

# Rational Cherednik Algebras and Link Invariants



Emanuele Ghedin  
Balliol College  
University of Oxford

A thesis submitted for the degree of  
*Doctor of Philosophy*  
Trinity 2015

*A mamma e papà*

## Acknowledgements

First and foremost, I would like to thank my supervisor Kevin McGerty for his enormous help. This thesis would not have been possible without his expert and patient guidance. I am grateful to Tony Licata for suggesting this research problem and providing useful advice. I have also much benefited from helpful conversations with Alberto Cazzaniga, Iordan Ganev, Iain Gordon, Justin Hilburn, Kobi Kremnizer, Jacob Matherne, Peng Shan, Balázs Szendroi and Ben Webster. I thank my friends in Oxford, especially Albe, Jack and Marta, for making this experience so enjoyable. Finally, thanks to Chiara for her unwavering support and to my family for always giving without ever asking.

# Abstract

Motivated by homological mirror symmetry, Smith and Thomas tried to construct a link invariant considering the derived category of coherent sheaves on the Hilbert scheme of  $n$  points on the minimal resolution of the Klenian singularity of type  $\mathbf{A}$ , and an object  $\mathcal{L}^{(n)}$  thereof. The braid group acts on this category by spherical twists, so one obtains a braid invariant by taking the Ext between  $\mathcal{L}^{(n)}$  and its image under the braid group action. Smith and Thomas proved that taking the plat closure of the braid, this cohomology does not produce a link invariant but is close to doing so, and they conjectured that, in order to fix the one knot relation that is not satisfied, one has to consider a deformation of the Hilbert scheme.

In this thesis, we give a non-commutative approach to this problem: the commutative picture can be quantised by considering modules for the rational Cherednik algebra of cyclotomic type. This algebra gives a quantisation of the Hilbert scheme and there is a localisation theorem which allows one to work in the algebraic setting. In this context, the role of  $\mathcal{L}^{(n)}$  turns out to be played by a certain module for the rational Cherednik algebra which we define for  $k = 0$ . We then show that this module deforms to non-zero values of  $k$ . There is an action of the braid group on the derived category of category  $\mathcal{O}$  by twisting functors, which is defined at all deformation parameters, whereas the existence of the action on deformed Hilbert schemes in the commutative setting has not been rigorously established. We prove the analogue of the Smith-Thomas theorem, and conjecture that the braid invariant given by the algebraic analogue of the Smith-Thomas construction yields a link invariant for certain non-zero values of the deformation parameter.

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# Introduction

In [SS], Seidel and Smith constructed an invariant of links applying Floer cohomology to a Lagrangian inside a fibre bundle obtained from a Slodowy slice to a nilpotent matrix with two equal Jordan blocks. This invariant has an obvious grading and they conjectured that Khovanov homology is recovered after collapsing its bigrading to a single grading. This inspired the work of Smith and Thomas [SmTh] (see also [Th]), who, motivated by homological mirror symmetry, tried to construct a link invariant, which, on the one hand, would coincide with Khovanov homology, and on the other, would be a remarkable evidence of the power of Kontsevich's mirror symmetry conjecture. The basic idea comes from Manolescu's isomorphism [Man] between an open subset of the Hilbert scheme of points on an ALE space and the Slodowy slice considered by Seidel and Smith. Their construction works as follows (see section 3.2 for precise notation and definitions).

Let  $X = \widetilde{\mathbb{C}^2/\mu_{2n}}$  be the minimal resolution of the Kleinian singularity of type  $\mathbf{A}_{2n-1}$ . The irreducible components  $C_i$  of the exceptional divisor of this resolution form a chain of  $2n - 1$  projective lines and one can consider  $\mathcal{L}_i := \mathcal{O}_{C_i}(-1)$ , an object of  $D(X)$ , the bounded derived category of coherent sheaves with compact support on  $X$ . The Hilbert scheme  $\text{Hilb}^n(X)$  of  $n$  points on the smooth surface  $X$  is a resolution of singularities of  $\mathbb{C}^{2n}/\Gamma_n$ , where  $\Gamma_n$  denotes the wreath product  $\Sigma_n \wr \mu_{2n}$ . Smith and Thomas define

$$\mathcal{L}^{(n)} := \Sigma_n \cdot (\mathcal{L}_1 \boxtimes \mathcal{L}_3 \boxtimes \dots \boxtimes \mathcal{L}_{2n-1}) = \bigoplus_{\sigma \in \Sigma_n} \sigma^* (\mathcal{L}_1 \boxtimes \mathcal{L}_3 \boxtimes \dots \boxtimes \mathcal{L}_{2n-1}),$$

an object in

$$\mathbf{D}_n := D(\text{Hilb}^n(X)) \cong D(X)^{\Sigma_n}. \quad (0.0.1)$$

The spherical twists  $T_i$  corresponding to  $\mathcal{L}_i$ , defined in [SeTh], generate a weak action of the Artin braid group  $B_{2n}$  on  $D(X)$ , which can also be lifted to an action on  $\mathbf{D}_n$ . Since the supports of the  $\mathcal{L}_i$ 's are pairwise disjoint, the two actions on  $\mathcal{L}$  are related by

$$T_i(\Sigma_n \cdot (\mathcal{L}_1 \boxtimes \dots \boxtimes \mathcal{L}_{2n-1})) = \Sigma_n \cdot (T_i \mathcal{L}_1 \boxtimes \dots \boxtimes T_i \mathcal{L}_{2n-1}). \quad (0.0.2)$$

For any  $\beta \in B_{2n}$ , denote the corresponding autoequivalence in  $D_n$ , shifted by 1, by  $T_\beta$ . Smith and Thomas braid invariant cohomology is the following:

$$ST^*(\beta) := \text{Ext}_{D_n}^* (T_\beta \mathcal{L}^{(n)}, \mathcal{L}^{(n)}[n]) . \quad (0.0.3)$$

In order for this to be an invariant of the link given by the plat closure of the braid  $\beta$ , one needs to check that it is invariant under the Markov moves, which can be rephrased as the following relations:

- (A)  $T_1 \mathcal{L}^{(n)} \cong \mathcal{L}^{(n)}$ ;
- (B)  $T_{2i-1} T_{2i} \mathcal{L}^{(n)} \cong T_{2i-1}^{-1} T_{2i}^{-1} \mathcal{L}^{(n)}$ ;
- (C)  $T_{2i} T_{2i-1} T_{2i+1} T_{2i} \mathcal{L}^{(n)} \cong \mathcal{L}^{(n)}$ ;
- (D)  $\text{Ext}_{D_n}^* (T_\beta \mathcal{L}^{(n)}, \mathcal{L}^{(n)}[n]) \cong \text{Ext}_{D_{n+1}}^* (T_\beta \mathcal{L}^{(n+1)}, T_{2n}^{\pm 1} \mathcal{L}^{(n+1)}[n+1])$ .

In [SmTh], it is proved that (A), (C) and (D) hold, but (B) does not. This says that (0.0.3) is not a link invariant, but is close to being one. Smith and Thomas's idea to obtain a link invariant is to deform the Hilbert scheme, where they can check that the relations (A) to (D) should all hold. Unfortunately, controlling the commutative (non-compact) deformation theory, in particular of the autoequivalences, seems unwieldy, so their argument has never been fully executed.

A possible approach to the solution of this problem comes from non-commutative geometry: the Hilbert scheme of points on a smooth surface is a symplectic resolution of a Poisson variety and admits a quantisation (i.e. a filtered non-commutative deformation). One way to see this, in our case, is to first notice that  $\text{Hilb}^n(X)$  is isomorphic to a Nakajima quiver variety  $\mathcal{M}_\lambda^\theta(n\delta)$  associated to the cyclic quiver with  $2n$  vertices, hence we have a realisation of the Hilbert scheme as the Hamiltonian reduction  $\mu^{-1}(0) //^\theta G_{n\delta}$  (see chapter 2). Taking the quantum Hamiltonian reduction

$$\mathcal{A}_\lambda^\theta(n\delta) = \left( D_{\text{Rep}(Q_\infty, n\delta)} / D_{\text{Rep}(Q_\infty, n\delta)} \{ \Phi(x) - \langle \lambda, x \rangle \mid x \in \mathfrak{gl}_v \} \Big|_{T^* \text{Rep}(Q_\infty, n\delta)^{\theta-ss}} \right)^{G_v} ,$$

we obtain a quantisation of  $\text{Hilb}^n(X)$ , namely a sheaf of non-commutative filtered algebras such that  $\text{gr } \mathcal{A}_\lambda^\theta(n\delta) \cong \mathcal{O}_{\text{Hilb}^n(X)}$ . By [Ob, Go1], the global sections of this sheaf are isomorphic to the spherical subalgebra  $eH_{k,c}(\Gamma_n)e$  of the cyclotomic rational Cherednik algebra (see section 1.4 for the definition). Moreover, by a result of McGerty and Nevins [MN1], the global section functor defines an equivalence between

the derived categories of coherent  $\mathcal{A}_\lambda^\theta(n\delta)$ -modules and finitely generated  $eH_{k,\mathbf{c}}(\Gamma_n)e$ -modules, for generic values of  $(k, \mathbf{c})$ . These are exactly the spherical values, namely the values for which  $H_{k,\mathbf{c}}(\Gamma_n)$  and its spherical subalgebra are Morita equivalent and, unlike in most other cases, they are known for the cyclotomic algebra by [DG].

This suggests that one can attempt to construct Khovanov's link homology using a noncommutative, rather than commutative, deformation of  $\text{Hilb}^n(X)$ , where the role of the coherent sheaf  $\mathcal{L}^{(n)}$  is played by a module for  $H_{k,\mathbf{c}}(\Gamma_n)$ . This has the advantage that the representation theory of  $H_{k,\mathbf{c}}(\Gamma_n)$  has been extensively studied via category  $\mathcal{O}$  and thus many of the structures required to construct the link invariant are already known. Note that a crucial point is that the coherent sheaf  $\mathcal{L}^{(n)}$  has Lagrangian support, so it is reasonable to hope it quantises.

The parameter  $k$  corresponds to the deformation direction considered by Smith and Thomas and, imposing  $k = 0$ , one obtains

$$H_{0,\mathbf{c}}(\Gamma_n) = \Sigma_n \# H_{\mathbf{c}}(\mu_{2n})^{\otimes n},$$

the analogue of (0.0.1). Here,  $H_{\mathbf{c}}(\mu_{2n})$  is the rational Cherednik algebra of cyclic type (note that this depends just on  $\mathbf{c}$ ), a quantisation of the surface  $X$ . Let  $L_i$  be the simple in category  $\mathcal{O}$  for  $H_{\mathbf{c}}(\mu_{2n})$  corresponding to the standard module induced from the  $(2n - i)^{\text{th}}$  irreducible representation of  $\mu_{2n}$ . By [Ku2], the singular support of  $L_i$  coincides with the divisor  $C_i$ , provided  $\mathbf{c}$  satisfies certain integrality conditions, and this motivates the following definition:

$$L_{0,\mathbf{c}}^{(n)} := \Sigma_n \cdot (L_1 \otimes L_3 \otimes \dots \otimes L_{2n-1}) = \bigoplus_{\sigma \in \Sigma_n} \sigma(L_1 \otimes L_3 \otimes \dots \otimes L_{2n-1}). \quad (0.0.4)$$

This is a finite dimensional module, which can be regarded as an object of the derived category  $D(\mathcal{O}(H_{\mathbf{c}}(\mu_{2n})))$ . The last part of the construction is the categorical braid group action. This is provided by the twisting functors, defined in [BLPW] for hyperbolic varieties, and hence for the cyclic case, and in [BPW] and [BL] for the general case, which includes the cyclotomic rational Cherednik algebra. Denote the generators of  $B_{2n}$  inside the autoequivalence group of  $D(\mathcal{O}(H_{\mathbf{c}}(\mu_{2n})))$  and  $D(\mathcal{O}(H_{k,\mathbf{c}}(\Gamma_n)))$  by  $\mathbb{T}_i^{\mathbf{c}}$  and  $\mathbb{T}_i^{k,\mathbf{c}}$ , respectively, and let  $T_i^{k,\mathbf{c}} := \mathbb{T}_i^{k,\mathbf{c}}[-1]$ . Note that these generators correspond to the inverse generators of the braid group action considered by Smith and Thomas, thus we need a negative shift. Finally, denote  $\Sigma_n \cdot (\mathbb{T}_i^{\mathbf{c}} L_1 \otimes \dots \otimes \mathbb{T}_i^{\mathbf{c}} L_{2n-1})$  by  $\hat{\mathbb{T}}_i^{\mathbf{c}} L_{0,\mathbf{c}}^{(n)}$  and  $\Sigma_n \cdot (\mathbb{T}_i^{\mathbf{c}} L_1 \otimes \dots \otimes \mathbb{T}_i^{\mathbf{c}} L_{2n-1})[-1]$  by  $\hat{T}_i^{\mathbf{c}} L_{0,\mathbf{c}}^{(n)}$ .

We prove that, on objects of the form (0.0.4), there is an isomorphism of functors  $\mathbb{T}_i^{0,\mathbf{c}} \cong \hat{\mathbb{T}}_i^{\mathbf{c}}$ , which can be regarded as the quantum analogue of (0.0.2). One can

therefore consider the Markov relations at the level of the cyclic rational Cherednik algebras, where they are explicitly computable. Our main result reads:

**Theorem 0.0.1.** *Replacing  $\mathcal{L}^{(n)}$  with  $L_{0,\mathbf{c}}^{(n)}$  and  $T_i$  with  $T_i^{0,\mathbf{c}}$ , relations (A), (C) and (D) hold, but (B) does not.*

This suggests that the non-commutative translation is accurate. Moreover, we are able to algebraically deform the module  $L_{0,\mathbf{c}}^{(n)}$  to an irreducible  $H_{k,\mathbf{c}}(\Gamma_n)$ -module  $L_{k,\mathbf{c}}^{(n)}$ , for some non-zero values of  $k$ . The simple modules in category  $\mathcal{O}(H_{k,\mathbf{c}}(\Gamma_n))$  are labelled by  $2n$ -multipartition of  $n$  (see section 1.4.2) and we identify  $L_{k,\mathbf{c}}^{(n)}$  among this collection.

Let  $\mathcal{H}(\mathbf{c}')$  be the linear subspace of the space of parameters  $\mathfrak{z}^*$  defined by the relations (3.6.6).

**Theorem 0.0.2.** *The module  $L_{0,\mathbf{c}}^{(n)}$  deforms uniquely to the simple in category  $\mathcal{O}(H_{k,\mathbf{c}'}(\Gamma_n))$  labelled by the multipartition  $((1), \emptyset, (1), \dots, (1), \emptyset)$ , for  $(k, \mathbf{c}')$  taking values in an open neighbourhood of  $(0, \mathbf{c})$  inside  $\mathcal{H}(\mathbf{c}')$ .*

It is clear that the main problem of deforming the autoequivalences is not an issue in the algebraic setting, because the  $T_i^{k,\mathbf{c}'}$ 's satisfy the braid relations. What is left to prove in order to get an invariant of links is that all the knot relations hold after deforming to values of  $k$  in a neighbourhood of zero. Note that this statement is true in the commutative setting and it is therefore reasonable to expect it to be true also in the quantised picture.

## Structure of the thesis

The thesis has four chapters. In chapter 1 we introduce symplectic reflection algebras and, in particular, rational Cherednik algebras. All the material is known and we refer the reader to [Be, EtMa, Go2, Go3, Ro1] for good expository notes. We start in section 1.1 with symplectic reflection algebras and we specialise to rational Cherednik algebras in section 1.2. We then define the KZ functor that we use in the final section of the chapter, where we outline some results about the cyclotomic rational Cherednik algebra needed in the rest of the paper.

Chapter 2 is devoted to the study of quiver varieties, their quantisation and their relation to symplectic reflection algebras. We begin with some generalities on symplectic resolutions that we use in the following two sections about quiver varieties. In section 2.4 we introduce the quantum Hamiltonian reduction and we specify to the

case of Nakajima quiver varieties. We state the derived localisation theorem and the isomorphism between the global sections of the quantum Hamiltonian reduction sheaf and the corresponding spherical symplectic reflection algebra. We end the section, using quantum reflection functors introduced earlier, to prove a generalisation of the latter theorem for any dimension vector of the quiver variety. In the final section of the chapter, we outline the proof of this theorem for the cyclotomic case, giving a more general approach to the case of arbitrary dimension vector. Along the way, we give a generalisation of the Chevalley restriction theorem and its double analogue for the cyclic quiver, which is the main result of [Ga].

In chapter 3, we prove our main results. We give a motivation in the first two sections, recalling the works of Seidel and Smith [SS], and Smith and Thomas [SmTh]. We then give a description of category  $\mathcal{O}$  for the cyclic rational Cherednik algebra in terms of the representations of a certain quiver and introduce twisting functors. We show that these are well-behaved when the parameter  $k$  equals zero and we use this result and the techniques developed in the previous sections to prove Theorem 0.0.1 in section 3.5. Finally, we generalise the work of [EtMo] about deforming modules of deformed preprojective algebras, and apply it to our case, to prove Theorem 0.0.2. In the final chapter, we outline a possible strategy to show that our construction does produce an invariant of links and formulate a conjecture that would yield the existence of such an invariant.

## Notation and conventions

In this thesis, we only work over the field of complex numbers. So, every algebra and variety/scheme is considered over  $\mathbb{C}$ . By convention, zero is included in  $\mathbb{N}$ , while the positive integers are denoted by  $\mathbb{Z}_{>0}$ . Given a category  $\mathcal{A}$ ,  $D(\mathcal{A})$  denotes the bounded derived category of  $\mathcal{A}$ , while, for a variety  $X$ ,  $D(X)$  denotes the bounded derived category of coherent sheaves with compact support on  $X$ . Given an algebra  $A$  (or a sheaf of algebras), the symbols  $A\text{-Mod}$ ,  $A\text{-mod}$  and  $A\text{-fdmod}$  stand for the category of  $A$ -modules, finitely generated  $A$ -modules and finite dimensional  $A$ -modules, respectively. For a finite group  $G$ ,  $\text{Irrep}(G)$  denotes the finite set of pairwise non-isomorphic irreducible representations of  $G$ .

# Chapter 1

## Symplectic reflection algebras

### 1.1 Generalities

#### 1.1.1 Motivation

One classic problem in algebraic geometry is to study *orbifolds*  $X//G := \text{Spec}[X]^G$ , where  $G$  is a finite group and  $X$  is a smooth algebraic variety. One would like to understand when this space is smooth and, if not, how singular it is.

This problem is interesting already when  $X = V$  is a vector space and  $G$  is a finite subgroup of the linear group  $GL(V)$ . In this case, we have a satisfying criterion, due to the work of Chevalley [Ch] and Shephard and Todd [ShTo], to determine when  $V//G$  is a smooth variety.

Let  $s \in G$  be an element fixing a hyperplane in  $V$ , namely  $\text{rk}(1 - s) = 1$ . Such an element is called **complex reflection** and the couple  $(G, V)$  is a **complex reflection group** if  $G$  is generated by complex reflections.

**Theorem 1.1.1.**  *$V//G$  is smooth if and only if  $G$  is a complex reflection group. In this case,  $V//G \cong \mathbb{C}^n$ , where  $n = \dim V$ .*

Every complex reflection group can be built out (i.e. is a direct product) of irreducible (as  $V$ -modules) complex reflection groups and the latter have been classified in [ShTo]. Hence, we can turn our attention to the problem of determining when a non-smooth orbifold  $V//G$  admits a *crepant* (i.e. whose canonical bundle is trivial) resolution of singularities. To have a chance for this to happen, we need to restrict to a certain class of groups defined similarly (but differently) to complex reflection groups.

Let  $(V, \omega)$  be a vector space equipped with a symplectic form. A finite group  $G$  preserving  $\omega$  ( $G < Sp(V)$ ) is a **symplectic reflection group** if it is generated by

**symplectic reflections**, namely the elements that fix a subspace of  $V$  of codimension 2. The following result is proved in [Ve, Theorem 3.2].

**Theorem 1.1.2.** *If  $V//G$  admits a crepant resolution of singularities, then  $G$  is a symplectic reflection group.*

**Remark 1.1.3.** (1) In this case, the concepts of crepant and symplectic resolution (see section 2.1) coincide.

(2) The converse of this result is false (see [Kal1, section 6]).

We have a classification of (irreducible) symplectic reflection groups  $(V, \omega, G)$ , due to Guralnick and Saxl [GuSa]:

- (1)  $V = \mathfrak{h} \oplus \mathfrak{h}^*$ ,  $G = W$  is such that  $(W, \mathfrak{h})$  is a complex reflection group, and  $\omega$  is the natural symplectic form on the cotangent bundle  $V = T^*\mathfrak{h} = \mathfrak{h} \oplus \mathfrak{h}^*$ ;
- (2)  $V = L^{\oplus n}$ , with  $L$  a 2-dimensional symplectic vector space, and  $G = \Sigma_n \wr \Gamma := \Sigma_n \rtimes \Gamma$  is the *wreath product* of the symmetric group  $\Sigma_n$  and a finite subgroup  $\Gamma < SL_2(\mathbb{C})$ ;
- (3)  $(V, \omega, G)$  is part of a finite list of exceptions. In this case, we have  $\dim V < 10$ .

A fruitful way to study the orbifold  $V//G$  is to look at its deformations, namely the generic fibre of a *flat* map  $\mu : X \rightarrow \mathbb{C}^m$  from an affine variety  $X$ , such that the zero fibre is isomorphic to  $V//G$ . This may turn out to be a difficult task because we cannot give a description of the ring of the invariants  $\mathbb{C}[V]^G$  by generators and relations, in general. The approach of non-commutative geometry is to replace  $\mathbb{C}[V]^G$  by the *skew-group ring*  $\mathbb{C}[V]\#G$ , in light of the fact that these two rings are Morita equivalent when the  $G$ -action is free (so that in the “good” cases the commutative and non-commutative theories coincide), but we know generators and relations of the latter (it equals  $\mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}[G]$  as a vector space and we multiply elements in the “wrong order” by  $(g \cdot f)(v) = f(g^{-1} \cdot v)g$ ,  $\forall g \in G, f \in \mathbb{C}[V], v \in V$ ). Thus, given a skew-symmetric bilinear form  $(-, -) : V \times V \rightarrow \mathbb{C}$ , we can deform the skew-group ring by twisting the multiplication in  $\mathbb{C}[V]$  and consider the ring defined by

$$H_{(-,-)} := TV^* \# G / \langle u \otimes v - v \otimes u - (u, v) \mid u, v \in V^* \rangle,$$

where  $TV^*$  denotes the tensor algebra of the dual vector space  $V^*$ . Notice that, if  $(-, -)$  is the zero form,  $H_{(-,-)}$  reduces to  $\text{Sym } V^* \# G = \mathbb{C}[V]\#G$ . Moreover,  $H_{(-,-)}$

has a natural increasing filtration setting  $G$  in degree 0 and  $V^*$  in degree 1, and, if  $\text{gr } H_{(-,-)}$  denotes the associated graded algebra with respect to this filtration, there is a surjective algebra homomorphism

$$\nu : \mathbb{C}[V]\#G \longrightarrow \text{gr } H_{(-,-)} .$$

We say that the **PBW-property** holds for  $H_{(-,-)}$ , if  $\nu$  is an isomorphism (this property is related to the flatness of the deformation map and with the idea of quantisation, section 2.4).

Let  $(V, \omega, G)$  be an irreducible symplectic reflection group and let  $S$  be the set of symplectic reflections of  $G$ . Note that, for all  $s \in S$ , we have a  $\omega$ -orthogonal decomposition  $V = \text{Im}(1 - s) \oplus \text{Ker}(1 - s)$ , and denote by  $\omega_s$  the 2-form on  $V$  coinciding with  $\omega$  on  $\text{Im}(1 - s)$  and whose restriction to  $\text{Ker}(1 - s)$  is zero.

**Theorem 1.1.4.** ([EG, Theorem 1.3]) *Let  $(V, \omega, G)$  be as before and  $(-, -)$  a skew-symmetric bilinear form. The PBW-property holds for  $H_{(-,-)}$  if and only if there exist a constant  $t \in \mathbb{C}$  and a conjugacy class invariant function  $\mathbf{c} : S \rightarrow \mathbb{C}$ , such that*

$$(u, v) = t\omega(u, v) - 2 \sum_{s \in S} \mathbf{c}(s)\omega_s(u, v)s . \quad (1.1.1)$$

**Definition 1.1.5.** If  $(-, -)$  is defined by (1.1.1), the algebra

$$H_{t,\mathbf{c}}(G) := H_{(-,-)} = TV^*\#G \left/ \left\langle u \otimes v - v \otimes u - t\omega(u, v) - 2 \sum_{s \in S} \mathbf{c}(s)\omega_s(u, v)s \mid u, v \in V^* \right\rangle \right.$$

is called a **symplectic reflection algebra**.

**Corollary 1.1.6.** (i)  $H_{t,\mathbf{c}}(G)$  is a prime (left and right) Noetherian ring.

(ii)  $H_{t,\mathbf{c}}(G)$  has finite global dimension.

*Proof.* These properties are well-known for the skew-group ring  $\mathbb{C}[V]\#G$ . It is a standard exercise to prove that (i) and (ii) hold for a filtered ring if and only if they hold for its associated graded (see, for example, [Le2]).  $\square$

### 1.1.2 The spherical subalgebra

We would like to use symplectic reflection algebras in order to find deformations of the ring of invariants  $\mathbb{C}[V]^G = \mathfrak{z}(\mathbb{C}[V]\#G)$ , the centre of the skew-group ring. Let  $e := \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{C}[G]$  be the trivial idempotent. The algebra  $eH_{t,\mathbf{c}}(G)e$  is called the **spherical subalgebra**. It inherits the filtration from  $H_{t,\mathbf{c}}(G)$  and, by Theorem 1.1.4, we have  $\text{gr } eH_{t,\mathbf{c}}(G)e \cong \mathbb{C}[V]^G$ . As a corollary, we immediately obtain that  $eH_{t,\mathbf{c}}(G)e$  is a (left and right) Noetherian integral domain. Moreover, we have the following *double centraliser* property.

**Theorem 1.1.7.** (*[EG, Theorem 1.5]*)

(i) *The maps*

$$\begin{array}{ccc} eH_{t,\mathbf{c}}(G) & \longrightarrow & \text{Hom}_{eH_{t,\mathbf{c}}(G)e}(H_{t,\mathbf{c}}(G)e, eH_{t,\mathbf{c}}(G)e) \\ eh & \longmapsto & (h'e \mapsto eh'h'e) \end{array}$$

$$\text{and} \quad \begin{array}{ccc} H_{t,\mathbf{c}}(G)e & \longrightarrow & \text{Hom}_{eH_{t,\mathbf{c}}(G)e}(eH_{t,\mathbf{c}}(G), eH_{t,\mathbf{c}}(G)e) \\ he & \longmapsto & (eh' \mapsto eh'he) \end{array}$$

*are isomorphisms of left and right modules, respectively. In particular,  $eH_{t,\mathbf{c}}(G)$  and  $H_{t,\mathbf{c}}(G)e$  are reflexive (left and right, respectively)  $eH_{t,\mathbf{c}}(G)e$ -modules.*

$$(ii) \quad \text{End}_{H_{t,\mathbf{c}}(G)}(H_{t,\mathbf{c}}(G)e)^{op} \cong eH_{t,\mathbf{c}}(G)e.$$

$$(iii) \quad \text{End}_{eH_{t,\mathbf{c}}(G)e^{op}}(H_{t,\mathbf{c}}(G)e) \cong H_{t,\mathbf{c}}(G).$$

This implies the following result, which is crucial in order to relate the representation theory of the symplectic reflection algebra  $H_{t,\mathbf{c}}(G)$  and the geometry of the orbifold  $V//G$ .

**Corollary 1.1.8.** *The algebras  $H_{t,\mathbf{c}}(G)$  and  $eH_{t,\mathbf{c}}(G)e$  are Morita equivalent if and only if  $eM = 0$  implies  $M = 0$ , for all finitely generated  $H_{t,\mathbf{c}}(G)$ -modules  $M$ .*

*Proof.* By general theory, the multiplication by the idempotent  $e$  defines a Morita equivalence if and only if  $H_{t,\mathbf{c}}(G)e$  is a projective generator of  $H_{t,\mathbf{c}}(G)$ -mod. It is always projective, while it is a generator if and only if  $H_{t,\mathbf{c}}(G)eH_{t,\mathbf{c}}(G) = H_{t,\mathbf{c}}(G)$ , which is equivalent to the required condition.  $\square$

**Definition 1.1.9.** The parameter  $(t, \mathbf{c})$  is called a **spherical value** if  $H_{t,\mathbf{c}}(G)$  and  $eH_{t,\mathbf{c}}(G)e$  are Morita equivalent. It is called **aspherical** if there exists a non-zero module  $M \in H_{t,\mathbf{c}}(G)$ -mod such that  $eM = 0$ .

The parameter  $t$  in the definition of symplectic reflection algebras is the *quantum parameter*. The following result tells us that the structure of symplectic reflection algebras is very different depending on whether  $t = 0$  or  $t \neq 0$ .

**Theorem 1.1.10.** (i) (Satake isomorphism) *The multiplication by the idempotent  $e$  induces a (Poisson) algebra isomorphism between The centres  $\mathfrak{z}(H_{t,\mathbf{c}}(G)) \cong \mathfrak{z}(eH_{t,\mathbf{c}}(G)e)$ .*

(ii) *If  $t \neq 0$ , the centre of  $eH_{t,\mathbf{c}}(G)e$ , and hence of  $H_{t,\mathbf{c}}(G)$ , equals  $\mathbb{C}$ .*

(iii) *If  $t = 0$ ,  $eH_{t,\mathbf{c}}(G)e$  is commutative and coincides with the centre of  $H_{t,\mathbf{c}}(G)$ . The latter is a finite module over its centre.*

*Proof.* (i) and (iii) are proved in [EG] (Theorems 3.1 and 1.6, respectively). (ii) is due to Brown and Gordon [BG, Proposition 7.2].  $\square$

Notice that, by definition, there is an algebra isomorphism  $H_{t,\mathbf{c}}(G) \cong H_{\lambda t, \lambda \mathbf{c}}(G)$ , for all  $\lambda \in \mathbb{C}$ ,  $\lambda \neq 0$ . Thus, one can restrict the attention to the cases  $t = 0$  and  $t = 1$ . In the former, the *semi-classical case*, Theorem 1.1.10 (iii), tells us that the symplectic reflection algebra produces a commutative deformation of the orbifold  $V//G$ , by considering the *generalised Calogero-Moser spaces*  $X_{\mathbf{c}}(G) := \text{Spec } \mathfrak{z}(H_{0,\mathbf{c}}(G))$ . The representation theory of  $H_{0,\mathbf{c}}(G)$  is closely related to the geometry (in particular, the grade of singularity) of  $X_{\mathbf{c}}(G)$  (see, for example, [Be, section 5]).

For the rest of the chapter, we will only consider the case  $t = 1$ . We will denote by  $\mathfrak{z}^*$  the space of parameters  $\mathbf{c}$  (which is isomorphic to  $\mathbb{C}^d$ , where  $d$  is the number of conjugacy classes in  $S$ ) and omit  $t$  from the notation.

## 1.2 Rational Cherednik algebras

In this section, we consider symplectic reflection algebras of type (1) in the classification above, namely those with  $G = W$ ,  $V = \mathfrak{h} \oplus \mathfrak{h}^*$  and  $(W, \mathfrak{h})$  is a complex reflection group (notice that the set of complex reflections of  $W$  acting on  $\mathfrak{h}$  coincides with  $\mathcal{S}$ , the set of symplectic reflections of  $W$  acting on  $\mathfrak{h} \oplus \mathfrak{h}^*$ ). It turns out that these algebras can also be obtained as certain rational degenerations of *double affine Hecke algebras*, introduced by Cherednik. For this reason, symplectic reflection algebras in this case are called **rational Cherednik algebras**.

By definition of symplectic reflections, the spaces  $\text{Im}(s - 1)|_{\mathfrak{h}^*}$  and  $\text{Im}(s - 1)|_{\mathfrak{h}}$  are one-dimensional, for all  $s \in \mathcal{S}$ . We can pick bases of these vector spaces  $\alpha_s \in \mathfrak{h}^*$  and  $\alpha_s^\vee \in \mathfrak{h}$ , respectively, such that  $(\alpha_s^\vee, \alpha_s) = 2$ . Here,  $(-, -) : \mathfrak{h} \times \mathfrak{h}^* \rightarrow \mathbb{C}$  is the perfect

pairing defined by  $(y, x) = x(y)$ ,  $\forall y \in \mathfrak{h}$ ,  $x \in \mathfrak{h}^*$ , so that the symplectic form  $\omega$  on  $\mathfrak{h} \oplus \mathfrak{h}^*$  is given by

$$\omega((y, x), (y', x')) = (y, x') - (y', x).$$

With this notation, relation (1.1.1) simplifies as follows

$$\begin{aligned} [x, x'] &= 0, \quad [y, y'] = 0, \quad \forall x, x' \in \mathfrak{h}^*, \quad y, y' \in \mathfrak{h}, \\ [y, x] &= t(y, x) - \sum_{s \in \mathcal{S}} \mathbf{c}(s)(y, \alpha_s)(\alpha_s^\vee, x)s, \quad \forall x \in \mathfrak{h}^*, \quad y \in \mathfrak{h}, \end{aligned} \quad (1.2.1)$$

and we notice immediately that  $\mathbb{C}[\mathfrak{h}]$  and  $\mathbb{C}[\mathfrak{h}^*]$  are polynomial subalgebras of  $H_{\mathbf{c}}(W)$ .

### 1.2.1 The Dunkl embedding

For each complex reflection  $s \in \mathcal{S}$ , consider the reflecting hyperplane  $\mathcal{H}_s := \text{Ker } \alpha_s \subset \mathfrak{h}$ . Let  $\mathfrak{h}^{\text{reg}} := \mathfrak{h} \setminus \bigcup_{s \in \mathcal{S}} \mathcal{H}_s$ , the affine open subspace of  $\mathfrak{h}$  on which  $W$  acts freely, and consider  $\delta := \prod_{s \in \mathcal{S}} \alpha_s \in \mathbb{C}[\mathfrak{h}]$ , so that  $\mathfrak{h}^{\text{reg}} = \{y \in \mathfrak{h} \mid \delta(y) \neq 0\}$ . Localising the ring of differential operators  $D(\mathfrak{h})$  to  $\mathfrak{h}^{\text{reg}}$ , we obtain  $D(\mathfrak{h}^{\text{reg}}) = D(\mathfrak{h})[\delta^{-1}]$ . Hence, if  $\lambda_s \in \mathbb{C}^\times$  is the non-trivial eigenvalue of  $s$  in  $\mathfrak{h}^*$  (namely,  $s.\alpha_s = \lambda_s \alpha_s$ ), the element

$$\mathcal{D}_y := \partial_y - \sum_{s \in \mathcal{S}} \frac{2\mathbf{c}(s)}{1 - \lambda_s} \frac{(y, \alpha_s)}{\alpha_s} (1 - s)$$

lies in  $D(\mathfrak{h}^{\text{reg}}) \# W$ ,  $\forall y \in \mathfrak{h}$ . Here, if  $y_1, \dots, y_n$  is a basis of  $\mathfrak{h} \cong \mathbb{C}^n$ , with dual basis  $x_1, \dots, x_n$ , and  $y = \sum_{i=1}^n a_i y_i$ , we put  $\partial_y := \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \in D(\mathfrak{h})$ . It is easy to check that the  $\mathcal{D}_y$ 's, called **Dunkl operators**, act on  $\mathbb{C}[\mathfrak{h}]$  and, since  $s.x = x - \frac{(\alpha_s^\vee, x)}{2} (1 - \lambda_s) \alpha_s$ ,  $\forall x \in \mathfrak{h}^*$ , a direct calculation shows that we have

$$[\mathcal{D}_y, x] = (y, x) - \sum_{s \in \mathcal{S}} \mathbf{c}(s)(y, \alpha_s)(\alpha_s^\vee, x)s,$$

for all  $x \in \mathfrak{h}^*$ ,  $y \in \mathfrak{h}$ . Thus, we have a well-defined algebra homomorphism  $\Theta_{\mathbf{c}} : H_{\mathbf{c}}(W) \rightarrow D(\mathfrak{h}^{\text{reg}}) \# W$  given by

$$x \longmapsto x, \quad y \longmapsto \mathcal{D}_y, \quad w \longmapsto w, \quad \forall x \in \mathfrak{h}^*, \quad y \in \mathfrak{h}, \quad w \in W.$$

$\Theta_{\mathbf{c}}$  is called **Dunkl embedding** because of the following:

**Theorem 1.2.1.**  *$\Theta_{\mathbf{c}}$  is injective and induces an isomorphism*

$$H_{\mathbf{c}}(W)[\delta^{-1}] \xrightarrow{\sim} D(\mathfrak{h}^{\text{reg}}) \# W. \quad (1.2.2)$$

*Proof.* The injectivity is [EG, Proposition 4.5]. For the second part, observe that  $\{\delta^i\}_{i=0}^\infty$  forms an Ore set in  $H_{\mathbf{c}}(W)$ . This is because  $\delta$  commutes with the  $x_i$ 's, it is  $W$ -semi-invariant and satisfies  $\delta^{a+b}y_i^b \in H_{\mathbf{c}}(W)\delta^a$ ,  $\forall a, b \in \mathbb{N}$ , by defining relations (it is enough to check the Ore relations on monomials because they form a basis of  $H_{\mathbf{c}}(W)$ , by Theorem 1.1.4). The term in the group ring  $\mathbb{C}[W]$  in the definition of  $\mathcal{D}_y$  clearly belongs to  $H_{\mathbf{c}}(W)[\delta^{-1}]$ , which implies the surjectivity of (1.2.2).  $\square$

**Remark 1.2.2.** One can use the Dunkl embedding together with some basic properties of the differential operators to give elementary proofs of Theorems 1.1.4 and 1.1.10 (iii), for rational Cherednik algebras.

