

The influence of dimension on the relaxation process of East-like models

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Abstract – We consider the relaxation process and the out-of-equilibrium dynamics of natural generalizations to arbitrary dimensions of the well known one dimensional East process. These facilitated models are supposed to catch some of the main features of the complex dynamics of fragile glasses. We focus on the low temperature regime (small density $c \approx e^{-\beta}$ of the facilitating sites). In the literature the relaxation process for the above models has been assumed to be *quasi-one dimensional* and, in particular, their equilibration time has been computed using the relaxation time of the East model ($d = 1$) on the equilibrium length scale $L_c = (1/c)^{1/d}$ in d -dimension. This led to the derivation of a super-Arrhenius scaling for the relaxation time of the form $T_{\text{rel}} \asymp \exp(\beta^2/d \log 2)$. In a companion paper, using mainly renormalization group ideas and electrical networks methods, we rigorously establish that instead $T_{\text{rel}} \asymp \exp(\beta^2/2d \log 2)$, a result showing that the relaxation process cannot be quasi-one-dimensional. The above scaling sharply confirms previous MCAMC (Monte Carlo with Absorbing Markov Chains) simulations. Next we compute the relaxation time at finite and mesoscopic length scales, and show a dramatic dependence on the boundary conditions, yet another indication of key dimensional effects. Our final result is related to the out-of-equilibrium dynamics. Starting with a single facilitating site at the origin we show that, up to length scales $L = O(L_c)$, its influence propagates much faster (on a logarithmic scale) along the diagonal direction than along the axes directions. Such unexpected result is due to a rather delicate balance between dynamical energy barriers and entropic effects in the constrained dynamics.

Introduction and main results. – Kinetically constrained spins models [2, 21, 26] are stochastic particle/spin models, usually defined in terms of a non-interacting Hamiltonian, whose dynamics is determined by local rules encoding a kinetic constraint. The main interest for these models (see e.g. [5, 17, 18, 20, 26–28]) stems from the fact that KCMs, in spite of their simplicity, display many key dynamical features of glass forming supercooled liquids: rapidly diverging relaxation times as the temperature drops, super-Arrhenius behavior, dynamic heterogeneity (*i.e.* non-trivial spatio-temporal fluctuations of the local relaxation to equilibrium) and aging, just to mention a few.

One of the simplest models, showing all the above features and yet being tractable even at a rigorous level, is the East model introduced in [21] and further analysed in [4, 6, 9–11, 14–16, 27, 28]. It is defined on a one dimensional integer lattice of L sites, each of which can be in state (or spin) 0 or 1, corresponding to *empty* or *occupied* respectively. The spin configuration η evolves under Glauber type dynamics in the presence of the kinetic constraint which forbids flips of those spins whose left neighbor has spin one. Each vertex x waits an independent mean one exponential time and then, provided that the current configuration η satisfies the constraint $\eta_{x-1} = 0$, the value of η_x is refreshed and set equal to 1 with probability $1 - c$ and

to 0 with probability c . The leftmost vertex $x = 1$ is unconstrained, equivalently we could think of a fixed 0 at the boundary $x = 0$. Since the constraint at site x does not depend on the spin at x , detailed balance is satisfied with respect to the product probability measure $\pi := \prod_x \pi_x$, where π_x is the Bernoulli probability measure with density c for the facilitating sites (the vacancies). Taking c equal to $e^{-\beta}/(1 + e^{-\beta})$, where β is the inverse temperature, then π is the Gibbs distribution associated to the non-interacting Hamiltonian given by the total number of vacancies. In particular, small values of c correspond to low temperatures.

