Contagious McKean–Vlasov Systems with Resets

Integrate-and-Fire Neuron Models with Common Noise



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A thesis submitted for the degree of

Doctor of Philosophy

Trinity 2024

To my mother, Judith.

Acknowledgements

I would like to express my sincere gratitude to the many individuals who have supported me throughout my doctoral journey, both within and outside of academia.

First and foremost, I extend my heartfelt thanks to my supervisors, Prof. Ben Hambly and Prof. Christoph Reisinger, for their invaluable guidance and mentorship. Their expertise has been instrumental in shaping the direction of this work and in developing my confidence as an independent researcher. I owe a deep debt of gratitude to Dr. Andreas Søjmark for the numerous fruitful and insightful discussions we have shared over the course of this project. His enthusiasm and passion for the subject have been a constant source of inspiration. My sincere appreciation goes to Philipp Jettkant for our innumerable discussions on McKean–Vlasov equations and various related models, which have greatly enriched my understanding of the field.

Outside of academia, I have been fortunate to have the support of many individuals who have played crucial roles in my journey. I am profoundly grateful to my mother, Judith, whose unwavering belief in me has been a constant source of strength and motivation. Her encouragement and support have been invaluable throughout this process. I would also like to express my appreciation to my friends who have been a source of joy and laughter throughout the course of this project. I am particularly indebted to Alice, who offered immeasurable support during the most challenging six months preceding my submission. My gratitude extends to Salah for our ever-joyful Thursday library sessions, to Fabrice who came to my rescue more than once with his expertise on the M_1-topology, to Maja for our endless mental health runs, to Sahra for our Saturday library sessions and explorations of Oxford's restaurants, and to Jayoti who kindly drove me to Oxford and helped me move in. A special mention goes to Luna, who, despite our relatively short acquaintance, became one of my closest

friends in Oxford. I am also grateful for the wonderful friends I made in the CDT of Mathematics of Random Systems, including Filippo, László, Andrea, Felix and Luca.

I would be remiss if I did not acknowledge the financial support that made this work possible. My sincere thanks go to the Engineering and Physical Sciences Research Council (EPSRC) for their funding, without which this research would not have been feasible.

To all those mentioned and the many others who have contributed to my growth and success during this journey, I offer my deepest thanks. This work would not have been possible without your support and encouragement.

Abstract

This thesis focuses on constructing solutions to McKean–Vlasov equations with resets and common noise, which serve as a model for infinite populations of neurons based on the leaky integrate-and-fire neuron model.

Our study progresses through three main stages. First, we examine the limiting behaviour of a simplified McKean–Vlasov system with a smoothed feedback mechanism as the smoothing becomes singular. These equations are employed in modelling systemic risk in large interconnected financial networks and serve as a simplification of the integrate-and-fire neuron model due to the absence of resets and their simplified feedback mechanism. We prove weak convergence of the system with smoothed feedback to a system with instantaneous feedback.

Next, we introduce a finite particle system that generalizes the integrate-and-fire neuron model, incorporating physiologically relevant features such as common noise, refractory periods, and delayed transmission of neuronal interactions. We rigorously show the tightness of empirical measures in this interacting particle system and prove that their limit points solve the natural limiting McKean–Vlasov problem. Furthermore, we establish uniqueness within a class of processes, allowing us to conclude that our solution is measurable with respect to the common noise.

Finally, we synthesize these results to construct relaxed solutions to a McKean–Vlasov system with resets, common noise, and an instantaneous feedback mechanism. This is achieved by taking limits of a parameterized system as the smoothing effects become singular and refractory periods approach zero.

Our work provides a novel approach to constructing solutions to singular McKean–Vlasov problems under more general coefficient conditions than previously considered in the literature.

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Chapter 1

Introduction

The study of large interacting particle systems that interact through their hitting times and their mean-field limits has been a topic of intense research in recent years, with applications spanning diverse fields. These range from finance, where they model systemic risk and defaults [31, 29, 56, 28], to physics, providing a probabilistic formulation of the super-cooled Stefan problem [14, 56], and to neuroscience, where they model the dynamics of voltage in neuronal systems [15, 16, 34]. This thesis focuses on the latter class of models: McKean–Vlasov equations with resets. Our primary objective is to extend the classical integrate-and-fire neuron models by incorporating common noise.

1.1 Background and Motivation

At the heart of our study lies the neuron, the elementary processing unit of the central nervous system. A representative neuron is composed of three functionally distinct sections:

- **Dendrites:** The input device of the neuron, consisting of several branches forming a tree-like structure. Other neurons connect to the dendritic tree through specialized cell sites called synapses.
- **Soma:** The processing unit of the neuron, which determines whether to generate an action potential (a rapid discharge of the membrane potential) and send a pulse down the axon.
- **Axon:** The output device of the neuron, connecting to the dendrites of other neurons via synapses to deliver signals produced by the soma.

The human brain contains an estimated 10^{10} to 10^{12} neurons, with a single neuron often connecting to more than 10^4 others [26]. For instance, Purkinje cells in the cerebellum have approximately 10^5 connections [70]. This vast number of neurons and their intricate interconnections give rise to complex behaviours and cognitive functions, making them ideally suited for modelling as large interacting particle systems and analysis through mean-field limits. We shall not provide a comprehensive overview of the physiological properties of neurons here and instead refer the reader to the literature for more detail, [68, 26, 69, 27, 67, 42, 36].

Various types of spiking neuron models exist, ranging from detailed biophysical models like the Hodgkin–Huxley model [32, 57, 13] to simpler variants, such as the FitzHugh–Nagumo model and Morris–Lecar model, [26, Chapter 3]. Our work focuses on leaky integrate-and-fire neuron models, which, while less complex than their biophysical counterparts, capture essential features of neuronal behaviour:

(i) The membrane potential of a neuron is the sum of all incoming currents [46, 7].

(ii) Action potentials, also called spikes, are stereotyped events. The shape of an action potential does not carry information; rather, it is the number and timing of these spikes that matter [26, 18].

(iii) There is a passive leak of potential driving the neuron towards its resting state due to the diffusion of ions across the cell membrane.

Our work builds upon and extends the models proposed by researchers such as Ostojic, Brunel, and Hakim [58]. They proposed a model of N neurons:

$$\begin{cases} X_t^i = -\lambda \int_0^t (V_{\text{rest}} - X_s^i)\,\mathrm{d}s + \mu t + \frac{\gamma}{N}\sum_{j\neq i}\int_0^t (X_s^j - X_s^i)\,\mathrm{d}s \\ \qquad\quad +\sigma W_t^i + \alpha F_t^N - (V_{\text{threshold}} - V_{\text{reset}})J_t^i \\ \tau_k^i = \inf\{t > \tau_{k-1}^i \ :\ X_{t-}^i + \alpha \Delta F_t^N \geq V_{\text{threshold}}\}, \qquad \tau_0^i = 0, \\ J_t^i = \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^i),\ F_t^N = \frac{1}{N}\sum_{j=1}^N \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^j). \end{cases} \tag{1.1.1}$$

Here, X_t^i represents the voltage of the i^{th} neuron at time t. The voltage of each neuron is driven by an independent Brownian motion W^i, while μ represents a homogeneous external input to the system. V_{rest} is the resting potential of the neuron, and $V_{\text{threshold}}$ is the firing threshold of the neuron. Once X_t^i hits $V_{\text{threshold}}$, it is immediately reset to its reset potential V_{reset} by the $(V_{\text{threshold}} - V_{\text{reset}})J_t^i$ term. The parameters α, λ, γ, and σ are calibrated to experimental data. ΔF_t^N denotes the jump of F_t^N at time t. We can readily see that (1.1.1) captures the core features described in (i), (ii), and (iii) above.

Without loss of generality, in the work that follows, we standardise this model by rescaling and shifting the system such that X_t^i lives on $(-\infty, 0]$, $V_{\text{threshold}} = 0$, and $V_{\text{reset}} = -1$. A notable feature of this model is the instantaneous feedback mechanism, represented by F^N and its inclusion in the τ_k^i term. This mechanism allows for the possibility of cascade effects: when one neuron reaches its threshold at 0, it can cause others in the network to reach their thresholds due to the instantaneous transmission of feedback. This process continues until all cascades have been resolved. Hence, due to the rescaling and shifting, to prevent this process from continuing *ad infinitum*, α is generally set to be less than 1.

While Ostojic, Brunel, and Hakim [58] studied the properties of the large network behaviour (i.e., the mean-field limit), they did not rigorously prove convergence to this large-scale limit. This gap in the literature was addressed by Delarue et al. [16]. Our work takes a significant step forward by incorporating common noise into these models, reflecting shared inputs or correlated stimuli that affect multiple neurons simultaneously. We also generalise the drift and diffusion coefficients, thereby encapsulating a broader range of dynamics in our models.

1.2 Research Objectives and Methodological Approach

The primary objective of this thesis is to prove the existence of solutions to a class of McKean–Vlasov equations with resets and common noise. To achieve this, we employ a structured approach that progresses from simpler models to more complex ones, developing and refining our mathematical techniques along the way. We delay a complete overview of the related literature for each model to the introduction of the relevant chapter but provide a brief overview of each model below.

1.2.1 Systemic Risk Model

We begin by examining a simplified model established by Hambly and Søjmark [31] in the context of systemic risk. This model serves as a foundation for our analysis and is described by the following system parameterised by $\varepsilon > 0$:

$$\begin{cases} \mathrm{d}X_t^\varepsilon = b(t, X_t^\varepsilon, \boldsymbol{\nu}_t^\varepsilon)\,\mathrm{d}t + \sigma(t, X_t^\varepsilon)\sqrt{1-\rho(t,\boldsymbol{\nu}_t^\varepsilon)^2}\,\mathrm{d}W_t + \sigma(t, X_t^\varepsilon)\rho(t,\boldsymbol{\nu}_t^\varepsilon)\,\mathrm{d}W_t^0 \\ \qquad\qquad -\alpha(t)\,\mathrm{d}\mathfrak{L}_t^\varepsilon, \\ \tau^\varepsilon = \inf\{t > 0 \ : \ X_t^\varepsilon \le 0\}, \\ \boldsymbol{\nu}_t^\varepsilon = \mathbb{P}\left[X_t^\varepsilon \in \cdot, \tau^\varepsilon > t \middle| W^0\right], \\ L_t^\varepsilon = \mathbb{P}\left[\tau^\varepsilon \le t \middle| W^0\right], \quad \mathfrak{L}_t^\varepsilon = \int_0^t \mathfrak{K}^\varepsilon(t-s) L_s^\varepsilon\,\mathrm{d}s. \end{cases} \tag{1.2.1}$$

In this model, X^ε represents the *distance-to-default* of a representative particle in large financial networks. The processes W and W^0 are independent Brownian motions, with W^0 referred to as the common noise. This system captures the phenomenon in an infinite network of interconnected institutions where the default of one institution (represented by X^ε hitting zero) can trigger defaults in other institutions, pushing the entire system towards the boundary. The random sub-probability measure $\boldsymbol{\nu}_t^\varepsilon$ represents the distribution of the system that has not yet defaulted at time t. L_t^ε is the mass of the system that has defaulted by time t. Precise conditions on the coefficients are provided in Chapter 2.

This model can be viewed as a simplified version of integrate-and-fire models in neuroscience, with two key differences. First, particles interact only through their first hitting time of the threshold and are not reset. Second, this model features a delay mechanism in the network feedback through the mollification with $\mathfrak{K}^\varepsilon$.

The mollifier $\mathfrak{K}^\varepsilon$ is parameterised by ε in such a way that it converges to the Dirac delta as ε approaches zero. By taking the limit as $\varepsilon \to 0$, we show that the system converges weakly to a McKean–Vlasov system where feedback is felt instantaneously.

1.2.2 Integrate-And-Fire Neuron Model

Building on the insights gained from the systemic risk model, we turn our attention to a more complex integrate-and-fire neuron model. We propose an interacting particle system that extends the models studied by Delarue et al. [16], Inglis and Talay [34], and Hambly and Søjmark [31]. Our model incorporates several important physiological features of real neurons:

- Common noise
- A refractory period - a time during which a neuron cannot fire again after an action potential
- Delays in the feedback mechanism

- Stochasticity in the reset value

For this model, we prove the existence and uniqueness of *strong* solutions to the following McKean–Vlasov system:

$$\begin{cases} \mathrm{d}X_t = b(t, X_t, \nu_t, \mathfrak{f}_t)\mathbb{1}_{\{X_t<0\}}\,\mathrm{d}t + \sigma(t, X_t)\sqrt{1-\rho^2(t,\nu_t,\mathfrak{f}_t)}\mathbb{1}_{\{X_t<0\}}\,\mathrm{d}W_t \\ \qquad\quad +\sigma(t, X_t)\rho(t,\nu_t,\mathfrak{f}_t)\mathbb{1}_{\{X_t<0\}}\,\mathrm{d}W_t^0 - \mathrm{d}\sum_{k\geq 1}\xi_k\mathbb{1}_{[0,t]}(\tau_k+\varsigma_k), \\ \tau_k = \inf\{t>\tau_{k-1}+\varsigma_{k-1}\ :\ X_{t-}\geq 0\}, \qquad \tau_0 = 0, \\ F_t = \sum_{k\geq 1}\mathbb{P}\left[\tau_k\leq t|\, W^0\right],\ \mathfrak{f}_t = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathfrak{K}*F\right)_t = \int_0^t \mathfrak{K}'(t-s)F_s\,\mathrm{d}s, \\ \nu_t = \mathbb{P}[X_t\in\cdot, X_t<0|\, W^0]. \end{cases} \tag{1.2.2}$$

Strong solutions are characterised by the property that the random sub-probability measure ν_t is adapted to the σ-algebra generated by the common noise. Here, ς_k represents the refractory period, and ξ_k denotes the stochasticity in the reset value. F_t is referred to as the firing function and represents the mass of the system that has fired once, twice, thrice, etc. up to time t. Here, F_t is the parallel of L_t in the systemic risk model, (1.2.1). We encapsulate the feedback into the drift coefficient b with the term $\mathfrak{f}_t$, which is the derivative of the mollified firing function F_t. The measure ν_t is the distribution of the system which is not in the refractory period. The coefficients b, σ, and ρ are assumed to satisfy suitable conditions, which are provided in Chapter 3.

1.2.3 Limit of the Neuronal Model

Our final objective is to extend the methods developed for the systemic risk model to this more complex neuronal model. We achieve this by first parameterizing $\mathfrak{K}$ in (1.2.2) with ε such that it converges to the Dirac delta. We also parameterise ς_k with ε so that it converges to zero in a suitable sense. Consequently, we consider the following system of equations parameterised by ε:

$$\begin{cases} X_t^\varepsilon = \int_0^t b(s, X_s^\varepsilon, \boldsymbol{\nu}_s^\varepsilon)\mathbb{1}_{\{X_s^\varepsilon<0\}}\,\mathrm{d}s + \int_0^t \sigma(s,\ X_s^\varepsilon)\sqrt{1-\rho^2(s,\boldsymbol{\nu}_s^\varepsilon)}\mathbb{1}_{\{X_s^\varepsilon<0\}}\,\mathrm{d}W_s \\ \qquad + \int_0^t \sigma(s,\ X_s^\varepsilon)\rho(s,\boldsymbol{\nu}_s^\varepsilon)\mathbb{1}_{\{X_s^\varepsilon<0\}}\,\mathrm{d}W_s^0 + \int_0^t \alpha(s)\mathbb{1}_{\{X_s^\varepsilon<0\}}\,\mathrm{d}\mathfrak{F}_s^\varepsilon \\ \qquad - \sum_{k\geq 1}\mathbb{1}_{[0,t]}(\tau_k^\varepsilon+\varsigma_k^\varepsilon), \\ \tau_k^\varepsilon = \inf\{t>\tau_{k-1}^\varepsilon+\varsigma_{k-1}^\varepsilon\ :\ X_t^\varepsilon\geq 0\}, \qquad \tau_0^\varepsilon = 0, \\ F_t^\varepsilon = \sum_{k\geq 1}\mathbb{P}\left[\tau_k^\varepsilon\leq t|\, W^0\right],\ \mathfrak{F}_t^\varepsilon = \left(\mathfrak{K}^\varepsilon * F^\varepsilon\right)_t = \int_0^t \mathfrak{K}^\varepsilon(t-s)F_s^\varepsilon\,\mathrm{d}s, \\ \boldsymbol{\nu}_t^\varepsilon = \mathbb{P}[X_t^\varepsilon\in\cdot, X_t^\varepsilon<0|\, W^0], \end{cases} \tag{1.2.3}$$

with suitable assumptions on the coefficients provided later in Chapter 4. By employing and generalising the methods developed in Chapter 2, loosely speaking, we show that $\mathrm{Law}(X_t^\varepsilon + \sum_{k\geq 1}\mathbb{1}_{[0,t]}(\tau_k^\varepsilon+\varsigma_k^\varepsilon)\ |\ W^0)$ converges weakly to some random

variable $\mathbf{P}$, as $\varepsilon \to 0$ along a suitable subsequence. As we are taking weak limits of W^0-measurable random variables, we cannot guarantee that the limit will be W^0-measurable. However, we can show that the limit is measurable with respect to the common noise and some extraneous source of randomness. Using this limit point, we construct *relaxed solutions* to the McKean–Vlasov system with resets and common noise:

$$\begin{cases} \mathrm{d}X_t = b(t, X_t, \boldsymbol{\nu}_t)\,\mathrm{d}t + \sigma(t, X_t)\sqrt{1-\rho(t,\boldsymbol{\nu}_t)^2}\,\mathrm{d}W_t + \sigma(t, X_t)\rho(t,\boldsymbol{\nu}_t)\,\mathrm{d}W_t^0 \\ \qquad\quad +\alpha(t)\,\mathrm{d}F_t - \mathrm{d}J_t, \\ \tau_k = \inf\{t > \tau_{k-1} \;:\; X_{t-} + \alpha(t)\Delta F_t \geq 0\}, \qquad \tau_0 = 0, \\ J_t = \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k), \\ \boldsymbol{\nu}_t = \mathbb{P}\left[X_t \in \cdot \mid W^0, \mathbf{P}\right], \\ F_t = \sum_{k\geq 1} \mathbb{P}\left[\tau_k \leq t \mid W^0, \mathbf{P}\right], \end{cases} \tag{1.2.4}$$

and show that the limit point $\mathbf{P}$ satisfies the fixed point equation $\mathbf{P} = \mathbb{P}\left[X + J \in \cdot \mid W^0, \mathbf{P}\right]$.

Through this structured approach, we aim to provide a comprehensive analysis of McKean–Vlasov equations with resets and common noise.

1.3 Main Contributions and Outline of the Thesis

The key contributions of this thesis are as follows:

In Chapter 2, we focus on a simplified system of equations employed to model systemic risk in large interconnected financial networks. We show the weak convergence of solutions of (1.2.1) to relaxed solutions of the comparable McKean–Vlasov equation where the feedback mechanism is instantaneous. Our approach provides a novel method to construct solutions to singular McKean–Vlasov problems employed in systemic risk models, allowing for more general drift and diffusion coefficients than those previously considered in the literature. Additionally, since the limiting system may exhibit jumps, we prove an upper bound on the sizes of these jumps. Under additional assumptions on the coefficients, we show that the jumps of the system satisfy the *physical jump condition*. This condition restricts the size of jumps in the limiting system, avoiding the construction of solutions that exhibit non-physical behaviour due to their origins as a particle system.

In Chapter 3 we introduce our generalised integrate-and-fire neuron model, incorporating common noise, refractory periods, delays in the feedback mechanism, and

stochasticity in the reset value. We consider the system of interacting particles:

$$\begin{cases} \mathrm{d}X_t^i = b(t, X_t^i, \nu_t^N, \mathfrak{f}_t^N)\mathbb{1}_{\{X_t^i<0\}}\,\mathrm{d}t + \sigma(t,\, X_t^i)\sqrt{1-\rho^2(t,\nu_t^N,\mathfrak{f}_t^N)}\mathbb{1}_{\{X_t^i<0\}}\,\mathrm{d}W_t^i \\ \qquad\quad +\sigma(t,\, X_t^i)\rho(t,\nu_t^N,\mathfrak{f}_t^N)\mathbb{1}_{\{X_t^i<0\}}\,\mathrm{d}W_t^0 - \mathrm{d}\sum_{k\geq 1}\xi_k^i\mathbb{1}_{[0,t]}(\tau_k^i+\varsigma_k^i) \\ \tau_k^i = \inf\{t>\tau_{k-1}^i+\varsigma_{k-1}^i \;:\; X_{t-}^i\geq 0\}, \qquad \tau_0^i=0, \\ F_t^N = \frac{1}{N}\sum_{i=1}^N J_t^i, \quad \mathfrak{f}_t^N = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathfrak{K}*F^N\right)_t = \int_0^t \mathfrak{K}'(t-s)F_s^N\,\mathrm{d}s, \\ \nu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}\mathbb{1}_{\{X_t^i<0\}}. \end{cases}$$

We rigorously prove that the empirical measures, ν^N, are tight and that the limit point of these measures is a solution to the McKean–Vlasov problem (1.2.2). Furthermore, we establish the uniqueness of the limit point within a class of processes, hence concluding that the limit point is measurable with respect to the common noise.

Lastly, in Chapter 4, we generalise the results from Chapter 2 and apply them to the McKean–Vlasov system constructed in Chapter 3. By considering the system of equations in (1.2.3) parameterised by ε, we show that we can construct a solution to the McKean–Vlasov system with resets and common noise, (1.2.4).

Chapter 2

Contagious McKean–Vlasov Problems With Common Noise: From Smooth to Singular Feedback Through Hitting Times

2.1 Introduction

In this chapter, we study the limiting behaviour of the family of conditional McKean–Vlasov equations

$$\begin{cases} \mathrm{d}X_t^\varepsilon = b(t, X_t^\varepsilon, \boldsymbol{\nu}_t^\varepsilon)\,\mathrm{d}t + \sigma(t, X_t^\varepsilon)\sqrt{1-\rho(t,\boldsymbol{\nu}_t^\varepsilon)^2}\,\mathrm{d}W_t + \sigma(t, X_t^\varepsilon)\rho(t,\boldsymbol{\nu}_t^\varepsilon)\,\mathrm{d}W_t^0 \\ \qquad\quad -\alpha(t)\,\mathrm{d}\mathfrak{L}_t^\varepsilon, \\ \tau^\varepsilon = \inf\{t > 0 \,:\, X_t^\varepsilon \le 0\}, \\ \mathbf{P}^\varepsilon = \mathbb{P}\left[X^\varepsilon \in \cdot \,|\, W^0\right], \quad \boldsymbol{\nu}_t^\varepsilon \coloneqq \mathbb{P}\left[X_t^\varepsilon \in \cdot, \tau^\varepsilon > t \,|\, W^0\right], \\ L_t^\varepsilon = \mathbb{P}\left[\tau^\varepsilon \le t \,|\, W^0\right], \quad \mathfrak{L}_t^\varepsilon = \int_0^t \mathfrak{K}^\varepsilon(t-s) L_s^\varepsilon\,\mathrm{d}s, \end{cases} \tag{2.1.1}$$

as ε tends to zero. Here, W and W^0 are independent standard Brownian motions, and $\mathfrak{K}^\varepsilon$ is a rescaled mollifier which converges to the Dirac delta as ε converges. Motivated by their origin as the limit of a particle system, W is usually referred to as the *idiosyncratic noise* (of a representative particle) and W^0 as the *common noise*. Also for the same reason, L^ε is referred to as the *loss* process and quantifies the amount of mass that has crossed the boundary at zero by time t. A solution to this system consists of the random probability measure $\mathbf{P}^\varepsilon$ and the loss process L^ε, conditional on W^0.

In addition to the more classical measure dependence of the coefficients that characterise McKean–Vlasov equations, there is a further feedback mechanism through the loss process L^ε: depending on the value of $\alpha(t) \ge 0$, L^ε pushes X^ε towards zero,

causing the value of L^ε to increase, hence pushing X^ε even closer to 0. The integral kernel $\mathfrak{K}^\varepsilon$, which is parameterised by some $\varepsilon > 0$, is a key element of the model and captures a latency in the transmission of L^ε to X^ε present in real-world systems. Precise conditions on the coefficient functions will be given later.

One motivation for this model arises in systemic risk, where X^ε represents the *distance-to-default* of a prototypical institution in a financial network with infinitely many entities, see for example [31]. In this setting, L^ε_t denotes the proportion of institutions that have defaulted by time t and is the cause of *endogenous contagion* through the feedback mechanism. In this model, we use the kernel $\mathfrak{K}^\varepsilon$ to capture feedback where, when a financial institution defaults and their positions are unwound, the counterparties experience a gradual decrease in the value of their assets over time.

This kernel structure has also been used in other settings. A model for bank runs with common noise and smooth transmission of boundary losses was analysed in [8]. Moreover, [34] study a related mean-field model for neurons interacting gradually through threshold hitting times, albeit without common noise.

If the support of $\mathfrak{K}^\varepsilon$ is contained in the interval $[0, \delta]$ with $\delta \ll 1$, then the transmission of shocks is almost instantaneous as the integral $\int_0^t \mathfrak{K}^\varepsilon(t-s) L^\varepsilon_s \, \mathrm{d}s$ is approximately equal to L^ε_t. In this chapter, we prove convergence in the following sense: if we fix a kernel $\mathfrak{K}$ and rescale it with a variable $\varepsilon > 0$ by $\mathfrak{K}^\varepsilon(t) = \varepsilon^{-1}\mathfrak{K}(\varepsilon^{-1}t)$, then we have convergence in the M_1-topology of X^ε to X, where X is a (relaxed) solution to

$$\begin{cases} \mathrm{d}X_t = b(t, X_t, \boldsymbol{\nu}_t)\,\mathrm{d}t + \sigma(t, X_t)\sqrt{1-\rho(t,\boldsymbol{\nu}_t)^2}\,\mathrm{d}W_t + \sigma(t,X_t)\rho(t,\boldsymbol{\nu}_t)\,\mathrm{d}W^0_t \\ \qquad\quad -\alpha(t)\,\mathrm{d}L_t, \\ \tau = \inf\{t > 0 \,:\, X_t \le 0\}, \\ \mathbf{P} = \mathbb{P}\left[X \in \cdot \,|\, W^0, \mathbf{P}\right], \quad \boldsymbol{\nu}_t \coloneqq \mathbb{P}\left[X_t \in \cdot, \tau > t \,|\, W^0, \mathbf{P}\right], \\ L_t = \mathbb{P}\left[\tau \le t \,|\, W^0, \mathbf{P}\right]. \end{cases} \tag{2.1.2}$$

It is well known that equations of the form (2.1.2) may develop jump discontinuities. Without common noise, given suitable assumptions, for $\alpha(t)$ sufficiently large, a jump must occur, as per [30, Theorem 1.1] and also [2]. With common noise, there is a set of paths of positive probability where a jump must happen, [48, Theorem 2.1]. This motivates our use of the M_1-topology as it is rich enough to facilitate the convergence of continuous functions to ones that jump. The equation (2.1.1) has been posed in slightly more generality (with $\alpha(t, X^\varepsilon_t, \boldsymbol{\nu}^\varepsilon_t)$) in [31]. As the convergence is strictly in the M_1-topology, not in the J_1-topology, we only consider α to be of the form $\alpha(t)$. If we considered it to also be a function of X_t and/or $\boldsymbol{\nu}_t$, then we cannot

expect to obtain an equation of the form (2.1.2) in the limit as $\varepsilon \downarrow 0$, due to X, $\boldsymbol{\nu}$, and L having jumps at the same time.

Intuitively, (2.1.1) is a smoothed approximation to (2.1.2). From a mathematical perspective, the advantage of smoothing out the interactions is that the well-posedness of (2.1.1) is well understood, see [31]. On the other hand, for the formulation (2.1.2) with instantaneous feedback, various questions concerning existence are yet to be addressed, and uniqueness remains a completely open problem. Even with constant coefficients, one cannot rely on the methods from [17], where uniqueness was treated successfully in the case of *no* common noise ($\rho \equiv 0$) and constant coefficients. Further discussion of the issue of existence of solutions to (2.1.2) will follow later, while we do not address uniqueness.

From a modelling perspective, the smoothing naturally captures the latency in real-world systems. Our motivation for taking the limit as $\varepsilon \to 0$ is to investigate the convergence to the system where the feedback is felt instantaneously. This captures the situation when the latency is small compared to the timescale of interest. The instantaneous transmission model has been used in applications to systemic risk, the supercooled Stefan problem, and leaky integrate-and-fire models in neuroscience.

Variants and special cases of (2.1.2) have been the subject of extensive research in the field. In the simplest scenario, where b and ρ are both zero, σ is equal to 1, and α is a positive constant, we obtain the probabilistic formulation of the *supercooled Stefan problem*. A version of the *Stefan problem*, introduced by Stefan in [65], can be described as follows:

$$\begin{cases} \partial_t u = \frac{1}{2}\partial_{xx} u, \quad x \geq \alpha L_t, \quad t \geq 0, \\ u(0,x) = f(x), \qquad x \geq 0, \quad \text{and} \quad u(t, \alpha L_t) = 0, \quad t \geq 0, \\ L_t' = \frac{1}{2}\partial_x u(t, \alpha L_t), \quad t \geq 0 \quad \text{and} \quad L_0 = 0. \end{cases} \tag{2.1.3}$$

The solution to the partial differential equation (PDE) describes the temperature and the boundary of a material undergoing a phase transition, typically from a solid to a liquid. The supercooled Stefan problem describes the freezing of a supercooled liquid (i.e. a liquid which is below its freezing point) on the semi-infinite strip $(\alpha L_t, \infty)$. Here, αL_t is the location of the liquid-solid boundary at time t. In the PDE literature, it was first established in [61] that L_t may explode in finite time, i.e., there exists a $t_* \in (0, \infty)$ such that $\lim_{s \uparrow t_*} L_s' = \infty$. While many authors constructed classical solutions to (2.1.3) for $t_* \in [0, \infty]$, [22, 23, 24, 45, 33], from the PDE perspective it was unclear how to restart a given solution after a finite time blow-up of L'.

Under suitable assumptions on the initial condition (see [56]), the process $X_t \mathbb{1}_{\tau > t}$ admits a density $p(t, \cdot)$ on the interval $(0, \infty)$, which satisfies the PDE:

$$\partial_t p = \frac{1}{2}\partial_{xx} p + \alpha L_t' \partial_x p, \qquad L_t' = \frac{1}{2}\partial_x p(t, 0), \qquad p(t, 0) = 0.$$

By setting $u(t, x) = -p(t, x - \alpha L_t)$, we recover the classical formulation of the Stefan problem. The probabilistic reformulation provided a way to restart the system following a blow-up. Blow-ups of L correspond to jumps in the probabilistic setting.

From a probabilistic perspective, when b and ρ are both zero, σ is equal to 1, and α is a positive constant, the stochastic differential equation (SDE) presented here yields a succinct model for studying contagion in large financial networks. With this motivation, extensive research has been conducted to investigate various properties of *physical solutions* to this equation [56, 55, 30, 49, 48, 51, 39, 14]. The paper [17] establishes that when X_{0-} possesses a bounded density that changes monotonicity finitely many times, then L is unique, and for any $t \geq 0$, L is continuously differentiable on $(t,\, t+h)$ for some $h > 0$. Additionally, in [30], it is demonstrated that for an initial condition with a bounded density that is Hölder continuous near the boundary, L is unique, continuous, and has a weak derivative until some explosion time. The work in [49] extends these results by showing that if the initial condition possesses an L^2-density, then we have uniqueness for a short time after the explosion time. Moreover, irrespective of the initial condition, there exists a minimal loss process that will be dominated by any other loss process that solves the equation [14].

For arbitrary initial conditions, see [30, Example 2.2], there may be infinitely many solutions. Furthermore, different solutions will take different jump sizes. Hence, it may be possible for two solutions to be equal up to the first jump time t and then take jumps of different sizes. To address this ambiguity that arises at a jump time, a condition is typically imposed to restrict to admissible jump sizes. This condition is known as the *physical jump condition*, defined as:

$$\Delta L_t = \inf\{x > 0 \,:\, \boldsymbol{\nu}_{t-}([0, \alpha x]) < x\}, \tag{2.1.4}$$

where $\Delta L_t \coloneqq L_t - L_{t-}$. The intuitive interpretation of (2.1.4) is that if we take the density of $X_{t-}\mathbb{1}_{\tau \geq t}$ and displace it by an αx amount towards 0, then the mass of the system below zero is exactly x. So it is the minimal amount by which we may displace our density such that the displacement and the mass below zero correspond. From a modelling perspective, the physical jump condition is the preferred choice of jump sizes due to its economic and physical interpretations. It has been established that minimal solutions are physical [14, Theorem 6.5]. However, it remains unclear

whether physical solutions are necessarily minimal due to the lack of uniqueness for general initial conditions.

Returning to (2.1.2), recent advances have been made in the study of general coefficients, specifically $t \mapsto b(t)$, $t \mapsto \sigma(t)$, and $t \mapsto \rho(t)$, in the presence of common noise. In Remark 2.5 from [55], a generalized Schauder fixed-point argument is presented to construct *strong solutions* in this setting. Strong solutions refer to the property $\mathbf{P} = \mathbb{P}(X \in \cdot \mid W^0)$, indicating that the random probability measure $\mathbf{P}$ is adapted to the σ-algebra generated by the common noise. In [48], an underlying finite particle system was shown to converge to *relaxed* (or *weak*) *solutions* (see Definition 2.2.1), satisfying the aforementioned physical jump condition (with coefficients $(t,x) \mapsto b(t,x)$, $t \mapsto \sigma(t)$, and $t \mapsto \rho(t)$). Weak/relaxed solutions are characterised by having $\mathbf{P} = \mathbb{P}(X \in \cdot \mid W^0, \mathbf{P})$, instead of $\mathbf{P} = \mathbb{P}(X \in \cdot \mid W^0)$, see Definition 2.2.1. As the empirical distributions of the finite particle systems converge weakly to $\mathbf{P}$, there is no guarantee that $\mathbf{P}$ will be adapted to the σ-algebra generated by the common noise. As regards to *strong* solutions in the sense just discussed, the existence of strong solutions for the common noise problem satisfying the physical jump condition (2.1.4) has not yet been addressed in the literature.

The main contributions and structure of this chapter are as follows. Firstly, in Section 2.2, we state Theorem 2.2.4 and Corollary 2.2.5, showing the weak convergence of solutions of (2.1.1) to relaxed solutions of (2.1.2) as $\varepsilon \to 0$, i.e., as the gradual feedback mechanism becomes instantaneous in the limit. As a by-product, this gives a novel method for establishing the existence of solutions to (2.1.2), avoiding time regularity assumptions on $\sigma\rho$ as needed in [48]. Furthermore, we derive an upper bound on the jump sizes, Theorem 2.2.4, and, under additional assumptions on the coefficients, Corollary 2.2.5, show that the loss process L satisfies the physical jump condition (2.1.4). In Section 2.3 we prove tightness of the smoothed system and hence the existence of a limit point. In Section 2.4 we prove continuity of the hitting times and subsequently can conclude that the smoothed loss function converges to the limiting loss function. Lastly, in Section 2.5 we prove that the limit point solves the system (2.1.2). For presentational purposes, we have deferred the proofs of some technical results to Section 2.6. The proofs in Section 2.3 to Section 2.5 will refer to these results as needed.

2.2 Weak Convergence of Smoothed Feedback Systems

Fix a finite time horizon $T > 0$, and let $\mathcal{P}(A)$ denote the set of probability measures on a measurable space $(A, \mathcal{A})$. When A is a metric space, $\mathcal{B}(A)$ denotes the Borel σ-algebra. Let $\mathbf{M}_{\leq 1}(A)$ denote the space of sub-probability measures, which we shall endow with the topology of weak convergence. For any interval I and metric space X, let $C(I, X)$ denote the space of continuous functions from I to X. Similarly, $D(I, X)$ denotes the space of càdlàg functions from I to X. We shall employ the shorthand notation $\mathcal{C}_X$ and D_X for $C(I, X)$ and $D(I, X)$, respectively, when the interval I is clear.

For every $\varepsilon > 0$, we fix a probability space $(\Omega^\varepsilon, \mathcal{F}^\varepsilon, \mathbb{P}^\varepsilon)$ that supports two independent Brownian motions. To simplify the notation, we will denote these Brownian motions by W and W^0; however, it is important to note that they may not be equal for different values of ε. Similarly, we adopt the simplified notations $\mathbb{P}$ and $\mathbb{E}$ to refer to $\mathbb{P}^\varepsilon$ and the expectation under the measure $\mathbb{P}^\varepsilon$ respectively. In this section, we characterise the weak limit of the system given by the following equation as ε tends to zero:

$$\begin{cases} \mathrm{d}X_t^\varepsilon = b(t, X_t^\varepsilon, \boldsymbol{\nu}_t^\varepsilon)\,\mathrm{d}t + \sigma(t, X_t^\varepsilon)\sqrt{1-\rho(t,\boldsymbol{\nu}_t^\varepsilon)^2}\,\mathrm{d}W_t + \sigma(t, X_t^\varepsilon)\rho(t, \boldsymbol{\nu}_t^\varepsilon)\,\mathrm{d}W_t^0 \\ \qquad\quad -\alpha(t)\,\mathrm{d}\mathfrak{L}_t^\varepsilon, \\ \tau^\varepsilon = \inf\{t > 0 \,:\, X_t^\varepsilon \leq 0\}, \\ \mathbf{P}^\varepsilon = \mathbb{P}\left[X^\varepsilon \in \cdot \,|\, W^0\right], \quad \boldsymbol{\nu}_t^\varepsilon \coloneqq \mathbb{P}\left[X_t^\varepsilon \in \cdot, \tau^\varepsilon > t \,|\, W^0\right], \\ L_t^\varepsilon = \mathbb{P}\left[\tau^\varepsilon \leq t \,|\, W^0\right], \quad \mathfrak{L}_t^\varepsilon = \int_0^t \mathfrak{K}^\varepsilon(t-s) L_s^\varepsilon\,\mathrm{d}s, \end{cases} \tag{2.2.1}$$

where $t \in [0, T]$. The coefficient b (σ, ρ, or α, respectively) is a measurable map from $[0, T] \times \mathbb{R} \times \mathbf{M}_{\leq 1}(\mathbb{R})$ ($[0, T] \times \mathbb{R}$, $[0, T] \times \mathbf{M}_{\leq 1}(\mathbb{R})$, or $[0, T]$, respectively) into $\mathbb{R}$. The initial condition, denoted by X_{0-}, is assumed to be independent of the Brownian motions and positive almost surely. Finally, we define $\mathfrak{K}^\varepsilon(t) \coloneqq \varepsilon^{-1}\mathfrak{K}(t\varepsilon^{-1})$.

One way to view X^ε is as the mean-field limit of an interacting particle system where particles interact through their first hitting time of zero. The interactions among particles are smoothed out over time by convolving with the kernel $\mathfrak{K}^\varepsilon$. As ε approaches zero, the effect of interactions occurs over increasingly smaller time intervals. As $\mathfrak{K}^\varepsilon$ is a mollifier, it is natural to expect, as ε tends to zero, $\mathfrak{L}_t^\varepsilon$ to converge to the instantaneous loss at time t. That is to say, along a suitable subsequence, the random tuple $\{(\mathbf{P}^\varepsilon, W^0, W)\}_{\varepsilon>0}$ would have a limit point $(\mathbf{P}, W^0, W)$ where

$\mathbf{P} = \mathbb{P}[X \in \cdot \mid W^0]$ and X solves

$$\begin{cases} \mathrm{d}X_t = b(t, X_t, \boldsymbol{\nu}_t)\,\mathrm{d}t + \sigma(t, X_t)\sqrt{1-\rho(t,\boldsymbol{\nu}_t)^2}\,\mathrm{d}W_t + \sigma(t, X_t)\rho(t,\boldsymbol{\nu}_t)\,\mathrm{d}W_t^0 \\ \qquad\quad -\alpha(t)\,\mathrm{d}L_t, \\ \tau = \inf\{t > 0 \ : \ X_t \le 0\}, \\ \mathbf{P} = \mathbb{P}[X \in \cdot \,|\, W^0], \quad \boldsymbol{\nu}_t \coloneqq \mathbb{P}[X_t \in \cdot, \tau > t |\, W^0], \\ L_t = \mathbb{P}[\tau \le t |\, W^0], \end{cases} \tag{2.2.2}$$

with $X_0 = X_{0-} + \alpha(0)L_0$. In this system, the feedback is felt instantaneously and is characterised by the common noise W^0. In what follows, for technical reasons, we construct an extension $\tilde{X}$ of the process X. For an arbitrary stochastic process H, we define its extended version as follows:

$$\tilde{H}_t = \begin{cases} H_{0-} & t \in [-1,\, 0), \\ H_t & t \in [0,\, T], \\ H_T + W_t - W_T & t \in (T,\, T+1]. \end{cases} \tag{2.2.3}$$

We artificially extend the processes to be constant on $[-1, 0)$ and by a pure Brownian noise term on $(T, T+1]$. Therefore, the extension to $\mathbf{P}^\varepsilon$ is $\tilde{\mathbf{P}}^\varepsilon \coloneqq \mathbb{P}(\tilde{X}^\varepsilon \in \cdot \mid W^0)$. Consequently, the random measure $\tilde{\mathbf{P}}^\varepsilon$ remains W^0-measurable. We show that the collection of measures $\{\tilde{\mathbf{P}}^\varepsilon\}_{\varepsilon>0}$ is tight; hence there exists a subsequence $(\varepsilon_n)_{n\ge1}$ that converges to zero such that $\tilde{\mathbf{P}}^{\varepsilon_n}$ converges weakly to the random measure $\mathbf{P}$. However, as the mode of convergence is weak, we cannot expect that the limit point $\mathbf{P}$ is also measurable with respect to W^0.

Hence, we relax our notion of solution to (2.2.2), which leads to the definition of *relaxed solutions* employed in the literature when studying the mean-field limit of particle systems with common noise [48] and also in the mean-field game literature with common source of noise [10].

Definition 2.2.1 (Relaxed solutions)**.** Let the coefficient functions b, σ, ρ, and α be given along with the initial condition X_{0-} at time $t = 0-$. We define a relaxed solution to (2.2.2) as a family $(X,\, W,\, W^0,\, \mathbf{P})$ on a filtered probability space $(\Omega,\, \mathcal{F},\, \mathbb{P})$ such that

$$\begin{cases} \mathrm{d}X_t = b(t, X_t, \boldsymbol{\nu}_t)\,\mathrm{d}t + \sigma(t, X_t)\sqrt{1-\rho(t,\boldsymbol{\nu}_t)^2}\,\mathrm{d}W_t + \sigma(t, X_t)\rho(t,\boldsymbol{\nu}_t)\,\mathrm{d}W_t^0 \\ \qquad\quad -\alpha(t)\,\mathrm{d}L_t, \\ \tau = \inf\{t > 0 \ : \ X_t \le 0\}, \\ \mathbf{P} = \mathbb{P}[X \in \cdot \,|\, W^0, \mathbf{P}], \quad \boldsymbol{\nu}_t \coloneqq \mathbb{P}[X_t \in \cdot, \tau > t |\, W^0, \mathbf{P}], \\ L_t = \mathbb{P}[\tau \le t \,|\, W^0, \mathbf{P}], \end{cases} \tag{2.2.4}$$

with $X_0 = X_{0-} + \alpha(0)L_0$, $L_{0-} = 0$, $X_{0-} \perp (W, W^0, \mathbf{P})$, and $(W^0, \mathbf{P}) \perp W$, where (W, W^0) is a two-dimensional Brownian motion, X is a càdlàg process, and $\mathbf{P}$ is a random probability measure on the space of càdlàg paths $D([-1, T+1], \mathbb{R})$.

As the drift and correlation function depend on a flow of measures, we still want them to satisfy some notion of linear growth and Lipschitzness in the measure component. We will also require some spatial and temporal regularity such that (2.2.1) is well-posed. We will suppose that our coefficients b, σ, ρ, $\mathfrak{K}$ and α satisfy the following assumptions.

Assumption 2.2.2. *(i) (Regularity of b) For all $t \in [0, T]$ and $\mu \in \mathbf{M}_{\leq 1}(\mathbb{R})$, the map $x \mapsto b(t, x, \mu)$ is $\mathcal{C}^2(\mathbb{R})$. Moreover, there exists a constant $C_b > 0$ such that*

$$|b(t,x,\mu)| \leq C_b\left(1 + |x| + \langle \mu, |\cdot| \rangle\right), \quad |\partial_x^{(n)} b(t,x,\mu)| \leq C_b, \quad n = 1, 2,$$
$$|b(t,x,\mu) - b(t,x,\tilde{\mu})| \leq C_b\left(1 + |x| + \langle \mu, |\cdot| \rangle\right) d_0(\mu, \tilde{\mu}),$$

where

$$d_0(\mu, \tilde{\mu}) = \sup\left\{ |\langle \mu - \tilde{\mu}, \psi \rangle| \,:\, \|\psi\|_{\text{Lip}} \leq 1, |\psi(0)| \leq 1 \right\}$$

for any $\mu, \tilde{\mu} \in \mathbf{M}_{\leq 1}(\mathbb{R})$.

(ii) (Space/time regularity of σ) The map $(t, x) \mapsto \sigma(t, x)$ is $\mathcal{C}^{1,2}([0, T] \times \mathbb{R})$. Moreover, there exists a constant $C_\sigma > 0$ such that

$$|\sigma(t,x)| \leq C_\sigma, \quad |\partial_t \sigma(t,x)| \leq C_\sigma, \quad \text{and} \quad |\partial_x^{(n)} \sigma(t,x)| \leq C_\sigma \quad \text{for} \quad n = 1, 2.$$

(iii) (d_1-Lipschitzness of ρ) For all $t \in [0, T]$, there exists a constant $C_\rho > 0$ such that

$$|\rho(t,\mu) - \rho(t,\tilde{\mu})| \leq C_\rho\left(1 + \langle \mu, |\cdot| \rangle\right) d_1(\mu, \tilde{\mu}),$$

where

$$d_1(\mu, \tilde{\mu}) = \sup\left\{ |\langle \mu - \tilde{\mu}, \psi \rangle| \,:\, \|\psi\|_{\text{Lip}} \leq 1, \|\psi\|_\infty \leq 1 \right\}$$

for any $\mu, \tilde{\mu} \in \mathbf{M}_{\leq 1}(\mathbb{R})$.

(iv) (Non-degeneracy) For all $t \in [0, T]$, $x \in \mathbb{R}$, and $\mu \in \mathbf{M}_{\leq 1}(\mathbb{R})$, the constants C_σ and C_ρ assumed above are such that $0 < C_\sigma^{-1} \leq \sigma(t, x)$ and $0 \leq \rho(t, \mu) \leq 1 - C_\rho^{-1}$.

(v) (Temporal regularity of α) The map $t \mapsto \alpha(t)$ is $\mathcal{C}^1([0, T])$ and increasing with $\alpha(0) \geq 0$.

(vi) (Sub-Gaussian initial law) The initial law, ν_{0-} is sub-Gaussian,

$$\exists\, \gamma > 0 \quad s.t. \quad \nu_{0-}(\lambda, \infty) = O(e^{-\gamma\lambda^2}) \quad as \quad \lambda \to \infty,$$

and has a density $V_{0-} \in L^2(0,\infty)$ such that $\|xV_{0-}\|_{L^2}^2 = \int_0^\infty |xV_{0-}(x)|^2 \,\mathrm{d}x < \infty$.

(vii) (Regularity of mollifier) The function $\mathfrak{K} \in \mathcal{W}_0^{1,1}(\mathbb{R}_+)$, the Sobolev space with one weak derivative in L^1 and zero at 0, such that $\mathfrak{K}$ is non-negative, and $\|\mathfrak{K}\|_1 = 1$.

Solutions of (2.2.1) are the matter of study in [31]. Under Assumption 2.2.2, the existence and uniqueness of solutions are guaranteed.

Theorem 2.2.3 ([31, Theorem 2.7])**.** *Let $(\boldsymbol{\nu}^\varepsilon, W^0)$ be the unique strong solution to the SPDE*

$$\begin{aligned}\mathrm{d}\langle \boldsymbol{\nu}_t^\varepsilon, \phi\rangle =& \langle \boldsymbol{\nu}_t^\varepsilon, b(t,\cdot,\boldsymbol{\nu}_t^\varepsilon)\partial_x\phi\rangle \,\mathrm{d}t + \frac{1}{2}\langle \boldsymbol{\nu}_t^\varepsilon, \sigma(t,\cdot)^2\partial_{xx}\phi\rangle \,\mathrm{d}t \\ &+ \langle \boldsymbol{\nu}_t^\varepsilon, \sigma(t,\cdot)\rho(t,\boldsymbol{\nu}_t^\varepsilon)\partial_x\phi\rangle \,\mathrm{d}W_t^0 - \langle \boldsymbol{\nu}_t^\varepsilon, \alpha(t)\partial_x\phi\rangle \,\mathrm{d}\mathfrak{L}_t^\varepsilon,\end{aligned}$$

where the coefficients b, σ, ρ, $\mathfrak{K}^\varepsilon$, and α satisfy Assumption 2.2.2, $\mathfrak{L}_t^\varepsilon := \int_0^t \mathfrak{K}^\varepsilon(t-s)L_s \,\mathrm{d}s$ with $L_t := 1 - \boldsymbol{\nu}_t^\varepsilon(0,\infty)$, and $\phi \in \mathscr{C}_0$, the set of Schwartz functions that are zero at 0. Then, for any Brownian motion $W \perp (X_{0-}, W^0)$, we have

$$\boldsymbol{\nu}_t^\varepsilon = \mathbb{P}\left[X_t^\varepsilon \in \cdot,\, \tau^\varepsilon > t \mid W^0\right] \quad \text{for} \quad \tau^\varepsilon := \inf\{t > 0 \,:\, X_t^\varepsilon \le 0\},$$

where X^ε is the solution to the conditional McKean–Vlasov diffusion

$$\begin{cases} \mathrm{d}X_t^\varepsilon = b(t, X_t^\varepsilon, \boldsymbol{\nu}_t^\varepsilon)\,\mathrm{d}t + \sigma(t, X_t^\varepsilon)\sqrt{1-\rho(t,\boldsymbol{\nu}_t^\varepsilon)^2}\,\mathrm{d}W_t + \sigma(t, X_t^\varepsilon)\rho(t,\boldsymbol{\nu}_t^\varepsilon)\,\mathrm{d}W_t^0 \\ \qquad\quad -\alpha(t)\,\mathrm{d}\mathfrak{L}_t^\varepsilon, \\ \mathbf{P}^\varepsilon = \mathbb{P}\left[X^\varepsilon \in \cdot \mid W^0\right], \\ L_t^\varepsilon = \mathbb{P}\left[\tau^\varepsilon \le t \mid W^0\right], \quad \mathfrak{L}_t^\varepsilon = \int_0^t \mathfrak{K}^\varepsilon(t-s)L_s^\varepsilon \,\mathrm{d}s, \end{cases}$$

with initial condition $X_{0-} \sim \nu_{0-}$.

The existence of solutions to (2.2.1) allows us to introduce the main result of this section, showing that solutions to (2.2.4) exist as limit points of the collection of smoothed equations.

Theorem 2.2.4 (Existence and convergence generalised)**.** *Let $\tilde{X}^\varepsilon$ be the extended version of X^ε in (2.2.1) and set $\tilde{\mathbf{P}}^\varepsilon = \mathrm{Law}(\tilde{X}^\varepsilon \mid W^0)$. Then, the family of random tuples $\{(\tilde{\mathbf{P}}^\varepsilon, W^0, W)\}_{\varepsilon>0}$ is tight. Any subsequence $\{(\tilde{\mathbf{P}}^{\varepsilon_n}, W^0, W)\}_{n\ge1}$, for a positive sequence $(\varepsilon_n)_{n\ge1}$ which converges to zero, has a further subsequence which converges*

weakly to some $(\mathbf{P}, W^0, W)$. *Here,* W^0 *and* W *are standard Brownian motions,* $\mathbf{P}$ *is a random probability measure* $\mathbf{P} : \Omega \to \mathcal{P}(D_{\mathbb{R}})$, *and* $(\mathbf{P}, W^0)$ *is independent of* W.

Given this limit point, there is a background space which preserves the independence and carries a stochastic process X *such that* $(X, W^0, W, \mathbf{P})$ *is a relaxed solution to* (2.2.4). *Moreover, we have the upper bound*

$$\Delta L_t \leq \inf\{x \geq 0 : \boldsymbol{\nu}_{t-}([0, \alpha(t)x]) < x\} \qquad \text{a.s.} \tag{2.2.5}$$

for all $t \geq 0$.

The notation $\text{Law}(\tilde{X}^\varepsilon \mid W^0)$ stands for the conditional law of $\tilde{X}^\varepsilon$ given W^0, which indeed defines a random W^0-measurable probability measure on $D([-1, T+1], \mathbb{R})$ ([35, Theorem 8.5]). Under stronger assumptions, namely b, σ, and ρ being of the form $(t, x) \mapsto b(t, x)$, $t \mapsto \sigma(t)$, $t \mapsto \rho(t)$ and α is a positive constant, there are established results in the literature for a lower bound on the jumps of the loss function. By Proposition 3.5 in [48], the jumps of the loss satisfy

$$\Delta L_t \geq \inf\{x \geq 0 : \boldsymbol{\nu}_{t-}([0, \alpha x]) < x\} \qquad \text{a.s.}$$

Due to the generality of the coefficients, we were not able to establish if (2.2.5) holds with equality. The primary reason is the lack of independence between the term driven by the idiosyncratic noise and the remainder of the terms that X is composed of. Hence, the technique employed in [48, Proposition 3.5] may not be readily applied or extended to our setting. Regardless, given these two results, under stronger assumptions, we have the following existence result.

Corollary 2.2.5 (Existence of physical solutions)**.** *Let the coefficients* b, σ, *and* ρ *be of the form* $(t, x) \mapsto b(t, x)$, $t \mapsto \sigma(t)$, $t \mapsto \rho(t)$ *and satisfy Assumption 2.2.2. Then provided* $\alpha(t) \equiv \alpha > 0$ *and constant, there exists a relaxed solution to*

$$\begin{cases} \mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t)\sqrt{1-\rho(t)^2}\,\mathrm{d}W_t + \sigma(t)\rho(t)\,\mathrm{d}W_t^0 - \alpha\,\mathrm{d}L_t, \\ \quad \tau = \inf\{t > 0 \ : \ X_t \leq 0\}, \\ \quad \mathbf{P} = \mathbb{P}\left[X \in \cdot \,|\, W^0, \mathbf{P}\right], \quad \boldsymbol{\nu}_t \coloneqq \mathbb{P}\left[X_t \in \cdot, \tau > t \,|\, W^0, \mathbf{P}\right], \\ \quad L_t = \mathbb{P}\left[\tau \leq t \,|\, W^0, \mathbf{P}\right]. \end{cases} \tag{2.2.6}$$

Moreover, we have the minimal jump constraint

$$\Delta L_t = \inf\{x \geq 0 : \boldsymbol{\nu}_{t-}([0, \alpha x]) < x\} \qquad \text{a.s.}$$

for all $t \geq 0$. *This determines the jump sizes of* L.

This work presents a minor generalisation of the results in [48]. In their work, the authors imposed the condition that $t \mapsto \sigma(t)\rho(t)$ must be Hölder continuous with an exponent strictly greater than $1/2$. Here we have made no explicit assumptions on the regularity of ρ, only requiring it to be non-degenerate. Consequently, we can consider Corollary 2.2.5 as an extension to Theorem 3.2 in [48].

2.3 Limit Points of the Smoothed System

In order to show the existence of a limit point of X^ε, we must first choose a suitable topology to establish convergence. By Theorem 2.4 in [31], L^ε is continuous for every $\varepsilon > 0$, but the loss of the limiting process may in fact jump. Skorokhod's M_1-topology is sufficiently rich to facilitate the convergence of continuous functions to those with jumps.

Convergence in the M_1-topology implies that we will have pointwise convergence at time $t = 0$ and $t = T$, [71, Theorem 12.5.1]. As we are starting from an arbitrary initial condition X_{0-} which is positive almost surely, the limiting process may exhibit a jump immediately at time $t = 0$ given sufficient mass near the boundary, while the approximating processes are equal to X_{0-} at $t = 0$. Furthermore, the limiting process may exhibit a jump at time $t = T$. For this reason, we shall embed the process from $D([0, T], \mathbb{R})$ into $D([-1, \bar{T}], \mathbb{R})$, where $\bar{T} = T + 1$, using the extension defined in (2.2.3). Unless stated otherwise, for notational convenience, we shall denote the latter space, $D([-1, \bar{T}], \mathbb{R})$, by $D_{\mathbb{R}}$. Recall that $\tilde{\mathbf{P}}^\varepsilon$ is defined to be the law of $\tilde{X}^\varepsilon$ conditional on W^0. That is, $\tilde{\mathbf{P}}^\varepsilon := \mathrm{Law}(\tilde{X}^\varepsilon \mid W^0)$.

To show tightness and convergence of the collection of random measures $\{\tilde{\mathbf{P}}^\varepsilon\}_{\varepsilon>0}$, we shall follow the ideas in [16] and [48]. To begin, we first derive a Grönwall-type estimate of the smoothed system uniformly in ε. These estimates are necessary to show the tightness of $\{\tilde{\mathbf{P}}^\varepsilon\}_{\varepsilon>0}$ and the existence of a limiting random measure. In the following Proposition and its sequels, C will denote a constant that may change from line to line, and we will denote the dependencies of the value of C in its subscript. To further simplify notation, we use Y_t^ε, $Y_t^{0,\varepsilon}$, and $\mathcal{Y}_t^\varepsilon$ to denote

$$\int_0^t \sigma(u, X_u^\varepsilon)\sqrt{1 - \rho^2(u, \boldsymbol{\nu}_u^\varepsilon)}\, \mathrm{d}W_u, \qquad \int_0^t \sigma(u, X_u^\varepsilon)\rho(u, \boldsymbol{\nu}_u^\varepsilon)\, \mathrm{d}W_u^0, \quad \text{and} \quad Y_t^\varepsilon + Y_t^{0,\varepsilon}$$

respectively. We shall use $\tilde{Y}_t^\varepsilon$, $\tilde{Y}_t^{0,\varepsilon}$, and $\tilde{\mathcal{Y}}_t^\varepsilon$ to denote their corresponding extensions as defined in (2.2.3).

Proposition 2.3.1 (Grönwall upper bound)**.** *For any $p \geq 1$ and $t \leq T$, there exists a constant $C_{\alpha,b,p,T,\sigma} > 0$ independent of $\varepsilon > 0$ such that*

$$\mathbb{E}\left[\sup_{s\leq T}|X_s^\varepsilon|^p\right] \leq C_{\alpha,b,p,T,\sigma}. \tag{2.3.1}$$

Proof. By the linear growth condition on b and the triangle inequality, we have

$$|X_t^\varepsilon| \leq |X_{0-}| + C_b\int_0^t 1 + \sup_{u\leq s}|X_u^\varepsilon| + \mathbb{E}\left[|X_{s\wedge\tau^\varepsilon}^\varepsilon|\,|\,W^0\right]\mathrm{d}s + \sup_{s\leq T}|\mathcal{Y}_s^\varepsilon| + \|\alpha\|_\infty.$$

By [31, Lemma A.3], $\int_0^T \mathbb{E}\,|X_{s\wedge\tau^\varepsilon}^\varepsilon|^p\,\mathrm{d}s < \infty$ for any $p \geq 1$. Therefore, a simple application of Grönwall's inequality shows that

$$\sup_{s\leq t}|X_s^\varepsilon| \leq C_{T,b,\alpha}\left(|X_{0-}| + \int_0^t \mathbb{E}\left[|X_{s\wedge\tau^\varepsilon}^\varepsilon|\,|\,W^0\right]\mathrm{d}s + \sup_{s\leq T}|\mathcal{Y}_s^\varepsilon| + 1\right).$$

By (vi) in Assumption 2.2.2, X_{0-} has finite L^p-moments for every $p > 0$. Furthermore, by employing the Burkholder-Davis-Gundy inequality to control $\mathbb{E}[\sup_{s\leq T}|\mathcal{Y}_s^\varepsilon|]$, we may deduce that $\mathbb{E}[\sup_{s\leq t}|X_s^\varepsilon|^p] < \infty$ for all $t \geq 0$ and $p \geq 1$. Now, observing that $|X_{t\wedge\tau^\varepsilon}^\varepsilon| \leq \sup_{s\leq t}|X_s^\varepsilon|$, we have by the monotonicity of expectation and Jensen's inequality that

$$\begin{aligned}\sup_{s\leq t}|X_s^\varepsilon|^p &\leq C_{T,b,\alpha}^p\left(|X_{0-}| + \int_0^t \mathbb{E}\left[|X_{s\wedge\tau^\varepsilon}^\varepsilon|\,|\,W^0\right]\mathrm{d}s + \sup_{s\leq T}|\mathcal{Y}_s^\varepsilon| + 1\right)^p \\ &\leq C_{T,b,\alpha,p}\left(|X_{0-}|^p + \int_0^t \mathbb{E}\left[\sup_{u\leq s}|X_u^\varepsilon|^p\,\middle|\,W^0\right]\mathrm{d}s + \sup_{s\leq T}|\mathcal{Y}_s^\varepsilon|^p + 1\right).\end{aligned}$$

Taking expectations and applying Fubini's theorem followed by Grönwall's inequality, we obtain

$$\mathbb{E}\left[\sup_{s\leq T}|X_s^\varepsilon|^p\right] \leq C_{T,b,\alpha,p}\mathbb{E}\left[|X_{0-}|^p + \sup_{s\leq T}|\mathcal{Y}_s^\varepsilon|^p + 1\right]. \tag{2.3.2}$$

Lastly, by the Burkholder-Davis-Gundy inequality and (ii) from Assumption 2.2.2, we may bound (2.3.2) independently of ε. This completes the proof. □

The collection of measures $\{\tilde{\mathbf{P}}^\varepsilon\}_{\varepsilon>0}$ are $\mathcal{P}(D_\mathbb{R})$-valued random measures. To show that this collection of random variables is tight, we need to look for compact sets in $\mathcal{P}(\mathcal{P}(D_\mathbb{R}))$. In fact, it will be sufficient to show that $\{\mathrm{Law}(\tilde{X}^\varepsilon)\}_{\varepsilon>0}$ is tight in $\mathcal{P}(D_\mathbb{R})$. Due to the extension of the process, the tightness of the collection of measures $\{\mathrm{Law}(\tilde{X}^\varepsilon)\}_{\varepsilon>0}$ follows easily from the properties of the M_1-topology and [1, Theorem 1].

Proposition 2.3.2 (Tightness of smoothed random measures)**.** *Let $\mathfrak{T}_{M_1}^{wk}$ denote the topology of weak convergence on $\mathcal{P}(D_{\mathbb{R}})$ induced by the M_1-topology on $D_{\mathbb{R}}$. Then, the collection $\{\mathrm{Law}(\tilde{X}^\varepsilon \mid W^0)\}_{\varepsilon>0}$ is tight on $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{wk})$ under Assumption 2.2.2.*

Proof. Define $\tilde{P}^\varepsilon := \mathrm{Law}(\tilde{X}^\varepsilon)$. By [71, Theorem 12.12.3], we need to verify two conditions to show the tightness of the measures on $D_{\mathbb{R}}$ endowed with the M_1-topology:

(i) $\lim_{\lambda\to\infty} \sup_{\varepsilon>0} \tilde{P}^\varepsilon \left(\{x \in D_{\mathbb{R}} \ : \ \|x\|_\infty > \lambda\}\right) = 0.$

(ii) For any $\eta > 0$, we have $\lim_{\delta\to 0} \sup_{\varepsilon>0} \tilde{P}^\varepsilon \left(\{x \in D_{\mathbb{R}} \ : \ w_{M_1}(x,\delta) \geq \eta\}\right) = 0$, where w_{M_1} is the oscillatory function of the M_1-topology, defined as in [71, Section 12, Equation 12.2].

To show the first condition, we observe that by the definition of the extension of our process, we have

$$\sup_{t\leq \bar{T}} |\tilde{X}_t^\varepsilon| \leq \sup_{t\leq T} |X_t^\varepsilon| + \sup_{t\leq 1} |W_{T+t} - W_T| + \|\alpha\|_\infty .$$

Then, it is clear by Markov's inequality and Proposition 2.3.1 that for any $\lambda > 0$,

$$\mathbb{P}\left[\sup_{t\leq \bar{T}} |\tilde{X}_t^\varepsilon| > \lambda\right] = O(\lambda^{-1})$$

uniformly in ε. Therefore, by taking the supremum over ε and then the lim sup over λ, the first condition holds. We shall not show the second condition directly. By [1, Theorem 1], the second condition is equivalent to showing:

(I) There exists some $C > 0$, uniformly in ε, such that $\mathbb{P}\left[\mathcal{H}_{\mathbb{R}}(\tilde{X}_{t_1}^\varepsilon, \tilde{X}_{t_2}^\varepsilon, \tilde{X}_{t_3}^\varepsilon) \geq \eta\right] \leq C\eta^{-4}|t_3 - t_1|^2$ for all $\eta > 0$ and $-1 \leq t_1 \leq t_2 \leq t_3 \leq \bar{T}$ where $\mathcal{H}_{\mathbb{R}}(x_1, x_2, x_3) = \inf_{\lambda\in[0,1]} |x_2 - (1-\lambda)x_1 - \lambda x_3|$.

(II) $\lim_{\delta\to 0} \sup_{\varepsilon>0} \mathbb{P}\left[\sup_{t\in(-1,-1+\delta)} |\tilde{X}_t^\varepsilon - \tilde{X}_{-1}| + \sup_{t\in(\bar{T}-\delta,\bar{T})} |\tilde{X}_t^\varepsilon - \tilde{X}_{\bar{T}}| \geq \eta\right] = 0$ for all $\eta > 0$.

Note that by Assumption 2.2.2, we have that α is non-decreasing and non-negative. Therefore, by the properties of Lebesgue-Stieltjes integration, $t \mapsto \int_0^t \alpha(s)\,\mathrm{d}\mathfrak{L}_s^\varepsilon$ is non-decreasing. As monotone functions are immaterial to the M_1-modulus of continuity, we have

$$\mathcal{H}_{\mathbb{R}}(\tilde{X}_{t_1}^\varepsilon, \tilde{X}_{t_2}^\varepsilon, \tilde{X}_{t_3}^\varepsilon) \leq |H_{t_1} - H_{t_2}| + |H_{t_2} - H_{t_3}|,$$

where H is given by

$$H_t = X_{0-} + \int_0^{t\wedge T} b(u, X_u^\varepsilon, \boldsymbol{\nu}_u^\varepsilon)\,\mathrm{d}u + \tilde{\mathcal{Y}}_t$$

for $t \geq 0$ and $H_t = X_{0-}$ for $t < 0$. Hence, to show (I), it is sufficient to bound the increments of H. Note that when $s < t < -1$, H is constant. Therefore, trivially we have $\mathbb{E}\left[|H_t - H_s|^4\right] \leq C(t-s)^2$ for any $C > 0$. When $0 \leq s < t$, by the formula above for H we have that

$$H_t - H_s = \int_{s\wedge T}^{t\wedge T} b(u, X_u^\varepsilon, \boldsymbol{\nu}_u^\varepsilon)\,\mathrm{d}u + \tilde{\mathcal{Y}}_t - \tilde{\mathcal{Y}}_s. \tag{2.3.3}$$

Employing the linear growth condition on b and Proposition 2.3.1,

$$\mathbb{E}\left[\left|\int_{s\wedge T}^{t\wedge T} b(u, X_u^\varepsilon, \boldsymbol{\nu}_u^\varepsilon)\,\mathrm{d}u\right|^4\right] \leq C(t-s)^4\left(1 + \mathbb{E}\left[\sup_{u\leq T}|X_u^\varepsilon|^4\right]\right) = O((t-s)^2) \tag{2.3.4}$$

uniformly in ε. By the Burkholder-Davis-Gundy inequality and the upper bound on σ, it is clear that

$$\mathbb{E}\left[\left|\tilde{\mathcal{Y}}_t - \tilde{\mathcal{Y}}_s\right|^4\right] = O((t-s)^2) \tag{2.3.5}$$

uniformly in ε. Therefore, by Markov's inequality,

$$\begin{aligned}
\mathbb{P}\left[\mathcal{H}_{\mathbb{R}}(\tilde{X}_{t_1}^\varepsilon, \tilde{X}_{t_2}^\varepsilon, \tilde{X}_{t_3}^\varepsilon) \geq \eta\right] &\leq \eta^{-4}\mathbb{E}\left[\mathcal{H}_{\mathbb{R}}(\tilde{X}_{t_1}^\varepsilon, \tilde{X}_{t_2}^\varepsilon, \tilde{X}_{t_3}^\varepsilon)^4\right] \\
&\leq C\eta^{-4}\mathbb{E}\left[|H_{t_1} - H_{t_2}|^4 + |H_{t_2} - H_{t_3}|^4\right] \\
&\leq C\eta^{-4}\left((t_2 - t_1)^2 + (t_3 - t_2)^2\right) \\
&\leq C\eta^{-4}(t_3 - t_1)^2,
\end{aligned}$$

where all the constants hold uniformly in ε. To verify the second condition, we observe that for any $\eta > 0$ and $\delta < 1$,

$$\begin{aligned}
\mathbb{P}\left[\sup_{t\in(-1,-1+\delta)}|\tilde{X}_t^\varepsilon - \tilde{X}_{-1}^\varepsilon| \geq \frac{\eta}{2}\right] &= \mathbb{P}\left[\sup_{t\in(-1,-1+\delta)}|X_{0-} - X_{0-}| \geq \frac{\eta}{2}\right] = 0, \text{ and} \\
\mathbb{P}\left[\sup_{t\in(\bar{T}-\delta,\bar{T})}|\tilde{X}_t^\varepsilon - \tilde{X}_{\bar{T}}| \geq \frac{\eta}{2}\right] &= \mathbb{P}\left[\sup_{t\in(\bar{T}-\delta,\bar{T})}|W_t - W_{\bar{T}}| \geq \frac{\eta}{2}\right] = O(\delta^2),
\end{aligned}$$

uniformly in ε. Hence, we have shown that

$$\mathbb{P}\left[\sup_{t\in(-1,-1+\delta)}|\tilde{X}_t^\varepsilon - \tilde{X}_{-1}^\varepsilon| + \sup_{t\in(\bar{T}-\delta,\bar{T})}|\tilde{X}_t^\varepsilon - \tilde{X}_{\bar{T}}^\varepsilon| \geq \eta\right] = O(\delta^2) \quad \text{for all} \quad \eta > 0,\, \delta < 1.$$

Therefore, together, conditions (I) and (II) show

$$\sup_{\varepsilon>0}\tilde{P}^\varepsilon\left(\{x \in D_{\mathbb{R}} \,:\, w_{M_1}(x,\delta) \geq \eta\}\right) = O(\delta^2),$$

for all $\delta < 1$ uniformly in ε. This shows condition (ii). Lastly, we shall employ Markov's inequality and Prokhorov's theorem to construct a compact set in $\mathcal{P}(\mathcal{P}(D_{\mathbb{R}}))$

to conclude that $\left\{\text{Law}(\tilde{\mathbf{P}}^\varepsilon)\right\}_{\varepsilon>0}$ is tight. To begin, fix a $\tilde{\delta} > 0$. Now for any $l, k \in \mathbb{N}$, we may find a λ_l, $\delta_{k,l} > 0$ such that

$$\tilde{P}^\varepsilon(A_{k,l}^\complement) < \tilde{\delta} 2^{-(k+2l+1)} \quad \forall k \in \mathbb{N} \qquad \text{uniformly in } \varepsilon,$$

where

$$A_{0,l} = \{x \in D_{\mathbb{R}} \ : \ \|x\| \leq \lambda_l\},$$

$$A_{k,l} = \left\{x \in D_{\mathbb{R}} \ : \ w_{M_1}(x, \delta_{k,l}) < \frac{1}{k+2l}\right\}.$$

We define $A_l = \cap_{k\geq 0} A_{k,l}$. By [71, Theorem 12.12.2], A_l has compact closure in the M_1-topology. The closure of A_l is denoted by $\bar{A}_l$. Furthermore, by construction $\tilde{P}^\varepsilon(A_l^\complement) \leq \sum_{k\geq 0} \tilde{P}^\varepsilon(A_{k,l}^\complement) \leq \tilde{\delta} 4^{-l}$. By the subadditivity of measures and Markov's inequality,

$$\mathbb{P}\left[\bigcup_{l=1}^{\infty}\left\{\tilde{\mathbf{P}}^\varepsilon(A_l^\complement) > 2^{-l}\right\}\right] \leq \sum_{l\geq 1} \mathbb{P}\left[\tilde{\mathbf{P}}^\varepsilon(A_l^\complement) > 2^{-l}\right] \leq \sum_{l\geq 1} 2^l \mathbb{E}\left[\tilde{\mathbf{P}}^\varepsilon(A_l^\complement)\right] \leq \sum_{l\geq 1} \frac{\tilde{\delta} 2^l}{4^l} = \tilde{\delta}. \tag{2.3.6}$$

Finally, we consider the set $K \coloneqq \{\mu \in \mathcal{P}(D_{\mathbb{R}}) \ : \ \mu(\bar{A}_l^\complement) \leq 2^{-l} \, \forall \, l \in \mathbb{N}\}$. As $D_{\mathbb{R}}$ endowed with the M_1-topology is a Polish space, Prokhorov's theorem may be applied, and it will be sufficient to show that the set of measures K is tight; hence, K will then have compact closure in $\mathcal{P}(D_{\mathbb{R}})$ by Prokhorov's theorem. It is clear by construction that the set of measures K is tight as the sets $\bar{A}_l$ are compact in $D_{\mathbb{R}}$ endowed with the M_1-topology. By (2.3.6), we have $\text{Law}(\tilde{\mathbf{P}}^\varepsilon)(\bar{K}^\complement) \leq \tilde{\delta}$, uniformly in ε. As $\tilde{\delta}$ was arbitrary, this completes the proof. □

2.4 Continuity of Hitting Times

Note that $(D_{\mathbb{R}}, M_1)$ is a Polish space by [71, Theorem 12.8.1] and its Borel σ-algebra is generated by the marginal projections, [71, Theorem 11.5.2]. Hence, the topological space $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{\text{wk}})$ is also a Polish space. Therefore, by invoking Prokhorov's Theorem, [3, Theorem 5.1], tightness is equivalent to being sequentially pre-compact. So, we may choose a weakly convergent subsequence $\{\tilde{\mathbf{P}}^{\varepsilon_n}\}_{n\geq 1}$ for a positive sequence $(\varepsilon_n)_{n\geq 1}$ which converges to zero. Let $\mathbf{P}^*$ denote the limit point of this sequence. Using this limit point, we will construct a probability space and a stochastic process that will be a solution to (2.2.4).

Before proceeding, we seek to show that for a co-countable set of times t, $L_t^{\varepsilon_n} = \tilde{\mathbf{P}}^{\varepsilon_n}(\tau_0 \leq t)$ converges weakly to $L_t^* \coloneqq \mathbf{P}^*(\tau_0 \leq t)$, where τ_0 is a function on $D_{\mathbb{R}}$ whose value is the first hitting time of 0. To be explicit, we define

$$\mathbb{T} \coloneqq \left\{t \in [-1, \bar{T}] \ : \ \mathbb{E}\left[\mathbf{P}^*(\eta_t = \eta_{t-})\right] = 1, \mathbb{E}\left[\mathbf{P}^*(\tau_0 = t)\right] = 0\right\},$$

and

$$\tau_0(\eta) \coloneqq \inf\{t \geq -1 \,:\, \eta_t \leq 0\} \tag{2.4.1}$$

with the convention that $\inf\{\emptyset\} = \bar{T}$. Our first result is that for $\mathrm{Law}(\mathbf{P}^*)$-almost every measure μ, μ-almost every path $\eta \in D_{\mathbb{R}}$ is constant on the interval $[-1, 0)$.

Lemma 2.4.1. *For* $\mathrm{Law}(\mathbf{P}^*)$*-almost every measure* μ*, we have that* $\sup_{s<0} |\eta_s - \eta_{-1}| = 0$ *for* μ*-almost every path* η*.*

Proof. As $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}^{\mathrm{wk}}_{M_1})$ is a Polish space, we may apply Skorokhod's Representation Theorem [3, Theorem 6.7]. Hence, there exists a common probability space, and $\mathcal{P}(D_{\mathbb{R}})$-valued random variables $(\mathbf{Q}^n)_{n\geq 1}$ and $\mathbf{Q}^*$ such that

$$\mathrm{Law}(\mathbf{Q}^n) = \mathrm{Law}(\tilde{\mathbf{P}}^{\varepsilon_n}), \qquad \mathrm{Law}(\mathbf{Q}^*) = \mathrm{Law}(\mathbf{P}^*), \quad \text{and} \quad \mathbf{Q}^n \to \mathbf{Q}^* \quad \text{a.s.}$$

It is straightforward to see, by [71, Theorem 13.4.1], that the following maps from $D_{\mathbb{R}}$ into itself

$$\eta \mapsto \left(t \mapsto \inf_{s\leq t}\{\eta_s - \eta_{-1}\}\right) \qquad \eta \mapsto \left(t \mapsto \sup_{s\leq t}\{\eta_s - \eta_{-1}\}\right)$$

are continuous. Now, for a $t \in \mathbb{T} \cap (-1, 0)$, the maps c_t and $\tilde{c}_t$ from $D_{\mathbb{R}}$ onto $\mathbb{R}$ such that

$$c_t(\eta) = \inf_{s\leq t}\{\eta_s - \eta_{-1}\} \qquad \tilde{c}_t(\eta) = \sup_{s\leq t}\{\eta_s - \eta_{-1}\}$$

are continuous. Therefore, by the Continuous Mapping Theorem, $c_t^{\#}\mathbf{Q}^n \to c_t^{\#}\mathbf{Q}^*$ and $\tilde{c}_t^{\#}\mathbf{Q}^n \to \tilde{c}_t^{\#}\mathbf{Q}^*$ almost surely in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}^{\mathrm{wk}}_{M_1})$.

Fix a $\delta > 0$. Then by the Portmanteau Theorem and Fatou's lemma, we have

$$\begin{aligned}
\mathbb{E}\left[\mathbf{P}^*\left(\inf_{s\leq t}\{\eta_s - \eta_{-1}\} < -\delta\right)\right] &= \mathbb{E}\left[\mathbf{Q}^*\left(\inf_{s\leq t}\{\eta_s - \eta_{-1}\} < -\delta\right)\right] \\
&\leq \mathbb{E}\left[\liminf_{n\to\infty} \mathbf{Q}^n\left(\inf_{s\leq t}\{\eta_s - \eta_{-1}\} < -\delta\right)\right] \\
&\leq \liminf_{n\to\infty} \mathbb{E}\left[\mathbf{Q}^n\left(\inf_{s\leq t}\{\eta_s - \eta_{-1}\} < -\delta\right)\right] \\
&= \liminf_{n\to\infty} \mathbb{E}\left[\tilde{\mathbf{P}}^{\varepsilon_n}\left(\inf_{s\leq t}\{\eta_s - \eta_{-1}\} < -\delta\right)\right] \\
&= \liminf_{n\to\infty} \tilde{P}^{\varepsilon_n}\left(\inf_{s\leq t}\{\eta_s - \eta_{-1}\} < -\delta\right) \\
&= 0,
\end{aligned}$$

where the last equality follows from the embedding of X^{ε_n} from $D([0,T],\mathbb{R})$ into $D([-1,\bar{T}],\mathbb{R})$. So, by continuity of measure and the Monotone Convergence Theorem, as δ was arbitrary, we have

$$\mathbb{E}\left[\mathbf{P}^*\left(\inf_{s\leq t}\{\eta_s-\eta_{-1}\}<0\right)\right]=0.$$

Similarly, $\mathbb{E}\left[\mathbf{P}^*\left(\sup_{s\leq t}\{\eta_s-\eta_{-1}\}>0\right)\right]=0.$ □

As $\tilde{X}^\varepsilon$ is fundamentally a time-changed Brownian motion with drift, it is not hard to show that, with probability one, $\tilde{X}^\varepsilon$ will take a negative value on any open neighbourhood of its first hitting time of zero. This property is preserved by weak convergence for almost every realisation of $\mathbf{P}^*$. Furthermore, as the Lebesgue-Stieltjes integral $\int_0^t \alpha(s)\,\mathrm{d}\mathfrak{L}_s^\varepsilon$ takes non-negative values, by weak convergence we expect that for Law($\mathbf{P}^*$)-almost every measure μ, μ-almost every path $\eta\in D_{\mathbb{R}}$ will have only downward jumps.

Lemma 2.4.2 (Strong crossing property)**.** *For any $h>0$, we have*

$$\mathbb{E}\left[\mathbf{P}^*\left(\inf_{s\in(\tau_0,(\tau_0+h)\wedge\bar{T})}\{\eta_s-\eta_{\tau_0}\}\geq 0,\ \tau_0<\bar{T}\right)\right]=0, \tag{2.4.2}$$

$$\mathbb{E}\left[\mathbf{P}^*\left(\eta\,:\,\Delta\eta_t\leq 0\quad\forall\, t\leq\bar{T}\right)\right]=1. \tag{2.4.3}$$

Proof. As with the space of càdlàg functions, we shall employ the shorthand notation $\mathcal{C}_{\mathbb{R}}$ for this proof to denote $\mathcal{C}([-1,\bar{T}],\mathbb{R})$. Now, as σ is non-degenerate and bounded by assumption, by the Kolmogorov-Chentsov Tightness Criterion, [43] and [12], we have that $(\tilde{\mathcal{Y}}^\varepsilon)_{\varepsilon>0}$ is tight. Additionally, we define the random variable $\tilde{H}^\varepsilon:=\langle\tilde{\mathbf{P}}^\varepsilon,\sup_{u\leq\bar{T}}|\eta_u|\rangle$. By definition of $\tilde{\mathbf{P}}^\varepsilon$, $\mathbb{E}[\tilde{H}^\varepsilon]=\mathbb{E}[\sup_{u\leq\bar{T}}|\tilde{X}_u^\varepsilon|]$. Therefore, $\mathbb{E}[\tilde{H}^\varepsilon]$ is uniformly bounded by Proposition 2.3.1 and hence $\{\tilde{H}^\varepsilon\}_{\varepsilon>0}$ is tight on $\mathbb{R}$.

As marginal tightness implies joint tightness, we have $\mathbb{P}^\varepsilon_{x,y,z}:=\mathrm{Law}(\tilde{X}^\varepsilon,\tilde{\mathcal{Y}}^\varepsilon,\tilde{H}^\varepsilon)$ is tight in $\mathcal{P}(D_{\mathbb{R}}\times\mathcal{C}_{\mathbb{R}}\times\mathbb{R})$ by Proposition 2.3.2. Given a suitable subsequence, also denoted by $(\varepsilon_n)_{n\geq1}$ for simplicity, we have $\tilde{\mathbf{P}}^{\varepsilon_n}\implies\mathbf{P}^*$ and $\mathbb{P}^{\varepsilon_n}_{x,y,z}\implies\mathbb{P}^*_{x,y,z}$. Here $\mathbb{P}^*_x$ and $\mathbb{P}^*_y$ are used to denote the first and second marginal respectively.

Intuitively, $\mathbb{E}[\mathbf{P}^*(\cdot)]$ and $\mathbb{P}^*_x$ should have the same law as we are averaging over the stochasticity inherited by the common noise. By definition of $\mathbf{P}^\varepsilon$ and $\mathbb{P}^\varepsilon_{x,y,z}$, for any continuous bounded function $f:D_{\mathbb{R}}\to\mathbb{R}$, we have that

$$\mathbb{E}\left[\langle\mathbf{P}^{\varepsilon_n},\,f\rangle\right]=\langle\mathbb{P}^{\varepsilon_n}_{x,y,z},\,f\rangle.$$

As $D_{\mathbb{R}}$ is a Polish space, by a Monotone Class Theorem argument and Dynkin's Lemma, we have

$$\mathbb{E}\left[\mathbf{P}^*(A)\right] = \mathbb{P}^*_x(A) = \mathbb{P}^*_{x,y,z}(A \times \mathcal{C}_{\mathbb{R}} \times \mathbb{R}) \qquad \forall\, A \in \mathcal{B}(D_{\mathbb{R}}). \tag{2.4.4}$$

Define the canonical processes X^*, Y^* and H^* on $(D_{\mathbb{R}}, M_1) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty) \times (\mathbb{R}, |\cdot|)$, where for $(\eta, \omega, z) \in D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}} \times \mathbb{R}$, $X^*(\eta, \omega, z) = \eta$, $Y^*(\eta, \omega, z) = \omega$ and $H^*(\eta, \omega, z) = z$. By considering the parametric representations, the map $\eta \mapsto \sup_{u \le \bar{T}} |\eta_u|$ is M_1-continuous for any $\eta \in D_{\mathbb{R}}$. Hence, by the linear growth condition on b, the Continuous Mapping Theorem, and the Portmanteau Theorem, for any s, $t \in \mathbb{T}$ with $s < t$ and $\gamma > 0$,

$$\begin{aligned}
&\mathbb{P}^*_{x,y,z}\left(X^*_t - X^*_s \le Y^*_t - Y^*_s + C_b(t-s)(1 + \sup_{u \le \bar{T}} |X^*_u| + H^*) + \gamma\right) \\
&\quad \ge \limsup_{n \to \infty} \mathbb{P}\left[\tilde{X}^{\varepsilon_n}_t - \tilde{X}^{\varepsilon_n}_s \le \tilde{\mathcal{Y}}^{\varepsilon_n}_t - \tilde{\mathcal{Y}}^{\varepsilon_n}_s + C_b(t-s)(1 + \sup_{u \le \bar{T}} |\tilde{X}^{\varepsilon_n}_u| + \tilde{H}^{\varepsilon_n}) + \gamma\right] \\
&\quad = 1.
\end{aligned} \tag{2.4.5}$$

The last equality follows from the fact that for any ε_n

$$\begin{aligned}
\tilde{X}^{\varepsilon_n}_t - \tilde{X}^{\varepsilon_n}_s &= \tilde{\mathcal{Y}}^{\varepsilon_n}_t - \tilde{\mathcal{Y}}^{\varepsilon_n}_s + \int_{(s \vee 0) \wedge T}^{(t \vee 0) \wedge T} b(u, X^{\varepsilon_n}_u, \boldsymbol{\nu}^{\varepsilon_n}_u) \mathbb{1}_{[0,T]}(u)\, \mathrm{d}u - \int_{(s \vee 0) \wedge T}^{(t \vee 0) \wedge T} \alpha(u)\, \mathrm{d}\mathfrak{L}^{\varepsilon_n}_u \\
&\le \tilde{\mathcal{Y}}^{\varepsilon_n}_t - \tilde{\mathcal{Y}}^{\varepsilon_n}_s + C_b(t-s)\left(1 + \sup_{u \le \bar{T}} |\tilde{X}^{\varepsilon_n}_u| + \tilde{H}^{\varepsilon_n}\right).
\end{aligned}$$

Sending $\gamma \to 0$ countably and employing the right continuity of X^* and Y^*, we deduce

$$X^*_t - X^*_s \le Y^*_t - Y^*_s + C_b(t-s)(1 + \sup_{u \le \bar{T}} |X^*_u| + H^*) \qquad \forall\, s < t \quad \mathbb{P}^*_{x,y,z}\text{-a.s.}$$

Furthermore, $\Delta X^*_t \le 0$ for all t $\mathbb{P}^*_{x,y,z}$-almost surely. By Lemma 2.6.1, Y^* is a continuous local martingale with respect to the filtration generated by (X^*, Y^*). It is clear that $\tau_0(X^*)$ is a stopping time with respect to the filtration generated by (X^*, Y^*). So the claim follows from Lemma 2.6.2 if $\tau_0(X^*) \ge 0$ and $\mathbb{E}[\sup_{u \le \bar{T}} |X^*_u| + H^*] < \infty$. For the former condition, it is sufficient to show

$$\mathbb{P}^*_x\left(\inf_{s<0} \eta_s \le 0\right) = 0.$$

As $\mathbb{P}^{\varepsilon_n}_x \implies \mathbb{P}^*_x$, then by Skorokhod's Representation Theorem, there exist (Z^n) and Z on a common probability space such that $\mathrm{Law}(Z^n) = \mathbb{P}^{\varepsilon_n}_x$, $\mathrm{Law}(Z) = \mathbb{P}^*_x$, and $Z^n \to Z$ almost surely in $(D_{\mathbb{R}}, M_1)$. By the Portmanteau Theorem, for any $\gamma > 0$

$$\mathbb{P}\left[Z_{-1} < \gamma\right] \le \liminf_{n \to \infty} \mathbb{P}\left[Z^n_{-1} < \gamma\right] = \mathbb{P}\left[X_{0-} < \gamma\right] = O(\gamma^{1/2}),$$

as X_{0-} has an L^2-density by Assumption 2.2.2 (vi). So $\mathbb{P}^*_x(\eta_{-1} \leq 0) = \mathbb{P}[Z_{-1} \leq 0] = 0$. By Lemma 2.4.1 and (2.4.4), we have

$$1 = \mathbb{E}\left[\mathbf{P}^*\left(\sup_{s<0}|\eta_s - \eta_{-1}| = 0\right)\right] = \mathbb{P}^*_x\left(\sup_{s<0}|\eta_s - \eta_{-1}| = 0\right).$$

Therefore, $X^*_s > 0$ for every $s \in [-1, 0)$ $\mathbb{P}^*_{x,y,z}$-almost surely. Hence, $\tau_0(X^*) \geq 0$ almost surely. Furthermore, as $\eta \mapsto \sup_{u\leq \bar{T}}|\eta_u|$ is an M_1-continuous map, $\mathbb{E}[\sup_{u\leq\bar{T}}|X^*_u| + H^*] < \infty$ follows from a simple application of the Continuous Mapping Theorem and Proposition 2.3.1. Therefore, we deduce,

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{P}^*\left(\inf_{s\in(\tau_0,(\tau_0+h)\wedge\bar{T})}\{\eta_s - \eta_{\tau_0}\} \geq 0,\, \tau_0 < \bar{T}\right)\right] \\
&\quad\leq \mathbb{P}^*_{x,y,z}\left(\inf_{s\in(\tau^*_0,(\tau^*_0+h)\wedge\bar{T})}\left\{Y^*_s - Y^*_{\tau^*_0} + C_b(s-\tau^*_0)(1+\sup_{u\leq\bar{T}}|X^*_u| + H^*)\right\} \geq 0,\, \tau^*_0 < \bar{T}\right) \\
&\quad= 0,
\end{aligned}$$

where $\tau^*_0 = \tau_0(X^*)$ and the final equality is due to Lemma 2.6.2. □

Now we have all the ingredients to show that τ_0 is an M_1-continuous map.