## 1.2.2 Category $\mathcal{O}$

Trying to understand the category of finitely generated  $H_{\mathbf{c}}(W)$ -modules is too much of a general question, so one needs to restrict to a class of modules that is small enough to be studied and large enough to give important information about the representation theory of  $H_{\mathbf{c}}(W)$ . This is obtained by considering the following:

**Definition 1.2.3.** The category  $\mathcal{O}(H_{\mathbf{c}}(W))$  is the full subcategory of  $H_{\mathbf{c}}(W)$ -mod whose objects are the modules on which the action of  $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*] \subset H_{\mathbf{c}}(W)$  is locally nilpotent (i.e. for each element  $a$  of the module, there exists a positive integer  $m$  such that  $\mathfrak{h}^m a = 0$ ).

The motivation for such a definition comes from category  $\mathcal{O}$  for semisimple Lie algebras, which is defined on the grounds of the triangular decomposition  $\mathcal{U}_{\mathfrak{g}} \cong \mathcal{U}_{\mathfrak{n}^-} \otimes_{\mathbb{C}} \mathcal{U}_{\mathfrak{h}} \otimes_{\mathbb{C}} \mathcal{U}_{\mathfrak{n}^+}$ , implied by the PBW theorem (where, in standard notation, we have the Cartan decomposition  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ ). Likewise, Theorem 1.1.4 implies the vector space isomorphism

$$H_{\mathbf{c}}(W) \cong \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*], \quad (1.2.3)$$

which allows one to define category  $\mathcal{O}$  for rational Cherednik algebras in a way analogous to the Lie algebra case. Observe that, in the latter, modules in category  $\mathcal{O}$  are required to decompose as direct sums of weight spaces, while the analogous property for Cherednik algebras is not part of the definition, because, as we are going to see, this is implied by Definition 1.2.3.

Given  $\rho$ , an irreducible representation of  $W$ , the **standard module**  $\Delta(\rho)$  is defined as the induced module

$$\Delta(\rho) := \text{Ind}_{\mathbb{C}[\mathfrak{h}^*] \# W}^{H_{\mathbf{c}}(W)} \rho = H_{\mathbf{c}}(W) \otimes_{\mathbb{C}[\mathfrak{h}^*] \# W} \rho,$$

where  $\rho$  is viewed as a  $\mathbb{C}[\mathfrak{h}^*]\#W$ -module by letting  $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$  act by zero. Note that (1.2.3) implies the  $\mathbb{C}[\mathfrak{h}]$ -module isomorphism  $\Delta(\rho) \cong \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \rho$ .

Let  $x_1, \dots, x_n$  be a basis of  $\mathfrak{h}^*$  with dual basis  $y_1, \dots, y_n \in \mathfrak{h}$ . The **Euler element** is defined by

$$\mathbf{eu} := \sum_{i=1}^n x_i y_i - \sum_{s \in \mathcal{S}} \frac{2\mathbf{c}(s)}{1 - \lambda_s} s.$$

By direct calculation, one sees that we have

$$[\mathbf{eu}, x] = x, \quad [\mathbf{eu}, y] = -y \quad \text{and} \quad [\mathbf{eu}, w] = 0, \quad \forall x \in \mathfrak{h}^*, y \in \mathfrak{h}, w \in W,$$

and hence the action of  $\mathbf{eu}$  defines a  $\mathbb{Z}$ -grading on  $H_{\mathbf{c}}(W)$ . Moreover,  $-\sum_{s \in \mathcal{S}} \frac{2\mathbf{c}(s)}{1 - \lambda_s} s$  is a central element in  $\mathbb{C}[W]$ , so it acts by a scalar (denoted by  $\mathbf{c}_\rho$ ) on an irreducible representation  $\rho$ . Thus, we have the *generalised  $\mathbf{eu}$ -eigenspace*

$$\Delta(\rho)_i := \{m \in \Delta(\rho) \mid \mathbf{eu}.m = (i + \mathbf{c}_\rho)m\}$$

and the standard module  $\Delta(\rho)$  is the direct sum of such subspaces. This implies the following lemma (see [GGOR] or [Be, Lemma 2.4.3] for a proof).

**Lemma 1.2.4.** *Every module  $M \in \mathcal{O}(H_{\mathbf{c}}(W))$  is the direct sum of its generalised  $\mathbf{eu}$ -eigenspaces, each of which is finite dimensional.*

It is easy to see that, if  $\mathbb{C}[\mathfrak{h}]_i$  denotes the space of homogeneous polynomials of degree  $i$ , we have  $\Delta(\rho)_i \cong \mathbb{C}[\mathfrak{h}]_i \otimes_{\mathbb{C}} \rho$ . Hence, the zero weight space  $\Delta(\rho)_0 = 1 \otimes_{\mathbb{C}} \rho$  is an irreducible  $W$ -module that generates  $\Delta(\rho)$ . Let  $M(\rho)$  be the sum of all proper submodules of  $\Delta(\rho)$ . If  $M(\rho)_0 \neq 0$ , then there would exist some proper submodule  $N$  of  $\Delta(\rho)$ , such that  $N_0 \neq 0$ , which gives a contradiction by the above. Thus,  $M(\rho)_0 = 0$  and  $M(\rho)$  is a proper maximal submodule of  $\Delta(\rho)$ . Let  $L(\rho) := \Delta(\rho)/M(\rho)$  be the simple quotient.

**Lemma 1.2.5.** *([GGOR], [Be, Lemma 2.6.2]) The set*

$$\{L(\rho) \mid \rho \in \text{Irrep}(W)\}$$

*is a complete collection of pairwise non-isomorphic simple modules in  $\mathcal{O}(H_{\mathbf{c}}(W))$ .*

When a category has a collection of standard and simple objects labelled by some set  $\Lambda$ , it is natural to ask if it displays an additional structure, which has many favourable properties analogous to category  $\mathcal{O}$  for semisimple Lie algebras. The following definition collects the hypotheses needed for this.

**Definition 1.2.6** ([CPS]). Let  $\mathcal{A}$  be an abelian category and  $\Lambda$  a poset.  $(\mathcal{A}, \Lambda)$  is called a **highest weight category** if the following conditions are satisfied.

- (1)  $\mathcal{A}$  is  $\mathbb{C}$ -linear and each object in  $\mathcal{A}$  has finite length.
- (2) There is a complete set of non-isomorphic simple objects  $\{L(\lambda) \mid \lambda \in \Lambda\}$  parametrised by  $\Lambda$ .
- (3) There is a complete set of objects  $\{\Delta(\lambda) \mid \lambda \in \Lambda\}$  parametrised by  $\Lambda$  and,  $\forall \lambda \in \Lambda$ , a surjection  $\pi_\lambda : \Delta(\lambda) \twoheadrightarrow L(\lambda)$  such that, if  $L(\mu)$  is a composition factor of  $\text{Ker } \pi_\lambda$ , for some  $\mu \in \Lambda$ , then we have  $\mu < \lambda$ .
- (4) Each simple  $L(\lambda)$  has a projective cover  $P(\lambda)$  which admits a (*standard*) filtration  $0 = F_0P(\lambda) \subset \dots \subset F_kP(\lambda) = P(\lambda)$ , such that  $F_kP(\lambda)/F_{k-1}P(\lambda) \cong \Delta(\lambda)$  and  $F_iP(\lambda)/F_{i-1}P(\lambda) \cong \Delta(\mu_i)$ , for some  $\mu_i > \lambda$  and for all  $1 \leq i \leq m-1$ .

Define the partial ordering  $<_{\mathbf{c}}$  on the set of non-isomorphic irreducible representations of  $W$ , by  $\lambda <_{\mathbf{c}} \mu$  if  $\mathbf{c}_\mu - \mathbf{c}_\lambda$  is a non-negative integer.

The following is proved in [GGOR].

**Theorem 1.2.7.**  $(\mathcal{O}(H_{\mathbf{c}}(W)), \text{Irrep}(W))$  is a highest weight category under the ordering  $<_{\mathbf{c}}$ .

*Proof.* The condition (1) in Definition 1.2.6 is [Be, Corollary 2.6.4]. (2) and (3) follow from Lemma 1.2.5 and the discussion above it. The proof of (4) is more involved; see, for example, [Be, Theorems 2.7.2 and 2.8.3].  $\square$

**Corollary 1.2.8.** (i)  $\mathcal{O}(H_{\mathbf{c}}(W))$  has finite global dimension.

(ii) (BGG reciprocity) For all  $\rho, \rho' \in \text{Irrep}(W)$ , we have

$$(P(\rho) : \Delta(\rho)) = [\Delta(\rho') : L(\rho)],$$

where the left-hand side denotes the multiplicity of  $\Delta(\rho')$  in the standard filtration of  $P(\rho)$  and the right-hand side is the multiplicity of  $L(\rho)$  in the Jordan-Hölder filtration of  $\Delta(\rho')$ .

(iii)  $\mathcal{O}(H_{\mathbf{c}}(W))$  is a semisimple category if and only if  $\Delta(\rho) = L(\rho)$ ,  $\forall \rho \in \text{Irrep}(W)$ . In particular, it is so for generic values of  $\mathbf{c}$ .

(iv) The standard modules form a basis of the Grothendieck group  $K_0(\mathcal{O}(H_{\mathbf{c}}(W)))$ .

*Proof.* (i) is [Be, Corollary 2.8.6]. (ii) is a consequence of [CPS, Theorem 3.11] and [GGOR, Proposition 3.3]. To prove (iii), first note that  $\mathcal{O}(H_{\mathbf{c}}(W))$  semisimple clearly implies  $\Delta(\rho) = L(\rho)$ ,  $\forall \rho \in \text{Irrep}(W)$ , because  $\Delta(\rho)$  is indecomposable. Conversely, if  $\Delta(\rho) = L(\rho)$ ,  $\forall \rho \in \text{Irrep}(W)$ , all simples are projective by BGG reciprocity, and hence  $\mathcal{O}(H_{\mathbf{c}}(W))$  is semisimple. This statement implies that  $\mathcal{O}(H_{\mathbf{c}}(W))$  is semisimple if and only if  $\mathbf{c}_{\rho} - \mathbf{c}_{\rho'} \notin \mathbb{Z}_{>0}$  for all  $\rho \in \text{Irrep}(W)$  ([GGOR, Corollary 2.20]), and so it is semisimple for generic values of  $\mathbf{c}$ .

Finally, observe that, by (1) and (2) in Definition 1.2.6, the images of the simples in  $K_0(\mathcal{O}(H_{\mathbf{c}}(W)))$  form a basis thereof. Hence, write  $[\Delta\rho'] = \sum m_{\rho',\rho}[L(\rho)]$  and pick the order of the basis  $[L(\rho_1)], \dots, [L(\rho_{|\text{Irrep}(W)|})]$  given by  $i < j$  if  $\rho_j <_{\mathbf{c}} \rho_i$ . By Definition 1.2.6 (3), the multiplicity matrix  $M := (m_{\rho_i, \rho_j})$  is upper-triangular with ones on the diagonal. Thus,  $M$  is invertible, which implies that the class of every simple, and hence of every module in  $\mathcal{O}(H_{\mathbf{c}}(W))$ , can be expressed as a  $\mathbb{C}$ -linear combination of  $[\Delta(\rho)]$ , for  $\rho \in \text{Irrep}(W)$ .  $\square$

## 1.3 The KZ functor

In this section, we study the *Knizhnik-Zamolodchikov functor*, introduced by Ginzburg, Guay, Opdam and Rouquier [GGOR], to relate in a topological way, category  $\mathcal{O}$  for the rational Cherednik algebra and the representation theory of the (finite) *Iwahori-Hecke algebra*.

The motivation comes, once again, from semisimple Lie algebras, in which case an analogous procedure, via the Beilinson-Bernstein localisation theorem (Theorem 2.4.5) and the Riemann-Hilbert correspondence, led to the proof of the celebrated Kazhdan-Lusztig conjecture. The KZ functor gives a fruitful approach to the problem, still open in the general case, of calculating the multiplicities of simples in standard modules.

### 1.3.1 The Iwahori-Hecke algebra

Let  $(W, \mathfrak{h})$  be a complex reflection group and recall  $\mathfrak{h}^{\text{reg}}$ , the open subset of  $\mathfrak{h}$  on which  $W$  acts freely. Denote by  $\mathcal{A}$  the set of reflecting hyperplanes and, given  $\mathcal{H} \in \mathcal{A}$ , let  $W_{\mathcal{H}}$  be the (cyclic) subgroup of  $W$  consisting of elements that fix  $\mathcal{H}$  pointwise and set  $n_{\mathcal{H}} := |W_{\mathcal{H}}|$ . The fundamental group  $B_W := \pi_1(\mathfrak{h}^{\text{reg}}/W)$  coincides with a certain braid group whose generators  $\{T_s \mid s \in \mathcal{S}\}$  correspond to the generators of the monodromy around the reflecting hyperplanes of  $W$  in  $\mathfrak{h}$ . Introduce the parameter

$$\mathbf{q} := \{q_{\mathcal{H},i} \in \mathbb{C}^{\times} \mid \mathcal{H} \in \mathcal{A}, 0 \leq i < n_{\mathcal{H}}\}$$

and consider the two-sided ideal  $\mathcal{I}_q$  in the group algebra  $\mathbb{C}[B_W]$ , generated by the elements

$$\prod_{i=0}^{n_{\mathcal{H}}-1} (T_s - q_{\mathcal{H},i}), \quad \forall s \in \mathcal{S}.$$

**Definition 1.3.1.** The algebra  $\mathcal{H}_q(W) := \mathbb{C}[B_W]/\mathcal{I}_q$  is called **Iwahori-Hecke algebra**.

In light of this definition, we will need to consider a different parametrisation for the rational Cherednik algebra  $H_c(W)$  (which is the choice made in [GGOR]).

Notice that, given  $\mathcal{H} \in \mathcal{A}$  and  $s, s' \in W_{\mathcal{H}}$ , we can assume  $\alpha_s = \alpha_{s'}$  and  $\alpha_s^{\vee} = \alpha_{s'}^{\vee}$  (and call them  $\alpha_{\mathcal{H}}$  and  $\alpha_{\mathcal{H}}^{\vee}$ , respectively). Moreover, we have the equality of sets

$$\mathcal{S} = \bigcup_{\mathcal{H} \in \mathcal{A}} (W_{\mathcal{H}} \setminus \{1\}).$$

Hence, the sum on the right-hand side of relation (1.2.1) can be decomposed as

$$\sum_{s \in \mathcal{S}} \mathbf{c}(s)(y, \alpha_s)(\alpha_s^{\vee}, x)s = \sum_{\mathcal{H} \in \mathcal{A}} (y, \alpha_{\mathcal{H}})(\alpha_{\mathcal{H}}^{\vee}, x) \left( \sum_{s \in W_{\mathcal{H}} \setminus \{1\}} \mathbf{c}(s)s \right).$$

For  $\mathcal{H} \in \mathcal{A}$  and  $0 \leq n_{\mathcal{H}}$ , let  $e_{\mathcal{H},i} := \frac{1}{n_{\mathcal{H}}} \sum_{w \in W_{\mathcal{H}}} (\det w)^i w$  be the primitive idempotents of  $\mathbb{C}[W_{\mathcal{H}}]$ , and introduce the parameter

$$\mathbf{k} := \{k_{\mathcal{H},i} \in \mathbb{C} \mid \mathcal{H} \in \mathcal{A}, 0 \leq i \leq n_{\mathcal{H}}, k_{\mathcal{H},0} = k_{\mathcal{H},n_{\mathcal{H}}} = 0, k_{\mathcal{H},i} = k_{w.\mathcal{H},i}, \forall w \in W\}$$

by imposing the condition

$$\mathbf{c}(s) = \sum_{i=0}^{n_{\mathcal{H}}-1} (\det s)^i (k_{\mathcal{H},i+1} - k_{\mathcal{H},i}), \quad (1.3.1)$$

Hence, we have

$$\sum_{s \in W_{\mathcal{H}}} \mathbf{c}(s)s = n_{\mathcal{H}} \sum_{i=0}^{n_{\mathcal{H}}-1} (k_{\mathcal{H},i+1} - k_{\mathcal{H},i})e_{\mathcal{H},i},$$

and the rational Cherednik algebra  $H_c(W)$  equals the quotient of  $\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle \# W$  by the relations

$$[x, x'] = 0, \quad [y, y'] = 0, \quad \forall x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h},$$

$$[y, x] = (y, x) - \sum_{\mathcal{H} \in \mathcal{A}} (y, \alpha_{\mathcal{H}})(\alpha_{\mathcal{H}}^{\vee}, x) n_{\mathcal{H}} \sum_{i=0}^{n_{\mathcal{H}}-1} (k_{\mathcal{H},i+1} - k_{\mathcal{H},i})e_{\mathcal{H},i}, \quad \forall x \in \mathfrak{h}^*, y \in \mathfrak{h}.$$

We denote the latter by  $H_{\mathbf{k}}(W)$ .

### 1.3.2 The Knizhnik-Zamolodchikov functor

Recall the Dunkl embedding. By Theorem 1.2.1, localising a  $H_{\mathbf{k}}(W)$ -module  $M \mapsto M[\delta^{-1}]$  yields a functor

$$\mathcal{O}(H_{\mathbf{k}}(W)) \longrightarrow D(\mathfrak{h}^{\text{reg}})\#W\text{-mod} . \quad (1.3.2)$$

Moreover, we have the following well-known result.

**Proposition 1.3.2.** (i) *There is an algebra isomorphism*

$$D(\mathfrak{h}^{\text{reg}})^W \xrightarrow{\sim} D(\mathfrak{h}^{\text{reg}}/W) .$$

(ii) *The assignment  $M \mapsto M^W$  gives an equivalence of categories*

$$D(\mathfrak{h}^{\text{reg}})\#W\text{-mod} \xrightarrow{\sim} D(\mathfrak{h}^{\text{reg}}/W)\text{-mod} . \quad (1.3.3)$$

*Proof.* (i) follows by the fact that  $D(\mathfrak{h}^{\text{reg}})^W$  is simple and the  $W$ -action on  $\mathfrak{h}^{\text{reg}}$  is free (see [Le1, Théorème 4.2]).

The multiplication by the trivial idempotent gives the isomorphism  $D(\mathfrak{h}^{\text{reg}}/W) \cong e(D(\mathfrak{h}^{\text{reg}})\#W)e$ . Since  $W$  acts freely on  $\mathfrak{h}^{\text{reg}}$ ,  $M^W = 0$  implies  $M = 0$ , and (ii) follows by the same argument of Corollary 1.1.8 and by (i).  $\square$

Starting with a module  $M \in \mathcal{O}(H_{\mathbf{k}}(W))$  and composing (1.3.2) and (1.3.3), we obtain the  $D(\mathfrak{h}^{\text{reg}}/W)$ -module  $M[\delta^{-1}]^W$ , which we can view as a flat connection on the smooth variety  $\mathfrak{h}^{\text{reg}}/W$ . As it is shown in [GGOR], such connection has regular singularities, so we can apply the *deRham functor* DR and obtain a representation of the fundamental group  $B_W = \pi_1(\mathfrak{h}^{\text{reg}}/W)$ , by the *Riemann-Hilbert correspondence* (as explained in [De]). Putting all the pieces together, we can define the following functor.

**Definition 1.3.3.** The KZ functor is defined by

$$\begin{aligned} \text{KZ} : \mathcal{O}(H_{\mathbf{k}}(W)) &\longrightarrow \mathbb{C}[B_W]\text{-mod} \\ M &\longmapsto \text{DR}(M[\delta^{-1}]^W) . \end{aligned}$$

Let  $\mathcal{O}_{\text{tor}}(H_{\mathbf{k}}(W))$  be the Serre subcategory of  $\mathcal{O}(H_{\mathbf{k}}(W))$  whose objects are torsion modules for the Ore set  $\{\delta^m\}_{m \in \mathbb{N}}$ . Given a parameter  $\mathbf{k}$  for the rational Cherednik algebra, set

$$q_{\mathcal{H},j} = (\det s_{\mathcal{H}})^j e^{2\pi i - k_{\mathcal{H},j}} .$$

**Theorem 1.3.4.** ([GGOR, Theorem 5.13]) *The KZ functor factors through*

$$\mathrm{KZ} : \mathcal{O}(H_{\mathbf{k}}(W))/\mathcal{O}_{\mathrm{tor}}(H_{\mathbf{k}}(W)) \longrightarrow \mathcal{H}_{\mathbf{q}}(W)\text{-mod}.$$

Since all the functors occurring in Definition 1.3.3 are exact, so is the KZ functor, thus there exists a projective module  $P_{\mathrm{KZ}} \in \mathcal{O}(H_{\mathbf{k}}(W))$  such that

$$\mathrm{KZ}(-) \cong \mathrm{Hom}_{\mathcal{O}(H_{\mathbf{k}}(W))}(P_{\mathrm{KZ}}, -). \quad (1.3.4)$$

By definition of projective cover, we have  $\dim \mathrm{Hom}_{\mathcal{O}(H_{\mathbf{k}}(W))}(P(\rho_i), L(\rho_j)) = \delta_{ij}$ . Since, by Theorem 1.2.7,  $\{P(\rho) \mid \rho \in \mathrm{Irrep}(W)\}$  forms a complete collection of indecomposable projective modules in  $\mathcal{O}(H_{\mathbf{k}}(W))$ , we can write  $P_{\mathrm{KZ}} = \bigoplus_{\rho} P(\rho)^{m_{\rho}}$ , where  $m_{\rho} = \dim \mathrm{Hom}_{\mathcal{O}(H_{\mathbf{k}}(W))}(P_{\mathrm{KZ}}, L(\rho))$ . Therefore  $P_{\mathrm{KZ}}$  has the form

$$P_{\mathrm{KZ}} = \bigoplus_{\rho \in \mathrm{Irrep}(W)} \dim \mathrm{KZ}(L(\rho)) P(\rho).$$

The main result of [GGOR] is the *double centraliser theorem* (cf. [GGOR, Theorems 5.14, 5.15 and 5.16]), which reads:

**Theorem 1.3.5.** (i) *The morphism  $\phi : \mathcal{H}_{\mathbf{q}}(W) \rightarrow \mathrm{End}_{\mathcal{O}(H_{\mathbf{k}}(W))}(P_{\mathrm{KZ}})^{\mathrm{op}}$  induced by Theorem 1.3.4, is an algebra isomorphism.*

(ii) *Equivalently, the KZ functor induces an equivalence of categories*

$$\mathrm{KZ} : \mathcal{O}(H_{\mathbf{k}}(W))/\mathcal{O}_{\mathrm{tor}}(H_{\mathbf{k}}(W)) \xrightarrow{\sim} \mathcal{H}_{\mathbf{q}}(W)\text{-mod}.$$

(iii) *Given a projective generator  $X$  of  $\mathcal{O}(H_{\mathbf{k}}(W))$  (e.g.  $X = \bigoplus_{\rho} P(\rho)$ ), we have an equivalence of categories*

$$\mathcal{O}(H_{\mathbf{k}}(W)) \cong \mathrm{End}_{\mathcal{H}_{\mathbf{q}}(W)}(\mathrm{KZ}(X))^{\mathrm{op}}\text{-mod}.$$

Theorem 1.3.5 provides an approach to the study of category  $\mathcal{O}(H_{\mathbf{k}}(W))$  via the Iwahori-Hecke algebra  $\mathcal{H}_{\mathbf{q}}(W)$ , whose representation theory displays a nice combinatorics. One useful consequence of this result is the following theorem, due to Rouquier [Ro2] (see also [Va, Theorem 2.1]).

**Theorem 1.3.6.** *The following are equivalent:*

- (1)  $H_{\mathbf{k}}(W)$  is a simple algebra.
- (2)  $\mathcal{O}(H_{\mathbf{k}}(W))$  is a semisimple category.
- (3)  $\mathcal{H}_{\mathbf{q}}(W)$  is a semisimple algebra.

## 1.4 The cyclotomic rational Cherednik algebra

Let  $\mu_\ell$  denote the cyclic group of order  $\ell$ . The semidirect product  $\Gamma := \Gamma_n = \Sigma_n \ltimes \mu_\ell^n$  is a somewhat special symplectic reflection group. Beyond being a wreath product ((2) in the classification of section 1.1.1), it can also be realised as a complex reflection group acting on two copies of its reflection representation ((1) in the classification). Hence, the corresponding symplectic reflection algebra  $H_c(\Gamma_n)$  is in fact a rational Cherednik algebra, yet its representation theory is related to the geometry of the Hilbert scheme of  $n$  points on the minimal resolution of the Kleinian singularity, which is a common feature of the symplectic reflection algebras associated to wreath products (see Theorem 2.4.10). Furthermore, it specialises to the cyclic rational Cherednik algebra, when  $n = 1$ , which is the only case that can be studied via direct computations (see section 3.3), and to the rational Cherednik algebra of type A, when  $\ell = 1$ , which was extensively studied from combinatorial ([VV]), geometric ([GG, GS1, GS2]) and microlocal ([KR]) point of view.

In this section, we collect some important facts about the cyclotomic rational Cherednik algebra, which will be our main object of study in the remaining chapters.

### 1.4.1 Definition

Denote by  $s_{ij}$  the transposition swapping  $i$  and  $j$  in  $\Sigma_n$ , embedded in  $\Gamma_n$  as the element  $(s_{ij}, 1, \dots, 1)$ , and by  $s_i$  a generator of the  $i$ -th copy of the cyclic group inside  $\Gamma_n$ , namely  $s_i = (1, \dots, s, \dots, 1)$ . The set of symplectic reflections  $\mathcal{S}$  has two types of conjugacy classes:

- (1) the elements of the form  $s_{ij}s_i^m s_j^{-m}$ , for  $1 \leq i, j \leq n$  and  $0 \leq m \leq \ell - 1$ ;
- (2) the elements  $s_i^m$ , for  $1 \leq i \leq n$  and  $0 < m \leq \ell - 1$ .

There is one conjugacy class of type (1) and  $\ell - 1$  of type (2) (one for each value of  $m$ ). Hence, the space of deformation parameters for the corresponding symplectic reflection algebra is  $\mathfrak{z}^* \cong \mathbb{C}^\ell$  and we can consider parameters of the form  $(k, \mathbf{c})$ , where  $k \in \mathbb{C}$  corresponds to the unique conjugacy class of type (1) and  $\mathbf{c} = (c_1, \dots, c_{\ell-1}) \in \mathbb{C}^{\ell-1}$  corresponds to the conjugacy classes of type (2).

Let  $\mathfrak{h}$  be the ( $n$ -dimensional) reflection representation of  $\Gamma_n$ ,  $\mathfrak{h}^*$  be its dual, and pick dual bases  $y_1, \dots, y_n \in \mathfrak{h}$ ,  $x_1, \dots, x_n \in \mathfrak{h}^*$ . Recall that  $\Sigma_n$  acts on  $\mathfrak{h} \oplus \mathfrak{h}^*$  by simultaneously permuting the  $x_i$ 's and the  $y_i$ 's, while the  $i$ -th copy of  $\mu_\ell$  inside  $\Gamma_n$  acts on the symplectic space spanned by  $\{x_i, y_i\}$ , by the inclusion  $s^m \mapsto \text{diag}(\zeta^m, \zeta^{-m}) \in SL_2(\mathbb{C})$ , where  $\zeta$  denotes a primitive  $\ell$ -th root of unity.

**Definition 1.4.1.** The **cyclotomic rational Cherednik algebra**  $H_{k,c}(\Gamma_n)$  is the quotient of  $T(\mathfrak{h} \oplus \mathfrak{h}^*) \# \Gamma_n$  by the relations

$$\begin{aligned} [x_i, x_j] &= 0, \quad [y_i, y_j] = 0, \quad i, j = 1, \dots, n, \\ [y_i, x_i] &= 1 - k \sum_{j \neq i} \sum_{m=0}^{\ell-1} s_{ij} s_i^m s_j^{-m} - 2 \sum_{m=1}^{\ell-1} c_m s_i^m, \quad i = 1, \dots, n, \\ [y_i, x_j] &= k \sum_{m=0}^{\ell-1} \zeta^m s_{ij} s_i^m s_j^{-m}, \quad i, j = 1, \dots, n, \quad i \neq j. \end{aligned}$$

Given a parameter  $\mathbf{k} = (k_{00}, k_1, \dots, k_{\ell-1})$  that satisfies the condition (1.3.1) (where  $k_{00}$  corresponds to the hyperplane of the elements of type (1) and  $k_i$ ,  $0 \leq i \leq \ell$  correspond to the hyperplane of elements of type (2), where, by convention,  $k_0 = k_\ell = 0$ ), the last two relations become

$$\begin{aligned} [y_i, x_i] &= 1 - k_{00} \sum_{j \neq i} \sum_{m=0}^{\ell-1} s_{ij} s_i^m s_j^{-m} - \sum_{r=0}^{\ell-1} (k_{r+1} - k_r) \sum_{m=0}^{\ell-1} \zeta^{rm} s_i^m, \quad i = 1, \dots, n, \\ [y_i, x_j] &= k_{00} \sum_{m=0}^{\ell-1} \zeta^m s_{ij} s_i^m s_j^{-m}, \quad i, j = 1, \dots, n, \quad i \neq j. \end{aligned}$$

## 1.4.2 Multipartitions

A partition of a positive integer  $n$  is a non-increasing sequence  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ , with  $\lambda_i \in \mathbb{Z}_{>0}$ , such that  $\sum_i \lambda_i = n$ . We write  $\lambda \vdash n$ ,  $|\lambda| = n$  and  $\text{ln}(\lambda)$  for the length of the partition (namely, the greatest  $i$  such that  $\lambda_i \neq 0$ ). It is well known that these are in bijection with the *Specht modules*, which, in characteristic zero, form a complete set of non-isomorphic irreducible modules of the symmetric group algebra  $\mathbb{C}[\Sigma_n]$ . By convention, the partition  $(n)$  will label the trivial representation and the partition  $(1, \dots, 1)$  will correspond to the sign representation.

Given another positive integer  $\ell$ , a **multipartition of  $n$  in  $\ell$  parts**, or  **$\ell$ -multipartition of  $n$** , is a  $\ell$ -tuple of partitions  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ , such that  $\sum_{i=1}^{\ell} |\lambda^{(i)}| = n$ . Let  $\mathcal{P}(\ell, n)$  be the set of  $\ell$ -multipartitions of  $n$ . It is known that the irreducible representations of  $\Gamma_n$  are labelled by  $\mathcal{P}(\ell, n)$  and are described as follows (we follow the presentation given in [Ro2, section 6.1.1]). For  $\lambda \in \mathcal{P}(\ell, n)$  and  $1 \leq r \leq \ell$ , consider

$$I_\lambda(r) := \left\{ \sum_{i=1}^{r-1} |\lambda^{(i)}| + 1, \sum_{i=1}^{r-1} |\lambda^{(i)}| + 2, \dots, \sum_{i=1}^r |\lambda^{(i)}| \right\},$$

an ordered, possibly empty subset of  $\{1, 2, \dots, n\}$ , and write  $\Sigma_\lambda := \Sigma_{I_\lambda(1)} \times \dots \times \Sigma_{I_\lambda(\ell)}$  for the corresponding subgroup of the symmetric group  $\Sigma_n$ . Finally, denote by  $\phi(r)$  the one-dimensional representation of  $\Sigma_{I_\lambda(r)} \times (\mu_\ell)^{I_\lambda(r)}$  with trivial  $\Sigma_{I_\lambda(r)}$ -action and whose character of the  $(\mu_\ell)^{I_\lambda(r)}$ -action equals  $\det^{r-1}$ , and by  $\chi_{\lambda(r)}$  the irreducible representation of  $\Sigma_{I_\lambda(r)}$  labelled by  $\lambda^{(r)}$ . Then,

$$\left\{ \rho_\lambda := \text{Ind}_{\Sigma_\lambda \times (\mu_\ell)^n}^{\Sigma_n \times (\mu_\ell)^n} (\phi(1)\chi_{\lambda(1)} \otimes \dots \otimes \phi(\ell)\chi_{\lambda(\ell)}) \mid \lambda \in \mathcal{P}(\ell, n) \right\}$$

is a complete set of non-isomorphic irreducible representations of  $\Gamma_n$ . Note that, with our choice, the trivial representation is labelled by the  $\ell$ -multipartition  $((n), \emptyset, \dots, \emptyset)$ .

### 1.4.3 The Ariki-Koike algebra

We give an explicit definition of the Iwahori-Hecke algebra in the case  $W = \Gamma_n$ .

**Definition 1.4.2.** Let  $\mathbf{q} = (q_{00}, q_1, \dots, q_{\ell-1}) \in \mathbb{C}^\ell$ . The **Ariki-Koike algebra**  $\mathcal{H}_{\mathbf{q}}(\Gamma_n)$  is the algebra generated by  $T_1, \dots, T_n$  and subject to the relations:

$$\begin{aligned} T_1 T_2 T_1 T_2 - T_2 T_1 T_2 T_1 &= 0, \\ T_i T_{i+1} T_1 - T_{i+1} T_i T_{i+1} &= 0, \quad 2 \leq i \leq n-1, \\ [T_i, T_j] &= 0, \quad |i-j| > 1, \\ (T_i - 1)(T_i + q_{00}) &= 0, \quad 2 \leq i \leq n, \\ (T_1 - 1) \prod_{i=1}^{\ell-1} (T_1 - q_i) &= 0. \end{aligned}$$

This algebra has a *Specht module*  $S^\lambda$  for all multipartitions  $\lambda \in \mathcal{P}(\ell, n)$ , each of which has a quotient  $D^\lambda$ , which is either zero or irreducible. By [Mat, Theorem 3.12],  $\{D^\lambda \mid D^\lambda \neq 0, \lambda \in \mathcal{P}(\ell, n)\}$  forms a complete set of non-isomorphic simple  $\mathcal{H}_{\mathbf{q}}(\Gamma_n)$ -modules, and by [Mat, Theorem 3.13], the algebra  $\mathcal{H}_{\mathbf{q}}(\Gamma_n)$  is semisimple if and only if  $S^\lambda = D^\lambda, \forall \lambda \in \mathcal{P}(\ell, n)$ . The set of parameters  $\mathbf{q}$  for which this happens has been explicitly calculated. Let  $[n]_{q_{00}}! := \prod_{i=1}^n (1 + q_{00} + \dots + q_{00}^{i-1})$ .

**Theorem 1.4.3.** (*[Ari, Main Theorem]*) *The algebra  $\mathcal{H}_{\mathbf{q}}(\Gamma_n)$  is semisimple if and only if*

$$[n]_{q_{00}}! \prod_{h \neq j} \prod_{r=-\ell}^{\ell} (q_h - q_{00}^r q_j) \neq 0.$$

Let  $\mathbf{k}$  be a parameter for the rational Cherednik algebra  $H_{\mathbf{k}}(\Gamma_n)$  and set  $q_{00} = e^{-2\pi i k_{00}}$  and  $q_j = \zeta^{-j} e^{2\pi i k_j}$ , for  $1 \leq j \leq \ell-1$ . By Theorem 1.3.6, we obtain a criterion for category  $\mathcal{O}(H_{\mathbf{k}}(\Gamma_n))$  to be semisimple.

**Corollary 1.4.4.** *The cyclotomic rational Cherednik algebra is simple (and thus category  $\mathcal{O}(H_{\mathbf{k}}(\Gamma_n))$  is semisimple) if and only if*

$$[n]_{e^{-2\pi i k_{00}}}! \prod_{h \neq j} \prod_{r=-\ell}^{\ell} (\zeta^{-h} e^{2\pi k_h} - \zeta^{-j} e^{-2\pi i(r k_{00} - k_j)}) \neq 0. \quad (1.4.1)$$

Corollary 1.4.4 tells us that  $H_{k,\mathbf{c}}(\Gamma_n)$  ( $= H_{\mathbf{k}}(\Gamma_n)$ ) is a simple algebra for generic values of  $(k, \mathbf{c}) \in \mathfrak{z}^*$ . As it is natural to expect, the most interesting case is when the parameters do not satisfy equation (1.4.1), which is the case we will consider in chapter 3.

### 1.4.4 Spherical values

We end this section with another useful theorem, due to Dunkl and Griffeth [DG], which describes the set of parameters for which the cyclotomic rational Cherednik algebra is Morita equivalent to its spherical subalgebra.

**Theorem 1.4.5.** *The parameters  $(k, \mathbf{c}) \in \mathfrak{z}^* \cong \mathbb{C}^\ell$  are aspherical values if and only if:*

- (1)  $k = -\frac{a}{b}$ , for some  $a, b \in \mathbb{Z}_{>0}$ ,  $1 \leq a < b \leq n$ , or
- (2) there exist a partition  $\lambda \vdash n$  and integers  $0 \leq d \leq \ell - 1$  and  $e \neq 0 \pmod{\ell}$ , such that  $1 \leq e \leq d + \ell(\ln(\lambda) - 1)$  and

$$\frac{2e}{\ell} - \left( \left[ \frac{e-d}{\ell} \right] + 1 \right) = \sum_{m=1}^e C_{d-e+m} + k(\lambda_1 - \ln(\lambda)),$$

where  $C_i := \ell^{-1} \left( 1 + 2 \sum_{m=1}^{\ell-1} \zeta^{mi} c_m \right)$ , for  $i = 1, \dots, \ell - 1$ .

# Chapter 2

## Quantisation of quiver varieties

### 2.1 Symplectic resolutions

In this section, we recall symplectic resolutions of singularities.

Let  $X$  be a possibly singular (affine, normal, irreducible) Poisson variety, namely the ring of regular sections  $\mathbb{C}[X]$  is equipped with a Lie bracket  $\{\cdot, \cdot\}$  that satisfies Leibniz rule. Let  $\pi : \tilde{X} \rightarrow X$  be a resolution of singularities (i.e. a birational, proper map), with  $(\tilde{X}, \omega)$  a smooth symplectic variety. The non-degenerate 2-form  $\omega$  induces a Poisson structure on  $\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}})$ , by  $\{f, g\} := \omega(\xi_f, \xi_g)$ , where  $\xi_f$  is a vector field satisfying  $\omega(\cdot, \xi_f) = df$ , for  $f \in \mathcal{O}(\tilde{X})$ .

**Definition 2.1.1.**  $\pi$  is a **symplectic resolution** if the pull-back  $\pi^* : \mathbb{C}[X] \rightarrow \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}})$  is a morphism of Poisson algebras.

**Remark 2.1.2.** Alternatively, one can define a symplectic resolution  $\pi : \tilde{X} \rightarrow X$ , with  $X$  an algebraic variety whose regular locus is equipped with a symplectic form  $\omega$ , by requiring that the pull-back  $\pi^*\omega$  extends to a symplectic form on  $\tilde{X}$  (cf. [Kal2]).

