity, and show that margins can be destabilizing and market liquidity and funding liquidity are mutually reinforcing, leading to liquidity spirals. Garleanu and Pedersen (2011) show how securities' required returns increase in both their betas and their margin requirements given heterogeneous-risk-aversion agents facing margin constraints. Brunnermeier and Sannikov (2014) show that endogenous leverage determines the distance to crisis, and that securitization and derivatives contracts that improve risk sharing may lead to higher leverage and more frequent crises.

The rest of the chapter is organized as follows. Section 2.2 introduces the basic collateral equilibrium in a two-period model with a continuum of payoff states, establishes the endogenous determination of the collateralized debt contract, and reveals the relationships among belief heterogeneity, asset prices, and leverage. Section 2.3 considers two extension of the basic model, by adding risk-free asset and endogenous trading volume of the security. Section 2.4 concludes the paper.

2.2 The Model

Consider a two period economy ($t = 0, 1$) with two types of traders, atomistic and of equal mass. There is a perishable consumption good which serves as numeraire, and a security with inelastic supply a from an outsider. The security sells at an endogenous price p in terms of the consumption good at $t = 0$, and pays a random amount of dividend $s \in (0, \bar{s})$ of the consumption good at $t = 1$. The traders disagree on the distribution of s, with subjective probability density function $f^o(s)$ for the optimistic traders and $f^p(s)$ for the pessimistic traders, where $\int_0^{\bar{s}} f^o(s)sds > \int_0^{\bar{s}} f^p(s)sds$, and the hazard rates satisfy $h^o(s) < h^p(s), \forall s \in (0, \bar{s})$[2]. There is no risk-free asset, and the traders are endowed with strictly positive amount of the consumption good at both periods $(e_0^h, e_1^h, h \in \{o, p\})$. The types and endowments are all public information.

The Collateralized Debt Contract I define the debt contract as in Geanakoplos and Zame (2014). Assume that borrowers cannot commit to repay, thus a debt always requires collateral. When the value of collateral falls below the debt's face value, the

[2]Hazard rate $h(s) = \frac{f(s)}{1-F(s)}$, where $f(s)$ is the probability density function, and $F(s)$ is the cumulative density function.

borrower simply walks away from the contract and gives up the collateral. Let there be a continuous menu of collateralized debt contracts with face value $j \in (0, \bar{s})$, each sells at a price $\pi(j)$ at t=0, repays at $t = 1$, taking 1 unit of the security as collateral. A face value above the highest possible future value of the collateral leads to default at all states and such contract has the same payoff as the one with face value of $\bar{s}$. Hence, all contracts with face value above $\bar{s}$ are redundant. The price of the debt contract, $\pi(j)$, or the amount that can be borrowed with one unit of collateral, increases with the face value. This is because, higher face value increases the payment in the upside states, while keeping the downside states payment unchanged, thus raising the price of the debt. If the security's realized payoff at $t = 1$ is high ($s \geqslant j$), the borrowers keep their promises, repay the debt, and get back the collateral. If the security's realized payoff is low ($s < j$), the borrowers walk away from the contract, and give up the collateral. There are no penalties, pecuniary or non-pecuniary, other than the foreclosure of the collateral.

Time Structure The optimistic traders borrow from the pessimistic traders to purchase the security. Assume that the market is competitive.

1) Both types of traders enter the economy at the beginning of $t = 0$;

2) The optimistic traders purchase the security from the outsider, and borrow from the pessimistic traders, pledging the security as collateral;

3) The traders consume at the end of $t = 0$;

4) At the beginning of $t = 1$, security return realizes, and the optimistic traders choose whether to repay or to default;

5) The traders consume at the end of $t = 1$.

2.2.1 The Optimization Problem

Table 2.1: List of Variables

Subscripts and Superscripts	
$t \in \{0, 1\}$	time periods
$\bar{s}$	highest possible payoff of the security
$s \in S = (0, \bar{s})$	the set of states
$h \in \{o, p\}$	the set of traders
Exogenous Variables and Functions	
e_t^h	trader h's endowments of consumption goods at time t
β^h	trader h's discount factor
a	security supply
$f^h(s)$	trader h's subjective PDF on the security payoff
Endogenous Variables and Functions	
p	security price
$\pi(\cdot)$	debt price as a function of the face value
j	debt face value
c_0^h	consumption of goods by trader h at $t = 0$
c_s^h	consumption of goods by trader h at state s
ϕ	number of debt contracts traded
r	contract required rate of return
LTV	loan-to-value ratio of the debt
FOD	first-order derivative of the contract menu curve
SOD	second-order derivative of the contract menu curve
FV	fundamental value of the security
CV	collateral value of the security
λ	Lagrangian multiplier of the collateral constraint

2.2.1.1 The pessimistic traders

The pessimistic traders are the lenders in the debt market. They choose the consumptions c_0^p, c_s^p and the unit of debt contract to offer ϕ^p to maximize the expected

utility,

$$\max_{c_0^p,\phi^p,c_s^p} \bar{U}^p = U(c_0^p) + \beta^p \int_0^{\bar{s}} U(c_s^p) f^p(s) ds \tag{2.2.1}$$

subject to the budget set $\mathcal{B}_1^p(p, j, \pi(\cdot))$ defined by inequalities (2.2.2)∼(2.2.3)

$$c_0^p + \pi(j)\phi^p \leqslant e_0^p \tag{2.2.2}$$

(consumption at $t = 0$ plus the lending should not exceed endowments)

$$c_s^p \leqslant e_1^p + \phi^p \min[s, j] \quad \forall s \in (0, \bar{s}) \tag{2.2.3}$$

(consumption at $t = 1$ should not exceed endowments plus the debt repayment)

2.2.1.2 The optimistic traders

The optimistic traders are the borrowers in the debt market. They choose the consumptions c_0^o, c_s^o, the holding of the security a^o, and the face value j^o and position of the debt contract ϕ^o to maximize the expected utility,

$$\max_{c_0^o,a^o,\phi^o,j^o,c_s^o} \bar{U}^o = U(c_0^o) + \beta^o \int_0^{\bar{s}} U(c_s^o) f^o(s) ds \tag{2.2.4}$$

subject to the budget set $\mathcal{B}_1^o(p, \pi(\cdot))$ defined by inequalities (2.2.5)∼(2.2.7)

$$c_0^o + pa^o \leqslant e_0^o + \pi(j)\phi^o \tag{2.2.5}$$

($t = 0$ consumption and the payment for the security should not exceed the endowments plus the borrowing)

$$\phi^o \leqslant a^o \tag{2.2.6}$$

(Collateral constraint: number of contracts entered, which equals the amount of collateral, should not exceed the security holding)

$$c_s^o \leqslant e_1^o + (a^o - \phi^o)s + \phi^o \max[s - j^o, 0] \quad \forall s \in (0, \bar{s}) \tag{2.2.7}$$

($t = 1$ consumption should not exceed the endowments plus the income from the security with and without using the collateralized debt contract)

2.2.2 The Equilibrium

2.2.2.1 Market Clearing Conditions

The markets for the security and the debt contract should clear,

$$a^o = a \tag{2.2.8}$$

(demand for the security equals supply)

$$\phi^o = \phi^p \equiv \phi \tag{2.2.9}$$

(demand for the debt contract equals the supply)

$$j^o \equiv j \tag{2.2.10}$$

(the lenders offer a menu of contracts, and the borrowers choose from the menu)

2.2.2.2 Contract Determination

The optimization problem above implies that the borrowers have all the bargaining power on choosing the debt contract, the market determines the prices, while the lenders accept the type of debt contract to be traded. Recall that the collateralized debt contract in this paper is characterized by the face value and the collateral level, with normalization that each unit of contract requires one unit of collateral. The market offers a menu of collateralized debt contracts, taking the form of a function $\pi(j), \forall j \in (0, \bar{s})$. For any face value j, the contract price will be $\pi(j)$. In this way, the debt's required interest rate is given as $r(j) = j/\pi(j) - 1, \forall j$. The borrowers choose one of the contract from the menu, by choosing the face value j, while taking the function $\pi(j)$ as given. When they choose different debt contracts, they are aware that different choices of j yield different prices $\pi(j)$. However, they are still considered to be price takers in the credit market, since they do not manipulate $\pi(j), \forall j$ by choosing the quantity of debt.

When solving for the equilibrium, I focus on the cases where the collateral constraint binds, so that the model can show how the scarcity of collateral affects the equilibrium prices and leverage. The collateral constraint and the market clearing conditions yield $\phi^p = \phi^o = a^o = a$ in equilibrium.

The **pessimistic traders** take the choice of j as given, and their Lagrangian is,

$$\begin{aligned}\mathcal{L}^p = U(e_0^p - \pi(j)\phi^p) + \beta^p \int_0^j U(e_1^p + \phi^p s) f^p(s) ds \\ + \beta^p \int_j^{\bar{s}} U(e_1^p + \phi^p j) f^p(s) ds \end{aligned} \tag{2.2.11}$$

Taking derivative of ϕ^p, we have the incentive compatibility (IC) constraint for the pessimists w.r.t. the debt contract quantity as below,

$$\frac{\partial \mathcal{L}^p}{\partial \phi^p} = -\pi(j)U'(e_0^p - \pi(j)\phi^p) + \beta^p \int_0^j sU'(e_1^p + \phi^p s) f^p(s) ds + \beta^p \int_j^{\bar{s}} jU'(e_1^p + \phi^p j) f^p(s) ds = 0 \tag{2.2.12}$$

In equilibrium, this equation becomes,

$$\pi(j)U'(e_0^p - \pi(j)a) = \beta^p \int_0^j sU'(e_1^p + as) f^p(s) ds + \beta^p \int_j^{\bar{s}} jU'(e_1^p + aj) f^p(s) ds \tag{2.2.13}$$

This equation is the Euler equation for the lenders' investment in the collateralized debt. The left hand side is the marginal cost issuing one additional unit of the loans, whereas the right hand side is the marginal benefit. In equilibrium, it gives the relationship between the debt's price and face value, thus represents the menu of debt contracts as a function $\pi(j), \forall j$ that the market offers. Equation (2.2.13) holds for all the values of j given to the lenders. Obviously this relationship is differentiable. Taking derivative w.r.t. j on both sides yields,

$$\begin{aligned}&\frac{\partial \pi(j)}{\partial j} \left(U'(e_0^p - \pi(j)a) - a\pi(j)U''(e_0^p - \pi(j)a)\right) \\ &= \beta^p \int_j^{\bar{s}} \left(U'(e_1^p + aj) + jaU''(e_1^p + aj)\right) f^p(s) ds \end{aligned} \tag{2.2.14}$$

which gives the slope of the *menu* $\frac{\partial \pi(j)}{\partial j}$. Moreover, the general equilibrium analysis requires convex preference and convex budget sets, yet in this model the default decision leads to non-convexities in the budget sets. Thus, the contract determination method in this model requires checking the second-order condition w.r.t. j, and detailed discussions are presented in Appendix 2.A.

The Lagrangian for the **optimistic traders** is,

$$\begin{aligned}\mathcal{L}^o = U(e_0^o + \pi(j)\phi^o - pa^o) + \beta^o \int_0^{j^o} U(e_1^o + (a - \phi^o)s) f^o(s) ds \\ + \beta^o \int_{j^o}^{\bar{s}} U(e_1^o + a^o s - \phi^o j^o) f^o(s) ds - \lambda(\phi^o - a)\end{aligned}$$

Taking derivative w.r.t. a^o, ϕ^o and j^o, respectively, we have the incentive compatibility (IC) constraints of the optimists w.r.t. the security holding, debt contract quantity, and the choice of the contract as below,

$$\begin{aligned}\frac{\partial \mathcal{L}^o}{\partial a^o} =& - pU'(e_0^o + \pi(j)\phi^o - pa^o) + \beta^o \int_0^{j^o} sU'(e_1^o + (a^o - \phi^o)s) f^o(s) ds \\ &+ \beta^o \int_{j^o}^{\bar{s}} sU'(e_1^o + a^o s - \phi^o j^o) f^o(s) ds + \lambda = 0 \qquad (2.2.15) \\ \frac{\partial \mathcal{L}^o}{\partial \phi^o} =& \pi(j) U'(e_0^o + \pi(j)\phi^o - pa^o) - \beta^o \int_0^{j^o} sU'(e_1^o + (a^o - \phi^o)s) f^o(s) ds \\ &- \beta^o \int_{j^o}^{\bar{s}} j^o U'(e_1^o + a^o s - \phi^o j^o) f^o(s) ds - \lambda = 0 \qquad (2.2.16) \\ \frac{\partial \mathcal{L}^o}{\partial j^o} =& \phi^o \frac{\partial \pi(j)}{\partial j^o} U'(e_0^o + \pi(j)\phi^o - pa^o) \\ &- \beta^o \int_{j^o}^{\bar{s}} \phi^o U'(e_1^o + a^o s - \phi^o j^o) f^o(s) ds = 0 \qquad (2.2.17)\end{aligned}$$

In equilibrium, these equations become,

$$(p-\pi(j))U'(e_0^o - a(p-\pi(j))) = \beta^o \int_j^{\bar{s}} (s-j)U'(e_1^o + a(s-j))f^o(s)ds \tag{2.2.18}$$

$$\frac{\partial \pi(j)}{\partial j} U'(e_0^o - a(p-\pi(j))) = \beta^o \int_j^{\bar{s}} U'(e_1^o + a(s-j))f^o(s)ds \tag{2.2.19}$$

Equation (2.2.18) is the Euler equation for optimistic traders' down-payments on the security. $p - \pi(j)$ is the security price minus the borrowed amount, and $s - j$ is the net pay-off of holding one unit of the security bought with leverage. Equation (2.2.19) implies the optimistic traders choice of contracts. By agreeing to repay one more unit of the consumption goods in each of the future scenario, $\frac{\partial \pi(j)}{\partial j}$ more can be borrowed using the same collateral. Note that both choices are truncated at j. When borrowers default, they no longer care about the payoff of the security or the debt repayment. Default leads to the asymmetries regarding optimists beliefs. Dividing equation (2.2.19) by (2.2.14) we have,

$$\begin{aligned} &\frac{U'(e_0^o - a(p-\pi(j)))}{U'(e_0^p - a\pi(j)) - a\pi(j)U''((e_0^p - a\pi(j))} \\ =&\frac{\beta^o}{\beta^p} \frac{\int_j^{\bar{s}} U'(e_1^o + a(s-j))f^o(s)ds}{\int_j^{\bar{s}} \left(U'(e_1^p + aj) + jaU''(e_1^p + aj)\right) f^p(s)ds} \end{aligned} \tag{2.2.20}$$

In equilibrium, only one debt contract j is selected. Now the equilibrium can be characterized by three variables $(p, j, \pi(j))$ and the three equations (2.2.13) (2.2.18) (2.2.20). The debt contract menu is given by equation (2.2.13).

Later, I will show that the equilibrium is unique under certain conditions[3]. The contract determination doesn't depend on the existence of a risk-free asset as in Simsek (2013). In that paper, the risk-neutral pessimistic traders must hold cash, which should have the same return as the debt contract, and this assumption consequently fixes the leverage level of the collateralized debt contract. In Section 2.3.1 I will discuss how the introduction of a risk-free asset affects the economy.

The equilibrium is a set of contract choice and price, security price, and consumptions for both types of traders at each state. The equilibrium defined later assumes that the pessimistic traders do not purchase the security themselves, and that the collateral constraint binds in equilibrium. Therefore, only the optimists trade in the security market, whereas both the optimists and the pessimists trade in the debt market. Moreover, I focus on equilibria where the collateral constraint binds so as to assess its real effects. The following two conditions are consequences of the traders rational optimization behaviour and fulfill the aforementioned requirements.

$$p > p^p \equiv \beta^p \int_0^{\bar{s}} \frac{U'(c_s^p)}{U'(c_0^p)} s f^p(s) ds \tag{2.2.21}$$

$$\lambda \equiv \pi(j) U'(c_0^o) - \beta^o \int_0^{\bar{s}} \min[s, j] U'(c_s^o) f^o(s) ds > 0 \tag{2.2.22}$$

Condition (2.2.21) ensures that, the equilibrium price of the security is higher than the pessimistic traders' subjective valuation of the security. Thus, they are not willing

[3]Moreover, if the traders' risk aversion approaches 0, the equations above would imply $F^o(j) = F^p(j)$, where $F^h(\cdot), h \in \{o, p\}$ is the cumulative probability function of the beliefs. In this way we can find the equilibrium face value of the debt. However, if the belief functions satisfy first order stochastic dominance condition, we always have $F^o(s) < F^p(s), s \in (0, \bar{s})$. So the only values of j satisfying the condition are 0 and $\bar{s}$, which imply no lending at all.

to directly purchase it. Condition (2.2.22) states that the Lagrangian multiplier for the collateral constraint is positive, which implies that the collateral constraint binds. The marginal benefit of borrowing one more contract is greater than the marginal cost of the repayment, but the borrowers cannot do so, due to the binding collateral constraint. Hence, the equilibria satisfying equations (2.2.13) (2.2.18) (2.2.20), can be formally defined as follows:

Definition 1. *The **Collateral Equilibrium** for the economy*

$$\mathcal{E}_1 = \left\{(\bar{U}^h, e_0^h, e_1^h)_{h\in\{o,p\},s\in(0,\bar{s})}, a\right\}$$

is a vector consisting of a security price, choice of the debt contract the contract menu, individual consumptions as a function of the state's realization, security holding, and contract trading,

$$\left(p, j, \pi(\cdot), (c_0^o, (c_s^o)_{s\in(0,\bar{s})}, a^o, \phi^o), (c_0^p, (c_s^p)_{s\in(0,\bar{s})}, \phi^p)\right)$$

if and only if

1. $(c_0^p, c_s^p, \phi^p) \in \mathcal{B}_1^p(p, j, \pi(\cdot))$, *and* $((\tilde{c}_0, \tilde{c}_s, \tilde{\phi}) \in \mathcal{B}_1^p(p, j, \pi(\cdot)) \Rightarrow \bar{U}^p(\tilde{c}_0, \tilde{c}_s, \tilde{\phi}) \leqslant \bar{U}^p(c_0^p, c_s^p, \phi^p)$
2. $(c_0^o, c_s^o, \phi^o, a^o, j^o) \in \mathcal{B}_1^o(p, \pi(\cdot))$, *and* $((\tilde{c}_0, \tilde{c}_s, \tilde{\phi}, \tilde{a}, \tilde{j}) \in \mathcal{B}_1^o(p, \pi(\cdot)) \Rightarrow \bar{U}^h(\tilde{c}_0, \tilde{c}_s, \tilde{\phi}, \tilde{a}, \tilde{j}) \leqslant \bar{U}^h(c_0^o, c_s^o, \phi^o, a^o, j^o)$
3. *Market clearing conditions (2.2.8)∼(2.2.10) hold.*
4. *Conditions (2.2.21) and (2.2.22) hold.*

where the budget sets $\mathcal{B}_1^p(p, j, \pi(\cdot))$ and $\mathcal{B}_1^o(p, \pi(\cdot))$ are defined by inequalities (2.2.2) $\sim$ (2.2.3) and (2.2.5) $\sim$ (2.2.7), respectively.

2.2.3 Equilibrium Characterization

Proposition 1 (Asymmetric Disciplining of Optimism)**.** *When the collateral constraints bind, changes in the optimistic traders beliefs concerning the probability of the downside states has no or little effect on asset prices, whereas the beliefs concerning the probability of the upside states have significant effects on asset prices.*

Proof. These results are obvious from equation (2.2.13) (2.2.18) (2.2.20). □

The stochastic discount factors for the two traders are,

$$m^o(s) = \beta^o \frac{U'(c_s^o)}{U'(c_0^o)} \tag{2.2.23}$$

$$m^p(s) = \beta^p \frac{U'(c_s^p)}{U'(c_0^p)} \tag{2.2.24}$$

Thus, equations (2.2.13) and (2.2.18) can be written as,

$$\pi = \mathbb{E}^p \left[m_s^p \cdot \min[s, j] \right] \tag{2.2.25}$$

$$p - \pi = \mathbb{E}^o_{s>j} \left[m_s^o \cdot (s - j) \right] \tag{2.2.26}$$

These two equations are no other than the Euler equations. More specifically, equation (2.2.25) is the Euler equation of the debt contract for the pessimistic traders, whereas equation (2.2.26) is the Euler equation of the down payments for the optimistic traders. After plugging in the market clearing conditions and focusing on the situations where

the collateral constraint binds, we have a system of three variables $(p, j, \pi(j))$, and three equations (2.2.13) (2.2.18) (2.2.20). Obviously, the optimistic trader's belief on the bad states ($s < j$) does not enter the calculation. Thus the main argument in Simsek (2013) about asymmetric discipline of beliefs is robust in this model, while I generalize with inter-temporal consumption trade-off and risk-aversion, and this result does not depend on the existence of a risk-free asset.

Proposition 2 (Collateral Constraint)**.** *Only if the collateral constraint binds then the security price is higher than its fundamental value to the buyers (the optimistic traders); the contract price is equal to the fundamental value to the lenders, but higher than that of the borrowers.*

Proof. It is easy to see that, in equilibrium,

$$p = \beta^o \int_0^{\bar{s}} \frac{U'(c_s^o)}{U'(c_0^o)} s f^o(s) ds + \frac{\lambda}{U'(c_0^o)} = \mathbb{E}\left[m_s^o \cdot s\right] + \frac{\lambda}{U'(c_0^o)} \tag{2.2.27}$$

$$\begin{aligned} \pi &= \beta^o \int_0^{\bar{s}} \frac{U'(c_s^o)}{U'(c_0^o)} \min[s, j] f^o(s) ds + \frac{\lambda}{U'(c_0^o)} = \mathbb{E}\left[m_s^o \cdot \min[s, j]\right] + \frac{\lambda}{U'(c_0^o)} \\ &= \beta^p \int_0^{\bar{s}} \frac{U'(c_s^p)}{U'(c_0^p)} \min[s, j] f^p(s) ds = \mathbb{E}\left[m_s^p \cdot \min[s, j]\right] \end{aligned} \tag{2.2.28}$$

Equation (2.2.27) implies that the security price can be decomposed into the fundamental value for the optimistic traders plus the collateral value, as suggested by Fostel and Geanakoplos (2008). Although the security price is higher than the optimistic traders' valuation, they still choose to buy it, since holding the security as collateral allows them to borrow more. The price of the debt contract equals the pessimistic traders' valuation of future income from the debt, but is greater than the optimistic traders' valuation of future payment. This implies that, the borrowers would like to

borrow more at the current interest rate, but could not do so due to lack of collateral. As for the lenders, the debt contract is fairly priced, and the collateral constraint has no direct effect on them. Note that the above holds only when the collateral constraint binds ($\lambda > 0$). This proposition summarizes the effect of collateral constraint on the security price and debt price. □

Theorem 1 (Uniqueness)**.** *The solution to the system of the three simultaneous equations (2.2.13) (2.2.18) (2.2.20) is unique, if*

1. *The utility function takes the form of* $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$*, and the relative risk aversion coefficient* γ *satisfies* $0 \leqslant \gamma \leqslant 1$.

2. *The hazard rates* $h^o(s), h^p(s)$ *of the two types of traders' belief function satisfies* $\frac{a\gamma}{(e_1^o + a(\bar{s}-j)^{\gamma+1})} + \frac{h^p(j)}{(e_1^o + a(\bar{s}-j))^{\gamma}} > \frac{h^o(h)}{(e_1^o)^{\gamma}}, \quad \forall j \in (0, \bar{s})$

The proof is provided in Appendix 2.B. Note that the assumptions about the utility function, the endowments and the beliefs are sufficient but not necessary to ensure uniqueness. Also, if $\gamma \to 0$, the traders become risk-neutral, and the second condition above reduces to $h^p(s) > h^o(s)$.

2.2.4 Numerical Examples

Consider two types of risk-averse traders with belief function

$$f^h(s) = k(s - \frac{\bar{s}}{2}) + \frac{1}{\bar{s}} \quad \forall s \in (0, \bar{s}), h \in \{o, p\} \tag{2.2.29}$$

where the slope k satisfies $k \in (-\frac{2}{\bar{s}^2}, \frac{2}{\bar{s}^2})$, so that it is linear on $(0, \bar{s})$ with no kink. It is easy to show that $\mathbb{E}^o(s) > \mathbb{E}^p(s)$, and $h^o(s) < h^p(s), \forall s \in (0, \bar{s})$, where $\mathbb{E}(s)$ is the

subjective expectation of the security payoff, and $h(s) = f(s)/(1-F(s))$ is the hazard rate. The rest of the parameters and functions are given in Table 2.2,

Table 2.2: Exogenous variables

Belief	$k^o = 0.5 \quad k^p = -0.5$
Utility	$U(\cdot) = \ln(\cdot)$
Endowment	$e_0^o = e_0^p = e_1^o = e_1^p = 1$
Security Supply	$a = 1 \quad \bar{s} = 2$
Discount factor	$\beta^o = \beta^p = 1$

Table 2.3 shows the endogenous variables of the Collateral Equilibrium, together with a no-lending benchmark, in which the two types of traders are forbidden from lending/borrowing. Notice that, in absence of the debt contract, both types of traders invest part of their endowment in the security, and the optimistic traders purchase more than the pessimistic traders do. Given that the traders have the same amount of endowment, same discount factor and same utility function, the difference in trading reflects the belief disagreement. The benchmark optimization problem is given in Appendix 2.C.

Table 2.3: Endogenous Variables

No-Lending Benchmark				
$p = 0.4511$	$a^o = 0.7052$	$a^p = 0.2948$	$\bar{U}^o = 0.2636$	$\bar{U}^p = 0.0302$
Collateral Equilibrium				
$p = 0.4825$	$\pi = 0.1560$	$j = 0.2647$	$LTV = 32.34\%$	$r = 0.6968$
$\bar{U}^o = 0.3027$	$\bar{U}^p = 0.0367$	$\lambda = 0.0968$	$p - p^p = 0.0359$	$\partial^2\mathcal{L}^o/(\partial j^o)^2 = -2.1419$

λ is the Lagrangian multiplier for the budget constraint. In equilibrium, the collateral constraint binds, thus λ should be positive.
p^p is the price the pessimistic traders willing to pay for the security, defined in equation (2.2.21). $p - p^p > 0$ means the equilibrium price of the security is higher than the pessimistic traders' bid price, so it is optimal not to purchase.
$\partial^2\mathcal{L}^o/(\partial j^o)^2$ is the second-order derivative w.r.t. j^o for the optimistic traders' optimization problem.

In the collateral equilibrium, the security sells at $p = 0.4825$, which is higher than the price 0.4511 in the no-lending benchmark. The debt contract sells at $\pi = 0.1560$, with face value $j = 0.2647$. This means that, with one unit of the security as collateral, an optimistic trader can borrow 0.1560 at $t = 0$, and promises to pay back 0.2647 at $t = 1$. This yields a Loan-to-Value ratio (LTV) of 32.34%, and an interest rate of $r = 0.6968$. The net marginal benefit if the borrowers borrow more is $\lambda = 0.0968 > 0$, which means it is optimal for them to borrow until the collateral constraint binds. The price of the security is greater than the pessimistic traders' offer ($p > p^p$), so it is optimal for them not to purchase the security, as assumed. The optimistic traders, who value the security more, buy all the security, financed by the collateralized debt. The expected utility of the optimistic traders is 0.3027, and 0.0367 of the pessimistic traders. Compared with the no-lending benchmark, both type' expected utilities are higher in the collateral equilibrium. The introduction of the collateralized debt contract provides better risk-sharing between the two traders, allocates the assets to those who value them more, thus boosts the asset price, and achieves Pareto-improvement.

More importantly, this example shows that we cannot construct a default-free equilibrium which is equivalent to the collateral equilibrium with default. As the security's payoff lies in the domain $(0, \bar{s})$, it is possible that the security may end up with value close to zero. According to Fostel and Geanakoplos (2015), this security cannot be used as collateral in a *binomial structure*, and consequently the default-free equilibrium is no other than the no-lending benchmark in the previous section. In the collateral equilibrium model, *potential* default affects the determination of the debt contract, while *actual* default re-distributes welfare between the households. The collateral

equilibrium and the default-free equilibrium are *not* equivalent.

2.2.5 Belief heterogeneity

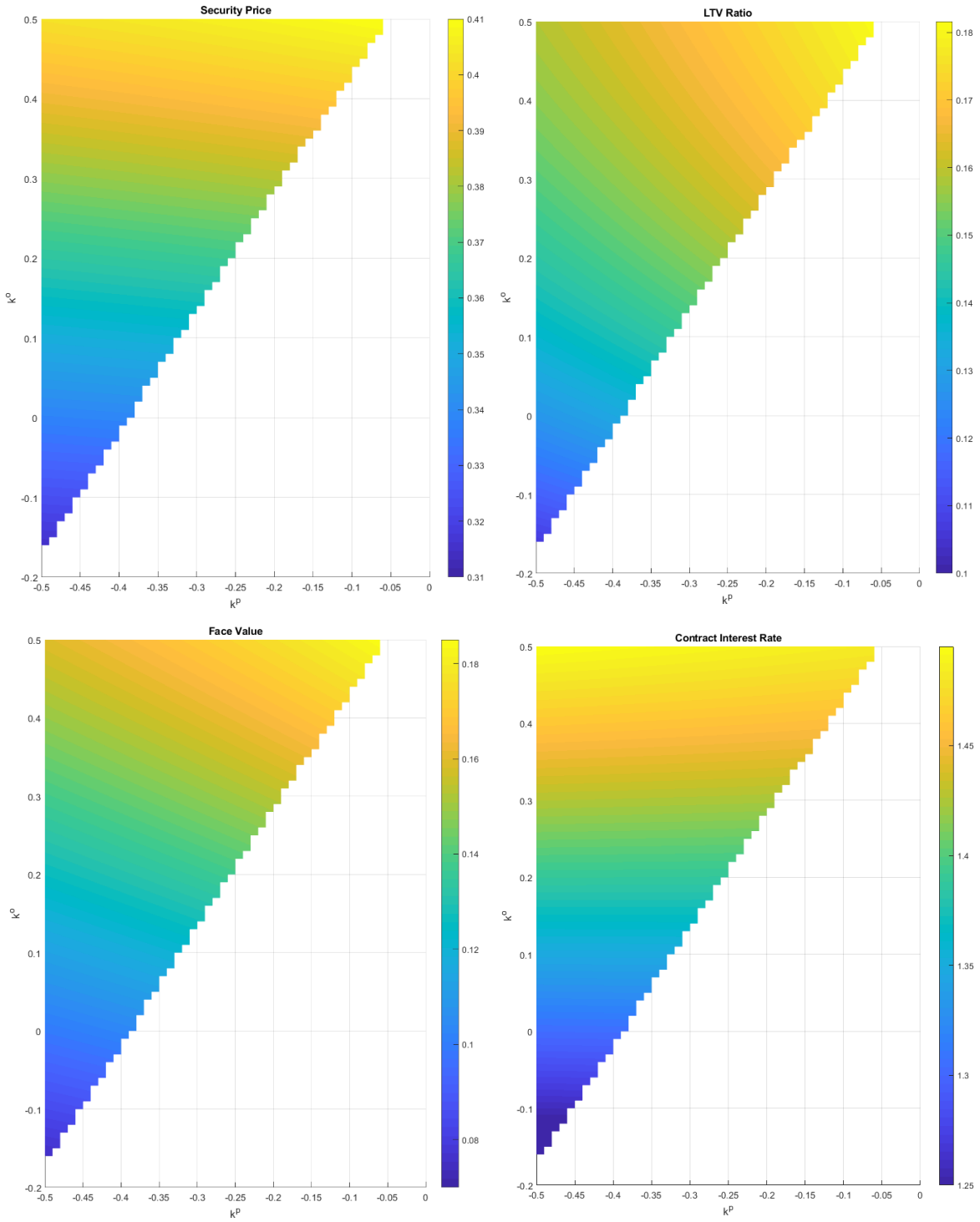
This section evaluates how changes in beliefs affect the equilibrium, without changes in the fundamentals. I fix the value of all other exogenous variables, and only change the value of k^o and k^p. Remember that these two coefficients show up in the traders' PDFs of belief (equation (2.2.29)), with higher value representing a more optimistic belief. More specifically, I compute the equilibrium with different combinations of $(k^o, k^p) \in \{-0.5, -0.49, -0.48, ..., 0.49, 0.5\}^2, k^o > k^p$ (5050 cases), fixing all other exogenous variables. I am particularly interested in the cases where the collateral constraint binds, and the pessimistic traders do not purchase the security. In order to generate more equilibrium satisfying the assumptions above, the endowments of the pessimistic traders need to be large so that they are not interested in purchasing the security. All the other exogenous variables are the same as in the previous case, except for the second period endowments of the pessimists, with $e_1^p = 2$ in this section.

Figure 2.1 shows the simulation results on belief heterogeneity, asset price, and contract leverage ratio. For each figure, the horizontal and vertical axis represents k^p and k^o, respectively. Remember that k is the slope of the linear belief functions $f^h(s) = k^h(s - \bar{s}/2) + 1/\bar{s}$. The higher value of the coefficients, the more optimistic the traders are. Also, the average of the two slopes, $(k^o + k^p)/2$, can be seen as a measure of average market optimism, while the difference $k^o - k^p$ can be taken as a measure of belief disagreement. All the data points are on the northwest part of each graph, since I only consider the cases where the borrowers are more optimistic

than the lenders ($k^o > k^p$). Moreover, I assume that in equilibrium the collateral constraint binds, and the pessimistic traders do not purchase the security directly. The calculation indicates that, when the belief disagreement between the two types is small, the security price falls below the subjective valuation of the pessimistic traders, incentivizing their purchase. However, in this paper I only focus on the equilibrium with properties mentioned in Section 2.2.3, thus in Figure 2.1 there is a blank area to the southeast. The zigzags are due to the discrete increment of k^o and k^p, while within the colored region the graph is smoothed by MATLAB graphic function.

The four figures show how changes in beliefs affect equilibrium security price, contract loan-t-value ratio, face value and required rate of return. The region with color closer to blue represents lower value, while the the region with color closer to yellow represents higher value. Obviously, more optimistic borrowers or lenders result in higher security price and contract leverage. Higher optimism of the borrowers increases the demand for the security, boosting up the security prices. They are willing to pay higher interest rate, thus are compensated for higher loan-to-value ratios. This leads to higher lending per contract, and consequently higher face value of the debt. More optimistic lenders value the collateral more, and believe that default is less likely to occur. Thus, they are willing to extend loans with a higher loan-to-value ratio. The increase in credit supply boosts up the security price.

Figure 2.1: Simulation of Baseline Model

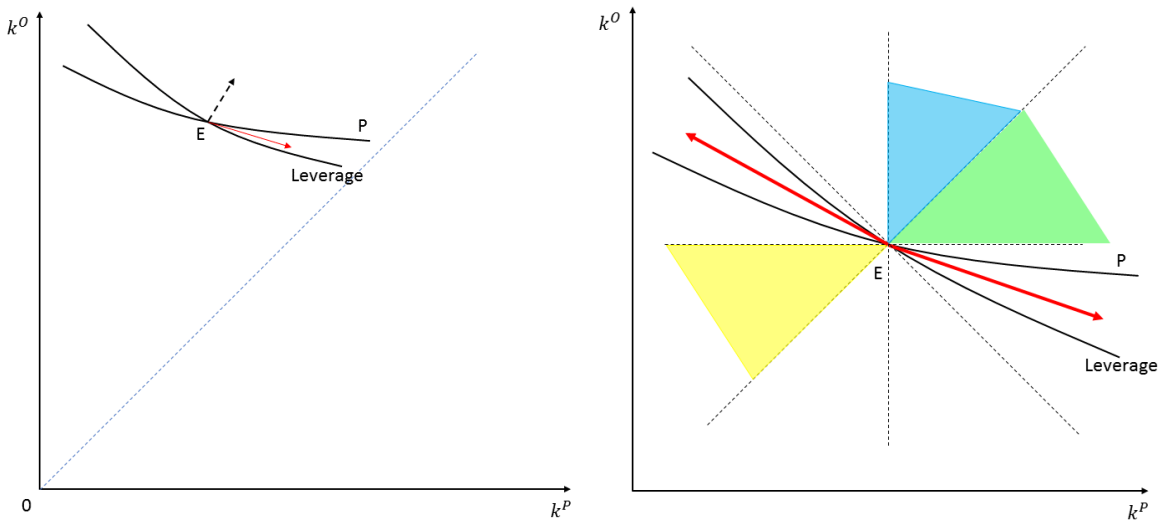


These figures show how beliefs affect equilibrium security prices, contract leverage, face value and required rate of return. For each figure, the horizontal axis is k^p which represents the lenders' belief, while the vertical axis is k^o which represents the borrowers' belief. Figures for security price, contract loan-to-value ratio, contract face value and required rate of return are on top-left, top-right, bottom-left and bottom-right, respectively. The region with color closer to blue represents lower value, while the region with color closer to yellow represents higher value.

Fixing other exogenous variables, we can always find combinations of (k^o, k^p) that keep either equilibrium asset prices or the contract LTV ratio unchanged. These combination are represented by region or curve of the same color in Figure 2.1. To clarify the idea, I draw the iso-value curves of price and leverage crossing at an arbitrary equilibrium E in Figure 2.2. The movement along the curve p represents (k^o, k^p) that keep the asset prices unchanged, while the movement along the curve *Leverage* represents (k^o, k^p) that keep the contract leverage ratio unchanged. At each initial equilibrium, the slopes of the two curves are different. It is obvious from the figure that, in most cases asset prices and leverage move in the same direction when the two types of traders' beliefs change. The right-hand-side figure zooms in at the initial equilibrium point E, and the dashed lines are the horizontal, the vertical and the $\pm 45°$ lines. Remember that I only consider the cases where the borrowers are more optimistic than the lenders, thus moving towards the area above the $45°$ line represents an increase in the belief disagreement, whereas moving towards the area blow the $45°$ line represents a decrease in the belief disagreement. Similarly, moving towards the area above the $-45°$ line represents an increase in the average belief, whereas moving towards the area below the $-45°$ line represents a decrease in the average belief. For instance, the green area represents the boom times in an economy, when belief disagreement narrows, and asset prices and leverage move up together. The yellow area represents the crisis times, when belief disagreement broadens, and asset prices and leverage move down together. These predictions are consistent with the procyclicality of asset prices and leverage (Adrian and Shin (2010),Gorton and Metrick (2012)), as well as the counter cyclicality of belief disagreement (Patton and Timmermann (2010)).

In both the blue and yellow area, belief disagreement increases, but the direction of asset prices and leverage comovement depends on the change in the average market beliefs.

Figure 2.2: Comovement of asset prices and leverage



The figure on the left summarizes the main message of Figure 2.1. Fixing other exogenous variables, we can always find combinations of k^o and k^p that keep either equilibrium asset prices or contract leverage ratio unchanged. Note that the two 'indifference curves' have different slopes at the initial equilibrium point. The figure on the right zooms in, with dashed horizontal, vertical and $\pm 45°$ lines.

The results above predict a narrow area where asset prices and leverage move in opposite directions due to change in beliefs. On the right hand side of Figure 2.2, if we move the initial equilibrium (point E) along one of the red arrows, we could have such consequences. The arrow pointing to the northwest resembles the case where lenders are getting less optimistic thus raising the required contract leverage, whereas the borrowers are still getting more optimistic thus boosting up the asset prices. The arrow pointing to the south east resembles the case where lenders are getting more optimistic thus decreasing the required contract leverage, yet the borrowers are getting less optimistic thus losing support for the asset prices. This paper takes beliefs as

exogenous.

