Zariski Structures in Noncommutative Algebraic Geometry and Representation Theory

Vinesh Solanki
Exeter College
University of Oxford

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To my parents, for their enduring support.

paśya me pārtha rūpāṇi śataśo 'tha sahasraśaḥ
nānā-vidhāni divyāni nānā-varṇākṛtīni ca

Bhagavad-Gita, XI. V

Abstract

A suitable subcategory of affine Azumaya algebras is defined and a functor from this category to the category of Zariski structures is constructed. The rudiments of a theory of presheaves of topological structures is developed and applied to construct examples of structures at a generic parameter. The category of equivariant algebras is defined and a first-order theory is associated to each object. For those theories satisfying a certain technical condition, uncountable categoricity and quantifier elimination results are established. Models are shown to be Zariski structures and a functor from the category of equivariant algebras to Zariski structures is constructed. The two functors obtained in the thesis are shown to agree on a nontrivial class of algebras.

Contents

Chapter 1

Introduction

Let CRing and AffSch denote the categories of commutative unital rings and affine schemes respectively. The equivalence of the categories CRing^{op} and AffSch is a fundamental result of algebraic geometry. On the one hand, we have an algebraic category CRing consisting of algebraic objects and appropriate morphisms. On the other hand, we have a category AffSch consisting of geometric objects and appropriate morphisms. An equivalence of categories is the appropriate language for facilitating a transfer of information between the algebraic and geometric worlds.

Now suppose that we take for our algebraic category the larger category Ring of unital rings. Is there a plausible candidate for a geometric category to which Ring^{op} is equivalent? Clearly one also demands such an equivalence to extend the usual algebro-geometric one, namely that the diagram

$$\begin{tikzcd}
\mathsf{CRing}^{op} \arrow[r, shift left] \arrow[d, hook] & \mathsf{AffSch} \arrow[l, shift left] \arrow[d, hook] \\
\mathsf{Ring}^{op} \arrow[r, shift left] & \mathsf{NSch} \arrow[l, shift left]
\end{tikzcd}$$

commutes, where NSch is our candidate for a geometric category of 'noncommutative schemes'. Ultimately, we find ourselves cast out of a commutative Eden and into the wider and harsher terrain of noncommutative algebraic geometry.

The contents of this thesis constitute a geometric model-theorist's attempts to explore this terrain using the tools of his trade. Specifically, the author has adopted the viewpoint that geometric model theory can only interact with noncommutative algebraic geometry via the theory of Zariski structures; and indeed does so most naturally. This is a viewpoint he now wishes to justify.

Firstly, an investigation of important applications of geometric model theory to questions of *commu-*

tative algebraic geometry (specifically diophantine geometry) indicates that a difference of approach is currently needed if our endeavours are to succeed. The methodology of such applications can be summarized as follows. One selects appropriate structures (e.g. algebraically closed fields, differentially closed fields, separably closed fields), establishes what 'stability class' the structures belong to and deduces results about definable sets by applying the relevant abstract model-theoretic tools associated with this stability class [1]. Crucially, the language and techniques of geometric model theory, with its emphasis on stability and appropriate generalizations of this notion, independence and ranks, working in a universal domain etc, is closer in spirit and language to Weil's foundations than scheme theory. One certainly does not work at the level of generality of an arbitrary commutative ring. For any hope of applying model theory to noncommutative algebraic geometry, it seems that one should remain (at the very least) in a suitably geometric setting and look for geometric counterparts to suitable classes of noncommutative k-algebras, where k is an algebraically closed field. Henceforth, this assumption on k shall be assumed throughout. But at the same time, there seems to be no reason for suspecting that there is a nice geometric structure whose definable subsets can be regarded as corresponding to coordinate rings of a sufficiently interesting and large class of noncommutative k-algebras. It is not possible to do any 'naive' noncommutative algebraic geometry in the manner that one can work with varieties as subsets of affine or projective space. The language of schemes and category-theoretic generalizations of it are indispensable for most of the popular existing approaches to noncommutative algebraic geometry ([Mah06], [Ros95]).

Rather, we are forced to

- find a systematic means of associating a structure to a given noncommutative k-algebra, suitably axiomatized in an appropriate language.
- ask whether these structures share any common geometry.

An association of structures to algebras should be functorial if it is to be systematic; thus we must work with a geometric category of structures not necessarily all defined in the same language. But dropping the requirement that all structures be formulated in the same language leaves the only possibility of morphisms between them being topological in nature. Thus there should be a topology on models. If we are aiming for an extension of commutative algebraic geometry then a basic dimension theory resembling the commutative case should exist. It transpires that these

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[1] The articles [Mac03] and [Hru98] contain an introduction to methods of geometric model theory; [B99] discusses the specific application of these methods to Mordell-Lang.

rather basic requirements (i.e. that we have 'topological' structures with a reasonable dimension theory) lead us to stipulate that the associated structures are Zariski structures. We work with the definition of [Zil10]:

Definition 1.1. *Let* $\mathbf{X}$ *be a set. A* ***Zariski structure*** [2] *on* $\mathbf{X}$ *consists of a Noetherian topology on* $\mathbf{X}^n$ *for every* $n > 0$ *and an* $\mathbb{N}$*-valued dimension function* $\dim$ *on non-empty projective subsets (finite unions of projections of closed subsets) satisfying the following properties:*

1. *The dimension of a point is* 0.
2. $\dim(\mathbf{P}_1 \cup \mathbf{P}_2) = \max\{\dim \mathbf{P}_1, \dim \mathbf{P}_2\}$ *for all projective subsets* $\mathbf{P}_1, \mathbf{P}_2$.
3. *For* $\mathbf{C}$ *closed and irreducible in* $\mathbf{X}^n$ *and* $\mathbf{C}_1$ *a closed subset of* $\mathbf{C}$*, if* $\mathbf{C}_1 \neq \mathbf{C}$ *then* $\dim \mathbf{C}_1 < \dim \mathbf{C}$.
4. *For* $\mathbf{C}$ *irreducible and closed in* $\mathbf{X}^n$*, if* $\pi : \mathbf{X}^n \to \mathbf{X}^m$ *is a projection then*
$$\dim \mathbf{C} = \dim \pi(\mathbf{C}) + \min_{a \in \pi(\mathbf{C})} \dim(\pi^{-1}(a) \cap \mathbf{C})$$
5. *For any irreducible closed* $\mathbf{C}$ *in* $\mathbf{X}^n$ *and projection map* $\pi : \mathbf{X}^n \to \mathbf{X}^m$*, there is a subset* $\mathbf{V}$ *relatively open in* $\pi(\mathbf{C})$ *such that*
$$\min_{a \in \pi(\mathbf{C})} \dim(\pi^{-1}(a) \cap \mathbf{C}) = \dim(\pi^{-1}(v) \cap \mathbf{C})$$
for every $v \in \mathbf{V} \cap \pi(\mathbf{C})$.

Moreover, projections must be semi-proper, i.e. for any closed irreducible subset $\mathbf{C}$ *of* $\mathbf{X}^n$ *and projection map* $\pi : \mathbf{X}^n \to \mathbf{X}^m$*, there is a proper closed subset* $\mathbf{D}$ *of* $\overline{\pi\mathbf{C}}$ *such that* $\overline{\pi\mathbf{C}} \setminus \mathbf{D} \subseteq \pi\mathbf{C}$*. A Zariski structure is said to be* ***presmooth*** *if for any closed irreducible subsets* $\mathbf{C}_1, \mathbf{C}_2$ *of* $\mathbf{X}^n$ *the dimension of any irreducible component of* $\mathbf{C}_1 \cap \mathbf{C}_2$ *is greater than or equal to*

$$\dim \mathbf{C}_1 + \dim \mathbf{C}_2 - \dim \mathbf{X}^n$$

A natural candidate for a morphism $f : \mathbf{X} \to \mathbf{Y}$ of Zariski structures is a function inducing a continuous map on $\mathbf{X}^n$ for every n. [3] Thus we have a category of Zariski structures with these morphisms, which we denote by Zar. Some familiarity with algebraic geometry (in particular results

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[2] Technically, according to the terminology of [Zil10] we shall be defining Noetherian Zariski structures as opposed to analytic Zariski structures. Because we do not deal with the latter, for the purposes of this thesis the adjective 'Noetherian' can be dropped.

[3] Specifically a morphism $f : \mathbf{X} \to \mathbf{Y}$ is a collection of maps $f_{m,n} : \mathbf{X}^m \to \mathbf{Y}^n$ such that $f_{m,n}^{-1}(\mathbf{C})$ is closed for each closed $\mathbf{C} \subseteq \mathbf{Y}^n$. We also require that these maps are compatible with projections, i.e. if $\pi_{1,2} : \mathbf{Y}^{n_1} \to \mathbf{Y}^{n_2}$ is a projection, then $\pi_{1,2} \circ f_{m,n_1} = f_{m,n_2}$.

on the dimensions of fibers, [Har77], II, Exercise 3.22) will allow one to conclude that varieties are Zariski structures, and are presmooth if the varieties are smooth. Hence the category of algebraic varieties is a subcategory of Zar. Moreover, like schemes, Zariski structures have the advantage of being abstractly given and not as sitting in some ambient structure.

However, Zariski structures were not introduced to fulfill the purpose of being a model-theorists' answer to algebraic manifolds. Rather, they first appeared in [HZ96] as a response to the failure of Zilber's trichotomy conjecture. Roughly speaking, the trichotomy conjecture proposed that the geometry of certain subsets of models (the so-called strongly minimal sets) fell into three mutually exclusive classes; such geometries were either trivial, linear, or that of an algebraically closed field. After the ingenious refutation of this conjecture by Hrushovski, it was natural to ask whether there was a natural class of structures for which the trichotomy conjecture did hold. One-dimensional Zariski structures [4] turned out to be such a class. For our purposes, two aspects of the work in [HZ96] are particularly important. Firstly, as already mentioned, projective algebraic curves are Zariski structures. Secondly, there are one-dimensional Zariski structures which are demonstrably not projective curves but are certain finite covers of them. These structures, rather than turning out to be mathematical pathologies, can be taken to be geometric objects corresponding to certain noncommutative algebras. In this regard, we mention the paper [ZS09] as providing an example of such a one-dimensional Zariski structure corresponding to a physically important algebra, namely the Heisenberg algebra. In short, Zariski structures corresponding to noncommutative algebras do exist that can be distinguished from projective curves by their geometry demonstrably not being reducible to them.

Given that there are one-dimensional **non-classical** Zariski structures (those not arising from algebraic curves) and that these correspond to certain noncommutative algebras, it is natural to expect that there are higher-dimensional Zariski structures corresponding to other noncommutative algebras. The paper [Zil06] establishes exactly this: that non-classical Zariski structures can be associated to a class of noncommutative algebras, described in the paper as 'quantum algebras at roots of unity'. The definition of such algebras can be simplified with some knowledge of ring theory and the results of [Zil06] shall be discussed in due course. The results of [ZS09] and [Zil06] provide

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[4] The definition of Zariski structures appearing in [HZ96] is less general than Definition 1.1 because it stipulates that the underlying set $\mathbf{X}$ is one-dimensional in a suitable model-theoretic sense. When $\mathbf{X}$ is one-dimensional, both definitions coincide. We shall be dealing with Zariski structures where $\mathbf{X}$ has dimension > 1. Such Zariski structures will be referred to as higher-dimensional.

sufficient evidence to propose the following conjecture.

Conjecture 1.1. *Let k be an algebraically closed field. Then there is a commutative diagram of functors*

$$\begin{tikzcd}
(\mathsf{CAlg}(k)_{fg,int})^{op} \arrow[r] \arrow[d, hook] & \mathsf{Zar}^c \arrow[d, hook] \\
\mathsf{Alg}(k)^{op} \arrow[r] & \mathsf{Zar}
\end{tikzcd}$$

$\mathsf{CAlg}(k)_{fg,int}$	Finitely generated, commutative k-algebras that are domains
$\mathsf{Alg}(k)$	k-algebras
Zar^c	Classical Zariski structures

where the functor $\mathsf{Alg}(k)^{op} \to \mathsf{Zar}$ *is an equivalence of categories.*

The conjectural functor is, of course, $\mathsf{Alg}(k)^{op} \to \mathsf{Zar}$ and the work in this thesis has the construction of this functor as a focal point. To date, a general means of constructing a suitable such functor has not been found. As far as the author's work is concerned, the most fruitful modus operandi (both conceptually and pragmatically) has been the following:

1. Rather than attempting to construct a general functor $\mathsf{Alg}(k)^{op} \to \mathsf{Zar}$, isolate an interesting subcategory of k-algebras A that contains a suitably large subcategory B of the category of affine commutative k-algebras that are domains.

2. Constrain the algebraic characterisation of A by those additional assumptions necessary to associate an $\mathcal{L}_A$-structure nSpec A to every object A of A (where the language $\mathcal{L}_A$ depends on the object A). Such assumptions may well be non-algebraic in character. The structure nSpec A should be a moduli space for certain representations of A, preferably those A-modules that 'generate' an interesting subcategory of the category of all left A-modules, ${}_A\mathsf{Mod}$.

3. Carry out an analysis of the definable subsets of nSpec A and conclude that nSpec A is a Zariski structure.

4. Extend the correspondence $A \mapsto \text{nSpec}\, A$ to a functor nSpec : $\mathsf{A}^{op} \to \mathsf{Zar}$ and verify that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{B}^{op} \arrow[r] \arrow[d, hook] & \mathsf{Zar}^c \arrow[d, hook] \\
\mathsf{A}^{op} \arrow[r, "\text{nSpec}"'] & \mathsf{Zar}
\end{tikzcd}$$

5. Finally analyze the relationship between nSpec A and ${}_A\mathsf{Mod}$ for every object A of A.

It is appropriate to be a little bit more specific about syntax and related issues at this juncture. Let A' be our category of k-algebras obtained after appropriate constraints are introduced in 2. Then to each algebra A in A' we associate an $\mathcal{L}_A$-theory T_A that is first-order axiomatizable. The structure nSpec A is then taken to be a large saturated model (universal domain) of T_A. In much the same way that the language of rings naturally axiomatizes the theory of algebraically closed fields, the language $\mathcal{L}_A$ is chosen to be 'natural' for T_A. Moreover, the Zariski structure obtained on nSpec A should respect the theory T_A, in the sense that it arises from a suitable quantifier-elimination result. This particular methodology is uniquely model-theoretic and results in a topology that is rather descriptive. Crucial to this is the insistence in 2 on the structure nSpec A being a moduli space for a class of A-modules. The topology on nSpec A typically incorporates the internal structure of the modules explicitly into the geometry.

We now summarize the contents of this thesis. Chapter 2 builds on the work of the paper [Zil06]. It is shown that the assumptions defining a quantum algebra A at a root of unity are equivalent to A being a prime finitely generated Azumaya k-algebra satisfying certain definability conditions. For practical purposes, these definability conditions can be simplified by using the notion of a Galois covering. The corresponding structure nSpec A is a first-order definable moduli space parametrizing isomorphism types of irreducible A-modules. Denoting a suitable subcategory of $\mathsf{Alg}(k)$ consisting of these Azumaya algebras by $\mathsf{Azum}(k)_{def}$, we do indeed obtain a functor nSpec : $\mathsf{Azum}(k)^{op}_{def} \to \mathsf{Zar}$ such that

$$\begin{tikzcd}
(\mathsf{CAlg}_{fg,int})^{op} \arrow[r] \arrow[d, hook] & \mathsf{Zar}^c \arrow[d, hook] \\
\mathsf{Azum}(k)^{op}_{def} \arrow[r] & \mathsf{Zar}
\end{tikzcd}$$

commutes. Results on imaginaries in the associated Zariski structures are derived using the model theory of modules.

Chapter 3 develops the rudiments of a theory of presheaves of topological structures. The guiding idea is one proposed by A. Macintyre; namely that ultraproducts of structures should be stalks of appropriate presheaves. A formalism is developed for which this holds. The theory is applied to some of the Azumaya structures of Chapter 2.

Chapters 4 - 7 contain the bulk of the work carried out in this thesis, and they are concerned with a class of algebras which the author has called **equivariant**. The choice of terminology here is motivated by important structures appearing in geometric representation theory; namely those line bundles L over a variety V endowed with an action of a linear algebraic group G, such that

$$\text{for all } g \in G,\ g(L_x) = L_{gx} \text{ and } g : L_x \to L_{gx} \text{ is a linear isomorphism}$$

where L_x denotes the fiber of L at $x \in V$. Such line bundles are said to be **G-equivariant** (see [RTT07]). The structure corresponding to the Heisenberg algebra introduced in [ZS09] looked, at least superficially, to be an equivariant line bundle. However, further examination revealed some crucial differences. Firstly, there was no claim on local triviality. Secondly, whereas certain operators ($\mathbf{a}$ and $\mathbf{a}^\dagger$ for those familiar with the paper) did move between fibers in a manner that introduced an action of a group on the base, these two operators themselves didn't generate a group because they were not mutually inverse. It is the author's contention (and no doubt that of B. Zilber also) that such phenomena are characteristic of 'quantum' objects. Additional examples worked out in a similar vein (the quantum 2-torus by Zilber, $U_q(\mathfrak{sl}_2(k))$ for generic q by the author) suggested that an appropriate formalism could be found that treated all of these examples (and more) collectively. The fruits of this labour comprise the contents of chapter 4. In accordance with points 1 and 2 of the proposed methodology on page 5, we introduce a category of 'equivariant k-algebras' which we denote by $\mathsf{Equiv}(k)$. It is not a full subcategory of $\mathsf{Alg}(k)$ and an appropriate notion of a morphism in this category is given. We also show that given an object A of $\mathsf{Equiv}(k)$, we can associate a first-order $\mathcal{L}_A$-theory T_A to A.

Chapters 5 and 6 are devoted to the model theory of T_A under an additional technical assumption on T_A (Γ-rigidity), which the key examples mentioned in the previous paragraph are shown to satisfy. Uncountable categoricity and quantifier elimination results are established thus leading to the expected consequences for the category of definable subsets; namely that every definable subset is constructible for an appropriate topology on models. With this topology, an appropriate dimension theory turns each model into a Zariski structure. The method of technical analysis is that of [Zil06]. It is worth remarking that the condition of Γ-rigidity encapsulates precisely what is required for T_A to possess a rich structure theory, i.e. it is only for Γ-rigid T_A that uncountable categoricity holds.

Chapter 7 concludes our excursion into equivariant algebras and their associated Zariski structures with the expected construction of a functor nSpec. The Azumaya and equivariant approaches are unified on a certain class of noncommutative algebras. The epilogue (Chapter 8) discusses possibilities for further investigation.

To the author's knowledge, the structures $\mathrm{nSpec}\,A$ for general equivariant A have no precedent. They are also unusual in being able to assign to certain noncommutative algebras parametrized at a generic parameter a bone fide topological space, in contrast to the approaches to noncommutative algebraic geometry surveyed. It is worth remarking that the notion of Zariski structure is the thread uniting the various functors considered in this thesis: they each have Zar as a target category. It is the author's belief that this flexibility to accommodate differing constructions under the banner of Zariski structures is a strength of our approach.

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Chapter 2

Azumaya Structures

As stated in the introduction, in this chapter we define the category $\mathsf{Azum}(k)_{def}$ and we construct a functor nSpec : $\mathsf{Azum}(k)_{def}^{op} \to \mathsf{Zar}$ such that the diagram

$$\begin{tikzcd}
(\mathsf{CAlg}_{fg,int})^{op} \arrow[r] \arrow[d, hook] & \mathsf{Zar}^c \arrow[d, hook] \\
\mathsf{Azum}(k)_{def}^{op} \arrow[r] & \mathsf{Zar}
\end{tikzcd}$$

commutes. Given an algebra A in $\mathsf{Azum}(k)_{def}$, imaginaries in the corresponding Zariski structure are essentially pp-imaginaries parametrized by definable subsets of the variety corresponding to the center of A. These imaginaries can therefore be handled most appropriately by adopting a functorial viewpoint. A basic elimination of imaginaries result is established when A is semisimple.

2.1 Preliminaries on Galois Coverings

In this section, all rings are assumed to be commutative. Recall that an R-module M is **flat** if the functor $\cdot \otimes_R M : {}_R\mathsf{Mod} \to {}_R\mathsf{Mod}$ is exact. A ring homomorphism $\varphi : R \to S$ is flat if S is flat as an R-module.

Definition 2.1. *A morphism $f : X \to Y$ of schemes (or varieties) is* ***flat*** *if for every $x \in X$, the induced map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. It is* ***faithfully flat*** *if it is flat and surjective.*

Here $\mathcal{O}_{X,x}$ denotes the stalk at x of the structure sheaf of X (similarly for $\mathcal{O}_{Y,f(x)}$). Because flatness is a local property ([AM69], Proposition 3.10), the scheme-theoretic definition agrees with that given for rings.

Theorem 2.1. *Let M be a finitely generated R-module. The following are equivalent:*

1. *M is flat.*

2. *$M_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$ of R.*

3. *Let $\tilde{M}$ be the $\mathcal{O}_X$-module corresponding to M (where $X = \operatorname{Spec} R$). Then $\tilde{M}$ is a locally free sheaf on $\operatorname{Spec} R$.*

4. *M is a projective R-module.*

In addition, if R is an integral domain then the above are equivalent to:

- *$\dim_{k(\mathfrak{p})}(M \otimes_R k(\mathfrak{p}))$ is the same for all prime ideals $\mathfrak{p}$ of R.*

where $k(\mathfrak{p})$ denotes the residue field at $\mathfrak{p}$.

Proof. [Mil80], Theorem 2.9. ☐

Proposition 2.1. *A finite morphism of varieties $f : X \to Y$ is flat if and only if each fiber $f^{-1}(y)$ has the same number of points (counting multiplicities).*

Proof. We regard X and Y as integral separated schemes of finite type over an algebraically closed field k. If Y is affine, it is the spectrum of an integral domain and the result is immediate by Theorem 2.1. The general case follows by glueing. ☐

Let $f : X \to Y$ be a morphism of schemes (or varieties), Γ a finite group. A **left action of** Γ **on** X **over** Y is a map $\alpha : \Gamma \to \operatorname{Aut}_Y(X)$ such that $\alpha(\gamma\delta) = \alpha(\gamma) \cdot \alpha(\delta)$. Here $\operatorname{Aut}_Y(X)$ denotes the group of those automorphisms σ of X such that the diagram

$$\begin{tikzcd}
X \arrow[r, "\sigma"] \arrow[dr, "f"'] & X \arrow[d, "f"] \\
& Y
\end{tikzcd}$$

commutes.

Definition 2.2. *Let $f : X \to Y$ be a faithfully flat morphism of schemes (or varieties), Γ a finite group acting on X over Y on the left via α. Then f is a* ***Galois covering of** Y **with group** Γ* *if the morphism $\Gamma \times X \to X \times_Y X$ given by $(\gamma, x) \mapsto (\alpha(\gamma)(x), x)$ is an isomorphism.*

The notation $\Gamma \times X$ denotes a disjoint union of $\coprod_{\gamma \in \Gamma} X$ of copies of X indexed by Γ. Definition 2.2 easily implies that Γ acts freely and transitively on each fiber of f over Y.

Definition 2.3. *A morphism $f : X \to Y$ of schemes which is locally of finite type is* ***unramified*** *at $x \in X$ if $\mathcal{O}_{Y,f(x)}/\mathfrak{m}_x\mathcal{O}_{Y,f(x)}$ is a finite separable field extension of $k(x)$ (where $\mathfrak{m}_x$ is the unique maximal ideal of $\mathcal{O}_{X,x}$). It is unramified if it is unramified at all $x \in X$. The morphism f is* ***étale*** *if it is flat and unramified.*

It follows straightforwardly from Definition 2.3 that if $f : X \to Y$ is a Galois covering of Y with group Γ then f is both finite and étale.

2.2 Towards Azumaya Structures

A fairly self-contained introduction to much of the noncommutative algebra required is given in Appendix A, to which the reader is referred for the relevant definitions and results. We recall the following definition from [Zil06].

Definition 2.4. *Let k be an algebraically closed field. A k-algebra A is a **quantum algebra at a root of unity** if*

1. *A is finitely generated (i.e. affine) as a k-algebra by some $\mathbf{U}_1, \ldots, \mathbf{U}_d$ with defining relations with parameters taken from a finite subset $C \subset k$.*
2. *A is finitely generated as a module over a central subalgebra Z_0.*
3. *The set of maximal ideals of Z_0, $\operatorname{Spec} Z_0$, can be identified with the k-points of an irreducible affine variety V over C.*
4. (a) *There is a positive integer N such that to every maximal ideal $\mathfrak{m} \in \operatorname{Spec} Z_0$, there corresponds an unique (up to isomorphism) irreducible A-module M_x (where $x \in V$ corresponds to $\mathfrak{m}$) of dimension N as a k-vector space, with the property that $\operatorname{ann}_{Z_0}(M_x) = \mathfrak{m}$.*

   (b) *The isomorphism type of the module M_x is determined uniformly by a solution to a system of polynomial equations Σ^A, i.e. Σ^A is such that for each $x \in V$, there are elements $t = \{t_{ijk} : i \leq d; j, k \leq N\}$ in k such that $\Sigma^A(t, x) = 0$ and there is a k-basis $e_1, \ldots, e_N$ of M_x such that*

   $$\bigwedge_{i \leq d, j \leq N} \mathbf{U}_i e_j = \sum_{k=1}^{N} t_{ijk} e_k$$

   *holds. Any such basis is called a **canonical basis** for M_x.*
5. *There is a finite group Γ and a partial map $g : V \times \Gamma \to GL_N(k)$ such that*
   (a) *For each $\gamma \in \Gamma$, there is a non-empty open subset U_γ of V such that $g(\cdot, \gamma) : U_\gamma \to GL_N(k)$ is a C-definable (total) map.*

   (b) *For each $x \in V$, there is a subgroup Γ_x of Γ such that $g(x, \cdot) : \Gamma_x \to GL_N(k)$ is a C-definable injective group homomorphism.*

(c) *Given $x \in V$, for any two canonical bases $e_1, \dots, e_N$ and $e'_1, \dots, e'_N$ of M_x there is $\lambda \in k^*$ and a unique $\gamma \in \Gamma_x$ such that*

$$e'_i = \lambda \sum_{j=1}^{N} g_{ij}(x,\gamma) e_j \qquad 1 \leq i \leq N$$

where $g_{ij}(x,\gamma)$ denotes the (i,j)-th entry of the matrix $g(x,\gamma)$.

Proposition 2.2. *Conditions 1 - 4 (a) of Definition 2.4 hold if and only if A is a prime finitely generated k-algebra which is Azumaya over its center.*

Proof. Suppose that A is a prime finitely generated k-algebra that is Azumaya over its center Z. Because A is finitely generated as a Z-module, (k, Z, A) is an affine PI triple (Definition A.3). By the remarks following Lemma A.1 Z is a Noetherian domain and a finitely generated k-algebra, thus 3 holds. Say A is finitely generated as a Z-module by $\mathbf{U}_1, \dots, \mathbf{U}_d$. Putting $F = \bigoplus_d Z$ we therefore have a surjection $\pi : F \to A$ of Z-modules and $\ker \pi$ is a submodule of F. But F is a Noetherian module, hence $\ker \pi$ is finitely generated over Z. Thus A has finite presentation as a Z-module and the defining relations involving the $\mathbf{U}_i$ take their parameters from a finite set $C \subseteq k$. Condition 4 (a) is immediate from Corollary A.1.

Conversely, suppose that A is a k-algebra satisfying 1 - 4 (a). Then (k, Z_0, A) is an affine PI triple. By 2, Z_0 is a domain hence so is A. If $\mathfrak{M}$ is a maximal ideal of A, put $\mathfrak{m} = \mathfrak{M} \cap Z_0$ and let M be an irreducible A-module such that $\mathfrak{M} = \operatorname{ann}_A(M)$. Then $\operatorname{ann}_{Z_0}(M) = \mathfrak{m}$ by Proposition A.1 and 4 (a) gives that M is unique up to isomorphism. Hence by Proposition A.4 all maximal ideals of A are regular. But then A is Azumaya over its center by Theorem A.3. □

Conditions 4 (b) and 5 of Definition 2.4 serve to further constrain the class of Azumaya k-algebras. Our aim will be to justify the use of the following definition in place of Definition 2.4.