All of the symplectic resolutions we will deal with in the rest of the paper are of the following type.

**Definition 2.1.3.** A symplectic resolution  $\pi^* : \tilde{X} \rightarrow X$  is called **conical** if  $\tilde{X}$  is equipped with a  $\mathbb{C}^\times$ -action such that  $\lambda^*\omega = \lambda^d\omega$ , for some  $d \geq 1$ ,  $X$  is a cone and this action contracts  $X$  to a (unique) point  $o$ . Algebraically, this is equivalent to giving a non-negative grading on the coordinate ring  $\mathbb{C}[X] = \bigoplus_{i \in \mathbb{N}} \mathbb{C}^i[X]$ , such that the 0-th component, coinciding with the  $\mathbb{C}^\times$ -invariant elements, consists of only the constant functions, and one has

$$\{\mathbb{C}^i[X], \mathbb{C}^j[X]\} \subset \mathbb{C}^{i+j-d}[X],$$

for some  $d \geq 1$ . We call  $\pi^{-1}(o)$ , the fibre over the  $\mathbb{C}^\times$ -fixed point of  $X$ , the **central fibre** of the resolution.

The following theorem collects some well-known useful properties of conical symplectic resolutions.

**Theorem 2.1.4.** 1.  $\dim \tilde{X} \times_X \tilde{X} = \dim X$ , and in particular  $\pi$  is semismall.

2.  $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ , for  $i > 0$ , and  $\pi^* : \mathbb{C}[X] \rightarrow \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}})$  is an isomorphism.

3.  $H^{2k+1}(\tilde{X}, \mathbb{C}) = 0$  and  $H^{2k}(\tilde{X}, \mathbb{C}) = H^{k,k}(\tilde{X}, \mathbb{C})$ , for  $k \geq 0$ .

4. There is a decomposition into finitely many symplectic leaves  $X = \bigsqcup X_i$ , where  $X_i$  is a smooth, locally closed subvariety in  $X$ .

*Proof.* (1) is Proposition 4.4 of [Kal2].

In (2), the fact the  $\pi^*$  is an isomorphism follows by a standard argument because  $\pi$  is proper and  $X$  is normal. Since  $\tilde{X}$  is symplectic, it has trivial canonical bundle and by the Grauert-Reimenschneider theorem the derived direct image  $\mathcal{R}^i \pi_* \mathcal{O}_{\tilde{X}}$  vanishes for  $i > 0$ , because  $\pi$  is proper. Thus  $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^i(X, \pi_* \mathcal{O}_{\tilde{X}}) = 0$ , for  $i > 0$ .

Let  $\beta : X \rightarrow \mathbb{R}_{\geq 0}$  an algebraic function such that  $\alpha := \beta \circ \pi$  is a rug function for the central fibre in  $\tilde{X}$  (see [Du, Definition 1.1]). Then, for  $t > 0$ ,  $\alpha^{-1}([0, t])$  is an algebraic neighbourhood of  $\pi^{-1}(o)$  in  $\tilde{X}$  and hence these two are homotopic by [Du, Proposition 1.6]. (3) follows now by [Kal3, Theorem 2.12].

(4) is proved in [Kal3], Theorem 2.3. □

**Example 2.1.5.** (a) Let  $\mathfrak{g}$  be a semisimple Lie algebra and  $G/B$  the corresponding flag variety. Then, the total space of the cotangent bundle  $T^*G/B$  is naturally a smooth symplectic variety. The subvariety  $\mathcal{N}$  of nilpotent elements of  $\mathfrak{g}$  is a cone with respect to an algebraic  $\mathbb{C}^\times$ -action and its regular locus  $\mathcal{N} \setminus \{0\}$  is equipped with the Kirillov-Konstant Poisson structure induced from  $\mathfrak{g}^*$  (identifying  $\mathfrak{g}$  and  $\mathfrak{g}^*$  by means of the Killing form). The Springer resolution  $\mu : T^*G/B \rightarrow \mathcal{N}$  is a conical symplectic resolution.

(b) More generally, let  $e \in \mathcal{N}$ . By the Jacobson-Morozov theorem, we can embed  $e$  in an  $\mathfrak{sl}_2$ -triple  $e, f, h$  satisfying the standard commutation relations. The intersection of the Slodowy slice with the nilpotent cone  $\mathcal{S}_e := (e + \ker \operatorname{ad} f) \cap \mathcal{N}$ , is singular Poisson and its preimage via the Springer resolution  $\tilde{\mathcal{S}}_e := \pi^{-1}(\mathcal{S}_e)$  gives a symplectic resolution  $\pi_e : \tilde{\mathcal{S}}_e \rightarrow \mathcal{S}_e$ .

- (c) Let  $\Gamma$  be a finite subgroup of  $SL_2(\mathbb{C})$ . By blowing up, the affine singular Poisson variety  $\mathbb{C}^2/\Gamma$  admits a crepant resolution  $\widetilde{\mathbb{C}^2}/\Gamma$ , whose exceptional divisor forms a chain of  $\mathbb{P}^1$ 's, giving rise to a Dynkin diagram of type A,D,E, by McKay correspondence. The  $\mathbb{C}^*$ -action is induced by the inverse of the diagonal action and  $\pi : \widetilde{\mathbb{C}^2}/\Gamma \rightarrow \mathbb{C}^2/\Gamma$  is a conical symplectic resolution.
- (d) The symmetric group  $\Sigma_n$  acts diagonally on  $\mathbb{C}^{2n}$ , two copies of the permutation representation. The symmetric  $n$ -th power of  $\mathbb{C}^2$  coincides with the singular Poisson variety  $\mathbb{C}^{2n}/\Sigma_n$  and it is resolved by the Hilbert scheme of  $n$  points on  $\mathbb{C}^2$  via the Hilbert-Chow morphism. The diagonal inverse  $\mathbb{C}^*$ -action on  $\mathbb{C}^2$  induces an action on the Hilbert scheme and  $\pi : \text{Hilb}^n(\mathbb{C}^2) \rightarrow \mathbb{C}^{2n}/\Sigma_n$  turns out to be a conical symplectic resolution.
- (e) Combining the last two constructions on gets another example of conical symplectic resolutions, namely  $\pi : \text{Hilb}^n(\widetilde{\mathbb{C}^2}/\Gamma) \rightarrow \mathbb{C}^{2n}/\Gamma_n$ , where  $\Gamma_n := \Sigma_n \wr \Gamma$  is the wreath product. As we will see in the next sections, these resolutions are closely related to the corresponding symplectic reflection algebras.

### 2.1.1 Hamiltonian reduction

Let  $V$  be a smooth, irreducible, affine algebraic variety and consider its cotangent bundle  $T^*V$ , a symplectic manifold. Given an action of a connected, reductive algebraic group  $G$  on  $V$ , consider the induced infinitesimal action of  $\mathfrak{g} := \text{Lie } G$ . The latter gives a Lie morphism  $\phi : \mathfrak{g} \rightarrow \text{Vect } V$ , where  $\text{Vect } V$  denotes the vector fields on  $V$ , and dualising this map we obtain the moment map  $\mu : T^*V \rightarrow \mathfrak{g}^*$ . More precisely, let  $\alpha \in T^*V$  and denote by  $\alpha_x \in T_x^*V$ , a covector at a point  $x \in V$ . Then,  $\mu(\alpha_x)$  is the linear functional on  $\mathfrak{g}$  defined by  $a \mapsto \langle \alpha_x, \phi(a)_x \rangle$ , for  $a \in \mathfrak{g}$ .

In the case when  $V$  is a vector space, the symplectic form on the cotangent bundle  $T^*V = V \times V^*$  is given by the formula

$$\omega((x, x^*), (y, y^*)) = \langle y^*, x \rangle - \langle x^*, y \rangle, \quad \forall x, y \in V, x^*, y^* \in V^*.$$

If a group  $G$  acts linearly on  $V$ , the resulting moment map is given by  $\mu(x, x^*)(a) = \langle x^*, a.x \rangle$ , for all  $x \in V$ ,  $x^* \in V^*$  and  $a \in \mathfrak{g} = \text{Lie } G$ .

If the  $G$ -action on  $V$  is free, the cotangent bundle  $T^*V/G$  coincides with  $\mu^{-1}(0)/G$ . This gives a motivation to study such a space and, more generally, its deformations  $\mu^{-1}(\lambda)/G$ , for  $\lambda \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$ , even when the  $G$ -action is not free (in such a case, we need to specify what kind of quotient we consider). This construction is called

**Hamiltonian reduction.**

Let  $W$  be a smooth algebraic variety acted on by  $G$ . We will be mainly interested in the case  $W = T^*V$ . Recall the GIT quotient: for a character  $\theta : G \rightarrow \mathbb{C}^\times$ , we define an action of  $G$  on the Cartesian product  $W \times \mathbb{C}$ , by  $g.(w, z) := (gw, \theta(g)^{-1}z)$ , and consider the ring of  $G$ -invariant regular functions  $\mathbb{C}[W \times \mathbb{C}]^G$ . This is a graded ring by

$$\mathbb{C}[W \times \mathbb{C}]^G = \bigoplus_{n \in \mathbb{N}} \mathbb{C}[W]^{\theta^n},$$

and we define  $W//^\theta G := \text{Proj } \mathbb{C}[W \times \mathbb{C}]^G$ , a quasi-projective, irreducible scheme. Since, for  $n = 0$ , we have  $\mathbb{C}[W]^{\theta^n} = \mathbb{C}[W]^G$ , there is a canonical embedding  $\mathbb{C}[W]^G \hookrightarrow \mathbb{C}[W \times \mathbb{C}]^G$  (induced by the projection  $W \times \mathbb{C} \rightarrow W$ ), which gives rise to a projective morphism

$$\pi : W//^\theta G := \text{Proj } \mathbb{C}[W \times \mathbb{C}]^G \longrightarrow \text{Spec } \mathbb{C}[W]^G =: W//G.$$

To understand the GIT quotient geometrically, consider the  $\theta$ -unstable locus of  $W$   $W^{\theta-un} := \{w \in W \mid f(w) = 0, \forall f \in \mathbb{C}[W]^{\theta^n}, \forall n \in \mathbb{N}\}$ . By general results (cf. [MFK]), if the  $G$ -action on  $W \setminus W^{\theta-un}$  is free, then there is an isomorphism between  $(W \setminus W^{\theta-un})/G$  and the closed points of  $W//^\theta G$ , which form, therefore, a smooth quasi-projective variety.

We are interested in Hamiltonian reductions performed via GIT and affine quotients.

**Example 2.1.6.** Let  $V = \mathbb{C}^2$  and  $G = \mathbb{C}^\times$  with the action on  $V$  induced by the inclusion  $\mathbb{C}^\times \hookrightarrow GL(V)$ ,  $\lambda \mapsto \lambda \text{Id}$ . The resulting action on the cotangent bundle  $W = T^*V = \mathbb{C}^4$  is given by  $\lambda.(x_1, x_2, y_1, y_2) = (\lambda x_1, \lambda x_2, \lambda^{-1}y_1, \lambda^{-1}y_2)$  and the moment map translates to  $\mu(x_1, x_2, y_1, y_2) = x_1y_1 + x_2y_2$  via the Killing form. The invariant regular functions on the zero fibre are

$$\mathbb{C}[\mu^{-1}(0)]^{\mathbb{C}^\times} = \mathbb{C}[x_1y_1, x_1y_2, x_2y_1, x_2y_2]/(x_1y_1 + x_2y_2) \cong \mathbb{C}[u, v, w]/(u^2 - vw).$$

Taking the spectrum  $X := \text{Spec } \mathbb{C}[u, v, w]/(u^2 - vw)$  we get the affine Hamiltonian reduction, a cubic in  $\mathbb{C}^3$  singular at 0. Consider now the trivial character  $\theta = 1$ . The unstable locus of  $W$  consists, by definition, of all the points that are annihilated by homogeneous functions in  $x_1, x_2$ . Hence,

$$(T^*\mathbb{C}^2)^{\theta-un} = \{(0, 0, y_1, y_2)\} = T_{(0,0)}^*\mathbb{C}^2$$

and taking the GIT quotient we obtain

$$\mu^{-1}(0)//^\theta \mathbb{C}^\times = T^*(\mathbb{C}^2 \setminus \{(0, 0)\}/\mathbb{C}^\times) = T^*\mathbb{P}^1.$$

The canonical map  $\pi : T^*\mathbb{P}^1 \rightarrow X$  is a conical symplectic resolution.

Notice that the previous example is a particular case of (a), (b), (c) and (e) in Example 2.1.5. As we will see in the next section, most instances of conical symplectic resolutions can, in fact, be obtained as canonical maps between GIT and affine Hamiltonian reductions with  $V$  a vector space and  $G \leq GL(V)$ .

## 2.2 Representations of quivers

To fix the notation, let  $Q = (I, E, s, t)$  be a quiver, with  $I$  the (finite) set of vertices,  $E$  the (finite) set of edges (we allow multiple arrows and loops) and  $s, t : Q \rightarrow I$  the source and target maps. The opposite quiver is given by  $Q^{op} = (I, E, s^{op}, t^{op})$ , with  $s^{op} = t$  and  $t^{op} = s$ . Let  $\text{Rep}(Q)$  denote the category of representations of  $Q$ , namely the objects are collections of vector spaces  $(V_i)_{i \in I}$  together with a family of linear maps  $f_\gamma^V : V_{s(\gamma)} \rightarrow V_{t(\gamma)}$ ,  $\forall \gamma \in E$ , and morphisms are linear maps  $(u_i : V_i \rightarrow W_i)_{i \in I}$  compatible with  $f_\gamma^V$  and  $f_\gamma^W$ ,  $\forall \gamma \in E$ . It is immediate to show that  $\text{Rep}(Q)$  is equivalent to the category of left  $\mathbb{C}Q$ -modules, where  $\mathbb{C}Q$  is the algebra of the quiver  $Q$ , namely it has generators  $e_i$ , for  $i \in I$  and  $\gamma \in E$ , and it is subject to the relations

$$e_i e_j = \delta_{ij} \quad \text{and} \quad e_{t(\gamma)} \gamma = \gamma e_{s(\gamma)} = \gamma$$

( $e_i$  and  $\gamma$  are paths of length 0 and 1, respectively). Hence,  $\text{Rep}(Q)$  is automatically an abelian  $\mathbb{C}$ -linear category. To a finite dimensional representation  $(V_i)_{i \in I} \in \text{Rep}(Q)$  we can assign its dimension vector  $\mathbf{v} = (v_i)_{i \in I} \in \mathbb{N}$ ,  $v_i := \dim V_i$ .

An important problem in quiver theory is to classify isomorphism classes of finite dimensional representations. Since two isomorphic finite dimensional representations of  $Q$  obviously have the same dimension vector, it makes sense to consider the space of representations with a given dimension vector  $\mathbf{v} \in \mathbb{N}$ :

$$\text{Rep}(Q, \mathbf{v}) := \bigoplus_{\gamma \in E} \text{Hom}_{\mathbb{C}}(\mathbb{C}^{v_{s(\gamma)}}, \mathbb{C}^{v_{t(\gamma)}}) = \bigoplus_{\gamma \in E} \text{Mat}_{v_{t(\gamma)} \times v_{s(\gamma)}}(\mathbb{C}).$$

This space is acted on by the reductive group  $G_{\mathbf{v}} := \prod_{i \in I} GL_{v_i}(\mathbb{C})$  by base change

$$(g_i)_{i \in I} \cdot (X_\gamma)_{\gamma \in E} = \left( g_{t(\gamma)} X_\gamma g_{s(\gamma)}^{-1} \right)_{\gamma \in E},$$

where  $X_\gamma \in \text{Mat}_{v_{t(\gamma)} \times v_{s(\gamma)}}(\mathbb{C})$  and this descends to an action of the projective linear group  $PG_{\mathbf{v}} := G_{\mathbf{v}}/\mathbb{C}^\times$ . It is immediate to see that there is a bijection between the orbits of  $PG_{\mathbf{v}}$  in  $\text{Rep}(Q, \mathbf{v})$  and the set of isomorphism classes of representations of  $Q$  with dimension vector  $\mathbf{v}$ . Thus, we can exploit the technology developed in the previous section to study the quotient  $\text{Rep}(Q, \mathbf{v})/PG_{\mathbf{v}}$  geometrically. We first need

a notion of stability for representations of quivers introduced by King (cf. [Ki]). His approach is purely algebraic and makes it easier to check when stability conditions hold for representations of quivers, yet it is equivalent to that of [MFK] in this case. Let  $V = (V_i)_{i \in I} \in \text{Rep}(Q, \mathbf{v})$  be a non-zero representation of  $Q$  and consider the character defined by

$$(g_i)_{i \in I} \mapsto \prod_{i \in I} \det(g_i)^{\theta_i}, \quad (2.2.1)$$

where  $\theta := (\theta_i)_{i \in I} \in \mathbb{Z}^I$  is an integral vector. By abuse of notation, we will also denote the character (2.2.1) by  $\theta$ .

As we will see, an interesting choice of  $\theta$  is  $\theta^+ := (-1, \dots, -1)$ .

**Definition 2.2.1.** The **slope** of  $V$  with respect to  $\theta$  is defined by the formula

$$\text{slope}_\theta(V) := (\theta \cdot \mathbf{v}) / (\theta^+ \cdot \mathbf{v}).$$

$V$  is called  **$\theta$ -semistable** if  $\text{slope}_\theta(U) \leq \text{slope}_\theta(V)$ , for all non-zero subrepresentations  $U$  of  $V$ . It is called  **$\theta$ -stable** if the inequality is strict whenever  $U$  is a proper subrepresentation.

The main result of King is the following.

**Theorem 2.2.2** ([Ki]). *Assume that  $Q$  has no edge loops and let  $\mathbf{v} \in \mathbb{N}$ ,  $\theta \in \mathbb{Z}^I$  such that  $\theta \cdot \mathbf{v} = 0$ . Then,  $PG_{\mathbf{v}}$  acts freely on the set  $\text{Rep}(Q, \mathbf{v})^{\theta-s}$  of  $\theta$ -stable points in  $\text{Rep}(Q, \mathbf{v})$ , and the orbit space  $\text{Rep}(Q, \mathbf{v})^{\theta-s} / PG_{\mathbf{v}}$  is a Zariski open (possibly empty) subset in  $\text{Rep}(Q, \mathbf{v}) //^\theta PG_{\mathbf{v}}$ . When the  $\theta$ -semistable and  $\theta$ -stable representations coincide,  $\text{Rep}(Q, \mathbf{v}) //^\theta PG_{\mathbf{v}}$  is a smooth quasi-projective variety of dimension  $1 + {}^t \mathbf{v} A_Q \mathbf{v} - \mathbf{v} \cdot \mathbf{v}$ , where  $A_Q \in \text{Mat}_{|I| \times |I|}(\mathbb{N})$  is the adjacency matrix of  $Q$  (namely, the  $(i, j)$ -th entry equals the number of arrows from  $i$  to  $j$ ).*

## 2.2.1 Reflection functors

We now introduce some functors that naturally arise when studying indecomposable representations of quivers and, as we will see in the next section, turn out to be useful to understand quiver varieties.

For a quiver  $Q = (I, E)$ , we say that a vertex  $i \in I$  is **+-admissible** if there is no arrow in  $E$  that has  $i$  as a source. We call  $i$  **--admissible** if there is no arrow in  $E$  that has  $i$  as a target. Let  $\varphi_i(Q)$  be the quiver obtained by reversing all the arrows that have  $i$  either as a source or a target. If  $i$  is +-admissible, define a functor

$$\begin{aligned} \varphi_i^+ : \text{Rep}(Q) &\longrightarrow \text{Rep}(\varphi_i(Q)) \\ ((V_j)_{j \in J}, (f_\gamma)_{\gamma \in E}) &\longmapsto ((\varphi_i^+(V_j))_{j \in J}, (\varphi_i^+(f_\gamma))_{\gamma \in E}), \end{aligned}$$

where  $\varphi_i^+(V_j) = V_j$ , if  $i \neq j$ ,  $\varphi_i^+(V_i)$  is the kernel of the obvious map

$$\beta_i^+ := \sum_{\substack{\gamma \in E \\ \gamma=j \rightarrow i}} f_\gamma^V : \bigoplus V_j \longrightarrow V_j,$$

$\varphi_i^+(f_\gamma) = f_\gamma$ , if  $t(\gamma) \neq i$ , and  $\varphi_i^+(f_\gamma)$  is the natural projection on  $V_j$ , when  $t(\gamma) = i$ . If  $i$  is --admissible, define  $\varphi_i^- : \text{Rep}(Q) \rightarrow \text{Rep}(\varphi_i(Q))$  by  $\varphi_i^-(V_j) = V_j$ , if  $i \neq j$ ,  $\varphi_i^-(V_i) = \text{Coker } \beta_i^-$ , where

$$\beta_i^- := \sum_{\substack{\gamma \in E \\ \gamma=i \rightarrow j}} f_\gamma^V : V_i \longrightarrow \bigoplus V_j,$$

$\varphi_i^-(f_\gamma) = f_\gamma$ , if  $s(\gamma) \neq i$ , and  $\varphi_i^-(f_\gamma)$  is the restriction of  $\bigoplus V_j \rightarrow \text{Coker } \beta_i^-$  to  $V_j$ , when  $s(\gamma) = i$ .

The importance of **reflection functors** is summarised in the next theorem. We use  $\varphi_i^\pm$  as a notation to indicate that a certain statement holds for both  $\varphi_i^+$  and  $\varphi_i^-$  whenever  $i$  is +-admissible and --admissible, respectively.

**Theorem 2.2.3.** *Let  $V$  be an indecomposable representation of  $Q$  and  $L^{(i)}$  be the 1-dimensional simple representation supported at the vertex  $i \in I$ .*

(i)  $\varphi_i^\pm(V) = 0$  if and only if  $V = L^{(i)}$ .

(ii) If  $V \neq L^{(i)}$ ,  $\varphi_i^\pm(V)$  is indecomposable and we have

$$\dim \varphi_i^\pm(V) = \sum_{\substack{j \rightarrow i \\ i \rightarrow j}} \dim V_j - \dim V_i.$$

(iii) If  $V \neq L^{(i)}$ ,  $\varphi_i^+ \varphi_i^-(V) \cong V$ , when  $i$  is --admissible, and  $\varphi_i^+ \varphi_i^-(V) \cong V$ , when  $i$  is +-admissible.

(iv)  $\varphi_i^+$  is left exact and  $\varphi_i^-$  is right exact.

Moreover, for any  $V \in \text{Rep}(Q)$  and  $U \in \text{Rep}(\varphi_i(Q))$ , we have

(v)  $\text{Hom}_Q(V, \varphi_i^+(U)) \cong \text{Hom}_{\varphi_i(Q)}(\varphi_i^-(V), U)$ .

*Proof.* (i) is trivial. For (ii) and (iii), assume first that  $i \in I$  is +-admissible. By indecomposability of  $V$ ,  $\beta_i^+$  is surjective and, taking the kernel, we can complete such a map to the short exact sequence

$$0 \longrightarrow (\varphi_i^+ V)_i \xrightarrow{\beta_i^-} \bigoplus_{j \rightarrow i} V_j \xrightarrow{\beta_i^+} V_i \longrightarrow 0, \quad (2.2.2)$$

because  $\beta_i^-$  is the identity inclusion, by definition of  $\varphi_i^+(f_\gamma)$ . Thus, (ii) follows by the injectivity of  $\beta_i^-$  and (iii) because  $V_i \cong \text{Coker } \beta_i^- \cong \varphi_i^- \varphi_i^+(V_i)$ . If  $i$  is  $--$ -admissible, we obtain the dual statement by applying  $D : \text{Rep}(Q) \rightarrow \text{Rep}(Q^{op})$  that reverses all the arrows of  $Q$ , and observing that  $D \circ \varphi_i^+ = \varphi_i^- \circ D$ .

If  $U \in \text{Rep}(Q)$  and  $\iota : U \hookrightarrow V$  is an injection, an easy calculation shows that, for  $u \in \text{Ker } \beta_i^+$ , one has  $\varphi_i^+(\iota)_i(u) = \iota(u)$  and thus  $\varphi_i^+$  is injective. The same way, one proves that  $\varphi_i^-$  takes surjective maps to surjective maps, and (iv) follows.

Finally, note that, if  $V$  is not indecomposable, we loose the surjectivity of  $\beta_i^+$ , but (2.2.2) is still left exact. Taking the cokernel of  $\beta_i^-$ , we can form a short exact sequence, which yields an injective map  $q : (\varphi_i^- \varphi_i^+ V)_i \rightarrow V_i$ . The same way, taking the cokernel of  $\beta_i^-$ , we get the right exact sequence

$$V_i \xrightarrow{\beta_i^-} \bigoplus_{i \rightarrow j} \xrightarrow{\beta_i^+} (\varphi_i^- V)_i \longrightarrow 0,$$

and the identification  $\text{Ker } \beta_i^+ = (\varphi_i^+ \varphi_i^- V)_i$  gives a surjection  $p : (\varphi_i^+ \varphi_i^- V)_i \rightarrow V_i$ . Let  $\mu \in \text{Hom}_Q(V, \varphi_i^+(U))$ , we define a map in  $\text{Hom}_{\varphi_i(Q)}(\varphi_i^-(V), U)$  given by the composition  $q \circ \varphi_i^-(\mu)$ . This yields a bijection with inverse  $\varphi_i^+(-) \circ p$ .  $\square$

**Corollary 2.2.4.**  $\varphi_i^+$  yields a bijection between indecomposable representations of  $Q$  (non-isomorphic to  $L^{(i)}$ ) and indecomposable representations of  $\varphi_i(Q)$  (non-isomorphic to  $L^{(i)}$ ), with inverse  $\varphi_i^-$ .

*Proof.* This is a straight consequence of Theorem 2.2.3, (ii) and (iii).  $\square$

The following lemma gives a relation with the Weyl group of a graph, which justifies the nomenclature.

Let  $R = \mathbb{Z}^I$  be the root lattice of the underlying graph of  $Q$ . We have a bilinear symmetric form  $\langle -, - \rangle$  on  $R$ , defined by  $\langle \alpha_i, \alpha_j \rangle = c_{ij}$ , where  $c_{ij}$  is the  $(i, j)$ -th entry of the Cartan matrix  $C_Q = 2\text{Id} - (A_Q + {}^t A_Q)$ , and  $\alpha_i$  is the simple root at  $i \in I$ . The Weyl group  $W(Q)$  is generated by the simple reflections

$$s_i : \mathbf{v} \longmapsto \mathbf{v} - \langle \mathbf{v}, \alpha_i \rangle \alpha_i.$$

By abuse of notation, here and in what follows, we write  $\mathbf{v}$  for both the vector  $(v_i)_{i \in I} \in \mathbb{Z}^I$  and the corresponding root  $\sum_{i \in I} v_i \alpha_i$ .

**Lemma 2.2.5.** Let  $i \in I$  and  $V$  be an indecomposable representation of  $Q$  non-isomorphic to  $L^{(i)}$  (so that  $\mathbf{v} = \dim V \neq \alpha_i$ ). Then,  $\dim \varphi_i^\pm(V) = s_i(\mathbf{v})$ .

*Proof.* The claim follows from Theorem 2.2.3 (ii) via an easy calculation.  $\square$

## 2.3 Nakajima quiver varieties

### 2.3.1 Framing

We will be interested in quiver varieties obtained as GIT quotients  $\text{Rep}(Q, \mathbf{v}) //^\theta G_{\mathbf{v}}$ . It turns out that in many cases such a space is empty, so we need to restrict our attention to particular quivers, which, on the other hand, give rise to a rich number of interesting examples. To do this, consider the following construction, which goes under the name of **framing**. Given a quiver  $Q = (I, E)$ , consider a new quiver  $Q_\infty = (I_\infty, E_\infty)$ , where, for every vertex in  $Q$ , we add another vertex and an arrow pointing from the new vertex to the corresponding old one. Namely,  $I_\infty = I \sqcup \tilde{I}$ , where  $\tilde{I}$  is a copy of  $I$  via a bijection  $i \mapsto \tilde{i}$ ,  $E_\infty = E \sqcup \{\gamma_i\}_{i \in I}$  and  $t(\gamma_i) = i \in I$ ,  $s(\gamma_i) = \tilde{i} \in \tilde{I}$ . Representations of the framed quiver  $Q_\infty$  consist of representations of the original quiver  $Q$  together with a collection of vector spaces  $(W_i)_{i \in I}$  and linear maps  $(\mathbf{i}_i : W_i \rightarrow V_i)_{i \in I}$ . Given a dimension vector  $(\mathbf{v}, \mathbf{w}) \in \mathbb{Z}^{I \sqcup \tilde{I}}$ , we denote the space of representations of  $Q_\infty$  with such dimensions by  $\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w})$ . The difference with quiver varieties of the previous section is that we allow the subgroup  $G_{\mathbf{w}}$  of the base change group  $G_{(\mathbf{v}, \mathbf{w})} = G_{\mathbf{v}} \times G_{\mathbf{w}}$  to act trivially, so that the action of  $G_{(\mathbf{v}, \mathbf{w})}$  coincides with that one of  $G_{\mathbf{v}}$ :

$$(g_i)_{i \in I} \cdot ((X_\gamma)_{\gamma \in E}, (\mathbf{i}_i)_{i \in I}) = \left( (g_{t(\gamma)} X_\gamma g_{s(\gamma)}^{-1})_{\gamma \in E}, (g_i \mathbf{i}_i)_{i \in I} \right). \quad (2.3.1)$$

Hence,  $\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w})$  is a  $G_{\mathbf{v}}$ -variety and it makes sense to consider the moduli space  $\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w}) / G_{\mathbf{v}}$  (notice that this action does not factor through  $PG_{\mathbf{v}}$  as before). By [CB1], p.261, this space is, nonetheless, isomorphic to the quiver variety obtained in the following way. Define a quiver  $Q^{\mathbf{w}}$  by adding to  $Q$  a new vertex  $\infty$  and  $w_i$  arrows from  $\infty$  to  $i$ , for all  $i \in I$ , and take the  $\mathbf{v}_\infty := (\mathbf{v}, 1)$  as a dimension vector. The group  $G_{\mathbf{v}_\infty} / \mathbb{C}^\times \cong G_{\mathbf{v}}$  acts on the space of representations of  $Q^{\mathbf{w}}$  with dimension  $\mathbf{v}_\infty$  and there is a  $G_{\mathbf{v}}$ -equivariant isomorphism  $\text{Rep}(Q^{\mathbf{w}}, \mathbf{v}_\infty) \cong \text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w})$ . Such an identification is also useful to understand stability conditions for framed representations. Let  $\theta \in \mathbb{Z}^I$  and consider  $\theta_\infty := (\theta, -\sum_{i \in I} \theta_i v_i)$ , a stability parameter for  $Q^{\mathbf{w}}$  such that  $\theta_\infty \cdot \mathbf{v}_\infty = 0$ . Then, by the above, we have an isomorphism of quasi-projective schemes

$$\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w}) //^\theta G_{\mathbf{v}} \cong \text{Rep}(Q^{\mathbf{w}}, \mathbf{v}_\infty) //^{\theta_\infty} G_{\mathbf{v}_\infty}.$$

A modification of Theorem 2.2.2 yields the following:

**Theorem 2.3.1** ([Ki]). *Assume that  $Q$  has no edge loops and  $\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w})^{\theta-s} \neq \emptyset$ , then we have*

$$\dim \text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w}) //^\theta G_{\mathbf{v}} = \mathbf{v} \cdot \mathbf{w} + {}^t \mathbf{v} A_Q \mathbf{v} - \mathbf{v} \cdot \mathbf{v}.$$

*If, moreover,  $\theta = (1, \dots, 1)$ , then every semistable representation in  $\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w})$  is stable and hence  $\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w}) //^\theta G_{\mathbf{v}}$  is a smooth quasi-projective variety.*

### 2.3.2 Nakajima varieties

The most interesting cases of quiver varieties appear as GIT Hamiltonian reductions of representations of framed quivers. Note that the duality  $D$  (defined in the proof of Theorem 2.2.3) yields an isomorphism  $\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w})^* \cong \text{Rep}(Q_\infty^{op}, \mathbf{v}, \mathbf{w})$  and hence the cotangent bundle  $T^*\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w})$  gets identified with  $\text{Rep}(\overline{Q_\infty}, \mathbf{v}, \mathbf{w})$ , where  $\overline{Q_\infty} = (I_\infty, E_\infty \sqcup E_\infty^{op})$  is the double of  $Q_\infty$ . The  $G_{\mathbf{v}}$  action on  $\text{Rep}(\overline{Q_\infty}, \mathbf{v}, \mathbf{w})$  is given by

$$g \cdot (X, Y, \mathbf{i}, \mathbf{j}) = (gXg^{-1}, gYg^{-1}, g\mathbf{i}, \mathbf{j}g^{-1}),$$

where we use a compact notation for the obvious analogue of (2.3.1).

Using the formula given in section 2.1.1, we can compute the moment map

$$\begin{aligned} \mu : \text{Rep}(\overline{Q_\infty}, \mathbf{v}, \mathbf{w}) &\longrightarrow \mathfrak{g}_{\mathbf{v}}^* \cong \mathfrak{g}_{\mathbf{v}} \\ (X, Y, \mathbf{i}, \mathbf{j}) &\longmapsto \sum [X, Y] + \mathbf{i} \otimes \mathbf{j}, \end{aligned}$$

where  $\mathfrak{g}_{\mathbf{v}} := \text{Lie } G_{\mathbf{v}}$ ,

$$\sum [X, Y] := \sum_{i \in I} \left( \sum_{\substack{\gamma \in E: \\ t(\gamma)=i}} X_\gamma Y_{\gamma^{op}} - \sum_{\substack{\gamma \in E: \\ s(\gamma)=i}} Y_{\gamma^{op}} X_\gamma \right),$$

and  $\mathbf{i} \otimes \mathbf{j} := \sum_{i \in I} \mathbf{i}_i \cdot \mathbf{j}_i$ .

The centre  $\mathfrak{z}(\mathfrak{g}_{\mathbf{v}})$  of  $\mathfrak{g}_{\mathbf{v}}$  is identified with  $\mathbb{C}^I$  and, since any character of  $\mathfrak{g}_{\mathbf{v}}$  is of the form

$$\mathfrak{g}_{\mathbf{v}} \longrightarrow \mathbb{C}, \quad (X_i)_{i \in I} \longmapsto \sum_{i \in I} \lambda_i \text{tr } X_i,$$

we also have  $\mathfrak{z}(\mathfrak{g}_{\mathbf{v}})^* = (\mathfrak{g}_{\mathbf{v}} / [\mathfrak{g}_{\mathbf{v}}, \mathfrak{g}_{\mathbf{v}}])^* \cong \mathbb{C}^I$ . Given  $\lambda \in \mathbb{C}^I$ , the quotient of the path algebra  $\mathbb{C}\overline{Q_\infty}$  by the two-sided ideal generated by  $\sum [X, Y] + \mathbf{i} \otimes \mathbf{j} - \sum_{i \in I} \lambda_i \text{Id}_i$  is denoted by  $\Pi_\lambda$  and is called **deformed preprojective algebra** (in general, this definition is given for unframed representations, but the one we give yields to the same theory via the identification explained in the last section). The space  $\text{Rep}(\Pi_\lambda, \mathbf{v}, \mathbf{w})$  of

representations of  $\Pi_\lambda$  of dimension  $(\mathbf{v}, \mathbf{w})$  is, by definition, isomorphic to the scheme-theoretic fibre

$$\mu^{-1}(\lambda) = \{(X, Y, \mathbf{i}, \mathbf{j}) \in \text{Rep}(\overline{Q_\infty}, \mathbf{v}, \mathbf{w}) \mid [X, Y] + \mathbf{i} \otimes \mathbf{j} = \lambda\} .$$

There is a combinatorial criterion, due to Crowley-Boevey, that tells us when the moment map  $\mu$  is flat. Let  $R_Q$  be the set of roots for  $Q$ , namely all the vectors in  $\mathbb{Z}^I$  that are in the  $W(Q)$ -orbit of a simple root  $\alpha_i$ , for some  $i \in I$ , (the real roots) or of a vector  $\alpha \in \mathbb{N}^I \setminus \{0\}$  with connected support and such that  $(\alpha, \alpha_i) \leq 0, \forall i \in I$ , (the imaginary roots). Define a form on  $\mathbb{Z}^I$  given by

$$p(\mathbf{v}) := 1 + {}^t \mathbf{v} A_Q \mathbf{v} - \mathbf{v} \cdot \mathbf{v} .$$

**Theorem 2.3.2.** *Fix  $\lambda \in \mathbb{C}^I$  and  $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^I$  such that  $\lambda \cdot \mathbf{v} = 0$ . The moment map  $\mu : \text{Rep}(\overline{Q_\infty}, \mathbf{v}, \mathbf{w}) \rightarrow \mathfrak{g}_\mathbf{v}$  is flat if and only if*

$$p(\mathbf{v}) + \mathbf{v} \cdot \mathbf{w} \geq \sum_{i=0}^k p(\mathbf{v}^{(i)}) + \mathbf{v}^{(0)} \cdot \mathbf{w} , \quad (2.3.2)$$

for all the decompositions  $\mathbf{v} = \mathbf{v}^{(0)} + \dots + \mathbf{v}^{(k)}$ , with  $\mathbf{v}^{(0)}, \dots, \mathbf{v}^{(k)} \in \{\alpha \in \mathbb{N}^I \mid \alpha \cdot \lambda = 0\}$ . In such a case,  $\mu^{-1}(\lambda)$  has dimension  $2\mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{v} - {}^t \mathbf{v} C_Q \mathbf{v}$ .