Corollary 2.4.3 (Hitting time continuity)**.** *For* Law$(\mathbf{P}^*)$*-almost every measure* μ*, we have that the hitting time map* $\tau_0 : D_{\mathbb{R}} \to \mathbb{R}$ *is continuous in the* M_1*-topology for* μ*-almost every* $\eta \in D_{\mathbb{R}}$*.*

Proof. By Lemma 2.4.2, for Law$(\mathbf{P}^*)$-almost every measure μ, μ-almost every path $\eta \in D_{\mathbb{R}}$ will have only downward jumps and one of the following conditions holds:

(i) $\tau_0(\eta) < \bar{T}$ and η takes a negative value on any neighbourhood of $\tau_0(\eta)$,

(ii) $\tau_0(\eta) = \bar{T}$ and $\inf_{s\leq\bar{T}}\eta_s > 0$,

(iii) $\tau_0(\eta) = \bar{T}$ and $\eta_{\bar{T}} = 0$.

If (i) holds, then by Lemma 2.6.3, τ_0 is M_1-continuous at η. If (ii) holds–$\tau_0(\eta) = \bar{T}$ and $\inf_{s\leq\bar{T}}\eta_s > 0$–then for any approximating sequence $(\eta^n)_{n\geq 1} \subset D_{\mathbb{R}}$ in the M_1-topology, we must have $\inf_{s\leq\bar{T}}\eta^n_s > 0$ eventually as the parametric representations get arbitrarily close in the uniform topology. Therefore, as $\inf_{s\leq\bar{T}}\eta^n_s > 0$ eventually, by definition $\tau_0(\eta^n) = \bar{T}$ eventually. Therefore, τ_0 is M_1-continuous at η. If (iii) holds, when $\tau_0(\eta) = \bar{T}$ and $\eta_{\bar{T}} = 0$, then for any $\gamma > 0$, with $\bar{T}-\gamma$ being a continuity point, we must have $\inf_{s\leq\bar{T}-\gamma}\eta_s > 0$ because η only jumps downwards. So for any approximating sequence $(\eta^n)_{n\geq 1} \subset D_{\mathbb{R}}$ in the M_1-topology, eventually $\inf_{s\leq\bar{T}-\gamma}\eta^n_s > 0$. Hence, eventually $\tau_0(\eta^n) > \bar{T} - \gamma$. As $\gamma > 0$ can be made arbitrarily close to zero, by definition, $\lim_{n\to\infty}\tau_0(\eta^n) = \tau_0(\eta) = \bar{T}$. Therefore, τ_0 is M_1-continuous at η. □

With the result stating that the hitting time is an M_1-continuous map, weak convergence of the loss function follows immediately.

Lemma 2.4.4 (Continuity of conditional feedback)**.** *For* Law($\mathbf{P}^*$)*-almost every measure* $\mu \in \mathcal{P}(D_{\mathbb{R}})$*, the map* $\mu \mapsto \mu(\tau_0 \leq t)$ *is continuous with respect to* $\mathfrak{T}^{wk}_{M_1}$ *for all* $t \in \mathbb{T}^\mu \cap [0, \bar{T})$*.* $\mathbb{T}^\mu$ *is the set of continuity points of* $t \mapsto \mu(\tau_0 \leq t)$*.*

Proof. By Corollary 2.4.3, for Law($\mathbf{P}^*$)-almost every measure μ, τ_0 is M_1-continuous for μ-almost every η. We fix such a μ and consider a sequence $(\mu^n)_{n\geq 1}$ converging to μ in $\mathcal{P}(D_{\mathbb{R}})$. By Skorokhod's Representation Theorem,

$$\mu^n(\tau_0 \leq t) = \mathbb{E}\left[\mathbb{1}_{\tau_0(H^n)\leq t}\right] \qquad \text{and} \qquad \mu(\tau_0 \leq t) = \mathbb{E}\left[\mathbb{1}_{\tau_0(H)\leq t}\right],$$

where τ_0 is continuous for almost all paths H and $H^n \to H$ almost surely in $(D_{\mathbb{R}}, M_1)$. Now, for any $t \in \mathbb{T}^\mu := \{t \in [-1, \bar{T}] : \mu(\tau_0 = t) = 0\}$, by the Monotone Convergence Theorem,

$$\mathbb{P}\left[\tau_0(H) = t\right] = \mu(\tau_0 \leq t) - \lim_{s\uparrow t} \mu(\tau_0 \leq s) = 0. \tag{2.4.6}$$

Therefore, employing the continuity of τ_0 and (2.4.6), we have

$$\mathbb{E}\left[\mathbb{1}_{\tau_0(H^n)\leq t}\right] \to \mathbb{E}\left[\mathbb{1}_{\tau_0(H)\leq t}\right],$$

by the Dominated Convergence Theorem. So, we conclude that

$$\mu^n(\tau_0 \leq t) \to \mu(\tau_0 \leq t) \qquad \forall\, t \in \mathbb{T}^\mu.$$

□

Furthermore, we have weak convergence of the mollified loss to the singular loss.

Corollary 2.4.5 (Convergence of delayed loss)**.** *For* Law($\mathbf{P}^*$)*-almost every measure* μ*,* $\int_0^t \mathfrak{K}^{\varepsilon_n}(t-s)\mu^n(\tau_0 \leq s)\,\mathrm{d}s$ *converges to* $\mu(\tau_0 \leq t)$ *for any* $t \in \mathbb{T}^\mu$ *and any sequence* $(\mu^n)_{n\geq 1}$ *that converges to* μ *in* $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}^{wk}_{M_1})$*.*

Proof. By Lemma 2.4.4, $\mu^n(\tau_0 \leq t)$ converges to $\mu(\tau_0 \leq t)$ for any $t \in \mathbb{T}^\mu$ when τ_0 is an M_1-continuous map μ-almost surely. Such measures μ have full Law($\mathbf{P}^*$)-measure by Corollary 2.4.3. Furthermore, for every such μ,

$$\left(s \mapsto \mu^n(\tau_0 \leq s)\mathbb{1}_{[0,t]}(s)\right) \xrightarrow{n\to\infty} \left(s \mapsto \mu(\tau_0 \leq s)\mathbb{1}_{[0,t]}(s)\right) \tag{2.4.7}$$

in the M_1-topology as functions from $[-1,\, t] \to \mathbb{R}$, as the functions are non-decreasing, [71, Corollary 12.5.1]. Now, for any $t \in \mathbb{T}^\mu \cap (0, \bar{T}]$,

$$\begin{aligned}
&\left| \int_0^t \mathfrak{K}^{\varepsilon_n}(s) \mu^n(\tau_0 \leq t - s)\, \mathrm{d}s - \mu(\tau_0 \leq t) \right| \\
&\qquad \leq \left| \int_0^t \mathfrak{K}^{\varepsilon_n}(s) (\mu^n(\tau_0 \leq t - s) - \mu(\tau_0 \leq t - s))\, \mathrm{d}s \right| \\
&\qquad\quad + \left| \int_0^t \mathfrak{K}^{\varepsilon_n}(s) (\mu(\tau_0 \leq t - s) - \mu(\tau_0 \leq t))\, \mathrm{d}s \right| \\
&\qquad\quad + \left| \int_0^t \mathfrak{K}^{\varepsilon_n}(s)\, \mathrm{d}s - 1 \right| \mu(\tau_0 \leq t) \\
&\qquad = I + II + III.
\end{aligned}$$

For any $\delta > 0$, we observe

$$\begin{aligned}
I &\leq \sup_{t-\delta \leq s \leq t} |\mu^n(\tau_0 \leq s) - \mu(\tau_0 \leq s)| \int_0^\delta \mathfrak{K}^{\varepsilon_n}(s)\, \mathrm{d}s + \int_\delta^\infty \mathfrak{K}^{\varepsilon_n}(s)\, \mathrm{d}s, \\
II &\leq \sup_{t-\delta \leq s \leq t} |\mu(\tau_0 \leq s) - \mu(\tau_0 \leq t)| \int_0^\delta \mathfrak{K}^{\varepsilon_n}(s)\, \mathrm{d}s + \int_\delta^\infty \mathfrak{K}^{\varepsilon_n}(s)\, \mathrm{d}s, \\
III &\leq \int_t^\infty \mathfrak{K}^{\varepsilon_n}(s)\, \mathrm{d}s.
\end{aligned}$$

As M_1-convergence implies local uniform convergence at continuity points, [71, Theorem 12.5.1], and t is a continuity point, by setting $\delta = \varepsilon_n^{1/2}$ and sending $n \to \infty$, we have that I, II, and III all go to zero. □

2.5 Martingale Arguments and Convergence

As marginal tightness implies joint tightness, $\{(\tilde{\mathbf{P}}^\varepsilon, W^0, W)\}_{\varepsilon > 0}$ is tight in $(\mathcal{P}(D_\mathbb{R}), \mathfrak{T}_{M_1}^{\mathrm{wk}}) \times (\mathcal{C}_\mathbb{R}, \|\cdot\|_\infty) \times (\mathcal{C}_\mathbb{R}, \|\cdot\|_\infty)$ where $(\mathcal{C}_\mathbb{R}, \|\cdot\|_\infty)$ is shorthand notation for $(\mathcal{C}([0,\, T]\,, \mathbb{R}), \|\cdot\|_\infty)$, the space of continuous functions from $[0,\, T]$ to $\mathbb{R}$ endowed with the topology of uniform convergence. From now on, we fix a weak limit point $(\mathbf{P}^*,\, W^0,\, W)$ along a subsequence $(\varepsilon_n)_{n \geq 1}$ for which ε_n converges to zero. Although we have fixed a limit point, all the following results will hold for any limit point.

Let $\mathbb{P}^n \coloneqq \mathrm{Law}(\tilde{\mathbf{P}}^{\varepsilon_n},\, W^0,\, W)$ and $\mathbb{P}^*_{\mu,\omega^0,\omega} \coloneqq \mathrm{Law}(\mathbf{P}^*,\, W^0,\, W)$. So $\mathbb{P}^n \implies \mathbb{P}^*_{\mu,\omega^0,\omega}$. For completeness, we will define the probability space $(\Omega^*,\, \mathcal{F}^*,\, \mathbb{P}^*_{\mu,\omega^0,\omega})$ where $\Omega^* = \mathcal{P}(D_\mathbb{R}) \times \mathcal{C}_\mathbb{R} \times \mathcal{C}_\mathbb{R}$ and $\mathcal{F}^*$ is the corresponding Borel σ-algebra. Define the random variables $\mathbf{P}^*$, W^0, and W on Ω^* such that for any tuple (μ, ω^0, ω),

$$\mathbf{P}^*(\mu, \omega^0, \omega) = \mu, \qquad W^0(\mu, \omega^0, \omega) = \omega^0, \qquad \text{and} \qquad W(\mu, \omega^0, \omega) = \omega.$$

Hence, the joint law of $(\mathbf{P}^*, W^0, W)$ is $\mathbb{P}^*_{\mu,\omega^0,\omega}$ and $\mathcal{F}^* = \sigma(\mathbf{P}^*, W^0, W)$. We also define the limiting loss function $L^* \coloneqq \mathbf{P}^*(\tau_0 \leq \cdot)$ and the co-countable set of times

$$\mathbb{T} \coloneqq \left\{ t \in [-1, \bar{T}] \,:\, \mathbb{P}^*_{\mu,\omega^0,\omega}(\eta_t = \eta_{t-}) = 1,\, \mathbb{P}^*_{\mu,\omega^0,\omega}(L^*_t = L^*_{t-}) = 1 \right\}. \tag{2.5.1}$$

Looking at the approximating system, we know $(\tilde{\mathbf{P}}^\varepsilon, W^0) \perp W$ for any $\varepsilon > 0$. Even though $\mathbf{P}^*$ is the weak limit of W^0-measurable random variables, weak convergence does not allow us to guarantee that limit points will be W^0-measurable. Regardless, we may exploit the independence from the approximating system to deduce the independence of $(\mathbf{P}^*, W^0)$ and W in the limit. To fix the notation, let $\mathbb{P}^*_{\mu,\omega^0}$ denote the projection of the measure $\mathbb{P}^*_{\mu,\omega^0,\omega}$ onto its first two coordinates and $\mathbb{P}^*_\omega$ denote the projection onto its final coordinate. Then we intuitively expect $\mathbb{P}^*_{\mu,\omega^0,\omega} = \mathbb{P}^*_{\mu,\omega^0} \otimes \mathbb{P}^*_\omega$.

Lemma 2.5.1 (Independence from idiosyncratic noise)**.** *Let $\mathbf{P}^*$, W^0, and W be random variables on the probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*_{\mu,\omega^0,\omega})$ defined above. Then, $(\mathbf{P}^*, W^0)$ is independent of W.*

Proof. As $(\mathcal{P}(D_\mathbb{R}), \mathfrak{T}^{\mathrm{wk}}_{M_1})$ and $(\mathcal{C}_\mathbb{R}, \|\cdot\|_\infty)$ are Polish spaces, it is sufficient to show that for any $f \in \mathcal{C}_b(\mathcal{P}(D_\mathbb{R}))$ and $g, h \in \mathcal{C}_b(\mathcal{C}_\mathbb{R})$,

$$\left\langle \mathbb{P}^*_{\mu,\omega^0,\omega},\, f \otimes g \otimes h \right\rangle = \left\langle \mathbb{P}^*_{\mu,\omega^0},\, f \otimes g \right\rangle \left\langle \mathbb{P}^*_\omega,\, h \right\rangle. \tag{2.5.2}$$

The result follows from the Dominated Convergence Theorem and Dynkin's Lemma. Now, (2.5.2) follows readily by weak convergence and the Portmanteau Theorem as

$$\begin{aligned}
\left\langle \mathbb{P}^*_{\mu,\omega^0,\omega},\, f \otimes g \otimes h \right\rangle &= \lim_{n\to\infty} \left\langle \mathbb{P}^n,\, f \otimes g \otimes h \right\rangle \\
&= \lim_{n\to\infty} \left\langle \mathbb{P}^n,\, f \otimes g \right\rangle \left\langle \mathbb{P}^n,\, h \right\rangle \\
&= \left\langle \mathbb{P}^*_{\mu,\omega^0},\, f \otimes g \right\rangle \left\langle \mathbb{P}^*_\omega,\, h \right\rangle.
\end{aligned}$$

The equality in the second line follows from the independence of $(\tilde{\mathbf{P}}^\varepsilon, W^0)$ from W. □

We shall use $\mathbb{P}^*_{\mu,\omega^0,\omega}$ to construct a probability space where we can define a process that will solve (2.2.2) in the sense of Definition 2.2.1. Prior to that, we need to define the map employed in the martingale arguments that follow. This allows us to deduce that the process we construct will be of the correct form. For any $\varepsilon > 0$, we define the following functionals $\mathcal{M}$, $\mathcal{M}^\varepsilon : \mathcal{P}(D_\mathbb{R}) \times D_\mathbb{R} \to D_\mathbb{R}$:

$$\mathcal{M}^\varepsilon(\mu, \eta) = \eta - \eta_{-1} - \int_0^\cdot b(s, \eta_s, \nu^\mu_s)\, \mathrm{d}s + \int_0^\cdot \alpha(s)\, \mathrm{d}\mathfrak{L}^{\mu,\varepsilon}_s, \tag{2.5.3}$$

$$\mathcal{M}(\mu, \eta) = \eta - \eta_{-1} - \int_0^\cdot b(s, \eta_s, \nu^\mu_s)\, \mathrm{d}s + \int_{[0,\cdot]} \alpha(s)\, \mathrm{d}L^\mu_s, \tag{2.5.4}$$

where for any $\mu \in \mathcal{P}(D_{\mathbb{R}})$,

$$\nu_t^\mu \coloneqq \mu(\eta_t \in \cdot, \tau_0 > t), \qquad L_t^\mu \coloneqq \mu(\tau_0 \le t), \qquad \mathfrak{L}_t^{\mu,\varepsilon} = \int_0^t \mathfrak{K}^\varepsilon(t-s) L_s^\mu \, \mathrm{d}s,$$

and b satisfies Assumption 2.2.2. For any $s_0, t_0 \in \mathbb{T}^\mu \cap [0, T)$ with $s_0 < t_0$ and $\{s_i\}_{i=1}^k \subset [0, s_0] \cap \mathbb{T}$, we define the function

$$G \ : \ D_{\mathbb{R}} \to \mathbb{R}, \quad \eta \mapsto (\eta_{t_0} - \eta_{s_0}) \prod_{i=1}^k g_i(\eta_{s_i}), \tag{2.5.5}$$

for arbitrary $g_i \in \mathcal{C}_b(\mathbb{R})$. We define the functionals:

$$\begin{cases} \Psi^\varepsilon(\mu) = \langle \mu, \eta \mapsto G\left(\mathcal{M}^\varepsilon(\mu, \eta)\right) \rangle, \\ \Upsilon^\varepsilon(\mu) = \left\langle \mu, \eta \mapsto G\left((\mathcal{M}^\varepsilon(\mu, \eta))^2 - \int_0^\cdot \sigma(s, \eta_s)^2 \, \mathrm{d}s \right) \right\rangle, \\ \Theta^\varepsilon(\mu, \omega) = \left\langle \mu, \eta \mapsto G\left(\mathcal{M}^\varepsilon(\mu, \eta) \times \omega - \int_0^\cdot \sigma(s, \eta_s) \sqrt{1 - \rho(s, \nu_s^\mu)^2} \, \mathrm{d}s \right) \right\rangle, \\ \Theta^{0,\varepsilon}(\mu, \omega^0) = \left\langle \mu, \eta \mapsto G\left(\mathcal{M}^\varepsilon(\mu, \eta) \times \omega^0 - \int_0^\cdot \sigma(s, \eta_s) \rho(s, \nu_s^\mu) \, \mathrm{d}s \right) \right\rangle. \end{cases} \tag{2.5.6}$$

Lastly, the corresponding functionals without the mollification, denoted by $\Psi(\mu)$, $\Upsilon(\mu)$, $\Theta(\mu, \omega)$ and $\Theta^0(\mu, \omega^0)$, are defined in exactly the same way as $\Psi^\varepsilon(\mu)$, $\Upsilon^\varepsilon(\mu)$, $\Theta^\varepsilon(\mu, \omega)$, and $\Theta^{0,\varepsilon}(\mu, \omega^0)$ with $\mathcal{M}^\varepsilon$ replaced by $\mathcal{M}$.

Remark 1 (Measurability of measure flows). In (2.5.3) and (2.5.4), we are taking a fixed measure, μ, and computing the integral with respect to the measure flow $t \mapsto \nu_t^\mu$. The measurability of the functions b and σ is sufficient for this integral to be well-defined.

Using Corollary 2.4.5, we have the following proposition.

Proposition 2.5.2 (Functional Continuity I Generalised). *For $\mathbb{P}^*_{\mu,\omega^0,\omega}$-almost every measure μ, we have that $\Psi^{\varepsilon_n}(\mu^n)$, $\Upsilon^{\varepsilon_n}(\mu^n)$, $\Theta^{\varepsilon_n}(\mu^n, \omega^n)$, and $\Theta^{0,\varepsilon_n}(\mu^n, \omega^{0,n})$ converge to $\Psi(\mu)$, $\Upsilon(\mu)$, $\Theta(\mu, \omega)$, and $\Theta^0(\mu, \omega^0)$ respectively, whenever $(\mu^n, \omega^{0,n}, \omega^n) \to (\mu, \omega^0, \omega)$ in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}^{wk}_{M_1}) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty)$, along a sequence for which $\sup_{n\ge1} \langle \mu^n, \sup_{s \le \bar{T}} |\tilde{\eta}_s|^p \rangle$ is bounded for some $p > 2$ and $(\varepsilon_n)_{n\ge1}$ that converges to zero.*

Proof. By Lemma 2.4.2 and the definition of $\mathbb{T}$, we have a set of μ's that have full $\mathbb{P}^*_{\mu,\omega^0,\omega}$ measure, such that

$$\mu\left(\inf_{s \in (\tau_0, (\tau_0 + h) \wedge \bar{T})} \{\eta_s - \eta_{\tau_0}\} \ge 0, \ \tau_0 < \bar{T} \right) = 0$$

for any $h > 0$, $\mu(\eta_{s_i} = \eta_{s_i -}) = 1$, $\mu(\eta_{t_0} = \eta_{t_0 -}) = 1$, and $\mu(\tau_0 = t_0) = 0$. First, we shall show that $\Psi^{\varepsilon_n}(\mu^n)$ converges to $\Psi(\mu)$. By Corollary 2.4.5, $\mathfrak{L}_t^{\mu^n, \varepsilon_n}$ converges

to L_t^μ. It is well-known that for any Borel measurable functions f and g of finite variation, we have for any $t > 0$

$$f_t g_t = f(0)g(0) + \int_{(0,t]} f_{s-}\,\mathrm{d}g_s + \int_{(0,t]} g_{s-}\,\mathrm{d}f_s + \sum_{s\le t} \Delta f_s \Delta g_s.$$

This, together with the continuous differentiability of α, implies

$$\begin{aligned}\int_0^t \alpha(s)\,\mathrm{d}\mathfrak{L}_s^{\mu,\varepsilon_n} &= \alpha(t)\mathfrak{L}_t^{\mu,\varepsilon_n} - \int_0^t \mathfrak{L}_s^{\mu,\varepsilon_n}\alpha'(s)\,\mathrm{d}s \to \alpha(t)L_t^\mu - \int_0^t L_s^\mu \alpha'(s)\,\mathrm{d}s \\ &= \int_{[0,t]} \alpha(s)\,\mathrm{d}L_s^\mu.\end{aligned}$$

As $\mu^n \implies \mu$, by Skorokhod's Representation Theorem, there exist $(H^n)_{n\ge1}$ and H defined on a common probability space such that $\mathrm{Law}(H^n) = \mu^n$, $\mathrm{Law}(H) = \mu$ and $H^n \to H$ almost surely in $(D_{\mathbb{R}}, M_1)$. Hence,

$$\Psi^{\varepsilon_n}(\mu^n) = \mathbb{E}\left[G(\mathcal{M}^{\varepsilon_n}(\mu^n, H^n))\right] \qquad \text{and} \qquad \Psi(\mu) = \mathbb{E}\left[G(\mathcal{M}(\mu, H))\right].$$

By Lemma 2.6.6,

$$\int_0^t b(s, H_s^n, \nu_s^{\mu^n})\,\mathrm{d}s \to \int_0^t b(s, H_s, \nu_s^\mu)\,\mathrm{d}s \tag{2.5.7}$$

almost surely for any $t \ge 0$. Since $\mathbb{T}^\mu$ contains all the almost sure continuity points of H, by the properties of M_1-convergence and (2.5.7), we have

$$H_t^n - H_{-1}^n - \int_0^t b(s, H_s^n, \nu_s^{\mu^n})\,\mathrm{d}s \to H_t - H_{-1} - \int_0^t b(s, H_s, \nu_s^\mu)\,\mathrm{d}s$$

almost surely for any $t \in \{t_0, s_0, \ldots, s_k\}$. Hence, we deduce that $G(\mathcal{M}^{\varepsilon_n}(\mu^n, H^n))$ converges almost surely to $G(\mathcal{M}(\mu, H))$ in $\mathbb{R}$. Lastly, we observe

$$\langle \mu^n, |\mathcal{M}_t^{\varepsilon_n}(\mu^n, \cdot)|^p \rangle \le C\left(\left\langle \mu^n, \sup_{s\le \bar{T}} |\tilde{\eta}_s|^p \right\rangle + 1\right), \tag{2.5.8}$$

for some constant C that depends on p and b only but is uniform in n. Therefore, $G(\mathcal{M}^{\varepsilon_n}(\mu^n, H^n))$ is uniformly L^p-bounded as

$$|G(\mathcal{M}^{\varepsilon_n}(\mu^n, H^n))|^p \le C\left(\left|\mathcal{M}_{t_0}^{\varepsilon_n}(\mu^n, H^n)\right|^p + \left|\mathcal{M}_{s_0}^{\varepsilon_n}(\mu^n, H^n)\right|^p\right),$$

and $\mathbb{E}[|\mathcal{M}_{t_0}^{\varepsilon_n}(\mu^n, H^n)|^p] = \langle \mu^n, |\mathcal{M}_{t_0}^{\varepsilon_n}(\mu^n, \cdot)|^p \rangle$ where the latter is uniformly bounded in n for some $p > 2$ by (2.5.8) and assumption. Therefore, by Vitali's Convergence Theorem, it follows that $\Psi^{\varepsilon_n}(\mu^n)$ converges to $\Psi(\mu)$.

The convergence of $\Upsilon^{\varepsilon_n}(\mu^n)$, $\Theta^{\varepsilon_n}(\mu^n, \omega^n)$, and $\Theta^{0,\varepsilon_n}(\mu^n, \omega^{0,n})$, to $\Upsilon(\mu)$, $\Theta(\mu, \omega)$ and $\Theta^0(\mu, \omega^0)$ respectively follows from similar arguments. As σ and ρ are totally

bounded by Assumption 2.2.2 (ii) and (iv), $\Upsilon^{\varepsilon_n}(\mu^n)$, $\Theta^{\varepsilon_n}(\mu^n, \omega^n)$, and $\Theta^{0,\varepsilon_n}(\mu^n, \omega^{0,n})$ are L^p-bounded uniformly in n. The continuity of σ and the almost sure convergence of H^n to H in the M_1-topology ensures that

$$\int_0^t \sigma(s, H_s^n)^2 \, \mathrm{d}s \to \int_0^t \sigma(s, H_s)^2 \, \mathrm{d}s$$

almost surely for all $t \geq 0$. Lastly, by the boundedness of σ and ρ, a straightforward calculation shows that

$$\begin{aligned} |\sigma(t,x)\rho(t,\mu) - \sigma(t,x)\rho(t,\tilde{\mu})| &\leq C\,(1 + \langle \mu, \, |\cdot| \rangle)\, d_1(\mu, \, \tilde{\mu}), \\ \left|\sigma(t,x)\sqrt{1-\rho(t,\mu)^2} - \sigma(t,x)\sqrt{1-\rho(t,\tilde{\mu})^2}\right| &\leq C\,(1 + \langle \mu, \, |\cdot| \rangle)\, d_1(\mu, \, \tilde{\mu}). \end{aligned}$$

As $d_1(\mu, \, \tilde{\mu}) \leq d_0(\mu, \, \tilde{\mu})$, we deduce that the functions $(t, x, \mu) \mapsto \sigma(t,x)\rho(t,\mu)$ and $(t, x, \mu) \mapsto \sigma(t,x)\sqrt{1-\rho(t,\mu)^2}$ satisfy Assumption 2.2.2 (i). Now we may apply Lemma 2.6.6 and conclude that

$$\begin{aligned} \int_0^t \sigma(s, H_s^n)\sqrt{1-\rho(s,\nu_s^{\mu^n})^2} \, \mathrm{d}s &\to \int_0^t \sigma(s, H_s)\sqrt{1-\rho(s,\nu_s^{\mu})^2} \, \mathrm{d}s, \\ \int_0^t \sigma(s, H_s^n)\rho(s,\nu_s^{\mu^n}) \, \mathrm{d}s &\to \int_0^t \sigma(s, H_s)\rho(s,\nu_s^{\mu}) \, \mathrm{d}s, \end{aligned}$$

almost surely for all $t \geq 0$. □

The remainder of this section aims to show that the conditional law of $\{\tilde{X}^{\varepsilon_n}\}$ converges weakly to a random variable X which will have the dynamics defined in (2.2.6). This is achieved in the following two steps:

(i) First, we construct a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ such that $\mathcal{M}_\cdot$, $\mathcal{M}_\cdot^2 - \int_0^\cdot \sigma(s, \eta_s)^2 \, \mathrm{d}s$, $\mathcal{M}_\cdot \times W - \int_0^\cdot \sigma(s, \eta_s)\sqrt{1-\rho(s,\nu_s^\mu)^2} \, \mathrm{d}s$, and $\mathcal{M}_\cdot \times W^0 - \int_0^\cdot \sigma(s, \eta_s)\rho(s, \nu_s^\mu) \, \mathrm{d}s$, defined as in (2.5.4), are continuous martingales.

(ii) Secondly, we construct a stochastic process X on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ such that the tuple $(X, W, W^0, \mathbf{P}^*)$ is a solution to (2.2.4) in the sense of Definition 2.2.1.

To this end, we now proceed to show the above two claims. We begin by defining the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ where $\bar{\Omega} = \Omega^* \times D_{\mathbb{R}} = \mathcal{P}(D_{\mathbb{R}}) \times \mathcal{C}_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}} \times D_{\mathbb{R}}$ and $\bar{\mathcal{F}}$ is the Borel σ-algebra $\mathcal{B}(\bar{\Omega})$. We define the probability measure

$$\bar{\mathbb{P}}(A) \coloneqq \int_{\mathcal{P}(D_{\mathbb{R}}) \times \mathcal{C}_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}} \mu\left(\{\eta \, : \, (\mu, \omega^0, \omega, \eta) \in A\}\right) \mathrm{d}\mathbb{P}^*_{\mu,\omega^0,\omega}(\mu, \omega^0, \omega), \tag{2.5.9}$$

for any $A \in \mathcal{B}(\bar{\Omega})$. Observe by construction, for any $A \in \mathcal{B}(\bar{\Omega})$,

$$\bar{\mathbb{P}}(A) = \mathbb{E}^*\left[\left\langle \mathbf{P}^*, \, \mathbb{1}_A(\mathbf{P}^*, \, W^0, \, W, \, \cdot)\right\rangle\right].$$

Furthermore, under $\bar{\mathbb{P}}$, W^0 and W are still Brownian motions, and $(\mathbf{P}^*, W^0)$ is independent of W. This is immediate, as for any $A \in \mathcal{B}(\mathcal{P}(D_{\mathbb{R}}) \times \mathcal{C}_{\mathbb{R}})$ and $B \in \mathcal{B}(\mathcal{C}_{\mathbb{R}})$,

$$\bar{\mathbb{P}}\left[(\mathbf{P}^*, W^0) \in A,\, W \in B\right] = \bar{\mathbb{P}}\left(A \times B \times D_{\mathbb{R}}\right) = \mathbb{P}^*_{\mu,\omega^0,\omega}(A \times B).$$

Given these ingredients, we may now show our first claim.

Proposition 2.5.3. *Let $\mathcal{M}$ be given as in (2.5.4). Then $\mathcal{M}_\cdot$, $\mathcal{M}_\cdot^2 - \int_0^\cdot \sigma(s, \pi_s(\cdot))^2 \,\mathrm{d}s$, $\mathcal{M}_\cdot \times W - \int_0^\cdot \sigma(s, \pi_s(\cdot))\sqrt{1-\rho(s,\nu_s^\cdot)^2}\,\mathrm{d}s$, and $\mathcal{M}_\cdot \times W^0 - \int_0^\cdot \sigma(s, \pi_s(\cdot))\rho(s, \nu_s^\cdot)\,\mathrm{d}s$ are all continuous martingales on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, where*

$$\pi_s : D_{\mathbb{R}} \to \mathbb{R},\ \pi_s(\eta) = \eta_s \qquad \text{and} \qquad \nu_s^\cdot : \mathcal{P}(D_{\mathbb{R}}) \to \mathbf{M}_{\leq 1}(\mathbb{R}),\ \nu_s^\cdot(\mu) = \nu_s^\mu.$$

Proof. If $\mathcal{M}$ is continuous, then the continuity of the other processes follows from the continuity of $\mathcal{M}$ and the continuity of integration. For the sake of conciseness, we shall employ $\mathcal{N}$ to denote any one of the following processes:

$$\begin{aligned}
&\mathcal{M}_\cdot, \\
&\mathcal{M}_\cdot \times W - \int_0^\cdot \sigma(s, \pi_s(\cdot))\sqrt{1-\rho(s,\nu_s^\cdot)^2}\,\mathrm{d}s, \\
&\mathcal{M}_\cdot \times W^0 - \int_0^\cdot \sigma(s, \pi_s(\cdot))\rho(s, \nu_s^\cdot)\,\mathrm{d}s, \\
&\mathcal{M}_\cdot^2 - \int_0^\cdot \sigma(s, \pi_s(\cdot))^2\,\mathrm{d}s.
\end{aligned} \tag{2.5.10}$$

Hence, to show that $\mathcal{N}$ is a martingale, it is sufficient by a Monotone Class argument that

$$\bar{\mathbb{E}}\left[G(\mathcal{N})\right] = 0, \tag{2.5.11}$$

To begin, recall that $\mathbb{P}^{\varepsilon_n} \implies \mathbb{P}^*$, where $\mathbb{P}^{\varepsilon_n} = \mathrm{Law}(\tilde{\mathbf{P}}^{\varepsilon_n}, W^0, W)$ and $\mathbb{P}^* = \mathrm{Law}(\mathbf{P}^*, W^0, W)$. By Skorokhod's Representation Theorem, we can find sequences $\{(\mathbf{Q}^n, B^n, \tilde{B}^n)\}_{n\geq 1}$ and $(\mathbf{Q}^*, B^*, \tilde{B}^*)$ defined on a common probability space such that:

$$\begin{aligned}
\mathbb{P}^{\varepsilon_n} &= \mathrm{Law}(\mathbf{Q}^n,\, B^n,\, \tilde{B}^n), \\
\mathbb{P}^* &= \mathrm{Law}(\mathbf{Q}^*,\, B^*,\, \tilde{B}^*),
\end{aligned}$$

and $(\mathbf{Q}^n,\, B^n,\, \tilde{B}^n) \to (\mathbf{Q}^*,\, B^*,\, \tilde{B}^*)$ almost surely in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}^{\mathrm{wk}}_{M_1}) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty)$.

By the definition of $\tilde{\mathbf{P}}^{\varepsilon_n}$, for any $p > 1$, we have:

$$\mathbb{E}\left[\left\langle \tilde{\mathbf{P}}^{\varepsilon_n},\, \sup_{s\leq \bar{T}} |\eta_s|^p \right\rangle\right] = \mathbb{E}\left[\sup_{s\leq \bar{T}} \left|\tilde{X}^{\varepsilon_n}\right|^p\right] \leq C_p, \tag{2.5.12}$$

where the constant C_p is derived from Proposition 2.3.1. As the map $\eta \mapsto \sup_{s\leq \bar{T}} |\eta_s|$ is M_1-continuous, the Portmanteau Theorem yields:

$$\mathbb{E}\left[\left\langle \mathbf{P}^*, \sup_{s\leq \bar{T}} |\eta_s|^p \right\rangle\right] \leq C_p. \tag{2.5.13}$$

Now, for any $K > 0$, we can write

$$\begin{aligned}\mathbb{E}[|\langle \mathbf{Q}^n, \sup_{s\leq \bar{T}} |\eta_s|^p\rangle - \langle \mathbf{Q}^*, \sup_{s\leq \bar{T}} |\eta_s|^p\rangle|] \leq & \mathbb{E}[|\langle \mathbf{Q}^n, \sup_{s\leq \bar{T}} |\eta_s|^p \wedge K\rangle - \langle \mathbf{Q}^*, \sup_{s\leq \bar{T}} |\eta_s|^p \wedge K\rangle|] \\ & + \mathbb{E}[|\langle \mathbf{Q}^n, (\sup_{s\leq \bar{T}} |\eta_s|^p - K)\mathbb{1}_{\sup_{s\leq \bar{T}} |\eta_s|^p > K}\rangle|] \\ & + \mathbb{E}[|\langle \mathbf{Q}^*, (\sup_{s\leq \bar{T}} |\eta_s|^p - K)\mathbb{1}_{\sup_{s\leq \bar{T}} |\eta_s|^p > K}\rangle|].\end{aligned}$$

By equation (2.5.12), and a combination of Hölder's and Markov's inequalities, the second term on the right-hand side above is $o(1)$ as $K \to \infty$ uniformly in n. Similarly, by (2.5.13), the third term on the right-hand side above is $o(1)$ as $K \to \infty$. For the first term, we recall that $\mathbf{Q}^n \to \mathbf{Q}^*$ almost surely in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}^{\mathrm{wk}}_{M_1})$. Hence, as $\eta \mapsto \sup_{s\leq \bar{T}} |\eta_s|^p \wedge K$ is an M_1-continuous and bounded function, we have by the almost sure convergence and the Portmanteau Theorem that $\langle \mathbf{Q}^n, \sup_{s\leq \bar{T}} |\eta_s|^p \wedge K\rangle$ converges almost surely towards $\langle \mathbf{Q}^*, \sup_{s\leq \bar{T}} |\eta_s|^p \wedge K\rangle$. Therefore, by the Dominated Convergence Theorem, the first term on the right-hand side above is $o(1)$ as $n \to \infty$. Therefore, we can make the right-hand side arbitrarily small by first fixing K sufficiently large, then choosing n sufficiently large. Hence, we have

$$\mathbb{E}[|\langle \mathbf{Q}^n, \sup_{s\leq \bar{T}} |\eta_s|^p\rangle - \langle \mathbf{Q}^*, \sup_{s\leq \bar{T}} |\eta_s|^p\rangle|] \to 0.$$

Now, fixing a $p > 2$, as we have L^1-convergence, we can find a subsequence (also denoted by n for simplicity) such that $\langle \mathbf{Q}^n, \sup_{s\leq \bar{T}} |\eta_s|^p\rangle$ converges almost surely to $\langle \mathbf{Q}^*, \sup_{s\leq \bar{T}} |\eta_s|^p\rangle$. By (2.5.13), $\langle \mathbf{Q}^*, \sup_{s\leq \bar{T}} |\eta_s|^p\rangle < +\infty$ almost surely. Therefore, restricting ourselves to this subsequence going forward, we have

$$\sup_{n\geq 1} \left\langle \mathbf{Q}^n, \sup_{s\leq \bar{T}} |\eta_s|^p \right\rangle < \infty$$

almost surely.

By definition of $\mathcal{M}$, for the same $p > 2$ fixed above

$$\begin{aligned}&\mathbb{E}\left[\left\langle \tilde{\mathbf{P}}^{\varepsilon_n}, \left|\mathcal{M}_t(\tilde{\mathbf{P}}^{\varepsilon_n}, \cdot)\right|^p \right\rangle\right] \\ &\quad = \mathbb{E}\left[\left|\int_0^t \sigma(s, X_s^{\varepsilon_n})\sqrt{1-\rho(s,\boldsymbol{\nu}_s^{\varepsilon_n})^2}\,\mathrm{d}W_s + \int_0^t \sigma(s, X_s^{\varepsilon_n})\rho(s,\boldsymbol{\nu}_s^{\varepsilon_n})\,\mathrm{d}W_s^0\right|^p\right] \\ &\quad \leq Ct^p,\end{aligned} \tag{2.5.14}$$

where the constant C depends on the constant from applying Burkholder-Davis-Gundy, p, and the bounds on σ but is independent of ε. Hence, $\mathbb{E}[\langle \mathbf{Q}^n, |\mathcal{M}_t(\mathbf{Q}^n, \cdot)|^p \rangle] < \infty$ uniformly in n.

Employing Proposition 2.5.3 and Vitali's Convergence Theorem, we have

$$\begin{aligned}\bar{\mathbb{E}}\left[G(\mathcal{N})\right] &= \mathbb{E}^*\left[\left\langle \mathbf{P}^*, G(\mathcal{N}(\mathbf{P}^*, W^0, W, \cdot))\right\rangle\right] \\ &= \lim_{n\to\infty} \mathbb{E}\left[\left\langle \mathbf{P}^{\varepsilon_n}, G(\mathcal{N}^{\varepsilon_n}(\mathbf{P}^{\varepsilon_n}, W^0, W, \cdot))\right\rangle\right],\end{aligned}$$

where $\mathcal{N}^{\varepsilon_n}$ represents one of the equations in (2.5.10) with $\mathcal{M}$ replaced by $\mathcal{M}^{\varepsilon_n}$ depending on $\mathcal{N}$. Recall that for arbitrary $g_i \in \mathcal{C}_b(\mathbb{R})$, $G(\eta) = (\eta_{t_0} - \eta_{s_0}) \prod_{i=1}^k g_i(\eta_{s_i})$. Consequently, we have

$$\mathbb{E}\left[\left\langle \mathbf{P}^{\varepsilon_n}, G(\mathcal{N}^{\varepsilon_n}(\mathbf{P}^{\varepsilon_n}, W^0, W, \cdot))\right\rangle\right] = \mathbb{E}\left[\left(\tilde{\mathcal{N}}^{\varepsilon_n}_{t_0} - \tilde{\mathcal{N}}^{\varepsilon_n}_{s_0}\right) \prod_{i=1}^k g_i(\tilde{\mathcal{N}}^{\varepsilon_n}_{s_i})\right], \tag{2.5.15}$$

where $\tilde{\mathcal{N}}^{\varepsilon_n}$ represents one of the following processes, depending on the choice of $\mathcal{N}$:

- $\tilde{\mathcal{Y}}^{\varepsilon_n}$
- $(\tilde{\mathcal{Y}}^{\varepsilon_n})^2 - \int_0^\cdot \sigma(s, \tilde{X}^{\varepsilon_n})^2 \, \mathrm{d}s$
- $\tilde{\mathcal{Y}}^{\varepsilon_n} \times W - \int_0^\cdot \sigma(s, \tilde{X}^{\varepsilon_n}) \sqrt{1 - \rho(s, \boldsymbol{\nu}^{\varepsilon_n}_s)^2} \, \mathrm{d}s$
- $\tilde{\mathcal{Y}}^{\varepsilon_n} \times W^0 - \int_0^\cdot \sigma(s, \tilde{X}^{\varepsilon_n}) \rho(s, \boldsymbol{\nu}^{\varepsilon_n}_s) \, \mathrm{d}s$

By the boundedness assumption on σ, Assumption 2.2.2 (ii), $\tilde{\mathcal{N}}^{\varepsilon_n}$ is a martingale. Hence, as $s_1 \leq \ldots \leq s_k \leq s_0 < t_0$, we have that (2.5.15) equals zero by the tower property. Hence, we have shown (2.5.11).

Lastly, to see the continuity of $\mathcal{M}$, define the function

$$\tilde{G} \ : \ D_{\mathbb{R}} \to \mathbb{R}, \quad \eta \mapsto |\eta_t - \eta_s|^4,$$

for $s, t \in \mathbb{T} \cap [0, T)$. As before, define the functionals

$$\tilde{\Psi}^{\varepsilon}(\mu) = \left\langle \mu, \tilde{G}(\mathcal{M}^{\varepsilon}(\mu, \cdot)) \right\rangle, \qquad \tilde{\Psi}(\mu) = \left\langle \mu, \tilde{G}(\mathcal{M}(\mu, \cdot)) \right\rangle.$$

Following the same proof as in Proposition 2.5.2, we have that for $\mathbb{P}^*_{\mu,\omega^0,\omega}$-almost every measure μ, $\tilde{\Psi}^{\varepsilon_n}(\mu^n)$ converges to $\tilde{\Psi}(\mu)$ whenever $\mu^n \to \mu$ in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}^{\mathrm{wk}}_{M_1})$ along a sequence for which

$$\sup_{n\geq 1} \langle \mu^n, \sup_{s \leq \bar{T}} |\eta_s|^p \rangle < \infty$$

for some $p > 4$. We have finite moments for any $p > 1$, by (2.5.12). Therefore, by functional continuity and Vitali's convergence theorem, for any $s, t \in \mathbb{T} \cap [0, T)$ we have

$$\begin{aligned}\bar{\mathbb{E}}\left|\mathcal{M}_t - \mathcal{M}_s\right|^4 &= \mathbb{E}^*\left[\left\langle \mathbf{P}^*,\, \left|\mathcal{M}_t(\mathbf{P}^*,\,\cdot) - \mathcal{M}_s(\mathbf{P}^*,\,\cdot)\right|^4 \right\rangle\right] \\ &= \lim_{n\to\infty} \mathbb{E}\left[\left\langle \tilde{\mathbf{P}}^{\varepsilon_n},\, \left|\mathcal{M}_t^{\varepsilon_n}(\tilde{\mathbf{P}}^{\varepsilon_n},\,\cdot) - \mathcal{M}_s^{\varepsilon_n}(\tilde{\mathbf{P}}^{\varepsilon_n},\,\cdot)\right|^4 \right\rangle\right].\end{aligned}$$

By the definition of $\tilde{\mathbf{P}}^{\varepsilon_n}$ and the Burkholder-Davis-Gundy inequality,

$$\mathbb{E}\left[\left\langle \tilde{\mathbf{P}}^{\varepsilon_n},\, \left|\mathcal{M}_t^{\varepsilon_n}(\mathbf{P}^{\varepsilon_n},\,\cdot) - \mathcal{M}_s^{\varepsilon_n}(\mathbf{P}^{\varepsilon_n},\,\cdot)\right|^4 \right\rangle\right] = \mathbb{E}\left|\tilde{\mathcal{Y}}_t^{\varepsilon_n} - \tilde{\mathcal{Y}}_s^{\varepsilon_n}\right|^4 \le C\left|t - s\right|^2,$$

where the constant C is uniform in n. As $\mathbb{T}$ is dense, by Kolmogorov's Criterion, there exists a continuous process that is a modification of $\mathcal{M}$. Since $\mathcal{M}$ is right continuous and $\mathbb{T}$ is dense, these two processes are indistinguishable. Hence, $\mathcal{M}$ has a continuous version. □

Now, we have all the ingredients to prove Theorem 2.2.4.

Proof of Theorem 2.2.4. By Proposition 2.3.2, $\{(\tilde{\mathbf{P}}^{\varepsilon},\, W^0,\, W)\}_{\varepsilon>0}$ is tight. By Prokhorov's Theorem, tightness on Polish spaces is equivalent to being sequentially precompact. Therefore, for any subsequence $\{(\mathbf{P}^{\varepsilon_n},\, W^0,\, W)\}_{n\ge1}$, where $(\varepsilon_n)_{n\ge1}$ is a positive sequence that converges to zero, we have a convergent sub-subsequence. Fix a limit point $(\mathbf{P}^*,\, W^0,\, W)$ of this subsequence. Having fixed $(\mathbf{P}^*,\, W^0,\, W)$, we define the probability space $(\bar{\Omega},\, \bar{\mathcal{F}},\, \bar{\mathbb{P}})$ exactly as in (2.5.9). Now, define the càdlàg process X by

$$X \,:\, \bar{\Omega} \to D_{\mathbb{R}}, \qquad (\mu,\, \omega^0,\, \omega,\, \eta) \mapsto \eta.$$

Then, by the construction of $\bar{\mathbb{P}}$ and the fact that $\mathbb{P}^*_{\mu,\omega^0,\omega} = \mathbb{P}^*_{\mu,\omega^0} \times \mathbb{P}^*_{\omega}$ by Lemma 2.5.1, for all $A \in \mathcal{B}(D_{\mathbb{R}})$, $S \in \mathcal{B}(\mathcal{P}(D_{\mathbb{R}}) \times \mathcal{C}_{\mathbb{R}})$, we have

$$\bar{\mathbb{P}}\left[X \in A,\, (\mathbf{P}^*,\, W^0) \in S\right] = \int_S \mu(A)\, \mathrm{d}\mathbb{P}^*_{\mu,\omega^0}.$$

Here, $\mathbb{P}^*_{\mu,\omega^0,\omega} = \mathrm{Law}(\mathbf{P}^*,\, W^0,\, W)$, $\mathbb{P}^*_{\mu,\omega^0} = \mathrm{Law}(\mathbf{P}^*,\, W^0)$, and $\mathbb{P}^*_{\omega} = \mathrm{Law}(W)$. Consequently,

$$\bar{\mathbb{P}}\left[X \in A \middle|\, \mathbf{P}^*,\, W^0\right] = \mathbf{P}^*(A) \qquad \forall \quad A \in \mathcal{B}(D_{\mathbb{R}}).$$

By Proposition 2.5.3,

$$\mathcal{M}_t = X_t - X_{-1} - \int_0^t b(s,\, X_s,\, \boldsymbol{\nu}_s^*)\, \mathrm{d}s - \int_{[0,t]} \alpha(s)\, \mathrm{d}\mathbf{P}^*(\tau_0(X) \le s)$$

is a continuous local martingale with

$$\langle \mathcal{M} \rangle_t = \int_0^t \sigma(s, X_s)^2 \, \mathrm{d}s, \quad \langle \mathcal{M}, W \rangle_t = \int_0^t \sigma(s, X_s) \sqrt{1 - \rho(s, \boldsymbol{\nu}_s^*)^2} \, \mathrm{d}s,$$
$$\langle \mathcal{M}, W^0 \rangle_t = \int_0^t \sigma(s, X_s) \rho(s, \boldsymbol{\nu}_s^*) \, \mathrm{d}s,$$

where $\boldsymbol{\nu}_s^* \coloneqq \mathbf{P}^*(X_s \in \cdot, \tau_0(X) > s)$. As W^0 and W are standard independent Brownian motions, by Lévy's Characterisation Theorem we have that

$$\mathcal{M}_t = \int_0^t \sigma(s, X_s) \left(\sqrt{1 - \rho(s, \boldsymbol{\nu}_s^*)^2} \, \mathrm{d}W_s + \rho(s, \boldsymbol{\nu}_s^*) \, \mathrm{d}W_s^0 \right).$$

Now, as $-1 \in \mathbb{T}$, the map $\eta \mapsto \eta_{-1}$ is μ-almost surely continuous for $\mathbb{P}_{\mu,\omega^0,\omega}$-almost every measure μ. A simple application of the Portmanteau Theorem shows that $X_{-1} \sim \nu_{0-}$. By Lemma 2.4.1, we deduce that $X_{0-} \sim \nu_{0-}$. The independence between $(\mathbf{P}^*, W^0)$ and W follows from Lemma 2.5.1. A similar argument as employed in Lemma 2.5.1 shows that $X_{0-} \perp (\mathbf{P}^*, W^0, W)$. Lastly, by Lemma 2.6.9,

$$\Delta L_t^* \leq \inf\{x \geq 0 : \boldsymbol{\nu}_{t-}^*([0, \alpha(t)x]) < x\} \qquad \text{a.s.}$$

for all $t \geq 0$. □

2.6 Technical Lemmas

This section contains the proofs of several technical lemmas which were deferred from the previous sections to aid with the presentation. These results are vital for the convergence results detailed above.

Lemma 2.6.1. *Suppose that $\tilde{P}^{\varepsilon_n} = \mathrm{Law}(\tilde{X}^{\varepsilon_n}, \tilde{\mathcal{Y}}^{\varepsilon_n})$ converges weakly in $\mathcal{P}(D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}})$ to $\tilde{P}^*$, where $\tilde{X}^{\varepsilon_n}$ and $\tilde{\mathcal{Y}}^{\varepsilon_n}$ are the extensions of X^{ε_n} and $\mathcal{Y}^{\varepsilon_n}$, respectively. Let X^* and Y^* be the canonical processes on $D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}$ such that for $(\eta, \omega) \in D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}$, $X^*(\eta, \omega) = \eta$ and $Y^*(\eta, \omega) = \omega$. Then, under $\tilde{P}^*$, Y^* is a martingale with respect to the filtration generated by (X^*, Y^*) with quadratic variation*

$$\langle Y^* \rangle_t = \begin{cases} 0 & t \in [-1, 0), \\ \displaystyle\int_0^t \sigma(s, X_s^*)^2 \, \mathrm{d}s & t \in [0, T], \\ \displaystyle\int_0^T \sigma(s, X_s^*)^2 \, \mathrm{d}s + (t - T) & t \in (T, \bar{T}]. \end{cases}$$

Proof. Set $\tilde{P}^*$ to be the limit point of $(\tilde{P}^{\varepsilon_n})_{n\geq 0}$ and

$$\mathbb{T}^{\tilde{P}^*} := \left\{ t \in [-1,\, \bar{T}] \;:\; \tilde{P}^*(\eta_t = \eta_{t-}) = 1,\, \tilde{P}^*(\tau_0 = t) = 0 \right\}.$$

Now for any $s_0,\, t_0 \in \mathbb{T}^{\tilde{P}^*}$ with $s_0 < t_0$ and $\{s_i\}_{i=1}^k \subset [-1,\, s_0] \cap \mathbb{T}^{\tilde{P}^*}$, we define the function

$$G \,:\, D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}} \to \mathbb{R}, \quad (\eta, \omega) \mapsto (\omega_{t_0} - \omega_{s_0}) \prod_{i=1}^{k} g_i(\eta_{s_i}, \omega_{s_i}),$$

for arbitrary $g_i \in \mathcal{C}_b(D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}})$. To show that Y^* is a martingale, it is sufficient to show that $\mathbb{E}^{\tilde{P}^*}[G(X^*,\, Y^*)] = 0$.

As $\tilde{P}^{\varepsilon_n} \implies \tilde{P}^*$, then by Skorokhod's Representation Theorem, see [3, Theorem 7.6], there exist $((x^n,\, y^n))_{n\geq 1}$ and $(x,\, y)$ defined on the same background space such that $(x^n,\, y^n)$ converges to $(x,\, y)$ almost surely in $(D_{\mathbb{R}},\, M_1) \times (\mathcal{C}_{\mathbb{R}},\, \|\cdot\|_\infty)$ with $\mathrm{Law}(x^n,\, y^n) = \tilde{P}^{\varepsilon_n}$ and $\mathrm{Law}(x, y) = \tilde{P}^*$. Now for any $p > 1$,

$$\mathbb{E}\left[|G(x^n,\, y^n)|^p\right] \leq C\mathbb{E}\left[\sup_{s\leq \bar{T}} |\tilde{\mathcal{Y}}_s^{\varepsilon_n}|^p\right] \leq C,$$

where C is a constant that changes from line to line and depends only on p, σ, T and the g_i's but is independent of ε. Therefore, $G(x^n,\, y^n)$ is L^p-bounded uniformly in n. For $t \in \{t_0,\, s_0,\, s_1,\, \ldots,\, s_k\}$, t is an almost sure continuity point of x. Therefore, by the properties of M_1-convergence, $(x_t^n,\, y_t^n)$ converges to $(x_t,\, y_t)$ almost surely. Hence, we have almost sure convergence of $G(x^n,\, y^n)$ to $G(x,\, y)$. Vitali's Convergence Theorem states that almost sure convergence and uniform integrability imply convergence of means, hence

$$\mathbb{E}^{\tilde{P}^*}[G(X^*,\, Y^*)] = \mathbb{E}[G(x,\, y)] = \lim_{n\to\infty} \mathbb{E}[G(x^n,\, y^n)] = \lim_{n\to\infty} \mathbb{E}[G(\tilde{X}^{\varepsilon_n},\, \tilde{\mathcal{Y}}^{\varepsilon_n})] = 0,$$

where the last equality follows from the fact that $\mathbb{E}[G(\tilde{X}^{\varepsilon_n},\, \tilde{\mathcal{Y}}^{\varepsilon_n})] = 0$ for all n as $\tilde{\mathcal{Y}}^{\varepsilon_n}$ is a martingale. Therefore, by a monotone class theorem argument, Y^* is a continuous local martingale.

Recall $x^n \to x$ almost surely in $(D_{\mathbb{R}},\, M_1)$, hence we have pointwise convergence at the continuity points of x, see [71, Theorem 12.5.1]. As σ is in $\mathcal{C}^{1,\,2}$ by Assumption 2.2.2, there exists a set of full probability such that $\sigma(s, x_s^n) \to \sigma(s,\, x_s)$ for a set of s's that have full Lebesgue measure in $[0,\, T]$. Furthermore, as σ is bounded, by the Bounded Convergence Theorem

$$\int_0^t \sigma(s,\, x_s^n)^2\, \mathrm{d}s \to \int_0^t \sigma(s,\, x_s)^2\, \mathrm{d}s \tag{2.6.1}$$

almost surely for any $t \in [0, T]$. Set

$$\langle Y \rangle_t = \begin{cases} 0 & t \in [-1, 0), \\ \displaystyle\int_0^t \sigma(s, X_s)^2 \, \mathrm{d}s & t \in [0, T], \\ \displaystyle\int_0^T \sigma(s, X_s)^2 \, \mathrm{d}s + (t - T) & t \in (T, \bar{T}], \end{cases}$$

where Y is any one of Y^*, y^n or $\tilde{\mathcal{Y}}^{\varepsilon_n}$ and X is the respective X^*, x^n or $\tilde{X}^{\varepsilon_n}$. Employing (2.6.1),

$$G(x^n, (y^n)^2 - \langle y^n \rangle) \to G(x, y^2 - \langle y \rangle) \qquad \text{almost surely.}$$

Also, by the above and the boundedness of σ by Assumption 2.2.2 (ii), $G(x^n, (y^n)^2 - \langle y^n \rangle)$ is L^p-bounded uniformly in n. Hence, by Vitali's Convergence Theorem,

$$\begin{aligned} \mathbb{E}^{\tilde{P}^*} \left[G(X^*, (Y^*)^2 - \langle Y^* \rangle) \right] &= \mathbb{E} \left[G(x, y^2 - \langle y \rangle) \right] \\ &= \lim_{n \to \infty} \mathbb{E} \left[G(x^n, (y^n)^2 - \langle y^n \rangle) \right] \\ &= \lim_{n \to \infty} \mathbb{E} \left[G(\tilde{X}^{\varepsilon_n}, (\tilde{\mathcal{Y}}^{\varepsilon_n})^2 - \langle \tilde{\mathcal{Y}}^{\varepsilon_n} \rangle) \right] = 0, \end{aligned}$$

where the last equality follows from the fact that $(\tilde{\mathcal{Y}}^{\varepsilon_n})^2 - \langle \tilde{\mathcal{Y}}^{\varepsilon_n} \rangle$ is a true martingale due to the boundedness of σ. This completes the proof. □

Lemma 2.6.2. *Consider the process $H_t = M_t + At$ for $t \in [-1, \bar{T}]$, where M_t is a continuous local martingale with $c_M(t - s) \leq \langle M \rangle_t - \langle M \rangle_s \leq C_M(t - s)$ for any $0 \leq s < t$ almost surely, and A is a non-negative random variable such that $\mathbb{E}[A] < \infty$. Then, for any stopping time τ such that $\tau \geq 0$ almost surely, we have*

$$\mathbb{P} \left[\inf_{s \in (\tau, (\tau+h) \wedge \bar{T})} \{H_s - H_\tau\} \geq 0, \, \tau < \bar{T} \right] = 0,$$

for any $h > 0$.

Proof. In the case when M is simply a Brownian motion, the result readily follows from the Strong Markov Property and the standard properties of Brownian motion. As M is a continuous local martingale, we may view it as a (random) time-changed Brownian motion. We exploit this fact to show the claim. To begin, fix a $\delta \in (0, h)$, $\lambda > 0$ and set $\bar{\tau} \coloneqq \tau \wedge (\bar{T} - \delta)$. Then, conditioning on the event $E \coloneqq \{\tau \leq \bar{T} - \delta\} \cap \{A \leq \lambda\}$ and its complement, we have

$$\begin{aligned} &\mathbb{P} \left[\inf_{s \in (\tau, (\tau+h) \wedge \bar{T})} \{H_s - H_\tau\} > -\delta, \, \tau < \bar{T} \right] \\ &\qquad \leq \mathbb{P} \left[\inf_{s \in (\bar{\tau}, \bar{\tau} + \delta)} \{M_s - M_{\bar{\tau}} + (s - \bar{\tau})\lambda\} > -\delta \right] + \mathbb{P} \left[E^{\complement}, \, \tau < \bar{T} \right]. \end{aligned} \tag{2.6.2}$$

Focusing on the first term, we observe that

$$\mathbb{P}\left[\inf_{s\in(\bar{\tau},\bar{\tau}+\delta)}\{M_s - M_{\bar{\tau}} + (s-\bar{\tau})\lambda\} > -\delta\right] \leq \mathbb{P}\left[\inf_{s\in(\bar{\tau},\bar{\tau}+\delta)}\{M_s - M_{\bar{\tau}}\} > -\delta(1+\lambda)\right].$$

By the Dubins-Schwarz Theorem, M is a time-changed Brownian motion. Therefore, there exists a Brownian motion B such that

$$\mathbb{P}\left[\inf_{s\in(\bar{\tau},\bar{\tau}+\delta)}\{M_s - M_{\bar{\tau}}\} > -\delta(1+\lambda)\right] = \mathbb{P}\left[\inf_{s\in(\bar{\tau},\bar{\tau}+\delta)}\{B_{\langle M\rangle_s - \langle M\rangle_{\bar{\tau}}}\} > -\delta(1+\lambda)\right].$$

Now as $\bar{\tau} = \tau \wedge (\bar{T} - \delta) > 0$ almost surely, $\langle M\rangle_s - \langle M\rangle_{\bar{\tau}} \geq c_M(s-\bar{\tau})$ for any $s > \bar{\tau}$ almost surely. So,

$$\mathbb{P}\left[\inf_{s\in(\bar{\tau},\bar{\tau}+\delta)}\{B_{\langle M\rangle_s - \langle M\rangle_{\bar{\tau}}}\} > -\delta(1+\lambda)\right] \leq \mathbb{P}\left[\inf_{s\in(0,\delta)}\{B_{c_M s}\} > -\delta(1+\lambda)\right].$$

By the reflection principle of Brownian motion, we have

$$\begin{aligned}
\mathbb{P}\left[\inf_{s\leq\delta} B_{c_M s} \leq -\delta(1+\lambda)\right] &= 2\mathbb{P}\left[B_{\delta c_M} \leq -\delta(1+\lambda)\right] \\
&= 2(2\pi)^{-1/2}\int_{-\infty}^{-\delta^{1/2}c_M^{-1/2}(1+\lambda)} e^{-\frac{y^2}{2}}\,\mathrm{d}y \\
&\geq 1 - 2\delta^{1/2}(1+\lambda)(2\pi c_M)^{-1/2}.
\end{aligned}$$

In conclusion, we have shown

$$\begin{aligned}
\mathbb{P}\left[\inf_{s\in(\bar{\tau},\bar{\tau}+\delta)}\{M_s - M_{\bar{\tau}} + (s-\bar{\tau})\lambda\} > -\delta\right] &\leq \mathbb{P}\left[\inf_{s\in(0,\delta)}\{B_{c_M s}\} > -\delta(1+\lambda)\right] \\
&\leq 2\delta^{1/2}(1+\lambda)(2\pi c_M)^{-1/2}.
\end{aligned}$$

Setting $\lambda = \delta^{-1/4}$, then by continuity of measure and the above, the expression in (2.6.2) converges to 0 as we send δ to zero. This completes the proof. □

Lemma 2.6.3 (Convergence of Stopping Times)**.** *Consider a sequence of functions $(\eta^n)_{n\geq 1}$ in $D_{\mathbb{R}}$ converging towards some $\eta \in D_{\mathbb{R}}$ with respect to the M_1-topology. We assume that η has the following crossing property:*

$$\forall\, h > 0\ \tau_0(\eta) < \bar{T} \qquad \Longrightarrow \qquad \inf_{s\in(\tau_0(\eta),(\tau_0(\eta)+h)\wedge\bar{T})}\left\{\eta_s - \eta_{\tau_0(\eta)}\right\} < 0, \tag{2.6.3}$$

where τ_0 is defined as in (2.4.1) and $\Delta\eta_t \leq 0$ for all $t \in [-1, \bar{T}]$. Then we have

$$\lim_{n\to\infty} \tau_0(\eta^n) = \tau_0(\eta).$$

Proof. The proof is composed of two steps. We shall show that $\limsup_{n\to\infty}\tau_0(\eta^n)\le\tau_0(\eta)\le\liminf_{n\to\infty}\tau_0(\eta^n)$. Hence, we will have equality and the claim follows.

Step 1: $\limsup_{n\to\infty}\tau_0(\eta^n)\le\tau_0(\eta)$

We define the set of continuity points of η to be $\mathbb{T}^\eta:=\{t\in[-1,\bar T]:\eta_t=\eta_{t-}\}$. We remark that $\mathbb{T}^\eta$ is co-countable by [3, Section 13]. As $\tau_0(\eta)<\bar T$, by (2.6.3) for any fixed $m\in\mathbb{N}$ there exists a $t\in(\tau_0(\eta),(\tau_0(\eta)+1/m)\wedge\bar T)\cap\mathbb{T}^\eta$ such that $\eta_t<0$. Now, as t is a continuity point of η, by [71, Theorem 12.5.1], we have that $\eta^n_t\to\eta_t$ in $\mathbb{R}$ as $n\to\infty$. Therefore, for large n, $\eta^n_t<0$ hence

$$\limsup_{n\to\infty}\tau_0(\eta^n)\le t\le\tau_0(\eta)+\frac{1}{m}.$$

As $m\in\mathbb{N}$ was arbitrary, the claim follows.

Step 2: $\liminf_{n\to\infty}\tau_0(\eta^n)\ge\tau_0(\eta)$

As $\eta^n\to\eta$ in the M_1-topology, we may find a sequence of parametric representations $((u^n,r^n))_{n\ge1}$ of $(\eta^n)_{n\ge1}$ which converges uniformly to a parametric representation (u,r) of η, see [71, Theorem 12.5.1]. Therefore, we may find an $s^n\in[0,1]$ such that $(u^n_{s^n},r^n_{s^n})=(\eta^n_{\tau_0(\eta^n)},\tau_0(\eta^n))$. By Step 1, since $\tau_0(\eta)<\bar T$, we have

$$\liminf_{n\to\infty}\tau_0(\eta^n)\le\limsup_{n\to\infty}\tau_0(\eta^n)\le\tau_0(\eta)<\bar T.$$

Therefore, by the finiteness of $\liminf_{n\to\infty}\tau_0(\eta^n)$ and compactness of $[0,1]$, we may find a subsequence n_k such that $\tau_0(\eta^{n_k})\to\liminf_{n\to\infty}\tau_0(\eta^n)$ and $s_{n_k}\to s$ for some $s\in[0,1]$. By the uniform convergence of the parametric representations

$$\begin{aligned}\liminf_{k\to\infty}\eta^{n_k}_{\tau_0(\eta^{n_k})}&=\liminf_{k\to\infty}u^{n_k}_{s^{n_k}}=u_s,\\ \liminf_{k\to\infty}\tau_0(\eta^{n_k})&=\liminf_{k\to\infty}r^{n_k}_{s^{n_k}}=r_s.\end{aligned}$$

As $r_s=\liminf_{n\to\infty}\tau_0(\eta^n)$, we may find $\gamma\in[0,1]$ such that $u_s=\gamma\eta_{(\liminf_{n\to\infty}\tau_0(\eta^n))-}+(1-\gamma)\eta_{\liminf_{n\to\infty}\tau_0(\eta^n)}$. We also note $u_s\le0$ as $\liminf_{k\to\infty}\eta^{n_k}_{\tau_0(\eta^{n_k})}\le0$. Lastly, as $\Delta\eta_t\le0$ for all t, we have $\eta_{\liminf_{n\to\infty}\tau_0(\eta^n)}\le0$. Therefore, $\tau_0(\eta)\le\liminf_{n\to\infty}\tau_0(\eta^n)$. This completes the proof. □

Lemma 2.6.4 (Functional Continuity II)**.** *Let $\mu\in\mathcal{P}(D_\mathbb{R})$ be any measure such that*

$$\mu\left(\inf_{s\in(\tau_0,\,(\tau_0+h)\wedge\bar T)}\{\eta_s-\eta_{\tau_0}\}\ge0,\ \tau_0<\bar T\right)=0,\tag{2.6.4}$$

for any $h>0$. Then, for any sequence of measures $(\mu^n)_{n\ge1}$ such that $\mu^n\implies\mu$ in $(\mathcal{P}(D_\mathbb{R}),\mathfrak{T}^{wk}_{M_1})$, we have

$$\nu^{\mu^n}_t:=\mu^n(\eta_t\in\cdot,\ \tau_0>t)\implies\nu^\mu_t:=\mu(\eta_t\in\cdot,\ \tau_0>t)$$

in $\mathbf{M}_{\leq 1}(\mathbb{R})$*, for any* $t \in \mathbb{T}^\mu$*, where* $\mathbb{T}^\mu$ *is defined as*

$$\mathbb{T}^\mu := \left\{ t \in [-1, \bar{T}] \ : \ \mu(\eta_t = \eta_{t-}) = 1, \, \mu(\tau_0 = t) = 0 \right\}.$$

Proof. The proof is an application of the Continuous Mapping Theorem [3, Theorem 2.7]. We only need to construct μ-almost sure continuous maps.

Step 1: Projection of measures from $D_{\mathbb{R}}$ to $\mathbb{R} \times \{0, 1\}$

Choose a $t \in \mathbb{T}^\mu$ and consider the map

$$(X_t, \, \mathbb{1}_{\{\tau_0 > t\}}) \ : \ D_{\mathbb{R}} \to \mathbb{R} \times \{0, 1\}, \qquad \eta \mapsto (\eta_t, \, \mathbb{1}_{\{\tau_0(\eta) > t\}}). \tag{2.6.5}$$

M_1-convergence implies pointwise convergence at continuity points, [71, Theorem 12.5.1]; therefore, as $t \in \mathbb{T}^\mu$, X_t is M_1-continuous μ-almost surely. Furthermore, as (2.6.4) holds, by Lemma 2.4.4, τ_0 is an M_1-continuous map μ-almost surely. Therefore, since $\mu(\tau_0 = t) = 0$, $\mathbb{1}_{\{\tau_0 > t\}}$ is M_1-continuous μ-almost surely. Hence, $(X_t, \, \mathbb{1}_{\{\tau_0 > t\}})$ is a μ-almost sure continuous map. By the Continuous Mapping Theorem,

$$(X_t, \, \mathbb{1}_{\{\tau_0 > t\}})^{\#}\mu^n \implies (X_t, \, \mathbb{1}_{\{\tau_0 > t\}})^{\#}\mu \quad \text{in} \quad \mathcal{P}(\mathbb{R} \times \{0, 1\}).$$

Step 2: Weak convergence of sub-probability measures

For any $f \in \mathcal{C}_b(\mathbb{R})$, define the map

$$\hat{f} \ : \ \mathbb{R} \times \{0, 1\} \to \mathbb{R}, \qquad (x, y) \mapsto f(x)\mathbb{1}_{\{1\}}(y).$$

It is clear that $\hat{f} \in \mathcal{C}_b(\mathbb{R} \times \{0, 1\})$. By Step 1,

$$\langle (X_t, \, \mathbb{1}_{\{\tau_0 > t\}})^{\#}\mu^n, \hat{f} \rangle \to \langle (X_t, \, \mathbb{1}_{\{\tau_0 > t\}})^{\#}\mu, \hat{f} \rangle.$$

By definition, $\langle (X_t, \, \mathbb{1}_{\{\tau_0 > t\}})^{\#}\mu^n, \hat{f} \rangle = \nu_t^{\mu^n}(f)$ and $\langle (X_t, \, \mathbb{1}_{\{\tau_0 > t\}})^{\#}\mu, \hat{f} \rangle = \nu_t^{\mu}(f)$. Thus, $\nu_t^{\mu^n}(f) \to \nu_t^{\mu}(f)$. The conclusion now follows from the Portmanteau Theorem. □

Lemma 2.6.5 (Weak convergence of sub-probability measures)**.** *Suppose that* $\tilde{\mathbf{P}}^{\varepsilon_n} \implies \mathbf{P}^*$ *on* $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{wk})$ *for a positive sequence* $(\varepsilon_n)_{n \geq 1}$ *which converges to zero. Set*

$$\mathbb{T} := \left\{ t \in [-1, \bar{T}] \ : \ \mathbb{E}\left[\mathbf{P}^*(\eta_t = \eta_{t-})\right] = 1, \, \mathbb{E}\left[\mathbf{P}^*(\tau_0 = t)\right] = 0 \right\}.$$

Then, for any $t \in \mathbb{T}$*,*

$$\boldsymbol{\nu}_t^{\varepsilon_n} \implies \boldsymbol{\nu}_t^* := \mathbf{P}^*(\eta_t \in \cdot, \, \tau_0 > t) \quad \textit{in} \quad \mathbf{M}_{\leq 1}(\mathbb{R}).$$

Proof. By the definition of $\mathbb{T}$ and Lemma 2.4.2, for any $t \in \mathbb{T}$, there exists a set of measures μ with full Law($\mathbf{P}^*$)-measure such that the following conditions hold simultaneously:

$$\begin{aligned} \mu\left(\inf_{s\in(\tau_0(\eta),\,(\tau_0(\eta)+h)\wedge\bar{T})}\{\eta_s-\eta_{\tau_0(\eta)}\}\geq 0,\ \tau_0(\eta)<\bar{T}\right)&=0,\\ \mu(\eta_t=\eta_{t-})&=1,\\ \mu(\tau_0=t)&=0. \end{aligned} \tag{2.6.6}$$

As $\tilde{\mathbf{P}}^{\varepsilon_n} \implies \mathbf{P}^*$, by Skorokhod's Representation Theorem there exist $(\mathbf{Q}^n)_{n\geq 1}$ and $\mathbf{Q}^*$ such that $\mathbf{Q}^n \to \mathbf{Q}^*$ almost surely, Law($\mathbf{Q}^n$) = Law($\tilde{\mathbf{P}}^{\varepsilon_n}$), Law($\mathbf{Q}^*$) = Law($\mathbf{P}^*$), and $\mathbf{Q}^*$ satisfies (2.6.6) almost surely. Set

$$\nu_t^{\mathbf{Q}^n} = \mathbf{Q}^n(\eta_t \in \cdot,\ \tau_0 > t) \quad \text{and} \quad \nu_t^{\mathbf{Q}^*} = \mathbf{Q}^*(\eta_t \in \cdot,\ \tau_0 > t).$$

By Lemma 2.6.4, $\nu_t^{\mathbf{Q}^n} \to \nu_t^{\mathbf{Q}^*}$ almost surely in $\mathbf{M}_{\leq 1}(\mathbb{R})$. Now, for any $G \in \mathcal{C}_b(\mathbf{M}_{\leq 1}(\mathbb{R}))$, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty}\mathbb{E}\left[G(\boldsymbol{\nu}_t^{\varepsilon_n})\right] = \lim_{n\to\infty}\mathbb{E}\left[G(\nu_t^{\mathbf{Q}^n})\right] = \mathbb{E}\left[G(\nu_t^{\mathbf{Q}^*})\right] = \mathbb{E}\left[G(\boldsymbol{\nu}_t^*)\right].$$

The result now follows from the Portmanteau Theorem. □

Lemma 2.6.6 (Functional Continuity III)**.** *Let $\mu \in \mathcal{P}(D_{\mathbb{R}})$ be any measure such that*

$$\mu\left(\inf_{s\in(\tau_0(\eta),\,(\tau_0(\eta)+h)\wedge\bar{T})}\{\eta_s-\eta_{\tau_0(\eta)}\}\geq 0,\ \tau_0(\eta)<\bar{T}\right)=0, \tag{2.6.7}$$

for any $h > 0$, and let $g(t,x,\nu)$ be any function satisfying Assumption 2.2.2 i. Then $\int_0^t g(s,\eta_s^n,\nu_s^{\mu^n})\,\mathrm{d}s$ converges to $\int_0^t g(s,\eta_s,\nu_s^{\mu})\,\mathrm{d}s$ for any $t \geq 0$ whenever $(\eta^n,\mu^n) \to (\eta,\mu)$ in $(D_{\mathbb{R}},M_1)\times(\mathcal{P}(D_{\mathbb{R}}),\mathfrak{T}_{M_1}^{wk})$ along a sequence for which $\langle\mu^n, \sup_{s\leq\bar{T}}|\tilde{\eta}_s|^p\rangle$ is bounded uniformly in n for some $p > 1$. For any measure $m \in \mathcal{P}(D_{\mathbb{R}})$, we define $\nu_s^m := m(\tilde{\eta}_s \in \cdot,\ \tau_0(\tilde{\eta}) > s)$.

Proof. By Assumption 2.2.2,

$$\left|g(s,\eta_s^n,\nu_s^{\mu^n})\right| \leq C\Big(1+\sup_{m\geq 1}\sup_{u\leq\bar{T}}|\eta_u^m| + \sup_{m\geq 1}\langle\mu^m, \sup_{u\leq\bar{T}}|\tilde{\eta}_u|\rangle\Big) \tag{2.6.8}$$

The right-hand side of (2.6.8) is finite because $\eta^n \to \eta$ in $(D_{\mathbb{R}},M_1)$ and by our assumption. Thus, it is sufficient to show that $g(s,\eta_s^n,\nu_s^{\mu^n})$ converges to $g(s,\eta_s,\nu_s^{\mu})$ on a set of full Lebesgue measure. The conclusion then follows from the Dominated Convergence Theorem.

By Assumption 2.2.2,

$$\left|g(s,\eta_s^n,\nu_s^{\mu^n}) - g(s,\eta_s,\nu_s^{\mu})\right| \le C\left|\eta_s^n - \eta_s\right| + C(1+|\eta_s|+\langle\nu_s^{\mu^n},|\cdot|\rangle)d_0(\nu_s^{\mu^n},\nu_s^{\mu}). \quad (2.6.9)$$

For any $s \in \mathbb{T}^{\mu} := \left\{t \in [-1,\bar{T}] \,:\, \mu(\eta_t = \eta_{t-}) = 1,\, \mu(\tau_0 = t) = 0\right\}$, the first term converges to zero as $\eta^n \to \eta$ in $(D_{\mathbb{R}}, M_1)$. For the second term in (2.6.9), it suffices to show $d_0(\nu_s^{\mu^n}, \nu_s^{\mu}) \to 0$, since $(1+|\eta_s|+\langle\nu_s^{\mu^n},|\cdot|\rangle)$ is bounded uniformly in n by our assumption.

Recall that

$$d_0(\nu_s^{\mu^n}, \nu_s^{\mu}) = \sup\left\{\left|\langle\nu_s^{\mu^n} - \nu_s^{\mu}, \psi\rangle\right| \,:\, \psi \in \mathcal{C}_{d_0}\right\}, \quad (2.6.10)$$

where $\mathcal{C}_{d_0} := \{\psi \in \mathcal{C}(\mathbb{R}) \,:\, \|\psi\|_{\text{Lip}} \le 1,\, |\psi(0)| \le 1\}$. Fix a $\delta > 0$, then for any $\lambda > 1$, by the Arzerlà-Ascoli Theorem, there exists a finite family of functions $\psi_1, \ldots, \psi_m \in \mathcal{C}_{d_0}$ supported on $[-\lambda-1, \lambda+1]$, for $m = m(\lambda) \in \mathbb{N}$ such that for any $\psi \in \mathcal{C}_{d_0}$,

$$\sup_{x\in[-\lambda,\lambda]} |\psi(x) - \psi_i(x)| < \delta/2 \quad (2.6.11)$$

for some $i \in \{1,\ldots,m\}$. Fixing any $\psi \in \mathcal{C}_{d_0}$ and choosing a ψ_i such that (2.6.11) holds, we have

$$\begin{aligned}\left|\langle\nu_s^{\mu^n} - \nu_s^{\mu}, \psi\rangle\right| \le &\left|\int_0^{\lambda} (\psi - \psi_i)\,\mathrm{d}(\nu_s^{\mu^n} - \nu_s^{\mu})\right| + \left|\int_{\lambda}^{\infty} (\psi - \psi_i)\,\mathrm{d}(\nu_s^{\mu^n} - \nu_s^{\mu})\right| \\ &+ \left|\langle\nu_s^{\mu^n} - \nu_s^{\mu}, \psi_i\rangle\right|.\end{aligned}$$

By our choice of ψ_i, the first term is bounded by δ uniformly in n. By the linear growth condition for functions in $\mathcal{C}_{d_0}$,

$$\left|\int_{\lambda}^{\infty} (\psi - \psi_i)\,\mathrm{d}(\nu_s^{\mu^n} - \nu_s^{\mu})\right| \le C\langle\nu_s^{\mu^n} + \nu_s^{\mu}, |\cdot|\,\mathbb{1}_{[\lambda,\infty)}\rangle$$

for a constant C independent of n. Consequently, by the definition of $\nu_s^{\mu^n}$,

$$\begin{aligned}\langle\nu_s^{\mu^n}, |\cdot|\,\mathbb{1}_{[\lambda,\infty)}\rangle \le \int_{D_{\mathbb{R}}} |\tilde{\eta}_s|\,\mathbb{1}_{\{\tilde{\eta}_s>\lambda\}}\,\mathrm{d}\mu^n(\tilde{\eta}) &\le \langle\mu^n, \sup_{u\le\bar{T}}|\tilde{\eta}_u|^p\rangle^{\frac{1}{p}}\mu^n\left(\sup_{u\le\bar{T}}|\tilde{\eta}_u| > \lambda\right)^{\frac{p-1}{p}} \\ &\le \frac{1}{\lambda^{p-1}}\langle\mu^n, \sup_{u\le\bar{T}}|\tilde{\eta}_u|^p\rangle.\end{aligned}$$

The second inequality follows from Hölder's inequality, while the last inequality follows from Markov's inequality. We may also employ the same argument to upper bound the $\langle\nu_s^{\mu}, |\cdot|\,\mathbb{1}_{[\lambda,\infty)}\rangle$ term in the above. Hence, by these upper bounds and as $\mu^n \implies \mu$, we deduce that

$$\left|\int_{\lambda}^{\infty} (\psi - \psi_i)\,\mathrm{d}(\nu_s^{\mu^n} - \nu_s^{\mu})\right| \le \frac{C}{\lambda^{p-1}}\sup_{m\ge1}\langle\mu^m, \sup_{u\le\bar{T}}|\tilde{\eta}_u|^p\rangle,$$

where C is independent of n. Returning to (2.6.10), we have shown that

$$d_0(\nu_s^{\mu^n}, \nu_s^{\mu}) \leq \delta + \frac{C}{\lambda^{p-1}} \sup_{m\geq 1} \langle \mu^m, \sup_{u\leq \bar{T}} |\tilde{\eta}_u|^p \rangle + \sum_{i=1}^{m} |\langle \nu_s^{\mu^n} - \nu_s^{\mu}, \psi_i \rangle|.$$

The middle term will vanish uniformly in n as $\lambda \to \infty$. Thus, we fix a λ sufficiently large such that the middle term is bounded by δ uniformly in n. As λ is now fixed, so is m. By Lemma 2.6.4, $\nu_s^{\mu^n} \implies \nu_s^{\mu}$ in $\mathbf{M}_{\leq 1}(\mathbb{R})$; hence, as the ψ_i's have compact support, they are continuous and bounded. Therefore, the final term will be smaller than δ for all n sufficiently large. Consequently, for all n sufficiently large, $d_0(\nu_s^{\mu^n}, \nu_s^{\mu}) \leq C\delta$ for some constant C independent of n. Hence, $d_0(\nu_s^{\mu^n}, \nu_s^{\mu}) \to 0$ as $n \to \infty$. This completes the proof. □

Lemma 2.6.7. *Fix any $t < T$. There exists a constant $C > 0$, independent of ε and t, such that for any $h < 1 \wedge (T - t)$ we have*

$$\mathbb{P}\Big[\boldsymbol{\nu}_t^{\varepsilon}\big[0, \alpha_t\lambda + Ch^{1/3} + \alpha_t(L_t^{\varepsilon} - \mathfrak{L}_t^{\varepsilon}) + (\alpha_{t+h} - \alpha_t)\big] \geq \lambda, \forall\, \lambda \leq L_{t+h}^{\varepsilon} - L_t^{\varepsilon} - Ch^{1/3}\Big] \geq 1 - Ch^{1/3}$$

Proof. To begin, fix an $h > 0$ such that $h < 1 \wedge (T - t)$ and fix a $\lambda \in \mathbb{R}$. Then we denote by E_1^{λ} the event

$$E_1^{\lambda} := \Bigg\{ 0 \geq X_t^{\varepsilon} - hC\Bigg(1 + \sup_{u\leq t+h} |X_u^{\varepsilon}| + \mathbb{E}\Big[\sup_{u\leq t+h} |X_u^{\varepsilon}| \,\Big|\, W^0\Big]\Bigg) - \sup_{u\leq h} \big|\mathcal{Y}_{t+u}^{\varepsilon} - \mathcal{Y}_t^{\varepsilon}\big| - \alpha_t\lambda - \alpha_t(L_t^{\varepsilon} - \mathfrak{L}_t^{\varepsilon}) - (\alpha_{t+h} - \alpha_t),\, \tau^{\varepsilon} > t \Bigg\},$$

where C is the constant from the linear growth condition on b. Now fix $\lambda \leq L_{t+h}^{\varepsilon} - L_t^{\varepsilon}$. By the continuity of the loss process, [31, Theorem 2.4], there exists an $s \leq h$ such that $\lambda = L_{t+s}^{\varepsilon} - L_t^{\varepsilon}$. Employing the integration by parts formula, we observe for any $u \in [t, t + s]$

$$\begin{aligned}\int_t^u \alpha_v \,\mathrm{d}\mathfrak{L}_v^{\varepsilon} &= \alpha_u\mathfrak{L}_u^{\varepsilon} - \alpha_t\mathfrak{L}_t^{\varepsilon} - \int_t^u \alpha'(v)\mathfrak{L}_v^{\varepsilon}\,\mathrm{d}v \leq \alpha_u\mathfrak{L}_u^{\varepsilon} - \alpha_t\mathfrak{L}_t^{\varepsilon} \\ &\leq \alpha_{t+s}L_{t+s}^{\varepsilon} - \alpha_t\mathfrak{L}_t^{\varepsilon} \leq \alpha_t L_{t+s}^{\varepsilon} + (\alpha_{t+s} - \alpha_t) - \alpha_t\mathfrak{L}_t^{\varepsilon},\end{aligned}$$

where to establish upper bounds we use the fact that α is non-negative and non-

decreasing, and $\mathfrak{L}^\varepsilon_v \le L^\varepsilon_v \le 1$ for any $v \ge 0$. Therefore, for any $u \in [t, t+s]$

$$\begin{aligned}
X^\varepsilon_u &= X^\varepsilon_t + (X^\varepsilon_u - X^\varepsilon_t) = X^\varepsilon_t + \int_t^u b(v, X^\varepsilon_v, \boldsymbol{\nu}^\varepsilon_v)\,\mathrm{d}v + \mathcal{Y}^\varepsilon_u - \mathcal{Y}^\varepsilon_t - \int_t^u \alpha_v\,\mathrm{d}\mathfrak{L}^\varepsilon_v \\
&\ge X^\varepsilon_t - hC\left(1 + \sup_{u\le t+h}|X^\varepsilon_u| + \mathbb{E}\left[\sup_{u\le t+h}|X^\varepsilon_u|\,|\,W^0\right]\right) - \sup_{u\le h}\left|\mathcal{Y}^\varepsilon_{t+u} - \mathcal{Y}^\varepsilon_t\right| \\
&\quad - \{\alpha_t L^\varepsilon_{t+s} + (\alpha_{t+s} - \alpha_t) - \alpha_t \mathfrak{L}^\varepsilon_t \pm \alpha_t L^\varepsilon_t\} \\
&\ge X^\varepsilon_t - hC\left(1 + \sup_{u\le t+h}|X^\varepsilon_u| + \mathbb{E}\left[\sup_{u\le t+h}|X^\varepsilon_u|\,|\,W^0\right]\right) - \sup_{u\le h}\left|\mathcal{Y}^\varepsilon_{t+u} - \mathcal{Y}^\varepsilon_t\right| \\
&\quad - \alpha_t\lambda - (\alpha_{t+s} - \alpha_t) - \alpha_t(L^\varepsilon_t - \mathfrak{L}^\varepsilon_t)
\end{aligned}$$

Therefore, as L^ε is W^0-measurable and conditioning on W^0 fixes L^ε, we have

$$\mathbb{P}\left[E^\lambda_1 \middle|\, W^0\right] \ge \mathbb{P}\left[\inf_{t\le u\le t+s} X^\varepsilon_u \le 0, \tau^\varepsilon > t \middle|\, W^0\right] = L^\varepsilon_{t+s} - L^\varepsilon_t = \lambda.$$

Now, for any fixed $\lambda \le L^\varepsilon_{t+h} - L^\varepsilon_t - 2h^{1/3}$, set $\lambda_0 = \lambda + 2h^{1/3}$. We define the event

$$E_2 \coloneqq \left\{hC(1 + \sup_{u\le t+h}|X^\varepsilon_u| + \mathbb{E}[\sup_{u\le t+h}|X^\varepsilon_u|\,|\,W^0]) + \sup_{u\le h}\left|\mathcal{Y}^\varepsilon_{t+u} - \mathcal{Y}^\varepsilon_t\right| \ge h^{1/3}\right\}.$$

Then on the event $E^{\lambda_0}_1 \cap E^{\complement}_2$

$$\begin{aligned}
X^\varepsilon_t - \alpha_t\lambda_0 &\le hC(1 + \sup_{u\le t+h}|X^\varepsilon_u| + \mathbb{E}[\sup_{u\le t+h}|X^\varepsilon_u|\,|\,W^0]) + \sup_{u\le h}\left|\mathcal{Y}^\varepsilon_{t+u} - \mathcal{Y}^\varepsilon_t\right| \\
&\quad + \alpha_t(L^\varepsilon_t - \mathfrak{L}^\varepsilon_t) + (\alpha_{t+h} - \alpha_t) \\
&\le h^{1/3} + \alpha_t(L^\varepsilon_t - \mathfrak{L}^\varepsilon_t) + (\alpha_{t+h} - \alpha_t).
\end{aligned}$$

Therefore, on the same event

$$X^\varepsilon_t - \alpha_t\lambda = X^\varepsilon_t - \alpha_t\lambda_0 + 2\alpha_t h^{1/3} \le (1 + 2\alpha_t)h^{1/3} + \alpha_t(L^\varepsilon_t - \mathfrak{L}^\varepsilon_t) + (\alpha_{t+h} - \alpha_t).$$

Consequently, we deduce

$$\begin{aligned}
\boldsymbol{\nu}^\varepsilon_t([0, \alpha_t\lambda + (1+2\alpha_t)h^{1/3} + \alpha_t(L^\varepsilon_t - \mathfrak{L}^\varepsilon_t) + (\alpha_{t+h} - \alpha_t)]) &\ge \mathbb{P}\left[E^{\lambda_0}_1 \cap E^{\complement}_2 \middle|\, W^0\right] \\
&\ge \mathbb{P}\left[E^{\lambda_0}_1 \middle|\, W^0\right] - \mathbb{P}\left[E_2 |\, W^0\right] \\
&\ge \lambda_0 - \mathbb{P}\left[E_2 |\, W^0\right].
\end{aligned}$$

Therefore, if we have control over the mass $\mathbb{P}\left[E_2 |\, W^0\right]$, we may estimate the mass with respect to $\boldsymbol{\nu}_t$ that is near the boundary. Defining the event $E_3 \coloneqq \{\mathbb{P}\left[E_2 |\, W^0\right] \le h^{1/3}\}$, we deduce on E_3 that

$$\boldsymbol{\nu}^\varepsilon_t([0, \alpha_t\lambda + (1+2\alpha_t)h^{1/3} + \alpha_t(L^\varepsilon_t - \mathfrak{L}^\varepsilon_t) + (\alpha_{t+h} - \alpha_t)]) \ge \lambda_0 - h^{1/3} \ge \lambda.$$

The last inequality follows from the fact that $\lambda_0 = \lambda + 2h^{1/3}$. Now we only need to find a C independent of ε, h, and t such that $\mathbb{P}[E_3^{\complement}] \leq Ch^{1/3}$. By applying Markov's inequality twice, we get

$$\begin{aligned}
\mathbb{P}\left[E_3^{\complement}\right] \leq & h^{-1/3}\mathbb{P}\left[E_2\right] \\
\leq & h^{-1/3}\mathbb{P}\left[hC(1+\sup_{u\leq t+h}|X_u^\varepsilon| + \mathbb{E}[\sup_{u\leq t+h}|X_u^\varepsilon||W^0]) \geq 2^{-1}h^{1/3}/2\right] \\
& + h^{-1/3}\mathbb{P}\left[\sup_{u\leq h}\left|\mathcal{Y}_{t+u}^\varepsilon - \mathcal{Y}_t^\varepsilon\right| \geq 2^{-1}h^{1/3}/2\right] \\
\leq & 2Ch^{1/3}\mathbb{E}\left[1+\sup_{u\leq t+h}|X_u^\varepsilon| + \mathbb{E}[\sup_{u\leq t+h}|X_u^\varepsilon||W^0]\right] \\
& + 2^8 h^{-3}\mathbb{E}\left[\sup_{u\leq h}\left|\mathcal{Y}_{t+u}^\varepsilon - \mathcal{Y}_t^\varepsilon\right|^8\right] \\
\leq & c_1(h^{1/3}+h) \leq 2c_1 h^{1/3},
\end{aligned}$$

where c_1 depends on the constant from Proposition 2.3.1, the constant from the Burkholder-Davis-Gundy inequality to bound the second term, and the uniform bounds on σ, but is notably independent of ε. Therefore, setting $C = \max\{1 + 2\alpha(T), c_1\}$ completes the proof. □

Lemma 2.6.8. *Suppose that* $\tilde{\mathbf{P}}^{\varepsilon_n} \implies \mathbf{P}^*$ *on* $(\mathcal{P}(D_\mathbb{R}), \mathfrak{T}_{M_1}^{wk})$ *for a positive sequence* $(\varepsilon_n)_{n\geq 1}$ *which converges to zero. Set*

$$\mathbb{T} := \left\{t \in [-1, \bar{T}] \ : \ \mathbb{E}\left[\mathbf{P}^*(\eta_t = \eta_{t-})\right] = 1, \mathbb{E}\left[\mathbf{P}^*(\tau_0 = t)\right] = 0\right\}.$$

Then, for any $t \in \mathbb{T} \cap [0, T)$ *and* $h > 0$ *such that* $t + h \in \mathbb{T} \cap [0, T)$*, we have*

$$\begin{aligned}
\mathbb{P}\Big[\boldsymbol{\nu}_t^*\left[0, \alpha(t)\lambda + Ch^{1/3} + \alpha(t+h) - \alpha(t)\right] \geq \lambda, \forall \lambda \leq L_{t+h}^* - L_t^* - Ch^{1/3}\Big] \\
\geq 1 - Ch^{1/3},
\end{aligned}$$

for some constant $C > 0$ *independent of* t *and* h*.*

Proof. As $\tilde{\mathbf{P}}^{\varepsilon_n} \implies \mathbf{P}^*$, by employing Skorokhod's Representation Theorem, there exist $(\boldsymbol{\mu}^n)_{n\geq 1}$ and $\boldsymbol{\mu}$ such that $\text{Law}(\boldsymbol{\mu}^n) = \text{Law}(\tilde{\mathbf{P}}^{\varepsilon_n})$, $\text{Law}(\boldsymbol{\mu}) = \text{Law}(\mathbf{P}^*)$, and $\boldsymbol{\mu}^n \to \boldsymbol{\mu}$ almost surely in $(\mathcal{P}(D_\mathbb{R}), \mathfrak{T}_{M_1}^{\text{wk}})$. As $\text{Law}(\mathbf{P}^*)$-almost every measure μ satisfies (2.6.4) by Lemma 2.4.2, then by Lemma 2.6.4, $\nu_t^{\boldsymbol{\mu}^n} \to \nu_t^{\boldsymbol{\mu}}$ almost surely for any $t \in \mathbb{T}$. Furthermore, for any $t \in \mathbb{T} \cap [0, T)$, by Lemma 2.4.4 and Corollary 2.4.5, we have

$$\boldsymbol{\mu}^n(\tau_0 \leq t) \to \boldsymbol{\mu}(\tau_0 \leq t)$$

and

$$\int_0^t \mathfrak{K}^{\varepsilon_n}(t-s)\boldsymbol{\mu}^n(\tau_0 \le s)\,\mathrm{d}s \to \int_0^t \mathfrak{K}^{\varepsilon_n}(t-s)\boldsymbol{\mu}(\tau_0 \le s)\,\mathrm{d}s$$

almost surely. Therefore, for simplicity and notational convenience in the remainder of this proof, we may suppose that

$$\begin{aligned} \boldsymbol{\nu}_t^{\varepsilon_n} &\to \boldsymbol{\nu}_t^* && \text{a.s. in } \mathbf{M}_{\le 1}(\mathbb{R}), \\ L_t^{\varepsilon_n} &\to L_t^* && \text{a.s. in } \mathbb{R}, \\ \mathfrak{L}_t^{\varepsilon_n} &\to L_t^* && \text{a.s. in } \mathbb{R}. \end{aligned}$$

Recall by Lemma 2.6.7,

$$\mathbb{P}\Big[\boldsymbol{\nu}_t^{\varepsilon}\big[0, \alpha_t\lambda + Ch^{1/3} + \alpha_t(L_t^{\varepsilon} - \mathfrak{L}_t^{\varepsilon}) + (\alpha_{t+h} - \alpha_t)\big] \ge \lambda,\ \forall\, \lambda \le L_{t+h}^{\varepsilon} - L_t^{\varepsilon} - Ch^{1/3}\Big] \ge 1 - Ch^{1/3}$$

for any $\varepsilon > 0$. It is well known that the Lévy-Prokhorov metric, d_L, metricizes weak convergence, [59, Theorem 1.11]. Fixing δ_1, δ_2, δ_3, $\delta_4 > 0$, we define the event

$$A_n := \left\{|L_t^{\varepsilon_n} - L_t^*| < \delta_1, \left|L_{t+h}^{\varepsilon_n} - L_{t+h}^*\right| < \delta_2, |\mathfrak{L}_t^{\varepsilon_n} - L_t^{\varepsilon_n}| < \delta_3, d_L(\boldsymbol{\nu}_t^{\varepsilon_n}, \boldsymbol{\nu}_t^*) < \delta_4\right\}.$$

Therefore,

$$\begin{aligned} &1 - Ch^{1/3} \\ &\le \mathbb{P}\Big[\boldsymbol{\nu}_t^{\varepsilon_n}\big[0, \alpha_t\lambda + Ch^{1/3} + \alpha_t(L_t^{\varepsilon_n} - \mathfrak{L}_t^{\varepsilon_n}) + (\alpha_{t+h} - \alpha_t)\big] \ge \lambda,\ \forall \lambda \le L_{t+h}^{\varepsilon_n} - L_t^{\varepsilon_n} - Ch^{1/3}, A_n\Big] \\ &\quad + \mathbb{P}[A_n^{\complement}] \\ &\le \mathbb{P}\Big[\delta_4 + \boldsymbol{\nu}_t^*\big(-\delta_4, \alpha_t\lambda + Ch^{1/3} + \alpha_t\delta_3 + (\alpha_{t+h} - \alpha_t) + \delta_4\big) \ge \lambda,\ \forall \lambda \le L_{t+h}^* - L_t^* - Ch^{1/3} - \delta_1 - \delta_2\Big] \\ &\quad + \mathbb{P}[A_n^{\complement}]. \end{aligned}$$

Sending $\varepsilon_n \to 0$, then $\mathbb{P}[A_n^{\complement}] \to 0$ by the Dominated Convergence Theorem as we have almost sure convergence. Lastly, by sending $\delta_1 \to 0$, $\delta_2 \to 0$, $\delta_3 \to 0$, $\delta_4 \to 0$ one at a time and in order, then by employing continuity of measure, we may conclude that

$$\mathbb{P}\left[\boldsymbol{\nu}_t^*[0, \alpha_t\lambda + Ch^{1/3} + (\alpha_{t+h} - \alpha_t)] \ge \lambda,\ \forall\, \lambda \le L_{t+h}^* - L_t^* - Ch^{1/3}\right] \ge 1 - Ch^{1/3}.$$

□

Lemma 2.6.9. *Suppose that* $\tilde{\mathbf{P}}^{\varepsilon_n} \implies \mathbf{P}^*$ *on* $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{wk})$ *for a positive sequence* $(\varepsilon_n)_{n\ge 1}$ *which converges to zero. Then, we have*

$$\Delta L_t^* \le \inf\left\{x \ge 0 \,:\, \boldsymbol{\nu}_{t-}^*([0,\, \alpha(t)x]) < x\right\} \tag{2.6.12}$$

almost surely for all $t \in [0,\, T)$.