Recently the problem of dynamic heterogeneities and time-scales separation in the East model was rigorously analyzed in [9,10]¹, where the picture obtained in previous non rigorous work was corrected and extended. Dynamical heterogeneity is strongly associated to a *broad spectrum* of relaxation time scales which emerges as the result of a subtle *energy-entropy* competition. Isolated vacancies with a string of N particles to their left, cannot be updated unless the system injects enough additional vacancies in a cooperative way in order to unblock the target one. Finding the correct time scale on which this unblocking process occurs requires a highly non-trivial analysis to correctly measure the energy contribution (how many extra vacancies are needed) and the entropic one (in how many ways the unblocking process may occur). The final outcome is a very non trivial dependence of the corresponding characteristic time scale on c and N . In particular it can be shown that the correct asymptotic of the relaxation time $T_{\text{rel}}^{\text{East}}(L; c)$ as $c \searrow 0$ takes the form ($n = \lceil \log_2 L \rceil$ and we also allow $L = +\infty$)

$$T_{\text{rel}}^{\text{East}}(L; c) \asymp \begin{cases} e^{\beta n} \times n! 2^{-\binom{n}{2}}, & L \leq 1/c, \\ e^{\beta^2/2 \log 2} & L \geq 1/c. \end{cases} \quad (1)$$

In the above formula $1/c$ is the mean inter-vacancy distance and n represents the minimal energy barrier between the ground state and the set of configurations with a vacancy at L . The above result came out as a surprise since the relaxation time is determined not only by the “standard” energy barrier contribution $1/c^n$ (cf. [11, 27, 28]), but rather by the interplay between energy and entropy (the latter being encoded in the new term $n! 2^{-\binom{n}{2}}$), a fact that was overlooked in previous work.

In several interesting contributions [13, 17, 22–24] a natural generalization of the East dynamics to higher dimensions $d > 1$, in the sequel referred to as the *East-like* process, appears to play a key role in *realistic* models of glass formers. In $d = 2$ the East-like process evolves similarly to the East process but now the kinetic constraint requires that the South or West neighbor of the updating vertex contains *at least one* vacancy. In general η_x can flip if $\eta_{x-\vec{e}} = 0$ for some \vec{e} in the canonical basis of \mathbb{Z}^d .

¹We warn the reader that in [8–10] the density c of the facilitating sites is denoted by q .

Infinite volume relaxation time. A simple comparison with the East model shows that for any value of the vacancy density c the relaxation time of the East-like model on \mathbb{Z}^d , in the sequel $T_{\text{rel}}(\mathbb{Z}^d; c)$, is finite. For example, in two dimensions the East-like model is in fact *less* constrained than a infinite array of East models, one for each line parallel to the first coordinate axis, and this immediately gives rise to the conclusion. More subtle is the problem of computing the asymptotic of $T_{\text{rel}}(\mathbb{Z}^d; c)$ at low temperature, a key intermediate step towards a qualitative and quantitative description of dynamic heterogeneities in dimension greater than one.

In [17] it was assumed that the relaxation process of the low temperature East-like models is *quasi-one-dimensional*, i.e. it is determined by that of the East model on the equilibrium scale² in d -dimension $L_c = (1/c)^{1/d}$. In particular, it was argued that the relaxation time of the East-like model scales like $T_{\text{rel}}^{\text{East}}(L_c; c)$. If one neglects the entropic contribution in (1), as it was done in [17], the above assumption leads to a super-Arrhenius law of the form (recall that $\beta \simeq \log(1/c)$)

$$T_{\text{rel}}(\mathbb{Z}^d; c) \asymp e^{\beta^2/d \log 2}. \quad (2)$$

Using the correct form (1) for $T_{\text{rel}}^{\text{East}}(L_c; c)$ gives instead

$$T_{\text{rel}}(\mathbb{Z}^d; c) \asymp e^{\beta^2(\frac{2d-1}{2d^2 \log 2})}.$$

It turns out that both results are wrong, because of important dimensional effects in the relaxation process of East-like models. Our first main result in [8] shows in fact that:

Theorem 1 *As $c \searrow 0$*

$$T_{\text{rel}}(\mathbb{Z}^d; c) = e^{\frac{\beta^2}{2d \log 2}(1+o(1))} \quad (3)$$

and the $o(1)$ correction³ is $\Omega(\beta^{-1} \log \beta)$ and $O(\beta^{-1/2})$. The above result can also be read as

$$T_{\text{rel}}(\mathbb{Z}^d; c) = T_{\text{rel}}^{\text{East}}(\mathbb{Z}; c)^{\frac{1}{d}(1+o(1))}.$$

We discuss some interesting aspects of the derivation of the above result later. Notice that if we write the coefficient of β^2 as b/d , then $b = 1/(2 \log 2) \approx 0.721$, a result that confirms the value $b \approx 0.8$ found in simulations [3, Fig.3] based on the “Monte Carlo with Absorbing Markov Chains” method [25]. In finite or infinite subsets of the lattice \mathbb{Z}^d the dimensional effects behind (3) depend strongly on the boundary conditions as explained below.