*Proof.* Recall the identification  $\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w}) \cong \text{Rep}(Q^\mathbf{w}, \mathbf{v}_\infty)$ . By definition of  $Q^\mathbf{w}$ , one calculates

$$p(\mathbf{v}_\infty) = p(\mathbf{v}) + \mathbf{v} \cdot \mathbf{w} - 1 .$$

The decompositions  $\mathbf{v} = \mathbf{v}^{(0)} + \dots + \mathbf{v}^{(k)}$  of  $\mathbf{v}$  are in bijection with the decompositions  $\mathbf{v}_\infty = \tilde{\mathbf{v}}^{(0)} + \dots + \mathbf{v}^{(k)}$  of  $\mathbf{v}_\infty$ , where the coefficient of  $\alpha_\infty$  in  $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(k)}$  equals zero, while we have  $\tilde{\mathbf{v}}^{(0)} = \mathbf{v}^{(0)} + \alpha_\infty$ . The statement follows then by [CB1, Theorem 1.1 and Theorem 1.2], and by the formula

$$1 + 2 {}^t \mathbf{v}_\infty A_{Q^\mathbf{w}} \mathbf{v}_\infty - \mathbf{v} \cdot \mathbf{v} = 2\mathbf{v} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{v} + 2 {}^t \mathbf{v} A_Q \mathbf{v} .$$

□

**Definition 2.3.3.** Given a quiver  $Q$ ,  $\mathbf{v}, \mathbf{w} \in \mathbb{N}^I$ ,  $\theta \in \mathbb{Z}^I$  and  $\lambda \in \mathbb{C}^I$ . The **Nakajima quiver variety** is defined as the GIT Hamiltonian reduction

$$\mathcal{M}_\lambda^\theta(\mathbf{v}, \mathbf{w}) := \mu^{-1}(\lambda) //^\theta G_\mathbf{v} .$$

The following proposition is useful to understand the elements of  $\mathcal{M}_\lambda^\theta(\mathbf{v}, \mathbf{w})$  and follows directly by applying Definition 2.2.1 and the isomorphism  $\text{Rep}(\overline{Q_\infty}, \mathbf{v}, \mathbf{w}) \cong \text{Rep}(Q^\mathbf{w}, \mathbf{v}_\infty)$ .

**Proposition 2.3.4.** *A point  $(X, Y, \mathbf{i}, \mathbf{j}) \in \mu^{-1}(\lambda)$  is  $\theta$ -semistable if and only if, for all the collections of vector subspaces  $(U_i)_{i \in I} \subset (V_i)_{i \in I}$  of dimension vector  $\mathbf{u} \in \mathbb{N}^I$ , stable under the linear maps  $X$  and  $Y$ , we have*

$$\begin{aligned} U_i \subset \text{Ker } \mathbf{j}_i, \forall i \in I &\implies \theta \cdot \mathbf{u} \geq 0, \\ U_i \supset \text{Im } \mathbf{i}_i, \forall i \in I &\implies \theta \cdot \mathbf{u} \geq \theta \cdot \mathbf{v}. \end{aligned}$$

*In particular, when  $\theta = \theta^+$  ( $-\theta^+$ ),  $(X, Y, \mathbf{i}, \mathbf{j}) \in \mu^{-1}(\lambda)^{\theta\text{-ss}}$  if and only if zero ( $V$  itself) is the only subrepresentation of  $V$  contained in the kernel of  $\mathbf{j}$  (containing the image of  $\mathbf{i}$ ).*

**Definition 2.3.5.** Given a dimension vector  $\mathbf{v} \in \mathbb{N}^I$ , the parameters  $(\lambda, \theta) \in \mathbb{C}^I \times \mathbb{Z}^I$  are called **generic** if there is no  $\mathbf{u} \in R_Q$  such that  $0 \leq \mathbf{u} \leq \mathbf{v}$  and  $\mathbf{u} \cdot \theta = \mathbf{u} \cdot \lambda = 0$ .

Note that the parameter  $(0, \theta^+)$  is always generic.

**Theorem 2.3.6.** *Let  $\mathbf{v} \in \mathbb{N}^I$ ,  $\lambda \in \mathbb{C}^I$  and  $\theta \in \mathbb{Z}^I$  such that  $\lambda \cdot \mathbf{v} = 0$ . Then, we have:*

- (i)  $\mathcal{M}_\lambda^0(\mathbf{v}, \mathbf{w}) = \mu^{-1}(\lambda) // G_{\mathbf{v}}$  is an affine Poisson variety and there is a projective morphism  $\pi : \mathcal{M}_\lambda^\theta(\mathbf{v}, \mathbf{w}) \rightarrow \mathcal{M}_\lambda^0(\mathbf{v}, \mathbf{w})$  which respects the Poisson brackets.
- (ii) If  $(\lambda, \theta)$  are generic, every semistable point in  $\mu^{-1}(\lambda)$  is stable and  $\mathcal{M}_\lambda^\theta(\mathbf{v}, \mathbf{w})$  is a connected, smooth, symplectic variety of dimension  $2\mathbf{w} \cdot \mathbf{v} - {}^t\mathbf{v}C_Q\mathbf{v}$ .

*Proof.* (i) is a general feature of GIT and affine Hamiltonian reductions.

For (ii), in [Na3] Nakajima proves that, for generic parameters, a point in  $\mu^{-1}(\lambda)$  that satisfies the conditions of Proposition 2.3.4 has trivial stabiliser. Hence, its  $G_{\mathbf{v}}$ -orbit has maximal dimension and is closed, which implies that such a point is stable. This proves that  $\mathcal{M}_\lambda^\theta(\mathbf{v}, \mathbf{w})$  is smooth (symplectic). It also implies the statement about the dimension, because the  $G_{\mathbf{v}}$ -action is free and, by Theorem 2.3.2, we have

$$\dim \mathcal{M}_\lambda^\theta(\mathbf{v}, \mathbf{w}) = \dim \mu^{-1}(\lambda)^{\theta\text{-ss}} / G_{\mathbf{v}} = 2\dim \mu^{-1}(\lambda) - \dim G_{\mathbf{v}} = 2\mathbf{v} \cdot \mathbf{w} - {}^t\mathbf{v}C_Q\mathbf{v}.$$

Finally, the connectedness is proved at the end of section 1 of [CB1]. □

Nakajima quiver varieties are equipped with three different  $\mathbb{C}^\times$ -actions. The first one comes from the centre of  $G_{\mathbf{w}}$ , the subgroup of  $G_{(\mathbf{v}, \mathbf{w})} = G_{\mathbf{v}} \times G_{\mathbf{w}}$  that acts naturally on  $\text{Rep}(\overline{Q_\infty}, \mathbf{v}, \mathbf{w})$  preserving the fibres of the moment map. The other two actions on  $\text{Rep}(\overline{Q_\infty}, \mathbf{v}, \mathbf{w})$  commute with the  $G_{\mathbf{v}}$ -action but only preserve the zero fibre. The first one is given by dilation along the fibres of the cotangent bundle  $T^*\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w}) \cong \text{Rep}(\overline{Q_\infty}, \mathbf{v}, \mathbf{w})$ . It is defined by  $t.(X, Y, \mathbf{i}, \mathbf{j}) = (X, tY, \mathbf{i}, t\mathbf{j})$  and

rescales the symplectic form as  $t^*\omega = t\omega$ . Finally, we have the action on  $\mathcal{M}_\lambda^\theta(\mathbf{v}, \mathbf{w})$  induced by  $t.(X, Y, \mathbf{i}, \mathbf{j}) = (tX, tY, t\mathbf{i}, t\mathbf{j})$ , for which we have  $t^*\omega = t^2\omega$ . From now on, every time we talk about the  $\mathbb{C}^\times$ -action for  $\mathcal{M}_0^\theta(\mathbf{v}, \mathbf{w})$ , we will refer to the latter.

**Theorem 2.3.7.** *Assume that  $Q$  has no edge loops and let  $(\lambda, \theta)$  be generic.*

*Let  $\mathcal{M}_\lambda^0(\mathbf{v}, \mathbf{w})^{reg}$  denote the image in  $\mu^{-1}(\lambda)//G_{\mathbf{v}}$  of the points in  $\mu^{-1}(\lambda)$  with trivial stabiliser. If  $\mathcal{M}_\lambda^0(\mathbf{v}, \mathbf{w})^{reg}$  is nonempty, it is dense in  $\mathcal{M}_\lambda^0(\mathbf{v}, \mathbf{w})$  and isomorphic to  $\pi^{-1}(\mathcal{M}_\lambda^0(\mathbf{v}, \mathbf{w})^{reg})$ , a dense subset of  $\mathcal{M}_\lambda^\theta(\mathbf{v}, \mathbf{w})$ . The morphism*

$$\pi : \mathcal{M}_\lambda^\theta(\mathbf{v}, \mathbf{w}) \rightarrow \mathcal{M}_\lambda^0(\mathbf{v}, \mathbf{w})$$

*is a symplectic resolution of singularities, which is conical when  $\lambda = 0$ .*

*Proof.* The first statement is proved in [Na1], Theorem 4.1. This makes  $\pi$  a birational morphism and thus a symplectic resolution, by Theorem 2.3.6 (i). The fact that this resolution is conical when  $\lambda = 0$  is immediate from the definition of the  $\mathbb{C}^\times$ -action.  $\square$

**Remark 2.3.8.** Notice that  $\overline{Q_\infty}$  and hence  $\text{Rep}(\overline{Q_\infty}, \mathbf{v}, \mathbf{w})$ ,  $\mu^{-1}(\lambda)$  and  $\mathcal{M}_\lambda^\theta(\mathbf{v}, \mathbf{w})$  do not depend on the orientation of the quiver  $Q$ .

### 2.3.3 Examples

Let  $Q = (I, E)$  be a quiver with an affine Dynkin diagram of type  $\tilde{A}, \tilde{D}, \tilde{E}$  as underlying graph and any orientation. Let  $\Gamma$  be the corresponding finite subgroup of  $SL_2(\mathbb{C})$  via McKay correspondence,  $\rho_0, \dots, \rho_{\ell-1}$  its irreducible representations (where  $\rho_0$  is the trivial one), and  $N$  the tautological representation. So, we have  $I = \{0, \dots, \ell-1\}$  and the number of edges between  $i$  and  $j$  of the double quiver  $\overline{Q}$  is given by the multiplicity of  $\rho_i$  in  $N \otimes \rho_j$ . Also, the dimension vector  $\delta = (\dim \rho_i)_{i \in I}$  turns out to be the minimal imaginary root of  $Q$ . The following is a result due to Kronheimer (cf. [Kr]), rephrased in the language of quiver varieties in [CS]. See also [CBH, Theorem 8.10].

**Theorem 2.3.9.** *There is an isomorphism of algebraic varieties  $\mathcal{M}_0^0(\delta, 0) \cong \mathbb{C}^2/\Gamma$ . Moreover, if  $\theta \in \mathbb{Z}^I$  does not lie in any of the root hyperplanes, the canonical map  $\pi : \mathcal{M}_0^\theta(\delta, 0) \rightarrow \mathcal{M}_0^0(\delta, 0)$  is the minimal resolution of the Klenian singularity.*

Notice that, since  $\delta_0 = 1$ , the result is also true if we take the framing  $\mathbf{w} = (1, 0, \dots, 0)$  instead of zero.

Consider now the Jordan quiver, namely the quiver with only one vertex with an edge loop. If we take  $(\mathbf{v}, \mathbf{w}) = (n, 0)$  as dimension vector and  $\lambda = 0$ , we have

$$\mu^{-1}(0) = \{(X, Y) \in \mathfrak{gl}_n \times \mathfrak{gl}_n \mid [X, Y] = 0\},$$

the **commuting variety**. This scheme is known to be irreducible (cf. [Ri]), but whether it is reduced is a long standing open question. The linear group  $GL_n$  acts on  $\mu^{-1}(0)$  by the adjoint action and we have the double analogue of the Chevalley restriction theorem

$$\mathbb{C}[\mu^{-1}(0)]^{GL_n} \xrightarrow{\sim} \mathbb{C}[\mathbb{C}^n \times \mathbb{C}^n]^{\Sigma_n},$$

induced by embedding couples of diagonal matrices. Hence, we have the isomorphism of algebraic varieties  $\mathcal{M}_0^0(n, 0) \cong \mathbb{C}^n \times \mathbb{C}^n / \Sigma_n$ . One can prove that this isomorphism still holds when we take  $\mathbf{w} = 1$  instead of zero as a framing vector (see [Gi, section 5.6]).

$$\begin{array}{c} \mathbb{C} \\ \begin{array}{c} \downarrow i \\ \uparrow j \end{array} \\ x \left( \mathbb{C}^n \right) y \end{array}$$

Consider the stability condition  $\theta = 1$ . Then, Proposition 2.3.4 and [Na2, Theorem 1.14] yield the following:

**Theorem 2.3.10.** *There is an isomorphism of algebraic varieties  $\mathcal{M}_0^\theta(n, 1) \cong \text{Hilb}^n(\mathbb{C}^2)$ . The canonical map  $\pi : \mathcal{M}_0^\theta(n, 1) \rightarrow \mathcal{M}_0^0(n, 1)$  is a resolution of singularities and coincide with the Hilbert-Chow morphism.*

We also have the combined version of these two cases. Namely, if  $Q$  and  $\Gamma$  are as in Theorem 2.3.9,  $\mathbf{w} = (1, 0, \dots, 0)$  and  $\Gamma_n = \Sigma_n \wr \Gamma$  denotes the wreath product, we have the following:

**Theorem 2.3.11.** *There is an isomorphism of algebraic varieties  $\mathcal{M}_0^0(n\delta, \mathbf{w}) \cong \mathbb{C}^{2n} / \Gamma_n$ . Moreover, for generic values of  $\theta$ , we have  $\mathcal{M}_0^\theta(n\delta, \mathbf{w}) \cong \text{Hilb}^n(\widetilde{\mathbb{C}^2 / \Gamma})$  and the canonical map  $\pi : \mathcal{M}_0^\theta(n\delta, \mathbf{w}) \rightarrow \mathcal{M}_0^0(n\delta, \mathbf{w})$  coincides with the resolution  $\text{Hilb}^n(\widetilde{\mathbb{C}^2 / \Gamma}) \rightarrow \mathbb{C}^{2n} / \Gamma_n$ .*

### 2.3.4 LMN isomorphisms

A modification of reflection functors introduced in section 2.2.1 gives rise to isomorphisms of quiver varieties. This was first conjectured by Nakajima [Na1] and then proved by Maffei [Maf] who generalised a work of Lusztig [Lu] and Nakajima himself [Na4]. In [BL], such isomorphisms are therefore called **LMN isomorphisms**.

Consider a quiver  $Q$ , dimension vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^I$ , deformation parameter  $\lambda \in \mathbb{C}^I$  and stability condition  $\theta \in \mathbb{Z}^I$ . Define a *dot action* of the Weyl group  $W(Q)$  on the dimension vector by

$$\sigma \cdot (\mathbf{v}, \mathbf{w}) = (\sigma(\mathbf{v} - \mathbf{w}) + \mathbf{w}, \mathbf{w}), \quad (2.3.3)$$

where  $\sigma(\mathbf{v})$  is defined in section 2.2.1.  $W(Q)$  also acts on  $\lambda$  and  $\theta$  by the dual action:

$$(s_i \lambda)_j = \lambda_j - c_{ij} \lambda_i \quad \text{and} \quad (s_i \theta)_j = \theta_j - c_{ij} \theta_i. \quad (2.3.4)$$

Let  $i \in I$  be a source for  $Q$  (we can always assume that thanks to Remark 2.3.8) and assume that either  $\lambda_i \neq 0$  or  $\theta_i > 0$ .

**Theorem 2.3.12.** (*[Maf, Theorem 26]*) *Under the assumptions above, we have the isomorphism of algebraic varieties*

$$\phi_{s_i}^\lambda : \mathcal{M}_\lambda^\theta(\mathbf{v}, \mathbf{w}) \xrightarrow{\sim} \mathcal{M}_{s_i \lambda}^{s_i \theta}(s_i \cdot (\mathbf{v}, \mathbf{w})).$$

Notice that it is enough to assume  $(\lambda_i, \theta_i) \neq (0, 0)$  because, if  $\lambda_i = 0$  and  $\theta_i < 0$ , we can construct the LMN isomorphism for  $s_i \theta > 0$ , instead.

**Remark 2.3.13.** The assumptions we make differ from the original ones in [Maf]. We follow the description given in [BL, section 2.1.3].

## 2.4 Quantisation

### 2.4.1 Generalities and motivating examples

Let  $A$  be an associative non-negatively increasingly filtered algebra  $A = \bigcup_{m \geq 0} A_m$  and let  $\text{gr } A = \bigoplus_{m \geq 0} A_m / A_{m-1}$  be its associated graded algebra. Suppose that  $\text{gr } A$  is commutative and let  $d \geq 1$  the maximal integer such that  $[A_m, A_n] \subset A_{m+n-d}$ , for all  $m$  and  $n$ . An easy calculation shows that  $\text{gr } A$  is canonically Poisson with bracket given by

$$\{\text{gr}_m a, \text{gr}_n b\} := \text{gr}_{m+n-d}(ab - ba),$$

for  $a \in A_m$  and  $b \in A_n$ .

**Definition 2.4.1.** Let  $B$  be a graded Poisson algebra with Poisson bracket of degree  $-d$ , for some  $d \geq 1$ . A (filtered) **quantisation** of  $B$  is an associative filtered algebra  $A$  such that  $\text{gr } A \cong B$  is an isomorphism of Poisson algebras.

If  $B$  is the ring of regular functions of a Poisson variety  $X$ , we also say that  $A$  is a quantisation of (or quantises)  $X$ .

**Remark 2.4.2.** More generally, one could define a (deformation) quantisation as a one-parameter deformation  $A_{\hbar} = (B[[\hbar]], *)$  such that

$$a * b = ab \pmod{\hbar} \quad \text{and} \quad a * b - b * a = \hbar\{a, b\} \pmod{\hbar^2},$$

for  $a, b \in B$  ( $\hbar$  is a formal parameter).

In what follows, we will only deal with filtered quantisations. For more on deformation quantisation, see [Ko2, Ko3, Ye].

The most basic example of quantisation is given by the universal enveloping algebra  $\mathcal{U}_{\mathfrak{g}}$  of a complex semisimple Lie algebra  $\mathfrak{g}$ . This is an associative filtered algebra by assigning degree 1 to the elements of  $\mathfrak{g}$ , so that

$$(\mathcal{U}_{\mathfrak{g}})_m = \langle x_1 \cdots x_j \mid x_1, \dots, x_j \in \mathfrak{g}, j \leq m \rangle.$$

The symmetric algebra  $\text{Sym } \mathfrak{g}$  is Poisson with bracket of degree  $-1$  given by

$$\{x_1 \cdots x_m, y_1 \cdots y_n\} = \sum_{i,j} [x_i, y_j] x_1 \cdots \hat{x}_i \cdots x_m y_1 \cdots \hat{y}_j \cdots y_n,$$

and the natural surjection  $\text{Sym } \mathfrak{g} \twoheadrightarrow \text{gr } \mathcal{U}_{\mathfrak{g}}$  is an isomorphism by the PBW theorem. One checks this is Poisson, so the enveloping algebra is a quantisation of  $\mathfrak{g}$  (via  $\text{Sym } \mathfrak{g} = \mathbb{C}[\mathfrak{g}^*]$  and  $\mathfrak{g} \cong \mathfrak{g}^*$ ), viewed as an affine variety.

Fix now a Cartan subalgebra  $\mathfrak{t} \subset \mathfrak{g}$ , let  $W$  be the Weyl group and  $\mathfrak{z}(\mathcal{U}_{\mathfrak{g}})$  the centre of the enveloping algebra. For a character  $\lambda \in \mathfrak{z}(\mathcal{U}_{\mathfrak{g}})^*$ , we can form the central reduction  $(\mathcal{U}_{\mathfrak{g}})_{\lambda} := \mathcal{U}_{\mathfrak{g}} / (\text{Ker } \lambda) \mathcal{U}_{\mathfrak{g}}$ . Note that this algebra inherits the filtration from  $\mathcal{U}_{\mathfrak{g}}$  and its associated graded equals  $\text{Sym } \mathfrak{g} / (\text{Sym } \mathfrak{g})_+^{\mathfrak{g}}$  (where  $(\text{Sym } \mathfrak{g})_+^{\mathfrak{g}}$  denotes the augmentation ideal), because  $\mathfrak{z}(\text{Sym } \mathfrak{g}) = (\text{Sym } \mathfrak{g})^{\mathfrak{g}}$  and  $(\text{Sym } \mathfrak{g})_0 = \mathbb{C}$ .

**Proposition 2.4.3.**  $(\mathcal{U}_{\mathfrak{g}})_{\lambda}$  is a quantisation of the nilpotent cone  $\mathcal{N}$  of  $\mathfrak{g}$ .

*Proof.* It is enough to show that  $\text{Sym } \mathfrak{g} / (\text{Sym } \mathfrak{g})_+^{\mathfrak{g}} \cong \mathbb{C}[\mathcal{N}]$ . Under the Harish-Chandra isomorphism  $(\text{Sym } \mathfrak{g})^{\mathfrak{g}} \cong \mathbb{C}[\mathfrak{t}^*]^W$  the augmentation ideal  $(\text{Sym } \mathfrak{g})_+^{\mathfrak{g}}$  is sent to  $\mathbb{C}[\mathfrak{t}^*]_+^W$ , the ideal defining zero in  $\mathfrak{t}^* \cong \mathfrak{t}$ . Thus,  $(\text{Sym } \mathfrak{g})_+^{\mathfrak{g}} \text{Sym } \mathfrak{g}$  is the defining ideal of

$$\{x \in \mathfrak{g} \mid \overline{G.x} \cap \mathfrak{t} = 0\},$$

the elements of  $\mathfrak{g}$  ( $\cong \mathfrak{g}^*$ ) with zero semisimple part. This coincides with  $\mathcal{N}$  and we are done.  $\square$

Another classic example of quantisation is the Weyl algebra

$$A_n := \mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle / ([x_i, y_j] = \delta_{ij}, [x_i, x_j] = 0 = [y_i, y_j]),$$

isomorphic to the algebra of global differential operators on  $\mathbb{C}^n$  via  $y_i \mapsto -\frac{\partial}{\partial x_i}$ . Setting  $x_i$  and  $y_i$  in degree 1 defines a filtration on  $A_n$  and the associated graded equals  $\mathbb{C}[\mathbb{C}^n]$ . More generally, let  $(V, \omega)$  be a symplectic vector space. Then, the Weyl algebra of  $(V, \omega)$

$$A(V) := TV / (vw - vw - \omega(v, w))$$

gives a quantisation of  $\text{Sym } V$ , a Poisson algebra with bracket  $\{v, w\} = \omega(v, w)$  of degree  $-2$ .

Further, let  $G$  be a finite group of  $Sp(V)$ . By definition,  $G$  acts on  $V$  preserving the symplectic form and hence each  $g \in G$  gives rise to a Poisson automorphism of  $V$ . The induced filtration on the  $G$ -invariants  $A(V)^G$  yields a quantisation of the orbifold  $V^*/G$  via

$$\text{gr}(A(V)^G) \cong \text{Sym}(V^G) = \mathbb{C}[V^*/G].$$

**Remark 2.4.4.** Notice that  $A(V)^G$  is nothing but the spherical symplectic reflection algebra  $eH_{t, \mathbf{c}}(G)e$ , with parameter  $(t, \mathbf{c}) = (1, 0)$ . One can prove that, by taking all possible values of  $\mathbf{c}$ ,  $eH_{1, \mathbf{c}}(G)e$  gives a *universal family of quantisations* of  $V^*/G$ , namely any quantisation of such an orbifold arises as the spherical symplectic reflection algebra with some deformation parameter (cf. [EG, Theorem 2.16]). It turns out that many interesting Poisson varieties (such as a rich class of quiver varieties) are isomorphic to orbifolds of the type  $V^*/G$  and admit quantisations via different constructions, which, by the above, get identified with the corresponding spherical symplectic reflection algebra. This yields, for example, some results that we will need in the next chapter, such as Theorem 2.4.10 and Proposition 3.3.4 (see [Lo]).

We can also work locally. Namely, consider  $D_X$ , the sheaf of differential operators on a smooth variety  $X$ , generated by the structure sheaf  $\mathcal{O}_X$  and the tangent sheaf  $\mathcal{T}_X$  inside  $\mathcal{E}nd_{\mathbb{C}_X}(\mathcal{O}_X)$ . It is filtered recursively by

$$(D_X)_m = \{P \in \mathcal{E}nd_{\mathbb{C}_X}(\mathcal{O}_X) \mid [P, f] \in (D_X)_{m-1}, \forall f \in \mathcal{O}_X\}, \text{ if } m \geq 0,$$

and  $(D_X)_m = 0$ , if  $m < 0$ , and we have

$$\text{gr } D_X \cong p_* \mathcal{O}_{T^*X} \cong \text{Sym } \mathcal{T}_X,$$

where  $p : T^*X \rightarrow X$  is the natural projection. Hence, differential operators on a smooth variety  $X$  quantise the cotangent bundle  $T^*X$ . Note that this remains true if

we replace  $D_X$  by  $D_{\mathcal{L}}$ , the sheaf of differential operators twisted by some line bundle  $\mathcal{L}$  on  $X$ , obtained by replacing  $\mathcal{O}_X$  with  $\mathcal{L}$  in the definition. We denote by  $D_X\text{-mod}$  (resp.  $D_{\mathcal{L}}\text{-mod}$ ) the category of *coherent*  $D_X$ -modules (resp.  $D_{\mathcal{L}}$ -modules), namely the modules equipped with a complete separated filtration (*good filtration*) such that the corresponding associated graded is a coherent  $\mathcal{O}_X$ -module.

We now state the celebrated Beilinson-Bernstein localisation theorem (cf. [BB1, BB2]).

Let  $\mathfrak{g} \supset \mathfrak{t}$  be as before,  $R^+$  the set of positive roots of  $\mathfrak{g}$  and  $\rho$  be half the sum of all positive roots. Denote by  $\mathcal{B} := G/B$  the flag variety. For  $\lambda \in \mathfrak{t}^*$ , we have a line bundle  $\mathcal{L}_\lambda$  on  $\mathcal{B}$ , and we set  $D_\lambda := D_{\mathcal{L}_\lambda}$ .

**Theorem 2.4.5** (Beilinson-Bernstein). *(i) The algebra of global sections  $\Gamma(\mathcal{B}, D_\lambda)$  equals the central reduction  $(\mathcal{U}_{\mathfrak{g}})_\lambda$  and we have  $H^k(\mathcal{B}, D_\lambda) = 0$ , for  $k > 0$ .*

*(ii) The global section functor  $\Gamma_\lambda : D_\lambda\text{-mod} \rightarrow (\mathcal{U}_{\mathfrak{g}})_\lambda\text{-mod}$  is an equivalence if  $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$ ,  $\forall \alpha \in R^+$ .*

*(iii) The derived functor  $\mathcal{R}\Gamma_\lambda : D(D_\lambda\text{-mod}) \rightarrow D((\mathcal{U}_{\mathfrak{g}})_\lambda\text{-mod})$  is an equivalence if  $\langle \lambda + \rho, \alpha^\vee \rangle \neq 0$ ,  $\forall \alpha \in R^+$ .*

By the discussion above, this statement can be interpreted as a quantised version of the Springer resolution  $\mu : T^*\mathcal{B} \rightarrow \mathcal{N}$ . Given a symplectic resolution, it makes therefore sense to try to understand when this can be quantised and give rise to a localisation result analogous to the Beilinson-Bernstein theorem.

## 2.4.2 Quantum Hamiltonian reduction

There is a standard procedure to quantise Poisson varieties obtained as Hamiltonian reductions.

Let  $D$  be an associative algebra acted on by an algebraic group  $G$  by algebra automorphisms. Consider  $\mathcal{U}_{\mathfrak{g}}$ , the enveloping algebra of the Lie algebra of  $G$  and let  $\rho : \mathcal{U}_{\mathfrak{g}} \rightarrow D$  be a  $G$ -equivariant algebra homomorphism such that the adjoint action  $\text{ad } x : a \mapsto \rho(x)a - a\rho(x)$ ,  $a \in D$ , equals the differential of the  $G$ -action. Then,  $D\rho(\mathfrak{g})$  is a  $G$ -stable left ideal in  $D$  and we have

$$(D/D\rho(\mathfrak{g}))^G \cong D^G / (D\rho(\mathfrak{g}))^G ,$$

if  $D$  is semisimple as a  $G$ -module. Although  $D\rho(\mathfrak{g})$  is only a left ideal, the ring of invariants  $(D\rho(\mathfrak{g}))^G$  is a two-sided ideal in  $D^G$  and  $(D/D\rho(\mathfrak{g}))^G$  acquires a well-defined

associative algebra structure by multiplication in  $D$ . Moreover, it is not hard to see that the right action of  $(D/D\rho(\mathfrak{g}))^G$  on  $D/D\rho(\mathfrak{g})$  induces an algebra isomorphism

$$(D/D\rho(\mathfrak{g}))^G \cong \text{End}_D(D/D\rho(\mathfrak{g}))^{op}. \quad (2.4.1)$$

Let  $M$  be a  $D$ -module equipped with a  $G$ -action such that the module map  $D \otimes M \rightarrow M$  is  $G$ -equivariant and  $M$  is semisimple as a  $G$ -module. If the differential of the  $G$ -action is induced by  $\rho$  and the  $D$ -action (in this case, we say that  $M$  is a  $(G, \rho)$ -equivariant  $D$ -module), we have

$$M^G \cong M^{\rho(\mathfrak{g})} \cong \text{Hom}_D(D/D\rho(\mathfrak{g}), M).$$

Hence,  $M^G$  acquires the structure of  $(D/D\rho(\mathfrak{g}))^G$ -module via (2.4.1) and the tautological right action of  $\text{End}_D(D/D\rho(\mathfrak{g}))$  on  $D/D\rho(\mathfrak{g})$ .

**Definition 2.4.6.** The algebra

$$\mathfrak{A}(D, G, \rho) := (D/D\rho(\mathfrak{g}))^G$$

is called **quantum Hamiltonian reduction**.

Let  $(D, G, \rho)\text{-Mod}$  be the category of  $(G, \rho)$ -equivariant  $D$ -modules. The functor defined by

$$\mathbb{H} : (D, G, \rho)\text{-Mod} \rightarrow \mathfrak{A}(D, G, \rho)\text{-Mod}, \quad \mathbb{H}(M) = M^G$$

is called **quantum Hamiltonian reduction functor**.

The reason for such a definition becomes clear in the case when  $D = D(X)$  is the algebra of differential operators on a smooth affine variety  $X$ ,  $G$  is a reductive group acting on  $X$  (and thus on  $D(X)$ ) and  $\rho = \Phi$  is the quantum comoment map, namely the extension to  $\mathcal{U}_{\mathfrak{g}}$  of the Lie morphism  $\phi : \mathfrak{g} \rightarrow \text{Vect } X$  induced by the infinitesimal action. Note that  $(D(X)/D(X)\Phi(\mathfrak{g}))^G = D(X)^G/(D(X)\Phi(\mathfrak{g}))^G$  is filtered by order of the differential operators. By [Ho, Proposition 2.4], if the moment map  $\mu : T^*X \rightarrow \mathfrak{g}^*$  is flat, there is an isomorphism

$$\text{gr}(D(X)/D(X)\Phi(\mathfrak{g})) \cong \text{gr } D(X)\mu^*(\mathfrak{g})$$

and hence  $\text{gr } \mathfrak{A}(D(X), G, \Phi) \cong \mathbb{C}[\mu^{-1}(0)]^G$ . Given  $\lambda \in \mathfrak{z}(\mathfrak{g})^*$ , we can also consider  $\Phi_\lambda := \Phi - \langle \lambda, - \rangle$  instead of  $\Phi$ , which still gives a quantisation of  $\mu^{-1}(0)//G$ , because  $\text{gr } \Phi_\lambda = \mu^*$ .

We now apply this construction to Nakajima quiver varieties. In the notation of the previous section, we define the quantum Hamiltonian reduction

$$\mathfrak{A}_\lambda(\mathbf{v}, \mathbf{w}) := \mathfrak{A}(D(\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w})), G_{\mathbf{v}}, \Phi_\lambda) = \\ (D(\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w})) / D(\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w})) \{\Phi(x) - \langle \lambda, x \rangle \mid x \in \mathfrak{g}\{\mathbf{v}\}\})^{G_{\mathbf{v}}}$$

and, for  $(\lambda, \theta)$  generic, its sheaf-theoretic version

$$\mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w}) := \left( D_{\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w})} / D_{\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w})} \{\Phi(x) - \langle \lambda, x \rangle \mid x \in \mathfrak{g}\{\mathbf{v}\}\} \Big|_{T^*\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w})^{\theta\text{-ss}}} \right)^{G_{\mathbf{v}}}. \quad (2.4.2)$$

In general, one has to consider this sheaf as a *microlocalisation* of the former algebra, namely a sheaf of formal series with respect to an extra parameter  $\hbar$  (see [KR] for the definition). Taking  $\mathbb{C}^*$ -invariants, however, yields an equivalence of good modules for the microlocal sheaf and coherent  $\mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w})$ -modules, analogously to [KR, section 2.3.3]. Notice that while Kashiwara and Rouquier work with holomorphic functions, this result is still true in the algebraic setting, as explained at the end of section 3.3 in [Lo]. Thus, we can consider (2.4.2) as a sheaf on the cotangent bundle  $T^*\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w})$  (restricted to the  $\theta$ -semistable locus) with the conical topology, where the opens are the Zariski opens that are stable for the  $\mathbb{C}^*$ -action. As before, we denote by  $\mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w})\text{-mod}$  the category of coherent  $\mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w})$ -modules.

We want to restrict ourselves to the case when the underlying graph of the quiver  $Q$  is of affine Dynkin type,  $\mathbf{v} = n\delta$ , for some  $n \in \mathbb{Z}_{>0}$  and  $\mathbf{w} = (1, 0, \dots, 0)$ . We have seen in the previous section that many interesting Poisson varieties arise in this setting. Moreover, in such cases, the corresponding Nakajima quiver variety enjoys some favourable properties.

**Proposition 2.4.7.** *Under the assumptions above, we have:*

- (i) *The moment map  $\mu : \text{Rep}(\overline{Q}_\infty, \mathbf{v}, \mathbf{w}) \rightarrow \mathfrak{g}_{\mathbf{v}}$  is flat.*
- (ii) *If  $(\lambda, \theta) \in \mathbb{C}^I \times \mathbb{Z}^I$  is generic, the canonical morphism  $\pi : \mathcal{M}_\lambda^\theta(\mathbf{v}, \mathbf{w}) \rightarrow \mathcal{M}_\lambda^0(\mathbf{v}, \mathbf{w})$  is a resolution of singularities.*

*Proof.* (i) Since we have  $p(\alpha) = 0$ , for all real roots  $\alpha$ , and  $p(m\delta) = 1$ , for  $m \in \mathbb{Z}$ , the right-hand side of (2.3.2) is maximised when  $\mathbf{v}^{(i)} = \delta$ , for all  $1 \leq i \leq k$ , and  $\mathbf{v}^{(0)} = m_0\delta$ , for some  $m_0 > 0$ . In such a case, that quantity is equal to  $k + 1 + v_0^{(0)} = k + 1 + m_0 \leq n + 1$ . Since the left-hand side equals  $n + 1$ , the claim follows by Theorem 2.3.2.

(ii) is proved in [BL, Proposition 2.3]. □

By Proposition 2.4.7 and Theorem 2.1.4, the isomorphism  $\mathrm{gr} \mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w}) \cong \mathcal{O}_{\mathcal{M}_0^\theta(\mathbf{v}, \mathbf{w})}$  implies

$$\mathrm{gr} \mathfrak{A}_\lambda(\mathbf{v}, \mathbf{w}) \cong \mathbb{C}[\mathcal{M}_0^0(\mathbf{v}, \mathbf{w})] \cong \mathbb{C}[\mathcal{M}_0^\theta(\mathbf{v}, \mathbf{w})] \cong \mathrm{gr} \Gamma(\mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w})) \quad (2.4.3)$$

and  $H^i(\mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w})) = 0, \forall i > 0$ . (2.4.3) also yields  $\mathfrak{A}_\lambda(\mathbf{v}, \mathbf{w}) \cong \Gamma(\mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w}))$  and, in particular, the ring of global sections of  $\mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w})$  does not depend on  $\theta$ .

**Remark 2.4.8.** (i) Proposition 2.4.7 holds also under some less restrictive assumptions on  $Q$  and the dimension vector. Namely, it suffices that  $Q$  is of finite or affine type and that  $\nu := \sum_{i \in I} w_i \omega_i - v_i \alpha_i$  is a dominant weight for  $\mathfrak{g}(Q)$ , the (finite or affine) Lie algebra associated to the graph of  $Q$ , where  $\omega_i$  denote the fundamental weights and  $\alpha_i$  are the real roots.

(ii) Notice that the sheaf isomorphism  $\mathrm{gr} \mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w}) \cong \mathcal{O}_{\mathcal{M}_0^\theta(\mathbf{v}, \mathbf{w})}$  is always true, while the global analogue  $\mathrm{gr} \mathfrak{A}_\lambda(\mathbf{v}, \mathbf{w}) \cong \mathbb{C}[\mathcal{M}_0^\theta(\mathbf{v}, \mathbf{w})]$  is false in general.

### 2.4.3 Localisation

Under our assumptions, there is a derived localisation theorem, due to McGerty and Nevins, cf. [MN1, Theorem 1.1] (see also [BPW, Theorem A]).

**Theorem 2.4.9.** *The derived functor*

$$\mathcal{R}\Gamma_\lambda^\theta : D(\mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w})\text{-mod}) \longrightarrow D(\mathfrak{A}_\lambda(\mathbf{v}, \mathbf{w})\text{-mod}) \quad (2.4.4)$$

*is an exact equivalence of bounded derived categories if and only if  $\mathfrak{A}_\lambda(\mathbf{v}, \mathbf{w})$  has finite global dimension. In such a case, the quasi-inverse is the localisation functor given by the left derived tensor  $\mathcal{L}\mathrm{Loc}_\lambda^\theta := \mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w}) \otimes_{\mathfrak{A}_\lambda(\mathbf{v}, \mathbf{w})}^{\mathcal{L}} -$ .*

The abelian version of this localisation result has also been studied in [MN2], by proving a vanishing theorem for the quantum Hamiltonian reduction functor, under a combinatorial assumption on the deformation parameter  $\lambda$  (see also [BPW, Corollary B.1]). For our purposes, however, we will only deal with the derived localisation.

We finally have the analogue of statement (i) in Theorem 2.4.5.