Definition 2.5. *A prime finitely generated k-algebra A which is Azumaya over its center Z is* ***definably irreducibly representable*** *if*

1. *4 (b) holds for $V = \operatorname{Spec} Z(A)$.*
2. *$p : \Sigma^A(k) \to V$ is a Galois covering with group Γ, where each $\gamma \in \Gamma$ acts by a linear automorphism.*

Here $\operatorname{Spec} Z(A)$ denotes the space of characters $Z(A) \to k$ (i.e. the maximal spectrum of $Z(A)$) and $\Sigma^A(k)$ denotes the variety defined by Σ^A in k.

Proposition 2.3. *Let A be a quantum algebra at a root of unity and suppose that $p : \Sigma^A(k) \to V$ is the projection $(t, x) \mapsto x$. Then p is a Galois covering of a non-empty open (dense) subset U of V where Γ acts linearly on $\Sigma^A(k)$.*

Proof. Put $U = \bigcap_{\gamma\in\Gamma} U_\gamma$. Because V is irreducible, U is a non-empty open (hence dense) subset of V.

Claim: There is a free and transitive action of Γ on $p^{-1}(y)$ for each $y \in U$.

Proof. First some notation. If $t = (t_{ijk})$ and $y \in V$ are such that $\Sigma^A(t, y) = 0$, we regard t as a tuple of $N\times N$ matrices $(\mathbf{A}_1, \dots, \mathbf{A}_d)$ where $\mathbf{A}_i = (t_{ijk})$ for each $1 \le i \le d$. Define $\cdot : \Gamma\times p^{-1}(U) \to p^{-1}(U)$ by conjugation, i.e.

$$\gamma \cdot (\mathbf{A}_1, \dots, \mathbf{A}_n) = (g(y, \gamma)^{-1}\mathbf{A}_1 g(y, \gamma), \dots, g(y, \gamma)^{-1}\mathbf{A}_d g(y, \gamma))$$

for $t = (\mathbf{A}_1, \dots, \mathbf{A}_d) \in p^{-1}(y)$ and $y \in U$. For any $t, t' \in k$ such that $\Sigma^A(t, y) = \Sigma^A(t', y) = 0$, there are corresponding canonical bases $e_1, \dots, e_N$ and $e'_1, \dots, e'_N$ of M_y such that

$$\mathbf{U}_i e_j = \sum_{k=1}^{N} t_{ijk} e_k \qquad \mathbf{U}_i e'_j = \sum_{k=1}^{N} t'_{ijk} e'_k$$

By condition 5 (c), there is $\lambda \in k^*$ and unique $\gamma \in \Gamma$ such that

$$e'_i = \lambda \sum_{j=1}^{N} g_{ij}(y, \gamma) e_j \qquad 1 \le i \le N$$

Let $\mathbf{A}_i$ (respectively $\mathbf{A}'_i$) be the $N \times N$ matrix (t_{ijk}) (respectively (t'_{ijk})). Then putting $\mathbf{G} = g(y, \gamma)$, we have

$$(\mathbf{A}'_1, \dots, \mathbf{A}'_d) = (\mathbf{G}\mathbf{A}_1\mathbf{G}^{-1}, \dots, \mathbf{G}\mathbf{A}_d\mathbf{G}^{-1})$$

hence the action is transitive. Now suppose that $\gamma \cdot t = t$. If $t = (\mathbf{A}_1, \dots, \mathbf{A}_d)$ and $g(y, \gamma) = \mathbf{G}$, then $\mathbf{A}_i\mathbf{G} = \mathbf{G}\mathbf{A}_i$ for each i. Because k is algebraically closed and $\mathbf{G}$ acts on M_y, $\mathbf{G}$ has a non-zero eigenvalue λ. But each $\mathbf{U}_i$ acts on M_y by the linear transformation $\mathbf{A}_i$ and this commutes with $\mathbf{G}$. Hence $\ker(\mathbf{G} - \lambda\mathbf{I})$ is a non-zero A-submodule of M_y. The latter is irreducible, thus $\mathbf{G} = \lambda\mathbf{I}$ on M_y. But $g(y, e) = \mathbf{I}$ where $e \in \Gamma$ is the identity element (this follows by 5 (b)). Thus if $e_1, \dots, e_N$ is a canonical basis corresponding to t, then we have both of

$$e_i = \lambda^{-1} \sum_{j=1}^{N} g_{ij}(y, \gamma) e_j \qquad e_i = \sum_{j=1}^{N} g_{ij}(y, e) e_i$$

holding. Uniqueness (5 (c)) now implies that $\gamma = e$, as required. □

The finiteness of Γ now implies that $|\Gamma| = |p^{-1}(y)|$ for every $y \in U$, hence the projection p is faithfully flat (Proposition 2.1). It is now clear by Definition 2.2 that p is a Galois covering over U where each $\gamma \in \Gamma$ acts linearly. $\square$

Thus a quantum algebra at a root of unity is definably irreducibly representable on a large open subset of V. We have a partial converse.

Proposition 2.4. *Let A be a definably irreducibly representable Azumaya k-algebra, μ_N the group of N-th roots of unity in k, where $N = \deg_{PI}(A)$. Then there is a definable map $g : V \times \Gamma \to GL_N(k)/\mu_N\mathbf{I}$ which satisfies condition 5 of Definition 2.4 (with all occurrences of $GL_N(k)$ replaced with $GL_N(k)/\mu_N\mathbf{I}$).*

Proof. Given $x \in V$, let $e_1, \ldots, e_N$ be any basis for $\pi^{-1}(x) = M_x$ as a vector space (i.e. it need not be canonical). We can find μ_{ijk} such that

$$\mathbf{U}_i e_j = \sum_{k=1}^{N} \mu_{ijk} e_k \qquad 1 \leq j, k \leq N$$

in M_x for each i. The elements μ_{ijk} can be assembled into an d-tuple of $N \times N$ matrices $\mathbf{A} = (\mathbf{A}_1, \ldots, \mathbf{A}_d)$ where $\mathbf{A}_i = (\mu_{ijk})$. Thus we obtain a bundle $p : E \to V$ where each fiber $p^{-1}(x)$ consists of such $\mathbf{A}$ (which can be thought of as elements of k^{dN^2}) for each basis of M_x. As in the proof of Proposition 2.3, there is an action of $\mathbf{G} \in G = \mathrm{GL}_N(k)$ by conjugation in each fiber, namely

$$\mathbf{G} \cdot \mathbf{A} = (\mathbf{G}\mathbf{A}_1\mathbf{G}^{-1}, \ldots, \mathbf{G}\mathbf{A}_d\mathbf{G}^{-1})$$

It is transitive and free up to multiplication by a non-zero scalar (by the same argument as in Proposition 2.3). Because $p' : \Sigma^A(k) \to V$ is a Galois covering with group Γ, there is an isomorphism $\Gamma \times \Sigma^A(k) \to \Sigma^A(k) \times_V \Sigma^A(k)$. Take $\gamma \in \Gamma$ and define the set

$$I_\gamma = \{(t, \mathbf{G}) : \gamma t = \mathbf{G} \cdot t \wedge \det(\mathbf{G}) = 1\}$$

Let $I_\gamma(t) = \{\mathbf{G} : (t, \mathbf{G}) \in I_\gamma\}$. Because γ acts linearly, $I_\gamma(t) = I_\gamma(t')$ for each $t, t' \in p'^{-1}(x)$. Thus we have a definable correspondence $x \mapsto I_\gamma(t)$ for every $x \in V$. Note that $|I_\gamma(t)| = N$: if $\mathbf{G}, \mathbf{G}' \in I_\gamma(t)$ then $\mathbf{G} = \lambda\mathbf{G}'$ for $\lambda \in k^*$. But then $\det(\mathbf{G}) = \det(\mathbf{G}') = 1$ implies that λ must be an N-th root of unity. Hence we obtain a definable map $V \to G/\mu_N\mathbf{I}$. Repeating this for each $\gamma \in \Gamma$ gives our required map $g : \Gamma \times V \to G/\mu_N\mathbf{I}$. Now let $e_1, \ldots, e_N$ and $e'_1, \ldots, e'_N$ be two canonical bases of M_x. Then the corresponding coefficients t_{ijk} and t'_{ijk} (respectively) are conjugated by some $\gamma \in \Gamma$. Thus there is $\lambda \in k^*$ such that

$$e'_i = \lambda \sum_{j=1}^{N} g_{ij}(x, \gamma) e_j$$

as required. □

Although definably irreducibly representable Azumaya algebras are not therefore quantum algebras at roots of unity, we remark that only a slight modification of the arguments in [Zil06] is required to show that one can associate Zariski structures to them. Given that the condition of being a Galois covering is easier to check than condition 5 of Definition 2.4, and because the definition of a definably irreducibly representable Azumaya algebra uses more widespread terminology, we stick with this definition in the sequel.

2.2.1 The associated theory

By virtue of the foregoing, it is particularly straightforward to associate a first-order theory T_A to a definably irreducibly representable Azumaya k-algebra A. The theory T_A is formulated in the three-sorted language

$$\mathcal{L}_A = (E, V, k, \pi, \mathbf{U}_i, C : 1 \leq i \leq d)$$

where

1. C is a finite set of constants from k.
2. $\pi : E \to V$ and $\mathbf{U}_i : E \to E$ are unary functions.
3. E, V, k are sorts, k has the language of rings and E is equipped with
    - a relation $+ \subseteq E^3$ where $+(v_1, v_2, v_3)$ holds if and only if there exists $x \in V$ such that $v_i \in \pi^{-1}(x)$ for $1 \leq i \leq 3$ and $v_1 + v_2 = v_3$.
    - a relation $\cdot \subseteq k \times E^2$, where $\cdot(\lambda, v_1, v_2)$ holds if and only if there is $x \in V$ such that $v_1, v_2 \in \pi^{-1}(x)$ and $\lambda v_1 = v_2$.

The $\mathcal{L}_A$-theory says:

1. k is an algebraically closed field.
2. $V = \phi(k)$ where ϕ defines V over C.
3. $\pi : E \to V$ is a surjective map and each fiber $\pi^{-1}(x)$ is an N-dimensional vector space, where $N = \deg_{PI}(A)$.
4. The operators $\mathbf{U}_i$ act according to the axiom

$$(\forall x \in k)(\exists e_1, \ldots, e_N \in E)(\exists t \in k) \left(\bigwedge_{i=1}^{N} \pi(e_i) = x \wedge \Sigma(t, x) = 0 \wedge \bigwedge_{i \leq d, j \leq N} \mathbf{U}_i e_j = \sum_{k=1}^{N} t_{ijk} e_k \right)$$

5. For the $g : V \times \Gamma \to \mathrm{GL}_N(k)/\mu_N\mathbf{I}$ of Definition 2.5, if e, e' are canonical bases for $\pi^{-1}(x)$ then

$$(\forall x \in k)(\exists! \gamma \in \Gamma)(\exists \lambda \in k^*) \left(\bigwedge_{i=1}^{N} e'_i = \lambda \sum_{j=1}^{N} g_{ij}(x, \gamma) e_j \right)$$

We note that the property of e being a canonical basis for $\pi^{-1}(x)$ is definable (just remove the existential quantification over $e_1, \ldots, e_N$ appearing in the formula of condition 4). Models of T_A will be denoted (E, k); the justification for suppressing the sort V from the notation being that V is in fact a definable subset of k.

2.2.2 Zariski structure on models of T_A

We fix a definably representable Azumaya k-algebra A. Let $(E, k) \models T_A$ and let $\mathbf{E}(e, x)$ express that e is a canonical basis for $\pi^{-1}(x)$. We restate the main results of the technical analysis of definable sets carried out in [Zil06] using our present notation. Let x be a tuple of variables from k, $v = (v_1, \ldots, v_m)$, $w = (w_1, \ldots, w_n)$ tuples of variables from E. Let e' be a tuple of canonical bases of length p. Then all definable subsets of (E, k) are boolean combinations of sets of the form

$$\exists e \exists y \exists t \exists \lambda \exists \mu \left(\bigwedge_{i=1}^{s} \bigwedge_{j=1}^{s_i} \mathbf{E}(e_i, y_i) \wedge v_{ij} = \sum_{k=1}^{N} \lambda_{ijk} e_{ik} \wedge \bigwedge_{i=1}^{p} \bigwedge_{j=1}^{p_i} w_{ij} = \sum_{k=1}^{N} \mu_{ijk} e'_{ik} \wedge S(t, x, y, \lambda, \mu) \right)$$

where

1. The only free variables are the tuples x, v and w.
2. $\{v_{ij} : 1 \leq i \leq s, 1 \leq j \leq s_i\}$ is an enumeration of v.
3. S is a constructible subset of $k^{r_1} \times V^s$ where $r_1 = m + n + l(t) + l(x)$ (where l denotes length).
4. $\{w_{ij} : 1 \leq i \leq p, 1 \leq j \leq p_i\}$ is an enumeration of w.

We shall denote such formulas by $\exists e S$ and refer to them as **general core formulas with Zariski constructible component** S. Besides quantifier elimination to the level of general core formulas, a key result of [Zil06] is that one obtains a topology on the sorts of (E, k) and their cartesian powers by taking finite unions and arbitrary intersections of subsets defined by formulas of the form $\exists e C$ where C is Zariski closed (with C satisfying some additional technical conditions). An appropriate dimension theory exists, making (E, k) a Zariski structure.

2.2.3 Functorial correspondence

Although every model of T_A is a Zariski structure, as stated in the introduction, we choose a model $\mathrm{nSpec}\, A = (E, \mathbb{K})$ where $\mathbb{K}$ is large (and saturated). This is done primarily to avoid complications using different fields. We recall the following definition from [Art69].

Definition 2.6. *A homomorphism of rings* $\varphi : R \to S$ *is an* ***extension*** *if* S *becomes an* R*-bimodule under* φ*.*

It is immediate that Definition 2.6 is equivalent to the assertion that S is generated as a ring by $\varphi(R)$ and its centralizer $Z_R(S) = \{s \in S : \varphi(r)s = s\varphi(r) \text{ for all } r \in R\}$. It is easy to see that an extension respects the centers, i.e. $\varphi : Z(R) \to Z(S)$. Let $\mathsf{Azum}(k)_{def}$ be the category consisting of definably irreducibly representable Azumaya k-algebras, with morphisms that are extensions.

Proposition 2.5. *If* $\varphi : A \to B$ *is a morphism in* $\mathsf{Azum}(k)_{def}$ *then there is corresponding morphism of Zariski structures* $\mathrm{nSpec}\,\varphi : \mathrm{nSpec}\,B \to \mathrm{nSpec}\,A$.

Proof. If $\mathrm{nSpec}\,A = (E_A, V_A, \mathbb{K})$ and $\mathrm{nSpec}\,B = (E_B, V_B, \mathbb{K})$, we have a corresponding map of varieties $f : V_B \to V_A$ because extensions respect centers. Let $x \in V_B$ and denote by $E_{B,x}$ the irreducible B-module lying over x. By Proposition A.1, $\mathfrak{m}_x = \mathrm{ann}_{Z(B)}(E_x)$ is a maximal ideal of $Z(B)$. Let $E^c_{B,x}$ denote the A-module obtained from $E_{B,x}$ by restriction of scalars. Then $\mathfrak{m}_{f(x)} = \varphi^{-1}(\mathfrak{m}_x)$ annihilates $E^c_{B,x}$, hence it is a $A/\mathfrak{m}_{f(x)}A$-module. But $A/\mathfrak{m}_{f(x)}A \simeq M_{N'}(k)$ for $N' = \deg_{PI}(A)$ (by Proposition A.3), hence $E^c_{B,x}$ projects onto the irreducible module $E_{A,f(x)}$ lying over $f(x)$ by semisimplicity. One can therefore define a map $\mathrm{nSpec}\,\varphi : E_B \to E_A$ to be the direct sum over all $x \in V_B$ of the projections $E^c_{B,x} \to E_{A,f(x)}$.

Suppose that A is generated as a k-algebra by $\mathbf{U}_{11}, \ldots, \mathbf{U}_{1d_1}$; B is generated by $\mathbf{U}_{21}, \ldots, \mathbf{U}_{2d_2}$. Without loss of generality, we can assume that $\varphi(\mathbf{U}_{11}), \ldots, \varphi(\mathbf{U}_{1d_1})$ occur amongst $\mathbf{U}_{21}, \ldots, \mathbf{U}_{2d_2}$ (otherwise, just extend the language $\mathcal{L}_B$ and note that the assumptions regarding Σ^B are not affected). Let $\phi(v, w, x)$ be the basic closed subset of $\mathrm{nSpec}\,A$ given by

$$\exists e \exists y \exists t \exists \lambda \exists \mu \left(\bigwedge_{i=1}^{s} \bigwedge_{j=1}^{s_i} \mathbf{E}(e_i, y_i) \wedge v_{ij} = \sum_{k=1}^{N} \lambda_{ijk} e_{ik} \wedge \bigwedge_{i=1}^{p} \bigwedge_{j=1}^{p_i} w_{ij} = \sum_{k=1}^{N} \mu_{ijk} e'_{ik} \wedge C(t, x, y, \lambda, \mu) \right)$$

where $N = \deg_{PI}(A)$.

Claim: Let $\pi_A : E_A \to V_A$ (similarly $\pi_B : E_B \to V_B$). The preimage of the set of realisations of ϕ is nSpec under the map nSpec is a finite union of closed subsets defined by formulas of the form

$$\exists e \exists e'' \exists y \exists t \exists \lambda \exists \mu \left(\begin{array}{c} \bigwedge_{i=1}^{s} \bigwedge_{j=1}^{s_i} \mathbf{E}'(e_i, y_i) \wedge v_{ij} = \sum_{k=1}^{N'} \lambda_{ijk} e_{ik} \\ \wedge \bigwedge_{i=1}^{p} \bigwedge_{j=1}^{p_i} \mathbf{E}'(e''_i, z'_i) \wedge w_{ij} = \sum_{k=1}^{N'} \mu_{ijk} e''_{ik} \wedge C'(t, x, y, z, \lambda, \mu) \end{array} \right)$$

where $C'(t, x, y, z, \lambda, \mu)$ holds if and only if

1. $C(\pi_t(t), x, f(y), \pi_\lambda(\lambda), \pi_\mu(\mu))$ holds in $\mathrm{nSpec}\,A$ for appropriate projections $\pi_t, \pi_\lambda, \pi_\mu$.

2. $z' = (z'_i : 1 \leq i \leq p) \in f^{-1}(z)$ where $z = (\pi_A(e'_i) : 1 \leq i \leq p)$.

Proof. Suppose that (v, w, x) lies in the preimage. Then it is clear that there are e, y, λ, t such that

$$\operatorname{nSpec} B \models \bigwedge_{i=1}^{s} \bigwedge_{j=1}^{s_i} \mathbf{E}'(e_i, y_i) \wedge v_{ij} = \sum_{k=1}^{N'} \lambda_{ijk} e_{ik}$$

where $\mathbf{E}'(e_i, y_i)$ is the formula

$$\bigwedge_{j=1}^{N'} \pi_B(e_{ij}) = y_i \wedge \Sigma^B(t_i, y_i) = 0 \wedge \bigwedge_{j \leq d_2, k \leq N'} \mathbf{U}_{2j} e_{ik} = \sum_{l=1}^{N'} t_{ijkl} e_{il}$$

We have assumed that $\varphi(\mathbf{U}_{1i})$ occur among the $\mathbf{U}_{2j}$. For each i, the image under nSpec of E_{B,y_i} will be a cyclic A-submodule of E^c_{B,y_i} generated by some e_{ij} for $1 \leq j \leq N'$. Suppose, for ease of notation, that $j = 1$ in each case. Then we have a subtuple $\pi_t(t)$ of t such that $t_{ijkl} \in \pi_t(t)$ if and only if

- $\mathbf{U}_{2j}$ lies in the image of φ.
- e_{ik} and e_{il} lie in the A-module generated by e_{i1}.

Likewise, we have a subtuple $\pi_\lambda(\lambda)$ of λ such that $\lambda_{ijk} \in \pi_\lambda(\lambda)$ if and only if e_{ik} lies in the A-submodule generated by e_{i1}. Now repeat the above for w, thus obtaining a tuple of canonical basis elements e'', elements μ and a projection π_μ (except that for w, we are not interested in how canonical basis elements are related to each other via the $\mathbf{U}_{2j}$, hence we do not have to worry about parameters like t). It is now clear that $C(\pi_t(t), x, f(y), \pi_\lambda(\lambda), \pi_\mu(\mu))$ must hold in $\operatorname{nSpec} A$ and that $z' = (\pi_B(e''_i) : 1 \leq i \leq p) \in f^{-1}(z)$. The choice of projections $\pi_t, \pi_\lambda, \pi_\mu$ is dependent on the choice of canonical basis in each of the finitely many fibers considered, but there can only be finitely many such choices in each fiber. The claim now follows. □

Being the intersection of a closed set with the preimage of C under a continuous map, C' is closed. Hence the formula in the claim defines a closed set in the topology on $\operatorname{nSpec} B$, as required. □

Corollary 2.1. *There is a functor* nSpec : $\mathsf{Azum}^{op}_{def} \to \mathsf{Zar}$ *extending* $(\mathsf{CAlg}_{fg,int})^{op} \to \mathsf{Zar}^c$

Proof. Immediate from Proposition 2.5. Note that if A is a commutative finitely generated k-algebra which is a domain, the structure we obtain is a line bundle L over the irreducible variety V corresponding to A. For each $x \in V$, the fiber $L_x = \pi^{-1}(x)$ is the one-dimensional A-module given by the character $\chi_x : A \to k$ at x, i.e.

$$\mathbf{A}v = \chi_x(\mathbf{A})v \qquad \text{for all } \mathbf{A} \in A \text{ and } v \in L_x$$

It is clear that this structure is definably interpretable in k, hence the associated Zariski structure is classical. According to Definition 2.6, any ring homomorphism $\varphi : A \to B$ of commutative rings is an extension. Under our present conditions of A and B being commutative finitely generated k-algebras which are domains, the proof of Proposition 2.5 takes the usual morphism of corresponding varieties $f : V_B \to V_A$ and constructs a direct sum over all $x \in V_B$ of the trivial A-linear maps $L^c_{B,x} \to L_{A,f(x)}$. □

We make some remarks on the relationship between definably irreducibly representable Azumaya algebras A and abstract theories which look like T_A. By the latter, we mean theories T formulated in a three-sorted language $\mathcal{L} = (E, V, k, \pi, \mathbf{U}_i, C : 1 \leq i \leq d)$ satisfying the axioms of subsection 2.2.1. Given such a theory T, it is possible to recover a certain amount of algebraic structure. Indeed, let F be the free k-algebra on generators $\mathbf{U}_1, \ldots, \mathbf{U}_d$. If $(E, k) \models T$, then each fiber $E_x = \pi^{-1}(x)$ has the structure of an $F/\operatorname{ann}_F(E_x)$-module. Thus the algebra $A = F/I$ where $I = \bigcap_{x \in V} \operatorname{ann}_F(E_x)$ is represented in (E, k). For uncountable algebraically closed fields k, the algebra A recovered is determined up to isomorphism by the cardinality of k (this follows by the categoricity result of [Zil06]). However, it need not be the case that T is T_A, as the following example demonstrates.

Example 2.1. *Let T be the three-sorted theory of the above kind, where $V = k$ (the affine line) and each fiber over V is an N-dimensional k-vector space. We have a presentation of $M_N(k)$ in terms of matrix units, i.e. take the generators $\{\mathbf{E}_{ij} : 1 \leq i, j \leq N\}$ over k subject to the relations $\sum_i \mathbf{E}_{ii} = 1$ and $\mathbf{E}_{ij}\mathbf{E}_{jk} = \mathbf{E}_{ik}$. Then each fiber has a canonical basis as a $M_N(k)$-module; namely a set of elements $e_1, \ldots, e_N$ such that $\mathbf{E}_{ij} e_k = \delta_{jk} e_i$ for all $1 \leq i, j, k \leq N$. Σ is then taken to be the set of polynomials $\{\lambda_{ijkl} - \delta_{jk}\delta_{il}, x = x : 1 \leq i, j, k, l \leq N\}$, which trivially gives a Galois cover of V. Now the algebra A recovered from T is $M_N(k)$, which is Azumaya. But the structure given by T_A is a single N-dimensional vector space over a point, hence $T_A \neq T$.*

In order for nSpec to provide an equivalence of categories, one would require a means of defining the image of nSpec as a suitable subcategory of Zar. The most plausible candidate would be the full subcategory whose objects are saturated models of T where T is an abstract theory which looks like T_A. But Example 2.1 demonstrates that there would be Zariski structures lying in this subcategory which do not lie in the image of nSpec. For this reason, it does not seem possible to establish an equivalence of categories using nSpec.

2.3 Imaginaries

Let A be a definably irreducibly representable Azumaya k-algebra, $(E,k) \models T_A$. All imaginaries defined using the field sort k alone are eliminable, by elimination of imaginaries for algebraically closed fields ([Mar02], [B99]). Can one expect such a result to hold for all imaginaries in (E,k)? Given that E consists of a disjoint union of A-modules, methods from the model theory of modules are applicable. In particular, it is more meaningful to consider those imaginaries which possess the structure of an abelian group, and this leads us naturally to the consideration of pp-conditions.

2.3.1 Pp-pairs

All material in this subsection is taken from [Pres09]. Let R be a ring. An R-module M can clearly be regarded as a first-order structure in the language $\mathcal{L}_R = (0, +, r : r \in R)$ containing a unary function symbol for each element of R. Positive-primitive formulas (pp-conditions) are positive existential formulas in the language, i.e. formulas of the form

$$\exists x_{k+1} \ldots x_n \bigwedge_{j=1}^{m} \sum_{i=1}^{n} r_{ij} x_i = 0$$

If ϕ is a pp-formula then the set of realizations $\phi(M)$ is an additive subgroup of M. The correspondence $M \mapsto \phi(M)$ is easily shown to extend to a functor $F_\phi : {}_R\mathsf{Mod} \to \mathsf{Ab}$, where Ab denotes the category of Abelian groups.

The category of pp-pairs

A **pp-pair** ϕ/ψ is a pair of pp-conditions ϕ and ψ such that $\psi \to \phi$. The **category of pp-pairs**, denoted $\mathbb{L}_R^{eq+}$, consists of objects that are pp-pairs. A morphism of pp-pairs $\phi(x)/\psi(x) \to \phi'(y)/\psi'(y)$ is given by a pp-condition $\rho(x,y)$ such that

1. $\rho(x,y) \wedge \phi(x) \to \phi'(y)$.
2. $\rho(x,y) \wedge \psi(x) \to \psi'(y)$.
3. $\phi(x) \to \exists! y \rho(x,y)$.

Proposition 2.6. *The category $\mathbb{L}_R^{eq+}$ is abelian.*

Proof. [Pres09] Proposition 3.2.10. We note the key facts:

1. The zero object is given by ϕ/ϕ for any formula ϕ.

2. The direct sum of ϕ/ψ and ϕ'/ψ' is given by

$$(\phi(x) \wedge \phi'(y))/(\psi(x) \wedge \psi'(y))$$

where the tuples of variables x and y have no common element.

3. If $\rho : \phi/\psi \to \phi'/\psi'$ is a morphism of pp-pairs then

   - $\ker \rho$ is the pp-pair ψ''/ψ where ψ'' is $\phi(x) \wedge \exists y(\rho(x, y) \wedge \psi(y))$.
   - $\operatorname{im} \rho$ is the pp-pair $\phi''/\psi' \wedge \phi''$ where $\phi''(y)$ is $\exists x(\phi(x) \wedge \rho(x, y))$.

□

The category $\mathbb{L}_R^{eq+}$ can be regarded as the language $\mathcal{L}_R$ enriched by certain imaginaries. Indeed, suppose one takes $\mathcal{L}_R$ and introduces

- A sort $s_\sim$ for every pp-definable equivalence relation $\sim$ on tuples and an appropriate projection map $p_\sim : s_=^n \to s_\sim$ for some $n > 0$. Here $s_=$ is the original sort of $\mathcal{L}_R$.
- A sort for every pp-definable subgroup.
- A function symbol for every pp-definable function between sorts.

Then each sort can clearly be identified with a pp-pair, and the pp-definable functions between sorts are the morphisms of pp-pairs. The augmentation of $\mathcal{L}_R$ by the additional sorts outlined differs from the construction of $\mathcal{L}_R^{eq}$ given by Shelah, particularly in the sense that we are only interested in quotients by equivalence relations which are defined by pp-conditions, not all 0-definable equivalence relations.