Proof. It is clear that (2.6.12) holds for any $t \in \mathbb{T}$. Hence, we must only show the upper bound for $t \notin \mathbb{T}$. We first consider the case when $t \in (0,T) \cap \mathbb{T}^{\complement}$. The case when $t = 0$ will be treated separately. As $\mathbb{T}$ is dense in $[0,T]$, we may find sequences $(t_n)_{n\geq 1}$, $(t_n + h_n)_{n\geq 1} \subset \mathbb{T}$ such that $t_n \uparrow t$, $t_n + h_n \downarrow t$ and $h_n < 2^{-3n}$. Now by the Borel-Cantelli Lemma, we have a set of full measure such that

$$\boldsymbol{\nu}^*_{t_n}([0, \alpha_{t_n}\lambda + Ch_n^{1/3} + (\alpha_{t_n+h_n} - \alpha_{t_n})]) \geq \lambda,\ \forall\, \lambda \leq L^*_{t_n+h_n} - L^*_{t_n} - Ch_n^{1/3}, \quad (2.6.13)$$

for all (possibly stochastic) n sufficiently large. Furthermore, by the Dominated Convergence Theorem, we have

$$\boldsymbol{\nu}^*_{t_n} \to \boldsymbol{\nu}^*_{t-}, \quad (2.6.14)$$

$\mathrm{Law}(\mathbf{P}^*)$-almost surely in $\mathbf{M}_{\leq 1}(\mathbb{R})$ as for any $\phi \in \mathcal{C}_b(\mathbb{R})$

$$\lim_{n\to\infty} \langle \boldsymbol{\nu}^*_{t_n}, \phi\rangle = \lim_{n\to\infty} \int_{D_{\mathbb{R}}} \phi(\eta_{t_n}) \mathbb{1}_{\{\tau_0 > t_n\}}\, \mathrm{d}\mathbf{P}^*(\eta) = \int_{D_{\mathbb{R}}} \phi(\eta_{t-}) \mathbb{1}_{\{\tau_0 \geq t\}}\, \mathrm{d}\mathbf{P}^*(\eta).$$

So on an event of full $\mathrm{Law}(\mathbf{P}^*)$-measure where (2.6.13) and (2.6.14) holds, by the Portmanteau Theorem for any $\delta > 0$, $\lambda < \Delta L^*_t$,

$$\begin{aligned}
\boldsymbol{\nu}^*_{t-}([0,\, \alpha(t)\lambda + \delta]) \geq& \limsup_{n\to\infty} \boldsymbol{\nu}^*_{t_n}([0,\, \alpha(t)\lambda + \delta]) \\
\geq& \limsup_{n\to\infty} \boldsymbol{\nu}^*_{t_n}([0, \alpha_{t_n}\lambda + Ch_n^{1/3} + (\alpha_{t_n+h_n} - \alpha_{t_n})]) \\
\geq& \lambda.
\end{aligned}$$

This holds as $\lambda < \Delta L^*_t$ and $L^*_{t_n+h_n} - L^*_{t_n} - Ch_n^{1/3} \to \Delta L^*_t$. Sending h to zero shows the claim for every $t > 0$.

In the case when $t = 0$, we have by Lemma 2.6.7

$$\mathbb{P}\left[\boldsymbol{\nu}^{\varepsilon}_{0-}[0, \alpha_0\lambda + Ch^{1/3} + (\alpha_h - \alpha_0)] \geq \lambda,\ \forall\, \lambda \leq L^{\varepsilon}_h - Ch^{1/3}\right] \geq 1 - Ch^{1/3}.$$

As $\boldsymbol{\nu}^{\varepsilon}_{0-} = \nu_{0-}$, where ν_{0-} is a deterministic measure, and $\boldsymbol{\nu}_{0-}$ is almost surely distributed as ν_{0-} almost surely, then we have by Lemma 2.6.8

$$\mathbb{P}\left[\boldsymbol{\nu}^{*}_{0-}[0, \alpha(0)\lambda + Ch^{1/3} + \alpha(h) - \alpha(0)] \geq \lambda,\ \forall\, \lambda \leq L^{*}_h - Ch^{1/3}\right] \geq 1 - Ch^{1/3}$$

for $h \in \mathbb{T}$. Now the rest of the proof follows similar arguments as above by choosing $h_n \in \mathbb{T}$ such that $h_n \downarrow 0$. □

Chapter 3

Mean–Field Limit of Integrate-And-Fire Neuronal Networks With Common Noise

3.1 Introduction

There has been a growing interest in the study of interacting particle systems with feedback through hitting times. They provide a simple and intuitive way to model complex systems ranging from finance, as models for systemic risk and defaults, [31, 29, 56, 28], to physics as a probabilistic formulation of the super-cooled Stefan problem, [14, 56], and in neuroscience to model the dynamics of the voltage of a system of neurons, [15, 16, 34].

In this chapter, we focus on the latter and study the mean-field limit of a system of noisy integrate-and-fire neurons. The noisy integrate-and-fire neuron model, introduced in [50, 58], models the dynamics of the voltage of a neuron as a stochastic process, X_t^i, with the following two rules:

- Whenever X_t^i hits its *firing threshold*, it is instantly reset to its resting potential. The firing threshold is a predefined voltage level that, once reached or exceeded by a neuron's voltage, triggers a neural action known as a *spike*. This event models the real-life process where a neuron, after reaching a certain electrical potential, fires an electrical signal through the nervous system.

- At such times when a spike occurs, all N neurons in the system receive an impulse proportional to $1/N$. This reflects the biological phenomenon where a single neuron's firing can slightly alter the state of numerous other neurons, contributing to the collective dynamics of the neuronal network.

These models aim to simplify conductance-based neuron models, such as the Hodgkin–Huxley model, which is composed of four non-linear coupled differential equations, [26, Section 2.2]. The noisy integrate-and-fire neuron model captures the feature that action potentials are stereotyped events: all spikes of a given neuron look the same, and the shape of the action potential does not carry any information. Rather, it is the number and timing of the spikes that matter, [26, 18]. Under these rules, this model has been studied from a probabilistic perspective in [15, 16] and from a Partial Differential Equation (PDE) perspective in [9, 11].

3.1.1 Extensions to the Noisy Integrate-And-Fire Neuron Model

This work expands on the models above to include various other aspects of neuron physiology to better represent the dynamics observed in the real world. We propose a model that includes the following features:

Delays: A common feature in the works above is the instantaneous transmission of the feedback from a neuron hitting its firing threshold, neglecting the effect of the dendritic tree, which is the primary site of synaptic integration. The dendritic tree is a complex structure, and the time taken for the signal to travel from the dendritic tree to the soma is non-negligible. In [4], the authors argue that the response of a neuron depends on the history of the input it has received due to the diffusive spread of activity along the dendritic tree. By employing the cable equation, one can rigorously justify how the delay in the impact of neuronal firings may occur; see [34, Section 3] for a detailed discussion.

Common noise: Finite systems of noisy integrate-and-fire neuronal models have been studied in the literature, namely in [19, 20, 41, 40]. Common noise can originate through a variety of sources. In [6, 5], it originates from the neurons forming a highly connected network where the common noise represents the shared connectivity. In [40], a finite system of uncoupled integrate-and-fire neurons with common noise is considered to model the synchrony induced by common stimuli due to overlapping receptive fields. Populations of uncoupled neurons with overlapping receptive fields can be found in auditory nerve cells [37], olfactory receptor neurons in insects, [72, 25], and electroreceptors in weakly electric fish [53]. The same system is studied in [41] to investigate the probability distribution of the number of synchronous action potentials.

Refractory period: For a short period after a spike (approximately 2 ms), a neuron enters a period called the absolute refractory period. During this period, the neuron

is unable to fire, [52]. This behaviour is exhibited in the Hodgkin–Huxley model, [26], but is generally not incorporated into the models above.

3.1.2 A Model With Delay, Common Noise and Refractory Period

We introduce a simplified extension to the integrate-and-fire neuron model, incorporating delays, common noise, and a refractory period, to better reflect the dynamics observed in neuronal systems. Our model is formulated on the negative half-line. When a neuron hits its firing threshold, identified here as 0, the neuron is temporarily held at 0, representing the absolute refractory period, before being reset to -1. The dynamics of the voltage of the neuron is given by the following stochastic differential equation (SDE):

$$\begin{cases} X_t^i = -\int_0^t \lambda(V_{\text{rest}} - X_s^i)\mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}s + \mu\int_0^t \mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}s \\ \qquad + \frac{\gamma}{N}\sum_{j=1}^N \int_0^t (X_s^j - X_s^i)\mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}s + \int_0^t \sigma(s,\,X_s^i)\sqrt{1-\rho^2(s,\nu_s^N)}\mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}W_s^i \\ \qquad + \int_0^t \sigma(s,\,X_s^i)\rho(s,\nu_s^N)\mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}W_s^0 + \alpha\int_0^t \mathfrak{f}_s^N \mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}s \\ \qquad - \sum_{k\geq 1}\mathbb{1}_{[0,t]}(\tau_k^i + \varsigma_k^i), \\ \tau_k^i = \inf\{t > \tau_{k-1}^i + \varsigma_{k-1}^i \;:\; X_{t-}^i \geq 0\}, \qquad \tau_0^i = 0, \\ F_t^N = \frac{1}{N}\sum_{i=1}^N \sum_{k\geq 1}\mathbb{1}_{[0,t]}(\tau_k^i),\ \mathfrak{f}_t^N = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathfrak{K} * F^N\right)_t = \int_0^t \mathfrak{K}'(t-s)F_s^N\,\mathrm{d}s, \\ \nu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}\mathbb{1}_{\{X_t^i<0\}}. \end{cases} \tag{3.1.1}$$

In this model:

- λ represents the membrane capacitance, influencing how quickly the voltage decays towards the resting potential, V_{rest}.
- γ measures the coupling strength between neurons, affecting how the voltage of one neuron impacts another.
- α indicates the feedback strength, affecting how the firing of one neuron influences the voltages of others, while the kernel $\mathfrak{K}$ smooths the impact of neuronal firing in time.
- μ denotes a constant base current, setting the baseline activity within the neuron.
- σ and ρ describe the system's noise volatility and its correlation with external stimuli or shared connectivity within the neuronal network, respectively.

- The random variables $(\varsigma_k^i)_{i,k\geq 1}$ are independent and identically distributed (i.i.d.) and represent the refractory period of the neuron.

This model amalgamates features from the works of [50, 58, 19, 20, 41, 40]. Our interest is to rigorously prove its convergence, under more general assumptions on the coefficients, to its mean-field limit:

$$\begin{cases} X_t = \int_0^t b(s, X_s, \nu_s)\mathbb{1}_{\{X_s<0\}}\,\mathrm{d}s + \int_0^t \sigma(s,\, X_s)\sqrt{1-\rho^2(s,\nu_s)}\mathbb{1}_{\{X_s<0\}}\,\mathrm{d}W_s \\ \qquad + \int_0^t \sigma(s,\, X_s)\rho(s,\nu_s)\mathbb{1}_{\{X_s<0\}}\,\mathrm{d}W_s^0 + \alpha\int_0^t \mathfrak{f}_s\mathbb{1}_{\{X_s<0\}}\,\mathrm{d}s \\ \qquad - \sum_{k\geq 1}\mathbb{1}_{[0,t]}(\tau_k+\varsigma_k), \\ \tau_k = \inf\{t>\tau_{k-1}+\varsigma_{k-1}\ :\ X_{t-}\geq 0\}, \qquad \tau_0 = 0, \\ F_t = \sum_{k\geq 1}\mathbb{P}\left[\tau_k\leq t|\, W^0\right],\ \mathfrak{f}_t = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathfrak{K}*F\right)_t = \int_0^t \mathfrak{K}'(t-s)F_s\,\mathrm{d}s, \\ \nu_t = \mathbb{P}[X_t\in\cdot, X_t<0|\, W^0], \end{cases} \tag{3.1.2}$$

where the function b is given by $b(s,x,\nu) = -\lambda(V_{\text{rest}}-x) - \gamma x + \gamma\langle\nu, \mathrm{Id}\rangle + \mu$.

3.1.3 Related Literature

Variants of (3.1.1) and (3.1.2) have been extensively studied in the literature, encompassing models with and without common noise, with and without delays, and with and without refractory periods.

In [15], the authors prove the existence of solutions to (3.1.2) where the firing function is continuously differentiable for all sufficiently small α when the initial condition is deterministic. In their model, b was a function solely of space, σ was constant, and $\mathfrak{K} = \delta_0$. Under these same assumptions on the coefficients, in [16] the convergence of (3.1.1) to its mean-field limit (3.1.2) was proven for any $\alpha < 1$. Inglis and Talay [34] provided a broader generalization, establishing the convergence of the particle system to the mean-field limit in scenarios incorporating delays, where b is a function of both time and space, and the feedback term is heterogeneous, reflecting variations in connectivity strengths among neurons.

In [40], the authors consider a system of N leaky integrate-and-fire neurons with common noise, where the SDE has Ornstein–Uhlenbeck type dynamics. They study the probability distribution of the number of synchronous action potentials in a model network consisting of a homogeneous neural population driven by a common time-dependent stimulus, i.e. a common noise. Notably, there is no interaction through the firing function in the proposed model. In [41], the authors consider the same model and focus on studying synchrony induced by common stimuli. In [20], the authors consider an extension of this model, where the neurons interact with each

other through the smoothing of the firing function. They investigate the effect of common noise on neuronal dynamics.

The McKean–Vlasov problem (3.1.2) can be rewritten as a partial differential equation (PDE) problem. Using V_t to denote the density of ν_t, which exists by Proposition 3.5.1 below, the density process will satisfy, in a weak sense, the (stochastic) Fokker–Planck equation

$$\begin{aligned}\partial_t V_t(x) =& \frac{1}{2}\partial_{xx}(\sigma(t,x)^2 V_t(x)) - \partial_x(b(t,x,\nu_t)V_t(x)) - \partial_x V_t(x)\alpha \int_0^t \mathfrak{K}'(t-s)F_s \,\mathrm{d}s \\ & - \rho(t,\nu_t)\partial_x(\sigma(t,x)V_t(x))\eta_t + \delta_{-1}\int_0^t r'(t-s)F_s \,\mathrm{d}s - \delta_0 F_t'\end{aligned} \tag{3.1.3}$$

where η_t is a white noise in time. Due to the conservation of mass, we must have $\int_{-\infty}^0 V_t(x)\,\mathrm{d}x + F_t - \int_0^t r(t-s)F_s\,\mathrm{d}s = 1$, where r is the density of the refractory period. Assuming that our density V_t has sufficient regularity, by integrating by parts we formally obtain the expression

$$F_t' = -\frac{1}{2}\sigma(t,0)^2\partial_x V_t(0). \tag{3.1.4}$$

Variants of this (stochastic) Fokker–Planck equation (3.1.3) have also been studied in the literature. In [9, 11], the authors consider the nonlinear Fokker–Planck equation in the setting where there is no common noise. They consider the equation

$$\partial_t V_t + \partial_x(-\lambda x + \alpha F_t' V_t) - \frac{1}{2}\partial_{xx}V_t = \delta_{-1}F_t'.$$

They prove that no global-in-time solution exists for certain parameter choices. Specifically, in the setting when $\alpha > 0$, there exists an initial density where a blow-up must occur, that is, $F_t' = +\infty$. In the setting when $\alpha \leq 0$, there exists a unique solution that does not blow up in finite time. In [5], the authors consider another variant of (3.1.3), [5, Eq. 32], and study the autocorrelation of global activity when the system is stationary. In [6, Eq. 3.24], the focus is on studying the autocorrelation of global activity in stationary and oscillatory states. In [58, Eq. 12], the authors investigate how synchrony can be generated or induced in networks of electrically coupled integrate-and-fire neurons.

3.1.4 Main Contributions and Outline of the Chapter

In the following, we consider the system of interacting particles defined by (3.1.1) under more general assumptions on the coefficients. Building on the ideas presented

in [29, 31], we rigorously prove that the empirical measures are tight and that the limit point of the empirical measures is a solution to the McKean–Vlasov problem (3.1.2). There are several key differences in our proofs compared to those in [31] due to the presence of the finite variation term in our system that comes from the resets.

In Section 3.2, we introduce (3.1.1) in its most general form and state our main results. We also give the assumptions on the coefficients that we require to prove our results. In Section 3.3, we show that the finite particle system is well-posed and collect some useful estimates and results on the empirical measure and the firing function of the finite particle system. In Section 3.4, we prove that the empirical measures are tight, and each limit point will satisfy the SPDE (3.2.3). We also establish in Lemma 3.4.10 that every limit point will satisfy a weak flux condition, which is a relaxation of the condition in (3.1.4). In Section 3.5, we prove the uniqueness of the limit point within a class of processes, hence we may conclude that the limit point is measurable with respect to the common noise. Lastly, in Section 3.6, we collect some technical lemmas that are necessary for the proofs of the particle system.

3.2 Main Results

To begin, we fix a finite time horizon $T > 0$ and let $\mathcal{P}(A)$ denote the set of probability measures on a measurable space $(A, \mathcal{A})$. When A is a metric space, $\mathcal{B}(A)$ denotes the Borel σ-algebra. Let further $\mathbf{M}_{\leq 1}(A)$ denote the space of sub-probability measures, which we shall endow with the topology of weak convergence. For any interval I and metric space X, let $C(I, X)$ denote the space of continuous functions from I to X. Similarly, $D(I, X)$ denotes the space of càdlàg functions from I to X. We shall employ the shorthand notation $\mathcal{C}_X$ and D_X for $C(I, X)$ and $D(I, X)$, respectively, when the interval I is clear.

From this point onward, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Furthermore, we will assume that the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions. We will suppose that the filtration supports an infinite collection of random variables, $\{W^i, \varsigma_k^i, \xi_k^i\}_{i,k\geq 0}$, which are all mutually independent of each other. W^i is a $\mathcal{F}_t$-standard Brownian motion. $\{\varsigma_k^i\}_{k,i}$ and $\{\xi_k^i\}_{k,i}$ are i.i.d. random variables on $\mathbb{R}$ that are $\mathcal{F}_0$-measurable and satisfy Assumption 3.2.1 below. For simplicity, when the indexing is not important, we will simply refer to ξ_k^i and ς_k^i by ξ and ς respectively. We will be considering a generalisation to the integrate-and-fire neuronal models discussed in Section 3.1. This model is an extension of the model introduced in [31] as

the diffusions are allowed to be reset. We shall consider the following finite particle system:

$$\begin{cases} \mathrm{d}X_t^i = b(t, X_t^i, \nu_t^N, \mathfrak{f}_t^N)\mathbb{1}_{\{X_t^i<0\}}\,\mathrm{d}t + \sigma(t,\, X_t^i)\sqrt{1-\rho^2(t,\nu_t^N,\mathfrak{f}_t^N)}\mathbb{1}_{\{X_t^i<0\}}\,\mathrm{d}W_t^i \\ \qquad\quad +\sigma(t,\, X_t^i)\rho(t,\nu_t^N,\mathfrak{f}_t^N)\mathbb{1}_{\{X_t^i<0\}}\,\mathrm{d}W_t^0 - \mathrm{d}\sum_{k\geq 1}\xi_k^i\mathbb{1}_{[0,t]}(\tau_k^i+\varsigma_k^i) \\ \tau_k^i = \inf\{t>\tau_{k-1}^i+\varsigma_{k-1}^i \;:\; X_{t-}^i\geq 0\}, \qquad \tau_0^i=0, \\ J_t^i = \sum_{k\geq 1}\mathbb{1}_{[0,t]}(\tau_k^i),\;\; J_t^{D,i} = \sum_{k\geq 1}\mathbb{1}_{[0,t]}(\tau_k^i+\varsigma_k^i), \\ F_t^N = \frac{1}{N}\sum_{i=1}^N J_t^i,\;\; F_t^{D,N} = \frac{1}{N}\sum_{i=1}^N J_t^{D,i}, \\ \nu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}\mathbb{1}_{\{X_t^i<0\}}, \\ \mathfrak{f}_t^N = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathfrak{K}*F^N\right)_t = \int_0^t \mathfrak{K}'(t-s)F_s^N\,\mathrm{d}s. \end{cases} \tag{3.2.1}$$

The variable ς_k^i represents the refractory period of the neuron, which is a random time during which a neuron is not affected by any external stimulus after it has fired. The term ξ_k^i captures the inherent stochasticity in the reset potential of the neuron. The empirical distribution of the system, denoted by ν_t^N, captures the distribution of the neurons that are not deactivated. The variable J_t^i represents the number of times that neuron i has fired by time t, while F_t^N captures the total number of firings that have occurred in the entire system by time t. We encapsulate the feedback into the drift coefficient b with the term $\mathfrak{f}_t^N$, which is the derivative of the mollified firing function F_t. We stress that although we have not explicitly included an N in the notation X_t^i for simplicity, the value of X_t^i is dependent on N.

We delay the proof of well-posedness and regularity properties of (3.2.1) to Section 3.3. Incorporating the empirical measure into the drift function b allows (3.2.1) to facilitate dynamics governed by the process's mean, denoted by $M_t^N := \langle \nu_t^N, \mathrm{Id}\rangle$. Additionally, it enables the model to encompass dynamics akin to those observed in Ornstein—Uhlenbeck processes. Moreover, the integration of the term $\mathfrak{f}^N$ into the drift function captures how neurons firing may affect the dynamics of the entire particle system.

3.2.1 Assumptions

In the context of the basic framework outlined in (3.1.1), we would require our coefficients to satisfy a notion of global Lipschitzness in X_t^i, ν_t^N and $\mathfrak{f}_t^N$. We must also allow for the possibility of linear growth in each of these components. Therefore, we will need to introduce an appropriate metric on the space of sub-probability measures, $\mathbf{M}_{\leq 1}(\mathbb{R})$. This is detailed below.

Assumption 3.2.1 (Structural assumptions)**.** *Let the coefficients of* (3.2.1) *be of the form* $b(t,x,\mu,f)$, $\sigma(t,x)$, *and* $\rho(t,\mu,f)$. *We assume:*

(i) (Linear growth and space/time regularity) The map $x \mapsto b(t,x,\mu,f)$ *is* $\mathcal{C}^2(\mathbb{R})$ *and* $(t,x) \mapsto \sigma(t,x)$ *is* $\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$. *Moreover, there exist* $C_b, C_\sigma > 0$ *such that*

$$|b(t,x,\mu,f)| \le C_b(1+|x|+\langle\mu,|\cdot|\rangle+|f|), \quad |\partial_x^{(n)} b(t,x,\mu,f)| \le C_b, \quad n=1,2,$$
$$|\sigma(t,x)| \le C_\sigma, \quad |\partial_t\sigma(t,x)| \le C_\sigma, \quad \text{and} \quad |\partial_x^{(n)}\sigma(t,x)| \le C_\sigma, \quad n=1,2.$$

(ii) (Lipschitzness for fixed t*) There exists* $C>0$ *such that*

$$\begin{aligned} |b(t,x,\mu,f)-b(t,\tilde{x},\tilde{\mu},\tilde{f})| &\le C(|x-\tilde{x}|+d_0(\mu,\tilde{\mu})+|f-\tilde{f}|),\\ |\sigma(t,x)-\sigma(t,\tilde{x})| &\le C(|x-\tilde{x}|),\\ |\rho(t,\mu,f)-\rho(t,\tilde{\mu},\tilde{f})| &\le C(d_1(\mu,\tilde{\mu})+|f-\tilde{f}|), \end{aligned}$$

where

$$d_0(\mu,\tilde{\mu}) = \sup\left\{|\langle\mu-\tilde{\mu},\psi\rangle| \,:\, \|\psi\|_{\mathrm{Lip}} \le 1,\, |\psi(0)| \le 1\right\} \qquad \text{and}$$
$$d_1(\mu,\tilde{\mu}) = \sup\left\{|\langle\mu-\tilde{\mu},\psi\rangle| \,:\, \|\psi\|_{\mathrm{Lip}} \le 1,\, \|\psi\|_{\infty} \le 1\right\}.$$

(iii) (Non-degeneracy) C_σ *above is chosen such that* $0 < C_\sigma^{-1} \le \sigma(t,x)$ *and there exists* $\gamma > 0$ *such that* $0 \le \rho(t,\mu,f) \le 1-\gamma$.

(iv) (Sub-Gaussian initial law) The collection of random variables $\{X_0^i, W^i, \varsigma_k^i, \xi_k^i\}_{i,k\ge 0}$ *are all mutually independent of each other. The sequence of random variables* $\{X_0^i\}_i$ *is i.i.d. with common law* ν_0, *which has a density,* V_0, *in* $L^2(-\infty,0)$ *and*

$$\exists\gamma>0 \quad s.t. \quad \nu_0(-\infty,-\lambda) = O(e^{-\gamma\lambda^2}) \quad as \quad \lambda\to\infty.$$

(v) (Delay and re-insertion) The function $\mathfrak{K} \in \mathcal{W}_0^{1,1}(\mathbb{R}_+)$, *the Sobolev space with one weak derivative in* L^1 *and zero at* 0, *is such that* $\mathfrak{K}$ *is non-negative, and* $\|\mathfrak{K}\|_1 = 1$. *Furthermore,* ς_k^i *has a density* $r \in \mathcal{W}_0^{1,1}(\mathbb{R}_+)$. *Lastly, there exist* $\gamma, R > 0$ *such that* $\mathrm{supp}(\xi_k^i) \subset [\gamma, R]$.

To establish uniqueness of the limiting Stochastic Partial Differential Equation (SPDE), introduced in (3.2.3) later on in this section, it is necessary to restrict ourselves to a class of processes that have sufficient regularity. Naturally, every limit point

of our particle system should also live in this class. Furthermore, as our ultimate goal is to show the existence of strong solutions, we must ensure that the identified limit point is measurable with respect to the filtration generated by the Brownian motion. Hence, the set in which these processes reside must be both convex and a subset of a Polish space. This then allows us to apply the results from [44]. We note that the conditions imposed are weaker than those satisfied by the limit point.

Assumption 3.2.2. *In Theorem 3.2.4, we consider the class of processes denoted by* $(\tilde{\nu}, \tilde{F})$ *which are elements of the product space* $D_{\mathbf{M}_{\leq 1}(\mathbb{R})} \times D_{\mathbb{R}}$ *and satisfy the following conditions:*

(i) (Support on $\mathbb{R}_-$*) For every* $t \in [0,T]$*,* $\tilde{\nu}_t$ *is supported in* $\mathbb{R}_- = (-\infty, 0]$*.*

(ii) (Increasing firing function) The firing function satisfies $\tilde{\nu}_t(-\infty, 0) + \tilde{F}_t - \int_0^t r(t-s)\tilde{F}_s \, \mathrm{d}s = 1$ *and is increasing.*

(iii) (Exponential tails) For every $\delta > 0$*, we have*

$$\mathbb{E}\left[\sup_{t \leq T} \tilde{\nu}_t(-\infty, -a)\right] = o(\exp\{-\delta a\}) \quad \text{as} \quad a \to \infty.$$

(iv) (Boundary decay) There exists a $\beta > 0$ *such that*

$$\mathbb{E}\int_0^T \tilde{\nu}_t(-\varepsilon, 0) \, \mathrm{d}t = o(\varepsilon^{1+\beta}) \quad \text{as} \quad \varepsilon \to 0.$$

(v) (Spatial concentration) There exists a $C, \delta > 0$ *such that*

$$\mathbb{E}\int_0^T (\tilde{\nu}_t(a,b))^2 \, \mathrm{d}t \leq C\,|a-b|^{\delta} \qquad \forall\, a, b \in \mathbb{R}.$$

Assumption 3.2.2 (ii) captures the fact that fired particles are slowly re-inserted into the system. This is a natural formulation for the limiting equation as $\tilde{F}_t$ represents the cumulative mass of particles that have fired up to time t. Similarly, $\int_0^t r(t-s)\tilde{F}_s \, \mathrm{d}s$ is the mass of the system that has been reinserted by time t. Hence, their difference represents the mass of particles that is currently in the refractory period. The mass of particles in their refractory period plus the mass of active particles must be equal to one. We also remark that Assumption 3.2.2 (ii) is equivalent to

$$\tilde{F}_t = 1 - \tilde{\nu}_t(-\infty, 0) + \sum_{n \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_n} (1 - \tilde{\nu}_{t_n}(-\infty, 0)) r(t - t_n) \prod_{j=1}^{n-1} r(t_j - t_{j+1}) \, \mathrm{d}t_1 \cdots \mathrm{d}t_n. \tag{3.2.2}$$

That is to say, we may view $\tilde{F}$ as a function of $\tilde{\nu}$, and we can characterise the limiting SPDE in terms of $\tilde{\nu}$ only.

3.2.2 The Limit SPDE

Building on the techniques introduced in [29] and [31], to establish tightness of the tuple $(\nu^N, F^N, W^0)_N$, we will work in the product topology $(D_{\mathscr{S}'}, M_1) \times (D_{\mathbb{R}}, M_1) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty)$. Here, $\mathscr{S}$ is the space of Schwartz functions, the space of rapidly decreasing infinitely differentiable functions, and $\mathscr{S}'$ is its dual, the space of tempered distributions. $(D_{\mathscr{S}'}, M_1)$ is the space of $\mathscr{S}'$-valued càdlàg processes on $[0,T]$, and it is equipped with the M_1-topology introduced in [47]. The M_1-topology is useful for our setting as monotone real-valued functions are automatically tight in the M_1-topology. With this in mind, it is natural to use weak convergence in $(D_{\mathscr{S}'}, M_1)$ with the view of exploiting the monotonicity of the firing function F^N. We refer the reader to [47] for more details.

The first main result of this chapter is:

Theorem 3.2.3. *(Limit SPDE) Suppose Assumption 3.2.1 is satisfied. Then $\{(\nu^N, F^N, W^0)\}_N$ is tight on $(D_{\mathscr{S}'}, M_1) \times (D_{\mathbb{R}}, M_1) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty)$. Given each limit point, (ν, F, W^0), ν is a càdlàg $\mathbf{M}_{\leq 1}(\mathbb{R}_-)$-valued process with probability 1 and (ν, F) satisfy Assumption 3.2.2. Moreover, ν obeys, with probability 1, the limit SPDE*

$$\begin{aligned} \mathrm{d}\langle \nu_t,\, \phi\rangle =& \langle \nu_t,\, b(t,\cdot,\nu_t,\mathfrak{f}_t)\partial_x\phi\rangle\,\mathrm{d}t + \frac{1}{2}\langle \nu_t, \sigma^2(t,\cdot)\partial_{xx}\phi\rangle\,\mathrm{d}t \\ &+ \langle \nu_t,\, \sigma(t,\cdot)\rho(t,\nu_t,\mathfrak{f}_t)\partial_x\phi\rangle\,\mathrm{d}W_t^0 + \mathbb{E}\left[\phi(-\xi)\right]\mathrm{d}\int_0^t r(t-s)F_s\,\mathrm{d}s \\ &- \phi(0)\,\mathrm{d}F_t, \\ \nu_0 \sim^d & V_0(x)\,\mathrm{d}x \end{aligned} \tag{3.2.3}$$

where $\phi \in \mathscr{S}$ and $\mathfrak{f} = (\mathfrak{K}' * F)$. Finally, if ν is obtained along a subsequence $\{(\nu^{N_j}, F^{N_j}, W^0)\}_{j\geq 1}$, then we have that (M^{N_k}, W^0) converges weakly to (M, W^0) on $(D_{\mathbb{R}}, M_1) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_{L^\infty})$, where $M_t = \langle \nu_t, \mathrm{Id}\rangle$.

We may then deduce that we have a pathwise unique solution to the SPDE (3.2.3) among the processes that satisfy Assumption 3.2.2.

Theorem 3.2.4 (Uniqueness)**.** *Let (ν, F, W^0) and $(\tilde{\nu}, \tilde{F}, W^0)$ be any two solutions to (3.2.3) satisfying Assumption 3.2.2 on the same probability space. Then, with probability 1,*

$$\nu_t(A) = \tilde{\nu}_t(A) \quad \text{and} \quad F_t = \tilde{F}_t \quad \forall\, t \in [0,T],\ A \in \mathcal{B}(\mathbb{R}).$$

With this result in hand, we may deduce that we also have uniqueness in law and convergence along the full sequence of the particle system.

Theorem 3.2.5 (Conditional LLN)**.** *The SPDE (3.2.3) has a unique law on* $(D_{\mathscr{S}'}, M_1) \times (D_{\mathbb{R}}, M_1) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty)$ *and* (ν^N, F^N, W^0) *converges weakly to this law. Furthermore, there exists a measurable map* $Q : \mathcal{C}_{\mathbb{R}} \to D_{\mathscr{S}'} \times D_{\mathbb{R}}$ *such that* $(\nu, F) = Q(W^0)$ *and the mean process* M^N *converges weakly to* M *as defined in Theorem 3.2.3.*

Now we may conclude that we have convergence to the limiting system and that it is written as a conditional law.

Theorem 3.2.6 (Conditional McKean–Vlasov formulation)**.** *Let* (ν, F, W^0) *be the unique strong solution to the SPDE (3.2.3). Then, for any Brownian motion* W *and random variables* $\{\xi_k\}_k$ *and* $\{\varsigma_k\}_k$*, all mutually independent and independent of* X_0 *and* W^0*, we have*

$$\nu_t = \mathbb{P}\left[X_t \in \cdot, X_t < 0 \middle| W^0\right],$$

where X_t *is the unique solution to the conditional McKean–Vlasov diffusion*

$$\begin{cases} \mathrm{d}X_t = b(t, X_t, \nu_t, \mathfrak{f}_t)\mathbb{1}_{\{X_t<0\}}\,\mathrm{d}t + \sigma(t, X_t)\sqrt{1-\rho^2(t,\nu_t,\mathfrak{f}_t)}\mathbb{1}_{\{X_t<0\}}\,\mathrm{d}W_t \\ \qquad\quad +\sigma(t, X_t)\rho(t,\nu_t,\mathfrak{f}_t)\mathbb{1}_{\{X_t<0\}}\,\mathrm{d}W^0_t - \mathrm{d}\sum_{k\geq 1}\xi_k\mathbb{1}_{[0,t]}(\tau_k+\varsigma_k), \\ \tau_k = \inf\{t > \tau_{k-1}+\varsigma_{k-1} \ : \ X_{t-} \geq 0\}, \qquad \tau_0 = 0, \\ F_t = \sum_{k\geq 1}\mathbb{P}\left[\tau_k \leq t \middle| W^0\right], \ \mathfrak{f}_t = \frac{\mathrm{d}}{\mathrm{d}t}(\mathfrak{K} * F)_t = \int_0^t \mathfrak{K}'(t-s)F_s\,\mathrm{d}s. \end{cases} \tag{3.2.4}$$

Proof. Proposition 3.3.1, presented later in this chapter, shows that we may iteratively construct a solution to (3.2.4). A simple modification of its proof allows us to iteratively construct a solution, $\tilde{X}$, to the SDE (3.2.4), where we insert the limit point of the particle system into the drift and correlation coefficients. Thus, we can define $\tilde{\nu}_t = \mathbb{P}[\tilde{X}_t \in \cdot, \tilde{X}_t < 0 \mid W^0]$ and $\tilde{F}$ to be the corresponding firing function.

It is clear that

$$1 = \mathbb{1}_{\{\tilde{X}_t=0\}} + \mathbb{1}_{\{\tilde{X}_t<0\}} = \sum_{k\geq 1}\mathbb{1}_{[0,t]}(\tilde{\tau}_k) - \sum_{k\geq 1}\mathbb{1}_{[0,t]}(\tilde{\tau}_k+\varsigma_k) + \mathbb{1}_{\{\tilde{X}_t<0\}}.$$

Taking expectations conditional on W^0, we have

$$\tilde{\nu}_t(-\infty, 0) + \tilde{F}_t - \int_0^t r(t-s)\tilde{F}_s\,\mathrm{d}s = 1.$$

Therefore, $(\tilde{\nu}, \tilde{F})$ satisfy Assumption 3.2.2 (ii), and it is an easy exercise to see that $(\tilde{\nu}, \tilde{F})$ satisfy Assumption 3.2.2 from the work later on in this chapter. Hence, as (ν, F) are W^0-measurable by Theorem 3.2.5, we may argue as in [29, Section 9] and apply Itô's formula to $\phi(\tilde{X}_t)$ for $\phi \in \mathscr{S}$ to show that $\tilde{\nu}$ solves a linear version of SPDE (3.2.3) with a fixed ν. Then, by uniqueness of the linear SPDE, which follows from a simplified proof of Theorem 3.2.4 and Theorem 3.2.5, we have $\tilde{\nu} = \nu$ and $\tilde{F} = F$. This completes the proof. □

3.3 Finite Particle System and Regularity Properties

We begin by showing that our finite particle system is well-posed and that there exists a pathwise unique solution to our particle system.

Proposition 3.3.1. *There exists a pathwise unique solution to the particle system* (3.2.1).

Proof. By Assumption 3.2.1, our coefficients are globally Lipschitz. Our strategy is as follows. As the value of J^i is constant between hitting times of the boundary at 0, we have that F^N is constant between hitting times. Therefore, we may iteratively construct a pathwise unique solution to (3.2.1) by viewing the coefficients as functions of time, t, and the diffusions that are below 0 on the intervals of constancy of J^i.

To be precise, we define the process $\mathbf{X}^N = (X^1, \dots, X^N)$ for $t \in [0, \tau^1 \wedge T]$ to be the solution to the vector-valued SDE

$$\mathrm{d}X_t^i = b(t, X_t^i, \nu_t^N, 0)\,\mathrm{d}t + \sigma(t,\, X_t^i)\sqrt{1-\rho^2(t, \nu_t^N, 0)}\,\mathrm{d}W_t^i + \sigma(t,\, X_t^i)\rho(t, \nu_t^N, 0)\,\mathrm{d}W_t^0.$$

This process is well-defined by standard results for Lipschitz coefficients, as guaranteed by Assumption 3.2.1 (ii) and (iii). Consequently, there exists a unique strong solution. Now, we set $\tau^{1,\,i} = \inf\{t \in [0,T]\,;\, X_t^i \geq 0\}$ ($\inf\{\emptyset\} = \infty$). Clearly, we have $0 < \tau^{1,i}$ a.s., so $0 < \inf\{\tau^{1,i}\}$ a.s. Set $\tau^1 = \inf\{\tau^{1,i}\}$ and $i^1 = \operatorname{argmin}\{\tau^{1,i}\}$. On the event $\{\tau^1 < \infty\}$ we set

$$J_t^i = 0,\ t \in [0,\, \tau^1), \qquad \text{and} \qquad J_t^i = \begin{cases} 1 & \text{if } i = i^1,\ t \geq \tau^1, \\ 0 & \text{if } i \neq i^1,\ t \geq \tau^1. \end{cases}$$

We define the random index set $\mathcal{I}_{\mathrm{ref}}^1 = \{i^1\}$, which represents the set of diffusions that are currently deactivated, and set $\tau^{2,i^1} = \tau^1 + \varsigma^{i^1}_{J^{i^1}_{\tau^1}}$, the time when particle i^1 will be reinserted back into the system.

For the induction step, we assume that we have defined a sequence of stopping times $\tau^1, \dots, \tau^k$ and $\tau^{k+1,i}$ for $i \in \mathcal{I}_{\mathrm{ref}}^k$. On the event $\{\tau^k < \infty\}$, we define $\mathbf{X}^N$ for $t \in (\tau^k, \tau^{k+1} \wedge T]$ to be the solution to the vector-valued SDE

$$\begin{aligned} X_t^i =& X_{\tau^k}^i + \int_{\tau^k}^t b(s, X_s^i, \nu_s^N, \mathfrak{f}_s^N)\,\mathrm{d}s + \int_{\tau^k}^t \sigma(s,\, X_s^i)\sqrt{1-\rho^2(s, \nu_s^N, \mathfrak{f}_s^N)}\,\mathrm{d}W_s^i \\ &+ \int_{\tau^k}^t \sigma(s,\, X_s^i)\rho(s, \nu_s^N, \mathfrak{f}_s^N)\,\mathrm{d}W_s^0 \end{aligned}$$

for $i \notin \mathcal{I}^k_{\text{ref}}$, where $\mathfrak{f}^N_t = (\mathfrak{K}' * F^N)_t$. For $i \in \mathcal{I}^k_{\text{ref}}$, we set $X^i_t = 0$. This process is well-defined by standard results for Lipschitz coefficients; therefore, there exists a unique strong solution.

For $i \notin \mathcal{I}^k_{\text{ref}}$, we set $\tau^{k+1,i} = \inf\{t \in [0,T] \ ; \ X^i_t \geq 0\}$ ($\inf\{\emptyset\} = \infty$). Set $\tau^{k+1} = \inf\{\tau^{k+1,i}\}$ and $i^{k+1} = \operatorname{argmin}\{\tau^{k+1,i}\}$. On the event $\{\tau^{k+1} = \infty\}$, we set $J^i_t = J^i_{\tau^k}$ for $t \geq \tau^k$. On the event $\{\tau^{k+1} < \infty\}$, we shall proceed in two cases.

Case 1: $i^{k+1} \in \mathcal{I}^k_{\text{ref}}$

In this case, we are reinserting a particle back into the system; therefore, we need to update the values J^i and the particle being reinserted, $X^{i^{k+1}}$. As no new active particles hit the boundary, set $J^i_t = J^i_{\tau^k}$ for $t \geq \tau^{k+1}$ for any i. Furthermore, we set $X^{i^{k+1}}_{\tau^{k+1}} = -\xi^{i^{k+1}}_{J^{i^{k+1}}_{\tau^{k+1}}}$. Next, we update the set of deactivated particles by defining $\mathcal{I}^{k+1}_{\text{ref}} = \mathcal{I}^k_{\text{ref}} \setminus \{i^{k+1}\}$. Lastly, we set $\tau^{k+2,i} = \tau^{k+1,i}$ for $i \in \mathcal{I}^{k+1}_{\text{ref}}$.

Case 2: $i^{k+1} \notin \mathcal{I}^k_{\text{ref}}$

In this case, an active particle has hit the boundary. Hence, we first update the values of J^i to

$$J^i_t = \begin{cases} J^i_{\tau^k} + 1 & \text{if } i = i^{k+1},\ t \geq \tau^{k+1}, \\ J^i_{\tau^k} & \text{if } i \neq i^{k+1},\ t \geq \tau^{k+1}, \end{cases}$$

and set $\mathcal{I}^{k+1}_{\text{ref}} = \mathcal{I}^k_{\text{ref}} \cup \{i^{k+1}\}$. Lastly, we set $\tau^{k+2,i^{k+1}} = \tau^{k+1} + \varsigma^{i^{k+1}}_{J^{i^{k+1}}_{\tau^{k+1}}}$ and $\tau^{k+2,i} = \tau^{k+1,i}$ for $i \in \mathcal{I}^{k+1}_{\text{ref}}$.

Hence, by the principle of induction, we have iteratively constructed a pathwise unique solution. In our construction, we acquired a sequence of increasing stopping times $\{\tau^k\}_k$. We would like to show that these stopping times do not accumulate in finite time, hence our construction will be a solution on $[0,T]$. Let $\tau^* = \lim_{k\to\infty} \tau^k$. On the event $\{\tau^* \leq T\}$, there must be a particle i which has hit 0 infinitely many times in finite time. That is

$$\{\tau^* \leq T\} \subset \bigcup_{i=1}^N \left\{ \lim_{k\to\infty} \tau^i_k \leq T \right\}.$$

As a particle needs to wait at least a ς^i_k amount of time between hitting times, it is clear that

$$\{\tau^* \leq T\} \subset \bigcup_{i=1}^N \left\{ \lim_{k\to\infty} \tau^i_k \leq T \right\} \subset \bigcup_{i=1}^N \left\{ \sum_{k\geq 1} \varsigma^i_k \leq T \right\}.$$

As $\{\varsigma^i_k\}$ are i.i.d. positive random variables with a density by Assumption 3.2.1 (v), the Strong Law of Large Numbers implies that $\left\{\sum_{k\geq 1} \varsigma^i_k \leq T\right\}$ is a null set for every i. Hence, $\tau^* = \infty$ almost surely. □

Remark 2 (Relationship between ν^N, F^N and $F^{D,N}$). We observe that by (3.2.1), for any $i = 1, \dots, N$, $J_t^i - J_t^{D,i} = \mathbb{1}_{\{X_t^i = 0\}}$. Taking an average over i, we have $\nu_t^N(-\infty, 0) + F_t^N - F_t^{D,N} = 1$ for any $N \in \mathbb{N}$.

Secondly, we make the observation that τ_k^i is independent of ς_k^i and ξ_k^i. To see this, consider the particle system $\tilde{\mathbf{X}}^N = (\tilde{X}^1, \dots, \tilde{X}^N)$ constructed in exactly the same fashion as $\mathbf{X}^N = (X^1, \dots, X^N)$, as outlined in Proposition 3.3.1. The only difference is that when $\tilde{X}^i$ hits the boundary for the k^{th} time, it is not reset. We also let $\tilde{\tau}_k^i$ denote the corresponding stopping times. As both $\tilde{\mathbf{X}}^N$ and $\mathbf{X}^N$ have a pathwise unique solution between the intervals of constancy of the firing function, we must have that $\tau_k^i = \tilde{\tau}_k^i$ almost surely. Now, it is clear that ς_k^i and ξ_k^i are independent of $\tilde{\tau}_k^i$ as $\tilde{X}^i$ isn't reset after it hits the boundary for the k^{th} time. Hence, we must have τ_k^i is independent of ς_k^i and ξ_k^i. Furthermore, as the paths of $\tilde{\mathbf{X}}^N$ and $\mathbf{X}^N$ must correspond before time τ_k^i, $\{\tau_\ell^j < \tau_k^i\} = \{\tilde{\tau}_\ell^j < \tilde{\tau}_k^i\}$. Hence, the event $\{\tau_\ell^j < \tau_k^i\}$ is independent of ς_k^i and ξ_k^i. We formalise these two observations below.

Lemma 3.3.2 (Independence of stopping times and refractory period)**.** *For any fixed i, j, ℓ, and k, we have that ς_k^i and ξ_k^i are independent of τ_k^i and the event $\{\tau_\ell^j < \tau_k^i\}$.*

For many of the proofs in the latter parts of this chapter, we require bounds that hold uniformly in N and i. Therefore, for completeness, we state the bounds here but reserve the proofs to Section 3.6.

Proposition 3.3.3. *For any $p \geq 1$, we have*

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t^i|^p + \left(J_T^i\right)^p + \left(F_T^i\right)^p + \left(F_T^N\right)^p\right] \leq C_{T,p},$$

where $C_{T,p} > 0$ is independent of i and N.

Proof. Follows directly from Proposition 3.6.1. □

3.3.1 Regularity Properties of the Particles

When passing to the limit of an infinite number of particles, we need to employ the regularity properties of the empirical measure near the boundary and the properties of the firing function for tightness.

Fundamentally, each particle is a Brownian motion with a drift that is reset to a random point every time it hits the boundary. Therefore, assuming we may apply Girsanov's Theorem, which we can by Proposition 3.6.7, then on an alternative probability space, we may view each particle X^i as essentially being driven by a Brownian

motion and being reset to a random position every time it hits the boundary. Hence, as the particle is reset to a position at least γ away from the boundary, formally one can see that the τ_k^i as being equivalent to the first time a Brownian motion with initial condition X_0^i is at least greater than $k \cdot \gamma$. As the tail probabilities of Brownian motion decay exponentially, we also intuitively expect the tail probabilities of our system to decay exponentially. We formalise the intuition of this in the next proposition.

Proposition 3.3.4 (Exponential decay of hitting times)**.** *For any $i, k \geq 1$ and for all $p > 1$, sufficiently close to 1, we have*

$$\mathbb{P}\left[\tau_k^i \leq T\right] \leq 2^{\frac{p-1}{p}} C^{\frac{1}{p}} e^{-\frac{(k-1)^2\gamma^2(p-1)}{2pTC_\sigma^2}},$$

for some constant $C > 0$ that depends on p and T but holds uniformly in i, k, and N, and γ is the constant in Assumption 3.2.1 (v).

We defer the proof of Proposition 3.3.4 to Section 3.6. Furthermore, using the intuition above, we also expect the probability that the mass of the particle system is in an interval to be similar to that of Brownian motion. Additionally, as we have exponential decay of the tail probabilities of the initial condition, there is only so much mass that can be pushed towards $-\infty$ on compact time intervals. Hence, we also intuitively expect the tail probability of the empirical distribution to decay exponentially.

Corollary 3.3.5 (Regularity of the empirical measure)**.** *The empirical measures ν^N satisfy, uniformly in $N \geq 1$ and $t \in (0, T]$,*

$$\begin{cases} \exists \varepsilon > 0 & \mathbb{E}\left[\nu_t^N(-\infty, -a)\right] = O(e^{-\varepsilon a^2}) \quad as \quad a \to \infty, \\ \exists \delta \in (0, 1] & \mathbb{E}\left[\nu_t^N(a, b)\right] \leq C t^{-\delta/2} |b - a|^\delta \quad for \quad a < b < 0, \\ \exists \delta \in (0, 1], \beta > 0 & \mathbb{E}\left[\nu_t^N(-\varepsilon, 0)\right] = t^{-\delta/2} O(\varepsilon^\beta) \quad as \quad \varepsilon \to 0. \end{cases} \tag{3.3.1}$$

Proof. By the sub-Gaussianity of X_t^i, which is uniform in i and N, the first claim is a direct consequence of Corollary 3.6.5. For the second and third claim, by Proposition 3.6.9

$$\mathbb{E}\left[\nu_t^N(a, b)\right] \leq C t^{-\frac{\delta}{2}} \operatorname{Leb}((a, b))^\delta \frac{1}{N} \sum_{i=1}^N \left[1 + \sum_{k \geq 1} \mathbb{P}\left[\tau_k^i \leq t\right]^\delta\right],$$

for all $\delta > 0$, close enough to 0. Lastly, $N^{-1} \sum_{i=1}^N \left[1 + \sum_{k \geq 1} \mathbb{P}\left[\tau_k^i \leq t\right]^\delta\right]$ is bounded independently of N by Proposition 3.3.4. □

On small-time scales, we can view our system as being parallel to one where particles are killed at the boundary. Therefore, we can deduce an even stronger condition on the regularity of ν^N near the boundary. This will be important in the uniqueness proofs.

Proposition 3.3.6 (Boundary decay)**.** *There exists a $\delta \in (0,1]$ and $\beta > 0$ such that uniformly in N and $t \in (0,T]$, we have*

$$\mathbb{E}\left[\nu_t^N(-\varepsilon, 0)\right] = t^{-\delta/2} O(\varepsilon^{1+\beta}) \quad \text{as} \quad \varepsilon \to 0.$$

3.3.2 Regularity of the Firing Function

We also observe that the probability of seeing a large proportion of neurons firing is small for all sufficiently large N, provided the time domain we are considering, $[t, t+h]$, is small enough.

Proposition 3.3.7. *For any given $T > 0$, consider $0 \leq t \leq t+h \leq T$ with $h \in (0,\, 1)$. Then, we may find a $C_p > 0$, independent of h, and an integer $N_0 = N_0(h)$ such that for all $N \geq N_0$, we have*

$$\mathbb{P}\left[F_{t+h}^N - F_{t-}^N > 1 + C_p h^{1/16}\right] \leq C_p h^p, \tag{3.3.2}$$

for any $p \geq 1$.

Proof. See Section 3.6, page 123. □

Proposition 3.3.8. *For every $t \in [0,\, T]$ and $\eta > 0$, we have*

$$\lim_{h\to 0} \limsup_{N\to+\infty} \mathbb{P}\left[F_{(t+h)\wedge T}^N - F_t^N \geq \eta\right] = 0 \quad \text{and} \quad \lim_{h\to 0} \limsup_{N\to+\infty} \mathbb{P}\left[F_t^N - F_{(t-h)\vee 0}^N \geq \eta\right] = 0$$

Proof. See Section 3.6, page 127. □

Furthermore, we require sufficient regularity on the increments of the firing function and their second moments to conclude that $F^{D,N}$ is tight and that the limiting process is continuous.

Proposition 3.3.9. *For any $s, t \in [0,T]$, we may find a $C > 0$, independent of s, t and N, such that*

$$\mathbb{E}\left[\left|F_t^{D,N} - F_s^{D,N}\right|\right] \leq C\left|t - s\right|. \tag{3.3.3}$$

Proof. Without loss of generality, we may suppose that $s < t$. By Lemma 3.3.2, ς_k^i is independent of τ_k^i for any i and k. Therefore,

$$\begin{aligned}\mathbb{P}\left[\tau_k^i + \varsigma_k^i \in (s,t]\right] &= \int_0^t \int_{s-u}^{t-u} r(u)\,\mathrm{d}\mathbb{P}\left[\tau_k^i \in \mathrm{d}v\right]\mathrm{d}u \\ &= \int_0^s \int_{s-v}^{t-v} r(u)\,\mathrm{d}u\,\mathrm{d}\mathbb{P}\left[\tau_k^i \in \mathrm{d}v\right] + \int_s^t \int_0^{t-v} r(u)\,\mathrm{d}u\,\mathrm{d}\mathbb{P}\left[\tau_k^i \in \mathrm{d}v\right] \\ &\leq 2\left\|r\right\|_\infty (t-s)\mathbb{P}\left[\tau_k^i \leq T\right].\end{aligned} \tag{3.3.4}$$

Hence, by definition of $F^{D,N}$,

$$\begin{aligned}\mathbb{E}\left[\left|F_t^{D,N} - F_s^{D,N}\right|\right] &= \frac{1}{N}\sum_{i=1}^N \sum_{k\geq 1} \mathbb{P}\left[\tau_k^i + \varsigma_k^i \in (s,t]\right] \\ &\leq 2\left\|r\right\|_\infty (t-s)\frac{1}{N}\sum_{i=1}^N \sum_{k\geq 1} \mathbb{P}\left[\tau_k^i \leq T\right].\end{aligned}$$

By Proposition 3.3.4, $N^{-1}\sum_{i=1}^N \sum_{k\geq 1} \mathbb{P}\left[\tau_k^i \leq T\right]$ is bounded uniformly in N. □

Proposition 3.3.10. *For any $s, t \in [0,T]$, there exist a $\beta > 0$ and a $C = C(\beta) > 0$, independent of s, t and N, such that*

$$\mathbb{E}\left[\left|F_t^{D,N} - F_s^{D,N}\right|^2\right] \leq C(|t-s|\,N^{-1} + |t-s|^{1+\beta}). \tag{3.3.5}$$

Proof. Without loss of generality, we may suppose that $s \leq t$. By definition of $F^{D,N}$, we have

$$\begin{aligned}\mathbb{E}\left[\left|F_t^{D,N} - F_s^{D,N}\right|^2\right] =& \frac{1}{N^2}\sum_{i=1}^N \sum_{k\geq 1} \mathbb{P}\left[\tau_k^i + \varsigma_k^i \in (s,t]\right] \\ &+ \frac{1}{N^2}\sum_{i=1}^N \sum_{k\neq \ell} \mathbb{P}\left[\tau_k^i + \varsigma_k^i \in (s,t], \tau_\ell^i + \varsigma_\ell^i \in (s,t]\right] \\ &+ \frac{1}{N^2}\sum_{\substack{i,j=1 \\ i\neq j}}^N \sum_{k,\ell\geq 1} \mathbb{P}\left[\tau_k^i + \varsigma_k^i \in (s,t], \tau_\ell^j + \varsigma_\ell^j \in (s,t]\right] \\ &= I + II + III.\end{aligned}$$

The aim is to analyse each term above separately. By (3.3.4), the first term is upper bounded by

$$I \leq \frac{c(t-s)}{N^2}\sum_{i=1}^N \sum_{k\geq 1} \mathbb{P}\left[\tau_k^i \leq T\right] \leq \frac{C(t-s)}{N}.$$

The constant C in the last inequality is independent of N by Proposition 3.3.4. For the second term, we observe that if $\ell > k$, then, by definition, $\tau_k^i + \varsigma_k^i \leq \tau_\ell^i$ almost surely. As the system receives no information from ς_ℓ^i until after time τ_ℓ^i, we have by construction that ς_ℓ^i is independent of τ_k^i, ς_k^i, and τ_ℓ^i, Lemma 3.3.2. Therefore,

$$\begin{aligned}
&\mathbb{P}\left[\tau_k^i + \varsigma_k^i \in (s,t], \tau_\ell^i + \varsigma_\ell^i \in (s,t]\right] \\
&\quad = \int_0^t \int_{s-u}^{t-u} r(u)\mathbb{P}\left[\tau_k^i + \varsigma_k^i \in (s,t], v+u \in (s,t] \mid \tau_\ell^i = v\right] \mathrm{d}\mathbb{P}\left[\tau_\ell^i \in \mathrm{d}v\right] \mathrm{d}u \\
&\quad = \int_0^s \int_{s-v}^{t-v} r(u)\mathbb{P}\left[\tau_k^i + \varsigma_k^i \in (s,t], v+u \in (s,t] \mid \tau_\ell^i = v\right] \mathrm{d}\mathbb{P}\left[\tau_\ell^i \in \mathrm{d}v\right] \mathrm{d}u \\
&\qquad + \int_s^t \int_0^{t-v} r(u)\mathbb{P}\left[\tau_k^i + \varsigma_k^i \in (s,t], v+u \in (s,t] \mid \tau_\ell^i = v\right] \mathrm{d}\mathbb{P}\left[\tau_\ell^i \in \mathrm{d}v\right] \mathrm{d}u \\
&\quad \leq 2\left\|r\right\|_\infty (t-s)\mathbb{P}\left[\tau_k^i + \varsigma_k^i \in (s,t], \tau_\ell^i \leq t\right] \\
&\quad \leq 2\left\|r\right\|_\infty (t-s)\mathbb{P}\left[\tau_k^i + \varsigma_k^i \in (s,t]\right]^{\frac{1}{p}} \mathbb{P}\left[\tau_\ell^i \leq t\right]^{\frac{1}{q}} \\
&\quad \leq C(t-s)^{1+\frac{1}{p}}\mathbb{P}\left[\tau_k^i \leq t\right]^{\frac{1}{p}} \mathbb{P}\left[\tau_\ell^i \leq t\right]^{\frac{1}{q}},
\end{aligned}$$

where the penultimate line follows from Hölder's inequality and the last line follows from (3.3.4). Therefore, by Proposition 3.3.4, there is a larger $C > 0$ independent of N but dependent on $p > 1$ such that $II \leq C(t-s)^{1+\frac{1}{p}}$. For the final term, as we have independence between the Brownian motions and the waiting times, on the event that τ_ℓ^j has not occurred yet, ς_ℓ^j must be independent of the other terms. That is, on the event $\{\tau_k^i + \varsigma_k^i \leq \tau_\ell^j\}$, ς_ℓ^j is independent of τ_k^i, ς_k^i and τ_ℓ^j, see Lemma 3.3.2. Therefore,

$$\begin{aligned}
&\mathbb{P}\left[\tau_k^i + \varsigma_k^i \in (s,t], \tau_\ell^j + \varsigma_\ell^j \in (s,t], \tau_k^i \leq \tau_\ell^j\right] \\
&\quad = \int_0^t \int_{s-u}^{t-u} r(u)\mathbb{P}\left[\tau_k^i + \varsigma_k^i \in (s,t], v+u \in (s,t], \tau_k^i \leq v \mid \tau_\ell^j = v\right] \mathrm{d}\mathbb{P}\left[\tau_\ell^j \in \mathrm{d}v\right] \mathrm{d}u \\
&\quad \leq 2\left\|r\right\|_\infty (t-s)\mathbb{P}\left[\tau_k^i + \varsigma_k^i \in (s,t], \tau_\ell^j \leq t\right] \\
&\quad \leq C(t-s)^{1+\frac{1}{p}}\mathbb{P}\left[\tau_k^i \leq t\right]^{\frac{1}{p}} \mathbb{P}\left[\tau_\ell^j \leq t\right]^{\frac{1}{q}}.
\end{aligned}$$

Therefore, by Proposition 3.3.4, there is a larger $C > 0$ independent of N but dependent on $p > 1$ such that $III \leq C(t-s)^{1+\frac{1}{p}}$. □

3.4 Tightness and Existence

The aim of this section is to show that the limit points are within the support of solutions to SPDE (3.2.3). To achieve this, we pass to the limit of the empirical distribution for the finite particle system (along a suitable subsequence) and then employ

results on the convergence of (stochastic) integrals. First, we present the following proposition, which describes the evolution equation for the empirical distribution of the finite particle system.

Proposition 3.4.1. *Given $N \geq 1$, for all $\phi \in \mathcal{C}^2$, the space of twice continuously differentiable functions on $\mathbb{R}$, we have that*

$$\begin{aligned}
\mathrm{d}\langle \nu_t^N,\, \phi\rangle =& \frac{1}{2}\langle \nu_t^N,\, \sigma^2(t,\,\cdot)\partial_{xx}\phi\rangle\,\mathrm{d}t + \langle \nu_t^N,\, b(t,\cdot,\nu_t^N,\mathfrak{f}_t^N)\partial_x\phi\rangle\,\mathrm{d}t \\
&+ \langle \nu_t^N,\, \sigma(t,\cdot)\rho(t,\nu_t^N,\mathfrak{f}_t^N)\partial_x\phi\rangle\,\mathrm{d}W_t^0 + \mathrm{d}\frac{1}{N}\sum_{i=1}^{N}\sum_{k\geq 1}\phi(-\xi_k^i)\mathbb{1}_{[0,t]}(\tau_k^i+\varsigma_k^i) \\
&- \phi(0)\,\mathrm{d}F_t^N + \frac{1}{N}\sum_{i=1}^{N}\sigma(t,X_t^i)\sqrt{1-\rho^2(t,\nu_t^N,\mathfrak{f}_t^N)}\partial_x\phi(X_t^i)\mathbb{1}_{\{X_t^i<0\}}\,\mathrm{d}W_t^i.
\end{aligned} \tag{3.4.1}$$

Proof. For any $\phi \in \mathcal{C}^2$, we can apply Itô's formula with jumps [60, Chapter II, Theorem 31] to $\phi(X_t^i)$. Since each X^i given by (3.2.1) has càdlàg paths and will jump to $-\xi_k^i$ only at times $\tau_k^i+\varsigma_k^i$, and is equal to 0 as we approach $(\tau_k^i+\varsigma_k^i)$ from below, by construction, this leads to

$$\begin{aligned}
\phi(X_t^i) =& \phi(X_0^i) + \int_0^t \phi'(X_s^i)b(s,\,X_s^i,\nu_s^N,\mathfrak{f}_s^N)\mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}s \\
&+ \int_0^t \phi'(X_s^i)\sigma(s,\,X_s^i)\mathbb{1}_{\{X_s^i<0\}}\left(\sqrt{1-\rho^2(s,\nu_s^N,\mathfrak{f}_s^N)}\,\mathrm{d}W_s^i + \rho(s,\nu_s^N,\mathfrak{f}_s^N)\,\mathrm{d}W_s^0\right) \\
&- \int_{0+}^t \phi'(X_{s-}^i)\,\mathrm{d}\sum_{k\geq 1}\xi_k^i\mathbb{1}_{[0,t]}(\tau_k^i+\varsigma_k^i) + \frac{1}{2}\int_0^t \phi''(X_s^i)\sigma^2(s,X_s^i)\mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}s \\
&+ \sum_{k\geq 1}\{\phi(-\xi_k^i) - \phi(0) - \phi'(0)\cdot -\xi_i^k\}\mathbb{1}_{[0,t]}(\tau_k^i+\varsigma_k^i).
\end{aligned} \tag{3.4.2}$$

Noting that $\int_{0+}^t \phi'(X_{s-}^i)\,\mathrm{d}\sum_{k\geq 1}\xi_k^i\mathbb{1}_{[0,t]}(\tau_k^i+\varsigma_k^i) = \sum_{k\geq 1}\phi'(0)\cdot\xi_i^k\mathbb{1}_{[0,t]}(\tau_k^i+\varsigma_k^i)$, (3.4.2) simplifies to

$$\begin{aligned}
\mathrm{d}\phi(X_t^i) =& \phi'(X_t^i)b(t,\,X_t^i,\nu_t^N,\mathfrak{f}_t^N)\mathbb{1}_{\{X_t^i<0\}}\,\mathrm{d}t + \frac{1}{2}\phi''(X_t^i)\sigma^2(t,X_t^i)\mathbb{1}_{\{X_t^i<0\}}\,\mathrm{d}t \\
&+ \phi'(X_t^i)\sigma(t,X_t^i)\mathbb{1}_{\{X_t^i<0\}}\left(\sqrt{1-\rho^2(t,\nu_t^N,\mathfrak{f}_t^N)}\,\mathrm{d}W_t^i + \rho(t,\nu_t^N,\mathfrak{f}_t^N)\,\mathrm{d}W_t^0\right) \\
&+ \mathrm{d}\sum_{k\geq 1}\phi(-\xi_k^i)\mathbb{1}_{[0,t]}(\tau_k^i+\varsigma_k^i) - \phi(0)\,\mathrm{d}J_t^{D,i}.
\end{aligned} \tag{3.4.3}$$

It is clear that $\phi(X_t^i)\mathbb{1}_{\{X_t^i<0\}} = \phi(X_t^i) - \phi(0)\mathbb{1}_{\{X_t^i=0\}}$ as X^i lives on $(-\infty, 0]$. Furthermore, by Remark 2, $J_t^i = \mathbb{1}_{\{X_t^i=0\}} + J_t^{D,i}$. Therefore, we may rewrite (3.4.3) as

$$\begin{aligned}\mathrm{d}\phi(X_t^i)\mathbb{1}_{\{X_t^i<0\}} &= \phi'(X_t^i)b(t,\, X_t^i, \nu_t^N, \mathfrak{f}_t^N)\mathbb{1}_{\{X_t^i<0\}}\,\mathrm{d}t + \frac{1}{2}\phi''(X_t^i)\sigma^2(t, X_t^i)\mathbb{1}_{\{X_t^i<0\}}\,\mathrm{d}t \\ &\quad + \phi'(X_t^i)\sigma(t, X_t^i)\mathbb{1}_{\{X_t^i<0\}}\left(\sqrt{1-\rho^2(t,\nu_t^N,\mathfrak{f}_t^N)}\,\mathrm{d}W_t^i + \rho(t,\nu_t^N,\mathfrak{f}_t^N)\,\mathrm{d}W_t^0\right) \\ &\quad + \mathrm{d}\sum_{k\geq 1}\phi(-\xi_k^i)\mathbb{1}_{[0,t]}(\tau_k^i + \varsigma_k^i) - \phi(0)\,\mathrm{d}J_t^i.\end{aligned} \tag{3.4.4}$$

After averaging over $i \in \{1, \ldots, N\}$, we obtain (3.4.1). □

The next step is to show the existence of limit points. We show that the set of empirical measures is tight in a suitable topological space; that is, the space of $\mathscr{S}'$-valued càdlàg processes on $[0, T]$, equipped with the M_1-topology, as done in [29, 31]. Having obtained the evolution equation (3.4.1) for the empirical distribution, observe that we can decompose it into a continuous part, the instantaneous firing function, and the delayed re-insertion firing function. We write this decomposition as

$$\langle \nu_t^N, \phi\rangle = \langle \hat{\nu}_t^N, \phi\rangle - \phi(0)F_t^N + \frac{1}{N}\sum_{i=1}^N\sum_{k\geq 1}\phi(-\xi_k^i)\mathbb{1}_{[0,t]}(\tau_k^i + \varsigma_k^i) \tag{3.4.5}$$

3.4.1 Tightness of the Finite Particle System

To prove tightness, we will first obtain a control on the increments of $\hat{\nu}_t^N$, as shown in the next proposition. We will then obtain the SPDE (3.2.3) by passing to the limit and confirming that the term involving the independent Brownian motions W^i in (3.4.1) converges to 0.

Proposition 3.4.2. *For all $s, t \in [0, T]$ and $\phi \in \mathcal{C}_b^2$, it holds uniformly in $N \geq 1$ and $i \in \{1, \ldots, N\}$ that*

$$\mathbb{E}\left[\left|\langle \hat{\nu}_t^N - \hat{\nu}_s^N, \phi\rangle\right|^4\right] \leq C_T\left\|\partial_x\phi\right\|_{\mathcal{C}^1}^4 |t-s|^2.$$

Furthermore, uniformly in N,

$$\mathbb{E}\left[\sup_{t\leq T}\left|\langle \hat{\nu}_t^N, \phi\rangle\right|^4\right] \leq C_T\left\|\partial_x\phi\right\|_{\mathcal{C}^1}^4.$$

Proof. The proof relies on repeated applications of the Burkholder-Davis-Gundy inequality and Jensen's inequality to control each component of $\hat{\nu}^N$ individually. By Jensen's inequality, we may control the increments of the first term in (3.4.1):

$$\begin{aligned}\mathbb{E}\left[\left(\int_s^t \langle \nu_u^N,\, \frac{1}{2}\sigma^2(u,\cdot)\partial_{xx}\phi\rangle \,\mathrm{d}u\right)^4\right] &\leq (t-s)^3\mathbb{E}\left[\int_s^t \langle \nu_u^N,\, \frac{1}{2}\sigma^2(u,\cdot)\partial_{xx}\phi\rangle^4 \,\mathrm{d}u\right] \\ &\leq \frac{(t-s)^4 C_\sigma^8 \left\|\partial_{xx}\phi\right\|_\infty^4}{2^4}.\end{aligned}$$

As the integrands are bounded, we may employ the Burkholder-Davis-Gundy inequality to control the increments of the stochastic terms. Therefore,

$$\begin{aligned}&\mathbb{E}\left[\left(\int_s^t \langle \nu_u^N,\, \sigma(u,\cdot)\rho(u,\nu_u^N,\mathfrak{f}_u^N)\partial_x\phi\rangle \,\mathrm{d}W_u^0\right)^4\right] \\ &\qquad\leq C\mathbb{E}\left[\left(\int_s^t \langle \nu_u^N,\, \sigma(u,\cdot)\rho(u,\nu_u^N,\mathfrak{f}_u^N)\partial_x\phi\rangle^2 \,\mathrm{d}u\right)^2\right] \\ &\qquad\leq C(t-s)\mathbb{E}\int_s^t \langle \nu_u^N,\, \sigma(u,\cdot)\rho(u,\nu_u^N,\mathfrak{f}_u^N)\partial_x\phi\rangle^4 \,\mathrm{d}u \\ &\qquad\leq C(t-s)^2 \left\|\partial_x\phi\right\|_\infty^4,\end{aligned}$$

where we first apply the Burkholder-Davis-Gundy inequality and then Jensen's inequality. By the independence of W^i and by similarly applying the Burkholder-Davis-Gundy inequality followed by Jensen's inequality, we have

$$\begin{aligned}&\mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^N \int_s^t \sigma(u,X_u^i)\sqrt{1-\rho^2(u,\nu_u^N,\mathfrak{f}_u^N)}\partial_x\phi(X_u^i)\mathbb{1}_{\{X_u^i<0\}} \,\mathrm{d}W_u^i\right)^4\right] \\ &\leq C\mathbb{E}\left[\left(\frac{1}{N^2}\sum_{i=1}^N \int_s^t \sigma(u,X_u^i)^2(1-\rho^2(u,\nu_u^N,\mathfrak{f}_u^N))(\partial_x\phi(X_u^i))^2 \,\mathrm{d}u\right)^2\right] \\ &\leq \frac{C(t-s)}{N^3}\mathbb{E}\left[\sum_{i=1}^N \int_s^t \sigma(u,X_u^i)^4(1-\rho^2(u,\nu_u^N,\mathfrak{f}_u^N))^2(\partial_x\phi(X_u^i))^4 \,\mathrm{d}u\right] \\ &\leq \frac{C(t-s)^2 \left\|\partial_x\phi\right\|_\infty^4 N}{N^3}.\end{aligned}$$

Lastly, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_s^t \langle \nu_u^N,\, b(u,\cdot,\nu_u^N,\mathfrak{f}_u^N)\partial_x\phi\rangle \,\mathrm{d}u\right)^4\right] \\
&\qquad\qquad \leq (t-s)^3\mathbb{E}\left[\int_s^t \langle \nu_u^N,\, b(u,\cdot,\nu_u^N,\mathfrak{f}_u^N)\partial_x\phi\rangle^4 \,\mathrm{d}u\right] \\
&\qquad\qquad \leq \frac{(t-s)^3}{N}\sum_{j=1}^N \mathbb{E}\left[\int_s^t \left(b(u,X_u^j,\nu_u^N,\mathfrak{f}_u^N)\partial_x\phi(X_u^j)\right)^4 \mathrm{d}u\right] \\
&\qquad\qquad \leq C(t-s)^4\left\|\partial_x\phi\right\|_\infty^4 \mathbb{E}\left[1+(F_T^N)^4 + N^{-1}\sum_{j=1}^N \sup_{u\leq T}\left|X_u^j\right|^4\right],
\end{aligned}$$

where the constant $C > 0$ is independent of N. The first inequality follows from Jensen's inequality, while the second inequality is a consequence of the definition of the empirical measure ν_u^N. The final inequality is obtained by applying the linear growth condition on b. The last term in the final inequality can be bounded uniformly in N by Corollary 3.6.5 from the appendix. The proof of the second claim is similar to the above with the obvious changes. □

With control on the increments of $\hat{\nu}^N$ that holds uniformly in N, we are able to show that the individual components from the decomposition in (3.4.5) are tight. This is the content of the following proposition.

Proposition 3.4.3. *The sequences* $\{N^{-1}\sum_{i=1}^N\sum_{k\geq 1}\phi(-\xi_k^i)\mathbb{1}_{[0,t]}(\tau_k^i+\varsigma_k^i)\}_{N\geq 1}$ *and* $\{\langle\hat{\nu}_t^N,\phi\rangle - \phi(0)F_t^N\}_{N\geq 1}$ *are tight on* $(D_{\mathbb{R}}, M_1)$.

Proof. To facilitate the presentation of this proof, we introduce the following notation:

$$\mathcal{G}_t^{1,N} \coloneqq \langle\hat{\nu}_t^N,\phi\rangle - \phi(0)F_t^N \quad \text{and} \quad \mathcal{G}_t^{2,N} \coloneqq \frac{1}{N}\sum_{i=1}^N\sum_{k\geq 1}\phi(-\xi_k^i)\mathbb{1}_{[0,t]}(\tau_k^i+\varsigma_k^i). \tag{3.4.6}$$

As ϕ is fixed, we do not include the explicit dependence of $\mathcal{G}^{1,N}$ and $\mathcal{G}^{2,N}$ on ϕ in this notation. We first consider the sequence $\{\mathcal{G}^{1,N}\}_{N\geq 1}$. We will verify that $\{\mathcal{G}^{1,N}\}_{N\geq 1}$ satisfies the conditions of [71, Theorem 12.12.3]. The first condition is satisfied, as for any $\lambda > 0$ we have

$$\mathbb{P}\left[\sup_{t\leq T}\left|\mathcal{G}_t^{1,N}\right| > 2\lambda\right] \leq \mathbb{P}\left[\sup_{t\leq T}\left|\langle\hat{\nu}_t^N,\phi\rangle\right| > \lambda\right] + \mathbb{P}\left[\sup_{t\leq T}\left|F_T^N\right| > \lambda/\left\|\phi\right\|_\infty\right] = O(\lambda^{-1}),$$

where the last equality follows from Markov's inequality and holds uniformly in N by Proposition 3.3.3 and Proposition 3.4.2. As F^N is monotone, it is immaterial to

the M_1-oscillatory function. Hence, for the second condition, by [1, Theorem 1], it is sufficient to verify that

$$\mathbb{E}\left[\left|\langle\hat{\nu}_t^N,\phi\rangle-\langle\hat{\nu}_s^N,\phi\rangle\right|^4\right]=O(|t-s|^2)\qquad\forall s,t\in[0,T] \tag{3.4.7}$$

and that, for every $\varepsilon>0$,

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbb{P}\left[\sup_{t\in(0,\delta)}\left|\mathcal{G}_t^{1,N}-\mathcal{G}_0^{1,N}\right|+\sup_{t\in(T-\delta,T)}\left|\mathcal{G}_t^{1,N}-\mathcal{G}_T^{1,N}\right|>\varepsilon\right]=0. \tag{3.4.8}$$

This approach is similar to that used in [31, Proposition 4.2], [47, Proposition 4.1, Proposition 4.2] and [29, Proposition 5.1]. Equation (3.4.7) has been shown in Proposition 3.4.2. By the decomposition of $\mathcal{G}^{1,N}$ given in (3.4.6),

$$\begin{aligned}\mathbb{P}\left[\sup_{t\in(0,\delta)}\left|\mathcal{G}_t^{1,N}-\mathcal{G}_0^{1,N}\right|>\varepsilon/2\right]\leq&\mathbb{P}\left[\sup_{t\in(0,\delta)}|\langle\hat{\nu}_t^N,\phi\rangle-\langle\hat{\nu}_0^N,\phi\rangle|>\varepsilon/4\right]\\&+\mathbb{P}\left[F_\delta^N>\frac{\varepsilon}{4\left\|\phi\right\|_\infty}\right]\end{aligned}$$

and likewise for the supremum over $t\in(T-\delta,T)$. Therefore, by first employing Markov's inequality, (3.4.8) follows from Proposition 3.3.8 and Proposition 3.4.2. Hence, $\{\mathcal{G}^{1,N}\}_{N\geq 1}$ is tight on $(D_{\mathbb{R}},M_1)$.

To show the tightness of $\{\mathcal{G}^{2,N}\}_{N\geq 1}$, we shall also show that the sequence satisfies the conditions of [71, Theorem 12.12.3]. However, here the method deviates from the literature cited above. We first observe that for any $t,s\geq 0$,

$$\left|\mathcal{G}_t^{2,N}\right|\leq\left\|\phi\right\|_\infty F_t^{D,N}\qquad\text{and}\qquad\left|\mathcal{G}_t^{2,N}-\mathcal{G}_s^{2,N}\right|\leq\left\|\phi\right\|_\infty\left|F_t^{D,N}-F_s^{D,N}\right|. \tag{3.4.9}$$

The first condition is satisfied, as for any $\lambda>0$ we have

$$\mathbb{P}\left[\sup_{t\leq T}\left|\mathcal{G}_t^{2,N}\right|>\lambda\right]\leq\mathbb{P}\left[F_T^{D,N}>\lambda/\left\|\phi\right\|_\infty\right]=O(\lambda^{-1}),$$

where the last equality follows from Markov's inequality and holds uniformly in N by Proposition 3.3.3. For the second condition, we need to work slightly harder as the increments are of order 1 in t and s by Proposition 3.3.10. Hence, we cannot immediately deduce that the condition is satisfied by applying [1, Theorem 1]. To verify that the second condition of [71, Theorem 12.12.3] is satisfied, we must show that

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbb{P}\left[w_\delta(\mathcal{G}^{2,N})\geq\varepsilon\right]=0, \tag{3.4.10}$$

and

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbb{P}\left[\sup_{t\in(0,\delta)}\left|\mathcal{G}_t^{2,N}-\mathcal{G}_0^{2,N}\right|+\sup_{t\in(T-\delta,T)}\left|\mathcal{G}_t^{2,N}-\mathcal{G}_T^{2,N}\right|>\varepsilon\right]=0 \tag{3.4.11}$$

for any $\varepsilon>0$, where w_δ is the M_1-oscillatory function defined by

$$w_\delta(x)\coloneqq\sup\{H_x(t_1,t_2,t_3)\,:\,(t-\delta)\vee 0\le t_1\le t_2\le t_3\le(t+\delta)\wedge T,\,0\le t\le T\},$$

where $H_x(t_1,t_2,t_3)\coloneqq\inf_{\lambda\in[0,1]}|x_{t_2}-(1-\lambda)x_{t_1}-\lambda x_{t_3}|$ for any $x\in D_{\mathbb{R}}$. Equation (3.4.11) follows immediately from (3.4.9) and Proposition 3.3.9.