Let  $\Gamma$  be the finite subgroup of  $SL_2(\mathbb{C})$  corresponding to  $Q$  via McKay correspondence (see section 2.3.3),  $\rho_0, \dots, \rho_{\ell-1}$  its irreducible representations, and consider  $\Gamma_n := \Sigma_n \wr \Gamma$ . As in the cyclotomic case (section 1.4.1), we have two types of conjugacy classes of symplectic reflections:

- (1) the elements of the form  $s_{ij}\gamma_i\gamma_j^{-1}$ , for  $1 \leq i, j \leq n$ , where  $s_{ij} \in \Sigma_n$  is the transposition swapping  $i$  and  $j$ , and  $\gamma \in \Gamma$ ;
- (2) the elements  $\gamma_i$ , for  $1 \leq i \leq n$ , and  $\gamma \in \Gamma \setminus \{1\}$ .

Thus, we can view the deformation parameter for the symplectic reflection algebra  $H_{t,k,\mathbf{c}}(\Gamma_n)$  as a triple  $(t, k, \mathbf{c})$ , where  $t$  is the quantum parameter,  $k \in \mathbb{C}$  corresponds to the unique conjugacy class of type (1) and  $\mathbf{c}$  (corresponding to type (2)) is a conjugacy class invariant function from  $\Gamma \setminus \{1\}$  to  $\mathbb{C}$ , viewed as the central element in the group algebra  $\mathbb{C}[\Gamma]$

$$\mathbf{c} = \sum_{\gamma \in \Gamma \setminus \{1\}} \mathbf{c}(\gamma)\gamma, \quad \mathbf{c}(\gamma) \in \mathbb{C}.$$

In this notation, the symplectic reflection algebra  $H_{t,k,\mathbf{c}}(\Gamma_n)$  is defined as the quotient of  $TL^{\oplus n} \# \mathbb{C}[\Gamma_n]$  by the relations

$$\begin{aligned} [x_i, x_j] &= 0, \quad [y_i, y_j] = 0, \quad i, j = 1, \dots, n, \\ [y_i, x_i] &= t - k \sum_{j \neq i} \sum_{\gamma \in \Gamma} s_{ij}\gamma_i\gamma_j^{-1} - 2 \sum_{\gamma \in \Gamma \setminus \{1\}} \mathbf{c}(\gamma)\gamma_i, \quad i = 1, \dots, n, \\ [y_i, x_j] &= k \sum_{\gamma \in \Gamma} \omega(\gamma \cdot y_i, x_j) s_{ij}\gamma_i\gamma_j^{-1}, \quad i, j = 1, \dots, n, \quad i \neq j, \end{aligned}$$

where  $(L, \omega)$  is a two-dimensional symplectic vector space, with a fixed basis  $\{x, y\}$  such that  $\omega(x, y) = 1$ .

Introduce the following vectors:

$$\partial = \{\partial_i\}_{i \in I} \in \mathbb{Z}^I, \quad \partial_i := n \left( -\delta_i + \sum_{\gamma \in E: s(\gamma)=i} \delta_t(\gamma) \right), \quad (2.4.5)$$

$$\mu = \{\mu_i\}_{i \in I} \in \mathbb{C}^I \quad \mu_i := \text{tr}_{\rho_i}(-2\mathbf{c}) + \delta_i/|\Gamma|,$$

$$\lambda = \{\lambda_i\}_{i \in I} \in \mathbb{C}^I \quad \lambda_0 := \mu_0 - \partial_0 + k|\Gamma|, \quad \lambda_i := \mu_i - \partial_i, \quad \forall i \in I \setminus \{0\}.$$

Finally, recall the spherical subalgebra  $eH_{t,k,\mathbf{c}}(\Gamma_n)e$ , where  $e = 1/|\Gamma_n| \sum_{g \in \Gamma_n} g$ .

**Theorem 2.4.10.** *With the settings above, there is an isomorphism of filtered algebras*

$$\mathfrak{H}_{k,\mathbf{c}} : \mathfrak{A}_\lambda(\mathbf{v}, \mathbf{w}) \xrightarrow{\sim} eH_{1,k,\mathbf{c}}(\Gamma_n)e.$$

*Its associated graded gives rise to an isomorphism of Poisson algebras*

$$\text{gr } \mathfrak{H}_{k,\mathbf{c}} : \mathbb{C}[\mu^{-1}(0)]^{G_{\mathbf{v}}} \xrightarrow{\sim} \mathbb{C}[L^{\oplus n}]^{\Gamma_n}.$$

Note that this theorem is the quantum analogue of Theorem 2.3.11. The proof has several authors. Holland [Ho] first proved the case  $n = 1$  (where symplectic reflection algebras coincide with deformed preprojective algebras). The general case was then conjectured (in a slightly different way) in [EG], where it is also proved for  $\Gamma = \{1\}$  and generic deformation parameter. The complete proof in this case is given in [GG]. When  $\Gamma = \mu_\ell$  is the cyclic group, Oblomkov [Ob] defined the map and Gordon [Go1] proved the isomorphism. In [EGGO], it is proved for  $Q$  a bi-partitive quiver, which includes all the cases except when it is of type  $\tilde{\mathbf{A}}_{2m}$ . Finally, Losev [Lo] gave another proof using different techniques based on canonical quantisations (see Remark 2.4.4).

**Remark 2.4.11.** Unlike quiver varieties (see Remark 2.3.8), the sheaf  $\mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w})$  and the algebra  $\mathfrak{A}_\lambda(\mathbf{v}, \mathbf{w})$  depend on the orientation of the quiver; changing orientation produces a shift of  $\lambda$  by an integral parameter. This is the reason why we have to subtract the vector  $\partial$  in the definition of  $\lambda$  (2.4.5). There is a way to avoid this dependence, that is to consider a *symmetrised* quantum comoment map  $\Phi^{sym}$  (which is independent of the orientation) instead of the usual comoment map  $\Phi$ . This is the choice made in [Lo] (compare the parameters (2.4.5) with [Lo, (6.2)]).

## 2.4.4 Quantum reflection functors

We end this section with a construction that will turn useful for us in the next chapter.

We now drop the assumptions on  $Q$ ,  $\mathbf{v}$  and  $\mathbf{w}$ . Recall that reflection functors (section 2.2.1) give rise to the LMN isomorphisms  $\phi_\sigma^\lambda : \mathcal{M}_\lambda^\theta(\mathbf{v}, \mathbf{w}) \xrightarrow{\sim} \mathcal{M}_{\sigma\bullet\lambda}^{\sigma\theta}(\sigma \cdot (\mathbf{v}, \mathbf{w}))$  (Theorem 2.3.12). In [Lo, section 6.4], and in [BL, section 2.2.4], these are quantised and give rise to sheaf isomorphisms between quantisations. They do not intertwine the quantum comoment map and send  $\Phi_\lambda$  to  $\Phi_{\sigma\bullet\lambda}$ . Here,  $\sigma \bullet \lambda$  is a *dot-action*, which differs from the usual action by a shift by a character  $\rho(\mathbf{v}, \mathbf{w}) \in \mathbb{Z}^I$  (which depends on the orientation of  $Q$ ), such that  $\Phi(x) - \Phi^{sym}(x) = \langle \rho(\mathbf{v}, \mathbf{w}), x \rangle$  (see Remark 2.4.11).  $\rho(\mathbf{v}, \mathbf{w})$  is computed in [BL] and it equals minus half the character of  $\mathfrak{g}_\mathbf{v}$  on  $\bigwedge^{\text{top}} \text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w})$ . We have

$$\rho(\mathbf{v}, \mathbf{w})_i = -\frac{1}{2} \left( \sum_{\gamma \in E: t(\gamma)=i} v_{s(\gamma)} - \sum_{\gamma \in E: s(\gamma)=i} v_{t(\gamma)} - w_i \right) \quad (2.4.6)$$

$$\text{and} \quad \sigma \bullet \lambda = \sigma(\lambda - \rho(\mathbf{v}, \mathbf{w})) + \rho(\sigma \cdot (\mathbf{v}, \mathbf{w})).$$

Hence, we have the following theorem [BL, section 2.2.4] (see also [BPW, Proposition 3.10]).

**Theorem 2.4.12.** *Let  $(\lambda, \theta)$  be generic and  $\sigma \in W(Q)$ . We have an isomorphism*

$$\mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w}) \cong \mathcal{A}_{\sigma \bullet \lambda}^{\sigma \theta}(\sigma \cdot (\mathbf{v}, \mathbf{w}))$$

*quantising the LMN isomorphism  $\mathcal{M}_0^\theta(\mathbf{v}, \mathbf{w}) \cong \mathcal{M}_0^{\sigma \theta}(\sigma \cdot (\mathbf{v}, \mathbf{w}))$ .*

**Corollary 2.4.13.** *Let  $Q$  be of affine Dynkin type,  $\mathbf{v} \in \mathbb{N}^I$  be any dimension vector and set  $\mathbf{w} = (1, 0, \dots, 0)$ . Let  $eH_{1,k,\mathbf{c}}(\Gamma_n)e$  as in Theorem 2.4.10, then there exists  $\lambda \in \mathbb{C}^I$  such that*

$$\Gamma(\mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w})) \cong eH_{1,k,\mathbf{c}}(\Gamma_n)e.$$

*Proof.* It is easy to see that, in the notation of Remark 2.4.8, for all  $n \in \mathbb{N}$ ,  $\omega_0 - \sum_{i \in I} n\alpha_i$  is a dominant weight for  $\mathfrak{g}(Q)$ . Thus, by [Ka2, Corollary 10.1],  $\omega_0 - \sum_{i \in I} v_i\alpha_i$  is in the same  $W(Q)$ -orbit (for the dot-action (2.3.3)) as  $\omega_0 - \sum_{i \in I} n\alpha_i$ , for some  $n \in \mathbb{N}$ , and let  $\sigma \in W(Q)$  be such that  $(\mathbf{v}, \mathbf{w}) = \sigma \cdot (n\delta, \mathbf{w})$ . By Theorems 2.4.10 and 2.4.12, we have

$$eH_{1,k,\mathbf{c}}(\Gamma_n)e \cong \Gamma(\mathcal{A}_{\sigma^{-1} \bullet \lambda}^{\sigma^{-1} \theta}(n\delta, \mathbf{w})) \cong \Gamma(\mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w}))$$

and we are done. □

Nakajima [Na1] defined an action of  $\mathfrak{g}(Q)$  on  $\bigoplus_{\mathbf{v}} H_{\text{mid}}(\mathcal{M}_0^\theta(\mathbf{v}, \mathbf{w}))$ , where  $\mathbf{w}$  is fixed and  $H_{\text{mid}}$  denotes the middle homology group. This way, this space becomes an irreducible integrable  $\mathfrak{g}(Q)$ -module of highest weight  $\mathbf{w}$ . By quantising this action, one could use Corollary 2.4.13, to study the representation theory of the corresponding spherical symplectic reflection algebra via categorical techniques.

## 2.5 The cyclotomic case

Defining the *deformed Harish-Chandra* homomorphism, the map  $\mathfrak{H}_{k,\mathbf{c}}$  in Theorem 2.4.10, and proving that this gives a filtered isomorphism is quite complicated in the general case. In particular, the proof in [EGGO] relies on a technical result about the commutativity of a certain diagram ([EGGO, Theorem 1.6.1]), while in [Lo], the isomorphism is proved without specifying what the map is. In the cyclotomic case, however, the situation simplifies remarkably because of the Dunkl embedding, which provides a direct link between the spherical symplectic reflection algebra (a rational Cherednik algebra in this case) and invariant differential operators.

In this last section, we review the strategy of the proof of Theorem 2.4.10 in this case and we use it to illustrate how Corollary 2.4.13 is quite a subtle statement. Along the way, we give a generalisation of the main result of [Ga].

### 2.5.1 Restriction theorem for the cyclic quiver

Let  $Q$  be the cyclic quiver with  $\ell$  vertices and cyclic orientation, where  $\ell$  is an integer greater than 1, and let  $\mathbf{v} = (v_0, \dots, v_{\ell-1}) \in \mathbb{N}^\ell$  be a dimension vector for  $Q$  (we consider the indices as elements of  $\mathbb{Z}/\ell\mathbb{Z}$ , so that  $v_\ell$  is identified with  $v_0$ ). Our first goal is to find a slice in  $\text{Rep}(Q, \mathbf{v})$  for the action of  $G_{\mathbf{v}}$ .

Let  $n := \min\{v_0, \dots, v_{\ell-1}\}$  and assume, without loss of generality,  $n = v_0$ . Recall  $\mathfrak{h}^{\text{reg}} \subset \mathfrak{h} = \mathbb{C}^n$  from section 1.2.1, the affine open subset of  $\mathbb{C}^n$  on which  $\Gamma_n = \Sigma_n \wr \mu_\ell$  acts freely. It is given by

$$\mathfrak{h}^{\text{reg}} = \{(x_1, \dots, x_n) \in (\mathbb{C}^\times)^n \mid x_i \neq \zeta^m x_j, \forall i \neq j, \forall m \in \mathbb{Z}\}, \quad (2.5.1)$$

where  $\zeta$  denotes a primitive  $\ell$ -th root of unity. We embed  $\mathfrak{h}^{\text{reg}}$  into  $\text{Rep}(Q, \mathbf{v})$  in the following way

$$\begin{aligned} \varphi : \mathfrak{h}^{\text{reg}} &\hookrightarrow \text{Rep}(Q, \mathbf{v}) & (2.5.2) \\ (x_1, \dots, x_n) &\mapsto (\underline{X}_0, \dots, \underline{X}_{\ell-1}), \end{aligned}$$

where  $\underline{X}_i$  is the  $v_{i+1} \times v_i$  matrix with the largest possible square block having  $x_1, \dots, x_n$  and 1's on the diagonal and 0's elsewhere, as shown below

$$\begin{pmatrix} \text{diag}(x_1, \dots, x_n) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \text{diag}(1, \dots, 1) & 0 & \cdots & 0 \end{pmatrix},$$

$$\begin{pmatrix} \text{diag}(x_1, \dots, x_n) & \cdots & 0 \\ \vdots & & \vdots \\ \text{diag}(x_1, \dots, x_n) & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \text{diag}(1, \dots, 1) \end{pmatrix} \text{ or } \begin{pmatrix} \text{diag}(x_1, \dots, x_n) & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \text{diag}(1, \dots, 1) \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

Note that  $\varphi$  is injective as  $n$  is the minimum of the components of the dimension vector.

Call  $\mathcal{S}$  the image of this mapping; we claim that this set is a slice for the  $G_{\mathbf{v}}$ -action. More precisely, let  $T := Z_{G_{\mathbf{v}}}(\mathcal{S}) = \{g \in G_{\mathbf{v}} \mid g.x = x, \forall x \in \mathcal{S}\}$  the stabiliser of  $\mathcal{S}$  and consider the following  $G_{\mathbf{v}}$ -equivariant morphism

$$\begin{aligned} \pi_{\mathbf{v}} : G_{\mathbf{v}}/T \times \mathfrak{h}^{\text{reg}} &\longrightarrow \text{Rep}(Q, \mathbf{v}) \\ (gT, x) &\longmapsto g.\varphi(x) \end{aligned}$$

We have the following lemma.

**Lemma 2.5.1.**  $\pi_{\mathbf{v}}$  is an étale mapping with covering group  $\Gamma_n$ .

*Proof.* Since the action of  $\Gamma_n$  on  $\mathfrak{h}^{\text{reg}}$  and therefore on  $G_{\mathbf{v}}/T \times \mathfrak{h}^{\text{reg}}$  is free, it suffices to show that the image  $\text{Rep}(Q, \mathbf{v})^{\text{reg}} := \pi(G/T \times \mathfrak{h}^{\text{reg}})$  is open in  $\text{Rep}(Q, \mathbf{v})$  and that we have an isomorphism

$$G_{\mathbf{v}}/T \times_{\Gamma_n} \mathfrak{h}^{\text{reg}} \xrightarrow{\sim} \text{Rep}(Q, \mathbf{v})^{\text{reg}} .$$

Let's start with the latter. As in the proof of [Go1, Lemma 2.4], the key point is to prove that there is an isomorphism  $\Gamma_n \cong N_{G_{\mathbf{v}}}(\mathcal{S})/Z_{G_{\mathbf{v}}}(\mathcal{S})$ , where  $N_{G_{\mathbf{v}}}(\mathcal{S}) := \{g \in G \mid g.\mathcal{S} = \mathcal{S}\}$  is the normaliser of  $\mathcal{S}$  in  $G_{\mathbf{v}}$ . Unlike in [Go1], however, in the general case we do not have a nice description of  $T$ , which, for example, may well not sit in the diagonal subgroup of  $G_{\mathbf{v}}$ . Suppose  $g.\varphi(x) = \varphi(y)$ , for some  $x, y \in \mathfrak{h}^{\text{reg}}$ , so that

$$(g_1 \underline{X}_0 g_0^{-1}, \dots, g_0 \underline{X}_{\ell-1} g_{\ell-1}^{-1}) = (\underline{Y}_0, \dots, \underline{Y}_{\ell-1}) .$$

Multiplying all the components on both sides of this equality, we obtain

$$g_i \text{diag}(x_1, \dots, x_n, 0, \dots, 0)^\ell g_i^{-1} = \text{diag}(y_1, \dots, y_n, 0, \dots, 0)^\ell, \quad \forall 0 \leq i \leq \ell - 1,$$

the length of the sequence of 0's being equal to  $v_i - n$ . This implies that  $g_i$  has two blocks, one of size  $n$  and the other one of size  $v_i - n$ . Call them  $\tilde{g}_i$  and  $\bar{g}_i$ , respectively. Since  $\text{diag}(x_1, \dots, x_n)^\ell$  and  $\text{diag}(y_1, \dots, y_n)^\ell$  are regular semisimple in  $\mathbb{C}^n$ , we must have  $\tilde{g}_i \in N_{GL_n(\mathbb{C})}(H) = H.\Sigma_n$ , where  $H$  is the diagonal subgroup of  $GL_n(\mathbb{C})$ . So, there exist  $\sigma \in \Sigma_n$  and  $h_i \in H$ , such that  $\tilde{g}_i = h_i \sigma$ , for all  $1 \leq i \leq n$ , and  $x_{\sigma^{-1}(r)}^\ell = y_r^\ell$ , for all  $1 \leq r \leq n$ , which implies  $x_{\sigma^{-1}(r)} = \zeta^{m_r} y_r$ , for some  $m_r \in \mathbb{Z}$ . As  $g_{i+1} \underline{X}_i g_i^{-1} = \underline{Y}_i$ , we obtain  $\tilde{g}_{i+1} \text{diag}(x_1, \dots, x_n) \tilde{g}_i^{-1} = \text{diag}(y_1, \dots, y_n)$  and hence  $\text{diag}(y_1, \dots, y_n) = h_{i+1} h_i^{-1} \text{diag}(\zeta^{m_1} y_1, \dots, \zeta^{m_n} y_n)$ . Since  $y_r \neq 0, \forall 1 \leq r \leq n$ , we have  $h_{i+1} = \text{diag}(\zeta^{m_1}, \dots, \zeta^{m_n}) h_i$  for all  $i$ .

Now, if  $t = (t_0, \dots, t_{\ell-1}) \in T = Z_{G_{\mathbf{v}}}(\mathcal{S})$ , there exists  $h \in H$  such that each  $t_i$  has two blocks, the first one being equal to  $h$  and the second one depending on  $i$ . Call the latter  $\bar{t}_i$ . For all  $i$ , we have one of the following conditions (depending on the size of  $\bar{t}_i$ )

$$\bar{t}_{i+1} \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} \bar{t}_i^{-1} = \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix},$$

$$\bar{t}_{i+1} = \bar{t}_i \quad \text{or} \quad \bar{t}_{i+1} \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \end{pmatrix} \bar{t}_i^{-1} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \end{pmatrix}.$$

If we consider the blocks  $\bar{g}_i$  of elements of  $N_{G_{\mathbf{v}}}(\mathcal{S})$ , they must satisfy the same conditions of  $\bar{t}_i$ . This means that any element of the group  $N_{G_{\mathbf{v}}}(\mathcal{S})/Z_{G_{\mathbf{v}}}(\mathcal{S})$  has the form

$$gT = (\sigma, \text{diag}(\zeta^{m_1}, \dots, \zeta^{m_n})\sigma, \dots, \text{diag}(\zeta^{m_1}, \dots, \zeta^{m_n})^{\ell-1}\sigma) T$$

and we have the isomorphism

$$\begin{aligned} \Gamma_n & \longrightarrow N_G(\mathcal{S})/Z_G(\mathcal{S}) \\ \sigma(s^{m_1}, \dots, s^{m_n}) & \longmapsto (\sigma, \text{diag}(\zeta^{m_1}, \dots, \zeta^{m_n})\sigma, \dots, \text{diag}(\zeta^{m_1}, \dots, \zeta^{m_n})^{\ell-1}\sigma) T. \end{aligned}$$

Therefore, if  $\pi_{\mathbf{v}}(gT, x) = \pi_{\mathbf{v}}(hT, y)$ , then  $(h^{-1}g)\varphi(x) = \varphi(y)$  and  $h^{-1}g \in N_{G_{\mathbf{v}}}(\mathcal{S})$ , which implies that  $\pi_{\mathbf{v}}$  factors through the following isomorphism

$$G_{\mathbf{v}}/T \times_{\Gamma_n} \mathfrak{h}^{\text{reg}} \xrightarrow{\sim} \text{Rep}(Q, \mathbf{v})^{\text{reg}}.$$

We must now show that  $\text{Rep}(Q, \mathbf{v})^{\text{reg}}$  is open in  $\text{Rep}(Q, \mathbf{v})$ . In [Ka1], it is shown that, given a quiver, a dimension vector  $\alpha$  has canonical decomposition  $\alpha = \sum \beta_i$  if and only if each  $\beta_i$  is a Schur root and  $\text{Ext}(\beta_i, \beta_j)$  vanishes generically. In the case of the cyclic quiver it follows by a direct computation that the canonical decomposition of the dimension vector  $\mathbf{v} = (n, v_1, \dots, v_{\ell-1})$  is given by

$$\mathbf{v} = \underbrace{\delta + \dots + \delta}_n + \gamma_1 + \dots + \gamma_p,$$

where  $\delta = (1, \dots, 1)$  and  $\gamma_j$  are of the form  $(0, \dots, 0, 1, \dots, 1, 0, \dots, 0)$  with the string of 1's being the longest possible. Call  $U$  the open subset of  $\text{Rep}(Q, \mathbf{v})$  consisting of representations whose decomposition in indecomposable components is canonical, so that  $X \in U$  if and only if

$$X = X_1 \oplus \dots \oplus X_n \oplus Y_1 \oplus \dots \oplus Y_p, \quad \dim X_i = \delta, \quad \dim Y_j = \gamma_j.$$

Then let  $O$  be the subset of representations of  $U$  such that  $X_i$  is simple for all  $1 \leq i \leq n$  and  $\dim \text{End}(X_1 \oplus \dots \oplus X_n) = n$ . Consider the continuous function

$$\begin{aligned} f : \text{Rep}(Q, \delta) & \longrightarrow \mathbb{C} \\ (\lambda_0, \dots, \lambda_{\ell-1}) & \longmapsto \lambda_0 \cdots \lambda_{\ell-1}. \end{aligned}$$

Since the open set  $f^{-1}(\mathbb{C}^\times)$  consists of the simple representations of dimension vector  $\delta$ , the subset  $V$  of  $U$  consisting of representations whose components  $X_i$  are simple is open. Now, consider the closed subvariety of  $\bigoplus \text{Mat}_{n,n}(\mathbb{C}) \times U$

$$\{(\phi, X) \mid \phi \in \text{End}(X_1 \oplus \dots \oplus X_n)\}$$

and its projection onto  $U$ . The fibre of such projection over a representation  $X$  is precisely  $\text{End}(X_1 \oplus \dots \oplus X_n)$  and the mapping  $U \rightarrow \mathbb{N}$ ,  $X \mapsto \dim \text{End}(X_1 \oplus \dots \oplus X_n)$  is upper semi-continuous by Chevalley's theorem (the fact that the projection is not proper does not create problems as we can embed  $\bigoplus \text{Mat}_{n,n}(\mathbb{C})$  into its projective space). This tells us that the set

$$W := \{X \in U \mid \dim \text{End}(X_1 \oplus \dots \oplus X_n) = n\}$$

is open. As we have  $O = V \cap W$ ,  $O$  is open in  $U$  and hence in  $\text{Rep}(Q, \mathbf{v})$ .

Now we show that  $\text{Rep}(Q, \mathbf{v})^{\text{reg}}$  is included in  $O$ . Every element in  $\text{Rep}(Q, \mathbf{v})^{\text{reg}}$  is isomorphic to a representation of the form  $\varphi(x) = (\underline{X}_0, \dots, \underline{X}_{\ell-1})$  which decomposes as the sum  $X_1 \oplus \dots \oplus X_n \oplus Y_1 \oplus \dots \oplus Y_p$  where each  $X_i$  has dimension vector  $\delta$  and is of the form  $(x_i, \dots, x_i)$  and  $Y_j$  has dimension vector  $\gamma_j$  and is of the form  $(0, \dots, 0, 1, \dots, 1, 0, \dots, 0)$ . First of all, this implies the inclusion  $\text{Rep}(Q, \mathbf{v})^{\text{reg}} \subset U$ . Then, we observe that, since  $x_i \neq 0$ , for all  $i$ , each  $X_i$  must be simple. Finally, every  $X_i$  is clearly isomorphic to a representation of the form  $(1, \dots, 1, x_i^\ell)$ , therefore, the condition  $x_i^\ell \neq y_i^\ell$ , for  $i \neq j$ , implies  $\dim \text{End}(X_1 \oplus \dots \oplus X_n) = n$ .

Now, if  $X \in O$ , we have  $X = X_1 \oplus \dots \oplus X_n \oplus Y_1 \oplus \dots \oplus Y_p$ . The hypothesis of simplicity forces  $X_i$  to be isomorphic to  $(1, \dots, 1, \nu_i)$  with  $\nu_i \neq 0$ , while the fact that  $\dim \text{End}(X_1 \oplus \dots \oplus X_n) = n$  implies  $\nu_i \neq \nu_j$  for  $i \neq j$ . Therefore, we have  $X_i \cong (\eta_i, \dots, \eta_i)$ , where  $\eta_i$  is a  $\ell$ -th root of  $\nu_i$ . Finally, being of dimension vector  $\gamma_j$ , each  $Y_j$  must be of the form  $(0, \dots, \mu_j^{(1)}, \dots, \mu_j^{(r_j)}, 0, \dots, 0)$ ,  $\mu_j^{(k)} \neq 0$ , which is clearly isomorphic to the representation  $(0, \dots, 0, 1, \dots, 1, 0, \dots, 0)$ . Putting the pieces together, we obtain  $X \cong \varphi(x)$ , where  $x = (\eta_1, \dots, \eta_n)$ . Hence  $O \subset \text{Rep}(Q, \mathbf{v})^{\text{reg}}$  and the proof is complete.  $\square$

Recall the moment map

$$\begin{aligned} \mu : \text{Rep}(\overline{Q}, \mathbf{v}) &\longrightarrow \mathfrak{g}^* \cong \mathfrak{g} \\ (X, Y) &\longmapsto [X, Y]. \end{aligned}$$

The following theorem, which is a particular case of Theorem 2.3.11, is due to Crawley-Boevey.

**Theorem 2.5.2.** *Let  $\mu^{-1}(0)$  be the scheme-theoretic fibre of  $\mu$  over zero. We have an isomorphism  $\mathbb{C}[\mu^{-1}(0)]^{G_{\mathbf{v}}} \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n}$ .*

*Proof.* Let  $\lambda \in \mathbb{C}^\ell$  such that  $\lambda \cdot \mathbf{v} = \sum_{i=0}^{\ell-1} \lambda_i v_i = 0$ , viewed inside  $\mathfrak{g}_{\mathbf{v}}$  as a multiple of the identity, and define the symplectic reduction

$$N(\lambda, \mathbf{v}) := \mu^{-1}(\lambda) // G_{\mathbf{v}} .$$

Write  $\mathbf{v} = n\delta + (v_1 - n)\alpha_1 + \dots + (v_{\ell-1} - n)\alpha_{\ell-1}$ , with  $\delta$  the minimal positive imaginary root and  $\alpha_i$  the real roots given by coordinate vectors. [CB2, Theorem 1.1] gives a description of the structure of  $N(\lambda, \mathbf{v})$  which, because of [CB2, Proposition 1.2 (1)], reduces to the following isomorphism for  $\lambda = 0$

$$N(0, \mathbf{v}) \cong S^m N(0, \delta) .$$

By [CBH, Theorem 8.10],  $N(0, \delta)$  is the Klenian singularity of type  $\tilde{\mathbf{A}}_{\ell-1}$  and we obtain the desired isomorphism.  $\square$

A straightforward consequence of Lemma 2.5.1 and Theorem 2.5.2 is the Chevalley restriction theorem for representations of the cyclic quiver with arbitrary dimension vector and its double analogue. The equidimensional case was treated by Gan [Ga] and we give the same proof.

As seen in the proof of Lemma 2.5.1, we can embed  $\Gamma_n$  into  $G_{\mathbf{v}}$ , by composition of  $\Gamma_n \hookrightarrow \prod_{i=0}^{\ell-1} GL_n$

$$(\sigma, s^{m_1}, \dots, s^{m_n}) \mapsto (\sigma, \text{diag}(\zeta^{m_1}, \dots, \zeta^{m_n})\sigma, \dots, \text{diag}(\zeta^{m_1}, \dots, \zeta^{m_n})^{\ell-1}\sigma) ,$$

where  $\sigma \in \Sigma_n$  is regarded as a permutation matrix in  $GL_n$ , and

$$\prod_{i=0}^{\ell-1} GL_n \hookrightarrow G_{\mathbf{v}} , \quad (g_0, \dots, g_{\ell-1}) \mapsto (\hat{g}_0, \dots, \hat{g}_{\ell-1}) ,$$

where  $\hat{g}_i$  is a block diagonal matrix with  $g_i$  and  $\text{diag}(1, \dots, 1)$  as blocks of size  $n$  and  $v_i - n$ , respectively. We can also embed  $\mathfrak{h}$  into  $\text{Rep}(Q, \mathbf{v})$  naturally extending  $\varphi$  in (2.5.2) and call its image  $\bar{\mathcal{S}}$ . Note that  $\bar{\mathcal{S}}$  is stable for the action of  $\Gamma_n$  (which is, in fact, the same as the action of  $\Gamma_n$  on  $\mathfrak{h}$ ).

**Theorem 2.5.3.** (1) *The restriction map from the space of functions on  $\text{Rep}(Q, \mathbf{v})$  to the space of functions on  $\bar{\mathcal{S}}$  produces an isomorphism*

$$\rho : \mathbb{C}[\text{Rep}(Q, \mathbf{v})]^G \xrightarrow{\sim} \mathbb{C}[\bar{\mathcal{S}}]^{\Gamma_n} .$$

(2) *The restriction map from the space of functions on  $\mu^{-1}(0) \subset \text{Rep}(\bar{Q}, \mathbf{v})$  to the space of functions on  $\bar{\mathcal{S}} \times \bar{\mathcal{S}}$  produces an isomorphism*

$$\phi : \mathbb{C}[\mu^{-1}(0)]^G \xrightarrow{\sim} \mathbb{C}[\bar{\mathcal{S}} \times \bar{\mathcal{S}}]^{\Gamma_n} .$$

*Proof.* (1) Let  $(X_0, \dots, X_{\ell-1}) \in \text{Rep}(Q, \mathbf{v})$  and let  $p$  be the characteristic polynomial of the  $n \times n$  matrix  $X_{\ell-1} \cdots X_0$ . This is clearly  $G_{\mathbf{v}}$ -invariant and its coefficients restrict to the elementary symmetric polynomials in  $x_1^\ell, \dots, x_n^\ell$ , where  $\varphi(x_1, \dots, x_n) \in \bar{\mathcal{S}}$ . Such polynomials generate  $\mathbb{C}[\bar{\mathcal{S}}]^{\Gamma_n}$  and hence  $\rho$  is surjective. Let now  $f$  be a regular  $G_{\mathbf{v}}$ -invariant function on  $\text{Rep}(Q, \mathbf{v})$  that equals zero when restricted to  $\bar{\mathcal{S}}$ . By Lemma 2.5.1,  $f$  is zero when restricted also to  $\text{Rep}(Q, \mathbf{v})^{\text{reg}}$  which is a Zariski dense in  $\text{Rep}(Q, \mathbf{v})$ . It follows that  $f$  is identically zero and  $\rho$  is injective.

(2) The proof of the double case follows immediately from Theorem 2.5.2. □

## 2.5.2 The radial part map

We now restrict ourselves to the case  $\mathbf{v} = n\delta$ .

Given  $k \in \mathbb{C}$  and  $\mathbf{c} = (c_1, \dots, c_{\ell-1}) \in \mathbb{C}^{\ell-1}$ , recall the cyclotomic rational Cherednik algebra  $H_{k, \mathbf{c}} = H_{1, k, \mathbf{c}}(\Gamma_n)$  defined in 1.4.1. In this case, the Dunkl embedding (section 1.2.1)  $\Theta_{k, \mathbf{c}} : H_{k, \mathbf{c}} \hookrightarrow D(\mathfrak{h}^{\text{reg}}) \# \Gamma_n$  is given by

$$x_i \mapsto x_i, \quad y_i \mapsto \mathcal{D}_i, \quad s_i \mapsto s_i, \quad i = 1, \dots, n,$$

where

$$\mathcal{D}_i := \frac{\partial}{\partial x_i} + k \sum_{j \neq i} \sum_{m=0}^{\ell-1} \frac{1}{x_i - \zeta^m x_j} (s_{ij} s_i^m s_j^{-m} - 1) - 2 \sum_{m=1}^{\ell-1} \frac{c_m}{(\zeta^m - 1) x_i} (s_i^m - 1).$$

Via the Dunkl embedding, the spherical subalgebra  $eH_{k, \mathbf{c}}e$  is sent to the algebra of  $\Gamma_n$ -invariant differential operators  $D(\mathfrak{h}^{\text{reg}})^{\Gamma_n}$ . We put  $\Theta_{k, \mathbf{c}}^{\text{sph}} := \Theta_{k, \mathbf{c}}|_{eH_{k, \mathbf{c}}e}$ .

Introduce the following complex values

$$C_0 := \ell^{-1} \left( 1 - \ell + 2 \sum_{m=1}^{\ell-1} c_m \right), \quad C_i := \ell^{-1} \left( 1 + 2 \sum_{m=1}^{\ell-1} \zeta^{mi} c_m \right), \quad \text{for } i = 1, \dots, \ell-1, \quad (2.5.3)$$

$$\sigma := \ell^{-1} \sum_{j=0}^{\ell-1} j C_j \quad \text{and} \quad r_i := \sum_{j=0}^i C_j + \sigma, \quad \text{for } i = 1, \dots, \ell-1.$$

Define the character  $\lambda_{\mathbf{c}} : \mathfrak{g}_{\mathbf{v}} \rightarrow \mathbb{C}$ , which sends an element  $g = (g_0, \dots, g_{\ell-1}) \in \mathfrak{g}_{\mathbf{v}}$  to  $\lambda_{\mathbf{c}}(g) := \sum_{j=0}^{\ell-1} C_j \text{tr} g_j$ .  $\lambda_{\mathbf{c}}$  is actually a character of  $\mathfrak{pg}_{\mathbf{v}} := \text{Lie } PG_{\mathbf{v}}$ , as we have  $\sum_{j=0}^{\ell-1} C_j = 0$ . We also define  $\lambda_k \in \mathfrak{g}_{\mathbf{v}}^*$  by  $\lambda_k(g) := -k \text{tr} g_0$  and put  $\lambda_{k, \mathbf{c}} := \lambda_k + \lambda_{\mathbf{c}}$ .

Let

$$W'_k := (y_1 \cdots y_n)^{-k} \mathbb{C}_{(0)}[y_1^{\pm 1}, \dots, y_n^{\pm 1}],$$

where  $\mathbb{C}_{(0)}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$  is the space of Laurent polynomials of degree 0 in the variables  $y_1, \dots, y_n$ . This is a  $\mathfrak{g}_{\mathbf{v}}$ -module with  $\mathfrak{g}_{\mathbf{v}}$ -action given by the projection on the first factor and the natural action  $\mathfrak{gl}_n \rightarrow \mathfrak{gl}(W'_k)$ ,  $E_{ij} \mapsto \frac{\partial}{\partial y_j}$ . We can also consider the  $\lambda_k$ -twisted version  $W_k := W'_k \otimes \lambda_k$ , which is a  $\mathfrak{pg}_{\mathbf{v}}$ -module since the identity acts as zero on  $W_k$ . Define the space

$$Fun' := \left\{ \tilde{f} \prod_{i=0}^{\ell-1} \det(X_i)^{r_i} \mid \tilde{f} \text{ is rational on } \text{Rep}(Q, \mathbf{v})^{\text{reg}}, \text{ regular on } \mathcal{S} \right\}.$$

A simple computation shows that  $Fun_{k,\mathbf{c}} := (Fun' \otimes W_k)^{\mathfrak{pg}_{\mathbf{v}}}$  is a space of  $W_k$ -valued,  $\lambda_{\mathbf{c}}$ -semi-invariants functions, defined on a neighbourhood of  $\mathcal{S}$ . Their restriction to  $\mathcal{S}$  takes values in the 1-dimensional zero-weight space of  $W_k$  and we can therefore regard  $Fun_{k,\mathbf{c}}$  as a space of scalar functions defined on  $\mathcal{S}$ . Lemma 2.5.1 implies, in fact, that, for a generic element  $X \in \text{Rep}(Q, \mathbf{v})$ , there exist a unique (up to  $\Gamma_n$ -action) element  $x \in \mathfrak{h}^{\text{reg}}$  and  $g \in PG_{\mathbf{v}}$ , such that  $g.X = \varphi(x) \in \mathcal{S}$ . Hence, by semi-invariance, every  $f \in Fun_{k,\mathbf{c}}$  is determined by its restriction to  $\mathcal{S}$  and every  $\Gamma_n$ -invariant function on  $\mathfrak{h}^{\text{reg}}$  is the restriction of some function in  $Fun_{k,\mathbf{c}}$ . In other words, the restriction to  $\mathcal{S}$  gives a  $\mathbb{C}[\mathfrak{h}^{\text{reg}}]^{\Gamma_n}$ -module isomorphism  $Fun_{k,\mathbf{c}} \cong \mathbb{C}[\mathfrak{h}^{\text{reg}}]^{\Gamma_n}$ . Thus, it is possible to define the mapping

$$\begin{aligned} \mathfrak{R}_{k,\mathbf{c}} : D(\text{Rep}(Q, \mathbf{v}))^{\mathfrak{pg}} &\longrightarrow D(\mathfrak{h}^{\text{reg}})^{\Gamma_n} \\ D &\longmapsto \mathfrak{R}_{k,\mathbf{c}}(D), \end{aligned} \quad (2.5.4)$$

where  $\mathfrak{R}_{k,\mathbf{c}}(D)(f) := D(\bar{f})|_{\mathcal{S}}$ , with  $\bar{f} \in Fun_{k,\mathbf{c}}$  the function such that  $\bar{f}|_{\mathcal{S}} = f$ . Let  $\delta := \prod_{1 \leq i < j \leq n} (x_i^\ell - x_j^\ell)$ ,  $\delta_{\Gamma} := \prod_{i=1}^n x_i$  and  $\delta_{k,\mathbf{c}} := \delta^{k-1} \delta_{\Gamma}^{\ell \sigma}$ . The twisted version of  $\mathfrak{R}_{k,\mathbf{c}}$  by  $\delta_{k,\mathbf{c}}$

$$\mathfrak{R}_{k,\mathbf{c}}^{tw}(D) := \delta_{k,\mathbf{c}} \circ \mathfrak{R}_{k,\mathbf{c}}(D) \circ \delta_{k,\mathbf{c}}^{-1}$$

is called the **radial part map**. The main result of Oblomkov [Ob] is the following.