2.3.2 Pp-imaginaries in models of T_A

A pp-condition should carve out a definable subset of any $(E, k) \models T_A$, and in order to express pp-conditions, the language $\mathcal{L}_A$ is enriched. Let $\mathcal{L}_A^+ = \mathcal{L}_A \cup \{\mathbf{A} : \mathbf{A} \in A\}$, where each $\mathbf{A}$ is a unary operator on the sort E. We shall refer to $\mathcal{L}_A^+$ as the **extended language**. The theory T_A canonically extends to the $\mathcal{L}_A^+$-theory T_A^+ by defining the actions of the additional operators $\mathbf{A} \in A$ in accordance with the axioms of T_A governing the actions of the generators $\mathbf{U}_i$ of A. Thus we do not obtain anything new by passing to T_A^+; it is just convenient to have a symbol for every element of A. Likewise, a model $(E, k) \models T_A$ extends to $(E, k)^+ \models T_A^+$. The following remark is trivial.

Remark 2.1. *All models of T_A^+ are of the form $(E, k)^+$ for $(E, k) \models T_A$.*

The extended language $\mathcal{L}_A^+$ is further enriched to the language $\mathcal{L}_A^{+eq+}$ by introducing

- A sort $s_{k,\sim}$ for every 0-definable equivalence relation on tuples of k and an appropriate projection map $p_{k,\sim} : s_k^n \to s_{k,\sim}$ for some $n > 0$.
- A sort for every 0-definable subset of k and its cartesian powers, along with function symbols for all 0-definable functions between sorts.
- All sorts of $\mathbb{L}_A^{eq+}$.

The theory T_A^+ extends canonically to an $\mathcal{L}_A^{+eq+}$-theory T_A^{+eq+} in an obvious way, and likewise so do structures. We adopt the notation $(E,k)^{+eq+}$ for such an extension of $(E,k)^+ \models T_A^+$. Note that for each pp-pair ϕ/ψ in $\mathbb{L}_A^{eq+}$ we have the sort $\coprod_{x\in V} \phi(\pi^{-1}(x))/\psi(\pi^{-1}(x))$ in $(E,k)^{+eq+}$ where $\pi : E \to V$. By construction, the structure $(E,k)^{+eq+}$ contains the category of all varieties defined over the prime subfield of k. We denote the category of all definable subsets of $(E,k)^{+eq+}$ (with definable maps as morphisms) by $\mathrm{Def}(E,k)^{+eq+}$.

Proposition 2.7. *Let $(E,k) \models T_A$, $\pi : E \to V$. Let W be a definable subset of V. Then W determines a functor $G_W : \mathbb{L}_A^{eq+} \to \mathrm{Def}(E,k)^{+eq+}$.*

Proof. If ϕ/ψ is a pp-pair we put $G_W(\phi/\psi) = \coprod_{x\in W} \phi(\pi^{-1}(x))/\psi(\pi^{-1}(x)) \times W$, which is a definable subset of $(E,k)^{+eq+}$. Let ϕ'/ψ' be another pp-pair, $\rho : \phi/\psi \to \phi'/\psi'$ a morphism. For each module $E_x = \pi^{-1}(x)$ we obtain a homomorphism of abelian groups

$$\phi(E_x)/\psi(E_x) \to \phi'(E_x)/\psi'(E_x) \qquad v + \psi(E_x) \mapsto \rho(v) + \psi'(E_x)$$

where $v' = \rho(v)$ is any element such that $\rho(v,v')$ holds. This map is easily checked to be well-defined. Taking the disjoint union of these over $x \in W$ gives us the required definable map $G_W(\phi/\psi) \to G_W(\phi'/\psi')$. □

Proposition 2.8. *Let W, W' be definable subsets of V with $f : W \to W'$ a definable map. Then there is a functor $f^*G_{W'} : \mathbb{L}_A^{eq+} \to \mathrm{Def}(E,k)^{+eq+}$ and a natural transformation α_f such that the diagram*

$$\begin{tikzcd} f^*G_{W'} \arrow[r, "\alpha_f"] \arrow[d, "\pi_2"'] & G_{W'} \arrow[d, "\pi_2"] \\ W \arrow[r, "f"'] & W' \end{tikzcd}$$

commutes for an appropriate functor π_2.

Proof. By abuse of notation, we make no distinction here between W and the constant functor $W : \mathbb{L}_A^{eq+} \to \mathrm{Def}(E,k)^{+eq+}$ which takes the value W at every pp-pair and where every morphism of pp-pairs maps to the identity map $W \to W$. The value of $f^*G_{W'}$ at the pp-pair ϕ/ψ is the pullback of the 'bundle' $G_{W'}(\phi/\psi)$, i.e.

$$f^*G_{W'}(\phi/\psi) = \{(v,x) \in \coprod_{x \in W'} \phi(\pi^{-1}(x))/\psi(\pi^{-1}(x)) \times W : f(x) = \pi(v)\}$$

which is definable in $(E,k)^{+eq+}$. The component of α_f at ϕ/ψ is defined by $(v,x) \mapsto (v, f(x))$. It is therefore clear that the diagram

$$\begin{tikzcd}
f^*G_{W'}(\phi/\psi) \arrow[r, "(\alpha_f)_{\phi/\psi}"] \arrow[d, "\pi_2"'] & G_{W'}(\phi/\psi) \arrow[d, "\pi_2"] \\
W \arrow[r, "f"'] & W'
\end{tikzcd}$$

commutes, where π_2 is the projection onto the second factor. If $\rho : \phi/\psi \to \phi'/\psi'$ is a morphism of pp-pairs then $G_{W'}(\rho)$ 'restricts' to a definable map $f^*G_{W'}(\rho) : f^*G_{W'}(\phi/\psi) \to f^*G_{W'}(\phi'/\psi')$ (after removing the product with W' and applying $\times W$). Thus the diagram

$$\begin{tikzcd}
f^*G_{W'}(\phi/\psi) \arrow[r, "(\alpha_f)_{\phi/\psi}"] \arrow[d, "f^*G_{W'}(\rho)"'] & G_{W'}(\phi/\psi) \arrow[d, "G_{W'}(\rho)"] \\
f^*G_{W'}(\phi'/\psi') \arrow[r, "(\alpha_f)_{\phi'/\psi'}"'] & G_{W'}(\phi'/\psi')
\end{tikzcd}$$

commutes and the result follows. ☐

Lemma 2.1. *Let* $(E,k) \models T_A$, $\pi : E \to V$. *For every* $x \in V$, *the functor* $G_x : \mathbb{L}_A^{eq+} \to \mathsf{Ab}$ *is exact.*

Proof. Any exact sequence in $\mathbb{L}_A^{eq+}$ is isomorphic to a canonical exact sequence of the form $0 \to \phi'/\psi \to \phi/\psi \to \phi/\phi' \to 0$ where $\psi \Rightarrow \phi' \Rightarrow \phi$. It is now immediate that evaluation at $E_x = \pi^{-1}(x)$ gives an exact sequence of Abelian groups. ☐

Definition 2.7. *Let* ϕ/ψ *be a pp-pair in* $\mathbb{L}_A^{eq+}$, $(E,k) \models T_A$, W *a closed subset of* V *where* $\pi : E \to V$. *We define the* ***support*** *of* ϕ/ψ *in* W *to be*

$$\mathrm{Supp}_W(\phi/\psi) = \{x \in W : \phi(\pi^{-1}(x))/\psi(\pi^{-1}(x)) \neq 0\}$$

Proposition 2.9. *Let* $0 \to \phi'/\psi' \xrightarrow{\rho'} \phi/\psi \xrightarrow{\rho''} \phi''/\psi'' \to 0$ *be an exact sequence in* $\mathbb{L}_A^{eq+}$. *Then* $\mathrm{Supp}_W(\phi/\psi) = \mathrm{Supp}_W(\phi'/\psi') \cup \mathrm{Supp}_W(\phi''/\psi'')$ *in any model* $(E,k)^{+eq+}$.

Proof. The result is immediate by Lemma 2.1 and the observation that for any exact sequence $0 \to A \to B \to C \to 0$ of Abelian groups, $B = 0$ if and only if $A = 0$ and $C = 0$. ☐

Proposition 2.9 demonstrates that our defined notion of support does indeed satisfy a key property required of any reasonable notion of support (the two properties required are summarized nicely in [Ros08]). The remaining property concerns those categories with Ab5, hence it does not apply to $\mathbb{L}_A^{eq+}$. We now explicate a link with localization at Serre subcategories.

Definition 2.8. *Let* A *be an abelian category.* S *is a **Serre subcategory** of* A *if for any exact sequence* $0 \to A \to B \to C \to 0$ *of objects of* A*,* $B \in \mathsf{S}$ *if and only if* A *and* C *are in* S*.*

Let $\mathsf{S}_A(W) = \{\phi/\psi \in \mathbb{L}_A^{eq+} : \mathrm{Supp}_W(\phi/\psi) = \emptyset\}$. Then $\mathsf{S}_A(W)$ can be regarded as a full subcategory of ${}_A\mathsf{Mod}$ and it is immediate from Proposition 2.9 that $S_A(W)$ is a Serre subcategory of $\mathbb{L}_A^{eq+}$. A key property of a Serre subcategory S of an abelian category A is that there is a 'localization functor' from A to a category of 'fractions' A/S.

Theorem 2.2. *Let* A *be an abelian category,* S *a Serre subcategory. There exists an abelian category* A/S *and an exact functor* $L : \mathsf{A} \to \mathsf{A}/\mathsf{S}$ *with the following properties:*

1. $\ker L = \{A \in \mathsf{A} : L(A) = 0\} = \mathsf{S}$.
2. *Every exact functor* $H : \mathsf{A} \to \mathsf{B}$ *(where* B *is abelian) such that* $\ker H \supseteq \mathsf{S}$ *factors uniquely through* L *via an exact and faithful functor.*

Proof. [Gab62] Chapter III.1. □

We now apply Theorem 2.2 to obtain the localization $\mathbb{L}_A^{eq+}/\mathsf{S}_A(W)$, which we denote by $(\mathbb{L}_A^{eq+})_W$ and call the **relativization** of the category of pp-pairs to W. It consists precisely of those pp-pairs and pp-conditions which define functions on the realizations at $E_x = \pi^{-1}(x)$ for all $x \in W$, but which do not necessarily define functions on all A-modules. We remark that $\mathsf{S}_A(W)$ carves out a definable (or elementary) subcategory of ${}_A\mathsf{Mod}$, using the terminology of [Pres09]. The objects of this full subcategory are precisely those modules M for which $\phi(M) = \psi(M)$ for every pp-pair ϕ/ψ in $\mathsf{S}_A(W)$. Clearly all of the irreducible modules parametrized by W are contained in this category.

2.3.3 Elimination of imaginaries

We now wish to investigate under what circumstances $(E, k)^{+eq+}$ is definably interpretable in $(E, k)^+$ (model-theoretically, this amounts to 'elimination of imaginaries'). By the foregoing, this can only be possible if for every pp-pair ϕ/ψ there exists a pp-condition θ such that $\theta(E_x) \simeq \phi(E_x)/\psi(E_x)$ for every $x \in V$, where the isomorphism is in $(\mathbb{L}_A^{eq+})_V$. It transpires that this holds if $A = M_n(k)$.

Proposition 2.10. *Let R be a ring and denote by $\mathsf{Irr}({}_R\mathsf{Mod})$ the set of irreducible R-modules. Let Σ consist of those pp-pairs ϕ/ψ such that $\phi(M) = \psi(M)$ for every $M \in \mathsf{Irr}({}_R\mathsf{Mod})$. If R is semisimple, then $\Sigma = 0$.*

Proof. This amounts to showing that if ϕ, ψ are pp-conditions such that $M \models \phi \leftrightarrow \psi$ for every $M \in \mathsf{Irr}({}_R\mathsf{Mod})$ then the theory of R-modules itself says this (i.e. it is true of every module). Let M be an R-module. Then $M = \bigoplus_i M_i$ where each of the M_i are irreducible. We have canonical injection and projection maps $\iota_j : M_j \to M$, $\pi_j : M \to M_j$ such that $1_M = \sum_j \iota_j \circ \pi_j$ where 1_M is the identity map on M. Now let ϕ, ψ be pp-conditions which are equivalent on the irreducible modules. Then

$$M \models \phi(x) \Rightarrow M_j \models \phi(\pi_j(x)) \Leftrightarrow M_j \models \psi(\pi_j(x)) \Rightarrow M \models \psi((\iota_j \circ \pi_j)(x))$$

Summing over all j gives that $M \models \psi(x)$, and the argument is completely reversible with ψ in place of ϕ. □

Corollary 2.2. *Under the hypotheses of Proposition 2.10, $\mathbb{L}_R^{eq+} \simeq \mathbb{L}_R^{eq+}/\Sigma$.*

Proof. By Theorem 2.2 the functor $L : \mathbb{L}_R^{eq+} \to \mathbb{L}_R^{eq+}/\Sigma$ has kernel $\Sigma = 0$ by Proposition 2.10. □

Definition 2.9. *A ring R is **von Neumann regular** if for every element $r \in R$ there is $s \in R$ such that $r = rsr$.*

Lemma 2.2. *A von Neumann regular ring R is left (respectively right) Noetherian if and only if R is semisimple.*

Proof. [Lam01] Theorem 4.25. □

Proposition 2.11. *Let k be an algebraically closed field, A a finitely generated prime k-algebra Azumaya over its center. Then A is a von Neumann regular ring if and only if $A = M_n(k)$ for some integer $n > 0$.*

Proof. If $A = M_n(k)$ then A is Azumaya. Let $\mathbf{A} \in A$ be a matrix of rank r, $\mathbf{I}_r$ the $r \times r$ identity matrix. Then there are invertible matrices $\mathbf{U}$ and $\mathbf{V}$ such that

$$\mathbf{A} = \mathbf{U} \begin{pmatrix} \mathbf{I}_r & 0 \\ 0 & 0 \end{pmatrix} \mathbf{V}$$

Putting $\mathbf{W} = \mathbf{V}^{-1}\mathbf{U}^{-1}$, it follows that $\mathbf{AWA} = \mathbf{A}$. Thus A is von Neumann regular. Conversely, suppose that A is finitely generated Azumaya and von Neumann regular. By the Artin-Tate lemma

(Lemma A.1), A is Noetherian. But a Noetherian von Neumann regular ring is semisimple by Lemma 2.2. Thus by Wedderburn's theorem,

$$A \simeq \bigoplus_{i=1}^{m} M_{n_i}(k)$$

for some integers $n_i > 0$. The Azumaya property (or Proposition A.4) now gives the result. ☐

Lemma 2.3. *Let $A = M_n(k)$ where k is an algebraically closed field. Then A is definably irreducibly representable.*

Proof. The structure (E, k) corresponding to A consists of an n-dimensional k-vector space E, and E projects to a single point in k. The remaining details are given in Example 2.1. ☐

Proposition 2.12. *A ring R is von Neumann regular if and only if for every pp-pair ϕ/ψ there is a pp-condition θ such that $F_{\phi/\psi} \simeq F_\theta$ in $\mathbb{L}_R^{eq+}$. In this case θ may be taken to be quantifier-free, i.e. a system of R-linear equations.*

Proof. [Pres09] Corollary 10.2.40. ☐

Theorem 2.3. *Let $A = M_n(k)$. If $(E, k) \models T_A$, then $(E, k)^{+eq+}$ is definably interpretable in $(E, k)^+$.*

Proof. By Lemma 2.3 A is definably irreducibly representable, thus there is a theory T_A. By Proposition 2.11, A is von Neumann regular. The result now follows by Proposition 2.12. ☐

Note that if $A = M_n(k)$, Proposition 2.10 gives that $\mathbb{L}_A^{eq+} \simeq (\mathbb{L}_A^{eq+})_V$ which is what one intuitively expects if A is semisimple; i.e. all pp-information for ${}_A\mathsf{Mod}$ is already determined on the irreducible A-modules. Because of the strength of the semisimplicity assumption, there is no reason to suspect that the converse to Theorem 2.3 holds. Rather, the author expects the following to be true.

Conjecture 2.1. *Let A be a definably irreducibly representable k-algebra, $(E, k) \models T_A$. Then $(E, k)^{+eq+}$ is definably interpretable in $(E, k)^+$ if and only if A is a finitely generated projective module over a commutative finitely generated k-algebra which is a domain.*

Conjecture 2.1 could be used to delineate those Zariski structures which are nonclassical. Intuitively, if (E, k) were definably interpretable in k then elimination of imaginaries for algebraically closed fields should imply that $(E, k)^{+eq+}$ is definably interpretable in $(E, k)^+$. The failure of this to hold for those definably irreducibly representable k-algebras which are not $M_n(Z)$ should therefore lead to the associated Zariski structures being nonclassical. If true, this would provide a sharper statement and more conceptual proof of nonclassicality than that found in [Zil06].

Chapter 3

Limits

Limits of topological structures are investigated using presheaves. Ultraproducts of topological structures become stalks of presheaves over appropriate bases. Using this formalism, ultraproducts of certain Azumaya structures can be taken which can be viewed as structures at a ‘generic parameter’, i.e. a parameter that is not a root of unity.

3.1 Ultraproducts of Topological Structures

By a **topological structure** we shall mean a set $\mathbf{X}$ together with a topology on each $\mathbf{X}^n$ for every $n > 0$ (compare with Definition 1.1). Syntactically, a topological structure has an associated canonical language $\mathcal{C}$ consisting of predicates for each closed subset; with elements satisfying the predicate if and only if they lie in the subset. Frequently, a finer language is used to describe the topological structures under consideration, with the topology only materializing after a quantifier-elimination result has been established. The models of T_A in Chapter 2 are examples of such topological structures.

The following constructions work equally well for languages finer than the canonical language $\mathcal{C}$, and even collections of structures which are not topological. Nevertheless, we shall stick with topological structures in the canonical language. Let X be a topological space, $\{M_x : x \in X\}$ a collection of (topological) $\mathcal{C}$-structures parametrized by X. We establish the following notation:

- $\mathsf{Open}(X)$ for the poset of open subsets of X ordered by inclusion.
- $\mathsf{Fil}(X)$ for the poset (ordered by inclusion) of filters on $(X, \subseteq)$.
- $\mathsf{Mod}(\mathcal{C})$ for the category of $\mathcal{C}$-structures with $\mathcal{C}$-homomorphisms.
- $\mathsf{Fun}(\mathsf{A}, \mathsf{B})$ for the category of functors $\mathsf{A} \to \mathsf{B}$, where A, B are categories.

Concerning $\mathsf{Mod}(\mathcal{C})$, recall that for any language $\mathcal{L}$, an $\mathcal{L}$-homomorphism between two $\mathcal{L}$-structures is a map which respects all symbols of $\mathcal{L}$. Thus if M and N are two $\mathcal{C}$-structures, a $\mathcal{C}$-homomorphism $f : M \to N$ is a map such that for any predicate $C \in \mathcal{C}$,

$$M \models C(a) \Rightarrow N \models C(f(a))$$

In particular, if M is an elementary extension of N then the notion of a $\mathcal{C}$-homomorphism $f : M \to N$ accords with the definition of **specialization** found in [Zil10].

Given a filter $\mathcal{F}$ on X it is possible to form the reduced product

$$M_{\mathcal{F}} = \prod_{x \in X} M_x / \mathcal{F}$$

Explicitly, one takes the set $\prod_{x \in X} M_x = \{f : X \to \bigcup M_x : f(x) \in M_x\}$ and forms the quotient of this by the equivalence relation $\sim_{\mathcal{F}}$ given by

$$f \sim_{\mathcal{F}} g \Leftrightarrow \{x : f(x) = g(x)\} \in \mathcal{F}$$

We denote the elements of $M_{\mathcal{F}}$ by $f/\mathcal{F}$, where $f \in \prod_{x \in X} M_x$.

Proposition 3.1. *Let ϕ be a positive $\mathcal{C}$-formula, $\mathcal{F}$ a filter on X. Then*

$$M_{\mathcal{F}} \models \phi(f/\mathcal{F}) \Leftrightarrow \{x : M_x \models \phi(f(x))\} \in \mathcal{F}$$

Proof. Easy induction on complexity of formulas. □

By Proposition 3.1, the correspondence $\mathcal{F} \mapsto M_{\mathcal{F}}$ extends to a functor $M : \mathsf{Fil}(X) \to \mathsf{Mod}(\mathcal{C})$. Indeed, if $\mathcal{F} \subseteq \mathcal{G}$ are filters on X then the equivalence relation $\sim_{\mathcal{F}}$ refines $\sim_{\mathcal{G}}$ and we have a well-defined map $M_{\mathcal{F}} \to M_{\mathcal{G}}$ given by $f/\mathcal{F} \mapsto f/\mathcal{G}$. It is immediate by Proposition 3.1 that this map is a $\mathcal{C}$-homomorphism. Proposition 3.1 is a restricted form of Łos's theorem, the latter applying to ultraproducts of structures. For topological structures, it allows for a transfer of information between basic projective sets (projections of closed sets) of the different M_x and the topological structures $M_{\mathcal{F}}$.

Definition 3.1. *Let $\mathbb{G} : \mathsf{Open}(X)^{op} \to \mathsf{Fil}(X)$ be a functor. We define the presheaf $M_{\mathbb{G}} : \mathsf{Open}(X)^{op} \to \mathsf{Mod}(\mathcal{C})$ by $M_{\mathbb{G}} = M \circ \mathbb{G}$ and call it the* ***presheaf of topological structures*** *associated with the data $(M, \mathbb{G})$.*

Let $\mathbb{G} : \mathsf{Open}(X)^{op} \to \mathsf{Fil}(X)$ be a functor. For $x \in X$, let N_x denote the directed system of open neighbourhoods of X. Then $\mathbb{G}(N_x)$ is a directed system in $\mathsf{Fil}(X)$ whose limit exists because $\mathsf{Fil}(X)$ has direct limits.

Definition 3.2. *The **stalk** of the presheaf* $M_{\mathbb{G}} : \mathsf{Open}(X)^{op} \to \mathsf{Mod}(\mathcal{C})$ *is defined to be*

$$M_{\mathbb{G}(x)} = M \circ \varinjlim \mathbb{G}(N_x)$$

Proposition 3.2. *For any* $\mathbb{G} : \mathsf{Open}(X)^{op} \to \mathsf{Fil}(X)$ *we have*

$$M_{\mathbb{G}(x)} = \varinjlim_{U \in N_x} M_{\mathbb{G}}(U)$$

Proof. We show that the left-hand side satisfies the required universal property. Put $\mathbb{G}(x) = \varinjlim \mathbb{G}(N_x)$. Because $\mathbb{G}(U) \subseteq \mathbb{G}(x)$ for every $U \in N_x$ we have a well-defined map $\sigma_U : M_{\mathbb{G}}(U) \to M(\mathbb{G}(x))$. If $i : V \to U$ is an inclusion of open sets then it is clear that the following diagram commutes:

$$\begin{tikzcd}
M_{\mathbb{G}}(U) \arrow[r, "\sigma_U"] \arrow[d, "M_{\mathbb{G}}(i)"'] & M(\mathbb{G}(x)) \\
M_{\mathbb{G}}(V) \arrow[ur, "\sigma_V"'] &
\end{tikzcd}$$

Let N be a $\mathcal{C}$-structure and suppose that there are maps $\tau_U : M_{\mathbb{G}}(U) \to N$ for each $U \in N_x$ such that

$$\begin{tikzcd}
M_{\mathbb{G}}(U) \arrow[r, "\tau_U"] \arrow[d, "M_{\mathbb{G}}(i)"'] & N \\
M_{\mathbb{G}}(V) \arrow[ur, "\tau_V"'] &
\end{tikzcd}$$

commutes for each inclusion $i : V \to U$ of open sets in N_x. Define the (unique) map $\eta : M(\mathbb{G}(x)) \to N$ by $\eta(f/\mathbb{G}(x)) = \tau_U(f/\mathbb{G}(U))$ for any open set U containing x, which is well-defined by an easy diagram chase. Then

$$\begin{tikzcd}
M_{\mathbb{G}}(U) \arrow[dd, "M_{\mathbb{G}}(i)"'] \arrow[dr, "\sigma_U"'] \arrow[drr, "\tau_U"] & & \\
& M(\mathbb{G}(x)) \arrow[r, dotted, "\eta"] & N \\
M_{\mathbb{G}}(V) \arrow[ur, "\sigma_V"] \arrow[urr, "\tau_V"'] & &
\end{tikzcd}$$

commutes for every inclusion $i : V \to U$ of open sets, as required. □

The fundamental presheaf and sheafification

Despite the above general treatment, for practical purposes we shall be dealing almost exclusively with one particular presheaf of topological structures, namely that obtained from $\mathbb{F} : \mathsf{Open}(X)^{op} \to \mathsf{Fil}(X)$ where $\mathbb{F}(U) = \{V \subseteq X : U \subseteq V\}$ for each open subset $U \subseteq X$. We call $M_{\mathbb{F}}$ the **fundamental presheaf of topological structures** associated with M. The sheafification $M_{\mathbb{F}}^{sh}$ of $M_{\mathbb{F}}$ is given by associating to each open subset U of X the set $M_{\mathbb{F}}^{sh}(U)$ consisting of functions $s : U \to \bigcup_{x \in U} M_{\mathbb{F}(x)}$ such that

- For each $x \in U$, $s(x) \in M_{\mathbb{F}(x)}$.
- For each $x \in U$ there is an open neighbourhood $V \subseteq U$ of x and $t \in M_{\mathbb{F}}(V)$ such that for every $y \in V$, $t_y = s(y)$, where t_y is the germ of t at y.

Recalling the proof of Proposition 3.2, t_y is merely the image of t under the map $\sigma_V : M(\mathbb{F}(V)) \to M(\mathbb{F}(y))$. It need not be the case that $M_{\mathbb{F}(x)} = M_x$ for every $x \in X$. This will be true if x is an open point of X or if one can get arbitrarily close to x (for example, if X is a metric space).

Remark 3.1. *If $M_{\mathbb{F}(x)} = M_x$ for every $x \in X$, then $M_{\mathbb{F}} = M_{\mathbb{F}}^{sh}$.*

Proof. Immediate once one notices that elements of $M_{\mathbb{F}}(U)$ can be regarded as functions $f : U \to \bigcup_{x \in U} M_x$ where $f(x) \in M_x$ for every $x \in U$. ☐

3.1.1 Compactifications of the base

Firstly, we investigate how presheaves of filters behave on change of base.

Proposition 3.3. *Let $f : X \to Y$ be a continuous map of topological spaces. Then f induces a pushforward functor $f_* : \mathsf{Fun}(\mathsf{Open}(X)^{op}, \mathsf{Fil}(X)) \to \mathsf{Fun}(\mathsf{Open}(Y)^{op}, \mathsf{Fil}(Y))$.*

Proof. By continuity, f gives a map of posets $f^{-1} : \mathsf{Open}(Y)^{op} \to \mathsf{Open}(X)^{op}$ by $U \mapsto f^{-1}(U)$. For $\mathcal{F} \in \mathsf{Fil}(X)$ define the pushforward under f by $f_*\mathcal{F} = \{V \subseteq Y : f^{-1}(V) \in \mathcal{F}\} \in \mathsf{Fil}(Y)$. This correspondence clearly extends to a functor $f_* : \mathsf{Fil}(X) \to \mathsf{Fil}(Y)$. Now the pushforward is obtained by the correspondence $\mathbb{G} \mapsto f_* \circ \mathbb{G} \circ f^{-1}$, which is easily seen to be functorial. ☐

Recall that if X is a topological space then a compact space X^* is a **compactification** of X if there is an embedding (a homeomorphism onto its image) $i : X \to X^*$ such that $i(X)$ is dense in X^*.