Finite volume relaxation time. Consider a finite box with L vertices on each edge in the positive quadrant $\mathbb{Z}_+^d = \{(x_1, \dots, x_d) \in \mathbb{Z}^d : x_i \geq 0\}$ and containing the origin. In order to ensure ergodicity of the dynamics at

²As equilibrium scale one can take the typical inter-vacancy distance under π .

³Recall that $f = O(g)$, $f = o(1)$ and $f = \Omega(g)$ mean that $|f| \leq c|g|$ for some constant c , $f \rightarrow 0$ and $\limsup |f|/|g| > 0$ respectively.

least the spin at the origin must be unconstrained, *i.e.* it should be free to flip. If this is the only unconstrained spin we say that we have *minimal* boundary conditions and we denote the associated relaxation time by $T_{\text{rel}}^{\text{min}}(L; c)$. *Maximal* boundary conditions correspond instead to the case in which all the spins in the box which belong to the coordinate hyperplanes $x_i = 0$ are unconstrained. In this case we write $T_{\text{rel}}^{\text{max}}(L; c)$ for the associated relaxation time. Our second main result in [8] pins down the asymptotics of the above relaxation times:

Theorem 2 *Take L depending on c such that $\lim_{c \searrow 0} L = +\infty$. Then, as $c \searrow 0$,*

$$T_{\text{rel}}^{\text{max}}(L; c) = \begin{cases} e^{(n\beta - d\binom{n}{2} \log 2)(1+o(1))}, & n \leq \frac{\beta}{d \log 2}, \\ e^{\frac{\beta^2}{2d \log 2}(1+o(1))} & \text{otherwise.} \end{cases} \quad (4)$$

$$T_{\text{rel}}^{\text{min}}(L; c) = \begin{cases} e^{n\beta - \binom{n}{2} \log 2 + n \log n + O(\beta)}, & n \leq \frac{\beta}{\log 2}, \\ e^{\frac{\beta^2}{2 \log 2} + \frac{\beta \log \beta}{\log 2} + O(\beta)} & \text{otherwise.} \end{cases} \quad (5)$$

For L fixed independent of c the energy barrier dominates and the entropic reduction does not contribute to the leading order. In particular for L independent of c as $c \searrow 0$,

$$T_{\text{rel}}^{\text{min}}(L; c) = e^{n\beta + O_n(1)},$$

where⁴ $n = \lceil \log_2(d(L-1) + 1) \rceil$.

Notice that with minimal boundary conditions the relaxation time scales exactly as the relaxation time of the East model given in (1). The slowest mode in this case occurs along the coordinate axes.

Persistence times. We conclude with a last result which highlights once again some non-trivial dimensional effects in the *out-of-equilibrium* East-like dynamics. Consider the model in the positive quadrant \mathbb{Z}_+^d with minimal boundary condition (only the spin at the origin is unconstrained) and starting from the configuration with no vacancies. Let $T(x; c)$ be the *persistence* time of $x \in \mathbb{Z}_+^d$, namely the mean time it takes to create a vacancy at x . The knowledge of the collection $T(x; c)$ as x varies in \mathbb{Z}_+^d gives some insight on how a wave of vacancies originating from a single one spreads out in space-time. Their analysis is therefore a key step in order to understand the more complex phenomenon of time scale separation and dynamic heterogeneities in a high-dimensional setting. In the last theorem we compute the low temperature scaling of $T(x; c)$ for sites x which either belong to the diagonal of \mathbb{Z}_+^d or to one of the coordinate axes. Although the original result in [8, Theorem 3] covers quite precisely *all* scales up to the equilibrium scale L_c , in order to outline our main finding we describe it only for $|x| \approx L_c$, $L_c = (1/c)^{1/d}$, and $|x| = O(1)$ and we only give the leading term in β in the asymptotics without specifying the error.