To show (3.4.10), the strategy is as follows. We shall construct a piecewise linear function that approximates $\mathcal{G}^{2,N}$. Our approximations will be sufficiently close to $\mathcal{G}^{2,N}$ using the sup-norm, and we will have sufficient control over the M_1-oscillatory function of the approximations. To begin, we define $t_j^N\coloneqq jT/N$. Therefore, the piecewise linear approximation to $\mathcal{G}^{2,N}$ is

$$\tilde{\mathcal{G}}^{2,N}\coloneqq\sum_{j=0}^{N-1}\mathcal{G}^{2,N}_{t_j^N}\mathbb{1}_{[t_j^N,t_{j+1}^N)}+\mathcal{G}_T^{2,N}\mathbb{1}_{\{T\}}.$$

We observe that for any $0\le t_1\le t_2\le t_3\le T$,

$$\begin{aligned}H_{\mathcal{G}^{2,N}}(t_1,t_2,t_3)&=\inf_{\lambda\in[0,1]}\left|\mathcal{G}_{t_2}^{2,N}-(1-\lambda)\mathcal{G}_{t_1}^{2,N}-\lambda\mathcal{G}_{t_3}^{2,N}\right|\\&\le\inf_{\lambda\in[0,1]}\left|\tilde{\mathcal{G}}_{t_2}^{2,N}-(1-\lambda)\tilde{\mathcal{G}}_{t_1}^{2,N}-\lambda\tilde{\mathcal{G}}_{t_3}^{2,N}\right|+3\left\|\mathcal{G}^{2,N}-\tilde{\mathcal{G}}^{2,N}\right\|_\infty\\&\le H_{\tilde{\mathcal{G}}^{2,N}}(t_1,t_2,t_3)+3\left\|\mathcal{G}^{2,N}-\tilde{\mathcal{G}}^{2,N}\right\|_\infty.\end{aligned}$$

Hence, $w_\delta(\mathcal{G}^{2,N})\le w_\delta(\tilde{\mathcal{G}}^{2,N})+3\left\|\mathcal{G}^{2,N}-\tilde{\mathcal{G}}^{2,N}\right\|_\infty$. Now, to estimate the second term, by definition, for any $t\in[t_j^N,t_{j+1}^N)$,

$$\left|\mathcal{G}_t^{2,N}-\tilde{\mathcal{G}}_t^{2,N}\right|=\left|\mathcal{G}_t^{2,N}-\mathcal{G}_{t_j^N}^{2,N}\right|\le\|\phi\|_\infty\left|F_t^{D,N}-F_{t_j^N}^{D,N}\right|\le\|\phi\|_\infty\left|F_{t_{j+1}^N}^{D,N}-F_{t_j^N}^{D,N}\right|,$$

where the penultimate inequality follows from (3.4.9) and the last inequality follows from the monotonicity of $F^{D,N}$. Therefore, we may conclude that

$$\begin{aligned}\mathbb{E}\left[\left\|\mathcal{G}^{2,N}-\tilde{\mathcal{G}}^{2,N}\right\|_\infty^2\right]&\le\sum_{j=0}^{N-1}\|\phi\|_\infty^2\,\mathbb{E}\left[\left|F_{t_{j+1}^N}^{D,N}-F_{t_j^N}^{D,N}\right|^2\right]\\&\le\sum_{j=0}^{N-1}C\,\|\phi\|_\infty^2\left(\left|t_{j+1}^N-t_j^N\right|N^{-1}+\left|t_{j+1}^N-t_j^N\right|^{1+\beta}\right)\\&\le C(N^{-1}+N^{-\beta}),\end{aligned}$$

where the penultimate inequality follows from Proposition 3.3.10 and the constant C is independent of N. By applying Markov's inequality,

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbb{P}\left[\left\|\mathcal{G}^{2,N}-\tilde{\mathcal{G}}^{2,N}\right\|_{\infty}\geq\varepsilon\right]=0 \tag{3.4.12}$$

for any $\varepsilon>0$. Now, to bound the oscillations of $\tilde{\mathcal{G}}^{2,N}$, choose a $t_1\leq t_2\leq t_3<T$. If we suppose there is a j such that $t_i\in[t_j^N,t_{j+2}^N)$ for every i, then it is immediate that $H_{\tilde{\mathcal{G}}^{2,N}}(t_1,t_2,t_3)=0$. Hence, trivially,

$$\mathbb{P}\left[H_{\tilde{\mathcal{G}}^{2,N}}(t_1,t_2,t_3)\geq\varepsilon\right]\leq 2\left\|\phi\right\|_{\infty}^2 C(T^{-1}+1)\varepsilon^{-2}\left|t_1-t_3\right|^{1+\beta} \tag{3.4.13}$$

uniformly in N for any $\varepsilon>0$, where C and β are the constants from Proposition 3.3.10. In the case when there is some $j,\tilde{j}\in[1:N]$ with $j<\tilde{j}$ and $m\geq 2$ such that $t_1\in[t_j^N,t_{j+1}^N)$, $t_3\in[t_{j+m}^N,t_{j+m+1}^N)$ and $t_2\in[t_{\tilde{j}}^N,t_{\tilde{j}+1}^N)\subset[t_j^N,t_{j+m+1}^N)$, then as $H_{\tilde{\mathcal{G}}^{2,N}}(t_1,t_2,t_3)$ only takes the values 0, $\left|\mathcal{G}_{t_1}^{2,N}-\mathcal{G}_{t_2}^{2,N}\right|$ or $\left|\mathcal{G}_{t_3}^{2,N}-\mathcal{G}_{t_2}^{2,N}\right|$, we observe that

$$\begin{aligned}
\mathbb{E}\left[H_{\tilde{\mathcal{G}}^{2,N}}(t_1,t_2,t_3)^2\right]&\leq\mathbb{E}\left[\left|\tilde{\mathcal{G}}_{t_1}^{2,N}-\tilde{\mathcal{G}}_{t_2}^{2,N}\right|^2+\left|\tilde{\mathcal{G}}_{t_3}^{2,N}-\tilde{\mathcal{G}}_{t_2}^{2,N}\right|^2\right]\\
&=\mathbb{E}\left[\left|\mathcal{G}_{t_j^N}^{2,N}-\mathcal{G}_{t_{\tilde{j}}^N}^{2,N}\right|^2+\left|\mathcal{G}_{t_{j+m}^N}^{2,N}-\mathcal{G}_{t_{\tilde{j}}^N}^{2,N}\right|^2\right]\\
&\leq\left\|\phi\right\|_{\infty}^2\mathbb{E}\left[\left|F_{t_j^N}^{D,N}-F_{t_{\tilde{j}}^N}^{D,N}\right|^2+\left|F_{t_{j+m}^N}^{D,N}-F_{t_{\tilde{j}}^N}^{D,N}\right|^2\right]\\
&\leq\left\|\phi\right\|_{\infty}^2 C\Bigg[N^{-1}\left|t_j^N-t_{\tilde{j}}^N\right|+N^{-1}\left|t_{j+m}^N-t_{\tilde{j}}^N\right|\\
&\qquad\qquad+\left|t_j^N-t_{\tilde{j}}^N\right|^{1+\beta}+\left|t_{j+m}^N-t_{\tilde{j}}^N\right|^{1+\beta}\Bigg]\\
&\leq 2\left\|\phi\right\|_{\infty}^2 C\left[N^{-1}\left(\frac{(m-1)T}{N}\right)+\left(\frac{(m-1)T}{N}\right)^{1+\beta}\right]\\
&\leq 2\left\|\phi\right\|_{\infty}^2 C\left[N^{-1}\left|t_3-t_1\right|+\left|t_3-t_1\right|^{1+\beta}\right]
\end{aligned}$$

As $(t_3-t_1)>TN^{-1}$ by assumption,

$$\begin{aligned}
&\leq 2\left\|\phi\right\|_{\infty}^2 C\left[T^{-1}\left|t_3-t_1\right|^2+\left|t_3-t_1\right|^{1+\beta}\right]\\
&\leq 2\left\|\phi\right\|_{\infty}^2 C(T^{-1}+1)\left|t_3-t_1\right|^{1+\beta}.
\end{aligned}$$

The above computation also holds in the case when $t_3=T$. Therefore, (3.4.13) holds for any $0\leq t_1\leq t_2\leq t_3\leq T$. In view of (3.4.13), the assumptions of [1, Theorem 1]

are satisfied, and we may conclude that

$$\mathbb{P}\left[w_\delta(\tilde{\mathcal{G}}^{2,N}) \geq \varepsilon\right] = O(\delta^\beta \varepsilon^{-2}) \tag{3.4.14}$$

uniformly in N for any $\varepsilon > 0$. By combining (3.4.12) and (3.4.14), we deduce that (3.4.10) holds. This verifies the second condition of [71, Theorem 12.12.3], and hence $\{\mathcal{G}^{2,N}\}_{N\geq 1}$ is tight. □

As the components from decomposition (3.4.5) are tight, we may use this result to deduce the tightness of their sum.

Proposition 3.4.4. *The sequence* $\{(\nu^N, F^N, W^0)\}_{N\geq 1}$ *is tight on* $(D_{\mathscr{S}'}, M_1) \times (D_{\mathbb{R}}, M_1) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty)$. *Furthermore, any subsequence of* $\{(\nu^N, F^N, W^0)\}_{N\geq 1}$ *has a further subsequence that converges weakly to a limit point* (ν^*, F^*, W^0).

Proof. As tightness of the marginals implies tightness of the joint distribution, it will be sufficient to show that each component is individually tight. Trivially, $\{W^0\}_N$ is tight as W^0 does not change with N. We now turn our attention to $\{F^N\}_N$. As monotone functions are immaterial to the M_1-oscillatory function, by Proposition 3.3.3 and Proposition 3.3.8, we may conclude that $\{F^N\}_N$ satisfies the assumptions of [71, Theorem 12.12.3]. Therefore, $\{F^N\}_N$ is tight.

For the tightness of $\{\nu^N\}_N$, it suffices to show that $\{\nu^N(\phi)\}_N$ is tight in $(D_{\mathbb{R}}, M_1)$ for any $\phi \in \mathscr{S}$. Since $(D_{\mathbb{R}}, M_1)$ is a Polish space, by Prokhorov's Theorem, tightness is equivalent to sequential pre-compactness. By (3.4.6), we may write $\nu^N(\phi) = \mathcal{G}^{1,N} + \mathcal{G}^{2,N}$. The sequence $\{(\mathcal{G}^{1,N}, \mathcal{G}^{2,N})\}_N$ is tight in $(D_{\mathbb{R}}, M_1) \times (D_{\mathbb{R}}, M_1)$ by Proposition 3.4.3. Therefore, for any subsequence N_j, we may choose a further subsequence, also denoted by N_j for simplicity, such that $(\mathcal{G}^{1,N}, \mathcal{G}^{2,N})$ converges weakly to $(\mathcal{G}^{1,*}, \mathcal{G}^{2,*})$. An application of the Continuous Mapping Theorem, the Portmanteau Theorem, and Proposition 3.3.10 allows us to deduce that

$$\mathbb{E}\left|\mathcal{G}^{2,*}_t - \mathcal{G}^{2,*}_s\right|^2 \leq C\left|t - s\right|^{1+\beta} \tag{3.4.15}$$

for a co-countable set of times t, s with constants C and β that are independent of t and s. By Kolmogorov's Continuity Theorem, $\mathcal{G}^{2,*}$ has continuous paths almost surely. Therefore, addition is a Law$((\mathcal{G}^{1,*}, \mathcal{G}^{2,*}))$-almost sure continuous map by the continuity of $\mathcal{G}^{2,*}$ and [71, Theorem 12.7.3]. Hence, by the Continuous Mapping Theorem,

$$\nu^{N_j}(\phi) = \mathcal{G}^{1,N_j} + \mathcal{G}^{2,N_j} \implies \mathcal{G}^{1,*} + \mathcal{G}^{2,*}.$$

Therefore, $\{\nu^N(\phi)\}_N$ is sequentially pre-compact and hence, by the Portmanteau Theorem, tight in $(D_{\mathbb{R}}, M_1)$.

For the second claim, the proof follows along the lines of [47, Theorem 3.2]. Both $(D_{\mathscr{S}'}, M_1)$ and any Polish space are topological Radon spaces, by [62, Section 3, Exercise 4] and [62, Section 3, Exercise 1], respectively. Hence, $(D_{\mathscr{S}'}, M_1) \times (D_{\mathbb{R}}, M_1) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty)$ is a topological Radon space. As every compact subset of $(D_{\mathscr{S}'}, M_1)$ is metrizable ([47, Theorem 3.1 (iii)]) and it is a completely regular space ([47, Proposition 2.7 (iii)]), it is clear that every compact subset of $(D_{\mathscr{S}'}, M_1) \times (D_{\mathbb{R}}, M_1) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty)$ is metrizable, and it is a completely regular space as Polish spaces are completely regular spaces and the product of completely regular spaces is a completely regular space. Therefore, the second claim follows from [62, Section 5, Theorem 2]. □

3.4.2 Limit Point Representation and Properties

In view of Proposition 3.4.4, we fix a limit point and denote it by (ν^*, F^*, W^0). We now aim to show that the effects from the reinsertion of the particles are smoothed out by the refractory period, and we see that particles are continuously reinserted into the system in the limit.

Proposition 3.4.5 (Limit point representation)**.** *Fix a $\phi \in \mathscr{S}$ and let (ν^N, F^N, W^0) be a subsequence, still indexed by N, along which $(\nu^N, F^N, W^0) \implies (\nu^*, F^*, W^0)$. Then on $(D_{\mathbb{R}}, M_1) \times (D_{\mathbb{R}}, M_1)$, we have $(F^N, N^{-1}\sum_{i=1}^N \sum_{k\geq 1} \phi(-\xi_k^i)\mathbb{1}_{[0,t]}(\tau_k^i + \varsigma_k^i)) \implies (F^*, \mathbb{E}[\phi(-\xi)] \int_0^\cdot r(\cdot - s)F_s^* \,\mathrm{d}s)$.*

Proof. To facilitate the presentation of this proof, we introduce the following notation:

$$F_t^N = \frac{1}{N}\sum_{i=1}^N \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^i) \qquad \tilde{F}_t^{D,N} = \frac{1}{N}\sum_{i=1}^N \sum_{k\geq 1} \phi(-\xi_k^i)\mathbb{1}_{[0,t]}(\tau_k^i + \varsigma_k^i)$$

$$F_t^{N,k} = \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{[0,t]}(\tau_k^i) \qquad \tilde{F}_t^{D,N,k} = \frac{1}{N}\sum_{i=1}^N \phi(-\xi_k^i)\mathbb{1}_{[0,t]}(\tau_k^i + \varsigma_k^i)$$

The strategy is as follows: we want to show that for any subsequence of $(F^N, \tilde{F}^{D,N})$, we have a further subsequence that converges weakly towards $(F^*, \mathbb{E}[\phi(-\xi)](r * F^*))$, where we use $(r * F^*)$ to denote $\int_0^\cdot r(\cdot - s)F_s^* \,\mathrm{d}s$. To achieve this, we employ a combination of the Portmanteau Theorem and the Continuous Mapping Theorem. We exploit the fact that the probability of τ_k^i being less than T decays exponentially in k; hence, we may fundamentally see F^N and $\tilde{F}^{D,N}$ as being finite sums.

We first turn our attention to $\{(F^{N,k})_{k\geq 1}\}_N$. We observe that for any $t \geq s$, $F_t^{N,k} - F_s^{N,k} \leq F_t^N - F_s^N$ and $F_T^{N,k} \leq F_T^N$. Hence,

$$\lim_{h\to 0}\limsup_{N\to+\infty} \mathbb{P}\left[F_{t+h}^{N,k} - F_t^{N,k} \geq \eta\right] = 0 \quad \text{and} \quad \lim_{h\to 0}\limsup_{N\to+\infty} \mathbb{P}\left[F_t^{N,k} - F_{t-h}^{N,k} \geq \eta\right] = 0 \tag{3.4.16}$$

for any $t \in [0,T]$ by Proposition 3.3.8. As monotone functions are immaterial to the M_1-oscillatory function, by Proposition 3.3.3 and (3.4.16), we may conclude that $\{F^{N,k}\}_N$ satisfies the assumptions of [71, Theorem 12.12.3]. Therefore, $\{F^{N,k}\}_N$ is tight in $(D_{\mathbb{R}}, M_1)$. By tightness, for every k (fixed) and $\varepsilon > 0$, we may find $A^k \subset D_{\mathbb{R}}$, compact, such that $\mathbb{P}\left[F^{N,k} \notin A^k\right] < \varepsilon 2^{-k}$, uniformly in N. The countable product of compact sets is compact in the product topology. Hence, $\prod_{k\geq 1} A^k$ is compact in $(\prod_{k\geq 1} D_{\mathbb{R}}, M_1)$, where $(\prod_{k\geq 1} D_{\mathbb{R}}, M_1)$ is endowed with the product topology, and $\mathbb{P}\left[(F^{N,k})_{k\geq 1} \notin \prod_{k\geq 1} A^k\right] < \varepsilon$ uniformly in N. Hence, $\{(F^{N,k})_{k\geq 1}\}_N$ is tight.

Furthermore, $\{F^N\}_N$ and $\{\tilde{F}^{D,N}\}_N$ are tight by Proposition 3.4.3 and Proposition 3.4.4 respectively. Hence, for any subsequence, N_j, there exists a further subsequence, also denoted by N_j for simplicity, such that $(F^{N_j}, \tilde{F}^{D,N_j})$ converges weakly to $(F^\dagger, \tilde{F}^{D,\dagger})$ on $(D_{\mathbb{R}}, M_1) \times (D_{\mathbb{R}}, M_1)$ and $\{(F^{N_j,k})_k\}_j$ converges weakly to $(F^{\dagger,k})_k$ on $(\prod_{k\geq 1} D_{\mathbb{R}}, M_1)$.

We now aim to show that $(F^\dagger, \tilde{F}^{D,\dagger})$ has the same distribution as $(F^*, \mathbb{E}[\phi(-\xi)](r * F^*))$. As the Kolmogorov σ-Algebra coincides with the Borel σ-Algebra on $D_{\mathbb{R}}$ [71, Theorem 11.5.1], it is sufficient to show that for a dense set of times, $\mathbb{T}$, the following holds for every $m \geq 1$:

$$\mathbb{E}\left[\prod_{\ell=1}^m g_\ell(F_{t_\ell}^\dagger) h_\ell(\tilde{F}_{t_\ell}^{D,\dagger})\right] = \mathbb{E}\left[\prod_{\ell=1}^m g_\ell(F_{t_\ell}^*) h_\ell(\mathbb{E}[\phi(-\xi)](r * F^*)_{t_\ell})\right],$$

for $t_1, \ldots, t_m \in \mathbb{T}$ and $g_1, h_1, \ldots, g_m, h_m \in \mathcal{C}_b(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$. To this end, we set

$$\mathbb{T} := \left\{t \in [0,T] \,:\, \mathbb{P}\left[F_t^\dagger = F_{t-}^\dagger\right] = \mathbb{P}\left[F_t^* = F_{t-}^*\right] = \mathbb{P}\left[F_t^{\dagger,k} = F_{t-}^{\dagger,k}\right] = 1 \,\forall\, k\right\}. \tag{3.4.17}$$

$\mathbb{T}$ is co-countable, hence dense in $[0,T]$. We observe that for any N_j and $M>0$,

$$
\begin{aligned}
&\left|\mathbb{E}\left[\prod_{\ell=1}^{m} g_\ell(F^{\dagger}_{t_\ell})h_\ell(\tilde{F}^{D,\dagger}_{t_\ell})\right]-\mathbb{E}\left[\prod_{\ell=1}^{m} g_\ell(F^{*}_{t_\ell})h_\ell(\mathbb{E}[\phi(-\xi)](r*F^*)_{t_\ell})\right]\right| \\
&\leq \left|\mathbb{E}\left[\prod_{\ell=1}^{m} g_\ell(F^{\dagger}_{t_\ell})h_\ell(\tilde{F}^{D,\dagger}_{t_\ell})\right]-\mathbb{E}\left[\prod_{\ell=1}^{m} g_\ell(F^{N_j}_{t_\ell})h_\ell(\tilde{F}^{D,N_j}_{t_\ell})\right]\right| \\
&+\left|\mathbb{E}\left[\prod_{\ell=1}^{m} g_\ell(F^{N_j}_{t_\ell})h_\ell(\tilde{F}^{D,N_j}_{t_\ell})\right]-\mathbb{E}\left[\prod_{\ell=1}^{m} g_\ell\left(F^{N_j,\Sigma}_{t_\ell}\right)h_\ell\left(\tilde{F}^{D,N_j,\Sigma}_{t_\ell}\right)\right]\right| \\
&+\left|\mathbb{E}\left[\prod_{\ell=1}^{m} g_\ell\left(F^{N_j,\Sigma}_{t_\ell}\right)h_\ell\left(\tilde{F}^{D,N_j,\Sigma}_{t_\ell}\right)\right]-\mathbb{E}\left[\prod_{\ell=1}^{m} g_\ell\left(F^{\dagger,\Sigma}_{t_\ell}\right)h_\ell\left(\mathbb{E}[\phi(-\xi)]\left(r*F^{\dagger,\Sigma}\right)_{t_\ell}\right)\right]\right| \\
&+\left|\mathbb{E}\left[\prod_{\ell=1}^{m} g_\ell\left(F^{\dagger,\Sigma}_{t_\ell}\right)h_\ell\left(\mathbb{E}[\phi(-\xi)]\left(r*F^{\dagger,\Sigma}\right)_{t_\ell}\right)\right]-\mathbb{E}\left[\prod_{\ell=1}^{m} g_\ell(F^{*}_{t_\ell})h_\ell(\mathbb{E}[\phi(-\xi)](r*F^*)_{t_\ell})\right]\right| \\
&= I+II+III+IV,
\end{aligned}
$$

where we employed the notation a^Σ to denote the finite sum $\sum_{k=1}^{M} a^k$ for any sequence $(a^k)_k$. We suppress the dependence on M for simplicity.

Now, fix an arbitrary $\varepsilon>0$. We have

$$
\begin{aligned}
II &\leq c\sum_{\ell=1}^{m}\mathbb{E}\left[\left|F^{N_j}_{t_\ell}-F^{N_j,\Sigma}_{t_\ell}\right|+\left|\tilde{F}^{D,N_j}_{t_\ell}-\tilde{F}^{D,N_j,\Sigma}_{t_\ell}\right|\right] \\
&\leq 2c(1+\|\phi\|_\infty)\mathbb{E}\left[\sum_{k>M} F^{N_j,k}_T\right],
\end{aligned}
\tag{3.4.18}
$$

where $c=\max_\ell\{\|g_\ell\|_\infty, \|g_\ell\|_{\mathrm{Lip}}, \|h_\ell\|_\infty, \|h_\ell\|_{\mathrm{Lip}}\}^{2m}$. By Proposition 3.3.4, we may fix an M large enough such that, independent of N_j, $\mathbb{E}\left[\sum_{k>M} F^{N_j,k}_T\right] < \varepsilon(8c+8c\|\phi\|_\infty)^{-1}$. Hence, $II<\varepsilon/4$. Now, as M is independent of N_j, we may choose a j large enough to control the other terms. As $t_1,\dots,t_m\in\mathbb{T}$, by the Continuous Mapping Theorem and weak convergence, there exists a J_1 such that for all $j\geq J_1$, we have $I<\varepsilon/4$.

To control III, we will consider the infinite tuple of (random) measures $(\mu^{N,k})_{k\geq 1}$ on $\prod_{k\geq 1}\mathcal{P}(\bar{\mathbb{R}}^3)$. Here $\mu^{N,k}:=N^{-1}\sum_{i=1}^{N}\delta_{\xi^i_k,\tau^i_k,\varsigma^i_k}$ and $\bar{\mathbb{R}}$ is the one-point compactification of $\mathbb{R}$. Trivially $\{(\mu^{N,k})_k\}$ is tight. Without loss of generality, we may suppose that our further subsequence, recall also denoted by N_j for simplicity, is such that $(\mu^{N_j,k})_k$ converges weakly towards $(\mu^{\dagger,k})_k$. By Proposition 3.6.10, we have that $\mu^{\dagger,k}=\mathbb{P}[\xi\in\mathrm{d}x]\otimes\mu_2^{\dagger,k}(\mathrm{d}y)\otimes\mathbb{P}[\varsigma\in\mathrm{d}z]$ almost surely. We now suppose further that the times $t_1,\dots,t_m\in\tilde{\mathbb{T}}$, where

$$
\tilde{\mathbb{T}}=\left\{t\in[0,T]\,:\,\mathbb{P}[\mu_2^{\dagger,k}(\{t\})=0]=1\,\forall\,k\right\}\cap\mathbb{T}.
$$

$\tilde{\mathbb{T}}$ is still co-countable, as for any probability measure μ, the map $t \mapsto \mu([0,t])$ is a càdlàg function. Therefore, by construction of $\tilde{\mathbb{T}}$, the map $\prod_{k\geq 1} \mathcal{P}(\bar{\mathbb{R}}^3) \to \prod_{k\geq 1} \mathbb{R}$, $(\mu^k)_k \mapsto (\langle \mu^k, \mathbb{1}_{[0,t]}(y)\rangle)_k$ is $\mathrm{Law}((\mu^{\dagger,k}))$-almost surely continuous for every $t \in \tilde{\mathbb{T}}$. Furthermore, as $\mu^{\dagger,k}$ may be written as a product of measures and as ς has a density by Assumption 3.2.1 (v), the map $\prod_{k\geq 1} \mathcal{P}(\bar{\mathbb{R}}^3) \to \prod_{k\geq 1} \mathbb{R}$, $(\mu^k)_k \mapsto (\langle \mu^k, \mathbb{1}_{[0,t]}(y+z)\rangle)_k$ is $\mathrm{Law}((\mu^{\dagger,k}))$-almost surely continuous for every $t \in [0,T]$. Therefore,

$$\begin{aligned}
&\lim_{j\to\infty} \mathbb{E}\left[\prod_{\ell=1}^m g_\ell\left(F_{t_\ell}^{N_j,\Sigma}\right) h_\ell\left(\tilde{F}_{t_\ell}^{D,N_j,\Sigma}\right)\right] \\
&\quad = \lim_{j\to\infty} \mathbb{E}\left[\prod_{\ell=1}^m g_\ell\left(\langle \mu^{N_j,\Sigma}, \mathbb{1}_{[0,t_\ell]}(y)\rangle\right) h_\ell\left(\langle \mu^{N_j,\Sigma}, \phi(-x)\mathbb{1}_{[0,t_\ell]}(y+z)\rangle\right)\right] \\
&\quad = \mathbb{E}\left[\prod_{\ell=1}^m g_\ell\left(\langle \mu^{\dagger,\Sigma}, \mathbb{1}_{[0,t_\ell]}(y)\rangle\right) h_\ell\left(\mathbb{E}[\phi(-\xi)]\int_0^{t_\ell} r(t_\ell - s)\langle \mu^{\dagger,\Sigma}, \mathbb{1}_{[0,s]}(y)\rangle \,\mathrm{d}s\right)\right],
\end{aligned} \tag{3.4.19}$$

where the first equality follows from the definition of $F^{N,k}$ and $\tilde{F}^{D,N,k}$, and the second equality follows from the Continuous Mapping Theorem. Furthermore, by the Portmanteau Theorem and the Dominated Convergence Theorem, it is easy to check that the map

$$\prod_{k\geq 1} \mathcal{P}(\bar{\mathbb{R}}^3) \to \prod_{k\geq 1} \mathbb{R}, \qquad (\mu^k)_k \mapsto \left(\int_0^{t_\ell} r(t_\ell - s)\langle \mu^k, \mathbb{1}_{[0,s]}(y)\rangle \,\mathrm{d}s\right)_k$$

is $\mathrm{Law}((\mu^{\dagger,k}))$-almost surely continuous. Therefore, going in the reverse direction in (3.4.19),

$$\begin{aligned}
&\mathbb{E}\left[\prod_{\ell=1}^m g_\ell\left(\langle \mu^{\dagger,\Sigma}, \mathbb{1}_{[0,t_\ell]}(y)\rangle\right) h_\ell\left(\mathbb{E}[\phi(-\xi)]\int_0^{t_\ell} r(t_\ell - s)\langle \mu^{\dagger,\Sigma}, \mathbb{1}_{[0,s]}(y)\rangle \,\mathrm{d}s\right)\right] \\
&= \lim_{j\to\infty} \mathbb{E}\left[\prod_{\ell=1}^m g_\ell\left(\langle \mu^{N_j,\Sigma}, \mathbb{1}_{[0,t_\ell]}(y)\rangle\right) h_\ell\left(\mathbb{E}[\phi(-\xi)]\int_0^{t_\ell} r(t_\ell - s)\langle \mu^{N_j,\Sigma}, \mathbb{1}_{[0,s]}(y)\rangle \,\mathrm{d}s\right)\right] \\
&= \lim_{j\to\infty} \mathbb{E}\left[\prod_{\ell=1}^m g_\ell\left(F_{t_\ell}^{N_j,\Sigma}\right) h_\ell\left(\mathbb{E}[\phi(-\xi)]\int_0^{t_\ell} r(t_\ell - s)F_s^{N_j,\Sigma} \,\mathrm{d}s\right)\right],
\end{aligned} \tag{3.4.20}$$

where the first equality follows from the Continuous Mapping Theorem, and the second equality follows from the definition of $F^{N,k}$. Lastly, by the properties of M_1-convergence, the map

$$\prod_{k\geq 1} D_{\mathbb{R}} \to \prod_{k\geq 1} \mathbb{R}, \qquad (f^k)_k \mapsto \left(\int_0^{t_\ell} r(t_\ell - s) f_s^k \,\mathrm{d}s\right)_k$$

is continuous. Therefore, putting (3.4.19) and (3.4.20) together, and applying the Continuous Mapping Theorem one last time, we deduce that

$$\lim_{j\to\infty} \mathbb{E}\left[\prod_{\ell=1}^{m} g_\ell\left(F_{t_\ell}^{N_j,\Sigma}\right) h_\ell\left(\tilde{F}_{t_\ell}^{D,N_j,\Sigma}\right)\right] = \mathbb{E}\left[\prod_{\ell=1}^{m} g_\ell\left(F_{t_\ell}^{\dagger,\Sigma}\right) h_\ell\left(\mathbb{E}[\phi(-\xi)]\int_0^{t_\ell} r(t_\ell - s)F_s^{\dagger,\Sigma}\,\mathrm{d}s\right)\right].$$

Therefore, we may find a J_2 such that for all $j \geq J_2$, $III < \varepsilon/4$. Setting $j = \max\{J_1, J_2\}$, we have $I + III < \varepsilon/2$.

Now, we only need to control IV. By the Continuous Mapping Theorem and the Portmanteau Theorem, IV is bounded by the lim sup over j of

$$\left|\mathbb{E}\left[\prod_{\ell=1}^{m} g_\ell\left(F_{t_\ell}^{N_j,\Sigma}\right) h_\ell\left(\mathbb{E}[\phi(-\xi)]\left(r * F^{N_j,\Sigma}\right)_{t_\ell}\right) - g_\ell(F_{t_\ell}^{N_j}) h_\ell(\mathbb{E}[\phi(-\xi)](r * F^{N_j})_{t_\ell})\right]\right|.$$

Therefore, proceeding similarly as in (3.4.18),

$$IV \leq 2c(1 + \|\phi\|_\infty) \limsup_j \mathbb{E}\left[\sum_{k>M} F_T^{N_j,k}\right].$$

By our choice of M earlier, independently of N_j, we have $\mathbb{E}\left[\sum_{k>M} F_T^{N_j,k}\right] < \varepsilon(8c + 8c\,\|\phi\|_\infty)^{-1}$. Therefore, $IV < \varepsilon/4$. Hence, we have shown that $I + II + III + IV < \varepsilon$. As $\varepsilon > 0$ was arbitrary, the proof is now complete. □

We now have all the ingredients to show that our limit point will satisfy Assumption 3.2.2.

Proposition 3.4.6 (Regularity conditions)**.** *Let (ν^N, F^N, W^0) be a subsequence, still indexed by N, along which $(\nu^N, F^N, W^0) \implies (\nu^*, F^*, W^0)$. Then (ν^*, F^*) satisfies Assumption 3.2.2. Furthermore, F_t^* is sub-Gaussian for every $t \in [0, T]$.*

Proof. By making the obvious changes to [29, Proposition 5.3], ν_t^* is a sub-probability measure supported on $(-\infty, 0]$ for every $t \in [0, T]$ with probability 1. For Assumption 3.2.2 (iv) and (v), see the proof of [29, Proposition 5.6] with the obvious changes. This also gives us that F^* is non-decreasing. The exponential decay of the tail probabilities, Assumption 3.2.1 (iii), follows directly from Proposition 3.6.12. This also shows that ν^* has càdlàg paths.

It remains to show Assumption 3.2.2 (ii). Let $\phi_n \in \mathcal{C}^\infty([0,1])$ be such that

$$\phi_n \begin{cases} = 1 & \text{on } [-n, -1/n], \\ \in (0,1) & \text{on } [-1-n, -n) \cup (-1/n, 0], \\ = 0 & \text{otherwise.} \end{cases}$$

We set $\mathbb{T} = \{t \in [0,T] : \mathbb{P}[\langle \nu_t^*, \phi_n \rangle = \langle \nu_{t-}^*, \phi_n \rangle] = \mathbb{P}[F_t^* = F_{t-}^*] = 1 \, \forall \, n\}$. By construction, $\mathbb{T}$ is a co-countable set in $[0,T]$ and, therefore, dense. By the Continuous Mapping Theorem, for all $t \in \mathbb{T}$, we have

$$\langle \nu^N, \phi_n \rangle_t + F_t^N - (r * F^N)_t \implies \langle \nu^*, \phi_n \rangle_t + F_t^* - (r * F^*)_t.$$

Therefore,

$$\begin{aligned} \mathbb{E}\left[\left| \langle \nu_t^*, \phi_n \rangle + F_t^* - (r * F^*)_t - 1 \right| \right] &\leq \limsup_{N \to \infty} \mathbb{E}\left[\left| \langle \nu_t^N, \phi_n \rangle + F_t^N - (r * F^N)_t - 1 \right| \right] \\ &\leq \limsup_{N \to \infty} \mathbb{E}\left[\left| \langle \nu_t^N, \phi_n \rangle - \nu_t^N(-\infty, 0) \right| \right] \\ &\quad + \limsup_{N \to \infty} \mathbb{E}\left[\left| (r * F^N)_t - F_t^{D,N} \right| \right] \\ &\leq 2 \limsup_{N \to \infty} \mathbb{E}\left[\nu_t^N(-\infty, -n) + \nu_t^N(-1/n, 0) \right] \\ &\quad + \limsup_{N \to \infty} \mathbb{E}\left[\left| (r * F^N)_t - F_t^{D,N} \right| \right], \end{aligned}$$

where the first inequality follows from the Portmanteau Theorem, and the second follows from the fact that $\nu_t^N(-\infty, 0) + F_t^N - F_t^{D,N} = 1$. By Proposition 3.4.5, a simple application of the Portmanteau Theorem allows us to deduce that $\mathbb{E}[|(r * F^N)_t - F_t^{D,N}|] \to 0$. By Corollary 3.3.5, there exist $\delta, \tilde{\delta}, C > 0$ such that

$$\limsup_{N \to \infty} \mathbb{E}\left[\nu_t^N(-\infty, -n) \right] \leq C e^{-\delta n^2} \quad \text{and} \quad \limsup_{N \to \infty} \mathbb{E}\left[\nu_t^N(-1/n, 0) \right] \leq C n^{-\tilde{\delta}}.$$

Therefore, sending $n \to \infty$ and employing the Dominated Convergence Theorem, we may conclude that

$$\mathbb{E}\left[|\nu_t^*(-\infty, 0) + F_t^* - (r * F^*)_t - 1| \right] = 0.$$

By choosing a countable, dense subset $\{t_n\}_n \subset \mathbb{T}$, we have

$$\nu_{t_n}^*(-\infty, 0) + F_{t_n}^* - (r * F^*)_{t_n} = 1 \quad \forall \, t_n \quad \text{a.s.}$$

As $\nu^*(-\infty, 0)$, F^*, and $(r * F^*)$ are all càdlàg, we have

$$\nu_t^*(-\infty, 0) + F_t^* - (r * F^*)_t = 1 \quad \forall \, t \geq 0 \quad \text{a.s.}$$

For the second claim, uniformly in N and $t \in [0,T]$, we have a $\delta > 0$ such that

$$\mathbb{P}\left[F_t^N > \lambda\right] = O(e^{-\delta\lambda^2}).$$

Since $F^N \implies F^*$ in $(D_{\mathbb{R}}, M_1)$ and T is an almost sure continuity point of F^*, then by the Continuous Mapping Theorem, $F_T^N \implies F_T^*$ in $\mathbb{R}$. Therefore, by the Portmanteau Theorem,

$$\mathbb{P}\left[F_T^* > \lambda\right] \leq \liminf_{N\to\infty} \mathbb{P}\left[F_T^N > \lambda\right] \leq Ce^{-\delta\lambda^2}.$$

Thus, F_T^* is sub-Gaussian. As $t \mapsto F_t^*$ is non-decreasing, F_t^* is sub-Gaussian uniformly in $t \in [0,T]$. □

3.4.3 Existence of Solutions to SPDE (3.2.3)

We would like to know if $\nu^N \implies \nu^*$ on $(D_{\mathscr{S}'}, M_1)$, then will we have $\langle \nu^N, \mathrm{Id}\rangle =: M^N \implies M := \langle \nu^*, \mathrm{Id}\rangle$, where Id is the identity map. Since $\mathrm{Id} \notin \mathscr{S}$, we cannot employ the continuous mapping theorem. Hence, we present the following result, which serves two purposes:

(i) It allows us to deduce the convergence of the mean process M^N to $M = \langle \nu^*, \mathrm{Id}\rangle$.

(ii) It provides a stepping stone for proving the convergence of certain integrals, which is essential in demonstrating that each limit ν^* satisfies the SPDE (3.2.3).

Proposition 3.4.7. *Let $\psi(t,x,\mu,f)$ be a measurable function that satisfies Assumption 3.2.1 (i) and (ii), and (ν^N, F^N, W^0) be a subsequence, still indexed by N, along which $(\nu^N, F^N, W^0) \implies (\nu^*, F^*, W^0)$. Then there is a co-countable set of times on which the finite-dimensional distributions of $\langle \nu_t^N, \psi(t,\cdot,\nu_t^N, \mathfrak{f}_t^N)\rangle$ converge weakly to those of $\langle \nu_t^*, \psi(t,\cdot,\nu_t^*, \mathfrak{f}_t^*)\rangle$.*

Proof. Our strategy is as follows. We shall construct a co-countable set $\mathbb{T}$ and show that if we fix a $t \in \mathbb{T}$, we have $\langle \nu_t^N, \psi(t,\cdot,\nu_t^N, \mathfrak{f}_t^N)\rangle \implies \langle \nu_t^*, \psi(t,\cdot,\nu_t^*, \mathfrak{f}_t^*)\rangle$. Then, by employing the techniques in Proposition 3.4.5, notably the method used to control II in (3.4.18), we may conclude that we have convergence of the finite-dimensional distributions. The proof relies on several technical results presented later in this chapter, namely Corollary 3.6.5, Proposition 3.6.11, and Proposition 3.6.12. Due to their technical nature, we omit the proofs of these results here and direct the reader to the relevant sections for detailed expositions.

To begin, we fix a bounded $h \in \mathrm{Lip}(\mathbb{R})$ and fix a $t \in \mathbb{T}$, where

$$\mathbb{T} \coloneqq \left\{ t \in [0,T] \,:\, \mathbb{P}[\langle \nu_t^*, \phi^n \rangle = \langle \nu_{t-}^*, \phi^n \rangle] = 1 \text{ for all } n \in \mathbb{N}, \text{ where } \phi^n \in \mathscr{S} \text{ dense} \right\}.$$

It is clear that $\mathbb{T}$ is co-countable. By [31, Remark 4.6], any subsequence of (ν^N, F^N) has a further subsequence, also denoted by N for simplicity, for which we may assume almost sure convergence on a common probability space. By the triangle inequality,

$$\begin{aligned}
&\left| \mathbb{E}\left[h(\langle \nu_t^N, \psi(t,\cdot,\nu_t^N, \mathfrak{f}_t^N)\rangle) \right] - \mathbb{E}\left[h(\langle \nu_t^*, \psi(t,\cdot,\nu_t^*, \mathfrak{f}_t^*)\rangle) \right] \right| \\
&\qquad \lesssim \mathbb{E}\left| \langle \nu_t^N - \nu_t^*, \psi(t,\cdot,\nu_t^*, \mathfrak{f}_t^*)\rangle \right| \\
&\qquad\quad + \mathbb{E}\left| \langle \nu_t^N, \psi(t,\cdot,\nu_t^*, \mathfrak{f}_t^N) - \psi(t,\cdot,\nu_t^*, \mathfrak{f}_t^*)\rangle \right| \\
&\qquad\quad + \mathbb{E}\left| \langle \nu_t^N, \psi(t,\cdot,\nu_t^N, \mathfrak{f}_t^N) - \psi(t,\cdot,\nu_t^*, \mathfrak{f}_t^N)\rangle \right| \\
&\qquad = \mathbb{E} I_1^N + \mathbb{E} I_2^N + \mathbb{E} I_3^N.
\end{aligned} \tag{3.4.21}$$

The rest of the proof is split into three parts. We will show that each term converges to 0 as $N \to \infty$.

Control on I_1^N:

For $\lambda > 0$, let $\chi_\lambda \in \mathcal{C}^\infty([0,1])$ denote the standard cut-off function which is equal to 1 on $[-\lambda, \lambda]$ and $\varphi_{1/\lambda} \in \mathcal{C}_c^\infty(\mathbb{R})$ be the standard mollifier rescaled by $1/\lambda$. We consider the (random) functions, indexed by $\lambda > 0$,

$$\psi_\lambda^*(x) = \chi_\lambda(x) \cdot (\psi(t,\cdot,\nu_t^*, \mathfrak{f}_t^*) * \varphi_{1/\lambda})(x).$$

By our assumptions on ψ, it is globally Lipschitz in x, and we have $|\psi(t,x,\mu,f_t)| \leq C_\psi(1 + |x| + \langle \mu, |\cdot| \rangle + |f_t|)$. Therefore, setting $C(\nu^*) = C_\psi(1 + \sup_{s \leq T} \langle \nu_s^*, |\cdot| \rangle + F_t^*)$, we may show

$$|\psi_\lambda^* - \psi(t,\cdot,\nu_t^*, \mathfrak{f}_t^*)| \leq \begin{cases} 2 \left\|\psi\right\|_{\mathrm{Lip}} \lambda^{-1} & \text{on } [-\lambda, \lambda], \\ 2 \left\|\psi\right\|_{\mathrm{Lip}} (\lambda^{-1} + |x|) + C(\nu^*) & \text{on } [-\lambda, \lambda]^{\complement}. \end{cases}$$

Therefore, for all $\lambda > 0$, by employing the upper bound above followed by Corollary 3.3.5 and Proposition 3.6.11

$$\begin{aligned}
\mathbb{E}|\langle \nu_t^N, \psi_\lambda^* - \psi(t,\cdot,\nu_t^*, \mathfrak{f}_t^*)\rangle| &\leq 2 \left\|\psi\right\|_{\mathrm{Lip}} \mathbb{E}\langle \nu_t^N, \lambda^{-1} \mathbb{1}_{\mathbb{R}} + (\lambda^{-1} + |x| + C(\nu^*)) \mathbb{1}_{(-\infty,-\lambda)} \rangle \\
&\leq C(\lambda^{-1} + \lambda^{-1} e^{-\delta\lambda^2} + e^{-a\lambda}) + \mathbb{E}[C(\nu^*) \nu_t^N(-\infty, -\lambda)],
\end{aligned}$$

where C, a, $\delta > 0$ are independent of N and t. We note $\mathbb{E}[C(\nu^*)^q]$ is finite by Proposition 3.4.6 and Proposition 3.6.12 for any $q \geq 1$. Therefore, by employing

Hölder's inequality and Corollary 3.3.5 to control the last term in the above, we may conclude that

$$\mathbb{E}|\langle \nu_t^N, \psi_\lambda^* - \psi(t,\cdot,\nu_t^*,\mathfrak{f}_t^*)\rangle| = O(\lambda^{-1}) \tag{3.4.22}$$

uniformly in N. By employing similar arguments, we may conclude that the bound in (3.4.22) holds when ν^N is replaced by ν^*. Therefore, by employing the triangle inequality,

$$\mathbb{E}I_1^N \leq \mathbb{E}|\langle \nu_t^N, \psi_\lambda^* - \psi(t,\cdot,\nu_t^*,\mathfrak{f}_t^*)\rangle| + \mathbb{E}\left|\langle \nu_t^N - \nu_t^*, \psi_\lambda^*\rangle\right| + \mathbb{E}|\langle \nu_t^N, \psi_\lambda^* - \psi(t,\cdot,\nu_t^*,\mathfrak{f}_t^*)\rangle|.$$

For any $\delta > 0$, we may fix a $\lambda > 0$ large enough such that the first term and the third term in the above is less than $\delta/3$. As the projection map $\Pi^{\psi_\lambda^*} : (D_{\mathscr{S}'}, M_1) \to (D_{\mathbb{R}}, M_1)$, where $\Pi^{\psi_\lambda^*}(f) = f(\psi_\lambda^*)$ is continuous ([47, Proposition 2.7]) and $t \in \mathbb{T}$, we have $\langle \nu_t^N - \nu_t^*, \psi_\lambda^*\rangle$ converges almost surely to 0 as $N \to \infty$. Hence, for this fixed λ, we may deduce that the second term is less than $\delta/3$ in expectation for all N large by the Dominated Convergence Theorem. Therefore, by first choosing a λ large enough, we have for all N large that $\mathbb{E}I_1^N \leq \delta$.

Control on I_2^N:

By assumption, ψ is globally Lipschitz in its fourth component. This property, along with the fact that ν_t^N is a sub-probability measure, gives us

$$\left|\langle \nu_t^N, \psi(t,\cdot,\nu_t^*,\mathfrak{f}_t^N) - \psi(t,\cdot,\nu_t^*,\mathfrak{f}_t^*)\rangle\right| \leq C\int_0^t |\mathfrak{K}'(t-s)|\left|F_s^N - F_s^*\right| \mathrm{d}s.$$

By Proposition 3.4.6 and Corollary 3.6.5, it is immediate that for any $p \geq 1$, there exists a constant $C_p > 0$ such that

$$\sup_N \mathbb{E}\left[\left(\int_0^t |\mathfrak{K}'(t-s)|\left|F_s^N - F_s^*\right| \mathrm{d}s\right)^p\right] < C_p.$$

Furthermore, as $F^N \to F^*$ almost surely in $(D_{\mathbb{R}}, M_1)$, by the properties of M_1-convergence, we have

$$\int_0^t |\mathfrak{K}'(t-s)|\left|F_s^N - F_s^*\right| \mathrm{d}s \to 0$$

almost surely. Hence, by Vitali's Convergence Theorem, we obtain

$$\mathbb{E}\int_0^t |\mathfrak{K}'(t-s)|\left|F_s^N - F_s^*\right| \mathrm{d}s \to 0.$$

This proves that $\mathbb{E}I_2^N \to 0$.

Control on I_3^N:

Similarly to the analysis of I_2^N, the global Lipschitzness of ψ in the measure component combined with the fact that $\nu_t^N \in \mathbf{M}_{\leqslant 1}(\mathbb{R})$ gives

$$I_3^N \leq C \sup\{\left|\langle \nu_t^N - \nu_t^*, \phi\rangle\right| \,:\, \phi \in \mathcal{C}_{d_0}\}, \tag{3.4.23}$$

where $\mathcal{C}_{d_0} \coloneqq \{\phi \in \mathcal{C}(\mathbb{R}) \,:\, \|\phi\|_{\mathrm{Lip}} \leq 1, |\phi(x)| \leq 1 + |x|\}$. Fix a $\delta > 0$, then for any $\lambda > 1$, by the Arzerlà-Ascoli Theorem, there is a finite family of functions $\phi_1, \ldots, \phi_m \in \mathcal{C}_{d_0}$ supported on $[-\lambda - 1, \lambda + 1]$, for $m = m(\lambda) \in \mathbb{N}$ such that for any $\phi \in \mathcal{C}_{d_0}$,

$$\sup_{x \in [-\lambda, \lambda]} |\phi(x) - \phi_i(x)| < \delta/2 \tag{3.4.24}$$

for some $i \in \{1, \ldots, m\}$. Fixing any $\phi \in \mathcal{C}_{d_0}$ and choosing a ϕ_i such that (3.4.24) holds, we have

$$\begin{aligned}\left|\langle \nu_t^N - \nu_t^*, \phi\rangle\right| \leq &\left|\int_{-\lambda}^{0} (\phi - \phi_i)\, \mathrm{d}(\nu_t^N - \nu_t^*)\right| + \left|\int_{-\infty}^{-\lambda} (\phi - \phi_i)\, \mathrm{d}(\nu_t^N - \nu_t^*)\right| \\ &+ \left|\langle \nu_t^N - \nu_t^*, \phi_i\rangle\right|.\end{aligned}$$

By choice of ϕ_i, the first term is bounded by δ uniformly in N. By the linear growth condition of functions in $\mathcal{C}_{d_0}$,

$$\left|\int_{-\infty}^{-\lambda} (\phi - \phi_i)\, \mathrm{d}(\nu_t^N - \nu_t^*)\right| \leq C\langle \nu_t^* + \nu_t^N, |\cdot|\, \mathbb{1}_{(-\infty, -\lambda]}\rangle.$$

Returning to (3.4.23), we have shown

$$\mathbb{E} I_3^N \leq C\delta + C\mathbb{E}\langle \nu_t^* + \nu_t^N, |\cdot|\, \mathbb{1}_{(-\infty, -\lambda]}\rangle + C \sum_{i=1}^{m} \mathbb{E}\left|\langle \nu_t^N - \nu_t^*, \phi_i\rangle\right|.$$

Proposition 3.6.11 gives that the middle term will vanish uniformly in N as $\lambda \to \infty$. Hence, we fix a λ sufficiently large such that the middle term is bounded by δ uniformly in N. As λ is now fixed, so is m. Since the ϕ_i's have compact support, we may apply the same mollification argument as in the case of I_1^N to deduce the final term will be smaller than δ for all N large. Therefore, for all N sufficiently large, $\mathbb{E} I_3^N < C\delta$ for some constant C independent of N. Hence, $\mathbb{E} I_3^N \to 0$.

This completes the proof. □

A particular case we may consider is $\psi(t, \cdot, \mu, f) = \phi \in \mathcal{C}^2(\mathbb{R})$ with $\|\partial_x \phi\|_{\mathcal{C}^1} < \infty$. Although ϕ itself may be unbounded as we only assume $\|\partial_x \phi\|_{\mathcal{C}^1} < \infty$, the statements and proofs of Proposition 3.4.2, Proposition 3.4.3, and Proposition 3.4.4 remain valid

for this ϕ if we replace $\|\phi\|_\infty$ with $\sup_{x\in[-R,R]}|\phi(x)|$, since ξ has compact support. As a consequence, $\{\langle\nu^N,\phi\rangle\}_N$ is tight and, by the proposition above, converges weakly in the M_1-topology towards $\langle\nu^*,\phi\rangle$. A special case arises when $\phi=\mathrm{Id}$, the identity function. In this situation, we obtain the convergence of the mean process.

Lemma 3.4.8. *Let $\psi(t,x,\mu,f)$ be a measurable function that satisfies Assumption 3.2.1 (i) and $\phi\in\mathscr{S}$. Then for a subsequence (ν^N,F^N,W^0), still indexed by N, along which $(\nu^N,F^N,W^0)\implies(\nu^*,F^*,W^0)$, we have*

$$\int_0^\cdot\langle\nu_s^N,\psi(s,\cdot,\nu_s^N,\mathfrak{f}_s^N)\phi\rangle\,\mathrm{d}s\implies\int_0^\cdot\langle\nu_s^*,\psi(s,\cdot,\nu_s^*,\mathfrak{f}_s^*)\phi\rangle\,\mathrm{d}s$$

on $D_{\mathbb{R}}$ endowed with the M_1-topology.

Proof. By [31, Remark 4.6], any subsequence of (ν^N,F^N) has a further subsequence, also denoted by N for simplicity, for which we may assume almost sure convergence on a common probability space. Fixing this subsequence and choosing a function $h\in\mathrm{Lip}(D_{\mathbb{R}};\mathbb{R})$, by the definition of the M_1-topology, we have

$$\begin{aligned}\left|\mathbb{E}\left[h\left(\int_0^\cdot\langle\nu_s^N,\psi(s,\cdot,\nu_s^N,\mathfrak{f}_s^N)\phi\rangle\,\mathrm{d}s\right)\right]-\mathbb{E}\left[h\left(\int_0^\cdot\langle\nu_s^*,\psi(s,\cdot,\nu_s^*,\mathfrak{f}_s^*)\phi\rangle\,\mathrm{d}s\right)\right]\right|\\ \leq\int_0^T\mathbb{E}\left[\left|\langle\nu_t^N,\psi(t,\cdot,\nu_t^N,\mathfrak{f}_t^N)\phi\rangle-\langle\nu_t^*,\psi(t,\cdot,\nu_t^*,\mathfrak{f}_t^*)\phi\rangle\right|\right]\mathrm{d}t.\end{aligned}\tag{3.4.25}$$

At this point, the aim is to apply the Dominated Convergence Theorem. We will show that the expectation in the above converges to 0 as $N\to\infty$ and is bounded uniformly in N. The rest of this proof is nearly verbatim to that of Proposition 3.4.7. The only difference lies in that, as $\psi(t,x,\mu,f)\phi(x)$ need not be globally Lipschitz in x, we must adapt the mollification argument to show that the first term in the decomposition in (3.4.21) is zero. First, decomposing the difference in the expectation in (3.4.25) into three parts as in (3.4.21) and choosing a $t\in\mathbb{T}$ from Proposition 3.4.7, we have

New control on I_1^N:

Fix a $\delta>0$. We have $|\psi(t,x,\mu,f_t)|\leq C(1+|x|+\langle\mu,|\cdot|\rangle+|f_t|)$. Therefore, setting $C(\nu^*)=1+\langle\nu_t^*,|\cdot|\rangle+F_T^*$, we observe there exists a $\lambda=\lambda(\delta)\gg1$ such that

$$\sup_{x\in[-\lambda,\lambda]^\complement}|\psi(t,x,\nu_t^*,\mathfrak{f}_t^*)\phi(x)|\leq C(\nu^*)\delta/4\qquad\forall\,t\in[0,T].$$

Now, we choose the standard family of mollifiers $\varphi^\varepsilon\in\mathcal{C}_c^\infty(\mathbb{R})$ and consider the (random) mollifications

$$\psi_t^\varepsilon=\psi(t,\cdot,\nu_t^*,\mathfrak{f}_t^*)*\varphi^\varepsilon\qquad\text{for}\qquad0<\varepsilon<1.$$

As ν^* and ν^N are sub-probability measures,

$$I_1^N \leq \left|\langle \nu_t^N - \nu_t^*, \psi_t^\varepsilon \phi\rangle\right| + 2\left\|\phi\right\|_\infty \sup_{|x|<2\lambda} |\psi_t^\varepsilon - \psi_t| + C(\nu^*)\delta.$$

As $\psi_t^\varepsilon \to \psi(t,\cdot,\nu_t^*,\mathfrak{f}_t^*)\phi$ uniformly in $x \in [-2\lambda, 2\lambda]$, by the Dominated Convergence Theorem we have for all $\varepsilon \ll 1$, the second term will be less than δ in expectation. Independent of ε, there is a set of full measure such that $\psi_t^\varepsilon \phi \in \mathscr{S}$. As $t \in \mathbb{T}$, we may deduce that the first term converges almost surely to 0 as $N \to \infty$. Hence, it will be less than δ in expectation by the Dominated Convergence Theorem for all N sufficiently large. Therefore, by first choosing an ε small enough, we have for all N large $EI_1^N \leq C(\delta + \delta + \delta)$, where C depends only on $\mathbb{E}[C(\nu^*)]$. We note $\mathbb{E}[C(\nu^*)]$ is finite by Proposition 3.4.6 and Proposition 3.6.12.

As $\psi(t,x,\mu,f)\phi(x)$ is still Lipschitz in μ and f, we still have $\mathbb{E}I_2^N$ and $\mathbb{E}I_3^N$ converge to 0 for all $t \in \mathbb{T}$ by the results in Proposition 3.4.7.

Lastly by the linear growth on ψ and the fact that $\phi \in \mathscr{S}$, we observe

$$\begin{aligned}\mathbb{E}\left[\left|\langle \nu_t^N, \psi(t,\cdot,\nu_t^N,\mathfrak{f}_t^N)\phi - \langle \nu_t^*, \psi(t,\cdot,\nu_t^*,\mathfrak{f}_t^*)\phi\rangle\right|\right] \\ \leq C\mathbb{E}\left[1 + \langle \nu_t^N, |\cdot|\rangle + F_T^N + \langle \nu_t^*, |\cdot|\rangle + F_T^*\right]\end{aligned}$$

where $C > 0$ is independent of t. The right-hand side in the above is bounded uniformly in N by Corollary 3.3.5 and Proposition 3.6.11. Therefore, by the Dominated Convergence Theorem, we have the integral in (3.4.25) converges to 0 as $N \to \infty$. This completes the proof. □

As σ and ρ are bounded by Assumption 3.2.1 (i) and (iii), we may deduce that the convergence result from the previous proposition also holds in the case when the integrands are squared. This is formalized in the following corollary.

Corollary 3.4.9. *Let $\phi \in \mathscr{S}$ and (ν^N, F^N, W^0) be a subsequence, still indexed by N, which converges weakly to (ν^*, F^*, W^0). Then, we have*

$$\int_0^\cdot \langle \nu_s^N, \sigma(s,\cdot)\rho(s,\nu_s^N,\mathfrak{f}_s^N)\phi\rangle^2 \,\mathrm{d}s \implies \int_0^\cdot \langle \nu_s^*, \sigma(s,\cdot)\rho(s,\nu_s^*,\mathfrak{f}_s^*)\phi\rangle^2 \,\mathrm{d}s$$

on $D_\mathbb{R}$ endowed with the M_1-topology.

We now have all the ingredients to prove Theorem 3.2.3.

Proof of Theorem 3.2.3. Fix an arbitrary $\phi \in \mathscr{S}$. Returning to (3.4.1), we first focus on the last term. By the Burkholder-Davis-Gundy inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\leq T}\left|\int_0^t \frac{1}{N}\sum_{i=1}^N \sigma(s,X_s^i)\sqrt{1-\rho^2(s,\nu_s^N,\mathfrak{f}_s^N)}\partial_x\phi(X_s^i)\mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}W_s^i\right|\right] \\
&\qquad\leq C^{\mathrm{BGD}}\mathbb{E}\left[\left(\frac{1}{N^2}\sum_{j=1}^N\int_0^T \sigma(s,X_s^i)^2(1-\rho^2(s,\nu_s^N,\mathfrak{f}_s^N))(\partial_x\phi(X_s^i))^2\,\mathrm{d}s\right)^{1/2}\right] \\
&\qquad\leq \frac{C^{\mathrm{BGD}}C_\sigma\left\|\partial_x\phi\right\|_\infty N^{1/2}T^{1/2}}{N}.
\end{aligned}$$

Therefore, we may conclude that this term converges weakly towards 0 in $D_{\mathbb{R}}$ endowed with the M_1-topology. By Proposition 3.4.4, we may find a subsequence $\{(\nu^{N_j},F^{N_j},W^0)\}$ which converges weakly toward (ν^*,F^*,W^0). By Lemma 3.4.8, individually, each of the $\mathrm{d}t$ integrals in (3.4.1) converge weakly along this subsequence towards their counterparts where ν^N is replaced by ν^* in $D_{\mathbb{R}}$ endowed with the M_1-topology. Given Lemma 3.4.8 and Corollary 3.4.9, we may apply [64, Corollary 3.24] to conclude that the stochastic integral with respect to W^0 in (3.4.1) converges weakly towards its analogous counterpart. Furthermore, by Proposition 3.4.5, we also have functional weak convergence of the firing function and the delayed re-insertion term. As all limiting terms are continuous apart from the F^* term, we may conclude by the continuous mapping theorem that the entire right-hand side of (3.4.1) converges weakly in $D_{\mathbb{R}}$ endowed with the M_1-topology towards the right-hand side of (3.2.3). Similarly, by the continuous mapping theorem, the left-hand side of (3.4.1) converges weakly in $D_{\mathbb{R}}$ endowed with the M_1-topology towards the left-hand side of (3.2.3). This precisely means that (ν^*,F^*,W^0) is a solution to the SPDE (3.2.3). Lastly, Proposition 3.4.6 gives us that ν^* is a càdlàg $\mathbf{M}_{\leq 1}(\mathbb{R}_-)$-valued process with probability 1 and (ν^*,F^*) satisfy Assumption 3.2.2. This completes the proof of Theorem 3.2.3. □

Having established that any limit point is a solution to the SPDE (3.2.3), we now turn our attention to establishing a relationship between the firing function and the flux of mass across the boundary. We provide a weaker formulation of the flux condition stated in (3.1.4) as we cannot assume that F_t^* is differentiable. The following lemma formalizes this weak flux condition:

Lemma 3.4.10 (Weak flux condition)**.** *Let ψ_ε be a smooth cut-off function approximating $\mathbb{1}_{\{0\}}$ and (ν^N,F^N,W^0) be a subsequence, still indexed by N, along which*

$(\nu^N, F^N, W^0) \implies (\nu^*, F^*, W^0)$. *Then, almost surely and in* L^1,

$$\mathbb{E}\left[F_t^* \middle| \mathcal{F}_s\right] - F_s^* = \frac{1}{2} \lim_{\varepsilon \downarrow 0} \int_s^t \mathbb{E}\left[\left| \langle \nu_u^*, \partial_x [\sigma(u,\cdot)^2 \partial_x \psi_\varepsilon] \rangle \right| \middle| \mathcal{F}_s \right] \mathrm{d}u.$$

Proof. Let $\psi_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}; [0,1])$ be a smooth approximation to the indicator function of $[-\varepsilon/2, \varepsilon/2]$, with $\psi_\varepsilon(x) = 1$ on $[-\varepsilon/2, \varepsilon/2]$ and $\psi_\varepsilon(x) = 0$ outside $[-\varepsilon, \varepsilon]$. Moreover, we can choose ψ_ε such that $|\partial_x \psi_\varepsilon| \leq C\varepsilon^{-1}$ for some constant $C > 0$ that is uniform in ε. Since $\psi_\varepsilon \in \mathscr{S}$, we can use it as a test function in the SPDE (3.2.3) to obtain the following equation:

$$\begin{aligned} F_t^* = &\int_0^t \langle \nu_u^*,\, b(u,\cdot,\nu_u^*, \mathfrak{f}_u^*) \partial_x \psi_\varepsilon \rangle \,\mathrm{d}u + \frac{1}{2} \int_0^t \langle \nu_u^*, \sigma^2(u,\cdot) \partial_{xx} \psi_\varepsilon \rangle \,\mathrm{d}u \\ &+ \int_0^t \langle \nu_u^*,\, \sigma(u,\cdot) \rho(u, \nu_u^*, \mathfrak{f}_u^*) \partial_x \psi_\varepsilon \rangle \,\mathrm{d}W_u^0 + \langle \nu_0,\, \psi_\varepsilon \rangle - \langle \nu_t^*,\, \psi_\varepsilon \rangle \end{aligned} \tag{3.4.26}$$

for all sufficiently small ε as $\psi_\varepsilon(-\xi) = 0$. Since ν_0 is deterministic and has an L^2 density, we have convergence in L^1 and almost surely of $\langle \nu_0,\, \psi_\varepsilon \rangle$ towards 0 as $\varepsilon \to 0$. By employing Proposition 3.3.6 and the continuous mapping theorem, we have

$$\mathbb{E}[\nu_t^*(-\varepsilon, 0)] = O(t^{-\delta/2} \varepsilon^{1+\beta}) \qquad \text{and} \qquad \mathbb{E} \int_0^t \nu_s^*(-\varepsilon, 0) \,\mathrm{d}s = O(\varepsilon^{1+\beta}), \tag{3.4.27}$$

for some constants $\delta,\, \beta > 0$ that are independent of ε and $\delta \leq 1$. By slightly augmenting the arguments in [31, Lemma A.4], we can employ a Borel-Cantelli argument to show that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-p} \nu_t^*(-\varepsilon, 0) = 0 \qquad \text{and} \qquad \lim_{\varepsilon \downarrow 0} \varepsilon^{-p} \int_0^t \nu_s^*(-\varepsilon, 0) \,\mathrm{d}s = 0 \tag{3.4.28}$$

almost surely for all $1 \leq p \leq 1 + \beta/2$, where β is the constant from (3.4.27). Therefore, by (3.4.27) and (3.4.28), $\langle \nu_t^*,\, \psi_\varepsilon \rangle$ converges to 0 almost surely and in L^1 as $\varepsilon \downarrow 0$. Since the stochastic integral in (3.4.26) is a martingale, it disappears when we take increments and conditional expectations. Therefore, we only need to control the terms on the first line of (3.4.26). For brevity, we show the almost sure convergence, but similar arguments apply to show convergence in L^1.

By Assumption 3.2.1, we have $|b(u, x, \nu_u^*, \mathfrak{f}_u^*)| \leq C_b(1 + |x| + \langle \nu_u^*, |\cdot| \rangle + F_T^*)$. Therefore,

$$\int_0^t |\langle \nu_u^*,\, b(u,\cdot,\nu_u^*, \mathfrak{f}_u^*) \partial_x \psi_\varepsilon \rangle| \,\mathrm{d}u \leq C \int_0^t (1 + \langle \nu_u^*, |\cdot| \rangle + F_T^*) \varepsilon^{-1} \nu_u^*(-\varepsilon, 0) \,\mathrm{d}u.$$

By applying Hölder's inequality, we get

$$\int_0^t |\langle \nu_u^*,\, b(u,\cdot,\nu_u^*,\mathfrak{f}_u^*)\partial_x\psi_\varepsilon\rangle|\,\mathrm{d}u \\ \leq C\left(\int_0^t (1+\langle \nu_u^*, |\cdot|^q\rangle + (F_T^*)^q)\,\mathrm{d}u\right)^{\frac{1}{q}}\left(\int_0^t \varepsilon^{-p}\nu_u^*(-\varepsilon,0)\,\mathrm{d}u\right)^{\frac{1}{p}}$$

for all conjugate exponents p and q. The first term above is finite in expectation for all $q \geq 1$ by Proposition 3.4.6, hence is finite almost surely. The second term in the above converges to 0 almost surely by (3.4.28) for all $p > 1$ close enough to 1. Therefore, by fixing an appropriate p, we have almost sure convergence to 0 of the term with b in (3.4.26).

Lastly, we note that we can write

$$\langle \nu_u^*, \sigma^2(u,\cdot)\partial_{xx}\psi_\varepsilon\rangle = \langle \nu_u^*, \partial_x[\sigma^2(u,\cdot)\partial_x\psi_\varepsilon]\rangle - \langle \nu_u^*, \partial_x[\sigma^2(u,\cdot)]\partial_x\psi_\varepsilon\rangle,$$

and the second term on the right-hand side converges to 0 almost surely as $\varepsilon \to 0$ by the same arguments as above. Therefore, we have shown

$$\mathbb{E}\left[F_t^* \middle| \mathcal{F}_s\right] - F_s^* - \frac{1}{2}\int_s^t \langle \nu_u^*, \partial_x[\sigma^2(u,\cdot)\partial_x\psi_\varepsilon]\rangle\,\mathrm{d}u = o(1) \qquad \text{as} \quad \varepsilon \to 0 \quad \text{almost surely.}$$

This completes the proof. □

3.5 Uniqueness

This section is devoted to the proofs of Theorem 3.2.4 and Theorem 3.2.5. Our approach builds upon the techniques developed in [29] and [31], with careful consideration given to maintaining control over the additional finite variation terms present in our system. We begin by showing that any process satisfying Assumption 3.2.2 and solving SPDE (3.2.3) will have a density. This is achieved by employing a kernel smoothing method. We mollify the measure ν with the Gaussian heat kernel

$$p_\varepsilon(x) \coloneqq (2\pi\varepsilon)^{-1/2}e^{-\frac{x^2}{2\varepsilon}}.$$

We denote the mollification with ν by

$$(\bar{\mathcal{T}}_\varepsilon\nu_t)(x) \coloneqq \int_{\mathbb{R}} p_\varepsilon(x-y)\,\mathrm{d}\nu_t(y).$$

By establishing uniform estimates on these functions, we show the existence of a density for the measure. With the existence of a density, we establish uniqueness in the class of processes that satisfy Assumption 3.2.2.

For this, we will work in a space with lower regularity, namely H^{-1}, the dual space of $H_0^1 = \mathcal{W}^{1,2}(\mathbb{R}_-)$. This is a natural space to work in, as the space of finite signed measures on $\mathbb{R}$ (with the topology induced by the finite variation norm) is embedded into H^{-1}. Hence, given any two candidate solutions, ν and $\tilde{\nu}$, which solve SPDE (3.2.3) and satisfy Assumption 3.2.2, we establish an energy estimate in H^{-1} on $\Delta_t = \nu_t - \tilde{\nu}_t$. We do not directly try to estimate the H^{-1} norm of Δ_t. Instead, we mollify Δ_t with the Dirichlet heat kernel

$$(\mathcal{T}_\varepsilon \Delta_t)(x) \coloneqq \int_{\mathbb{R}} G_\varepsilon(x,y)\,\mathrm{d}\Delta_t(y), \qquad \text{where} \qquad G_\varepsilon(x,y) = p_\varepsilon(x-y) - p_\varepsilon(x+y).$$

Then, by [29, Proposition 6.5], we have

$$\|\Delta_t\|_{H^{-1}} \leq \liminf_{\varepsilon \to 0} \left\|\partial_x^{-1}\mathcal{T}_\varepsilon \Delta_t\right\|_2, \qquad \text{where} \qquad (\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t)(x) \coloneqq \int_{-\infty}^{x} (\mathcal{T}_\varepsilon \Delta_t)(y)\,\mathrm{d}y. \tag{3.5.1}$$

For simplicity of notation, we use $\|\cdot\|_2$ to denote either the $L^2(-\infty,0)$ or $L^2(\mathbb{R})$ norm. Generally, the space will be clear from context.

3.5.1 Existence of an L^2 Density

Proposition 3.5.1 (L^2 energy estimate)**.** *Let (ν, F, W^0) be any process that solves the SPDE (3.2.3) and satisfies Assumption 3.2.2. Then,*

$$\sup_{t\in[0,T]} \sup_{\varepsilon>0} \|\mathcal{T}_\varepsilon \nu_t\|_2^2 < \infty \qquad \textit{with probability } 1.$$

Proof. The proof follows the ideas from [31, Proposition 5.3] and [29, Proposition 7.1]. The primary deviation lies in the fact that we must control the two added finite variation terms in our setting due to the resetting. To begin, we fix an $x \in \mathbb{R}$ and apply the function $\phi(y) = p_\varepsilon(x-y)$ to (3.2.3) to obtain

$$\begin{aligned}\mathrm{d}(\bar{\mathcal{T}}_\varepsilon \nu_t)(x) =& \langle \nu_t, b_t(y)\partial_y p_\varepsilon(x-y)\rangle\,\mathrm{d}t + \frac{1}{2}\langle \nu_t, \sigma_t(y)^2 \partial_{yy} p_\varepsilon(x-y)\rangle\,\mathrm{d}t \\ &+ \langle \nu_t, \sigma_t(y)\rho_t \partial_y p_\varepsilon(x-y)\rangle\,\mathrm{d}W_t^0 + \mathbb{E}\left[p_\varepsilon(x+\xi)\right]\mathrm{d}\int_0^t r(t-s)F_s\,\mathrm{d}s \\ &- p_\varepsilon(x)\,\mathrm{d}F_t,\end{aligned} \tag{3.5.2}$$

First, we would like to move the diffusion coefficients and the derivatives out of the integrals against ν. As ν satisfies the spatial regularity and tail decay of [31,

Assumption 2.3] on the negative half-line, [31, Lemma A.2] and [29, Lemma 8.2] hold in our setting. Therefore, applying [31, Lemma A.2], (3.5.2) becomes

$$\begin{aligned}\mathrm{d}(\bar{\mathcal{T}}_\varepsilon \nu_t) = &- (b_t \partial_x (\bar{\mathcal{T}}_\varepsilon \nu_t) - \partial_x b_t \bar{\mathcal{H}}^b_{t,\varepsilon} + \bar{\mathcal{E}}^b_{t,\varepsilon})\,\mathrm{d}t \\ &+ \frac{1}{2}\partial_x(\sigma_t^2 \partial_x(\bar{\mathcal{T}}_\varepsilon \nu_t) - \partial_x \sigma_t^2 \bar{\mathcal{H}}^{\sigma^2}_{t,\varepsilon} + \bar{\mathcal{E}}^{\sigma^2}_{t,\varepsilon})\,\mathrm{d}t \\ &- \rho_t(\sigma_t \partial_x(\bar{\mathcal{T}}_\varepsilon \nu_t) - \partial_x \sigma_t \bar{\mathcal{H}}^{\sigma}_{t,\varepsilon} + \bar{\mathcal{E}}^{\sigma}_{t,\varepsilon})\,\mathrm{d}W_t^0 \\ &+ \mathbb{E}\left[p_\varepsilon(x+\xi)\right]\mathrm{d}\int_0^t r(t-s)F_s\,\mathrm{d}s - p_\varepsilon\,\mathrm{d}F_t,\end{aligned} \tag{3.5.3}$$

where the error terms $\bar{\mathcal{E}}$ and $\bar{\mathcal{H}}$ are defined in [31, Lemma A.2]. For brevity, the dependence on x in these terms is omitted. Applying Itô's formula to $(\bar{\mathcal{T}}_\varepsilon \nu_t)^2$, we obtain the following stochastic differential equation:

$$\begin{aligned}\mathrm{d}(\bar{\mathcal{T}}_\varepsilon \nu_t)^2 = &- 2(\bar{\mathcal{T}}_\varepsilon \nu_t)(b_t \partial_x (\bar{\mathcal{T}}_\varepsilon \nu_t) - \partial_x b_t \bar{\mathcal{H}}^b_{t,\varepsilon} + \bar{\mathcal{E}}^b_{t,\varepsilon})\,\mathrm{d}t \\ &+ (\bar{\mathcal{T}}_\varepsilon \nu_t)\partial_x(\sigma_t^2 \partial_x(\bar{\mathcal{T}}_\varepsilon \nu_t) - \partial_x \sigma_t^2 \bar{\mathcal{H}}^{\sigma^2}_{t,\varepsilon} + \bar{\mathcal{E}}^{\sigma^2}_{t,\varepsilon})\,\mathrm{d}t \\ &- 2\rho_t(\bar{\mathcal{T}}_\varepsilon \nu_t)(\sigma_t \partial_x(\bar{\mathcal{T}}_\varepsilon \nu_t) - \partial_x \sigma_t \bar{\mathcal{H}}^{\sigma}_{t,\varepsilon} + \bar{\mathcal{E}}^{\sigma}_{t,\varepsilon})\,\mathrm{d}W_t^0 \\ &+ \rho_t^2(\sigma_t \partial_x(\bar{\mathcal{T}}_\varepsilon \nu_t) - \partial_x \sigma_t \bar{\mathcal{H}}^{\sigma}_{t,\varepsilon} + \bar{\mathcal{E}}^{\sigma}_{t,\varepsilon})^2\,\mathrm{d}t \\ &+ 2(\bar{\mathcal{T}}_\varepsilon \nu_t)\mathbb{E}\left[p_\varepsilon(x+\xi)\right]\mathrm{d}\int_0^t r(t-s)F_s\,\mathrm{d}s - 2(\bar{\mathcal{T}}_\varepsilon \nu_t)p_\varepsilon\,\mathrm{d}F_t.\end{aligned} \tag{3.5.4}$$

Our goal is now to establish an upper bound for each term in the above equation. As a consequence, we will obtain an estimate on the L^2-norm of $\bar{\mathcal{T}}_\varepsilon \nu_t$. As in [31, Proposition 3.5], we may integrate by parts to control the first term on the first line in (3.5.4). The tails of $\bar{\mathcal{T}}_\varepsilon \nu_t$ vanish at $\pm\infty$ by Assumption 3.2.2 (iii); therefore, we can conclude that

$$\begin{aligned}-\int_{\mathbb{R}}\int_0^t 2b_s(\bar{\mathcal{T}}_\varepsilon \nu_s)\partial_x(\bar{\mathcal{T}}_\varepsilon \nu_s)\,\mathrm{d}s\,\mathrm{d}x &= -\int_0^t\int_{\mathbb{R}} b_s \partial_x(\bar{\mathcal{T}}_\varepsilon \nu_s)^2\,\mathrm{d}s\,\mathrm{d}x \\ &= \int_0^t\int_{\mathbb{R}} \partial_x b_s(\bar{\mathcal{T}}_\varepsilon \nu_s)^2\,\mathrm{d}s\,\mathrm{d}x \le C_b\int_0^t\int_{\mathbb{R}} \left\|\bar{\mathcal{T}}_\varepsilon \nu_s\right\|_2^2\,\mathrm{d}s.\end{aligned} \tag{3.5.5}$$

Turning our attention to the first term in the last line of (3.5.4), by integrating over $\mathbb{R}$, we need to control a term of the form

$$\int_{\mathbb{R}}\int_0^t (\bar{\mathcal{T}}_\varepsilon \nu_s)(x)p_\varepsilon(x+z)\int_0^s r'(s-u)F_u\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}x \tag{3.5.6}$$

uniformly in $z \in [\gamma, R]$. As $\bar{\mathcal{T}}_\varepsilon \nu_s = (p_\varepsilon * \nu_s)$, we first observe that

$$\int_{\mathbb{R}} (\bar{\mathcal{T}}_\varepsilon \nu_s)(x)p_\varepsilon(x+z)\,\mathrm{d}x = (p_\varepsilon * (p_\varepsilon * \nu_s))(-z) = \bar{\mathcal{T}}_{2\varepsilon}\nu_s(-z), \tag{3.5.7}$$

where the last line follows from the associativity of the convolution operator and the fact that $p_\varepsilon * p_\varepsilon = p_{2\varepsilon}$. By the Fundamental Theorem of Calculus,

$$|\bar{\mathcal{T}}_{2\varepsilon}\nu_s(-z)| = \left| -\int_{-z}^{\varepsilon^{1/4}} \partial_x \bar{\mathcal{T}}_{2\varepsilon}\nu_s(x)\,\mathrm{d}x + \bar{\mathcal{T}}_{2\varepsilon}\nu_s(\varepsilon^{1/4}) \right| \leq C \left\|\partial_x \bar{\mathcal{T}}_{2\varepsilon}\nu_s\right\|_2 + |\bar{\mathcal{T}}_{2\varepsilon}\nu_s(\varepsilon^{1/4})|,$$

where the last line follows from an application of Hölder's inequality. The constant C holds uniformly in $\varepsilon < 1$ and $z \in [\gamma, R]$. By the definition of p_ε, $|\bar{\mathcal{T}}_{2\varepsilon}\nu_s(\varepsilon^{1/4})| \leq (4\pi\varepsilon)^{-1/2} e^{-\frac{1}{4\varepsilon^{1/2}}}$. Furthermore, as $\partial_x \bar{\mathcal{T}}_{2\varepsilon}\nu_s = p_\varepsilon * \partial_x \bar{\mathcal{T}}_\varepsilon \nu_s$, convolution with p_ε contracts the L^2-norm, [29, Proposition 6.1], and $\partial_x \bar{\mathcal{T}}_\varepsilon \nu_s$ is in $L^2(\mathbb{R})$, we have $\left\|\partial_x \bar{\mathcal{T}}_{2\varepsilon}\nu_s\right\|_2 \leq \left\|\partial_x \bar{\mathcal{T}}_\varepsilon \nu_s\right\|_2$. Therefore, we have shown that there is a constant C, which holds uniformly in $\varepsilon < 1$ and $z \in [\gamma, R]$, such that

$$|\bar{\mathcal{T}}_{2\varepsilon}\nu_s(-z)| \leq C\left(\left\|\partial_x \bar{\mathcal{T}}_\varepsilon \nu_s\right\|_2 + \varepsilon^{-\frac{1}{2}} e^{-\frac{1}{4\varepsilon^{1/2}}}\right). \tag{3.5.8}$$

Returning to (3.5.5), and employing the upper bounds in (3.5.7) and (3.5.8), we obtain

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \int_0^t \int_0^s (\bar{\mathcal{T}}_\varepsilon \nu_s)(x) p_\varepsilon(x+z) r'(s-u) F_u \,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}x \right| \\
&\qquad \leq C \int_0^t \left(\left\|\partial_x \bar{\mathcal{T}}_\varepsilon \nu_s\right\|_2 + \varepsilon^{-\frac{1}{2}} e^{-\frac{1}{4\varepsilon^{1/2}}}\right) F_s \left\|r'\right\|_1 \mathrm{d}s \\
&\qquad \leq \theta \int_0^t \left\|\partial_x \bar{\mathcal{T}}_\varepsilon \nu_s\right\|_2^2 \mathrm{d}s + C_\theta F_t^2 \left\|r'\right\|_1^2 \\
&\qquad\quad + \varepsilon^{-\frac{1}{2}} e^{-\frac{1}{4\varepsilon^{1/2}}} F_t \left\|r'\right\|_1,
\end{aligned} \tag{3.5.9}$$

where the last line follows from applying Young's inequality with free parameter θ. Therefore, returning to (3.5.4), integrating over x, employing the bounds (3.5.5) and (3.5.9), using the fact that $|\bar{\mathcal{H}}^g_{t,\varepsilon}| \lesssim \bar{\mathcal{T}}_{2\varepsilon}\nu_t$ by [29, Lemma 8.2], and employing Young's inequality with free parameter θ, we obtain

$$\begin{aligned}
\left\|\bar{\mathcal{T}}_\varepsilon \nu_t\right\|_2^2 \leq{}& \left\|\bar{\mathcal{T}}_\varepsilon \nu_0\right\|_2^2 + C_\theta \int_0^t \left\|\bar{\mathcal{T}}_\varepsilon \nu_s\right\|_2^2 \mathrm{d}s + C_\theta \int_0^t \left\|\bar{\mathcal{T}}_{2\varepsilon} \nu_s\right\|_2^2 \mathrm{d}s \\
&+ C_\theta \int_0^t \left(\left\|\bar{\mathcal{E}}^b_{s,\varepsilon}\right\|_2^2 + \left\|\bar{\mathcal{E}}^{\sigma^2}_{s,\varepsilon}\right\|_2^2 + \left\|\bar{\mathcal{E}}^\sigma_{s,\varepsilon}\right\|_2^2 \right) \mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{R}} (\rho_s^2 \sigma_s^2 + \theta \rho_s^2 \sigma_s^2 - \sigma_s^2 + \theta) |\partial_x (\bar{\mathcal{T}}_\varepsilon \nu_s)|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{R}} \rho_s \partial_x \sigma_s (\bar{\mathcal{T}}_\varepsilon \nu_s)^2 + 2\rho_s (\bar{\mathcal{T}}_\varepsilon \nu_s)(\partial_x \sigma_s \bar{\mathcal{H}}^\sigma_{s,\varepsilon} - \bar{\mathcal{E}}^\sigma_{s,\varepsilon})\,\mathrm{d}x\,\mathrm{d}W^0_s \\
&+ C_\theta F_t^2 \left\|r'\right\|_1^2 + \varepsilon^{-\frac{1}{2}} e^{-\frac{1}{4\varepsilon^{1/2}}} F_t \left\|r'\right\|_1.
\end{aligned} \tag{3.5.10}$$

We used the Stochastic Fubini Theorem ([29, Lemma 3.8]), as ν satisfies the same tail decay, to swap the order between the stochastic integral and the $\mathrm{d}x$-integral. Then

we applied integration by parts on the $\mathrm{d}x$-integral. Furthermore, as the last term in (3.5.4) is negative, it was simply discarded.

Now, as ρ is bounded away from 1 and σ is bounded away from 0, we may find a θ sufficiently small such that the third term in (3.5.10) is negative. Hence, we may discard it. Now, first taking the supremum over $r \leq t$ and then expectations, by [29, Lemma 8.5], the stochastic integral in (3.5.10) is bounded by

$$\begin{aligned}\mathbb{E}\sup_{r\leq t}\int_0^r\int_{\mathbb{R}}\rho_s\partial_x\sigma_s(\bar{\mathcal{T}}_\varepsilon\nu_s)^2+2\rho_s(\bar{\mathcal{T}}_\varepsilon\nu_s)(\partial_x\sigma_s\bar{\mathcal{H}}^\sigma_{s,\varepsilon}-\bar{\mathcal{E}}^\sigma_{s,\varepsilon})\,\mathrm{d}x\,\mathrm{d}W^0_s\\ \leq\frac{1}{2}\mathbb{E}\sup_{r\leq t}\left\|\bar{\mathcal{T}}_\varepsilon\nu_r\right\|_2^2+C\mathbb{E}\int_0^t\left\|\bar{\mathcal{T}}_\varepsilon\nu_s\right\|_2^2+\left\|\bar{\mathcal{T}}_{2\varepsilon}\nu_s\right\|_2^2+\left\|\bar{\mathcal{E}}^\sigma_{s,\varepsilon}\right\|_2^2\mathrm{d}s.\end{aligned}$$

Therefore, taking the supremum over $r \leq t$ and then expectations, (3.5.10) becomes

$$\begin{aligned}\mathbb{E}\sup_{r\leq t}\left\|\bar{\mathcal{T}}_\varepsilon\nu_r\right\|_2^2\leq&C_0\left\|\bar{\mathcal{T}}_\varepsilon\nu_0\right\|_2^2+tC_0\mathbb{E}\sup_{r\leq t}\left\|\bar{\mathcal{T}}_\varepsilon\nu_r\right\|_2^2+tC_0\mathbb{E}\sup_{r\leq t}\left\|\bar{\mathcal{T}}_{2\varepsilon}\nu_r\right\|_2^2\\&+C_0\mathbb{E}\int_0^t\left\|\bar{\mathcal{E}}^b_{s,\varepsilon}\right\|_2^2+\left\|\bar{\mathcal{E}}^{\sigma^2}_{s,\varepsilon}\right\|_2^2+\left\|\bar{\mathcal{E}}^\sigma_{s,\varepsilon}\right\|_2^2\mathrm{d}s\\&+C_0\mathbb{E}[F_t^2]+C_0\varepsilon^{-\frac{1}{2}}e^{-\frac{1}{4\varepsilon^{1/2}}}\mathbb{E}[F_t].\end{aligned}\tag{3.5.11}$$

Taking the $\liminf$ as $\varepsilon\to 0$ in (3.5.11) yields

$$\begin{aligned}f(t)\coloneqq\liminf_{\varepsilon\to0}\mathbb{E}\sup_{r\leq t}\left\|\bar{\mathcal{T}}_\varepsilon\nu_r\right\|_2^2&\leq C_0\left\|V_0\right\|_2^2+2tC_0\liminf_{\varepsilon\to0}\mathbb{E}\sup_{r\leq t}\left\|\bar{\mathcal{T}}_\varepsilon\nu_r\right\|_2^2+C_0\mathbb{E}[F_t^2]\\&=C_0\left(\left\|V_0\right\|_2^2+\mathbb{E}[F_t^2]\right)+2tC_0f(t),\end{aligned}\tag{3.5.12}$$

where the inequality follows from applying [31, Lemma A.2] and [29, Proposition 6.1], as $V_0 \in L^2(\mathbb{R})$ by Assumption 3.2.1 (iv). Therefore, for $t \leq T_0 \coloneqq 1/4C_0$, $\liminf_{\varepsilon\to0}\mathbb{E}\sup_{r\leq T_0}\left\|\bar{\mathcal{T}}_\varepsilon\nu_r\right\|_2^2$ is finite. By iteratively repeating the argument for $t \in [T_k, T_{k+1} \wedge T]$, where $T_k = k/4C_0$ and $k = 1, \ldots, \lceil 4C_0T\rceil$ (as shown in the proof of [29, Proposition 7.1]), we conclude that $\liminf_{\varepsilon\to0}\mathbb{E}\sup_{r\leq T}\left\|\bar{\mathcal{T}}_\varepsilon\nu_r\right\|_2^2<\infty$. As $\bar{\mathcal{T}}_{\tilde{\varepsilon}}\nu_t \geq \bar{\mathcal{T}}_\varepsilon\nu_t$ for any $\tilde{\varepsilon} < \varepsilon$, we may conclude that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\sup_{\varepsilon>0}\left\|\bar{\mathcal{T}}_\varepsilon\nu_t\right\|_2^2\right]<\infty.$$

Lastly, we note that by the definition of $\mathcal{T}_\varepsilon\nu_t$ and $\bar{\mathcal{T}}_\varepsilon\nu_t$, we have $0 \leq \mathcal{T}_\varepsilon\nu_t \leq \bar{\mathcal{T}}_\varepsilon\nu_t$. This completes the proof. □

Hence, we may conclude that any process that solves the SPDE (3.2.3) and satisfies Assumption 3.2.2 has a density.

Corollary 3.5.2 (L^2-regularity)**.** *Let (ν, F, W^0) be any process that solves the SPDE (3.2.3) and satisfies Assumption 3.2.2. Then, with probability 1, for every $t \in [0,T]$, there exists $V_t \in L^2(\mathbb{R})$ such that V_t is supported on $(-\infty, 0]$ and is a density of ν_t. Furthermore, $\sup_{t\in[0,T]} \|V_t\|_2 < \infty$ with probability 1.*

Proof. The result follows directly from the last line of Proposition 3.5.1 and [29, Proposition 6.2]. □

Before proceeding, we introduce the following notation to simplify the computations to follow.

Remark 3 (Notation). We will use the notation $o_{\text{sq}}(1)$ to denote any family of $L^2(-\infty, 0)$-valued processes $\{(f_{t,\varepsilon})_{t\in[0,T]}\}_{\varepsilon>0}$ satisfying

$$\mathbb{E}\int_0^T \|f_{t,\varepsilon}\|_{L^2(-\infty,0)}\,\mathrm{d}t \to 0 \qquad \text{as} \qquad \varepsilon \to 0.$$

The following lemma is crucial for handling the boundary effects in the smoothed estimate of $\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_T$. It ensures that these terms vanish as $\varepsilon \to 0$, which is essential to deduce uniqueness. The result employs the behaviour of the solutions' mass near the boundary, Assumption 3.2.2 (iii).

Lemma 3.5.3 (Boundary estimate)**.** *Let μ be any measure satisfying Assumption 3.2.2 and let $g_t(y)$ be a (stochastic) function with $|g_t(y)| \lesssim 1 + |y| + M_t + F_t^\mu$, where $M_t = \langle \mu_t, \psi\rangle$ for some $\psi \in \mathcal{C}^2(\mathbb{R})$ with $\|\partial_x\psi\|_{\mathcal{C}^1} < +\infty$. Then*

$$\mathbb{E}\int_0^T \int_{-\infty}^0 |\langle \mu_t, g_t(\cdot)p_\varepsilon(x+\cdot)\rangle|^2\,\mathrm{d}x\,\mathrm{d}t \to 0 \qquad \textit{as} \qquad \varepsilon \to 0.$$

Proof. By Assumption 3.2.2 (ii) and employing Grönwall's inequality, there is a constant $C > 0$ such that $F_T^\mu < M$. Therefore, $|g_t(y)| \lesssim 1 + |y| + M_t$. Now, as μ satisfies the boundary regularity of [31, Assumption 2.3], the proof follows directly from [31, Lemma 5.5]. □

3.5.2 The Smoothed H^{-1} Estimate

Equation (3.2.2) shows that the firing function may be reformulated exclusively in terms of the measure flow. As a result, we may compare two firing functions and rewrite it as a function of the distance between the corresponding measure flows using a suitable metric. We observe by Assumption 3.2.2 (ii) and employing Jensen's inequality,

$$\left|F_t - \tilde{F}_t\right|^2 \lesssim |\Delta_t(-\infty, 0)|^2 + t\int_0^t \left|F_s - \tilde{F}_s\right|^2 \mathrm{d}s$$

for any $t > 0$. Consequently, a straightforward application of Grönwall's inequality gives us

$$\left|F_t - \tilde{F}_t\right|^2 \lesssim |\Delta_t(-\infty, 0)|^2 + t \int_0^t |\Delta_s(-\infty, 0)|^2 \, e^{t-s} \, \mathrm{d}s. \tag{3.5.13}$$

Given that $|\Delta_t(-\infty, 0)| \leq d_1(\nu_t, \tilde{\nu}_t)$ by definition of the d_1-metric, (3.5.13) can be simplified to

$$\left|F_t - \tilde{F}_t\right|^2 \lesssim d_1(\nu_t, \tilde{\nu}_t)^2 + \int_0^t d_1(\nu_s, \tilde{\nu}_s)^2 \, \mathrm{d}s. \tag{3.5.14}$$

Finally, integrating (3.5.14) over the interval $[0, T]$, we obtain

$$\int_0^T |F_s - \tilde{F}_s|^2 \, \mathrm{d}s \leq C_T \int_0^T d_1(\nu_s, \tilde{\nu}_s)^2 \, \mathrm{d}s. \tag{3.5.15}$$

This result effectively bounds the difference between firing functions in terms of the d_1-metric distance between their corresponding measure flows and is employed in the proof of the following proposition.