Proposition 3.4. *Let $i : X \to X^*$ be a compactification. Then a functor $M_{\mathbb{G}} : \mathsf{Open}(X)^{op} \to \mathsf{Mod}(\mathcal{C})$ extends to a functor $M_{\mathbb{G}^*} : \mathsf{Open}(X^*)^{op} \to \mathsf{Mod}(\mathcal{C})$.*

Proof. Without loss we assume that $X \subseteq X^*$. For every $y \in X^* \setminus X$ and $U \in N_y$, $U \cap X \neq \emptyset$ by density. Hence $i^{-1}N_y = \{U \cap X : U \in N_y\}$ is a directed set. Let D be an ultrafilter extending $\varinjlim \mathbb{G}(i^{-1}N_y)$ and put

$$M_y = \prod_{x \in X} M_x / D$$

Repeating this construction for each $y \in X^* \setminus X$ gives us a collection of structures $\{M_y : y \in X^*\}$, allowing us to define a functor $\mathsf{Fil}(X^*) \to \mathsf{Mod}(\mathcal{C})$. The result follows by applying Proposition 3.3. □

The reason for using an ultrafilter to define a stalk comes from the observation that for the fundamental presheaf $\mathbb{F}$, when the topology on X is sufficiently fine, stalks are ultraproducts (by the comments preceding Remark 3.1). So the additional stalks over X^* should also be ultraproducts. We note that for the specific case of $\mathbb{F}$, the pushforward of D in the proof of Proposition 3.4 is in fact the principal filter on $y \in X^*$.

Stone-Čech compactification

There is an important compactification worth mentioning because of its relationship with ultrafilters. Recall that the **Stone-Čech compactification** of a topological space X is an embedding $i : X \to \beta X$ where βX is compact and Hausdorff; and such that any continuous map $f : X \to Y$ to a compact Hausdorff space factors uniquely through i, i.e. the diagram

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr, "f"'] & \beta X \arrow[d, dotted, "\beta f"] \\
& Y
\end{tikzcd}$$

commutes. For lack of a suitable reference, we summarize the details of the construction as follows:

1. Let X' be the set of ultrafilters in X with the Stone topology, i.e. the topology generated by sets of the form $\{D \in X' : U \in D\}$ for every subset U of X.
2. Form the quotient $\beta X = X'/\sim$ where $\sim$ identifies those ultrafilters which cannot be distinguished by closed subsets of X.

If $\{M_x : x \in X\}$ is a collection of topological structures parametrized by X then the Stone-Čech compactification is the most 'ideal' compactification in the following sense. Suppose that $\mathbb{G}$ is a presheaf of filters on X such that there is a natural transformation $\mathbb{F} \to \mathbb{G}$ from the fundamental presheaf. Adopting the notation of Proposition 3.4, for any $y \in \beta X \backslash X$ the only ultrafilters extending $\varinjlim \mathbb{G}(i^{-1}N_y)$ are exactly those which the topology on X cannot distinguish. I.e. if one has a choice of structures $\prod_{x \in X} M_x/D$ and $\prod_{x \in X} M_x/D'$ for M_y, it is precisely because $D \sim D'$ in βX. Evidently, if X has the discrete topology then there is no such choice (because $X' = \beta X$ in this case).

3.2 Examples

Both of the following examples apply the above theory to collections of structures which are associated to algebras parametrized by primitive l-th roots of unity ϵ. In each case, we shall be interested in what happens to the structures when $l \to \infty$. It is therefore appropriate to parametrize these structures by $\mathbb{N}$ equipped with the standard (metric) topology given by the absolute value.

3.2.1 Quantum torus

For $l \in \mathbb{N}$ with $l \neq 0$, put

$$A^l_\epsilon = k\langle \mathbf{U}, \mathbf{V} \rangle / \langle \mathbf{UV} - \epsilon \mathbf{VU} \rangle \qquad \epsilon^l = 1 \qquad \epsilon \text{ primitive}$$

where k is an algebraically closed field of 0 characteristic. It is known that A^l_ϵ satisfies the conditions of Definition 2.4 (for a proof, see [Zil06]), thus there is a corresponding Zariski structure $\operatorname{nSpec} A^l_\epsilon = (E_l, V_l, \mathbb{K})$ where $\mathbb{K}$ is a large algebraically closed field. It is worth noting that the theories corresponding to each A^l_ϵ are formulated in the same language, namely

$$\mathcal{L} = (E, V, k, \pi, \mathbf{U}, \mathbf{V}, \epsilon)$$

hence it makes sense to consider reduced products of the collection $\{\operatorname{nSpec} A^l_\epsilon : l \in \mathbb{N}\}$. The above formalism allows for ‘singular’ fibers because no continuity conditions are imposed on the presheaves. Hence we can take any arbitrary $\mathcal{L}$-structure (which is topological) to associate to $l = 0$. Let $i : \mathbb{N} \to \mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ denote the one-point compactification of $\mathbb{N}$ in the metric topology. We consider the fundamental presheaf $\mathbb{F}$ on $\mathbb{N}$. By the proof (and notation) of Proposition 3.4 we wish to consider ultrafilters extending

$$\varinjlim \mathbb{F}(i^{-1}N_\infty) = \{S^{>n} : n \in \mathbb{N}\}$$

where $S^{>n} = \{p \in \mathbb{N} : p > n\}$. For such an ultrafilter D, put

$$\operatorname{nSpec} A^\infty_\epsilon = \prod_{l \in \mathbb{N}} \operatorname{nSpec} A^l_\epsilon / D$$

Proposition 3.5. $\operatorname{nSpec} A_\epsilon^\infty = (E_\infty, V_\infty, \mathbb{K})$ *where*

1. $V_\infty = (\mathbb{K}^\times)^2$.

2. *There is a surjective map* $\pi_\infty : E_\infty \to V_\infty$. *For each* $(a,b) \in V_\infty$, *there is a basis* $\{e_i : i \geq 0\}$ *of* $\pi_\infty^{-1}(a,b)$ *such that*

$$\mathbf{U}e_i = a'\epsilon^i e_i \qquad \mathbf{V}e_i = e_{i+1}$$

for every i, *where* ϵ *is generic and* $(a')^l \neq a$ *for any* $l \in \mathbb{N}$.

Proof. These follow from Łos's theorem. First we state the relevant facts from [Zil06] concerning the structures $\operatorname{nSpec} A_\epsilon^l$. We have $\operatorname{nSpec} A_\epsilon^l = (E_l, V_l, \mathbb{K})$ where

1. $V_l = (\mathbb{K}^\times)^2$ and we have a surjective map $\pi_l : E_l \to V_l$ with the property that for each $(a,b) \in V_l$, the fiber $\pi_l^{-1}(a,b)$ is an l-dimensional vector space over $\mathbb{K}$.

2. Each fiber $\pi_l^{-1}(a,b)$ has a canonical basis $e_0^l, \dots, e_{l-1}^l$ such that

$$\mathbf{U}e_i^l = a_l \epsilon_l^i e_i^l \qquad \mathbf{V}e_i^l = e_{i+1}^l \qquad \mathbf{V}^l e_0^l = b e_0^l$$

where $a_l^l = a$ and ϵ_l is a primitive l-th root of unity.

There is a surjective map $\pi_\infty : E_\infty \to V_\infty$ because there is such a map in every $\operatorname{nSpec} A_\epsilon^l$. Let $(a,b) \in V$. In each $\operatorname{nSpec} A_\epsilon^l$, the fiber $\pi^{-1}(a,b)$ has a basis $e_0^l, \dots, e_{l-1}^l$ such that

$$\mathbf{U}e_i^l = a_l \epsilon_l^i e_i^l \qquad \mathbf{V}e_i^l = e_{i+1}^l \qquad \mathbf{V}^l e_0^l = b e_0^l$$

Define

- $\epsilon : \mathbb{N} \to \mathbb{K}$ by $\epsilon(l) = \epsilon_l$.
- $a', b' : \mathbb{N} \to \mathbb{K}$ by $a'(l) = a_l$ and $b'(l) = b_l$
- $e_i : \mathbb{N} \to \mathbb{K}$ by $e_i(l) = e_i^l$ where this exists and 0 otherwise.

Then $\mathbf{U}e_i = a'\epsilon^i e_i$ holds in $\operatorname{nSpec} \mathbb{A}_\epsilon^\infty$ if and only if $\{l : \operatorname{nSpec} A_\epsilon^l \models \mathbf{U}e_i(l) = a'(l)\epsilon(l)^i e_i(l)\} \in D$. But this set certainly contains $S^{>i}$, as required. Similarly $\mathbf{V}e_i = e_{i+1}$ holds. Suppose for contradiction that $\epsilon^m = 1$ for some m. Then $\{l : \operatorname{nSpec} A_\epsilon^l \models \epsilon_l^m = 1\} = \{l : l|m\} \in D$. But then this set does not intersect $S^{>m} \in D$, which is forbidden. Likewise, $(a')^l \neq a$ for any $l \in \mathbb{N}$. ☐

3.2.2 Quantized Weyl algebras

Recall that the **quantized n-th Weyl algebra over a field** k at a primitive l-th root of unity ϵ is defined to be

$$A_n^\epsilon(k) = \langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle / I_\epsilon$$

where I_ϵ is the ideal generated by $\partial_i x_j - \epsilon^{\delta_{ij}} \partial_j x_i - \delta_{ij}$, $x_i x_j - x_j x_i$ and $\partial_i \partial_j - \partial_j \partial_i$ for $1 \le i, j \le n$. Define

$$f_\epsilon = \prod_{i=1}^{n} [\partial_i, x_i]$$

Proposition 3.6. *The Ore-localization $A_{n,f}^\epsilon(k)$ of $A_n^\epsilon(k)$ at f is a definably irreducibly representable Azumaya k-algebra, for $\epsilon \neq 1$.*

Proof. Proofs of the following enumerated facts can be found in [Back10].

1. $\{f^n : n \ge 0\}$ is a left Ore set in $A_n^\epsilon(k)$, hence $A_{n,f_\epsilon}^\epsilon(k)$ exists.
2. The center of $A_n^\epsilon(k)$ is $Z_n^\epsilon = k[x_1^l, \dots, x_n^l, \partial_1^l, \dots, \partial_n^l]$. The element $f_\epsilon^l = \prod_{i=1}^n 1 - (1-\epsilon)^l x_i^l \partial_i^l \in Z_n^\epsilon$, hence one can localize Z_n^ϵ at f_ϵ^l to obtain $Z_{n,f_\epsilon^l}^\epsilon$. Moreover,

   $$A_{n,f_\epsilon}^\epsilon(k) \simeq A_n^\epsilon(k) \otimes_{Z_n^\epsilon} Z_{n,f_\epsilon^l}^\epsilon$$

   from which it is immediate that the center of $A_{n,f_\epsilon}^\epsilon(k)$ is $Z_{n,f_\epsilon^l}^\epsilon$.
3. $A_{n,f_\epsilon}^\epsilon(k)$ is Azumaya of rank l^{2n} over its center.

It follows by 3 that all irreducible $A_{n,f_\epsilon}^\epsilon(k)$-modules have dimension l^n over k. Let $\mathfrak{m}$ be the maximal ideal of Z_n^ϵ given by

$$(x_1^l - a_1, \dots x_n^l - a_n, \partial_1^l - b_1, \dots \partial_n^l - b_n)$$

with the property that $f_\epsilon^l \neq 0$ at the point $(a_1, \dots, a_n, b_1, \dots, b_n)$, i.e. that $a_i b_i \neq (1-\epsilon)^{-l}$ for any $1 \le i \le n$. Let $k(a_i, \lambda_i)$ be the one-dimensional $Z' = k[x_1^l, \dots, x_n^l, \partial_1, \dots, \partial_n]$-module given by

$$x_i^l v = a_i v \qquad \partial_i v = \lambda_i v \qquad 1 \le i \le n \qquad v \in k(a_i, \lambda_i)$$

where $\lambda_i^l = b_i$ for every i.

Claim: The induced module $M(a_i, \lambda_i) = A_n^\epsilon(k) \otimes_{Z'} k(a_i, \lambda_i)$ has a k-vector space basis $\{e(i_1, \dots, i_n) : 0 \le i_j \le l-1, 1 \le j \le n\}$ with relations

$$x_j e(i_1, \dots, i_n) = \begin{cases} e(i_1, \dots, i_j + 1, \dots i_n) & i_j < l-1 \\ a_j e(i_1, \dots, i_{j-1}, 0, i_{j+1}, \dots, i_n) & i_j = l-1 \end{cases}$$

$$\partial_j e(i_1,\dots,i_n) = [i_j]_\epsilon e(i_1,\dots,i_j-1,\dots,i_n) + \lambda_j \epsilon^{i_j} e(i_1,\dots,i_n)$$

where one defines $[m]_\epsilon = (1-\epsilon^m)/(1-\epsilon)$ for every $m \in \mathbb{N}$.

Proof. Straightforward calculation using the following relation:

$$\partial_i x_i^m = [m]_\epsilon x_i^{m-1} + \epsilon^m x_i^m \partial_i \qquad m \geq 0, 1 \leq i \leq n$$

□

Clearly $M(a_i, \lambda_i)$ is an irreducible $A^\epsilon_{n,f_\epsilon}(k)$-module by the assumptions on $\mathfrak{m}$ and fact 2. It is annihilated by $\mathfrak{m}$, hence it must be the unique irreducible module (up to isomorphism) with this property. Moreover, all of the parameters involved in defining the actions of x_j, ∂_j are solutions to $\Sigma = \{y_i^l - b_i, z_i - a_i : 1 \leq i \leq n\}$ and letting the a_i, b_i vary over the variety corresponding to $Z^\epsilon_{n,f^l_\epsilon}$ gives us a Galois covering of it with group Γ equal to the group of l-th roots of unity. □

By Proposition 3.6, we obtain a collection of structures $\operatorname{nSpec} A^\epsilon_{n,f^l_\epsilon}$ for $l \neq 0, 1$. By inserting arbitrary structures at 0 and 1, we have a collection parametrized by $\mathbb{N}$. Now consider the compactification $i : \mathbb{N} \to \mathbb{N}^*$ and form the ultraproduct

$$\operatorname{nSpec} A^\infty_n = \prod_{l \in \mathbb{N}} \operatorname{nSpec} A^\epsilon_{n,f^l_\epsilon} / D$$

where D is an ultrafilter extending $\{S^{>n} : n \in \mathbb{N}\}$.

Proposition 3.7. $\operatorname{nSpec} A^\infty_n = (E_\infty, V_\infty, \mathbb{K})$ *where*

1. $V_\infty = \{(a_1,\dots,a_n,b_1,\dots,b_n) : a_i b_i \neq (1-\epsilon)^{-l}$ *for any* $i \leq n$ *and* $l \in \mathbb{N}$ $\}$.
2. *There is a surjective map* $\pi_\infty : E_\infty \to V_\infty$ *such that each* $(a,b) \in V_\infty$ *there is a basis* $\{e(i_1,\dots,i_n) : i_j \geq 0, 1 \leq j \leq n\}$ *of* $\pi^{-1}_\infty(a,b)$ *such that*

$$x_j e(i_1,\dots,i_n) = e(e_i,\dots,i_j+1,\dots,i_n)$$

$$\partial_j e(i_1,\dots,i_n) = [i_j]_\epsilon e(i_1,\dots,i_j-1,\dots i_n) + b'_j \epsilon^{i_j} e(i_1,\dots,i_n)$$

where ϵ *is generic and* $(b'_j)^l \neq b_j$ *for any* $l \in \mathbb{N}$, $1 \leq j \leq n$.

Proof. Analogous to the proof of Proposition 3.5, using the claim in the proof of Proposition 3.6. □

We remark that the limit which is of interest in [Back10] is when $\epsilon \to 1$, and this limit of structures cannot be obtained using the methods above. In essence, we do not have enough structures parametrized finely enough for this limit to exist. However, when $\epsilon = 1$ we obtain the classical n-th Weyl algebra, and at least for $n = 1$ we demonstrate (in the next chapter) that this algebra falls into another class of algebras for which there are associated Zariski structures.

3.3 Remarks

The formalism established is flexible enough to also be applicable inside certain individual structures. For example, if A is a definably irreducibly representable Azumaya algebra then $\mathrm{nSpec}\, A = (E_A, V_A, \mathbb{K})$ is itself a collection of modules parametrized by a variety. In this manner, given a presheaf $\mathbb{G} : \mathsf{Open}(V)^{op} \to \mathsf{Fil}(V)$, it is possible to construct the associated presheaf $E_{A,\mathbb{G}}$ of A-modules over V. If one works in the extended language $\mathcal{L}_A^+$, then depending on the presheaf $\mathbb{G}$ chosen, it will be possible to realize certain pp-types in the presheaf which are not realizable in the individual modules $E_{A,x} = \pi^{-1}(x)$ for $x \in V_A$ (this is just model-theoretic compactness). And more is possible if V has a compactification.

The above formalism remains defective insofar as the author is unaware of a suitable general notion of a ‘continuously varying family of topological structures’. Intuitively, it seems that the reason one doesn’t obtain nonsense when taking the limit $l \to \infty$ in both of the above examples is because we do have a continuously varying family of structures already, in both cases (discarding the arbitrarily inserted structures at finitely many points). Related to this issue is the question of whether there is a suitable dimension theory for the structures obtained at generic parameter, arising naturally from the Zariski structures which are parametrized by $\mathbb{N}$.

Chapter 4

Equivariant Algebras and Their Associated Theories

We now embark on the task of associating first-order theories to a class of k-algebras, which we call equivariant. The next two chapters will contain model-theoretic results. In the present chapter, we shall discuss some important examples, define this class of algebras and demonstrate how to associate a theory to each such algebra.

4.1 Some examples

Three examples of noncommutative algebras are discussed, occupying a central place in physics, the theory of quantum groups and noncommutative geometry respectively.

4.1.1 Weyl Algebra

Recall that for a commutative ring R, the **n-th Weyl algebra** $A_n(R)$ (for $n > 0$) is defined to be

$$R\langle x_1, \dots x_n, \partial_1, \dots, \partial_n \rangle / I$$

where I is the ideal generated by

$$\partial_i x_j - x_j \partial_i - \delta_{ij} \qquad x_i x_j - x_j x_i \qquad \partial_i \partial_j - \partial_j \partial_i \qquad \text{for } 1 \leq i, j \leq n$$

We shall concentrate on the first Weyl algebra $A_1(k)$ for k an algebraically closed field of characteristic 0. Firstly, we note that $A_1(k)$ can be redefined in terms of three operators $H, \mathbf{a}, \mathbf{a}^\dagger$:

$$\mathbf{H} = \frac{1}{2}(x^2 - \partial_x^2) \qquad \mathbf{a} = \frac{1}{\sqrt{2}}(x + \partial_x) \qquad \mathbf{a}^\dagger = \frac{1}{\sqrt{2}}(x - \partial_x)$$

Because we are working in an arbitrary algebraically closed field, $\sqrt{2}$ represents an element that squares to 2. The operator $\mathbf{a}^\dagger$ is a formal adjoint to $\mathbf{a}$, as an element of the differential ring $A_1(k)$.

Proposition 4.1. *The following relations hold between* $H, \mathbf{a}, \mathbf{a}^\dagger$*:*

1. $\mathbf{a}^\dagger\mathbf{a} = \mathbf{H} - 1/2$, $\mathbf{a}\mathbf{a}^\dagger = \mathbf{H} + 1/2$.
2. $[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = 1$.
3. *Putting* $\mathbf{N} = \mathbf{H} - 1/2$, *we have* $[\mathbf{N}, \mathbf{a}^\dagger] = \mathbf{a}^\dagger$ *and* $[\mathbf{N}, \mathbf{a}] = -\mathbf{a}$. *Thus we also have*

$$[\mathbf{H}, \mathbf{a}^\dagger] = \mathbf{a}^\dagger \qquad [\mathbf{H}, \mathbf{a}] = -\mathbf{a}$$

Proof. 1 and 2 are easy verification. For 3, use the fact that for any three operators A, B, C, we have the relation $[A, BC] = [A, B]C + B[A, C]$. □

In [ZS09], a Zariski structure was associated to the Heisenberg algebra $k\langle \mathbf{P}, \mathbf{Q}\rangle/I$ where I is generated by $[\mathbf{P}, \mathbf{Q}] + i$. Similarly this algebra was also re-expressed in terms of operators $\mathbf{H}, \mathbf{a}, \mathbf{a}^\dagger$ defined slightly differently to the above, namely by

$$\mathbf{H} = \frac{1}{2}(\mathbf{P}^2 + \mathbf{Q}^2) \qquad \mathbf{a} = \frac{1}{\sqrt{2}}(\mathbf{P} - i\mathbf{Q}) \qquad \mathbf{a}^\dagger = \frac{1}{\sqrt{2}}(\mathbf{P} + i\mathbf{Q})$$

The relations satisfied by these are, however, the same as in Proposition 4.1. The structure we define below is different to that of [ZS09]; indeed the following structure originally appeared as a quotient of an initial (also Zariski) structure in that paper. The latter was important insofar as it provided another example of a one-dimensional Zariski geometry (a finite cover of the projective line) not definable in an algebraically closed field. For our purposes, we can start directly with the quotient.

Definition 4.1. *We consider a two-sorted language* $\mathcal{L}_{A_1} = (k, L, \pi, \mathbf{E}, \mathbf{H}, \mathbf{a}, \mathbf{a}^\dagger)$ *where*

1. $\pi : L \to k$ *and* $\mathbf{H}, \mathbf{a}, \mathbf{a}^\dagger : L \to L$ *are maps.*
2. $\mathbf{E} \subseteq L \times k$ *is a relation.*
3. *The sort* k *has the language of rings. The sort* L *comes equipped with*
   - *a relation* $+ \subseteq L^3$ *where* $+(v_1, v_2, v_3)$ *holds if and only if there exists* $x \in k$ *such that* $v_i \in \pi^{-1}(x)$ *for* $1 \leq i \leq 3$ *and* $v_1 + v_2 = v_3$.
   - *a relation* $\cdot \subseteq k \times L^2$ *where* $\cdot(\lambda, v_1, v_2)$ *holds if and only if there is* $x \in k$ *such that* $v_1, v_2 \in \pi^{-1}(x)$ *and* $\lambda v_1 = v_2$.

The $\mathcal{L}_{A_1}$*-theory* T_{A_1} *says the following:*

1. k *is an algebraically closed field of characteristic* 0.

2. *$\pi : L \to k$ is a surjective map; each fiber $\pi^{-1}(x)$ for $x \in k$ is a one-dimensional k-vector space.*

3. *For each $x \in k$, the subset $\mathbf{E}(L, x)$ is non-empty and $\mathbf{E}(L, x) \subseteq \pi^{-1}(x)$.*

4. *We fix an $l \in \mathbb{Z}$, $l > 0$, l even. If $\Gamma[l]$ denotes the group of l-th roots of unity of k, then there is a free and transitive action of $\Gamma[l]$ on $\mathbf{E}(L, x)$ induced by the vector space action on the fiber $\pi^{-1}(x)$.*

5. *The map $\mathbf{H}$ is linear on each fiber and satisfies the following axiom*

$$(\forall v \in \pi^{-1}(x))(\mathbf{H}v = xv)$$

6. *The maps $\mathbf{a}$ and $\mathbf{a}^\dagger$ are linear and move between fibers according to the following axiom:*

$$(\forall v \in \pi^{-1}(x))(\mathbf{E}(v, x) \to (\exists v' \in \pi^{-1}(x + 1))(\exists y \in k)(y^2 = x \wedge \mathbf{a}^\dagger v = yv' \wedge \mathbf{a}v' = yv))$$

Let $(L, k) \models T_{A_1}$. Then L is a 'line bundle' over the base k, though we do not claim local triviality. Each fiber $\pi^{-1}(x)$ is an x-eigenspace for $\mathbf{H}$. The elements $\mathbf{E}(L, x) \subseteq \pi^{-1}(x)$ are to be regarded as normal basis elements of the fiber $\pi^{-1}(x)$ which can be permuted by the group of l-th roots of unity $\Gamma[l]$. Consequently, l is part of the data of the theory T_{A_1} although this dependence has not been explicitly indicated. This setup serves as a discrete (and algebraic) model for a well-known phenomenon encountered when dealing with normed vector spaces. If V is a normed vector space over $\mathbb{C}$ and $v \in V$ is an element of norm 1, then so is αv for any $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$. Of course, in this case there is an infinite group S^1 acting on the elements in V of norm 1. It will be seen, in the next chapter, that being able to permute the normal basis elements by a finite group is also essential for a decent structure theory for T_{A_1}.

Proposition 4.2. *Let $(L, k) \models T_{A_1}$. Then (L, k) is a representation of the associated Lie algebra of $A_1(k)$; namely the algebra over k generated by $\mathbf{H}, \mathbf{a}, \mathbf{a}^\dagger$ subject to the relations*

$$[\mathbf{H}, \mathbf{a}] = -\mathbf{a} \qquad [\mathbf{H}, \mathbf{a}^\dagger] = \mathbf{a}^\dagger \qquad [\mathbf{a}, \mathbf{a}^\dagger] = 1$$

Proof. Let $e \in \pi^{-1}(x)$ such that $\mathbf{E}(e, x)$ holds. Then there is $e' \in \pi^{-1}(x + 1)$ such that

$$\mathbf{H}\mathbf{a}^\dagger e = y\mathbf{H}e' = (x + 1)ye' \qquad y^2 = x$$

But

$$(x + 1)ye' = (x + 1)\mathbf{a}^\dagger e = \mathbf{a}^\dagger \mathbf{H}e + \mathbf{a}^\dagger e$$

Thus $[\mathbf{H}, \mathbf{a}^\dagger]e = \mathbf{a}^\dagger e$. Similarly, we obtain that $[\mathbf{H}, \mathbf{a}]e = -\mathbf{a}e$. Now

$$\mathbf{a}\mathbf{a}^\dagger e = y\mathbf{a}e' = xe$$

whereas

$$\mathbf{a}^\dagger \mathbf{a}e = z\mathbf{a}^\dagger e'' = (x-1)e$$

where $z^2 = x - 1$ and $e'' \in \pi^{-1}(x-1)$. Thus $[\mathbf{a}, \mathbf{a}^\dagger]e = e$ as required. □

4.1.2 $U_q(\mathfrak{sl}_2(k))$ for generic q

Let k be an algebraically closed field of characteristic 0, $q \in k$ with $q \neq 0, \pm 1$. The **quantized enveloping algebra** of $sl_2(k)$, denoted $U_q(sl_2(k))$, is defined to be the k-algebra with generators $E, F, K^{\pm 1}$ subject to the following relations

$$KEK^{-1} = q^2 E \qquad KFK^{-1} = q^{-2}F \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

along with $KK^{-1} = K^{-1}K = 1$. We associate a structure to this algebra when q is a generic parameter; namely when q is not a root of unity.

Definition 4.2. *Consider the two-sorted language $\mathcal{L}_q = (k, L, \pi, \mathbf{E}, E, F, K^{\pm 1}, q)$ where*

1. *$\pi : L \to k$ and $E, F, K^{\pm 1} : L \to L$ are maps.*

2. *$\mathbf{E} \subseteq L \times k^*$ is a relation.*

3. *q is a constant from the sort k.*

The sorts k and L are equipped with the same language as in condition 3 of Definition 4.1 (with the obvious amendments made to the relations $+$ and $\cdot$). The first-order $\mathcal{L}_q$-theory T_q states the following:

1. *k is an algebraically closed field of characteristic 0.*

2. *$q^n \neq 1$ for every $n \in \mathbb{N}$.*

3. *The map $\pi : L \to k^*$ is surjective and each fiber $\pi^{-1}(x)$ for $x \in k^*$ is a one-dimensional k-vector space.*

4. *For each $x \in k^*$, the set $\mathbf{E}(L, x)$ is non-empty and $\mathbf{E}(L, x) \subseteq \pi^{-1}(x)$.*

5. *Fix an $l \in \mathbb{Z}$, $l > 0$, l even. If $\Gamma[l]$ denotes the group of l-th roots of unity of k, then there is a free and transitive action of $\Gamma[l]$ on $\mathbf{E}(L, x)$ induced by the vector space action on the fiber $\pi^{-1}(x)$.*

6. *The $K^{\pm 1}$ act on each fiber according to the following axiom:*

$$(\forall v \in \pi^{-1})(\overline{x})(Kv = xv \wedge K^{-1}v = x^{-1}v)$$

The maps $K^{\pm 1}$ are linear.