Theorem 3 *Let $v_* = (dL_c, 0, \dots, 0)$ and $v^* = (L_c, L_c, \dots, L_c)$ so that v_* belongs to the first coordinate axis and v^* to the diagonal. Then, as $c \searrow 0$, the persistence time satisfies*

$$T(v_*; c) \asymp T_{\text{rel}}^{\text{East}}(L_c; c) \asymp T_{\text{rel}}^{\text{East}}(\mathbb{Z}; c)^{(2d-1)/d^2}, \quad (6)$$

$$T(v^*; c) \asymp T_{\text{rel}}(\mathbb{Z}^d; c) \asymp T_{\text{rel}}^{\text{East}}(\mathbb{Z}; c)^{1/d}. \quad (7)$$

Fix $n \in \mathbb{N}$ and let $x \in \mathbb{Z}_+^d$ be such that⁵ $\|x\|_1 + 1 \in [2^{n-1}, 2^n)$. Then,

$$T(x; c) = e^{n\beta + O_n(1)}. \quad (8)$$

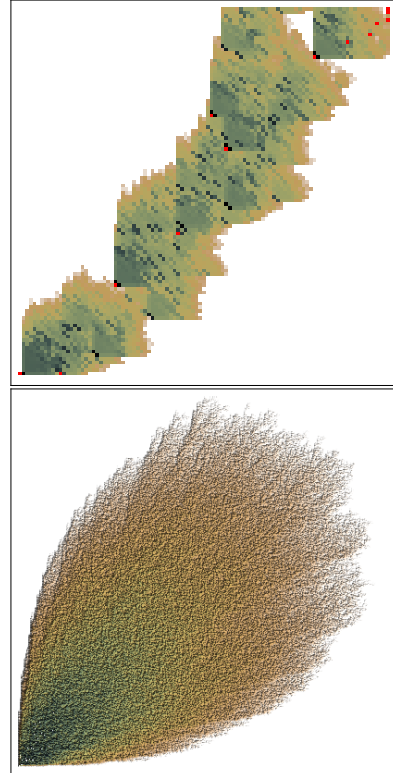


Fig. 1: A snapshot of the East-like process with minimal boundary conditions and initial condition with no vacancies. White vertices have never been updated, the darker the color the more times the site has refreshed. (TOP) $c = 0.002$, snapshot taken at the first time a vacancy is present at the top-right corner $(100, 100)$ ($t \simeq 2.5 \times 10^{12}$). Red dots denote the vacancies present in the snapshot. (BOTTOM) $c = 0.25$, $L = 400$ and $t = 9 \times 10^3$.

We conclude this first part with two observations. Notice that we (intentionally) always compare the persistence time of vertices with equal ℓ_1 -norm. If in fact the relaxation process for the East-like model was quasi-one dimensional, then the asymptotics of $T(x; c)$ should be determined by the minimal length of a path connecting the origin to x , at least at the logarithmic level. When the distance of x from the initially unconstrained spin at

⁴The notation $O_n(1)$ means that the constant may depend on n .

⁵The ℓ_1 -norm of $x = (x_1, \dots, x_d)$ is given by $\|x\|_1 := \sum_{i=1}^d |x_i|$.

the origin is $O(1)$, that is indeed correct as shown by (8). Instead, when the distance of x from the origin is large (diverging as $c \searrow 0$), the ℓ_1 -norm is no longer sufficient and the direction matters dramatically. One could wonder whether another norm would be more relevant at this scale. Indeed v^* has much smaller Euclidean norm than v_* ($\sqrt{d}L_c$ compared to dL_c) and that might be responsible for the difference in the corresponding persistence times. However, the full result of [8, Theorem 3] shows that this is not the case and that, for any $\lambda, \lambda' > 0$ independent of c , $T(\lambda v_*; c)$ is (logarithmically) much shorter than $T(\lambda' v^*; c)$ as $c \searrow 0$. Of course, if λ, λ' depend on c the situation may change completely. Theorem 3 in [8] indeed implies for example that $T(v^*; c) \asymp T(c^\alpha v_*; c)$ if $\alpha = 1/d + \sqrt{1 - 1/d} - 1$. Lastly, in case (1) we observe that the persistence time of the vertex v^* is much shorter (on a logarithmic scale) than the relaxation time of a finite box of side dL_c and minimal boundary conditions (cf. Theorem (2)). In this case, contrary to what happens for the East model, the system is able to bring a vacancy very far from the initial one *without* reaching equilibrium in the corresponding box (see Figure 1).