Proposition 3.5.4 (Smoothed H^{-1} estimate)**.** *As $\varepsilon \downarrow 0$, we have*

$$\begin{aligned}\mathbb{E}\left\|\partial_x^{-1}\mathcal{T}_\varepsilon \Delta_{t\wedge t_n}\right\|_2^2 + c_0 \mathbb{E}\int_0^{t\wedge t_n} \left\|\mathcal{T}_\varepsilon \Delta_s\right\|_2^2 \, \mathrm{d}s \leq & c_n \mathbb{E}\int_0^{t\wedge t_n} d_0(\nu_s, \tilde{\nu}_s) \left\|\partial_x^{-1}\mathcal{T}_\varepsilon \Delta_s\right\|_2 \mathrm{d}s \\ & + c_n \mathbb{E}\int_0^{t\wedge t_n} d_1(\nu_s, \tilde{\nu}_s)^2 \, \mathrm{d}s + o(1)\end{aligned} \tag{3.5.16}$$

for a fixed $c_0 > 0$ and with c_n depending only on n, and $(t_n)_n$ is a sequence of stopping times such that $t_n \uparrow T$ as $n \to \infty$.

Proof. As for any (fixed) $x < 0$, $y \mapsto G_\varepsilon(x, y) \in \mathscr{S}$ with $G_\varepsilon(x, 0) = 0$, then by (3.2.3),

$$\begin{aligned}\mathrm{d}\langle \nu_t, G_\varepsilon(x, \cdot)\rangle = & \langle \nu_t, b_t \partial_y G_\varepsilon(x, \cdot)\rangle \, \mathrm{d}t + \frac{1}{2}\langle \nu_t, \sigma_t^2 \partial_{yy} G_\varepsilon(x, \cdot)\rangle \, \mathrm{d}t \\ & + \rho_t \langle \nu_t, \sigma_t \partial_y G_\varepsilon(x, \cdot)\rangle \, \mathrm{d}W_t^0 + \mathbb{E}\left[G_\varepsilon(x, -\xi)\right] \mathrm{d}\int_0^t r(t-s) F_s \, \mathrm{d}s.\end{aligned} \tag{3.5.17}$$

A straightforward computation shows that

$$\partial_y G_\varepsilon(x, y) = -\partial_x G_\varepsilon(x, y) - 2\partial_x p_\varepsilon(x + y) \qquad \text{and} \qquad \partial_{yy} G_\varepsilon(x, y) = \partial_{xx} G_\varepsilon(x, y).$$

Therefore, (3.5.17) becomes

$$\begin{aligned}\mathrm{d}\mathcal{T}_\varepsilon \nu_t = & -\partial_x \langle \nu_t, b_t G_\varepsilon(x, \cdot)\rangle \, \mathrm{d}t + \frac{1}{2}\partial_{xx}\langle \nu_t, \sigma_t^2 G_\varepsilon(x, \cdot)\rangle \, \mathrm{d}t \\ & - \rho_t \partial_x \langle \nu_t, \sigma_t G_\varepsilon(x, \cdot)\rangle \, \mathrm{d}W_t^0 + \mathbb{E}\left[G_\varepsilon(x, -\xi)\right] \mathrm{d}\int_0^t r(t-s) F_s \, \mathrm{d}s \\ & - 2\partial_x \langle \nu_t, b_t p_\varepsilon(x + \cdot)\rangle \, \mathrm{d}t - 2\rho_t \partial_x \langle \nu_t, \sigma_t p_\varepsilon(x + \cdot)\rangle \, \mathrm{d}W_t^0.\end{aligned} \tag{3.5.18}$$

Now, integrating over the domain $(-\infty, x)$ to introduce the antiderivative and invoking [29, Lemma 8.3] to swap the order of the space and time integrals, (3.5.18) becomes

$$\begin{aligned}\mathrm{d}\partial_x^{-1}\mathcal{T}_\varepsilon\nu_t =& -\langle\nu_t, b_t G_\varepsilon(x,\cdot)\rangle\,\mathrm{d}t + \frac{1}{2}\partial_x\langle\nu_t, \sigma_t^2 G_\varepsilon(x,\cdot)\rangle\,\mathrm{d}t \\ &- \rho_t\langle\nu_t, \sigma_t G_\varepsilon(x,\cdot)\rangle\,\mathrm{d}W_t^0 + \mathbb{E}\left[\partial_x^{-1}G_\varepsilon(x,-\xi)\right]\mathrm{d}\int_0^t r(t-s)F_s\,\mathrm{d}s \\ &- 2\langle\nu_t, b_t p_\varepsilon(x+\cdot)\rangle\,\mathrm{d}t - 2\rho_t\langle\nu_t, \sigma_t p_\varepsilon(x+\cdot)\rangle\,\mathrm{d}W_t^0.\end{aligned} \tag{3.5.19}$$

Although the model in this chapter is more general than that proposed in [31], [31, Lemma A.1] is still valid in our setting, as the proof of [31, Lemma A.1] only requires that ν satisfies the spatial regularity and tail decay of [31, Assumption 2.3]. Therefore, by employing [31, Lemma A.1], we may remove the diffusion coefficients out of the integrals against ν and obtain

$$\begin{aligned}\mathrm{d}\partial_x^{-1}\mathcal{T}_\varepsilon\nu_t =& -b_t\mathcal{T}_\varepsilon\nu_t\,\mathrm{d}t + \frac{1}{2}\partial_x\left(\sigma_t^2\mathcal{T}_\varepsilon\nu_t + \mathcal{E}_{t,\varepsilon}^{\sigma^2}\right)\mathrm{d}t - \rho_t\sigma_t\mathcal{T}_\varepsilon\nu_t\,\mathrm{d}W_t^0 \\ &+ \mathbb{E}\left[\partial_x^{-1}G_\varepsilon(x,-\xi)\right]\mathrm{d}\int_0^t r(t-s)F_s\,\mathrm{d}s + o_{\mathrm{sq}}(1)\,\mathrm{d}t + o_{\mathrm{sq}}(1)\,\mathrm{d}W_t^0,\end{aligned} \tag{3.5.20}$$

where we employed the $o_{\mathrm{sq}}(1)$ notation defined in Remark 3, and $\mathcal{E}_{t,\varepsilon}^g$ are the error terms from applying this transformation. Note that the $\mathcal{E}_{t,\varepsilon}^b$ and $\mathcal{E}_{t,\varepsilon}^\sigma$ terms are $o_{\mathrm{sq}}(1)$ and are captured in the last line.

Equation (3.5.20) is satisfied by any equation that satisfies Assumption 3.2.2. Writing $\Delta \coloneqq \nu - \tilde{\nu}$, where $\tilde{\nu}$ is any other solution to (3.2.3) satisfying Assumption 3.2.2, and defining $\tilde{\mathcal{E}}_{t,\varepsilon}^b$ analogously to $\mathcal{E}_{t,\varepsilon}^b$ but with $\tilde{\nu}$ instead, we obtain

$$\begin{aligned}\mathrm{d}\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t =& -(\tilde{b}_t\mathcal{T}_\varepsilon\Delta_t + \delta_t^b\mathcal{T}_\varepsilon\nu_t)\,\mathrm{d}t + \frac{1}{2}\partial_x(\sigma_t^2\mathcal{T}_\varepsilon\Delta_t + \mathcal{E}_{t,\varepsilon}^{\sigma^2} - \mathcal{E}_{t,\varepsilon}^{\tilde{\sigma}^2})\,\mathrm{d}t \\ &- \sigma_t(\tilde{\rho}_t\mathcal{T}_\varepsilon\Delta_t + \delta_t^\rho\mathcal{T}_\varepsilon\nu_t)\,\mathrm{d}W_t^0 + \mathbb{E}\left[\partial_x^{-1}G_\varepsilon(x,-\xi)\right]\mathrm{d}\int_0^t r(t-s)(F_s - \tilde{F}_s)\,\mathrm{d}s \\ &+ o_{\mathrm{sq}}(1)\,\mathrm{d}t + o_{\mathrm{sq}}(1)\,\mathrm{d}W_t^0,\end{aligned} \tag{3.5.21}$$

where $\tilde{g}_t = g(t,x,\tilde{\nu}_t,\tilde{f}_t)$, and we set $\delta_t^g \coloneqq g(t,x,\nu_t,\tilde{f}_t) - g(t,x,\tilde{\nu}_t,\tilde{f}_t)$. Applying Itô's

formula to $(\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t)^2$, we obtain the following stochastic differential equation:

$$\begin{aligned}
\mathrm{d}(\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t)^2 =& -2(\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t)(\tilde{b}_t\mathcal{T}_\varepsilon\Delta_t + \delta_t^b\mathcal{T}_\varepsilon\nu_t)\,\mathrm{d}t \\
&+ (\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t)\partial_x(\sigma_t^2\mathcal{T}_\varepsilon\Delta_t + \mathcal{E}_{t,\varepsilon}^{\sigma^2} - \mathcal{E}_{t,\varepsilon}^{\tilde{\sigma}^2})\,\mathrm{d}t \\
&+ 2(\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t)\mathbb{E}\left[\partial_x^{-1}G_\varepsilon(x,-\xi)\right]\mathrm{d}\int_0^t r(t-s)(F_s-\tilde{F}_s)\,\mathrm{d}s \\
&+ \sigma_t^2(\tilde{\rho}_t\mathcal{T}_\varepsilon\Delta_t + \delta_t^\rho\mathcal{T}_\varepsilon\nu_t)^2\,\mathrm{d}t + \sigma_t(\tilde{\rho}_t\mathcal{T}_\varepsilon\Delta_t + \delta_t^\rho\mathcal{T}_\varepsilon\nu_t)\cdot o_{\mathrm{sq}}(1)\,\mathrm{d}t \\
&+ (\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t)\cdot o_{\mathrm{sq}}(1)\,\mathrm{d}t + o_{\mathrm{sq}}(1)^2\,\mathrm{d}t \\
&- 2\sigma_t(\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t)(\tilde{\rho}_t\mathcal{T}_\varepsilon\Delta_t + \delta_t^\rho\mathcal{T}_\varepsilon\nu_t) + (\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t)\cdot o_{\mathrm{sq}}(1)\,\mathrm{d}W_t^0.
\end{aligned} \tag{3.5.22}$$

The goal is now to integrate over $\mathbb{R}_-$ and, hence, by gaining control of all the terms on the right-hand side of (3.5.22), we will have control over the L^2-norm of $\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t$. For simplicity, we introduce the notation

$$\begin{aligned}
|M|_{t,*} &:= \sup_{s\le t}\{1+|M_s|+|\tilde{M}_s|\}, \quad |F|_{t,*} := \sup_{s\le t}\{1+|F_s|+|\tilde{F}_s|\}, \quad \text{and} \\
\|\mathcal{T}\nu\|_{t,*} &:= \sup_{s\le t}\sup_{\varepsilon>0}\|\mathcal{T}_\varepsilon\nu_s\|_2\,.
\end{aligned}$$

We proceed in 6 steps.

Step 1. Note that $2(\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t)(\mathcal{T}_\varepsilon\Delta_t) = \partial_x(\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t)^2$ and, by the linear growth condition on b, $\tilde{b}_t(0) \le C_b(1+\|\mathfrak{K}'\|_1)(|M|_{t,*}+|F|_{t,*})$. Therefore, integrating by parts and applying Young's inequality with parameter θ to the first term in (3.5.22), we obtain

$$\begin{aligned}
-2\int_{-\infty}^0(\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t)\tilde{b}_t\mathcal{T}_\varepsilon\Delta_t\,\mathrm{d}x =& -\tilde{b}_t(0)\int_{-\infty}^0(\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t)\mathcal{T}_\varepsilon\Delta_t\,\mathrm{d}x \\
&+\int_{-\infty}^0(\partial_x\tilde{b}_t)(\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t)^2\,\mathrm{d}x \\
\le& C_b\left\|\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t\right\|_2^2 + C_\theta C_b(|M|_{t,*}^2+|F|_{t,*}^2)\left\|\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t\right\|_2^2 \\
&+\theta\left\|\mathcal{T}_\varepsilon\Delta_t\right\|_2^2.
\end{aligned}$$

For the second integral, we note that $|\delta_t^b| \le C_b(d_0(\nu_t,\tilde{\nu}_t)+|\mathfrak{f}_t-\tilde{\mathfrak{f}}_t|)$. By applying the Cauchy-Schwarz inequality to the d_0 term and Young's inequality with parameter $1/2$ to the other term, we obtain

$$\begin{aligned}
\left|\int_{-\infty}^0\delta_t^b(\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t)\mathcal{T}_\varepsilon\nu_t\,\mathrm{d}x\right| \lesssim& \|\mathcal{T}\nu\|_{t,*}\,d_0(\nu_t,\tilde{\nu}_t)\left\|\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t\right\|_2 + \left\|\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t\right\|_2^2 \\
&+|\mathfrak{f}_t-\tilde{\mathfrak{f}}_t|^2\left\|\mathcal{T}\nu\right\|_{t,*}^2.
\end{aligned}$$

We note that

$$\begin{aligned}
\int_0^t\|\mathcal{T}\nu\|_{s,*}^2|\mathfrak{f}_s-\tilde{\mathfrak{f}}_s|^2\,\mathrm{d}s &\le \|\mathcal{T}\nu\|_{t,*}^2\int_0^t\left|\int_0^s\mathfrak{K}'(s-u)(F_u-\tilde{F}_u)\,\mathrm{d}u\right|^2\mathrm{d}s \\
&\le \|\mathcal{T}\nu\|_{t,*}^2\|\mathfrak{K}'\|_1^2\int_0^t|F_s-\tilde{F}_s|^2\,\mathrm{d}s.
\end{aligned}$$

Step 2. We now turn our attention to the second line in (3.5.22). As $G_\varepsilon(x,0) = G_\varepsilon(0,y) = 0$ for all x and y, $\mathcal{E}^g_{t,\varepsilon}(0) = \tilde{\mathcal{E}}^g_{t,\varepsilon}(0) = \mathcal{T}_\varepsilon\Delta_t(0) = 0$. Therefore, by first integrating by parts and then applying Young's inequality with parameter θ,

$$\begin{aligned}\int_{-\infty}^0 (\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_t)\partial_x(\sigma_t^2\mathcal{T}_\varepsilon\Delta_t + \mathcal{E}^{\sigma^2}_{t,\varepsilon} - \mathcal{E}^{\tilde{\sigma}^2}_{t,\varepsilon})\,\mathrm{d}x &= -\int_{-\infty}^0 \sigma_t^2(\mathcal{T}_\varepsilon\Delta_t)^2 + \mathcal{T}_\varepsilon\Delta_t(\mathcal{E}^{\sigma^2}_{t,\varepsilon} - \mathcal{E}^{\tilde{\sigma}^2}_{t,\varepsilon})\,\mathrm{d}x \\ &\leq -\left\|\sigma_t\mathcal{T}_\varepsilon\Delta_t\right\|_2^2 + \theta\left\|\mathcal{T}_\varepsilon\Delta_t\right\|_2^2 \\ &\quad + C_\theta\left\|\mathcal{E}^{\sigma^2}_{t,\varepsilon}\right\|_2^2 + C_\theta\left\|\mathcal{E}^{\tilde{\sigma}^2}_{t,\varepsilon}\right\|_2^2.\end{aligned}$$

Step 3. To gain control over the third term in (3.5.22), we first seek to obtain control on the $L^1(\mathbb{R}_-)$-norm of $\partial_x^{-1}G_\varepsilon(x,-z)$ uniformly in $\varepsilon < 1$ and $z \in [\gamma, R]$. First, we observe that for any $z > 0$,

$$\int_{-\infty}^0 \partial_x^{-1}G_\varepsilon(x,-z)\,\mathrm{d}x = \int_{-\infty}^0\int_{-\infty}^x G_\varepsilon(y,-z)\,\mathrm{d}y\,\mathrm{d}x \leq \int_{-\infty}^0\int_{-\infty}^x p_\varepsilon(y+z)\,\mathrm{d}y\,\mathrm{d}x,$$

where the inequality follows from the definition of G_ε. The inner integral above represents the probability that a normal random variable with mean $-z$ and variance ε is smaller than x. By employing the well-known result $\mathbb{P}[\mathcal{N}(0,1) > x] \leq \frac{1}{2}e^{-\frac{x^2}{2}}$ for any $x > 0$, we know that if $x < -R$, then $\mathbb{P}[\mathcal{N}(-z,\varepsilon) < x] \leq 2^{-1}e^{-\frac{(x+z)^2}{2\varepsilon}}$ for any $z \in [\gamma, R]$. Therefore,

$$\int_{-\infty}^0 \partial_x^{-1}G_\varepsilon(x,-z)\,\mathrm{d}x \leq R + \int_{-\infty}^{-R} 2^{-1}e^{-\frac{(x+z)^2}{2\varepsilon}}\,\mathrm{d}x < R + \pi$$

for any $\varepsilon < 1$ and $z \in [\gamma, R]$. Hence, $\left\|\mathbb{E}\left[\partial_x^{-1}G_\varepsilon(x,-\xi)\right]\right\|_{L^1(\mathbb{R}_-)} < R + \pi$. Now, as $\mathbb{E}\left[\partial_x^{-1}G_\varepsilon(x,-\xi)\right]$ is non-negative and bounded by 1, by first applying the Cauchy-Schwarz inequality to the space integral and then Young's inequality with parameter $1/2$ to the $\mathrm{d}s$-integral, we have

$$\begin{aligned}2\left|\int_0^t\int_{-\infty}^0 (\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_s)\mathbb{E}\left[\partial_x^{-1}G_\varepsilon(x,-\xi)\right]\int_0^s r'(s-u)(F_u - \tilde{F}_u)\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}s\right| &\\ \lesssim \int_0^t \left\|\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_s\right\|_2 \left\|\mathbb{E}\left[\partial_x^{-1}G_\varepsilon(x,-\xi)\right]\right\|_{L^1(\mathbb{R}_-)} &\\ \times\left|\int_0^s r'(s-u)(F_u - \tilde{F}_u)\,\mathrm{d}u\right|\mathrm{d}s &\\ \lesssim \int_0^t \left\|\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_s\right\|_2^2\,\mathrm{d}s + \int_0^t\left|\int_0^s r'(s-u)(F_u - \tilde{F}_u)\,\mathrm{d}u\right|^2\mathrm{d}s &\\ \lesssim \int_0^t \left\|\partial_x^{-1}\mathcal{T}_\varepsilon\Delta_s\right\|_2^2\,\mathrm{d}s + \|r'\|_1^2\int_0^t\left|F_s - \tilde{F}_s\right|^2\mathrm{d}s.&\end{aligned}$$

Step 4. To gain control over the fourth term in (3.5.22), we observe that by expanding the brackets and applying Young's inequality with parameter θ to control the cross terms, we have

$$\begin{aligned}\int_{-\infty}^{0} \sigma_t^2(\tilde{\rho}_t \mathcal{T}_\varepsilon \Delta_t + \delta_t^\rho \mathcal{T}_\varepsilon \nu_t)^2 \,\mathrm{d}x \leq & C_\theta \left\|\mathcal{T}\nu\right\|_{t,*}^2 (d_1(\nu_t, \tilde{\nu}_t)^2 + |\mathfrak{f}_t - \tilde{\mathfrak{f}}_t|^2) \\ & + \left\|\sigma_t \tilde{\rho}_t \mathcal{T}_\varepsilon \Delta_t\right\|_2^2 + \theta \left\|\mathcal{T}_\varepsilon \Delta_t\right\|_2^2, \\ \int_{-\infty}^{0} |\sigma_t(\tilde{\rho}_t \mathcal{T}_\varepsilon \Delta_t + \delta_t^\rho \mathcal{T}_\varepsilon \nu_t) \cdot o_{\mathrm{sq}}(1)| \,\mathrm{d}x \leq & C_\theta \left\|\mathcal{T}\nu\right\|_{t,*}^2 (d_1(\nu_t, \tilde{\nu}_t)^2 + |\mathfrak{f}_t - \tilde{\mathfrak{f}}_t|^2) \\ & + \theta \left\|\mathcal{T}_\varepsilon \Delta_t\right\|_2^2 + \left\|o_{\mathrm{sq}}(1)\right\|_2^2.\end{aligned}$$

We used the fact that $|\delta_t^\rho|^2 \lesssim d_1(\nu_t, \tilde{\nu}_t)^2 + |\mathfrak{f}_t - \tilde{\mathfrak{f}}_t|^2$ in the above.

Step 5. As the stochastic integral in (3.5.22) is a true martingale, it vanishes when taking expectations. Therefore, by first taking expectations, then integrating over $x < 0$, and then applying Fubini's Theorem, it follows from Steps 1-4 and Young's inequality with parameter θ that

$$\begin{aligned}\mathbb{E}\left\|\partial_x^{-1}\mathcal{T}_\varepsilon \Delta_t\right\|_2^2 \leq & C_\theta \mathbb{E}\int_0^t (|M|_{s,*}^2 + |F|_{s,*}^2)\left\|\partial_x^{-1}\mathcal{T}_\varepsilon \Delta_s\right\|_2^2 \mathrm{d}s \\ & + C_\theta \mathbb{E}\int_0^t \left\|\mathcal{T}\nu\right\|_{s,*}^2 |\mathfrak{f}_s - \tilde{\mathfrak{f}}_s|^2 \,\mathrm{d}s + C\left\|r'\right\|_1^2 \mathbb{E}\int_0^t \left|F_s - \tilde{F}_s\right|^2 \mathrm{d}s \\ & + C_b \mathbb{E}\int_0^t \left\|\mathcal{T}\nu\right\|_{s,*} d_0(\nu_s, \tilde{\nu}_s)\left\|\partial_x^{-1}\mathcal{T}_\varepsilon \Delta_s\right\|_2 \mathrm{d}s \\ & + C_\theta \mathbb{E}\int_0^t \left\|\mathcal{T}\nu\right\|_{s,*}^2 d_1(\nu_s, \tilde{\nu}_s)^2 \,\mathrm{d}s \\ & + \mathbb{E}\int_0^t \int_{-\infty}^0 \{\sigma_s^2 \tilde{\rho}_s^2 - \sigma_s^2 + \theta\}|\mathcal{T}_\varepsilon \Delta_s|^2 \,\mathrm{d}x \,\mathrm{d}s + o(1)\end{aligned} \tag{3.5.23}$$

as $\varepsilon \downarrow 0$ for a constant C_θ which depends only on the parameter θ. We also remark that, as the stochastic integral in (3.5.22) is a true martingale, (3.5.23) holds for any bounded stopping time by Steps 1-4 and the Optional Stopping Theorem.

Step 6. As ρ is bounded away from 1 and σ is bounded away from 0, we may take the free parameter θ to be small enough such that for every s and x,

$$\sigma(s,x)^2 \rho(s, \tilde{\nu}, \tilde{\mathfrak{f}}_s)^2 - \sigma(s,x)^2 + \theta \leq -c_0$$

for a fixed constant $c_0 > 0$. Now, we consider the stopping times

$$t_n \coloneqq \inf\{t > 0 \,:\, |M|_{t,*}^2 > n,\, |F|_{t,*}^2 > n, \text{ or } \left\|\mathcal{T}\nu\right\|_{t,*}^2 > n\} \wedge T$$

for $n \geq 1$. By Proposition 3.5.1, $t_n \uparrow T$ almost surely. We observe that at the time $t \wedge t_n$,

$$C_\theta \mathbb{E} \int_0^{t\wedge t_n} \|\mathcal{T}\nu\|_{s,*}^2 \, |\mathfrak{f}_s - \tilde{\mathfrak{f}}_s|^2 \, \mathrm{d}s \leq C_\theta n \mathbb{E} \int_0^{t\wedge t_n} \left| \int_0^s \mathfrak{K}'(s-u)(F_u - \tilde{F}_u) \, \mathrm{d}u \right|^2 \mathrm{d}s$$
$$\leq C_\theta n \, \|\mathfrak{K}'\|_1^2 \, \mathbb{E} \int_0^{t\wedge t_n} |F_s - \tilde{F}_s|^2 \, \mathrm{d}s,$$

where the last inequality follows from an application of Fubini's Theorem and Jensen's inequality. Hence, evaluating (3.5.23) at $t \wedge t_n$, we obtain

$$\begin{aligned} &\mathbb{E} \left\| \partial_x^{-1} \mathcal{T}_\varepsilon \Delta_{t\wedge t_n} \right\|_2^2 + c_0 \mathbb{E} \int_0^{t\wedge t_n} \|\mathcal{T}_\varepsilon \Delta_s\|_2^2 \, \mathrm{d}s \\ &\leq Cn\mathbb{E} \int_0^{t\wedge t_n} \left\| \partial_x^{-1} \mathcal{T}_\varepsilon \Delta_s \right\|_2^2 \mathrm{d}s + Cn\mathbb{E} \int_0^{t\wedge t_n} \left| F_s - \tilde{F}_s \right|^2 + d_1(\nu_s, \tilde{\nu}_s)^2 \, \mathrm{d}s \\ &\quad + Cn\mathbb{E} \int_0^{t\wedge t_n} d_0(\nu_s, \tilde{\nu}_s) \left\| \partial_x^{-1} \mathcal{T}_\varepsilon \Delta_s \right\|_2 \mathrm{d}s + o(1). \end{aligned} \tag{3.5.24}$$

Then (3.5.16) follows from employing (3.5.15) and a simple application of Grönwall's inequality, where $c_n := Cne^{CnT}$ (for a possibly larger C than that in (3.5.24)). □

3.5.3 Uniqueness of the SPDE – Proof of Theorem 3.2.4

At this point, we have all the ingredients to prove Theorem 3.2.4. We have already shown that every solution to the SPDE (3.2.3) will have a density in L^2. By a simple modification of [31, Lemma 5.2], as our process lives on $(-\infty, 0)$, we have a universal constant $c > 0$ such that for any $s \leq T$, $\delta \in (0,1)$, and $\lambda > 0$,

$$\begin{aligned} d_0(\nu_s, \tilde{\nu}_s) &\leq c\lambda(1+\delta^{-1}) \left\| \partial_x^{-1} \mathcal{T}_\varepsilon \Delta_s \right\|_2 + c\delta^{\frac{1}{2}} \left\| \frac{\mathrm{d}\Delta_s}{\mathrm{d}x} \right\|_2 + f_s(\lambda) + g_s(\varepsilon), \\ d_1(\nu_s, \tilde{\nu}_s) &\leq c\sqrt{\lambda + \delta^{-1}} \left\| \partial_x^{-1} \mathcal{T}_\varepsilon \Delta_s \right\|_2 + c\delta^{\frac{1}{2}} \left\| \frac{\mathrm{d}\Delta_s}{\mathrm{d}x} \right\|_2 + f_s(\lambda) + g_s(\varepsilon), \end{aligned}$$

where $(f_s(\lambda))_{s\leq T}$ is a process such that for all $a > 0$, $\mathbb{E}\int_0^T f_s(\lambda)^2 \, \mathrm{d}s \leq c_a e^{-a\lambda}$ for some $c_a > 0$, and $(g_s(\varepsilon))_{s\leq T}$ is a process such that $\mathbb{E}\int_0^T g_s(\varepsilon) \, \mathrm{d}s = o(1)$ as $\varepsilon \downarrow 0$. Substituting the bounds for $d_0(\nu_s, \tilde{\nu}_s)$ and $d_1(\nu_s, \tilde{\nu}_s)$ into (3.5.16) and then applying Young's inequality to all the terms except for the $\|\partial_x^{-1}\mathcal{T}_\varepsilon \Delta_s\|_2^2$ term, we obtain

$$\begin{aligned} \mathbb{E} \left\| \partial_x^{-1} \mathcal{T}_\varepsilon \Delta_{t\wedge t_n} \right\|_2^2 &\leq \delta c_n' \mathbb{E} \int_0^{t\wedge t_n} \left\| \frac{\mathrm{d}\Delta_s}{\mathrm{d}x} \right\|_2^2 \mathrm{d}s - c_0 \mathbb{E} \int_0^{t\wedge t_n} \|\mathcal{T}_\varepsilon \Delta_s\|_2^2 \, \mathrm{d}s \\ &\quad + c_n'(1 + \lambda + \lambda\delta) \int_0^t \mathbb{E} \left\| \partial_x^{-1} \mathcal{T}_\varepsilon \Delta_{s\wedge t_n} \right\|_2^2 \mathrm{d}s + c_{a,n}' e^{-a\lambda} + o(1) \end{aligned} \tag{3.5.25}$$

as $\varepsilon \downarrow 0$. The Monotone Convergence Theorem gives

$$\mathbb{E}\int_0^{t\wedge t_n} \|\mathcal{T}_\varepsilon \Delta_s\|_2^2 \,\mathrm{d}s \to \mathbb{E}\int_0^{t\wedge t_n} \left\|\frac{\mathrm{d}\Delta_s}{\mathrm{d}x}\right\|_2^2 \mathrm{d}s$$

as $\varepsilon \downarrow 0$ because $\|\mathcal{T}_\varepsilon \Delta_s\|_2 \nearrow \left\|\frac{\mathrm{d}\Delta_s}{\mathrm{d}x}\right\|_2$. Therefore, fixing $\delta = c_0/2c_n'$, we have that for all $\varepsilon > 0$ small enough, the difference of the first two terms will be negative. Thus, applying Grönwall's inequality to (3.5.25), we have

$$\mathbb{E}\left\|\partial_x^{-1}\mathcal{T}_\varepsilon \Delta_{t\wedge t_n}\right\|_2^2 \le c_{a,n}' e^{c_n'(1+\lambda+\lambda\delta)T} e^{-a\lambda} + o(1)$$

as $\varepsilon \downarrow 0$. Hence, it follows from Fatou's lemma and (3.5.1) that

$$\mathbb{E}\left\|\Delta_{t\wedge t_n}\right\|_{H^{-1}} \le c_{a,n,T}' e^{\lambda(1+\delta)T - a\lambda}.$$

Therefore, setting $a = 2(1+\delta^{-1})T$ and sending $\lambda \to \infty$, we obtain $\mathbb{E}\left\|\Delta_{t\wedge t_n}\right\|_{H^{-1}} = 0$ for all $t \in [0,T]$. Lastly, sending $n \uparrow \infty$, we have $t_n \uparrow \infty$, and consequently, we have $\nu_t = \tilde{\nu}_t$ for all $t \in [0,T]$. Finally, by (3.5.15), we must have $F = \tilde{F}$. This completes the proof of Theorem 3.2.4. □

3.5.4 Strong Solutions of the SPDE – Proof of Theorem 3.2.5

With pathwise uniqueness of the SPDE in hand, we can show that we also have uniqueness in law and therefore conclude that our solution to the SPDE (3.2.3) is, in fact, a strong solution. Suppose that (ν, F, W^0) and $(\tilde{\nu}, \tilde{F}, \tilde{W}^0)$ are any two solutions to the SPDE (3.2.3) satisfying Assumption 3.2.2 on possibly different probability spaces. As we have endowed $\mathbf{M}_{\le 1}$ with the topology of weak convergence, it is a Polish space ([59, Theorem 1.11]). Therefore, $D_{\mathbf{M}_{\le 1}}$ endowed with the Skorokhod topology is a Polish space ([21, Theorem 3.5.6]). Hence, by [44, Lemma A.1], we may find two maps

$$F^1, F^2 : \mathcal{C}_{\mathbb{R}} \times [0,1] \to D_{\mathbf{M}_{\le 1}} \times D_{\mathbb{R}}$$

such that given a $\Xi \sim \mathrm{Unif}[0,1]$ and an independent Brownian motion, W, we have $\mathrm{Law}((\nu, F, W^0)) = \mathrm{Law}((F^1(W,\Xi), W))$ and $\mathrm{Law}((\tilde{\nu}, \tilde{F}, \tilde{W}^0)) = \mathrm{Law}((F^2(W,\Xi), W))$.

As $(F^1(W,\Xi), W)$ and $(F^2(W,\Xi), W)$ are on the same probability space, by Theorem 3.2.4, they must be equal, and hence, we have uniqueness in law. Lastly, by employing [44, Lemma 3.2], [44, Lemma 3.14], and [44, Lemma A.2], we may fix a measurable map $Q : \mathcal{C}_{\mathbb{R}} \to D_{\mathbf{M}_{\le 1}} \times D_{\mathbb{R}}$ such that $\mathrm{Law}((\nu, F, W^0)) = \mathrm{Law}((Q(W), W))$ for any Brownian motion W. In particular, by Theorem 3.2.4, we must have $(\nu, F) = Q(W^0)$ almost surely.

Now, given any limit point (ν^*, F^*, W^0) of the particle system (3.2.1), let $\mathbb{P}^* := \text{Law}((\nu^*, F^*, W^0)) \in \mathcal{P}(D_{\mathscr{S}'} \times D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}})$. By Theorem 3.2.3 and Assumption 3.2.2 (iii), we have

(I) $\mathbb{E}\left[\sup_{s\le T} \nu_s^*(-\infty, -\lambda)\right] = o(\exp -a\lambda)$ as $\lambda \to \infty$ for any $a > 0$,

(II) $\nu_t^* \in \mathbf{M}_{\le 1}(\mathbb{R}_-)$ for all $t \in [0, T]$ a.s.

By (I) and (II), along with employing the Borel-Cantelli Lemma, we may conclude that there exists a set $A \subset D_{\mathscr{S}'}$ such that $A \times D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}$ has $\mathbb{P}^*$-measure 1 and ν^* is an element of $D_{\mathbf{M}_{\le 1}}$ on $A \times D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}$. We now observe that the map

$$\begin{aligned} \nu^* \mathbb{1}_{A\times D_{\mathbb{R}}\times \mathcal{C}_{\mathbb{R}}} &: D_{\mathscr{S}'} \times D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}} \to D_{\mathbf{M}_{\le 1}} \\ (\nu, F, \omega) &\mapsto \begin{cases} \nu \text{ if } \nu \in A \\ 0 \text{ otherwise} \end{cases} \end{aligned}$$

is measurable. Because $D_{\mathbf{M}_{\le 1}}$ is endowed with the Skorokhod topology, the Borel σ-algebra coincides with the Kolmogorov σ-algebra ([21, Proposition 3.7.1]). As $\mathbf{M}_{\le 1}$ is endowed with the topology of weak convergence, the finite intersections of the sets $U_{t,\phi,x,\delta} = \{\mu \in \mathbf{M}_{\le 1} : |\mu_t(\phi) - x| < \delta\}$ for $\phi \in \mathcal{C}_b$, $x \in \mathbb{R}$, $t \in [0, T]$, and $\delta > 0$ form a π-system which generates the σ-algebra on $D_{\mathbf{M}_{\le 1}}$. Therefore, for any $\phi \in \mathcal{C}_b$, we may find a sequence $\phi^n \in \mathscr{S}$ which converges pointwise to ϕ. Thus, when $x \ge \delta$,

$$\begin{aligned} \{\nu^* \mathbb{1}_{A\times D_{\mathbb{R}}\times \mathcal{C}_{\mathbb{R}}} \in U_{t,\phi,x,\delta}\} = (A \cap B) \times D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}} \qquad &\text{where} \\ B = \bigcup_{M\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{n>N} \{\nu \in D_{\mathscr{S}'} : |\nu_t(\phi^n) - x| < \delta - 1/M\} & \end{aligned}$$

and similarly when $x < \delta$. Sets of this form are measurable in $D_{\mathscr{S}'} \times D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}$.

Now, setting $(\hat{\nu}, \hat{F}, \hat{W}^0) = (\nu^*, F^*, W^0)\mathbb{1}_{A\times D_{\mathbb{R}}\times \mathcal{C}_{\mathbb{R}}}$, $(\hat{\nu}, \hat{F}, \hat{W}^0)$ is a random variable on $D_{\mathbf{M}_{\le 1}} \times D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}$. $(\hat{\nu}, \hat{F}, \hat{W}^0)$ satisfies the assumptions and solves the SPDE (3.2.3). Therefore, we have $(\hat{\nu}, \hat{F}, \hat{W}^0) = (Q(\hat{W}^0), \hat{W}^0)$ by the above. On $D_{\mathscr{S}'}$, the Borel σ-algebra coincides with the Kolmogorov σ-algebra ([47, Proposition 2.7 (iv)]); therefore, the identity map $\text{Id} : D_{\mathbf{M}_{\le 1}} \times D_{\mathbb{R}} \to D_{\mathscr{S}'} \times D_{\mathbb{R}}$ is measurable, and we have $\mathbb{P}^* = \text{Law}((Q(W^0), W^0))$. As the limit point was arbitrary and Q is fixed, we may conclude that the SPDE (3.2.3) has a unique law on $(D_{\mathscr{S}'}, M_1) \times (D_{\mathbb{R}}, M_1) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty)$ and that (ν^N, F^N, W^0) converges weakly to this law.

Lastly, as there is a set of full measure such that ν^* is an element of $D_{\mathbf{M}_{\le 1}}$, the proof of Theorem 3.2.4 applies, and we have $(\nu^*, F^*) = Q(W^0)$. This completes the proof. □

3.6 Technical Lemmas and Proofs

The objective of this section is to provide the proofs of the propositions and lemmas previously introduced and deferred. Firstly, we will show that if the initial law of our process is sub-Gaussian, then all the other processes of interest are also sub-Gaussian. To do this, it is more convenient to work with the continuous version of our process, denoted by Z^i. The idea here is that the Z^i process does not reset but lives on the whole space. This reformulation allows us to employ Grönwall-type arguments and, in turn, gives us information about X^i.

To be explicit, $Z_t^i \coloneqq X_t^i + \sum_{k\geq 1} \xi_k^i \mathbb{1}_{[0,t]}(\tau_k^i + \varsigma_k^i)$. It is clear that the K^{th} time X^i hits the boundary at 0 corresponds to the first time Z^i hits $\sum_{k=1}^{K-1} \xi_k^i$. Therefore, we observe

$$\sum_{k\geq 1} \xi_k^i \mathbb{1}_{[0,t]}(\tau_k^i + \varsigma_k^i) \leq \sum_{k\geq 1} \xi_k^i \mathbb{1}_{[0,t]}(\tau_k^i) \leq \sup_{s\leq t} Z_s^i. \tag{3.6.1}$$

Furthermore, if X^i has hit the boundary K times by time t, we have $J_t^i = K$. Therefore, as ξ_k^i is lower bounded by $\gamma > 0$, we have $K \leq \gamma^{-1} \sum_{k=1}^{K-1} \xi_k^i + 1$. Thus, by the above reasoning and (3.6.1), we deduce

$$J_t^i \leq \gamma^{-1} \sup_{s\leq t} Z_s^i + 1 \qquad \text{and} \qquad F_t^N \leq \frac{1}{N\gamma} \sum_{i=1}^{N} \sup_{s\leq t} Z_s^i + 1. \tag{3.6.2}$$

Using this observation, we may deduce by a Grönwall argument that $\sup_{s\leq T} |Z_s|^p$ is finite for any $p \geq 1$. This is the content of the following proposition.

3.6.1 Tail Estimates and Change of Measure

Proposition 3.6.1. *For any $p \geq 1$, there exists a constant $C > 0$ that depends on p, b, $\mathfrak{K}$, σ and T but is independent of N and i such that $\mathbb{E}\left[\sup_{s\leq T} |Z_s^i|^p\right] \leq C$.*

Proof. By (3.6.1), we have

- $\left|X_s^i\right| \leq 2 \sup_{u\leq s} \left|Z_u^i\right|$,
- $\frac{1}{N} \sum_{i=1}^{N} \left|X_s^i\right| \leq \frac{2}{N} \sum_{i=1}^{N} \sup_{u\leq s} \left|Z_u^i\right|$,
- $\left|\mathfrak{f}_s^N\right| \leq \frac{\|\mathfrak{K}'\|_1}{N\gamma} \sum_{i=1}^{N} \sup_{u\leq s} \left|Z_u^i\right| + \|\mathfrak{K}'\|_1$.

Setting $c_1 = \max\{\gamma^{-1}\|\mathfrak{K}'\|_1 + 2, \|\mathfrak{K}'\|_1 + 1, 2\}C_b$, by the linear growth condition on b,

$$\begin{aligned} |b(s,\, X_s^i, \nu_s^N, \mathfrak{f}_s^N)| &\le C_b\left(1 + \left|X_s^i\right| + \langle \nu_s^N, |\cdot|\rangle + \left|\mathfrak{f}_s^N\right|\right) \\ &\le c_1\left(1 + \sup_{u\le s}\left|Z_u^i\right| + \frac{1}{N}\sum_{j=1}^N \sup_{u\le s}\left|Z_u^j\right|\right). \end{aligned} \tag{3.6.3}$$

By taking norms in (3.2.1) and using the triangle inequality, we get for any $t \le T$,

$$\begin{aligned} |Z_t^i| &= \left|X_0^i + \int_0^t b(s,\, X_s^i, \nu_s^N, \mathfrak{f}_s^N)\,\mathrm{d}s + Y_t^i\right| \\ &\le \left|X_0^i\right| + c_1\int_0^t 1 + \sup_{u\le s}\left|Z_u^i\right| + \frac{1}{N}\sum_{j=1}^N \sup_{u\le s}\left|Z_u^j\right|\mathrm{d}s + |Y_t^i| \\ &\le |X_0^i| + tc_1 + c_1\int_0^t \sup_{u\le s}|Z_u^i| + \frac{1}{N}\sum_{j=1}^N \sup_{u\le s}|Z_u^j|\,\mathrm{d}s + \sup_{s\le t}\left|Y_s^i\right|, \end{aligned} \tag{3.6.4}$$

where $Y_t^i = \int_0^t \sigma(s, X_s^i)\mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}B_s^i$ with $\mathrm{d}B_s^i = \sqrt{1-\rho_s^2}\,\mathrm{d}W_s^i + \rho_s\,\mathrm{d}W_s^0$ and $\rho_s = \rho(s, \nu_s^N \mathfrak{f}_s^N)$. By taking the empirical mean above, we deduce

$$\frac{1}{N}\sum_{j=1}^N \sup_{s\le t}|Z_s^j| \le tc_1 + \frac{1}{N}\sum_{j=1}^N\left(|X_0^j| + \sup_{s\le t}|Y_s^j|\right) + 2c_1\int_0^t \frac{1}{N}\sum_{j=1}^N \sup_{u\le s}|Z_u^j|\,\mathrm{d}s.$$

Then, by Grönwall's integral lemma, we obtain

$$\frac{1}{N}\sum_{j=1}^N \sup_{s\le t}|Z_s^j| \le C\left[1 + \frac{1}{N}\sum_{j=1}^N\left(|X_0^j| + \sup_{s\le t}|Y_s^j|\right)\right] \quad \text{a.s.} \tag{3.6.5}$$

for some possibly larger constant $C > 0$ that depends on T, b, and $\mathfrak{K}$ only. Now, going back to (3.6.4) and substituting (3.6.5) into it, we obtain

$$\left|Z_t^i\right| \le |X_0^i| + \sup_{s\le t}\left|Y_s^i\right| + C\left[1 + \frac{1}{N}\sum_{j=1}^N\left(|X_0^j| + \sup_{s\le t}|Y_s^j|\right)\right] + c_1\int_0^t \sup_{u\le s}|Z_u^i|\,\mathrm{d}s \quad \text{a.s.}$$

for some possibly larger constant $C > 0$ that still depends only on T, b, and $\mathfrak{K}$. By using Grönwall's lemma again, there is a constant $C_{T,b,\mathfrak{K}} > 0$ such that for any $t \le T$,

$$\sup_{s\le t}|Z_s^i| \le C_{T,b,\mathfrak{K}}\left[1 + \left|X_0^i\right| + \sup_{s\le t}\left|Y_s^i\right| + \frac{1}{N}\sum_{j=1}^N\left(\left|X_0^j\right| + \sup_{s\le t}\left|Y_s^j\right|\right)\right] \quad \text{a.s.}$$

Jensen's inequality and the equation above gives, for any $p \ge 1$,

$$\sup_{s\le t}|Z_s^i|^p \le C_{T,b,\mathfrak{K},p}\left[1 + \left|X_0^i\right|^p + \sup_{s\le t}\left|Y_s^i\right|^p + \frac{1}{N}\sum_{j=1}^N\left(\left|X_0^j\right|^p + \sup_{s\le t}\left|Y_s^j\right|^p\right)\right] \quad \text{a.s.} \tag{3.6.6}$$

By Burkholder-Davis-Gundy, for any j

$$\mathbb{E}\left[\sup_{0\le s\le t}\left|\int_0^s \sigma(u,X_u^j)\mathbb{1}_{\{X_u^j<0\}}\,\mathrm{d}B_u^j\right|^p\right] \le \mathbb{E}\left[\left(\int_0^T \sigma^2(s,X_s^j)\,\mathrm{d}s\right)^{p/2}\right] \le C_{\sigma,p}T^{p/2}.$$

Hence,

$$\mathbb{E}\left[\sup_{0\le s\le t}\left|Z_s^i\right|^p\right] \le C_{T,b,\mathfrak{K},p}\left[1+\mathbb{E}\left|X_0^i\right|^p + 2C_{\sigma,p}T^{p/2} + \frac{1}{N}\sum_{j=1}^N \mathbb{E}\left|X_0^j\right|^p\right]. \tag{3.6.7}$$

As X_0 has finite p^{th}-moments for every $p>0$ by Assumption 3.2.1 (iii), the R.H.S. of (3.6.7) is bounded by a constant that depends on p, b, $\mathfrak{K}$, σ, and T but is independent of N. □

Next, we need two results regarding the exponential moments of the supremum of Brownian motion and sub-Gaussian random variables.

Proposition 3.6.2. *For any $p=1,2$, $t\in[0,T]$, and $\delta<T/2$, there exists a constant $C_{p,\delta,T}>0$ that depends on p, T, and δ such that*

$$\mathbb{E}\left[e^{\delta\sup_{s\le t}|B_s|^p}\right] \le C_{p,\delta,T}.$$

Proof. For any $x\in\mathbb{R}$, we have $|x|\le 1+|x|^2$. Therefore, it is sufficient to show the claim for $p=2$, as

$$\mathbb{E}\left[e^{\delta\sup_{s\le t}|B_s|}\right] \le e^{\delta}\mathbb{E}\left[e^{\delta\sup_{s\le t}|B_s|^2}\right].$$

A straightforward computation shows

$$\begin{aligned}\mathbb{E}\left[e^{\delta\sup_{s\le t}|B_s|^2}\right] &= \int_0^\infty \mathbb{P}\left[e^{\delta\sup_{s\le t}|B_s|^2}>x\right]\mathrm{d}x\\ &= 1+\int_1^\infty \mathbb{P}\left[\sup_{s\le t}|B_s| > \left(\frac{\log(x)}{\delta}\right)^{1/2}\right]\mathrm{d}x\end{aligned}$$

Now, as $\mathbb{P}[\sup_{s\le t}|B_s|>x]\le 2\mathbb{P}[\sup_{s\le t}B_s>x]$ and $\mathbb{P}[\sup_{s\le t}B_s>x]=2\mathbb{P}[B_t>x]$ for any $x>0$, we have

$$= 1+4\int_1^\infty \mathbb{P}\left[B_t > \left(\frac{\log(x)}{\delta}\right)^{1/2}\right]\mathrm{d}x$$

Employing the well-known result $\mathbb{P}[\mathcal{N}(0,1)>x]\le\frac{1}{2}e^{-\frac{x^2}{2}}$ for any $x>0$, we have

$$\begin{aligned}&\le 1+2\int_1^\infty e^{-\frac{\log(x)}{2t\delta}}\,\mathrm{d}x = 1+2\int_1^\infty x^{-\frac{1}{2t\delta}}\,\mathrm{d}x\\ &= 1+\frac{2}{(2t\delta)^{-1}-1} \le 1+\frac{2}{(2T\delta)^{-1}-1}.\end{aligned}$$

The last line follows from the fact that $2t\delta<2T\delta<1$. □

Proposition 3.6.3. *Fix any $p = 1, 2$ and $\delta < \gamma$, where γ is the constant in Assumption 3.2.1 (iv). Then, there exists a constant $C_{p,\delta,X_0} > 0$ that depends on p, δ, and X_0 only such that*

$$\mathbb{E}\left[e^{\delta|X_0|^p}\right] \leq C_{p,\delta,X_0}.$$

Proof. For any $x \in \mathbb{R}$, we have $|x| \leq 1 + |x|^2$. Therefore, it is sufficient to show the claim for $p = 2$. By Assumption 3.2.1 (iv), there exists a $\tilde{\lambda} > 0$ such that $\nu_0(-\infty, -\lambda) < e^{-\gamma\lambda^2}$ for all $\lambda > \tilde{\lambda}$. Therefore, for any $\delta < \gamma$, a straightforward computation shows

$$\begin{aligned}
\mathbb{E}\left[e^{\delta|X_0|^2}\right] &= \int_0^\infty \mathbb{P}\left[e^{\delta|X_0|^2} > x\right] \mathrm{d}x \\
&= 1 + \int_1^\infty \mathbb{P}\left[|X_0| > \left(\frac{\log(x)}{\delta}\right)^{1/2}\right] \mathrm{d}x \\
&\leq 1 + e^{\delta\tilde{\lambda}^2} + \int_{e^{\delta\tilde{\lambda}^2}}^\infty \mathbb{P}\left[|X_0| > \left(\frac{\log(x)}{\delta}\right)^{1/2}\right] \mathrm{d}x \\
&\leq 1 + e^{\delta\tilde{\lambda}^2} + \int_{e^{\delta\tilde{\lambda}^2}}^\infty e^{-\frac{\gamma\log(x)}{\delta}} \mathrm{d}x \\
&= 1 + e^{\delta\tilde{\lambda}^2} + \int_{e^{\delta\tilde{\lambda}^2}}^\infty x^{-\frac{\gamma}{\delta}} \mathrm{d}x \\
&= 1 + e^{\delta\tilde{\lambda}^2} + (\gamma/\delta - 1)^{-1} e^{-\tilde{\lambda}^2(\gamma-\delta)}.
\end{aligned}$$

The last line follows from the fact that $0 < \delta < \gamma$. Hence, we set C_{2,δ,X_0} to be the value in the ultimate line above. □

Our aim is to show that X_t^i, J_t^i, and F_t^N are all sub-Gaussian uniformly in N. To this end, it will be sufficient to show that the processes

$$\Lambda_t^i := \left|Z_t^i\right| + \frac{1}{N}\sum_{j=1}^N \left|Z_t^j\right|, \qquad \Gamma_t^i := \left|Z_t^i\right|^2 + \frac{1}{N}\sum_{j=1}^N \left|Z_t^j\right|^2, \tag{3.6.8}$$

$$\hat{\Lambda}_t^i := \sup_{0\leq s\leq t}\left|Z_s^i\right| + \frac{1}{N}\sum_{j=1}^N \sup_{0\leq s\leq t}\left|Z_s^j\right|, \quad \hat{\Gamma}_t^i := \sup_{0\leq s\leq t}\left|Z_s^i\right|^2 + \frac{1}{N}\sum_{j=1}^N \sup_{0\leq s\leq t}\left|Z_s^j\right|^2. \tag{3.6.9}$$

all have finite exponential moments. We introduce this notation here and consider these random variables, as they play a role in Girsanov-type arguments presented later. These show that we have a probability space where we can view our particle as being driven by only a Brownian motion. This notation is exactly that presented in [31, Section 6.1].

Proposition 3.6.4. *For any $\delta < \min\{\gamma/4, T/8\}$, where γ is the constant in Assumption 3.2.1 (iv), and any $i = 1, \dots, N$, there exists a constant $C > 0$ such that*

$$\mathbb{E}\left[e^{\delta \hat{\Gamma}_T^i}\right] \leq C.$$

The constant C depends on δ, T, $\mathfrak{K}$, the initial distribution X_0, and the bounds on b and σ, but is independent of i and N.

Proof. By Proposition 3.6.1, for any $i = 1, \dots, N$ fixed, we have by (3.6.6),

$$\sup_{s\leq T} |Z_s^i|^2 \leq C_{T,b,\mathfrak{K}} \left[1 + \left|X_0^i\right|^2 + \sup_{s\leq T}\left|Y_s^i\right|^2 + \frac{1}{N}\sum_{j=1}^N \left(\left|X_0^j\right|^2 + \sup_{s\leq T}\left|Y_s^j\right|^2\right)\right] \quad \text{a.s.},$$

where $C_{T,b,\mathfrak{K}}$ is a constant that depends on T, b, and $\mathfrak{K}$ only, and $Y_t^i = \int_0^t \sigma(s, X_s^i)\mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}B_s^i$ with $\mathrm{d}B_s^i = \sqrt{1-\rho_s^2}\,\mathrm{d}W_s^i + \rho_s\,\mathrm{d}W_s^0$ and $\rho_s = \rho(s, \nu_s^N, \mathfrak{f}_s^N)$. By the Generalised Hölder inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[e^{\delta\hat{\Gamma}_t^i}\right] \\
&\leq e^{C_{T,b,\mathfrak{K}}}\mathbb{E}\left[e^{4\delta\left|X_0^i\right|^2}\right]^{\frac{1}{4}}\mathbb{E}\left[e^{4\delta\sup_{s\leq T}\left|Y_s^i\right|^2}\right]^{\frac{1}{4}}\mathbb{E}\left[e^{\frac{4\delta}{N}\sum_{j=1}^N\left|X_0^j\right|^2}\right]^{\frac{1}{4}}\mathbb{E}\left[e^{\frac{4\delta}{N}\sum_{j=1}^N\sup_{s\leq T}\left|Y_s^j\right|^2}\right]^{\frac{1}{4}}.
\end{aligned}$$

The goal is now to upper bound each term on the R.H.S. above by a constant that is independent of N and i. As $\delta < \gamma/4$ by assumption, then by Proposition 3.6.3,

$$\mathbb{E}\left[e^{4\delta\left|X_0^j\right|^2}\right] \leq C_{2,4\delta,X_0} \qquad \forall\, j \in \{1, \dots, N\}.$$

Therefore, by Generalised Hölder inequality and the above, we have

$$\mathbb{E}\left[e^{\frac{4\delta}{N}\sum_{j=1}^N\left|X_0^j\right|^2}\right] \leq \prod_{j=1}^N \mathbb{E}\left[e^{4\delta\left|X_0^j\right|^2}\right]^{\frac{1}{N}} \leq C_{2,4\delta,X_0}.$$

For any j, as Y_t^j is a continuous local martingale, by the Dubins-Schwarz Theorem there is a Brownian motion $\hat{B}$ such that $Y_t^j = \hat{B}_{\int_0^t \sigma^2(s,X_s^j)\mathbb{1}_{\{X_s^j<0\}}\,\mathrm{d}s}$. Therefore,

$$\begin{aligned}
\mathbb{E}\left[e^{4\delta\sup_{t\leq T}\left|Y_t^j\right|^2}\right] &= \mathbb{E}\left[e^{4\delta\sup_{t\leq T}\left|\hat{B}_{\int_0^t \sigma^2(s,X_s^j)\mathbb{1}_{\{X_s^j<0\}}\,\mathrm{d}s}\right|^2}\right] \\
&\leq \mathbb{E}\left[e^{4\delta\sup_{t\leq T}\left|\hat{B}_{C_\sigma^2 t}\right|^2}\right] \\
&\leq C_{p,4\delta,C_\sigma^2 T},
\end{aligned}$$

where the penultimate line follows from the bounds on σ, Assumption 3.2.1 (i), and the last line follows from Proposition 3.6.2. By the Generalised Hölder inequality and the above, we have

$$\mathbb{E}\left[e^{\frac{4\delta}{N}\sum_{j=1}^{N}\sup_{s\leq T}\left|Y_s^j\right|^2}\right] \leq \prod_{j=1}^{N}\mathbb{E}\left[e^{4\delta\sup_{s\leq T}\left|Y_s^j\right|^2}\right]^{\frac{1}{N}} \leq C_{p,4\delta,C_\sigma^2 T}.$$

Therefore, we have shown that

$$\mathbb{E}\left[e^{\delta\hat{\Gamma}_t^i}\right] \leq e^{C_{T,b,\mathfrak{K}}}\cdot\left(C_{2,4\delta,X_0}\right)^{1/2}\cdot\left(C_{p,4\delta,C_\sigma^2 T}\right)^{1/2}.$$

□

Corollary 3.6.5. *X_t^i, Z_t^i, J_t^i, F_t^N, and $\hat{\Lambda}_t^i$ are all sub-Gaussian uniformly in $N \geq 1$.*

Proof. The claim follows directly from Proposition 3.6.4. □

Similarly, as in [31, Section 6.2], we may transform each particle into a Brownian motion with drift, which pauses for a random amount of time when the particle hits 0. Then, the particle is reset to a random value below 0. Additionally, this transformation is such that the times the transformed particle hits 0 are equal to the times the original particle hits 0. This transformation is crucial for the Girsanov-type arguments that follow.

Proposition 3.6.6. *Define the transformation $\Upsilon \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ by*

$$(t,x) \mapsto \Upsilon_t(x) \coloneqq \int_0^x \frac{\mathrm{d}y}{\sigma(t,y)}.$$

Fixing an arbitrary index $i \in \{1,\ldots,N\}$, then

$$\mathrm{d}\Upsilon_t\left(X_t^i\right) = \hat{b}_t^i \mathbb{1}_{\{\Upsilon_t\left(X_t^i\right)<0\}}\,\mathrm{d}t + \mathbb{1}_{\{\Upsilon_t\left(X_t^i\right)<0\}}\,\mathrm{d}B_t^i + \mathrm{d}\sum_{k\geq 1}\Upsilon_{\tau_k^i+\varsigma_k^i}(-\xi_k^i)\mathbb{1}_{[0,t]}(\tau_k^i+\varsigma_k^i),$$

where B^i is a Brownian motion and the (stochastic) drift $\hat{b}_t^i$ satisfies the growth condition

$$|\hat{b}_t^i| \leq C_{b,\sigma,\mathfrak{K}}\left(1+\left|X_t^i\right|+\langle\nu_t^N,|\cdot|\rangle+\left|F_t^N\right|\right). \tag{3.6.10}$$

Furthermore, the transformed process $\Upsilon_t\left(X_t^i\right)$ satisfies

$$\operatorname{sgn}\left(\Upsilon_t\left(X_t^i\right)\right) = \operatorname{sgn}\left(X_t^i\right) \quad \textit{and} \quad \left|\Upsilon_t\left(X_t^i\right)\right| \leq C\left|X_t^i\right|. \tag{3.6.11}$$

Proof. As before, we define $\mathrm{d}B_t^i \coloneqq \rho(t, \nu_t^N, \mathfrak{f}_t^N)\,\mathrm{d}W_t^0 + \sqrt{1-\rho^2(t, \nu_t^N, \mathfrak{f}_t^N)}\,\mathrm{d}W_t^i$. Therefore, by applying Itô's formula with jumps [60, Chapter II, Theorem 33], we have

$$\begin{aligned}\Upsilon_t(X_t^i) =& \Upsilon_0(X_0^i) + \int_0^t \partial_s\Upsilon_s(X_s^i)\,\mathrm{d}s + \int_{0+}^t \partial_x\Upsilon_{s-}(X_{s-}^i)\,\mathrm{d}X_s^i \\ &+ \frac{1}{2}\int_{0+}^t \partial_x^2\Upsilon_{s-}(X_{s-}^i)\,\mathrm{d}[X^i]_s^c \\ &+ \sum_{\substack{0<s\le t \\ s \text{ is jump time}}} \{\Upsilon_s(X_s^i) - \Upsilon_{s-}(X_{s-}^i) - \partial_x\Upsilon_s(X_{s-}^i)\Delta X_s^i\}.\end{aligned} \tag{3.6.12}$$

Since X^i has càdlàg paths and will jump to $-\xi_k^i$ only at times $\tau_k^i + \varsigma_k^i$, and X^i, by construction, is 0 at time $(\tau_k^i + \varsigma_k^i)-$, and also noting that $\Upsilon_s(0) = 0$ for every s, (3.6.12) expands to

$$\begin{aligned}\Upsilon_t(X_t^i) = \Upsilon_0(X_0^i) &+ \int_0^t \partial_s\Upsilon_s(X_s^i)\,\mathrm{d}s + \int_0^t \partial_x\Upsilon_s(X_s^i)b(s,\, X_s^i, \nu_s^N, \mathfrak{f}_s^N)\mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}s \\ &+ \int_0^t \partial_x\Upsilon_s(X_s^i)\sigma(s,\, X_s^i)\mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}B_s^i - \int_{0+}^t \partial_x\Upsilon_{s-}(X_{s-}^i)\,\mathrm{d}\left(\sum_{k\ge1}\xi_k^i\mathbb{1}_{[0,s]}(\tau_k^i+\varsigma_k^i)\right) \\ &+ \frac{1}{2}\int_0^t \partial_x^2\Upsilon_s(X_s^i)\sigma^2(s,\, X_s^i)\mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}s + \sum_{\substack{0<s\le t \\ s \text{ is jump time}}} \{\Upsilon_s(X_s^i) - \partial_x\Upsilon_s(X_{s-}^i)\Delta X_s^i\}.\end{aligned}$$

As $\sum_{k\ge1}\xi_k^i\mathbb{1}_{[0,s]}(\tau_k^i+\varsigma_k^i)$ is a pure jump process, we observe that

$$-\int_{0+}^t \partial_x\Upsilon_{s-}(X_{s-}^i)\,\mathrm{d}\left(\sum_{k\ge1}\xi_k^i\mathbb{1}_{[0,s]}(\tau_k^i+\varsigma_k^i)\right) = \sum_{\substack{0<s\le t \\ s \text{ is jump time}}} \partial_x\Upsilon_s(X_{s-}^i)\Delta X_s^i.$$

Therefore,

$$\begin{aligned}\Upsilon_t(X_t^i) = \Upsilon_0(X_0^i) &+ \int_0^t \partial_s\Upsilon_s(X_s^i)\,\mathrm{d}s + \int_0^t \partial_x\Upsilon_s(X_s^i)b(s,\, X_s^i, \nu_s^N, \mathfrak{f}_s^N)\mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}s \\ &+ \int_0^t \partial_x\Upsilon_s(X_s^i)\sigma(s,\, X_s^i)\mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}B_s^i + \frac{1}{2}\int_0^t \partial_x^2\Upsilon_s(X_s^i)\sigma^2(s,\, X_s^i)\mathbb{1}_{\{X_s^i<0\}}\,\mathrm{d}s \\ &+ \sum_{\substack{0<s\le t \\ s \text{ is jump time}}} \Upsilon_s(X_s^i).\end{aligned} \tag{3.6.13}$$

A straightforward computation shows that

$$\partial_t\Upsilon_t(x) = -\int_0^x \frac{\partial_t\sigma(t,\, y)}{\sigma(t,\, y)^2}\,\mathrm{d}y, \qquad \partial_x\Upsilon_t(x) = \frac{1}{\sigma(t,\, x)}, \qquad \partial_x^2\Upsilon_t(x) = -\frac{\partial_x\sigma(t,\, x)}{\sigma(t,\, x)^2}.$$

We observe that $\partial_t \Upsilon_t(0) = 0$; therefore, it follows from (3.6.13) that

$$\mathrm{d}\Upsilon_t\left(X_t^i\right) = \hat{b}_t^i \mathbb{1}_{\{X_t^i<0\}}\,\mathrm{d}t + \mathbb{1}_{\{X_t^i<0\}}\,\mathrm{d}B_t^i + \mathrm{d}\sum_{k\geq 1}\Upsilon_{\tau_k^i+\varsigma_k^i}(-\xi_k^i)\mathbb{1}_{[0,t]}(\tau_k^i+\varsigma_k^i),$$

with

$$\hat{b}_t^i := \frac{b(t, X_t^i, \nu_t^N, \mathfrak{f}_t^N)}{\sigma(t, X_t^i)} - \frac{1}{2}\partial_x\sigma(t, X_t^i) - \int_0^{X_t^i}\frac{\partial_t\sigma(t,y)}{\sigma(t,y)^2}\,\mathrm{d}y.$$

Now, the bound on $\Upsilon_t\left(X_t^i\right)$ and the statement about its sign in (3.6.11) follow directly from the definition of Υ, since σ is strictly positive and bounded away from zero. Similarly, the growth condition in (3.6.10) follows from the Lipschitz continuity of b and the fact that $\left\|\mathfrak{f}_t^N\right\| \leq \|\mathfrak{K}'\|_{L^1} F_t^N$. □

Building upon the scale transformation introduced in Proposition 3.6.6, we employ the sub-Gaussianity to introduce a change of measure that will entirely remove the drift, which will allow us to view the transformed process as driven by a standard Brownian motion under the new measure.

Proposition 3.6.7. *Fix $i \in \{1, \ldots, N\}$ and define the stochastic exponential to be*

$$\mathcal{E}_t := \exp\left\{-\int_0^t \hat{b}_s^i \mathbb{1}_{\{\Upsilon_s(X_s^i)<0\}}\,\mathrm{d}B_s^i - \frac{1}{2}\int_0^t \left(\hat{b}_s^i\right)^2 \mathbb{1}_{\{\Upsilon_s(X_s^i)<0\}}\,\mathrm{d}s\right\},$$

where $\hat{b}_t^i \mathbb{1}_{\{\Upsilon_t(X_t^i)<0\}}$ is the drift of $\Upsilon_t\left(X_t^i\right)$ defined as in Proposition 3.6.6. Then $\int_0^t \hat{b}_s^i \mathbb{1}_{\{\Upsilon_s(X_s^i)<0\}}\,\mathrm{d}s + B_t^i$ is a standard Brownian motion under the probability measure $\mathbb{Q}$ given by the Radon-Nikodym derivative

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}_t. \tag{3.6.14}$$

Proof. The proof follows exactly that of [31, Lemma 6.4] with the only change being $\left|\hat{b}_s^i \mathbb{1}_{\{\Upsilon_s(X_s^i)<0\}}\right| \leq C(1+\hat{\Lambda}_s^i)$. □

In addition to the change of measure, we will also need an estimate on the expected value of $\mathcal{E}_t^{1-p}$ for $p > 1$ close to 1, so that we may employ Hölder-type estimates to bound the probabilities of X^i for any i.

Proposition 3.6.8 (Radon-Nikodym Estimate)**.** *Let $\mathcal{E}_t$ be the stochastic exponential defined as in Proposition 3.6.7. Then, for all $p > 1$ close enough to 1,*

$$\mathbb{E}\left[\mathcal{E}_t^{1-p}\right] \leq C,$$

where $C > 0$ is a constant that depends on p, T, $\mathfrak{K}$, the initial distribution X_0, and the bounds on b and σ, but is independent of i and N. Moreover, for any $q > 1$, we have

$$\mathbb{E}\left[|\hat{b}_t^i|^q\right] \leq \tilde{C},$$

where $\tilde{C} > 0$ is a constant that depends on q, T, $\mathfrak{K}$, the initial distribution X_0, and the bounds on b and σ, but is independent of i and N.

Proof. We define $\mathrm{d}Y_t = -\hat{b}_t^i \mathbb{1}_{\{\Upsilon_t(X_t^i)<0\}} \mathrm{d}B_t^i$. Then, for any $p > 1$, letting $\mathcal{E}(pY)$ denote the stochastic exponential of pY, we observe

$$\begin{aligned}
\mathbb{E}\left[\mathcal{E}_t^{1-p}\right] &= \mathbb{E}\left[\exp\left\{-(1-p)Y_t - \frac{1-p}{2}\langle Y\rangle_t\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{pY_t - \frac{1}{2}\langle pY\rangle_t\right\}^{\frac{p-1}{p}} \exp\left\{\frac{p(p+1)(p-1)}{2}\langle Y\rangle_t\right\}^{\frac{1}{p}}\right] \\
&\leq \mathbb{E}\left[\mathcal{E}(pY)_t\right]^{\frac{p-1}{p}} \mathbb{E}\left[\exp\left\{\frac{p(p+1)(p-1)}{2}\langle Y\rangle_t\right\}\right]^{\frac{1}{p}},
\end{aligned}$$

where the last line follows from Hölder's inequality. Using the same proof as in Proposition 3.6.7, we have that $\mathcal{E}(pY)$ is a martingale. Therefore, the first term is bounded by 1. Hence, we deduce

$$\mathbb{E}\left[\mathcal{E}_t^{1-p}\right] \leq \mathbb{E}\left[\exp\left\{C_p \int_0^t (\hat{b}_s^i)^2 \mathbb{1}_{\{\Upsilon_s(X_s^i)<0\}} \mathrm{d}s\right\}\right]^{\frac{1}{p}},$$

with $C_p = p(p-1)(p+1)/2$. Observing that $|\hat{b}_s^i| \leq C_{b,\sigma,\mathfrak{K}}(1+\hat{\Lambda}_t^i) \leq C_{b,\sigma,\mathfrak{K}}(2+(\hat{\Lambda}_t^i)^2)$ by (3.6.10), we have

$$\mathbb{E}\left[\mathcal{E}_t^{1-p}\right] \leq e^{\frac{2TC_{b,\sigma,\mathfrak{K}}C_p}{p}} \mathbb{E}\left[\exp\left\{TC_{b,\sigma,\mathfrak{K}}C_p(\hat{\Lambda}_t^i)^2\right\}\right]^{\frac{1}{p}}.$$

Now, by choosing $p \in (1, p_0)$, where $p_0 := \sup\{p > 1 : 2TC_pC < \min\{\gamma/4, T/8\}\}$ and γ is the constant in Assumption 3.2.1 (iv), we have

$$\begin{aligned}
\mathbb{E}\left[\mathcal{E}_t^{1-p}\right] &\leq e^{\frac{2TC_{b,\sigma,\mathfrak{K}}C_p}{p}} \mathbb{E}\left[\exp\left\{TC_pC(\hat{\Lambda}_t^i)^2\right\}\right]^{\frac{1}{p}} \\
&\leq e^{\frac{2TC_{b,\sigma,\mathfrak{K}}C_p}{p}} \mathbb{E}\left[\exp\left\{2TC_pC(\hat{\Gamma}_t^i)^2\right\}\right]^{\frac{1}{p}} \\
&\leq C.
\end{aligned}$$

The last line follows from Proposition 3.6.4, and the constant depends on p, T, $\mathfrak{K}$, the initial distribution X_0, and the bounds on b and σ, but is independent of i and N.

For the second claim, by Proposition 3.6.6 we deduce that $|\hat{b}_s^i| \leq C(1+\hat{\Lambda}_s^i)$. Hence, the claim follows from applying Corollary 3.6.5. □

3.6.2 Exponential Decay of Hitting Probabilities – Proof of Proposition 3.3.4

Now that we have established the necessary estimates, we can proceed to prove the exponential decay of the hitting time probabilities.

Proof of Proposition 3.3.4. To begin, we fix i, $k \geq 1$. By the definition of $\Upsilon_t(x)$, the k^{th} hitting time of X^i is the k^{th} hitting time of $\Upsilon_t(X_t^i)$. As

$$\mathrm{d}\Upsilon_t\left(X_t^i\right) = \hat{b}_t^i \mathbb{1}_{\{\Upsilon_t(X_t^i)<0\}}\,\mathrm{d}t + \mathbb{1}_{\{\Upsilon_t(X_t^i)<0\}}\,\mathrm{d}B_t^i + \mathrm{d}\sum_{\tilde{k}\geq 1}\Upsilon_{\tau_{\tilde{k}}^i+\varsigma_{\tilde{k}}^i}(-\xi_{\tilde{k}}^i)\mathbb{1}_{[0,t]}(\tau_{\tilde{k}}^i+\varsigma_{\tilde{k}}^i)$$

by Proposition 3.6.6, then τ_k^i corresponds to the first time the process $\Upsilon_0(X_0^i) + \int_0^t \mathbb{1}_{\{\Upsilon_s(X_s^i)<0\}}\,\mathrm{d}\tilde{B}_s^i = \Upsilon_t\left(X_t^i\right) - \sum_{\tilde{k}\geq 1}\Upsilon_{\tau_{\tilde{k}}^i+\varsigma_{\tilde{k}}^i}(-\xi_{\tilde{k}}^i)\mathbb{1}_{[0,t]}(\tau_{\tilde{k}}^i+\varsigma_{\tilde{k}}^i)$ hits the level $-\sum_{\tilde{k}=1}^{k-1}\Upsilon_{\tau_{\tilde{k}}^i+\varsigma_{\tilde{k}}^i}(-\xi_{\tilde{k}}^i)\mathbb{1}_{[0,t]}(\tau_{\tilde{k}}^i+\varsigma_{\tilde{k}}^i)$. Here, $\tilde{B}_t^i = \int_0^t \hat{b}_s^i \mathbb{1}_{\{\Upsilon_s(X_s^i)<0\}}\,\mathrm{d}s + B_t^i$ is a $\mathbb{Q}$-standard Brownian motion by Proposition 3.6.7. Thus,

$$\mathbb{P}\left[\tau_k^i \leq T\right] = \mathbb{P}\left[\Upsilon_0(X_0^i) + \sup_{t\leq T}\int_0^t \mathbb{1}_{\{\Upsilon_s(X_s^i)<0\}}\,\mathrm{d}\tilde{B}_s^i \geq -\sum_{\tilde{k}=1}^{k-1}\Upsilon_{\tau_{\tilde{k}}^i+\varsigma_{\tilde{k}}^i}(-\xi_{\tilde{k}}^i)\mathbb{1}_{[0,t]}(\tau_{\tilde{k}}^i+\varsigma_{\tilde{k}}^i)\right].$$

By the bounds on σ and $\xi_{\tilde{k}}^i$ given in Assumption 3.2.1 (i) and (v), we have

$$-\Upsilon_{\tau_{\tilde{k}}^i+\varsigma_{\tilde{k}}^i}(-\xi_{\tilde{k}}^i) = \int_{-\xi_{\tilde{k}}^i}^0 \frac{\mathrm{d}y}{\sigma(\tau_{\tilde{k}}^i+\varsigma_{\tilde{k}}^i, y)} \geq \xi_{\tilde{k}}^i C_\sigma^{-1} \geq \gamma C_\sigma^{-1}.$$

Therefore,

$$\begin{aligned}
\mathbb{P}\left[\tau_k^i \leq T\right] &= \mathbb{P}\left[\Upsilon_0(X_0^i) + \sup_{t\leq T}\int_0^t \mathbb{1}_{\{\Upsilon_s(X_s^i)<0\}}\,\mathrm{d}\tilde{B}_s^i \geq -\sum_{\tilde{k}=1}^{k-1}\Upsilon_{\tau_{\tilde{k}}^i+\varsigma_{\tilde{k}}^i}(-\xi_{\tilde{k}}^i)\mathbb{1}_{[0,t]}(\tau_{\tilde{k}}^i+\varsigma_{\tilde{k}}^i)\right] \\
&\leq \mathbb{P}\left[\Upsilon_0(X_0^i) + \sup_{t\leq T}\int_0^t \mathbb{1}_{\{\Upsilon_s(X_s^i)<0\}}\,\mathrm{d}\tilde{B}_s^i \geq (k-1)\gamma C_\sigma^{-1}\right] \\
&\leq \mathbb{P}\left[\sup_{t\leq T}\int_0^t \mathbb{1}_{\{\Upsilon_s(X_s^i)<0\}}\,\mathrm{d}\tilde{B}_s^i \geq (k-1)\gamma C_\sigma^{-1}\right],
\end{aligned}$$

where the last line follows from the fact that $\Upsilon_0(X_0^i) < 0$ a.s. because $X_0^i < 0$ a.s. Then, for all $p > 1$ and close enough to 1, by Hölder's inequality,

$$\begin{aligned}
&\leq \mathbb{E}\left[\mathcal{E}_t^{1-p}\right]^{\frac{1}{p}}\mathbb{Q}\left[\sup_{t\leq T}\int_0^t \mathbb{1}_{\{\Upsilon_s(X_s^i)<0\}}\,\mathrm{d}\tilde{B}_s^i \geq (k-1)\gamma C_\sigma^{-1}\right]^{\frac{p-1}{p}} \\
&\leq C^{\frac{1}{p}}\mathbb{Q}\left[\sup_{t\leq T}\tilde{B}_t^i \geq (k-1)\gamma C_\sigma^{-1}\right]^{\frac{p-1}{p}}
\end{aligned}$$

where the second term in the last line follows from the Dubins-Schwarz Theorem, and C is the constant from Proposition 3.6.8. By the Reflection Principle for Brownian motion, $\mathbb{Q}\left[\sup_{t\leq T}\tilde{B}_t^i \geq (k-1)\gamma C_\sigma^{-1}\right] = 2\mathbb{Q}\left[\tilde{B}_T^i \geq (k-1)\gamma C_\sigma^{-1}\right]$. Therefore,

$$\mathbb{P}\left[\tau_k^i \leq T\right] \leq C^{\frac{1}{p}}\mathbb{Q}\left[\sup_{t\leq T}\tilde{B}_t^i \geq (k-1)\gamma C_\sigma^{-1}\right]^{\frac{p-1}{p}} \leq 2^{\frac{p-1}{p}}C^{\frac{1}{p}}\mathbb{Q}\left[\tilde{B}_T^i \geq (k-1)\gamma C_\sigma^{-1}\right]^{\frac{p-1}{p}}.$$

The claim now follows from employing the well-known result $\mathbb{P}[\mathcal{N}(0,1) > x] \leq \frac{1}{2}e^{-\frac{x^2}{2}}$ for any $x > 0$. □

3.6.3 Probabilistic Bounds on the Particle System

Furthermore, by employing the Radon-Nikodym estimate from Proposition 3.6.8, we may acquire bounds on the probabilities of X^i conditioned on the event $\{X_t^i < 0\}$, which hold uniformly in i and N.

Proposition 3.6.9. *For any set $S \in \mathcal{B}(\mathbb{R})$, $i = 1, \ldots, N$, and $\delta > 0$ sufficiently close to 0, we have*

$$\mathbb{P}\left[X_t^i \in S, X_t^i < 0\right] \leq Ct^{-\frac{\delta}{2}}\left[1 + \sum_{k\geq 1}\mathbb{P}\left[\tau_k^i \leq t\right]^\delta\right]\mathrm{Leb}(S)^\delta,$$

where $C > 0$ is a constant that depends on p, T, $\mathfrak{K}$, r, the initial distribution X_0, and the bounds on b and σ, but is independent of i and N.

Proof. To begin, we fix $i = 1, \ldots, N$ and $t \in [0, T]$. Then, by the construction of X_t^i, $X_t^i < 0$ if and only if $t \in [0, \tau_1^i) \cup (\cup_{k\geq 1}[\tau_k^i + \varsigma_k^i, \tau_{k+1}^i))$. Therefore, by the law of total probability,

$$\mathbb{P}\left[X_t^i \in S, X_t^i < 0\right] = \mathbb{P}\left[X_t^i \in S, t < \tau_1^i\right] + \sum_{k\geq 1}\mathbb{P}\left[X_t^i \in S, \tau_k^i + \varsigma_k^i \leq t < \tau_{k+1}^i\right]. \tag{3.6.15}$$

The goal now is to apply the scale transform and employ the change of measure to rewrite the above as the probability of a Brownian motion with random jumps. As the scale transform is an injective function, we have

$$\begin{aligned}\mathbb{P}\left[X_t^i \in S, X_t^i < 0\right] =& \mathbb{P}\left[\Upsilon_t(X_t^i) \in \Upsilon_t(S), t < \tau_1^i\right] \\ &+ \sum_{k\geq 1}\mathbb{P}\left[\Upsilon_t(X_t^i) \in \Upsilon_t(S), \tau_k^i + \varsigma_k^i \leq t < \tau_{k+1}^i\right].\end{aligned}$$

We shall analyse each term above individually. On the event $\{t < \tau_1^i\}$, we have

$$\Upsilon_t(X_t^i) = \Upsilon_0(X_0^i) + \tilde{B}_t,$$

where $\tilde{B}$ is a $\mathbb{Q}$-standard Brownian motion by Proposition 3.6.7. Therefore, by Hölder's inequality, we have

$$\begin{aligned}\mathbb{P}\left[\Upsilon_t(X_t^i) \in \Upsilon_t(S), t < \tau_1^i\right] &\leq \mathbb{P}\left[\Upsilon_0(X_0^i) + \tilde{B}_t \in \Upsilon_t(S)\right] \\ &\leq \mathbb{E}^{\mathbb{P}}\left[\mathcal{E}_t^{1-p}\right]^{\frac{1}{p}} \mathbb{Q}\left[\Upsilon_0(X_0^i) + \tilde{B}_t \in \Upsilon_t(S)\right]^{\frac{p-1}{p}}.\end{aligned} \tag{3.6.16}$$

The right-hand side of the above is finite for all $p > 1$ sufficiently close to 1 by Proposition 3.6.8. Girsanov's theorem, [38, Chapter 3, Theorem 5.1], gives us that $\tilde{B}$ is a Brownian motion with respect to the same filtration of our original probability space. Therefore, as X_0^i is $\mathcal{F}_0$-measurable and $\tilde{B}$ is independent of $\mathcal{F}_0$,

$$\begin{aligned}\mathbb{Q}\left[\Upsilon_0(X_0^i) + \tilde{B}_t \in \Upsilon_t(S)\right] &= \int_{\mathbb{R}} \mathbb{Q}\left[\Upsilon_0(x) + \tilde{B}_t \in \Upsilon_t(S)\right] \mathrm{d}\nu_0(x) \\ &= \int_{\mathbb{R}} \int_{\Upsilon_t(S)-\Upsilon_0(x)} (2\pi t)^{-1/2} e^{-\frac{y^2}{2t}} \,\mathrm{d}y \,\mathrm{d}\nu_0(x) \\ &= \int_{\mathbb{R}} \int_S (2\pi t)^{-1/2} e^{-\frac{(\Upsilon_t(\tilde{y})-\Upsilon_0(x))^2}{2t}} \Upsilon_t'(\tilde{y}) \,\mathrm{d}\tilde{y} \,\mathrm{d}\nu_0(x) \\ &\leq C_\sigma (2\pi t)^{-1/2} \,\mathrm{Leb}\,(S) \int_{\mathbb{R}} \mathrm{d}\nu_0(x) \\ &= C_\sigma (2\pi t)^{-1/2} \,\mathrm{Leb}\,(S)\end{aligned}$$

where the third line follows from employing the substitution $\tilde{y} = \Upsilon_t^{-1}(y + \Upsilon_0(x))$, and the fourth line follows from upper bounding the exponential term and the $\Upsilon_t'(\tilde{y})$ term. Setting $\delta = (p-1)p^{-1}$, by (3.6.16), we have the bound

$$\mathbb{P}\left[\Upsilon_t(X_t^i) \in \Upsilon_t(S), t < \tau_1^i\right] \leq C_{\sigma,\delta} \mathbb{E}^{\mathbb{P}}\left[\mathcal{E}_t^{1-p}\right]^{\frac{1}{p}} t^{-\frac{\delta}{2}} \,\mathrm{Leb}(S)^\delta. \tag{3.6.17}$$

Now, we turn our attention to the second term. By construction, τ_k^i is independent of ς_k^i under the measure $\mathbb{P}$, but independence need not be preserved under a change of measure. Hence, we apply the tower property of conditional expectation to preserve the independence between τ_k^i and ς_k^i and estimate conditional probabilities of $\tilde{B}$ under $\mathbb{Q}$. We observe that on the event $\{\tau_k^i + \varsigma_k^i \leq t < \tau_{k+1}^i\}$, we have

$$\Upsilon_t(X_t^i) = \Upsilon_{\tau_k^i+\varsigma_k^i}(-\xi_k^i) + \tilde{B}_t - \tilde{B}_{\tau_k^i+\varsigma_k^i}.$$

Therefore,

$$\begin{aligned}&\mathbb{P}\left[\Upsilon_t(X_t^i) \in \Upsilon_t(S), \tau_k^i + \varsigma_k^i \leq t < \tau_{k+1}^i\right] \\ &\qquad \leq \mathbb{P}\left[\Upsilon_{\tau_k^i+\varsigma_k^i}(-\xi_k^i) + \tilde{B}_t - \tilde{B}_{\tau_k^i+\varsigma_k^i} \in \Upsilon_t(S), \tau_k^i + \varsigma_k^i \leq t\right].\end{aligned}$$

Letting the event on the R.H.S. above be denoted by A, we deduce

$$\begin{aligned}
\mathbb{P}[A] &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_A \mid \tau_k^i + \varsigma_k^i, \xi_k^i\right]\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[\mathcal{E}_t^{-1}\mathbb{1}_A \mid \tau_k^i + \varsigma_k^i, \xi_k^i\right]\right] \\
&\leq \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\mathcal{E}_t^{1-p} \mid \tau_k^i + \varsigma_k^i, \xi_k^i\right]^{\frac{1}{p}} \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_A \mid \tau_k^i + \varsigma_k^i, \xi_k^i\right]^{\frac{p-1}{p}}\right] \\
&\leq \mathbb{E}^{\mathbb{P}}\left[\mathcal{E}_t^{1-p}\right]^{\frac{1}{p}} \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_A \mid \tau_k^i + \varsigma_k^i, \xi_k^i\right]\right]^{\frac{p-1}{p}}, \qquad (3.6.18)
\end{aligned}$$

where the penultimate line follows from the Conditional Hölder inequality and the last line follows from Hölder's inequality. The right-hand side of the above is finite for all $p > 1$ sufficiently close to 1 by Proposition 3.6.8.

We now turn our attention to estimating the second term in the product found in (3.6.18). Let τ be any stopping time such that $0 < \tau \leq t$ almost surely and ξ be an $\mathcal{F}_0$-measurable random variable supported on $(0, R]$ for some $R > 0$. Furthermore, let τ^n (ξ^n) denote the discrete approximation to τ (ξ, respectively) defined by

$$\begin{aligned}
\tau^n &= \sum_{j=1}^{2^n} tj2^{-n}\mathbb{1}_{((j-1)t2^{-n}, jt2^{-n}]}(\tau), \\
\xi^n &= \sum_{l=1}^{2^n} Rl2^{-n}\mathbb{1}_{((l-1)R2^{-n}, lR2^{-n}]}(\xi).
\end{aligned}$$

Then for any open sets $A, B, C \subset \mathbb{R}$,

$$\begin{aligned}
&\mathbb{Q}\left[\Upsilon_{\tau^n}(\xi^n) + \tilde{B}_t - \tilde{B}_{\tau^n} \in A, \tau^n \in B, \xi^n \in C\right] \\
&= \sum_{j,l=1}^{2^n} \mathbb{Q}\left[\Upsilon_{\tau^n}(-\xi^n) + \tilde{B}_t - \tilde{B}_{\tau^n} \in A, \tau^n = tj2^{-n}, \xi^n = Rl2^{-n}\right] \mathbb{1}_B(tj2^{-n})\mathbb{1}_C(Rl2^{-n}) \\
&= \sum_{j,l=1}^{2^n} \mathbb{Q}\left[\Upsilon_{\tau^n}(-\xi^n) + \tilde{B}_t - \tilde{B}_{\tau^n} \in A\right] \mathbb{1}_B(tj2^{-n})\mathbb{1}_C(Rl2^{-n})\mathbb{Q}[\tau^n = tj2^{-n}, \xi^n = Rl2^{-n}] \\
&= \int_{\mathbb{R}^2} \mathbb{Q}\left[\Upsilon_s(-x) + \tilde{B}_t - \tilde{B}_s \in A\right] \mathbb{1}_B(s)\mathbb{1}_C(x)\mathbb{Q}\left[\tau^n \in \mathrm{d}s, \xi^n \in \mathrm{d}x\right] \\
&= \int_{\Omega} \mathbb{Q}\left[\Upsilon_{\tau^n(\omega)}(-\xi^n(\omega)) + \tilde{B}_t - \tilde{B}_{\tau^n(\omega)} \in A\right] \mathbb{1}_B(\tau^n(\omega))\mathbb{1}_C(\xi^n(\omega))\,\mathrm{d}\mathbb{Q}\left[\omega\right]
\end{aligned}$$

Therefore, by first taking limits as $n \to \infty$ and then applying a monotone class theorem argument, we have

$$\mathbb{Q}\left[\Upsilon_\tau(-\xi) + \tilde{B}_t - \tilde{B}_\tau \in A \mid \tau, \xi\right] = \mathbb{Q}\left[\Upsilon_{\tau(\cdot)}(-\xi(\cdot)) + \tilde{B}_t - \tilde{B}_{\tau(\cdot)} \in A\right] \qquad (3.6.19)$$

for all measurable sets A. Girsanov's theorem, [38, Chapter 3, Theorem 5.1], gives us that $\tilde{B}$ is a Brownian motion with respect to the same filtration as our original

probability space. As ξ_k^i and ς_k^i are $\mathcal{F}_0$-measurable and $\tau_k^i + \varsigma_k^i$ is a stopping time, returning to the second term in (3.6.18) and employing (3.6.19), we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_A \mid \tau_k^i + \varsigma_k^i, \xi_k^i\right]\right] \\
&\quad = \mathbb{E}^{\mathbb{P}}\left[\mathbb{Q}\left[\Upsilon_{\tau_k^i+\varsigma_k^i}(-\xi_k^i) + \tilde{B}_t - \tilde{B}_{\tau_k^i+\varsigma_k^i} \in \Upsilon_t(S), \tau_k^i + \varsigma_k^i \leq t \mid \tau_k^i + \varsigma_k^i, \xi_k^i\right]\right] \\
&\quad = \int_{[0,t]\times\mathbb{R}} \mathbb{Q}\left[\Upsilon_s(-x) + \tilde{B}_t - \tilde{B}_s \in \Upsilon_t(S)\right] \mathbb{P}\left[\tau_k^i + \varsigma_k^i \in \mathrm{d}s, \xi_k^i \in \mathrm{d}x\right] \\
&\quad \leq C_\sigma (2\pi)^{-1/2} \operatorname{Leb}(S) \int_0^t (t-s)^{-1/2} \mathbb{P}\left[\tau_k^i + \varsigma_k^i \in \mathrm{d}s\right] \\
&\quad = C_\sigma (2\pi)^{-1/2} \operatorname{Leb}(S) \int_0^t \int_0^s (t-s)^{-1/2} r(s-u) \mathbb{P}\left[\tau_k^i \in \mathrm{d}u\right] \mathrm{d}s
\end{aligned}$$

where the third line follows from $\mathbb{Q}[\Upsilon_s(-x) + \tilde{B}_t - \tilde{B}_s \in \Upsilon_t(S)] \leq C_\sigma(2\pi)^{-1/2}(t-s)^{-1/2}\operatorname{Leb}(S)$ and the last line is due to the independence between τ_k^i and ς_k^i. Lastly, by first applying Fubini's Theorem followed by using the fact that r is continuous, we compute

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_A \mid \tau_k^i + \varsigma_k^i, \xi_k^i\right]\right] \\
&\qquad \leq C_\sigma (2\pi)^{-1/2} \operatorname{Leb}(S) \int_0^t \int_u^t (t-s)^{-1/2} r(s-u)\, \mathrm{d}s \mathbb{P}\left[\tau_k^i \in \mathrm{d}u\right] \\
&\qquad \leq C_\sigma (2\pi)^{-1/2} \|r\|_\infty \operatorname{Leb}(S) \int_0^t \int_u^t (t-s)^{-1/2}\, \mathrm{d}s \mathbb{P}\left[\tau_k^i \in \mathrm{d}u\right] \\
&\qquad = 2C_\sigma (2\pi)^{-1/2} \|r\|_\infty \operatorname{Leb}(S) \int_0^t (t-u)^{1/2} \mathbb{P}\left[\tau_k^i \in \mathrm{d}u\right] \\
&\qquad \leq 2C_\sigma (2\pi)^{-1/2} \|r\|_\infty t^{1/2} \mathbb{P}\left[\tau_k^i \leq t\right] \operatorname{Leb}(S).
\end{aligned}$$

Returning to (3.6.18), and enlarging C_σ when necessary, we have

$$\begin{aligned}
&\mathbb{P}\left[\Upsilon_t(X_t^i) \in \Upsilon_t(S), \tau_k^i + \varsigma_k^i \leq t < \tau_{k+1}^i\right] \\
&\qquad \leq C_{\sigma,\delta} \|r\|_\infty t^{\delta/2} \mathbb{E}^{\mathbb{P}}\left[\mathcal{E}_t^{1-p}\right]^{\frac{1}{p}} \mathbb{P}\left[\tau_k^i \leq t\right]^\delta \operatorname{Leb}(S)^\delta,
\end{aligned} \tag{3.6.20}$$

where $\delta = (p-1)p^{-1}$. Combining the upper bounds in (3.6.17) and (3.6.20) into (3.6.15), we obtain

$$\begin{aligned}
\mathbb{P}\left[X_t^i \in S, X_t^i < 0\right] \leq & C_{\sigma,\delta} \mathbb{E}^{\mathbb{P}}\left[\mathcal{E}_t^{1-p}\right]^{\frac{1}{p}} t^{-\frac{\delta}{2}} \operatorname{Leb}(S)^\delta \\
& + \sum_{k\geq 1} C_{\sigma,\delta} \|r\|_\infty t^{\delta/2} \mathbb{E}^{\mathbb{P}}\left[\mathcal{E}_t^{1-p}\right]^{\frac{1}{p}} \mathbb{P}\left[\tau_k^i \leq t\right]^\delta \operatorname{Leb}(S)^\delta \\
\leq & C_{\sigma,\delta,T,r} t^{-\frac{\delta}{2}} \mathbb{E}^{\mathbb{P}}\left[\mathcal{E}_t^{1-p}\right]^{\frac{1}{p}} \left[1 + \sum_{k\geq 1} \mathbb{P}\left[\tau_k^i \leq t\right]^\delta\right] \operatorname{Leb}(S)^\delta.
\end{aligned}$$

The right-hand side of the above is finite for all $p > 1$ sufficiently close to 1 by Proposition 3.6.8. □

3.6.4 Boundary Estimate – Proof of Proposition 3.3.6

The aim of this section is to prove the boundary estimate in Proposition 3.3.6. This will be achieved by controlling $\mathbb{P}(X_t^i \in S,\, X_t^i < 0)$, uniformly in i, where S is defined to be the interval $(-\varepsilon, 0)$ for some $\varepsilon > 0$. The strategy is outlined as follows:

By the definition of the scale transform in Proposition 3.6.6, the function $x \mapsto \Upsilon_t(x)$ is a bijection for any fixed t. In addition, $\Upsilon_t(x) \leq 0$ if and only if $x \leq 0$. Consequently, we have

$$\mathbb{P}\left[X_t^i \in S,\, X_t^i < 0\right] = \mathbb{P}\left[\hat{X}_t \in S_t,\, \hat{X}_t < 0\right],$$

where $\hat{X}_t = \Upsilon_t(X_t^i)$ and $S_t = \Upsilon_t(S)$. By applying a change of measure, we may view $\hat{X}$ as a neuronal particle driven solely by a standard Brownian motion. This particle is paused at 0 for a random duration ς_k^i after reaching 0 for the k^{th} time. Subsequently, it is reset to a random value in the negative half-line after this waiting period.