7. *The linear maps E and F move between the fibers according to the following axiom:*

$$\begin{aligned}(\forall v \in \pi^{-1}(x))(\mathbf{E}(v,x) \to \quad & (\exists v' \in \pi^{-1}(q^2x))(\exists y \in k) \\ & (y^2 = x \wedge Ev = \lambda(y)v' \wedge Fv' = -\lambda(qx)v))\end{aligned}$$

where $\lambda : k^* \to k$ is defined by

$$\lambda(y) = \frac{y^{-1}+y}{q-q^{-1}}$$

If $(L,k) \models T_q$ then each fiber $\pi^{-1}(x)$ is an eigenspace for K with eigenvalue x. Each eigenspace contains a finite set of normal basis elements selected by $\mathbf{E}$, and permuted by $\Gamma[l]$.

Proposition 4.3. *Let $(L,k) \models T_q$. Then (L,k) is a representation of $U_q(\mathfrak{sl}_2(k))$.*

Proof. Consider $e \in \pi^{-1}(x)$ such that $\mathbf{E}(e,x)$ holds. Then there are y such that $y^2 = x$ and $e' \in \pi^{-1}(q^2x)$ such that

$$KEe = \lambda(y)Ke' = \lambda(y)q^2xe'$$

But

$$EKe = xEe = x\lambda(y)e'$$

Thus $KEe = q^2EKe$. Similarly $KFe = q^{-2}FKe$. We shall now adopt the more intuitive notation $x^{1/2}$ for the element y such that $Ee = \lambda(y)e'$. Thus

$$Ee = \frac{x^{-1/2}+x^{1/2}}{q-q^{-1}}e'$$

and by the linearity of F,

$$FEe = -\frac{(x^{-1/2}+x^{1/2})(q^{-1}x^{-1/2}+qx^{1/2})}{(q-q^{-1})^2}e$$

Whereas applying F first,

$$Fe = -\frac{x^{-1/2}+x^{1/2}}{q-q^{-1}}e'' \qquad e'' \in \pi^{-1}(q^{-2}x)$$

hence

$$EFe = -\frac{(x^{-1/2}+x^{1/2})(qx^{-1/2}+q^{-1}x^{1/2})}{(q-q^{-1})^2}e$$

After some expansion and rearrangement,

$$(EF-FE)e = \frac{x-x^{-1}}{q-q^{-1}}e$$

as required. □

4.1.3 Quantum torus

Our final example will be a certain multi-parameter quantum torus $\mathcal{O}_{\mathbf{q}}((k^{\times})^n)$ where the parameters $\mathbf{q}$ will depend on some generic q. Recall that this is the k-algebra with generators $\{\mathbf{U}_i^{\pm 1} : 1 \leq i \leq n\}$ subject to the relations

$$\mathbf{U}_i\mathbf{U}_i^{-1} = \mathbf{U}_i^{-1}\mathbf{U}_i = 1 \qquad \mathbf{U}_i\mathbf{U}_j = q_{ij}\mathbf{U}_j\mathbf{U}_i \qquad \text{for } i < j$$

We shall consider the following specific parameters

$$q_{ij} = q^{j-i}$$

which are best visualized as being upper triangular elements of a multiplicatively anti-symmetric $n \times n$ matrix:

$$\begin{pmatrix} * & q & q^2 & \dots & q^{n-1} \\ & * & q & \dots & q^{n-2} \\ \vdots & & \ddots & & q \\ & & & & * \end{pmatrix}$$

The base of our line bundle will parametrize eigenvalues of $\mathbf{U}_1$ and the remaining operators will move between fibers. We eliminate some linguistic preliminaries from the following definition, which should now be clear by referring to Definitions 4.1 and 4.2.

Definition 4.3. *We work with a two-sorted language* $\mathcal{L}_q = (k, L, \pi, q, \mathbf{E}, \mathbf{U}_i^{\pm 1} : 1 \leq i \leq n)$. *The* $\mathcal{L}_q$*-theory* T_q *says the following:*

1. k *is an algebraically closed field of characteristic* 0.
2. $q^n \neq 1$ *for every* $n \in \mathbb{N}$.
3. $\pi : L \to k^*$ *is a surjective map and each fiber* $\pi^{-1}(x)$ *is a one-dimensional vector space for* $x \in k$.
4. *For each* $x \in k$, $\mathbf{E}(L, x)$ *is non-empty and* $\mathbf{E}(L, x) \subseteq \pi^{-1}(x)$.
5. *Fix an integer* l *(not necessarily odd). Then there is a free and transitive action of* $\Gamma[l]$ *on each* $\mathbf{E}(L, x)$ *induced by the vector space structure on* $\pi^{-1}(x)$.
6. $\mathbf{U}_1^{\pm 1}$ *are linear and we have*

$$(\forall v \in \pi^{-1}(x))(\mathbf{U}_1 v = xv \wedge \mathbf{U}_1^{-1} v = x^{-1} v)$$

7. *The linear maps* $\mathbf{U}_2^{\pm 1}, \dots \mathbf{U}_n^{\pm 1}$ *move between fibers according to*

$$(\forall v \in \pi^{-1}(x))(\mathbf{E}(v,x) \to \bigwedge_{i=2}^{n} (\exists v_i \in \pi^{-1}(q^{i-1}x))(\mathbf{E}(v, q^{i-1}x) \wedge \mathbf{U}_i v = x v_i \wedge \mathbf{U}_i^{-1} v_i = x^{-1} v))$$

8. *For each* $i < j$ *with* $i \neq 1$ *we have the following axiom:*

$$(\forall v \in \pi^{-1}(x))(\mathbf{E}(v,x) \to \mathbf{U}_i \mathbf{U}_j v = q^{j-i} \mathbf{U}_j \mathbf{U}_i v)$$

There are some points of difference with the previous examples worth noting. The first is that we have allowed our group $\Gamma[l]$ to be finite and cyclic of any order. The reasons for this are again model-theoretic. The second point is axiom 8 stipulating explicitly some good behaviour of basis elements with respect to the relations satisfied. This good behaviour was actually coded into the definitions of how the operators act between fibers in the previous two examples.

Proposition 4.4. *Let* $(L,k) \models T_q$. *Then* (L,k) *is a representation of* $\mathcal{O}_{\mathbf{q}}((k^{\times})^n)$.

Proof. Let $e \in \pi^{-1}(x)$ for some $x \in k^*$. By the axioms there is a $e_i \in \pi^{-1}(q^{i-1}x)$ such that $\mathbf{U}_i e = x e_i$. Thus

$$\mathbf{U}_1 \mathbf{U}_i e = x \mathbf{U}_1 e_i = x^2 q^{i-1} e_i$$

whereas

$$\mathbf{U}_i \mathbf{U}_1 e = x \mathbf{U}_i e = x^2 e_i$$

But $q^{i-1} = q_{1i}$, thus $\mathbf{U}_1 \mathbf{U}_i e = q_{1i} \mathbf{U}_i \mathbf{U}_1 e$. For $i < j$ and $i \neq 1$, we have $\mathbf{U}_i \mathbf{U}_j e = q^{j-i} \mathbf{U}_j \mathbf{U}_i e$ by definition. □

Remark 4.1. *Note that for* $i < j$ *and* $i \neq 1$, *if* $\mathbf{E}(e,x)$ *holds then*

$$\mathbf{U}_i \mathbf{U}_j e = x \mathbf{U}_i e_j = q^{j-1} x^2 e_{ji}$$

for some $e_j \in \pi^{-1}(q^{j-1}x)$ *and* $e_{ji} \in \pi^{-1}(q^{i+j-2}x)$. *On the other hand*

$$\mathbf{U}_j \mathbf{U}_i e = x \mathbf{U}_j e_i = q^{i-1} x^2 e_{ij}$$

where $e_i \in \pi^{-1}(q^{i-1}x)$ *and* $e_{ij} \in \pi^{-1}(q^{i+j-2}x)$. *Thus the stipulation that* $\mathbf{U}_i \mathbf{U}_j e = q^{j-i} \mathbf{U}_j \mathbf{U}_i e$ *implies that* $e_{ij} = e_{ji}$.

4.2 Equivariant Algebras

We now define the class of equivariant algebras over an algebraically closed field k of characteristic 0. As indicated in the introduction, there are two parts to this definition. Firstly, we have to isolate a suitable class of algebras with an algebraic characterization, a class which we call 'semi-equivariant'. An equivariant algebra is then defined to be a semi-equivariant algebra satisfying some additional (though rather cumbersome) assumptions. The reader is referred to Appendix C for basic definitions and notations concerning Hopf algebras.

Definition 4.4. *A prime k-algebra A is said to be **semi-equivariant** if*

1. *There is a maximal commutative affine k-subalgebra H of A that is a Hopf algebra.*
2. *A is generated as a k-algebra by the generators of H and finitely many eigenvectors $\mathbf{U}_1, \dots \mathbf{U}_n$ of the left adjoint action of H on A; namely the action defined by*

$$h \cdot a = \sum_{(h)} h' a S(h'') \qquad \textit{for all } a \in A \textit{ and } h \in H$$

3. *There are generators $h_1, \dots h_m$ of H such that A has a presentation in terms of the h_i and $\mathbf{U}_j$ and finitely many relations between them. All relations not expressing the adjoint action of h_i on $\mathbf{U}_j$ have the form*

$$c \prod_{k=0}^{p-1} \mathbf{U}_{i_{p-k}} - d \prod_{k=0}^{q-1} \mathbf{U}_{j_{q-k}} = f(h_1, \dots, h_m)$$

where $c, d \in k$ and f is a polynomial over k.

If H and $\mathbf{U}_i$ exist as above, then A is said to be semi-equivariant with respect to H and the elements $\mathbf{U}_1, \dots \mathbf{U}_n$. It should be noted that although one typically defines the adjoint action of a Hopf algebra on itself, the definition makes sense in the current setting, thus turning A into a H-module.

Definition 4.4 takes its inspiration from the basic result that a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$ possesses a Cartan decomposition (see Appendix B). Recall that if $\mathfrak{g}$ is such a Lie algebra then there is a direct sum decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\mathfrak{h}$ is an abelian Lie subalgebra (the Cartan subalgebra) and the $\mathfrak{g}_\alpha$ are eigenspaces of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. Regarding $U(\mathfrak{g})$ (the universal enveloping algebra of $\mathfrak{g}$) as a Hopf algebra, the Lie algebra and Hopf algebra adjoint actions agree. Thus, heuristically, Definition 4.4 says that a semi-equivariant algebra A has a 'generalized Cartan decomposition' where the eigenvectors of the adjoint action satisfy some manageable relations amongst themselves.

4.2.1 Towards equivariant algebras

Let A be a semi-equivariant k-algebra. Because A is prime, H is a domain and by the assumption that H is an affine k-algebra, it is therefore the coordinate ring of an affine variety V. Suppose that $V \subseteq k^m$. Clearly there is a bijective correspondence between points of V and characters (k-algebra homomorphisms) on H, and we denote the character corresponding to $x \in V$ by χ_x. Let L_x be a one-dimensional k-vector space endowed with the structure of a H-module by the character χ_x, i.e.

$$hv = \chi_x(h)v \qquad \text{for all } h \in H \text{ and } v \in L_x$$

The H-modules L_x will form the fibers of a line bundle over V; thus we form the disjoint union

$$L = \coprod_{x \in V} L_x$$

and define the surjective map $\pi : L \to V$ by $\pi(v) = x$ if $v \in L_x$.

Lemma 4.1. *V is a group.*

Proof. This is a consequence of H being a Hopf algebra. First note that if $\chi_x, \chi_y : H \to k$ are the characters corresponding to the points $x, y \in V$ then $\chi_x \otimes \chi_y$ is a character on $H \otimes H$ by

$$(\chi_x \otimes \chi_y)(h_1 \otimes h_2) = \chi_x(h_1)\chi_y(h_2) \qquad h_1, h_2 \in H$$

extended to an algebra homomorphism. Now $\chi = (\chi_x \otimes \chi_y) \circ \Delta$ is a character on H and the kernel of χ is a maximal ideal of H, thus corresponding to a point $z \in V$. This allows us to define a map $\cdot : (y, z) \mapsto z$. The coassociativity of Δ easily implies that $\cdot$ is associative. Similarly, we define $\chi_{x^{-1}} = \chi_x \circ S$ and put x^{-1} as the point in V corresponding to the kernel of $\chi_{x^{-1}}$. □

Because A is semi-equivariant, we have

$$h \cdot \mathbf{U}_i = \chi_i(h)\mathbf{U}_i \qquad \text{for all } h \in H$$

for some characters $\chi_i : H \to k$. In particular, we obtain constants $\alpha_{ji} = \chi_i(h_j)$ such that $h_j \cdot \mathbf{U}_i = \alpha_{ji}\mathbf{U}_i$ for all $1 \leq j \leq m$.

Lemma 4.2. *Let $v \in L_x$. Then*

$$\sum_{(h)} \chi_{x^{-1}}(h_j'')h_j'\mathbf{U}_i v = \alpha_{ji}\mathbf{U}_i v$$

Proof. By $h_j \cdot \mathbf{U}_i = \alpha_{ji}\mathbf{U}_i$ we obtain that

$$\sum_{(h)} h'_j \mathbf{U}_i S(h''_j) v = \alpha_{ji}\mathbf{U}_i v$$

But $S(h''_j)$ acts on v by the scalar $\chi_{x^{-1}}(h''_j)$ and the result follows. □

Corollary 4.1. *Suppose that for each $1 \leq j \leq m$, $\Delta(h_j) = \sum_{(h)} h'_j \otimes h''_j$ is such that*

1. *$h'_j = h_j$ or 1 for every element h'_j in the sum.*
2. *There is at least one h'_j with $h'_j = h_j$.*

Then there are regular maps $r_j, s_j : V \to k$ such that for each $v \in L_x$,

$$h_j \mathbf{U}_i v = \frac{\alpha_{ji} - r_j(x)}{s_j(x)} \mathbf{U}_i v$$

Proof. Immediate from Lemma 4.2. □

Assuming that the coproduct satisfies the assumptions of Corollary 4.1, we have functions $\eta_{ji}(x) = (\alpha_{ji} - r_j(x))/s_j(x)$. Given $x \in V$, put $\eta_i(x) = (\eta_{1i}(x), \ldots, \eta_{mi}(x))$ for each $1 \leq i \leq n$. Then for each i, $\eta_i : V \to V$ because $\eta_i(x)$ defines a character on H for each $x \in V$. We obtain that $\mathbf{U}_i : L_x \to L_{\eta_i(x)}$ for every $x \in V$ and $1 \leq i \leq n$. Let Π be the semigroup generated by the η_i (under composition). Thus Π acts on V in an obvious way. We shall, henceforth, treat the elements of Π as functions on V and adopt the usual convention for composition of functions; namely for $\eta_1, \eta_2 \in \Pi$, $\eta_1\eta_2 = \eta_1 \circ \eta_2$ (apply η_2 first, then η_1).

We can now define the class of equivariant algebras. As stated previously, the definition has the sole purpose of narrowing down the class of semi-equivariant algebras to those which have a suitably geometric first-order definable space which is a representation of A.

Definition 4.5. *We define a semi-equivariant algebra A to be* ***equivariant*** *if*

1. *For each $1 \leq j \leq m$, $\Delta(h_j) = \sum_{(h)} h'_j \otimes h'_j$ is such that*
    - *(a) $h'_j = h_j$ or 1 for each element in the sum.*
    - *(b) $h'_j = h_j$ for at least one h'_j.*
2. *Π is a group such that for each $1 \leq i \leq n$, $\eta_i^{-1} = \eta_j$ for some $j \leq n$.*
3. *V is defined over $\mathbb{Q}$.*

4. *There exist regular functions* $\lambda_i : V \to k$ *and polynomials*

$$P_i(x,y) := y^{n_i} - \mu_i(x) \qquad n_i \in \mathbb{N}, n_i > 0 \qquad \mu_i \in \Pi$$

for each $1 \leq i \leq n$ *such that for each relation of the form*

$$c\prod_{k=0}^{p-1} \mathbf{U}_{i_{p-k}} - d\prod_{k=0}^{q-1} \mathbf{U}_{j_{q-k}} = f(h_1, \dots, h_m) \tag{4.1}$$

we have

   (a) $\eta_{i_p} \dots \eta_{i_1} = \eta_{j_q} \dots \eta_{j_1}$ *and* $\eta_{j_q} \dots \eta_{j_1} = 1$ *if* $f \neq 0$.

   (b)

$$c\prod_{k=0}^{p-1} \lambda_{i_{p-k}}(y_{i_{p-k}}) - d\prod_{k=0}^{q-1} \lambda_{j_{q-k}}(y_{j_{q-k}}) = f(x) \tag{4.2}$$

   holding for every $x \in V$*, where*

      i. y_{i_1} *(respectively* y_{j_1}*) is a solution to the polynomial equation* $P_{i_1}(x, y_{i_1}) = 0$ *(respectively* $P_{j_1}(x, y_{j_1}) = 0$*).*

      ii. $y_{i_{p-k}}$ *(respectively* $y_{j_{q-k}}$*) for* $k < p-1$ *is a solution to the polynomial equation*

$$P_{i_{p-k}}((\prod_{r>k} \eta_{i_{p-r}})(x), y_{i_{p-k}}) = 0 \qquad \left(P_{j_{q-k}}((\prod_{r>k} \eta_{j_{q-r}})(x), y_{j_{q-k}}) = 0\right)$$

Moreover

- *the roots* y_{i_k} *can be chosen compatibly for all conditions of the form* (4.2) *and for all* $x \in V$.
- *for each condition of the form* (4.2)*, the roots* $y_{i_1}, \dots, y_{i_p}$ *(respectively* $y_{j_1}, \dots, y_{j_q}$*) are all related to each other by scalars type-definable over* $\mathbb{Q}$ *(i.e. by equalities of the from* $y_{i_{k_1}} = \alpha y_{i_{k_2}}$*).*

5. *The parameters appearing in all* λ_i, η_i *and* f*, along with the constants* c, d*, are type-definable over* $\mathbb{Q}$.

6. *The maps* λ_i *are* Γ*-linear, where* Γ *is the group of roots of unity of order* l *for some* $l > 0$ *such that* $n_i | l$ *for all* $1 \leq i \leq n$.

The significance of many of these conditions will only become apparent when we commence doing some model theory, although 4 can be presently motivated. If an algebra A is semi-equivariant then the relations of the form (4.1) may not have much to do with the adjoint action of H on A. Condition

4 effectively remedies this. We already know that $\mathbf{U}_i : L_x \to L_{\eta_i(x)}$ for every $x \in V$, $1 \leq i \leq n$. So 4 (combined with 5) states that we can define the action of the $\mathbf{U}_i$, respecting the way they move between fibers, in such a way that all of these relations are satisfied regardless of what fiber L_x we start at. The use of the polynomials P_i and the functions λ_i is to allow a certain amount of definitional flexibility. For two of the preliminary examples considered (the Weyl algebra and $U_q(\mathfrak{sl}_2(k))$), this extra flexibility is necessary. We now define the category $\mathsf{Equiv}(k)$.

Definition 4.6. *Let A and B be two equivariant k-algebras. A k-algebra homomorphism $\varphi : A \to B$ is defined to be* ***equivariant*** *if for any Hopf algebra H such that A is equivariant with respect to H and elements $\mathbf{U}_{11}, \ldots \mathbf{U}_{1n_1}$ of A, there is a Hopf subalgebra H' of B such that*

1. *B is equivariant with respect to H' and $\mathbf{U}_{21}, \ldots, \mathbf{U}_{2n_2} \in B$.*
2. *$\varphi(\mathbf{U}_{1i})$ is a monomial in $\mathbf{U}_{21}, \ldots \mathbf{U}_{2n_2}$ for each i.*
3. *$\varphi|_H : H \to H'$.*

Thus the category $\mathsf{Equiv}(k)$ is defined to consist of equivariant k-algebras and equivariant morphisms.

4.2.2 Application to initial examples

We now show that our initial examples are equivariant algebras.

First Weyl algebra

We consider $A = A_1(k)$. Put $H = k[\mathbf{H}]$ and endow H with the Hopf algebra structure associated to universal enveloping algebras, namely

$$\Delta(\mathbf{H}) = 1 \otimes \mathbf{H} + \mathbf{H} \otimes 1 \qquad \epsilon(\mathbf{H}) = 0 \qquad S(\mathbf{H}) = -\mathbf{H}$$

The variety corresponding to H is the affine line k. For each $x \in k$, we have the one-dimensional H-module L_x given by the character $\chi_x : H \to k$ with $\chi_x(\mathbf{H}) = x$. Of course, the module L_x is merely an x-eigenspace for $\mathbf{H}$. It is easy to see that V is the group $(k, +)$. Indeed

$$\chi_{xy}(\mathbf{H}) = [(\chi_x \otimes \chi_y) \circ \Delta](\mathbf{H}) = \chi_x(1)\chi_y(\mathbf{H}) + \chi_x(\mathbf{H})\chi_y(1) = x + y$$

and

$$\chi_{x^{-1}}(\mathbf{H}) = -\chi_x(\mathbf{H}) = -x$$

For any $a \in A$,

$$\mathbf{H} \cdot a = \mathbf{H}a - a\mathbf{H} = [\mathbf{H}, a]$$

But $[\mathbf{H},\mathbf{a}^\dagger]=\mathbf{a}^\dagger$ and $[\mathbf{H},\mathbf{a}]=-\mathbf{a}$, thus $\mathbf{a}^\dagger$ and $\mathbf{a}$ are eigenvectors for the adjoint action of H on A. Moreover, we have the additional relation

$$\mathbf{a}\mathbf{a}^\dagger-\mathbf{a}^\dagger\mathbf{a}=1$$

which is of the required form. Hence A is semi-equivariant. By inspection, the coproduct has the required form. Now we determine Π. Suppose that $v\in L_x$. By Corollary 4.1 we obtain

$$\mathbf{H}\mathbf{a}^\dagger v=(x+1)\mathbf{a}^\dagger v \qquad \mathbf{H}\mathbf{a}v=(x-1)\mathbf{a}v$$

Thus the semigroup Π is generated by two functions:

$$\eta_\dagger(x)=x+1 \qquad \eta(x)=x-1$$

Hence $\Pi=\mathbb{Z}$ is a group such that the inverse of $\eta_\dagger$ is η (and vice versa). Furthermore, the parameters appearing $\eta_\dagger$ and η are integral. For the relation $[\mathbf{a},\mathbf{a}^\dagger]=1$, we note that for every $x\in k$,

$$(x^{1/2})^2-((x-1)^{1/2})^2=1$$

where $x^{1/2}$ denotes some y such that $y^2=x$. So we put $\lambda(y)=\lambda_\dagger(y)=y$ and

$$P(x,y)=y^2-\eta(x) \qquad P_\dagger(x,y)=y^2-x=P(\eta(x),y)$$

It is then clear that all of the roots y of $P,P_\dagger$ can be chosen compatibly for all $x\in k$ (for any $x\in k$ we just pick, once and for all, any y such that $P(x,y)=0$ and everything works). By doing this, our requirements for the roots being related are automatically satisfied. We can take Γ to be the group of roots of unity of order l for any even l. Trivially, λ_1 and λ_2 are Γ-linear.

$U_q(\mathfrak{sl}_2(k))$

Put $A=U_q(\mathfrak{sl}_2(k))$ and consider $H=k[K,K^{-1}]$. Then $V=k^*$ (which is definable over $\mathbb{Q}$) and we endow H with the group Hopf algebra structure, namely

$$\Delta(K)=K\otimes K \qquad \epsilon(K)=1 \qquad S(K)=K^{-1}$$

with analogous relations for K^{-1}. For $x\in k^*$, L_x is therefore an x-eigenspace for K. Now V is the group $(k^*,\cdot)$. To see this, note that

$$\chi_{xy}=[(\chi_x\otimes\chi_y)\circ\Delta](K)=\chi_x(K)\chi_y(K)=xy$$

and

$$\chi_{x^{-1}}=\chi_x(K^{-1})=x^{-1}$$

Now for any $a \in A$,

$$K \cdot a = KaK^{-1}$$

But $KEK^{-1} = q^2E$ and $KFK^{-1} = q^{-2}F$, thus E and F are eigenvectors of the adjoint action of H on A. We have the additional relation

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

which of the required form, thus giving that A is semi-equivariant. Let $v \in L_x$. Then by Corollary 4.1,

$$KEv = q^2xEv \qquad KFv = q^{-2}xFv$$

Thus Π is generated by the functions

$$\eta_E(x) = q^2x \qquad \eta_F(x) = q^{-2}x$$

hence $\Pi = q^{2\mathbb{Z}} = \{q^l : l \in 2\mathbb{Z}\}$ (also a group) and η_E, η_F are mutually inverse. The parameter q appearing in the definition of η_E and η_F satisfies the type

$$\{x^n \neq 1 : n \in \mathbb{N}, n > 0\}$$

By reference to Proposition 4.3, we take

$$\lambda_E(y) = -\lambda_F(y) = \frac{y^{-1} + y}{q - q^{-1}}$$

and

$$P_E(x, y) = P_F(x, y) = y^2 - x$$

By the calculation performed in Proposition 4.3, we obtain

$$\lambda_E(y_2)\lambda_F(y_1) - \lambda_F(z_2)\lambda_E(z_1) = \frac{x - x^{-1}}{q - q^{-1}}$$

for appropriate y_i, z_i. By contrast with the previous example, not any y_i, z_i will do and we have to be careful about picking them compatibly. For this purpose, we partition k^* into cosets of $q^{2\mathbb{Z}}$:

$$k^* = \bigcup_{x \in \Lambda} q^{2\mathbb{Z}}x$$

where Λ is a set of representatives. Given $x \in \Lambda$, choose any square root y of x. For any other $z \in q^{2\mathbb{Z}}x$, there is $l \in \mathbb{Z}$ such that $z = q^{2l}x$ and we choose the square root q^ly of z. Now repeat this for every coset representative in Λ. The result is a compatible set of square roots which are related to each other as required. Clearly our polynomial P_E satisfies the required conditions involving parameters. Consider $\Gamma = \{\pm 1\}$. Then λ_E is Γ-linear, for

$$\lambda_E(-y) = \frac{-y^{-1} - y}{q - q^{-1}} = -\lambda_E(y)$$

hence so is λ_F.

The multiparameter quantum torus

Put $A = \mathcal{O}_{\mathbf{q}}((k^{\times})^n)$. We take the same Hopf algebra H as for $U_q(\mathfrak{sl}_2(k))$, hence we obtain a line bundle L over the base $(k^*, \cdot)$. Again, for any $a \in A$ we have

$$\mathbf{U}_1 \cdot a = \mathbf{U}_1 a \mathbf{U}_1^{-1}$$

But $\mathbf{U}_1 \mathbf{U}_i \mathbf{U}_1^{-1} = q^{i-1}\mathbf{U}_i$ for $i > 1$. Moreover, the remaining relations are

$$\mathbf{U}_i \mathbf{U}_j - q^{j-i}\mathbf{U}_j \mathbf{U}_i = 0 \qquad i < j$$

giving that A is semi-equivariant. By Corollary 4.1, for $v \in L_x$,

$$\mathbf{U}_1 \mathbf{U}_i v = x q^{i-1} \mathbf{U}_i v$$

Thus we have functions

$$\eta_i(x) = q^{i-1}x \qquad i > 1$$

and Π is generated by these η_i and their inverses (hence Π is a group). Again, the single parameter q appearing in the definition of the η_i is generic, hence satisfies $\{x^n \neq 1 : n \in \mathbb{N}, n > 0\}$. We take $\lambda_i(y) = y$ and $P_i(x, y) = y - x$ for all $i > 1$ and obtain what is required. On this occasion, the P_i are linear and we do not have to worry about roots. Because the λ_i are also linear, they are automatically Γ-linear for any group of roots of unity Γ.

4.2.3 Associating a theory to an equivariant algebra

Let an equivariant algebra A be given. By Definitions 4.4 and 4.5 we have the following data:

1. A maximal commutative affine k-subalgebra H of A which is generated as a k-algebra by $h_1, \ldots, h_m \in H$.
2. Elements $\mathbf{U}_1, \ldots, \mathbf{U}_n$ of A such that A is semi-equivariant with respect to H and these elements.

3. A group Π generated (under composition) by the functions $\eta_1, \ldots, \eta_n : V \to V$ with the property that $\mathbf{U}_i : L_x \to L_{\eta_i x}$ for each i, in the associated 'line bundle' structure. Notably, by condition 5 of Definition 4.5, these functions are type-definable over $\mathbb{Q}$.