Outline of the methods. – We use several techniques to derive the above results. As a preliminary step we exploit the monotonicity in the constraints of the quadratic form associated to the generator of the master equation to derive the inequalities

$$T_{\text{rel}}(\mathbb{Z}^d; c) \leq T_{\text{rel}}^{\text{East}}(\mathbb{Z}; c), \quad (9)$$

$$T_{\text{rel}}^{\text{max}}(L; c) \leq T_{\text{rel}}^{\text{min}}(L; c), \quad T_{\text{rel}}^{\text{East}}(L; c) \leq T_{\text{rel}}^{\text{min}}(L; c), \quad (10)$$

and the fact that $T_{\text{rel}}^{\text{max}}(L; c)$, $T_{\text{rel}}^{\text{min}}(L; c)$ are both increasing in L . It is worth noticing that (10) together with (1) leads to the correct lower bound asymptotics in (5). Next we use [6, Prop. 2.13] together with the monotonicity in L of $T_{\text{rel}}^{\text{max}}(L; c)$ to get that⁶ $T_{\text{rel}}(\mathbb{Z}^d; q) = \lim_{L \rightarrow \infty} T_{\text{rel}}^{\text{max}}(L; q)$. Unfortunately no monotonicity argument is available for the persistence times since they are not characterized uniquely by a variational principle involving the quadratic form of the generator. The proof of our results then rely on four main ingredients:

- comparison with the East model on a spanning tree to *upper bound* $T_{\text{rel}}^{\text{min}}(L; c)$;
- a renormalization group approach, involving a sequence of coarse-grained auxiliary dynamics, to *upper bound* $T_{\text{rel}}(\mathbb{Z}^d; c)$ and $T_{\text{rel}}^{\text{max}}(L; c)$;
- an algorithmic construction of an efficient bottleneck in the configuration space to *lower bound* $T_{\text{rel}}(\mathbb{Z}^d; c)$, $T_{\text{rel}}^{\text{max}}(L; c)$ and the persistence time $T(x; c)$;
- resistors network techniques to *upper bound* the persistence times.

East model on a spanning tree. Choose a rooted, oriented spanning tree \mathcal{T} for the box $\Lambda = [0, L - 1]^d$ with

⁶Notice that the infinite volume relaxation time is *not* given by the large system limit with minimal boundary conditions (cf. (3) and (5)).

root at the origin and edges oriented as the canonical basis of \mathbb{Z}^d . On \mathcal{T} we consider a constrained East dynamics in which the spin at x is free to flip if either x is the root or there is a vacancy at the \mathcal{T} -ancestor of x . This new dynamics is more constrained than the East-like dynamics in Λ with minimal boundary conditions. Hence $T_{\text{rel}}^{\text{min}}(L; c) \leq T_{\text{rel}}(\mathcal{T}; c)$, where $T_{\text{rel}}(\mathcal{T}; c)$ denotes the relaxation time of the \mathcal{T} -chain. In turn, as shown in [7, Th. 6.1 and eq. (6.3)], $T_{\text{rel}}^{\text{min}}(\mathcal{T}; c)$ is smaller than the relaxation time of East process on the longest branch of \mathcal{T} . To upper bound the latter it is enough to apply (1).