2.3 Extensions

2.3.1 Risk-free asset

One merit of the contract determination method above is that a risk-free asset is not required to pin down the leverage. It is not to say that this model cannot tolerate the existence of a risk-free asset. In the previous section, the only way the pessimists can *save* their first period endowments is to issue the collateralized debt and lend to the optimists. In this section I allow the traders to save costlessly from $t = 0$ to $t = 1$, and present a numerical example to show how the results change. The economy is almost the same as in Section 2.2, and the only difference is that both traders now can choose to save the endowments throughout time. In line with Simsek (2013), I assume that only the pessimistic traders save, since the optimistic traders have limited endowments at $t = 0$ and they leverage to purchase the security.

2.3.1.1 The optimization problem

The pessimistic traders The pessimistic traders are the lenders in the debt market. They choose the consumption c_0^p, c_s^p, saving B^p, and the unit of debt contract to offer ϕ^p to maximize the expected utility,

$$\max_{c_0^p, \phi^p, B^p, c_s^p} \bar{U}^p = U(c_0^p) + \beta^p \int_0^{\bar{s}} U(c_s^p) f^p(s) ds \tag{2.3.1}$$

subject to the budget set $\mathcal{B}_2^p(p, j, \pi(\cdot))$ defined by inequalities (2.3.2)$\sim$(2.3.3)

$$c_0^p + \pi(j)\phi^p + B^p \leqslant e_0^p \tag{2.3.2}$$

(consumption at $t = 0$ plus the lending and saving should not exceed endowments)

$$c_s^p \leqslant e_1^p + B^P + \phi^p \min[s, j] \quad \forall s \in (0, \bar{s}) \tag{2.3.3}$$

(consumption at $t = 1$ should not exceed the sum of endowments, saving and the debt repayment)

The optimistic traders The optimistic traders' optimization problem is the same as in Section 2.2.1, and I copy it below. They choose the consumptions c_0^o, c_s^o, the holding of the security a^o, and the face value j^o and position of the debt contract ϕ^o to maximize the expected utility,

$$\max_{c_0^o, a^o, \phi^o, j^o, c_s^o} \bar{U}^o = U(c_0^o) + \beta^o \int_0^{\bar{s}} U(c_s^o) f^o(s) ds \tag{2.3.4}$$

subject to the budget set $\mathcal{B}_2^o(p, \pi(\cdot))$ defined by inequalities (2.3.5)$\sim$(2.3.7)

$$c_0^o + pa^o \leqslant e_0^o + \pi(j)\phi^o \tag{2.3.5}$$

($t = 0$ consumption and the payment for the security should not exceed the endowments plus the borrowing)

$$\phi^o \leqslant a^o \tag{2.3.6}$$

(Collateral constraint: number of contracts entered, which equals the amount of collateral, should not exceed the security holding)

$$c_s^o \leqslant e_1^o + (a^o - \phi^o)s + \phi^o \max[s - j^o, 0] \quad \forall s \in (0, \bar{s}) \tag{2.3.7}$$

($t = 1$ consumption should not exceed the endowments plus the income from the security with and without using the collateralized debt contract)

2.3.1.2 The equilibrium

In equilibrium the security and debt market should clear, and the contract is determined in the same way as in the previous section. I skip the equilibrium definition, and leave all the calculations into Appendix 2.D. The equilibrium can be characterized by four variables $(p, j, \pi(j), B)$ and the following four equations.

$$(p - \pi(j))U'(e_0^o - a(p - \pi(j))) = \beta^o \int_j^{\bar{s}} (s - j)U'(e_1^o + a(s - j))f^o(s)ds \tag{2.3.8}$$

$$U'(e_0^p - \pi(j)\phi - B) = \beta^p \int_0^{\bar{s}} U'(e_1^p + B + \phi \min[s, j])f^p(s)ds \tag{2.3.9}$$

$$\begin{aligned} &\pi(j)U'(e_0^p - \pi(j)a - B) \\ =&\beta^p \int_0^j sU'(e_1^p + B + as)f^p(s)ds + \beta^p \int_j^{\bar{s}} jU'(e_1^p + B + aj)f^p(s)ds \end{aligned} \tag{2.3.10}$$

$$\begin{aligned} &\frac{U'(e_0^o - a(p - \pi(j)))}{(U'(e_0^p - B - \pi(j)a) - \pi(j)aU''(e_0^p - B - \pi a))} \\ =&\frac{\beta^o}{\beta^p} \frac{\int_j^{\bar{s}} U'(e_1^o + a(s - j))f^o(s)ds}{\int_j^{\bar{s}} (U'(e_1^p + B + aj) + ajU''(e_1^p + B + aj)f^p(s)ds)} \end{aligned} \tag{2.3.11}$$

Note that equation (2.3.10) is the new contract menu, and it is different from the one in the previous section (equation (2.2.13)), since now the pessimists have the option to

save. I will show how this new menu changes the effects of beliefs on the equilibrium by the numerical examples in the following section. We need to check the second order condition for the optimists, and the following conditions to ensure that in equilibrium the collateral constraint binds, the pessimists do not want to purchase the security directly, and the optimists do not want to save through the risk-free asset,

$$\lambda \equiv \pi(j)U'(c_0^o) - \beta^o \int_0^{\bar{s}} \min[s, j]U'(c_s^o)f^o(s)ds > 0 \tag{2.3.12}$$

$$p > p^p \equiv \beta^p \int_0^{\bar{s}} \frac{U'(c_s^p)}{U'(c_0^p)} s f^p(s)ds \tag{2.3.13}$$

$$U'(c_0^o) - \beta^o \int_0^{\bar{s}} U'(c_s^o)f^o(s)ds > 0 \tag{2.3.14}$$

It is not difficult to see from the equilibrium conditions that the main properties from the previous section on asymmetric disciplining of optimism and the collateral constraints remain true when we add risk-free asset into the economy. Now let's check how the change in beliefs affect the equilibrium with the presence of the risk-free asset.

2.3.1.3 Numerical example

Consider the exogenous variables in Table 2.4. To ensure that the pessimistic traders are interested in saving, let their endowments at $t = 0$ to be high. The choices of all other parameters, the utility function and the belief function are the same as before.

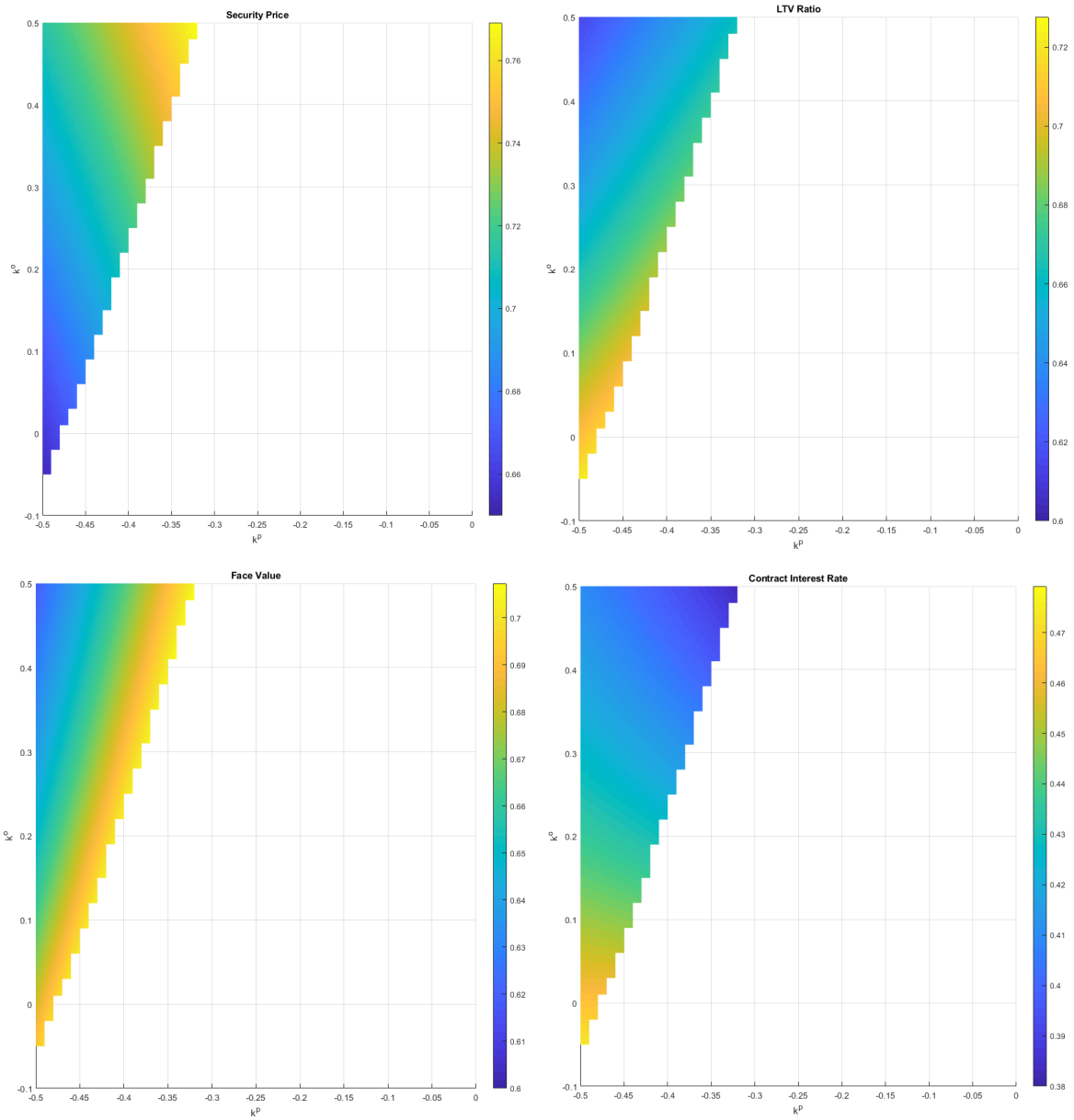
Table 2.4: Exogenous variables

$e_0^o = 1$	$e_1^o = 1$	$e_0^p = 10$	$e_1^p = 1$
$a = 1$	$\bar{s} = 2$	$\beta^o = 1$	$\beta^p = 1$

I run a similar simulation as in Section 2.2.4, and the results are presented in Figure 2.3. Remember that I focus on the equilibrium where the collateral constraint binds, the optimists do not save, and the pessimists do not purchase the security directly. These conditions limits the number of "satisfactory" equilibria, thus the colored area is only a small fraction on each graph. The axes and the color present in the same we as before, i.e. the vertical axis is the optimists' belief, while the horizontal axis is the pessimists' belief. The color closer to yellow represents higher value, and blue represents lower value.

Now the pessimists have a choice over risk-free saving and risky debt, so they can charge higher rate if the face value is higher. The optimists are aware of this, and may choose to borrow with lower leverage so as to avoid high interest rate. More specifically, more optimistic lenders accept debt with higher leverage and face value, as well as lower interest requirement. This extends credit supply and boosts up the security price. Lenders' saving is consequently lower. More optimistic borrowers demand more security, increasing the price. Confident about the security payoff, the optimists refuse to pay high interest rates on the debt, which is a default-premium. Thus, they choose debt with lower leverage and face value to avoid the high interest payment. Note that in the numerical example, I make the pessimists much richer than the optimists at $t = 0$, so it is not surprising that p, π, j are influenced more by k^p than by k^o. The average belief still increases equilibrium security price as in the previous model, but belief disagreement now decreases the equilibrium leverage.

Figure 2.3: Simulation with risk-free asset



These figures show how changes in beliefs affect equilibrium asset prices, contract leverage, face value and interest rate. For each figure, the horizontal axis is k^p which represents the lenders' belief, while the vertical axis is k^o which represents the borrowers' belief. Fixing other exogenous variables, equilibrium changes when the beliefs change. Figures for asset price, contract leverage, saving through risk-free asset and contract required rate of return are on top-left, top-right, bottom-left and bottom-right, respectively. The region with color closer to blue represents lower value, while color closer to yellow represents higher value.

The model in this section differs from Simsek (2013) by considering inter-temporal consumption smoothing. With risk-neutrality, agents are indifferent toward the time to consume, as long as the discount factor equals the inverse of the interest rate. Risk aversion is not the reason why this model delivers different results. When the lenders become more optimistic, they save less and lend more, increasing the leverage while decreasing the required rate of return. This is only possible with risk aversion. The lender's preference over intertemporal consumption smoothing affects the shape of contract menu that the borrowers face, as well as their expected payoff from leveraged purchase of security. As explained previously, the borrowers now would like to avoid the high debt repayment by choosing a lower leverage when they become more optimistic, which leads to the result that overall leverage decreases with belief disagreement in this section.

2.3.2 Endogenous Trading Volume

Now I consider the case where the security is initially owned by the pessimistic traders. Such ownership is inefficient, since the security is owned by someone who values it less. Thus, the financial contract is supposed to increase efficiency in the market. Again, assume that the pessimistic traders offer a menu of collateralized debt contract, and the optimistic traders choose from the menu. Both types are price takers in the security market and credit market.

2.3.2.1 The optimization problem

The pessimistic traders Assume that initially the pessimistic traders own w securities, and sell part of them to the optimistic traders, using a single collateralized debt contract. They choose the consumptions c_0^p, c_s^p, the volume of security to sell a^p, the number ϕ^p of the debt contract, to maximize the expected utility,

$$\max_{c_0^p, c_s^p, a^p, \phi^p} \bar{U}^p = U^p(c_0^p) + \beta^p \int_0^{\bar{s}} U^p(c_s^p) f^p(s) ds \tag{2.3.15}$$

subject to the budget set $\mathcal{B}_3^p(p, j, \pi(\cdot))$ defined by inequalities (2.3.16)$\sim$(2.3.18)

$$c_0^p + \pi(j)\phi^p \leqslant e_0^p + pa^p \tag{2.3.16}$$

(consumption at $t = 0$ plus the lending should not exceed the endowments plus the income from security sales)

$$a^p \leqslant w \tag{2.3.17}$$

(the pessimistic traders cannot sell more security than the endowment)

$$c_s^p \leqslant e_1^p + (w - a^p)s + \phi^p \min[s, j^p] \tag{2.3.18}$$

(the consumption at $t = 1$ should not exceed the endowments plus the payoff of the remaining security and the repayment of the debt)

The optimistic traders The optimistic traders are the borrowers in the debt market. They choose the consumptions c_0^o, c_s^o, the volume of security to purchase a^o, the

number ϕ^o and face value j^o of the debt contract, to maximize the expected utility,

$$\max_{c_0^o, c_s^o, a^o, \phi^o, j^o} \bar{U}^o = U^o(c_o^o) + \beta^o \int_0^{\bar{s}} U^o(c_s^o) f^o(s) ds \tag{2.3.19}$$

subject to the budget set $\mathcal{B}_3^o(p, \pi(\cdot))$ defined by inequalities (2.3.20)$\sim$(2.3.22)

$$c_0^o + pa^o \leqslant e_0^o + \pi(j)\phi^o \tag{2.3.20}$$

(the consumption at $t = 0$ plus the payment for the security should not exceed the endowments plus the borrowing)

$$\phi^o \leqslant a^o \tag{2.3.21}$$

(collateral constraint)

$$c_s^o \leqslant e_1^o + (a^o - \phi^o)s + \phi^o \max[s - j^o, 0] \tag{2.3.22}$$

(the consumption at $t = 1$ should not exceed the payoff of the security with and without using the debt.

2.3.2.2 The equilibrium

In equilibrium the security and debt market should clear, and the contract is determined in the same way as in the previous section. After some algebra (see Appendix 2.D), the equilibrium can be characterized by four variables $(p, j, \pi(j), a)$ and the fol-

lowing four equations,

$$(p-\pi(j))U'(e_0^o-(p-\pi(j))a)=\beta^o\int_j^{\bar{s}}(s-j)U'\left(e_1^o+a(s-j)\right)f^o(s)ds \quad (2.3.23)$$

$$\begin{aligned}&pU'(e_0^p+(p-\pi(j))a)\\ =&\beta^p\int_0^j sU'\left(e_1^p+ws\right)f^p(s)ds+\beta^p\int_j^{\bar{s}}sU'\left(e_1^p+(w-a)s+aj\right)f^p(s)ds\end{aligned} \quad (2.3.24)$$

$$\begin{aligned}&\pi(j)U'(e_0^p+(p-\pi(j))a)\\ =&\beta^p\int_0^j sU'\left(e_1^p+ws\right)f^p(s)ds+\beta^p\int_j^{\bar{s}}jU'\left(e_1^p+(w-a)s+aj\right)f^p(s)ds\end{aligned} \quad (2.3.25)$$

$$\begin{aligned}&\frac{U'(e_0^o-(p-\pi(j))a)}{U'(e_0^p+a(p-\pi(j)))-a\pi(j)U''(e_0^p+a(p-\pi(j))}\\ =&\frac{\beta^o}{\beta^p}\frac{\int_j^{\bar{s}}U'(e_1^o+a(s-j))f^o(s)ds}{\int_j^{\bar{s}}\left(U'(e_1^p+(w-a)s+aj)+ajU''(e_1^p+(w-a)s+aj)\right)f^p(s)ds}\end{aligned} \quad (2.3.26)$$

Now equation (2.3.25) is the contract menu. We need to check the second order condition for the optimists, and the following conditions to ensure that in equilibrium the collateral constraint binds, and the pessimists do not sell more securities than they initial own,

$$\lambda\equiv\pi(j)U'(c_0^o)-\beta^o\int_0^{\bar{s}}\min[s,j]U'(c_s^o)f^o(s)ds>0 \quad (2.3.27)$$

$$a\leqslant w \quad (2.3.28)$$

2.3.2.3 Numerical example

Consider the exogenous variables in Table 2.4. To ensure that the pessimistic traders do not sell all their initial security holding, let their security endowment at $t=0$ to be high ($w=5$). The choices of all other parameters, the utility function and the belief

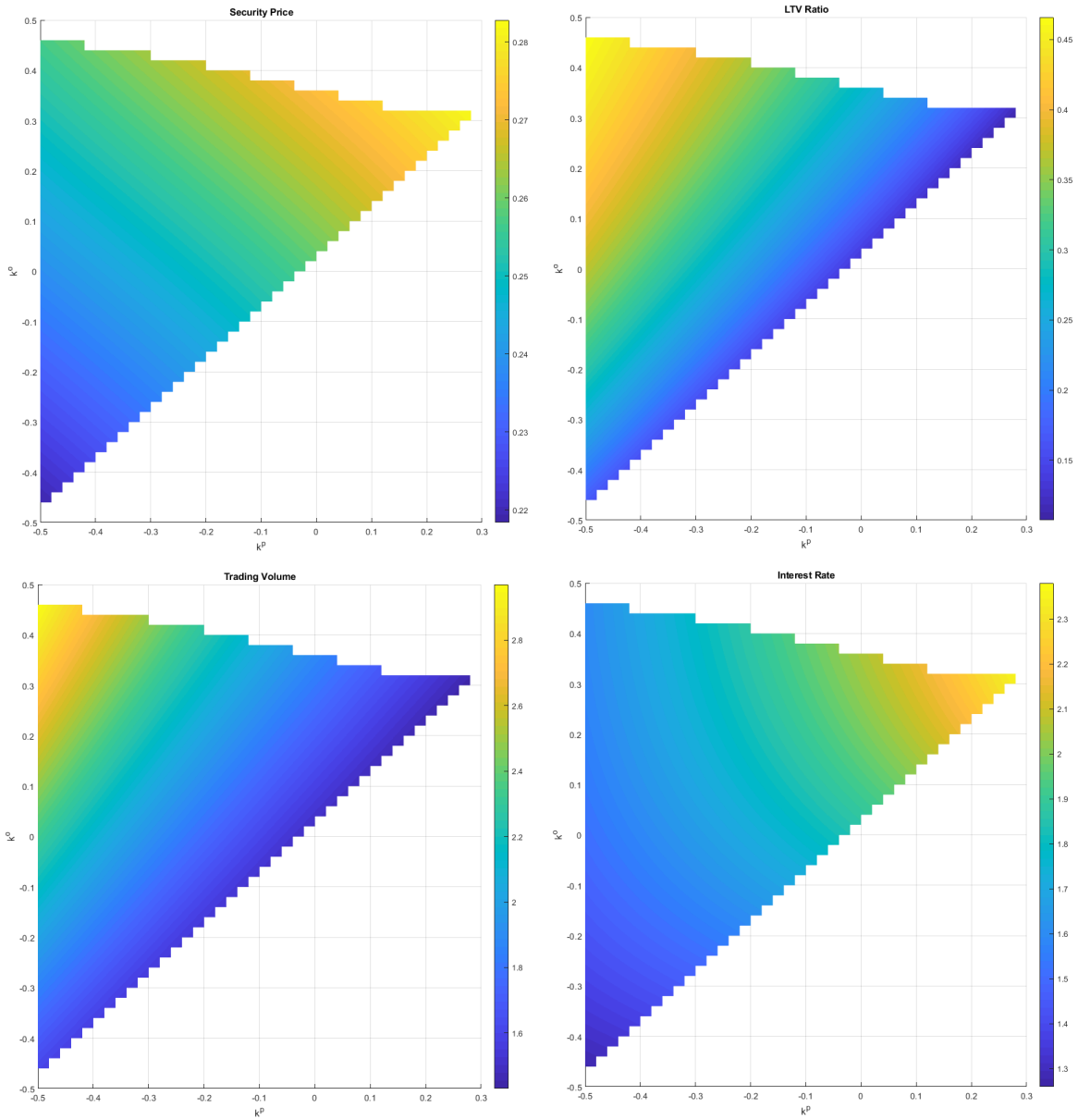
function are the same as before.

Table 2.5: Exogenous variables

$e_0^o = 1$	$e_1^o = 1$	$e_0^p = 1$	$e_1^p = 1$
$w = 5$	$\bar{s} = 2$	$\beta^o = 1$	$\beta^p = 1$

Again I run a similar simulation as in the previous sections, and the results are presented in Figure 2.4. Now I only look at the equilibrium in which the optimists' collateral constraint binds, and the pessimists do not sell more than their initial security endowments. The blank on top-right of each graph represents a violation of either the above two conditions, or the second-order condition of the optimists. Now the trading volume is endogenous, and traders' beliefs affect not only the collateralized debt contract, but the security trading as well. More optimistic borrowers demand more security, increasing the price and trading volume. They choose higher leverage, and are willing to pay higher interest rate. More optimistic lenders want to hold the security rather than to sell, decreasing the trading volume. The lower supply increases the security price in equilibrium. They also demand higher interest rate on the debt and tolerate lower leverage. The endogenous trading volume of the security changes the contract menu, and it is easy from the figures that security price still increases with average belief, while leverage increases with belief disagreement.

Figure 2.4: Simulation with endogenous trading volume



These figures show how changes in beliefs affect equilibrium asset prices, contract loan-to-value ratio and interest rate, and trading volume. For each figure, the horizontal axis is k^p which represents the lenders' belief, while the vertical axis is k^o which represents the borrowers' belief. Fixing other exogenous variables, equilibrium changes when the beliefs change. Figures for asset price, contract LTV, security trading volume and contract required rate of return are on top-left, top-right, bottom-left and bottom-right, respectively. The region with color closer to blue represents lower value, while color closer to yellow represents higher value.

2.4 Conclusion

To recapitulate, this chapter provides a novel approach within the general equilibrium paradigm to assess how belief heterogeneity affects equilibrium and contract origination under different model settings. The chapter establishes that changes in market *average* belief rather than *belief disagreement* determine the comovement of asset prices and leverage when there is no risk-free asset and the total supply of the underlying asset is fixed. Under mild technical conditions, asset prices and leverage could move in opposite directions. Optimism concerning the probability of downside states has no or little effect on asset prices, whereas optimism concerning the relative probability of upside states may have a significant impact on asset prices. Asset prices are above the fundamental value due to their usefulness as collateral. Extensions of the model considers different market structures. When a risk-free asset exists, the leverage decreases with belief disagreement. When the trading volume of the underlying asset is endogenous, the leverage increases with belief disagreement.

2.A Non-convexity

Traditional general equilibrium analysis requires convex preference and convex budget sets. In this model, traders can default on the collateralized debt contract, which leads to non-convexities in the budget sets. To avoid this problem, let the borrowers choose from a menu of contracts, and the lenders take the choice as given. There are two reasons why the borrowers make the choice rather than the lenders. First, it is more realistic. In the mortgage market, the bank or other financial intermediaries would offer several combinations of down-payments and interest rates, and the house buyers choose one contract from the menu. Second, the borrowers' budget constraint (2.2.3) is more likely to be concave rather than convex, compared with the borrowers' budget constraint (2.2.7), if both types are allowed to choose the value of j. The calculation for the lender-choosing case violates the second-order conditions. In this model, the lenders' budget set is linear. But we still need to deal with the non-convexity in the borrowers' budget set. To ensure that we are maximizing rather than minimizing, we need to check the second-order condition w.r.t. j for the borrowers. In (2.2.7), the non-linearity comes from the term $\phi^o \max[s - j^o, 0]$, thus we check the following two second-order derivatives,

$$\frac{\partial^2 \mathcal{L}^o}{(\partial \phi^o)^2} = \pi^2 U''(c_0^o) + \beta^o \int_0^{j^o} s^2 U''(c_s^o) f^o(s) ds + \beta^o \int_{j^o}^{\bar{s}} j^2 U''(c_s^o) f^o(s) ds \tag{2.A.1}$$

$$\begin{aligned}\frac{\partial^2 \mathcal{L}^o}{(\partial j^o)^2} =& \phi^o \frac{\partial^2 \pi}{(\partial j^o)^2} U'(c_0^o) + \left(\frac{\partial \pi}{\partial j^o} \phi^o\right)^2 U''(c_0^o) + \beta^p \phi^o U'(e_1^o + a^o j^o - \phi^o j^o) f^o(j^o) \\ &+ \beta^p \int_{j^o}^{\bar{s}} \phi^o U''(c_s^o) f^o(s) ds\end{aligned} \tag{2.A.2}$$

Obviously $\partial^2 \mathcal{L}^o / (\partial \phi^o)^2$ is negative, given that the marginal utility is decreasing. The sign of $\partial^2 \mathcal{L}^o / (\partial j^o)^2$ is not that obvious, and we need to check this value numerically.

2.B Proof of Theorem 1

The equilibrium of the economy in Section 2.2.1 can be summarized by the following three equations.

$$(p-\pi)U'(e_0^o - a(p-\pi)) = \beta^o \int_j^{\bar{s}} (s-j)U'(e_1^o + a(s-j))f^o(s)ds \tag{2.B.1}$$

$$\pi U'(e_0^p - \pi a) = \beta^p \int_0^j sU'(e_1^p + as)f^p(s)ds + \beta^p \int_j^{\bar{s}} jU'(e_1^p + aj)f^p(s)ds \tag{2.B.2}$$

$$\frac{U'(e_0^o - a(p-\pi))}{U'(e_0^p - a\pi) - a\pi U''((e_0^p - a\pi)} = \frac{\beta^o}{\beta^p} \frac{\int_j^{\bar{s}} U'(e_1^o + a(s-j))f^o(s)ds}{\int_j^{\bar{s}} \left(U'(e_1^p + aj) + jaU''(e_1^p + aj)\right) f^p(s)ds} \tag{2.B.3}$$

Lemma 1. *$\forall j$, $\exists! \pi \in \mathbb{R}^+$ s.t. equation (2.B.3) holds. Given power utility function $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, π increases as j increases if $\gamma \leqslant 1$.*

Proof. Take derivative of the left hand side (LHS) of equation (2.B.3) w.r.t. π,

$$\frac{\partial LHS}{\partial \pi} = U'(e_0^p - \pi a) - \pi a U''(e_0^p - \pi a) > 0$$

thus the left hand side increases as π increases. Also,

$$\lim_{\pi \to 0} \pi U'(e_0^p - \pi a) = 0$$

$$\lim_{\pi \to \frac{e_0^p}{a}} \pi U'(e_0^p - \pi a) = \infty$$

$\forall j \in (0, \bar{s})$, the right hand side of equation (2.B.3) is a fixed positive number. Thus, there must be a unique and positive π that satisfies equation (2.B.3), $\forall j \in (0, \bar{s})$. Take derivative of the right hand side (RHS) of equation (2.B.3) w.r.t. j,

$$\frac{\partial RHS}{\partial j} = \beta^p jU'(e_1^p + aj)f^p(j) + \beta^p \left(-jU'(e_1^p + aj)f^p(j) + \int_j^{\bar{s}} (U'(e_1^p + aj) + ajU''(e_1^p + aj))f^p(s)ds \right)$$

$$= \beta^p \int_j^{\bar{s}} (U'(e_1^p + aj) + ajU''(e_1^p + aj)) f^p(s) ds$$
$$= \beta^p \int_j^{\bar{s}} \frac{e_1^p + aj(1-\gamma)}{(e_1^p + aj)^{\gamma+1}} f^p(s) ds > 0$$

thus the right hand side increases as j increases. So π increases as j increases, given power utility function and that $\gamma \leqslant 1$. Thus equation (2.B.3) presents π as an increasing function of j. □

Lemma 2. *$\forall j$, $\exists! p \in \mathbb{R}^+$, $p > \pi$ s.t. equation (2.B.1) holds. Given power utility function $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, p decreases as j increases if $\gamma \leqslant 1$.*

Proof. It's obvious that $p - \pi > 0$ always holds. Take derivative of the left hand side of equation (2.B.1) w.r.t. p,

$$\frac{\partial LHS}{\partial p} = U'(e_0^o - a(p-\pi)) - a(p-\pi)U''(e_0^o - a(p-\pi)) > 0$$

thus the left hand side increases as p increases. Also,

$$\lim_{p-\pi \to 0} (p-\pi)U'(e_0^o - a(p-\pi)) = 0$$
$$\lim_{p-\pi \to \frac{e_0^o}{a}} (p-\pi)U'(e_0^o - a(p-\pi)) = \infty$$

$\forall j \in (0, \bar{s})$, the right hand side of equation 2.B.1 is a fixed positive number. Thus, there must be a unique and positive $p-\pi$ that satisfies equation (2.B.1), $\forall j \in (0, \bar{s})$. Given Lemma 1, there must be a unique and positive p. Take derivative of the right hand side of equation (2.B.1) w.r.t. j,

$$\frac{\partial RHS}{\partial j} = \beta^o \int_j^{\bar{s}} \left(-U'(e_1^o + a(s-j)) - a(s-j)U''(e_1^o + a(s-j))\right) f^o(s) ds$$

$$= -\beta^o \int_j^{\bar{s}} \frac{e_1^o + a(s-j)(1-\gamma)}{(e_1^o + a(s-j))^{\gamma+1}} f^o(s)ds < 0$$

thus the right hand side decreases as j increases. So p decreases as j increases, given power utility function and that $\gamma \leqslant 1$. Thus equation (2.B.1) presents p as a decreasing function of j. □

Theorem 1. *The solution to the system of the four simultaneous equations (2.B.1) ∼ (2.B.3) is unique, if*

1. *The utility function takes the form of* $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$*, and the relative risk aversion coefficient* γ *satisfies* $0 \leqslant \gamma \leqslant 1$.

2. *The hazard rates* $h^o(s), h^p(s)$ *of the two types of traders' belief function satisfies* $\frac{a\gamma}{(e_1^o + a(\bar{s}-j)^{\gamma+1})} + \frac{h^p(j)}{(e_1^o + a(\bar{s}-j))^{\gamma}} > \frac{h^o(h)}{(e_1^o)^{\gamma}}, \quad \forall j \in (0, \bar{s})$

Proof. Plug in the power utility function into equation (2.B.3), and rearrange,

$$\frac{1}{(e_0^o - a(p-\pi))^{\gamma}} \frac{(e_0^p - a\pi)^{1+\gamma}}{e_0^o - a\pi(1-\gamma)} \frac{e_1^p + aj(1-\gamma)}{(e_1^p + aj)^{1+\gamma}} = \frac{\beta^o}{\beta^p} \frac{\int_j^{\bar{s}} \frac{f^o(s)ds}{(e_1^o + a(s-j))^{\gamma}}}{\int_j^{\bar{s}} f^p(s)ds} \tag{2.B.4}$$

Now look at the three terms on the left hand side separately. From Lemma 2 $p - \pi$ is decreasing with j, so the first term is decreasing with j. Take derivative w.r.t. j of the third term yields,

$$\begin{aligned}
\left(\frac{e_1^p + aj(1-\gamma)}{(e_1^p + aj)^{1+\gamma}}\right)' &= \frac{a(1-\gamma)(e_1^p + aj)^{1+\gamma} - (e_1^p + aj(1-\gamma))(1+\gamma)a(e_1^p + aj)^{\gamma}}{(e_1^p + aj)^{2+2\gamma}} \\
&= \frac{a}{(e_1^p + aj)^{2+\gamma}} \Big((1-\gamma)(e_1^p + aj) - (e_1^p + aj(1-\gamma))(1+\gamma) \Big) \\
&= \frac{-\gamma a}{(e_1^p + aj)^{2+\gamma}} (2e_1^p + aj(1-\gamma)) \leqslant 0
\end{aligned}$$

thus the third term is decreasing with j. The second term takes similar form with the third

term. Given Lemma 1, π increases with j, thus the second term decreases with j. All the three terms on the left hand side decrease with j, thus the left hand side decreases with j.

Take derivative w.r.t. j of equation (2.B.4)'s right hand side yields,

$$
\begin{aligned}
&\left(\frac{\int_j^{\bar{s}} \frac{f^o(s)ds}{(e_1^o + a(s-j))^\gamma}}{\int_j^{\bar{s}} f^p(s)ds}\right)' \\
=&\left[\left(-\frac{f^o(j)}{(e_1^o)^\gamma} + \int_j^{\bar{s}} \frac{a\gamma f^o(s)ds}{(e_1^o + a(s-j))^{\gamma+1}}\right)\int_j^{\bar{s}} f^p(s)ds + \int_j^{\bar{s}} \frac{f^o(s)ds}{(e_1^o + a(s-j))^\gamma} f^p(j)\right] \Big/ \left(\int_j^{\bar{s}} f^p(s)ds\right)^2 \\
=&\int_j^{\bar{s}}\int_j^{\bar{s}} \left(-\frac{h^o(j)}{(e_1^o)^\gamma} + \frac{a\gamma}{(e_1^o + a(s-j))^{\gamma+1}} + \frac{h^p(j)}{(e_1^o + a(s-j))^\gamma}\right) f^o(s_1) f^p(s_2) ds_1 ds_2 \\
>&\int_j^{\bar{s}}\int_j^{\bar{s}} \left(-\frac{h^o(j)}{(e_1^o)^\gamma} + \frac{a\gamma}{(e_1^o + a(\bar{s}-j))^{\gamma+1}} + \frac{h^p(j)}{(e_1^o + a(\bar{s}-j))^\gamma}\right) f^o(s_1) f^p(s_2) ds_1 ds_2 > 0
\end{aligned}
$$

thus the right hand side of equation (2.B.4) increases with j. So if the system has a solution, it must be unique. □

2.C No-lending benchmark for the collateralized equilibrium

Consider the case in which the two types of traders are forbidden from lending/borrowing. The traders maximize the expected utility by choosing the consumptions and asset holdings,

$$\max_{c_0^h, a^h, c_s^h} \bar{U}^h = U(c_0^h) + \beta^h \int_0^{\bar{s}} U(c_s^h) f^h(s) ds, \quad h \in \{o, p\} \tag{2.C.1}$$

with budget set defined by (2.C.2)(2.C.3):

$$c_0^h + p a^h \leqslant e_0^h \tag{2.C.2}$$

(Consumption at $t = 0$ plus the payment for the security should not exceed the endowment.)

$$c_s^h \leqslant e_1^h + a^h s \quad \forall s \in (0, \bar{s}) \tag{2.C.3}$$

(Consumption at $t = 1$ should not exceed the endowment plus the security's payoff)

Market clears if total demand for the security equals the supply,

$$a^o + a^p = a \tag{2.C.4}$$

Thus we have,

$$p = \beta \int_0^{\bar{s}} \frac{U'(c_s^h)}{U'(c_0^h)} s f^h(s) ds, \quad h \in \{o, p\} \tag{2.C.5}$$

2.D Collateral equilibrium with risk-free asset

In equilibrium the markets for the security and the debt contract should clear,

$$a^o = a \tag{2.D.1}$$

(demand for the security equals supply)

$$\phi^o = \phi^p \equiv \phi \tag{2.D.2}$$

(demand for the debt contract equals the supply)

$$j^o \equiv j \tag{2.D.3}$$

(the lenders offer a menu of contracts, and the borrowers choose from the menu)

The optimization problem for the optimists is the same as in the case without risk-free asset. In equilibrium we have,

$$(p - \pi(j))U'(e_0^o - a(p - \pi(j))) = \beta^o \int_j^{\bar{s}} (s - j)U'(e_1^o + a(s - j))f^o(s)ds \tag{2.D.4}$$

$$\frac{\partial \pi(j)}{\partial j} U'(e_0^o - a(p - \pi(j))) = \beta^o \int_j^{\bar{s}} U'(e_1^o + a(s - j))f^o(s)ds \tag{2.D.5}$$

The Lagrangian for the pessimistic traders is,

$$\mathcal{L}^p = U(e_0^p - \pi(j)\phi^p - B^p) + \beta^p \int_0^{\bar{s}} U(e_1^p + B^p + \phi^p \min[s, j])f^p(s)ds$$

First-order condition w.r.t. B^p is,

$$\frac{\partial \mathcal{L}^p}{\partial B^p} = -U'(e_0^p - \pi(j)\phi^p - B^p) + \beta^p \int_0^{\bar{s}} U'(e_1^p + B^p + \phi^p \min[s, j])f^p(s)ds = 0$$

First-order condition w.r.t. ϕ^p is,

$$\frac{\partial \mathcal{L}^p}{\partial \phi^p} = -\pi(j)U'(e_0^p - \pi(j)\phi^p - B^p) + \beta^p \int_0^j sU'(e_1^p + B^p + \phi^p s)f^p(s)ds + \beta^p \int_j^{\bar{s}} jU'(e_1^p + B^p + \phi^p j)f^p(s)ds = 0$$

In equilibrium, these equations become,

$$U'(e_0^p - \pi(j)\phi - B) = \beta^p \int_0^{\bar{s}} U'(e_1^p + B + \phi \min[s, j])f^p(s)ds \tag{2.D.6}$$

$$\pi(j)U'(e_0^p - \pi(j)a - B) = \beta^p \int_0^j sU'(e_1^p + B + as)f^p(s)ds + \beta^p \int_j^{\bar{s}} jU'(e_1^p + B + aj)f^p(s)ds \tag{2.D.7}$$

Equation (2.D.7) represents the contract menu in equilibrium. Taking derivative w.r.t. j on both sides yields,

$$\begin{aligned}&\frac{\partial \pi(j)}{\partial j}\left(U'(e_0^p - B - \pi(j)a) - \pi(j)aU''(e_0^p - B - \pi(j)a)\right)\\ =&\beta^p \int_j^{\bar{s}} \left(U'(e_1^p + B + aj) + ajU''(e_1^p + B + aj)f^p(s)ds\right)\end{aligned} \tag{2.D.8}$$

Divide (2.D.5) by (2.D.8) we have,

$$\begin{aligned}&\frac{U'(e_0^o - a(p - \pi(j)))}{(U'(e_0^p - B - \pi(j)a) - \pi(j)aU''(e_0^p - B - \pi(j)a))}\\ =&\frac{\beta^o}{\beta^p}\frac{\int_j^{\bar{s}} U'(e_1^o + a(s-j))f^o(s)ds}{\int_j^{\bar{s}}\left(U'(e_1^p + B + aj) + ajU''(e_1^p + B + aj)f^p(s)ds\right)}\end{aligned} \tag{2.D.9}$$

and now we have four variables $(p, j, \pi(j), B)$ and four equations (2.D.4)(2.D.6)(2.D.7)(2.D.9).