**Theorem 2.5.4.** *The images of the Dunkl embedding  $\Theta_{k,\mathbf{c}}^{sph}$  and of the radial part map  $\mathfrak{R}_{k,\mathbf{c}}^{tw}$  coincide in  $D(\mathfrak{h}^{\text{reg}})^{\Gamma_n}$ .*

We now move from  $Q$  to the framed quiver  $Q_{\infty}$ , with framing vector  $\mathbf{w} = (1, 0, \dots, 0)$ . Define  $\mathfrak{X}$  to be the quotient of the open subvariety  $U := \{(X, \mathbf{i}) \in \text{Rep}(Q_{\infty}, \mathbf{v}, \mathbf{w}) \mid \mathbf{i} \neq 0\}$  by the action of  $\mathbb{C}^{\times}$ . Hence, we have  $\mathfrak{X} \cong \text{Rep}(Q, \mathbf{v}) \times \mathbb{P}^{n-1}$  and the restriction of the  $G_{\mathbf{v}}$ -action on  $\text{Rep}(Q_{\infty}, \mathbf{v}, \mathbf{w})$  to  $U$  descends to an action of  $PG_{\mathbf{v}}$  on  $\mathfrak{X}$ . Denote by  $D_{\mathfrak{X},k}$  the sheaf of differential operators on  $\mathfrak{X}$  twisted by  $\lambda_k$ . Namely, if  $\hat{\Phi}$  denotes the quantum comoment map of the  $G_{\mathbf{v}}$ -action on  $U$ , we have  $D_{\mathfrak{X},k} =$

$(D_U/D_U(\hat{\Phi} - \lambda_k)(\mathbb{C}\text{Id}))^{\mathbb{C}^\times}$ . It is not difficult to see that this yields an isomorphism of algebras (cf. [Go1, Lemma 4.2])

$$\left( \frac{D(\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w}))}{D(\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w}))(\hat{\Phi} - \lambda_{k,\mathbf{c}})(\mathfrak{g}_{\mathbf{v}})} \right)^{G_{\mathbf{v}}} \cong \left( \frac{D_{\mathfrak{X},k}(\mathfrak{X})}{D_{\mathfrak{X},k}(\mathfrak{X})(\Phi_k - \lambda_{\mathbf{c}})(\mathfrak{pg}_{\mathbf{v}})} \right)^{PG_{\mathbf{v}}}, \quad (2.5.5)$$

where  $\Phi_k : \mathfrak{pg}_{\mathbf{v}} \rightarrow D_{\mathfrak{X},k}$  is the twisted comoment map induced by the action of  $PG_{\mathbf{v}}$  on  $\mathfrak{X}$ . Gordon [Go1, Theorem 3.13] proves that the radial part map  $\mathfrak{R}_{k,\mathbf{c}}^{tw}$  factors through the right-hand side of (2.5.5) and its kernel equals the ideal generated by  $(\Phi_k - \lambda_{\mathbf{c}})(\mathfrak{pg}_{\mathbf{v}})$ . Then, via (2.5.5) and setting  $\mathfrak{H}_{k,\mathbf{c}} := (\Theta_{k,\mathbf{c}}^{sph})^{-1} \circ \mathfrak{R}_{k,\mathbf{c}}^{tw}$ , we obtain the following theorem, a particular case of Theorem 2.4.10.

**Theorem 2.5.5.** *There is an isomorphism of filtered algebras*

$$\left( \frac{D(\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w}))}{D(\text{Rep}(Q_\infty, \mathbf{v}, \mathbf{w}))(\hat{\Phi} - \lambda_{k,\mathbf{c}})(\mathfrak{g}_{\mathbf{v}})} \right)^{G_{\mathbf{v}}} \cong eH_{k,\mathbf{c}}e.$$

**Remark 2.5.6.** Since Lemma 2.5.1 holds for a general dimension vector  $\mathbf{v} \in \mathbb{N}$ , it is possible to define a radial part map analogous to (2.5.4) in the general case. The same strategy as in the equidimensional case does not yield, however, another proof of Corollary 2.4.13. This is because  $\mathfrak{A}_\lambda(\mathbf{v}, \mathbf{w}) \not\cong \Gamma(\mathcal{A}_\lambda^\theta(\mathbf{v}, \mathbf{w}))$  unless  $\mathbf{v} = n\delta$  (see Remark 2.4.8 (ii)). In fact, using Theorem 2.3.2, one can prove that Proposition 2.4.7 (i) is, in this case, an if and only if, so that, if  $\mathbf{v} \neq n\delta$ , the moment map  $\mu$  is not flat and  $\mathfrak{A}_\lambda(\mathbf{v}, \mathbf{w})$  is not a quantisation of  $\mathbb{C}[\mu^{-1}(0)]$ .

# Chapter 3

## Invariants of links

### 3.1 Homological mirror symmetry

The expression mirror symmetry refers to a phenomenon, observed by physicists in string theory, which allows two different Calabi-Yau manifolds to give rise to equivalent quantum field theories. This remarkable connection between different manifolds remained without a mathematical explanation until 1994, when Kontsevich formulated a rigorous framework with the attempt to translate mirror symmetry in mathematical language. His conjecture [Ko1], slightly modified in the following years, asserts that, given two Calabi-Yau manifolds  $X$  and  $Y$ ,  $X$  is mirror dual to  $Y$  if the derived category of coherent sheaves of  $X$  is equivalent to the derived Fukaya category of  $Y$ . This statement, today known as homological mirror symmetry, relates the symplectic structure of  $Y$  and the complex (or algebraic) structure of  $X$ , which should therefore give the mathematical counterpart of the so called A and B model topological string theories.

Although proved for elliptic curves ([PZ]), quartic surfaces ([Se]) and other particular cases, this conjecture remains far from being solved and even understood in general. It is not clear, in particular, what definition of the Fukaya category to consider in order to obtain the desired equivalence.

Despite this unwieldy issue, one of the powerful aspects of homological mirror symmetry lies in the possibility, via the heuristics of this equivalence, to make predictions in derived algebraic geometry, based on observations in symplectic geometry, and viceversa. One example is given by the following construction.

### 3.1.1 Link invariants and Slodowy slices

Let  $\mathfrak{g} = \mathfrak{sl}_{2n}$ ,  $\mathfrak{t}$  be the Cartan subalgebra of traceless diagonal matrices and  $W = \Sigma_{2n}$  be the Weyl group. Consider the Slodowy slice  $\mathcal{S}_n := e + \mathfrak{z}(f)$ , transverse to the orbit  $G.e$ , where  $e$  is the nilpotent matrix with two Jordan blocks of size  $n$ ,  $G = SL_{2n}$  acts by adjunction and  $e, f, h$  are an  $\mathfrak{sl}_2$  triple by the Jacobson-Morozov theorem. The unordered  $2n$ -tuples of distinct eigenvalues  $\mathfrak{t}^{reg}/W$  are identified with the configuration space  $\text{Conf}_{2n}(\mathbb{C})$  and the quotient map  $\mathfrak{g} \rightarrow \mathfrak{g}/G \cong \mathfrak{t}/W$  restricted to  $\mathcal{S}_n$  gives rise to a fibre bundle over  $\text{Conf}_{2n}(\mathbb{C})$ . Denote by  $\mathcal{Y}_{n,t}$  the fibre of this map at a point  $t$ . Let  $L_\beta$  be a link obtained as the plat closure of  $\beta := b \times 1^n$ , where  $b$  is an element of the braid group on  $n$  strands. We can view  $L_\beta$  as a loop in the configuration space  $\text{Conf}_{2n}(\mathbb{C})$  starting at a basepoint  $t$ ; taking monodromy along this loop yields a symplectic automorphism  $\phi_\beta$ . Seidel and Smith [SS] defined the **symplectic Khovanov homology** by

$$Kh_{\text{symp}}^*(\beta) := HF^{*+w+n}(\mathcal{L}, \phi_\beta \mathcal{L}) ,$$

where  $\mathcal{L}$  is a certain Lagrangian in  $\mathcal{Y}_{n,t}$ ,  $w$  is the writhe of the braid diagram of  $\beta$ , and  $HF^*$  denotes the Lagrangian Floer cohomology. They prove the following theorem.

**Theorem 3.1.1.** ([SS, Theorem 1])  $Kh_{\text{symp}}^*(\beta)$  is an invariant of the isotopy class of the link  $L_\beta$ .

They also formulated the following remarkable conjecture, now a theorem by the work of Abouzaid and Smith ([AbSm]), which justifies the notation. Let  $Kh^{*,*}(L)$  be the Khovanov homology of a link  $L$  ([Kh]).

**Theorem 3.1.2.** *We have the isomorphism*

$$Kh_{\text{symp}}^k(\beta) \cong \bigoplus_{i-j=k} Kh^{i,j}(L_\beta) .$$

In an attempt to shed some light on this (at that time) conjecture, or, more specifically, try to relate symplectic Khovanov cohomology to the Jones polynomials so as to give a geometric interpretation thereof, Manolescu [Man] proved that there is an isomorphism between  $\mathcal{Y}_{n,t}$  and an open subset of the Hilbert scheme of  $n$  points on the ALE surface. The latter is equipped with a  $\mathbb{C}^*$ -action, which should provide a second grading supposed to match with the Khovanov homology bigrading. Motivated by this result and the fact that Lagrangian submanifolds of general fibres can become holomorphic when passing to the resolution of the central fibre (using symplectic parallel transport and simultaneous resolution, as explained in [Th]), hence swapping

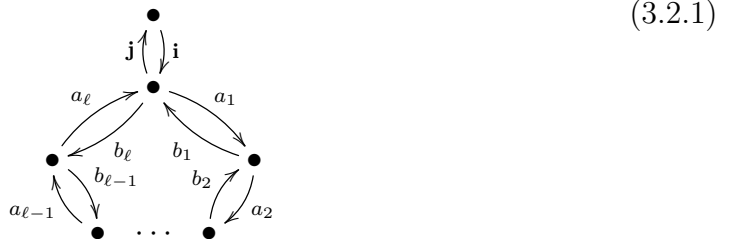
complex and symplectic structures, Smith and Thomas conjectured that the mirror of  $\mathcal{Y}_{n,t}$  might be related to such a resolution. This way, replacing Lagrangian Floer homology with  $\text{Ext}^*$  in the derived category of coherent sheaves on the resolution, one should be able to obtain a bigraded link invariant in algebro-geometric setting, coinciding with Khovanov homology.

We review this construction in the following section.

## 3.2 ALE spaces and braid invariants

### 3.2.1 The singularity of type $\mathbf{A}_{\ell-1}$

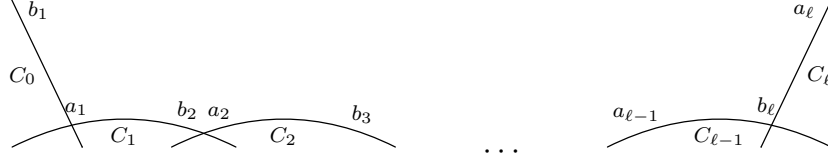
Let  $Q$  be the cyclic quiver with  $\ell$  vertices and  $\delta := (1, \dots, 1)$  the minimal imaginary root of the corresponding affine Dynkin diagram. Consider the framing with dimension vector  $\mathbf{w} := (1, 0, \dots, 0)$  and denote by  $Q_\infty$  the resulting quiver. The space of representations of the double  $\overline{Q_\infty}$  coincides with the cotangent bundle of  $\text{Rep}(Q_\infty, \delta, \mathbf{w})$  and we have the moment map  $\mu : \text{Rep}(\overline{Q_\infty}, \delta, \mathbf{w}) \rightarrow \mathfrak{g}_\delta^* \cong \mathfrak{g}_\delta$ , where  $G_\delta = (\mathbb{C}^\times)^\ell$  and  $\mathfrak{g}_\delta = \text{Lie } G_\delta$ . From now on, we will only consider this choice of  $\mathbf{w}$  and will omit it from the notation.



By Theorem 2.3.9, the affine quotient  $\mathcal{M}_0^0(\delta) = \mu^{-1}(0)//G_\delta$  is isomorphic to  $\mathbb{C}^2/\mu_\ell$ , the Klenian singularity of type  $\mathbf{A}_{\ell-1}$ , where  $\mu_\ell$  is the cyclic group whose generator  $s$  acts by means of the inclusion  $s \mapsto \text{diag}(e^{2\pi i/\ell}, e^{-2\pi i/\ell}) \in SL_2(\mathbb{C})$ . The corresponding Nakajima quiver variety  $X := \mathcal{M}_0^\theta(\delta)$  with stability parameter  $\theta := (-1, \dots, -1, \ell-1)$  is a minimal resolution  $\pi : X \rightarrow \mathbb{C}^2/\mu_\ell$  (we use this choice of stability as we will need to be consistent with [Ku2]). Let  $((a_i, b_i)_{1 \leq i \leq \ell}, \mathbf{i}, \mathbf{j})$  be a representation of  $\overline{Q_\infty}$  as in picture (3.2.1), and denote by  $(\bar{a}, \bar{b})$  the image of  $[a_i, b_i]_{1 \leq i \leq \ell}$  via the isomorphism  $\mu^{-1}(0)//G_\delta \xrightarrow{\sim} \mathbb{C}^2/\mu_\ell$  (note that the class  $[a_i, b_i]$  is independent of  $\mathbf{i}$  and  $\mathbf{j}$ ). The Lagrangian  $\pi^{-1}(\{\bar{a} = 0\} \cup \{\bar{b} = 0\})$  has  $\ell+1$  irreducible components  $C_0, \dots, C_\ell$  such that  $C_0, C_\ell \cong \mathbb{A}^1$  and  $C_i \cong \mathbb{P}^1$ , for  $1 \leq i \leq \ell-1$ . Their description in coordinates is given by

$$C_i = \overline{\left\{ [a_j, b_j]_{1 \leq j \leq \ell} \mid \begin{array}{ll} a_j = 0, b_j \neq 0, & \text{for } j > i \\ a_j \neq 0, b_j = 0, & \text{for } j \leq i \end{array} \right\}} \quad \text{for } 1 \leq i \leq \ell,$$

and  $C_0 = \overline{\{[a_j, b_j]_{1 \leq j \leq \ell} \mid b_j \neq 0\}}$ .



The  $C_i$ 's for  $1 \leq i \leq \ell - 1$  are the irreducible components of the exceptional divisor of  $\pi$  and form a chain of  $\mathbb{P}^1$ 's of Dynkin type  $\mathbf{A}_{\ell-1}$ . Set  $\mathcal{L}_i := \mathcal{O}_{C_i}(-1)$ , a line bundle supported on  $C_i$ . For a variety  $Y$  (and a finite group  $W$  acting on  $Y$ ), we will denote by  $D(Y)$  (resp.  $D(Y)^W$ ) the bounded (resp.  $W$ -equivariant) derived category of coherent sheaves with compact support on  $Y$ . We can regard  $\mathcal{L}_i$  as an object of  $D(X)$ . Fix  $n > 1$  and set  $\ell = 2n$ . The  $\Sigma_n$ -linearised external tensor product

$$\mathcal{L}^{(n)} := \Sigma_n.(\mathcal{L}_1 \boxtimes \mathcal{L}_3 \boxtimes \dots \boxtimes \mathcal{L}_{2n-1}) = \bigoplus_{\sigma \in \Sigma_n} \sigma^*(\mathcal{L}_1 \boxtimes \mathcal{L}_3 \boxtimes \dots \boxtimes \mathcal{L}_{2n-1})$$

defines an object of  $D(X)^{\Sigma_n}$ , where the symmetric group  $\Sigma_n$  acts by permuting the  $n$  copies of  $X$ . We call  $\mathcal{L}^{(n)}$  **Smith-Thomas sheaf**. Let  $\text{Hilb}^n(X)$  be the Hilbert scheme of  $n$  points on  $X$ , a resolution of singularity of  $\mathbb{C}^{2n}/\Gamma_n$ , where  $\Gamma_n$  denotes the wreath product  $\Sigma_n \wr \mu_{2n}$ , and set  $\mathbf{D}_n := D(\text{Hilb}^n(X))$ . By the derived McKay correspondence [BKR] and a result of Haiman [Ha], we have a derived equivalence  $\mathbf{D}_n \cong D(X)^{\Sigma_n}$  and hence  $\mathcal{L}^{(n)}$  can be viewed as an object of  $\mathbf{D}_n$ .

### 3.2.2 Spherical twists

We denote by  $B_\ell$  the braid group on  $\ell$  strands, i.e. the group generated by  $\beta_1, \dots, \beta_{\ell-1}$ , subject to the braid relations

$$\begin{aligned} \beta_i \beta_{i+1} \beta_i &= \beta_{i+1} \beta_i \beta_{i+1} & \text{for } 1 \leq i \leq \ell - 2, \\ \beta_i \beta_j &= \beta_j \beta_i & \text{for } |i - j| > 1. \end{aligned} \quad (3.2.2)$$

Let  $Y$  be a smooth projective variety, an object  $\mathcal{E} \in D(Y)$  is called **spherical** if it satisfies

- $\text{Ext}^i(\mathcal{E}, \mathcal{E}) = \begin{cases} \mathbb{C} & \text{if } i = 0, \dim(\mathcal{E}) \\ 0 & \text{otherwise} \end{cases}$ ,
- $\mathcal{E} \otimes \omega_Y \cong \mathcal{E}$ , where  $\omega_Y$  is the canonical bundle of  $Y$ .

An  $\mathbf{A}_{\ell-1}$ -configuration of spherical objects is a collection of spherical objects  $\mathcal{E}_1, \dots, \mathcal{E}_{\ell-1} \in D(Y)$  such that

$$\bigoplus_k \text{Ext}^k(\mathcal{E}_i, \mathcal{E}_j) = \begin{cases} \mathbb{C} & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}.$$

Seidel and Thomas defined, for a spherical object  $\mathcal{E} \in D(Y)$ , an autoequivalence  $T_{\mathcal{E}} \in \text{Aut}(D(X))$  as Fourier-Mukai transform with kernel the cone of a certain morphism, which fits in the distinguished triangle

$$\text{Ext}^*(\mathcal{E}, \mathcal{F}) \otimes_{\mathbb{C}} \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow T_{\mathcal{E}}(\mathcal{F}),$$

for  $\mathcal{F} \in D(Y)$ . They proved the following ([SeTh], Theorem 1.2 and 1.3):

**Theorem 3.2.1.** 1.  $T_{\mathcal{E}}$  is an exact autoequivalence of  $D(Y)$ .

2. For an  $\mathbf{A}_{\ell-1}$ -configuration of spherical objects  $\mathcal{E}_1, \dots, \mathcal{E}_{\ell-1}$ , the spherical twists  $T_{\mathcal{E}_i}$  satisfy the braid relations (3.2.2).

3. The resulting group action on  $\text{Aut}(D(Y))$  is faithful.

### 3.2.3 A braid invariant cohomology

It is easy to see that  $\mathcal{L}_1, \dots, \mathcal{L}_{2n-1}$  form an  $\mathbf{A}_{2n-1}$ -configuration of spherical objects in  $D(X)$ . Moreover, by [Pl], there is a canonical injective homomorphism  $\Phi : \text{Aut}(D(X)) \hookrightarrow \text{Aut}(D(X^n))^{\Sigma_n}$ , such that

$$\Phi(T)(\Sigma_n \cdot (\mathcal{L}_1 \boxtimes \mathcal{L}_3 \boxtimes \dots \boxtimes \mathcal{L}_{2n-1})) = \Sigma_n \cdot (T\mathcal{L}_1 \boxtimes T\mathcal{L}_3 \boxtimes \dots \boxtimes T\mathcal{L}_{2n-1}), \quad (3.2.3)$$

for  $T \in \text{Aut}(D(X))$ . Thus we have a (faithful) action of  $B_{2n}$  on  $\mathbf{D}_n$  and we set  $T_i := \Phi(T_{\mathcal{L}_i})[1] \in \text{Aut}(\mathbf{D}_n)$  (the shift does not affect the braid relations, since they are homogeneous), corresponding to the generators  $\beta_i \in B_{2n}$ , and  $T_{\beta} = T_{i_1} \circ \dots \circ T_{i_r}$  corresponding to any element  $\beta = \beta_{i_1} \cdot \dots \cdot \beta_{i_r}$ . Smith and Thomas defined a braid invariant cohomology by

$$ST^*(\beta) := \text{Ext}_{\mathbf{D}_n}^*(T_{\beta}\mathcal{L}^{(n)}, \mathcal{L}^{(n)}[n]). \quad (3.2.4)$$

In order for this to be an invariant of the isotopy class of the link given by the plat closure of the braid  $\beta$ , one needs to check that it is invariant under certain moves. By [Bi, Lemma 5.2] and by the functoriality of  $\text{Ext}^*$ , these moves can be rephrased as the following relations:

- (A)  $T_1 \mathcal{L}^{(n)} \cong \mathcal{L}^{(n)}$ ;
- (B)  $T_{2i-1} T_{2i} \mathcal{L}^{(n)} \cong T_{2i-1}^{-1} T_{2i}^{-1} \mathcal{L}^{(n)}$ ;
- (C)  $T_{2i} T_{2i-1} T_{2i+1} T_{2i} \mathcal{L}^{(n)} \cong \mathcal{L}^{(n)}$ ;
- (D)  $\text{Ext}_{\mathbb{D}_n}^* (T_\beta \mathcal{L}^{(n)}, \mathcal{L}^{(n)}[n]) \cong \text{Ext}_{\mathbb{D}_{n+1}}^* (T_\beta \mathcal{L}^{(n+1)}, T_{2n}^{\pm 1} \mathcal{L}^{(n+1)}[n+1])$ .

**Theorem 3.2.2.** (*[SmTh, Theorem 3.10]*) *Relations (A), (C) and (D) hold, but (B) does not.*

### 3.3 The cyclic rational Cherednik algebra

Let  $W = \mu_\ell$ , a complex reflection group whose reflection representation is  $\mathfrak{h} = \mathbb{C}$ , where a generator  $s$  acts by a primitive  $\ell$ -th root of unity  $\zeta$ . Fix a parameter  $\mathbf{c} = (c_1, \dots, c_{\ell-1}) \in \mathbb{C}^{\ell-1}$  (we set  $t = 1$  in the general definition). The rational Cherednik algebra of cyclic type  $H_{\mathbf{c}} = H_{\mathbf{c}}(\mu_\ell)$  is the algebra generated by  $x, y, s$ , subject to the following relations

$$sx = \zeta^{-1}xs, \quad sy = \zeta ys, \quad s^\ell = 1,$$

$$[y, x] = 1 - 2 \sum_{j=1}^{\ell-1} c_j s^j.$$

Denote by  $\rho_j$  the irreducible representation of  $\mu_\ell$  where the generator  $s$  acts by multiplication by  $\zeta^j$ , and consider the standard module  $\Delta(\rho_j) = H_{\mathbf{c}} \otimes_{\mathbb{C}[\mathfrak{h}^*] \# \mu_\ell} \rho_j$  and the corresponding simple module  $L(\rho_j)$ . Let  $f_{\mathbf{c}}(z) := \sum_{j=1}^{\ell-1} \frac{2c_j}{1-\zeta^{-j}} z^j$ , so that  $\mathbf{e}\mathbf{u} = xy - f_{\mathbf{c}}(\zeta)$ . The multiplicities  $[\Delta(\rho_p) : L(\rho_m)]$  of  $L(\rho_m)$  in  $\Delta(\rho_p)$  are known in this case and easy to calculate (cf. [CE]).

**Theorem 3.3.1.** (i)  $L(\rho_m) = \Delta(\rho_m)$  if and only if  $L(\rho_m)$  is infinite dimensional.

(ii)  $[\Delta(\rho_p) : L(\rho_m)]$  equals 1 if  $f(p, m) := f_{\mathbf{c}}(\zeta^p) - f_{\mathbf{c}}(\zeta^m)$  is a non-negative integer congruent to  $p - m$  modulo  $\ell$ , and 0 otherwise.

(iii) If  $L(\rho_m) \neq \Delta(\rho_m)$ , let  $b > 0$  be the minimum of the set  $\{f(p, m) \mid f(p, m) \equiv p - m \pmod{\ell}, m \geq 0\}$ . The character of  $L(\rho_m)$  is given by

$$\text{tr}_{L(\rho_m)}(s^j) = \zeta^{jm} \frac{1 - \zeta^{-jb}}{1 - \zeta^{-j}}.$$

By Theorem 2.5.5, the spherical subalgebra  $eH_{\mathbf{c}}e$  of the cyclic rational Cherednik algebra is isomorphic to the global sections of  $\mathcal{A}_{\lambda}^{\theta}(\delta)$  (see (2.4.2)), a sheaf of non-commutative filtered algebras quantising the resolution of the Klenian singularity  $X = \mathcal{M}_0^{\theta}(\delta)$ . Thus, it makes sense to consider the singular support of modules for the spherical subalgebras. We would like to understand the singular support of  $H_{\mathbf{c}}$ -modules via Morita equivalence, but we first have to exclude the aspherical parameters. We need an alternative parametrisation. Let  $\tilde{c}_1, \dots, \tilde{c}_{\ell-1} \in \mathbb{C}$ ,  $\tilde{c}_0 = \tilde{c}_{\ell} = -(\tilde{c}_1 + \dots + \tilde{c}_{\ell-1})$ , such that

$$c_j = -\frac{1}{2} + \frac{1}{2} \sum_{i=0}^{\ell-1} \left( \tilde{c}_i + \frac{1}{\ell} \right) \zeta^{-ij}, \quad \text{for } 1 \leq j \leq \ell.$$

We have the following.

**Lemma 3.3.2.** (*[Ku1, Proposition 4.4]*) *Assume that  $\tilde{c}_i + \tilde{c}_{i+1} + \dots + \tilde{c}_{j-1} \neq 0$ , for  $0 < i < j \leq \ell$ . Then, the cyclic rational Cherednik algebra  $H_{\mathbf{c}}$  is Morita equivalent to its spherical subalgebra.*

From now on, we assume that the hypothesis of Lemma 3.3.2 holds. We can therefore consider the singular support of standards and simple modules for  $H_{\mathbf{c}}$ . Set  $\Delta_i := \Delta(\rho_{\ell-i})$  and  $L_i := L(\rho_{\ell-i})$ . The following theorem is the main result of [Ku2].

**Theorem 3.3.3.** *Suppose  $\tilde{c}_i + \tilde{c}_{i+1} + \dots + \tilde{c}_{j-1} \notin \mathbb{Z}$  for  $1 \leq i < j \leq \ell$ . Then, in the sense above, we have:*

1. *the singular support of  $\Delta_i$  coincides with the Lagrangian in  $X$  given by the union  $C_i \cup C_{i+1} \cup \dots \cup C_{\ell}$ ;*
2. *let  $\epsilon(i)$  be the unique index in  $\{i+1, \dots, \ell+1\}$  such that  $\tilde{c}_i + \tilde{c}_{i+1} + \dots + \tilde{c}_{\epsilon(i)-1} \in \mathbb{Z}$  and  $\tilde{c}_i + \tilde{c}_{i+1} + \dots + \tilde{c}_{j-1} \notin \mathbb{Z}$  for any  $i < j < \epsilon(i)$ . The singular support of  $L_i$  coincides with the Lagrangian in  $X$  given by the union  $C_i \cup C_{i+1} \cup \dots \cup C_{\epsilon(i)-1}$ .*

As a corollary of Theorem 3.3.3, we obtain that  $\text{SS}(L_i) = C_i$ , whenever  $\tilde{c}_i, \dots, \tilde{c}_{\ell-1} \in \mathbb{Z}_{>0}$ . In the following sections, we will assume this integrality condition on the deformation parameters.

### 3.3.1 Quiver description

For our purpose, it will be more convenient to work with a category equivalent to category  $\mathcal{O}$  for the cyclic rational Cherednik algebra. Let  $\mathcal{O}(\mathfrak{sl}_{\ell})$  be the BGG category  $\mathcal{O}$  for the Lie algebra  $\mathfrak{sl}_{\ell}$ . It is well known that such a category breaks up

into a direct sum of blocks  $\mathcal{O}_\mu$ . Here, if  $\mathfrak{t}$  denotes a Cartan subalgebra,  $\mu \in \mathfrak{t}^*/\Sigma_\ell$  is a weight corresponding to a central character  $\chi$  via the Harish-Chandra isomorphism, and the objects of  $\mathcal{O}_\mu$  are those killed by some power of  $\chi$ .

**Proposition 3.3.4.** *There exists a weight  $\mu = \mu(\mathbf{c})$ , whose stabiliser under the dot-action is a  $\Sigma_{\ell-1}$  generated by  $\ell - 2$  transpositions in  $\Sigma_\ell$ , such that there is an equivalence of categories  $\mathcal{O}(H_{\mathbf{c}}) \cong \mathcal{O}_\mu(\mathfrak{sl}_\ell)$ .*

*Proof.* By Theorem 2.5.5 and [Lo, Theorem 5.3.1],  $eH_{\mathbf{c}}e$  is isomorphic to a W-algebra in type  $\mathbf{A}_{\ell-1}$  with regular and integral parameter (see [BLPW] for the definition). W-algebras in type A with such parameters have equivalent categories  $\mathcal{O}$ , which, by [We1], are equivalent to a singular block of category  $\mathcal{O}(\mathfrak{sl}_\ell)$ . That the weight  $\mu$  must be of the claimed form follows by comparing the number of standard modules.  $\square$

Any block of  $\mathcal{O}(\mathfrak{sl}_\ell)$  corresponding to a weight  $\mu$  is known to be equivalent to the category of finitely generated modules for a certain finite dimensional basic algebra  $A$ . We can recover the latter by considering a projective generator

$$P := \bigoplus_{\sigma \in \Sigma_\ell/\Sigma_\mu} P(\sigma \cdot \mu),$$

where  $P(\sigma \cdot \mu)$  is the projective cover of the simple  $L(\sigma \cdot \mu)$ . Here  $\Sigma_\mu$  denotes the stabiliser of  $\mu$ , so that  $\{L(\sigma \cdot \mu) \mid \sigma \in \Sigma_\ell/\Sigma_\mu\}$  is the set of all the non-isomorphic irreducible modules in  $\mathcal{O}_\mu(\mathfrak{sl}_\ell)$ . Therefore, we have  $A = \text{End}_{\mathfrak{sl}_\ell}(P)^{op}$  and  $\mathcal{O}_\mu(\mathfrak{sl}_\ell) \cong A\text{-mod}$ . The description of  $A$  as a path algebra in the case of our choice of  $\mu$  is given by the quiver

$$\bullet \begin{array}{c} \xrightarrow{c_1} \\ \xleftarrow{d_1} \end{array} \bullet \begin{array}{c} \xrightarrow{c_2} \\ \xleftarrow{d_2} \end{array} \bullet \cdots \bullet \begin{array}{c} \xrightarrow{c_{\ell-1}} \\ \xleftarrow{d_{\ell-1}} \end{array} \bullet$$

with relations  $d_1c_1 = 0$ , and  $c_id_i = d_{i+1}c_{i+1}$ , for  $1 \leq i \leq \ell - 2$ . Via the former equivalence, a simple  $L_i \in \mathcal{O}(H_{\mathbf{c}})$  corresponds to  $\mathbb{C}\{e_i\}$ , the 1-dimensional  $A$ -module generated by the idempotent  $e_i$  at a vertex  $i$ , and its projective cover  $P_i \in \mathcal{O}(H_{\mathbf{c}})$  corresponds to the projective spanned by all paths with source  $i$ . With a slight abuse of notation, we will also denote by  $L_i$  and  $P_i$  the corresponding  $A$ -modules just described.

### 3.4 Twisting functors

We now introduce twisting functors, defined for conical symplectic resolutions in [BPW] and for certain Nakajima quiver varieties in [BL]. We will follow Bezrukavnikov

and Losev's description but, since the two definitions are equivalent, we will use results from both papers.

### 3.4.1 Quantum reflection and wall-crossing functors

Consider a Nakajima quiver variety  $\mathcal{M}_\lambda^\theta(\mathbf{v})$  and the corresponding sheaf  $\mathcal{A}_\lambda^\theta(\mathbf{v})$ , with a generic choice of  $(\lambda, \theta)$ . Recall quantum reflection functors from section 2.4.4. For  $(\lambda, \theta)$  generic and  $\sigma \in W(Q)$ , the Weyl group associated to the underlying graph of  $Q$ , we have the LMN isomorphisms  $\mathcal{M}_\lambda^\theta(\mathbf{v}) \cong \mathcal{M}_{\sigma\lambda}^{\sigma\theta}(\sigma \cdot \mathbf{v})$ . By quantising their construction, we get isomorphisms  $\mathcal{A}_\lambda^\theta(\mathbf{v}) \cong \mathcal{A}_{\sigma\lambda}^{\sigma\theta}(\sigma \cdot \mathbf{v})$ , which yield functors  $\mathcal{A}_\lambda^\theta(\mathbf{v})\text{-mod} \rightarrow \mathcal{A}_{\sigma\lambda}^{\sigma\theta}(\sigma \cdot \mathbf{v})\text{-mod}$  and their derived analogues

$$\Phi_\sigma^\lambda : D(\mathcal{A}_\lambda^\theta(\mathbf{v})\text{-mod}) \longrightarrow D(\mathcal{A}_{\sigma\lambda}^{\sigma\theta}(\sigma \cdot \mathbf{v})\text{-mod}) .$$

Let  $\chi \in \mathbb{Z}^I$ , where  $I$  is the set of vertices of the quiver, and consider the bimodule of  $\chi$ -semi-invariants ( $\chi$  is regarded as a  $G_{\mathbf{v}}$ -character via (2.2.1))

$$\mathcal{A}_{\lambda,\chi}^\theta(\mathbf{v}) := \left( D_{\text{Rep}(Q_\infty, \mathbf{v})} / D_{\text{Rep}(Q_\infty, \mathbf{v})} \{ \Phi(x) - \langle \lambda, x \rangle \mid x \in \mathfrak{gl}_{\mathbf{v}} \} \Big|_{T^*\text{Rep}(Q_\infty, \mathbf{v})^{\theta-ss}} \right)^{G_{\mathbf{v}}, \chi} ,$$

where we use the notation of chapter 2. This is a Harish-Chandra bimodule, namely it is finitely generated and admits a filtration such that the associated graded is scheme-theoretically supported on the diagonal ([BPW, Proposition 6.22]). If  $\chi = \lambda' - \lambda$ , with  $\lambda, \lambda'$  such that the algebras of global sections of  $\mathcal{A}_\lambda^\theta(\mathbf{v})$  and  $\mathcal{A}_{\lambda'}^\theta(\mathbf{v})$  have finite homological dimension, taking the derived tensor product by  $\mathcal{A}_{\lambda,\chi}^\theta(\mathbf{v})$  gives an equivalence of (bounded) derived categories ([BPW, Corollary 6.31])

$$\Phi^{\lambda', \lambda} : D(\mathcal{A}_\lambda^\theta(\mathbf{v})\text{-mod}) \xrightarrow{\sim} D(\mathcal{A}_{\lambda'}^\theta(\mathbf{v})\text{-mod}) .$$

We call  $\Phi^{\lambda', \lambda}$  **wall-crossing functors**. Notice that in [BL] the expression wall-crossing functors refers to their composition with quantum reflection functors.

### 3.4.2 Braid group action

Consider now representations of the cyclic quiver with dimension vector  $n\delta$ ,  $n \geq 1$ , and let  $H_{k,\mathbf{c}} = H_{1,k,\mathbf{c}}(\Gamma_n)$  be the cyclotomic rational Cherednik algebra (Definition 1.4.1), where  $k \in \mathbb{C}$  and  $\mathbf{c}$  is as in section 3.3. For  $\sigma \in W(Q)$ , write  $\sigma \bullet (k, \mathbf{c})$  for the parameters corresponding to  $\sigma \bullet \lambda_{k,\mathbf{c}}$ , via the isomorphism of Theorem 2.5.5, so that  $\sigma \bullet \lambda_{k,\mathbf{c}} = \lambda_{\sigma \bullet (k,\mathbf{c})}$ . By Theorem 2.4.9 and Corollary 2.4.13, we get an action of the affine Weyl group  $W(Q)$  (and hence of the symmetric group  $\Sigma_\ell$ ) on the (bounded) derived category  $D(H_{k,\mathbf{c}}\text{-mod})$ , provided that  $\Sigma_\ell \bullet (k, \mathbf{c})$  is a set of spherical values.