Block dynamics and renormalization group. Given $\ell = 2^n$ and $x \in \mathbb{Z}^d$ let $\Lambda(x, \ell) := \ell x + [0, \ell - 1]^d$. We partition the lattice \mathbb{Z}^d into (disjoint) blocks $\Lambda(x, \ell)$ and introduce an auxiliary constrained *block dynamics* on $\{0, 1\}^{\mathbb{Z}^d}$ which mimics the East-like dynamics on a coarse-grained scale. In each block $\Lambda(x, \ell)$ with rate one the spin configuration inside the block is refreshed to a new one sampled from the equilibrium distribution π provided that at least one neighboring block $\Lambda(x - \vec{e}, \ell)$, \vec{e} being a vector in the canonical basis of \mathbb{Z}^d , contains a vacancy. Note that the block dynamics reduces to the East-like process when $\ell = 1$. Writing $c_* \simeq e^{-\beta_*}$ for the probability $1 - (1 - c)^{\ell^d}$ that the block constraint is satisfied at x , it is simple to verify that the relaxation time $T_{\text{rel}}^{\text{block}}(\mathbb{Z}^d; c)$ of the block dynamics coincides with $T_{\text{rel}}(\mathbb{Z}^d; c_*)$. The idea at this point is to search for an explicit rescaling function $F(\ell; c)$ such that

$$T_{\text{rel}}(\mathbb{Z}^d; c) \simeq F(\ell, c) T_{\text{rel}}^{\text{block}}(\mathbb{Z}^d; c) = F(\ell, c) T_{\text{rel}}(\mathbb{Z}^d; c_*).$$

Once such an equation is available then, by choosing appropriately the block side ℓ as a function of the density c , one could hope to solve for $T_{\text{rel}}(\mathbb{Z}^d; c)$.

Clearly the sought function $F(\ell, c)$ should (roughly) be the time scale over which a single vacancy in one of the blocks $\Lambda(x - \vec{e}, \ell)$, whose presence is guaranteed by the block constraint, is able to induce equilibrium inside the block $\Lambda(x, \ell)$. Although we could not implement exactly the above program, we rigorously show in [8] that

$$T_{\text{rel}}(\mathbb{Z}^d; c) \leq \kappa T_{\text{rel}}^{\text{min}}(3\ell; c) T_{\text{rel}}(\mathbb{Z}^d; c_*) \quad (11)$$

for some constant κ depending only on d . Using (5) of Theorem 2 we know that

$$T_{\text{rel}}^{\text{min}}(3\ell; c) \leq e^{(n\beta - \frac{\log 2}{2} n^2)(1+o(1))} \quad (12)$$

for any $\ell \leq L_c$ large enough. Suppose now that

$$T_{\text{rel}}(\mathbb{Z}^d; c) \leq e^{\lambda \frac{\beta^2}{2 \log 2} (1+o(1))} \quad (13)$$

for some $\lambda \in (1/d, 1]$. Using (9) and the exact asymptotics of $T_{\text{rel}}^{\text{East}}(\mathbb{Z}; c)$ (cf. (1)) we indeed know that (13) holds for $\lambda = 1$. Then we plug (12) and (13)⁷ into the r.h.s. of (11)

⁷With c replaced by c_* in (13) and therefore with β replaced by $\beta_* \simeq \beta - d n \log 2$ for n large and c small.

and optimize over the free scale $\ell \leq L_c$. The conclusion is that (13) is actually valid for a smaller constant

$$\lambda' = \frac{2d\lambda - 1 - \lambda}{d^2\lambda - 1} \in (1/d, 1].$$

It is easy to verify that the map $\lambda \mapsto \lambda'$, $\lambda \in [1/d, 1]$, has an attractive quadratic fixed point at $\lambda_c = 1/d$. Thus, starting from $\lambda = 1$ and by iterating a large number of times the above recursion we get (13) with $\lambda = 1/d$.

We point out that in the rigorous derivation of (11) we were forced to use a slightly different block dynamics. Indeed, the presence of vacancies only inside some box $\Lambda(x - \vec{e}, \ell)$ is, in general, not enough to guarantee equilibration of the East-like process inside the block $\Lambda(x, \ell)$. Therefore, in some sense the East-like process and the above block dynamics are not directly comparable. To overcome this problem the block constraint for $\Lambda(x, \ell)$ was modified to require the presence of some vacancy in (at least) one block of the form $\Lambda(x - (1, 1, \dots, 1) - \vec{e}, \ell)$. In e.g. $d = 2$ we thus ask for some vacancy in $\Lambda(x - (2, 1), \ell)$ or in $\Lambda(x - (1, 2), \ell)$. In analogy with the chess piece, we call the resulting dynamics the *Knight Chain*. It is easy to check that the Knight Chain consists of $d + 1$ independent chains, each one isomorphic to the previously defined block dynamics (see Figure 2). In particular, the infinite volume relaxation time of the Knight Chain coincides with $T_{\text{rel}}^{\text{block}}(\mathbb{Z}^d; c) = T_{\text{rel}}(\mathbb{Z}^d; c_*)$.