4. Relations of the form
$$c \prod_{k=0}^{p-1} \mathbf{U}_{i_{p-k}} - d \prod_{k=0}^{q-1} \mathbf{U}_{j_{q-k}} = f(h_1, \ldots, h_m)$$
satisfying the requirements of condition 4 of Definition 4.5.

We wish to incorporate this data into a first-order theory which can be associated to A. Models of this theory will turn out to be the associated line bundles we have already discussed. First, we establish some notation. Let $\mathbf{i} \in n^{<\omega}$ be a finite sequence of elements from $\{1, \ldots, n\}$. Say $\mathbf{i} = (i_1, \ldots, i_p)$. Put

$$\eta_{\mathbf{i}} = \eta_{i_p} \ldots \eta_{i_1} \qquad \mathbf{U}_{\mathbf{i}} = \mathbf{U}_{i_p} \ldots \mathbf{U}_{i_1}$$

Definition 4.7. *We consider the three-sorted language*

$$\mathcal{L}_A = \{L, V, \Gamma, k, \pi, \mathbf{E}, \mathbf{U}_i, h_j, C : 1 \leq j \leq m, 1 \leq i \leq n)$$

where

1. *C is a finite set of constants.*

2. *$\pi : L \to V$ and $\mathbf{U}_i, h_j : L \to L$ are functions.*

3. *$\mathbf{E} \subseteq L \times V$ is a relation.*

4. *L, V, k are sorts, k has the language of rings and L comes equipped with*
    - *a relation $+ \subseteq L^3$ where $+(v_1, v_2, v_3)$ holds if and only if there exists $x \in V$ such that $v_i \in \pi^{-1}(x)$ for $1 \leq i \leq 3$ and $v_1 + v_2 = v_3$.*
    - *a relation $\cdot \subseteq k \times L^2$ where $\cdot(\lambda, v_1, v_2)$ holds if and only if there is $x \in V$ such that $v_1, v_2 \in \pi^{-1}(x)$ and $\lambda v_1 = v_2$.*

The $\mathcal{L}_A$-theory T_A says the following:

1. *k is an algebraically closed field of characteristic 0.*

2. *$\Sigma_c(c)$ holds for each $c \in C$, where Σ_c is the type that c satisfies.*

3. *$V = \phi(k)$ where ϕ is the formula over $\mathbb{Q}$ defining V.*

4. *$\pi : L \to V$ is a surjective map and each fiber $\pi^{-1}(x)$ is a one-dimensional k-vector space for each $x \in V$.*

5. *$\mathbf{E}(L, x)$ is non-empty for each $x \in V$ and $\mathbf{E}(L, x) \subseteq \pi^{-1}(x)$.*

6. *Let Γ be the group of l-th roots of unity in k, for some l satisfying condition 6 of Definition 4.5. Then Γ acts faithfully and transitively on $\mathbf{E}(L, x)$.*

7. *The operators h_j are linear and act on each fiber according to the following axiom:*
$$(\forall v \in \pi^{-1}(x)) \left(\bigwedge_{j=1}^{m} h_j v = x_j v \right)$$

8. *The linear operators $\mathbf{U}_i$ act according to*
$$(\forall v \in \pi^{-1}(x)) \exists v_i (\exists y_i \in k)(\mathbf{E}(v, x) \to \left(\bigwedge_{i=1}^{n} \mathbf{E}(v_i, \eta_i x) \wedge \mathbf{U}_i v = \lambda_i(y_i) v_i \wedge P_i(y_i, x) = 0 \right)$$
We shall denote the formula $\mathbf{E}(v_i, \eta_i x) \wedge \mathbf{U}_i v = \lambda_i(y_i) v_i \wedge P_i(y_i, x) = 0$ by $\varphi(v, v_i, y_i)$.

9. *All of the basis elements obtained on repeated application of axiom 8, if they should agree (according to the relations not expressing the adjoint action of the h_i on the $\mathbf{U}_j$), do agree. Specifically, enumerate these relations as*
$$c_i \mathbf{U}_{\mathbf{j}_i} - d_i \mathbf{U}_{\mathbf{k}_i} = f_i(h_1, \ldots, h_m) \qquad 1 \leq i \leq r$$
where $\mathbf{j}_i, \mathbf{k}_i \in n^{<\omega}$ for each i. For any $\mathbf{i} = (i_1, \ldots, i_p) \in n^{<\omega}$ we define the formula $\phi_{\mathbf{i}}(e_0, e, y)$ to be
$$\left(\bigwedge_{k=1}^{p} \mathbf{E}(e^k, \eta_{i_k} \pi(e^{k-1})) \wedge \mathbf{U}_{i_k} e^{k-1} = \lambda_{i_k}(y_k) e^k \wedge P_{i_k}(y_k, \pi(e^{k-1})) = 0 \right)$$
where $e^0 = e_0$. Then the following axiom holds:
$$\begin{array}{ll} (\forall v \in \pi^{-1}(x))(\forall_{s=1}^{n} v^{(s)} \in \pi^{-1}(\eta_s x))(& \\ \quad \mathbf{E}(v, x) \wedge \exists a^{(s)} \varphi(v, v^{(s)}, a^{(s)}) \to & \forall_{i=1}^{r} v_i \forall_{i=1}^{r} w_i \forall_{i=1}^{r} y_i \forall_{i=1}^{r} z_i ((\bigwedge_{i=1}^{r} \phi_{\mathbf{j}_i}(v, v_i, y_i) \\ & \wedge \phi_{\mathbf{k}_i}(v, w_i, z_i) \to (\theta_1 \wedge \psi)) \wedge \\ & \forall_t v_t^{(s)} \forall_t w_t^{(s)} \forall_t y_t^{(s)} \forall_t z_t^{(s)} (\bigwedge_{s=1}^{n} \bigwedge_{t=1}^{r} \phi_{\mathbf{j}_t}(v^{(s)}, v_t^{(s)}, y_t^{(s)}) \\ & \wedge \phi_{\mathbf{k}_t}(v^{(s)}, w_t^{(s)}, z_t^{(s)}) \to \theta_2))) \end{array}$$
where
    - *$\psi(y_i, z_i)$ is a conjunction of linear conditions which isolates the type $\mathrm{tp}^k(y_i, z_i/\mathbb{Q} \cup C)$ formulated in the language of the sort k, for any instantiation of such y_i, z_i.*
    - *θ_1 is the conjunction (over i, j, k, m) of*
$$\bigwedge_{\pi(v)=\pi(v_i^k)} v = v_i^k \wedge \bigwedge_{\pi(v)=\pi(w_i^k)} v = w_i^k \wedge \bigwedge_{\pi(v_i^k)=\pi(w_j^m)} v_i^k = w_j^m$$

and θ_2 is the conjunction (over i, j, k, m, u) of

$$\begin{array}{l}
\bigwedge_{\pi(v^{(s)})=\pi(v)} v^{(s)} = v \wedge \bigwedge_{\pi(v^{(s)})=\pi(v_i^k)} v^{(s)} = v_i^k \wedge \bigwedge_{\pi(v^{(s)})=\pi(w_i^k)} v^{(s)} = w_i^k \wedge \\
\bigwedge_{\pi(v_t^{(s)j})=\pi(v_i^k)} v_t^{(s)j} = v_i^k \wedge \bigwedge_{\pi(v_t^{(s)j})=\pi(w_i^k)} v_t^{(s)j} = w_i^k \wedge \bigwedge_{\pi(w_t^{(s)j})=\pi(w_i^k)} w_t^{(s)j} = w_i^k \wedge \\
\bigwedge_{\pi(w_t^{(s)j})=\pi(v_i^k)} w_t^{(s)j} = v_i^k \wedge \bigwedge_{\pi(v_t^{(s)j})=\pi(w_m^{(u)k})} v_t^{(s)j} = w_m^{(u)k}
\end{array}$$

Axiom 9 is important: given a defining relation of the form $c\mathbf{U_j} - d\mathbf{U_k} = f(h_1, \ldots, h_m)$, one expects the basis elements involved in defining the action of $\mathbf{U_j}$ and $\mathbf{U_k}$ to eventually coincide in their respective terminal fibers. And if $f \neq 0$ then there should be more; namely that these terminal basis elements coincide with the basis element we start with. Only with such a stipulation is it possible to make sense of expressions like

$$c\mathbf{U_i}e - d\mathbf{U}_j e = f(x)e$$

where $e \in \pi^{-1}(x)$ is a basis element. In this manner, every model of T_A is indeed a representation of A. The formula ψ is a rigidity condition, ensuring that those roots of polynomials P_i which can be related to each other are indeed related to each other. In the initial examples discussed, ψ was implicitly incorporated into the axioms.

Example 4.1. *Recall Definition 4.2 of the theory T_q associated to $U_q(\mathfrak{sl}_2(k))$. The actions of E and F were specified by*

$$\begin{array}{ll}
(\forall v \in \pi^{-1}(x))(\mathbf{E}(v, x) \rightarrow & (\exists v' \in \pi^{-1}(q^2 x))(\exists y \in k) \\
& (y^2 = x \wedge Ev = \lambda(y)v' \wedge Fv' = -\lambda(qx)v))
\end{array}$$

where $\lambda : k^* \rightarrow k$ is defined by

$$\lambda(y) = \frac{y^{-1} + y}{q - q^{-1}}$$

Axiom 9 of Definition 4.7 would state that for every $e \in \pi^{-1}(x)$, $e^{(E)} \in \pi^{-1}(q^2 x))$ and $e^{(F)} \in \pi^{-1}(q^{-2}x)$ which are canonical basis elements, the following holds:

$$\begin{array}{ll}
\forall v \forall w \forall y \forall z(& (Ee = \lambda_E(y_1)v^1 \wedge y_1^2 = x \wedge Fv^1 = \lambda_F(y_2)v^2 \wedge y_2^2 = q^2 x \wedge \\
& Fe = \lambda_F(z_1)w^1 \wedge z_1^2 = x \wedge Ew^1 = \lambda_E(z_2)w^2 \wedge z_2^2 = q^{-2}x \\
& \rightarrow (v^2 = w^2 = e \wedge \psi)) \wedge \\
& \forall v^{(E)} \forall w^{(E)} \forall y^{(E)} \forall z^{(E)} \forall v^{(F)} \forall w^{(F)} \forall y^{(F)} \forall z^{(F)}(\\
& Ee^{(E)} = \lambda_E(y_1^{(E)})v^{(E),1} \wedge (y_1^{(E)})^2 = q^2 x \wedge Fv^{(E),1} = \lambda_F(y_2^{(E)})v^{(E),2} \wedge (y_2^{(E)})^2 = q^4 x \wedge \\
& Fe^{(E)} = \lambda_F(z_1^{(E)})w^{(E),1} \wedge (z_1^{(E)})^2 = q^2 x \wedge Ew^{(E),1} = \lambda_E(z_2^{(E)})w^{(E),2} \wedge (z_2^{(E)})^2 = x \wedge \\
& Ee^{(F)} = \lambda_E(y_1^{(F)})v^{(F),1} \wedge (y_1^{(F)})^2 = q^{-2} x \wedge Fv^{(F),1} = \lambda_F(y_2^{(F)})v^{(F),2} \wedge (y_2^{(F)})^2 = x \wedge \\
& Fe^{(F)} = \lambda_F(z_1^{(F)})w^{(F),1} \wedge (z_1^{(F)})^2 = q^{-2} x \wedge Ew^{(F),1} = \lambda_E(z_2^{(F)})w^{(F),2} \wedge (z_2^{(F)})^2 = q^{-4} x \\
& \rightarrow (e^{(E)} = v^1 \wedge e^{(F)} = w^1 \wedge w^{(E),1} = e \wedge v^{(F),1} = e \wedge w^{(E),2} = v^{(E),2} = e^{(E)} \wedge \\
& w^{(F),2} = v^{(F),2} = e^{(F)})))
\end{array}$$

where ψ is chosen to be the formula

$$y_2 = qy_1 \wedge y_1 = z_1 \wedge z_1 = qz_2$$

After much simplification, this is indeed equivalent to the shorter axiom of Definition 4.2 when combined with axiom 8.

Remark 4.2. *If A is equivariant, we may have some choice of possible λ_i, f and P_i. Nevertheless, in defining T_A a particular choice of these functions and polynomials is fixed once and for all. If one was being pedantic, the dependence of T_A on these functions and polynomials could have been indicated.*

A model of T_A is therefore a three-sorted structure, which shall be denoted by a tuple (L, k). We have suppressed V from the notation because it is evident that V is in fact definable in k. As stated in the introduction, such structures bear a striking resemblance to the G-equivariant line bundles (for G a connected algebraic group over $\mathbb{C}$) found in geometric representation theory (see [RTT07]). The proof of the following result takes its inspiration from the construction of the line bundle $L_\lambda = G \times_B \mathbb{C}_\lambda$, where B is a Borel subgroup of G and $\mathbb{C}_\lambda$ is a one-dimensional representation of B corresponding to the weight λ of the Cartan subgroup H of G. The difference below is that we have $(\Gamma \times_\Gamma k) \times V$ instead. The resulting structure is then equipped with linear operators $\mathbf{U}_i$ between fibers that give us some kind of equivariance, in the sense that $\mathbf{U}_i : L_x \to L_{\eta_i(x)}$ is a linear isomorphism, for each $x \in V$.

Proposition 4.5. *T_A is consistent.*

Proof. We construct a model of T_A. Let k be an algebraically closed field of characteristic 0. Introduce the equivalence relation

$$(\delta_1, a_1, x_1) \sim (\delta_2, a_2, x_2) \Leftrightarrow (\exists \gamma \in \Gamma)(a_2 = \gamma \cdot a_1 \wedge \delta_2 = \gamma^{-1}\delta_1)$$

on $\Gamma \times k \times V$ and consider the quotient $\Gamma \times k \times V/\sim$. We shall denote the equivalence class of (γ, a, x) by $\overline{(\gamma, a, x)}$. Put

$$L_x = \{\overline{(\gamma, a, x)} : \gamma \in \Gamma, a \in k\} \qquad L = \coprod_{x \in V} L_x$$

Then there is a projection map $\pi : L \to V$ given by $\pi(\overline{(\gamma, a, x)}) = x$. When L_x is understood, we suppress x from the notation and write $\overline{(\gamma, a)}$ for $\overline{(\gamma, a, x)}$.

Claim: Each L_x has the structure of a one-dimensional k-vector space by

$$\overline{(\delta_1, a_1)} + \overline{(\delta_2, a_2)} := \overline{(\delta_2, \gamma^{-1}a_1 + a_2)} \text{ where } \delta_2 = \gamma\delta_1$$

$$\lambda\overline{(\delta, a)} := \overline{(\delta, \lambda a)}$$

Proof. Suppose that $(\delta_1, a_1) \sim (\delta_1', a_1')$ and $(\delta_2, a_2) \sim (\delta_2', a_2')$ and that $\delta_2 = \gamma\delta_1$. There are γ_1, γ_2 such that $\delta_1' = \gamma_1\delta_1$ and $\delta_2' = \gamma_2\delta_2$. Thus

$$\delta_2' = \gamma_2\gamma\gamma_1^{-1}\delta_1'$$

So it remains to prove that

$$(\delta_2, \gamma^{-1}a_1 + a_2) \sim (\delta_2', \gamma_1\gamma^{-1}\gamma_2^{-1}a_1' + a_2')$$

But $\gamma_1^{-1}a_1 = a_1'$ and $\gamma_2^{-1}a_2 = a_2'$. So

$$\gamma_1\gamma\gamma_2^{-1}a_1' + a_2' = \gamma_2^{-1}(\gamma^{-1}a_1 + a_2)$$

as required. Scalar multiplication is trivially well-defined and $\overline{(1,1)}$ is a basis element for L_x. $\square$

Normal basis elements are designated as those of the form $\overline{(\gamma, 1)}$ for $\gamma \in \Gamma$ and it is clear that Γ acts faithfully and transitively on the set of normal basis elements of L_x. We now define maps by

$$\mathbf{U}_i\overline{(1,1,x)} = \lambda_i(y_i)\overline{(1,1,\eta_i(x))} \qquad 1 \leq i \leq n$$

and extend linearly. By condition 4 of Definition 4.5, the roots y_i can be chosen compatibly so that all listed relations of the form

$$c\prod_{k=0}^{p-1}\lambda_{i_{p-k}}(y_{i_{p-k}}) - d\prod_{k=0}^{q-1}\lambda_{j_{q-k}}(y_{j_{q-k}}) = f(x)$$

are satisfied for every $x \in V$. So we use these y_i when defining the actions of the $\mathbf{U}_i$. It is now clear that our resulting structure satisfies T_A. $\square$

We conclude this subsection with a remark about the types Σ_c. The theory T_A will only be adequate insofar as each Σ_c contains all of the information that is required of c. The examples considered above contained at most one constant q, and all that was required of q in these cases was that q was generic; namely that it satisfied the type $\Sigma_q = \{x^n \neq 1 : n \in \mathbb{N}, n > 0\}$.

Chapter 5

Model Theory of Equivariant Structures: I

In this chapter and the next, we build on many of the results of [ZS09]. There it was proved that an uncountably categorical first-order theory can be associated to the Heisenberg algebra and a quantifier elimination result (down to the level of existential formulas) was established. Analogous results are proved here for T_A where A is an equivariant algebra with the property that models of T_A are, roughly speaking, 'rigid up to Γ'.

5.1 Categoricity and Quantifier Elimination for T_A

5.1.1 Categoricity

We shall fix an equivariant algebra A. As part of the definition, we have a certain amount of data (the regular functions λ_i, polynomials P_i and f, functions η_i). In the language $\mathcal{L}_A$, all these entities become definable over $\mathbb{Q}$ and the constants C. We recall that each of these constants has its properties fixed by a type Σ_c over the prime subfield. For ease of reference we recall axiom 9 of Definition 4.7:

$$\begin{array}{ll}
(\forall v \in \pi^{-1}(x))(\forall_{s=1}^{n} v^{(s)} \in \pi^{-1}(\eta_s x))(& \\
\quad \mathbf{E}(v,x) \wedge \exists a^{(s)} \varphi(v, v^{(s)}, a^{(s)}) \rightarrow & \forall_{i=1}^{r} v_i \forall_{i=1}^{r} w_i \forall_{i=1}^{r} y_i \forall_{i=1}^{r} z_i ((\bigwedge_{i=1}^{r} \phi_{\mathbf{j}_i}(v, v_i, y_i) \\
& \wedge \phi_{\mathbf{k}_i}(v, w_i, z_i) \rightarrow (\theta_1 \wedge \psi)) \wedge \\
& \forall_t v_t^{(s)} \forall_t w_t^{(s)} \forall_t y_t^{(s)} \forall_t z_t^{(s)} (\bigwedge_{s=1}^{n} \bigwedge_{t=1}^{r} \phi_{\mathbf{j}_t}(v^{(s)}, v_t^{(s)}, y_t^{(s)}) \\
& \wedge \phi_{\mathbf{k}_t}(v^{(s)}, w_t^{(s)}, z_t^{(s)}) \rightarrow \theta_2)))
\end{array}$$

where

- The relations not expressing the adjoint action of H on the $\mathbf{U}_i$ are enumerated as

  $$c_i \mathbf{U}_{\mathbf{j}_i} - d_i \mathbf{U}_{\mathbf{k}_i} = f_i(h_1, \ldots, h_m) \qquad 1 \leq i \leq r$$

  where $\mathbf{j}_i, \mathbf{k}_i \in n^{<\omega}$ for each i.

- For any $\mathbf{i} = (i_1, \ldots, i_p) \in n^{<\omega}$ we define the formula $\phi_{\mathbf{i}}(e_0, e, y)$ to be
$$\left(\bigwedge_{k=1}^{p} \mathbf{E}(e^k, \eta_{i_k}\pi(e^{k-1})) \wedge \mathbf{U}_{i_k} e^{k-1} = \lambda_{i_k}(y_k)e^k \wedge P_{i_k}(y_k, \pi(e^{k-1})) = 0\right)$$
where $e^0 = e_0$.

- $\psi(y_i, z_i)$ is a conjunction of linear conditions which isolates the type $\mathrm{tp}^k(y_i, z_i/\mathbb{Q} \cup C)$ formulated in the language of the sort k, for any instantiation of such y_i, z_i.

- θ_1 and θ_2 combined express that those basis elements which should agree (i.e. those which lie in the same fibers), do agree.

In order for T_A to be categorical in uncountable cardinals, given any model $(L, k) \models T_A$ where k is an uncountable field, one requires 'rigidity up to Γ'. Specifically, axioms 8 and 9 provide us with a set of basis elements (one for each fiber) over the orbit Πx. The requirement is that the truth of axiom 9 should not be affected by shifting these basis elements by certain $\gamma \in \Gamma$. The following technical restriction is designed to achieve this. First some notation. Let Ξ consist of those pairs $(\mathbf{i}, \mathbf{j}) \in (n^{<\omega})^2$ selected by $\theta_1 \wedge \theta_2$ in axiom 9 with $\eta_{\mathbf{i}} = \eta_{\mathbf{j}}$, i.e. for any $x \in V$ and basis element $e \in \pi^{-1}(x)$, $\theta_1 \wedge \theta_2$ says that the basis elements used to define $\mathbf{U}_{\mathbf{i}}e$ and $\mathbf{U}_{\mathbf{j}}e$ lying in $\pi^{-1}(\eta_{\mathbf{i}}x)$ and $\pi^{-1}(\eta_{\mathbf{j}}x)$ respectively, are equal.

Definition 5.1. *The theory T_A is Γ-**rigid** if for every model $(L, k) \models T_A$ and $(\mathbf{i}, \mathbf{j}) \in \Xi$ with $\mathbf{i} = (i_1, \ldots, i_p)$, $\mathbf{j} = (j_1, \ldots, j_q)$ we have that*

$$\gamma^{n_{i_1}} = \delta^{n_{j_1}} = 1 \Rightarrow \gamma^p = \delta^q$$

for every $\gamma, \delta \in \Gamma$.

Proposition 5.1. *Let T_A be Γ-rigid.*

1. *For every polynomial $P_i(x, y) = y^{n_i} - \mu_i(x)$, we have $n_i \leq 2$.*

2. *For every relation of the form $c\mathbf{U}_{\mathbf{i}} - d\mathbf{U}_{\mathbf{j}} = f(h_1, \ldots, h_m)$, if $\mathbf{i} = (i_1, \ldots, i_p)$ and $\mathbf{j} = (j_1, \ldots, j_q)$ then one of the following holds:*

    (a) *$p = q$.*

    (b) *$p = 2q$.*

    (c) *$q = 2p$.*

Proof. 1. A given $\eta_i \in \Pi$ has an inverse η_j and $(i,j) \in \Xi$ by condition 2 of Definition 4.5. Thus for any $\gamma, \delta \in \Gamma$ such that $\gamma^{n_i} = \delta^{n_j} = 1$, it follows by Γ-rigidity (applied with $\eta_i\eta_j = \eta_j\eta_i$ and $\eta_i\eta_j = 1$) that $\gamma^2 = \delta^2 = 1$. In particular, this holds if γ and δ are primitive n_i-th and n_j-th roots of unity respectively. Thus $n_i, n_j \leq 2$.

2. Suppose that $\gamma^{n_{i_1}} = \delta^{n_{j_1}} = 1$ where γ and δ are primitive n_{i_1}-th and n_{j_1}-th roots of unity respectively. Then $\gamma^{qn_{i_1}} = \delta^{qn_{j_1}} = \gamma^{pn_{j_1}} = 1$ (by $\gamma^p = \delta^q$). Thus $1 = \gamma^{pn_{j_1}/q}$, hence $n_{i_1} \leq pn_{j_1}/q$, implying $qn_{i_1} \leq pn_{j_1}$. The reverse inequality follows by a symmetrical argument, hence $qn_{i_1} = pn_{j_1}$. The result is now immediate by 1.

□

Proposition 5.1 gives some indication of the strength of the assumption of Γ-rigidity. The following results show that Γ-rigidity is equivalent to uncountable categoricity.

Proposition 5.2. *Suppose that $(L,k) \models T_A$ witnesses the failure of Γ-rigidity, where k is uncountable. Then there is an automorphism σ of k which does not extend to an automorphism of (L,k).*

Proof. There is $(\mathbf{i},\mathbf{j}) \in \Xi$ such that for some basis element $e \in \pi^{-1}(x)$, there are e_1, e_2, y_1, y_2 with

$$(L,k) \models \phi_{\mathbf{i}}(e, e_1, y_1) \wedge \phi_{\mathbf{j}}(e, e_2, y_2)$$

but there are $\gamma, \delta \in \Gamma$ such that $\gamma^{n_{i_1}} = \delta^{n_{j_1}} = 1$, and $\gamma^p \neq \delta^q$ (where $\mathbf{i} = (i_1, \ldots, i_p)$, $\mathbf{j} = (j_1, \ldots, j_q)$). Let y_1', y_2' be tuples obtained by transforming $y_{1i} \mapsto \gamma y_{1i}$ and $y_{2i} \mapsto \delta y_{2i}$. Now ψ implies a formula $\psi_{\mathbf{i},\mathbf{j}}$ which isolates the type $\mathrm{tp}^k(y_1, y_2/\mathbb{Q} \cup C)$; indeed $\psi_{\mathbf{i},\mathbf{j}}$ is just a suitable subformula of ψ. In particular it is a conjunction of linear conditions. Because the y_{1i} (respectively y_{2i}) are all related to each other via $\psi_{\mathbf{i},\mathbf{j}}$, $\psi_{\mathbf{i},\mathbf{j}}$ also holds of y_1', y_2' (this follows by condition 4 of Definition 4.5). Thus $\mathrm{tp}_k(y_1, y_2/\mathbb{Q} \cup C) = \mathrm{tp}_k(y_1', y_2'/\mathbb{Q} \cup C)$. By saturation of k, there is an automorphism σ of k such that $\sigma(y_1, y_2) = (y_1', y_2')$. Suppose for contradiction that σ does extend to an automorphism $\tilde{\sigma}$ of (L,k). Decomposing $\phi_{\mathbf{i}}(e, e_1, y_1)$ we obtain

$$(L,k) \models \mathbf{U_i} e = \prod_{k=1}^{p} \lambda_{i_k}(y_{1k}) e_1^p \Rightarrow (L,k) \models \mathbf{U_i} \tilde{\sigma}(e) = \prod_{k=1}^{p} \lambda_{i_k}(\sigma(y_{1k})) \tilde{\sigma}(e_1^p)$$

Now $\prod_{k=1}^{p} \lambda_{i_k}(\sigma(y_{1k})) = \gamma^p \prod_{k=1}^{p} \lambda_{i_k}(y_{1k})$. But $y_{11}^{n_{i_1}} = \mu_{i_1} x$, hence

$$\mu_{i_1} x = y_{11}^{n_{i_1}} = \gamma^{n_{i_1}} y_{11}^{n_{i_1}} = \mu_{i_1} \sigma(x) \Rightarrow x = \sigma(x)$$

the implication holding because Π is a group. Hence $\tilde{\sigma}(e)$ and e lie in the same fiber and $\tilde{\sigma}(e) = \mu e$ for some $\mu \in k$. Now we apply the same argument to $\phi_{\mathbf{j}}(e, e_2, y_2)$. By axiom 9, $e_1^p = e_2^p$. But then

$$\tilde{\sigma}(e_1^p) = \gamma^{-p} \mu e_1^p = \delta^{-q} \mu e_2^p = \tilde{\sigma}(e_2^p)$$

hence $\gamma^p = \delta^q$, resulting in contradiction. □

Theorem 5.1. *T_A is Γ-rigid if and only if for any models $(L,k),(L',k') \models T_A$ with k,k' uncountable, if $\sigma : k \to k'$ is an isomorphism then σ extends to an isomorphism $\tilde{\sigma} : L \to L'$.*

Proof. Suppose that T_A is Γ-rigid. Because V is defined over the prime subfield, $\sigma : V \to V' = \phi(k')$. Similarly $\sigma : \Gamma \to \Gamma'$ where Γ' is the group of l-th roots of unity in k'. We can assume that σ maps constants to constants (Σ_c are over $\mathbb{Q}$, hence are preserved by σ. Because all of the information we require of a constant is contained in Σ_c, we may as well reinterpret the constants of (L',k') so that they lie in the image of $C(k)$ under σ). Partition V up into orbits of the group Π, thus obtaining

$$V = \bigcup_{x \in \Lambda} \Pi x$$

for some set of representatives Λ of each orbit. It is then clear that we have a corresponding partition for V'

$$V' = \bigcup_{\sigma(x) \in \sigma(\Lambda)} \Pi' \sigma(x)$$

where Π' is the group generated by $\eta'_i = \sigma(\eta_i)$. Define the **length** of y (with respect to the representative x), $l(y)$, to be the length of the smallest sequence $\mathbf{i}$ such that $y = \eta_{\mathbf{i}}(x)$. We extend $\tilde{\sigma}$ to the rest of the orbit Πx by induction on length.