We then approximate $\hat{X}$ by allowing it to evolve up to time s, followed by running an independent absorbed Brownian motion for time $t - s$. By employing the bounds on the density of an absorbed Brownian motion, we may estimate the probability that this approximate particle is in S_t. Lastly, by sending s towards t from below, we may obtain an estimate on the probability that $\hat{X}$ is in S_t. This strategy mirrors the approach in [31, Section 6.3].

Proof of Proposition 3.3.6. By Proposition 3.6.6,

$$\begin{aligned}
\mathrm{d}\hat{X}_t &= \hat{b}_t^i \mathbb{1}_{\{\hat{X}_t<0\}}\,\mathrm{d}t + \mathbb{1}_{\{\hat{X}_t<0\}}\,\mathrm{d}W_t^i + \mathrm{d}\sum_{k\geq 1} \Upsilon_{\tau_k^i+\varsigma_k^i}(-\xi_k^i)\mathbb{1}_{[0,t]}(\tau_k^i+\varsigma_k^i), \\
&= \mathbb{1}_{\{\hat{X}_t<0\}}\,\mathrm{d}\tilde{B}_t + \mathrm{d}\sum_{k\geq 1} \Upsilon_{\tau_k^i+\varsigma_k^i}(-\xi_k^i)\mathbb{1}_{[0,t]}(\tau_k^i+\varsigma_k^i) \qquad (3.6.21)
\end{aligned}$$

where $\tilde{B}$ is a $\mathbb{Q}$-Brownian motion. Consider a given interval $[0,t]$ and define

$$u(s,x) = \int_{S_t} G_{t-s}(y,x)\,\mathrm{d}y$$

for $s \in [0,t]$, where $G_t(y,x) = p_t(x-y) - p_t(x+y)$, with $p_t(x) = (2\pi t)^{-1/2}e^{-x^2/2t}$, is the Dirichlet heat kernel on the domain $(-\infty, 0)$. We remark that u is a classical solution to the terminal-boundary value problem

$$\begin{cases}
\partial_s u(s,x) + \dfrac{1}{2}\Delta u(s,x) = 0 & \text{on} \quad [0,t)\times(-\infty,0), \\
u(t,x) = \mathbb{1}_{S_t}(x) & \text{on} \quad \{t\}\times(-\infty,0), \\
u(s,0) = 0 & \text{on} \quad [0,t)\times\{0\}.
\end{cases} \qquad (3.6.22)$$

Now, fix any $t_0 < t$ and let

$$v(s) \coloneqq \mathbb{E}\left[u(s, \hat{X}_s)\right].$$

Applying Itô's formula with jumps to $u(s, \hat{X}_s)$ and utilizing (3.6.22), we obtain

$$v(t_0) = v(0) + \int_0^{t_0} \mathbb{E}\left[\hat{b}^i_s \mathbb{1}_{\{\hat{X}_s<0\}} \partial_x u(s, \hat{X}_s)\right] \mathrm{d}s + \mathbb{E}\left[\sum_{s\le t_0} u(s, \hat{X}_s) - u(s, \hat{X}_{s-})\right]. \tag{3.6.23}$$

The underlying idea is that by taking the limit $t_0 \uparrow t$, the term on the left-hand side converges to $\mathbb{P}[\hat{X}_t \in S_t]$, which is our quantity of interest. Thus, we need to obtain control of the three terms on the right-hand side. The first term equates to $\mathbb{P}[\Upsilon_0(X^i_0) + B_t \in S_t,\, t < \tau^B]$ for some standard Brownian motion B independent of X^i_0, where $\tau^B \coloneqq \inf\{s \ge 0 \,:\, \Upsilon_0(X^i_0) + B_s \ge 0\}$. As $S_t \approx (-\varepsilon, 0)$, it follows that

$$\mathbb{P}[\Upsilon_0(X^i_0) + B_t \in S_t,\, t < \tau^B] = O(t^{-1/2}\varepsilon^2) \tag{3.6.24}$$

uniformly in i.

To estimate the third term on the right-hand side of (3.6.23), we begin by noting that if s is a reset time, i.e. $s = \tau^i_k + \varsigma^i_k$, $\hat{X}_{s-} = \Upsilon_s(0) = 0$ and $\hat{X}_s = \Upsilon_{\tau^i_k+\varsigma^i_k}(-\xi^i_k)$. Therefore, we can express

$$\mathbb{E}\left[\sum_{s\le t_0} u(s, \hat{X}_s) - u(s, \hat{X}_{s-})\right] = \sum_{k\ge 1} \mathbb{E}\left[\mathbb{1}_{t_0\ge\tau^i_k+\varsigma^i_k} u(\tau^i_k + \varsigma^i_k, \Upsilon_{\tau^i_k+\varsigma^i_k}(-\xi^i_k))\right]. \tag{3.6.25}$$

Proceeding, we find that

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{t_0\ge\tau^i_k+\varsigma^i_k} u(\tau^i_k + \varsigma^i_k, \Upsilon_{\tau^i_k+\varsigma^i_k}(-\xi^i_k))\right] \\
&\quad = \int_\gamma^R \int_0^{t_0} \int_{S_t} G_{t-s}(y, \Upsilon_s(-x))\, \mathrm{d}y \mathbb{P}[\tau^i_k + \varsigma^i_k \in \mathrm{d}s] \mathbb{P}[\xi^i_k \in \mathrm{d}x]
\end{aligned}$$

Applying the change of variables $\tilde{y} = -y$ and using the fact that $G_t(-y, -x) = G_t(y, x)$ for any x, y,

$$= \int_\gamma^R \int_0^{t_0} \int_{-S_t} G_{t-s}(\tilde{y}, -\Upsilon_s(-x))\, \mathrm{d}\tilde{y} \mathbb{P}[\tau^i_k + \varsigma^i_k \in \mathrm{d}s] \mathbb{P}[\xi^i_k \in \mathrm{d}x]$$

Using the change of variables $\tilde{y} = -\Upsilon_t(-\tilde{x})$,

$$= \int_\gamma^R \int_0^{t_0} \int_0^{\varepsilon} G_{t-s}(-\Upsilon_t(-\tilde{x}), -\Upsilon_s(-x)) \Upsilon'_t(-\tilde{x})\, \mathrm{d}\tilde{x} \mathbb{P}[\tau^i_k + \varsigma^i_k \in \mathrm{d}s] \mathbb{P}[\xi^i_k \in \mathrm{d}x].$$

By [63, Lemma 2.2.1], for any $x, y \geq 0$, we have

$$G_t(x,y) \leq \frac{C}{\sqrt{t}} \left(\frac{x}{\sqrt{t}} \wedge 1 \right) \left(\frac{y}{\sqrt{t}} \wedge 1 \right) e^{-\frac{(x-y)^2}{4t}}, \tag{3.6.26}$$

for some constant $C > 0$. Furthermore, by definition of Υ_t, we may find $c, C > 0$ such that

$$|\Upsilon_t(-\tilde{x})| \leq C\,|\tilde{x}|\,, \qquad |\Upsilon_t(-\tilde{x}) - \Upsilon_t(-x)| \geq c(\gamma - \varepsilon), \qquad |\Upsilon_t'(-\tilde{x})| \leq C, \tag{3.6.27}$$

for all $\tilde{x} \in (0, \varepsilon)$ and $x \in [\gamma, R]$. Employing (3.6.26) and (3.6.27),

$$\begin{aligned} \mathbb{E}\left[\mathbb{1}_{t_0 \geq \tau_k^i + \varsigma_k^i} u(\tau_k^i + \varsigma_k^i, \Upsilon_{\tau_k^i + \varsigma_k^i}(-\xi_k^i)) \right] &\lesssim \int_0^{t_0} \int_0^{\varepsilon} \frac{\tilde{x}}{t-s} e^{-\frac{c(\gamma-\varepsilon)^2}{4(t-s)}} \,\mathrm{d}\tilde{x} \mathbb{P}[\tau_k^i + \varsigma_k^i \in \mathrm{d}s] \\ &\lesssim (\gamma - \varepsilon)^{-2} \int_0^{t_0} \int_0^{\varepsilon} \tilde{x}\,\mathrm{d}\tilde{x} \mathbb{P}[\tau_k^i + \varsigma_k^i \in \mathrm{d}s] \end{aligned} \tag{3.6.28}$$

where the ultimate line follows from the fact that for any fixed $z > 0$, the function $t \mapsto t^{-1} e^{-z^2/4t}$ is bounded by $4e^{-1}z^{-2}$. Substituting (3.6.28) into (3.6.25),

$$\mathbb{E}\left[\sum_{s \leq t_0} u(s, \hat{X}_s) - u(s, \hat{X}_{s-}) \right] \leq C\varepsilon^2 \sum_{k \geq 1} \mathbb{P}[\tau_k^i \leq T], \tag{3.6.29}$$

for all $\varepsilon \ll 1$ and some constant C independent of i.

Now we turn our attention to the middle term. By definition of u,

$$\int_0^{t_0} \mathbb{E}\left[|\hat{b}_s^i| \mathbb{1}_{\{\hat{X}_s < 0\}} |\partial_x u(s, \hat{X}_s)| \right] \mathrm{d}s \tag{3.6.30}$$

$$\begin{aligned} &\leq \int_0^t \int_{S_t} \mathbb{E}\left[|\hat{b}_s^i| \mathbb{1}_{\{\hat{X}_s < 0\}} |\partial_x G_{t-s}(y, \hat{X}_s)| \right] \mathrm{d}y\,\mathrm{d}s \\ &\leq \int_0^t \int_{S_t} \mathbb{E}\left[|\hat{b}_s^i| \mathbb{1}_{\{s < \hat{\tau}_1\}} |\partial_x G_{t-s}(y, \hat{X}_s)| \right] \mathrm{d}y\,\mathrm{d}s \\ &+ \sum_{k \geq 1} \int_0^t \int_{S_t} \mathbb{E}\left[|\hat{b}_s^i| \mathbb{1}_{\{\hat{\tau}_k + \varsigma_k^i \leq s < \hat{\tau}_{k+1}\}} |\partial_x G_{t-s}(y, \hat{X}_s)| \right] \mathrm{d}y\,\mathrm{d}s \\ &= I + \sum_{k \geq 1} I_k, \end{aligned} \tag{3.6.31}$$

where $\hat{\tau}_k = \inf\{t > \hat{\tau}_{k-1} + \varsigma_{k-1}^i \,:\, \hat{X}_{t-} \geq 0\}$ with $\hat{\tau}_0 = \varsigma_0^i = 0$. To begin with I, by applying the tower property and Fubini's Theorem, we have

$$I = \mathbb{E} \int_0^t \int_{S_t} \mathbb{E}\left[|\hat{b}_s^i| \mathbb{1}_{\{s < \hat{\tau}_1\}} |\partial_x G_{t-s}(y, \hat{X}_s)| \,\middle|\, X_0^i \right] \mathrm{d}y\,\mathrm{d}s. \tag{3.6.32}$$

Now for any $p, q > 1$, we may apply Hölder's inequality to the conditional expectation in (3.6.32). This yields

$$\begin{aligned} I \leq \mathbb{E} \int_0^t \int_{S_t} \mathbb{E}\left[|\hat{b}_s^i|^{\frac{q}{q-1}} \middle| X_0^i \right]^{\frac{q-1}{q}} \mathbb{E}\left[\mathcal{E}_T^{1-p} \middle| X_0^i \right]^{\frac{1}{pq}} \\ \times \mathbb{E}^{\mathbb{Q}}\left[|\partial_x G_{t-s}(y, \hat{X}_s)|^{\frac{pq}{p-1}} \mathbb{1}_{\{s<\hat{\tau}_1\}} \middle| X_0^i \right]^{\frac{p-1}{pq}} \mathrm{d}y \, \mathrm{d}s. \end{aligned} \tag{3.6.33}$$

By Proposition 3.6.6, we deduce that $|\hat{b}_s^i| \leq C(1 + \hat{\Lambda}_s^i) \leq C(1 + \hat{\Lambda}_T^i)$. Therefore,

$$\begin{aligned} I \lesssim \mathbb{E} \int_0^t \int_{S_t} \mathbb{E}\left[|1 + \hat{\Lambda}_T^i|^{\frac{q}{q-1}} \middle| X_0^i \right]^{\frac{q-1}{q}} \mathbb{E}\left[\mathcal{E}_T^{1-p} \middle| X_0^i \right]^{\frac{1}{pq}} \\ \times \mathbb{E}^{\mathbb{Q}}\left[|\partial_x G_{t-s}(y, \hat{X}_s)|^{\frac{pq}{p-1}} \mathbb{1}_{\{s<\hat{\tau}_1\}} \middle| X_0^i \right]^{\frac{p-1}{pq}} \mathrm{d}y \, \mathrm{d}s. \end{aligned} \tag{3.6.34}$$

Since the sum of the exponents $(q-1)/q + 1/pq + (p-1)/pq = 1$ and the last term is the only term that depends on y and s, applying Hölder's inequality one more time to (3.6.34) yields

$$\begin{aligned} I \lesssim & \mathbb{E}\left[|1 + \hat{\Lambda}_T^i|^{\frac{q}{q-1}} \right]^{\frac{q-1}{q}} \mathbb{E}\left[\mathcal{E}_T^{1-p} \right]^{\frac{1}{pq}} \\ & \times \mathbb{E}\left[\left(\int_0^t \int_{S_t} \mathbb{E}^{\mathbb{Q}}\left[|\partial_x G_{t-s}(y, \hat{X}_s)|^a \mathbb{1}_{\{s<\hat{\tau}_1\}} \middle| X_0^i \right]^{\frac{1}{a}} \mathrm{d}y \, \mathrm{d}s \right)^a \right]^{\frac{1}{a}}, \end{aligned} \tag{3.6.35}$$

where $a = pq/(p-1)$. By Corollary 3.6.5, the first term in (3.6.35) is bounded for all q uniformly in N and the second term is bounded for all p close enough to 1 by Proposition 3.6.8. Thus, it remains to establish control over the last term in (3.6.35).

To this end, we shall employ the techniques in [63, Chapter 2]. For any $x_0 < 0$, we have $\mathbb{Q}[\hat{X}_s \in \cdot,\, s < \hat{\tau}_1 \mid X_0^i = x_0] = \mathbb{Q}[\Upsilon_0(x_0) + \tilde{B}_s \in \cdot,\, s < \hat{\tau}_1 \mid X_0^i = x_0]$. This is the law of a stopped Brownian Motion starting at $\Upsilon_0(x_0)$. Therefore,

$$\begin{aligned} \int_{S_t} \mathbb{E}^{\mathbb{Q}}\left[|\partial_x G_{t-s}(y, \hat{X}_s)|^a \mathbb{1}_{\{s<\hat{\tau}_1\}} \middle| X_0^i = x_0 \right]^{\frac{1}{a}} \mathrm{d}y \\ = \int_{S_t} \left(\int_{-\infty}^0 |\partial_x G_{t-s}(y, x)|^a G_s(z, x) \, \mathrm{d}x \right)^{\frac{1}{a}} \mathrm{d}y, \end{aligned} \tag{3.6.36}$$

where $z = \Upsilon_0(x_0)$. By applying the change of variables $\tilde{x} = -x$ and $\tilde{y} = -y$, and considering the expression for $G_t(y, x)$, we get

$$\begin{aligned} \int_{S_t} \mathbb{E}^{\mathbb{Q}}\left[|\partial_x G_{t-s}(y, \hat{X}_s)|^a \mathbb{1}_{\{s<\hat{\tau}_1\}} \middle| X_0^i = x_0 \right]^{\frac{1}{a}} \mathrm{d}y \\ = \int_{-S_t} \left(\int_0^\infty |\partial_{\tilde{x}} G_{t-s}(\tilde{y}, \tilde{x})|^a G_s(|z|, \tilde{x}) \, \mathrm{d}\tilde{x} \right)^{\frac{1}{a}} \mathrm{d}\tilde{y}, \end{aligned} \tag{3.6.37}$$

Writing out the expression for $\partial_{\tilde{x}} G_{t-s}$ and G_s, and using the upper bound

$$e^{-\frac{(\tilde{x}^2-|z|)^2}{2s}} - e^{-\frac{(\tilde{x}^2+|z|)^2}{2s}} \leq \left(\frac{2\tilde{x}|z|}{s} \wedge 1\right) e^{-\frac{(\tilde{x}^2-|z|)^2}{2s}},$$

we obtain $\int_{S_t} \mathbb{E}^{\mathbb{Q}}\left[|\partial_x G_{t-s}(y, \hat{X}_s)|^a \mathbb{1}_{\{s<\hat{\tau}_1\}} \middle| X_0^i = x_0\right]^{\frac{1}{a}} \mathrm{d}y \lesssim (t-s)^{-1/2} s^{-1/2a} \tilde{I}(s)$, where

$$\tilde{I}(s) := \int_{-S_t} \left(\int_0^\infty \left| \frac{(\tilde{y}-\tilde{x})}{t-s} e^{-\frac{(\tilde{y}-\tilde{x})^2}{2(t-s)}} + \frac{(\tilde{y}+\tilde{x})}{t-s} e^{-\frac{(\tilde{y}+\tilde{x})^2}{2(t-s)}} \right|^a \frac{\tilde{x}|z|}{s} e^{-\frac{(\tilde{x}-|z|)^2}{2s}} \mathrm{d}\tilde{x} \right)^{\frac{1}{a}} \mathrm{d}\tilde{y} \tag{3.6.38}$$

Now (3.6.38) mirrors the form of $\tilde{I}(s)$ as delineated in the proof of [63, Proposition 2.4.3]. Therefore, we have the bound

$$\int_0^t \int_{S_t} \mathbb{E}^{\mathbb{Q}}\left[|\partial_x G_{t-s}(y, \hat{X}_s)|^a \mathbb{1}_{\{s<\hat{\tau}_1\}} \middle| X_0^i = x_0\right]^{\frac{1}{a}} \mathrm{d}y \, \mathrm{d}s \leq C_a \int_{-S_t} t^{-\frac{1}{a}} |z|^{\frac{1}{a}} \tilde{y}^{\frac{1}{a}} e^{-\frac{(\tilde{y}-|z|)^2}{4at}} \mathrm{d}\tilde{y}. \tag{3.6.39}$$

Recalling $S_t = \Upsilon_t(S)$ and $\Upsilon_0(x_0) = z$, employing the change of variable $\tilde{y} = -\Upsilon_t(-\tilde{x})$ and the upper bounds in (3.6.27) which hold for all $\tilde{x}$, we have

$$\begin{aligned}\int_0^t \int_{S_t} \mathbb{E}^{\mathbb{Q}}\left[|\partial_x G_{t-s}(y, \hat{X}_s)|^a \mathbb{1}_{\{s<\hat{\tau}_1\}} \middle| X_0^i = x_0\right]^{\frac{1}{a}} \mathrm{d}y \, \mathrm{d}s &\leq C_{a,\sigma} \int_0^\varepsilon t^{-\frac{1}{a}} |x_0|^{\frac{1}{a}} \tilde{x}^{\frac{1}{a}} \mathrm{d}\tilde{x} \\ &= C_{a,\sigma} t^{-\frac{1}{a}} |x_0|^{\frac{1}{a}} \varepsilon^{1+\frac{1}{a}}.\end{aligned}$$

Returning to (3.6.35), using the upper bound above,

$$I \leq C_{a,p,q,\sigma} t^{-\frac{1}{a}} \varepsilon^{1+\frac{1}{a}} \mathbb{E}\left[|X_0|\right]^{\frac{1}{a}} \tag{3.6.40}$$

Now considering the term I_k in (3.6.31), performing analogous computations as in (3.6.32) - (3.6.35) but conditioning on $\hat{\tau}_k^i + \varsigma_k^i$ and ξ_k^i, we obtain

$$I_k \lesssim C_{p,q} \mathbb{E}\left[\left(\int_0^t \int_{S_t} \mathbb{E}^{\mathbb{Q}}\left[|\partial_x G_{t-s}(y, \hat{X}_s)|^a \mathbb{1}_{\{\hat{\tau}_k+\varsigma_k^i \leq s < \hat{\tau}_{k+1}\}} \middle| \hat{\tau}_k^i + \varsigma_k^i, \xi_k^i\right]^{\frac{1}{a}} \mathrm{d}y \, \mathrm{d}s \right)^a \right]^{\frac{1}{a}}, \tag{3.6.41}$$

For any $r < s < t$ and $x_0 \in [\gamma, R]$, $\mathbb{Q}[\hat{X}_s \in \cdot, s < \hat{\tau}_{k+1} \mid \hat{\tau}_k^i + \varsigma_k^i = r, \xi_k^i = x_0] = \mathbb{Q}[\Upsilon_r(-x_0) + \tilde{B}_s - \tilde{B}_r \in \cdot, s < \hat{\tau}_{k+1}]$. This is the law of a stopped Brownian motion starting at $\Upsilon_r(-x_0)$ at time $s-r$. Therefore,

$$\begin{aligned}&\int_0^t \int_{S_t} \mathbb{E}^{\mathbb{Q}}\left[|\partial_x G_{t-s}(y, \hat{X}_s)|^a \mathbb{1}_{\{\hat{\tau}_k+\varsigma_k^i \leq s < \hat{\tau}_{k+1}\}} \middle| \hat{\tau}_k^i + \varsigma_k^i = r, \xi_k^i = x_0\right]^{\frac{1}{a}} \mathrm{d}y \, \mathrm{d}s \\ &\qquad = \int_r^t \int_{S_t} \left(\int_{-\infty}^0 |\partial_x G_{t-s}(y, x)|^a G_{s-r}(z, x) \, \mathrm{d}x \right)^{\frac{1}{a}} \mathrm{d}y \, \mathrm{d}s \\ &\qquad = \int_0^{\tilde{t}} \int_{S_t} \left(\int_{-\infty}^0 |\partial_x G_{\tilde{t}-\tilde{s}}(y, x)|^a G_{\tilde{s}}(z, x) \, \mathrm{d}x \right)^{\frac{1}{a}} \mathrm{d}y \, \mathrm{d}\tilde{s}\end{aligned} \tag{3.6.42}$$

with $z = \Upsilon_r(-x_0)$, $\tilde{t} = t - r$ and $\tilde{s} = s - r$. (3.6.42) is the exact same equation as in (3.6.36). Therefore, repeating the calculations in (3.6.36) - (3.6.39), we have the bound

$$\begin{aligned}\int_0^t \int_{S_t} \mathbb{E}^{\mathbb{Q}}\left[|\partial_x G_{t-s}(y,\hat{X}_s)|^a \mathbb{1}_{\{\hat{\tau}_k+\varsigma_k^i \le s < \hat{\tau}_{k+1}\}} \middle| \hat{\tau}_k^i + \varsigma_k^i = r, \xi_k^i = x_0\right]^{\frac{1}{a}} \mathrm{d}y\,\mathrm{d}s \\ \le C_a \int_{-S_t} \tilde{t}^{-\frac{1}{a}} |z|^{\frac{1}{a}} \tilde{y}^{\frac{1}{a}} e^{-\frac{(\tilde{y}-|z|)^2}{4a\tilde{t}}} \mathrm{d}\tilde{y}.\end{aligned} \tag{3.6.43}$$

Recalling $S_t = \Upsilon_t(S)$ and $\Upsilon_r(-x_0) = z$, employing the change of variable $\tilde{y} = -\Upsilon_t(-\tilde{x})$ and the bounds in (3.6.27) which hold for all $\tilde{x}$, we have

$$\begin{aligned}\int_0^t \int_{S_t} \mathbb{E}^{\mathbb{Q}}\left[|\partial_x G_{t-s}(y,\hat{X}_s)|^a \mathbb{1}_{\{\hat{\tau}_k+\varsigma_k^i \le s < \hat{\tau}_{k+1}\}} \middle| \hat{\tau}_k^i + \varsigma_k^i = r, \xi_k^i = x_0\right]^{\frac{1}{a}} \mathrm{d}y\,\mathrm{d}s \\ \le C_{a,\sigma} \int_0^{\varepsilon} \tilde{t}^{-\frac{1}{a}} x_0^{\frac{1}{a}} \tilde{x}^{\frac{1}{a}} e^{-\frac{c(\gamma-\varepsilon)^2}{4a\tilde{t}}} \mathrm{d}\tilde{x} \\ \le C_{a,\sigma} R^{\frac{1}{a}} \varepsilon^{1+\frac{1}{a}},\end{aligned} \tag{3.6.44}$$

where the ultimate line follows from the fact that for any fixed $z > 0$, the function $t \mapsto t^{-1}e^{-z^2/4t}$ is bounded by $4e^{-1}z^{-2}$ and $x_0 \in [\gamma, R]$. Inserting this upper bound into (3.6.41), we obtain

$$I_k \le C_{a,p,q,\sigma} R^{\frac{1}{a}} \varepsilon^{1+\frac{1}{a}} \mathbb{P}[\tau_k^i \le T]^{\frac{1}{a}}. \tag{3.6.45}$$

Therefore, inserting the upper bounds (3.6.40) and (3.6.45) into (3.6.31), we obtain the bound

$$\int_0^{t_0} \mathbb{E}\left[|\hat{b}_s^i| \mathbb{1}_{\{\hat{X}_s<0\}} |\partial_x u(s,\hat{X}_s)|\right] \mathrm{d}s \le C\varepsilon^{1+\frac{1}{a}} \left(t^{-\frac{1}{a}} \mathbb{E}[|X_0|]^{\frac{1}{a}} + \sum_{k\ge 1} \mathbb{P}[\tau_k^i \le T]^{\frac{1}{a}} \right). \tag{3.6.46}$$

Inserting the bounds (3.6.24), (3.6.29) and (3.6.46) into (3.6.23) and using Proposition 3.3.4 to control the sum of the hitting time probabilities, we obtain

$$\mathbb{P}\left[X_t^i \in S,\, X_t^i < 0\right] = O(t^{-\frac{\delta}{2}} \varepsilon^{1+\beta}) \tag{3.6.47}$$

for some $\delta \in (0,1]$, $\beta > 0$ and all $\varepsilon \ll 1$. This completes the proof. □

3.6.5 Control on the Rate of Firing– Proof of Proposition 3.3.7

Step 1:

We will first show that the proportion of particles that spike twice in a small interval

tends to 0 with the length of the interval uniformly in N for N large. To be more precise, given an interval $[t, t+h]$ and $\beta \in (0,1)$ we define

$$\tau(\beta) \coloneqq \inf\{s \in [t, t+h] \,:\, \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{J^i_s - J^i_{t-} \geq 2\}} \geq \beta\}, \qquad \inf\{\emptyset\} = +\infty.$$

We shall show for $\beta = h^{1/4}$, $N > h^{-1/2}$ and $p \geq 1$ there exists C_p such that $\mathbb{P}[\tau(\beta) \leq t+h] \leq C_p h^p$.

Now let $\mathcal{I}^{(2)} = \{i \in \{1, \dots, N\} \,:\, J^i_{t+h} - J^i_{t-} \geq 2\}$, i.e. the random set of particles that spike at least twice on $[t, t+h]$. For the rest of this step, we will be working on the event $\{\tau(\beta) \leq t+h\}$. By right-continuity of J^i, we have $\left|\mathcal{I}^{(2)}\right| \geq N\beta$. Choose $\mathcal{I}^{(2)}(\beta) \subset \mathcal{I}^{(2)}$ such that $N\beta \leq \left|\mathcal{I}^{(2)}(\beta)\right| < N\beta + 1$, e.g. $\mathcal{I}^{(2)}(\beta)$ is the first $\lceil N\beta \rceil$ particles to spike twice.

Now for any $i \in \mathcal{I}^{(2)}(\beta)$, when X^i spikes for the second time, the particle Z^i must have had $\sup_{s \leq t} Z^i_s \leq \sum_{\tilde{k}=1}^{k-1} \xi^i_{\tilde{k}}$ and $\sup_{s \leq t+h} Z^i_s \geq \sum_{\tilde{k}=1}^{k} \xi^i_{\tilde{k}}$ for some k (stochastic). Furthermore, as this is the second spike in $[t, t+h]$, the value of Z^i must have increased by at least γ compared with its value just before time t. Hence, $\gamma \leq \sup_{t \leq s \leq t+h} \left|Z^i_s - Z^i_{t-}\right|$. Therefore,

$$\begin{aligned}
\gamma &\leq \sup_{t \leq s \leq t+h} \left|Z^i_s - Z^i_{t-}\right| \\
&\leq \int_t^{t+h} \left|b(s, X^i_s, \nu^N_s, \mathfrak{f}^N_s)\right| \mathrm{d}s + \sup_{t \leq s \leq t+h} \left|\int_t^s \sigma(u, X^i_u)\, \mathrm{d}B^i_u\right| \\
&\leq Ch\left[1 + \sup_{s \leq t+h} \left|Z^i_s\right| + \frac{1}{N}\sum_{j=1}^{N} \sup_{s \leq t+h} \left|Z^j_s\right|\right] + \sup_{t \leq s \leq t+h} \left|\int_t^s \sigma(u, X^i_u)\, \mathrm{d}B^i_u\right|,
\end{aligned}$$

where the last inequality is due to the linear growth condition on b, see (3.6.3). Now summing over the particles in $\mathcal{I}^{(2)}(\beta)$ and dividing by N, we obtain

$$\begin{aligned}
\frac{\gamma\left|\mathcal{I}^{(2)}(\beta)\right|}{N} \leq \frac{1}{N}\sum_{i \in \mathcal{I}^{(2)}(\beta)} Ch\left[1 + \sup_{s \leq t+h} \left|Z^i_s\right| + \frac{1}{N}\sum_{j=1}^{N} \sup_{s \leq t+h} \left|Z^j_s\right|\right] \\
+ \frac{1}{N}\sum_{i \in \mathcal{I}^{(2)}(\beta)} \sup_{t \leq s \leq t+h} \left|\int_t^s \sigma(u, X^i_u)\, \mathrm{d}B^i_u\right|.
\end{aligned}$$

By using $N\beta \le \left|\mathcal{I}^{(2)}(\beta)\right| < N\beta + 1$ and the Cauchy-Schwarz inequality we have

$$\begin{aligned}
\gamma\beta \le Ch &\left(\frac{\left|\mathcal{I}^{(2)}(\beta)\right|}{N}\right)^{1/2} \left(\frac{1}{N}\sum_{i=1}^{N}\left[1 + \sup_{s\le t+h}\left|Z_s^i\right| + \frac{1}{N}\sum_{j=1}^{N}\sup_{s\le t+h}\left|Z_s^j\right|\right]^2\right)^{1/2} \\
&+ \left(\frac{\left|\mathcal{I}^{(2)}(\beta)\right|}{N}\right)^{1/2} \left(\frac{1}{N}\sum_{i=1}^{N}\sup_{t\le s\le t+h}\left|\int_t^s \sigma(u, X_u^i)\,\mathrm{d}B_u^i\right|^2\right)^{1/2} \\
\le Ch &\left(\beta + \frac{1}{N}\right)^{1/2} \left(\frac{1}{N}\sum_{i=1}^{N}\left[1 + \sup_{s\le t+h}\left|Z_s^i\right| + \frac{1}{N}\sum_{j=1}^{N}\sup_{s\le t+h}\left|Z_s^j\right|\right]^2\right)^{1/2} \\
&+ \left(\beta + \frac{1}{N}\right)^{1/2} \left(\frac{1}{N}\sum_{i=1}^{N}\sup_{t\le s\le t+h}\left|\int_t^s \sigma(u, X_u^i)\,\mathrm{d}B_u^i\right|^2\right)^{1/2}.
\end{aligned}$$

As $\beta = h^{1/4} \ge N^{-1/2}$, we have $1/(\beta N) \le N^{-1/2} \le 1$. Therefore, dividing the above by β, we obtain

$$\begin{aligned}
\gamma \le& Ch\beta^{-1/2} \left(\frac{1}{N}\sum_{i=1}^{N}\left[1 + \sup_{s\le t+h}\left|Z_s^i\right| + \frac{1}{N}\sum_{j=1}^{N}\sup_{s\le t+h}\left|Z_s^j\right|\right]^2\right)^{1/2} \\
&+ 2\beta^{-1/2} \left(\frac{1}{N}\sum_{i=1}^{N}\sup_{t\le s\le t+h}\left|\int_t^s \sigma(u, X_u^i)\,\mathrm{d}B_u^i\right|^2\right)^{1/2} \\
\le& Ch\beta^{-1/2} \left(1 + \frac{1}{N}\sum_{i=1}^{N}\sup_{s\le t+h}\left|Z_s^i\right|^2\right)^{1/2} \\
&+ 2\beta^{-1/2} \left(\frac{1}{N}\sum_{i=1}^{N}\sup_{t\le s\le t+h}\left|\int_t^s \sigma(u, X_u^i)\,\mathrm{d}B_u^i\right|^2\right)^{1/2},
\end{aligned}$$

where C becomes larger between inequalities. Hence, by Markov's inequality, we have shown

$$\begin{aligned}
\mathbb{P}\left[\tau(\beta) \le t + h\right] \le C^p h^p \gamma^p \beta^{-p/2} &\mathbb{E}\left[\left(1 + \frac{1}{N}\sum_{i=1}^{N}\sup_{s\le t+h}\left|Z_s^i\right|^2\right)^{p/2}\right] \\
&+ 2^p \gamma^p \beta^{-p/2} \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N}\sup_{t\le s\le t+h}\left|\int_t^s \sigma(u, X_u^i)\,\mathrm{d}B_u^i\right|^2\right)^{p/2}\right].
\end{aligned} \tag{3.6.48}$$

For $p \le 2$, by Jensen's inequality and Proposition 3.6.1,

$$\mathbb{E}\left[\left(1 + \frac{1}{N}\sum_{i=1}^{N}\sup_{s\le t+h}\left|Z_s^i\right|^2\right)^{p/2}\right] \le \left(\mathbb{E}\left[1 + \frac{1}{N}\sum_{i=1}^{N}\sup_{s\le t+h}\left|Z_s^i\right|^2\right]\right)^{p/2} \le C.$$

By Jensen's inequality and the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}\mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N}\sup_{t\le s\le t+h}\left|\int_t^s \sigma(u,X_u^i)\,\mathrm{d}B_u^i\right|^2\right)^{p/2}\right]&\\ \le\left(\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\sup_{t\le s\le t+h}\left|\int_t^s \sigma(u,X_u^i)\,\mathrm{d}B_u^i\right|^2\right]\right)^{p/2}&\\ \le Ch^{p/2}.&\end{aligned}$$

Similarly, for $p>2$, by Jensen's inequality,

$$\mathbb{E}\left[\left(1+\frac{1}{N}\sum_{i=1}^{N}\sup_{s\le t+h}\left|Z_s^i\right|^2\right)^{p/2}\right]\le C_p\mathbb{E}\left[1+\frac{1}{N}\sum_{i=1}^{N}\sup_{s\le t+h}\left|Z_s^i\right|^p\right]\le C.$$

By Jensen's and the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}\mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N}\sup_{t\le s\le t+h}\left|\int_t^s \sigma(u,X_u^i)\,\mathrm{d}B_u^i\right|^2\right)^{p/2}\right]&\\ \le\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\sup_{t\le s\le t+h}\left|\int_t^s \sigma(u,X_u^i)\,\mathrm{d}B_u^i\right|^p\right]&\\ \le Ch^{p/2}.&\end{aligned}$$

Therefore,

$$\mathbb{P}\left[\tau(\beta)\le t+h\right]\le C_p(h^p\beta^{-p/2}+h^{p/2}\beta^{-p/2})=C_p(h^{7p/8}+h^{3p/8}).$$

Now recall $\mathcal{I}^{(2)}=\{i\in\{1,\dots,N\}\,:\,J_{t+h}^i-J_{t-}^i\ge 2\}$. On the event $\{\tau(\beta)>t+h\}\cap\{N^{-1}\sum_{i=1}^N(J_{t+h}^i)^2\le h^{-1/8}\}$ we have $\left|\mathcal{I}^{(2)}\right|<N\beta$, therefore

$$\begin{aligned}F_{t+h}^N-F_{t-}^N=\frac{1}{N}\sum_{i=1}^{N}(J_{t+h}^i-J_{t-}^i)&=\frac{1}{N}\sum_{i\in(\mathcal{I}^{(2)})^{\complement}}(J_{t+h}^i-J_{t-}^i)+\frac{1}{N}\sum_{i\in\mathcal{I}^{(2)}}(J_{t+h}^i-J_{t-}^i)\\ &\le 1+\left(\frac{\left|\mathcal{I}^{(2)}\right|}{N}\right)^{1/2}\left(\frac{1}{N}\sum_{i=1}^{N}\left(J_{t+h}^i\right)^2\right)^{1/2}\\ &\le 1+\beta\cdot h^{-1/16}=1+h^{1/16}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{P}\left[F_{t+h}^N-F_{t-}^N>1+h^{1/16}\right]&\le\mathbb{P}\left[\tau(\beta)\le t+h\right]+\mathbb{P}\left[N^{-1}\sum_{i=1}^{N}(J_{t+h}^i)^2>h^{-1/8}\right]\\ &\le C_p(h^{7p/8}+h^{3p/8}+h^{p/8}).\end{aligned}$$

By first replacing p with $8p$ in the above and then employing the fact that $h\in(0,1)$, the result follows. □

3.6.6 Control on the Increments of the Firing Function – Proof of Proposition 3.3.8

By Proposition 3.3.7, the probability that a large proportion of neurons fire more than twice in a small time interval is small uniformly for all N large. Hence, we would like to gain some control over the number of neurons that fire provided a small proportion of them fire twice. As the p-th moments of the number of firings are uniformly bounded in N, we may exploit this using Markov's inequality to control the contribution to F^N by the neurons that fire twice or more. Lastly, to control the neurons that fire at most once, we employ the techniques in [31, Appendix A.2].

Step 1:

Choose $h < 1$, $t \in [0, T]$, let $N > h^{-1/2}$ and $\beta = h^{1/4}$. As in Proposition 3.3.7, we define the set $\mathcal{I}^{(2)} = \{i \in \{1, \dots, N\} : J^i_{t+h} - J^i_{t-} \geq 2\}$, i.e. the set of particles which jump twice on the interval $[t, t+h]$. Choose a deterministic set $\mathcal{I} \subset \{1, \dots, N\}$. On the event $A(\mathcal{I}) \coloneqq \{\mathcal{I}^{(2)} = \mathcal{I}\} \cap \{\left|\mathcal{I}^{(2)}\right| < N\beta\}$, we have

$$\begin{aligned} F^N_{t+h} - F^N_t &= \frac{1}{N}\sum_{i=1}^N J^i_{t+h} - J^i_t = \frac{1}{N}\sum_{i\in\mathcal{I}} J^i_{t+h} - J^i_t + \frac{1}{N}\sum_{i\in\mathcal{I}^{\complement}} J^i_{t+h} - J^i_t \\ &\leq \left(\frac{|\mathcal{I}|}{N}\right)^{1/2}\left(\frac{1}{N}\sum_{i\in\mathcal{I}}\left(J^i_{t+h}\right)^2\right)^{1/2} + \frac{1}{N}\sum_{i\in\mathcal{I}^{\complement}}\left(J^i_{t+h} - J^i_t\right)\mathbb{1}_{\{J^i_{t+h} - J^i_{t-} \leq 1\}} \\ &\leq \beta^{1/2}\left(\frac{1}{N}\sum_{i=1}^N\left(J^i_{t+h}\right)^2\right)^{1/2} + \frac{1}{N}\sum_{i=1}^N\left(J^i_{t+h} - J^i_t\right)\mathbb{1}_{\{J^i_{t+h} - J^i_t \leq 1\}} \end{aligned}$$

where we used the fact that $J^i_{t+h} - J^i_t \leq J^i_{t+h} - J^i_{t-}$ to drop the dependence on $t-$ in the indicator function in the last inequality. Therefore, by the above and the law of

total probability

$$
\begin{aligned}
&\mathbb{P}\left[F^N_{t+h} - F^N_t \geq 2\eta, \left|\mathcal{I}^{(2)}\right| < N\beta\right] \\
&= \sum_{\mathcal{I}\subset\{1,\dots,N\}} \mathbb{P}\left[F^N_{t+h} - F^N_t \geq 2\eta, A(\mathcal{I})\right] \\
&\leq \sum_{\mathcal{I}\subset\{1,\dots,N\}} \mathbb{P}\left[\beta^{1/2}\left(\frac{1}{N}\sum_{i=1}^N \left(J^i_{t+h}\right)^2\right)^{1/2} + \frac{1}{N}\sum_{i=1}^N \left(J^i_{t+h} - J^i_t\right)\mathbb{1}_{\{J^i_{t+h}-J^i_t\leq 1\}} \geq 2\eta, A(\mathcal{I})\right] \\
&\leq \sum_{\mathcal{I}\subset\{1,\dots,N\}} \mathbb{P}\left[\beta^{1/2}\left(\frac{1}{N}\sum_{i=1}^N \left(J^i_{t+h}\right)^2\right)^{1/2} \geq \eta, A(\mathcal{I})\right] \\
&\qquad + \sum_{\mathcal{I}\subset\{1,\dots,N\}} \mathbb{P}\left[\frac{1}{N}\sum_{i=1}^N \left(J^i_{t+h} - J^i_t\right)\mathbb{1}_{\{J^i_{t+h}-J^i_t\leq 1\}} \geq \eta, A(\mathcal{I})\right] \\
&\leq \frac{\beta^{1/2}}{\eta}\mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^N \left(J^i_{t+h}\right)^2\right)^{1/2}\right] + \mathbb{P}\left[\frac{1}{N}\sum_{i=1}^N \left(J^i_{t+h} - J^i_t\right)\mathbb{1}_{\{J^i_{t+h}-J^i_t\leq 1\}} \geq \eta\right]
\end{aligned}
$$

where the first term is bounded uniformly in N by Jensen's inequality and Proposition 3.6.1.

Step 2:

We now look to analyse $\mathbb{P}\left[\frac{1}{N}\sum_{i=1}^N \left(J^i_{t+h} - J^i_t\right)\mathbb{1}_{\{J^i_{t+h}-J^i_t\leq 1\}} \geq \eta\right]$ obtained in step 1. The rest of this proof shall follow similarly as in [31, Appendix A.2]. By fixing an $\varepsilon = \varepsilon(h)$ and $a = a(h)$, to be specified later, such that $\varepsilon(h) \to 0$ and $a(h)/h \to \infty$ as $h \to 0$, then by Markov's inequality, Corollary 3.3.5, and the uniform in N sub-Gaussianity of the particles, we have

$$
\begin{aligned}
\mathbb{P}\left[\nu^N_t(-\varepsilon, 0) \geq \eta/4\right] &= o(1) \qquad \text{as } h \to 0, \\
\mathbb{P}\left[\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\sup_{s\leq t+h}|Z^i_s|\geq \frac{a}{2h}} \geq \eta/4\right] &= o(1) \qquad \text{as } h \to 0,
\end{aligned}
$$

uniformly in N. Hence, we now look to control

$$
\mathbb{P}\left[\frac{1}{N}\sum_{i=1}^N \left(J^i_{t+h} - J^i_t\right)\mathbb{1}_{\{J^i_{t+h}-J^i_t\leq 1\}} \geq \eta,\, \nu^N_t(-\varepsilon, 0) < \eta/4, \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\sup_{s\leq t+h}|Z^i_s|\geq \frac{a}{2h}} < \eta/4\right]. \tag{3.6.49}
$$

First, we let E denote the event in (3.6.49), and we define two sets:

$$
\begin{aligned}
\mathcal{I} &= \left\{i \in \{1,\dots,N\}\,;\, X^i_t \leq -\varepsilon,\, \sup_{s\leq t+h}\left|Z^i_s\right| < \frac{a}{2h}\right\} \qquad \text{and} \\
\mathcal{I}^{\text{ref}} &= \left\{i \in \{1,\dots,N\}\,;\, X^i_t = 0,\, \sup_{s\leq t+h}\left|Z^i_s\right| < \frac{a}{2h}\right\}.
\end{aligned}
$$

$\mathcal{I}$ is the set of indices of the particles that are not too close to the boundary but also not too far away. $\mathcal{I}^{\text{ref}}$ represents the set of indices of the particles that are in their refractory period at time t. We observe that by the countable additivity of probability measures

$$\mathbb{P}[E] \leq \sum_{\substack{\mathcal{I}_0, \mathcal{I}_0^{\text{ref}} \subset \{1,\dots,N\} \\ \mathcal{I}_0 \cap \mathcal{I}_0^{\text{ref}} = \emptyset}} \mathbb{P}\left[E \cap \{\mathcal{I} = \mathcal{I}_0\} \cap \{\mathcal{I}^{\text{ref}} = \mathcal{I}_0^{\text{ref}}\}\right]. \tag{3.6.50}$$

We observe that $E \cap \{\mathcal{I} = \mathcal{I}_0\} \cap \{\mathcal{I}^{\text{ref}} = \mathcal{I}_0^{\text{ref}}\}$ is a subset of the event

$$\begin{aligned} &E \cap \{\mathcal{I} = \mathcal{I}_0\} \cap \{\mathcal{I}^{\text{ref}} = \mathcal{I}_0^{\text{ref}}\} \cap \Big\{\#\{i \in \mathcal{I}_0 : \sup_{t \leq s \leq t+h} X_s^i \geq 0, X_t^i \leq -\varepsilon\} \geq N\eta/4\Big\} \\ &\bigcup E \cap \{\mathcal{I} = \mathcal{I}_0\} \cap \{\mathcal{I}^{\text{ref}} = \mathcal{I}_0^{\text{ref}}\} \cap \Big\{\#\{i \in \mathcal{I}_0^{\text{ref}} : \sup_{\hat{\tau}_t^i \leq s \leq t+h} X_s^i \geq 0, \hat{\tau}^i \leq t+h\} \geq N\eta/4\Big\}, \end{aligned} \tag{3.6.51}$$

where $\hat{\tau}^i$ is the time that particle i leaves its refractory period, i.e.

$$\hat{\tau}^i := \inf\{s \in [t, t+h]\,;\, X_s^i < 0\} \qquad \text{with} \qquad \inf \emptyset = \infty.$$

Therefore, replacing the event (3.6.51) into (3.6.50) and employing the subadditivity of the probability measures, we deduce

$$\begin{aligned} \mathbb{P}[E] \leq &\sum_{\mathcal{I}_0 \subset \{1,\dots,N\}} \mathbb{P}\left[\#\{i \in \mathcal{I}_0 : \sup_{t \leq s \leq t+h} X_s^i \geq 0, X_t^i \leq -\varepsilon\} \geq N\eta/4 \,\middle|\, \mathcal{I} = \mathcal{I}_0\right] \\ &\qquad \times \mathbb{P}[\mathcal{I} = \mathcal{I}_0] \\ &+ \sum_{\mathcal{I}_0^{\text{ref}} \subset \{1,\dots,N\}} \mathbb{P}\left[\#\{i \in \mathcal{I}_0^{\text{ref}} : \sup_{\hat{\tau}_t^i \leq s \leq t+h} X_s^i \geq 0, \hat{\tau}^i \leq t+h\} \geq N\eta/4 \,\middle|\, \mathcal{I}^{\text{ref}} = \mathcal{I}_0^{\text{ref}}\right] \\ &\qquad \times \mathbb{P}\left[\mathcal{I}^{\text{ref}} = \mathcal{I}_0^{\text{ref}}\right]. \end{aligned}$$

Now, employing the fact that particles in their refractory period are reset to at least γ away from the boundary by Assumption 3.2.1 (v), we have

$$\begin{aligned} \mathbb{P}[E] \leq &\sum_{\mathcal{I}_0 \subset \{1,\dots,N\}} \mathbb{P}\left[\#\{i \in \mathcal{I}_0 : \sup_{t \leq s \leq t+h} \{X_s^i - X_t^i\} \geq \varepsilon\} \geq N\eta/4 \,\middle|\, \mathcal{I} = \mathcal{I}_0\right] \\ &\qquad \times \mathbb{P}[\mathcal{I} = \mathcal{I}_0] \\ &+ \sum_{\mathcal{I}_0^{\text{ref}} \subset \{1,\dots,N\}} \mathbb{P}\left[\#\{i \in \mathcal{I}_0^{\text{ref}} : \sup_{\hat{\tau}^i \leq s \leq t+h} \{X_s^i - X_{\hat{\tau}^i}^i\} \geq \varepsilon\} \geq N\eta/4 \,\middle|\, \mathcal{I}^{\text{ref}} = \mathcal{I}_0^{\text{ref}}\right] \\ &\qquad \times \mathbb{P}\left[\mathcal{I}^{\text{ref}} = \mathcal{I}_0^{\text{ref}}\right]. \end{aligned} \tag{3.6.52}$$

Now we will look to control each of the terms in (3.6.52) individually. First turning our attention to the first term on the R.H.S. of (3.6.52), we shall apply the scale transform as in Proposition 3.6.6 to remove the general diffusive term in front of the Brownian motion. We introduce $U^i_s = \Upsilon_{t+s}(X^i_{t+s} - X^i_t)$ and note that, as in Proposition 3.6.6 we have

$$\begin{aligned}
\mathrm{d}U^i_s &= u^i_s \mathbb{1}_{\{X^i_{t+s}<0\}}\,\mathrm{d}s + \mathbb{1}_{\{X^i_{t+s}<0\}}\sqrt{1-\rho^2_{t+s}}\,\mathrm{d}B^i_{t+s} + \mathbb{1}_{\{X^i_{t+s}<0\}}\rho_{t+s}\,\mathrm{d}B^0_{t+s} \\
&\quad + \mathrm{d}\sum_{k\geq 1}\Upsilon_{\tau^i_k+\varsigma^i_k}(-\xi^i_k)\mathbb{1}_{(t,t+s]}(\tau^i_k+\varsigma^i_k) \\
&=: u^i_s\,\mathrm{d}s + \mathrm{d}\tilde{I}^i_s + \mathrm{d}I_s + \mathrm{d}\sum_{k\geq 1}\Upsilon_{\tau^i_k+\varsigma^i_k}(-\xi^i_k)\mathbb{1}_{(t,t+s]}(\tau^i_k+\varsigma^i_k)
\end{aligned}$$

where $\rho_{t+s} = \rho(t+s, \nu^N_{t+s}, \mathfrak{f}^N_{t+s})$ and the drift satisfies $|u^i_s| \leq c_1(1+\hat{\Lambda}^i_{t+s})$. We observe by the monotonicity of Υ in its second component and lower bound of σ away from 0,

$$X^i_{t+s} - X^i_t \geq \varepsilon \implies \Upsilon_{t+s}(X^i_{t+s} - X^i_t) \geq \Upsilon_{t+s}(\varepsilon) \geq \varepsilon C^{-1}_\sigma \geq c_2\varepsilon$$

where the last inequality holds for any $C^{-1}_\sigma \geq c_2 > 0$. Therefore,

$$\sup_{t\leq s\leq t+h}\left\{X^i_s - X^i_t\right\} \geq \varepsilon \implies \sup_{t\leq s\leq t+h}\left\{\Upsilon_{t+s}(X^i_{t+s} - X^i_t)\right\} \geq c_2\varepsilon$$

It follows that we can bound the first term on the R.H.S. of (3.6.52) by

$$\sum_{\mathcal{I}_0\subset\{1,\ldots,N\}} \mathbb{P}\left[\#\{i\in\mathcal{I}_0 : \sup_{s\leq h}\left\{U^i_s\right\} \geq c_2\varepsilon\} \geq N\eta/2 \,\middle|\, \mathcal{I}=\mathcal{I}_0\right]\mathbb{P}\left[\mathcal{I}=\mathcal{I}_0\right]$$

Using the decomposition of U^i_s, the fact that $-\sum_{k\geq 1}\Upsilon_{\tau^i_k+\varsigma^i_k}(-\xi^i_k)\mathbb{1}_{(t,t+s]}(\tau^i_k+\varsigma^i_k) \geq 0$, and the growth estimate on the drift u^i_s,

$$\begin{aligned}
\tilde{I}^i_s &= U^i_s - \int_0^s u^i_v\,\mathrm{d}v - I_s - \sum_{k\geq 1}\Upsilon_{\tau^i_k+\varsigma^i_k}(-\xi^i_k)\mathbb{1}_{(t,t+s]}(\tau^i_k+\varsigma^i_k) \geq U^i_s - h\sup_{s\leq h}\left|u^i_s\right| - \sup_{s\leq h}|I_s| \\
&\geq U^i_s - c_1h(1+\hat{\Lambda}^{i,N}_{t+h}) - \sup_{s\leq h}|I_s| \geq U^i_s - c_1h(1+\sup_{s\leq t+h}\left|Z^i_s\right| + a/2h) - a
\end{aligned}$$

on the event

$$\left\{\frac{1}{N}\sum_{j=1}^N \sup_{s\leq t+h}\left|Z^j_s\right| < \frac{a}{2h}\right\} \cap \left\{\sup_{s\leq h}|I_s| < a\right\}.$$

Hence, on the same event, if $i\in\mathcal{I}_0$ such that $\sup_{s\leq h}U^i_s \geq c_2\varepsilon$ then

$$\sup_{s\leq h}\tilde{I}^i_s \geq \sup_{s\leq h}U^i_s - c_1h(1+a/h) - a \geq c_2\varepsilon - c_1h(1+a/h) - a = c_2\varepsilon - c_1h - a(1+c_1).$$

Therefore, by splitting up the probability on this event and its complement and using the fact that the largest set $\mathcal{I}_0$ may be is $\{1, \ldots, N\}$, it follows that we can bound the first term on the R.H.S. of (3.6.52) by

$$\mathbb{P}\left[\#\{i \leq N \,:\, \sup_{s\leq h} \tilde{I}^i_s \geq c_2\varepsilon - c_1 h - a(1+c_1)\} \geq N\eta/4\right] + \mathbb{P}\left[\frac{1}{N}\sum_{j=1}^{N} \sup_{s\leq t+h} \left|Z^j_s\right| \geq \frac{a}{2h}\right]$$
$$+ \mathbb{P}\left[\sup_{s\leq h} |I_s| > a\right].$$

By Markov's inequality then Burkholder-Davis-Gundy,

$$\mathbb{P}\left[\sup_{s\leq h} |I_s| > a\right] \leq a^{-2}\mathbb{E}\left[\sup_{s\leq h} I_s^2\right] \lesssim ha^{-2}.$$

Also, by Markov's inequality and uniform sub-Gaussianity of Z^i, we have

$$\mathbb{P}\left[\frac{1}{N}\sum_{j=1}^{N} \sup_{s\leq t+h} \left|Z^j_s\right| \geq \frac{a}{2h}\right] \lesssim ha^{-1},$$

with a constant that holds uniformly in N. Hence, as $\tilde{I}^i$ is a time-changed Brownian motion, we have

$$\begin{aligned}
\mathbb{P}[E] \leq\ & \mathbb{P}\left[\#\{i \leq N \,:\, \sup_{s\leq h} B^i_s \geq c_2\varepsilon - c_1 h - a(1+c_1)\} \geq N\eta/4\right] + O(ha^{-2} + ha^{-1}) \\
& + \sum_{\mathcal{I}_0^{\text{ref}} \subset \{1,\ldots,N\}} \mathbb{P}\left[\#\{i \in \mathcal{I}_0^{\text{ref}} \,:\, \sup_{\hat{\tau}^i \leq s \leq t+h} \left\{X^i_s - X^i_{\hat{\tau}^i}\right\} \geq \varepsilon\} \geq N\eta/4 \,\middle|\, \mathcal{I}^{\text{ref}} = \mathcal{I}_0^{\text{ref}}\right] \\
& \qquad \times \mathbb{P}\left[\mathcal{I}^{\text{ref}} = \mathcal{I}_0^{\text{ref}}\right] \\
=\ & \mathbb{P}\left[\frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{\sup_{s\leq h} B^i_s \geq c_2\varepsilon - c_1 h - a(1+c_1)\}} \geq \eta/4\right] + O(ha^{-2} + ha^{-1}) \\
& + \sum_{\mathcal{I}_0^{\text{ref}} \subset \{1,\ldots,N\}} \mathbb{P}\left[\#\{i \in \mathcal{I}_0^{\text{ref}} \,:\, \sup_{\hat{\tau}^i \leq s \leq t+h} \left\{X^i_s - X^i_{\hat{\tau}^i}\right\} \geq \varepsilon\} \geq N\eta/4 \,\middle|\, \mathcal{I}^{\text{ref}} = \mathcal{I}_0^{\text{ref}}\right] \\
& \qquad \times \mathbb{P}\left[\mathcal{I}^{\text{ref}} = \mathcal{I}_0^{\text{ref}}\right]
\end{aligned}$$

We now turn our attention to the second term on the R.H.S. of (3.6.52). We note that the same argument as above can be applied to this term. Therefore, we have

$$\begin{aligned}
\mathbb{P}[E] \leq & \mathbb{P}\left[\frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{\sup_{s\leq h} B^i_s \geq c_2\varepsilon - c_1 h - a(1+c_1)\}} \geq \eta/4\right] + O(ha^{-2} + ha^{-1}) \\
& + \mathbb{P}\left[\frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{\sup_{s\leq h} B^i_s \geq c_2\gamma - c_1 h - a(1+c_1)\}} \geq \eta/4\right]
\end{aligned}$$

Supposing $\varepsilon < \gamma$, then the law of large numbers and the distribution of the minimum of Brownian motion gives

$$\limsup_{N\to\infty} \mathbb{P}[E] \leq 2\mathbb{1}_{\{\phi(-c_2\varepsilon h^{-1/2}+c_1h^{1/2}+ah^{-1/2}(1+c_1))\geq\eta/4\}} + O(ha^{-2}+ha^{-1})$$

provided $c_2\varepsilon - c_1h - a(1+c_1) > 0$, where ϕ is the normal c.d.f. Making the choice

$$\varepsilon(h) = h^{1/2}\log(1/h) \quad \text{and} \quad a(h) = h^{1/2}\log\log(1/h)$$

ensures that

$$\varepsilon(h) \to 0, \qquad ha^{-1} \to 0, \qquad ha^{-2} \to 0,$$

and

$$-c_2\varepsilon h^{-1/2}+c_1h^{1/2}+ah^{-1/2}(1+c_1) = -c_2\log(1/h)+c_1h^{1/2}+\log\log(1/h)(1+c_1) \to -\infty$$

as $h \to 0$.

Step 3: To bring it all together we have, for h small enough

$$\begin{aligned}
&\limsup_{N\to\infty} \mathbb{P}\left[F^N_{t+h} - F^N_t \geq 2\eta\right] \\
&\qquad \leq \limsup_{N\to\infty} \mathbb{P}\left[F^N_{t+h} - F^N_t \geq 2\eta, \left|\mathcal{I}^{(2)}\right| < N\beta\right] + \mathbb{P}\left[\left|\mathcal{I}^{(2)}\right| \geq N\beta\right] \\
&\qquad \lesssim h^{1/8}\eta^{-1} + 2\mathbb{1}_{\{\phi(-c_2\varepsilon h^{-1/2}+c_1h^{1/2}+ah^{-1/2}(1+c_1))\geq\eta/4\}} + ha^{-2} + ha^{-1} \\
&\qquad + 4\eta^{-1}\limsup_{N\to\infty} \mathbb{E}\nu^N_t(-\varepsilon,\, 0) + h.
\end{aligned}$$

The right-hand side converges to 0 as h converges to 0. Hence, the claim has been shown. □

3.6.7 Regularity Properties of Limit Points

Proposition 3.6.10 (Asymptotic decorrelation)**.** *Let $\mu^{N,k} := N^{-1}\sum_{i=1}^N \delta_{\xi^i_k,\tau^i_k,\varsigma^i_k}$. Then $\{\mu^{N,k}\}_N$ is tight on $\mathcal{P}(\bar{\mathbb{R}}^3)$, where $\bar{\mathbb{R}}$ is the one-point compactification of $\mathbb{R}$. For any limit point, denoted by $\mu^{*,k}$, we have $\mu^{*,k} = \mathbb{P}[\xi \in \mathrm{d}x] \otimes \mu_2^{*,k}(\mathrm{d}y) \otimes \mathbb{P}[\varsigma \in \mathrm{d}z]$ almost surely.*

Proof. As $\bar{\mathbb{R}}$ is compact and the product of compact sets is compact in the product topology, $\bar{\mathbb{R}}^3$ is a compact metric space. Consequently, $\mathcal{P}(\bar{\mathbb{R}}^3)$ is a compact metric space. Trivially, $\{\mu^{N,k}\}_N$ is tight on $\mathcal{P}(\bar{\mathbb{R}}^3)$.

We fix a limit point, $\mu^{*,k}$, along a subsequence, denoted by N for simplicity. Let $\Pi^{*,k}$ denote the law of $\mu^{*,k}$ and $\pi_i : \bar{\mathbb{R}}^3 \to \bar{\mathbb{R}}$ be the projection onto the i^{th} component. Then by the Continuous Mapping Theorem, $\pi_i^{\#}\mu^{N,k} \implies \pi_i^{\#}\mu^{*,k}$ for every i. First, we

fix $i = 1$ and choose a function $f : \bar{\mathbb{R}} \to \mathbb{R}$, continuous and bounded. Let $G : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then by the Continuous Mapping Theorem,

$$\mathbb{E}\left[G(\langle \pi_1^{\#}\mu^{N,k}, f\rangle)\right] \to \mathbb{E}\left[G(\langle \pi_1^{\#}\mu^{*,k}, f\rangle)\right]. \tag{3.6.53}$$

By definition, $\mathbb{E}[G(\langle \pi_1^{\#}\mu^{N,k}, f\rangle)] = \mathbb{E}[G(N^{-1}\sum_{i=1}^{N} f(\xi_k^i))]$. Therefore, by the Law of Large Numbers, as ξ_k^i are i.i.d., and the Dominated Convergence Theorem,

$$\mathbb{E}\left[G(N^{-1}\sum_{i=1}^{N} f(\xi_k^i))\right] \to \mathbb{E}\left[G(\mathbb{E}[f(\xi)])\right] = G(\mathbb{E}[f(\xi)]). \tag{3.6.54}$$

Therefore, we have shown by (3.6.53) and (3.6.54),

$$\mathbb{E}\left[G(\langle \pi_1^{\#}\mu^{*,k}, f\rangle)\right] = G(\mathbb{E}[f(\xi)]). \tag{3.6.55}$$

As (3.6.55) holds for any continuous function G, by replacing G with $\tilde{G}(\cdot) = (\cdot - \mathbb{E}[f(\xi)])^2$, it follows from (3.6.55) that $\langle \pi_1^{\#}\mu^{*,k}, f\rangle = \mathbb{E}[f(\xi)]$ almost surely. Therefore, by employing a Dynkin's Lemma argument, we deduce $(\pi_1^{\#}\mu^{*,k})(\mathrm{d}x) = \mathbb{P}[\xi \in \mathrm{d}x]$. Similarly, we deduce $(\pi_3^{\#}\mu^{*,k})(\mathrm{d}z) = \mathbb{P}[\varsigma \in \mathrm{d}z]$.

It remains only to show that the limiting measure is a product measure. For every $a_1, a_2, a_3 : \bar{\mathbb{R}} \to \mathbb{R}$, continuous and bounded,

$$\begin{aligned}
&\mathbb{E}\left[\left|\langle \mu^{*,k}, a_1 \otimes a_2 \otimes a_3\rangle - \prod_{i=1}^{3}\langle \pi_i^{\#}\mu^{*,k}, a_i\rangle\right|^2\right] \\
&\quad = \mathbb{E}\left[\left|\langle \mu^{*,k}, a_1 \otimes a_2 \otimes a_3\rangle - \mathbb{E}[a_1(\xi)]\langle \pi_2^{\#}\mu^{*,k}, a_2\rangle\mathbb{E}[a_3(\varsigma)]\right|^2\right] \\
&\quad = \lim_{N\to\infty} \mathbb{E}\left[\left|\langle \mu^{N,k}, a_1 \otimes a_2 \otimes a_3\rangle - \mathbb{E}[a_1(\xi)]\langle \pi_2^{\#}\mu^{N,k}, a_2\rangle\mathbb{E}[a_3(\varsigma)]\right|^2\right] \\
&\quad = \lim_{N\to\infty} \mathbb{E}\left[\left|N^{-1}\sum_{i=1}^{N} a_1(\xi_k^i)a_2(\tau_k^i)a_3(\varsigma_k^i) - N^{-1}\sum_{i=1}^{N} a_2(\tau_k^i)\mathbb{E}[a_1(\xi)]\mathbb{E}[a_3(\varsigma)]\right|^2\right] \\
&\quad = \lim_{N\to\infty} N^{-2}\sum_{i,j=1}^{N} \mathbb{E}\left[a_2(\tau_k^i)a_2(\tau_k^j)\Delta_i\Delta_j\right],
\end{aligned}$$

where $\Delta_i \coloneqq a_1(\xi_k^i)a_3(\varsigma_k^i) - \mathbb{E}[a_1(\xi)]\mathbb{E}[a_3(\varsigma)]$. Recall that by construction, on the event that τ_k^j has not occurred yet, i.e. $\{\tau_k^i < \tau_k^j\}$, (ξ_k^j, ς_k^j) is independent of $(\tau_k^i, \tau_k^j, \xi_k^i, \varsigma_k^i)$ for $i \neq j$. Therefore,

$$\mathbb{E}\left[a_2(\tau_k^i)a_2(\tau_k^j)\Delta_i\Delta_j\mathbb{1}_{\{\tau_k^i<\tau_k^j\}}\right] = \mathbb{E}\left[a_2(\tau_k^i)a_2(\tau_k^j)\Delta_i\mathbb{1}_{\{\tau_k^i<\tau_k^j\}}\right]\mathbb{E}\left[\Delta_j\right] = 0. \tag{3.6.56}$$

The last equality follows from the independence between ξ_k^i and ς_k^i. Equation (3.6.56) holds true if we swap i and j. Hence,

$$\mathbb{E}\left[\left|\langle \mu^{*,k}, a_1 \otimes a_2 \otimes a_3\rangle - \prod_{i=1}^{3}\langle \pi_i^{\#}\mu^{*,k}, a_i\rangle\right|^2\right] = \lim_{N\to\infty} N^{-2}\sum_{i=1}^{N}\mathbb{E}\left[a_2(\tau_k^i)^2\Delta_i^2\right] = 0.$$

The last equality follows from the boundedness of the a_i's. Therefore, by the Monotone Class Theorem, we may conclude that $\mu^{*,k} = \mathbb{P}[\xi \in \mathrm{d}x] \otimes \mu_2^{*,k}(\mathrm{d}y) \otimes \mathbb{P}[\varsigma \in \mathrm{d}z]$ almost surely, where $\mu_2^{*,k} = \pi_2^{\#}\mu^{*,k}$. □

Proposition 3.6.11. *For every $a > 0$, we have uniformly in $N \geq 1$ and $t \in [0,T]$ that*

$$\mathbb{E}\langle \nu_t^N, |x|^k \mathbb{1}_{(-\infty,-\lambda]}(x)\rangle = o(\lambda^k e^{-a\lambda}) \quad \text{and} \quad \mathbb{E}\int_0^T \langle \nu_t^N, |x|^k \mathbb{1}_{(-\infty,-\lambda]}(x)\rangle \,\mathrm{d}t = o(\lambda^k e^{-a\lambda}). \tag{3.6.57}$$

Furthermore, for any process ν that satisfies Assumption 3.2.2, we have

$$\mathbb{E}\int_0^T \langle \nu_t, |x|^k \mathbb{1}_{(-\infty,-\lambda]}(x)\rangle \,\mathrm{d}t = o(\lambda^k e^{-a\lambda}). \tag{3.6.58}$$

Proof. We are in an analogous setting to that of [31]. The proof follows along the same lines as [31, Lemma A.3] with the obvious changes. □

Proposition 3.6.12. *Suppose ν^N converges weakly towards ν^* on $(D_{\mathscr{S}'}, M_1)$. Then for any polynomial f, we have that $t \mapsto \langle \nu_t^*, f\rangle$ has paths in $D_{\mathbb{R}}$ almost surely. Furthermore, $\mathbb{E}\sup_{t\leq T}\langle \nu_t^*, |x|^p \mathbb{1}_{\{|x|>\lambda\}}\rangle = o(e^{-a\lambda})$ as $\lambda \to \infty$, for any $a > 0$.*

Proof. Without loss of generality, it suffices to show the claim for $f(x) = x^p$ for some $p \in \mathbb{N}$. To begin, for R, $n \in \mathbb{N}$ we choose $\chi_R, \phi_n \in \mathcal{C}^\infty([0,1])$ such that

$$\chi_R = \begin{cases} 1 & \text{on } [-R,R], \\ 0 & \text{on } (-R-1,R+1)^{\complement}, \end{cases} \quad \text{and} \quad \phi_n = \begin{cases} 1 & \text{on } [-2n,-R-1/n], \\ 0 & \text{on } (-2n-1,-R)^{\complement}. \end{cases}$$

Here, χ_R is a standard cut-off function and ϕ_n approximates $\mathbb{1}_{\{|x|>R\}}$ pointwise. Then for any $s,t \in [0,T]$,

$$\begin{aligned} |\langle \nu_t^*, x^p\rangle - \langle \nu_s^*, x^p\rangle| \leq{}& |\langle \nu_t^*, x^p\rangle - \langle \nu_t^*, x^p\chi_R\rangle| + |\langle \nu_s^*, x^p\rangle - \langle \nu_s^*, x^p\chi_R\rangle| \\ &+ |\langle \nu_t^*, x^p\chi_R\rangle - \langle \nu_s^*, x^p\chi_R\rangle| \\ ={}& I + II + III. \end{aligned}$$

The strategy will be to show that with probability one, we can choose a (stochastic) R, independent of t and s, such that we may make I and II as small as we

like. Therefore, as $t \mapsto \langle \nu_t^*, x^p \chi_R \rangle$ has càdlàg paths, the claim follows. By definition of χ_R, $|\langle \nu_t^*, x^p \rangle - \langle \nu_t^*, x^p \chi_R \rangle| \leq 2\langle \nu_t^*, |x|^p \mathbb{1}_{\{|x|>R\}} \rangle$. Therefore, we aim to gain control over $\sup_{t\leq T} \langle \nu_t^*, |x|^p \mathbb{1}_{\{|x|>R\}} \rangle$. To this end, by the definition of ϕ_n, we have $\lim_{n\to\infty} \sup_{t\leq T} \langle \nu_t^*, |x|^p \phi_n \rangle = \sup_{t\leq T} \langle \nu_t^*, |x|^p \mathbb{1}_{\{|x|>R\}} \rangle$. Therefore, by applying Fatou's Lemma and then the Portmanteau Theorem,

$$\mathbb{E} \sup_{t\leq T} \langle \nu_t^*, |x|^p \mathbb{1}_{\{|x|>R\}} \rangle \leq \liminf_{n\to\infty} \mathbb{E} \sup_{t\leq T} \langle \nu_t^*, |x|^p \phi_n \rangle = \liminf_{n\to\infty} \limsup_{N\to\infty} \mathbb{E} \sup_{t\leq T} \langle \nu_t^N, |x|^p \phi_n \rangle. \tag{3.6.59}$$

Furthermore, we may bound the last term in (3.6.59) by

$$\begin{aligned} \mathbb{E} \sup_{t\leq T} \langle \nu_t^N, |x|^p \phi_n \rangle &\leq \mathbb{E} \left[\sup_{t\leq T} N^{-1} \sum_{i=1}^N \left| X_t^i \right|^p \mathbb{1}_{\{|X_t^i|>R\}} \right] \\ &\leq N^{-1} \sum_{i=1}^N \mathbb{E} \left[\sup_{t\leq T} \left| X_t^i \right|^p \mathbb{1}_{\{\sup_{t\leq T}|X_t^i|>R\}} \right], \end{aligned}$$

where the first inequality follows from the definition of ν^N and ϕ_n. By Corollary 3.6.5, $\mathbb{E}[\sup_{t\leq T} |X_t^i|^p]$ is uniformly bounded in i and N for every p (fixed). Therefore, by applying Hölder's inequality to the final expression above and then employing Corollary 3.6.5, we deduce there exist $C, \delta > 0$, independent of N, R and i, such that $\mathbb{E}[\sup_{t\leq T} |X_t^i|^p \mathbb{1}_{\{\sup_{t\leq T}|X_t^i|>R\}}] \leq C \exp\{-\delta R^2\}$. Hence, we have shown $\mathbb{E} \sup_{t\leq T} \langle \nu_t^*, |x|^p \mathbb{1}_{\{|x|>R\}} \rangle \leq C \exp\{-\delta R^2\}$. This proves the second claim. By the Borel-Cantelli Lemma,

$$\mathbb{P} \left[\sup_{t\leq T} \langle \nu_t^*, |x|^p \mathbb{1}_{\{|x|>R\}} \rangle > \exp\{-\delta R^2/2\} \quad \text{i.o.} \right] = 0.$$

Hence, the set

$$\Omega^* := \bigcup_{R\in\mathbb{N}} \bigcap_{\substack{\tilde{R}\geq R \\ \tilde{R}\in\mathbb{N}}} \left\{ \sup_{t\leq T} \langle \nu_t^*, |x|^p \mathbb{1}_{\{|x|>R\}} \rangle \leq \exp\{-\delta R^2/2\} \right\}$$

has full measure. Now, for any $\omega \in \Omega^*$ and $\varepsilon > 0$, we may choose an R large enough such that $I + II < \varepsilon$. Furthermore, for any fixed R, $t \mapsto \langle \nu_t^*, x^p \chi_R \rangle$ has càdlàg paths. Combining these two facts, it is routine to show that $t \mapsto \langle \nu_t^*, x^p \rangle(\omega)$ is càdlàg. □

Chapter 4

Global Solvability of a Networked Integrate-And-Fire Model: From Smooth to Singular Feedback Through Hitting Times

4.1 Introduction

The central object of study in this chapter is the limiting behaviour of the family of McKean–Vlasov stochastic differential equations (SDEs) with jumps:

$$\begin{cases} X_t^\varepsilon = \int_0^t b(s, X_s^\varepsilon, \nu_s^\varepsilon)\mathbb{1}_{\{X_s^\varepsilon<0\}}\,\mathrm{d}s + \int_0^t \sigma(s,\, X_s^\varepsilon)\sqrt{1-\rho^2(s,\nu_s^\varepsilon)}\mathbb{1}_{\{X_s^\varepsilon<0\}}\,\mathrm{d}W_s \\ \qquad + \int_0^t \sigma(s,\, X_s^\varepsilon)\rho(s,\nu_s^\varepsilon)\mathbb{1}_{\{X_s^\varepsilon<0\}}\,\mathrm{d}W_s^0 + \int_0^t \alpha(s)\mathbb{1}_{\{X_s^\varepsilon<0\}}\,\mathrm{d}\mathfrak{F}_s^\varepsilon \\ \qquad - \sum_{k\geq 1}\mathbb{1}_{[0,t]}(\tau_k^\varepsilon + \varsigma_k^\varepsilon), \\ \tau_k^\varepsilon = \inf\{t > \tau_{k-1}^\varepsilon + \varsigma_{k-1}^\varepsilon \;:\; X_t^\varepsilon \geq 0\}, \qquad \tau_0^\varepsilon = 0, \\ F_t^\varepsilon = \sum_{k\geq 1}\mathbb{P}\left[\tau_k^\varepsilon \leq t \middle| W^0\right], \; \mathfrak{F}_t^\varepsilon = \left(\mathfrak{K}^\varepsilon * F^\varepsilon\right)_t = \int_0^t \mathfrak{K}^\varepsilon(t-s)F_s^\varepsilon\,\mathrm{d}s, \\ \boldsymbol{\nu}_t^\varepsilon = \mathbb{P}[X_t^\varepsilon \in \cdot, X_t^\varepsilon < 0 | W^0], \end{cases} \tag{4.1.1}$$

where $X_t^\varepsilon \leq 0$ almost surely. Here, W and W^0 are independent standard Brownian motions, $\mathfrak{K}^\varepsilon$ is a rescaled mollifier that converges to the Dirac delta, and ς_k^ε converges to zero as ε converges to zero. The precise conditions on the coefficients are provided in Assumption 4.2.2 below.

Motivated by its origin as the limit of a particle system, W is referred to as the *idiosyncratic noise* (of a representative particle) and W^0 as the *common noise*. Also for this reason, we say X^ε has *spiked* whenever it reaches its *firing threshold* at zero at time τ_k^ε for some k. After waiting for a duration of ς_k^ε at 0, X^ε resets to -1. The function F_t^ε, referred to as the firing function, represents the *theoretical* expected number of threshold crossings up to time t, conditional on the common noise.

In essence, X^ε models the voltage of a prototypical neuron in a large-scale system with an infinite number of neurons. It is an extension of the noisy integrate-and-fire neuron model introduced in [50, 58]. At its core, in the simplest setting, the noisy integrate-and-fire neuron model aims to capture two main features of neuronal dynamics:

- Whenever X_t^ε reaches its *firing threshold*, it resets to its resting potential. The firing threshold is a predefined voltage level that, once reached or exceeded, triggers a neural action known as a *spike*. This event models the real-life process where a neuron, upon reaching a certain electrical potential, fires an electrical signal through the nervous system.

- When a spike occurs, all neurons in the system receive an impulse proportional to the proportion of neurons that have spiked. This reflects the biological phenomenon where a single neuron's firing can slightly alter the state of numerous other neurons, contributing to the overall dynamics of the neuronal network.

The equation presented in (4.1.1) expands on these fundamental principles by incorporating additional features:

- A refractory period: After a spike, a neuron enters an absolute refractory period of approximately 2 ms, during which it cannot fire again [52]. This is captured by the ς_k^ε term in (4.1.1). When X^ε is at zero, it is deactivated and unable to spike again until this period has elapsed.

- Delayed impulse transmission: Due to the complex structure of neurons, particularly the dendritic tree, there is a natural delay in impulse transmission. This delay is represented by the convolution term $\mathfrak{K}^\varepsilon * F^\varepsilon$ in (4.1.1). Unlike simpler models that assume instantaneous transmission, this approach accounts for the time taken for a signal to travel from the dendritic tree to the soma. Bressloff and Coombes [4] argue that a neuron's response depends on the history of its input due to the diffusive spread of activity along the dendritic tree. Inglis and Talay [34] rigorously show, using the cable equation, that this convolution is the natural way to represent this delay. They emphasize that in reality, it takes time for the arrival of an action potential at a synapse to be felt at the soma, as the signal is transmitted along the dendritic tree.

- Common noise: This is a natural addition to these models and can originate from various sources. In [6, 5], it arises from neurons forming a highly connected

network where the common noise represents shared connectivity. Kruscha and Lindner [40] consider a finite system of uncoupled integrate-and-fire neurons with common noise to model the synchrony induced by common stimuli due to overlapping receptive fields. Such populations of uncoupled neurons with overlapping receptive fields can be found in auditory nerve cells [37], olfactory receptor neurons in insects [72, 25], and electroreceptors in weakly electric fish [53].

From a modelling perspective, the smoothing naturally captures the latency in real-world systems. However, in scenarios where the timescales of the refractory period and the kernel are negligible compared to the timescales of interest, it is natural to consider the limiting behaviour of the system. This motivates why we want to take $\varepsilon \to 0$ and investigate the limiting system's behaviour.

We fix a kernel $\mathfrak{K}$ and rescale it with a variable $\varepsilon > 0$, defining $\mathfrak{K}^\varepsilon(t) \coloneqq \varepsilon^{-1}\mathfrak{K}(\varepsilon^{-1}t)$. For sufficiently small ε, the integral $\mathfrak{F}_t^\varepsilon = \int_0^t \mathfrak{K}^\varepsilon(t-s)F_s^\varepsilon \,\mathrm{d}s$ is approximately equal to F_t^ε. Furthermore, as ς_k^ε converges to zero when ε approaches zero, the time spent at the boundary becomes negligible compared to the time spent in the interior. Consequently, by considering the *continuous reformulation* of X^ε, given by $X^\varepsilon + \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^\varepsilon + \varsigma_k^\varepsilon)$, we find that this reformulation converges weakly in the M_1-topology. We use this limit point to construct a (relaxed) solution to the system where the feedback is instantaneous and the refractory period is absent:

$$\begin{cases} \mathrm{d}X_t = b(t, X_t, \boldsymbol{\nu}_t)\,\mathrm{d}t + \sigma(t, X_t)\sqrt{1-\rho(t,\boldsymbol{\nu}_t)^2}\,\mathrm{d}W_t + \sigma(t,X_t)\rho(t,\boldsymbol{\nu}_t)\,\mathrm{d}W_t^0 \\ \qquad\qquad +\alpha(t)\,\mathrm{d}F_t - \mathrm{d}J_t, \\ \tau_k = \inf\{t > \tau_{k-1} \;:\; X_{t-} + \alpha(t)\Delta F_t \geq 0\}, \qquad \tau_0 = 0, \\ J_t = \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k), \\ \mathbf{P} = \mathbb{P}\left[X + J \in \cdot \mid W^0, \mathbf{P}\right], \quad \boldsymbol{\nu}_t = \mathbb{P}\left[X_t \in \cdot \mid W^0, \mathbf{P}\right], \\ F_t = \sum_{k\geq 1} \mathbb{P}\left[\tau_k \leq t \mid W^0, \mathbf{P}\right]. \end{cases} \tag{4.1.2}$$

This system exhibits two sources of discontinuities: resets when X hits the boundary at 0, and potential jumps in the firing function F. In the absence of common noise and under suitable assumptions on the coefficients, there exists a distribution X_0 such that a blow-up occurs in finite time [9], that is $F_t' = +\infty$ for some $t \geq 0$. Moreover, using probabilistic methods, it can be shown that if $\alpha \geq -2\mathbb{E}X_0$, then F_t cannot be continuous for all time [30, Theorem 1.1]. The presence of these two sources of discontinuities in (4.1.2) motivates our use of the continuous reformulation.

The J_1-topology might have sufficed to prove convergence of X^ε to X if F_t were guaranteed to be continuous. However, since jump discontinuities in F_t cannot be

ruled out, we employ the M_1-topology instead. This topology is rich enough to facilitate the convergence of continuous functions to those with jumps, making it crucial for establishing the convergence of the continuous reformulation of the family of processes X^ε.

Intuitively, (4.1.1) is a smoothed approximation to (4.1.2). From a mathematical perspective, the advantage of smoothing out the interactions and adding the refractory period is that (4.1.1) is well-posed (see Theorem 3.2.6). In contrast, for the formulation (4.1.2) with instantaneous feedback, various questions concerning existence in the setting with common noise remain open. Moreover, a general theory on uniqueness remains a completely open problem. We will discuss these issues of existence and uniqueness of solutions to (4.1.2) in more detail later. Notably, a by-product of our technique is that we obtain the existence of solutions in a setting not previously considered.

4.1.1 Related Literature

Variants of (4.1.2) have been extensively studied. In the simplest scenario, without common noise, where b is a function of space only, σ equals 1, and α is a positive constant, (4.1.2) becomes

$$\begin{cases} \mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + \alpha\,\mathrm{d}F_t - \mathrm{d}J_t, \\ \quad \tau_k = \inf\{t > \tau_{k-1} \ : \ X_{t-} + \alpha\Delta F_t \geq 0\}, \qquad \tau_0 = 0, \\ \quad J_t = \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k), \quad F_t = \sum_{k\geq 1} \mathbb{P}\left[\tau_k \leq t\right]. \end{cases} \tag{4.1.3}$$

This equation has been well-studied in the literature from both probabilistic [15, 16] and Partial Differential Equation (PDE) [9, 11] perspectives.

In contrast with classical McKean–Vlasov equations, which are characterised by the dependence of the law of the process in the coefficients, the dependence in this model is through the feedback mechanism F. Depending on the value of $\alpha \geq 0$, F pushes X towards zero, causing the value of F to increase, which in turn pushes X even closer to 0. Consequently, equations of this type, (4.1.2) and (4.1.3), are non-trivial to solve as the non-linearity in the equation is not standard for the application of classical McKean–Vlasov theory [54, 66]. In fact, the dependence on the law is incredibly singular, as it has been proven in [15] that $F_t' = -(1/2)\partial_y p(t,0)$, where $p(t,y)$ is the marginal density of X_t.

The condition $\tau_k = \inf\{t > \tau_{k-1} \ : \ X_{t-} + \alpha(t)\Delta F_t \geq 0\}$ ensures that X is càdlàg, where $\Delta F_t \coloneqq F_t - F_{t-}$ is the jump of F at time t. If ΔF_t is zero, then this condition is equivalent to saying that X spikes at the first time it touches 0; that is, X spikes

if and only if $X_{t-} \geq 0$. However, when F exhibits jumps, X may spike even if it is negative at time t. This is due to the self-interaction described above and the instantaneous feedback mechanism. In this case, after a spike, i.e. $X_{t-} + \alpha\Delta F_t \geq 0$, we have $X_t = X_{t-} - 1 + \alpha\Delta F_t$. This quantity may be positive, and hence X will spike again at time t. Such behaviour is deemed to be non-physical. To address this issue, motivated by the model's origins in particle systems, two conditions are imposed: firstly, $\alpha < 1$, and secondly, we look for solutions that satisfy the *physical jump condition* defined as:

$$\Delta F_t = \inf\{x \geq 0 \,:\, \boldsymbol{\nu}_{t-}([-\alpha x, 0]) < x\}\,. \tag{4.1.4}$$

Solutions where F satisfies (4.1.4) are called *physical* solutions.

Delarue et al. [15] show that for any deterministic initial condition $x_0 < 0$, then for all $\alpha > 0$ sufficiently small, (4.1.3) is well-posed, given suitable assumptions on the coefficient b. Using a Banach Fixed Point Theorem argument, they prove that the firing function F is continuously differentiable and that their solution is unique within the class of solutions with continuously differentiable firing functions F. In [16], this result is extended by means of a particle system approach to show that there is a physical solution for any $\alpha < 1$, provided the initial condition X_0 is L^p-integrable and its law has support that is bounded away from the boundary. Inglis and Talay [34] showed the existence of solutions in the setting where the drift is a function of both space and time, the diffusion coefficient depends only on space, and the feedback is smoothed by a kernel $\mathfrak{K}$.

Also in the probabilistic setting, several studies have focused on the finite N-particle system with common noise, described by:

$$X_t^i = \int_0^t (\lambda - X_s^i)\,\mathrm{d}s + \sigma\sqrt{1-\rho^2}W_t^i + \rho W_t^0 - J_t^i,$$

where J_t^i increases by 1 whenever X_t^i reaches its firing threshold. This model has been explored in various contexts. Kruscha and Lindner [41] investigated the probability distribution of synchronous spikes in a homogeneous neural population driven by a common time-dependent stimulus, providing insights into the collective behaviour of neurons under shared stimuli. Building on this work, Kruscha and Lindner [40] focused on synchrony induced by common stimuli due to overlapping receptive fields, rather than direct coupling between neurons. Doiron et al. [20] further extended the model by incorporating a refractory period and neuron interactions through a

smoothed firing function. Their theoretical analysis of a stochastic leaky integrate-and-fire neuron network with delayed feedback revealed that spatial correlations induce network oscillations, with the oscillation strength proportional to the degree of stimulus correlation at constant total stimulus power.

In the PDE setting, Cáceres, Carrillo, and Perthame [9] and Carrillo et al. [11], considered the non-linear Fokker–Planck equation corresponding to (4.1.3) with a linear drift. They analysed the equation

$$\partial_t p_t + \partial_x(-\lambda x + \alpha F_t' p_t) - \frac{1}{2}\partial_{xx} p_t = \delta_{-1} F_t'.$$

They proved that for certain parameter choices, no global-in-time solution exists. Specifically, when $\alpha > 0$, there exists an initial density for which a blow-up must occur, meaning $F_t' = +\infty$ at some finite time. Conversely, when $\alpha \leq 0$, they established the existence of a unique solution that does not blow up in finite time.