Definition 2. *The **Collateral Equilibrium** for the economy*

$$\mathcal{E}_2 = \left\{(\bar{U}^h, e_0^h, e_1^h)_{h\in\{o,p\}, s\in(0,\bar{s})}, a\right\}$$

is a vector consisting of a security price, choice of the debt contract and the prices, individual consumptions as a function of the state's realization, security holding, contract trading, and saving,

$$\left(p, j, \pi(\cdot), (c_0^o, (c_s^o)_{s\in(0,\bar{s})}, a^o, \phi^o), (c_0^p, (c_s^p)_{s\in(0,\bar{s})}, \phi^p, B^p)\right)$$

if and only if

1. $(c_0^p, c_s^p, \phi^p, B^p) \in \mathcal{B}_2^p(p, j, \pi(\cdot))$, *and* $((\tilde{c}_0, \tilde{c}_s, \tilde{\phi}, \tilde{B}) \in \mathcal{B}_2^p(p, j, \pi(\cdot)) \Rightarrow \bar{U}^p(\tilde{c}_0, \tilde{c}_s, \tilde{\phi}, \tilde{B}) \leqslant \bar{U}^p(c_0^p, c_s^p, \phi^p, B^p)$
2. $(c_0^o, c_s^o, \phi^o, a^o, j^o) \in \mathcal{B}_2^o(p, \pi(\cdot))$, *and* $((\tilde{c}_0, \tilde{c}_s, \tilde{\phi}, \tilde{a}, \tilde{j}) \in \mathcal{B}_2^o(p, \pi(\cdot)) \Rightarrow \bar{U}^h(\tilde{c}_0, \tilde{c}_s, \tilde{\phi}, \tilde{a}, \tilde{j}) \leqslant \bar{U}^h(c_0^o, c_s^o, \phi^o, a^o, j^o)$

3. Market clearing conditions (2.D.1) ∼ (2.D.3) hold.

4. Conditions (2.3.12) ∼ (2.3.14) hold.

where the budget sets $\mathcal{B}_2^p(p, j, \pi(\cdot))$ and $\mathcal{B}_2^o(p, \pi(\cdot))$ are defined by inequalities (2.3.2) ∼ (2.3.3) and (2.3.5) ∼ (2.3.7), respectively.

2.E Collateral equilibrium with endogenous trading volume

In equilibrium the markets for the security and the debt contract should clear,

$$a^o = a^p \equiv a \tag{2.E.1}$$

(demand for the security equals supply)

$$\phi^o = \phi^p \equiv \phi \tag{2.E.2}$$

(demand for the debt contract equals the supply)

$$j^o \equiv j \tag{2.E.3}$$

(the lenders offer a menu of contracts, and the borrowers choose from the menu)

The optimization problem for the optimists is the same as in previous cases, and in equilibrium we have,

$$(p - \pi(j))U'(e_0^o - (p - \pi(j))a) = \beta^o \int_j^{\bar{s}} (s - j)U'\left(e_1^o + a(s - j)\right) f^o(s)ds \tag{2.E.4}$$

$$\frac{\partial \pi(j)}{\partial j} U'(e_0^o - (p - \pi(j))a) = \beta^o \int_j^{\bar{s}} U'(e_1^o + a(s-j)) f^o(s) ds = 0 \tag{2.E.5}$$

The Lagrangian for the pessimists is,

$$\begin{aligned}\mathcal{L}^p = & U(e_0^p + pa^p - \pi(j)\phi^p) + \beta^p \int_0^{j^p} U(e_1^p + (w - a^p + \phi^p)s) f^p(s) ds \\ & + \beta^p \int_{j^p}^{\bar{s}} U(e_1^p + (w - a^p)s + \phi^p j^p) f^p(s) ds - \mu^p (a^p - w)\end{aligned}$$

First-order condition w.r.t. a^p and ϕ^p are as below,

$$\begin{aligned}\frac{\partial \mathcal{L}^p}{\partial a^p} = & pU'(e_0^p + pa^p - \pi(j)\phi^p) - \beta^p \int_0^{j^p} sU'(e_1^p + (w - a^p + \phi^p)s) f^p(s) ds \\ & - \beta^p \int_{j^p}^{\bar{s}} sU'(e_1^p + (w - a^p)s + \phi^p j^p) f^p(s) ds - \mu^p = 0 \\ \frac{\partial \mathcal{L}^p}{\partial \phi^p} = & -\pi(j) U'(e_0^p + pa^p - \pi(j)\phi^p) + \beta^p \int_0^{j^p} sU'(e_1^p + (w - a^p + \phi^p)s) f^p(s) ds \\ & + \beta^p \int_{j^p}^{\bar{s}} j^p U'(e_1^p + (w - a^p)s + \phi^p j^p) f^p(s) ds\end{aligned}$$

where μ^p is the KTT multiplier for the condition that the pessimists cannot sell more securities than they initially have. I only consider the cases in which the no-short-selling constraint is slack, so $\mu^p = 0$. In equilibrium, these equations become,

$$pU'(e_0^p + (p - \pi(j))a) = \beta^p \int_0^j sU'\left(e_1^p + ws\right) f^p(s) ds + \beta^p \int_j^{\bar{s}} sU'\left(e_1^p + (w-a)s + aj\right) f^p(s) ds \tag{2.E.6}$$

$$\pi(j) U'(e_0^p + (p - \pi(j))a) = \beta^p \int_0^j sU'\left(e_1^p + ws\right) f^p(s) ds + \beta^p \int_j^{\bar{s}} jU'\left(e_1^p + (w-a)s + aj\right) f^p(s) ds \tag{2.E.7}$$

Equation (2.E.7) represents the contract menu in equilibrium. Taking derivative w.r.t. j on both sides yields,

$$\begin{aligned}&\frac{\partial \pi(j)}{\partial j}\left(U'(e_0^p+a(p-\pi(j)))-a\pi(j)U''(e_0^p+a(p-\pi(j))\right)\\=&\beta^p\int_j^{\bar{s}}\left(U'(e_1^p+(w-a)s+aj)+ajU''(e_1^p+(w-a)s+aj)\right)f^p(s)ds\end{aligned}\tag{2.E.8}$$

Divide equation (2.E.5) by (2.E.8) we have,

$$\begin{aligned}&\frac{U'(e_0^o-(p-\pi(j))a)}{U'(e_0^p+a(p-\pi(j)))-a\pi(j)U''(e_0^p+a(p-\pi(j))}\\=&\frac{\beta^o}{\beta^p}\frac{\int_j^{\bar{s}}U'(e_1^o+a(s-j))f^o(s)ds}{\int_j^{\bar{s}}\left(U'(e_1^p+(w-a)s+aj)+ajU''(e_1^p+(w-a)s+aj)\right)f^p(s)ds}\end{aligned}\tag{2.E.9}$$

and now we have four variables $(p, j, \pi(j), a)$ and four equations (2.E.4)(2.E.6)(2.E.7)(2.E.9).

Definition 3. *The **Collateral Equilibrium** for the economy*

$$\mathcal{E}_3=\left\{(\bar{U}^h,e_0^h,e_1^h)_{h\in\{o,p\},s\in(0,\bar{s})},w\right\}$$

is a vector consisting of a security price, choice of the debt contract and the prices, individual consumptions as a function of the state's realization, security trading, and contract trading,

$$\left(p,j,\pi(\cdot),(c_0^o,(c_s^o)_{s\in(0,\bar{s})},a^o,\phi^o),(c_0^p,(c_s^p)_{s\in(0,\bar{s})},a^p,\phi^p)\right)$$

if and only if

1. $(c_0^p,c_s^p,a^p,\phi^p)\in\mathcal{B}_3^p(p,j,\pi(\cdot))$, *and* $((\tilde{c}_0,\tilde{c}_s,\tilde{a},\tilde{\phi})\in\mathcal{B}_3^p(p,j,\pi(\cdot))\Rightarrow\bar{U}^p(\tilde{c}_0,\tilde{c}_s,\tilde{a},\tilde{\phi})\leqslant\bar{U}^p(c_0^p,c_s^p,a^p,\phi^p)$
2. $(c_0^o,c_s^o,\phi^o,a^o,j^o)\in\mathcal{B}_3^o(p,\pi)$, *and* $((\tilde{c}_0,\tilde{c}_s,\tilde{\phi},\tilde{a},\tilde{j})\in\mathcal{B}_3^o(p,\pi(\cdot))\Rightarrow\bar{U}^h(\tilde{c}_0,\tilde{c}_s,\tilde{\phi},\tilde{a},\tilde{j})\leqslant$

$\bar{U}^h(c_0^o, c_s^o, \phi^o, a^o, j^o)$

3. *Market clearing conditions (2.E.1) $\sim$ (2.E.3) hold.*

4. *Conditions (2.3.27) $\sim$ (2.3.28) hold.*

where the budget sets $\mathcal{B}_2^p(p, j, \pi(\cdot))$ and $\mathcal{B}_2^o(p, \pi(\cdot))$ are defined by inequalities (2.3.2) $\sim$ (2.3.3) and (2.3.5) $\sim$ (2.3.7), respectively.

Chapter 3

Mortgages, Financial Intermediation and Optimism

This chapter[1] presents a micro-founded framework to study the macroeconomic effect of mortgage default, with endogenous mortgage origination. Our model shows that, collateral constraints raise the house price above the fundamental value. The endogenous leverage of the mortgage contract together with the bank capital determine the comovement of prices, leverage and credit extension. Default is more severe after good states than after bad states, due to high leverage and risk-taking. Minsky effect amplifies the financial instability. When financial intermediaries rationally update their expectations during good times and become more optimistic about future economic prospects, they issue more risky loans with looser terms. Opposite is true during recessions and the more pessimistic financial intermediaries over-deleverage, dampening the credit condition.

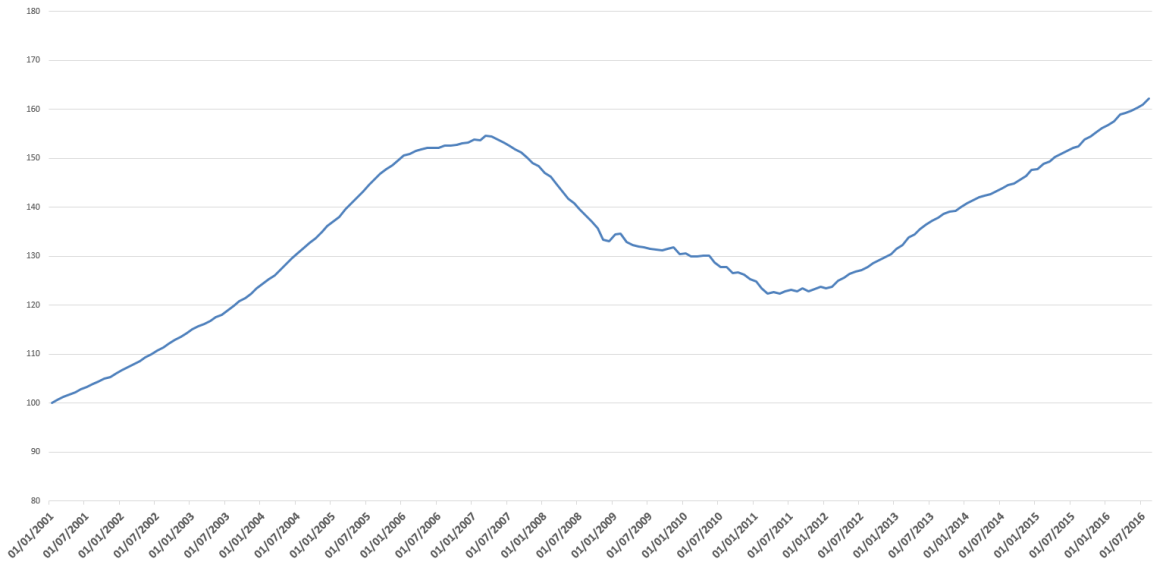
[1]This chapter is a joint with with Prof. Dimitrios Tsomocos

3.1 Introduction

The global financial crisis of 2007-2009 is considered the worst since the Great Depression in the 1930s. During the crisis, the burst of the U.S. housing bubble caused the values of securities tied to U.S. real estate pricing to plummet, damaging the financial institutions globally. However, the mainstream macroeconomic models are not very successful incorporating default, thus are not powerful enough to either prediction the crisis, or give valuable policy implications. This paper introduces general equilibrium models with collateral and default to study the mortgage market, financial intermediation and optimism. We offer an equilibrium approach to determine the leverage and interest rate of mortgage loans simultaneously. The model shows that, collateral constraints raise the housing price above the fundamental value; Mortgage leverage and housing price move in the same direction when the market condition changes; The changes in bank's balance sheet size and the endogenous leverage jointly induce reinforcing price and credit cycles; Default after a good state is more severe that after a bad state; Belief updating amplifies the financial instability.

Figure 3.1 shows the monthly House Price Index (HPI) for U.S. from Jan. 2004 to July 2016, with the data for Jan. 2004 normalized to 100. The house price kept increasing before the mortgage crisis, until reaching the peak around March to April 2007, then began to slide down significantly since Oct. 2007. The decreasing trend did not stop until 2011. Figure 3.2 shows the changes in the U.S. residential real estate loans from commercial banks, from Q4 2004 to Q3 2016. Obviously, there is a sharp decrease in change of the loan extension in Q4 2007, which becomes almost zero in

Figure 3.1: House Price Index

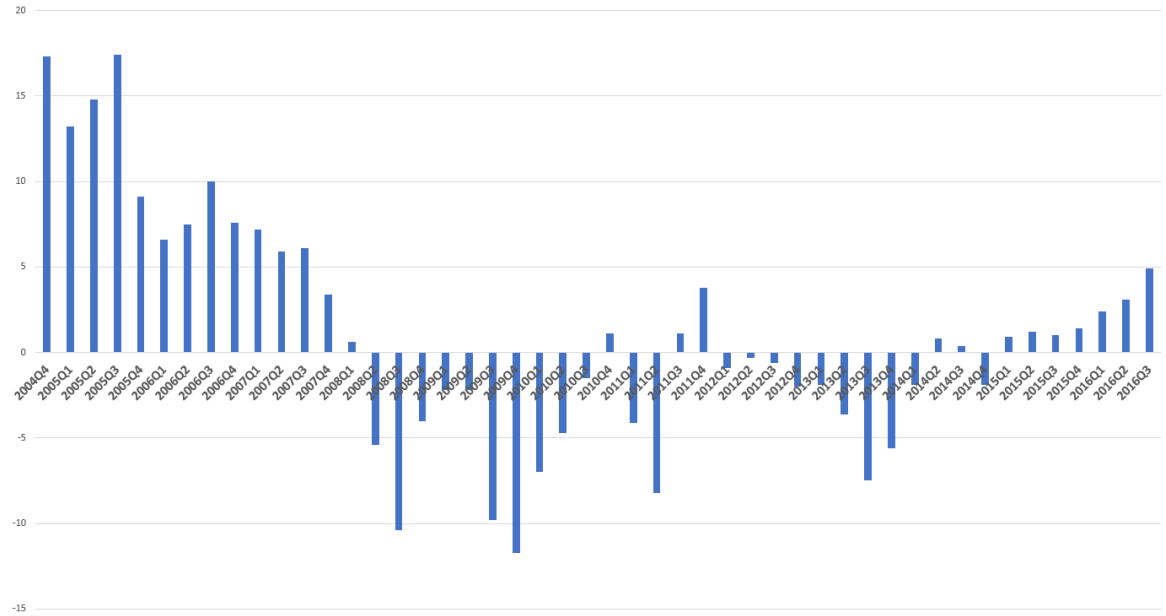


Monthly House Price index for U.S., purchase-only index, seasonally adjusted. Index at Jan. 2004 is normalized to 100. Data source: Federal Housing Finance Agency.

Q1 2008, and negative in Q2 2008. The residential real estate loans' extension did not stop decreasing until Q1 2015. Figure 3.3 presents the VIX index from Jan. 2004 to Nov. 2016. Although challenged and criticized, VIX is still a good measure for market fear or anxiety. The index started to fluctuate more since later 2007, and was boosted in Sep. 2008, reaching the peak around Oct. to Nov. 2008.

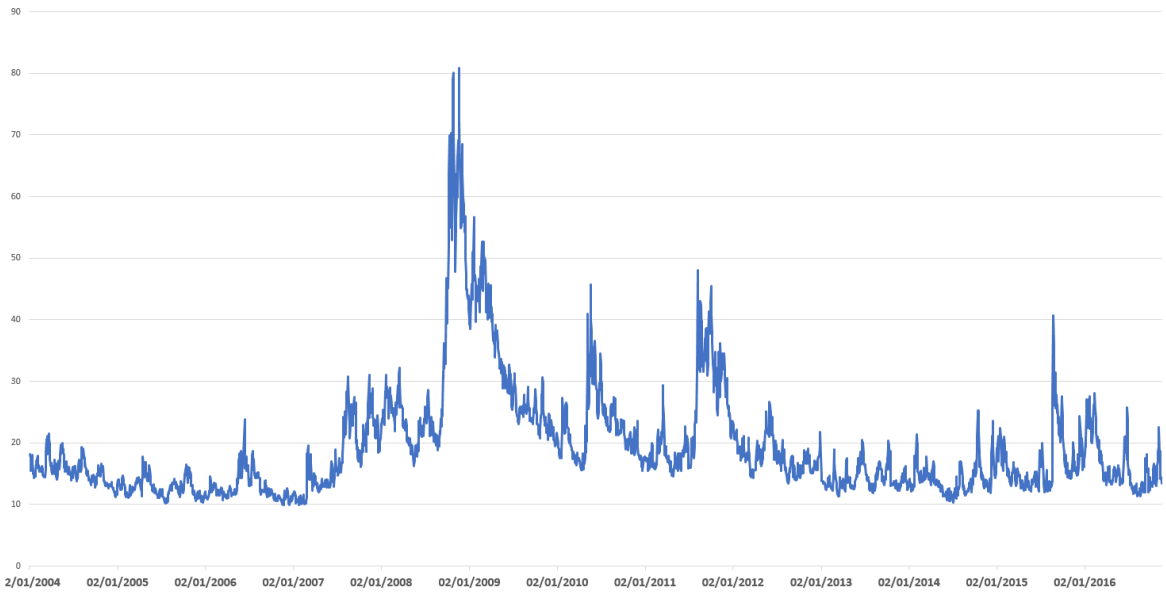
The change in the leverage may have played an important role. As documented by Fostel and Geanakoplos (2016), the leverage, prices and investment in the US housing market moved together during the crisis. This implies some interesting relationship among those economic variables. When demand in the housing market starts to drop, the house price declines, depressing demand for mortgage loans. But why does this lead to an unanticipated crisis? We know that due to the low credit quality in sub-prime mortgages, defaults occurred inevitably when the house price plummeted. Due to financial innovation, mainly securitization, the crisis in the mortgage market quickly

Figure 3.2: Residential real estate loans, percentage change per year



Percentage change per year in U.S. residential real estate loans, all commercial banks, seasonally adjusted, annual growth rate (break adjusted). Data source: Federal Reserve Bank, economics research & data.

Figure 3.3: Volatility S&P 500 (VIX)



Data source: Yahoo Finance.

spread into the whole financial sector. The anxiety in the market crystallized, persisting and amplifying the crisis. This paper aims at offering an coherent model to include house price, leverage, default, optimism, and financial intermediaries.

We first introduce a mortgage contract, and a collateral equilibrium with endogenous default. Due to lack of ability to commit, loans are possible only when backed by collaterals. The mortgage contract in the paper is similar to the collateralized security in Geanakoplos and Zame (2014). One unit of mortgage contract requires one unit of house as collateral. The price of the contract, π, represents how much can be borrowed per contract. If the house price is P_0 when the mortgage is issued, the Loan-to-Value ratio (LTV) is π/P_0. Consequently the margin $m \equiv 1 - LTV = 1 - \pi/P_0$, and the loan's leverage is $1/m = 1/(1 - \pi/P_0)$. The face value of the loan A is the required repayment of the loan. If the realized future house price is higher than A, it is optimal for the borrower to fully repay the loan. Otherwise, the house buyer defaults on the mortgage loan, gives up the collateral, and purchases some new houses at the new equilibrium price. We offer a new method to determine the equilibrium mortgage contract. The market offers a menu of mortgage contracts with different face value, thus different leverage and required rate of return accordingly. The borrowers choose the contract form the menu, taking the menu as given. The lenders take the choice of the borrowers as given. Using this method we are able to endogenously determine the price, leverage and the required rate of return simultaneously.

We show that the dynamic interactions between credit limits and asset prices transmit a temporary endowment shock into a persistent and amplified fluctuation by extending the model into a multi-period setting. "Overlapping generation" households trade consumption goods and houses with each other, using mortgage loans and deposit contracts to smooth consumption and share risk. A long-lived bank offers financial contracts to the households and is operated by bankers that maximise short term

profits. Aggregate risk is modelled via future endowment shocks of house buyers. The mortgage loan contract is determined using the same method as in the previous model and the equilibrium is characterized by house prices and leverage, as well as mortgage default in bad states. This result conforms with empirical data during the mortgage crisis. The spirit of the procyclical effect is similar with the double spiral in Brunnermeier and Pedersen (2009). Low endowment of house buyers decreases house prices, and triggers mortgage loan defaults. The banks balance sheet shrinks accordingly, thus, less credit is available for new mortgage loans. Meanwhile, the bank is forced to resell the foreclosed collateral in the housing market where the demand is already low. The Shleifer and Vishny (1992) fire sale mechanism further lowers the housing price. Therefore, this stressed financial condition makes the bank require tighter mortgage loan terms, i.e., higher collateral requirement and interest rates. This drop of housing collateral value precipitates in lower house prices. The credit cycle that is similar to Kiyotaki and Moore (1997) together with fire sale mechanism reinforce each other, hence, amplifying the effect of low endowments. Expected default limits credit extension through the endogenous mortgage contract origination, and actual default reduces the banks net worth, and subsequently the ability to extend future credit. Moreover, the model predicts that default after a good state is more severe than after a bad one, which captures Minsky's Financial Instability Hypothesis, as in Bhattacharya et al. (2015) who examine the effect of leverage, as a path dependent process, on financial stability by linking learning to risk-taking behavior.

We also consider the belief updating and the Minsky effect. When the bankers observe the realization of state, they update their beliefs. The occurrence of a good

state makes the bankers more optimistic, while a bad state makes them less optimistic. Minsky (1992) states that, "over a period in which the economy does well, optimism changes, and agents tend to invest more in the riskier asset. The tendency to transform doing well into a speculative investment boom is the basic instability in the economy". Minsky's argument elaborates how optimism grows during a long time of prosperity, while this paper also considers the situation when the economy is going down. The model shows that belief updating amplifies the cyclicality of price, leverage and credit extension. Thus, policies aiming at preventing the deterioration of financial crisis should take the change in the sentiment into consideration.

3.1.1 Related literature

This paper is related to literature on collateral and default in general equilibrium models with incomplete markets (GEI). Dubey et al. (2005) extend the standard GEI model with default and punishment by thinking of assets as pools. Their model encompasses a broad rage of adverse selection and signalling phenomena, and proves the existence of the refined equilibrium. Araujo et al. (2012) examine the effects of default and collateral on risk sharing, and demonstrate that the scarcity of collateralizable durable goods leads to large welfare losses. Regulation of collateral requirements never leads to a Pareto improvement for all agents, although it would often benefit a large group of agents. Geanakoplos and Zame (2014) define and prove the existence of the collateral equilibrium, and show that collateral requirements inhibit lending, limit borrowing, and distort consumption decisions. The shortage in supply creates incentive to create collateral by the authority and to stretch existing collateral by the

market through financial innovation. However, the possibility of robust inefficiency is inescapable.

There is a wide range of literature on borrowing constraints and credit cycles. Shleifer and Vishny (1992) identify the fire-sale mechanism in which a firm in financial distress sells assets when its industry peers are likely to be experiencing problems. In our model, the bank re-sells the collateralized houses when default happens in the mortgage loan market, which is similar to a fire-sale. Holmstrom and Tirole (1997) show that when firms and financial intermediaries are capital constrained, capital tightening hits poorly capitalized firms the hardest, but that interest rate effects and the intensity of monitoring will depend on relative changes in the various components of capital. Kiyotaki and Moore (1997) construct a dynamic economy in which durable assets are both factors of production and collateral for loans, and show that the dynamic interaction between credit limits and asset prices is a powerful transmission mechanism by which the effects of shocks persist, amplify and spill over to other sectors. Bernanke et al. (1999) develop a DSGE model which clarifies the role of credit market frictions in business fluctuations. The framework exhibits a financial accelerator, where endogenous developments in credit markets work to amplify and propagate shocks to the macroeconomy. Fostel and Geanakoplos (2008) show how leverage cycles can cause contagion, flight to collateral and issuance rationing in an anxious economy. Brunnermeier and Pedersen (2009) link an asset's market liquidity and traders' funding liquidity, and show that under certain conditions, margins are destabilizing and market liquidity and funding liquidity are mutually reinforcing, leading to liquidity spirals. Their model explains why market liquidity can suddenly dry up, has common-

ality across securities, is related to volatility, and is subject to flight to quality and co-moves with the market. Adrian and Shin (2010) document evidence that there is a strongly positive relationship between changes in leverage and changes in balance sheet size. Financial intermediaries adjust their balance sheets actively, and doing so in such a way that leverage is high during booms and low during busts. So leverage is procyclical, and this affects volatility and particularly the price of risk.

The mortgage model in the paper is closely related with Goodhart et al. (2010), which incorporates heterogeneous banks and households, as well as collateral backed mortgages. My model highlights the importance of collateral constraints. Other mortgage models include Campbell and Cocco (2003) who study the household choice between a fixed-rate and an adjustable-rate mortgage. Iacoviello (2005) develops a monetary business cycle model with nominal loans and collateral constraints tied to housing values. Forlati and Lambertini (2011) develop a DSGE model with housing, risky mortgages and endogenous default. Various explanations have been offered for the housing bubble and the mortgage crisis. Lack of incentive to monitor due to securitization, which leads to the loosening of lending standard and weakening of credit quality is one of the most popular explanations, such as in Demyanyk and Van Hemert (2009), Duca et al. (2010), Keys et al. (2010), Purnanandam (2010), DellAriccia et al. (2012), etc. Others argue that the housing bubble is rather rational, and based on expectation about housing price, without the finance insiders deceiving the uninformed mortgage borrowers and investors (Foote et al. (2012), Case et al. (2012)). Greenlaw et al. (2008), Mian and Sufi (2009, 2011) provide evidence on the bank credit and leverage. Iacoviello and Neri (2010) suggest that housing demand and housing

technological shock also explain the volatility in the housing investment and prices.

This paper is also related to literature on financial stability (Tsomocos (2003), Goodhart et al. (2004, 2006a), Goodhart et al. (2012, 2013)) and the Minsky effect (Minsky (1977, 1992), Bhattacharya et al. (2015)).

The rest of the paper is organized as follows: Section 3.2 models a mortgage contract directly between the house buyers and sellers in a simple binomial framework. Section 3.3 extends the model into multi-period settings, with overlapping generation households and a long-living bank, and shows that a credit and price cycle reinforce each other through the credit channel. Section 3.4 shows that a belief updating bankers over-react, and such behavior amplifies the credit cycle, deteriorating the financial condition in the bad times. Section 3.5 concludes the paper.

3.2 A mortgage model

This section shows how a mortgage contract is chosen between representative house sellers and buyers. Assume at this stage that the households can create mortgage contracts directly with each other. Consider a two-period ($t = 0, 1$) binomial economy with two types of representative households, the house sellers (R) and buyers (P), who are atomistic and of equal mass. A house buyer is endowed with perishable consumption goods at each state, while a house seller is endowed with some durable houses at the beginning, and some consumption goods at each state as well. The house buyers' endowments are high if a good state occurs at $t = 1$, and low if a bad state occurs at $t = 1$. Both types of households derive utility from consuming the goods and

living in the houses, and the house provides some salvage value in the end. Households trade with each other to maximize expected utility. There is no fiat money, and the consumption good is used as numeraire.

The Mortgage Contract The households cannot commit to repay their obligations, thus they have to pledge collateral for the loans. Only when the value of the collateral is higher than the loan's face value shall a borrower fully repay, otherwise the lender seizes the collateral, without further penalties, pecuniary or non-pecuniary. Assume for now that the households can create mortgage contracts directly with each other. Following Geanakoplos and Zame (2014), a collateralized loan contract is defined as a pair of variables: (A, C), where A is the face value of the contract, and C is the required collateral per contract. The contract's price, π, represents how much the house buyer can borrow per contract. Normalize C to 1, which means each unit of mortgage contract requires one unit of house as collateral, and assume that the contract is infinitely divisible. Given the current house price P, the Loan-to-Value ratio LTV is easily defined as π/P, the margin or haircut $m = 1 - LTV$, and the leverage equals $1/m$. If the future house price is higher than A, it is optimal for the borrower to repay the mortgage loan in full. Otherwise default occurs, and the lenders seize the collateralized houses. Partial default is not allowed, so the buyers either repay in full, or default on all the mortgage loans. The households are atomistic, so they do not internalized the impact on the loan's price of their the choice of quantity in the mortgage contracts to use, nor do they realize their impact on the house price. To deal with the potential non-convexity issue due to default, we assume that the lenders offer a menu of contracts with different face values, while the borrowers choose one

contract from it. In equilibrium, the contract is jointly determined by both parties' preferences.

Time Structure At $t = 0$, the two types of households enter the economy. The buyers purchase the houses and borrow from the sellers, pledging the houses as collateral. At $t = 1$, if a good state U occurs, the buyers' new endowments are high, and the realized house price is high enough such that it is optimal for the buyers to fully repay the mortgage loans. The buyers may purchase some more houses. If a bad state occurs, the buyers' new endowments are low. If the endowments are so low that either the buyer cannot afford to repay the mortgage loan, or the realized house price is lower than the face value of the mortgage loan, the buyers would default on the mortgage loan, give up the collateral, and purchase some houses at the new equilibrium price. The two cases represent forced default and strategic default, respectively.

3.2.1 The optimization problem

The house buyers (P) The house buyers are the borrowers in the mortgage contract, who derive utility from consuming the consumption goods and living in houses. The house also provides some salvage value in the end. Endowed with consumption goods at each state ($e^P = \{e_0^P, e_U^P, e_D^P\}$), the buyers choose the consumption ($c^p = \{c_0^P, c_U^P, c_D^P\}$), the quantity of houses to buy ($q^P = \{q_0^P, q_U^P, q_D^P\}$), the numbers of mortgage contract to enter (ϕ^P), and the mortgage loan's face value (A^P) to maximize

Table 3.1: List of Variables

Subscripts and Superscripts	
$t \in \{0, 1\}$	time periods
$s \in S \equiv \{0, U, D\}$	the set of states
$a \in \{R, P\}$	the set of households
Exogenous Variables	
E	house sellers' endowment of houses
e_s^a	agents' endowments of consumption goods
β^a	agents' discount factor
θ	probability of state U
τ	coefficient for house salvage value
Endogenous Variables	
P_s	house price
q_s^a	house trading quantity by agent a at state s
h_s^a	house owned by agent a at state s
c_s^a	consumption of goods by agent a at state s
π	price of the mortgage contract
A	face value of the mortgage contract
ϕ	quantity of mortgage contracts traded
r^M	mortgage interest rate
LTV	loan-to-value ratio of the mortgage
FOD	first-order derivative of the contract menu curve
SOD	second-order derivative of the contract menu curve
FV	fundamental value of the house
CV	collateral value of the house
λ	KKT multiplier of the collateral constraint
$\bar{U}^a$	agent a's expected utility
Utility Functions	
$U(c, h)$	instantaneous utility as a function of consumption and housing
$S(h)$	salvage value as a function of housing

the expected utility,

$$\bar{U}^P = U(c_0^P, h_0^P) + \beta^P \theta^P \left(U(c_U^P, h_U^P) + S(h_U^P)\right) + \beta^P (1 - \theta^P) \left(U(c_D^P, h_D^P) + S(h_D^P)\right) \tag{3.2.1}$$

where $h_0^p = q_0^p$

$$h_U^p = q_0^P + q_U^P$$

$$h_D^p = q_0^P + q_D^P - \phi^P$$

subject to the budget set $\mathcal{B}_1^P(P, \pi(A))$ defined by inequalities (3.2.2)$\sim$(3.2.5). At $t = 0$, the consumption plus the payment for the houses should not exceed the endowments plus the borrowed amount,

$$c_0^P + P_0 q_0^P \leqslant e_0^P + \pi\phi^P \tag{3.2.2}$$

Meanwhile, the collateral constraint is,

$$\phi^P \leqslant q_0^P \tag{3.2.3}$$

which states that one should have enough house as collateral in order to enter the mortgage loan contracts. Remember that one unit of contract requires one unit of houses as collateral, thus ϕ here represents both the unit of contracts entered and the unit of collateral required. If the good state U occurs, the sum of consumption, payment for some new houses, and the mortgage loan repayment should not exceed the endowments,

$$c_U^P + P_U q_U^P + A^P \phi^P \leqslant e_U^P \tag{3.2.4}$$

If the bad state D occurs, the endowments are low, and in equilibrium the house price will be low enough such that the buyers choose to default on the mortgage loans. The consumption plus the payment for the new houses should not exceed the endowments,

$$c_D^P + P_D q_D^P \leqslant e_D^P \tag{3.2.5}$$

We should check that in equilibrium the house price at state D is lower than the mortgage loan's face value so as to trigger default.

The house sellers (R) The house sellers are the lenders in the mortgage contract, who derive utility from consuming the consumption goods and living in houses. The house also provides some salvage value in the end. Endowed with some houses (E) at the beginning and the consumption goods at each state ($e^R \equiv \{e_0^R, e_U^R, e_D^R\}$), the house sellers choose the consumption ($c^R \equiv \{c_0^R, c_U^R, c_D^R\}$), the quantity of houses to sell ($q^R \equiv \{q_0^R, q_U^R, q_D^R\}$), the numbers of mortgage contract to issue (ϕ^R) for the borrowers' choices of contracts, to maximize the expected utility,

$$\bar{U}^R = U(c_0^R, h_0^R) + \beta^R \theta^R \left(U(c_U^R, h_U^R) + S(h_U^R)\right) + \beta^R (1-\theta^R) \left(U(c_D^R, h_D^R) + S(h_D^R)\right) \tag{3.2.6}$$

$$\begin{aligned} \text{where} \quad & h_0^R = E - q_0^R \\ & h_U^R = E - q_0^R - q_U^R \\ & h_D^R = E - q_0^R - q_D^R + \phi^R \end{aligned}$$

subject to the budget set $\mathcal{B}_1^R(P, \pi)$ defined by inequalities (3.2.7)$\sim$(3.2.9). At $t = 0$, the consumption plus the loans should not exceed the endowments plus the income from selling the house,

$$c_0^R + \pi \phi^R \leqslant e_0^R + P_0 q_0^R \tag{3.2.7}$$

If state U occurs at $t = 1$, the consumption should not exceed the sum of the endowments, income from selling some more houses, and the mortgage loan repayment,

$$c_U^R \leqslant e_U^R + P_U q_U^R + A\phi^R \tag{3.2.8}$$

If state D occurs at $t = 1$, the consumption should not exceed the endowments plus the income from selling some more houses.

$$c_D^R \leqslant e_D^R + P_D q_D^R \tag{3.2.9}$$

Note that default on the mortgage loans happens when $P_D < A$, and the lenders seize the collateralized houses.

3.2.2 The equilibrium

In equilibrium the house markets and the mortgage loan market clear, namely the supply of the houses equals the demand for the houses at each state,

$$q_0^R = q_0^P \equiv q_0 \tag{3.2.10}$$

$$q_U^R = q_U^P \equiv q_U \tag{3.2.11}$$

$$q_D^R = q_D^P \equiv q_D \tag{3.2.12}$$

the mortgage contract's supply equals demand,

$$\phi^R = \phi^P \equiv \phi \tag{3.2.13}$$

the equilibrium face value of the contract is the choice of the borrower,

$$A^P \equiv A \tag{3.2.14}$$

Contract Determination Here we introduce the contract selection mechanism: the market gives a menu of mortgage contracts, which takes the form of a function $\pi(A)$, each requiring one unit of house as collateral. The borrowers take the function as given, and choose one contract from the menu by choosing the face value A, aware that different A corresponds to different π. The lenders take the choice of the borrowers as given, and offer the contract accordingly. Consequently the interest rate and the leverage ratio are determined. When the borrowers choose the face value, they are in fact choosing the combination of leverage and required rate of return that the market offers. In equilibrium, the mortgage contract, as well as the function $\pi(A)$, should be consistent with both types' expectations and optimizations. The first-order conditions of the lenders give the following equation,

$$\pi U'(c_0^R) = \beta^R \theta^R A U'(c_U^R) + \beta^R (1-\theta^R)(U'(E-q_D^R) + S'(E-q_D^R)) \tag{3.2.15}$$

Plugging in the budget constraints, the collateral constraint and the market clearing conditions, the equation becomes,

$$\pi U'(e_0^R + (P_0 - \pi)q_0) = \beta^R \theta^R A U'(e_U^R + P_U q_U + A q_0) + \beta^R (1-\theta^R)(U'(E-q_D) + S'(E-q_D)) \tag{3.2.16}$$

In equilibrium, the equations implies the function $\pi(A)$, which represents the menu of mortgage contracts. As household R takes the choice of A as given, the above equation should hold for all the values of A. As long as the instantaneous utility function is second-order differentiable, the equation above is differentiable. Taking derivative w.r.t. A, we have,

$$\begin{aligned}&\frac{\partial \pi}{\partial A}\left(U'(e_0^R+(P_0-\pi)q_0)-\pi q_0 U''(e_0^R+(P_0-\pi)q_0)\right)\\ =&\beta^R\theta^R\left(U'(e_U^R+P_Uq_U+Aq_0)+Aq_0U''(e_U^R+P_Uq_U+Aq_0)\right)\end{aligned} \tag{3.2.17}$$

and Equation 3.2.17 gives the value of $\frac{\partial \pi}{\partial A}$ in equilibrium. The lenders' budget constraints are linear, while the borrowers' budget constraints are non-linear, thus we have a problem with non-convex budget set. The non-convexity comes from the term $\pi\phi$ and $A\phi$. It is easy to see that we only need to check if the following second-order condition,

$$\begin{aligned}\frac{\partial^2\mathcal{L}^P}{(\partial A^P)^2}=&\phi\left(\frac{\partial^2\pi}{(\partial A^P)^2}U'(e_0^P+\pi\phi-P_0q_0)+(\frac{\partial\pi}{\partial A})^2\phi U''(e_0^P+\pi\phi-P_0q_0)\right)\\ &+\beta^P\theta\phi^2U''(e_U^P-P_Uq_U-A\phi)<0\end{aligned} \tag{3.2.18}$$

Definition 4. *The **Collateral Equilibrium** for the economy*

$$\mathcal{E}_1=\left\{(\bar{U}^h,e^h,E)_{h\in\{R,P\}}\right\}$$

is a vector consisting of house prices, mortgage contract price, consumptions, house

trading, mortgage contract's face value and traded volume,

$$\big((P,\pi),(c^a,q^a)_{a\in\{R,P\}},A,\phi\big)$$

if and only if

1. $(c^P,q^P,\phi^P,A^P)\in\mathcal{B}_1^P(P,\pi(A))$, *and* $(\tilde{c},\tilde{q},\tilde{\phi},\tilde{A})\in\mathcal{B}_1^h(P,\pi(A))\Rightarrow\bar{U}^P(\tilde{c},\tilde{q})\leqslant\bar{U}^h(c^P,q^P)$.