Suppose this assumption holds and let  $s_i \in \Sigma_\ell$  be the transposition corresponding to the vertex  $i \in I$ , for  $1 \leq i \leq \ell - 1$ . We can consider the composition

$$\mathbb{T}_i^{k,\mathbf{c}} := \mathcal{R}\Gamma_{\lambda_{k,\mathbf{c}}}^{s_i\theta} \circ \Phi_{s_i}^{\lambda_{s_i \bullet (k,\mathbf{c})}} \circ \Phi^{\lambda_{s_i \bullet (k,\mathbf{c})}, \lambda_{k,\mathbf{c}}} \circ \mathcal{L}\text{Loc}_{\lambda_{k,\mathbf{c}}}^\theta : D(H_{k,\mathbf{c}}\text{-mod}) \xrightarrow{\sim} D(H_{k,\mathbf{c}}\text{-mod}) ,$$

where

$$\mathcal{L}\text{Loc}_{\lambda_{k,\mathbf{c}}}^\theta : D(\mathfrak{A}_{\lambda_{k,\mathbf{c}}}(n\delta)\text{-mod}) \xrightarrow{\sim} D(\mathcal{A}_{\lambda_{k,\mathbf{c}}}^\theta(n\delta)\text{-mod}) \quad \text{and}$$

$$\mathcal{R}\Gamma_{\lambda_{k,\mathbf{c}}}^{s_i\theta} : D(\mathcal{A}_{\lambda_{k,\mathbf{c}}}^{s_i\theta}(s_i \bullet n\delta)\text{-mod}) \xrightarrow{\sim} D(\Gamma(\mathcal{A}_{\lambda_{k,\mathbf{c}}}^{s_i\theta}(s_i \bullet n\delta))\text{-mod}) ,$$

are the equivalences of categories of Theorem 2.4.9, and we implicitly used the isomorphism of Corollary 2.4.13 and the Morita equivalence between  $H_{k,\mathbf{c}}$  and its spherical subalgebra.

By [BL, section 3], the functors  $\mathbb{T}_i^{k,\mathbf{c}}$  preserve  $D_{\pi^{-1}(0)}(H_{k,\mathbf{c}}\text{-mod})$ , the subcategory of objects whose cohomology is supported on  $\pi^{-1}(0) \in \mathcal{M}_0^\theta(\mathbf{v})$ . In particular, via equivalence (2.4.4), they preserve finite dimensional modules. By applying [BPW, Theorem 6.34], we get the following theorem.

**Theorem 3.4.1.** *The autoequivalences  $\mathbb{T}_i^{k,\mathbf{c}}$  generate an action of the braid group  $B_\ell$  on  $D(\mathcal{O}(H_{k,\mathbf{c}}))$ , which restricts to an action on  $D(H_{k,\mathbf{c}}\text{-fdmod})$ .*

For all  $\beta \in B_\ell$ , we denote by  $\mathbb{T}_\beta^{k,\mathbf{c}}$  the corresponding functor. The  $\mathbb{T}_\beta^{k,\mathbf{c}}$  are called **twisting functors**.

### 3.4.3 Compatibility at $k = 0$

Let  $\ell = 2n$ . We want to relate the action of twisting functors on modules for the cyclic and cyclotomic rational Cherednik algebras, so as to obtain the quantum analogue of (3.2.3).

We will need the following lemma.

**Lemma 3.4.2.** *Let  $A, A', B, B'$  be algebras,  $M \in A\text{-Mod}$ ,  $N \in B\text{-Mod}$  and  $\mathcal{A}, \mathcal{B}$  be a  $A\text{-}A'$ -bimodule and a  $B\text{-}B'$ -bimodule, respectively. Then, there is an isomorphism of  $A' \otimes B'$ -bimodules*

$$(\mathcal{A} \otimes \mathcal{B}) \otimes_{\mathcal{A} \otimes B} (M \otimes N) \cong (\mathcal{A} \otimes_A M) \otimes (\mathcal{B} \otimes_B N) .$$

*Proof.* The isomorphism follows by an easy calculation. □

Imposing  $k = 0$ , where  $k$  is the deformation parameter of the cyclotomic rational Cherednik algebra  $H_{k,\mathbf{c}}(\Gamma_n)$ , we have the isomorphism

$$H_{0,\mathbf{c}}(\Gamma_n) \cong \Sigma_n \# H_{\mathbf{c}}(\mu_{2n})^{\otimes n}. \quad (3.4.1)$$

Thus, for  $M_1, \dots, M_n$  finite dimensional  $H_{\mathbf{c}}(\mu_{2n})$ -modules,

$$M := \Sigma. (M_1 \otimes \dots \otimes M_n) = \bigoplus_{\sigma \in \Sigma} \sigma (M_1 \otimes \dots \otimes M_n) \quad (3.4.2)$$

is a finite dimensional induced  $H_{0,\mathbf{c}}(\Gamma_n)$ -module and we can consider  $\mathbb{T}_{\beta}^{0,\mathbf{c}} M$ , for  $\beta \in B_{2n}$ . On the other hand, we can define  $\hat{\mathbb{T}}_{\beta}^{\mathbf{c}} M := \Sigma. (\mathbb{T}_{\beta}^{\mathbf{c}} M_1 \otimes, \dots, \otimes \mathbb{T}_{\beta}^{\mathbf{c}} M_n)$ . We have the following compatibility result.

**Proposition 3.4.3.** *There is an isomorphism of objects in  $D(\mathcal{O}(H_{0,\mathbf{c}}(\Gamma_n)))$*

$$\hat{\mathbb{T}}_{\beta}^{\mathbf{c}} M \cong \mathbb{T}_{\beta}^{0,\mathbf{c}} M.$$

*Proof.* Let  $\varphi_{\mathbf{c}} : H_{0,\mathbf{c}}(\Gamma_n) \rightarrow \Sigma_n \# H_{\mathbf{c}}(\mu_{2n})^{\otimes n}$  be the map giving the isomorphism (3.4.1). We show that  $\mathbb{T}_i^{0,\mathbf{c}} \circ \varphi_{\mathbf{c}}^* \cong \varphi_{\mathbf{c}}^* \circ \hat{\mathbb{T}}_i^{\mathbf{c}}$ , for all  $1 \leq i < 2n$ , on objects of the form (3.4.2).

By (2.4.6), we have

$$\begin{aligned} s_i \bullet \lambda_{0,\mathbf{c}} &= s_i(\lambda_{0,\mathbf{c}} - \rho(n\delta)) + \rho(s_i \cdot (n\delta)) \\ &= s_i \left( \lambda_{0,\mathbf{c}} - \left( \frac{1}{2}, 0, \dots, 0 \right) \right) + \left( \frac{1}{2}, 0, \dots, 0 \right) \\ &= s_i(\lambda_{0,\mathbf{c}}). \end{aligned}$$

Since  $\lambda_{k,\mathbf{c}} = \lambda_k + \lambda_{\mathbf{c}}$  and looking at formulae (2.5.3),  $\lambda_{0,\mathbf{c}}$  can be written as the  $2n$ -tuple  $(-(\lambda_{\mathbf{c}}^{(1)} + \dots + \lambda_{\mathbf{c}}^{(2n-1)}), \lambda_{\mathbf{c}}^{(1)}, \dots, \lambda_{\mathbf{c}}^{(2n-1)})$  and so, by (2.3.4), we have  $s_i(\lambda_{0,\mathbf{c}}) = \lambda_{0,s_i \bullet \mathbf{c}}$ . Hence, by the localisation theorem (Theorem 2.4.9), there is a commutative diagram of isomorphisms

$$\begin{array}{ccc} eH_{0,\mathbf{c}}(\Gamma_n)e & \xrightarrow{s_i} & eH_{s_i \bullet (0,\mathbf{c})}(\Gamma_n)e \\ \varphi_{\mathbf{c}} \downarrow & & \downarrow \varphi_{s_i \bullet \mathbf{c}} \\ ((eH_{\mathbf{c}}(\mu_{2n})e)^{\otimes n})^{\Sigma_n} & \xrightarrow{s_i^{\otimes n}} & ((eH_{s_i \bullet \mathbf{c}}(\mu_{2n})e)^{\otimes n})^{\Sigma_n} \end{array}$$

which yields  $\mathbb{T}_{s_i}^{\lambda_{0,\mathbf{c}}} \circ \varphi_{\mathbf{c}}^* \cong \varphi_{s_i \bullet \mathbf{c}}^* \circ \hat{\mathbb{T}}_{s_i}^{\lambda_{\mathbf{c}}}$ , where  $\hat{\mathbb{T}}_{s_i}^{\lambda_{\mathbf{c}}} M := \Sigma_n. (\mathbb{T}_{s_i}^{\lambda_{\mathbf{c}}} M_1 \otimes \dots \otimes \mathbb{T}_{s_i}^{\lambda_{\mathbf{c}}} M_n)$ .

We now prove the analogous statement for wall-crossing functors, namely

$$\Phi^{\lambda_{s_i \bullet (0,\mathbf{c})}, \lambda_{0,\mathbf{c}}} \circ \varphi_{\mathbf{c}}^* \cong \varphi_{s_i \bullet \mathbf{c}}^* \circ \hat{\Phi}^{\lambda_{s_i \bullet \mathbf{c}}, \lambda_{\mathbf{c}}}. \quad (3.4.3)$$

Let  $\chi_{0,\mathbf{c}} := \lambda_{s_i \bullet (0,\mathbf{c})} - \lambda_{0,\mathbf{c}}$ . Notice that with our choice of the parameter  $\mathbf{c}$ , this is an integer vector. By definition,  $\mathcal{A}_{\lambda_{0,\mathbf{c}},\chi_{0,\mathbf{c}}}^\theta(n\delta)$  is a quantisation of the line bundle  $\mathcal{O}(\chi_{0,\mathbf{c}})$  on  $\mathcal{M}_0^\theta(n\delta)$ , whose sections are the  $\chi_{0,\mathbf{c}}$ -semi-invariant functions

$$\Gamma(U, \mathcal{O}(\chi_{0,\mathbf{c}})) := \mathbb{C}[p^{-1}(U)]^{G_{n\delta,\chi_{0,\mathbf{c}}}},$$

where  $U$  is an open subset of  $\mathcal{M}_0^\theta(n\delta)$  and  $p : \mu^{-1}(0)^{\theta-ss} \rightarrow \mathcal{M}_0^\theta(n\delta)$  denotes the quotient projection. The resolution  $\pi : \mathcal{M}_0^\theta(n\delta) \rightarrow \mathcal{M}_0^0(n\delta)$  factors through  $\mathcal{M}_0^\theta(\delta)^n/\Sigma_n$

$$\begin{array}{ccc} \mathcal{M}_0^\theta(n\delta) & \xrightarrow{\pi} & \mathcal{M}_0^0(n\delta) \\ & \searrow \rho & \nearrow \\ & \mathcal{M}_0^\theta(\delta)^n/\Sigma_n & \end{array}$$

and we have

$$\mathcal{O}(\chi_{0,\mathbf{c}}) \cong \rho^* \left( (\mathcal{O}_1(\chi_{\mathbf{c}}) \boxtimes \dots \boxtimes \mathcal{O}_1(\chi_{\mathbf{c}}))^{\Sigma_n} \right),$$

where  $\chi_{\mathbf{c}} := \lambda_{s_i \bullet \mathbf{c}} - \lambda_{\mathbf{c}}$  and the subscript of  $\mathcal{O}$  in the right-hand side indicates that the line bundle is over the surface (namely  $n = 1$ ). Notice that  $eH_{k,\mathbf{c}}(\Gamma_n)e$  quantises  $\mathcal{M}_0^\theta(n\delta)$ ,  $(eH_{\mathbf{c}}(\mu_{2n})e^{\otimes n})^{\Sigma_n}$  quantises  $\mathcal{M}_0^\theta(\delta)^n/\Sigma_n$  and these two algebras are isomorphic when  $k = 0$ . Hence, in such a case, there is an isomorphism

$$\mathcal{A}_{\lambda_{0,\mathbf{c}},\chi_{0,\mathbf{c}}}^\theta(n\delta) \cong \varphi_{\mathbf{c}}^* \left( (\mathcal{A}_{\lambda_{\mathbf{c}},\chi_{\mathbf{c}}}^\theta(\delta) \otimes \dots \otimes \mathcal{A}_{\lambda_{\mathbf{c}},\chi_{\mathbf{c}}}^\theta(\delta))^{\Sigma_n} \right),$$

due to the uniqueness of line bundle quantisations shown in [BPW, Proposition 5.2]. Thus, (3.4.3) follows by Lemma 3.4.2 and the  $\Sigma_n$ -linearity of the tensor product.  $\square$

We would like to understand the action of twisting functors on simple modules of the cyclic rational Cherednik algebra  $H_{\mathbf{c}}$ , or, equivalently, of the corresponding basic algebra  $A$ . As explained in [BLPW, Remark 8.3], for all  $1 \leq i \leq 2n - 1$ , there is a natural transformation  $\mathbb{T}_i^{\mathbf{c}} \rightarrow \text{Id}_{\mathcal{O}_{\mu(\mathbf{c})}(\mathfrak{sl}_\ell)}$ , which is an isomorphism when applied to the projective  $P_j$ , for  $j \neq i$ , and gives a short exact sequence

$$0 \longrightarrow \mathbb{T}_i^{\mathbf{c}} P_i \longrightarrow P_i \longrightarrow L_i \longrightarrow 0, \quad (3.4.4)$$

for  $i = j$ . This property is also satisfied by Arkhipov's twisting functors on blocks of category  $\mathcal{O}$  for a Lie algebra [Ark]. By applying the dual of [KM, Lemma 1], one shows that twisting functors and Arkhipov's twisting functors are identified under the equivalence of Proposition 3.3.4. Here and in what follows, we use the same notation for twisting functors and their underived analogue.

### 3.5 A non-commutative analogue of Smith-Thomas sheaf

In this section, we will deal with the category of  $A$ -modules. In terms of the path algebra of the quiver, the projective modules  $P_i$ , for  $1 \leq i \leq 2n$ , have the following form:

$$\begin{array}{ccccccc}
 P_i = \mathbb{C}\{d_1 \dots d_{i-1}\} \oplus_{\mathbb{C}} & \mathbb{C}\{d_2 \dots d_{i-1}\} \oplus_{\mathbb{C}} & \cdots \oplus_{\mathbb{C}} & & \mathbb{C}\{d_{i-1}\} & \oplus_{\mathbb{C}} & \searrow \\
 & \oplus_{\mathbb{C}} & & & \oplus_{\mathbb{C}} & & \\
 & \mathbb{C}\{c_1 d_1 \dots d_{i-1}\} & & & \mathbb{C}\{c_{i-2} d_{i-2} d_{i-1}\} & & \\
 & & & & \oplus_{\mathbb{C}} & & \\
 & & & & \vdots & & \\
 & & & & \oplus_{\mathbb{C}} & & \\
 & & & & \mathbb{C}\{c_{i-2} \dots c_1 d_1 \dots d_{i-1}\} & & \\
 \searrow \oplus_{\mathbb{C}} & \mathbb{C}\{e_i\} & \oplus_{\mathbb{C}} & \mathbb{C}\{c_i\} & \oplus_{\mathbb{C}} \cdots \oplus_{\mathbb{C}} & \mathbb{C}\{c_{2n-1} \dots c_i\} & \\
 & \oplus_{\mathbb{C}} & & \oplus_{\mathbb{C}} & & \oplus_{\mathbb{C}} & \\
 & \mathbb{C}\{c_{i-1} d_{i-1}\} & & \mathbb{C}\{c_i c_{i-1} d_{i-1}\} & & \mathbb{C}\{c_{2n-1} \dots c_{i-1} d_{i-1}\} & \\
 & \oplus_{\mathbb{C}} & & \oplus_{\mathbb{C}} & & \oplus_{\mathbb{C}} & \\
 & \vdots & & \vdots & & \vdots & \\
 & \oplus_{\mathbb{C}} & & \oplus_{\mathbb{C}} & & \oplus_{\mathbb{C}} & \\
 & \mathbb{C}\{c_{i-1} \dots c_1 d_1 \dots d_{i-1}\} & \mathbb{C}\{c_i \dots c_1 d_1 \dots d_{i-1}\} & & & \mathbb{C}\{c_{2n-1} \dots c_1 d_1 \dots d_{i-1}\} & 
 \end{array}$$

Here we use the symbol  $\oplus_{\mathbb{C}}$  to denote that direct sums hold only as graded vector spaces (elements with the same grading appear in the same column) and not as modules. The projection  $pr_i^e$  onto  $\mathbb{C}\{e_i\} = L_i$  gives the projective cover and it is easy to see that this can be completed to a projective resolution of the simple  $L_i$

$$0 \longrightarrow P_i \xrightarrow{\begin{pmatrix} c_{i-1} \\ -d_i \end{pmatrix}} P_{i-1} \oplus P_{i+1} \xrightarrow{(d_{i-1}, c_i)} P_i \xrightarrow{pr_i^e} L_i \longrightarrow 0, \quad (3.5.1)$$

where we write  $c_j$  and  $d_j$  for  $\cdot c_j$  and  $\cdot d_j$ , the concatenation of paths on the right. Using the short exact sequence (3.4.4), we can now compute twisting functors. Since we have made a precise choice of  $\mathbf{c}$ , we will omit it from the notation and write  $\mathbb{T}_i$  for  $\mathbb{T}_i^{\mathbf{c}}$ .

**Lemma 3.5.1.** *We have the isomorphisms  $\mathbb{T}_i L_i \cong L_i[1]$ ,  $\mathbb{T}_{i-1} L_i \cong \mathbb{C}\{d_{i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{e_i\}$ ,  $\mathbb{T}_{i+1} L_i \cong \mathbb{C}\{e_i\} \oplus_{\mathbb{C}} \mathbb{C}\{c_i\}$  and  $\mathbb{T}_j L_i \cong L_i$ , for  $j \neq i-1, 1, i+1$ .*

*Proof.* If  $j \neq i-1, 1, i+1$ , applying  $\mathbb{T}_j$  to

$$P_i \longrightarrow P_{i-1} \oplus P_{i+1} \longrightarrow P_i \quad (3.5.2)$$

amounts to applying the identity, by the remark at the end of the previous section. Hence taking cohomology produces again  $L_i$  in degree 0.

Suppose now  $j = i$ . By applying  $\mathbb{T}_i$  to (3.5.2) and by (3.4.4), we obtain  $\text{Ker } pr_i^e \rightarrow P_{i-1} \oplus P_{i+1} \rightarrow \text{Ker } pr_i^e$ . Taking cohomology, we are left with  $0 \rightarrow \mathbb{C}\{c_{i-1} - d_i\} \rightarrow 0$ , which is isomorphic to  $L_i[1]$  in  $D(A\text{-mod})$ .

Finally, repeating the same procedure for  $j = i - 1$ , we get

$$\left(0 \rightarrow 0 \rightarrow P_i / (\mathbb{T}_{i-1}(d_{i-1} \oplus c_i)(\text{Ker } pr_{i-1}^e) \oplus P_{i+1}) \right) \cong \mathbb{C}\{d_{i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{e_i\}.$$

Identically, one obtains  $\mathbb{T}_{i+1}L_i \cong \mathbb{C}\{e_i\} \oplus_{\mathbb{C}} \mathbb{C}\{c_i\}$ .  $\square$

To justify next theorem, let us go back to for a moment to rational Cherednik algebras and impose  $k = 0$ . Define

$$L_{0,\mathbf{c}}^{(n)} := \Sigma. (L_1 \otimes L_3 \otimes \dots \otimes L_{2n-1}) = \bigoplus_{\sigma \in \Sigma_n} \sigma (L_1 \otimes L_3 \otimes \dots \otimes L_{2n-1}),$$

an induced  $H_{0,\mathbf{c}}(\Gamma_n)$ -module. This module obviously lies in  $\mathcal{O}(H_{0,\mathbf{c}}(\Gamma_n))$ . By our choice of parameters  $\mathbf{c}$ ,  $L_1, \dots, L_{2n-1}$  are microlocally supported on  $\pi^{-1}(0)$  and  $L_{0,\mathbf{c}}^{(n)}$  is, in fact, finite dimensional.

Write  $\hat{\mathbb{T}}_i L_{0,\mathbf{c}}^{(n)} := \hat{\mathbb{T}}_i L_{0,\mathbf{c}}^{(n)}[-1]$  and denote by  $D_1$  and  $D_n$  the bounded derived category of  $H_{\mathbf{c}}(\mu_{2(n+1)})\text{-mod}$  and  $H_{0,\mathbf{c}}(\Gamma_n)\text{-mod}$ , respectively. We now prove the non-commutative analogue of Theorem 3.2.2. We will need the following lemmata.

**Lemma 3.5.2.** *Let  $E_1, \dots, E_n, F_1, \dots, F_n$  be  $H_{\mathbf{c}}(\mu_{2n})$ -modules. Then, we have the isomorphism*

$$\text{Ext}_{D_n}^*(\Sigma_n. (E_1 \otimes, \dots, \otimes E_n), \Sigma_n. (F_1 \otimes, \dots, \otimes F_n)) \cong \bigoplus_{\sigma \in \Sigma_n} \bigotimes_{i=1}^n \text{Ext}_{D_1}^*(E_i, F_{\sigma(i)}).$$

*Proof.* The statement follows by an easy calculation in the derived category, using Shapiro's lemma and the Kunneth formula.  $\square$

**Lemma 3.5.3.** *The morphisms in the category of  $A$ -modules between indecomposable projectives are given by*

$$\text{Hom}_A(P_i, P_j) = \mathbb{C}^{\min\{i,j\}}$$

*Proof.* It is more convenient to regard an  $A$ -module as a representation of the quiver obtained by assigning a  $\mathbb{C}$ -vector space  $V_h$  to every vertex  $h$ , and a  $\mathbb{C}$ -linear map  $e_{h,k} : V_h \rightarrow V_k$  to every edge from  $h$  to  $k$ . It is easy to see that the projective  $P_i$  is

the representation given by  $V_h = \mathbb{C}^h$  for  $1 \leq h < i$ , and  $V_h = \mathbb{C}^i$  for  $i \leq h \leq n$ , and the following maps

$$e_{h,h+1} = \begin{pmatrix} 0 & \cdots & 0 \\ & & \text{Id}_h \\ & & & 0 \end{pmatrix} \quad \text{and} \quad e_{h+1,h} = \begin{pmatrix} 0 \\ \text{Id}_h \\ \vdots \\ 0 \end{pmatrix}, \quad \text{for } 1 \leq h < i,$$

$$e_{h,h+1} = \text{Id}_i \quad \text{and} \quad e_{h+1,h} = \begin{pmatrix} 0 & \cdots & 0 \\ & & \text{Id}_{i-1} \\ & & & \vdots \\ & & & & 0 \end{pmatrix}, \quad \text{for } i \leq h \leq n.$$

A morphism  $f \in \text{Hom}_A(P_i, P_j)$  is a collection of linear maps  $f_h : V_h^{(i)} \rightarrow V_h^{(j)}$  compatible with  $e_{h,h+1}$  and  $e_{h+1,h}$ . We prove the statement for  $i \leq j$ , the case  $i > j$  follows from a similar calculation. By commutativity of the diagrams

$$\begin{array}{ccc} \mathbb{C}^h & \xrightarrow{e_{h,h+1}} & \mathbb{C}^{h+1} \\ f_h \downarrow & & \downarrow f_{h+1} \\ \mathbb{C}^h & \xrightarrow{e_{h,h+1}} & \mathbb{C}^{h+1} \end{array} \quad \begin{array}{ccc} \mathbb{C}^h & \xrightarrow{e_{h,h+1}} & \mathbb{C}^{h+1} \\ f_h \downarrow & & \downarrow f_{h+1} \\ \mathbb{C}^j & \xrightarrow{e_{h,h+1}} & \mathbb{C}^j \end{array} \quad \begin{array}{ccc} \mathbb{C}^i & \xrightarrow{e_{h,h+1}} & \mathbb{C}^i \\ f_h \downarrow & & \downarrow f_{h+1} \\ \mathbb{C}^j & \xrightarrow{e_{h,h+1}} & \mathbb{C}^j, \end{array}$$

for  $1 \leq h < j$ ,  $j \leq h < i$  and  $i \leq h \leq n$  respectively,  $f_h$  is a minor in  $f_{h+1}$  and hence  $f$  is uniquely determined by  $f_n$ . We also notice that  $f_i = f_n$  and, by commutativity of

$$\begin{array}{ccc} \mathbb{C}^{i-1} & \xleftarrow{e_{i,i-1}} & \mathbb{C}^i \\ f_{i-1} \downarrow & & \downarrow f_i \\ \mathbb{C}^j & \xleftarrow{e_{i,i-1}} & \mathbb{C}^j, \end{array}$$

$f_i$  must be a lower triangular matrix with equal entries in each diagonal. The claim follows. □

In our computations we will consider the natural maps

$$P_{i+1} \xrightarrow{c_i} P_i \quad \text{and} \quad P_i \xrightarrow{d_i} P_{i+1}.$$

In the notation of the previous lemma, these are expressed by the  $2n$ -tuples  $((c_i)_1, \dots, (c_i)_{2n})$  and  $((d_i)_1, \dots, (d_i)_{2n})$ , respectively, where

$$(c_i)_j = \begin{pmatrix} 0 & \cdots & 0 \\ & & \text{Id}_{j-1} \\ & & & \vdots \\ & & & & 0 \end{pmatrix} \quad \text{and} \quad (d_i)_j = \text{Id}_j, \quad \text{for } 1 \leq j \leq i, \quad (3.5.3)$$

$$(c_i)_j = \begin{pmatrix} 0 \\ \text{Id}_i \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad (d_i)_j = \begin{pmatrix} 0 \cdots 0 \\ \text{Id}_i \end{pmatrix}, \quad \text{for } i < j \leq 2n. \quad (3.5.4)$$

Consider the following relations:

- (A)  $\hat{T}_1 L_{0,\mathbf{c}}^{(n)} \cong L_{0,\mathbf{c}}^{(n)}$ ;
- (B)  $\hat{T}_{2i-1} \hat{T}_{2i} L_{0,\mathbf{c}}^{(n)} \cong \hat{T}_{2i-1}^{-1} \hat{T}_{2i}^{-1} L_{0,\mathbf{c}}^{(n)}$ ;
- (C)  $\hat{T}_{2i} \hat{T}_{2i-1} \hat{T}_{2i+1} \hat{T}_{2i} L_{0,\mathbf{c}}^{(n)} \cong L_{0,\mathbf{c}}^{(n)}$ ;
- (D)  $\text{Ext}_{\mathbb{D}_n}^* \left( \hat{T}_\beta L_{0,\mathbf{c}}^{(n)}, L_{0,\mathbf{c}}^{(n)}[n] \right) \cong \text{Ext}_{\mathbb{D}_{n+1}}^* \left( \hat{T}_\beta L_{0,\mathbf{c}}^{(n+1)}, \hat{T}_{2n}^{\pm 1} L_{0,\mathbf{c}}^{(n+1)}[n+1] \right)$ .

**Theorem 3.5.4.** (A), (C) and (D) hold, but (B) does not.

*Proof.* By Lemma 3.5.1, we have  $\mathbb{T}_1 L_1 = L_1[1]$  and  $\mathbb{T}_1 L_{2i+1} = L_{2i+1}$ , for  $i \geq 1$ . (A) follows then by the shift in the definition of  $\hat{T}_i$ .

We now calculate  $\mathbb{T}_{2i} \mathbb{T}_{2i-1} \mathbb{T}_{2i-1} \mathbb{T}_{2i} L_{2j+1}$ . By Lemma 3.5.1, this is isomorphic to  $L_{2j+1}$  if  $j \neq i-1, i$ . We also have  $\mathbb{T}_{2i} \mathbb{T}_{2i-1} \mathbb{T}_{2i-1} \mathbb{T}_{2i} L_{2i-1} \cong \mathbb{T}_{2i} \mathbb{T}_{2i-1} \mathbb{T}_{2i-1} (\mathbb{C}\{e_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i-1}\})$ . A projective resolution of  $\mathbb{C}\{e_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i-1}\}$  is

$$\cdots \longrightarrow P_{2i} \xrightarrow{\begin{pmatrix} c_{2i-1}c_{2i-2} \\ -d_{2i} \end{pmatrix}} P_{2i-2} \oplus P_{2i+1} \xrightarrow{(d_{2i-2}, c_{2i}c_{2i-1})} P_{2i-1} \xrightarrow{pr_{2i-1}^{e,c}} \mathbb{C}\{e_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i-1}\} \longrightarrow 0,$$

where  $pr_{2i-1}^{e,c}$  is the projection onto  $\mathbb{C}\{e_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i-1}\}$ . We do not care about smaller degrees as  $\mathcal{L}^m \mathbb{T}_k = 0$ , for all  $m > 1$  and all  $k$  ([AnSt, Theorem 2.2]). By applying  $\mathbb{T}_{2i-1}$ , we get  $\cdots \rightarrow P_{2i} \rightarrow P_{2i-2} \oplus P_{2i+1} \rightarrow \text{Ker } p_{2i-1}^e$ , whose cohomology gives  $\mathbb{C}\{c_{2i-1}\} \cong L_{2i}$ . Finally,  $\mathbb{T}_{2i} \mathbb{T}_{2i-1} L_{2i} \cong \mathbb{T}_{2i} (\mathbb{C}\{d_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{e_{2i}\})$ , which is isomorphic to  $\mathbb{C}\{d_{2i-1}\} \cong L_{2i-1}$ , by the same argument, using

$$\cdots \longrightarrow P_{2i-1} \xrightarrow{\begin{pmatrix} c_{2i-2} \\ -d_{2i-1}d_{2i} \end{pmatrix}} P_{2i-2} \oplus P_{2i+1} \xrightarrow{(d_{2i-2}d_{2i-1}, c_{2i})} P_{2i} \xrightarrow{pr_{2i}^{d,e}} \mathbb{C}\{d_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{e_{2i}\} \longrightarrow 0 \quad (3.5.5)$$

as a projective resolution.

For  $j = i$ , we have  $\mathbb{T}_{2i-1} \mathbb{T}_{2i} L_{2i+1} \cong \mathbb{T}_{2i-1} (\mathbb{C}\{d_{2i}\} \oplus_{\mathbb{C}} \mathbb{C}\{e_{2i+1}\})$ . Considering the projective resolution (3.5.5) with  $2i$  replaced by  $2i+1$  and applying  $\mathbb{T}_{2i-1}$ , one gets  $\cdots \rightarrow P_{2i} \rightarrow \text{Ker } pr_{2i-1}^e \oplus P_{2i+2} \rightarrow P_{2i+1}$ , whose cohomology gives  $\mathbb{C}\{d_{2i-1}d_{2i}\} \oplus_{\mathbb{C}} \mathbb{C}\{d_{2i}\} \oplus_{\mathbb{C}} \mathbb{C}\{e_{2i+1}\}$ .

We now prove that  $\mathbb{T}_{2i-1}^{-1}\mathbb{T}_{2i}^{-1}L_{2i+1} \cong \mathbb{C}\{e_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i}c_{2i-1}\}$ . After applying  $\mathbb{T}_{2i-1}$  to

$$\cdots \longrightarrow P_{2i+1} \xrightarrow{\begin{pmatrix} c_{2i}c_{2i-1}c_{2i-2} \\ -d_{2i+1} \end{pmatrix}} P_{2i-2} \oplus P_{2i+2} \xrightarrow{(d_{2i-2}, c_{2i+1}c_{2i}c_{2i-1})} P_{2i-1} \longrightarrow 0 ,$$

a projective resolution of the right-hand side, and taking cohomology, we are left with  $\mathbb{C}\{c_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i}c_{2i-1}\}$ . The same way, one calculates  $\mathbb{T}_{2i}(\mathbb{C}\{e_{2i}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i}\}) \cong L_{2i+1}$  and the claim follows. Notice that  $\mathbb{C}\{d_{2i-1}d_{2i}\} \oplus_{\mathbb{C}} \mathbb{C}\{d_{2i}\} \oplus_{\mathbb{C}} \mathbb{C}\{e_{2i+1}\}$  and  $\mathbb{C}\{e_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i}c_{2i-1}\}$  are clearly not isomorphic, because

$$\mathbb{C} \xleftarrow{1} \mathbb{C} \xleftarrow{1} \mathbb{C} \quad \not\cong \quad \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C} ,$$

so relation (B) does not hold.

To prove relation (C), one reasons analogously as before and checks that we have the isomorphisms

$$\begin{aligned} \mathbb{T}_{2i}\mathbb{T}_{2i-1}\mathbb{T}_{2i+1}\mathbb{T}_{2i}L_{2i-1} &\cong \mathbb{T}_{2i}\mathbb{T}_{2i-1}\mathbb{T}_{2i+1}(\mathbb{C}\{e_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i-1}\}) \\ &\cong \mathbb{T}_{2i}\mathbb{T}_{2i-1}(\mathbb{C}\{e_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i}c_{2i-1}\}) \\ &\cong \mathbb{T}_{2i}(\mathbb{C}\{e_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i}c_{2i-1}\}) \\ &\cong (\mathbb{C}\{c_{2i}c_{2i-1}\}) \cong L_{2i+1} \end{aligned}$$

and

$$\begin{aligned} \mathbb{T}_{2i}\mathbb{T}_{2i-1}\mathbb{T}_{2i+1}\mathbb{T}_{2i}L_{2i+1} &\cong \mathbb{T}_{2i}\mathbb{T}_{2i-1}\mathbb{T}_{2i+1}(\mathbb{C}\{d_{2i}\} \oplus_{\mathbb{C}} \mathbb{C}\{e_{2i+1}\}) \\ &\cong \mathbb{T}_{2i}\mathbb{T}_{2i-1}L_{2i} \\ &\cong \mathbb{T}_{2i}(\mathbb{C}\{d_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{e_{2i}\}) \\ &\cong (\mathbb{C}\{d_{2i-1}\}) \cong L_{2i-1} . \end{aligned}$$

Thus,  $\mathbb{T}_{2i}\mathbb{T}_{2i-1}\mathbb{T}_{2i+1}\mathbb{T}_{2i}$  acts as the identity on  $L_{2j+1}$ , for  $i \neq i-1, i$ , and swaps  $L_{2i-1}$  and  $L_{2i+1}$ , which proves the claim.

The proof of relation (D) is slightly more involved and we basically follow [SmTh]. We prove it for  $\hat{T}_{2n}^{-1}$ ; the case  $\hat{T}_{2n}$  is analogue. By Lemma 3.5.2 and the shift in the definition of  $\hat{T}_i$ , the right-hand side is isomorphic to

$$\bigoplus_{\tilde{\sigma} \in \Sigma_{n+1}} \left( \left( \bigotimes_{i=1}^n \text{Ext}_{D_1}^* (\mathbb{T}_{\beta} L_{2i-1}, \mathbb{T}_{2n}^{-1} L_{2\tilde{\sigma}(i)-1}) \right) \otimes \text{Ext}_{D_1}^{*+n} (L_{2n+1}, \mathbb{T}_{2n}^{-1} L_{2\tilde{\sigma}(n+1)-1}) \right) \quad (3.5.6)$$

By Lemma 3.5.1, we see that  $\mathbb{T}_{2n}^{-1}L_{2\tilde{\sigma}(n+1)-1}$  equals  $\mathbb{C}\{e_{2n}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2n}\}$ , if  $\tilde{\sigma}(n+1) = n+1$ ,  $\mathbb{C}\{d_{2n-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{e_{2n}\}$ , if  $\tilde{\sigma}(n+1) = n$ , and  $L_{2j-1}$ , if  $\tilde{\sigma}(n+1) = j \neq n, n+1$ .

Using the resolution (3.5.1) and noticing that  $\dim \operatorname{Hom}_A(P_h, L_k) = \delta_{hk}$ , one sees immediately that  $\operatorname{Ext}_{D_1}^*(L_{2n+1}, L_{2j-1}) = 0$ , if  $j < n$ . Hence, the summands of (3.5.6) are zero if  $\tilde{\sigma} \neq \sigma$  and  $\tilde{\sigma} \neq (n, n+1)\sigma$ , for all  $\sigma \in \Sigma_n$ .

We show that  $\operatorname{Ext}_{D_1}^*(L_{2n+1}, \mathbb{C}\{e_{2n}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2n}\}) \cong \mathbb{C}$ . We need to take the cohomology of the following sequence

$$(3.5.7) \quad \begin{aligned} 0 &\longrightarrow \operatorname{Hom}_A(P_{2n+1}, \mathbb{C}\{e_{2n}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2n}\}) \longrightarrow \operatorname{Hom}_A(P_{2n}, \mathbb{C}\{e_{2n}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2n}\}) \oplus \\ &\operatorname{Hom}_A(P_{2n+2}, \mathbb{C}\{e_{2n}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2n}\}) \longrightarrow \operatorname{Hom}_A(P_{2n+1}, \mathbb{C}\{e_{2n}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2n}\}) \longrightarrow 0. \end{aligned}$$

Let us calculate  $\operatorname{Hom}_A(P_{2n+1}, \mathbb{C}\{e_{2n}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2n}\})$ . An element  $f$  thereof is identified with a couple  $(f_{2n}, f_{2n+1})$  which fits into the diagrams (in the notation of Lemma 3.5.3)

$$\begin{array}{ccc} \mathbb{C}^{2n} & \xrightarrow{e_{2n, 2n+1}} & \mathbb{C}^{2n+1} \\ f_{2n} \downarrow & & \downarrow f_{2n+1} \\ \mathbb{C} & \xrightarrow{1} & \mathbb{C} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{C}^{2n} & \xleftarrow{e_{2n+1, 2n}} & \mathbb{C}^{2n+1} \\ f_{2n} \downarrow & & \downarrow f_{2n+1} \\ \mathbb{C} & \xleftarrow{0} & \mathbb{C} \end{array} .$$

By commutativity, we must have  $f = (0, (a, 0, \dots, 0))$ , with  $a \in \mathbb{C}$ , which proves  $\operatorname{Hom}_A(P_{2n+1}, \mathbb{C}\{e_{2n}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2n}\}) \cong \mathbb{C}$ . Similarly one shows

$$\operatorname{Hom}_A(P_{2n+2}, \mathbb{C}\{e_{2n}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2n}\}) = 0 \quad \text{and}$$

$$\operatorname{Hom}_A(P_{2n}, \mathbb{C}\{e_{2n}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2n}\}) \cong \mathbb{C},$$

where an element  $g$  of the latter is of the form  $g = ((b, 0, \dots, 0), (b, 0, \dots, 0))$ , with  $b \in \mathbb{C}$ . The maps in (3.5.7) are given by

$$\operatorname{Hom}_A(-, \mathbb{C}\{e_{2n}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2n}\})(d_{2n})(f) = f \circ d_{2n} \quad \text{and}$$

$$\operatorname{Hom}_A(-, \mathbb{C}\{e_{2n}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2n}\})(c_{2n})(g) = g \circ c_{2n}$$

(note that the functor  $\operatorname{Hom}_A(-, \mathbb{C}\{e_{2n}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2n}\})$  is contravariant). Using (3.5.3) and (3.5.4), one easily sees that (3.5.7) reduces to  $0 \longrightarrow \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{1} \mathbb{C} \longrightarrow 0$  and by taking cohomology we get the claim.