4.1.2 Outline of the Chapter

The remainder of this chapter is structured as follows. In Section 4.2, we outline the main results of this chapter. We state the assumptions on the coefficients in Assumption 4.2.2 and present the main result, Theorem 4.2.3. This theorem establishes that, in a suitable sense, we have weak convergence of the system in (4.1.1) towards (4.1.2) as $\varepsilon \to 0$.

In Section 4.3, we prove tightness of the smoothed system and hence the existence of a limit point. In Section 4.4 we prove continuity of the hitting times, allowing us to conclude that the smoothed firing function converges to the limiting firing function. Finally, in Section 4.5, we prove that the limit point solves the system (4.1.2).

For presentational purposes, we have deferred the proofs of some technical results to Section 4.6. The proofs in Section 4.2 to Section 4.5 will refer to these results as needed.

4.2 Main results

Fix a finite time horizon $T > 0$, and let $\mathcal{P}(A)$ denote the set of probability measures on a measurable space $(A, \mathcal{A})$. When A is a metric space, $\mathcal{B}(A)$ denotes the Borel σ-algebra. Let $\mathbf{M}_{\leq 1}(A)$ denote the space of sub-probability measures, which we shall endow with the topology of weak convergence. For any interval I and metric space X, let $C(I, X)$ denote the space of continuous functions from I to X. Similarly, $D(I, X)$

denotes the space of càdlàg functions from I to X. We shall employ the shorthand notation $\mathcal{C}_X$ and D_X for $C(I,X)$ and $D(I,X)$ when the interval I is clear.

For every $\varepsilon > 0$, we fix a probability space $(\Omega^\varepsilon, \mathcal{F}^\varepsilon, \mathbb{P}^\varepsilon)$ that supports two independent Brownian motions. To simplify the notation, we will denote these Brownian motions by W and W^0; however, it is important to note that they may not be equal for different values of ε. Similarly, we adopt the simplified notations $\mathbb{P}$ and $\mathbb{E}$ to refer to $\mathbb{P}^\varepsilon$ and the expectation under the measure $\mathbb{P}^\varepsilon$, respectively. In this section, we characterise the weak limit of the system given by the following equation as ε tends to zero:

$$\begin{cases} X_t^\varepsilon = \int_0^t b(s, X_s^\varepsilon, \nu_s^\varepsilon)\mathbb{1}_{\{X_s^\varepsilon<0\}}\,\mathrm{d}s + \int_0^t \sigma(s,\, X_s^\varepsilon)\sqrt{1-\rho^2(s,\nu_s^\varepsilon)}\mathbb{1}_{\{X_s^\varepsilon<0\}}\,\mathrm{d}W_s \\ \qquad + \int_0^t \sigma(s,\, X_s^\varepsilon)\rho(s,\nu_s^\varepsilon)\mathbb{1}_{\{X_s^\varepsilon<0\}}\,\mathrm{d}W_s^0 + \int_0^t \alpha(s)\mathbb{1}_{\{X_s^\varepsilon<0\}}\,\mathrm{d}\mathfrak{F}_s^\varepsilon \\ \qquad - \sum_{k\geq 1}\mathbb{1}_{[0,t]}(\tau_k^\varepsilon + \varsigma_k^\varepsilon), \\ \tau_k^\varepsilon = \inf\{t > \tau_{k-1}^\varepsilon + \varsigma_{k-1}^\varepsilon \;:\; X_t^\varepsilon \geq 0\}, \qquad \tau_0^\varepsilon = 0, \\ F_t^\varepsilon = \sum_{k\geq 1}\mathbb{P}\left[\tau_k^\varepsilon \leq t \middle| W^0\right], \; \mathfrak{F}_t^\varepsilon = (\mathfrak{K}^\varepsilon * F^\varepsilon)_t = \int_0^t \mathfrak{K}^\varepsilon(t-s)F_s^\varepsilon\,\mathrm{d}s, \\ \boldsymbol{\nu}_t^\varepsilon = \mathbb{P}[X_t^\varepsilon \in \cdot, X_t^\varepsilon < 0 | W^0], \end{cases} \tag{4.2.1}$$

where $t \in [0,T]$. The coefficients b, σ, ρ, and α are measurable maps from $[0,T]\times\mathbb{R}\times\mathbf{M}_{\leq 1}(\mathbb{R})$, $[0,T]\times\mathbb{R}$, $[0,T]\times\mathbf{M}_{\leq 1}(\mathbb{R})$, and $[0,T]$, respectively, into $\mathbb{R}$.

The initial condition, denoted by X_{0-}, is assumed to be independent of the Brownian motions and negative almost surely. We define $\mathfrak{K}^\varepsilon(t) \coloneqq \varepsilon^{-1}\mathfrak{K}(t\varepsilon^{-1})$, where $\mathfrak{K}$ is a mollifier satisfying Assumption 4.2.2 below. Lastly, $(\varsigma_k^\varepsilon)_k$ is a sequence of i.i.d. random variables which are independent of the Brownian motions W^0 and W and the initial condition X_{0-}. The random variables ς_k^ε are such that for each k, ς_k^ε converges weakly to 0 as $\varepsilon \to 0$.

As shown in Chapter 3, (4.2.1) can be viewed as the mean-field limit of an interacting particle system. The kernel $\mathfrak{K}^\varepsilon$ smooths the interactions between particles, while the random variables ς_k^ε represent the time when the system is deactivated. As ε decreases, the time-scales over which the full impact of inter-particle interactions is felt decrease, and the duration for which a particle remains deactivated diminishes. Intuitively, as ε converges to zero, we expect X^ε to converge to a system where interactions are felt instantaneously and the system never enters a state of deactivation.

In other words, one would expect convergence to the following system:

$$\begin{cases} \mathrm{d}X_t = b(t, X_t, \boldsymbol{\nu}_t)\,\mathrm{d}t + \sigma(t, X_t)\sqrt{1-\rho(t,\boldsymbol{\nu}_t)^2}\,\mathrm{d}W_t + \sigma(t, X_t)\rho(t,\boldsymbol{\nu}_t)\,\mathrm{d}W_t^0 \\ \qquad\quad +\alpha(t)\,\mathrm{d}F_t - \mathrm{d}J_t, \\ \tau_k = \inf\{t > \tau_{k-1} \;:\; X_{t-} + \alpha(t)\Delta F_t \geq 0\}, \qquad \tau_0 = 0, \\ J_t = \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k), \\ \boldsymbol{\nu}_t = \mathbb{P}\left[X_t \in \cdot \,|\, W^0\right], \\ F_t = \sum_{k\geq 1} \mathbb{P}\left[\tau_k \leq t \mid W^0\right], \end{cases} \tag{4.2.2}$$

with $X_0 = X_{0-} + \alpha(0)F_0$. In this system, the feedback from the firing function, F, is instantaneous, and the system never enters a deactivated state.

The goal of this section is to rigorously establish this convergence. We employ the techniques developed in Chapter 2 to show that we have weak convergence. However, there is no guarantee that the limiting process will be W^0-measurable. Therefore, we relax our notion of solution to (4.2.2). Our definition below is analogous to the notion of *relaxed solutions* employed in the literature when studying the mean-field limit of particle systems with common noise [48], and also in the mean-field game literature with common source of noise [10].

Definition 4.2.1 (Relaxed solutions)**.** Let the coefficient functions b, σ, ρ, and α be given along with the initial condition X_{0-} at time $t = 0-$. We define a relaxed solution to (4.2.2) as a family $(X,\, W,\, W^0,\, \mathbf{P})$ on a filtered probability space $(\Omega,\, \mathcal{F},\, \mathbb{P})$ such that

$$\begin{cases} \mathrm{d}X_t = b(t, X_t, \boldsymbol{\nu}_t)\,\mathrm{d}t + \sigma(t, X_t)\sqrt{1-\rho(t,\boldsymbol{\nu}_t)^2}\,\mathrm{d}W_t + \sigma(t, X_t)\rho(t,\boldsymbol{\nu}_t)\,\mathrm{d}W_t^0 \\ \qquad\quad +\alpha(t)\,\mathrm{d}F_t - \mathrm{d}J_t, \\ \tau_k = \inf\{t > \tau_{k-1} \;:\; X_{t-} + \alpha(t)\Delta F_t \geq 0\}, \qquad \tau_0 = 0, \\ J_t = \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k), \\ \mathbf{P} = \mathbb{P}\left[X + J \in \cdot \,|\, W^0, \mathbf{P}\right], \quad \boldsymbol{\nu}_t = \mathbb{P}\left[X_t \in \cdot \,|\, W^0, \mathbf{P}\right], \\ F_t = \sum_{k\geq 1} \mathbb{P}\left[\tau_k \leq t \mid W^0, \mathbf{P}\right], \end{cases} \tag{4.2.3}$$

with $X_0 = X_{0-} + \alpha(0)F_0$, $F_{0-} = 0$, $X_{0-} \perp (W,\, W^0,\, \mathbf{P})$, and $(W^0, \mathbf{P}) \perp W$, where $(W,\, W^0)$ is a two-dimensional Brownian motion, X is a càdlàg process, and $\mathbf{P}$ is a random probability measure on the space of càdlàg paths $D([-1, T+1], \mathbb{R})$. Furthermore, $\Delta F_t \leq 1$ for all $t \in [0,\, T]$ almost surely.

As the drift and correlation functions depend on a flow of measures, we require them to satisfy some notion of linear growth and Lipschitz continuity in the measure component. Additionally, we need spatial and temporal regularity conditions to

ensure that (4.2.1) is well-posed. We assume that our coefficients b, σ, ρ, $\mathfrak{K}$ and α satisfy the following assumptions:

Assumption 4.2.2. *(i) (Regularity of b) For all $t \in [0, T]$ and $\mu \in \mathbf{M}_{\leq 1}(\mathbb{R})$, the map $x \mapsto b(t, x, \mu)$ is $\mathcal{C}^2(\mathbb{R})$. Moreover, there exists a constant $C_b > 0$ such that*

$$|b(t,x,\mu)| \leq C_b\left(1 + |x| + \langle \mu, |\cdot|\rangle\right), \quad |\partial_x^{(n)} b(t,x,\mu)| \leq C_b, \quad n = 1, 2,$$
$$|b(t,x,\mu) - b(t,\tilde{x},\tilde{\mu})| \leq C_b\left(|x-\tilde{x}| + d_0(\mu, \tilde{\mu})\right),$$

where

$$d_0(\mu, \tilde{\mu}) = \sup\left\{|\langle \mu - \tilde{\mu}, \psi\rangle| \,:\, \|\psi\|_{\mathrm{Lip}} \leq 1, |\psi(0)| \leq 1\right\}$$

for any $\mu, \tilde{\mu} \in \mathbf{M}_{\leq 1}(\mathbb{R})$.

(ii) (Space/time regularity of σ) The map $(t, x) \mapsto \sigma(t, x)$ is $\mathcal{C}^{1,2}([0, T] \times \mathbb{R})$. Moreover, there exists a constant $C_\sigma > 0$ such that

$$|\sigma(t,x)| \leq C_\sigma, \quad |\partial_t \sigma(t,x)| \leq C_\sigma, \quad \text{and} \quad |\partial_x^{(n)}\sigma(t,x)| \leq C_\sigma \quad \text{for} \quad n = 1, 2.$$

(iii) (d_1-Lipschitzness of ρ) For all $t \in [0, T]$, there exists a constant $C_\rho > 0$ such that

$$|\rho(t,\mu) - \rho(t,\tilde{\mu})| \leq C_\rho d_1(\mu, \tilde{\mu}),$$

where

$$d_1(\mu, \tilde{\mu}) = \sup\left\{|\langle \mu - \tilde{\mu}, \psi\rangle| \,:\, \|\psi\|_{\mathrm{Lip}} \leq 1, \|\psi\|_\infty \leq 1\right\}$$

for any $\mu, \tilde{\mu} \in \mathbf{M}_{\leq 1}(\mathbb{R})$.

(iv) (Non-degeneracy) For all $t \in [0, T]$, $x \in \mathbb{R}$, and $\mu \in \mathbf{M}_{\leq 1}(\mathbb{R})$, the constants C_σ and C_ρ assumed above are such that $0 < C_\sigma^{-1} \leq \sigma(t, x)$ and $0 \leq \rho(t, \mu) \leq 1 - C_\rho^{-1}$.

(v) (Temporal regularity of α) The map $t \mapsto \alpha(t)$ is $\mathcal{C}^1([0, T])$ and increasing with $\alpha(t) \in [0, 1)$ for all t.

(vi) (Sub-Gaussian initial law) The initial law, ν_{0-}, is sub-Gaussian, that is,

$$\exists\, \gamma > 0 \quad s.t. \quad \nu_{0-}(-\infty, -\lambda) = O(e^{-\gamma\lambda^2}) \quad as \quad \lambda \to \infty.$$

and has a density $V_{0-} \in L^2(-\infty, 0)$.

(vii) (Delay and re-insertion) The function $\mathfrak{K} \in \mathcal{W}_0^{1,1}(\mathbb{R}_+)$, the Sobolev space with one weak derivative in L^1 and zero at 0, is such that $\mathfrak{K}$ is non-negative, and $\|\mathfrak{K}\|_1 = 1$. Furthermore, ς has a density $r \in \mathcal{W}_0^{1,1}(\mathbb{R}_+)$ and $(\varsigma_k^\varepsilon)_k$ is a sequence of i.i.d. random variables with distribution $\varepsilon^{1+\gamma}\varsigma$ and $\mathbb{E}[\varsigma^4] < +\infty$, i.e. finite fourth moments.

Solutions of (4.2.1) are the subject of study in Chapter 3. Under Assumption 4.2.2, the existence and uniqueness of solutions are guaranteed by Theorem 3.2.6.

It is more convenient to work with the continuous version of our process, denoted by Z^ε. The idea here is that the Z^ε process does not reset but lives on the whole space. To be explicit, $Z_t^\varepsilon := X_t^\varepsilon + \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^\varepsilon + \varsigma_k^\varepsilon)$. This reformulation allows us to employ Grönwall-type arguments which are essential to prove convergence. Furthermore, starting from the continuous reformulation, we may construct a solution to the system resets. In what follows, for technical reasons, we construct an extension $\tilde{Z}$ of the process Z. For an arbitrary stochastic process H, we define its extended version as follows:

$$\tilde{H}_t = \begin{cases} H_{0-} & t \in [-1,\, 0), \\ H_t & t \in [0,\, T], \\ H_T + W_t - W_T & t \in (T,\, T+1]. \end{cases} \tag{4.2.4}$$

Theorem 4.2.3 (Existence and convergence generalised)**.** *Let Z^ε be the continuous reformulation of (4.2.1) and $\tilde{Z}^\varepsilon$ be the extended version of Z^ε. We set $\tilde{\mathbf{P}}^\varepsilon = \mathrm{Law}(\tilde{Z}^\varepsilon \mid W^0)$. Then, the family of random tuples $\{(\tilde{\mathbf{P}}^\varepsilon,\, W^0,\, W)\}_{\varepsilon>0}$ is tight. Any subsequence $\{(\tilde{\mathbf{P}}^{\varepsilon_n},\, W^0,\, W)\}_{n\geq 1}$, for a positive sequence $(\varepsilon_n)_{n\geq 1}$ which converges to zero, has a further subsequence which converges weakly to some $(\mathbf{P},\, W^0,\, W)$. W^0 and W are standard Brownian motions, $\mathbf{P}$ is a random probability measure $\mathbf{P} : \Omega \to \mathcal{P}(D_{\mathbb{R}})$, and $(\mathbf{P},\, W^0)$ is independent of W.*

Given this limit point, there is a background space which preserves the independence and carries a stochastic process X such that $(X,\, W^0,\, W,\, \mathbf{P})$ is a relaxed solution to (4.2.3). Moreover, we have the upper bound

$$\Delta F_t \leq \inf\{x \geq 0 : \boldsymbol{\nu}_{t-}([-\alpha(t)x, 0]) < x\} \qquad a.s. \tag{4.2.5}$$

for all $t \geq 0$.

The notation $\mathrm{Law}(\tilde{Z}^\varepsilon \mid W^0)$ stands for the conditional law of $\tilde{Z}^\varepsilon$ given W^0, which indeed defines a random W^0-measurable probability measure on $D([-1, T+1], \mathbb{R})$ ([35, Theorem 8.5]). Under stronger assumptions, namely b, σ, and ρ be of the form $(t,x) \mapsto b(t,x)$, $t \mapsto \sigma(t)$, $t \mapsto \rho(t)$ and α is constant, by a simple modification of Proposition 3.5 in [48], the jumps of the firing function satisfy

$$\Delta F_t \geq \inf\{x \geq 0 : \boldsymbol{\nu}_{t-}([-\alpha x, 0]) < x\} \qquad \text{a.s.}$$

Due to the generality of the coefficients, we were not able to establish if (4.2.5) holds with equality. The primary reason is the lack of independence between the

term driven by the idiosyncratic noise and the remainder of the terms that X is composed of. Hence, the technique employed in [48, Proposition 3.5] may not be readily applied or extended to our setting. Regardless, given these two results, under stronger assumptions, we have the following existence result.

Corollary 4.2.4 (Existence of physical solutions)**.** *Let the coefficients b, σ, and ρ be of the form $(t,x) \mapsto b(t,x)$, $t \mapsto \sigma(t)$, $t \mapsto \rho(t)$ and satisfy Assumption 4.2.2. Then provided $\alpha(t) \equiv \alpha > 0$ and constant, there exists a relaxed solution to*

$$\begin{cases} \mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t)\sqrt{1-\rho(t)^2}\,\mathrm{d}W_t + \sigma(t)\rho(t)\,\mathrm{d}W_t^0 + \alpha\,\mathrm{d}F_t - \mathrm{d}J_t, \\ \tau_k = \inf\{t > \tau_{k-1} \,:\, X_{t-} + \alpha\Delta F_t \geq 0\}, \qquad \tau_0 = 0, \\ \mathbf{P} = \mathbb{P}\left[X \in \cdot \,|\, W^0, \mathbf{P}\right], \quad \boldsymbol{\nu}_t \coloneqq \mathbb{P}\left[X_t \in \cdot \,|\, W^0, \mathbf{P}\right], \\ F_t = \sum_{k\geq 1} \mathbb{P}\left[\tau_k \leq t \mid W^0, \mathbf{P}\right]. \end{cases} \tag{4.2.6}$$

Moreover, we have the physical jump condition

$$\Delta F_t = \inf\left\{x \geq 0 \,:\, \boldsymbol{\nu}_{t-}([-\alpha x, 0]) < x\right\} \qquad a.s.$$

for all $t \geq 0$. This determines the jump sizes of F.

This corollary provides a generalisation of the results in [16]. In their work, the authors imposed more restrictive conditions: the drift is a function of space only, the diffusivity is constant, and there is no common noise. The proof of Theorem 4.2.3 is the focus of the next sections.

4.3 Tightness of the Smoothed System

One of the benefits of the continuous reformulation is that the jump process of the neuronal particle is a function of its continuous reformulation. To begin, we will define the following maps:

$$\begin{aligned} \tilde{J}_t \,&:\, D_{\mathbb{R}} \to \mathbb{R}, \qquad \eta \mapsto \lfloor (\sup_{s\leq t} \eta_s + 1)_+ \rfloor, \qquad \text{and} \\ \tilde{J}_t^D \,&:\, D_{\mathbb{R}} \to \mathbb{R}, \qquad \eta \mapsto \lfloor (\sup_{s\leq t} \eta_s + 1)_+ \rfloor - \sum_{k\geq 1} \mathbb{1}_{\{\sup_{s\leq t}\eta_s + 1 = k\}}, \end{aligned}$$

where for any $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the integer part of x and $(x)_+ = \max\{x, 0\}$. Throughout this section, for the sake of simplicity, it is convenient to work with the continuous reformulation of X^ε. Recall that we denoted the continuous reformulation of X^ε by the process Z_t^ε, which is defined by

$$Z_t^\varepsilon \coloneqq X_t^\varepsilon + \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^\varepsilon + \varsigma_k^\varepsilon).$$

It is immediate from this definition that we have

$$\tau_k^\varepsilon = \inf\{t \geq \tau_{k-1}^\varepsilon + \varsigma_{k-1}^\varepsilon \,:\, X_t^\varepsilon \geq 0\} = \inf\{t \geq 0 \,:\, Z_t^\varepsilon \geq k-1\}.$$

This equality between the hitting times of the processes motivates the first result of this section.

Lemma 4.3.1 (Jump Process Reformulation)**.** *Let X^ε be a stochastic process defined in (4.2.1), and let Z^ε be its continuous reformulation. Then, the jump processes of X^ε can be expressed solely in terms of Z^ε as follows:*

$$\sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^\varepsilon) = \tilde{J}_t(Z^\varepsilon) \quad \textit{and} \quad \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^\varepsilon + \varsigma_k^\varepsilon) = \tilde{J}_t^D(Z^\varepsilon) \quad \forall\, t \quad \textit{almost surely.} \tag{4.3.1}$$

Proof. For the first claim, as $X_t^\varepsilon < 0$ and $\sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^\varepsilon + \varsigma_k^\varepsilon) \leq \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^\varepsilon)$, it is clear from the definitions of Z^ε and $\tilde{J}$ that $\tilde{J}_t(Z^\varepsilon) \leq \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^\varepsilon)$. Conversely, for any $K \geq 0$, if $\sum_{k\geq 1} \mathbb{1}_{[0,\tau_K]}(\tau_k^\varepsilon) = K$, then

$$\sum_{k\geq 1} \mathbb{1}_{[0,\tau_K^\varepsilon]}(\tau_k^\varepsilon) = 0 + K - 1 + 1 = X_{\tau_K^\varepsilon}^\varepsilon + \sum_{k\geq 1} \mathbb{1}_{[0,\tau_K^\varepsilon]}(\tau_k^\varepsilon + \varsigma_k^\varepsilon) + 1 = Z_{\tau_k^\varepsilon}^\varepsilon + 1.$$

Therefore, for any $t \in [\tau_K^\varepsilon, \tau_{K+1}^\varepsilon)$, as $\sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^\varepsilon)$ is constant on this interval, we may readily deduce from the above that $\sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^\varepsilon) \leq \tilde{J}_t(Z^\varepsilon)$. Hence, we have equality.

For the second claim in (4.3.1), we shall proceed by considering four cases:

Case 1: $t \in [0, \tau_1^\varepsilon)$

As $t \in [0, \tau_1^\varepsilon)$, we have by definition of τ_1^ε that $\sup_{s\leq t} Z_s^\varepsilon < 0$. Therefore, it is clear that $\tilde{J}_t^D(Z^\varepsilon) = 0 = \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^\varepsilon + \varsigma_k^\varepsilon)$.

Case 2: $t \in [\tau_K^\varepsilon, \tau_K^\varepsilon + \varsigma_K^\varepsilon)$ for some $K \geq 1$

As $Z_{\tau_K^\varepsilon} = K - 1$ and $(\tau_K^\varepsilon, \tau_K^\varepsilon + \varsigma_K^\varepsilon)$ is an interval of constancy for Z^ε, we have that $\sup_{s\leq t} Z_s^\varepsilon = K - 1$. Therefore, $\tilde{J}_t^D(Z^\varepsilon) = K - 1$. Since $t \in [\tau_K^\varepsilon, \tau_K^\varepsilon + \varsigma_K^\varepsilon)$, we have that $\sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^\varepsilon + \varsigma_k^\varepsilon) = K - 1$ by its definition. Hence, we have equality.

Case 3: $t \in (\tau_K^\varepsilon + \varsigma_K^\varepsilon, \tau_{K+1}^\varepsilon)$ for some $K \geq 1$

In this case, we have that $\sup_{s\leq t} Z_s^\varepsilon \in (K-1, K)$; therefore, $\lfloor (\sup_{s\leq t} Z_s^\varepsilon + 1)_+ \rfloor = K$ and $\sum_{k\geq 1} \mathbb{1}_{\{\sup_{s\leq t} Z_s^\varepsilon + 1 = k\}} = 0$. Hence, $\tilde{J}_t^D(Z^\varepsilon) = K = \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^\varepsilon + \varsigma_k^\varepsilon)$.

Case 4: $t = \tau_K^\varepsilon + \varsigma_K^\varepsilon$ for some $K \geq 1$

In this case,

$$\mathbb{P}[\tau_K^\varepsilon + \varsigma_K^\varepsilon = t] = \int_0^t \mathbb{P}[\varsigma_K^\varepsilon = t - s]\mathbb{P}[\tau_K^\varepsilon = s]\,\mathrm{d}s = 0.$$

The first equality follows from the fact that ς_K^ε is independent of τ_K^ε, and the second equality follows from the fact that ς_K^ε has a density by Assumption 4.2.2 (vii). Therefore, $\mathbb{P}[\tau_K^\varepsilon + \varsigma_K^\varepsilon = t \text{ for some } K \geq 1] = 0$.

Thus, we have shown that $\tilde{J}_t^D(Z^\varepsilon) = \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^\varepsilon + \varsigma_k^\varepsilon)$ almost surely for any $t > 0$. As both processes are càdlàg, we can conclude by right continuity that $\tilde{J}_t^D(Z^\varepsilon) = \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^\varepsilon + \varsigma_k^\varepsilon)$ for any $t \geq 0$ almost surely. □

By Lemma 4.3.1, we have the relationship

$$Z_t^\varepsilon = X_t^\varepsilon + \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^\varepsilon + \varsigma_k^\varepsilon) = X_t^\varepsilon + \tilde{J}_t^D(Z^\varepsilon). \tag{4.3.2}$$

Furthermore, by the definitions of $\tilde{J}$ and $\tilde{J}^D$, we have

$$\tilde{J}_t^D(Z^\varepsilon) \leq \tilde{J}_t(Z^\varepsilon) \leq \sup_{s\leq t} |Z_s^\varepsilon| + 1, \quad \text{and} \quad F_t^\varepsilon = \mathbb{E}\left[J_t(Z^\varepsilon) | W^0 \right] \leq \mathbb{E}\left[\sup_{s\leq t} |Z_s^\varepsilon| \,\middle|\, W^0 \right] + 1. \tag{4.3.3}$$

To show the existence of a limit point of Z^ε, we must first choose a suitable topology to establish convergence. By the definition of X^ε in (4.2.1) and the relationship above, we see that Z^ε has continuous paths almost surely. In the singular system (4.2.2), under more restrictive assumptions on the initial condition, it is shown in [9, Theorem 2.2] that there always exists an initial condition X_{0-} such that $F_t' = +\infty$ for some $t > 0$. Therefore, it is reasonable to suppose that the limiting system may exhibit jumps.

For this reason, we will work on Skorokhod's M_1-topology. This topology is sufficiently rich to facilitate the convergence of continuous functions to those with jumps. Convergence in the M_1-topology implies that we will have pointwise convergence at time $t = 0$ and $t = T$, [71, Theorem 12.5.1]. As we are starting from an arbitrary initial condition X_{0-} which is negative almost surely, the limiting process may exhibit a jump immediately at time $t = 0$ given sufficient mass near the boundary, while the approximating processes are equal to X_{0-} at $t = 0$. Furthermore, the limiting process may exhibit a jump at time $t = T$. For this reason, we shall embed the process from $D([0,T],\mathbb{R})$ into $D([-1,\bar{T}],\mathbb{R})$, where $\bar{T} = T + 1$, using the extension defined in (4.2.4).

Unless stated otherwise, for notational convenience, we shall denote the latter space, $D([-1,\bar{T}],\mathbb{R})$, by $D_{\mathbb{R}}$. Recall, $\tilde{\mathbf{P}}^{\varepsilon}$ is defined to be the law of $\tilde{Z}^{\varepsilon}$ conditional on W^0. That is, $\tilde{\mathbf{P}}^{\varepsilon} := \mathrm{Law}(\tilde{Z}^{\varepsilon} \mid W^0)$. To show tightness and convergence of the collection of random measures $\{\tilde{\mathbf{P}}^{\varepsilon}\}_{\varepsilon>0}$, we shall follow the ideas in [16] and [48]. We first derive a Grönwall-type estimate of the smoothed system uniformly in ε. These estimates are necessary to show the tightness of $\{\tilde{\mathbf{P}}^{\varepsilon}\}_{\varepsilon>0}$ and the existence of a limiting random measure.

In the following Proposition and its sequels, C will denote a constant that may change from line to line, and we will denote the dependencies on the value of C in its subscript. To further simplify notation, we shall use Y_t^{ε}, $Y_t^{0,\varepsilon}$, and $\mathcal{Y}_t^{\varepsilon}$ to denote

$$\int_0^t \sigma(u, X_u^{\varepsilon})\sqrt{1-\rho^2(u,\boldsymbol{\nu}_u^{\varepsilon})}\,\mathrm{d}W_u, \qquad \int_0^t \sigma(u, X_u^{\varepsilon})\rho(u,\boldsymbol{\nu}_t^{\varepsilon})\,\mathrm{d}W_u^0, \quad \text{and} \quad Y_t^{\varepsilon}+Y_t^{0,\varepsilon}$$

respectively. We shall use $\tilde{Y}_t^{\varepsilon}$, $\tilde{Y}_t^{0,\varepsilon}$, and $\tilde{\mathcal{Y}}_t^{\varepsilon}$ to denote their corresponding extensions as defined in (4.2.4).

Proposition 4.3.2 (Gronwall upper bound)**.** *For any $p \geq 1$ and $t \leq T$, there exists a constant $C_{\alpha,b,p,T,\sigma} > 0$ independent of $\varepsilon > 0$ such that*

$$\mathbb{E}\left[\sup_{s\leq T}|Z_s^{\varepsilon}|^p\right] \leq C_{\alpha,b,p,T,\sigma}. \tag{4.3.4}$$

Proof. By (4.3.2) and (4.3.3), we have

$$|X_s^{\varepsilon}| \leq 2\sup_{u\leq s}|Z_u^{\varepsilon}| + 1, \qquad \text{and} \qquad |\mathfrak{F}_s^{\varepsilon}| \leq \mathbb{E}\left[\sup_{u\leq s}|Z_u^{\varepsilon}| \;\middle|\; W^0\right] + 1.$$

By the linear growth condition on b in Assumption 4.2.2 (i), we have

$$|b(s,\, X_s^{\varepsilon}, \nu_s^{\varepsilon})| \leq C_b\left(1 + |X_s^{\varepsilon}| + \langle \nu_s^{\varepsilon}, |\cdot| \rangle\right) \leq 2C_b\Big(1 + \sup_{u\leq s}|Z_u^{\varepsilon}| + \mathbb{E}\big[\sup_{u\leq s}|Z_u^{\varepsilon}| \;\big|\; W^0\big]\Big). \tag{4.3.5}$$

Therefore, taking the absolute value in (4.2.1) and employing (4.3.5), we obtain

$$\begin{aligned}
|Z_t^{\varepsilon}| &\leq |X_{0-}| + 2C_b\int_0^t 1 + \sup_{u\leq s}|Z_u^{\varepsilon}| + \mathbb{E}\big[\sup_{u\leq s}|Z_u^{\varepsilon}| \;\big|\; W^0\big]\,\mathrm{d}s \\
&\quad + \sup_{s\leq T}|\mathcal{Y}_s^{\varepsilon}| + \|\alpha\|_{\infty}\,\mathfrak{F}_t \\
&\leq |X_{0-}| + 2C_b\int_0^t 1 + \sup_{u\leq s}|Z_u^{\varepsilon}| + \mathbb{E}\big[\sup_{u\leq s}|Z_u^{\varepsilon}| \;\big|\; W^0\big]\,\mathrm{d}s \\
&\quad + \sup_{s\leq T}|\mathcal{Y}_s^{\varepsilon}| + \|\alpha\|_{\infty}\,\mathbb{E}\big[\sup_{u\leq t}|Z_u^{\varepsilon}| \;\big|\; W^0\big] + \|\alpha\|_{\infty}.
\end{aligned}$$

By raising both sides to the power p and employing Jensen's inequality and Proposition 4.6.1 with $a = \|\alpha\|_\infty \mathbb{E}\big[\sup_{u\le s}|Z^\varepsilon_u| \;\big|\; W^0\big]$, we obtain for any $p \ge 1$

$$|Z^\varepsilon_t|^p \le C_{p,h,b,T}\left(1 + |X_{0-}|^p + \int_0^t \sup_{u\le s}|Z^\varepsilon_u|^p + \mathbb{E}\big[\sup_{u\le s}|Z^\varepsilon_u|^p \;\big|\; W^0\big]\,\mathrm{d}s + \sup_{s\le T}|\mathcal{Y}^\varepsilon_s|^p\right) + \|\alpha\|^p_\infty (1+h)\mathbb{E}\big[\sup_{u\le t}|Z^\varepsilon_u|^p \;\big|\; W^0\big]$$

for any $h > 0$. Taking the supremum on the left-hand side above followed by an expectation on both sides, we obtain

$$\mathbb{E}\big[\sup_{u\le t}|Z^\varepsilon_u|^p\big] \le C_{p,h,b,T}\left(1 + \mathbb{E}\,|X_{0-}|^p + \int_0^t \mathbb{E}\sup_{u\le s}|Z^\varepsilon_u|^p\,\mathrm{d}s + \mathbb{E}\sup_{s\le T}|\mathcal{Y}^\varepsilon_s|^p\right) + \|\alpha\|^p_\infty (1+h)\mathbb{E}\big[\sup_{u\le t}|Z^\varepsilon_u|^p\big].$$

As $\|\alpha\|_\infty < 1$ by Assumption 4.2.2 (v), we may choose an h small enough such that $\|\alpha\|^p_\infty (1+h) < 1$. Fixing such an h, we deduce

$$\mathbb{E}\big[\sup_{u\le t}|Z^\varepsilon_u|^p\big] \le C_{\alpha,p,h,b,T}\left(1 + \mathbb{E}\big[|X_{0-}|^p\big] + \int_0^t \mathbb{E}\big[\sup_{u\le s}|Z^\varepsilon_u|^p\big]\,\mathrm{d}s + \mathbb{E}\big[\sup_{s\le T}|\mathcal{Y}^\varepsilon_s|^p\big]\right)$$

for some constant $C_{\alpha,p,h,b,T} > 0$. Now applying Grönwall's inequality to the above, we deduce

$$\mathbb{E}\left[\sup_{s\le T}|Z^\varepsilon_s|^p\right] \le C_{\alpha,p,h,b,T}\mathbb{E}\left[1 + |X_{0-}|^p + \sup_{s\le T}|\mathcal{Y}^\varepsilon_s|^p\right]$$

Lastly, by the Burkholder-Davis-Gundy inequality and Assumption 4.2.2 (ii), we may bound the above independently of ε. This completes the proof. □

The collection of measures $\{\tilde{\mathbf{P}}^\varepsilon\}_{\varepsilon>0}$ are $\mathcal{P}(D_\mathbb{R})$-valued random measures. To show that this collection of random variables is tight, we will need to look for compact sets in $\mathcal{P}(\mathcal{P}(D_\mathbb{R}))$. In fact, it is sufficient to show that $\{\mathrm{Law}(\tilde{Z}^\varepsilon)\}_{\varepsilon>0}$ is tight in $\mathcal{P}(D_\mathbb{R})$. With the result from Proposition 4.3.2 in hand, the tightness follows immediately by adapting the proof of Proposition 2.3.2 with the necessary modifications.

Proposition 4.3.3 (Tightness of smoothed random measures)**.** *Let $\mathfrak{T}^{wk}_{M_1}$ denote the topology of weak convergence on $\mathcal{P}(D_\mathbb{R})$ induced by the M_1-topology on $D_\mathbb{R}$. Then, under Assumption 4.2.2, the collection $\{\mathrm{Law}(\tilde{Z}^\varepsilon \mid W^0)\}_{\varepsilon>0}$ is tight on $(\mathcal{P}(D_\mathbb{R}), \mathfrak{T}^{wk}_{M_1})$.*

4.4 Continuity of the Hitting Times

The space $(D_\mathbb{R}, M_1)$ is Polish [71, Theorem 12.8.1], and its Borel σ-algebra is generated by the marginal projections [71, Theorem 11.5.2]. Consequently, $(\mathcal{P}(D_\mathbb{R}), \mathfrak{T}^{\mathrm{wk}}_{M_1})$ is

also Polish. Hence, we can apply Prokhorov's Theorem [3, Theorem 5.1], which establishes that tightness is equivalent to being sequentially precompact, to select a weakly convergent subsequence $\{\tilde{\mathbf{P}}^{\varepsilon_n}\}_{n\geq 1}$ for a positive sequence $(\varepsilon_n)_{n\geq 1}$ converging to zero. We denote the limit of this sequence by $\mathbf{P}^*$. This limit point will be used to construct a probability space and a stochastic process that solves (4.2.3).

Before proceeding with this construction, we will show that for a co-countable set of times t, the function $F_t^{\varepsilon_n} = \sum_{k\geq 1} \tilde{\mathbf{P}}^{\varepsilon_n}(\tau_k \leq t)$ converges weakly to $F_t^* = \sum_{k\geq 1} \mathbf{P}^*(\tau_k \leq t)$, where τ_k represents a function on $D_{\mathbb{R}}$ whose value is the first hitting time of $k-1$. Specifically, we define:

$$\mathbb{T} \coloneqq \left\{ t \in [-1, \bar{T}] \ : \ \mathbb{E}\left[\mathbf{P}^*(\eta_t = \eta_{t-})\right] = 1, \, \mathbb{E}\left[\mathbf{P}^*(\tau_k = t)\right] = 0 \ \forall k \right\},$$

and

$$\tau_k(\eta) \coloneqq \inf\{t \geq -1 \ : \ \eta_t \geq k-1\} \tag{4.4.1}$$

with the convention that $\inf\{\emptyset\} = \bar{T}$. Our first result establishes that for $\mathrm{Law}(\mathbf{P}^*)$-almost every measure μ, μ-almost every path $\eta \in D_{\mathbb{R}}$ is constant on the interval $[-1, 0)$. This result parallels Lemma 2.4.1, and we can adapt its proof to our current context. We state the result here and refer the reader to the aforementioned lemma for a detailed proof.

Lemma 4.4.1. *For* $\mathrm{Law}(\mathbf{P}^*)$*-almost every measure* μ*, we have that* $\sup_{s<0} |\eta_s - \eta_{-1}| = 0$ *for* μ*-almost every path* η*.*

To establish the weak convergence of F^{ε_n} to F^*, we will show that for $\mathrm{Law}(\mathbf{P}^*)$-almost every measure μ, τ_k is a μ-almost sure continuous map. This will follow readily from applying Lemma 4.6.2, proved later in this work, provided μ-almost every path $\eta \in D_{\mathbb{R}}$ only jumps upwards and, on any neighbourhood of the hitting time, attains a value strictly greater than the threshold. To this end, we first prove the effects on the system due to the refractory period decay in the limit.

Proposition 4.4.2. *Let* $\mu^{\varepsilon,k} \coloneqq \delta_{\tau_k^\varepsilon, \varsigma_k^\varepsilon}$*. The family* $\{\mu^{\varepsilon,k}\}_{\varepsilon>0}$ *is tight on* $\mathcal{P}(\bar{\mathbb{R}}^2)$*, where* $\bar{\mathbb{R}}$ *denotes the one-point compactification of* $\mathbb{R}$*. For any limit point, denoted by* $\mu^{*,k}$*, obtained along a subsequence* $\{\varepsilon_n\}_{n\geq 1}$ *with* $\varepsilon_n \to 0$*, we have* $\mu^{*,k} = \mu_1^{*,k} \otimes \delta_0$ *almost surely.*

Proof. As $\bar{\mathbb{R}}$ is compact and the product of compact sets is compact in the product topology, $\bar{\mathbb{R}}^2$ is a compact metric space. Consequently, $\mathcal{P}(\bar{\mathbb{R}}^2)$ is a compact metric space. Therefore, $\{\mu^{\varepsilon,k}\}_{\varepsilon>0}$ is trivially tight on $\mathcal{P}(\bar{\mathbb{R}}^2)$.

Fix a limit point, $\mu^{*,k}$, along a subsequence $\{\varepsilon_n\}_{n\geq 1}$ for which $\varepsilon_n \to 0$. Let $\Pi^{*,k}$ denote the law of $\mu^{*,k}$ and $\pi_i : \bar{\mathbb{R}}^2 \to \bar{\mathbb{R}}$ be the projection onto the i^{th} component. By the Continuous Mapping Theorem, $\pi_i^{\#}\mu^{\varepsilon_n,k} \implies \pi_i^{\#}\mu^{*,k}$ for both $i = 1, 2$.

For $i = 2$, choose any continuous bounded function $f : \bar{\mathbb{R}} \to \mathbb{R}$ and any continuous function $G : \mathbb{R} \to \mathbb{R}$. By the Continuous Mapping Theorem,

$$\mathbb{E}\left[G(\langle \pi_2^{\#}\mu^{\varepsilon_n,k}, f\rangle)\right] \to \mathbb{E}\left[G(\langle \pi_2^{\#}\mu^{*,k}, f\rangle)\right]. \tag{4.4.2}$$

By definition and Assumption 4.2.2 (vii), $\mathbb{E}[G(\langle \pi_2^{\#}\mu^{\varepsilon_n,k}, f\rangle)] = \mathbb{E}[G(f(\varepsilon_n^{1+\gamma}\varsigma))]$. Therefore, by the Dominated Convergence Theorem, we have

$$\mathbb{E}\left[G(f(\varepsilon_n^{1+\gamma}\varsigma))\right] \to \mathbb{E}\left[G(f(0))\right] = G(f(0)). \tag{4.4.3}$$

Combining (4.4.2) and (4.4.3), we obtain

$$\mathbb{E}\left[G(\langle \pi_2^{\#}\mu^{*,k}, f\rangle)\right] = G(f(0)). \tag{4.4.4}$$

As (4.4.4) holds for any continuous function G, by replacing G with $\tilde{G}(\cdot) = (\cdot - f(0))^2$, it follows from (4.4.4) that $\langle \pi_2^{\#}\mu^{*,k}, f\rangle = f(0)$ almost surely. Therefore, by employing a Dynkin's Lemma argument, we deduce $\pi_2^{\#}\mu^{*,k} = \delta_0$ almost surely.

It remains to show that the limiting measure is a product measure. For every $a_1, a_2 : \bar{\mathbb{R}} \to \mathbb{R}$, continuous and bounded,

$$\begin{aligned}
&\mathbb{E}\left[\left|\langle \mu^{*,k}, a_1 \otimes a_2\rangle - \prod_{i=1}^{2}\langle \pi_i^{\#}\mu^{*,k}, a_i\rangle\right|^2\right] \\
&\qquad = \mathbb{E}\left[\left|\langle \mu^{*,k}, a_1 \otimes a_2\rangle - \langle \pi_1^{\#}\mu^{*,k}, a_1\rangle a_2(0)\right|^2\right] \\
&\qquad = \lim_{n\to\infty} \mathbb{E}\left[\left|\langle \mu^{\varepsilon_n,k}, a_1 \otimes a_2\rangle - \langle \pi_1^{\#}\mu^{\varepsilon_n,k}, a_1\rangle a_2(0)\right|^2\right] \\
&\qquad = \lim_{n\to\infty} \mathbb{E}\left[|a_1(\tau_k^{\varepsilon_n})a_2(\varsigma_k^{\varepsilon_n}) - a_1(\tau_k^{\varepsilon_n})a_2(0)|^2\right] \\
&\qquad = \lim_{n\to\infty} \mathbb{E}\left[a_1(\tau_k^{\varepsilon_n})^2\big(a_2(\varsigma_k^{\varepsilon_n}) - a_2(0)\big)^2\right].
\end{aligned}$$

As a_1 is bounded and $\varsigma_k^{\varepsilon_n} \sim^d \varepsilon_n^{1+\gamma}\varsigma$, we have

$$\lim_{n\to\infty} \mathbb{E}\left[a_1(\tau_k^{\varepsilon_n})^2\big(a_2(\varsigma_k^{\varepsilon_n}) - a_2(0)\big)^2\right] \leq \limsup_{n\to\infty} \|a_1\|_\infty^2\, \mathbb{E}\left[\big(a_2(\varepsilon_n^{1+\gamma}\varsigma) - a_2(0)\big)^2\right] = 0,$$

where the equality follows from the Dominated Convergence Theorem. Therefore, by the Monotone Class Theorem, we may conclude that $\mu^{*,k} = \mu_1^{*,k} \otimes \delta_0$ almost surely, where $\mu_1^{*,k} = \pi_1^{\#}\mu^{*,k}$. □

Building on the previous result, we can show that for a co-countable set of times, the limiting system will take values in $(-\infty, 0)$ with probability 1.

Proposition 4.4.3. *For any sequence $\{\varepsilon_n\}_{n\geq 1}$ converging to zero, there exists a subsequence $\{\varepsilon_{n_j}\}_{j\geq 1}$ and a co-countable set of times $\tilde{\mathbb{T}}$ (dependent on $\{\varepsilon_{n_j}\}_{j\geq 1}$), such that for all $t \in \tilde{\mathbb{T}}$,*

$$\lim_{j\to\infty} \mathbb{P}\big[X_t^{\varepsilon_{n_j}} < 0\big] = 1.$$

Proof. As $\mathbb{P}[X_t^\varepsilon \leq 0] = 1$ for any fixed $\varepsilon > 0$ by construction of X^ε, it suffices to consider the limit of $\mathbb{P}[X_t^\varepsilon = 0]$ along a suitable sequence. Since $X_t^\varepsilon = 0$ only during its refractory period, we have

$$\begin{aligned}
\mathbb{P}\big[X_t^\varepsilon = 0\big] &= \mathbb{P}\big[\exists\, k \geq 1 \text{ such that } t \in [\tau_k^\varepsilon, \tau_k^\varepsilon + \varsigma_k^\varepsilon)\big] \\
&\leq \sum_{k\geq 1} \mathbb{P}\big[t \in [\tau_k^\varepsilon, \tau_k^\varepsilon + \varsigma_k^\varepsilon)\big] \\
&\leq \sum_{k=1}^{M} \mathbb{P}\big[t \in [\tau_k^\varepsilon, \tau_k^\varepsilon + \varsigma_k^\varepsilon)\big] + \sum_{k>M} \mathbb{P}\big[\tau_k^\varepsilon \leq t\big] \\
&\leq \sum_{k=1}^{M} \mathbb{E}\big[\mathbb{1}_{[0,t]}(\tau_k^\varepsilon) - \mathbb{1}_{[0,t]}(\tau_k^\varepsilon + \varsigma_k^\varepsilon)\big] + \sum_{k>M} \mathbb{P}\big[\tau_k^\varepsilon \leq t\big], \qquad (4.4.5)
\end{aligned}$$

where $M \in \mathbb{N}$ is fixed. By the definition of τ_k^ε and Markov's inequality,

$$\sum_{k>M} \mathbb{P}\big[\tau_k^\varepsilon \leq t\big] = \sum_{k>M} \mathbb{P}\big[\sup_{s\leq t} Z_s^\varepsilon \geq k-1\big] \leq \mathbb{E}\big[\sup_{s\leq T} |Z_s^\varepsilon|^2\big] \sum_{k>M} \frac{1}{(k-1)^2}.$$

By Proposition 4.3.2, $\mathbb{E}\big[\sup_{s\leq T} |Z_s^\varepsilon|^2\big] = O(1)$ uniformly in ε. Thus, from (4.4.5),

$$\mathbb{P}\big[X_t^\varepsilon = 0\big] \leq \sum_{k=1}^{M} \mathbb{E}\big[\mathbb{1}_{[0,t]}(\tau_k^\varepsilon) - \mathbb{1}_{[0,t]}(\tau_k^\varepsilon + \varsigma_k^\varepsilon)\big] + C \sum_{k>M} \frac{1}{(k-1)^2}. \qquad (4.4.6)$$

To find a subsequence where the first term on the right-hand side of (4.4.6) converges to zero, consider the infinite tuple of (random) measures $(\mu^{\varepsilon,k})_{k\geq 1}$ on $\prod_{k\geq 1} \mathcal{P}(\bar{\mathbb{R}}^2)$, where $\mu^{\varepsilon,k} := \delta_{\tau_k^\varepsilon, \varsigma_k^\varepsilon}$ and $\bar{\mathbb{R}}$ is the one-point compactification of $\mathbb{R}$. We endow $\mathcal{P}(\bar{\mathbb{R}}^2)$ with the topology of weak convergence and $\prod_{k\geq 1} \mathcal{P}(\bar{\mathbb{R}}^2)$ with the product topology. The countable product of compact sets is compact in the product topology. Hence, $\prod_{k\geq 1} \mathcal{P}(\bar{\mathbb{R}}^2)$ is compact, and, trivially, $\{(\mu^{\varepsilon,k})_k\}_{\varepsilon>0}$ is tight.

For any sequence $\{\varepsilon_n\}_{n\geq 1}$ converging to 0, there exists a subsequence $\{\varepsilon_{n_j}\}_{j\geq 1}$ such that $(\mu^{\varepsilon_{n_j},k})_k$ converges weakly to $(\mu^{\dagger,k})_k$. By Proposition 4.4.2, $\mu^{\dagger,k} = \mu_1^{\dagger,k} \otimes \delta_0$ almost surely. Define

$$\tilde{\mathbb{T}} = \Big\{t \in [0,T] \,:\, \mathbb{P}[\mu_2^{\dagger,k}(\{t\}) = 0] = 1 \;\forall\, k\Big\}.$$

$\tilde{\mathbb{T}}$ is co-countable, as $t \mapsto \mu([0,t])$ is càdlàg for any probability measure μ. Therefore, by construction of $\tilde{\mathbb{T}}$, the map $\prod_{k\geq 1}\mathcal{P}(\bar{\mathbb{R}}^3) \to \prod_{k\geq 1}\mathbb{R}$, $(\mu^k)_k \mapsto (\langle \mu^k, \mathbb{1}_{[0,t]}(y)\rangle)_k$ is $\mathrm{Law}((\mu^{\dagger,k})_k)$-almost surely continuous for every $t \in \tilde{\mathbb{T}}$. Furthermore, as $\mu^{\dagger,k}$ may be written as a product of measures, the map $\prod_{k\geq 1}\mathcal{P}(\bar{\mathbb{R}}^3) \to \prod_{k\geq 1}\mathbb{R}$, $(\mu^k)_k \mapsto (\langle \mu^k, \mathbb{1}_{[0,t]}(y+z)\rangle)_k$ is $\mathrm{Law}((\mu^{\dagger,k})_k)$-almost surely continuous for every $t \in \tilde{\mathbb{T}}$. Thus, for $t \in \tilde{\mathbb{T}}$, by the Continuous Mapping Theorem,

$$\begin{aligned}
\lim_{j\to\infty} \mathbb{E}\big[\mathbb{1}_{[0,t]}(\tau_k^{\varepsilon_{n_j}}) - \mathbb{1}_{[0,t]}(\tau_k^{\varepsilon_{n_j}} + \varsigma_k^{\varepsilon_{n_j}})\big] \\
&= \lim_{j\to\infty} \mathbb{E}\big[\langle \mu^{\varepsilon_{n_j},k}, \mathbb{1}_{[0,t]}(x)\rangle - \langle \mu^{\varepsilon_{n_j},k}, \mathbb{1}_{[0,t]}(x+y)\rangle\big] \\
&= \mathbb{E}\big[\langle \mu^{\dagger,k}, \mathbb{1}_{[0,t]}(x)\rangle - \langle \mu^{\dagger,k}, \mathbb{1}_{[0,t]}(x+y)\rangle\big] \\
&= \mathbb{E}\big[\langle \mu_1^{\dagger,k}, \mathbb{1}_{[0,t]}(x)\rangle - \langle \mu_1^{\dagger,k}, \mathbb{1}_{[0,t]}(x)\rangle\big] \\
&= 0.
\end{aligned}$$

Inserting the above into (4.4.6), we have

$$\lim_{j\to\infty} \mathbb{P}\big[X_t^{\varepsilon_{n_j}} = 0\big] \leq C \sum_{k>M} \frac{1}{(k-1)^2}$$

for all $t \in \tilde{\mathbb{T}}$. Taking $M \to \infty$ yields the desired result. □

Employing the above result, we now have all the ingredients to show that the limit process of $\tilde{\mathcal{Y}}^{\varepsilon_n}$ is a continuous local martingale with a strictly increasing quadratic variation.

Lemma 4.4.4. *Suppose that $\tilde{P}^{\varepsilon_n} = \mathrm{Law}(\tilde{Z}^{\varepsilon_n}, \tilde{\mathcal{Y}}^{\varepsilon_n})$ converges weakly in $\mathcal{P}(D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}})$ to $\tilde{P}^*$, where $\tilde{Z}^{\varepsilon_n}$ and $\tilde{\mathcal{Y}}^{\varepsilon_n}$ are the extensions of Z^{ε_n} and $\mathcal{Y}^{\varepsilon_n}$, respectively. Let Z^* and Y^* be the canonical processes on $D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}$ such that for $(\eta, \omega) \in D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}$, $Z^*(\eta, \omega) = \eta$ and $Y^*(\eta, \omega) = \omega$. Then, under $\tilde{P}^*$, Y^* is a martingale with respect to the filtration generated by (Z^*, Y^*). Furthermore, there exist two constants $C_1, C_2 > 0$ such that $\tilde{P}^*$-almost surely,*

$$C_1(t-s) \leq \langle Y^* \rangle_t - \langle Y^* \rangle_s \leq C_2(t-s) \quad \textit{for all} \quad 0 \leq s \leq t.$$

Proof. Define the set of times when Z^* is continuous by

$$\mathbb{T}^{\tilde{P}^*} := \Big\{ t \in [-1, \bar{T}] \,:\, \tilde{P}^*(\eta_t = \eta_{t-}) = 1 \Big\}.$$

For any $s_0, t_0 \in \mathbb{T}^{\tilde{P}^*}$ with $s_0 < t_0$ and $\{s_i\}_{i=1}^k \subset [-1, s_0] \cap \mathbb{T}^{\tilde{P}^*}$, define the function G by

$$G \,:\, D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}} \to \mathbb{R}, \quad (\eta, \omega) \mapsto (\omega_{t_0} - \omega_{s_0}) \prod_{i=1}^k g_i(\eta_{s_i}, \omega_{s_i}),$$

for arbitrary $g_i \in \mathcal{C}_b(D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}})$. To show that Y^* is a martingale, it suffices to prove $\mathbb{E}^{\tilde{P}^*}[G(Z^*, Y^*)] = 0$.

As $\tilde{P}^{\varepsilon_n} \implies \tilde{P}^*$, by Skorokhod's Representation Theorem [3, Theorem 7.6], there exist $((z^n, y^n))_{n\geq 1}$ and (z, y) on the same probability space such that (z^n, y^n) converges to (z, y) almost surely in $(D_{\mathbb{R}}, M_1) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty)$ with $\text{Law}(z^n, y^n) = \tilde{P}^{\varepsilon_n}$ and $\text{Law}(z, y) = \tilde{P}^*$. For any $p > 1$,

$$\mathbb{E}\Big[\, |G(z^n, y^n)|^p \,\Big] \leq C\mathbb{E}\Big[\sup_{s\leq \bar{T}} |\tilde{\mathcal{Y}}_s^{\varepsilon_n}|^p\Big] \leq C,$$

where C depends only on p, σ, T and the g_i's but is independent of ε. Thus, $G(z^n, y^n)$ is uniformly L^p-bounded.

For $t \in \{t_0, s_0, s_1, \ldots, s_k\}$, t is an almost sure continuity point of z. Therefore, by the properties of M_1-convergence, (z_t^n, y_t^n) converges to (z_t, y_t) almost surely. Hence, $G(z^n, y^n)$ converges to $G(z, y)$ almost surely. By Vitali's Convergence Theorem, almost sure convergence and uniform integrability imply convergence of means, hence

$$\mathbb{E}^{\tilde{P}^*}[G(Z^*, Y^*)] = \mathbb{E}[G(z, y)] = \lim_{n\to\infty} \mathbb{E}[G(z^n, y^n)] = \lim_{n\to\infty} \mathbb{E}[G(\tilde{Z}^{\varepsilon_n}, \tilde{\mathcal{Y}}^{\varepsilon_n})] = 0,$$

where the last equality follows from $\tilde{\mathcal{Y}}^{\varepsilon_n}$ being a martingale. Therefore, by a monotone class argument, Y^* is a continuous local martingale.

To characterize the quadratic variation of Y^*, we use properties of $\tilde{\mathcal{Y}}^{\varepsilon_n}$. By Proposition 4.3.3, $\{\tilde{Z}^\varepsilon\}_{\varepsilon>0}$ is tight. For any $p > 2$,

$$\mathbb{E}\left[\left(\tilde{\mathcal{Y}}_t^\varepsilon - \tilde{\mathcal{Y}}_s^\varepsilon\right)^p\right] \leq C(t-s)^{p/2} \quad \text{and} \quad \mathbb{E}\left[\left(\langle\tilde{\mathcal{Y}}^\varepsilon\rangle_t - \langle\tilde{\mathcal{Y}}^\varepsilon\rangle_s\right)^p\right] \leq C(t-s)^p \tag{4.4.7}$$

for some constant $C > 0$ independent of ε. As $\tilde{\mathcal{Y}}^\varepsilon$ and $\langle\tilde{\mathcal{Y}}^\varepsilon\rangle$ are continuous processes, by Kolmogorov's criterion and (4.4.7), $\{\tilde{\mathcal{Y}}^\varepsilon\}_{\varepsilon>0}$ and $\{\langle\tilde{\mathcal{Y}}^\varepsilon\rangle\}_{\varepsilon>0}$ are tight. As tightness of the marginals implies joint tightness, we can find a suitable subsequence of $\{\varepsilon_n\}$ such that $\text{Law}(\tilde{Z}^{\varepsilon_{n_j}}, \tilde{\mathcal{Y}}^{\varepsilon_{n_j}}, \langle\tilde{\mathcal{Y}}^{\varepsilon_{n_j}}\rangle) \implies \tilde{P}^*_{z,y,\langle y\rangle}$, where under $\tilde{P}^*_{z,y,\langle y\rangle}$, (Z^*, Y^*) is distributed according to $\tilde{P}^*$.

Let $\langle Y^*\rangle$ denote the canonical process on $D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}$, where $\langle Y^*\rangle(\eta, \omega^1, \omega^2) = \omega^2$. Using similar arguments as above, Y^* and $(Y^*)^2 - \langle Y^*\rangle$ are continuous local martingales on $(D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}, \mathcal{B}(D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}), \tilde{P}^*_{z,y,\langle y\rangle})$. By uniqueness of quadratic variation, $\langle Y^*\rangle$ must be the quadratic variation of Y^*. As the quadratic variation of a continuous local martingale is a measurable function of the continuous local martingale, $\langle Y^*\rangle$ under $\tilde{P}^*_{z,y,\langle y\rangle}$ has the same law as the quadratic variation of Y^* under $\tilde{P}^*$.

We now show that there exist $C_1, C_2 > 0$ such that for all $0 \le s \le t$,

$$\tilde{P}_{z,y,\langle y\rangle}\left[\langle Y^*\rangle_t - \langle Y^*\rangle_s < C_1(t-s)\right] = 0 \quad \text{and} \quad \tilde{P}_{z,y,\langle y\rangle}\left[\langle Y^*\rangle_t - \langle Y^*\rangle_s > C_2(t-s)\right] = 0. \tag{4.4.8}$$

Fix $h > 0$ and $0 \le s \le t$. Set $C_1 = \inf_{u,x}\{\sigma(u,x)\} \wedge 1$. By the Portmanteau Theorem,

$$\begin{aligned}\tilde{P}_{z,y,\langle y\rangle}&\left[\langle Y^*\rangle_t - \langle Y^*\rangle_s < C_1(t-s) - h\right] \\ &\le \liminf_{j\to\infty} \mathbb{P}\left[\langle \tilde{\mathcal{Y}}^{\varepsilon_{n_j}}\rangle_t - \langle \tilde{\mathcal{Y}}^{\varepsilon_{n_j}}\rangle_s < C_1(t-s) - h\right].\end{aligned} \tag{4.4.9}$$

Recall, by the definition of $\tilde{\mathcal{Y}}^{\varepsilon_{n_j}}$, we have $\langle \tilde{\mathcal{Y}}^{\varepsilon_{n_j}}\rangle_t = \int_0^{t\wedge T} \sigma(u, X_u^{\varepsilon_{n_j}})^2 \mathbb{1}_{\{X_u^{\varepsilon_{n_j}} < 0\}}\,\mathrm{d}u + (t \vee T - T)$. Therefore,

$$\langle Y^*\rangle_t - \langle Y^*\rangle_s - C_1(t-s) \ge C_1 \int_{s\wedge T}^{t\wedge T} \mathbb{1}_{\{X_u^{\varepsilon_{n_j}} < 0\}} - 1\,\mathrm{d}u \tag{4.4.10}$$

Inserting (4.4.10) into (4.4.9) and applying Markov's inequality,

$$\begin{aligned}\tilde{P}_{z,y,\langle y\rangle}\left[\langle Y^*\rangle_t - \langle Y^*\rangle_s < C_1(t-s) - h\right] &\le \liminf_{j\to\infty} \mathbb{P}\left[C_1 \int_{s\wedge T}^{t\wedge T} \mathbb{1}_{\{X_u^{\varepsilon_{n_j}} < 0\}} - 1\,\mathrm{d}u < -h\right] \\ &= \liminf_{j\to\infty} \mathbb{P}\left[C_1 \int_{s\wedge T}^{t\wedge T} 1 - \mathbb{1}_{\{X_u^{\varepsilon_{n_j}} < 0\}}\,\mathrm{d}u > h\right] \\ &\le C_1 h^{-1} \liminf_{j\to\infty} \int_{s\wedge T}^{t\wedge T} 1 - \mathbb{P}\left[X_u^{\varepsilon_{n_j}} < 0\right]\mathrm{d}u.\end{aligned}$$

Without loss of generality, we may assume our subsequence $\{\varepsilon_{n_j}\}_j$ is such that the conclusion from Proposition 4.4.3 holds. Therefore, by the Dominated Convergence Theorem

$$\tilde{P}_{z,y,\langle y\rangle}\left[\langle Y^*\rangle_t - \langle Y^*\rangle_s < C_1(t-s) - h\right] = 0.$$

Letting $h \to 0$ from above, we obtain the first claim in (4.4.8). The second claim follows similarly by setting $C_2 = \sup_{u,x}\{\sigma(u,x)\} \vee 1$. This completes the proof, establishing that Y^* is a martingale with the desired quadratic variation properties. □

With the results of Lemma 4.4.4 established, we now have all the ingredients to show that the limit point satisfies the strong crossing property. The proof follows closely that of Lemma 2.4.2, with appropriate modifications. We state the result without proof due to its similarity.

Lemma 4.4.5 (Strong crossing property)**.** *For any $h > 0$, we have that*

$$\begin{aligned}&\mathbb{E}\left[\mathbf{P}^*\left(\bigcup_{k\ge 1}\left\{\eta \,:\, \sup_{s\in(\tau_k,(\tau_k+h)\wedge\bar{T})}\{\eta_s - \eta_{\tau_k}\} \le 0,\, \tau_k < \bar{T}\right\}\right)\right] = 0, \\ &\mathbb{E}\left[\mathbf{P}^*\left(\eta \,:\, \Delta\eta_t \ge 0 \quad \forall\ t \le \bar{T}\right)\right] = 1.\end{aligned} \tag{4.4.11}$$

Now we have all the ingredients to show that τ_k is an M_1-continuous map for every $k \geq 1$.

Corollary 4.4.6 (Hitting time continuity)**.** *For* Law($\mathbf{P}^*$)*-almost every measure* μ*, we have that the hitting time map* $\tau_k : D_{\mathbb{R}} \to \mathbb{R}$ *is continuous in the* M_1*-topology for every* $k \geq 1$ μ*-almost surely.*

Proof. By Lemma 4.4.5, for Law($\mathbf{P}^*$)-almost every measure μ, μ-almost every path $\eta \in D_{\mathbb{R}}$ will have only upward jumps and for each k, one of the following conditions holds:

(i) $\tau_k < \bar{T}$ and η takes a positive value on any neighbourhood of τ_k,

(ii) $\tau_k = \bar{T}$ and $\sup_{s \leq \bar{T}} \eta_s > k - 1$,

(iii) $\tau_k = \bar{T}$ and $\eta_{\bar{T}} = k - 1$.

If (i) holds, then by Lemma 4.6.2, τ_k is M_1-continuous at η. If (ii) holds, then for any approximating sequence $(\eta^n)_{n \geq 1} \subset D_{\mathbb{R}}$ in the M_1-topology, we must have $\sup_{s \leq \bar{T}} \eta^n_s > k - 1$ eventually as the parametric representations get arbitrarily close in the uniform topology. Therefore, as $\sup_{s \leq \bar{T}} \eta^n_s > k - 1$ eventually, by definition, $\tau_k(\eta^n) = \bar{T}$ eventually. Thus, τ_k is M_1-continuous at η. If (iii) holds, then for any $h > 0$, with $\bar{T} - h$ being a continuity point, we must have $\sup_{s \leq \bar{T}-h} \eta_s < k-1$ because η only jumps upwards. So for any approximating sequence $(\eta^n)_{n \geq 1} \subset D_{\mathbb{R}}$ in the M_1-topology, eventually $\sup_{s \leq \bar{T}-h} \eta^n_s < k - 1$. Hence, eventually $\tau_k(\eta^n) > \bar{T} - h$. As $h > 0$ can be made arbitrarily close to zero, by definition $\lim_{n \to \infty} \tau_k(\eta^n) = \tau_k = \bar{T}$. Therefore, τ_k is M_1-continuous at η. □

Having established the continuity of the hitting time map, we now turn our attention to the convergence of hitting time probabilities.

Lemma 4.4.7 (Continuity of conditional feedback)**.** *For* Law($\mathbf{P}^*$)*-almost every measure* $\mu \in \mathcal{P}(D_{\mathbb{R}})$*, the map* $\mu \mapsto \mu(\tau_k \leq t)$ *is continuous with respect to* $\mathfrak{T}^{wk}_{M_1}$ *for every* $k \geq 1$ *and* $t \in \mathbb{T}^\mu \cap [0, \bar{T})$*, where* $\mathbb{T}^\mu$ *is defined as*

$$\mathbb{T}^\mu := \{t \in [-1, \bar{T}] \, : \, \mu(\eta_t = \eta_{t-}) = 1, \, \mu(\tau_k = t) = 0 \; \forall \, k\}.$$

In other words, $\mathbb{T}^\mu$ *is the set of times for which* $t \mapsto \mu(\tau_k \leq t)$ *is continuous for all* $k \geq 1$ *simultaneously.*

Proof. By Corollary 4.4.6, for Law($\mathbf{P}^*$)-almost every measure μ, τ_k is M_1-continuous for every $k \geq 1$ for μ-almost every η. We fix such a μ and consider a sequence $(\mu^n)_{n\geq 1}$ converging to μ in $\mathcal{P}(D_{\mathbb{R}})$. By Skorokhod's Representation Theorem,

$$\mu^n(\tau_k \leq t) = \mathbb{E}\left[\mathbb{1}_{\{\tau_k(H^n)\leq t\}}\right] \qquad \text{and} \qquad \mu(\tau_k \leq t) = \mathbb{E}\left[\mathbb{1}_{\{\tau_k(H)\leq t\}}\right],$$

where τ_k is continuous for almost all paths H and $H^n \to H$ almost surely in $(D_{\mathbb{R}}, M_1)$. For any $t \in \mathbb{T}^\mu$, by the Monotone Convergence Theorem,

$$\mathbb{P}\left[\tau_k(H) = t\right] = \mu(\tau_k \leq t) - \lim_{s\uparrow t} \mu(\tau_k \leq s) = \mu(\tau_k = t) = 0,$$

where the last equality follows from the definition of $\mathbb{T}^\mu$. Therefore, employing the continuity of τ_k and the equation above, we have

$$\mathbb{E}\left[\mathbb{1}_{\{\tau_k(H^n)\leq t\}}\right] \to \mathbb{E}\left[\mathbb{1}_{\{\tau_k(H)\leq t\}}\right],$$

by the Dominated Convergence Theorem for any $t \in \mathbb{T}^\mu$. Thus, we conclude that

$$\mu^n(\tau_k \leq t) \to \mu(\tau_k \leq t) \qquad \forall\, t \in \mathbb{T}^\mu.$$

□

For any measure μ, we define its instantaneous and delayed firing functions respectively as

$$F_t^\mu := \sum_{k\geq 1} \mu(\tau_k \leq t) \quad \text{and} \quad \mathfrak{F}_t^{\mu,\varepsilon} = \int_0^t \kappa^\varepsilon(t-s) F_s^\mu \,\mathrm{d}s. \tag{4.4.12}$$

The continuity of the firing function as a map with respect to μ is not immediately apparent, as it is an infinite sum of hitting time probabilities. However, if these probabilities decay sufficiently rapidly with respect to k, we can establish that the map is continuous as a function of μ for a co-countable set of times.

Lemma 4.4.8. *Let $(\mu^n)_{n\geq 1}$ be a sequence of measures converging to μ in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{wk})$, and let $(\varepsilon_n)_{n\geq 1}$ be a sequence converging to zero. Suppose that for some $p > 2$, $\sup_{n\geq 1}\langle \mu^n, \eta \mapsto \sup_{s\leq \bar{T}} |\eta_s|^p\rangle < \infty$. Then, for* Law($\mathbf{P}^*$)*-almost every measure μ and for any $t \in \mathbb{T}^\mu$, we have*

$$F_t^{\mu^n} \to F_t^\mu \quad \textit{and} \quad \mathfrak{F}_t^{\mu^n,\varepsilon_n} \to F_t^\mu \quad \textit{as } n \to \infty.$$

Proof. By the properties of M_1-convergence, the map $\eta \mapsto \sup_{s\leq \bar{T}} |\eta_s|$ is an M_1-continuous map. Thus, by the Continuous Mapping Theorem, Vitali's Convergence Theorem, and Proposition 4.3.2, we deduce that for any $q \geq 1$,

$$\mathbb{E}\left[\langle \mathbf{P}^*, \eta \mapsto \sup_{s\leq \bar{T}} |\eta_s|^q \rangle\right] < +\infty. \tag{4.4.13}$$

Therefore, the set of μ's such that $\langle \mu, \eta \mapsto \sup_{s\leq \bar{T}} |\eta_s|^p \rangle < +\infty$ and Lemma 4.4.7 holds has full Law($\mathbf{P}^*$)-measure. For such a μ, F^μ_t is well-defined:

$$\begin{aligned} F^\mu_t &= \sum_{k\geq 1} \mu(\tau_k \leq t) \\ &= \sum_{k=1}^{M} \mu(\tau_k \leq t) + \sum_{k>M} \mu\left(\left\{\eta \ : \ \sup_{s\leq t} \eta_s \geq k-1\right\}\right) \\ &\leq \sum_{k=1}^{M} \mu(\tau_k \leq t) + \langle \mu, \eta \mapsto \sup_{s\leq \bar{T}} |\eta_s|^p \rangle \sum_{k>M} \frac{1}{(k-1)^p} < +\infty, \end{aligned} \tag{4.4.14}$$

where the penultimate inequality follows from Markov's inequality. As $t \mapsto \mu(\tau_k \leq t)$ is càdlàg and the second term in (4.4.14) can be made arbitrarily small for M sufficiently large, F^μ is càdlàg. Similarly, since $\sup_{n\geq 1} \langle \mu^n, \eta \mapsto \sup_{s\leq \bar{T}} |\eta_s|^p \rangle < +\infty$, we conclude that F^{μ^n} is càdlàg for every n and $\sup_n \|F^{\mu^n}\|_\infty < \infty$.

By (4.4.14), we have

$$\left|F^{\mu^n}_t - F^\mu_t\right| \leq \sum_{k=1}^{M} |\mu^n(\tau_k \leq t) - \mu(\tau_k \leq t)| + \langle \mu^n + \mu, \eta \mapsto \sup_{s\leq \bar{T}} |\eta_s|^p \rangle \sum_{k>M} \frac{1}{(k-1)^p}.$$

The second term on the right-hand side can be made arbitrarily small uniformly in n for all M large enough. For every $t \in \mathbb{T}^\mu$, by Lemma 4.4.7, the first term converges to zero as $n \to \infty$. Therefore, we have $F^{\mu^n}_t \to F^\mu_t$, proving the first claim. Moreover, as $\mathbb{T}^\mu$ is co-countable, for any $t \in \mathbb{T}^\mu$,

$$\left(s \mapsto F^{\mu^n}_s \mathbb{1}_{[0,\,t]}(s)\right) \xrightarrow{n\to\infty} \left(s \mapsto F^\mu_s \mathbb{1}_{[0,\,t]}(s)\right)$$

in the M_1-topology as functions from $[-1, t]$ to $\mathbb{R}$ by [71, Corollary 12.5.1]. The second claim then follows from Proposition 4.6.3. □

4.5 Martingale Arguments and Convergence

By Proposition 4.3.3, $\{(\tilde{\mathbf{P}}^\varepsilon, W^0, W)\}_\varepsilon$ is tight in $(\mathcal{P}(D_\mathbb{R}), \mathfrak{T}^{\mathrm{wk}}_{M_1}) \times (\mathcal{C}_\mathbb{R}, \|\cdot\|_\infty) \times (\mathcal{C}_\mathbb{R}, \|\cdot\|_\infty)$, where $(\mathcal{C}_\mathbb{R}, \|\cdot\|_\infty)$ denotes the space of continuous functions from $[0, T]$

to $\mathbb{R}$ endowed with the topology of uniform convergence. Henceforth, we fix a weak limit point $(\mathbf{P}^*, W^0, W)$ along a subsequence $(\varepsilon_n)_{n\geq 1}$ for which ε_n converges to zero. Although we have fixed a limit point, all the subsequent results will hold for any limit point.

Let $\mathbb{P}^n := \text{Law}(\tilde{\mathbf{P}}^{\varepsilon_n}, W^0, W)$ and $\mathbb{P}^*_{\mu,\omega^0,\omega} := \text{Law}(\mathbf{P}^*, W^0, W)$. Thus, $\mathbb{P}^n \implies \mathbb{P}^*_{\mu,\omega^0,\omega}$. For completeness, we define the probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*_{\mu,\omega^0,\omega})$, where $\Omega^* = \mathcal{P}(D_{\mathbb{R}}) \times \mathcal{C}_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}$ and $\mathcal{F}^*$ is the corresponding Borel σ-algebra. Define the random variables $\mathbf{P}^*$, W^0, and W on Ω^* such that for any tuple (μ, ω^0, ω),

$$\mathbf{P}^*(\mu, \omega^0, \omega) = \mu, \qquad W^0(\mu, \omega^0, \omega) = \omega^0, \qquad \text{and} \qquad W(\mu, \omega^0, \omega) = \omega.$$

Hence, the joint law of $(\mathbf{P}^*, W^0, W)$ is $\mathbb{P}^*_{\mu,\omega^0,\omega}$, and $\mathcal{F}^* = \sigma(\mathbf{P}^*, W^0, W)$. We also recall the definition of the limiting firing function $F^* := \sum_{k\geq 1} \mathbf{P}^*(\tau_k \leq t)$ and the co-countable set of times

$$\mathbb{T} := \left\{ t \in [-1, \bar{T}] \,:\, \mathbb{P}^*_{\mu,\omega^0,\omega}(\eta_t = \eta_{t-}) = 1,\, \mathbb{P}^*_{\mu,\omega^0,\omega}(\tau_k = t) = 0 \;\forall\, k \right\}. \tag{4.5.1}$$

We shall use $\mathbb{P}^*_{\mu,\omega^0,\omega}$ to construct a probability space on which we can define a process that solves (4.2.3). Before proceeding, we aim to determine the form of our limiting process through standard martingale arguments. To this end, for any $\varepsilon > 0$, we define the following functionals $\mathcal{M}$, $\mathcal{M}^{\varepsilon,D} : \mathcal{P}(D_{\mathbb{R}}) \times D_{\mathbb{R}} \to D_{\mathbb{R}}$:

$$\mathcal{M}^{\varepsilon,D}(\mu, \eta) = \eta - \eta_{-1} - \int_0^{\cdot} b(s, \eta_s - J^D_s, \nu^{\mu,D}_s)\, \mathrm{d}s - \int_0^{\cdot} \alpha(s)\, \mathrm{d}\mathfrak{F}^{\mu,\varepsilon}_s, \tag{4.5.2}$$

$$\mathcal{M}(\mu, \eta) = \eta - \eta_{-1} - \int_0^{\cdot} b(s, \eta_s - J_s, \nu^{\mu}_s)\, \mathrm{d}s - \int_{[0,\cdot]} \alpha(s)\, \mathrm{d}F^{\mu}_s, \tag{4.5.3}$$

where F^μ_t and $\mathfrak{F}^{\mu,\varepsilon}_t$ are defined in (4.4.12). We define the processes $\nu^{\mu^n,D}$ and ν^μ as follows:

$$\nu^{\mu^n,D}_t := \mu^n(\eta_t - \tilde{J}^D_t \in \cdot,\, \eta_t - \tilde{J}^D_t < 0), \quad \text{and} \quad \nu^\mu_t := \mu(\eta_t - \tilde{J}_t \in \cdot).$$

For $s_0, t_0 \in \mathbb{T}^\mu \cap [0, T)$ with $s_0 < t_0$ and $\{s_i\}_{i=1}^k \subset [0, s_0] \cap \mathbb{T}$, we introduce the following function:

$$G \,:\, D_{\mathbb{R}} \to \mathbb{R}, \quad \eta \mapsto (\eta_{t_0} - \eta_{s_0}) \prod_{i=1}^k g_i(\eta_{s_i}), \tag{4.5.4}$$

where $g_i \in \mathcal{C}_b(\mathbb{R})$ are arbitrary bounded continuous functions. We further define the following functionals:

$$\begin{cases} \Psi^{\varepsilon,D}(\mu) = \left\langle \mu, \eta \mapsto G\left(\mathcal{M}^{\varepsilon,D}(\mu,\eta)\right)\right\rangle, \\ \Upsilon^{\varepsilon,D}(\mu) = \left\langle \mu, \eta \mapsto G\left((\mathcal{M}^{\varepsilon,D}(\mu,\eta))^2 - \int_0^{\cdot} \sigma(s,\eta_s - J_s^D)^2 \,\mathrm{d}s\right)\right\rangle, \\ \Theta^{\varepsilon,D}(\mu,\omega) = \left\langle \mu, \eta \mapsto G\left(\mathcal{M}^{\varepsilon,D}(\mu,\eta) \times \omega - \int_0^{\cdot} \sigma(s,\eta_s - J_s^D)\sqrt{1-\rho(s,\nu_s^\mu)^2}\,\mathrm{d}s\right)\right\rangle, \\ \Theta^{0,\varepsilon,D}(\mu,\omega^0) = \left\langle \mu, \eta \mapsto G\left(\mathcal{M}^{\varepsilon,D}(\mu,\eta) \times \omega^0 - \int_0^{\cdot} \sigma(s,\eta_s - J_s^D)\rho(s,\nu_s^\mu)\,\mathrm{d}s\right)\right\rangle. \end{cases} \tag{4.5.5}$$

We also consider the corresponding functionals without mollification, denoted by $\Psi(\mu)$, $\Upsilon(\mu)$, $\Theta(\mu,\omega)$, and $\Theta^0(\mu,\omega^0)$. These are defined analogously to their mollified counterparts, with $\mathcal{M}^{\varepsilon,D}$ replaced by $\mathcal{M}$ throughout. With these definitions in place, we can now state the following proposition:

Proposition 4.5.1 (Functional Continuity Generalised)**.** *For $\mathbb{P}^*_{\mu,\omega^0,\omega}$-almost every measure μ, we have that $\Psi^{\varepsilon_n,D}(\mu^n)$, $\Upsilon^{\varepsilon_n,D}(\mu^n)$, $\Theta^{\varepsilon_n,D}(\mu^n,\omega^n)$, and $\Theta^{0,\varepsilon_n,D}(\mu^n,\omega^{0,n})$ converge to $\Psi(\mu)$, $\Upsilon(\mu)$, $\Theta(\mu,\omega)$, and $\Theta^0(\mu,\omega^0)$, respectively, whenever $(\mu^n,\omega^{0,n},\omega^n) \to (\mu,\,\omega^0,\,\omega)$ in $(\mathcal{P}(D_\mathbb{R}),\mathfrak{T}^{wk}_{M_1}) \times (\mathcal{C}_\mathbb{R},\|\cdot\|_\infty) \times (\mathcal{C}_\mathbb{R},\|\cdot\|_\infty)$, along a sequence for which $\sup_{n\geq 1}\langle \mu^n, \tilde{\eta} \mapsto \sup_{s\leq \bar{T}} |\tilde{\eta}_s|^p\rangle < \infty$ for some $p > 2$ and $(\varepsilon_n)_{n\geq 1}$ that converges to zero.*

Proof. By Lemma 4.4.5 and the definition of $\mathbb{T}$, there exists a set of measures μ with full $\mathbb{P}^*_{\mu,\omega^0,\omega}$-measure, such that for any $h > 0$,

$$\mu\left(\bigcup_{k\geq 1}\left\{\eta \,:\, \sup_{s\in(\tau_k,\,(\tau_k+h)\wedge\bar{T})}\{\eta_s - \eta_{\tau_k}\} \leq 0,\, \tau_k < \bar{T}\right\}\right) = 0,$$

and additionally, $\mu(\eta_{s_i} = \eta_{s_i-}) = 1$, $\mu(\eta_{t_0} = \eta_{t_0-}) = 1$, and $\mu(\tau_k = t_0) = 0$ for all k. We begin by showing the convergence of $\Psi^{\varepsilon_n}(\mu^n)$ to $\Psi(\mu)$. Recall the well-known result for Borel measurable functions f and g of finite variation: for any $t > 0$,

$$f_t g_t = f_0 g_0 + \int_{(0,t]} f_{s-}\,\mathrm{d}g_s + \int_{(0,t]} g_{s-}\,\mathrm{d}f_s + \sum_{s\leq t} \Delta f_s \Delta g_s.$$

Combining this with the continuous differentiability of α and the result from Lemma 4.4.8, which shows that $\mathfrak{F}_t^{\mu^n,\varepsilon_n}$ converges to F_t^μ on a co-countable set of times, we obtain:

$$\begin{aligned} \int_0^t \alpha(s)\,\mathrm{d}\mathfrak{F}_s^{\mu,\varepsilon_n} &= \alpha(t)\mathfrak{F}_t^{\mu,\varepsilon_n} - \int_0^t \mathfrak{F}_s^{\mu,\varepsilon_n}\alpha'(s)\,\mathrm{d}s \to \alpha(t)F_t^\mu - \int_0^t F_s^\mu \alpha'(s)\,\mathrm{d}s \\ &= \int_{[0,t]} \alpha(s)\,\mathrm{d}F_s^\mu. \end{aligned}$$

As $\mu^n \implies \mu$, Skorokhod's Representation Theorem allows us to construct $(H^n)_{n\geq 1}$ and H on a common probability space such that $\mathrm{Law}(H^n) = \mu^n$, $\mathrm{Law}(H) = \mu$, and $H^n \to H$ almost surely in $(D_{\mathbb{R}}, M_1)$. Consequently,

$$\Psi^{\varepsilon_n}(\mu^n) = \mathbb{E}\left[G(\mathcal{M}^{\varepsilon_n,D}(\mu^n, H^n))\right] \quad \text{and} \quad \Psi(\mu) = \mathbb{E}\left[G(\mathcal{M}(\mu, H))\right].$$

By Lemma 4.6.8, we have

$$\int_0^t b(s, H_s^n - \tilde{J}_s^D(H^n), \nu_s^{\mu^n,D})\,\mathrm{d}s \to \int_0^t b(s, H_s - \tilde{J}_s(H), \nu_s^{\mu})\,\mathrm{d}s \tag{4.5.6}$$

almost surely for any $t \geq 0$. Since $\mathbb{T}^{\mu}$ contains all the almost sure continuity points of Z, by the properties of M_1-convergence and (4.5.6), we have

$$H_t^n - H_{-1}^n - \int_0^t b(s, H_s^n - \tilde{J}_s^D(H^n), \nu_s^{\mu^n,D})\,\mathrm{d}s \to H_t - H_{-1} - \int_0^t b(s, H_s - \tilde{J}_s(H), \nu_s^{\mu})\,\mathrm{d}s$$

almost surely for any $t \in \{t_0, s_0, \ldots, s_k\}$. Hence, we can conclude that $G(\mathcal{M}^{\varepsilon_n,D}(\mu^n, H^n))$ converges almost surely to $G(\mathcal{M}(\mu, H))$ in $\mathbb{R}$. Furthermore, we observe that

$$\left\langle \mu^n,\, |\mathcal{M}^{\varepsilon_n,D}(\mu^n, \cdot)|^p \right\rangle \leq C\left(\left\langle \mu^n,\, \tilde{\eta} \mapsto \sup_{s\leq \bar{T}} |\tilde{\eta}_s|^p \right\rangle + 1\right), \tag{4.5.7}$$

for some constant C that depends only on p and b, and is uniform in n. Consequently, $G(\mathcal{M}^{\varepsilon_n,D}(\mu^n, H^n))$ is L^p-bounded uniformly in n, as

$$\left|G(\mathcal{M}^{\varepsilon_n,D}(\mu^n, H^n))\right|^p \leq C\left(\left|\mathcal{M}_{t_0}^{\varepsilon_n,D}(\mu^n, H^n)\right|^p + \left|\mathcal{M}_{s_0}^{\varepsilon_n,D}(\mu^n, H^n)\right|^p\right),$$

and $\mathbb{E}[|\mathcal{M}_{t_0}^{\varepsilon_n,D}(\mu^n, H^n)|^p] = \langle \mu^n, |\mathcal{M}_{t_0}^{\varepsilon_n,D}(\mu^n, \cdot)|^p \rangle$, where the latter is uniformly bounded in n for some $p > 2$ by (4.5.7) and our assumption. Therefore, by Vitali's Convergence Theorem, we can conclude that $\Psi^{\varepsilon_n}(\mu^n)$ converges to $\Psi(\mu)$.