1. $l(y) = 0$, i.e. $y = x$. By axiom 5 of T_A there is $e \in \pi^{-1}(x)$ such that $\mathbf{E}(e,x)$ holds in (L,k). Likewise, there is $e' \in \pi^{-1}(\sigma(x))$ such that $(L',k') \models \mathbf{E}(e',\sigma(x))$ and we define $\tilde{\sigma} : L_x \to L'_{\sigma(x)}$ by mapping $e \mapsto e'$ and extending linearly. By repeated application of axiom 8 we obtain basis elements e_{1i}, e_{2i} and elements y_{1i}, y_{2i} of k such that

$$(L,k) \models \bigwedge_{i=1}^{r} \phi_{\mathbf{j}_i}(e, e_{1i}, y_{1i}) \wedge \phi_{\mathbf{k}_i}(e, e_{2i}, y_{2i})$$

where $\mathbf{j}_i, \mathbf{k}_i$ are such that $c_i \mathbf{U}_{\mathbf{j}_i} - d_i \mathbf{U}_{\mathbf{k}_i} = f(h_1, \ldots, h_m)$. Similarly we obtain basis elements e'_{1i}, e'_{2i} and $y'_{1i}, y'_{2i} \in k'$ such that

$$(L',k') \models \bigwedge_{i=1}^{r} \phi_{\mathbf{j}_i}(e', e'_{1i}, y'_{1i}) \wedge \phi_{\mathbf{k}_i}(e', e'_{2i}, y'_{2i})$$

where ψ' is ψ with all parameters from k transformed to their images under σ. Fix some $1 \leq i \leq r$. Suppose that on decomposing $\phi_{\mathbf{j}_i}(e, e_{1i}, y_{1i})$ we obtain

$$(L,k) \models \bigwedge_{k=1}^{p_i} \mathbf{E}(e_{1i}^{k}, \eta_{j_k} \pi(e_{1i}^{k-1})) \wedge \mathbf{U}_{j_k} e_{1i}^{k-1} = \lambda_{j_k}(y_{1ik}) e_{1i}^{k} \wedge P_{j_k}(y_{1ik}, \pi(e_{1i}^{k-1})) = 0$$

Put $P'_l = \sigma(P_l)$ for each $1 \leq l \leq n$. Note that $\eta'_l(\sigma(x)) = \sigma(\eta_l(x))$ for every such l and $x \in V$. We also have

$$(L', k') \models \bigwedge_{k=1}^{p_i} \mathbf{E}(e_{1i}^{'k}, \eta'_{j_k}\pi(e_{1i}^{'k-1})) \wedge \mathbf{U}_{j_k} e_{1i}^{'k-1} = \lambda'_{j_k}(y'_{1ik})e_{1i}^{'k} \wedge P'_{j_k}(y'_{1ik}, \pi(e_{1i}^{'k-1})) = 0$$

Thus $P'_{j_k}(y'_{1ik}, \pi(e_{1i}^{'k-1})) = P'_{j_k}(\sigma(y_{1ik}), \pi(e_{1i}^{'k-1})) = 0$ for every k. Recalling that $P_{j_k}(y, x) = y^{n_{j_k}} - \mu_{j_k}(x)$ and that n_{j_k} divides the order of Γ, it follows that there are $\gamma_{1ik} \in \Gamma$ such that $y'_{1ik} = \gamma_{1ik}\sigma(y_{1ik})$ for every k. But then by Γ-linearity of the λ_{j_k}, it follows that

$$(L', k') \models \mathbf{U}_{j_k} e_{1i}^{'k-1} = \gamma_{1ik}\lambda'_{j_k}(\sigma(y_{1ik}))e_{1i}^{'k}$$

Thus we define $\tilde{\sigma}$ on the fibers containing the e_{1i}^k by mapping $e_{1i}^k \mapsto \gamma_{1ik}e_{1i}^{'k}$ and extending linearly. Repeat the above for $\phi_{\mathbf{k}_i}(e, e_{2i}, y_{2i})$ and every $1 \leq i \leq r$. By axiom 9, $\psi \in \mathrm{tp}^k(y_{1i}, y_{2i}/\mathbb{Q} \cup C) \Rightarrow \psi' \in \mathrm{tp}^k(\sigma(y_{1i}), \sigma(y_{2i})/\mathbb{Q} \cup C')$. Thus any roots of the various polynomials P_l related by ψ transform by the same element of Γ under σ. Now we invoke Γ-rigidity in (L', k'). Let $(\mathbf{i}, \mathbf{j}) \in \Xi$. Recall from the proof of Proposition 5.2 that there is a conjunction of linear conditions $\psi'_{\mathbf{i},\mathbf{j}}$ implied by ψ' that relates all those roots used to define the action of $\mathbf{U}_{\mathbf{i}}e'$ (respectively $\mathbf{U}_{\mathbf{j}}e'$). Concentrating on $\mathbf{U}_{\mathbf{i}}e'$, for the sake of notational clarity, we assume that these roots are the first p elements of the tuple y'_{1i}. Thus there is γ_{1i} such that $\gamma_{1i} = \gamma_{1ik}$ for $1 \leq k \leq p$ and

$$(L', k') \models \mathbf{U}_{\mathbf{j}_i} e' = \prod_{k=1}^{p} \lambda'_{j_k}(y'_{1ik})e_{1i}^{'p} = \gamma_{1i}^p \prod_{i=1} \lambda'_{j_k}(\sigma(y_{1ik}))e_{1i}^{'p}$$

By Γ-rigidity, $\gamma_{1i}^p = 1$. Because the γ_{1i} get successively absorbed under $\tilde{\sigma}$ we also have that $\mathbf{U}_{\mathbf{j}_i}e' = \prod_{k=1}^p \lambda'_{j_k}(\sigma(y_{1ik}))\tilde{\sigma}(e_{1i}^p)$, hence $\tilde{\sigma}(e_{1i}^p) = e_{1i}^{'p}$. Exactly the same argument applies to $\mathbf{U}_{\mathbf{j}}e'$, hence applying $\tilde{\sigma}$ as defined still preserves the truth of axiom 9 in (L', k'), as required.

2. $l(y) > 0$; thus $\tilde{\sigma}$ has been extended to all those L_z for which $l(z) \leq l(y) - 1$. Let $\mathbf{l} = (l_1, \ldots, l_s) \in n^{<\omega}$ witness the length of y. Put $z = \eta_{l_{s-1}} \ldots \eta_{l_1} x$. Then $l(z) \leq l(y) - 1$ and by induction $\tilde{\sigma}$ has already been extended to $\pi^{-1}(z)$. Let $e^{(1)} \in \pi^{-1}(z)$ be the basis element used to define $\mathbf{U}_{l_{s-1}} \ldots \mathbf{U}_{l_1} e$. Now apply axiom 8 to $\pi^{-1}(z)$ with $e^{(1)}$ to obtain $e^{(2)} \in \pi^{-1}(y)$ in terms of which $\mathbf{U}_{l_s} e^{(1)}$ is defined and extend as in the base case. Now repeat the whole argument for the base case with $e^{(2)}$ and the induction step follows.

Repeating the above construction for each orbit completes the construction of $\tilde{\sigma}$. The converse is immediate by Proposition 5.2. □

Proposition 5.3. *The theories associated to* $A_1(k)$*,* $U_q(\mathfrak{sl}_2(k))$ *and* $\mathcal{O}_{\mathbf{q}}((k^\times)^n)$ *are* Γ*-rigid.*

Proof. We start with $A_1(k)$. Recall that for this algebra Π is generated by two functions

$$\eta_\dagger(x) = x + 1 \qquad \eta(x) = x - 1$$

The only relation not expressing the adjoint action of $\mathbf{H}$ is $\mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = 1$, hence $\eta_\dagger\eta = \eta\eta_\dagger$. Now the corresponding polynomials are

$$P(x, y) = y^2 - \eta(x) \qquad P_\dagger(x, y) = y^2 - x$$

so we must demonstrate that for any $\gamma, \delta \in \Gamma$, $\gamma^2 = \delta^2 = 1 \Rightarrow \gamma^2 = \delta^2$ which is trivially true! For $U_q(\mathfrak{sl}_2(k))$, Π is generated by

$$\eta_E(x) = q^2 x \qquad \eta_F(x) = q^{-2} x$$

and similarly $\eta_E\eta_F = \eta_F\eta_E$. The corresponding polynomials are

$$P_E(x, y) = P_F(x, y) = y^2 - x$$

so again, Γ-rigidity trivially holds. Γ-rigidity for $\mathcal{O}_{\mathbf{q}}((k^\times)^n)$ is immediate because all of the polynomials involved are $P_i(x, y) = y - x$, which are linear. ☐

5.1.2 Quantifier elimination

From now on, we assume that T_A is Γ-rigid. Firstly, we provide some motivation for the definable sets we wish to consider. Fix a model $(L, k) \models T_A$ and suppose that $v = (v_1, \ldots, v_s)$ is a tuple from the sort L. We can re-index the v_i according to the fibers of π in which they appear. Namely, we fix an enumeration $\{v_{ij} : 1 \leq i \leq t; 1 \leq j \leq s_i, \sum_i s_i = s\}$ so that given v_{ij}, v_{kl}, we have $i = k$ if and only if $\pi(v_{ij}) = \pi(v_{kl})$. By the axioms, we can find m-tuples $a_i \in V \subseteq k^m$ such that $\pi(v_{ij}) = a_i$ for every $1 \leq i \leq t$. Moreover, because each $\pi^{-1}(a_i)$ is one-dimensional and we have basis elements $e_i \in \mathbf{E}(L, a_i)$, we can find scalars $\lambda_{ij} \in k$ such that

$$\bigwedge_{i=1}^{t} \bigwedge_{j=1}^{s_i} \lambda_{ij} e_i = v_{ij}$$

holds in (L, k). Thus one expects all the sentences satisfied by v to be determined by all the interrelationships between the e_i. But the relationships between the e_i depend on the orbits of Π on V. We set up some notation to describe these relationships. Suppose that e_i and e_j lie in the same orbit of Π. Then there is a 'path' in the structure connecting the fiber containing e_i to the fiber containing e_j, i.e. there is $\mathbf{i} \in n^{<\omega}$ such that $\mathbf{U}_{\mathbf{i}} e_i \in \pi^{-1}(\pi(e_j))$. We wish to construct an existential sentence θ_{ij} that codes this path. Writing e_i^0 for e_i, our candidate for θ_{ij} is the following:

$$\exists \gamma_{ij} \exists_{k=1}^{p} b_{ijk} \exists_{k=1}^{p} e_i^k \left(\begin{array}{l} \bigwedge_{k=1}^{p} \mathbf{E}(e_i^k, \eta_{i_k} \pi(e_i^{k-1})) \wedge \bigwedge_{k=1}^{p} \mathbf{U}_{i_k} e_i^{k-1} = \lambda_{i_k}(b_{ijk}) e_i^k \\ \wedge P_{i_k}(b_{ijk}, \pi(e_i^{k-1})) \wedge e_j = \gamma_{ij} e_i^p \end{array} \right)$$

This sentence is, of course, satisfied in (L, k). We now have enough information to construct a class of formulas with which to prove quantifier elimination.

Definition 5.2. *Let $\{v_{ij} : 1 \leq i \leq t; 1 \leq j \leq s_i, \sum_i s_i = s\}$ and $x = (x_1, \ldots x_r)$ be tuples of variables from the sorts L and k respectively. A **core formula** with variables (v, x) is defined to be a formula of the following shape:*

$$\exists \lambda \exists_{i=1}^{t} e_i \exists_{i=1}^{t} y_i \exists \gamma \exists b \left(\begin{array}{l} \bigwedge_{i=1}^{t} \bigwedge_{j=1}^{s_i} \pi(v_{ij}) = y_i \wedge \lambda_{ij} e_i = v_{ij} \wedge \mathbf{E}(e_i, y_i) \wedge \bigwedge_{(i,j) \in \Theta} \phi_{ij}(e_i, e_j, b_{ij}, \gamma_{ij}) \\ \wedge S(\lambda, y, \gamma, b, x) \end{array} \right)$$

where

1. *Θ is a subset of $\{(i, j) : 1 \leq i, j \leq t\}$.*
2. *S defines a Zariski constructible subset of $k^{r_1} \times V^t \times \Gamma^{r_2}$ where*
   - *(a) $r_1 = l(x) + l(b) + s$ (l denotes length)*
   - *(b) $r_2 = l(\gamma)$*
3. *ϕ_{ij} is θ_{ij} with the existential quantification over γ_{ij}, b_{ik} removed.*

*A **core type** is defined to be a consistent collection of core formulas. If (v, a) is a tuple of elements from $L^s \times k^r$, the **core type of** (v, a) (denoted $\operatorname{ctp}(v, a)$) is defined to be the set of all core formulas satisfied by (v, a).*

We now wish to demonstrate that the type of a tuple is determined by its core type.

Proposition 5.4. *Let $(L, k) \models T_A$ be $\aleph_0$-saturated. Suppose that $(v, c), (w, d)$ are both tuples from $L^s \times k^r$ with the property that $\operatorname{ctp}(v, c) = \operatorname{ctp}(w, d)$. Then $\operatorname{tp}(v, c) = \operatorname{tp}(w, d)$.*

Proof. We shall construct an automorphism $\tilde{\sigma}$ of (L, k) with the property that $\tilde{\sigma} : (v, c) \mapsto (w, d)$. Re-index the tuple v as $\{v_{ij} : 1 \leq i \leq t; 1 \leq j \leq s_i, \sum_i s_i = s\}$ so that given v_{ij}, v_{kl}, we have $i = k$ if and only if $\pi(v_{ij}) = \pi(v_{kl})$. By what has already been discussed, the axioms provide us with:

1. Tuples a_i^1 such that $\pi(v_{ij}) = a_i^1$ for every $1 \leq i \leq t$.

2. Basis elements $e_i^1 \in \mathbf{E}(L, a_i)$ and scalars λ_{ij}^1 such that

$$\bigwedge_{i=1}^{t} \bigwedge_{j=1}^{s_i} \lambda_{ij}^1 e_i^1 = v_{ij}$$

   holds.

Now we construct the set Θ so that $(i, j) \in \Theta$ if and only if there is a path from $\pi^{-1}(a_i^1)$ to $\pi^{-1}(a_j^1)$; namely there is a sequence $\mathbf{i} \in n^{<\omega}$ such that $\mathbf{U_i} e_i^1 \in \pi^{-1}(a_j^1)$. Note then that there is a corresponding existential sentence θ_{ij} that codes this path. Thus the following conjunct holds in (L, k):

$$\bigwedge_{i=1}^{t} \bigwedge_{j=1}^{s_i} \pi(v_{ij}) = a_i^1 \wedge \lambda_{ij}^1 e_i^1 = v_{ij} \wedge \mathbf{E}(e_i^1, a_i^1) \wedge \bigwedge_{(i,j) \in \Theta} \phi_{ij}(e_i^1, e_j^1, b_{ij}^1, \gamma_i^1)$$

We shall denote the above formula by $\phi(v, e^1, a^1, \lambda^1, \gamma^1, b^1)$. Consider the following set of formulas:

$$\Sigma = \begin{array}{c} \{\phi(w, e', x', \lambda', \gamma', b') \wedge S(x', \lambda', \gamma', b', d) : \\ (L, k) \models \phi(v, e^1, a^1, \lambda^1, \gamma^1, b^1) \wedge S(a^1, \lambda^1, \gamma^1, b^1, c)\} \end{array}$$

Here the variables have been primed to distinguish them from actual parameters. The S range over all constructible subsets of an appropriate cartesian power of k.

Claim: Σ is consistent.

Proof. We show that Σ is finitely consistent. By definition Σ is closed under finite conjunctions, so let $\phi \wedge S \in \Sigma$. Then

$$(L, k) \models \phi(v, e^1, a^1, \lambda^1, \gamma^1, b^1) \wedge S(a^1, \lambda^1, \gamma^1, b^1, c)$$

Existentially quantifying out $e^1, a^1, \lambda^1, \gamma^1$ and b^1, we obtain a core formula satisfied by (v, c). But $\mathrm{ctp}(v, c) = \mathrm{ctp}(w, d)$, so there are $e^2, a^2, \lambda^2, \gamma^2, b^2$ such that

$$(L, k) \models \phi(w, e^1, a^1, \lambda^1, \gamma^1, b^1) \wedge S(a^1, \lambda^1, \gamma^1, b^1, d)$$

as required. □

By saturation, the type Σ is satisfied by a tuple $(e^2, a^2, \lambda^2, \gamma^2, b^2)$ say. In particular, we have that

$$\mathrm{tp}^k(a^1, \lambda^1, \gamma^1, b^1, c) = \mathrm{tp}^k(a^2, \lambda^2, \gamma^2, b^2, d)$$

and by saturation of k we therefore obtain an isomorphism σ of k such that

$$\sigma : (a^1, \lambda^1, \gamma^1, b^1, c) \mapsto (a^2, \lambda^2, \gamma^2, b^2, d)$$

It remains to extend σ to the whole of (L, k). Partition V up into orbits of Π:

$$V = \coprod_{x \in \Lambda} \Pi x$$

for some set of representatives Λ. Let $x \in \Lambda$ be such that Πx contains $a_{i_1} \dots a_{i_q}$ and no other a_i. By re-indexing if necessary, we can assume that the a_{i_j} are listed in order of increasing length with respect to x. We may also assume that $x = a^1_{i_1}$ by changing the set of representatives if necessary. We carry out an induction on length which is identical to the proof of Theorem 5.1. One only has to note that the construction automatically maps $e^1_{i_j} \mapsto e^2_{i_j}$ for every $1 \leq j \leq q$. Indeed, suppose that there is a path from $\pi^{-1}(a^1_{i_1})$ to $\pi^{-1}(a^1_{i_j})$. Then it is coded in θ_{i_1,i_j}. But $\phi_{i_1,i_j}(e^1_{i_1}, e^1_{i_j}, b^1_{i_1,i_j}, \gamma^1_{i_1,i_j})$ holds, hence so does $\phi_{i_1,i_j}(e^2_{i_1}, e^2_{i_j}, b^2_{i_1,i_j}, \gamma^2_{i_1,i_j})$ by the fact that Σ is realized by $(w, e^2, a^2, \lambda^2, \gamma^2, b^2)$. Thus the following conjunctions hold in (L, k) for $l = 1, 2$:

$$\begin{aligned} &\textstyle\bigwedge_{k=1}^n \mathbf{E}(e^{lk}_{i_1}, \eta'_{j_k} \pi(e^{l,k-1}_{i_1})) \wedge \bigwedge_{k=1}^n \mathbf{U}_{j_k} e^{l,k-1}_{i_1} = \lambda_{j_k}(b^l_{i_1,i_j,k}) e^{lk}_{i_1} \wedge P_{j_k}(b^l_{i_1,i_j,k}, \pi(e^{l,k-1}_{i_1})) \\ &\wedge e^l_{i_j} = \gamma^l_{i_1,i_j} e^{ln}_{i_1} \end{aligned}$$

where $e^{l0}_{i_1} = e^l_{i_1}$.

Claim: $\tilde{\sigma}(e^{1k}_{i_1}) = e^{2k}_{i_1}$ for every $1 \leq k \leq n$.

Proof. This holds by fiat for $k = 0$. In constructing $\tilde{\sigma}$, we will have selected $e' \in \pi^{-1}(\eta_{j_1}(a^2_{i_1}))$ such that

$$\mathbf{U}_{j_1} e^2_{i_1} = \lambda_{j_1}(b^2_{i_1,i_j,1}) e' \wedge P_{j_1}(b^2_{i_1,i_j,1}, a^2_{i_1})$$

holds. But $\sigma(b^1_{i_1,i_j,,1}) = b^2_{i_1,i_j,1}$, hence we have $\tilde{\sigma}(e^{11}_{i_1}) = e'$. But $\mathbf{U}_{j_1} e^2_{i_1} = \lambda_{j_1}(b^2_{i_1,i_j,1}) e^{21}_{i_1}$ by the long conjunct, hence $\tilde{\sigma}(e^{11}_{i_1}) = e^{21}_{i_1}$. Now we repeat the argument till we reach $k = n$. ☐

Because $\sigma(\gamma^1_{i_1,i_j}) = \gamma^2_{i_1,i_j}$, it now follows that $\tilde{\sigma}(e^1_{i_j}) = e^2_{i_j}$ as required. ☐

It follows by compactness that every $\mathcal{L}_A$-formula with parameters from k is equivalent to a boolean combination of core formulas. Some further analysis reveals the structure of subsets of (L, k) defined using parameters from both L and k. Intuitively, these should be determined by a class of formulas similar to core formulas, the only difference being that these formulas can also express information about how bases from the fibers containing these parameters from L are connected to other fibers via paths.

Definition 5.3. *Let e' be a tuple of elements from L with length p such that all e'_i are basis elements. Let $v = (v_1, \dots, v_m)$, $w = (w_1, \dots, w_n)$ be tuples of variables from L. A* ***general core formula***

with variables (v, w, x) over e' is defined to be a formula of the following shape:

$$\exists\lambda\exists\mu\exists_{i=1}^{s}e_i\exists_{i=1}^{s}y_i\exists\gamma\exists b\left(\bigwedge_{i=1}^{s}\bigwedge_{j=1}^{s_i}\pi(v_{ij})=y_i\wedge\lambda_{ij}e_i=v_{ij}\wedge\mathbf{E}(e_i,y_i)\wedge\phi\wedge\bigwedge_{(i,j)\in\Theta}\phi_{ij}\wedge S(\lambda,\mu,y,\gamma,b,x)\right)$$

where

1. *$\{v_{ij} : 1 \le i \le s, 1 \le j \le s_i\}$ is an appropriate enumeration of v*

2. *$\Theta \subseteq \{(i,j) : 1 \le i, j \le s\}$*

3. *S is a constructible subset of $k^{r_1} \times V^s \times \Gamma^{r_2}$ where*

   (a) $r_1 = l(x) + l(b) + m + n$

   (b) $r_2 = l(\gamma)$

4. *ϕ is defined to be*

$$\bigwedge_{i=1}^{p}\bigwedge_{j=1}^{p_i}\mu_{ij}e'_i = w_{ij}\wedge\bigwedge_{(i,j)\in\Theta_1}\phi_{ij}(e'_i,e_j,b_i,\gamma_{ij})\wedge\bigwedge_{(i,j)\in\Theta_2}\phi_{ij}(e_i,e'_j,b_i,\gamma_{ij})$$

   where

$$\Theta_1 \subseteq \{(i,j) : 1 \le i \le p, 1 \le j \le s\} \qquad \Theta_2 \subseteq \{(i,j) : 1 \le i \le s, 1 \le j \le p\}$$

   and $\{w_{ij} : 1 \le i \le p, 1 \le j \le p_i\}$ is an appropriate enumeration of w.

We shall denote such a formula by $\exists eS$ and call S the ***Zariski constructible component*** *of $\exists eS$.*

Proposition 5.5. *Let $(L, k) \models T_A$. If ϕ is formula with parameters from L, k then it is equivalent to a boolean combination of general core formulas.*

Proof. Suppose that $\phi(v, x)$ is a formula with free variables (v, x) over a finite set of parameters $w = (w_1, \dots, w_p)$ of L and some unspecified parameters from k. Then $\phi(v, x)$ is equivalent to some $\phi_1(v, w, x)$ where $\phi_1(v, w', x)$ is a formula with free variables (v, w', x) merely over some set of parameters from k. Hence ϕ_1 is equivalent to a boolean combination of core formulas over k by Proposition 5.4. Thus it suffices to prove that a core formula over k with free variables (v, w', x) is equivalent to a boolean combination of general core formulas after substituting w' with the tuple w. It transpires that we have the stronger result that every core formula is equivalent to a finite disjunction of general core formulas after substitution, which we now show.

So let $\varphi(v, w', x)$ be a core formula. We can fix an enumeration $\{v_{ij} : 1 \le i \le s, 1 \le j \le s_i, \sum_i s_i = n\}$ of (v, w') such that

1. n is the length of (v, w').
2. $\pi(v_{ij}) = \pi(v_{kl})$ if and only if $i = k$.
3. Those v_{ij} for which v_{ij} is not in w' for any j are listed first, i.e. there is a maximum $m \leq s$ such that $v_{ij} \notin w'$ for all $i \leq m$.
4. For $i > m$, the w' variables are listed last, i.e. there is a minimum $t_i \leq s_i$ such that $v_{ij} \in w'$ for all $j > t_i$.

Now $\varphi(v, w', x)$ looks like

$$\exists\lambda\exists_{i=1}^{t}e_i\exists_{i=1}^{t}y_i\exists\gamma\exists b \left(\begin{array}{l} \bigwedge_{i=1}^{t}\bigwedge_{j=1}^{s_i} \pi(v_{ij}) = y_i \wedge \lambda_{ij}e_i = v_{ij} \wedge \mathbf{E}(e_i, y_i) \wedge \bigwedge_{(i,j)\in\Theta} \phi_{ij}(e_i, e_j, b_{ij}, \gamma_{ij}) \\ \wedge S(\lambda, y, \gamma, b, x) \end{array} \right)$$

for some $\Theta \subseteq \{(i,j) : 1 \leq i, j \leq s\}$ and S over k. Substitute w for w'. The resulting formula can be simplified by noting that some of the information it expresses is already contained in the theory. If $k > m$, then $y_k = \pi(w_{kl})$ is determined, thus $\exists y_k$ and such conjuncts can be dropped for $k > m$. Moreover, $\exists e_k$ can be dropped by replacing the formula with a finite disjunction, where each disjunct contains e'_k for e_k and e'_k ranges over the finitely many canonical basis elements of $\pi^{-1}(y_k)$. This allows us to make further deletions from each disjunct, namely $\mathbf{E}(e'_k, y_k)$ (which trivially holds) and $\lambda_{kl}e'_k = w_{kl}$ for $l > t_k$ (because λ_{kl} is determined), and we can therefore drop $\exists\lambda_{kl}$. This leaves us with the formula

$$\bigvee_{\substack{e' = (e_{t+1}, \ldots, e_s) \\ e'_k \in \pi^{-1}(\pi(w_{kl})) \wedge \mathbf{E}(e'_k, \pi(w_{kl}))}} \exists\lambda\exists_{i=1}^{m}e_i\exists_{i=1}^{m}y_i\exists\gamma\exists b \left(\begin{array}{l} \bigwedge_{i=1}^{m}\bigwedge_{j=1}^{s_i} \pi(v_{ij}) = y_i \wedge \lambda_{ij}e_i = v_{ij} \\ \wedge\mathbf{E}(e_i, y_i) \wedge \phi' \end{array} \right)$$

for appropriate ϕ' which we now determine. Clearly in

$$\bigwedge_{(i,j)\in\Theta} \phi_{ij}(e_i, e_j, b_{ij}, \gamma_{ij})$$

if we substitute the parameters e'_k for $k > m$ then some conjuncts are eliminable; namely those $\phi_{kl}(e'_k, e'_l, b_{kl}, \gamma_{kl})$ for $k, l > m$ (the theory itself tells us about paths that connect the fibers containing these e'_k, e'_l). Hence the quantifiers $\exists b_{kl}$ and $\exists\gamma_{kl}$ can also be eliminated from each disjunct. Define the sets

$$\Theta_1 = \{(i,j) \in \Theta : 1 \leq i \leq m, m < j \leq s\} \qquad \Theta_2 = \{(i,j) \in \Theta : m < i \leq s, 1 \leq j \leq m\}$$

$$\Phi = \{(i,j) \in \Theta : 1 \leq i, j \leq m\}$$

Then we have ϕ' as the formula

$$\bigwedge_{i=m+1}^{s}\bigwedge_{j=1}^{t_i}\lambda_{ij}e'_i = v_{ij} \wedge \bigwedge_{(i,j)\in\Phi}\phi_{ij} \wedge \bigwedge_{i=1}^{2}\bigwedge_{(i,j)\in\Theta_i}\phi_{ij} \wedge S'(\lambda, y, \gamma, b, x)$$

where S' is S with the determined parameters $\lambda_{kl}, y_k, b_{kl}$ and γ_{kl} substituted for the appropriate variables. Now re-label, putting $\mu_{ij} = \lambda_{i+m,j}$. We see that each disjunct is a general core formula as required. □

5.2 Some Further Analysis of Definable Sets

From now on, we fix an equivariant algebra A, a Γ-rigid theory T_A and model (L, k) of T_A.