2. $(c^R,q^R,\phi^R)\in\mathcal{B}_1^R(P,\pi,A)$, *and* $(\tilde{c},\tilde{q},\tilde{\phi})\in\mathcal{B}_1^R(P,\pi)\Rightarrow\bar{U}^R(\tilde{c},\tilde{q})\leqslant\bar{U}^R(c^R,q^R)$.

3. *Market clearing conditions (3.2.10)∼(3.2.14) hold.*

4. *Equation 3.2.17 holds.*

5. *In equilibrium* $\min\left[\frac{e_D^P}{\phi},P_D\right]<A<\min\left[\frac{e_U^P}{\phi},P_U\right]$

where the budget sets $\mathcal{B}_1^P(P,\pi(A))$ *and* $\mathcal{B}_1^R(P,\pi,A)$ *are defined by equations (3.2.2)∼(3.2.5) and (3.2.7)∼(3.2.9), respectively.*

Note that the last condition implies that at state U, it is optimal and affordable for the house buyers to repay the mortgage loan, while at state D, it is either optimal to default, or the debt is unaffordable.

Proposition 3. *When the house is used as collateral in the mortgage loans, and the collateral constraint binds, the house price is higher than the fundamental value for the buyer. The difference represents the collateral value of the house.*

Proof. From Appendix 3.A,

$$P_0=\frac{U'(h_0^P)+\beta^P\theta^P(U'(h_U^P)+S'(h_U^P))+\beta^P(1-\theta^P)(U'(h_D^P)+S'(h_D^P))}{U'(c_0^P)}+\frac{\lambda}{U'(c_0^P)}$$

where $h_s^P, s \in \{0, U, D\}$ is the house owed by P at state s. The first term represents the house's fundamental value to the house buyer, which is the sum of the marginal rates of substitution, and the second term represents the collateral value, with the KKT multiplier $\mu > 0$, and

$$\lambda = \pi U'(c_0^P) - \theta A U'(c_U^P) - (1-\theta)U'(h_D^P)$$

Combining the two equations we have,

$$(P_0 - \pi)U'(c_0^P) + \beta^P \theta^P A U'(c_U^P) = U'(q_0^P) + \beta^P \theta^P \left(U'(q_0^P + q_U^P) + S'(q_0^P + q_U^P)\right)$$

which implies that the marginal cost of down payment plus the marginal cost of the repayment should equal to the marginal benefit from living in the house. These results are consistent with Fostel and Geanakoplos (2008). □

Proposition 4. *We **cannot** construct another equilibrium with the same house and debt contract price and the same consumptions, in which a debt contract that eliminate default is the only contract traded.*

Proof. We prove the proposition by contradiction. Assume that there is a collateral equilibrium without default, where the prices of houses and debt contracts, as well as consumptions are the same as in Section 3.2.2. In the default-free collateral equilibrium, we should have,

$$c_D^R = e_D^R + P_D \tilde{q}_D + \tilde{A} q_0$$

$$h_D^R = E - q_0 - \tilde{q}_D$$

$$c_D^P = e_D^P - P_D\tilde{q}_D - \tilde{A}q_0$$

$$h_D^P = q_0 + \tilde{q}_D$$

which indicate that in the bad state, the borrowers also fully repay their obligations. Since the consumptions are the same between the two equilibrium, we should have,

$$P_D q_D = P_D\tilde{q}_D + \tilde{A}q_0$$

$$q_D = \tilde{q}_D + q_0$$

Thus $A = P_D$, which implies that what Fostel and Geanakoplos (2015) called as the "max min debt contract" is the only one traded in equilibrium. However, this leads to a contradiction, for the consumptions in the good state cannot be the same between the two equilibrium. Thus, we cannot construct a collateral equilibrium without default, where the prices of houses and debt contracts, as well as consumptions are the same with the collateral equilibrium with default. Default is non-trivial in our model. □

3.2.3 Numerical Example

Table 3.2 presents the simulation for the collateral equilibrium of the two-period mortgage model. The exogenous variables are chosen as below: the house sellers are endowed with 1 unit of houses at $t = 0$, and the endowments in consumption goods are the same in state U and D. The house buyers' endowments are 20 in the good state, and only 5 in the bad state, which leads to low house price and default in equilibrium. Let the utility function be logarithm, and separable across time and goods, with the

coefficient τ for salvage value.

$$\bar{U}^R = \ln(c_0^R) + \ln(h_0^R) + \beta^R\theta^R\left(\ln(c_U^R) + (1+\tau)\ln(h_U^R)\right) + \beta^R(1-\theta^R)\left(\ln(c_D^R) + (1+\tau)\ln(h_D^R)\right) \tag{3.2.19}$$

$$\bar{U}^P = \ln(c_0^P) + \ln(q_0^P) + \beta^P\theta^P\left(\ln(c_U^P) + (1+\tau)\ln(h_U^P)\right) + \beta^P(1-\theta^P)\left(\ln(c_D^P) + (1+\tau)\ln(h_D^P)\right) \tag{3.2.20}$$

Table 3.2: Simulation, two-period mortgage model

Exogenous Variables					
$E = 1$	$e_0^R = 5$ $e_0^P = 5$	$e_U^R = 10$ $e_U^P = 20$	$e_D^R = 10$ $e_D^P = 5$	$\beta^R = 1$ $\beta^P = 1$	$\theta = 0.9$ $\tau = 0$
Endogenous Variables, Collateral Equilibrium					
$P_0 = 21.357$ $c_0^R = 7.628$ $\pi = 9.51$ $\lambda = 1.359$	$P_U = 30$ $c_U^R = 19.361$ $A = 24.237$ $CV = 3.224$	$P_D = 15$ $c_D^R = 12.5$ $FOD = 0.201$ $FV = 18.132$	$q_0 = 0.222$ $c_0^P = 2.372$ $SOD = -0.007$ $U^R = 4.288$	$q_U = 0.133$ $c_U^P = 10.639$ $r = 1.549$ $U^P = 0.465$	$q_D = 0.167$ $c_D^P = 2.5$ $LTV = 0.445$
Endogenous Variables, Benchmark Equilibrium					
$P_0 = 21.885$ $c_0^R = 7.875$ $U^R = 4.285$	$P_U = 30$ $c_U^R = 18.03$ $U^P = 0.11$	$P_D = 15$ $c_D^R = 11.515$	$q_0 = 0.131$ $c_0^P = 2.125$	$q_U = 0.268$ $c_U^P = 11.97$	$q_D = 0.101$ $c_D^P = 3.485$

The table presents a simulation result of the two-period mortgage model. The Collateral Equilibrium is defined in Section 3.2.2, and the Benchmark Equilibrium is defined in Appendix 3.B.

In the Collateral Equilibrium, the house buyer's collateral constraint binds (The KKT multiplier $\lambda > 0$). The house buyer would like to borrow more, but is not able to do so due to lack of collateral. The marginal benefit of borrowing one more unit of mortgage loan is $\pi U'(c_0^P)$, while the marginal cost is $\beta^P\theta^P AU'(c_U^P) + \beta^P(1-\theta^P)(1+\tau)U'(q_D^P)$. In equilibrium the marginal benefit is greater than the marginal cost, but the binding collateral constraint limits the house buyer's ability to borrow. Note that

the budget set of the house buyers is not convex due to the choices on the contract menu. Thus we need to check the second-order condition (3.2.18). With concave utility functions, as long as $\partial^2\pi/(\partial A)^2 < 0$, the second-order condition is satisfied. The simulation shows that $\partial^2\pi/(\partial A)^2 = -0.007 < 0$.[2]

The initial house price is $P_0 = 21.357$, and the mortgage contract price is $\pi = 9.510$, resulting in a Loan-to-Value ratio of 44.5%. The mortgage contract's face value is $A = 24.237$, so the interest rate of the mortgage loan is $r^M \equiv A/\pi - 1 = 1.549$. The house price at state U is $P_U = 30$, and we have $A\phi < \min[e_U^P, P_U\phi]$, so the house buyers are optimal and able to repay the mortgage loan in full. The house price at state D is $P_D = 15$, which is smaller than the mortgage loan repayment. So the house buyers choose to default. The shock occurs to house buyers' endowments at state D. The direct effect is, the buyers have lower consumption at state D than in state U. Moreover, the low endowments lead to a low house price due to both income effect and substitution effect. The lower house price together with the endogenous default decrease R's income further at state D. The house price at $t = 0$ can be decomposed into two elements: the fundamental value which represents the houses' value to provide directly utility (accommodation), and the collateral value which allows the owner to borrow.

Introducing the mortgage market partially complete the market. Appendix 3.B provides a benchmark economy where mortgage contracts are prohibited, and agents

[2]In equilibrium, we have the relationship (3.2.17). Taking all other parameters as given, this equation can be seen as the relationship between π and A in equilibrium. The we can use this curve to find the first and second order derivatives $\partial\pi/\partial A$ and $\partial^2\pi/(\partial A)^2$. The two values are calculated numerically.

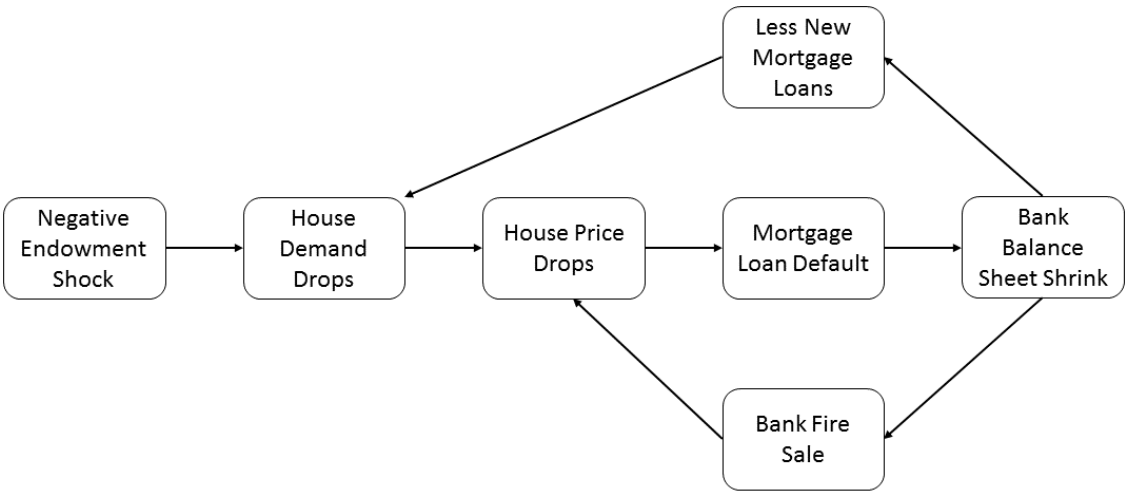
cannot borrow from each other. Table 3.2 also presents the benchmark equilibrium with the same set of exogenous variables. In this example, the mortgage market Pareto improves the allocation, since both types of the households enjoy higher level of utility in the collateral equilibrium than in the benchmark equilibrium. However, this is not always the case. It is not difficult to provide examples where the mortgage market fails to achieve Pareto-improvement. Araujo et al. (2012) assume that there are as many collateralized debts as the number of the states, thus in their model the market is complete. In our model, however, we restricted the type of contracts actively traded to be 1. The collateral equilibrium in our model should be Pareto-inferior than theirs. Appendix 3.C briefly presents the economy with two collateralized debt, each requiring 1 unit of house as collateral, and promises P_U and P_D, respectively. The numerical example shows that, the collateral equilibrium in our model with incomplete market is indeed Pareto-inferior than in the complete market case.

3.3 Credit cycle

This section extends the mortgage model above into a multi-period binomial framework with overlapping generation households and a long-lived bank operated by different bankers in different periods. We show that the endogenous changes of market leverage and banks capital lead to reinforcing credit and price cycles. Assume now that households do not borrow from each other directly, but through a bank. When a bad state occurs, the house buyers' endowment is low, decreasing the demand for houses. Thus the house price decreases, which triggers mortgage loan default. The bank's balance sheet shrinks accordingly, and less credit is available for new mortgage

loans. This further decreases the house demand and price, which forms a credit cycle. Also, the bank needs to resell the seized collateral at the housing market, resembling the Shleifer and Vishny (1992) fire sale, which further decrease the house price. Similar to Brunnermeier and Pedersen (2009)'s argument, the two cycles reinforce each other, persisting and amplifying the effect of low endowment. Figure 3.4 illustrates how the credit cycle and the price cycle reinforce each other, amplifying the endowment shock. As proved in Section 3.2, the house provides both fundamental value and collateral value. When the house price drops in the bad states, the value of loans backed by one unit of house also drops, which decreases the collateral value of the house. As in Kiyotaki and Moore (1997), the durable houses play a dual role of fundamental use and collateral.

Figure 3.4: Credit Cycle



Our model predicts that the leverage of the mortgage contract co-moves with the house price. This is because the shrinkage of the bank's balance sheet limits the credit for new mortgage loans thus the demand for new houses, while the fire sale keeps the

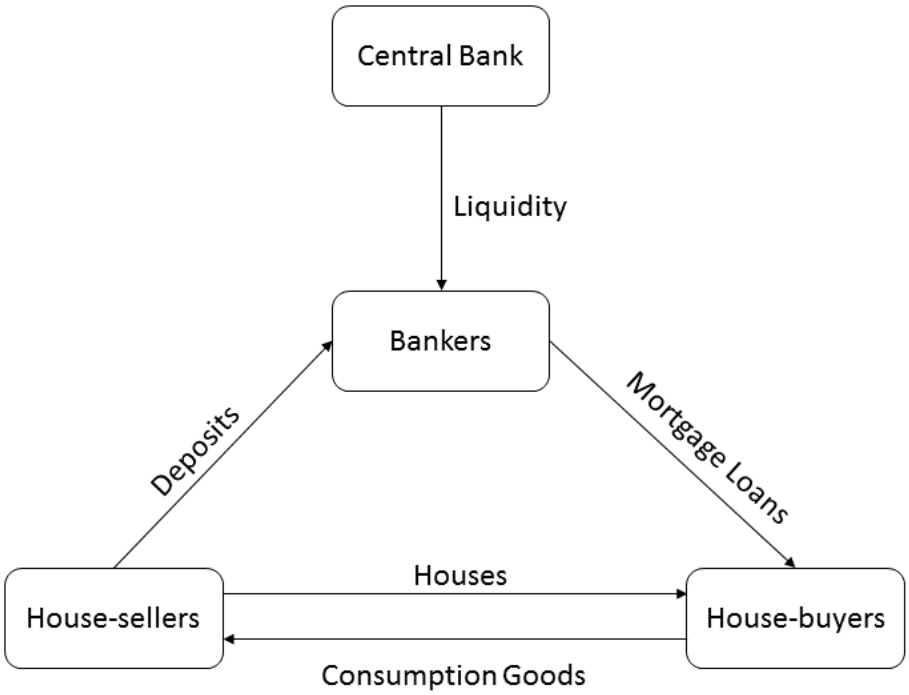
supply of houses high. Thus the low house price accompanies low credit, and a risk-averse banker would demand higher proportion of down-payment, thus low leverage as well. Geanakoplos (2010) describes such situation as leverage cycle, and suggests that the in the bad times, the belief disagreement among the households increases, while the overall optimism decreases due to the removal of the most-optimistic, highly leveraged agents after default. Our model discusses the change in optimism in the next section, and shows that if the banker becomes more pessimistic, the effect of the initial shock will be exacerbated. Adrian and Shin (2010) documents a strong positive relationship between changes in leverage and changes in balance sheet size, and implies that financial intermediaries, especially security brokers and dealers, adjust their balance sheets actively, leading to a procylical leverage. Our model suggests that, when the balance sheet of the bank is actively marked to market, the financial contract created by the banker has a pro-cyclical leverage ratio.

3.3.1 The Economy

Consider a three period ($t = 0, 1, 2$) economy with non-recombining binomial structure, perishable consumption goods and durable houses. There are two types of representative overlapping generation households, the house sellers (R) and house buyers (P), atomistic and of equal mass. Each of them only stays in the economy for two periods. Same as in the previous section, the house buyers are endowed with perishable consumption goods at each state, whereas the house sellers are endowed with some durable houses when they enter the economy, and some consumption goods at each state. Aggregate uncertainty is characterized by the second period endowments of the

house buyers. After each node, a good state with high endowments of the house buyers occurs with probability θ, and a bad state $1-\theta$. Both types of households derive utility from consuming the goods and living in the houses, and the house provides some salvage value when they exit the economy. Different from the previous model, a bank serves as the financial intermediary in the economy, offering mortgage loans to the house buyers and taking deposits from the house sellers. The banks is operated by two generations of bankers, who only care about short term profits. A Central Bank provides non-defaultable loans to the bank within each period with exogenous interest rates to provide liquidity. There is no fiat money in the economy, and the consumption good is used as numeraire. Figure 3.5 presents the structure of the economy.

Figure 3.5: The Economy

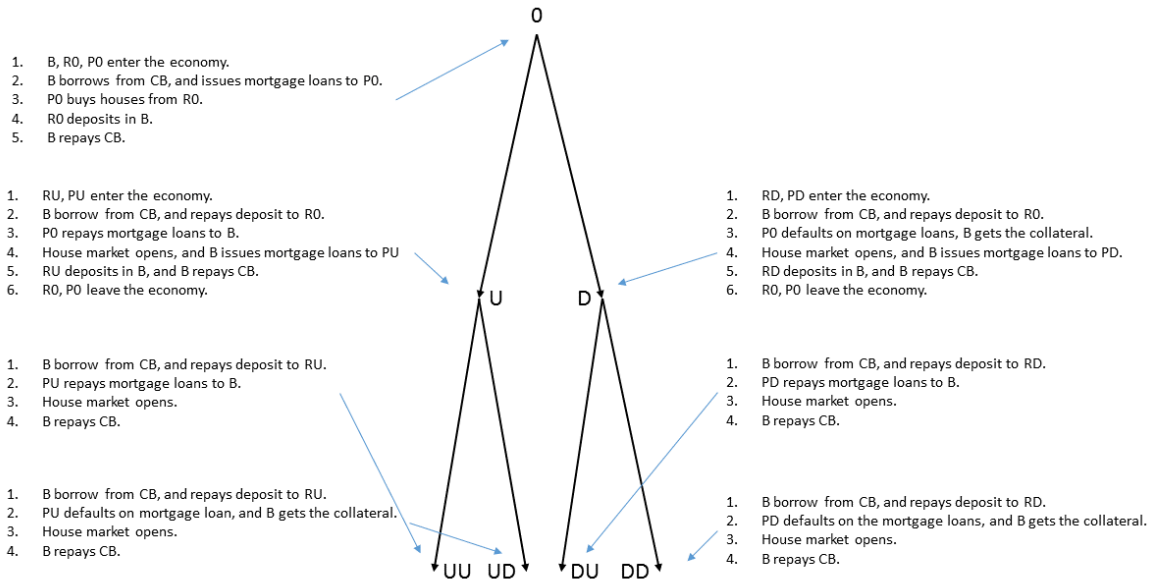


Let $Rs, Ps, Bs, \forall s \in \{0, U, D\}$ represent the house sellers, the house buyers and the bankers who enter the economy at the beginning of state $0, U, D$, respectively. Each

agent stays in the economy for two periods, and leaves the economy at the end of the subsequent state. Figure 3.6 illustrates the time structure of the economy. At the beginning of the initial state, the bank (B), the house seller ($R0$), and the house buyer ($P0$) enter the economy. B first borrows from the central bank, and issues mortgage loans to $P0$. $R0$ and $P0$ then trade houses with consumption goods, and $R0$ makes deposit to the bank. B repays fully to the central bank, and the households consume. If state U occurs at $t = 1$, $P0$'s endowments are high. At the beginning of state U, RU and PU enter the economy. B borrows from the central bank, and repays deposit to $R0$. Then $P0$ repays the mortgage loan, and B issues new mortgage loan to PU. Households trade houses with consumption goods, and RU deposit in B. By the end of the state, households consume and the bank repays the central bank. $R0$ and $P0$ leave the economy. If state UU occurs after U, PU's endowment is high. B borrows from the central bank, and repays deposit to RU. Then PU repays the mortgage loan, and households trade and consume. B repays the central bank by the end of the state, and realizes the profit. If state UD occurs after U, PU's endowments are low. B borrows from the central bank, and repays deposit to RU. PU's endowments are so low such that either they cannot afford to repay the mortgage loan, or the equilibrium house price is lower than the mortgage repayment. So PU defaults on the mortgage loan, B gets the collateral, and resells on the housing market. Then households and the bank trade and consume. B repays the central bank by the end of the state, and realizes the profit. The time structure is similar if state D occurs at $t = 1$. The only difference is, $P0$ defaults at D, the bank seizes the collateral, and resells on the housing market. When households leave the economy, they take away the houses owned at the end of

the period, and the houses provide some salvage value to the households.

Figure 3.6: Time Structure



Mortgage Contract Instead of creating loans between the households directly, now the bank issues mortgage loan contracts to the house buyers. The definition of the mortgage contract and the default conditions are exactly the same as in the previous section model. The difference is, the bank issues three different mortgage loans at state $0, U, D$.

Deposit The bank also provides a safe deposit contract, promising a risk-free interest rate of r. Assume that the bank faces a cash-in-advance constraint. Within each period, it can borrow from the central bank for liquidity at the beginning of the state, and is obliged to repay at the end of the state, with an exogenous interest rate at each state. This assumption leaves some room for research on monetary policy. More importantly, the bank can only leverage at a cost, making the bank's equity level matter for the whole economy. However, this model can only partially reflect the

situation in the mortgage market. As mentioned in Gorton and Metrick (2012), the banks do not underwrite the mortgage loan itself, but outsource this to a direct lender "with a specialization of underwriting loans to be held for only a short time before being sold to banks". Moreover, this model leaves out securitization, which might be the most important character of the crisis. Nevertheless, we can think of the bank in this model serving all the roles of financial intermediation, without complicating the model too much.

3.3.2 The Optimization Problem

The house buyers (Ps) A representative house buyer is the borrower in the mortgage contract, who derives utility from consuming the consumption goods and living in houses. The house also provides some salvage value in the end. She is only endowed with some consumption goods at each state ($e^{Ps} \equiv \{e_s^{Ps}, e_{sU}^{Ps}, e_{sD}^{Ps}\}$), and e_{sD}^{Ps} is low[3]. The buyer chooses her consumption ($c^{Ps} \equiv \{c_0^{Ps}, c_{sU}^{Ps}, c_{sD}^{Ps}\}$), the quantity of houses to buy ($q^{Ps} \equiv \{q_0^{Ps}, q_{sU}^{Ps}, q_{sD}^{Ps}\}$), the numbers of mortgage contract to use (ϕ_s^{Ps}), and the mortgage loan's face value (A_s^{Ps}) to maximize her expected utility,

$$\begin{aligned}\bar{U}^{Ps} =& U(c_s^{Ps}, h_s^{Ps}) + \beta^{Ps}\theta\left(U(c_{sU}^{Ps}, h_{sU}^{Ps}) + S(h_{sU}^{Ps})\right) \\ &+ \beta^{Ps}(1-\theta)\left(U(c_{sD}^{Ps}, h_{sD}^{Ps}) + S(h_{sD}^{Ps})\right) \qquad (3.3.1)\end{aligned}$$

$$\begin{aligned}\text{where} \quad & h_s^{Ps} = q_s^{Ps} \\ & h_{sU}^{Ps} = q_s^{Ps} + q_{sU}^{Ps}\end{aligned}$$

[3]The economy is complicated so that I will neither provide an analytical solution nor the threshold value of e_{sD}^{Ps} for default, however, e_{sD}^{Ps} should be low enough to trigger mortgage default in the bad states and this condition is satisfied in all the simulation presented hereafter.

Table 3.3: List of Variables and Notations

Subscripts and Superscripts	
$t \in \{0, 1, 2\}$	time periods
$s \in S \equiv \{0, U, UU, UD, D, DU, DD\}$	the set of states
sU, sD	the subsequent state of state $s \in \{0, U, D\}$
$R0, RU, RD$	house sellers entering the economy at state $0, U, D$
$P0, PU, PD$	house buyers entering the economy at state $0, U, D$
$B0, BU, BD$	bankers entering the economy at state $0, U, D$
$a \in \{R0, RU, RD, P0, PU, PD\}$	the set of households
$b \in \{B0, BU, BD\}$	the set of bankers
Exogenous Variables	
E^a	house sellers' endowment of houses
$e^a \equiv \{e^a_s, s \in S\}$	agents' endowments of consumption goods
β^a	agents' discount factor
θ	probability of state U
τ	coefficient for house salvage value
Endogenous Variables	
$P \equiv \{P_s, s \in S\}$	the house prices
q^a_s	house trading quantity by agent a at state s
h^a_s	house owned by agent a at state s
c^a_s	consumption of goods by agent a at state s
$Dep \equiv \{Dep_s, s \in \{0, U, D\}\}$	the deposit
$r \equiv \{r_s, s \in \{0, U, D\}\}$	deposit interest rate
$M \equiv \{M_s, s \in S\}$	bank's loan from the central bank
$\bar{r} \equiv \{P_s, s \in S\}$	central bank interest rate
$\pi \equiv \{\pi_s, s \in \{0, U, D\}\}$	the mortgage contract prices
$A \equiv \{A_s, s \in \{0, U, D\}\}$	face value of the mortgage contract
$\phi \equiv \{\pi_s, s \in \{0, U, D\}\}$	number of mortgage contracts traded
$r^M \equiv \{r^M_s, s \in S\}$	mortgage interest rate
LTV_s	loan-to-value ratio of the mortgage
FOD_s	first-order derivative of the contract menu curve
SOD_s	second-order derivative of the contract menu curve
FV_s	fundamental value of the house
CV_s	collateral value of the house
λ_s	KKT multiplier of the collateral constraint
δ_s	default at state s

$$h^{Ps}_{sD} = q^{Ps}_s - \phi^{Ps}_s + q^{Ps}_{sD}$$

subject to the budget set $\mathcal{B}^{Ps}_4(P, \pi(A))$ defined by inequalities (3.3.2)$\sim$(3.3.5). After entering the economy, the buyer's consumptions plus the payment for the houses should

not exceed the endowments plus the loans,

$$c_s^{Ps} + P_s q_s^{Ps} \leqslant e_s^{Ps} + \pi_s \phi_s^{Ps} \tag{3.3.2}$$

Meanwhile, the collateral constraint is,

$$\phi_s^{Ps} \leqslant q_s^{Ps} \tag{3.3.3}$$

and ϕ here represents both the number of contracts and the required amount of collateral, just as in the previous model. If the a good state occurs, the sum of the consumption, payment for new houses, and the mortgage loan repayment should not exceed the endowments,

$$c_{sU}^{Ps} + P_{sU} q_{sU}^{Ps} + A_s^{Ps} \phi_s^{Ps} \leqslant e_{sU}^{Ps} \tag{3.3.4}$$

If a bad state occurs, the buyer defaults on the mortgage loan, and the consumption plus the payment for new houses should not exceed the endowments at the bad state,

$$c_{sD}^{Ps} + P_{sD} q_{sD}^{Ps} \leqslant e_{sD}^{Ps} \tag{3.3.5}$$

The house seller (Rs) A representative house seller sells the houses to the house buyers and deposits in the bank to smooth consumptions, and derives utility from consuming the consumption foods and living in houses. Endowed with some houses (E_s^{Rs}) at the beginning and the consumption goods at each state ($e^{Rs} \equiv \{e_s^{Rs}, e_{sU}^{Rs}, e_{sD}^{Rs}\}$), the house seller chooses the consumption ($c^{Rs} \equiv \{c_s^{Rs}, c_{sU}^{Rs}, c_{sD}^{Rs}\}$), the quantity of houses

to sell ($q^{sR} \equiv \{q_s^{Rs}, q_{sU}^{Rs}, q_{sD}^{Rs}\}$), and the amount to deposit (Dep_s^{Rs}) to maximize her expected utility,

$$\begin{aligned}\bar{U}^{Rs} =& U(c_s^{Rs}, h_s^{Rs}) + \beta^{Rs}\theta\left(U(c_{sU}^{Rs}, h_{sU}^{Rs}) + S(h_{sU}^{Rs})\right) \\ &+ \beta^{Rs}(1-\theta)\left(U(c_{sD}^{Rs}, h_{sD}^{Rs}) + S(h_{sD}^{Rs})\right) \qquad (3.3.6)\end{aligned}$$

$$\begin{aligned}\text{where} \quad & h_s^{Rs} = E_s^{Rs} - q_s^{Rs} \\ & h_{sU}^{Rs} = E_s^{Rs} - q_s^{Rs} - q_{sU}^{Rs} \\ & h_{sD}^{Rs} = E_s^{Rs} - q_s^{Rs} - q_{sD}^{Rs}\end{aligned}$$

subject to the budget set $\mathcal{B}_4^{Rs}(P, r)$ defined by inequalities (3.3.7)$\sim$(3.3.9). After entering the economy, the consumption plus the deposit should not exceed the endowments plus the income from selling houses,

$$c_s^{Rs} + Dep_s^{Rs} \leqslant e_s^{Rs} + P_s q_s^{Rs} \qquad (3.3.7)$$

If a good state occurs, the consumption should not exceed the sum of the endowments, income from selling more houses, and the deposit repayment,

$$c_{sU}^{Rs} \leqslant e_{sU}^{Rs} + P_{sU} q_{sU}^{Rs} + Dep_s^{Rs}(1 + r_s) \qquad (3.3.8)$$

If a bad state occurs, the consumption should not exceed the sum of the endowments, income from selling some more houses, and the deposit repayment,

$$c_{sD}^{Rs} \leqslant e_{sD}^{Rs} + P_{sD} q_{sD}^{Rs} + Dep_s^{Rs}(1 + r_s) \qquad (3.3.9)$$

Notice that if $q_{sD}^{Rs} < 0$, the house seller buys back some houses at the bad state sD.

The bank The bank is endowed with some initial capital (e_0^B) in terms of the consumption goods at $t = 0$, and it offers mortgage loans to house buyers and absorbs deposits from house sellers. Assume that the bank is the price taker in all the markets, and different generations of bankers run the bank at different time periods. The budget set $\mathcal{B}_4^B(P, r, \pi, A, \bar{r})$ of the bank is defined by inequalities (3.3.10)$\sim$(3.3.20). At $t = 0$, the total mortgage loan issued should not exceed the initial capital plus the liquidity borrowed from the central bank,

$$\pi_0 \phi_0^B \leqslant e_0^B + M_0 \tag{3.3.10}$$

and the bank should absorb enough deposit to repay the central bank,

$$M_0(1 + \bar{r}_0) \leqslant Dep_0^B \tag{3.3.11}$$

At state U, the bank first borrows from the central bank to repay the deposit,

$$Dep_0^B(1 + r_0) \leqslant M_U \tag{3.3.12}$$

then the issuance of new mortgage loans plus the repayment to the central bank should not exceed the collected mortgage repayment plus the new deposit absorbed,

$$\pi_U \phi_U^B + M_U(1 + \bar{r}_U) \leqslant A_0^B \phi_0^B + Dep_U^B \tag{3.3.13}$$

At state UU, the bank first borrows from the central bank to repay the deposit,

$$Dep_U^B(1+r_U) \leqslant M_{UU} \tag{3.3.14}$$

At state UD, the bank first borrows from the central bank to repay the deposit,

$$Dep_U^B(1+r_U) \leqslant M_{UD} \tag{3.3.15}$$

At state D, the bank first borrows from the central bank to repay the deposit,

$$Dep_0^B(1+r_0) \leqslant M_D \tag{3.3.16}$$

then the issuance of new mortgage loans plus the repayment to the central bank should not exceed the resale of collateral plus the new deposit absorbed,

$$\pi_D \phi_D^B + M_D(1+\bar{r}_D) \leqslant P_D q_D^B + Dep_D^B \tag{3.3.17}$$

notice that the bank cannot sell more than the seized collateral,

$$q_D^B \leqslant \phi_0^B \tag{3.3.18}$$

At state DU, the bank first borrows from the central bank to repay the deposit,

$$Dep_D^B(1+r_D) \leqslant M_{DU} \tag{3.3.19}$$

At state DD, the bank first borrows from the central bank to repay the deposit,

$$Dep_D^B(1+r_D) \leqslant M_{DD} \tag{3.3.20}$$

Bankers Two generations of bankers operate the bank and they overlap in the interim period. The first generation banker $B0$ makes the investment decision by choosing ϕ^B, M_0, Dep_0^B, M_U and M_D, whereas the second generation bankers $Bs, s \in \{U, D\}$ choose ϕ_s^B, Dep_s^B, M_{sU} and M_{sD}. All the bankers maximize the expected utility from profits in their second period,

$$\bar{U}^{Bs} = \theta U(Y_{sU}) + (1-\theta)U(Y_{sD}), s \in \{0, U, D\} \tag{3.3.21}$$

where

$$Y_{sU} = A_s\phi_s^B - M_{sU}(1+r_{sU}^M)$$

$$Y_{sD} = P_{sD}\phi_s^B - M_{sD}(1+r_{sD}^M)$$

Different generations of the bankers cannot affect each other's decisions directly, while the first generation banker may affect the "endowments" of the second generation, which is the profit in the first period. Breaking down the bank into two generation of bankers reflect the myopia in the banking industry and simplify the computation, but ignores the potential benefit of investment smoothing. Goodhart et al. (2006b) proves that, under certain conditions, an infinite horizon utility maximising problem for bankers is equivalent to a finite horizon problem, and such horizon can be chosen

as one period. One potential reason behind such behaviour is that, the bankers may depart from the current bank for better alternative contracts after a finite period of time.

Contract Determination I use the same method as in the previous section to determine the equilibrium mortgage contract. Assume that the market offers a menu of mortgage contracts with different face value A, each requiring one unit of house as collateral. The menu is given as a function of $\pi(A)$, which specifies the price of the loan depending on the face value. Depending on the current market condition, the house buyers choose the face value A, taking the function as given, and the banker takes the choice of face value as given. In this way, the banker's budget constraint is linear, and there is only non-convexity problem with the house buyers. It is only necessary to check the second order condition $\partial^2 \mathcal{L}^P/(\partial A^p)^2 < 0$, where $\mathcal{L}^P$ is the house buyers' Lagrangian. In equilibrium, the mortgage contract and the function $\pi(A)$ should be consistent with the expectations of both the bankers and the house buyers. The first-order condition of the banker $Bs, s \in \{0, U, D\}$ gives the following equation,

$$\begin{aligned} &\theta U'(Y_{sU}) \left(A_s - \pi_s(1+r_s)(1+\bar{r}_s)(1+\bar{r}_{sU})\right) \\ &+ (1-\theta)U'(Y_{sD}) \left(P_{sD} - \pi_s(1+r_s)(1+\bar{r}_s)(1+\bar{r}_{sD})\right) = 0 \end{aligned} \tag{3.3.22}$$

This equation holds for all the value of A given to the bankers, thus in equilibrium, it represents the relationship between π and A. Just like in the previous model, we can take partial derivative w.r.t. A to get the value of $\dfrac{\partial \pi}{\partial A}$ at each point on the contract

menu in equilibrium.

Equilibrium The mortgage equilibrium is similar to the definition provided earlier, where all agents optimize by choosing their choice variables, and all the markets clear. The formal definition of the equilibrium is given in Appendix 3.D. In equilibrium, house buyers default at state D, DU, DD since the house price falls below the face value of the mortgage loan. Similar as in the previous section, the house price at state $0, U, D$ in equilibrium can be decompose into fundamental value and collateral value when collateral constraints bind. Moreover, the numerical example shows that the collateral value tends to be more volatile than the fundamental value, contributing to the price volatility.

3.3.3 Numerical Example

The following section presents a numerical example of the economy. The exogenous variables are given in Table 3.4. Each house seller has the same endowments in consumption goods across time, while each house buyer has relatively higher endowments in the good state and lower endowments in the bad state. As we assume, RU and RD are the same household entering the economy at different states of the same time period, thus RU and RD have the same endowment. So are PU and PD. Assume that all the households and the bankers have the same and constant belief on the states (same θ). Also, let all the instantaneous utility functions be logarithm, and the salvage value function be $S(\cdot) = \tau U(\cdot)$.

Table 3.4: Exogenous variables

$e_0^B = 1$	$r_0^M = 0.01$	$r_U^M = 0.01$	$r_{UU}^M = 0.01$	$r_{UD}^M = 0.01$
$\theta = 0.95$	$\tau = 1$	$r_D^M = 0.01$	$r_{DU}^M = 0.01$	$r_{DD}^M = 0.01$
$E_0^{R0} = 1$	$e_0^{R0} = 10$	$e_U^{R0} = 10$	$e_D^{R0} = 10$	$\beta^{R0} = 1$
	$e_0^{P0} = 10$	$e_U^{P0} = 20$	$e_D^{P0} = 5$	$\beta^{P0} = 1$
$E_U^{RU} = 1$	$e_U^{RU} = 10$	$e_{UU}^{RU} = 10$	$e_{UD}^{RU} = 10$	$\beta^{RU} = 1$
	$e_U^{PU} = 10$	$e_{UU}^{PU} = 20$	$e_{UD}^{PU} = 5$	$\beta^{PU} = 1$
$E_D^{RD} = 1$	$e_D^{RD} = 10$	$e_{DU}^{RD} = 10$	$e_{DD}^{RD} = 10$	$\beta^{RD} = 1$
	$e_D^{PD} = 10$	$e_{DU}^{PD} = 20$	$e_{DD}^{PD} = 5$	$\beta^{PD} = 1$

With the given choice of exogenous variables and utility functions, the equilibrium has the following characters: 1) the bank is solvent in all the states; 2) all the budget constraints bind; 3) all the collateral constraints for the house buyers bind; 4) the house buyers default on the mortgage loans at the bad states (D, UD, DD), and repays fully in the good states (U, UU, DU); 5) the banker $B0$ resells all the collateral as soon as possible. These properties are satisfied when all the households and the bankers optimize over their choice variables. Table 3.5 presents the endogenous variables given the choice of the parameters and utility functions. The house price is initially high, and remains in a high level at the good state U, but crashes at the bad state D, which leads to the default in the mortgage market. The overall trend of a decreasing house price is due to the choice of parameters, while the difference between state U and D is robust. As the model has only three periods, boundary conditions affect variables at different time significantly. However, variables in states sharing the same predecessor are comparable, namely the variables in state U and D. As the probability of state D is only 5%, the low endowment can be seen as a negative shock, although it is expected and affects the decision makings of the rational households and bankers.