Analogously one computes

$$\operatorname{Ext}_{D_1}^*(L_{2n+1}, \mathbb{C}\{d_{2n-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{e_{2n}\}) \cong \mathbb{C}[-1].$$

We can therefore write (3.5.6) as the direct sum

$$\bigoplus_{\sigma \in \Sigma_n} \left( \bigotimes_{\substack{i=1 \\ i \neq \sigma^{-1}(n)}}^n \text{Ext}_{D_1}^* (\mathbb{T}_\beta L_{2i-1}, L_{2\sigma(i)-1}) \otimes \right. \\ \left. \otimes \text{Ext}_{D_1}^* (\mathbb{T}_\beta L_{2\sigma^{-1}(n)-1}, \mathbb{T}_{2n}^{-1} L_{2n-1} \oplus \mathbb{T}_{2n}^{-1} L_{2n+1}[-1]) [n] \right). \quad (3.5.8)$$

The exact sequence

$$0 \longrightarrow \mathbb{C}\{d_{2n-1}\} \longrightarrow \mathbb{C}\{d_{2n-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{e_{2n}\} \longrightarrow \mathbb{C}\{e_{2n}\} \longrightarrow 0$$

gives rise to the distinguished triangle  $L_{2n-1} \xrightarrow{\varphi} \mathbb{T}_{2n}^{-1} L_{2n-1} \xrightarrow{\psi} L_{2n}$  in  $D_1$ . We stress that the direct sum of the middle term is just notational and holds only as vector spaces (in fact the exact sequence yields a generator of  $\text{Ext}_A^1(L_{2n-1}, L_{2n})$ ). We therefore have the following sequence in  $D_1$

$$\text{Ext}_{D_1}^* (\mathbb{T}_\beta L_{2\sigma^{-1}(n)-1}, L_{2n}) [-1] \longrightarrow \text{Ext}_{D_1}^* (\mathbb{T}_\beta L_{2\sigma^{-1}(n)-1}, L_{2n-1}) \longrightarrow \\ \longrightarrow \text{Ext}_{D_1}^* (\mathbb{T}_\beta L_{2\sigma^{-1}(n)-1}, \mathbb{T}_{2n}^{-1} L_{2n-1}) \longrightarrow \text{Ext}_{D_1}^* (\mathbb{T}_\beta L_{2\sigma^{-1}(n)-1}, L_{2n}). \quad (3.5.9)$$

$\mathbb{T}_\beta L_{2\sigma^{-1}(n)-1}$  lies in the category generated by  $L_j$ , for  $j \leq 2n-1$ . Since the generator of  $\mathcal{R}\text{Hom}(L_{2n-1}, \mathbb{T}_{2n}^{-1} L_{2n-1}) \cong \mathbb{C}$  is the image of the identity in  $\mathcal{R}\text{Hom}(L_{2n-1}, L_{2n-1})$  under the map  $\text{Ext}_{D_1}^*(L_{2n-1}, -)(\varphi)$  and  $\text{Ext}_{D_1}^*(L_j, L_{2n}) = 0$ , for  $j < 2n-1$ , the last map in (3.5.9) is zero, giving the splitting

$$\text{Ext}_{D_1}^* (\mathbb{T}_\beta L_{2\sigma^{-1}(n)-1}, L_{2n-1}) \cong \\ \text{Ext}_{D_1}^* (\mathbb{T}_\beta L_{2\sigma^{-1}(n)-1}, L_{2n}) [-1] \oplus \text{Ext}_{D_1}^* (\mathbb{T}_\beta L_{2\sigma^{-1}(n)-1}, \mathbb{T}_{2n}^{-1} L_{2n-1}). \quad (3.5.10)$$

For the same reason as before, we have  $\text{Ext}_{D_1}^* (\mathbb{T}_\beta L_{2\sigma^{-1}(n)-1}, L_{2n+1}) = 0$  and from the distinguished triangle  $L_{2n+1} \longrightarrow \mathbb{T}_{2n}^{-1} L_{2n+1} \longrightarrow L_{2n}$ , we obtain the isomorphism

$$\text{Ext}_{D_1}^* (\mathbb{T}_\beta L_{2\sigma^{-1}(n)-1}, \mathbb{T}_{2n}^{-1} L_{2n+1}) \cong \text{Ext}_{D_1}^* (\mathbb{T}_\beta L_{2\sigma^{-1}(n)-1}, L_{2n}). \quad (3.5.11)$$

Observe now that all of the previous computations work in  $D_1 = D(H_{\mathbf{c}}(\mu_{2(n+1)}))\text{-mod}$  as well as in  $D(H_{\mathbf{c}}(\mu_{2n}))\text{-mod}$ , thus relation (D) follows by substituting (3.5.10) and (3.5.11) in (3.5.8) and applying again Lemma 3.5.2.  $\square$

## 3.6 Deformation theory

### 3.6.1 Hochschild cohomology

We now recall deformation theory of representations of algebras and apply it to  $H_{\mathbb{C}}(\mu_{2n})$  and  $H_{0,\mathbb{C}}(\Gamma_n)$ . All preparatory material is taken from [EtMo] and we finally prove a slight generalisation of their main result, applied to our case.

Given an associative algebra  $R$  over  $\mathbb{C}$  and a  $R$ -bimodule  $E$ , one defines the Hochschild complex

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, E) \xrightarrow{d} \mathrm{Hom}_{\mathbb{C}}(R, E) \xrightarrow{d} \cdots \xrightarrow{d} \mathrm{Hom}_{\mathbb{C}}(R^{\otimes \ell}, E) \xrightarrow{d} \cdots$$

with differential given by

$$\begin{aligned} d\varphi(r_1, \dots, r_{\ell+1}) &:= r_1\varphi(r_2, \dots, r_{\ell+1}) + \\ &+ \sum_{i=1}^{\ell} (-1)^i \varphi(r_i, \dots, r_{i-1}, r_i r_{i+1}, r_{i+2}, \dots, r_{\ell+1}) + \\ &+ (-1)^{\ell+1} \varphi(r_1 \cdots, r_n) r_{\ell+1}, \end{aligned}$$

and shows that the cohomology of this complex  $H^*(R, E)$  equals  $\mathrm{Ext}_{R \otimes R^{op}}^*(R, E)$ .

Let  $U$  be a  $\ell$ -dimensional vector space and  $t_1, \dots, t_{\ell}$  be a basis of its dual. Let  $R_U$  be a  $\mathbb{C}[[U]] = \mathbb{C}[[t_1, \dots, t_{\ell}]]$ -algebra isomorphic to  $R[[U]]$  as  $\mathbb{C}[[U]]$ -module and such that  $R_U/\mathfrak{M} \cong R$ , where  $\mathfrak{M}$  is the maximal ideal in  $\mathbb{C}[[U]]$ . This is a formal flat deformation of  $R$  over the formal neighbourhood of the origin in  $U$ , and we have a map  $\phi : U \rightarrow H^2(R, R)$ , given by the assignment

$$(t_1, \dots, t_{\ell}) \longmapsto \sum_{j=1}^{\ell} t_j [c_{0, \dots, 1_j, \dots, 0}], \quad (3.6.1)$$

where  $c_{0, \dots, 1_j, \dots, 0}$  is the 2-cocycle sending  $r_1 \otimes r_2 \in R \otimes R$  to the coefficient of  $t_j$  in the expression defining the multiplication of  $r_1$  and  $r_2$  inside  $R[[t_1, \dots, t_{\ell}]]$ . For a  $R$ -module  $M$ , composing the latter map with  $H^2(R, R) \rightarrow H^2(R, \mathrm{End} M)$ , we obtain a map  $\eta : U \rightarrow H^2(R, \mathrm{End} M)$  which ‘‘controls’’ the deformations of the module  $M$  to a  $R_U$ -module. More precisely, we have the following ([EtMo, Proposition 4.1]).

**Proposition 3.6.1.** *Suppose that  $H^1(R, \mathrm{End} M) = 0$  and  $\eta$  is surjective. Then  $M$  deforms uniquely to a representation  $M_S$  of the algebra  $R_S \hat{\otimes}_{\mathbb{C}[[U]]} \mathbb{C}[[S]]$ , over a formal subscheme  $S$  of the formal neighbourhood of the origin in  $U$ , whose tangent space equals the kernel of  $\eta$ .*

### 3.6.2 Deformation of $H_{0,\mathbf{c}}(\Gamma_n)$ -modules

We need to calculate Hochschild cohomology for the cyclic rational Cherednik algebra and its simple finite dimensional modules. This is done in [EtMo], Corollary 4.6 and Proposition 4.7, for a general deformed preprojective algebra, using the fact that such an algebra belongs to the class  $VB(2)$  ([EtMo, Proposition 4.5]) and hence  $H_i(H_{\mathbf{c}}(\Gamma), E) \cong H^{2-i}(H_{\mathbf{c}}(\Gamma), E)$ , for any bimodule  $E$ , where  $H_i$  denotes Hochschild homology ([VB1, VB2]). Our proof for  $H_{\mathbf{c}}(\mu_{2n})$  is a direct computation and does not rely on this property.

**Lemma 3.6.2.** *For  $1 \leq i \leq 2n-1$ ,  $H^1(H_{\mathbf{c}}(\mu_{2n}), \text{End } L_i) = 0$ ,  $H^0(H_{\mathbf{c}}(\mu_{2n}), \text{End } L_i) \cong H^2(H_{\mathbf{c}}(\mu_{2n}), \text{End } L_i) \cong \mathbb{C}$  and  $H^j(H_{\mathbf{c}}(\mu_{2n}), \text{End } L_i) = 0$ , for  $j > 2$ .*

*Proof.* We first observe that, for an algebra  $R$  and modules  $M, N$ , we have

$$H^i(R, \text{Hom}(M, N)) \cong \text{Ext}_R^i(M, N).$$

This is shown by resolving  $R$  with the bar complex and using the tensor-hom adjunction. Reasoning as in the proof of relation (D) in Theorem 3.5.4, one shows that

$$0 \longrightarrow \text{Hom}_A(P_i, L_i) \longrightarrow \text{Hom}_A(P_{i-1}, L_i) \oplus \text{Hom}_A(P_{i+1}, L_i) \longrightarrow \text{Hom}_A(P_i, L_i) \longrightarrow 0$$

is isomorphic to  $0 \rightarrow \mathbb{C} \rightarrow 0 \rightarrow \mathbb{C} \rightarrow 0$ , and the claim follows.  $\square$

We now consider the cyclotomic rational Cherednik algebra  $H_{0,\mathbf{c}}(\Gamma_n)$  and the finite dimensional module  $L_{0,\mathbf{c}}^{(n)}$ , defined in the previous section.

**Proposition 3.6.3.** *We have the isomorphisms  $H^2(H_{0,\mathbf{c}}(\Gamma_n), \text{End } L_{0,\mathbf{c}}^{(n)}) \cong \mathbb{C}^n$  and  $H^1(H_{0,\mathbf{c}}(\Gamma_n), \text{End } L_{0,\mathbf{c}}^{(n)}) = 0$ .*

*Proof.* By definition,

$$\begin{aligned} H^2(H_{0,\mathbf{c}}(\Gamma_n), \text{End } L_{0,\mathbf{c}}^{(n)}) &\cong \text{Ext}_{H_{0,\mathbf{c}}(\Gamma_n) \otimes H_{0,\mathbf{c}}(\Gamma_n)^{op}}^2(H_{0,\mathbf{c}}(\Gamma_n), \text{End } L_{0,\mathbf{c}}^{(n)}) \cong \\ &\cong \text{Ext}_{\Sigma_n \# H_{\mathbf{c}}^{\otimes n} \otimes \Sigma_n \# H_{\mathbf{c}}^{op \otimes n}}^2\left(\Sigma_n \# H_{\mathbf{c}}^{\otimes n}, \text{End}\left(\bigoplus_{\sigma \in \Sigma_n} \sigma(L_1 \otimes \dots \otimes L_{2n-1})\right)\right), \end{aligned}$$

where we write  $H_{\mathbf{c}}$  for  $H_{\mathbf{c}}(\mu_{2n})$ . Identifying  $\Sigma_n \# (H_{\mathbf{c}}^{\otimes n} \otimes H_{\mathbf{c}}^{op \otimes n})$  with the  $\Sigma_n \times \Sigma_n \# (H_{\mathbf{c}}^{\otimes n} \otimes H_{\mathbf{c}}^{op \otimes n})$ -subalgebra  $\Delta \# (H_{\mathbf{c}}^{\otimes n} \otimes H_{\mathbf{c}}^{op \otimes n})$ , where  $\Delta$  denotes the diagonal in  $\Sigma_n \times \Sigma_n$ , we realise  $\Sigma_n \# H_{\mathbf{c}}^{\otimes n}$  as an induced  $\Sigma_n \times \Sigma_n \# (H_{\mathbf{c}}^{\otimes n} \otimes H_{\mathbf{c}}^{op \otimes n})$ -module over

such a subalgebra. By Shapiro's lemma and using the fact that taking  $\Sigma_n$ -invariants is an exact functor, the right-hand side equals

$$\left( \bigoplus_{\sigma \in \Sigma_n} \text{Ext}_{H_{\mathbf{c}}^{\otimes n} \otimes H_{\mathbf{c}}^{op \otimes n}}^2 (H_{\mathbf{c}}^{\otimes n}, \text{End}(\sigma(L_1 \otimes \dots \otimes L_{2n-1}))) \oplus \bigoplus_{\substack{\sigma, \tau \in \Sigma_n \\ \sigma \neq \tau}} \text{Ext}_{H_{\mathbf{c}}^{\otimes n} \otimes H_{\mathbf{c}}^{op \otimes n}}^2 (H_{\mathbf{c}}^{\otimes n}, \text{Hom}(\sigma(L_1 \otimes \dots \otimes L_{2n-1}), \tau(L_1 \otimes \dots \otimes L_{2n-1}))) \right)^{\Sigma_n}. \quad (3.6.2)$$

By the Kunneth formula, we have

$$\begin{aligned} \text{Ext}_{H_{\mathbf{c}}^{\otimes n} \otimes H_{\mathbf{c}}^{op \otimes n}}^2 (H_{\mathbf{c}}^{\otimes n}, \text{End}(\sigma(L_1 \otimes \dots \otimes L_{2n-1}))) &\cong \\ &\cong \bigoplus_{i+j=2} \left( \text{Ext}_{H_{\mathbf{c}} \otimes H_{\mathbf{c}}^{op}}^i (H_{\mathbf{c}}, \text{End} L_{2\sigma^{-1}(i)-1}) \otimes \right. \\ &\left. \otimes \text{Ext}_{H_{\mathbf{c}}^{\otimes n-1} \otimes H_{\mathbf{c}}^{op \otimes n-1}}^j (H_{\mathbf{c}}^{\otimes n-1}, \text{End}(L_{2\sigma^{-1}(2)-1} \otimes \dots \otimes L_{2\sigma^{-1}(n)-1})) \right). \end{aligned} \quad (3.6.3)$$

Iterating and using  $\text{Ext}_{H_{\mathbf{c}} \otimes H_{\mathbf{c}}^{op}}^0 (H_{\mathbf{c}}, \text{End} L_i) \cong \mathbb{C}$  and  $\text{Ext}_{H_{\mathbf{c}} \otimes H_{\mathbf{c}}^{op}}^1 (H_{\mathbf{c}}, \text{End} L_i) = 0$  shows that the left-hand side of (3.6.3) is isomorphic to

$$\bigoplus_{i=1}^n \text{Ext}_{H_{\mathbf{c}} \otimes H_{\mathbf{c}}^{op}}^2 (H_{\mathbf{c}}, \text{End} L_{2\sigma^{-1}(i)-1}).$$

By the same argument, the right summand of (3.6.2) vanishes, because

$$\text{Ext}_{H_{\mathbf{c}} \otimes H_{\mathbf{c}}^{op}}^i (H_{\mathbf{c}}, \text{Hom}(L_{2\sigma^{-1}(j)-1}, L_{2\tau^{-1}(j)-1})) \cong \text{Ext}^i(L_{2\sigma^{-1}(j)-1}, L_{2\tau^{-1}(j)-1}) = 0,$$

if  $i = 0, 1$  and  $\sigma \neq \tau$ . Therefore, we have

$$H^2(H_{0, \mathbf{c}}(\Gamma_n), \text{End} L_{0, \mathbf{c}}^{(n)}) \cong \left( \bigoplus_{\sigma \in \Sigma_n} \left( \bigoplus_{i=1}^n H^2(H_{\mathbf{c}}, \text{End} L_{2\sigma^{-1}(i)-1}) \right) \right)^{\Sigma_n} \quad (3.6.4)$$

which is isomorphic to  $\mathbb{C}^n$  by Lemma 3.6.2.

The second identity follows similarly by observing that at least one factor in every summand of the Kunneth formula vanishes.  $\square$

**Corollary 3.6.4.** *Let  $U = \mathfrak{z}^* \cong \mathbb{C}^{2n}$  be the vector space of deformation parameters of the cyclotomic rational Cherednik algebra  $H_{k, \mathbf{c}}(\Gamma_n)$ . The map  $\eta : U \rightarrow H^2(H_{0, \mathbf{c}}(\Gamma_n), \text{End} L_{0, \mathbf{c}}^{(n)})$  is surjective.*

*Proof.* Let  $U_0 := \{(k, \mathbf{c}) \in U \mid k = 0\}$ , a subspace of  $U$ . By the proof of Proposition 3.6.3,  $\eta_0$ , the restriction of  $\eta$  to  $U_0$ , is given by the following composition

$$U_0 \xrightarrow{\phi} H^2(H_{\mathbf{c}}, H_{\mathbf{c}}) \xrightarrow{\xi} H^2(H_{0,\mathbf{c}}(\Gamma_n), H_{0,\mathbf{c}}(\Gamma_n)) \xrightarrow{\psi} H^2\left(H_{0,\mathbf{c}}(\Gamma_n), \text{End } L_{0,\mathbf{c}}^{(n)}\right),$$

where  $\phi$  and  $\psi$  are as before and  $\xi$  is induced by the Kunneth formula. (3.6.4) also tells us that  $\psi \circ \xi$  coincides with the map

$$\psi_0 : H^2(H_{\mathbf{c}}, H_{\mathbf{c}}) \longrightarrow \bigoplus_{i=1}^n H^2(H_{\mathbf{c}}, \text{End } L_{2i-1})$$

coming from the representations  $H_{\mathbf{c}} \rightarrow \text{End } L_{2i-1}$ , so it is enough to show that

$$\dim \{u \in U_0 \mid \phi(u) \notin \text{Ker } \psi_0\} \geq n. \quad (3.6.5)$$

By Proposition 3.6.1 and [EtMo, Corollary 4.9], each module  $L_{2i-1}$  admits a deformation over a subscheme of codimension 1 in  $U_0$ , given by the equation

$$\text{tr}_{L_{2i-1}} \left( 1 - 2 \sum_{j=1}^{2n-1} c_j s^j \right) = 0,$$

since  $L_{2i-1}$  is finite dimensional, for  $1 \leq i \leq n$ . The characters of  $L_{2i-1}$  are obviously linearly independent and the kernel of  $\eta_0$  lies in the tangent space of the intersection of these  $n$  spaces defined by linearly independent equations. This proves (3.6.5), which shows that  $\eta_0$ , and hence  $\eta$ , is surjective.  $\square$

We now write  $\mathbf{c}' = (c'_1, \dots, c'_{2n-1})$  for an unspecified deformation parameter of the cyclic rational Cherednik algebra, while  $\mathbf{c}$  still denotes the specific choice made in Theorem 3.3.3. Let  $\mathcal{H}_i(\mathbf{c}')$  be the hyperplane in  $\mathfrak{z}^*$  defined by the equation

$$\dim L_{2i-1} - 2 \sum_{m=1}^{2n-1} c'_m \zeta^{-m(2i-1)} \frac{1 - \zeta^{-m \dim L_{2i-1}}}{1 - \zeta^{-m}} = 0, \quad (3.6.6)$$

for  $1 \leq i \leq n$ . Note that these equations are linearly independent and the intersection

$$\mathcal{H}(\mathbf{c}') := \bigcap_{i=1}^n \mathcal{H}_i(\mathbf{c}')$$

is a linear subspace of  $\mathfrak{z}^*$  of dimension  $n$ .

**Theorem 3.6.5.** *The representation  $L_{0,\mathbf{c}}^{(n)}$  deforms uniquely to a formal representation of  $H_{k,\mathbf{c}'}(\Gamma_n)$ , for  $(k, \mathbf{c}') \in \mathcal{H}(\mathbf{c}')$ . Moreover, there exists a Zariski open subset  $\mathcal{U}$  of  $\mathcal{H}(\mathbf{c}')$  such that  $H_{k,\mathbf{c}'}(\Gamma_n)$  admits an irreducible finite dimensional representation isomorphic to  $L_{0,\mathbf{c}}^{(n)}$  as a  $\Gamma_n$ -module.*

*Proof.* By linearly independence of the characters of  $L_{2j-1}$ , we can find elements  $\gamma_i \in \mathbb{C}[\mu_{2n}]$  such that  $\mathrm{tr}_{L_{2j-1}}(\gamma_i) = 1 - \delta_{ij}$ , for  $1 \leq i, j \leq n$ . Let

$$z_i := (\mathbf{1}_{\Sigma_n}, (\gamma_i, \dots, \gamma_i, \overbrace{\mathbf{1}}^{i\text{-th}}, \gamma_i, \dots, \gamma_i)),$$

an element of  $H_{k, \mathbf{c}'}(\Gamma_n)$  that commutes with  $x_i$  and  $y_i$ . Any finite dimensional representation  $L$  of  $H_{k, \mathbf{c}'}(\Gamma_n)$  must satisfy

$$\mathrm{tr}_L z_i - k \sum_{j \neq i} \sum_{m=0}^{2n-1} \mathrm{tr}_L s_{ij} s_i^m s_j^{-m} z_i - 2 \sum_{m=1}^{2n-1} c'_m \mathrm{tr}_L s_i^m z_i = 0, \quad (3.6.7)$$

obtained by multiplying the defining relation of the commutator  $[y_i, x_i]$  with  $z_i$  and observing that  $[y_i, x_i]z_i = [y_i, x_i z_i]$ . Since we want  $L$  to be isomorphic to  $L_{0, \mathbf{c}}^{(n)}$  as a  $\Gamma_n$ -module, we have

$$\mathrm{tr}_L z_i = \sum_{\substack{\sigma \in \Sigma_n \\ \sigma(i)=i}} \dim L_{2i-1},$$

$$\mathrm{tr}_L s_{ij} s_i^m s_j^{-m} z_i = \mathrm{tr}_L s_{ij} z_i = 0 \quad \text{and}$$

$$\mathrm{tr}_L s_i^m z_i = \mathrm{tr}_L (\gamma_i \dots, \gamma_i, s_i^m, \gamma_i, \dots, \gamma_i) = (n-1)! \mathrm{tr}_{L_{2i-1}}(s^m).$$

Substituting such equalities in (3.6.7) and by Theorem 3.3.1, we obtain the equations defining the hyperplanes  $\mathcal{H}_i(\mathbf{c}')$ . By Proposition 3.6.3 and Corollary 3.6.4, we can apply Proposition 3.6.1 and we obtain a formal deformation along a subscheme  $S \subset \mathfrak{z}^*$ , whose tangent space has dimension equal to  $\dim \mathrm{Ker} \eta = n$ . From the above, we have  $(0, \mathbf{c}) + S \subset \mathcal{H}(\mathbf{c}')$ , which proves that  $S$  is the formal neighbourhood of zero in  $\mathcal{H}(\mathbf{c}') - (0, \mathbf{c})$  because they have the same dimension.

The second claim follows as in [EtMo].  $\square$

**Remark 3.6.6.** Note that, with our choice of  $\mathbf{c}$ ,  $(k, \mathbf{c}) \in \mathcal{H}(\mathbf{c})$ , for all  $k \in \mathbb{C}$ . We denote by  $L_{k, \mathbf{c}}^{(n)}$  the representation of  $H_{k, \mathbf{c}}(\Gamma_n)$  obtained as deformation of  $L_{0, \mathbf{c}}^{(n)}$ . Being finite dimensional,  $L_{k, \mathbf{c}}^{(n)}$  belongs to  $\mathcal{O}(H_{k, \mathbf{c}}(\Gamma_n))$  and thus it has to coincide with a simple quotient by a maximal submodule of a standard module induced from an irreducible representation of  $\Gamma_n$ . Recall all such representations from section 1.4.2; since we have  $L_{k, \mathbf{c}}^{(n)} \cong L_{0, \mathbf{c}}^{(n)}$  as  $\Gamma_n$ -modules, we obtain that  $L_{k, \mathbf{c}}^{(n)}$  coincides with the simple corresponding to the  $2n$ -multipartition of  $n$   $((1), \emptyset, (1), \dots, (1), \emptyset)$ .

# Chapter 4

## A conjecture

We expect relation (B) of Theorem 3.5.4 to hold after deforming  $L_{0,\mathbf{c}}^{(n)}$  to  $L_{k,\mathbf{c}}^{(n)}$ , for values of  $k$  in a neighbourhood of zero. Evidence for this comes from the commutative setting, where Smith and Thomas prove that relations (A) to (D) all hold after deforming  $\mathcal{L}^{(n)}$  along a certain direction (cf. [SmTh, Theorem 5.1]). This is the only deformation direction of the Hilbert scheme  $\widetilde{\text{Hilb}^n(\mathbb{C}^2/\mu_{2n})}$  not coming from the surface  $\widetilde{\mathbb{C}^2/\mu_{2n}}$  and its non-commutative analogue is expressed by the parameter  $k$ . Hence, it is reasonable to formulate the following conjecture.

Let  $(k, \mathbf{c}) \in \mathfrak{z}^*$ , such that  $k \in Z \cap \mathbb{C} \setminus \mathbb{Q}$ , where  $Z$  is a neighbourhood of the origin, and  $\mathbf{c} \in \mathbb{C}^{2n-1}$  as described after Theorem 3.3.3. Notice that, with this choice,  $(k, \mathbf{c})$  is a spherical value for  $H_{k,\mathbf{c}}(\Gamma_n)$  (Lemma 3.3.2 and Theorem 1.4.5) which belongs, in addition, to the hyperplane  $\mathcal{H}(\mathbf{c})$  of Theorem 3.6.5. As before, we write  $D_n := D(H_{k,\mathbf{c}}(\Gamma_n)\text{-mod})$  and set  $T_i^{k,\mathbf{c}} := \mathbb{T}_i^{k,\mathbf{c}}[-1]$ .

**Conjecture 4.0.7.** The graded vector space

$$H^*(\beta) := \text{Ext}_{D_n}^* \left( T_\beta^{k,\mathbf{c}} L_{k,\mathbf{c}}^{(n)}, L_{k,\mathbf{c}}^{(n)}[n] \right)$$

is an invariant of the isotopy class of the link given by the plat closure of the braid  $\beta$ .

We outline a possible strategy to prove this conjecture, based on Theorems 3.5.4 and 3.6.5.

The main issue in proving that  $ST^*$  gives a link invariant, the commutative analogue of Conjecture 4.0.7, is to show that deforming the spherical twists still produces a braid group action on the bounded derived category of coherent sheaves on the deformation of  $\mathcal{L}^{(n)}$ . In our case, this is assured by Theorem 3.4.1, for any value of the deformation parameter.

As explained at the end of section 3.4.2, there is a natural transformation  $\mathbb{T}_i^{\mathbf{c}} \rightarrow \text{Id}$ , which, in light of Proposition 3.4.3, yields a natural transformation  $\mathbb{T}_\beta^{0,\mathbf{c}} \rightarrow \text{Id}$ . Thus,

it is possible to define a non-zero morphism between the two terms of all relations of Theorem 3.5.4, by applying such transformation to  $L_{0,\mathbf{c}}^{(n)}$ . In relations (A), (C) and (D), this map is an isomorphism for  $k = 0$  (by Theorem 3.5.4 and simplicity of  $L_{0,\mathbf{c}}^{(n)}$ ) and remains so for values of  $k$  in a neighbourhood of the origin, since being an isomorphism is an open condition. We believe that this should also give an isomorphism in relation (B), for  $k \neq 0$ , mirroring what happens in the commutative case ([SmTh, Theorem 5.1]).

Indeed, in light of the calculations of Theorem 3.5.4, such a map coincides with the obvious morphism

$$\Phi_i : \mathbb{C}\{e_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i-1}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i}c_{2i-1}\} \longrightarrow \mathbb{C}\{d_{2i-1}d_{2i}\} \oplus_{\mathbb{C}} \mathbb{C}\{d_{2i}\} \oplus_{\mathbb{C}} \mathbb{C}\{e_{2i+1}\}$$

factoring through  $L_{2i-1}$ . Consider its kernel and cokernel, given by  $\mathbb{C}\{e_{2i}\} \oplus_{\mathbb{C}} \mathbb{C}\{c_{2i}\}$  and  $\mathbb{C}\{d_{2i}\} \oplus_{\mathbb{C}} \mathbb{C}\{e_{2i+1}\}$ , respectively, and let

$$L_{0,\mathbf{c}}^{\text{Ker}} := \Sigma_n \cdot (L_1 \otimes \dots \otimes L_{2i-1} \otimes \text{Ker } \Phi_i \otimes L_{2i+3} \otimes \dots \otimes L_{2n-1}) \quad \text{and}$$

$$L_{0,\mathbf{c}}^{\text{Coker}} := \Sigma_n \cdot (L_1 \otimes \dots \otimes L_{2i-1} \otimes \text{Coker } \Phi_i \otimes L_{2i+3} \otimes \dots \otimes L_{2n-1}) .$$

**Proposition 4.0.8.** *Let  $U = \mathfrak{z}^* \cong \mathbb{C}^{2n}$  be the vector space of deformation parameters of the cyclotomic rational Cherednik algebra  $H_{k,\mathbf{c}}(\Gamma_n)$ . The deformation maps  $\eta^{\text{Ker}} : U \longrightarrow H^2(H_{0,\mathbf{c}}(\Gamma_n), \text{End } L_{0,\mathbf{c}}^{\text{Ker}})$  and  $\eta^{\text{Coker}} : U \longrightarrow H^2(H_{0,\mathbf{c}}(\Gamma_n), \text{End } L_{0,\mathbf{c}}^{\text{Coker}})$  are not surjective.*

*Proof.* We prove the statement for  $L_{0,\mathbf{c}}^{\text{Ker}}$ ; the case  $L_{0,\mathbf{c}}^{\text{Coker}}$  is identical.

Let us first calculate  $H^2(H_{0,\mathbf{c}}(\Gamma_n), \text{End } L_{0,\mathbf{c}}^{\text{Ker}})$ . Some standard computations give

$$\text{Ext}_A^*(\text{Ker } \Phi_i, \text{Ker } \Phi_i) \cong \mathbb{C}[0] \oplus \mathbb{C}[-2],$$

$$\text{Ext}_A^*(L_{2i-1}, \text{Ker } \Phi_i) \cong \mathbb{C}[-1] \cong \text{Ext}_A^*(\text{Ker } \Phi_i, L_{2i-1}) \quad \text{and}$$

$$\text{Ext}_A^*(L_{2j-1}, \text{Ker } \Phi_i) = 0 = \text{Ext}_A^*(\text{Ker } \Phi_i, L_{2j-1}), \quad \text{for } j \neq i, i+1.$$

Thus, following the proof of Proposition 3.6.3, we obtain

$$H^2(H_{0,\mathbf{c}}, \text{End } L_{0,\mathbf{c}}^{\text{Ker}}) \cong \left( \bigoplus_{\sigma \in \Sigma_n} \left( \bigoplus_{\substack{i=1 \\ i \neq \sigma(i+1)}}^n H^2(H_{\mathbf{c}}(\mu_{2n}), \text{End } L_{2\sigma^{-1}(i)-1}) \oplus \right. \right. \\ \left. \left. \oplus H^2(H_{\mathbf{c}}(\mu_{2n}), \text{End } \text{Ker } \Phi_i) \oplus (H^1(L_{2i-1}, \text{Ker } \Phi_i) \otimes H^1(\text{Ker } \Phi_i, L_{2i-1})) \right) \right)^{\Sigma_n} \cong \mathbb{C}^{n+1}.$$

Now, as in Corollary 3.6.4,  $\dim \text{Ker } \eta^{\text{Ker}} = n$ , because the characters of  $L_1, \dots, L_{2i-1}, \text{Ker } \Phi_i, L_{2i+3}, \dots, L_{2n-1}$  are linearly independent, hence  $\eta^{\text{Ker}}$  fails to be surjective.  $\square$

In the proof of Proposition 3.6.1, given an algebra  $R$ , the existence of a deformation of a  $R$ -module  $M$  relies on the surjectivity of the map  $\eta : U \rightarrow H^2(R, \text{End } M)$ . Indeed, such a deformation is defined by a formal series

$$\tilde{\rho}(r) = \sum_{p_1, \dots, p_n} \rho_{p_1, \dots, p_n}(r) h_1^{p_1} \cdot \dots \cdot h_n^{p_n},$$

where  $\rho_{0, \dots, 0} = \rho : R \rightarrow \text{End } M$  is the representation map and  $h_1, \dots, h_n$  is a coordinate system of the formal neighbourhood of zero  $D$  in the tangent space at the origin of  $\text{Ker } \eta$ . Recall the notation of section 3.6.1 and consider the map  $D \rightarrow U$ ,  $(h_1, \dots, h_n) \mapsto (t_1, \dots, t_\ell)$ , where  $t_i = h_i$ , for  $1 \leq i \leq n$ , and

$$t_i = \sum_{p_1, \dots, p_n} t_{i, p_1, \dots, p_n} h^{p_1} \cdot \dots \cdot h^{p_n}, \quad \text{for } n < i \leq \ell,$$

with  $t_{i, p_1, \dots, p_n} \in \mathbb{C}$  (we have supposed, without loss of generality, that the tangent space at zero of  $\text{Ker } \eta$  is defined by the equations  $t_{n+1} = \dots = t_\ell = 0$ ). Imposing the algebra homomorphism condition on  $\tilde{\rho}$ , we obtain

$$d\rho_{p_1, \dots, p_n} = \sum_{j=1}^{\ell} t_{j, p_1, \dots, p_n} \rho(c_{0, \dots, 1_j, \dots, 0}) + C_{p_1, \dots, p_n}, \quad (4.0.1)$$

where  $d$  is the differential in the Hochschild cohomology,  $c_{0, \dots, 1_j, \dots, 0}$  is as in (3.6.1) and  $t_{j, p_1, \dots, p_n}$  equals 1, if  $(p_1, \dots, p_n) = (0, \dots, 1_j, \dots, 0)$ , and zero otherwise. Here,  $C_{p_1, \dots, p_n}$  is a 2-cocycle defined as a certain combination of elements of the form  $\rho_{q_1, \dots, q_n}(c_{u_1, \dots, u_\ell})$ , with  $q_1 + \dots + q_n < p_1 + \dots + p_n$ , and where  $c_{u_1, \dots, u_\ell}$  is the obvious analogue of  $c_{0, \dots, 1_j, \dots, 0}$ , for  $(u_1, \dots, u_\ell) \in \mathbb{N}^\ell$ . Recall that  $\eta((t_1, \dots, t_\ell)) = \sum_{j=1}^n t_j \rho(c_{0, \dots, 1_j, \dots, 0}) + d\rho'$ , for some 1-cocycle  $\rho'$ .

In our case,  $\eta = \eta^{\text{Ker}}$  (resp.  $\eta^{\text{Coker}}$ ) is not surjective and hence it makes sense to formulate the following conjecture.

**Conjecture 4.0.9.** Let  $R = H_{0, \mathbf{c}}(\Gamma_n)$  and  $M = L_{0, \mathbf{n}}^{\text{Ker}}$  (resp.  $L_{0, \mathbf{n}}^{\text{Coker}}$ ). There exist some indices  $p_1, \dots, p_n \in \mathbb{N}^n$ , such that the classes of the 2-cocycles  $\{[C_{q_1, \dots, q_n}]\}_{q_1, \dots, q_n}$ , with  $q_1 + \dots + q_n < p_1 + \dots + p_n$ , span the vector space  $H^2(H_{0, \mathbf{c}}(\Gamma_n), \text{End } L_{0, \mathbf{c}}^{\text{Ker}})$  (resp.  $H^2(H_{0, \mathbf{c}}(\Gamma_n), \text{End } L_{0, \mathbf{c}}^{\text{Coker}})$ ).

**Proposition 4.0.10.** *Suppose Conjecture 4.0.9 is true. Then, Conjecture 4.0.7 is true.*

*Proof.* If the 2-cocycles  $\{[C_{q_1, \dots, q_n}]\}_{q_1, \dots, q_n}$  span the codomain of  $\eta^{\text{Ker}}$  (resp.  $\eta^{\text{Coker}}$ ), by Proposition 4.0.8, there exist some indices  $q'_1, \dots, q'_n$ , such that  $C_{q'_1, \dots, q'_n}$  is not in

the image of  $\eta^{\text{Ker}}$  (resp.  $\eta^{\text{Coker}}$ ). For such a 2-cocycle, there do not exist coefficients  $t_{j,q'_1,\dots,q'_n}$ , such that the equation (4.0.1) is satisfied. Thus,  $L_{0,\mathbf{c}}^{\text{Ker}}$  (resp.  $L_{0,\mathbf{c}}^{\text{Coker}}$ ) does not deform to a  $H_{k,\mathbf{c}}(\Gamma_n)$ -module, for  $k \neq 0$ . This turns  $\Phi_i$  into an isomorphism and makes relation (B) hold.  $\square$

**Remark 4.0.11.** Alternatively, one could try to prove Conjecture 4.0.7 directly showing relations (A) to (D), using the fact that  $L_{k,\mathbf{c}}^{(n)}$  is a specific finite dimensional simple module of  $H_{k,\mathbf{c}}(\Gamma_n)$  (see Remark 3.6.6). Unlike in the case  $n = 1$ , we do not know how to study the action of twisting functors on simple modules for the cyclotomic rational Cherednik algebra.

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