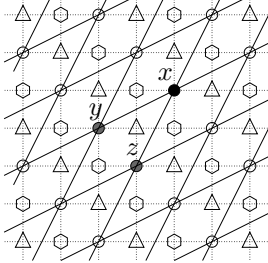


Fig. 2: Partition of \mathbb{Z}^2 in three sub-lattices distinguished by different geometric shapes. For circles, also the edges of the sub-lattice have been drawn. In the Knight Chain, the block $\Lambda(x, \ell)$ associated to x only looks at the blocks $\Lambda(y, \ell)$, $\Lambda(z, \ell)$, with y, z in the same sub-lattice of x .

Bottleneck inequality. To get a lower bound on $T_{\text{rel}}^{\text{max}}(L; c)$ we use the Rayleigh-Ritz variational principle for the spectral gap of the generator of the master equation. Restricting to test functions of the form $f(\eta) = 1$ if $\eta \in A$ and $f(\eta) = 0$ otherwise, for some subset A in the spin configuration space, one easily gets

$$T_{\text{rel}}^{\text{max}}(L) \geq L^{-d} \frac{\pi(A)\pi(A^c)}{\pi(\partial A)}, \quad (14)$$

where the boundary ∂A is given by the configurations η in A such that the East-like process can jump from η to A^c . When $\pi(A) \leq 1/2$ and $\pi(\partial A) \ll \pi(A)$, one says that ∂A is a bottleneck for the dynamics. The problem is then to find a set A leading to the correct asymptotics in (4).

We provide an algorithmic construction of such a set A . Given a configuration η on $[1, L]^d$ with a vacancy at $x = (x_1, x_2, \dots, x_d)$, we define the gap of this vacancy as the minimal ℓ_1 -distance between x and $y \in \prod_{i=1}^d [1, x_i]$ such that there is a vacancy at y (in η or in the maximal boundary conditions). The algorithm proceeds as follows. Starting from a spin configuration η remove from η all vacancies with gap equal to 1, then remove from the resulting configuration all vacancies with gap equal to 2, and so on until all vacancies with gap equal to $L - 1$ have been removed. At this point the algorithm stops. Due to the maximal boundary conditions there are only two possible final spin configurations: the fully occupied one and **10**, the configuration with a single vacancy at (L, L, \dots, L) . Our set A is then given by those spin configurations η for which the algorithm outputs **10**. The estimate of the ratio $\pi(A)\pi(A^c)/\pi(\partial A)$ is not trivial and is based on the fact that configurations in ∂A must have at least $\lceil \log_2 L \rceil$ vacancies whose locations have special geometric properties.

Resistor network. We use a standard analogy between electrical networks and Markov processes satisfying the detailed balance condition [12]. In particular, we map the East-like process on the box $[1, L]^d$, with minimal boundary conditions, to a resistor network whose nodes are given by the configurations η on $[1, L]^d$. Calling $\mathcal{K}(\eta, \eta')$ the probability rate for a jump from η to η' , to each unordered pair η, η' with $\mathcal{K}(\eta, \eta') > 0$ we attach a resistor with conductance $\mathcal{C}(\eta, \eta')$ given by

$$\mathcal{C}(\eta, \eta') = \pi(\eta)\mathcal{K}(\eta, \eta') = \pi(\eta')\mathcal{K}(\eta', \eta). \quad (15)$$