Table 3.5: Endogenous Variables, Credit Cycles

	0	U	UU	UD	D	DU	DD
P	63.914	59.639	53.098	36.510	47.724	51.740	35.816
π	36.842	32.404			24.679		
ϕ	0.202	0.205			0.248		
A	51.508	42.806			40.810		
$\pi\phi$	7.460	6.651			6.122		
r^M	0.398	0.321			0.654		
LTV	0.576	0.543			0.517		
Dep	6.525	5.187			5.424		
r	0.361	0.284			0.584		
CV		7.262			1.948		
δ				1.292			1.239

CV represents the collateral value of the houses, whereas δ is the level of default at a bad state.

The numerical example exhibits the following properties, which are consistent with the model's prediction. First, the house price and the leverage move in the same direction. At state U, both the house price and the leverage are higher than those at state D. As explained previously, the low endowments of house buyers at state D decreases the demand for houses, thus the house price drops and triggers mortgage default. The bank generates less profits at state D, thus the capital available for future investment is also low, shrinking the bank's balance sheet size. The endogenous mortgage contract selection method together with the difference in the bank's capital determines the comovement of house price and leverage. Second, default at state D leads to less credit extension. $\pi\phi$, which equals lending per unit of contract π multiplied by the number of contracts issued ϕ, is low after default occurs. Moreover, low credit extension is accompanied with stricter terms of the mortgage contract, which offers lower leverage (LTV) but demands higher required rate of return (r^M). Third, collateral value is more volatile than the price of the house. We calculate the collateral value (CV) following the definition from Proposition 3. Obviously, the

percentage difference of the collateral value between state U and D is much larger than that of the house price itself. When house price is low, its capacity to back loans is naturally low. When leverage comoves with the house price, the capacity is even lower. The comovement of leverage and house price leads to the high volatility of the collateral value, which in turn contributes to the volatility of house price. Last but not the least, the default in state UD is stronger than in state DD. This observation is consistent with the Minsky effect, which states that after a period of prosperity, risk accumulates. If a "Minsky moment" occurs, economy would fall down much more severely than in the case without the previous prosperity. In this model, leverage is endogenously higher at state U due to the bank's balance sheet effect. Thus, when default occurs in the subsequent state UD, it is more severe than in state DD, which is after a relatively lower leveraged state D.

3.4 The Minsky effect

Minsky (1977) states that, "over a period in which the economy does well, optimism changes, and agents tend to invest more in the riskier asset. The tendency to transform doing well into a speculative investment boom is the basic instability in the economy". In this section we study how the change in beliefs affect equilibrium asset price, leverage, credit extension and default. The economy and optimization problems are the same as in Section 3.3. The only difference is that, the true state of nature, which is the probability of a good state to occur in the next period (θ), is unknown. The bank holds a prior belief on θ at $t = 0$, and gives this belief to the first generation banker. At $t = 1$, the bankers observe the realization of states, learn new information

about the true state of nature, and update their beliefs by the Bayes' rule. The bank passes the information through generations of bankers. Assume that the households do not learn from observing the states, since they live for only two periods, or simply because they lack such an ability.

3.4.1 Beliefs

We model the belief updating in the same way as Bhattacharya et al. (2015), which follows Cogley and Sargent (2008). Assume that the true state of the nature θ, can only take two values, θ_1 or θ_2, with $0 < \theta_2 < \theta_1 < 1$. At a given state, the subsequent state can be either good (U), with probability θ, or bad (D), with probability $1 - \theta$. Also assume that θ does not change throughout time. The households and the bankers have rational expectation in the sense that they are aware of what is going to happen in each of the state, i.e. they know the prices, the interest rate and the consumption level if a certain state occurs. The households do not learn about θ from the realization of states, and assume that their beliefs are constant and homogeneous. The bankers know the set $\{\theta_1, \theta_2\}$, and have the subjective prior belief ρ on θ at $t = 0$. This means that at $t = 0$ they believe that state U will occur at $t = 1$ with probability

$$\theta_0^B = \rho\theta_1 + (1 - \rho)\theta_2 \tag{3.4.1}$$

If state U occurs, the bankers update their beliefs following the Bayes' rule,

$$\theta_U^B = \theta_1 Prob(\theta = \theta_1|U) + \theta_2 Prob(\theta = \theta_2|U) = \frac{\theta_1^2\rho + \theta_2^2(1 - \rho)}{\theta_1\rho + \theta_2(1 - \rho)} \tag{3.4.2}$$

If state D occurs, the bankers update their beliefs following the Bayes' rule,

$$\theta_D^B = \theta_1 Prob(\theta = \theta_1|D) + \theta_2 Prob(\theta = \theta_2|D) = \frac{\theta_1(1-\theta_1)\rho + \theta_2(1-\theta_2)(1-\rho)}{(1-\theta_1)\rho + (1-\theta_2)(1-\rho)} \quad (3.4.3)$$

It is easy to show that $\theta_D^B < \theta_0^B < \theta_U^B$, which means the realization of a good state increases optimism, whereas the realization of a bad state decreases optimism. The calculations are in Appendix 3.E. Using the same parameters as in Section 3.3.3, and the subjective probabilities given in Table 3.6 below, we can see how belief updating affects the economy.

Table 3.6: Beliefs

$\theta_1 = 0.99$	$\theta_2 = 0.91$	$\rho = 0.5$	$\theta_0^B = 0.95$	$\theta_U^B = 0.952$	$\theta_D^B = 0.918$
$\theta^{R0} = 0.95$	$\theta^{P0} = 0.95$	$\theta^{RU} = 0.95$	$\theta^{PU} = 0.95$	$\theta^{RD} = 0.95$	$\theta^{PD} = 0.95$

Assume that the true state of nature θ can take two values θ_1 and θ_2. Chooses the parameters such that the bankers subjective belief θ_0^B is the same as in the previous section. Now the bankers are more optimistic at state U and more pessimistic at state D. The households do not update their beliefs, and the subjective beliefs take the same value as in the previous section.

3.4.2 Comparative Statics

The endogenous variables in equilibrium after considering the bankers' belief updating are given in Table 3.7. Given that the true state of nature θ can only take two values, $\theta_1 = 0.99$ and $\theta_2 = 0.91$, and that the banker's prior belief $\rho = 0.5$, we have $\theta_0^B = 0.95$, $\theta_U^B = 0.952$ and $\theta_D^B = 0.918$. The banker is slightly more optimistic at the good state, and less optimistic at the bad state. The main results of the previous example without belief updating remain in this simulation. However, bankers extend more mortgage loans at state U after becoming more optimistic, and extend

less mortgage loans at state D after becoming more pessimistic. Such changes have the following consequences. When the banker do not learn from observing the realization of states, the total mortgage loan at state U is 6.651, while at state D it is 6.122. The low endowment decreases the total mortgage loan by 7.96%, and the Loan-to-Ratio by 4.83%. If the bankers can learn about the true state of nature by observing the realization of states, the low endowments at state D decreases the total mortgage loan, the Loan-to-Value ratio by 11.96% and 7.42%, respectively. Belief updating amplifies the cyclicality in credit extension and leverage. This is because, when a bad state occurs, the bank's balance sheet shrinks, providing less credit to the economic activities. Meanwhile, the bankers become more pessimistic, and "rationally over-react" to the low endowment shock.

It is worth noticing that, the difference in default levels in the subsequent states is also amplified by the belief updating. More specifically, we are comparing default level after a good state (state UD) and after a bad state (state DD). Without belief updating, default at state UD is 1.292, whereas default at state DD is 1.239, with 4.14% difference. However, if bankers update their beliefs, default at state UD is 1.304, whereas default at state DD is 0.943, with 27.64% difference. This result captures the main idea about Minsky's Financial Instability Hypothesis. At the good state (U), the bankers become more optimistic, thus invest more in the risky mortgage loans and extend more credit. When default occurs at the subsequent state UD, it is more severe than in the case with less credit extension. The opposite is true as well. At the bad state (D), the bankers become more pessimistic, thus less credit is extended. But less risk is accumulated, and default level is lower. Our simulation shows that, the effects

of the low endowment is amplified by the bankers belief updating through the credit extension. The comparisons are given in Table 3.8.

Table 3.7: Endogenous variables, the Minsky effect

	0	U	UU	UD	D	DU	DD
P	63.896	59.635	53.100	36.534	47.845	51.910	35.460
π	36.892	32.453			24.104		
ϕ	0.203	0.206			0.244		
A	51.602	42.875			39.331		
$\pi\phi$	7.483	6.671			5.874		
r^M	0.399	0.321			0.632		
LTV	0.577	0.544			0.504		
Dep	6.548	5.207			5.172		
r	0.361	0.285			0.579		
CV		7.248			2.830		
δ				1.304			0.943

Table 3.8: Bank's learning effects on financial contracts

	No Learning			Bank Learns		
	U	D	(D-U)/U%	U	D	(D-U)/U%
$\pi\phi$	6.651	6.122	-7.96%	6.671	5.874	-11.96%
LTV	0.543	0.517	-4.83%	0.544	0.504	-7.42%
δ_{sD}	1.292	1.239	-4.14%	1.304	0.943	-27.64%

This table presents how the belief updating amplifies the cyclicality in credit extension, leverage and default. $\pi\phi$ is the loans per contract multiplied by number of mortgage contracts issued; LTV is the Loan-to-Value ratio, measuring the leverage; δ_{sD} is the level of default in a subsequent bad state of $s \in \{U, D\}$.

The bankers are less confident in the economy at state D, and thus invests less in the risky assets. The change in optimism amplifies the effect from the shrinking balance sheet, and exacerbates the financial stress for the new house buyers. Minsky's Financial Instability hypothesis focuses on the building up of risk together with optimism during a prolonged history of prosperity, while this model shows that, when negative shocks occur, the bankers over-react to the endowment shock, deteriorating the financial

condition further. Geanakoplos (2010) elaborates a situation where over-leveraged optimistic traders default and are removed from the economy in bad times, lowering the overall optimism, asset prices, and leverage level. This model demonstrates a similar mechanism, with shrinking bank balance sheet and lower optimism in bad times. The results above suggest that, when financial crisis happens, the regulator should consider the change in optimism in the market, in addition to the financial conditions presented in the financial reports.

It is natural to think about the case in which all the agents can learn about the true state of nature by observing state realizations. Table 3.9 and Table 3.10 present the results when the second generation households can learn in the same way as the bankers thus share the same subjective probability over the future states. Obviously, learning still amplifies the difference between state U and D compared with the no learning case. However, the overall effect is smaller compared with the case when only the bank learns and updates. Now the differences in credit extension, leverage and default between state U and D are 9.10%, 5.58% and 17.55%, respectively. When the house sellers become more optimistic at state U, their expected house price is higher in the future, making them more likely to keep the house rather than to sell it. The opposite is true at state D when they get more pessimistic. Such actions mitigate the effects from bank's belief updating, thus reducing the cyclicality of credit extension and leverage.

Table 3.9: Endogenous variables, the Minsky effect

	0	U	UU	UD	D	DU	DD
P	63.912	59.639	53.099	36.528	47.737	51.826	35.601
π	36.847	32.430			24.510		
ϕ	0.203	0.205			0.247		
A	51.519	42.861			39.942		
$\pi\phi$	7.462	6.662			6.056		
r^M	0.398	0.322			0.630		
LTV	0.577	0.544			0.513		
Dep	6.527	5.198			5.357		
r	0.361	0.285			0.565		
CV		7.240			2.565		
δ				1.301			1.073

Table 3.10: All agents' learning effects on financial contracts

	No Learning			All Learn		
	U	D	(D-U)/U%	U	D	(D-U)/U%
$\pi\phi$	6.651	6.122	-7.96%	6.662	6.056	-9.10%
LTV	0.543	0.517	-4.83%	0.544	0.513	-5.58%
δ_{sD}	1.292	1.239	-4.14%	1.301	1.073	-17.55%

3.5 Conclusion

This chapter introduces general equilibrium models with collateral and default to study the mortgage market, financial intermediation and belief updating. Collateral is required when borrowers cannot commit to repay. We consider the following mechanism to determine the equilibrium leverage and interest rate of a mortgage contract in a binomial structure. The market offers a menu of mortgage contracts, each requiring one unit of house as collateral, but different face values and consequently different interest rates. Borrowers choose from the menu by choosing the face value, knowing that different face value correspond to different required rate of return. Lenders take the menu and the choice as given. The model shows that, when collateral is scarce,

the price of the underlying asset is higher than its fundamental value, and the difference represents the collateral value, which is the asset's borrowing capacity to borrow against. Default affects the economy in a non-trivial way, even in the simple binomial structure.

Then we extend the model into a multi-period setting to study how the collateral constraints affect the credit cycle. When a negative shock happens to house buyer's endowments, the decreased house price triggers mortgage default, and less credit is available for new mortgage borrowers. Decreased demand and fire sale further decreases house price, and a downward spiral of credit and prices materializes. Leverage and house price move in the same direction through time, due to the change in the bank's balance sheet size. The co-movement of price and leverage leads to a more volatile collateral value of the house, since not only the underlying asset's value but also the fraction that one can borrow against decrease when price and leverage decrease together. Default after a good state is more severe than that after a bad one, which resembles the main idea from Minsky's Financial Instability Hypothesis. The bank's belief updating amplifies the credit cycle, increasing the volatility of credit extension and default. The recent sub-prime crisis is obviously more complex than our model's description. Securitization and regulation related issues play major roles in the crystallization of the crisis. Nevertheless, our model provides a useful framework for future research on such topics.

3.A First order conditions: Two-period mortgage model

$$P_0 U'(c_0^R) = U'(h_0^R) + \beta^R \theta^R \left(U'(h_U^R) + S'(h_U^R)\right) + \beta^R (1-\theta^R) \left(U'(h_D^R) + S'(h_D^R)\right) \tag{3.A.1}$$

$$P_U U'(c_U^R) = U'(h_U^R) + S'(h_U^R) \tag{3.A.2}$$

$$P_D U'(c_D^R) = U'(h_D^R) + S'(h_D^R) \tag{3.A.3}$$

$$\pi U(c_0^R) = \beta^R \theta^R A U'(c_U^R) + \beta^R (1-\theta^R)(U'(h_D^R) + S'(h_D^R)) \tag{3.A.4}$$

$$(P_0 - \pi) U'(c_0^P) + \beta^P \theta^P A U'(c_U^R) = U'(h_0^P) + \beta^P \theta^P \left(U'(q_0^P + q_U^P) + S'(q_0^P + q_U^P)\right) \tag{3.A.5}$$

$$P_U U'(c_U^P) = U'(h_U^P) + S'(h_U^P) \tag{3.A.6}$$

$$P_D U'(c_D^P) = U'(h_D^P) + S'(h_D^P) \tag{3.A.7}$$

$$\frac{\partial \pi}{\partial A} U'(c_0^P) = \beta^P \theta^P U'(c_U^P) \tag{3.A.8}$$

$$\lambda = P_0 U'(c_0^P) - U'(h_0^P) - \beta^P \theta^P (U'(h_U^P) + S'(h_U^P)) - \beta^P (1-\theta^P)(U'(h_D^P) + S'(h_D^P)) \tag{3.A.9}$$

3.B Benchmark economy

Consider a two-period economy similar with the one in Section 3.2. Now we do not allow lending, and the agents can only trade consumption goods and houses directly. The optimization problems of the house buyers and sellers are below,

$$\begin{aligned} \max_{(c_s^R, q_s^R)_{s\in\{0,U,D\}}} \bar{U}^R = {} & U(c_0^R, E - q_0^R) + \beta^R \theta^R \left(U(c_U^R, E - q_0^R - q_U^R) + S(E - q_0^R - q_U^R)\right) \\ & + \beta^R (1-\theta^R) \left(U(c_D^R, E - q_0^R - q_D^R) + S(E - q_0^R - q_D^R)\right) \end{aligned} \tag{3.B.1}$$

$$c_0^R \leqslant e_0^R + P_0 q_0^R \tag{3.B.2}$$

$$c_U^R \leqslant e_U^R + P_U q_U^R \tag{3.B.3}$$

$$c_D^R \leqslant e_D^R + P_D q_D^R \tag{3.B.4}$$

$$\max_{(c_s^P, q_s^P)_{s\in\{0,U,D\}}} \bar{U}^P = U(c_0^P, q_0^P) + \beta^P \theta^P \left(U(c_U^P, q_0^P + q_U^P) + S(q_0^P + q_U^P)\right)$$

$$+ \beta^P(1-\theta^P)\left(U(c_D^P, q_0^P + q_D^P) + S(q_0^P + q_D^P)\right) \tag{3.B.5}$$

$$c_0^P + P_0 q_0^P \leqslant e_0^P \tag{3.B.6}$$

$$c_U^P + P_U q_U^P \leqslant e_U^P \tag{3.B.7}$$

$$c_D^P + P_D q_D^P \leqslant e_D^P \tag{3.B.8}$$

In equilibrium, the market for the houses should clear,

$$q_0^R = q_0^P \tag{3.B.9}$$

$$q_U^R = q_U^P \tag{3.B.10}$$

$$q_D^R = q_D^P \tag{3.B.11}$$

And the market for the consumption goods clears automatically due to the Walrus' Law.

Definition 5. *The **Benchmark Equilibrium** for the economy*

$$\mathcal{E}_2 = \left\{(\bar{U}^a, e^a, E)_{a\in\{R,P\}}\right\}$$

is a vector consisting of house prices, consumptions, and house trading volume,

$$\left(P, (c^h, q^h)_{h\in\{R,P\}}\right)$$

if and only if

1. $(c^P, q^P) \in \mathcal{B}_2^P(P)$, *and* $(\tilde{c}, \tilde{q}) \in \mathcal{B}_2^h(P) \Rightarrow \bar{U}^P(\tilde{c}, \tilde{q}) \leqslant \bar{U}^h(c^P, q^P)$.
2. $(c^R, q^R) \in \mathcal{B}_2^R(P)$, *and* $(\tilde{c}, \tilde{q}) \in \mathcal{B}_2^R(P) \Rightarrow \bar{U}^R(\tilde{c}, \tilde{q}) \leqslant \bar{U}^R(c^R, q^R)$.
3. *Market clearing conditions (3.B.9)∼(3.B.11) hold.*

where the budget sets $\mathcal{B}_2^P(P)$ *and* $\mathcal{B}_2^R(P)$ *are defined by equations (3.B.2)∼(3.B.4) and (3.B.6)∼(3.B.8), respectively.*

3.C Araujo's collateral equilibrium

Araujo et al. (2012) assume that there are as many collateralized contracts as the number of states at $t = 1$. In equilibrium, for each state, there is one contract for which the borrower is indifferent from defaulting or not. In our framework, there should be two mortgage contracts, with face value P_U and P_D, respectively. The optimization problem is as follows.

$$\max_{(c_s^R, q_s^R, \phi_U^R, \phi_D^R)_{s\in\{0,U,D\}}} \bar{U}^R = U(c_0^R, h_0^R) + \beta^R \theta^R \left(U(c_U^R, h_U^R) + S(h_U^R)\right) + \beta^R (1-\theta^R)\left(U(c_D^R, h_D^R) + S(h_D^R)\right) \tag{3.C.1}$$

$$h_0^R = E - q_0^R \tag{3.C.2}$$

$$h_U^R = E - q_0^R - q_U^R \tag{3.C.3}$$

$$h_D^R = E - q_0^R - q_D^R + \phi_U^R \tag{3.C.4}$$

$$c_0^R + \pi_U \phi_U^R + \pi_D \phi_D^R \leqslant e_0^R + P_0 q_0^R \tag{3.C.5}$$

$$c_U^R \leqslant e_U^R + P_U q_U^R + P_U \phi_U^R + P_D \phi_D^R \tag{3.C.6}$$

$$c_D^R \leqslant e_D^R + P_D q_D^R + P_D \phi_D \tag{3.C.7}$$

$$\max_{(c_s^P, q_s^P, \phi_U^P, \phi_D^P)_{s\in\{0,U,D\}}} \bar{U}^P = U(c_0^P, h_0^P) + \beta^P \theta^P \left(U(c_U^P, h_U^P) + S(h_U^P)\right) + \beta^P (1-\theta^P)\left(U(c_D^P, h_D^P) + S(h_D^P)\right) \tag{3.C.8}$$

$$h_0^P = q_0^P \tag{3.C.9}$$

$$h_U^P = q_0^P + q_U^P \tag{3.C.10}$$

$$h_D^P = q_0^P + q_D^P - \phi_U^P - \phi_D^P \tag{3.C.11}$$

$$c_0^P + P_0 q_0^P \leqslant e_0^P + P_U \phi_U^P + P_D \phi_D^P \tag{3.C.12}$$

$$\phi_U^P + \phi_D^P \leqslant q_0^P \tag{3.C.13}$$

$$c_U^P + P_U q_U^P + P_U \phi_U^P + P_D \phi_D^P \leqslant e_U^P \tag{3.C.14}$$

$$c_D^P + P_D q_D^P + P_D \phi_D^P \leqslant e_D^P \tag{3.C.15}$$

In equilibrium, the market for the houses and contracts should clear,

$$q_0^R = q_0^P \tag{3.C.16}$$

$$q_U^R = q_U^P \tag{3.C.17}$$

$$q_D^R = q_D^P \tag{3.C.18}$$

$$\phi_U^R = \phi_U^P \tag{3.C.19}$$

$$\phi_D^R = \phi_D^P \tag{3.C.20}$$

And the market for the consumption goods clears automatically due to the Walrus' Law.

Definition 6. *The **Benchmark Equilibrium** for the economy*

$$\mathcal{E}_3 = \left\{(\bar{U}^h, e^h, E)_{h\in\{R,P\}}\right\}$$

is a vector consisting of house prices, consumptions, house and mortgage contract trading volume,

$$\left(P, (c^a, q^a)_{a\in\{R,P\}}, \phi_U, \phi_D\right)$$

if and only if

1. $(c^P, q^P) \in \mathcal{B}_3^P(P)$, *and* $(\tilde{c}, \tilde{q}) \in \mathcal{B}_3^h(P) \Rightarrow \bar{U}^P(\tilde{c}, \tilde{q}) \leqslant \bar{U}^h(c^P, q^P)$.
2. $(c^R, q^R) \in \mathcal{B}_3^R(P)$, *and* $(\tilde{c}, \tilde{q}) \in \mathcal{B}_3^R(P) \Rightarrow \bar{U}^R(\tilde{c}, \tilde{q}) \leqslant \bar{U}^R(c^R, q^R)$.
3. *Market clearing conditions (3.C.16)∼(3.C.20) hold.*

where the budget sets $\mathcal{B}_3^P(P)$ *and* $\mathcal{B}_3^R(P)$ *are defined by equations (3.C.5)∼(3.C.7) and (3.C.12)∼(3.C.15), respectively.*

Table 3.11: Comparison, two-period mortgage model

Exogenous Variables					
$E = 1$	$e_0^R = 5$ $e_0^P = 5$	$e_U^R = 10$ $e_U^P = 20$	$e_D^R = 10$ $e_D^P = 15$	$\beta^R = 1$ $\beta^P = 1$	$\theta = 0.9$ $\tau = 0$
Endogenous Variables, Collateral Equilibrium					
$P_0 = 21.534$	$P_U = 30$	$P_D = 25$	$q_0 = 0.222$	$q_U = 0.133$	$q_D = 0.300$
$c_0^R = 7.629$	$c_U^R = 19.354$	$c_D^R = 17.500$	$c_0^P = 2.371$	$c_U^P = 10.646$	$c_D^P = 7.500$
$\pi = 9.664$	$A = 24.168$	$FOD = 0.200$	$SOD = -0.007$	$r = 1.501$	$LTV = 0.449$
$\lambda = 1.700$	$CV = 4.031$	$FV = 17.503$	$U^R = 4.304$	$U^P = 0.633$	
Endogenous Variables, Araujo et al. (2012) Equilibrium					
$P_0 = 20.015$	$P_U = 30$	$P_D = 25$	$q_0 = 0.289$	$q_U = 0$	$q_D = 0.826$
$c_0^R = 7.105$	$c_U^R = 21.316$	$c_D^R = 17.500$	$c_0^P = 2.895$	$c_U^P = 8.684$	$c_D^P = 7.500$
$\phi_U = 0.816$	$\phi_D = -0.526$	$\pi_U = 10.015$	$\pi_D = 8.515$	$r_U^M = 1.995$	$r_D^M = 1.936$
$\lambda = 0.017$	$CV = 0.050$	$FV = 19.965$	$LTV_U = 0.500$	$LTV_D = 0.425$	
$U^R = 4.316$	$U^P = 0.734$				

The table presents a comparison between the collateral equilibrium with incomplete market in our model, and the one with complete market in Araujo et al. (2012).

3.D Definition of equilibrium, credit cycle model in Section 3.3

In equilibrium, the markets for the houses, mortgage loans, and deposit should all clear. More specifically, the supply and demand of houses should equal in equilibrium at all states,

$$q_0^{R0} = q_0^{P0} \tag{3.D.1}$$

$$q_U^{R0} + q_U^{RU} = q_U^{P0} + q_U^{PU} \tag{3.D.2}$$

$$q_{UU}^{RU} = q_{UU}^{PU} \tag{3.D.3}$$

$$q_{UD}^{RU} + \phi_U^B = q_{UD}^{PU} \tag{3.D.4}$$

$$q_D^{R0} + q_D^{RD} + q_D^B = q_D^{P0} + q_D^{PD} \tag{3.D.5}$$

$$q_{DU}^{RD} + \phi_0^B - q_D^B = q_{DU}^{PD} \tag{3.D.6}$$

$$q_{DD}^{RD} + \phi_0^B - q_D^B + \phi_D^B = q_{UD}^{PD} \tag{3.D.7}$$

The mortgage loan supply from the banker equals the demand from the house buyers,

$$\phi_0^B = \phi_0^{P0} \equiv \phi_0 \tag{3.D.8}$$

$$\phi_U^B = \phi_U^{PU} \equiv \phi_U \tag{3.D.9}$$

$$\phi_D^B = \phi_D^{PD} \equiv \phi_D \tag{3.D.10}$$

The face value of the mortgage contract chosen by the banker and the house buyers should be the same,

$$A_0^B = A_0^{P0} \equiv A_0 \tag{3.D.11}$$

$$A_U^B = A_U^{PU} \equiv A_U \tag{3.D.12}$$

$$A_D^B = A_D^{PD} \equiv A_D \tag{3.D.13}$$

The demand for deposit of the banker should equal the supply of deposit of the house sellers,

$$Dep_0^B = Dep_0^{R0} \equiv Dep_0 \tag{3.D.14}$$

$$Dep_U^B = Dep_U^{RU} \equiv Dep_U \tag{3.D.15}$$

$$Dep_D^B = Dep_D^{RD} \equiv Dep_D \tag{3.D.16}$$

The contracts are determined in the same way as in previous sections. Three contracts are determined independently in at state 0, U, and D. Consequently, we need to check three second order conditions separately.

$$\begin{aligned}
&\theta\Bigg(\big(1-(1+r_0)(1+\bar{r}_0)(1+\bar{r}_U)\frac{\partial\pi_0}{\partial A_0}\big)U'(Y_U)\\
&+\big(A-(1+r_0)(1+\bar{r}_0)(1+\bar{r}_U)\pi\big)\big(\phi_0-(1+r_0)(1+\bar{r}_0)(1+\bar{r}_U)\phi_0\big)\frac{\partial\pi_0}{\partial A_0}U''(Y_U)\Bigg)\\
&+(1-\theta)\big(1-(1+r_0)(1+\bar{r}_0)(1+\bar{r}_D)\frac{\partial\pi_0}{\partial A_0}\big)\Bigg(U'(Y_D)
\end{aligned}$$

$$+\phi_0(A-(1+r_0)(1+\bar{r}_0)(1+\bar{r}_D)\frac{\partial\pi_0}{\partial A_0})U''(Y_D)\Big)=0 \tag{3.D.17}$$

$$\theta\Big(\big(1-(1+r_U)(1+\bar{r}_{UU})\frac{\partial\pi_U}{\partial A_U}\big)U'(Y_{UU})$$
$$+\big(A_U-(1+r_U)(1+\bar{r}_{UU})\pi_U\big)\big(\phi_U-(1+r_U)(1+\bar{r}_{UU})\phi_U\big)\frac{\partial\pi_U}{\partial A_U}U''(Y_{UU})\Big)$$
$$+(1-\theta)\big(1-(1+r_U)(1+\bar{r}_{UD})\frac{\partial\pi_U}{\partial A_U}\big)\Big(U'(Y_{UD})$$
$$+\phi_U(A_U-(1+r_U)(1+\bar{r}_{UD})\frac{\partial\pi_U}{\partial A_U})U''(Y_{UD})\Big)=0 \tag{3.D.18}$$

$$\theta\Big(\big(1-(1+r_D)(1+\bar{r}_{DU})\frac{\partial\pi_D}{\partial A_D}\big)U'(Y_{DU})$$
$$+\big(A_D-(1+r_D)(1+\bar{r}_{DU})\pi_D\big)\big(\phi_D-(1+r_D)(1+\bar{r}_{DU})\phi_D\big)\frac{\partial\pi_D}{\partial A_D}U''(Y_{DU})\Big)$$
$$+(1-\theta)\big(1-(1+r_D)(1+\bar{r}_{DD})\frac{\partial\pi_D}{\partial A_D}\big)\Big(U'(Y_{DD})$$
$$+\phi_D(A_D-(1+r_D)(1+\bar{r}_{DD})\frac{\partial\pi_D}{\partial A_D})U''(Y_{DD})\Big)=0 \tag{3.D.19}$$

Definition 7. *The **Collateral Equilibrium** for the economy*

$$\mathcal{E}_4=\big\{(\bar{U}^a,e^a)_{a\in\{R,P,B\}},\bar{r}\big\}$$

is a vector consisting of house prices, mortgage contract prices, deposit interest rates, consumptions, house trading, mortgage loan face value and traded volumes, and deposit volumes,

$$\big((P,\pi,r),(c^a,q^a)_{a\in\{R,P\}},q_D^B,A,\phi,Dep,M\big)$$

if and only if

1. $(c^{Rs},q^{Rs},Dep^{Rs})\in\mathcal{B}_3^{Rs}(P,r),s\in\{0,U,D\}$

   $(\tilde{c},\tilde{q},\tilde{Dep})\in\mathcal{B}_3^{Rs}(P,r)\Rightarrow\bar{U}^{Rs}(\tilde{c},\tilde{q})\leqslant\bar{U}^{Rs}(c^{Rs},q^{Rs})$

2. $(c^{Ps},q^{Ps},\phi^{Ps},A^{Ps})\in\mathcal{B}_3^{Ps}(P,\pi),s\in\{0,U,D\}$

   $(\tilde{c},\tilde{q},\tilde{\phi},\tilde{A})\in\mathcal{B}_3^{Ps}(P,\pi)\Rightarrow\bar{U}^{Ps}(\tilde{c},\tilde{q})\leqslant\bar{U}^{Ps}(c^{Ps},q^{Ps})$

3. *Bankers optimize accordingly, subject to the bank's budget constraints.*

4. *Market clearing conditions (3.D.1)∼(3.D.16) hold.*

5. *Equations (3.D.17) (3.D.18)(3.D.19) hold.*

6. *In equilibrium* $\min\left[\frac{e_{sD}^{Ps}}{\phi_s}, P_{sD}\right] < A_s < \min\left[\frac{e_{sU}^{Ps}}{\phi_s}, P_{sU}\right], s \in \{0, U, D\}$.

where

$\{R0, RU, RD, P0, PU, PD, B\}$ *is the set of all the agents in the economy,*

$e^{Rs} \equiv \{E_s, e_s^{Rs}, e_{sU}^{Rs}, e_{sD}^{Rs}\}, s \in \{0, U, D\}$ *is the vector of the endowment of the house sellers,*

$e^{Ps} \equiv \{e_s^{Ps}, e_{sU}^{Ps}, e_{sD}^{Ps}\}, s \in \{0, U, D\}$ *is the vector of the endowment of the house buyers,*

e^B *is the bankers endowment,*

$\bar{r} \equiv \{\bar{r}_0, \bar{r}_U, \bar{r}_{UU}, \bar{r}_{UD}, \bar{r}_D, \bar{r}_{DU}, \bar{r}_{DD}\}$ *is the vector of central bank lending interest rates,*

$P \equiv \{P_0, P_U, P_{UU}, P_{UD}, P_D, P_{DU}, P_{DD}\}$ *is the vector of house prices,*

$\pi \equiv \{\pi_0, \pi_U, \pi_D\}$ *is the vector of mortgage contract prices,*

$A \equiv \{A_0, A_U, A_D\}$ *is the vector of mortgage contract face values,*

$Dep \equiv \{Dep_0, Dep_U, Dep_D\}$ *is the vector of deposits,*

$r \equiv \{r_0, r_U, r_D\}$ *is the vector of deposit interest rates,*

$M \equiv \{M_0, M_U, M_{UU}, M_{UD}, M_D, M_{DU}, M_{DD}\}$ *is the vector of the banker's lending from the central bank.*

3.E Bayesian updating beliefs

$$
\begin{aligned}
\theta_U^B &= \theta_1 Prob(\theta = \theta_1|U) + \theta_2 Prob(\theta = \theta_2|U) \\
&= \theta_1 \frac{Prob(U|\theta = \theta_1)Prob(\theta = \theta_1)}{Prob(U|\theta = \theta_1)Prob(\theta = \theta_1) + Prob(U|\theta = \theta_2)Prob(\theta = \theta_2)} \\
&+ \theta_2 \frac{Prob(U|\theta = \theta_2)Prob(\theta = \theta_2)}{Prob(U|\theta = \theta_1)Prob(\theta = \theta_1) + Prob(U|\theta = \theta_2)Prob(\theta = \theta_2)} \\
&= \theta_1 \frac{\theta_1 \rho}{\theta_1 \rho + \theta_2(1-\rho)} + \theta_2 \frac{\theta_2(1-\rho)}{\theta_1 \rho + \theta_2(1-\rho)} \\
&= \frac{\theta_1^2 \rho + \theta_2^2(1-\rho)}{\theta_1 \rho + \theta_2(1-\rho)}
\end{aligned}
$$

$$
\begin{aligned}
\theta_D^B &= \theta_1 Prob(\theta=\theta_1|D) + \theta_2 Prob(\theta=\theta_2|D) \\
&= \theta_1 \frac{Prob(D|\theta=\theta_1)Prob(\theta=\theta_1)}{Prob(D|\theta=\theta_1)Prob(\theta=\theta_1)+Prob(D|\theta=\theta_2)Prob(\theta=\theta_2)} \\
&+ \theta_2 \frac{Prob(D|\theta=\theta_2)Prob(\theta=\theta_2)}{Prob(D|\theta=\theta_1)Prob(\theta=\theta_1)+Prob(D|\theta=\theta_2)Prob(\theta=\theta_2)} \\
&= \theta_1 \frac{(1-\theta_1)\rho}{(1-\theta_1)\rho+(1-\theta_2)(1-\rho)} + \theta_2 \frac{(1-\theta_2)(1-\rho)}{(1-\theta_1)\rho+(1-\theta_2)(1-\rho)} \\
&= \frac{\theta_1(1-\theta_1)\rho+\theta_2(1-\theta_2)(1-\rho)}{(1-\theta_1)\rho+(1-\theta_2)(1-\rho)}
\end{aligned}
$$

$$
\begin{aligned}
&\theta_U^B - \theta_0^B \\
=&\frac{\theta_1^2\rho+\theta_2^2(1-\rho)}{\theta_1\rho+\theta_2(1-\rho)} - \rho\theta_1 - (1-\rho)\theta_2 \\
=&\frac{\theta_1^2\rho+\theta_2^2(1-\rho)-\rho^2\theta_1^2-(1-\rho)^2\theta_2^2-2\rho(1-\rho)\theta_1\theta_2}{\theta_1\rho+\theta_2(1-\rho)} \\
=&\frac{\rho(1-\rho)(\theta_1-\theta_2)^2}{\theta_1\rho+\theta_2(1-\rho)} > 0
\end{aligned}
$$

$$
\begin{aligned}
&\theta_0^B - \theta_D^B \\
=&\rho\theta_1 + (1-\rho)\theta_2 - \frac{\theta_1(1-\theta_1)\rho+\theta_2(1-\theta_2)(1-\rho)}{(1-\theta_1)\rho+(1-\theta_2)(1-\rho)} \\
=&\frac{\rho^2\theta_1(1-\theta_1)+\rho(1-\rho)\theta_1(1-\theta_2)+\rho(1-\rho)\theta_2(1-\theta_1)+(1-\rho)^2\theta_2(1-\theta_2)-\theta_1(1-\theta_1)\rho-\theta_2(1-\theta_2)(1-\rho)}{(1-\theta_1)\rho+(1-\theta_2)(1-\rho)} \\
=&\frac{\rho(1-\rho)(\theta_1-\theta_2)^2}{(1-\theta_1)\rho+(1-\theta_2)(1-\rho)} > 0
\end{aligned}
$$

Chapter 4

An equilibrium framework for systemic risk: How economic theory meets statistics

This chapter[1] presents a calibration method for equilibrium models of banking systems, identifies channels that co-determine the distress dependence structures in the banking system, and the results suggest that the "Minsky effect", in addition to the widely accepted feedback channels such as the interbank lending and common exposure, serves as a systemic risk amplification mechanism. We construct an equilibrium model for financial system and assess deep parameters on risk-taking, regulation and the Minsky effect, using the Consistent Information Multivariate Density Optimizing Methodology with UK banking data from 2005 to 2007. We argue that before the financial crisis, large UK commercial banks exhibited increasing optimism, which led

[1]This chapter is a joint work with Dr. Raphael Espinoza and Dr. Miguel Segoviano

to over-investment and risk build-up.

4.1 Introduction

This paper provides a novel approach combining theoretical and statistical methods to measure the systemic risk, and identify different channels that contribute to the distress dependence structure in the banking system. We modify and extend the model by Goodhart, Sunirand, and Tsomocos (2005) into a multi-period setting, incorporate a systemic risk amplification mechanism (the Minsky effect), and offer a calibration method using Segoviano (2006)'s CIMDO approach to measure the deep parameters related with the banks' risk-taking, altitude towards default, and the Minsky effect. Our results show that large commercial banks in UK became more optimistic before the financial crisis. Such a change in optimism lead to over-investment, and accounts for a large proportion of banks' expected shortfalls in profits. Thus the co-movement of optimism among the banks contributes to the distress dependence structure of the banking system, thus the systemic risk. The model's prediction on the banks' leverage ratios gets closer to data after considering the Minsky effect.

Years before the financial crisis, there are signals in the market indicating that the banks may have been experiencing a period of increasing optimism and risk-taking. The Business Confidence Index (BCI) and the Consumer Confidence Index (CCI) remained high until Q3 2007[2]. The Fed started raising interest rates to cool off the overheated market since late 2004. It raised the Fed Fund rate five times, reaching

[2]The data for CCI and BCI is available on https://data.oecd.org/

2.25% by Dec. 2004, and raised it eight times in 2005 to 4.25 % by Dec. 2005. In 2006, the Fed raised the rate four times, hitting 5.25 % by June 2006. Rajan (2006) warned that the developments in the financial sector had led to an expansion in the ability to spread risks, and that the economies might be more exposed to financial-sector-induced turmoil than in the past.