The convergence of $\Upsilon^{\varepsilon_n,D}(\mu^n)$, $\Theta^{\varepsilon_n,D}(\mu^n, \omega^n)$, and $\Theta^{0,\varepsilon_n,D}(\mu^n, \omega^{0,n})$ to $\Upsilon(\mu)$, $\Theta(\mu, \omega)$, and $\Theta^0(\mu, \omega^0)$, respectively, follows from similar arguments. By Assumption 4.2.2 (ii) and (iv), σ and ρ are totally bounded. Consequently, $\Upsilon^{\varepsilon_n,D}(\mu^n)$, $\Theta^{\varepsilon_n,D}(\mu^n, \omega^n)$, and $\Theta^{0,\varepsilon_n,D}(\mu^n, \omega^{0,n})$ are L^p-bounded uniformly in n. The continuity of σ and the almost sure convergence of H^n to H in the M_1-topology ensure that

$$\int_0^t \sigma(s, H_s^n - \tilde{J}_s^D(H^n))^2\,\mathrm{d}s \to \int_0^t \sigma(s, H_s - \tilde{J}_s(H))^2\,\mathrm{d}s$$

almost surely for all $t \geq 0$. Furthermore, the bounds on σ and the boundedness of ρ yield the following:

$$|\sigma(t,x)\rho(t,\mu) - \sigma(t,\tilde{x})\rho(t,\tilde{\mu})| \leq C\left(|x-\tilde{x}| + d_1(\mu,\,\tilde{\mu})\right),$$
$$\left|\sigma(t,x)\sqrt{1-\rho(t,\mu)^2} - \sigma(t,\tilde{x})\sqrt{1-\rho(t,\tilde{\mu})^2}\right| \leq C\left(|x-\tilde{x}| + d_1(\mu,\,\tilde{\mu})\right).$$

As $d_1(\mu,\,\tilde{\mu}) \leq d_0(\mu,\,\tilde{\mu})$, we deduce that the functions $(t,x,\mu) \mapsto \sigma(t,x)\rho(t,\mu)$ and $(t,x,\mu) \mapsto \sigma(t,x)\sqrt{1-\rho(t,\mu)^2}$ satisfy Assumption 4.2.2 (i). Applying Lemma 4.6.8, we can conclude that:

$$\int_0^t \sigma(s, H_s^n - \tilde{J}_s^D(H^n))\sqrt{1-\rho(s,\nu_s^{\mu^n,D})^2}\,\mathrm{d}s \to \int_0^t \sigma(s, H_s - \tilde{J}_s(H))\sqrt{1-\rho(s,\nu_s^{\mu})^2}\,\mathrm{d}s,$$
$$\int_0^t \sigma(s, H_s^n - \tilde{J}_s^D(H^n))\rho(s,\nu_s^{\mu^n,D})\,\mathrm{d}s \to \int_0^t \sigma(s, H_s - \tilde{J}_s(H))\rho(s,\nu_s^{\mu})\,\mathrm{d}s,$$

almost surely for all $t \geq 0$. □

With this convergence result established, we can now proceed to employ the properties of the smoothed approximating processes to show that $\mathcal{M}$ is a martingale and deduce the form of the limiting process on a probability space we will construct. To this end, we define the probability space $(\bar{\Omega},\,\bar{\mathcal{F}},\,\bar{\mathbb{P}})$, where $\bar{\Omega} = \Omega^* \times D_{\mathbb{R}} = \mathcal{P}(D_{\mathbb{R}}) \times \mathcal{C}_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}} \times D_{\mathbb{R}}$ and $\bar{\mathcal{F}}$ is the Borel σ-algebra $\mathcal{B}(\bar{\Omega})$. We define the probability measure

$$\bar{\mathbb{P}}(A) \coloneqq \int_{\mathcal{P}(D_{\mathbb{R}})\times\mathcal{C}_{\mathbb{R}}\times\mathcal{C}_{\mathbb{R}}} \mu\left(\{\eta \,:\, (\mu,\omega^0,\omega,\eta) \in A\}\right)\mathrm{d}\mathbb{P}^*_{\mu,\omega^0,\omega}(\mu,\omega^0,\omega), \tag{4.5.8}$$

for any $A \in \mathcal{B}(\bar{\Omega})$. Observe that, by construction, for any $A \in \mathcal{B}(\bar{\Omega})$,

$$\bar{\mathbb{P}}(A) = \mathbb{E}^*\left[\left\langle \mathbf{P}^*,\, \mathbb{1}_A(\mathbf{P}^*,\, W^0,\, W,\, \cdot)\right\rangle\right].$$

Under $\bar{\mathbb{P}}$, W^0 and W remain Brownian motions. As established in Lemma 2.5.1, the independence between the idiosyncratic noise and the common noise persists in the limit. Consequently, we have $\mathbb{P}^*_{\mu,\omega^0,\omega} = \mathbb{P}^*_{\mu,\omega^0} \otimes \mathbb{P}^*_{\omega}$, and hence, for any $A \in \mathcal{B}(\mathcal{P}(D_{\mathbb{R}}) \times \mathcal{C}_{\mathbb{R}})$ and $B \in \mathcal{B}(\mathcal{C}_{\mathbb{R}})$,

$$\bar{\mathbb{P}}\left[(\mathbf{P}^*,\, W^0) \in A,\, W \in B\right] = \bar{\mathbb{P}}\left(A \times B \times D_{\mathbb{R}}\right) = \mathbb{P}^*_{\mu,\omega^0,\omega}(A \times B) = \mathbb{P}^*_{\mu,\omega^0}(A)\mathbb{P}^*_{\omega}(B).$$

Therefore, $(\mathbf{P}^*,\, W^0)$ is independent of W. This leads us to the following result:

Proposition 4.5.2. *Let $\mathcal{M}$ be given as in (4.5.3). Then, $\mathcal{M}_\cdot$, $\mathcal{M}_\cdot^2 - \int_0^\cdot \sigma(s,\pi_s - J_s)^2\,\mathrm{d}s$, $\mathcal{M}_\cdot \times W_\cdot - \int_0^\cdot \sigma(s,\pi_s - J_s)\sqrt{1-\rho(s,\nu_s^\cdot)^2}\,\mathrm{d}s$, and $\mathcal{M}_\cdot \times W_\cdot^0 - \int_0^\cdot \sigma(s,\pi_s - J_s)\rho(s,\nu_s^\cdot)\,\mathrm{d}s$ are all continuous martingales on $(\bar{\Omega},\,\bar{\mathcal{F}},\,\bar{\mathbb{P}})$ for $t \in [0,T]$, where $\pi_s \colon D_{\mathbb{R}} \to \mathbb{R}$, $\pi_s(\eta) = \eta_s$.*

Proof. If $\mathcal{M}$ is continuous, then the continuity of the other processes follows from the continuity of $\mathcal{M}$ and the continuity of integration. By first considering $\mathcal{M}$, to show that $\mathcal{M}$ is a martingale, it is sufficient by a Monotone Class argument that

$$\bar{\mathbb{E}}\left[G(\mathcal{M})\right] = 0. \tag{4.5.9}$$

To begin, we recall that $\mathbb{P}^{\varepsilon_n} \implies \mathbb{P}^*$ where $\mathbb{P}^{\varepsilon_n} = \text{Law}(\tilde{\mathbf{P}}^{\varepsilon_n}, W^0, W)$ and $\mathbb{P}^* = \text{Law}(\mathbf{P}^*, W^0, W)$. By Skorokhod's Representation Theorem, we can find $\{(\mathbf{Q}^n, B^n, \tilde{B}^n)\}_{n\geq 1}$ and $(\mathbf{Q}^*, B^*, \tilde{B}^*)$ defined on some common probability space such that $\mathbb{P}^{\varepsilon_n} = \text{Law}(\mathbf{Q}^n,\, B^n,\, \tilde{B}^n)$, $\mathbb{P}^* = \text{Law}(\mathbf{Q}^*,\, B^*,\, \tilde{B}^*)$, and $(\mathbf{Q}^n,\, B^n,\, \tilde{B}^n) \to (\mathbf{Q}^*,\, B^*,\, \tilde{B}^*)$ almost surely. By definition of $\mathcal{M}$, for any $p > 1$, we have:

$$\mathbb{E}\left[\left\langle \tilde{\mathbf{P}}^{\varepsilon_n},\, \eta \mapsto \sup_{s\leq \bar{T}} |\eta_s|^p \right\rangle\right] = \mathbb{E}\left[\sup_{s\leq \bar{T}} \left|\tilde{Z}^{\varepsilon}\right|^p\right] \leq C, \tag{4.5.10}$$

where the constant C is from Proposition 4.3.2. Hence, by employing the arguments from Proposition 2.5.3, for any $p > 2$ fixed, we can extract a subsequence (also denoted by n for simplicity) such that $\langle \mathbf{Q}^n,\, \sup_{s\leq \bar{T}} |\eta_s|^p\rangle$ converges almost surely to $\langle \mathbf{Q}^*,\, \sup_{s\leq \bar{T}} |\eta_s|^p\rangle$. By (4.5.10), we can deduce that $\langle \mathbf{Q}^*,\, \sup_{s\leq \bar{T}} |\eta_s|^p\rangle < +\infty$ almost surely. Therefore, restricting ourselves to this subsequence henceforth, we have

$$\sup_{n\geq 1} \left\langle \mathbf{Q}^n,\, \sup_{s\leq T} |\eta_s|^p \right\rangle < \infty$$

almost surely. By definition of $\mathcal{M}$, for the same fixed $p > 2$, we have:

$$\begin{aligned}
\mathbb{E}\left[\left\langle \tilde{\mathbf{P}}^{\varepsilon_n},\, \left|\mathcal{M}_t^{\varepsilon_n, D}(\tilde{\mathbf{P}}^{\varepsilon_n}, \cdot)\right|^p \right\rangle\right] \leq & C_p \mathbb{E}\left[\left|\int_0^t \sigma(s,\, X_s^{\varepsilon_n}) \sqrt{1 - \rho^2(s, \boldsymbol{\nu}_s^{\varepsilon_n})}\, \mathbb{1}_{\{X_s^{\varepsilon_n} < 0\}}\, \mathrm{d}W_s \right|^p\right] \\
& + C_p \mathbb{E}\left[\left|\int_0^t \sigma(s,\, X_s^{\varepsilon_n}) \rho(s, \boldsymbol{\nu}_s^{\varepsilon_n})\, \mathbb{1}_{\{X_s^{\varepsilon_n} < 0\}}\, \mathrm{d}W_s^0 \right|^p\right] \\
& + C_p \mathbb{E}\left[\left|\int_0^t b(s, X_s^{\varepsilon_n}, \boldsymbol{\nu}_s^{\varepsilon_n})\, \mathbb{1}_{\{X_s^{\varepsilon_n} = 0\}}\, \mathrm{d}s \right|^p\right] \\
& + C_p \mathbb{E}\left[\left|\int_0^t \alpha(s)\, \mathbb{1}_{\{X_s^{\varepsilon_n} = 0\}}\, \mathrm{d}\mathfrak{F}_s^{\varepsilon_n} \right|^p\right] \\
\leq & C_{\alpha, b, p, T, \sigma},
\end{aligned} \tag{4.5.11}$$

where $C_{\alpha,b,p,T,\sigma} > 0$ depends on the constant from applying the Burkholder-Davis-Gundy inequality, p, and the bounds on σ but is independent of ε. The constant C_p depends only on p and the constant from Proposition 4.6.7. Hence, $\mathbb{E}[\langle \tilde{\mathbf{P}}^{\varepsilon_n},\, |\mathcal{M}_t^{\varepsilon_n, D}(\tilde{\mathbf{P}}^{\varepsilon_n}, \cdot)|^p\rangle]$ is bounded uniformly in $\varepsilon > 0$, and therefore

$\mathbb{E}[\langle \mathbf{Q}^n, |\mathcal{M}_t^{\varepsilon_n,D}(\mathbf{Q}^n,\cdot)|^p\rangle] < \infty$ uniformly in n. Employing Proposition 4.5.1 and Vitali's Convergence Theorem, we obtain:

$$\begin{aligned}\bar{\mathbb{E}}\left[G(\mathcal{M})\right] &= \mathbb{E}^*\left[\left\langle \mathbf{P}^*,\, G(\mathcal{M}(\mathbf{P}^*,\, W^0,\, W,\, \cdot))\right\rangle\right] \\ &= \lim_{n\to\infty} \mathbb{E}\left[\left\langle \mathbf{P}^{\varepsilon_n},\, G(\mathcal{M}^{\varepsilon_n,D}(\mathbf{P}^{\varepsilon_n},\, W^0,\, W,\, \cdot))\right\rangle\right].\end{aligned}$$

Recall that for any $s_0,\, t_0 \in \mathbb{T}^\mu \cap [0,\, T)$ with $s_0 < t_0$, $\{s_i\}_{i=1}^k \subset [0, s_0] \cap \mathbb{T}$, and $g_i \in \mathcal{C}_b(\mathbb{R})$, we define $G(\eta) = (\eta_{t_0} - \eta_{s_0}) \prod_{i=1}^k g_i(\eta_{s_i})$. Consequently,

$$\begin{aligned}&\mathbb{E}\left[\left\langle \mathbf{P}^{\varepsilon_n},\, G(\mathcal{M}^{\varepsilon_n,D}(\mathbf{P}^{\varepsilon_n},\, W^0,\, W,\, \cdot))\right\rangle\right] \\ &\qquad = \mathbb{E}\left[\left(\mathcal{Y}_{t_0}^{\varepsilon_n} - \mathcal{E}_{t_0}^{\varepsilon_n} - \mathcal{Y}_{s_0}^{\varepsilon_n} + \mathcal{E}_{s_0}^{\varepsilon_n}\right) \prod_{i=1}^k g_i(\mathcal{Y}_{s_i}^{\varepsilon_n} - \mathcal{E}_{s_i}^{\varepsilon_n})\right],\end{aligned} \tag{4.5.12}$$

where $\mathcal{E}_t^{\varepsilon_n}$ is the error due to the process being at zero, defined as

$$\mathcal{E}_t^{\varepsilon_n} := \int_0^t b(s, X_s^{\varepsilon_n}, \boldsymbol{\nu}_s^{\varepsilon_n}) \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}}\, \mathrm{d}s + \int_0^t \alpha(s) \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}}\, \mathrm{d}\mathfrak{F}_s^{\varepsilon_n}.$$

By the boundedness assumption on σ, Assumption 4.2.2 (ii), $\mathcal{Y}^{\varepsilon_n}$ is a martingale. As $s_1 \le \ldots \le s_k \le s_0 < t_0$, applying the tower property to (4.5.12) yields:

$$\mathbb{E}\left[\left\langle \mathbf{P}^{\varepsilon_n},\, G(\mathcal{M}^{\varepsilon_n,D}(\mathbf{P}^{\varepsilon_n},\, W^0,\, W,\, \cdot))\right\rangle\right] = \mathbb{E}\left[\left(-\mathcal{E}_{t_0}^{\varepsilon_n} + \mathcal{E}_{s_0}^{\varepsilon_n}\right) \prod_{i=1}^k g_i(\mathcal{Y}_{s_i}^{\varepsilon_n} - \mathcal{E}_{s_i}^{\varepsilon_n})\right]. \tag{4.5.13}$$

Employing Proposition 4.6.7, we find that (4.5.13) converges to zero. Thus, we have established (4.5.9). For the other three processes, the proof follows an identical structure. The only difference lies in the terms of (4.5.12), which take one of the following forms:

- $(\mathcal{Y}_t^{\varepsilon_n})^2 - \int_0^t \sigma(s,\, X_s^{\varepsilon_n})^2 (\mathbb{1}_{\{X_s^{\varepsilon_n}<0\}} + \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}})\, \mathrm{d}s - 2\mathcal{Y}_t^{\varepsilon_n}\mathcal{E}_t^{\varepsilon_n} - (\mathcal{E}_t^{\varepsilon_n})^2$
- $\mathcal{Y}_t^{\varepsilon_n} \times W_t - \int_0^t \sigma(s,\, X_s^{\varepsilon_n}) \sqrt{1-\rho^2(s, \boldsymbol{\nu}_s^{\varepsilon_n})} (\mathbb{1}_{\{X_s^{\varepsilon_n}<0\}} + \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}})\, \mathrm{d}s - \mathcal{E}_t^{\varepsilon_n} \times W_t$
- $\mathcal{Y}_t^{\varepsilon_n} \times W_t^0 - \int_0^t \sigma(s,\, X_s^{\varepsilon_n}) \rho(s, \boldsymbol{\nu}_s^{\varepsilon_n}) (\mathbb{1}_{\{X_s^{\varepsilon_n}<0\}} + \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}})\, \mathrm{d}s - \mathcal{E}_t^{\varepsilon_n} \times W_t^0$

In each of these cases, we have a martingale term and a term that converges to zero by Proposition 4.6.7. Consequently, the other three terms in the statement are also martingales.

To establish the continuity of $\mathcal{M}$, we consider the function

$$\tilde{G} :\ D_{\mathbb{R}} \to \mathbb{R}, \quad \eta \mapsto |\eta_t - \eta_s|^4,$$

for $s,\, t \in \mathbb{T} \cap [0, T)$. Furthermore, we define the functionals

$$\tilde{\Psi}^{\varepsilon, D}(\mu) = \left\langle \mu,\, \tilde{G}(\mathcal{M}^{\varepsilon, \mathcal{D}}(\mu,\, \cdot)) \right\rangle, \qquad \tilde{\Psi}(\mu) = \left\langle \mu,\, \tilde{G}(\mathcal{M}(\mu,\, \cdot)) \right\rangle.$$

Following the same proof as in Proposition 4.5.1, we have for $\mathbb{P}^*_{\mu,\omega^0,\omega}$-almost every measure μ, $\tilde{\Psi}^{\varepsilon_n, D}(\mu^n)$ converges to $\tilde{\Psi}(\mu)$ whenever $\mu^n \to \mu$ in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}^{\mathrm{wk}}_{M_1})$ along a sequence for which $\sup_{n\geq 1} \langle \mu^n,\, \eta \mapsto \sup_{s\leq \bar{T}} |\eta_s|^p \rangle < \infty$ for some $p > 4$. We have finite moments for any $p > 1$, by (4.5.11). Therefore, by the above reasoning and Vitali's convergence theorem, for any $s, t \in \mathbb{T} \cap [0,\, T)$ we have

$$\begin{aligned}
\bar{\mathbb{E}}\, |\mathcal{M}_t - \mathcal{M}_s|^4 &= \mathbb{E}^* \left[\left\langle \mathbf{P}^*,\, |\mathcal{M}_t(\mathbf{P}^*,\, \cdot) - \mathcal{M}_s(\mathbf{P}^*,\, \cdot)|^4 \right\rangle \right] \\
&= \lim_{n\to\infty} \mathbb{E} \left[\left\langle \tilde{\mathbf{P}}^{\varepsilon_n},\, \left| \mathcal{M}^{\varepsilon_n, D}_t(\tilde{\mathbf{P}}^{\varepsilon_n},\, \cdot) - \mathcal{M}^{\varepsilon_n, D}_s(\tilde{\mathbf{P}}^{\varepsilon_n},\, \cdot) \right|^4 \right\rangle \right].
\end{aligned}$$

By the definition of $\tilde{\mathbf{P}}^{\varepsilon_n}$ and the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
\mathbb{E} \left[\left\langle \tilde{\mathbf{P}}^{\varepsilon_n},\, \left| \mathcal{M}^{\varepsilon_n, D}_t(\tilde{\mathbf{P}}^{\varepsilon_n},\, \cdot) - \mathcal{M}^{\varepsilon_n, D}_s(\tilde{\mathbf{P}}^{\varepsilon_n},\, \cdot) \right|^4 \right\rangle \right] &= \mathbb{E}\, |\mathcal{Y}^{\varepsilon_n}_t - \mathcal{E}^{\varepsilon_n}_t - \mathcal{Y}^{\varepsilon_n}_s + \mathcal{E}^{\varepsilon_n}_s|^4 \\
&\leq C\, |t - s|^2 + C\mathbb{E}\, |\mathcal{E}^{\varepsilon_n}_t|^4 + C\mathbb{E}\, |\mathcal{E}^{\varepsilon_n}_s|^4,
\end{aligned}$$

where the constant C is uniform in n. By employing Proposition 4.6.7, the expectations above converge to zero as $n \to \infty$. Therefore,

$$\bar{\mathbb{E}}\, |\mathcal{M}_t - \mathcal{M}_s|^4 \leq C\, |t - s|^2.$$

As $\mathbb{T}$ is dense, by Kolmogorov's Continuity Criterion, there exists a continuous process that is a modification of $\mathcal{M}$. Since $\mathcal{M}$ is right-continuous and $\mathbb{T}$ is dense, these two processes are indistinguishable. Hence, $\mathcal{M}$ has a continuous version. □

Now, we have all the ingredients to prove Theorem 4.2.3.

Proof of Theorem 4.2.3. By Proposition 4.3.3, $\{(\tilde{\mathbf{P}}^{\varepsilon}, W^0, W)\}_{\varepsilon>0}$ is tight. As we are on a Polish space, by Prokhorov's Theorem, tightness is equivalent to being sequentially pre-compact. Therefore, for any subsequence $\{(\mathbf{P}^{\varepsilon_n}, W^0, W)\}_{n\geq 1}$, where $(\varepsilon_n)_{n\geq 1}$ is a positive sequence that converges to zero, we have a convergent subsubsequence. Fix a limit point $(\mathbf{P}^*, W^0, W)$ of this subsequence. With $(\mathbf{P}^*, W^0, W)$ fixed, we define the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ exactly as in (4.5.8). Now, define the càdlàg processes X and Z by

$$\begin{aligned}
X : \bar{\Omega} \to D_{\mathbb{R}}, &\qquad (\mu,\, \omega^0,\, \omega,\, \eta) \mapsto \eta - \tilde{J}(\eta), \\
Z : \bar{\Omega} \to D_{\mathbb{R}}, &\qquad (\mu,\, \omega^0,\, \omega,\, \eta) \mapsto \eta.
\end{aligned}$$

Then, by the construction of $\bar{\mathbb{P}}$ and that $\mathbb{P}^*_{\mu,\omega^0,\omega} = \mathbb{P}^*_{\mu,\omega^0} \times \mathbb{P}^*_{\omega}$ by Lemma 2.5.1, for all $A \in \mathcal{B}(D_{\mathbb{R}})$, $S \in \mathcal{B}(\mathcal{P}(D_{\mathbb{R}}) \times \mathcal{C}_{\mathbb{R}})$,

$$\bar{\mathbb{P}}\left[Z \in A,\, (\mathbf{P}^*,\, W^0) \in S\right] = \int_S \mu(A)\, \mathrm{d}\mathbb{P}^*_{\mu,\omega^0}.$$

Note that $\mathbb{P}^*_{\mu,\omega^0,\omega} = \mathrm{Law}(\mathbf{P}^*,\, W^0,\, W)$, $\mathbb{P}^*_{\mu,\omega^0} = \mathrm{Law}(\mathbf{P}^*,\, W^0)$ and $\mathbb{P}^*_{\omega} = \mathrm{Law}(W)$. Consequently,

$$\bar{\mathbb{P}}\left[Z \in A |\, W^0, \mathbf{P}^*\right] = \mathbf{P}^*(A) \qquad \forall \quad A \in \mathcal{B}(D_{\mathbb{R}}).$$

By Proposition 4.5.2,

$$\mathcal{M}_t = Z_t - Z_{-1} - \int_0^t b(s,\, X_s,\, \boldsymbol{\nu}^*_s)\, \mathrm{d}s - \int_{[0,t]} \alpha(s)\, \mathrm{d}F^*_s$$

is a continuous local martingale with

$$\langle \mathcal{M} \rangle_t = \int_0^t \sigma(s, X_s)^2\, \mathrm{d}s, \quad \langle \mathcal{M},\, W \rangle_t = \int_0^t \sigma(s, X_s)\sqrt{1 - \rho(s, \boldsymbol{\nu}^*_s)^2}\, \mathrm{d}s,$$
$$\left\langle \mathcal{M},\, W^0 \right\rangle_t = \int_0^t \sigma(s, X_s)\rho(s, \boldsymbol{\nu}^*_s)\, \mathrm{d}s,$$

where $\boldsymbol{\nu}^*_s \coloneqq \mathbf{P}^*(X_s \in \cdot)$. As W^0 and W are standard independent Brownian motions, by Lévy's Characterisation Theorem we have that

$$\mathcal{M}_t = \int_0^t \sigma(s, X_s) \left(\sqrt{1 - \rho(s, \boldsymbol{\nu}^*_s)^2}\, \mathrm{d}W_s + \rho(s, \boldsymbol{\nu}^*_s)\, \mathrm{d}W^0_s \right).$$

Now, as $-1 \in \mathbb{T}$, the map $\eta \mapsto \eta_{-1}$ is μ-almost surely continuous for $\mathbb{P}_{\mu,\omega^0,\omega}$-almost every measure μ. A simple application of the Portmanteau Theorem shows that $Z_{-1} \sim \nu_{0-}$. By Lemma 4.4.1, we deduce that $Z_{0-} \sim \nu_{0-}$. The independence between $(\mathbf{P}^*,\, W^0)$ and W follows from Lemma 2.5.1. A similar argument as one employed in Lemma 2.5.1 shows $Z_{0-} \perp (\mathbf{P}^*,\, W^0,\, W)$.

Now we note that $\tilde{J}_{0-}(Z) = 0$, therefore $X_{0-} = Z_{0-} \sim \nu_{0-}$. By Proposition 4.6.12, on a set of full probability, the jumps of F^* are bounded by 1 on $[0, T)$. Therefore, we can conclude that

$$\tau^X_1 \coloneqq \inf\{t \geq 0 \,:\, X_{t-} + \alpha_t \Delta F_t \geq 0\} = \inf\{t \geq 0 \,:\, Z_t \geq 0\} = \tau_1(Z) \qquad \text{a.s.}$$

and

$$\tau^X_{k+1} = \inf\{t \geq \tilde{\tau}_k \,:\, X_{t-} + \alpha_t \Delta F_t \geq 0\} = \inf\{t \geq \tau_k \,:\, Z_t \geq k - 1\} = \tau_{k+1}(Z) \qquad \text{a.s.}$$

Therefore, we have shown that the stopping times of X and Z are the same. Hence, we may conclude that $F_t^* = \sum_k \mathbf{P}^*(\tau_k \leq t) = \sum_{k\geq 1} \bar{\mathbb{P}}(\tau_k^X \leq t)$ and $\tilde{J}_t(X) = \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^X)$. Lastly, by Proposition 4.6.11,

$$\Delta F_t \leq \inf\{x \geq 0 : \boldsymbol{\nu}_{t-}([-\alpha(t)x, 0]) < x\} \qquad \text{a.s.}$$

for all $t \geq 0$. Hence, we have shown that X is a solution in the sense of Definition 4.2.1. □

4.6 Technical Results

This section contains the proofs of several technical lemmas which were deferred from the previous sections to aid with presentation. These results are vital for the tightness and convergence results detailed above.

Proposition 4.6.1. *For any $p \in \mathbb{N}$ and $a, b, h > 0$, we have*

$$(a+b)^p \leq a^p(1+h) + C_{p,h}b^p,$$

where $C_{p,h} > 0$ is a constant depending only on p and h.

Proof. By the Binomial Theorem, we have

$$(a+b)^p = \sum_{k=0}^{p} \binom{p}{k} a^{p-k}b^k = a^p + \sum_{k=1}^{p-1} \binom{p}{k} a^{p-k}b^k + b^p. \tag{4.6.1}$$

Turning our attention to the cross terms, we note that $(p/k)^{-1} + (p/(p-k))^{-1} = k/p + (p-k)/p = 1$. This means that p/k and $p/(p-k)$ are conjugate exponents. Therefore, by Young's Inequality, we have for any $h > 0$:

$$a^{p-k}b^k = ha^{p-k}\frac{b^k}{h} \leq \frac{h^{\frac{p}{p-k}}a^p}{p/(p-k)} + \frac{b^p}{h^{\frac{p}{k}}p/k} = \frac{h^{1+\frac{k}{p-k}}a^p}{p/(p-k)} + \frac{b^p}{h^{\frac{p}{k}}p/k}.$$

Inserting the above into (4.6.1), we have

$$\begin{aligned}(a+b)^p &\leq a^p + \sum_{k=1}^{p-1} \binom{p}{k} \left(\frac{h^{1+\frac{k}{p-k}}a^p}{p/(p-k)} + \frac{b^p}{h^{\frac{p}{k}}p/k} \right) + b^p \\ &= a^p \left(1 + h\sum_{k=1}^{p-1} \binom{p}{k} \frac{h^{\frac{k}{p-k}}}{p/(p-k)} \right) + b^p \left(1 + \sum_{k=1}^{p-1} \binom{p}{k} \frac{k}{h^{\frac{p}{k}}p} \right).\end{aligned}$$

As h was arbitrary, the result follows. □

Lemma 4.6.2 (Convergence of Stopping Times)**.** *Let $k \in \mathbb{N}$ and consider a sequence of functions $(\eta^n)_{n\geq 1}$ in $D_{\mathbb{R}}$ converging to some $\eta \in D_{\mathbb{R}}$ with respect to the M_1-topology. We assume that η has the following crossing property:*

$$\forall\, h > 0,\ \tau_k(\eta) < \bar{T} \qquad \Longrightarrow \qquad \sup_{s\in(\tau_k(\eta),(\tau_k(\eta)+h)\wedge\bar{T})} \left\{\eta_s - \eta_{\tau_k(\eta)}\right\} > 0, \tag{4.6.2}$$

where τ_k is defined as in (4.4.1) and $\Delta\eta_t \geq 0$ for all $t \in [-1, \bar{T}]$. Then we have

$$\lim_{n\to\infty} \tau_k(\eta^n) = \tau_k(\eta).$$

Proof. The proof consists of two steps. We shall show that $\limsup_{n\to\infty} \tau_k(\eta^n) \leq \tau_k(\eta) \leq \liminf_{n\to\infty} \tau_k(\eta^n)$. Hence, we will have equality, and the claim follows.

Step 1: $\limsup_{n\to\infty} \tau_k(\eta^n) \leq \tau_k(\eta)$

We define the set of continuity points of η as $\mathbb{T}^\eta := \{t \in [-1, \bar{T}] : \eta_t = \eta_{t-}\}$. We note that $\mathbb{T}^\eta$ is co-countable by [3, Section 13]. As $\tau_k(\eta) < \bar{T}$, by (4.6.2) for any fixed $m \in \mathbb{N}$, there exists a $t \in (\tau_k(\eta), \tau_k(\eta) + 1/m) \cap \mathbb{T}^\eta$ such that $\eta_t > k - 1$. Now, as t is a continuity point of η, by [71, Theorem 12.5.1], we have that $\eta^n_t \to \eta_t$ in $\mathbb{R}$ as $n \to \infty$. Therefore, for large n, $\eta^n_t > k - 1$, hence

$$\limsup_{n\to\infty} \tau_k(\eta^n) \leq t \leq \tau_k(\eta) + \frac{1}{m}.$$

As $m \in \mathbb{N}$ was arbitrary, the claim follows.

Step 2: $\liminf_{n\to\infty} \tau_k(\eta^n) \geq \tau_k(\eta)$

As $\eta^n \to \eta$ in the M_1-topology, we can find a sequence of parametric representations $((u^n, r^n))_{n\geq 1}$ of $(\eta^n)_{n\geq 1}$ which converges uniformly to a parametric representation (u, r) of η, see [71, Theorem 12.5.1]. Thus, we can find an $s^n \in [0, 1]$ such that $(u^n_{s^n}, r^n_{s^n}) = (\eta^n_{\tau_k(\eta^n)}, \tau_k(\eta^n))$. By Step 1 and the fact that $\tau_k(\eta) < \bar{T}$, we have

$$\liminf_{n\to\infty} \tau_k(\eta^n) \leq \limsup_{n\to\infty} \tau_k(\eta^n) \leq \tau_k(\eta) < \bar{T}.$$

Therefore, by the finiteness of $\liminf_{n\to\infty} \tau_k(\eta^n)$ and compactness of $[0, 1]$, we can find a subsequence n_j such that $\tau_k(\eta^{n_j}) \to \liminf_{n\to\infty} \tau_k(\eta^n)$ and $s_{n_j} \to s$ for some $s \in [0, 1]$. By the uniform convergence of the parametric representations, we have

$$\begin{aligned}
\liminf_{j\to\infty} \eta^{n_j}_{\tau_k(\eta^{n_j})} &= \liminf_{j\to\infty} u^{n_j}_{s^{n_j}} = u_s, \\
\liminf_{j\to\infty} \tau_k(\eta^{n_j}) &= \liminf_{j\to\infty} r^{n_j}_{s^{n_j}} = r_s.
\end{aligned}$$

As $r_s = \liminf_{n\to\infty} \tau_k(\eta^n)$, we may find $\gamma \in [0,1]$ such that $u_s = \gamma\eta_{(\liminf_{n\to\infty}\tau_k(\eta^n))-} + (1-\gamma)\eta_{\liminf_{n\to\infty}\tau_k(\eta^n)}$. We also note that $u_s \geq k-1$ as $\liminf_{j\to\infty} \eta^{n_j}_{\tau_k(\eta^{n_j})} \geq k-1$. Lastly, since $\Delta\eta_t \geq 0$ for all t, we must have that $\eta_{\liminf_{n\to\infty}\tau_k(\eta^n)} \geq k-1$. Therefore, $\tau_k(\eta) \leq \liminf_{n\to\infty}\tau_k(\eta^n)$. This completes the proof. □

Proposition 4.6.3 (Convergence of smoothed functions)**.** *For any sequence of càdlàg functions $\{f^n\}_n$ which converges to a càdlàg function f in $D_\mathbb{R}$ endowed with the M_1-topology, and such that $\sup_n \|f^n\|_\infty < \infty$, we have that $\int_0^t \kappa^{\varepsilon_n}(t-s) f^n_s \,\mathrm{d}s$ converges to f_t for all $t > 0$ that are continuity points of f.*

Proof. First, we fix any $t > 0$ that is a continuity point of f. Then it is clear that

$$\begin{aligned}\left|\int_0^t \kappa^{\varepsilon_n}(s) f^n_{t-s}\,\mathrm{d}s - f_t\right| \leq& \left|\int_0^t \kappa^{\varepsilon_n}(s)(f^n_{t-s} - f_{t-s})\,\mathrm{d}s\right| + \left|\int_0^t \kappa^{\varepsilon_n}(s)(f_{t-s} - f_t)\,\mathrm{d}s\right| \\ &+ \left|\int_0^t \kappa^{\varepsilon_n}(s)\,\mathrm{d}s - 1\right| |f_t| \\ =& I + II + III.\end{aligned}$$

For any $\delta > 0$, we observe

$$\begin{aligned} I &\leq \sup_{t-\delta\leq s\leq t} |f^n_s - f_s| \int_0^\delta \kappa^{\varepsilon_n}(s)\,\mathrm{d}s + 2\sup_n \|f^n\|_\infty \int_\delta^\infty \kappa^{\varepsilon_n}(s)\,\mathrm{d}s, \\ II &\leq \sup_{t-\delta\leq s\leq t} |f_s - f_t| \int_0^\delta \kappa^{\varepsilon_n}(s)\,\mathrm{d}s + 2\|f\|_\infty \int_\delta^\infty \kappa^{\varepsilon_n}(s)\,\mathrm{d}s, \\ III &\leq \|f\|_\infty \int_t^\infty \kappa^{\varepsilon_n}(s)\,\mathrm{d}s. \end{aligned}$$

As M_1-convergence implies local uniform convergence at continuity points [71, Theorem 12.5.1], and t is a continuity point, by setting $\delta = \varepsilon_n^{1/2}$ and sending $n \to \infty$, we have that I, II, and III all go to zero. □

Lemma 4.6.4 (Functional continuity of measures)**.** *Let $\mu \in \mathcal{P}(D_\mathbb{R})$ be any measure such that*

$$\mu\left(\bigcup_{k\geq 1}\left\{\eta \,:\, \sup_{s\in(\tau_k,(\tau_k+h)\wedge\bar{T})}\{\eta_s - \eta_{\tau_k}\} \leq 0,\, \tau_k < \bar{T}\right\}\right) = 0, \tag{4.6.3}$$

for any $h > 0$. Then, for any sequence of measures $(\mu^n)_{n\geq 1}$ such that $\mu^n \implies \mu$ in $(\mathcal{P}(D_\mathbb{R}), \mathfrak{T}^{wk}_{M_1})$, we have

$$\nu^{\mu^n,D}_t := \mu^n(\eta_t - \tilde{J}^D_t \in \cdot,\, \eta_t - \tilde{J}^D_t < 0) \implies \nu^\mu_t := \mu(\eta_t - \tilde{J}_t \in \cdot),$$

in $\mathbf{M}_{\leq 1}(\mathbb{R})$, for any $t \in \mathbb{T}^\mu$, where $\mathbb{T}^\mu$ is defined as

$$\mathbb{T}^\mu := \{t \in [-1,\bar{T}] \,:\, \mu(\eta_t = \eta_{t-}) = 1,\, \mu(\tau_k = t) = 0\ \forall k\}.$$

Proof. The proof is an application of the Continuous Mapping Theorem [3, Theorem 2.7]. We only need to construct μ-almost surely continuous maps.

Step 1: Projection of measures from $D_{\mathbb{R}}$ to $\mathbb{R} \times \{0, 1\}$

As μ satisfies (4.6.3) and $t \in \mathbb{T}^{\mu}$ is a continuity point of μ, by the proof within [16, Lemma 5.6], we have that the map $\eta \mapsto \sup_{s \le t} \eta_s$ is M_1-continuous μ-almost surely. Hence, we may deduce that the map

$$\eta \mapsto \sum_{k \ge 1} \mathbb{1}_{\{\sup_{s \le t} \eta_s + 1 = k\}}$$

is M_1-continuous μ-almost surely, as the only points of discontinuity are at the points where $\sup_{s \le t} \eta_s = k - 1$, but since $t \in \mathbb{T}^{\mu}$, these points are of μ-measure zero because $\mu(\sup_{s \le t} \eta_s = k - 1) = \mu(\tau_k = t) = 0$. In fact, this map is 0 μ-almost surely. By [16, Lemma 5.6], as $\mu(\eta_t = \eta_{t-}) = 1$ and $\mu(\tau_k = t) = 0$ for every $k \ge 1$, the map $\eta \mapsto \lfloor (\sup_{s \le t} \eta_s + 1)_+ \rfloor$ is M_1-continuous μ-almost surely. It is also clear that $\eta_t - \lfloor (\sup_{s \le t} \eta_s + 1)_+ \rfloor < 0$ μ-almost surely. Hence, the map

$$(X_t,\, \mathbb{1}_{\{X_t < 0\}}) \,:\, D_{\mathbb{R}} \to \mathbb{R} \times \{0, 1\}, \qquad \eta \mapsto (\eta_t - \tilde{J}_t^D(\eta),\, \mathbb{1}_{\{\eta_t - \tilde{J}_t^D(\eta) < 0\}}) \tag{4.6.4}$$

is a μ-almost surely continuous map. Thus, by the Continuous Mapping Theorem,

$$(X_t,\, \mathbb{1}_{\{X_t < 0\}})^{\#}\mu^n \implies (X_t,\, \mathbb{1}_{\{X_t < 0\}})^{\#}\mu \quad \text{in} \quad \mathcal{P}(\mathbb{R} \times \{0,\, 1\}).$$

Step 2: Weak convergence of sub-probability measures

For any $f \in \mathcal{C}_b(\mathbb{R})$, define the map

$$\hat{f} \,:\, \mathbb{R} \times \{0,\, 1\} \to \mathbb{R}, \qquad (x, y) \mapsto f(x)\mathbb{1}_{\{1\}}(y).$$

It is clear that $\hat{f} \in \mathcal{C}_b(\mathbb{R} \times \{0,\, 1\})$. By Step 1,

$$\langle (X_t,\, \mathbb{1}_{\{X_t < 0\}})^{\#}\mu^n, \hat{f} \rangle \to \langle (X_t,\, \mathbb{1}_{\{X_t < 0\}})^{\#}\mu, \hat{f} \rangle.$$

However, by definition, $\langle (X_t,\, \mathbb{1}_{\{X_t < 0\}})^{\#}\mu^n, \hat{f} \rangle = \nu_t^{\mu^n, D}(f)$ and $\langle (X_t,\, \mathbb{1}_{\{X_t < 0\}})^{\#}\mu, \hat{f} \rangle = \nu_t^{\mu, D}(f)$. Thus, $\nu_t^{\mu^n, D}(f) \to \nu_t^{\mu, D}(f)$. But, as shown in Step 1, $\sum_{k \ge 1} \mathbb{1}_{\{\sup_{s \le t} \eta_s + 1 = k\}} = 0$ μ-almost surely and $\mu(\eta_t - \lfloor (\sup_{s \le t} \eta_s + 1)_+ \rfloor < 0) = 1$. Therefore, $\nu_t^{\mu, D}(f) = \nu_t^{\mu}(f)$. Hence, $\nu_t^{\mu^n, D}(f) \to \nu_t^{\mu}(f)$, and the conclusion now follows from the Portmanteau Theorem. □

Lemma 4.6.5 (Weak convergence of sub-probability measures)**.** *Suppose that* $\tilde{\mathbf{P}}^{\varepsilon_n} \implies \mathbf{P}^*$ *on* $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{wk})$ *for a positive sequence* $(\varepsilon_n)_{n \ge 1}$ *which converges to zero. Set*

$$\mathbb{T} \coloneqq \left\{ t \in [-1, \bar{T}] \,:\, \mathbb{E}\left[\mathbf{P}^*(\eta_t = \eta_{t-})\right] = 1,\, \mathbb{E}\left[\mathbf{P}^*(\tau_k = t)\right] = 0 \;\forall k \right\},$$

Then, for any $t \in \mathbb{T}$,

$$\boldsymbol{\nu}_t^{\varepsilon_n} \implies \boldsymbol{\nu}_t^* := \mathbf{P}^*(\eta_t - \tilde{J}_t \in \cdot) \quad \textit{in} \quad \mathbf{M}_{\leq 1}(\mathbb{R}).$$

Proof. By Lemma 4.4.5 and the definition of $\mathbb{T}$, for any fixed $t \in \mathbb{T}$, there is a set of μ's of full Law($\mathbf{P}^*$)-measure such that $\mu(\eta_t = \eta_{t-}) = 1$, $\mu(\tau_k = t) = 0$ for all $k \geq 1$, and

$$\mu\Bigg(\bigcup_{k\geq 1} \Big\{ \eta \ : \sup_{s\in(\tau_k,\,(\tau_k+h)\wedge\bar{T})} \{\eta_s - \eta_{\tau_k}\} \leq 0,\ \tau_k < \bar{T} \Big\} \Bigg) = 0,$$

for any $h > 0$. Since $\tilde{\mathbf{P}}^{\varepsilon_n} \implies \mathbf{P}^*$, by Skorokhod's Representation Theorem, there exist $(\mathbf{Q}^n)_{n\geq 1}$ and $\mathbf{Q}^*$ such that $\mathbf{Q}^n \to \mathbf{Q}^*$ almost surely, Law($\mathbf{Q}^n$) = Law($\tilde{\mathbf{P}}^{\varepsilon_n}$), Law($\mathbf{Q}^*$) = Law($\mathbf{P}^*$), and $\mathbf{Q}^*$ satisfies the properties above almost surely. Set

$$\nu_t^{\mathbf{Q}^n,D} = \mathbf{Q}^n(\eta_t - \tilde{J}_t^D \in \cdot,\ \eta_t - \tilde{J}_t^D < 0), \quad \text{and} \quad \nu_t^{\mathbf{Q}^*} = \mathbf{Q}^*(\eta_t - \tilde{J}_t \in \cdot).$$

By Lemma 4.6.4, $\nu_t^{\mathbf{Q}^n,D} \to \nu_t^{\mathbf{Q}^*}$ almost surely in $\mathbf{M}_{\leq 1}(\mathbb{R})$. Now, for any $G \in \mathcal{C}_b(\mathbf{M}_{\leq 1}(\mathbb{R}))$, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \mathbb{E}\left[G(\boldsymbol{\nu}_t^{\varepsilon_n})\right] = \lim_{n\to\infty} \mathbb{E}\left[G(\nu_t^{\mathbf{Q}^n,D})\right] = \mathbb{E}\left[G(\nu_t^{\mathbf{Q}^*})\right] = \mathbb{E}\left[G(\boldsymbol{\nu}_t^*)\right].$$

The result now follows from the Portmanteau Theorem. □

Corollary 4.6.6. *Suppose that* $\tilde{\mathbf{P}}^{\varepsilon_n} \implies \mathbf{P}^*$ *on* $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{wk})$ *for a positive sequence* $(\varepsilon_n)_{n\geq 1}$ *which converges to zero. Set*

$$\mathbb{T} := \left\{ t \in [-1, \bar{T}] \ : \ \mathbb{E}\left[\mathbf{P}^*(\eta_t = \eta_{t-})\right] = 1,\ \mathbb{E}\left[\mathbf{P}^*(\tau_k = t)\right] = 0\ \forall k \right\},$$

Then, for any $t \in \mathbb{T} \cap (0, T]$,

$$\mathrm{Law}(X_t^{\varepsilon_n} \mid W^0) \implies \mathbf{P}^*(\eta_t - \tilde{J}_t \in \cdot) \quad \textit{in} \quad \mathbf{M}_{\leq 1}(\mathbb{R}).$$

Furthermore,

$$\lim_{n\to\infty} \mathbb{P}\big[X_t^{\varepsilon_n} < 0\big] = 1.$$

Proof. By following the proofs of Lemma 4.6.4 and Lemma 4.6.5, it is immediate that

$$\tilde{\mathbf{P}}^{\varepsilon_n}(\eta_t - \tilde{J}_t^D \in \cdot) \implies \mathbf{P}^*(\eta_t - \tilde{J}_t \in \cdot) \quad \text{in} \quad \mathbf{M}_{\leq 1}(\mathbb{R}),$$

for any $t \in \mathbb{T}$. Now, if $t \in \mathbb{T} \cap (0, T]$, we can rewrite the above equation as

$$\mathrm{Law}(X_t^{\varepsilon_n} \mid W^0) \implies \mathbf{P}^*(\eta_t - \tilde{J}_t \in \cdot) \quad \text{in} \quad \mathbf{M}_{\leq 1}(\mathbb{R}). \tag{4.6.5}$$

However, by the definition of $\tilde{J}_t$, $\mathbf{P}^*(\eta_t - \tilde{J}_t < 0) = 1$ almost surely. Therefore, by (4.6.5) and a simple application of the Portmanteau Theorem, the second claim follows. □

Proposition 4.6.7. *Suppose that* $\tilde{\mathbf{P}}^{\varepsilon_n} \implies \mathbf{P}^*$ *on* $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}^{wk}_{M_1})$ *for a positive sequence* $(\varepsilon_n)_{n\geq 1}$ *which converges to zero. Set*

$$\mathbb{T} \coloneqq \left\{ t \in [-1, \bar{T}] \ : \ \mathbb{E}\left[\mathbf{P}^*(\eta_t = \eta_{t-})\right] = 1, \, \mathbb{E}\left[\mathbf{P}^*(\tau_k = t)\right] = 0 \ \forall k \right\}.$$

Then, for any $t \in \mathbb{T} \cap (0, T]$, $p \geq 1$, *and function* g *satisfying Assumption 4.2.2 (i) we have*

$$\mathbb{E}\left[\left| \int_0^t g(s, X_s^{\varepsilon_n}, \boldsymbol{\nu}_s^{\varepsilon_n}) \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}} \, \mathrm{d}s \right|^p + \left| \int_0^t \alpha(s) \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}} \, \mathrm{d}\mathfrak{F}_s^{\varepsilon_n} \right|^p \right] \leq C_{\alpha,g,p,T,\sigma},$$

for some constant $C_{\alpha,g,p,T,\sigma} > 0$ *independent of* n. *Furthermore, if* $p \in [1, 4]$, *then*

$$\lim_{n\to\infty} \mathbb{E}\left[\left| \int_0^t g(s, X_s^{\varepsilon_n}, \boldsymbol{\nu}_s^{\varepsilon_n}) \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}} \, \mathrm{d}s \right|^p + \left| \int_0^t \alpha(s) \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}} \, \mathrm{d}\mathfrak{F}_s^{\varepsilon_n} \right|^p \right] = 0.$$

Proof. By (4.3.2) and (4.3.3), we have

$$|X_s^\varepsilon| \leq 2 \sup_{u\leq s} |Z_u^\varepsilon| + 1, \qquad \text{and} \qquad |\mathfrak{F}_s^\varepsilon| \leq \mathbb{E}\left[\sup_{u\leq s} |Z_u^\varepsilon| \ \middle| \ W^0 \right] + 1. \tag{4.6.6}$$

Employing these bounds and the linear growth condition on g in Assumption 4.2.2 (i), we have

$$|g(s, X_s^\varepsilon, \nu_s^\varepsilon)| \leq C_g \left(1 + |X_s^\varepsilon| + \langle \nu_s^\varepsilon, |\cdot| \rangle\right) \leq 2C_g \Big(1 + \sup_{u\leq s} |Z_u^\varepsilon| + \mathbb{E}\big[\sup_{u\leq s} |Z_u^\varepsilon| \ \big| \ W^0 \big] \Big). \tag{4.6.7}$$

Therefore, by fixing a $p \in (1, 4]$ and applying Jensen's Inequality and employing the upper bound (4.6.7), we have

$$\begin{aligned} &\left| \int_0^t g(s, X_s^{\varepsilon_n}, \boldsymbol{\nu}_s^{\varepsilon_n}) \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}} \, \mathrm{d}s \right|^p \\ &\qquad \leq C_{g,p,T} \left(1 + \sup_{u\leq T} |Z_u^{\varepsilon_n}|^p + \mathbb{E}\big[\sup_{u\leq T} |Z_u^{\varepsilon_n}|^p \ \big| \ W^0 \big] \right) \int_0^t \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}} \, \mathrm{d}s. \end{aligned} \tag{4.6.8}$$

Hence, by the above and Proposition 4.3.2, it is immediate that

$$\mathbb{E}\left[\left| \int_0^t g(s, X_s^{\varepsilon_n}, \boldsymbol{\nu}_s^{\varepsilon_n}) \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}} \, \mathrm{d}s \right|^p \right] \leq C_{\alpha,g,p,T,\sigma}.$$

Now, taking the expectation in (4.6.8) and applying the Cauchy-Schwarz Inequality, Jensen's Inequality, and Fubini's Theorem, we have

$$\begin{aligned} &\mathbb{E}\left[\left| \int_0^t g(s, X_s^{\varepsilon_n}, \boldsymbol{\nu}_s^{\varepsilon_n}) \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}} \, \mathrm{d}s \right|^p \right] \\ &\qquad \leq C_{g,p,T} \mathbb{E}\left[1 + \sup_{u\leq T} |Z_u^{\varepsilon_n}|^{2p} \right]^{1/2} \left(\int_0^t \mathbb{P}(X_s^{\varepsilon_n} = 0) \, \mathrm{d}s \right)^{1/2}, \end{aligned}$$

for a possibly larger constant $C_{g,p,T} > 0$. By employing Corollary 4.6.6 and the Dominated Convergence Theorem, the right-hand side in the above equation converges to zero as $n \to \infty$.

For the second integral in our claim, we observe from (4.6.6) and employing the bounds on α from Assumption 4.2.2 (v) that

$$\mathbb{E}\left[\left|\int_0^t \alpha(s)\mathbb{1}_{\{X_s^{\varepsilon_n}=0\}}\,\mathrm{d}\mathfrak{F}_s^{\varepsilon_n}\right|^p\right] \leq C_{\alpha,p}\mathbb{E}\left[1+\sup_{u\leq T}|Z_u^{\varepsilon_n}|^p\right].$$

The right-hand side in the above inequality is bounded uniformly in n by Proposition 4.3.2.

For the convergence claim, more care needs to be taken, and we will need to employ the modulus of continuity of $\mathfrak{F}^{\varepsilon_n}$. For any $s,t \in [0,T]$, a straightforward computation shows that

$$|\mathfrak{F}_t^{\varepsilon_n} - \mathfrak{F}_s^{\varepsilon_n}| \leq \varepsilon_n^{-1}\left\|\mathfrak{K}'\right\|_{L^1} F_T^{\varepsilon_n}\,|t-s|\,.$$

Now, fixing any integer $K \geq 1$, we have that

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t \alpha(s)\mathbb{1}_{\{X_s^{\varepsilon_n}=0\}}\,\mathrm{d}\mathfrak{F}_s^{\varepsilon_n}\right|^p\right] &\leq \mathbb{E}\left[\left|\int_0^t \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}}\,\mathrm{d}\mathfrak{F}_s^{\varepsilon_n}\right|^p\right]\\ &\leq \mathbb{E}\left[\left|\int_0^t \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}}\,\mathrm{d}\mathfrak{F}_s^{\varepsilon_n}\right|^p \mathbb{1}_{\{\tau_K\leq t\}}\right]\\ &\quad + \mathbb{E}\left[\left|\int_0^t \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}}\,\mathrm{d}\mathfrak{F}_s^{\varepsilon_n}\right|^p \mathbb{1}_{\{\tau_K>t\}}\right],\end{aligned} \tag{4.6.9}$$

where the first inequality follows from the fact that $\|\alpha\|_\infty < 1$. Turning our attention to the first term on the right-hand side, we have, by the Cauchy-Schwarz Inequality,

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}}\,\mathrm{d}\mathfrak{F}_s^{\varepsilon_n}\right|^p \mathbb{1}_{\{\tau_K<t\}}\right] &\leq \mathbb{E}\left[(\mathfrak{F}_T^{\varepsilon_n})^{2p}\right]^{1/2}\mathbb{P}\left(\tau_K<t\right)^{1/2}\\ &= \mathbb{E}\left[(\mathfrak{F}_T^{\varepsilon_n})^{2p}\right]^{1/2}\mathbb{P}\left(\sup_{s\leq t} Z_s^{\varepsilon_n}\geq K\right)^{1/2}\\ &\leq K^{-1/2}\mathbb{E}\left[(\mathfrak{F}_T^{\varepsilon_n})^{2p}\right]^{1/2}\mathbb{E}\left[\sup_{s\leq T}|Z_s^{\varepsilon_n}|\right]^{1/2},\end{aligned} \tag{4.6.10}$$

where the last inequality follows from Markov's Inequality. For the second term in (4.6.9), since X^{ε_n} equals zero only on the intervals $[\tau_k^{\varepsilon_n}, \tau_k^{\varepsilon_n}+\varsigma_k^{\varepsilon_n})$ for $k \geq 1$, we have

$$\mathbb{E}\left[\left|\int_0^t \mathbb{1}_{\{X_s^{\varepsilon_n}=0\}}\,\mathrm{d}\mathfrak{F}_s^{\varepsilon_n}\right|^p \mathbb{1}_{\{\tau_K>t\}}\right] \leq \mathbb{E}\left[\left|\sum_{k=1}^K \mathfrak{F}^{\varepsilon_n}_{(\tau_k^{\varepsilon_n}+\varsigma_k^{\varepsilon_n})\wedge t} - \mathfrak{F}^{\varepsilon_n}_{\tau_k^{\varepsilon_n}\wedge t}\right|^p\right]$$

Employing Jensen's Inequality and the modulus of continuity of $\mathfrak{F}^{\varepsilon_n}$, we have

$$\leq K^{p-1}\varepsilon_n^{-p}\left\|\mathfrak{K}'\right\|_{L^1}^p \sum_{k=1}^{K} \mathbb{E}\left[(F_T^{\varepsilon_n}\varsigma_k^{\varepsilon_n})^p\right]$$

As F^{ε_n} is W^0-measurable and $\varsigma_k^{\varepsilon_n}$ is independent of W^0, we have

$$= K^{p-1}\varepsilon_n^{-p}\left\|\mathfrak{K}'\right\|_{L^1}^p \sum_{k=1}^{K} \mathbb{E}\left[(F_T^{\varepsilon_n})^p\right]\mathbb{E}\left[(\varsigma_k^{\varepsilon_n})^p\right]$$

By Assumption 4.2.2 (vii), we have $\varsigma_k^{\varepsilon_n} \sim^d \varepsilon_n^{1+\gamma}\varsigma$ for some $\gamma > 0$. Therefore, the above inequality becomes

$$\leq K^p\varepsilon_n^{\gamma p}\left\|\mathfrak{K}'\right\|_{L^1}^p \mathbb{E}\left[(F_T^{\varepsilon_n})^p\right]\mathbb{E}\left[\varsigma^p\right], \tag{4.6.11}$$

Therefore, inserting (4.6.10) and (4.6.11) into (4.6.9), we have

$$\mathbb{E}\left[\left|\int_0^t \alpha(s)\mathbb{1}_{\{X_s^{\varepsilon_n}=0\}}\,\mathrm{d}\mathfrak{F}_s^{\varepsilon_n}\right|^p\right] \leq K^{-1/2}\mathbb{E}\left[(\mathfrak{F}_T^{\varepsilon_n})^{2p}\right]^{1/2}\mathbb{E}\left[\sup_{s\leq T}|Z_s^{\varepsilon_n}|\right]^{1/2} + K^p\varepsilon_n^{\gamma p}\left\|\mathfrak{K}'\right\|_{L^1}^p \mathbb{E}\left[(F_T^{\varepsilon_n})^p\right]\mathbb{E}\left[\varsigma^p\right].$$

All the expectations on the right-hand side of the above inequality are bounded uniformly in n by Proposition 4.3.2 and Assumption 4.2.2 (vii). Therefore, setting $K = \lfloor \varepsilon_n^{-\gamma p/2} \rfloor$ and taking the limit as $n \to \infty$, the last claim follows. □

Lemma 4.6.8 (Functional Continuity)**.** *Let $\mu \in \mathcal{P}(D_{\mathbb{R}})$ be any measure such that*

$$\mu\left(\bigcup_{k\geq 1}\left\{\eta \;:\; \sup_{s\in(\tau_k,\,(\tau_k+h)\wedge\bar{T})}\{\eta_s - \eta_{\tau_k}\} \leq 0,\ \tau_k < \bar{T}\right\}\right) = 0, \tag{4.6.12}$$

for any $h > 0$, and let g be any function satisfying Assumption 4.2.2 (i). Then, for any $t \geq 0$, $\int_0^t g(s,\eta_s^n,\nu_s^{\mu^n,D})\,\mathrm{d}s$ converges to $\int_0^t g(s,\eta_s,\nu_s^{\mu})\,\mathrm{d}s$ whenever $(\eta^n,\mu^n) \to (\eta,\mu)$ in $(D_{\mathbb{R}},M_1)\times(\mathcal{P}(D_{\mathbb{R}}),\mathfrak{T}_{M_1}^{wk})$ along a sequence for which $\sup_{n\geq 1}\langle \mu^n,\, \tilde{\eta} \mapsto \sup_{s\leq\bar{T}}|\tilde{\eta}_s|^p\rangle < \infty$ for some $p > 1$. Here,

$$\nu_t^{\mu^n,D} := \mu^n(\eta_t - \tilde{J}_t^D \in \cdot,\ \eta_t - \tilde{J}_t^D < 0) \qquad \text{and} \qquad \nu_t^{\mu} := \mu(\eta_t - \tilde{J}_t \in \cdot).$$

Proof. By Assumption 4.2.2 (i) and the definition of $\tilde{J}_t^D$,

$$\left|g(s,\eta_s^n,\nu_s^{\mu^n,D})\right| \leq C\left(1 + \sup_{n\geq 1}\sup_{u\leq\bar{T}}|\eta_u^n| + \sup_{n\geq 1}\langle \mu^n,\, \tilde{\eta} \mapsto \sup_{u\leq\bar{T}}|\tilde{\eta}_u|\rangle\right) \tag{4.6.13}$$

for some constant $C > 0$ independent of n. The second term on the right-hand side of (4.6.13) is finite because $\eta^n \to \eta$ in $(D_{\mathbb{R}},M_1)$, and the third term is finite

by assumption. Therefore, it is sufficient to show that $g(s, \eta_s^n, \nu_s^{\mu^n,D})$ converges to $g(s, \eta_s, \nu_s^\mu)$ on a set of full Lebesgue measure. The conclusion then follows from the Dominated Convergence Theorem.

By Assumption 4.2.2 (i),

$$\left|g(s, \eta_s^n, \nu_s^{\mu^n,D}) - g(s, \eta_s, \nu_s^\mu)\right| \leq C\Big(|\eta_s^n - \eta_s| + d_0(\nu_s^{\mu^n,D}, \nu_s^\mu)\Big).$$

For any $s \in \mathbb{T}^\mu := \big\{t \in [-1, \bar{T}] : \mu(\eta_t = \eta_{t-}) = 1, \mu(\tau_0 = t) = 0\big\}$, the first term converges to zero as $\eta^n \to \eta$ in $(D_\mathbb{R}, M_1)$. For the second term, recall that

$$d_0(\nu_s^{\mu^n}, \nu_s^\mu) = \sup\left\{\left|\langle \nu_s^{\mu^n} - \nu_s^\mu, \psi\rangle\right| : \psi \in \mathcal{C}_{d_0}\right\}, \tag{4.6.14}$$

where $\mathcal{C}_{d_0} := \{\psi \in \mathcal{C}(\mathbb{R}) : \|\psi\|_{\text{Lip}} \leq 1, |\psi(0)| \leq 1\}$. Fix a $\delta > 0$, then for any $\lambda > 1$, by the Arzerlà-Ascoli Theorem, there exists a finite family of functions $\psi_1, \ldots, \psi_m \in \mathcal{C}_{d_0}$ supported on $[-\lambda - 1, \lambda + 1]$, for $m = m(\lambda) \in \mathbb{N}$ such that for any $\psi \in \mathcal{C}_{d_0}$,

$$\sup_{x \in [-\lambda, \lambda]} |\psi(x) - \psi_i(x)| < \delta/2 \tag{4.6.15}$$

for some $i \in \{1, \ldots, m\}$. Fixing any $\psi \in \mathcal{C}_{d_0}$ and choosing a ψ_i such that (4.6.15) holds, we have

$$\begin{aligned}\left|\langle \nu_s^{\mu^n,D} - \nu_s^\mu, \psi\rangle\right| \leq &\left|\int_{-\lambda}^0 (\psi - \psi_i)\, \mathrm{d}(\nu_s^{\mu^n,D} - \nu_s^\mu)\right| + \left|\int_{-\infty}^{-\lambda} (\psi - \psi_i)\, \mathrm{d}(\nu_s^{\mu^n,D} - \nu_s^\mu)\right| \\ &+ \left|\langle \nu_s^{\mu^n,D} - \nu_s^\mu, \psi_i\rangle\right|.\end{aligned}$$

By our choice of ψ_i, the first term is bounded by δ uniformly in n. By the linear growth condition for functions in $\mathcal{C}_{d_0}$,

$$\left|\int_{-\infty}^{-\lambda} (\psi - \psi_i)\, \mathrm{d}(\nu_s^{\mu^n,D} - \nu_s^\mu)\right| \leq C\langle \nu_s^{\mu^n,D} + \nu_s^\mu, |\cdot|\, \mathbb{1}_{(-\infty,-\lambda]}\rangle$$

for a constant C independent of n. Therefore, by the definition of $\nu_s^{\mu^n,D}$ and $\tilde{J}$,

$$\begin{aligned}\langle \nu_s^{\mu^n,D}, |\cdot|\, \mathbb{1}_{(-\infty,-\lambda]}\rangle &\leq \int_{D_\mathbb{R}} \left|\tilde{\eta}_s - \tilde{J}_s(\eta)\right| \mathbb{1}_{\{\tilde{\eta}_s - \tilde{J}_s(\eta) < -\lambda\}} \mathbb{1}_{\{\tilde{\eta}_s - \tilde{J}_s(\eta) < 0\}}\, \mathrm{d}\mu^n(\tilde{\eta}) \\ &\leq 4\left(1 + \langle \mu^n, \sup_{u \leq \bar{T}} |\tilde{\eta}_u|^p\rangle\right)^{\frac{1}{p}} \mu^n\left(2 + 2\sup_{u \leq \bar{T}} |\tilde{\eta}_u| > \lambda\right)^{\frac{p-1}{p}} \\ &\leq \frac{C_p}{\lambda^{p-1}}\left(1 + \langle \mu^n, \sup_{u \leq \bar{T}} |\tilde{\eta}_u|^p\rangle\right),\end{aligned}$$

for some constant C_p independent of n. The second inequality follows from Hölder's inequality, while the last inequality follows from Markov's inequality. We may also

employ the same argument to upper bound the $\langle \nu_s^\mu, |\cdot| \, \mathbb{1}_{(-\infty,-\lambda]} \rangle$ term in the above. Hence, by these upper bounds and as $\mu^n \implies \mu$, we deduce that

$$\left| \int_{-\infty}^{-\lambda} (\psi - \psi_i) \, \mathrm{d}(\nu_s^{\mu^n, D} - \nu_s^\mu) \right| \leq \frac{C_p}{\lambda^{p-1}} \left(1 + \sup_{m \geq 1} \langle \mu^n, \sup_{u \leq \bar{T}} |\tilde{\eta}_u|^p \rangle \right),$$

where C_p is independent of n. Returning to (4.6.14), we have shown that

$$d_0(\nu_s^{\mu^n, D}, \nu_s^\mu) \leq \delta + \frac{C_p}{\lambda^{p-1}} \left(1 + \sup_{m \geq 1} \langle \mu^n, \sup_{u \leq \bar{T}} |\tilde{\eta}_u|^p \rangle \right) + \sum_{i=1}^m |\langle \nu_s^{\mu^n, D} - \nu_s^\mu, \psi_i \rangle|.$$

The middle term will vanish uniformly in n as $\lambda \to \infty$. Thus, we fix a λ sufficiently large such that the middle term is bounded by δ uniformly in n. As λ is now fixed, so is m. By Lemma 4.6.4, $\nu_s^{\mu^n, D} \implies \nu_s^\mu$ in $\mathbf{M}_{\leq 1}(\mathbb{R})$; hence, as the ψ_i's have compact support, they are continuous and bounded. Therefore, the final term will be smaller than δ for all n sufficiently large. Consequently, for all n sufficiently large, $d_0(\nu_s^{\mu^n, D}, \nu_s^\mu) \leq C\delta$ for some constant C independent of n. Hence, $d_0(\nu_s^{\mu^n, D}, \nu_s^\mu) \to 0$ as $n \to \infty$. This completes the proof. □

Proposition 4.6.9. *For any $0 \leq t \leq t+h \leq T$, with $h \in (0,1)$ such that $t, t+h \in \mathbb{T}^\varepsilon$, where*

$$\mathbb{T}^\varepsilon := \left\{ t \in [0,T] \, : \, \mathbb{P}(F_t^\varepsilon = F_{t-}^\varepsilon) = 1, \, \mathbb{P}[\langle \nu_t^\varepsilon, \phi^n \rangle = \langle \nu_{t-}^\varepsilon, \phi^n \rangle] = 1 \text{ for } \phi^n \in \mathscr{S} \text{ dense} \right\},$$

we can find a $C > 0$, independent of ε and h, such that

$$\mathbb{P}\Big(\forall \, \lambda \leq F_{t+h}^\varepsilon - F_t^\varepsilon - Ch^{1/16} - (F_t^\varepsilon - \mathfrak{F}_t^\varepsilon), \, \boldsymbol{\nu}_t^\varepsilon\big([-\alpha_{t+h}\lambda - Ch^{1/16}, 0]\big) \geq \lambda \Big) \\ \geq 1 - Ch - Ch^{-2}\mathbb{E}[(F_t^\varepsilon - \mathfrak{F}_t^\varepsilon)^{16}].$$

Proof. We begin by returning to the particle system:

$$\begin{cases} \mathrm{d}X_t^{i,\varepsilon} = b(t, X_t^{i,\varepsilon}, \nu_t^{N,\varepsilon}) \mathbb{1}_{\{X_t^{i,\varepsilon} < 0\}} \, \mathrm{d}t + \sigma(t, X_t^{i,\varepsilon}) \sqrt{1 - \rho^2(t, \nu_t^{N,\varepsilon})} \mathbb{1}_{\{X_t^{i,\varepsilon} < 0\}} \, \mathrm{d}W_t^i \\ \qquad\qquad + \sigma(t, X_t^{i,\varepsilon}) \rho(t, \nu_t^{N,\varepsilon}) \mathbb{1}_{\{X_t^{i,\varepsilon} < 0\}} \, \mathrm{d}W_t^0 + \alpha(t) \mathbb{1}_{\{X_t^{i,\varepsilon} < 0\}} \, \mathrm{d}\mathfrak{F}_t^{N,\varepsilon} \\ \qquad\qquad - \mathrm{d} \sum_{k \geq 1} \mathbb{1}_{[0,t]}(\tau_k^{i,\varepsilon} + \varsigma_k^{i,\varepsilon}), \\ \tau_k^{i,\varepsilon} = \inf\{ t > \tau_{k-1}^{i,\varepsilon} + \varsigma_{k-1}^{i,\varepsilon} \, : \, X_{t-}^{i,\varepsilon} \geq 0 \}, \qquad \tau_0^{i,\varepsilon} = 0, \\ J_t^{i,\varepsilon} = \sum_{k \geq 1} \mathbb{1}_{[0,t]}(\tau_k^{i,\varepsilon}), \\ F_t^{N,\varepsilon} = \frac{1}{N} \sum_{i=1}^N J_t^{i,\varepsilon}, \; \mathfrak{F}_t^{N,\varepsilon} = \int_0^t \mathfrak{K}^\varepsilon(t-s) F_s^{N,\varepsilon} \, \mathrm{d}s, \\ \nu_t^{N,\varepsilon} = \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i,\varepsilon}} \mathbb{1}_{\{X_t^{i,\varepsilon} < 0\}}. \end{cases} \tag{4.6.16}$$

We shall obtain a comparable estimate for the particle system. Then, we will take limits to conclude that our claim holds for the smoothed system. As all constants we

obtain throughout this proof will be independent of the mollifiers, we will suppress the ε in the notation above for convenience throughout this proof. The proof proceeds in the following two steps:

Step 1: Analogous particle system estimate

To begin, we will show that a comparable estimate holds for the particle system. Specifically, we will show

$$\mathbb{P}\Big(\forall \lambda \leq F^N_{t+h} - F^N_t - Ch^{1/16} - (F^N_t - \mathfrak{F}^N_t),\, \nu^N_t\big([-\alpha_{t+h}\lambda - Ch^{1/16}, 0]\big) \geq \lambda\Big) \\ \geq 1 - Ch - Ch^{-2}\mathbb{E}[(F^N_t - \mathfrak{F}^N_t)^{16}]. \tag{4.6.17}$$

We define the set

$$\mathcal{I}^1 \coloneqq \big\{i \in \{1, \dots, N\} \,:\, J^i_{t+h} - J^i_t = 1,\, X^i_t < 0\big\}, \tag{4.6.18}$$

which contains all the particles that spike between times t and $t+h$ and are not in their refractory period at time t. Choosing $i_1 \in \mathcal{I}^1$, for any time $s \in [t, t+h]$ before X^{i_1} spikes, we have

$$X^{i_1}_s = X^{i_1}_t + \int_t^s b(v, X^{i_1}_v, \nu^N_v)\,\mathrm{d}v + \int_t^s \sigma(v,\, X^{i_1}_v)\,\mathrm{d}B^{i_1}_v + \int_t^s \alpha(v)\,\mathrm{d}\mathfrak{F}^N_v,$$

where $\mathrm{d}B^{i_1}_u = \sqrt{1-\rho^2(u,\nu^N_u)}\,\mathrm{d}W^{i_1}_u + \rho(u,\nu^N_u)\,\mathrm{d}W^0_u$. We similarly define the continuous reformulation of the particle X^i as $Z^i_t \coloneqq X^i_t + \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau^i_k + \varsigma^i_k)$. From this, we can deduce that

$$\big|X^i_s\big| \leq 2\sup_{u\leq s}\big|Z^i_u\big| + 1, \quad \text{and} \quad \frac{1}{N}\sum_{i=1}^N \big|X^i_s\big| \leq \frac{2}{N}\sum_{i=1}^N \sup_{u\leq s}\big|Z^i_u\big| + 1.$$

Therefore, by the linear growth condition on b (Assumption 4.2.2 (i)), we obtain the upper bound

$$\begin{aligned} X^{i_1}_s \leq & X^{i_1}_t + Ch\Big(1 + \sup_{u\leq t+h}\big|Z^{i_1}_u\big| + N^{-1}\sum_{i=1}^N \sup_{u\leq t+h}\big|Z^i_u\big|\Big) \\ & + \sup_{u\in[t,t+h]}\left|\int_t^u \sigma(v,\, X^{i_1}_v)\,\mathrm{d}B^{i_1}_v\right| + \int_t^s \alpha(v)\,\mathrm{d}\mathfrak{F}^N_v. \end{aligned}$$

For the Lebesgue-Stieltjes integral, noting that the function α is increasing (Assumption 4.2.2 (v)), we have

$$\int_t^s \alpha(v)\,\mathrm{d}\mathfrak{F}^N_v \leq \alpha_{t+h}(\mathfrak{F}^N_s - \mathfrak{F}^N_t) \leq \alpha_{t+h}(F^N_s - \mathfrak{F}^N_t) \leq \alpha_{t+h}(F^N_s - F^N_t) + F^N_t - \mathfrak{F}^N_t,$$

where the last inequality follows from the fact that α is upper bounded by 1 and $\mathfrak{F}_t^N \leq F_t^N$. Consequently, we arrive at the upper bound

$$\begin{aligned} X_s^{i_1} \leq & X_t^{i_1} + Ch\left(1 + \sup_{u \leq t+h} \left|Z_u^{i_1}\right| + N^{-1} \sum_{i=1}^{N} \sup_{u \leq t+h} \left|Z_u^i\right|\right) \\ & + \sup_{u \in [t,t+h]} \left| \int_t^u \sigma(v,\, X_v^{i_1})\, \mathrm{d}B_v^{i_1} \right| + \alpha_{t+h}(F_s^N - F_t^N) + F_t^N - \mathfrak{F}_t^N. \end{aligned}$$

The goal now is to split the firing function into the contributions from three groups of particles: those in $\mathcal{I}^1$, those that fire more than once, and those in their refractory period. We define the set

$$\mathcal{I}^{(2)} \coloneqq \{i \in \{1, \ldots, N\} \,:\, J_{t+h}^i - J_{t-}^i \geq 2\},$$

and note that the mass of particles in the refractory period at time t is given by $F_t^N - \mathfrak{F}_t^N$. If k particles in $\mathcal{I}^1$ spike before i_1, then we must have that

$$F_s^N - F_t^N \leq \frac{k}{N} + \left(\frac{\left|\mathcal{I}^{(2)}\right|}{N^2} \sum_{i=1}^{N} (J_{t+h}^i)^2\right)^{1/2} + F_t^N - \mathfrak{F}_t^N.$$

Consequently, if k particles in $\mathcal{I}^1$ spike before i_1, we must have

$$\begin{aligned} 0 \leq & X_t^{i_1} + Ch\left(1 + \sup_{u \leq t+h} \left|Z_u^{i_1}\right| + N^{-1} \sum_{i=1}^{N} \sup_{u \leq t+h} \left|Z_u^i\right|\right) + \sup_{u \in [t,t+h]} \left| \int_t^u \sigma(v,\, X_v^{i_1})\, \mathrm{d}B_v^{i_1} \right| \\ & + \alpha_{t+h} \frac{k}{N} + \left(\frac{\left|\mathcal{I}^{(2)}\right|}{N^2} \sum_{i=1}^{N} (J_{t+h}^i)^2\right)^{1/2} + 2(F_t^N - \mathfrak{F}_t^N). \end{aligned} \tag{4.6.19}$$

Clearly, the number of particles for which the above must hold is at least $k+1$, as it must be true for the k particles that spiked before i_1 and for i_1 itself. Furthermore, since

$$\frac{|\mathcal{I}^1|}{N} \leq F_{t+h}^N - F_t^N \leq \frac{|\mathcal{I}^1|}{N} + \left(\frac{\left|\mathcal{I}^{(2)}\right|}{N^2} \sum_{i=1}^{N} (J_{t+h}^i)^2\right)^{1/2} + F_t^N - \mathfrak{F}_t^N,$$

for any integer $k < N(F_{t+h}^N - F_t^N - F_t^N + \mathfrak{F}_t^N) - (\left|\mathcal{I}^{(2)}\right| \sum_{i=1}^{N} (J_{t+h}^i)^2)^{1/2}$, we have $k < |\mathcal{I}^1|$. Therefore, by (4.6.19), defining the set

$$\begin{aligned} B^{i,1}(k) = \Bigg\{ & 0 \leq X_t^i + Ch\left(1 + \sup_{u \leq t+h} \left|Z_u^i\right| + N^{-1} \sum_{j=1}^{N} \sup_{u \leq t+h} \left|Z_u^j\right|\right) \\ & + \sup_{u \in [t,t+h]} \left| \int_t^u \sigma(v,\, X_v^i)\, \mathrm{d}B_v^i \right| + \alpha_{t+h} \frac{k}{N} \\ & + \left(\frac{\left|\mathcal{I}^{(2)}\right|}{N^2} \sum_{j=1}^{N} (J_{t+h}^j)^2\right)^{1/2} + 2(F_t^N - \mathfrak{F}_t^N),\, X_t^i < 0 \Bigg\}, \end{aligned} \tag{4.6.20}$$

we can deduce that $N^{-1}\sum_{i=1}^{N}\mathbb{1}_{B^{i,1}(k)} \geq k/N$. We now define the following events:

$$\begin{aligned}
B^0 :=& \{\tau(\beta) > t+h\} \cap \left\{ N^{-1}\sum_{i=1}^{N}(J^i_{t+h})^2 \leq h^{-1/8} \right\},\\
\tau(\beta) :=& \inf\left\{ s \in [t, t+h] \,:\, N^{-1}\sum_{i=1}^{N}\mathbb{1}_{\{J^i_s - J^i_{t-} \geq 2\}} \geq \beta \right\},\\
B^2 :=& \left\{ N^{-1}\sum_{i=1}^{N}\mathbb{1}_{B^{i,2}} < h \right\},\\
B^{i,2} :=& \left\{ h^{1/16} \leq Ch\left(1 + \sup_{u\leq t+h}\left|Z^i_u\right| + N^{-1}\sum_{j=1}^{N}\sup_{u\leq t+h}\left|Z^j_u\right|\right)\right.\\
&\left. + \sup_{u\in[t,t+h]}\left|\int_t^u \sigma(v,\, X^i_v)\,\mathrm{d}B^i_v\right| + 2(F^N_t - \mathfrak{F}^N_t) \right\}.
\end{aligned}$$

On the event $\{\tau(\beta) > t+h\}$, we have $\left|\mathcal{I}^{(2)}\right|/N < \beta = h^{1/4}$. Therefore, on the event B^0, we have $(\left|\mathcal{I}^{(2)}\right|/N^2\sum_{i=1}^{N}(J^i_{t+h})^2)^{1/2} \leq h^{1/16}$. Consequently, on the event B^0, for any $\lambda \leq F^N_{t+h} - F^N_t - F^N_t + \mathfrak{F}^N_t - 3h^{1/16}$, we have

$$\lfloor \lambda N + 2Nh^{1/16} \rfloor \leq N(F^N_{t+h} - F^N_t - F^N_t + \mathfrak{F}^N_t) - \left(|\mathcal{I}^{(2)}|\sum_{i=1}^{N}(J^i_{t+h})^2\right)^{1/2}.$$

For this fixed λ, we set $k = \lfloor \lambda N + 2Nh^{1/16} \rfloor$. Thus, for such a k, we have $N^{-1}\sum_{i=1}^{N}\mathbb{1}_{B^{i,1}(k)} \geq k/N$. Hence, on the event $B^0 \cap B^{i,1}(k) \cap (B^{i,2})^{\complement}$, we have $X_t + \alpha k/N \geq -2h^{1/16}$. Consequently, on the same event,

$$X^i_t + \alpha_{t+h}\lambda \geq X^i_t + \alpha_{t+h}k/N - 2h^{1/16} \geq -4h^{1/16}.$$

Therefore, on the event $B^0 \cap B^2$ and for $N > h^{-1/2}$, we have

$$\begin{aligned}
\nu_t([-\alpha_{t+h}\lambda - 4h^{1/16}, 0]) =& N^{-1}\sum_{i=1}^{N}\mathbb{1}_{\{X^i_t \geq -\alpha_{t+h}\lambda - 4h^{1/16}\}}\mathbb{1}_{\{X^i_t < 0\}}\\
&\geq N^{-1}\sum_{i=1}^{N}\mathbb{1}_{B^0\cap B^{i,1}(k)\cap(B^{i,2})^{\complement}}\\
&\geq N^{-1}\sum_{i=1}^{N}\mathbb{1}_{B^0\cap B^{i,1}(k)} - N^{-1}\sum_{i=1}^{N}\mathbb{1}_{B^0\cap B^{i,1}(k)\cap B^{i,2}}
\end{aligned}$$

as we are on the event $B^0 \cap B^2$

$$\begin{aligned}
&\geq N^{-1}\sum_{i=1}^{N}\mathbb{1}_{B^{i,1}(k)} - N^{-1}\sum_{i=1}^{N}\mathbb{1}_{B^{i,2}}\\
&\geq k/N - h \geq \lambda.
\end{aligned}$$

The last inequality follows from the fact that since $N > h^{-1/2}$ and $h \leq 1$, we have $k/N - h \geq \lambda + 2h^{1/16} - 1/N - h \geq \lambda$. To complete the proof, we only need a bound on $\mathbb{P}((B^0 \cap B^2)^\complement)$ that is uniform in N and independent of the smoothing kernel. We observe that

$$\begin{aligned}\mathbb{P}((B^0 \cap B^2)^\complement) \leq & \mathbb{P}(\tau(\beta) \leq t + h) + \mathbb{P}\left(N^{-1}\sum_{i=1}^{N}(J^i_{t+h})^2 \geq h^{-1/8}\right) \\ & + \mathbb{P}\left(N^{-1}\sum_{i=1}^{N}\mathbb{1}_{B^{i,2}} \geq h\right) \\ & \leq \mathbb{P}(\tau(\beta) \leq t + h) + h^{p/8}N^{-1}\sum_{i=1}^{N}\mathbb{E}[(J^i_{t+h})^{2p}] + h^{-1}N^{-1}\sum_{i=1}^{N}\mathbb{P}(B^{i,2}).\end{aligned}$$

Our particle system (4.6.16) is simpler than that presented in Chapter 3, as we do not have contributions from the firing function in the drift b. Therefore, by a simple modification of the proofs of Proposition 3.6.1 and Proposition 3.3.7, we can find a constant $\tilde{C} > 0$, independent of the mollifier $\mathfrak{K}$, such that the first two terms are bounded by $\tilde{C}h$ for all $N > h^{-1/2}$. Additionally, since σ is bounded,

$$\mathbb{P}(B^{i,2}) \leq \tilde{C}h + \tilde{C}h^{-1}\mathbb{E}[(F^N_t - \mathfrak{F}^N_t)^{16}] \tag{4.6.21}$$

for a possibly larger constant $\tilde{C} > 0$, still independent of $\mathfrak{K}$. The last term results from applying Markov's inequality. Thus, we have shown (4.6.17).

Step 2: Weak convergence

By Theorem 3.2.3 and Theorem 3.2.5, we have that (ν^N, F^N) converges weakly towards $(\nu^\varepsilon, F^\varepsilon)$ on $(D_{\mathscr{S}'}, M_1) \times (D_{\mathbb{R}}, M_1)$. Now, we define the function

$$\Psi_{C,\lambda,\Delta}(x)\begin{cases} = 1 & \text{if } x \in [-\alpha_{t+h}\lambda - Ch^{1/16} - \Delta,\, 0], \\ = 0 & \text{if } x \in (-\infty, -\alpha_{t+h}\lambda - Ch^{1/16} - \Delta] \cup [\Delta, +\infty), \\ \in [0,1] & \text{otherwise.}\end{cases} \tag{4.6.22}$$

Then, by (4.6.17), we have

$$\begin{aligned}\mathbb{P}\left(\forall\, \lambda \leq F^N_{t+h} - F^N_t - Ch^{1/16} - (F^N_t - \mathfrak{F}^N_t),\, \langle \nu^N_t, \Psi_{C,\lambda,\Delta}\rangle \geq \lambda\right) \\ \geq 1 - Ch - Ch^{-2}\mathbb{E}[(F^N_t - \mathfrak{F}^N_t)^{16}].\end{aligned} \tag{4.6.23}$$

Therefore, for any $t \in \mathbb{T}^\varepsilon$ and by applying the Portmanteau theorem, we obtain

$$\begin{aligned}\mathbb{P}\left(\forall\, \lambda \leq F^\varepsilon_{t+h} - F^\varepsilon_t - 2Ch^{1/16} - (F^\varepsilon_t - \mathfrak{F}^\varepsilon_t),\, \langle \nu^\varepsilon_t, \Psi_{2C,\lambda,\Delta}\rangle \geq \lambda\right) \\ \geq 1 - 2Ch - 2Ch^{-2}\mathbb{E}[(F^\varepsilon_t - \mathfrak{F}^\varepsilon_t)^{16}].\end{aligned}$$

Finally, by taking the limit as $\Delta \downarrow 0$, the result follows from the continuity of the measure. □

Proposition 4.6.10. *Suppose that* $\tilde{\mathbf{P}}^{\varepsilon_n} \implies \mathbf{P}^*$ *on* $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{wk})$ *for a positive sequence* $(\varepsilon_n)_{n\geq 1}$ *which converges to zero. For any* $t \in [0,T)$ *and* $h \in (0,1)$ *such that* $t, t+h \in \cap_{n\geq 1}\mathbb{T}^{\varepsilon_n} \cap \mathbb{T} \cap [0,T)$*, we have*

$$\mathbb{P}\left(\forall\, \lambda \leq F^*_{t+h} - F^*_t - Ch^{1/16},\, \boldsymbol{\nu}^*_t([-\alpha_{t+h}\lambda - Ch^{1/16}, 0]) \geq \lambda\right) \geq 1 - Ch$$

for some constant $C > 0$*.*

Proof. The proof follows directly from Lemma 2.6.8 with the obvious changes. □

Proposition 4.6.11. *Suppose that* $\tilde{\mathbf{P}}^{\varepsilon_n} \implies \mathbf{P}^*$ *on* $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{wk})$ *for a positive sequence* $(\varepsilon_n)_{n\geq 1}$ *which converges to zero. Then, we have*

$$\Delta F^*_t \leq \inf\{x \geq 0 \,:\, \boldsymbol{\nu}^*_{t-}([-\alpha_t x, 0]) < x\} \tag{4.6.24}$$

almost surely, for any $t \in [0,T)$*.*

Proof. It is clear that (4.6.24) holds for any $t \in \mathbb{T}$. Hence, we must only show the upper bound for $t \notin \mathbb{T}$. We first consider the case when $t \in (0,T) \cap \mathbb{T}^{\complement}$. The case when $t = 0$ will be treated separately. Now, as $\cap_{n\geq 1}\mathbb{T}^{\varepsilon_n} \cap \mathbb{T} \cap [0,T)$ is dense in $[0,T]$, we may find sequences $(t_n)_{n\geq 1}$, $(t_n + h_n)_{n\geq 1} \subset \cap_{n\geq 1}\mathbb{T}^{\varepsilon_n} \cap \mathbb{T} \cap [0,T)$ such that $t_n \uparrow t$, $t_n + h_n \downarrow t$, and $h_n < 2^{-3n}$. By the Borel-Cantelli Lemma, we have a set of full measure such that

$$\boldsymbol{\nu}^*_{t_n}([-\alpha_{t_n+h_n}\lambda - Ch_n{}^{1/16}, 0]) \geq \lambda, \quad \forall\, \lambda \leq F^*_{t_n+h_n} - F^*_{t_n} - Ch_n{}^{1/16}, \tag{4.6.25}$$

for all (possibly stochastic) n sufficiently large. Furthermore, by the dominated convergence theorem, we have

$$\boldsymbol{\nu}^*_{t_n} \to \boldsymbol{\nu}^*_{t-}, \tag{4.6.26}$$

$\mathrm{Law}(\tilde{\mathbf{P}}^*)$-almost surely in $\mathbf{M}_{\leq 1}(\mathbb{R})$, as for any $\phi \in \mathcal{C}_b(\mathbb{R})$

$$\lim_{n\to\infty} \langle \boldsymbol{\nu}^*_{t_n}, \phi\rangle = \lim_{n\to\infty} \int_{D_{\mathbb{R}}} \phi(\eta_{t_n} - \tilde{J}_{t_n}(\eta))\, \mathrm{d}\tilde{\mathbf{P}}^*(\eta) = \int_{D_{\mathbb{R}}} \phi(\eta_{t-} - \tilde{J}_{t-}(\eta))\, \mathrm{d}\tilde{\mathbf{P}}^*(\eta).$$

So, on an event of full $\mathrm{Law}(\mathbf{P}^*)$-measure where (4.6.25) and (4.6.26) hold, by the Portmanteau Theorem for any $\delta > 0$ and $\lambda < \Delta F^*_t$, we have

$$\begin{aligned}
\boldsymbol{\nu}_{t-}([-\alpha_t\lambda - \delta, 0]) &\geq \limsup_{n\to\infty} \boldsymbol{\nu}_{t_n}([-\alpha_t\lambda - \delta, 0]) \\
&\geq \limsup_{n\to\infty} \boldsymbol{\nu}_{t_n}([-\alpha_{t_n+h_n}\lambda - Ch_n{}^{1/16}, 0]) \\
&\geq \lambda.
\end{aligned}$$

This holds as $\lambda < \Delta F_t^*$ and $F^*_{t_n+h_n} - F^*_{t_n} - C{h_n}^{1/16} \to \Delta F_t^*$. Sending δ to zero shows the claim for every $t > 0$.

In the case when $t = 0$, we have by Proposition 4.6.9 that

$$\mathbb{P}\left(\forall\, \lambda \leq F_h^\varepsilon - Ch^{1/16},\, \boldsymbol{\nu}^\varepsilon_{0-}([-\alpha_h\lambda - Ch^{1/16}, 0]) \geq \lambda\right) \geq 1 - Ch.$$

As $\boldsymbol{\nu}^\varepsilon_{0-} = \nu_{0-}$, where ν_{0-} is a deterministic measure, and $\boldsymbol{\nu}^*_{0-}$ has distribution ν_{0-} almost surely, then we have by Proposition 4.6.10

$$\mathbb{P}\left(\forall\, \lambda \leq F_h^* - Ch^{1/16},\, \boldsymbol{\nu}^*_{0-}([-\alpha_h\lambda - Ch^{1/16}, 0]) \geq \lambda\right) \geq 1 - Ch$$

for $h \in \mathbb{T}$. Now, the rest of the proof follows similar arguments as above by choosing $h_n \in \cap_{n\geq 1}\mathbb{T}^{\varepsilon_n} \cap \mathbb{T}$ such that $h_n \downarrow 0$. □

Proposition 4.6.12. *Suppose that* $\tilde{\mathbf{P}}^{\varepsilon_n} \implies \mathbf{P}^*$ *on* $(\mathcal{P}(D_\mathbb{R}), \mathfrak{T}^{wk}_{M_1})$ *for a positive sequence* $(\varepsilon_n)_{n\geq 1}$ *which converges to zero. Then,*

$$\mathbb{P}\left[\Delta F_t^* \leq 1 \quad \forall\, t \in [0, T)\right] = 1.$$

Proof. We begin by returning to the particle system defined in (4.6.16). Our particle system (4.6.16) is simpler than that presented in Chapter 3, as we do not have the contributions from the firing function in the drift b. Therefore, by a simple modification of the proofs of Proposition 3.6.1 and Proposition 3.3.7, we can find a constant $c_1 > 0$ that is independent of the mollifier $\mathfrak{K}^\varepsilon$, such that for all $t \in [0, T]$, $h \in (0, 1)$, and N sufficiently large, we have

$$\mathbb{P}\left[F^{N,\varepsilon}_{t+h} - F^{N,\varepsilon}_{t-} > 1 + c_1h^{1/16}\right] \leq c_1h^3.$$

By the Portmanteau Theorem, we may deduce that for all $t, t + h \in [0, T) \cap \mathbb{T}^\varepsilon$, we have

$$\mathbb{P}\left[F^\varepsilon_{t+h} - F^\varepsilon_t > 1 + 2c_1h^{1/16}\right] \leq 2c_1h^3,$$

where

$$\mathbb{T}^\varepsilon := \left\{t \in [0, T] \,:\, \mathbb{P}(F^\varepsilon_t = F^\varepsilon_{t-}) = 1,\, \mathbb{P}[\langle \nu^\varepsilon_t, \phi^n\rangle = \langle \nu^\varepsilon_{t-}, \phi^n\rangle] = 1 \text{ for } \phi^n \in \mathscr{S} \text{ dense}\right\}.$$

Lastly, by exploiting the right continuity of the firing function, we may conclude that for all $t, t + h \in [0, T)$ with $h \in (0, 1)$, we have

$$\mathbb{P}\left[F^\varepsilon_{t+h} - F^\varepsilon_t > 1 + 4c_1h^{1/16}\right] \leq 4c_1h^3. \tag{4.6.27}$$

As $\tilde{\mathbf{P}}^{\varepsilon_n} \implies \mathbf{P}^*$ on $(\mathcal{P}(D_\mathbb{R}), \mathfrak{T}^{\mathrm{wk}}_{M_1})$, by Lemma 4.4.8 and employing Skorokhod's Representation Theorem, we may deduce that $F^{\varepsilon_n}_{t+h} - F^{\varepsilon_n}_t \implies F^*_{t+h} - F^*_t$ for all

$t, t+h \in [0,T) \cap \mathbb{T}$. Therefore, from (4.6.27), first by applying the Portmanteau Theorem and then by employing the right continuity of the firing function, we have

$$\mathbb{P}\left[F^*_{t+h} - F^*_t > 1 + 16c_1 h^{1/16}\right] \leq 16c_1 h^3, \tag{4.6.28}$$

for all $t, t+h \in [0,T)$ with $h \in (0,1)$.

Now, we define the event

$$A^M := \bigcup_{j=1}^{M-1} \left\{F^*_{t_j^M} - F^*_{t_{j-1}^M} > 1 + 16c_1 M^{-1/16} T^{1/16}\right\}, \tag{4.6.29}$$

where $t_j^M = jT/M$. Therefore, for any $M > \lceil T \rceil$, we have by (4.6.28) that $\mathbb{P}[A^M] \leq 16c_1 T^3 M^{-2}$. By the Borel-Cantelli lemma, we have that $\mathbb{P}[A^M \text{ i.o.}] = 0$. Consequently, $\mathbb{P}[A^M \text{ ult.}] = 1$. Choosing any $\omega \in \{A^M \text{ ult.}\}$, there exists an $M_0(\omega) = M_0 > \lceil T \rceil$ such that for all $M \geq M_0$, we have

$$F^*_{t_j^M}(\omega) - F^*_{t_{j-1}^M}(\omega) \leq 1 + 16c_1 M^{-1/16} T^{1/16},$$

for all $j = 1, \ldots, M-1$. Hence, for any $t \in (0,T)$, we can find a sequence $(t_{j_k}^{M_k})_k$ such that $t_{j_k}^{M_k} \downarrow t$ and $t_{j_{k-1}}^{M_k} \uparrow t$. Therefore,

$$\Delta F^*_t(\omega) = \lim_{k\to\infty} F^*_{t_{j_k}^{M_k}}(\omega) - F^*_{t_{j_{k-1}}^{M_k}}(\omega) \leq \lim_{k\to\infty} 1 + 16c_1 M_k^{-1/16} T^{1/16} = 1.$$

Hence, the jumps of $F^*(\omega)$ are bounded by 1 on $(0,T)$.

For the case when $t = 0$, as $F^\varepsilon_0 = 0$ for all $\varepsilon > 0$, we have by (4.6.27) that $\mathbb{P}\left[F^\varepsilon_h > 1 + 4c_1 h^{1/16}\right] \leq 4c_1 h^3$. Employing the same arguments as above, we have that $\mathbb{P}\left[F^*_h > 1 + 16c_1 h^{1/16}\right] \leq 16c_1 h^3$ for all $h \in (0,1)$. Therefore, by employing the Borel-Cantelli lemma, we may conclude that $\Delta F^*_0 \leq 1$ almost surely. This completes the proof. □

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