Given $x \in [1, L]^d$ we set $B_x := \{\eta : \eta_x = 0\}$ and we let **1** denote the configuration with no vacancy. We write $R(x)$ for the effective resistance between **1** and B_x , given by the inverse current intensity when putting potential one on **1** and zero on B_x . Then the equilibrium potential at η equals the probability $\mathbb{P}_\eta(\tau_1 < \tau_{B_x})$ for the East-like process starting from η to hit **1** before B_x and the persistence time $T(x; c)$ satisfies

$$T(x; c) = R(x) \sum_{\eta \notin B_x} \pi(\eta) \mathbb{P}_\eta(\tau_1 < \tau_{B_x}). \quad (16)$$

The above identity implies that $T(x; c) \asymp R(x)$ if $L \leq L_c$. To bound $R(x)$ above we use Thomson's principle which states that $R(x)$ is the minimal energy $\mathcal{E}(\theta) = \frac{1}{2} \sum_{\eta, \eta'} \theta(\eta, \eta')^2 / \mathcal{C}(\eta, \eta')$ over unit flows θ from **1** to B_x . A unit flow θ is an antisymmetric function on the set of ordered pairs (η, η') with $\mathcal{K}(\eta, \eta') > 0$, which is divergence free on $(\mathbf{1} \cup B_x)^c$ and such that the flow exiting **1** is unitary. Thomson's principle reduces the problem to finding a unit flow whose energy has the expected asymptotic.

To construct such a flow we use a hierarchical procedure which, for $L = O(1)$ and $d = 1$, resembles the hierarchical method used to estimate the energy barrier that must be overcome in order to bring a vacancy at distance L [11, 28].

First we introduce a unit flow θ_y from **1** to B_x (passing through a point y around $x/2$) in three steps. Consider the

equilibrium unit flow ϕ_y from $\mathbf{1}$ to B_y which has energy $R(y)$. Consider then the reversed flow $\hat{\phi}_y$ keeping the vacancy at y fixed and removing all other vacancies, defined as $\hat{\phi}_y(\eta, \eta') = \phi_y(\eta'^y, \eta^y)$ if $\eta, \eta' \in B_y$ and zero otherwise (σ^y is obtained from σ by a single spin-flip at y). Note that $\phi_y + \hat{\phi}_y$ is a unit flow from $\mathbf{1}$ to the configuration 10_y with a single vacancy at y . Finally, let $\tilde{\phi}_y$ be the unit flow from 10_y to B_x which mimics on $\Delta = [y+1, x] \times [y, x]^{d-1}$ the equilibrium unit flow from the filled configuration on Δ (no vacancy) to the set of configurations on Δ with a vacancy at x . Finally we define the flow $\theta_y = \phi_y + \hat{\phi}_y + \tilde{\phi}_y$, which is indeed a unit flow from $\mathbf{1}$ to B_x . Since $\phi_y, \hat{\phi}_y, \tilde{\phi}_y$ have disjoint supports, the energy of θ_y is the sum of the individual energy, which is related to $R(y)$. At the end, at least for $y \approx x/2$, we get

$$R(x) \leq \mathcal{E}(\theta_y) \leq R(y) + \frac{2}{c}R(y) \quad (17)$$

(the factor $1/c$ is due to the vacancy at y).

The above recursive inequality (17) is indeed not sufficient to get the correct asymptotic for the upper bound of $R(x)$, since it disregards relevant entropic effects. In fact, to create a vacancy at x , the system can firstly bring a zero to any point close to the midpoint of x , potentially at the cost of a small number of extra vacancies, which gives rise to many more paths. We therefore replace θ_y with a local average $N^{-1} \sum_{y \in V_x} \theta_y$, where V_x is a box around $x/2$ with N points. Then, instead of (17), we get (see [8, Lemma 7.1])

$$R(x) \leq \frac{C}{N} \sum_{y \in V_x} R(y) + \frac{C}{cN^2} \sum_{y \in V_x} R(y) \quad (18)$$

for some universal constant C . It then remains to find a good choice of the box V_x : increasing the size N of the box V_x accounts for more entropy, on the other hand if V_x is too large then some of the points y become close to the coordinate axes, thus leading to a very large resistance $R(y)$ since there is a smaller entropy ($R(y)$ would approach the very large 1D persistence time $T(y; c)$). Applying the above bound recursively and choosing the box V_x carefully we arrive at the upper bound on the hitting time stated in Theorem 3.

* * *

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