We modify and extend the general equilibrium framework from Goodhart, Sunirand, and Tsomocos (2005) to model the connectedness of a banking system. The model considers a binomial economy of heterogeneous commercial banks with different risk-aversion and altitude towards default. The banks optimize by choosing their portfolio of risk-free assets and risky consumer loans. They may also borrow from each other in the interbank market. In the bad state the households can partially default on the consumer loans, and the repayment rates depend on the aggregate level of credit extension from the banking system. After the consumer loans repayments realize, the banks choose the repayment level of their deposit and interbank obligations, subject to non-pecuniary default penalties. By varying the penalties imposed on default, the model can generate 100% default, no default or an equilibrium level of default between 0 and 100 %, following Shubik and Wilson (1977). We further interpret a bank's default rate as "a probability that such bank chooses to shut down, and hence in the short run to default completely on its financial obligations" (Goodhart et al. (2005))[3]. The default decisions here are endogenous, whereas in Merton (1974), the value of a company's asset follows an exogenous stochastic process, and the default

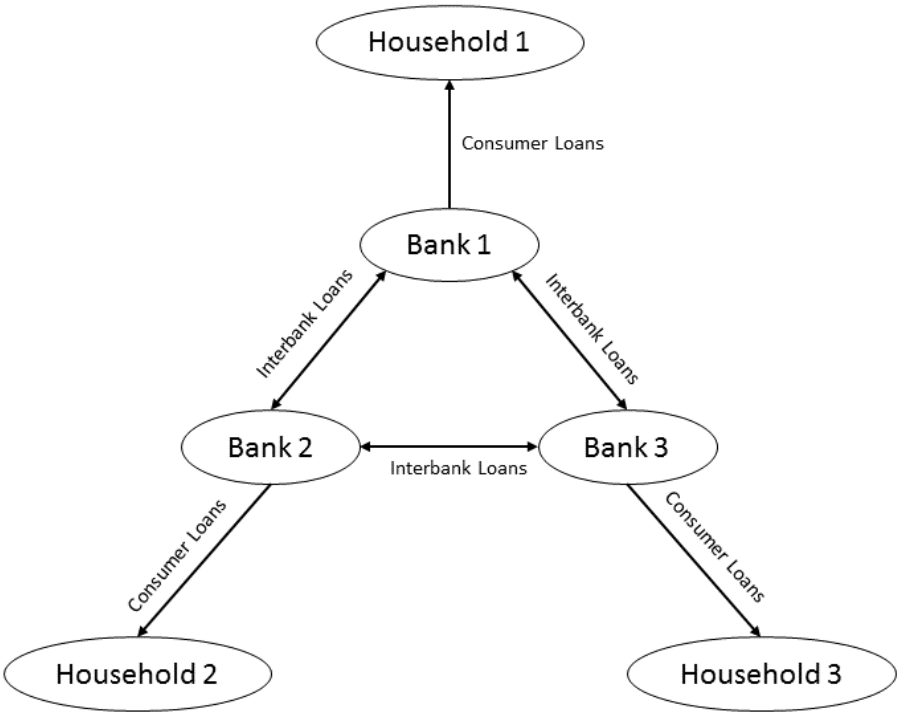
[3]Such an interpretation is a direct application of the law of large numbers, yet it may ignore some non-linear properties. However, our main interest is the systemic risk, rather than the individual banks themselves, thus this simplification is still a valid treatment.

conditions are exogenous.

The model considers different channels through which a shock can transmit in the banking system, such as the interactions in the interbank market and the feedback effects from assets with correlated return. Thus, the actual risk that an individual bank bears depends on not only its own investment decisions but other banks and investors as well. Figure 4.1 describes a simplified version of the economy. The commercial banks extend consumer loans to the households, and can borrow or lend in the interbank market, while a Central Bank supplies base money and determines the interbank rate (not in the figure). Figure 4.2 captures the contagion through the interbank lending channel. When a negative shock occurs to a borrowing bank's asset, its profit decreases, and it may default on its obligations to the households and interbank deposits. The default decreases the profitability of the lending banks, reduces its credit supply to households and the interbank market in the future, and may cause the lending banks to default as well. The model also considers the feedbacks from the correlated loan markets. As mentioned previously, the consumer loan repayment rate depends on the aggregate level of credit extension. When a bank suffers from a loss, it reduces its position in the future consumer loans. This decreases the total credit supply from the banking system, holding all other conditions constant. The model assumes that each household's repayment rate is a positive function of the aggregate credit supply. This resembles a "credit crunch", where a fall in the overall credit supply increases interest rates, which ultimately increases the liquidity constrained households' debt obligations, thus default rates. The shock to one bank in the banking system reduces the expected return of the risky investment opportunities for all the banks, and dis-

incentivize all the banks of their credit extension. Figure 4.3 illustrates this common asset exposure channel of the distress dependence.

Figure 4.1: Simplified structure of banking system



We further incorporate the Minsky effect into the model, which serves as an additional systemic risk amplification mechanism. Minsky (1977, 1992) propose the Financial Instability Hypothesis, which illustrates how the change in optimism affect risk-building and financial stability. In this paper, we refer to the effects of the changes in the commercial banks' optimism as the systemic risk amplification mechanism, or simply the Minsky effect. When a bank becomes more optimistic, it tends to invest more in risky assets. As mentioned above, such an adjustment affects other banks' profitability and riskiness through the interbank lending and the common asset exposure channel. If all the banks adjust their beliefs/optimism in the same direction,

Figure 4.2: Interbank channel of distress dependence

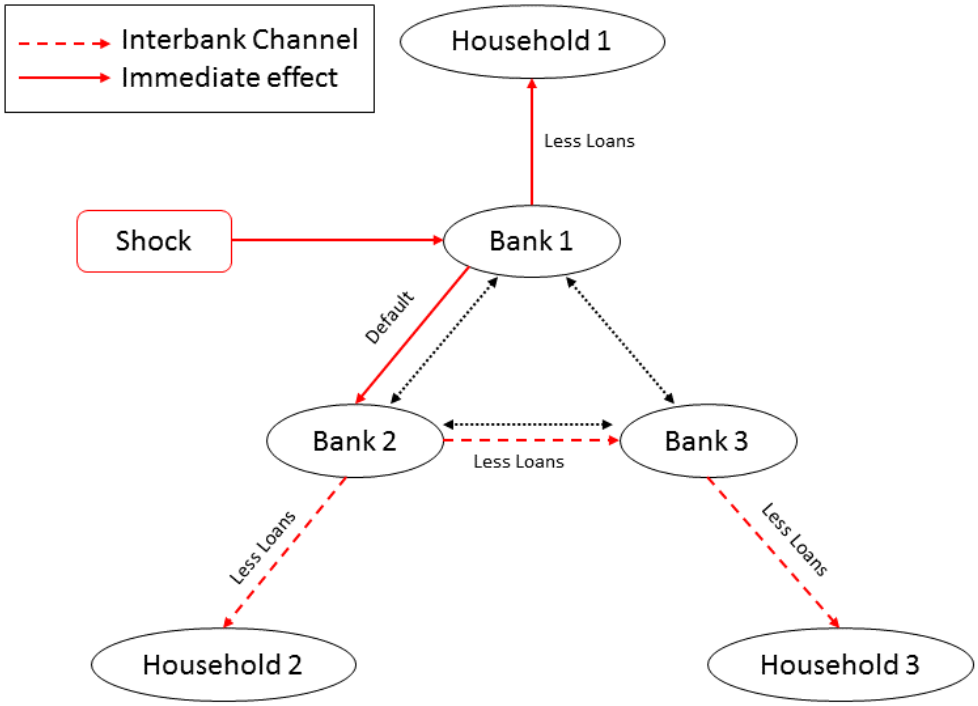
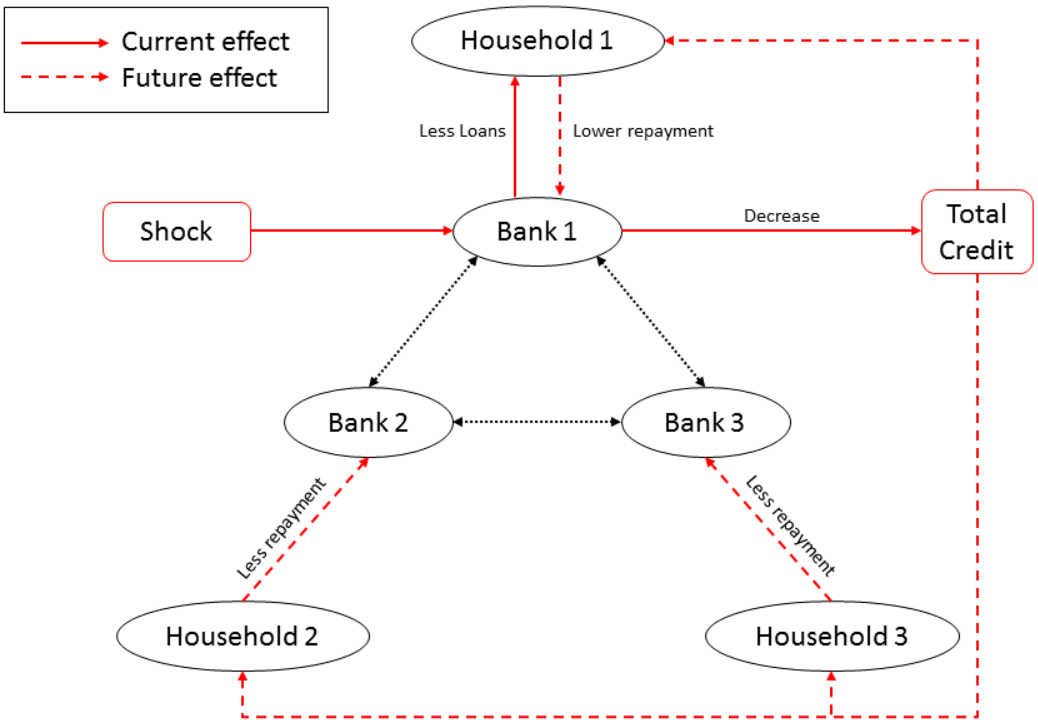
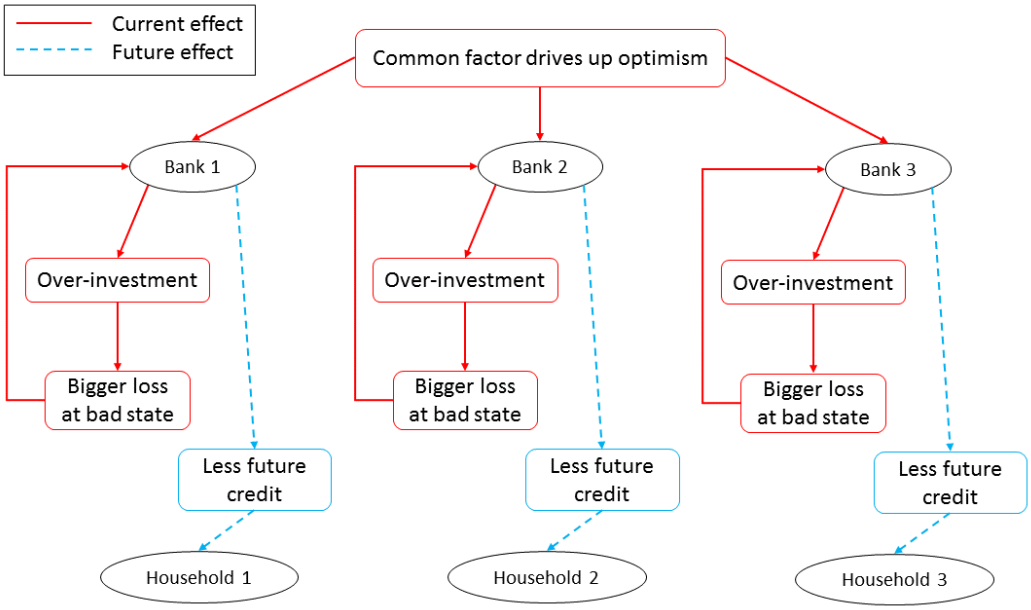


Figure 4.3: Common asset exposure channel of distress dependence



the effects are amplified. If, on the other hand, some banks become more optimistic, whereas others become more pessimistic, the aggregated effect will be obscure. The change in the optimism in the banking sector through time contributes to the systemic risk and the financial instability, and the co-movement of optimism in different banks amplifies the systemic risk. Figure 4.4 describes how the changes in beliefs affect the banking system. At a certain time, if some factors drive up a bank's optimism, the bank would believe that the economic condition in the future is better, thus engage in more risky investments, resulting in higher leverage, and a more risky asset allocation. When a bad outcome occurs, the bank suffers from a larger loss, leaving the future net worth lower. This decreases the profitability of the bank over time. If all the banks' optimism are driven by some common factors in a similar way, the effect would be amplified.

Figure 4.4: Systemic Risk Amplification / Minsky effect



In order to calibrate the model with real data, we interpret the binomial time structure in the following way. Let the good state in the theoretical model be the scenario where all the banks are healthy, whereas the bad state be the case where at least one bank is in distress. For instance, in an economy with five commercial banks which can be either healthy or in distress, there should be $2^5 = 32$ scenarios in total. The good state in the model represents the case where all the banks are healthy, while the bad state represents a combination of the remaining 31 scenarios where at least one of the five banks are in distress. Due to the interactions among the banking system, the distress in one bank decreases its profitability, and increases the default probabilities of all the other banks in the system. In particular, we look at a bank's expected shortfall in profits at the bad state. In the model, it is defined as the difference between the profit/loss across the states, and is affected by the bank's financial conditions. Thus, we believe that it captures the information of the distress dependence structure in the banking system. Furthermore, we measure the expected shortfalls using a statistical method (CIMDO), and connect the statistical method with the theoretical model. This approach shed light on the identification and evaluation of different contagion channels in a financial system.

The Consistent Information Multivariate Density Optimizing (CIMDO) by Segoviano (2006), is a non-parametric method to construct a consistent multivariate density from a very limited amount of information. Based on the Kullback (1959) cross-entropy approach, CIMDO infers the unknown multivariate distribution of asset values in a system of firms containing different risk quality. The recovered distribution is closest to the prior distribution but is also consistent with the empirically observed proba-

bilities of default. Segoviano and Espinoza (2016) apply CIMDO to estimate several indicators of systemic risk in a network of institutions representing the global banking and shadow banking system. In this paper, we use their estimates for expected equity growth rate to measure the conditional profit/loss in different states, and the difference across states to measure the expected shortfalls in profits. Both CIMDO and the equilibrium model are able to estimate the expected shortfalls in profits, thus providing us with a valid metric to compare the two approaches, and enabling us to identify the implied channels that contributes to the distress dependence structure of the banking system.

We take the following steps to identify and calibrate the corresponding coefficients for risk-aversion, altitude towards default, and the Minsky effect. First, we take the balance sheet data of the largest six UK commercial banks in 2005 by asset value, and predict the expected shortfalls in profits in 2006 using CIMDO approach. These expected shortfalls are used as input variables for the theoretical model to calibrate coefficients such as the marginal disutility of default and risk aversion coefficients. Second, we move on to the year of 2006, and simulate the conditional profit/loss from the model. This time the results will be different from CIMDO's prediction. The reason is that, the model only considers the interbank lending channel and the common exposure channel for the inter-connectedness of the banking system, and ignores other potential systemic risk amplification mechanisms. While the CIMDO method, as a non-parametric approach, is able to capture all the relevant information from the market data. We use CIMDO's prediction as input variables in the second-stage calibration, and solve for the change in the banks' beliefs. In this way, we attribute

all other systemic risk amplification mechanisms to the Minsky effect. The simulation results show that, this approach improves the models' performance on predicting the banks' leverage levels.

Figure 4.5, 4.6 and 4.7 present the input and output variables of our simulations. The equilibrium model takes the balance sheet items, interest rates, risk aversion, subjective probabilities, marginal disutility of default and household sector coefficients as input, and generates the implied optimal asset allocation, default rates of banks and households, conditional GDP growth, banks' conditional profits/losses and expected shortfalls in profits. CIMDO method takes the market data of stock prices and probabilities of distress as inputs, and generates banks' expected shortfalls in profits. As Figure 4.7 illustrates, we take CIMDO's prediction of banks' expected shortfalls in profits as input variables for the equilibrium model, and solve for the implied change in subjective optimism. The calculation results are consistent with our conjecture that all the banks become more optimistic before the financial crisis, and such a co-movement of optimism incentivizes risk-taking, and destabilized the financial system. More detailed discussion is presented in Section 4.4.

The calibration results show that increasing optimism are observed in the banking system before the financial crisis, which is consistent with the stylized facts mentioned previously. After considering the Minsky effect, the model predicts that all the banks assign higher subjective probabilities to the good state in 2006 than in 2005. The banks invest more in the risky asset, i.e. they extend more loans to the households. Although the increasing total credit extension slightly raises the household loan repayment rate,

Figure 4.5: Input and output variables in the equilibrium model

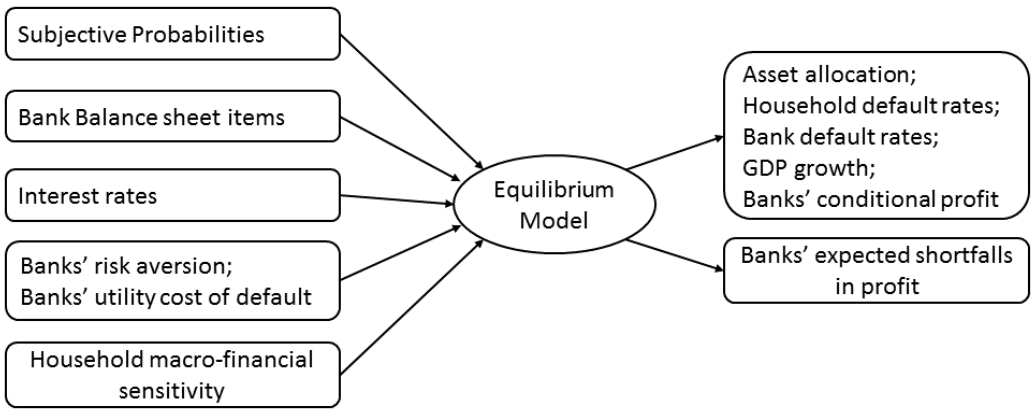
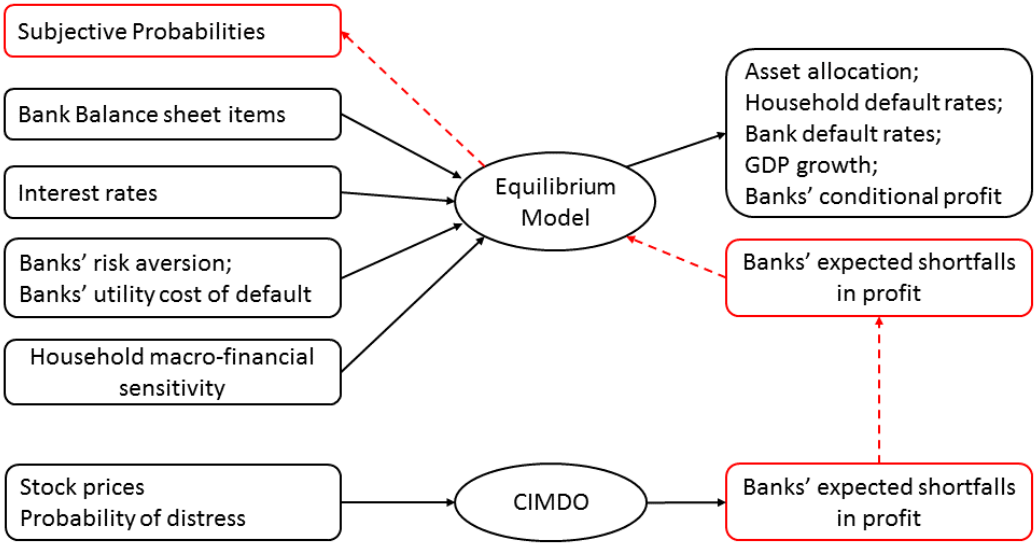


Figure 4.6: Input and output variables in CIMDO



Figure 4.7: Calibrating the Minsky effect



the higher leverage leads to a larger loss for each bank in the bad state, thus worsens the condition should a crisis occur. Our simulation also measures the hypothetical proportional contribution of the changes in beliefs to systemic risk, compared with the interbank lending channel and the common asset exposure. Moreover, the our approach measures the banks' leverage levels closer to the real data, once the Minsky effect is taken into consideration.

4.1.1 Related literature

Our paper is related to the literature on financial stability and systemic risk amplification mechanisms. Following the general equilibrium framework as in Tsomocos (2003), Goodhart et al. (2006a) and Dubey et al. (2005), our paper endogenizes default choices by banks, and incorporates the macro-financial feedback within the banking system as well as from the asset market. Although we did not explicitly model the Shleifer and Vishny (1992) type of fire-sale mechanism, the default decisions, the interbank market and the common asset exposure in our model transmit the shock through the financial system, and lead to the cyclical behaviour of leverage, credit and profits among the banks. Kiyotaki and Moore (1997) attribute the credit cycle to the role of durable assets as collateral for loans, and shows how the dynamic interaction between credit limits and asset prices becomes the transmission mechanism of shocks. Bernanke et al. (1999) show that the negative correlation between the net worth and the external finance premium serves as a financial accelerator, which works to amplify and propagate shocks to the macroeconomy. Brunnermeier and Pedersen (2009) link an assets market liquidity with traders funding liquidity, and margins are destabilizing and market liquidity and funding liquidity are mutually reinforcing, leading to liquidity spirals. Geanakoplos (2010) shows that heterogeneous beliefs leads to over-leverage during normal times, and over-deleveraging during the crisis, which leads to a leverage cycle. Adrian and Shin (2010) document evidence that marked-to-market leverage is strongly procyclical, and show expansions and contractions of balance sheets have asset pricing consequences through shifts in risk appetite. Brunnermeier and Sannikov (2014) consider nonliear amplification effects, and show that endogenous leverage de-

termines the distance to crisis, while securitization and derivatives contracts may lead to higher leverage and more frequent crises.

Minsky (1977, 1992) propose the Financial Instability Hypothesis, and point out that, over a prolonged prosperity, optimism changes, and agents tend to invest more in the riskier assets. This behavior is the source of financial instability in the economy. Bhattacharya et al. (2015) provide a formalized model, by considering investors updating beliefs following the Bayes' rule. In their paper, endogenous default introduces a pecuniary externality, since default decisions affect the borrowing cost. Whenever a good state realizes, investors become more optimistic. After several good realizations, the investors become so optimistic that they invest in assets which are considered too risky were it not for the belief updating.

The CIMDO approach is related to the CoVaR approach. Adrian and Brunnermeier (2016) propose a measure of systemic risk, ΔCoVaR, defined as the change in the value at risk of the financial system conditional on an institution being under distress relative to its median state. Reversing the conditioning illustrates how much a particular institution's risk increases given that the whole financial system is in distress, which is an important metric in our paper. In our simulation, we measure the conditional profits/losses for each bank, given the various systemic risk transmission and amplification channels. The differences between the profits and the losses share the same logic as ΔCoVaR. Instead of using a quantile regression of time-varying ΔCoVaR on lagged institutional characteristics and common risk factors as in Adrian and Brunnermeier (2016), the CIMDO method implements the cross-entropy approach, and constructs

the distribution closest to a prior that is also consistent with the available data.

Literature provides different measures of systemic risk. Lehar (2005) measures the risk of a regulator's portfolio by modelling the individual liabilities that the regulator has to each bank as contingent claims on the bank's asset. Billio et al. (2012) propose econometric measures of connectedness based on principal-components analysis and Granger-causality networks, and such measures can identify and quantify financial crisis periods. Huang et al. (2009, 2012) measure the systemic risk a banking sector as a hypothetical distress insurance premium, which is the insurance cost to protect against distressed losses. The measure indicates market perceived risk, which reflects expected default risk of individual banks, risk premia as well as correlated defaults. Acharya et al. (2017) measure each financial institution's contribution of systemic risk as its systemic expected shortfall in profit, which is the propensity to be undercapitalized when the system as a whole is undercapitalized.

The rest of the paper is organized as follows. Section 4.2 presents a modified version of the GST model, including an extension for Minsky effect. Section 4.3 briefly illustrates the CIMDO method, and the metrics of systemic risk we use in this paper. Section 4.4 presents our calibration approach to measure the risk-related and Minsky effect coefficients, and shows the simulation results. Section 4.5 concludes the paper.

4.2 The model

In this section we introduce a multi-period model with heterogeneous banks, a central bank and a regulator to simulate the banking system. In each period, the banks

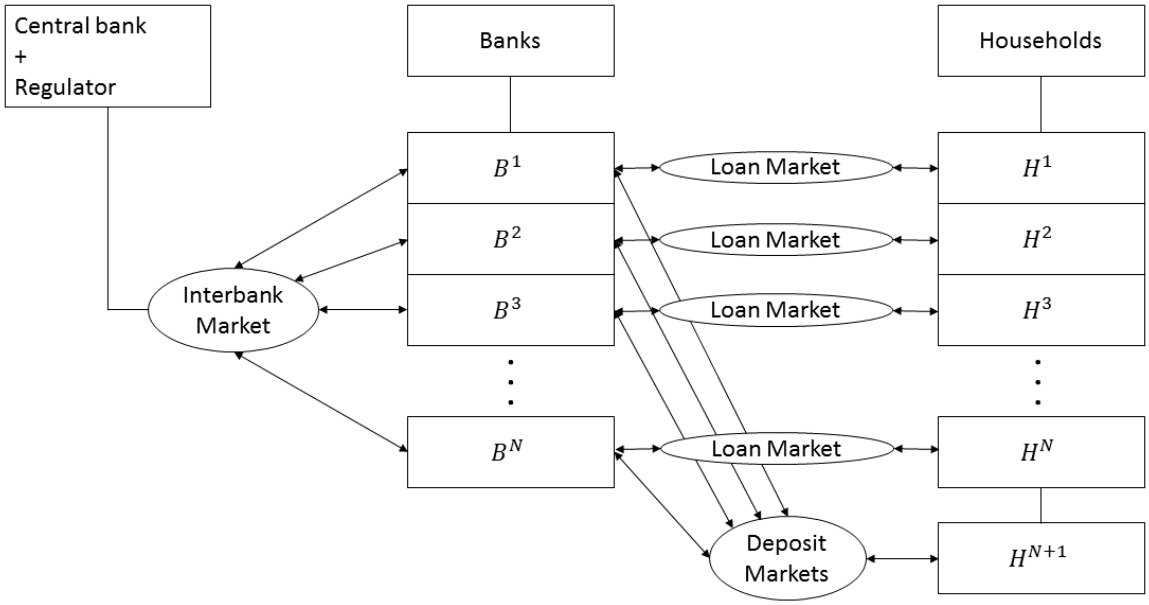
optimize their expected utilities by choosing the asset allocation *ex ante*, and the default level *ex post*. The endogenous default is a main feature of the model, where the banks choose the level/severity of default, depending on the realization of state and the repayment of the consumer loans. The model considers an interbank market, where the banks may choose to either lend or borrow *ex ante*, and choose to default a fraction *ex post*. Thus, a loss to one bank transmits through the interbank market and spreads across the entire banking system. The banks hold risky assets with correlated return, which also contribute to the transmission and amplification of systemic risk. Finally, we incorporate the Minsky effect, and prove that when a bank becomes more optimistic, its profit/loss becomes more volatile. If the optimism in the banking sector move in the same direction, the common change in optimism provides another source of systemic risk amplification. Our model is closely related with Goodhart et al. (2005).

4.2.1 The economy

The economy consists of N banks, $N+1$ households, a central bank, and a regulator. All agents interact in financial markets. The time horizon is modelled as discrete, multi-period ($t \in T = \{0, 1, 2, ...\}$), with two possible states at each future date. The banks are heterogeneous in the sense that they have different initial endowments, risk-aversion, and altitude toward default. Each household is assigned to borrow from a single bank. This limited participation assumption may come from history or informational constraint. Let the household $H^1, H^2, H^3, ..., H^N$ borrow from bank $B^1, B^2, B^3, ..., B^N$, respectively, and the remaining household H^{N+1} supplies de-

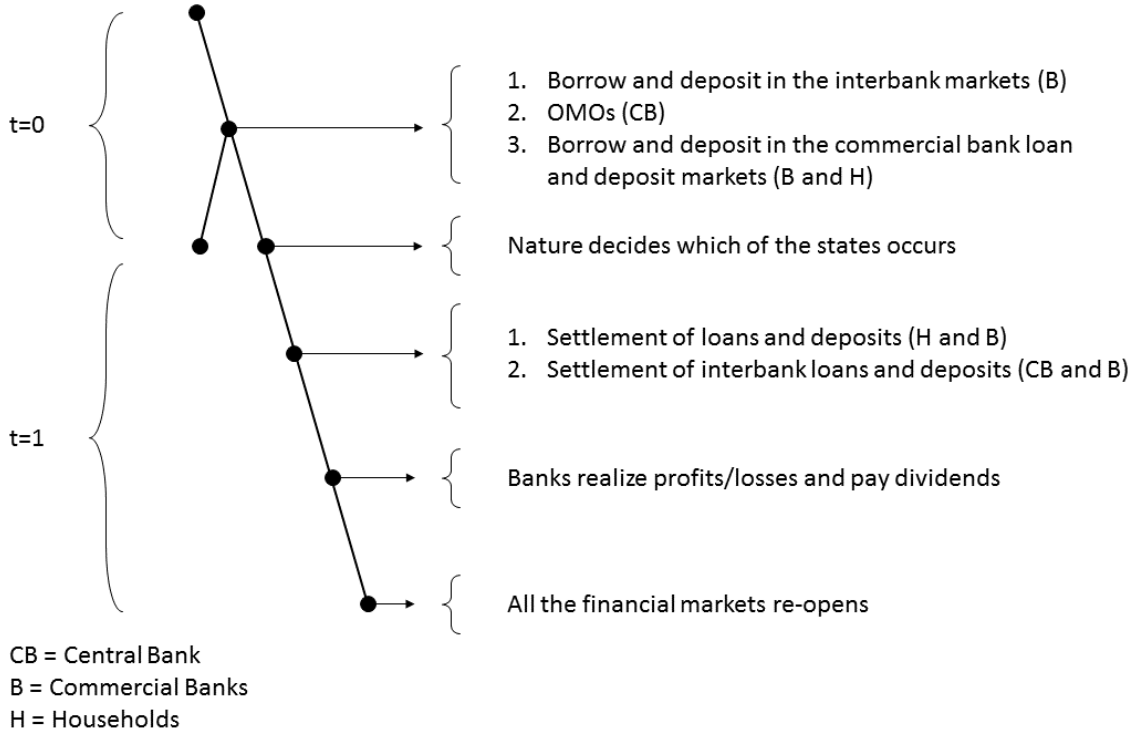
posits to all the banks. Both the loan rates and deposit rates are exogenous, and the repayment rates are different across the states. In addition, there is an interbank market where the banks are allowed to borrow from and deposit in other banks, and the Central Bank conducts open market operations (OMOs) to set the interbank lending rate. Figure 4.8 describes the economy.

Figure 4.8: The Economy



The economy operates in the time structure described by Figure 4.9. At the initial period, interbank, loan and deposit markets open. Banks determine the asset allocation in each market, and the central bank conducts OMOs in the interbank market. Nature decides which of the succeeding state will occur. At the beginning of the new period, financial contracts are settled, while the households and the banks choose the default level. Then financial markets re-open. Goodhart et al. (2006b) offers a similar extension with multiple periods, and proves that the infinite-period optimization is equivalent to a series of single-period optimizations. All the banks in the model are

Figure 4.9: The Time Structure



assumed to operate in a competitive environment, thus they take all interest rates as given. The structure of their balance sheets are shown in Table 4.1 below.

Table 4.1: Commercial bank's balance sheet

Assets	Liabilities
Loans to agents	Deposits from Agent η
Interbank deposits	Interbank Borrowing
Market book	Equity
	Others

The banks endogenize their decisions in the credit extension of consumer loans, deposit to take, and default level in their obligations. Assume that banks can partially default on their financial obligations, subject to non-pecuniary default penalties set by the regulator. The proportion of default here is interpreted as a probability that a bank chooses to shut down, hence in the short run, to default completely on its financial

obligations. We also assume that a bank's default proportion in the deposits and interbank markets are the same. The banks have ex-ante and ex-post trade-off with respect to asset allocation and repayment. The asset allocation trade-off is ex-ante, where the banks expect higher return from investing in riskier assets (consumer loans) than in safer ones (interbank loans and market book), but consequently face greater risks. The default decision is ex-post trade-off, where higher default level results in higher profits, but leads to higher level of default penalty. Details are provided in the following section.

4.2.2 The banks' optimization problem

The banks' optimization problems are given below. Assume that different bankers operate a bank in different time periods, thus within each period, the bank only cares about the profit in the subsequent period. As proved in Goodhart et al. (2006b), under certain conditions, the infinite horizon maximization problem is equivalent to a finite horizon maximization problem, and the horizon can be chosen as one period. At time t, bank b maximizes its expected utility, which is a quadratic function of its expected profit at time $t+1$ ($U(\pi) = \pi - c\pi^2$) minus non-pecuniary penalties that incurs if the bank defaults on its deposit and interbank obligations. The default penalty also takes the form of a quadratic function ($V(x) = \lambda x^2$), and we will see from the first-order conditions that the endogenous default varies with the bank's financial condition. Let B^B be the set of banks which are net borrowers in the interbank market, while B^L be the set of banks which are net lenders in the interbank market. Table 4.2 presents the notation for all the variables in the model.

Table 4.2: Variables in the model

Subscripts and Superscripts	
$t \in T \equiv \{0, 1, 2, ...\}$	time periods
$s \in S \equiv \{U, D\}$	the set of possible states after each node
$b \in B$	the set of all the banks
$b \in B^B$	the set of all the net borrowing banks
$b \in B^L$	the set of all the net lending banks
Exogenous Variables	
p_s	probability of state $s \in S$
γ^b	risk-aversion coefficient in the utility function of bank $b \in B$
λ^b	bank $b \in B$'s non-pecuniary cost of default coefficient for bank $b \in B$
A_t^b	the value of risk-free asset held by bank $b \in B$ at time t
$D_t^{I,b}$	interbank deposit taken by bank $b \in B^B$ at time t
$L_t^{I,b}$	interbank loan issued by bank $b \in B^L$ at time t
e_t^b	amount of capital that bank $b \in B$ holds at time t
ϵ_t^b	the un-modelled other items in the balance sheet of bank $b \in B$ at time t
r_t^b	interest rate of the consumer loans that bank $b \in B$ extends at time t
$r_t^{D,b}$	deposit rate offered by bank $b \in B$ at time t
r_t^A	the rate of return on the safe asset at time t
ρ_t	interbank rate at time t
$\delta_{t+1,s}^b$	dividend distribution of bank $b \in B$ at time $t+1$ in state $s \in S$
$g_{b,1}, g_{b,2}, g_{b,3}$	coefficients for consumer loans repayment rate
$u_{s,1}, u_{s,2}$	coefficients for GDP level
Endogenous Variables	
U_{t+1}^b	utility of bank $b \in B$ at $t+1$
$\Pi_{t+1,s}^b$	profit/loss of bank $b \in B$ at time $t+1$ in state $s \in S$
$\Delta\Pi_{t+1,D}^b$	expected shortfall in profit of bank $b \in B$ at time $t+1$
$\Lambda_{t+1,s}^b$	default of bank $b \in B$ at time $t+1$ in state $s \in S$
L_t^b	consumer loans of bank $b \in B$ at time t
D_t^b	deposit taken by bank $b \in B$ at time t
$v_{t+1,s}^b$	repayment rate of bank $b \in B$ at time $t+1$
$v_{t+1,s}^{b,h}$	repayment rate of bank b's consumer loans in state $s \in S$
$Y_{t+1,s}$	GDP level at time $t+1$ in state $s \in S$
M_t	central bank's supply of base money at time t
$\tilde{R}_{t+1}$	pooled repayment rate in the interbank market

A net borrowing bank $b \in B^B$ maximizes the following expected utility, which consists of a quadratic utility from profits, and quadratic disutility from default,

$$\max_{\sigma_t^b = \{L_t^b, D_t^b, v_{t+1,s}^b\}} \mathbb{E}_t(U_{t+1}^b) = \sum_{s \in S} p_s \left(U^b(\Pi_{t+1,s}^b) - V^b(\Lambda_{t+1,s}) \right) \tag{4.2.1}$$

$$\text{where} \qquad U^b(\Pi) = \Pi - \gamma^b \Pi^2 \tag{4.2.2}$$

$$V^b(\Pi) = \lambda^b \Lambda^2 \tag{4.2.3}$$

subject to the time t budget constraint $\mathcal{B}_t^b(r_t^b, r_t^{D,b})$ defined by equation (4.2.4) - (4.2.9). At time t, the sum of risky asset and safe asset equals the sum of deposits, interbank deposits, equity, and the other items that are not explicitly modelled,

$$L_t^b + A_t^b = D_t^b + D_t^{I,b} + e_t^b + \epsilon_t^b \tag{4.2.4}$$

The profit/loss at state s of time $t+1$ equals the proportional repayments from the consumer loans and the risk-free asset, minus the proportional repayments for the deposit and the interbank deposit, as well as the initial equity and some other exogenous items,

$$\Pi_{t+1,s}^b = v_{t+1,s}^{b,h} L_t^b (1 + r_t^b) + A_t^b (1 + r_t^A) - v_{t+1,s}^b \left(D_t^b (1 + r_t^{D,b}) + D_t^{I,b} (1 + \rho_t) \right) - e_t^b - \epsilon_t^b \tag{4.2.5}$$

and the expected shortfall in profit is given as,

$$\Delta \Pi_{t+1,D}^b = \Pi_{t+1,U}^b - \Pi_{t+1,D}^b \tag{4.2.6}$$

The bank chooses the level of default v_{t+1}^b after the state realizes, which should be a real number between zero and one,

$$0 \leqslant v_{t+1,s}^b \leqslant 1 \tag{4.2.7}$$

Consequently, the default at state s equals the unpaid obligation of the deposit and the interbank deposit,

$$\Lambda^b_{t+1,s} = (1 - v^b_{t+1}) \left(D^b_t(1 + r^{D,b}_t) + D^{I,b}_t(1 + \rho_t) \right) \tag{4.2.8}$$

The equity for the new period equals the equity in the previous period, plus profits, minus the dividend distribution,

$$e^b_{t+1,s} = e^b_t + \Pi^b_{t+1,s} - \delta^b_{t+1,s} \tag{4.2.9}$$

Similarly, a net lending bank $b \in B^L$, maximizes the following utility function,

$$\max_{\sigma^b_t = \{L^b_t, D^b_t, v^b_{t+1,s}\}} \mathbb{E}_t(U^b_{t+1}) = \sum_{s \in S} p_s \left(U(\Pi^b_{t+1,s}) - V(\Lambda_{t+1,s}) \right) \tag{4.2.10}$$

$$\text{where} \qquad U^b(\Pi) = \Pi - \gamma^b \Pi^2 \tag{4.2.11}$$

$$V^b(\Pi) = \lambda^b \Lambda^2 \tag{4.2.12}$$

subject to the time t budget constraint $\mathbb{B}^b_t(r^b_t, r^{D,b}_t)$ defined by equation (4.2.13) - (4.2.17). At time t, the sum of risky consumer loans, the interbank loans and the safe asset equals the sum of deposits, equity, and the other items that are not explicitly modelled,

$$L^b_t + L^{I,b}_t + A^b_t = D^b_t + e^b_t + \epsilon^b_t \tag{4.2.13}$$

The profit/loss at state s of time $T + 1$ equals the proportional repayments from the consumer loans and the interbank loan, plus the repayment of the the risk-free asset,

minus the sum of the proportional repayments for the deposit, the initial equity and some other exogenous items,

$$\Pi^b_{t+1,s} = v^{b,h}_{t+1,s} L^b_t (1 + r^b_t) + A^b_t (1 + r^A_t) + \tilde{R}_{t+1,s} L^{I,b}_t (1 + \rho_t) - v^b_{t+1,s} D^b_t (1 + r^{D,b}_t) - e^b_t - \epsilon^b_t \tag{4.2.14}$$

The bank chooses the level of default $v^b_{t+1,s}$ after the state realizes, which should be a real number between zero and one,

$$0 \leqslant v^b_{t+1,s} \leqslant 1 \tag{4.2.15}$$

Consequently, the default at state s equals the unpaid obligation of the deposit and the interbank deposit,

$$\Lambda^b_{t+1,s} = (1 - v^b_{t+1}) D^b_t (1 + r^{D,b}_t) \tag{4.2.16}$$

The equity for the new period should be equal to the equity in the previous period, plus profits, minus dividend distribution,

$$e^b_{t+1,s} = e^b_t + \Pi^b_{t+1,s} - \delta^b_{t+1,s} \tag{4.2.17}$$

4.2.3 Central bank and regulator

The central bank and the regulator are considered as strategic dummies in this paper, thus their decisions are taken as exogenous. The central bank conducts monetary policy by engaging in open market operations in the interbank market. In the model we assume that the central bank sets the interbank rate (ρ_t) as the monetary policy

instrument, and endogenize the base money (M_t). The regulator sets the default penalty functions and ratios (λ^b) for each bank.

4.2.4 Private agent sector

Each household borrower demands consumer loans from his nature-selected bank and chooses the default level in state $s \in S$. More specifically, the i^{th} household borrows from the i^{th} bank ($i = 1, 2, 3, ..., N$), and the $(N+1)^{th}$ household supplies deposit to each bank. Here we do not explicitly model the optimization problem of the households, but assume them to be strategic dummies on the loan and deposit market. This assumption simplifies the model, and our main interests focus on the potential risk in the banking sector. The repayment rates of the consumer loans, however, endogenously depend on the GDP level and the aggregated credit supply.

$$\ln(v_{t+1}^{b,h}) = g_{b,1} + g_{b,2} \ln(Y_{t+1,b}) + g_{b,3} \sum_{b \in B} \ln(L_t^b) \tag{4.2.18}$$

4.2.5 GDP

In this model, GDP in each state is assumed to be a positive function of the aggregate credit supply available in the previous period. As Goodhart et al. (2005) states, higher credit extension as a result of loosening monetary policy, or any other shocks, generates positive effects that raises consumption demand and ultimately GDP. Thus the following functional form for GDP holds.

$$\ln(Y_{t+1,s}) = u_{s,1} + u_{s,2} \sum_{b \in B} \ln(L_t^b) \tag{4.2.19}$$

4.2.6 Market clearing conditions

The only market we explicitly model in this paper is the interbank loan market. In equilibrium, market clears such that the sum of the interbank deposits should equal the sum of the interbank loans from both the banks and the central bank:

$$\sum_{b \in B^B} D_t^{I,b} = M_t + \sum_{b \in B^L} L_t^{I,b} \tag{4.2.20}$$

The interbank loans are pooled, so the repayment rate at the bad state should be determined by the following equation:

$$\tilde{R}_{t+1,s} = \frac{\sum_{b \in B^B} v_{t+1,s}^b D_t^{I,b}}{\sum_{b \in B^B} D_t^{I,b}} \tag{4.2.21}$$

The equilibrium is defined as a set of asset allocations and repayment rates such that the banks optimize their expected utility under the given constraints, and the interbank market clears. The first-order conditions are given in Appendix 4.A.

Definition 8. *The Equilibrium of the economy*

$$\mathcal{E}_t = \{(e_t^b, \epsilon_t^b, A_t^b, L_t^{I,t}, D_t^{I,t})_{b \in B}; M_t; (\lambda_t^b, \gamma_t^b)_{b \in B}; r_t^A; p; (g_{b,i})_{b \in B, I \in \{1,2,3\}}, (u_{s,j})_{s \in S, j \in \{1,2\}}\}$$

is

$$\{(\sigma_t^b)_{b \in B}; (Y_{t+1,s}) s \in S\}$$

iff:

1. $\sigma_t^b \in \underset{\sigma_t^b \in \mathcal{B}_t^b}{argmax} \quad \mathbb{E}_t(U_{t+1}^b) \quad b \in B$
2. *Household repayments satisfy equation* (4.2.18)

3. *GDP evolves by equation* (4.2.19)

4. *Interbank market clears by equation* (4.2.20)

5. *Banks are correct in their expectation about the repayment rates on the interbank market, given by equation* (4.2.21)

4.3 CIMDO

Segoviano (2006) proposes the consistent information multivariate density optimization (CIMDO) method to estimate dependence in a financial system. CIMDO is a non-parametric method and a general procedure that constructs multivariate densities from a prior distribution and a reduced set of information. It ensures that the posterior distribution is the closest distribution to the prior that is also consistent with the information set. The CIMDO method is based on the Kullback (1959) cross-entropy approach. Instead of assuming parametric probabilities to characterize the information contained in the data, the entropy approach uses the data information to infer values for the unknown probability density. Recovering multivariate densities representing the asset valuations of a system of firms is needed to evaluate the risks of joint distress. CIMDO constructs measures of distress dependence for financial firms, and captures not only linear but also non-linear distress dependencies among the firms in the system. In this section we present the CIMDO method, and the metrics we use to measure systemic risk dependence.

4.3.1 Computation method

Based on the Kullback (1959) cross-entropy approach, CIMDO infers the unknown multivariate distribution of system of firms containing M different risk-qualities, $p(l^1, ..., l^M)$ (the posterior distribution), from the observed Probability of Distress (PoD) of the assets making up the portfolio and from a prior multivariate distribution $q(l^1, ..., l^M)$. The cross-entropy approach recovers the distribution that is closest to the prior distribution but also consistent with the empirically observed probabilities fo default of the loans making up the portfolio.

We consider the financial system as a portfolio, and each firm/bank as an asset in the portfolio. For a portfolio, containing assets given to M different risk-qualities, whose logarithmic returns are characterized by the random variables $l^1, ..., l^M$, finding a multivariate distribution $p(l^1, ..., l^M)$ consistent with a set of observations is equivalent to solving the constrained minimization problem

$$\min_{p(.,...,.)\in S} C\left[p(l^1, ..., l^M), q(l^1, ..., l^M)\right] = \int_{l^M} .. \int_{l^1} p(l^1, ..., l^M) \ln\left[\frac{p(l^1, ..., l^M)}{q(l^1, ..., l^M)}\right] dl^1..dl^M \tag{4.3.1}$$

where the set of constraints S is the set of conditions given by the available information and the condition that the posterior probability distribution sums to 1. We illustrate an example of two assets below, while all results are directly applicable when $M > 2$. The two assets are characterized by their logarithmic asset returns x and y and the minimization problem is simply defined as

$$\min_{p(.,.)\in S} C\left[p(x, y), q(x, y)\right] = \int \int p(x, y) \ln\left[\frac{p(x, y)}{q(x, y)}\right] dxdy \tag{4.3.2}$$

where $q(x, y) \in \mathbb{R}^2$ is the prior distribution and $p(x, y) \in \mathbb{R}^2$ the posterior distribution. The Kullback cross-entropy criteria $C\left[p(x, y), q(x, y)\right]$ can be thought of as the weighted average of the relative distance between p and q and is a measure of distance between the prior distribution q and the posterior distribution p.

The moment-consistency constraints incorporate information about the probabilities of default of each type of asset, and imposing theses constraints on the optimization problem guarantees that the posterior multivariate distribution contains marginal densities that in the region of default ($[X_d^m, \infty)$ for $m \in \{x, y\}$) are equal to each of the borrowers' observed probabilities of default:

$$\int\int p(x, y)\mathbb{1}_{x \geqslant X_d^x} dxdy = PoD_t^x \quad \text{and} \quad \int\int p(x, y)\mathbb{1}_{y \geqslant X_d^y} dxdy = PoD_t^y \tag{4.3.3}$$

where $-x, -y$ are the equity annualized returns, and $\mathbb{1}$ is the indicator function. Segoviano and Espinoza (2016) solve the optimization problem, and give the posterior multivariate density as

$$\hat{p}(x, y) = q(x, y) \exp\left\{-\left[1 + \mu + \left(\lambda_x \mathbb{1}_{x \geqslant X_d^x}\right) + \left(\lambda_y \mathbb{1}_{y \geqslant X_d^y}\right)\right]\right\} \tag{4.3.4}$$

where μ, λ_x and λ_y are solutions of the system

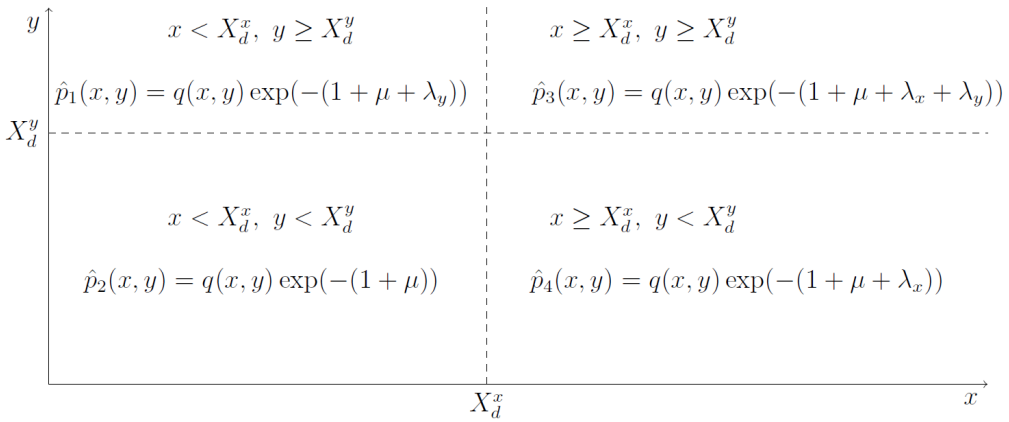
$$\begin{cases} \int\int \hat{p}(x, y)\mathbb{1}_{x \geqslant X_d^x} dxdy = PoD_t^x \\ \int\int \hat{p}(x, y)\mathbb{1}_{y \geqslant X_d^y} dxdy = PoD_t^y \\ \int\int \hat{p}(x, y) dxdy = 1 \end{cases} \tag{4.3.5}$$

The prior can be calibrated using information on asset or equity returns. For example, the stock market returns' moments can be used to calibrate a prior's normal or t-distribution. The observed PoDs are crucial inputs to CIMDO. For the application to systemic risk as in this paper, we could use CDS spreads as proxies, a Merton model's assessment of default frequency, Moody's KMV EDF, or bond spreads. The default-threshold (X_d^x and X_d^y in the example above) is fixed to an average through time that is consistent with the historical average of the probability of default for each asset.

4.3.2 Dependence structure

CIMDO adjusts prior distributions with available information on moments and incorporates the dependence structure. Figure 4.10 shows how the adjustment factor between the prior and the posterior, $\exp\{-[1+\mu+(\lambda_x \mathbb{1}_{x \geqslant X_d^x})+(\lambda_y \mathbb{1}_{y \geqslant X_d^y})]\}$, depends on the domain. Segoviano and Espinoza (2016) shows that CIMDO tends to strengthen dependence when marginal PoDs are underestimated by the prior.

Figure 4.10: CIMDO-density, adjustment factor



4.3.3 Expected shortfalls in profits

CIMDO is useful in estimating several financial stability measures to characterize distress dependence in the financial system. Segoviano and Espinoza (2016) proposed CIMDO's application on several financial stability measures, such as the Joint Probability of Distress (JPoD), the Distress Dependence Matrix (DiDe), the Financial Stability Index (FSI), the Probability of Cascade Effects (PCE), and the Marginal Contribution to Systemic Risk (MCSR). In this paper we consider the expected shortfall in profit for each bank, which can be estimated by both the model in Section 4.2 and CIMDO method.

The expected shortfall in profit is an output variable in the equilibrium model with binomial structure, which is the difference of the conditional profits/losses between a good state and a bad state. In the good state, no default occurs in the consumer loans, and all the banks are healthy; while in the bad state household partially default on their loans, damaging the banks' profitability and financial stability. The model solves for the conditional P/L at each state thus the expected shortfall in profits, given the banks' initial endowments, the risk-related coefficients, and the market condition. Interbank lending and common asset exposure channels for systemic risk are incorporated in the model, and we can also add the Minsky effect. Thus, the output variables, i.e. the expected shortfalls in profits of the banks should capture the information on the distress dependence structure accordingly.

We use CIMDO to compute the expected shortfalls in the following way. Below is an example with only two banks, and we can easily apply the result to a system with

N banks. Let the CIMDO posterior density of bank A and B be $p(x, y)$, which is the joint density function of equity annualized returns. Given this distribution, the expected equity value of bank A given the distress of bank B is,

$$\mathbb{E}_t(EQ_{A,t+1}|B) = \frac{1}{\mathbb{P}(B)}\mathbb{E}_t(EQ_{A,t+1}\mathbb{1}_B) = \frac{1}{\mathbb{P}(B)}\int\int EQ_{A,t}e^{-x}p(x,y)\mathbb{1}_{y>X_d^y}dxdy \tag{4.3.6}$$

where $EQ_{A,t}$ is the equity value of bank A at time t, and event B represents the case where bank B is in distress. Given the initial value of equity, we can calculate the annualized return on equity. The change of equity value through time represents the profit/loss in each scenario. The only question remains is to choose the distress threshold X_d^x and X_d^y. The thresholds delimiting the region of distress for each marginal need to be fixed so that changes in PoD_t^x and in PoD_t^y affect the shape of the posterior distribution rather than the thresholds themselves. The distress-threshold is fixed to an average through time that is consistent with the historical average of the probability of default for each asset/bank, and with the prior distribution. More precisely, if we assume that the prior distribution is bivariate standard normal, the historical average of the distress threshold for each borrower is set to $X_d^x = \Phi^{-1}(\overline{\alpha}^x)$ and $X_d^y = \Phi^{-1}(\overline{\alpha}^y)$, where $\Phi(\cdot)$ is the standard normal *cdf* and $\bar{\alpha}^x = 1 - \overline{PoD}^x$ and $\bar{\alpha}^y = 1 - \overline{PoD}^y$.

Now we consider N banks in the financial system, each of which can be either healthy or in distress. Thus there are 2^N scenarios in total. Let S represent all the 2^N scenarios, and H represent the case where no bank is in distress. Equation (4.3.7) gives the expected value of equity for each bank in cases where at least one bank is in distress, whereas equation (4.3.8) gives the expected value of equity for each bank

in the case where no bank is in distress. The difference between the two values is the expected shortfall of profits ($\Delta\Pi$) due to systemic risk.

$$\begin{aligned}&\mathbb{E}_t(e^b_{t+1}|S\backslash H)\\=&\frac{e^b_t}{1-\mathbb{P}(H)}\int\cdots\int e^{-x^b}p(x^1,\cdots,x^N)\left(1-\mathbb{1}(x^{b'}<X^{b'}_d,\forall b'\in B)\right)dx^1\cdots dx^N\end{aligned} \tag{4.3.7}$$

$$\begin{aligned}&\mathbb{E}_t(e^b_{t+1}|H)\\=&\frac{e^b_t}{\mathbb{P}(H)}\int\cdots\int e^{-x^b}p(x^1,\cdots,x^N)\mathbb{1}(x^{b'}<X^{b'}_d,\forall b'\in B)dx^1\cdots dx^N\end{aligned} \tag{4.3.8}$$

$$\Delta\Pi^b_{t+1}=\mathbb{E}_t(e^b_{t+1}|H)-\mathbb{E}_t(e^b_{t+1}|S\backslash H) \tag{4.3.9}$$

4.4 Model assessment

The paper aims at introducing a method to relate the statistical measurement of the systemic risk with a theoretical model for the financial system, and measuring the Minsky coefficients. The equilibrium model of banking system in Section 4.2 considers different risk transmission channels, and the CIMDO approach in Section 4.3 empirically measures the distress dependence of a financial system. Now we calibrate the equilibrium model against UK banking data from 2005 to 2007, and check if all the UK banks become more optimistic before the financial crisis, leading to more investment in risky assets, and higher downside risk.

The equilibrium model and the CIMDO method can be used separately to estimate the expected shortfalls in profits. For the equilibrium model, we can adopt the same

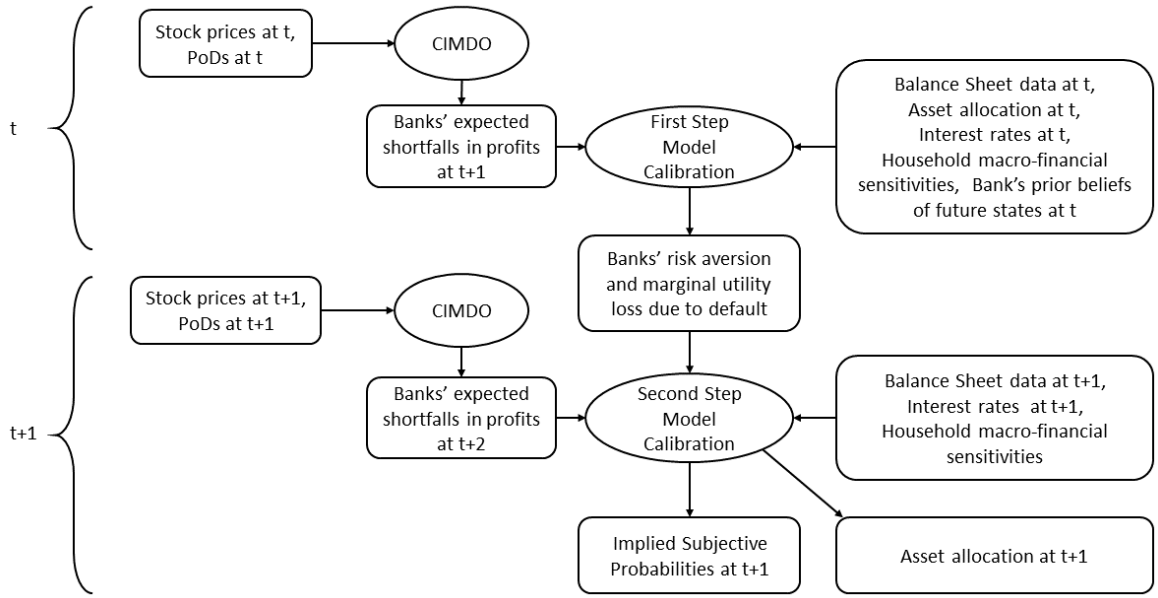
sets of exogenous and endogenous variables as in Section 4.2, taking the risk-aversion and default penalty ratios as given, and solve for the equilibrium asset allocation, the conditional profit/loss at each of the future state and the expected shortfalls. As put in Goodhart et al. (2005), the exogenous variables when solving the system of simultaneous equation do not necessarily have to be those which are exogenous in the model. Thus we may use the balance sheet data of the asset allocation, and the estimated expected shortfalls as exogenous, and solve for the risk-aversion coefficients and the default penalty ratios, which are not directly observable. For the CIMDO method, we use the stock prices and the probabilities of default for each bank as input, and compute the estimated conditional growth rate of bank equity, thus the expected shortfalls of profits.

Using a two-step calibration approach, we measure the risk-related deep parameters and the Minsky effect coefficients for the UK banks. The first step considers the period from June 2005 to June 2006, and the second step considers June 2006 to June 2007[4]. Market expected default frequency (EDF) is used a proxy for the probabilities of default (PoD) for each bank. We first take the stock prices and the probabilities of default for each bank up until June 2005 into CIMDO model, and compute for the expected shortfalls in profits for the banks in June 2006. Together with the banks' balance sheet data, interest rates, household macro-financial sensitivity coefficients and banks' prior beliefs, we calibrate the marginal disutility of default and the risk aversion coefficients. These variables are risk-related, but not directly observable. With these

[4]We have quarterly balance sheet data of the UK commercial banks, and the CIMDO approach can make prediction only one year ahead of time. In order to measure the system before the crisis burst out, without stepping into it, we use the balance sheet data in the middle of the year.

coefficients, we can predict the banks' profitability at each state in 2007 within the equilibrium model, thus the expected shortfalls in profits. However, the results are different from CIMDO's predictions for the same period. We conjecture that the difference are due to the changes in the banks' beliefs, which is the Minsky effect. In order to measure the Minsky coefficients, we take CIMDO's new prediction of expected shortfalls in profits for 2007 as input variables, together with the estimated default penalty ratios and risk-aversion coefficients from the previous step, as well as the new balance sheet data to calculate for the changes in implied subjective probabilities, and the asset allocation in the risky assets accordingly. Figure 4.11 illustrates the idea.

Figure 4.11: Model Assessment



The graph shows how we assess the model parameters in a two-step calibration approach. The ovals represent the models, while the rectangles represent the variables. The directions of the arrows indicate whether a variable is input or output for the model. The first-step calibration of the equilibrium model takes the banks' expected shortfalls in profits for time $t+1$ as inputs, and solve for the banks' risk aversion and default penalty coefficients. These two solved sets of variables are used as inputs for the second-step calibration, from which we get the implied changes in the subjective probability for the future states of each bank.

4.4.1 Equilibrium model revisited

Before we describe the data and present the simulation results, we have to see how the equilibrium model can be properly used to incorporate default and expected shortfalls in profits.

4.4.1.1 Default and distress dependence

The model endogenizes default by allowing the banks to choose the repayment level on their obligations. The lending banks' first order condition w.r.t. the repayment rate is below,

$$1 - 2\gamma^b \Pi^b_{t+1,s} = 2\lambda^b (1 - v^b_{t+1,s}) D^b_t (1 + r^{D,b}_t), \quad b \in B^L \tag{4.4.1}$$

and the equation for a borrowing bank is similar. The left-hand-side is the marginal benefit of default; while the right-hand-side is the non-pecuniary marginal cost of default, which can be considered as a reputation loss. After the consumer loans repayment rates realize, the banks choose the default level (v^b_{t+1}) to optimize, balancing the marginal benefit and marginal cost of default. Obviously, the repayment rate is higher when the profit is higher, holding other conditions constant. As in Goodhart et al. (2005), a bank's default rate is considered to be a proxy for the probability that such bank chooses to shut down and default completely on its financial obligations.

We interpret the good state in the model as a scenario where all the banks are healthy, while in the bad state at least one bank is in distress. For simplicity, assume that the consumer loans fully repay in the good state so it is reasonable to believe that

no banks should default on their obligations.[5] However, in the bad state, the repayment rates of the consumer loans are less than 1, so it is possible that certain banks are in distress, with positive probability of default. Given an N bank system, each of which can be either in distress or not, the total number of scenarios is 2^N. The good state in the equilibrium model represents the scenario where no banks default, while the bad state represents the combination/average of the remaining $2^N - 1$ scenarios, where at least one bank is in distress. Due to the existence of the interbank market, common asset exposure, and other systemic risk amplification mechanisms, the financial distress in one bank may affect other banks. Thus all the banks have positive probabilities of default in the bad state.

The equilibrium model explicitly incorporates an interbank lending market, and a common asset exposure channel to describe the distress dependence structure in the banking system. As illustrated by Figure 4.2, a borrowing bank suffers from a loss form the consumer loan in the bad state ($v^{b,h}_{t+1,D} < 1$), and chooses a default level ($v^{b}_{t+1,D} < 1$) on its obligations in the deposits and interbank market. Given the equilibrium condition (4.2.21), the repayment rate in the interbank market $\tilde{R}_{t+1} < 1$. Thus a lending bank in the interbank market suffers from losses in addition to its consumer loan positions. This affects the lending bank's profit ($\Pi^{b'}_{t+1,D}$), and its capital available for the next period ($e^{b'}_{t+1}$). The model also considers two dimensions of the common asset exposure channel. One is that we assume the repayment rate of the consumer loans are correlated, in the sense that no default occurs in the good state, while each of the consumer loans have positive probability of default in the bad state.

[5] We can assume that the realization of the states are observed at no cost, and the default penalty at the good state is infinitely large.

The other is that, when a bank suffers from a loss, it extends less credit, and the aggregate level of credit extension from the banking sector decreases. Given equation (4.2.18), the consumer loan repayment rate in the next period is reduced as well. The change in the asset payoff affects the banks' asset allocation, as well as the future profitability. Figure 4.3 demonstrates the common asset exposure channel of the model.

4.4.1.2 Incorporating the Minsky effect

Minsky effect is an important channel that affects the financial stability and the distress dependence structure in a financial system. Minsky (1977) states that, "over a period in which the economy does well, optimism changes, and agents tend to invest more in the riskier asset. The speculative investment boom is the basic instability in the economy". In this paper, the banks face the portfolio choices between the risky consumer loans and the safe market book. Limited by the binomial structure, the banks in the model only choose the level of risky loan and deposit taken, but the model still preserves the property that the banks choose the leverage and the proportion of investment in the risky asset. The question remains is how to model the change in beliefs. Bhattacharya et al. (2015) considers a portfolio choice problem in a similar binomial structure, with two possible true states of nature representing the likelihood of a good state to occur. Each agent holds her own subjective belief on the realization of future states, and adjust the belief when a new state realizes following the Bayes' rule. Such a setting allows the agents to become more optimistic in good states and more pessimistic in bad states.

Our model can be easily extended to incorporate the Minsky effect. Note that in Section 4.2.2, all the banks hold the same belief p of the states. In contrast, we may let this variable be heterogeneous among the banks, and change throughout time. It is not necessary to consider a specific rule of evolution for the optimism as in Bhattacharya et al. (2015), as long as we can show that a higher level of optimism (higher value of p) indicates a more risky investment strategy (higher value of m_t^b). The following proposition shows how the banks adjust the asset allocation when subjective beliefs change.

Proposition 5. *The banks described in Section 4.2.2 invest more in the risky consumer loans when becoming more optimistic, i.e.* $\frac{\partial L_t^b}{\partial p} > 0$. *This leads to a higher volatility in the banks' profit/loss, i.e.* $\frac{\partial \Pi_{t+1,U}^b}{\partial p} > 0$ *and* $\frac{\partial \Pi_{t+1,D}^b}{\partial p} < 0$, *thus larger expected shortfalls in profit.*

The proof is given in Appendix 4.B.

There are two ways we can quantify the Minsky effect in our model. One is to assume a certain belief formation process, as in Bhattacharya et al. (2015). The subjective probabilities $p^b, \forall b \in B$ update by the Bayes' rule as new states realize. As proved above, the asset allocations of each bank adjust as they update the beliefs. If the optimism of all the banks are going up simultaneously, they all invest more in the risky asset. This behavior increases the volatility of the banks' profit/loss, and leads to a bigger loss if a bad state occurs. This is the Minsky effect in the banking system, which amplifies the systemic risk. The other way is to estimate the change in the banks' profit/loss, and backwardly solve for the change in the subjective probabilities,

which we adopt in this paper.

4.4.2 Data description

Our specification of the financial system is based on the UK banking sector, which we assume to comprise of six largest UK banks by asset value in 2005: Barclays plc, HSBC Bank plc, the Royal Bank of Scotland plc, HBOS plc, Lloyds Bank plc, and Standard Chartered plc. The first period of the model is from June 2005 to June 2006, while the second period is from June 2006 to June 2007. This is a period before the financial crisis, during which the overall optimism in the banking sector is considered to have increased, and the banks took more risks. Thus, we do not use the end-of-year balance sheet data.

We collect the semi-annual balance sheet data, daily stock prices and EDFs for the banks from FitchConnect. For the balance sheet data, we use the total gross loans in the balance sheet as L^b in the equilibrium model, total securities as A^b, total deposits as D^b, total equity as e^b, and the difference between loans to banks and deposits from banks as L^I or D^I. The collected data are shown in Table 4.7 in Appendix 4.C, while the unavailable data points are linearly interpolated. Let bank $1, 2, 3, 5$ represent Barclays, Lloyds, RBS, and HSBC, respectively, whereas bank 4 represents the aggregation of the remaining two banks: HBOS and Standard Chartered. We directly add the balance sheet items of the two banks to get the variables for bank 4. We take the stock prices for the holding companies of each bank, and the expected default frequency (EDF) since 2004 as the input variables for CIMDO computation. The EDFs are uses as a measure of probability of default for each bank. Bank 4

aggregates the stock prices and EDFs by taking an weighted average of HBOS and Standard Chartered's data by market capitalization.

4.4.3 Calibrating the risk-related deep parameters

We first calibrate the model with UK bank data in June 2005 to solve for the risk-related coefficients. In the equilibrium model, the endogenous variables are the profits/losses of the banks in each state, the banks' investment in the risky consumer loans, the deposits taken, the repayment rates of the banks as well as the households, and the GDP levels in each state, while the exogenous variables are the interest rates, the risk-aversion coefficients and the default penalty ratios of the banks, and other fixed bank balance sheet items. However, the exogenous variables when solving the system of simultaneous equations do not have to be the exogenous ones in the model, which allows as to calibrate the model with real data, and find the implied deep parameters.

Table 4.3 summarizes the inputs and outputs of the model. The columns show whether a variable is exogenous or endogenous in the equilibrium model, while the rows show whether a variable is endogenously solved, empirically observed, estimated by CIMDO, or arbitrarily selected when we match the data into the equilibrium model. The variables on the left-hand-side of the table are endogenous in the model, and the model's exogenous variables are on the right. When solving the system of simultaneous equations, we calibrate the loans extended (L), deposits taken (D), safe asset holdings (A), capital level (e), and interbank loan/deposit (L^I/D^I) against the real balance sheet data in June 2005, the GDP in the good state against the nominal GDP in 2006, and the interbank rate against the 12-month LIBOR at June 2005. The expected

Table 4.3: Variables for first-step model calibration

Equilibrium Model		
	Model Endogenous Variables	Model Exogenous Variables
Endogenously Solved	$\Pi_U^b, \forall b \in B$: banks' conditional P/L in good state $\Delta\Pi^b, \forall b \in B$: banks' expected shortfalls in profits $v^{b,h}, \forall b \in B$: repayment level of the consumer loans M: Money supply by the central bank $\tilde{R}$: Pooled interbank deposit repayment rate in the bad state	$\lambda^b, \forall b \in B$: the non-pecuniary default penalty coefficient $\gamma^b, \forall b \in B$: each bank's risk-aversion coefficient $\epsilon^b, \forall b \in B$: un-modelled balance sheet items $g_1^b, \forall b \in B$: the elasticity of consumer loan repayment rate with respect to GDP $u_{s,1}, s \in \{U, D\}$: coefficients for GDP
Empirically Observed	$L^b, \forall b \in B$: banks' holding of consumer loans $D^b, \forall b \in B$: banks' liability of deposits Y_U: the GDP level at the good state	$A^b, \forall b \in B$: banks' holding of safe assets $e^b, \forall b \in B$: banks' initial equity level $D^{I,b}, \forall b \in B^B$: borrowing banks' liability of interbank deposit $L^{I,b}, \forall b \in B^L$: lending banks' holding of interbank loan ρ: the interbank interest rate
Estimated by CIMDO	$\Pi_D^b, \forall b \in B$: banks' conditional profits/losses in the bad state	
Manually Selected	$v^b, \forall b \in B$: banks' repayment rate in the bad state Y_D: the GDP level at the bad state	$r^{D,b}, \forall b \in B$: deposit interest rate $r^b, \forall b \in B$: consumer loan interest rate r^A: banks' market book return p: probability of good state g_2^b, g_3^b: coefficients for consumer loan repayment $u_{s,2}, s \in \{U, D\}$: coefficients for GDP

This table presents the variables for the equilibrium model. The values of these variables are presented in Table 4.8 in Appendix 4.C. Remember that the variables which are exogenous when solving the system of simultaneous equations, do not necessarily have to those which are exogenous in the model. The column on the left shows the endogenous variables in the equilibrium model, while the column on the right shows the exogenous variables. This is consistent with the classification of the variables in Table 4.2.

When solving the simultaneous equations, the value for some exogenous variables in the model such as the default penalty coefficient ($\lambda^b, b \in B$) and the risk aversion coefficients ($\gamma^b, b \in B$) cannot be empirically observed or estimated, while some endogenous variables in the model such as the position in the consumer loans ($L^b, b \in B$) and deposit taken ($D^b, b \in B$) can be observed empirically.

Variables in the first row are endogenously computed when solving the simultaneous equations; the balance sheet items and the macroeconomic data in the second row are empirically observed; the conditional expected profits/losses in the third row are estimated by CIMDO, which captures the information about distress dependence structure; the variables in the forth row are manually selected.

shortfalls in profits for each bank ($\Delta\Pi$) are estimated by CIMDO. In the model the deposits have the same payoff as the interbank loan, thus we assume the rate r^D is the same as the interbank rate ρ. Average spread between loans and deposits for commercial banks in 2005 is 3.66%[6], so we assume r to be r^D plus the spread. The probability of a good state to occur (p) is manually selected, and we run a robustness

[6]FISIM data.

check afterwards. The banks' repayment rate (v) is set to be consistent with the empirically observed Expected default frequency (EDFs)[7]. The banks' market book return (r^A), GDP level at the bad state (Y_D), and the household macro-financial sensitivities (g_2, g_3, u_2) are arbitrarily selected, and are consistent with the choices in Goodhart et al. (2005). The rest of the variables are endogenously solved from the simulation. The values of the input variables and the simulation results are present in Table 4.8 of Appendix 4.C.

We estimate the conditional profit/loss for each bank at the bad state using the CIMDO method. More specifically, CIMDO takes the stock prices and the EDFs from Jan. 2nd, 2004 to June 30th, 2005 as input variables, and generates the expected percentage change of equity value for each bank, under the condition that at least one bank is in distress in June 2006. As we explained previously, CIMDO re-constructs the joint distribution of banks' equity values, which is consistent with the probability of default, and captures the information of the distress dependence structure. Thus we can calculate for the conditional expected profits/losses in June 2006, using CIMDO's results and the initial capital value of each bank in June 2005.

The calibration at this step reveals the banks' risk aversion, and the altitude towards default. The risk aversion coefficients ($\gamma^b, b \in B$) and the default penalty ratio ($\lambda^b, b \in B$) are given in Table 4.3. These variables are important in the banks' portfolio decision process, but they are not directly observable. We calibrate the model in this way, such that these implied risk-related deep parameters are consistent with the banks' actual

[7] v is set such that $EDF = (1-p)(1-v)$

portfolio choices. We also assume that these coefficients are constant through time. Given these parameters, we can predict the banks' portfolio choices and expected profits/losses through time.

4.4.4 Model prediction without Minsky effect

Given the risk-related coefficients, the equilibrium model predicts the banks' conditional profits/losses in the next period. Now we take the data available in June 2006, and make predictions for the next year in June 2007. Table 4.4 summarizes the input and output variables for the second-period simulation, and the data are presented in Table 4.8 in the appendix. Similar as before, the column on the left shows the endogenous variables for the model, while the column on the right shows the exogenous variables for the model. We have calibrated the model previously, and obtained the risk-aversion coefficients ($\gamma^b, b \in B$) and the default penalty coefficients ($\lambda^b, b \in B$) for each bank, as well as the coefficients for consumer loan repayment rates ($g_1^b, b \in B$) and for GDP determination ($u_{s,1}, s \in \{U, D\}$). The balance sheet items such as the banks' holding of safe assets (A^b), equity for the new period (e^b), position in the interbank market ($D^{I,b}, L^{I,b}$), and un-modelled items (ϵ^b), as well as the interbank rate (ρ) should be updated to the values on 30th June, 2006. The arbitrarily selected variables take the same values as in the previous step calibration. Now the model endogenously solves for the following variables: the banks conditional expected P/L at the bad state ($\Pi_U^b, \Pi_D^b, b \in B$), thus the expected shortfalls in profits ($\Delta\Pi^b, b \in B$); the banks' positions in the risk-free asset ($A^b, b \in B$), consumer loans ($L^b, b \in B$) and deposits ($D^b, b \in B$); banks' probabilities of default ($v^b, b \in B$); default level of the consumer

loans ($v^{b,h}, b \in B$); money supply (M); the pooled interbank deposit repayment rate in the bad state ($\tilde{R}$); and the GDP level at each state ($Y_s, s \in \{U, D\}$). The values of these variables are presented in Table 4.9 of Appendix 4.C. The model predicts that all the banks are making profits in the good state, while some banks suffer from losses in the bad state. We will compare this result with the one from a model with Minsky effect later.

CIDMO also generates new predictions for the expected shortfalls in profits. Section 4.3 shows how CIMDO can generate prediction of the percentage changes in the banks' equity values at each state. Together with the initial equity level, we can compute for the conditional expected profit/loss for each bank at each state, thus the expected shortfalls. Table 4.5 presents the results, while Figure 4.12 compares the predictions by the two methods. It is easy to see that, CIMDO in general predicts larger expected shortfalls in profits. At this step, the equilibrium model only considers the interbank lending channel and the common asset exposure channel for distress dependence and contagion, thus we believe that CIMDO must have captured some additional information, which also contributes to the distress dependence structure of the banking system. There are many candidate channels that can help explain the distress dependence and contagion in a financial system. For example, the fire-sale mechanism, maturity mismatch, liquidity mismatch, etc. We believe that the Minsky effect is the most important one, and the following section shows how we measure the relevant coefficients.

Table 4.4: Variables for the second period model prediction without Minsky effect

Equilibrium Model		
	Model Endogenous Variables	Model Exogenous Variables
Endogenously Solved	$\Pi_U^b, \forall b \in B$: banks' conditional P/L in the good state $\Pi_D^b, \forall b \in B$: banks' conditional P/L in the bad state $\Delta\Pi^b, \forall b \in B$: banks' expected shortfalls in profits $v^{b,h}, \forall b \in B$: default level of the consumer loans M: Money supply by the central bank $\tilde{R}$: Pooled interbank deposit repayment rate in the bad state $L^b, \forall b \in B$: banks' holding of consumer loans $D^b, \forall b \in B$: banks' liability of deposits $v^b, \forall b \in B$: banks' probability of default in the bad state Y_U: the GDP level at the good state Y_D: the GDP level at the bad state	
Empirically Observed		$A^b, \forall b \in B$: banks' holding of safe assets $e^b, \forall b \in B$: banks' initial equity level $D^{I,b}, \forall b \in B^B$: borrowing banks' liability of interbank deposit $L^{I,b}, \forall b \in B^L$: lending banks' holding of interbank loan $\epsilon^b, \forall b \in B$: the other balance sheet items that are not explicitly modelled ρ: the interbank interest rate
Manually Selected		$r^{D,b}, \forall b \in B$: deposit interest rate $r^b, \forall b \in B$: consumer loan interest rate r^A: banks' market book return p: probability of good state g_2^b, g_3^b: coefficients for consumer loan repayment $u_{s,2}, s \in \{U, D\}$: coefficients for GDP
Calibrated		$\lambda^b, \forall b \in B$: the non-pecuniary default penalty coefficient for each bank $\gamma^b, \forall b \in B$: each bank's risk-aversion coefficient $g_1^b, \forall b \in B$: the elasticity of consumer loan repayment rate with respect to GDP $u_{s,1}, s \in \{U, D\}$: coefficients for GDP

This table presents the variables for the theoretical model in the second period, without Minsky effect. The main difference from Table 4.3 is that, the default penalty coefficient (λ^b), the risk-aversion coefficient (γ^b), the coefficient for the consumer loan repayment rate (g_1^b), and the coefficient for GDP determination ($u_{s,1}, s \in \{U, D\}$) have been calibrated in the previous step.

The values of the balance sheet items such as the banks' holding of safe assets (A^b), equity for the new period (e^b), position in the interbank market ($D^{I,b}, L^{I,b}$), and un-modelled items (ϵ^b), as well as the interbank rate (ρ) should be updated to 30th June, 2006.

Banks' holding of consumer loans (L^b), deposits taken (D^b), probability of default (v^b), the expected shortfalls in profits ($\Delta\Pi^b$), and the GDP levels (Y_s) are now endogenously solved from the simultaneous equations.

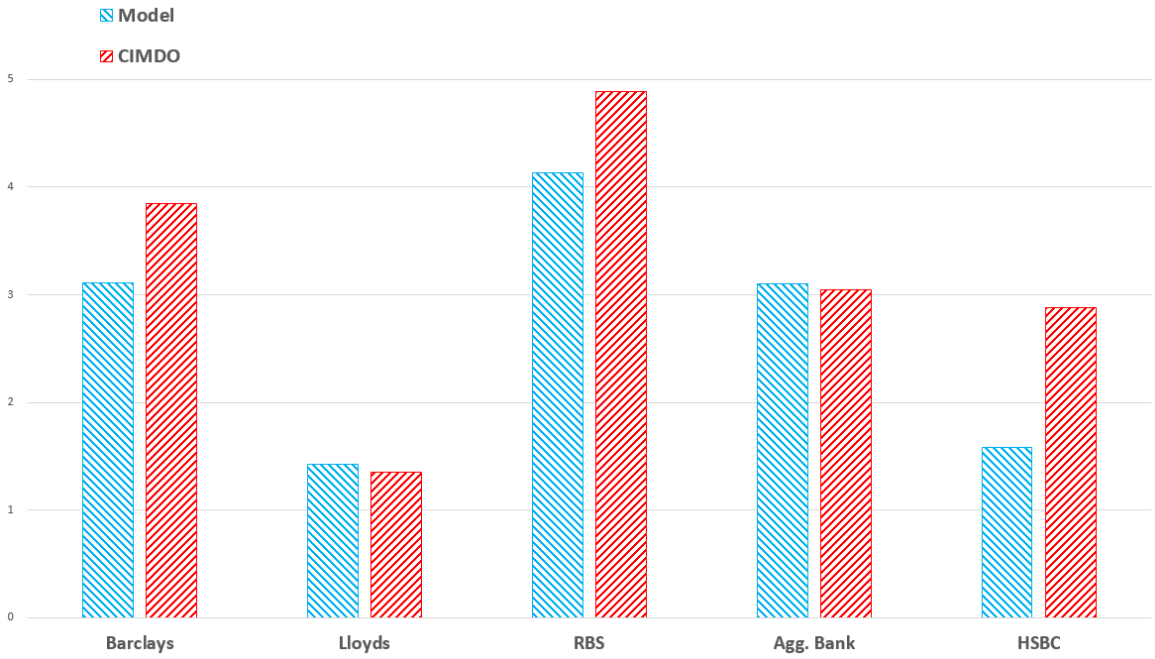
The values of the variables in this table are present in Table 4.9 in Appendix 4.C.

Table 4.5: Banks' expected shortfalls in profits in June 2007, measured by CIMDO

$\Delta\Pi^1 = 3.8507$	$\Delta\Pi^2 = 1.3481$	$\Delta\Pi^3 = 4.8913$	$\Delta\Pi^4 = 3.0423$	$\Delta\Pi^5 = 2.8821$

CIMDO generates the percentage change in equity value for each period. In this simulation, CIMDO predicts the change from June 2006 to June 2007.
Given the balance sheet data of equity level for each bank in June 2006, the profit at the end of the period is easily computed. The difference across states represents the expected shortfall in profit.

Figure 4.12: Expected Shortfalls in Profits, June 2007



This figure compares the prediction of the theoretical model without Minsky effect and CIMDO on the expected shortfalls in profits for each bank at June 2007. Agg. Bank represents the combination of HBOS and Standard Chartered. For each bank, the bar on the left is the theoretical model's result, while the bar on the right is CIMDO's result. Obviously CIMDO predicts larger expected shortfalls in profits in general. We know that the theoretical model incorporates the interbank lending channel and the common asset exposure channel for systemic risk transmission and amplification, while the CIMDO model should be able to capture the effects of all the channels, given the market data. Thus the difference between the two methods represents the missing channels of the theoretical model, which we attributes to the Minsky effect.

4.4.5 The Minsky effect

Minsky effect explains the relationship between optimism and risk building. In his early works, Minsky states that "over a prolonged period of prosperity, optimism changes, and agents tend to invest more in the riskier assets. The tendency to transform doing well into a speculative investment boom is the basic instability in the

economy". Bhattacharya et al. (2015) consider a Bayesian updating mechanism for investors during good times, which makes the investors increase their leverage and shift their portfolios towards projects that would previously have been considered too risky. We refer to the effect of changing beliefs as the Minsky effect, measure the banks' expected shortfalls in profits, and backwardly solve for the changes in the banks' optimism.

We use the following method to measure the change in beliefs. Section 4.4.1.2 shows how we incorporate time-varying expectations for each bank in the equilibrium model. In the previous step, we calibrated the model against the real data for the period form June 2005 to June 2006, and solved for the risk-aversion coefficients and the default penalty ratios. Each bank is assumed to hold a prior subjective probability for the future states $(p_0^b, b \in B)$. However, the equilibrium model and CIMDO made different prediction about the conditional expected profits/losses. We conjecture that the Minsky effect, namely the change in the subjective optimism, is the most important reason for such a difference, and attributes all the additional effects on the distress dependence structure to the Minsky effect. Thus we take CIMDO's results as input variable for the simulation of the equilibrium model for the period from June 2006 to June 2007, and solve for the implied change in the subjective probabilities $(p_1^b, b \in B)$. Table 4.6 presents the input and output variables for the second period model simulation with Minsky effect. Note that the only difference between this simulation and the previous one without Minsky effect is that, now we estimate the banks' expected shortfalls in profits $(\Delta\Pi^b, b \in B)$ by CIMDO, and solve for the implied subjective probability on the state for each bank $(p_1^b, b \in B)$. Table 4.10 in Appendix 4.C presents the values for

Table 4.6: Variables for the second period with Minsky effect

Equilibrium Model		
	Model Endogenous Variables	Model Exogenous Variables
Endogenously Solved	$\Pi_U^b, \forall b \in B$: banks' conditional P/L in the good state $\Pi_D^b, \forall b \in B$: banks' conditional P/L in the bad state $v^{b,h}, \forall b \in B$: default level of the consumer loans M: Money supply by the central bank $\tilde{R}$: Pooled interbank deposit repayment rate in the bad state $L^b, \forall b \in B$: banks' holding of consumer loans $D^b, \forall b \in B$: banks' liability of deposits $v^b, \forall b \in B$: banks' probability of default in the bad state Y_U: the GDP level at the good state Y_D: the GDP level at the bad state	$p^b, b \in B$: banks' implied subjective probability of the good state in the second period
Estimated by CIMDO	$\Delta\Pi^b, \forall b \in B$: banks' expected shortfalls in profits	
Empirically Observed		$A^b, \forall b \in B$: banks' holding of safe assets $e^b, \forall b \in B$: banks' initial equity level $D^{I,b}, \forall b \in B^B$: borrowing banks' liability of interbank deposit $L^{I,b}, \forall b \in B^L$: lending banks' holding of interbank loan $\epsilon^b, \forall b \in B$: the other balance sheet items that are not explicitly modelled ρ: the interbank interest rate
Manually Selected		$r^{D,b}, \forall b \in B$: deposit interest rate $r^b, \forall b \in B$: consumer loan interest rate r^A: banks' market book return p: prior probability of good state g_2^b, g_3^b: coefficients for consumer loan repayment $u_{s,2}, s \in \{U, D\}$: coefficients for GDP
Calibrated		$\lambda^b, \forall b \in B$: the non-pecuniary default penalty coefficient for each bank $\gamma^b, \forall b \in B$: each bank's risk-aversion coefficient $g_1^b, \forall b \in B$: the elasticity of consumer loan repayment rate with respect to GDP $u_{s,1}, s \in \{U, D\}$: coefficients for GDP

This table presents the variables for the theoretical model in the second period with Minsky effect. Compared with the case without Minsky effect (Table 4.4), we estimate the banks' expected shortfalls in profits by CIMDO, and solve for the banks' implied subjective probability of the good state in the second period.
The values of the variables in this table are presented in Table 4.10 in Appendix 4.C.

the variables. Obviously, $p_1^b > p_0^b, \forall b \in B$, i.e. all the five banks in the model become more optimistic in June 2006 than in June 2005. Consequently the asset allocation in the risky consumer loans ($L^b, b \in B$) is higher than those predicted without the Minsky effect. This result is consistent with Proposition 5. Individually, the banks

rationally increase their investment in the risky asset as well as their leverage levels. In addition, the potential default induces externality, since the banks do not factor in the impact of their decision on the future repayment rate of the consumer loans. When all the banks change their beliefs in the same direction simultaneously, they all suffer from more losses should a bad state occur, and the total credit extension decreases. This would decrease the future repayment rate of the consumer loans in the bad state, thus the expected profitability from the risky loans.

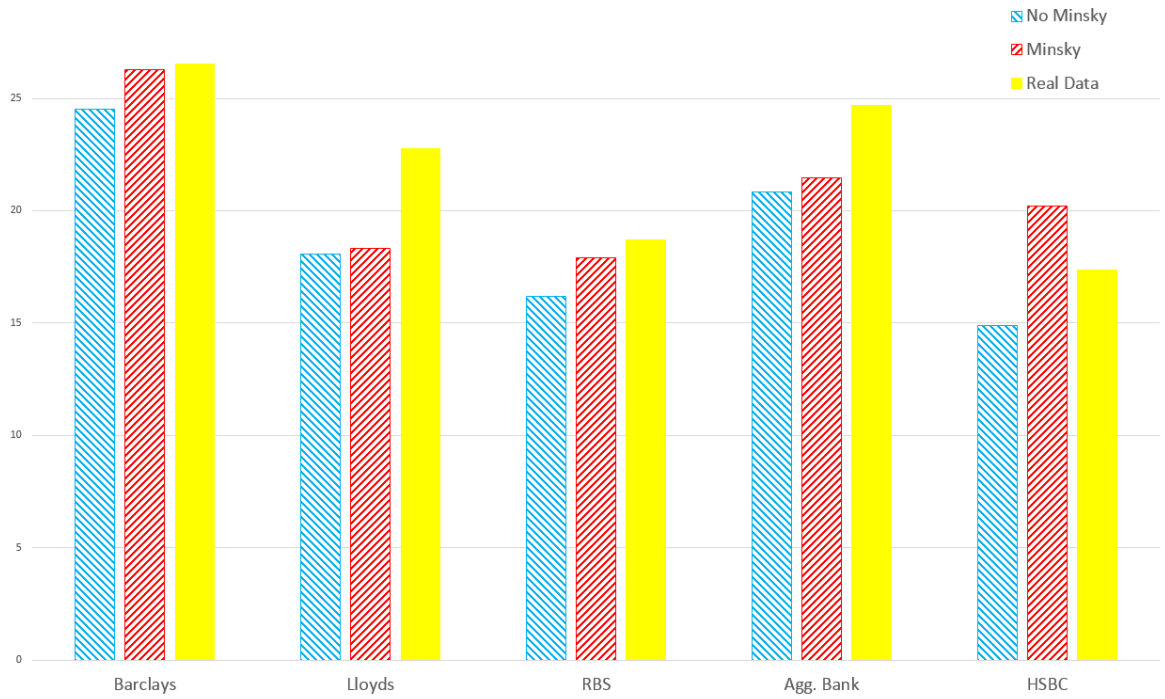
Minsky effect is stronger if we consider longer time horizon. The model assumes that the banks update their beliefs once in each period, and the calibration results show that they become more optimistic after the realization of a good state. If we allow for longer time horizon, the banks would be able to update the beliefs for more than once. Consider the scenario with several consecutive realization of good states, during which the banks become more optimistic year after year. From the previous analysis, we know that the banks would take more and more risk through out time. The build-up of optimism and risk is consistent with the idea from Minsky's early work.

The model's prediction on the banks' leverage ratios gets closer to real data after considering the Minsky effect. Based on the balance sheet structure in the equilibrium model, the leverage ratio is defined as,

$$\mathbb{L}_t^b \equiv \frac{\text{Total Asset}}{\text{Initial Equity}} = \begin{cases} \frac{L_t^b + A_t^b}{e_t^b}, & \text{for borrowing banks} \\ \frac{L_t^b + A_t^b + L_t^{I,b}}{e_t^b}, & \text{for lending banks} \end{cases} \tag{4.4.2}$$

Figure 4.13 presents the comparison among the banks' leverage ratios, following the definition above. We compute the leverage ratios using the data from the equilibrium model without Minsky effect, with Minsky effect, and with real data, and show on the graph from left to right for each bank. It is obvious that, by considering the Minsky effect, the prediction of the equilibrium model gets closer to the real data. The inclusion of the Minsky effect improves the performance of the equilibrium model for the banking system.

Figure 4.13: Leverage



This figure presents the comparison among the banks' leverage ratios, following the definition in Equation 4.4.2. For each bank, the bar on the left, in the middle, and on the right represents the leverage computed with data from GST model without Minsky effect, with Minsky effect, and with real data, respectively.

4.4.6 Robustness

In the previous simulations we manually selected the banks' prior subjective probability of states p. The initial choice is $p = 0.95$, which implies under the binomial

structure that on average every twenty years a crisis is expected. This is a quite optimistic view. We test the model with different values for p from 0.9 to 0.98, and all the results from above remain.

4.5 Conclusion

This chapter has two major contributions. First, it provides a valid method to combine a equilibrium model with an empirical approach (CIMDO) to estimate the systemic risk and distress dependence structure in a financial system. Second, the chapter highlights the importance of the Minsky effect: when the banks become more optimistic altogether, they increase the risk-taking of the financial system, and amplifies the systemic risk. We modify and extend the equilibrium model from Goodhart et al. (2005) of a banking system with the interbank lending and common asset exposure channel for systemic risk, and prove that, under our assumptions, the volatility in the banks' profits/losses increases when they become more optimistic, which is a typical result of the Minsky effect. Then we take the data from UK banking system form 2005 to 2007, and applied the CIMDO method to estimate the risk-related coefficients and the Minsky effect coefficients for the banks using a two-step calibration approach. Our results show that the Minsky effect accounts for a significant part of the distress dependence structure.

4.A First order conditions of the equilibrium model

Borrowing banks

$$p(1-2\gamma^b\Pi^b_{t+1,U})+(1-p)(1-2\gamma^b\Pi^b_{t+1,D})v^{b,h}_{t+1}=\mathcal{L}^b/(1+r^b_t) \tag{4.A.1}$$

$$p(1-2\gamma^b\Pi^b_{t+1,U})+(1-p)\Big((1-2\gamma^b\Pi^b_{t+1,D})v^b_{t+1}+2\lambda^b(1-v^b_{t+1})^2[D^b_{I,b}(1+\rho_t)+D^b_t(1+r^{D,b}_t)]\Big)$$
$$=\mathcal{L}^b/(1+r^b_t) \tag{4.A.2}$$

$$1-2\gamma^b\pi^b_{t+1,D}=2\lambda^b(1-v^b_{t+1})[D^{I,b}_t(1+\rho_t)+D^b_t(1+r^{D,b}_t)] \tag{4.A.3}$$

Lending banks

$$p(1-2\gamma^b\Pi^b_{t+1,U})+(1-p)(1-2\gamma^b\Pi^b_{t+1,D})v^{b,h}_{t+1}=\mathcal{L}^b/(1+r^b_t) \tag{4.A.4}$$

$$p(1-2\gamma^b\Pi^b_{t+1,U})+(1-p)\Big((1-2\gamma^b\Pi^b_{t+1,D})v^b_{t+1}+2\lambda^b(1-v^b_{t+1})^2D^b_t(1+r^{D,b}_t)\Big)$$
$$=\mathcal{L}^b/(1+r^b_t) \tag{4.A.5}$$

$$1-2\gamma^b\Pi^b_{t+1,D}=2\lambda^b(1-v^b_{t+1})D^b_t(1+r^{D,b}_t) \tag{4.A.6}$$

4.B Proof of Proposition 5

Proof. We rewrite the first-order conditions and the budget constraints of the lending banks as below. To simplify the notation, remove the subscript for time and the superscript for the bank.

$$p(1-2\gamma\Pi_U)+(1-p)(1-2\gamma\Pi_D)v^h=\mathcal{L}/(1+r) \tag{4.B.1}$$

$$p(1-2\gamma\Pi_U)+(1-p)\left((1-2\gamma\Pi_D)v+2\lambda(1-v)^2D(1+r^D)\right)=\mathcal{L}/(1+r^D) \tag{4.B.2}$$

$$1-2\gamma\Pi_D=2\lambda^b(1-v)D(1+r^D) \tag{4.B.3}$$

$$L+L^I+A=D+e+\epsilon \tag{4.B.4}$$

$$\Pi_U=L(1+r)+A(1+r^A)+L^I(1+\rho)-D(1+r^D)-e-\epsilon \tag{4.B.5}$$

$$\Pi_D=v^hL(1+r)+A(1+r^A)+\tilde{R}IBL(1+\rho)-vD(1+r^D)-e-\epsilon \tag{4.B.6}$$

For a lending bank, L, D, v, Π_U, Π_D and the Lagrangian multiplier $\mathcal{L}$ are endogenous, while all the other variables are exogenous. Eliminate the Lagrangian multiplier first, and take derivative w.r.t. p to the system, we have,

$$\left[1-2\gamma\Pi_U-p2\gamma\frac{\partial\Pi_U}{\partial p}\right](r-r^D)=-\left[1-2\gamma\Pi_D-(1-p)2\gamma\frac{\partial\Pi_D}{\partial p}\right](1+r^D-v^h(1+r))$$

$$-2\gamma\frac{\partial\Pi_D}{\partial p}=2\lambda(1+r^D)\left[-\frac{\partial v}{\partial p}D+(1-v)\frac{\partial D}{\partial p}\right]$$

$$\frac{\partial L}{\partial p}=\frac{\partial D}{\partial p}$$

$$\frac{\partial\Pi_U}{\partial p}=r\frac{\partial L}{\partial p}-r^D\frac{\partial D}{\partial p}$$

$$\frac{\partial\Pi_D}{\partial p}=\frac{\partial L}{\partial p}\left(v^h(1+r)-v(1+r^D)\right)-D(1+r^D)\frac{\partial v}{\partial p}$$

After some algebra, we have,

$$\frac{\partial L}{\partial p} = \frac{(1-2\gamma\Pi_U)(r-r^D) + (1-2\gamma\Pi_D)\left(1+r^D-v^h(1+r)\right)}{2p\gamma(r-r^D)^2 + 2(1-p)\frac{\gamma\lambda}{\gamma+\lambda}\left(1+r^D-v^h(1+r)\right)^2} \tag{4.B.7}$$

$$\frac{\partial \Pi_U}{\partial p} = (r-r^D)\frac{\partial L}{\partial p} \tag{4.B.8}$$

$$\frac{\partial \Pi_D}{\partial p} = -\frac{\gamma\lambda}{\gamma+\lambda}\left(1+r^D-v^h(1+r)\right)\frac{\partial L}{\partial p} \tag{4.B.9}$$

The denominator of equation (4.B.7) is obviously positive. For the numerator, $(1-2\gamma\Pi_U)$ and $(1-2\gamma\Pi_D)$ are marginal utility, thus should be positive in equilibrium. $r-r^D$ is the spread between the consumer loan rate and the deposit rate. As long as the consumer loan does not strictly dominate the deposit, which provides an arbitrage opportunity for the banks, the interest rates should satisfy $r-r^D>0$ and $1+r^D-v^h(1+r)>0$. Thus, we can have $\frac{\partial L}{\partial p}>0$, $\frac{\partial \Pi_U}{\partial p}>0$, and $\frac{\partial \Pi_D}{\partial p}<0$. The proof for the borrowing banks is similar. □

4.C Data

Table 4.7: Balance Sheet data of the UK banks

Barclays plc							
	31/12/2004	30/06/2005	31/12/2005	30/06/2006	31/12/2006	30/06/2007	31/12/2007
Loans to Banks	82404	184625	193404	207199	207424	233737	225031
Total Gross Loans	265097	239951	272342	285497	285631	324517	349167
Total Securities	166971	339495	359493	391867	399950	485961	541857
Total Deposits	329721	427138	437330	488286	475493	563172	387325
Deposits from Banks	112229	209423	144407	235086	161616	270728	92338
Total Equity	16445	21785	24430	25539	27390	28721	25240
The Royal Bank of Scotland PLC							
	31/12/2004	30/06/2005	31/12/2005	30/06/2006	31/12/2006	30/06/2007	31/12/2007
Loans to Banks	57369		66568		78536	87926	96346
Total Gross Loans	352314		422806		472435	504175	555684
Total Securities	115370		221561		248281	327327	419346
Total Deposits	383669		452729		516462	558099	594490
Deposits from Banks	99325		109889		131742	139084	151508
Total Equity	34999		34614		38332	39663	47835
HBOS plc							
	31/12/2004	30/06/2005	31/12/2005	30/06/2006	31/12/2006	30/06/2007	31/12/2007
Loans to Banks	21025	16133	18109	17730	12473	10075	9001
Total Gross Loans	289300	329434	346706	361631	379897	395210	430007
Total Securities	113251	142134	157058	165213	175383	192112	198158
Total Deposits	232889	227095	232989	241942	242414	264647	284734
Deposits from Banks	37395	30855	32041	33805	30557	37530	41513
Total Equity	17157	17249	17531	18026	20973	21881	22036
HSBC Bank plc							
	31/12/2004	30/06/2005	31/12/2005	30/06/2006	31/12/2006	30/06/2007	31/12/2007
Loans to Banks	36465		35173		47365		63198
Total Gross Loans	143888		184509		202178		229480
Total Securities	91857		145870		167372		298291
Total Deposits	194571		241941		268291		319030
Deposits from Banks	33502		36466		40941		50761
Total Equity	17208		19634		20938		23999
Lloyds Bank plc							
	31/12/2004	30/06/2005	31/12/2005	30/06/2006	31/12/2006	30/06/2007	31/12/2007
Loans to Banks	31848		31655		40638	33599	34845
Total Gross Loans	158721		178707		192328	202419	212222
Total Securities	43577		81385		92671	97377	86980
Total Deposits	160457		163535		177942	185610	196472
Deposits from Banks	40354		32185		37175	40744	39759
Total Equity	12783		11620		12476	12724	13439
Standard Chartered PLC							
	31/12/2004	30/06/2005	31/12/2005	30/06/2006	31/12/2006	30/06/2007	31/12/2007
Loans to Banks		11684	13333	9131	10730	10528	17651
Total Gross Loans		61278	66152	66125	72695	76793	77896
Total Securities		13750	31817	38300	37387	45026	53264
Total Deposits		72720	80588	89780	88443	93316	102635
Deposits from Banks		12073	10938	11990	13364	13390	12917
Total Equity		6965	7161	7549	8862	9768	6257

Source: FitchConnect, in Millions of GBP.

Table 4.8: Values of variables in the first-step model calibration

Equilibrium Model		
	Model Endogenous Variables	Model Exogenous Variables
Endogenously Solved	$\Pi_U^1 = 1.2746$ $M = -1.092$ $\Pi_D^1 = -1.8162$ $\tilde{R} = 0.8697$ $v^{1,h} = 0.7047$ $\Pi_U^2 = 0.8774$ $\Pi_D^2 = -0.8636$ $v^{2,h} = 0.8793$ $\Pi_U^3 = 1.7599$ $\Pi_D^3 = -2.6048$ $v^{3,h} = 0.7815$ $\Pi_U^4 = 2.3973$ $\Pi_D^4 = -1.092$ $v^{4,h} = 0.8697$ $\Pi_U^5 = 0.835$ $\Pi_D^5 = -0.9082$ $v^{5,h} = 0.8342$	$\lambda^1 = 0.1677$ $g_1^1 = 0.4372$ $\gamma^1 = 0.1473$ $g_1^2 = 4.722$ $\epsilon^1 = 10.5725$ $g_1^3 = 0.835$ $\lambda^2 = 0.7047$ $g_1^4 = -0.9082$ $\gamma^2 = 1.8911$ $g_1^5 = 0.8342$ $\epsilon^2 = 0.3267$ $u_{1,U} = -0.56$ $\lambda^3 = -0.8636$ $u_{1,D} = -0.3387$ $\gamma^3 = 0.8793$ $\epsilon^3 = 0.1421$ $\lambda^4 = 1.7599$ $\gamma^4 = -2.6048$ $\epsilon^4 = 0.3375$ $\lambda^5 = 20.7455$ $\gamma^5 = 2.3973$ $\epsilon^5 = 0.8697$
Estimated by CIMDO	$\Delta\Pi^1 = 3.0908$ $\Delta\Pi^4 = 3.4893$ $\Delta\Pi^2 = 1.741$ $\Delta\Pi^5 = 1.7432$ $\Delta\Pi^3 = 4.3647$	
Empirically Observed	$L^1 = 23.9951$ $L^4 = 39.0712$ $D^1 = 42.7138$ $D^4 = 29.9815$ $L^2 = 16.8714$ $L^5 = 16.4199$ $D^2 = 16.1996$ $D^5 = 21.8256$ $L^3 = 38.756$ $Y_U = 145.5644$ $D^3 = 41.8199$	$A^1 = 33.9495$ $A^4 = 15.5884$ $e^1 = 2.1785$ $e^4 = 2.4214$ $D^{I,1} = 2.4798$ $D^{I,4} = 1.5111$ $A^2 = 6.2481$ $A^5 = 11.8864$ $e^2 = 1.22015$ $e^5 = 1.8421$ $D^{I,2} = 0.4518$ $L^{I,5} = 0.0835$ $A^3 = 16.84655$ $\rho = 0.0454$ $e^3 = 3.48065$ $D^{I,3} = 4.26385$
Manually Selected	$v^1 = 0.9031$ $v^2 = 0.9734$ $v^3 = 0.9004$ $v^4 = 0.9387$ $v^5 = 0.947$ $Y_D = 131.008$	$r^{1,d} = 0.0454$ $g_2^1 = 0.01$ $r^1 = 0.082$ $g_2^2 = 0.01$ $r^{2,d} = 0.0454$ $g_2^3 = 0.01$ $r^2 = 0.082$ $g_2^4 = 0.01$ $r^{3,d} = 0.0454$ $g_2^5 = 0.01$ $r^3 = 0.082$ $g_3^1 = 0.01$ $r^{4,d} = 0.0454$ $g_3^2 = 0.01$ $r^4 = 0.082$ $g_3^3 = 0.01$ $r^{5,d} = 0.0454$ $g_3^4 = 0.01$ $r^5 = 0.082$ $g_3^5 = 0.01$ $r^A = 0.04$ $u_2 = 0.1$ $p = 0.95$

This table presents the values of the variables in the first-step model calibration, which covers the period from June 2005 to June 2006. The description of the variables are in Table 4.3.

Bank $1, 2, 3, 4, 5$ represents Barclays plc, Lloyds plc, RBS plc, the aggregated bank, and HSBC plc, respectively.

Negative value of $\Pi_D^b, b \in B$ indicates that the bank suffers from a loss.

Table 4.9: Values of variables in th second period, without Minsky effect

Equilibrium Model		
	Model Endogenous Variables	Model Exogenous Variables
Endogenously Solved	$L^1 = 23.4647$ $L^4 = 32.9579$ $D^1 = 43.7436$ $D^4 = 23.3545$ $v^1 = 0.9065$ $v^4 = 0.932$ $\Pi^1_U = 1.2998$ $\Pi^4_U = 2.4458$ $\Pi^1_D = -1.8067$ $\Pi^4_D = -0.6567$ $\Delta\Pi^1 = 3.1065$ $\Delta\Pi^4 = 3.1025$ $v^{1,h} = 0.6994$ $v^4 = 0.8631$ $L^2 = 12.9107$ $L^5 = 14.2459$ $D^2 = 11.4328$ $D^5 = 20.4231$ $v^2 = 0.9683$ $v^5 = 0.9485$ $\Pi^2_U = 0.9175$ $\Pi^5_U = 0.8686$ $\Pi^2_D = -0.5051$ $\Pi^5_D = -0.7179$ $\Delta\Pi^2 = 1.4226$ $\Delta\Pi^5 = 1.5865$ $v^{2,h} = 0.8726$ $v^5 = 0.8279$ $L^3 = 35.507$ $Y_U = 135.8639$ $D^3 = 39.2045$ $Y_D = 122.2775$ $v^3 = 0.9019$ $M = 9.1052$ $\Pi^3_U = 1.8552$ $\tilde{R} = 0.9092$ $\Pi^3_D = -2.2743$ $\Delta\Pi^3 = 4.1294$ $v^{3,h} = 0.7756$	
Empirically Observed		$A^1 = 39.1867$ $A^4 = 20.3513$ $e^1 = 2.5539$ $e^4 = 2.5575$ $D^{I,1} = 2.7887$ $D^{I,4} = 1.8934$ $\delta^1 = 13.5652$ $\delta^4 = 25.5038$ $A^2 = 8.7028$ $A^5 = 15.6621$ $e^2 = 1.2048$ $e^5 = 2.0286$ $L^{I,2} = 0.1467$ $L^{I,5} = 0.2566$ $\delta^2 = 9.1226$ $\delta^5 = 7.7128$ $A^3 = 23.4921$ $\rho = 0.0501$ $e^3 = 3.6473$ $L^{I,3} = 4.8264$ $\delta^3 = 11.321$
Manually Selected		$r^{d,1} = 0.0501$ $g^1_2 = 0.01$ $r^1 = 0.0879$ $g^2_2 = 0.01$ $r^{d,2} = 0.0501$ $g^3_2 = 0.01$ $r^2 = 0.0879$ $g^4_2 = 0.01$ $r^{d,3} = 0.0501$ $g^5_2 = 0.01$ $r^3 = 0.0879$ $g^1_3 = 0.01$ $r^{d,4} = 0.0501$ $g^2_3 = 0.01$ $r^4 = 0.0879$ $g^3_3 = 0.01$ $r^{d,5} = 0.0501$ $g^4_3 = 0.01$ $r^5 = 0.0879$ $g^5_3 = 0.01$ $r^A = 0.04$ $u_2 = 0.1$ $p = 0.95$
Calibrated		$\lambda^1 = 0.1677$ $g^1_1 = -0.5600$ $\gamma^1 = 0.1473$ $g^2_1 = -0.3387$ $\lambda^2 = 1.8911$ $g^3_1 = -0.4565$ $\gamma^2 = 0.4349$ $g^4_1 = -0.3496$ $\lambda^3 = 0.1814$ $g^5_1 = -0.3913$ $\gamma^3 = 0.1421$ $u_{1,U} = 3.3682$ $\lambda^4 = 0.3375$ $u_{1,D} = 3.2628$ $\gamma^4 = 0.1659$ $\lambda^5 = 0.7051$ $\gamma^5 = 0.389$

This table presents the values of the variables in the second period model simulation without Minsky effect, which covers the period from June 2006 to June 2007. The description of the variables are in Table 4.4.
Bank $1, 2, 3, 4, 5$ represents Barclays plc, Lloyds plc, RBS plc, the aggregated bank, and HSBC plc, respectively.
Negative value of $\Pi^b_D, b \in B$ indicates that the bank suffers from a loss.

Table 4.10: Values of variables in the second period, with Minsky effect

Equilibrium Model		
	Model Endogenous Variables	Model Exogenous Variables
Endogenously Solved	$L^1 = 27.9824$ $L^4 = 34.5094$ $D^1 = 48.2613$ $D^4 = 24.906$ $v^1 = 0.9054$ $v^4 = 0.938$ $\Pi^1_U = 1.4705$ $\Pi^4_U = 2.5044$ $\Pi^1_D = -2.3802$ $\Pi^4_D = -0.5379$ $v^{1,h} = 0.7069$ $v^4 = 0.8725$ $L^2 = 13.2112$ $L^5 = 25.083$ $D^2 = 11.7333$ $D^5 = 31.2603$ $v^2 = 0.9707$ $v^5 = 0.9514$ $\Pi^2_U = 0.9288$ $\Pi^5_U = 1.2783$ $\Pi^2_D = -0.4193$ $\Pi^5_D = -1.6039$ $v^{2,h} = 0.882$ $v^5 = 0.8368$ $L^3 = 41.8631$ $Y_U = 149.7852$ $D^3 = 45.5606$ $Y_D = 134.8067$ $v^3 = 0.9065$ $M = 9.1052$ $\Pi^3_U = 2.0954$ $\tilde{R} = 0.9125$ $\Pi^3_D = -2.7959$ $v^{3,h} = 0.784$	$p^1 = 0.9571$ $p^2 = 0.9445$ $p^3 = 0.9586$ $p^4 = 0.9491$ $p^5 = 0.9993$
Estimated by CIMDO	$\Delta\Pi^1 = 3.8507$ $\Delta\Pi^4 = 3.0423$ $\Delta\Pi^2 = 1.3481$ $\Delta\Pi^5 = 2.8821$ $\Delta\Pi^3 = 4.8913$	
Empirically Observed		$A^1 = 39.1867$ $A^4 = 20.3513$ $e^1 = 2.5539$ $e^4 = 2.5575$ $D^{I,1} = 2.7887$ $D^{I,4} = 1.8934$ $\delta^1 = 13.5652$ $\delta^4 = 25.5038$ $A^2 = 8.7028$ $A^5 = 15.6621$ $e^2 = 1.2048$ $e^5 = 2.0286$ $L^{I,2} = 0.1467$ $L^{I,5} = 0.2566$ $\delta^2 = 9.1226$ $\delta^5 = 7.7128$ $A^3 = 23.4921$ $\rho = 0.0501$ $e^3 = 3.6473$ $L^{I,3} = 4.8264$ $\delta^3 = 11.321$
Manually Selected		$r^{d,1} = 0.0501$ $g^1_2 = 0.01$ $r^1 = 0.0879$ $g^2_2 = 0.01$ $r^{d,2} = 0.0501$ $g^3_2 = 0.01$ $r^2 = 0.0879$ $g^4_2 = 0.01$ $r^{d,3} = 0.0501$ $g^5_2 = 0.01$ $r^3 = 0.0879$ $g^1_3 = 0.01$ $r^{d,4} = 0.0501$ $g^2_3 = 0.01$ $r^4 = 0.0879$ $g^3_3 = 0.01$ $r^{d,5} = 0.0501$ $g^4_3 = 0.01$ $r^5 = 0.0879$ $g^5_3 = 0.01$ $r^A = 0.04$ $u_2 = 0.1$ $p = 0.95$
Calibrated		$\lambda^1 = 0.1677$ $g^1_1 = -0.5600$ $\gamma^1 = 0.1473$ $g^2_1 = -0.3387$ $\lambda^2 = 1.8911$ $g^3_1 = -0.4565$ $\gamma^2 = 0.4349$ $g^4_1 = -0.3496$ $\lambda^3 = 0.1814$ $g^5_1 = -0.3913$ $\gamma^3 = 0.1421$ $u_{1,U} = 3.3682$ $\lambda^4 = 0.3375$ $u_{1,D} = 3.2628$ $\gamma^4 = 0.1659$ $\lambda^5 = 0.7051$ $\gamma^5 = 0.389$

This table presents the values of the variables in the second period model simulation with Minsky effect, which covers the period from June 2006 to June 2007. The description of the variables are in Table 4.6.
Bank $1, 2, 3, 4, 5$ represents Barclays plc, Lloyds plc, RBS plc, the aggregated bank, and HSBC plc, respectively.
Negative value of $\Pi^b_D, b \in B$ indicates that the bank suffers from a loss.

Chapter 5

Conclusions

This thesis consists of three papers on topics including default, collateral, credit cycles, financial stability and systemic risk. Chapter Two, "Belief disagreement, collateral and endogenous leverage", provides a novel approach within the general equilibrium paradigm to assess how belief heterogeneity affects equilibrium and contract origination under different model settings. The chapter establishes that changes in market *average* belief rather than *belief disagreement* determine the comovement of asset prices and leverage when there is no risk-free asset and the total supply of the underlying asset is fixed. Under mild technical conditions, asset prices and leverage could move in opposite directions. Optimism concerning the probability of downside states has no or little effect on asset prices, whereas optimism concerning the relative probability of upside states may have a significant impact on asset prices. Asset prices are above the fundamental value due to their usefulness as collateral. Extensions of the model considers different market structures. When a risk-free asset exists, the leverage decreases with belief disagreement. When the trading volume of the underlying asset

is endogenous, the leverage increases with belief disagreement.

Chapter Three, "Mortgages, financial intermediaries and optimism", introduces general equilibrium models with collateral and default to study the mortgage market, financial intermediation and belief updating. Collateral is required when borrowers cannot commit to repay. We consider the following mechanism to determine the equilibrium leverage and interest rate of a mortgage contract in a binomial structure. The market offers a menu of mortgage contracts, each requiring one unit of house as collateral, but different face values and consequently different interest rates. Borrowers choose from the menu by choosing the face values, knowing that different face values correspond to different required rate of return. Lenders take the menu and the choice as given. The model shows that, when collateral is scarce, the price of the underlying asset is higher than its fundamental value, and the difference represents the collateral value, which is the asset's borrowing capacity to borrow against. Default affects the economy in a non-trivial way, even in the simple binomial structure.

The multi-period extension of the model studies how the collateral constraints affect the credit cycle. When a negative shock happens to house buyer's endowment, the decreased house price triggers mortgage default, and less credit is available for new mortgage borrowers. Decreased demand and fire sale further decreases house price, and a downward spiral of credit and prices materializes. Leverage and house price move in the same direction through time, due to the change in the bank's balance sheet size. The co-movement of price and leverage leads to a more volatile collateral value of the house, since not only the underlying asset's value but also the fraction that

one can borrow against decrease when price and leverage decrease together. Default after a good state is more severe than that after a bad one, which resembles the main idea from Minsky's Financial Instability Hypothesis. The bank's belief updating amplifies the credit cycle, increasing the volatility of credit extension and default. The recent sub-prime crisis is obviously more complex than our model's description. Securitization and regulation related issues play major roles in the crystallization of the crisis. Nevertheless, our model provides a useful framework for future research on such topics.

Chapter Four, "An equilibrium framework for systemic risk: How economic theory meets statistics" has two major contributions. First, it provides a valid method to combine a equilibrium model with an empirical approach (CIMDO) to estimate the systemic risk and distress dependence structure in a financial system. Second, the paper highlights the importance of the Minsky effect: when the banks become more optimistic altogether, they increase the risk-taking of the financial system, and amplifies the systemic risk. We modify and extend the equilibrium model from Goodhart et al. (2005) of a banking system with the interbank lending and common asset exposure channel for systemic risk, and prove that, under our assumptions, the volatility in the banks' profits/losses increases when they become more optimistic, which is a typical result of the Minsky effect. Then we take the data from UK banking system form 2005 to 2007, and applied the CIMDO method to estimate the risk-related coefficients and the Minsky effect coefficients for the banks using a two-step calibration approach. Our results show that the Minsky effect accounts for a significant part of the distress dependence structure.

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