5.2.1 Constructibility

Proposition 5.5 suggests taking sets of the form $\exists eC$ (where C defines a closed subset of a cartesian power of k) as giving us the closed subsets of a topology on the sorts of (L, k) and their cartesian powers. As expected, it is possible to prove that all definable subsets are constructible after taking some technicalities (adapted suitably from [Zil06]) into account. Namely, given that basis elements of L may occur as parameters in S for some general core formula $\exists eS$, we require some formalism dealing with how S transforms under applications of Γ to basis elements in the fibers.

Definition 5.4. *Let $\exists eC$ be a general core formula with C giving a closed subset of $k^{r_1} \times V^s \times \Gamma^{r_2}$. We define the **action** of $\delta \in \Gamma^{r_2}$ on C to be*

$$C^\delta = \{(\lambda_{ij}, \mu, y, \gamma, b, a) : (\delta_i^{-1}\lambda_{ij}, \mu, y, \delta\cdot\gamma, b, a) \in C\}$$

where

$$\delta\cdot\gamma = \begin{cases} \delta_i^{-1}\gamma_{ij}\delta_j & (i,j)\in\Theta \\ \gamma_{ij}\delta_j & (i,j)\in\Theta_1 \\ \delta_i^{-1}\gamma_{ij} & (i,j)\in\Theta_2 \end{cases}$$

*C is defined to be Γ**-invariant** if $C^\delta = C$ for every $\delta \in \Gamma^{r_2}$.*

The motivation for this definition is as follows. If a tuple (v, w, a) satisfies $\exists eC$, then there are $\lambda, \mu, e, y, \gamma, b$ such that

$$\bigwedge_{i=1}^{s}\bigwedge_{j=1}^{s_i}\pi(v_{ij}) = y_i \wedge \lambda_{ij}e_i = v_{ij} \wedge \mathbf{E}(e_i, y_i) \wedge \phi \wedge \bigwedge_{(i,j)\in\Theta}\phi_{ij} \wedge C(\lambda, \mu, y, \gamma, b, a)$$

holds. Put $e'_i = \delta_i e_i$ for every i. Then one can see that (v, w, a) satisfies

$$\bigwedge_{i=1}^{s}\bigwedge_{j=1}^{s_i}\pi(v_{ij}) = y_i \wedge \lambda'_{ij}e_i = v_{ij} \wedge \mathbf{E}(e_i, y_i) \wedge \phi \wedge \bigwedge_{(i,j)\in\Theta}\phi_{ij} \wedge C(\lambda', \mu, y, \delta\cdot\gamma, b, a)$$

where $\lambda'_{ij} = \delta_i^{-1}\lambda_{ij}$ and $\delta \cdot \gamma$ is defined as above, only if $C^\delta(\lambda, \mu, y, \gamma, b, a)$ holds. In particular, if C is Γ-invariant then for any $\delta \in \Gamma^{r_2}$ a base change of this kind can be carried out without affecting validity.

Lemma 5.1. *Let* $\exists e C_1, \exists e C_2$ *be general core formulas with the same enumeration of variables. Let* C_1, C_2 *be Zariski closed and suppose that* C_2 Γ*-invariant. Then*

1. $(L, k) \models \exists e(C_1 \wedge C_2) \leftrightarrow \exists e C_1 \wedge \exists e C_2$
2. $(L, k) \models \exists e(\neg C_2) \leftrightarrow \neg \exists e C_2$

Proof.

1. Left to right is trivial. Conversely, if the right-hand side holds for a tuple (v, w, a), then we may obtain different basis elements e and e' as witnesses to $\exists e C_1$ and $\exists e C_2$ respectively. But the Γ-invariance of C_2 means that we can transform e' to e without affecting validity. So the left-hand side holds.
2. Right to left is easy. Conversely, suppose that (v, w, a) satisfies $\exists e(\neg C_2)$ and that e is a tuple of basis elements witnessing this. If some basis elements e' witness $\exists e C_2$ then we can transform e' to e, and using the Γ-invariance of C_2 we get a contradiction.

□

Lemma 5.2. *If* $\exists e S$ *is a general core formula then it is equivalent to a disjunction of general core formulas of the type* $\exists e(C_1 \wedge \neg C_2)$ *where* C_1, C_2 *are Zariski closed and* C_2 *is* Γ*-invariant.*

Proof. More or less the same as [Zil06]. Fix a tuple $a \in k$ and recall that $tp^k(a)$ denotes the type of a in the language of fields. Put

$$\Sigma(a) = \{C_1 \wedge \neg C_2 : (L, k) \models (C_1 \wedge \neg C_2)(a) \text{ and } C_2 \text{ is } \Gamma\text{-invariant}\}$$

Then it suffices to prove (by Propositions 5.4 and 5.5) that $\Sigma(a) \models tp^k(a)$. By quantifier-elimination for k and noting that every constructible subset is a disjunction of conjuncts of the kind $C_1 \wedge \neg C_2$, it remains to prove that C_1, C_2 (where $C_1 \wedge \neg C_2 \in tp^k(a)$) can be replaced with $\tilde{C}_1, \tilde{C}_2$ (respectively) such that $\tilde{C}_2$ is Γ-invariant and $(L, k) \models (\tilde{C}_1 \wedge \neg \tilde{C}_2) \rightarrow (C_1 \wedge \neg C_2)$. Put

$$\tilde{C}_2 = \bigvee_{\delta \in \Gamma^{r_2}} C_2^\delta$$

Evidently $\tilde{C}_2$ is closed, Γ-invariant and $\neg \tilde{C}_2$ implies $\neg C_2$. If $\neg \tilde{C}_2 \in p = tp^k(a)$ then we are done. Otherwise $\neg C_2 \wedge \tilde{C}_2 \in p$. Let Δ be the maximal (hence proper) subset of Γ^{r_2} consisting of those δ

such that

$$\neg D = \bigwedge_{\delta \in \Delta} \neg C_2^\delta \in p$$

Δ is non-empty because $1 \in \Delta$. Put

$$\mathrm{Stab}(\Delta) = \{\delta \in \Gamma^{r_2} : \delta\Delta = \Delta\}$$

If $\delta \notin \mathrm{Stab}(\Delta)$ then by maximality of Δ we have $\neg D^\delta \wedge \neg D \notin p$, hence $D^\delta \in p$. Thus

$$\bigwedge_{\delta \in \Gamma^{r_2} \setminus \mathrm{Stab}(\Delta)} D^\delta \in p$$

Claim: We have

$$(L, k) \models \bigwedge_{\delta \in \Gamma^{r_2} \setminus \mathrm{Stab}(\Delta)} D^\delta \wedge \bigvee_{\delta \in \Gamma^{r_2}} \neg D^\delta \to \bigvee_{\delta \in \mathrm{Stab}(\Delta)} \neg D^\delta$$

Proof. Suppose that $b \in k$ is such that $D^\delta(b)$ holds for every $\delta \in \Gamma^{r_2} \setminus \mathrm{Stab}(\Delta)$ and $\neg D^{\delta_1}(b)$ holds for some $\delta_1 \in \Gamma^{r_2}$. Then $\delta_1 \in \mathrm{Stab}(\Delta)$ and the claim follows. □

The latter disjunct is clearly equivalent to $\neg D$ and $\neg D$ implies $\neg C_2$. So we take

$$\tilde{C}_1 = C_1 \wedge \bigwedge_{\delta \in \Gamma^{r_2} \setminus \mathrm{Stab}(\Delta)} D^\delta$$

and replace $\tilde{C}_2$ with $\bigwedge_{\delta \in \Gamma^{r_2}} D^\delta$. The result now follows. □

Proposition 5.6. *All definable subsets of (L, k) are constructible, namely every definable subset of (L, k) is a boolean combination of those defined by general core formulas $\exists e C$ where C is Zariski closed and Γ-invariant.*

Proof. This is almost immediate by Proposition 5.5 and Lemmas 5.1 and 5.2. One has to note, additionally that if we have a general core formula $\exists e C$ where C is just closed, we can replace C with

$$\tilde{C} = \bigvee_{\delta \in \Gamma^{r_2}} C^\delta$$

to obtain something closed and Γ-invariant. □

Chapter 6

Model Theory of Equivariant Structures: II

We conclude our analysis of models of Γ-rigid T_A by demonstrating that they are Zariski structures. Intuitively, by inspection of general core formulas one expects all the relevant properties to be verified to reduce predictably to the corresponding properties for algebraic varieties. Vaguely speaking, the Zariski constructible components 'dominate' the geometry. We fix an equivariant algebra A, Γ-rigid theory T_A and model $(L, k) \models T_A$.

6.1 Topology on (L, k)

We introduce a topology on $L^n \times k^m$ by taking as a basis of closed subsets those subsets that are defined by general core formulas $\exists e C(v, w, x)$ ((v, w, x) a tuple of variables from $L^n \times k^m$) where C is Zariski closed and Γ-invariant. Closed subsets are given by finite unions and arbitrary intersections of basic closed subsets. Note that if $n = 0$, then these formulas reduce to those of the form $C(x)$ where C defines a Zariski closed subset of k^m. Thus the topology on (L, k) gives us the classical Zariski topology on the sort k and its cartesian powers.

Lemma 6.1. *Let $\exists e C_1$, $\exists e C_2$ be general core formulas defining basic closed subsets and suppose that both formulas have the same enumeration of v variables. Then*

$$(L, k) \models \exists e C_1 \leftrightarrow \exists e C_2 \Rightarrow (L, k) \models C_1 \leftrightarrow C_2$$

Proof. By Lemma 5.1, $\exists e C_1 \wedge \neg \exists e C_2$ is equivalent to $\exists e (C_1 \wedge \neg C_2)$ hence $C_1 \wedge \neg C_2$ must be inconsistent. The rest of the lemma follows by symmetry. □

Although a general core formula $\exists e S$ was defined with respect to two tuples of variables $v = (v_1, \ldots, v_m)$ and $w = (w_1, \ldots, w_n)$, we shall henceforth amalgamate these into one tuple which we enumerate as $\{v_{ij} : 1 \leq i \leq s, 1 \leq j \leq s_i\}$ where there is $t \leq s$ for which $v_{ij} \in w$ for all $i > t$.

Proposition 6.1. *The topology defined on* (L, k) *is Noetherian.*

Proof. Suppose for contradiction that $(\exists e C_i : i \in \mathbb{N})$ defines an infinite descending chain of basic closed subsets, i.e. we have proper inclusions $\exists e C_i(L,k) \supset \exists e C_{i+1}(L,k)$ for every i. Because there are only finitely many ways of enumerating the variables v as $\{v_{ij} : 1 \leq i \leq s, 1 \leq j \leq s_i\}$, there are infinitely many $\exists e C_i$ with the same enumeration. Hence we can assume, without loss of generality, that all $\exists e C_i$ have the same enumeration of v variables. By Lemma 5.1,

$$\exists e C_{i+1}(L,k) = (\exists e C_i \wedge \exists e C_{i+1})(L,k) = \exists e(C_i \wedge C_{i+1})(L,k)$$

By Lemma 6.1 it follows that $C_i(k) \supseteq C_{i+1}(k)$. Because $\exists e C_i(L,k) \supset \exists e C_{i+1}(L,k)$, Lemma 5.1 gives that $\exists e(C_i \wedge \neg C_{i+1})$ is satisfiable. Thus we have proper inclusions $C_i(k) \supset C_{i+1}(k)$ for every i, contradicting that the Zariski topology is Noetherian. $\square$

6.2 Finer Results on Projections and Intersections

We now work towards the proof that (L, k) is a Zariski structure. A quick glance at the axioms to be verified will indicate that we require more detailed results about projections and intersections of subsets defined by general core formulas.

6.2.1 Projections

By the results of the previous chapter, it is immediate that a projection of a constructible set is constructible. For a subset defined by a general core formula, we have more.

Proposition 6.2. *Let* $\exists e S$ *be a general core formula with the aforementioned convention on enumeration of variables. For a fixed* $1 \leq i \leq s$, *let* j *range over a subset* $J \subseteq \{1, \ldots, s_i\}$. *Then* $\exists_{j \in J} v_{ij} \exists e S$ *is a general core formula with Zariski constructible component equivalent to one of the following:*

1. $\exists_{j \in J} \lambda_{ij} S$.

2. $\exists_{j \in J} \mu_{i-t,j} S$

3. $\exists \mu_{i-t,1} \exists_{(i-t,j) \in \Theta_1} b_{i-t,j} \exists_{(i-t,j) \in \Theta_1} \gamma_{i-t,j} \exists_{(k,i-t) \in \Theta_2} b_{k,i-t} \exists_{(k,i-t) \in \Theta_2} \gamma_{i-t,k} S$

4. $\exists y_i \exists_{(i,k) \in \Theta} b_{ik} \exists_{(i,k) \in \Theta} \gamma_{ik} \exists_{(j,i) \in \Theta} b_{ji} \exists_{(j,i) \in \Theta} \gamma_{ji}$
$\exists_{(j-t,i) \in \Theta_1} b_{j-t,i} \exists_{(j-t,i) \in \Theta_1} \gamma_{j-t,i} \exists_{(i,j-t) \in \Theta_2} b_{i,j-t} \exists_{(i,j-t) \in \Theta_2} \gamma_{i,j-t} S$

Proof. The proof divides into four cases:

1. $1 \leq i \leq t$.
2. $t+1 \leq i \leq s$.
3. $t+1 \leq i \leq s$ and $s_i = 1$.
4. $1 \leq i \leq t$ and $s_i = 1$.

We deal with each of these in turn.

1. In this case the v_{ij} do not occur in ϕ and we can eliminate the conjuncts $\lambda_{ij} e_i = v_{ij}$, thus moving the quantifiers $\exists_{j \in J} \lambda_{ij}$ to S.

2. In this case $\exists v_{ij} \exists e S$ is equivalent to

   $$\exists \lambda \ldots \exists b \left(\bigwedge_{i=1}^{t} \bigwedge_{j=1}^{s_i} \pi(v_{ij}) = y_i \wedge \lambda_{ij} e_i = v_{ij} \wedge \mathbf{E}(e_i, y_i) \wedge \exists v_{ij} \phi \wedge \bigwedge_{(i,j) \in \Theta} \phi_{ij} \wedge S(\lambda, \mu, y, \gamma, b, a) \right)$$

   Recall that ϕ is

   $$\begin{array}{l} \bigwedge_{i=t+1}^{p} \bigwedge_{j=1}^{p_i} \mu_{i-t,j} e'_{i-t} = v_{ij} \wedge \bigwedge_{(i-t,j) \in \Theta_1} \phi_{i-t,j}(e'_{i-t}, e_j, b_{i-t,j}, \gamma_{i-t,j}) \wedge \\ \bigwedge_{(i,j-t) \in \Theta_2} \phi_{i-t,j}(e_i, e'_{j-t}, b_{i,j-t}, \gamma_{i,j-t}) \end{array}$$

   Thus $\exists v_{ij} \phi$ is equivalent to ϕ', where the latter is ϕ but with the conjuncts $\mu_{i-t,j} e'_{i-t} = v_{ij}$ removed for $j \in J$. It follows that we can move the quantifiers $\exists_{j \in J} \mu_{i-t,j}$ to S as required.

3. This case is similar to 2, but more is eliminable from ϕ because we can get rid of the parameter e'_i. Hence we can eliminate $\phi_{i-t,k}(e'_{i-t}, e_k, b_{i-t,k}, \gamma_{i-t,k})$ and $\phi_{k,i-t}(e_k, e'_{i-t}, b_{k,i-t}, \gamma_{k,i-t})$. The quantifiers

   $$\exists \mu_{i-t,1} \exists_{(i-t,j) \in \Theta_1} b_{i-t,j} \exists_{(i-t,j) \in \Theta_1} \gamma_{i-t,j} \exists_{(k,i-t) \in \Theta_2} b_{k,i-t} \exists_{(k,i-t) \in \Theta_2} \gamma_{k,i-t}$$

   can then be moved to S.

4. The most is eliminable in this case. We no longer require $\mathbf{E}(e_i, y_i)$ and those conjuncts ϕ_{jk} with $(j,k) \in \Theta$ and j or k equal to i. But we can also eliminate conjuncts from ϕ, namely $\phi_{j-t,i}$ for $(j-t, i) \in \Theta_1$ and $\phi_{i,j-t}$ for $(i, j-t) \in \Theta_2$. Thus we move the quantifiers

   $$\begin{array}{l} \exists y_i \exists_{(i,k) \in \Theta} b_{ik} \exists_{(i,k) \in \Theta} \gamma_{ik} \exists_{(j,i) \in \Theta} b_{ji} \exists_{(j,i) \in \Theta} \gamma_{ji} \\ \exists_{(j-t,i) \in \Theta_1} b_{j-t,i} \exists_{(j-t,i) \in \Theta_1} \gamma_{j-t,i} \exists_{(i,j-t) \in \Theta_2} b_{i,j-t} \exists_{(i,j-t) \in \Theta_2} \gamma_{i,j-t} \end{array}$$

   to S.

□

6.2.2 Intersections

What if two general core formulas defining basic closed subsets of $L^n \times k^m$ each have different enumerations of variables, but we wish to determine the intersection of the subsets they define? In this case, we require a common enumeration of both formulas and Lemma 5.1 should apply, providing that the resulting Zariski constructible components (after re-enumeration) are Γ-invariant.

Lemma 6.2. *Let $\exists e C_1$, $\exists e C_2$ be general core formulas defining basic closed subsets of $L^n \times k^m$. Then $\exists e C_1 \wedge \exists e C_2$ is equivalent to a general core formula $\exists e D$ where D is equivalent to*

$$\bigwedge_{p=1}^{2} C_p(\lambda, \mu, y, \gamma, b, x) \wedge \bigwedge_{\alpha_p(i,1) \sim_{12} \alpha_p(j,1)} y_i = y_j \wedge \bigwedge_{(\alpha_p(i,1), \alpha_p(j,1)) \sim_{12} (\alpha_p(k,1), \alpha_p(l,1))} b_{ij} = b_{kl} \wedge \gamma_{ij} = \gamma_{kl}$$

for some equivalence relation $\sim_{12}$ on $\{1, \ldots, n\}$.

Proof. Suppose that $\exists e C_1$ and $\exists e C_2$ have enumerations

$$\{v_{ij} : 1 \leq i \leq s_1, 1 \leq j \leq s_{1i}\} \qquad \{v_{ij} : 1 \leq i \leq s_2, 1 \leq j \leq s_{2i}\}$$

respectively. Linearly enumerate the v_{ij} as $v = (v_1, \ldots, v_n)$. Thus we obtain bijective maps

$$\alpha_p : \{(i,j) : 1 \leq i \leq s_p, 1 \leq j \leq s_{p_i}\} \to \{1, \ldots, n\}$$

for $p = 1, 2$. Now introduce the equivalence relations $\sim_p$ on $\{1, \ldots, n\}$ by

$$i \sim_p j \leftrightarrow \pi_1(\alpha_p^{-1}(i)) = \pi_1(\alpha_p^{-1}(j))$$

where π_1 is the projection onto the first coordinate. Let $\sim_{12}$ denote the symmetric closure of the composition $\sim_2 \circ \sim_1$ (hence $\sim_{12}$ is an equivalence relation). It is easy to see that each $\sim_p$ refines $\sim_{12}$. Each equivalence class $[i]_{12}$ of $\sim_{12}$ has a canonical representative (take the smallest i in the class) and we let $\Lambda = \{k_1, \ldots, k_s : k_i < k_{i+1} \text{ for all } i < t\}$ be the set of such representatives. Then one can define a map $\tau : \{1, \ldots, n\} \to \{(i,j) : 1 \leq i \leq t, 1 \leq j \leq t_i\}$ such that

1. $\pi_1(\tau(j)) = i$ if and only if $j \sim_{12} k_i$
2. $\pi_2(\tau(j)) < \pi_2(\tau(j'))$ if and only if $j < j'$

where π_2 denotes the projection onto the second coordinate, and this gives us an new enumeration of v.

Claim: For $p = 1, 2$, $\exists e C_p$ is equivalent to

$$\exists \lambda \exists \mu \exists e \exists \gamma \exists b \left(\bigwedge_{i=1}^{t} \bigwedge_{j=1}^{t_i} \pi(v_{ij}) = y_i \wedge \lambda_{ij} e_i = v_{ij} \wedge \mathbf{E}(e_i, y_i) \wedge \phi \bigwedge_{(i,j) \in \Theta} \phi_{ij} \wedge C'_p(\lambda, \mu, y, \gamma, b, x) \right)$$

where C'_p is equivalent to

$$C_p(\lambda,\mu,y,\gamma,b,x)\wedge\bigwedge_{\alpha_p(i,1)\sim_{12}\alpha_p(j,1)} y_i=y_j\wedge\bigwedge_{(\alpha_p(i,1),\alpha_p(j,1))\sim_{12}(\alpha_p(k,1),\alpha_p(l,1))} b_{ij}=b_{kl}\wedge\gamma_{ij}=\gamma_{kl}$$

Proof. First we obtain a formula that is equivalent to $\exists eC_p$ using the linear enumeration of v given by α_p. Define $C_{\alpha_p}(\lambda,\mu,y,\gamma,b,x)$ to be C_p with the variables enumerated as follows:

1. $\lambda_{ij}\mapsto\lambda_{\alpha_p(i,j)}$
2. $y_i\mapsto y_{\alpha_p(i,1)}$
3. $\gamma_{ij}\mapsto\gamma_{\alpha_p(i,1),\alpha_p(j,1)}$
4. $b_{ijk}\mapsto b_{\alpha_p(i,1),\alpha_p(j,1),k}$

Clearly $\exists eC_p$ is then equivalent to

$$\exists_{i=1}^n\lambda\exists\mu\exists_{i=1}^n e_i\exists\gamma\exists b\left(\begin{array}{l}\bigwedge_{i=1}^n\pi(v_i)=y_i\wedge\lambda_ie_i=v_i\wedge\mathbf{E}(e_i,y_i)\wedge\bigwedge_{i\sim_pj}e_i=e_j\wedge\phi\wedge\bigwedge_{(i,j)\in\Theta'}\phi_{ij}\\ \wedge C_{\alpha_p}(\lambda,\mu,y,\gamma,b,x)\wedge\bigwedge_{i\sim_pj}y_i=y_j\wedge\bigwedge_{(i,j)\sim_p(k,l)}\gamma_{ij}=\gamma_{kl}\wedge b_{ij}=b_{kl}\end{array}\right)$$

where $(i,j)\sim_p(k,l)$ is defined to hold if and only if $i\sim_p k$ and $j\sim_p l$; and Θ' is an appropriate subset of $\{(i,j):1\leq i,j\leq n\}$. Now we define $C_{\tau^{-1}\circ\alpha_p}(\lambda,\mu,y,\gamma,b,x)$ to be C_{α_p} but with the following enumeration of variables

1. $\lambda_i\mapsto\lambda_{\tau(i)}$
2. $y_i\mapsto y_{\pi_1(\tau(i))}$
3. $\gamma_{ij}\mapsto\gamma_{\pi_1(\tau(i)),\pi_1(\tau(j))}$
4. $b_{ijk}\mapsto b_{\pi_1(\tau(i)),\pi_1(\tau(j)),k}$

Because $\sim_p$ refines $\sim_{12}$, $i\sim_p j$ implies that $\pi_1(\tau(i))=\pi_1(\tau(j))$. So using the enumeration of v given by τ, we see that the above formula is equivalent to

$$\exists\lambda\exists\mu\exists e\exists\gamma\exists b\left(\bigwedge_{i=1}^{t}\bigwedge_{j=1}^{t_i}\pi(v_{ij})=y_i\wedge\lambda_{ij}e_i=v_{ij}\wedge\mathbf{E}(e_i,y_i)\wedge\phi\wedge\bigwedge_{(i,j)\in\Theta''}\phi_{ij}\wedge C_{\tau^{-1}\circ\alpha_p}(\lambda,\mu,y,\gamma,b,x)\right)$$

where Θ'' is an appropriate subset of $\{(i,j):1\leq i,j\leq t\}$. It is easy to see that $C_{\tau^{-1}\circ\alpha_p}$ is equivalent to C_p with the exception that some of the y_i,b_{ij},γ_{ij} become identified according to $\sim_{12}$, and hence the claim follows. □

Clearly C'_p is also Γ-invariant, hence by Lemma 5.1 we obtain the required result. □

By definition of the topology on (L,k), it follows by Lemma 6.2 that all closed subsets are finite unions of basic closed subsets. Thus if a closed subset $\mathbf{C}$ is irreducible, it is basic closed.

6.3 Zariski Structure

Let $\exists eC$ be a general core formula defining a basic closed subset. By Lemma 6.2, changing the enumeration of the variables can potentially affect C by introducing identifications. Hence if $\exists eC$ is equivalent to $\exists eC'$ where the latter has a different enumeration of variables, it is possible that $\dim C(k) > \dim C'(k)$. Sticking to our philosophy (and corroborative results) that the geometry on k 'dominates' the geometry on (L, k), we wish to define the dimension of $\exists eC(L, k)$ to be $\dim C(k)$ for suitable C. For this purpose we take the general core formula $\exists e\hat{C}$ defining $\exists eC(L, k)$ with the finest enumeration of v variables possible; namely such that if $\exists eC'$ also defines $\exists eC(L, k)$ then $\sim_{\hat{C}}$ refines $\sim_{C'}$ (with $\sim_{\hat{C}}$ and $\sim_{C'}$ defined as in Lemma 6.2). Such an enumeration is clearly possible because there are only finitely many possible enumerations, and we shall call $\exists e\hat{C}$ the **canonical presentation** of $\exists eC$.

Definition 6.1. *Let $\exists eC$ define a closed irreducible subset of $L^n \times k^m$. We define the **dimension** of $\exists eC(L, k)$ to be*

$$\dim \exists eC(L, k) := \dim \hat{C}(k)$$

For $\exists eC(L, k)$ a closed subset, $\dim \exists eC(L, k) := \max\{\mathbf{C}_i\}$ where $\mathbf{C}_i$ are the irreducible components of $\exists eC(L, k)$. If $\exists eS(L, k)$ is constructible, its dimension is defined to be the dimension of its closure.

Lemma 6.3. *Let $\exists e\hat{C}(L, k)$ be basic closed and irreducible. Then $\hat{C} = \bigvee D^\delta$ where D defines some closed irreducible subset of $\hat{C}(k)$.*

Proof. Let D be any irreducible component of $\hat{C}$. Then $\bigvee_{\delta \in \Gamma^{r_2}} D^\delta$ is closed and Γ-invariant. Hence $\exists e \bigvee D^\delta$ defines a basic closed subset. Because $\exists e\hat{C}(L, k)$ is irreducible, it follows that $\exists e\hat{C}(L, k) = \exists e \bigvee D^\delta(L, k)$ and by Lemma 6.1 the result follows. □

Lemma 6.4. *The notion of dimension defined in Definition 6.1 satisfies conditions $1 - 5$ of the definition of a Zariski structure (Definition 1.1).*

Proof. For ease of reference, we restate the conditions to be verified:

1. The dimension of a point is 0.
2. $\dim(\mathbf{P}_1 \cup \mathbf{P}_2) = \max\{\dim \mathbf{P}_1, \dim \mathbf{P}_2\}$ for all projective subsets $\mathbf{P}_1, \mathbf{P}_2$.
3. For $\mathbf{C}$ closed and irreducible in $\mathbf{X}^n$ and $\mathbf{C}_1$ a closed subset of $\mathbf{C}$, if $\mathbf{C}_1 \neq \mathbf{C}$ then $\dim \mathbf{C}_1 < \dim \mathbf{C}$.

4. For $\mathbf{C}$ irreducible and closed in $\mathbf{X}^n$, if $\pi : \mathbf{X}^n \to \mathbf{X}^m$ is a projection then

$$\dim \mathbf{C} = \dim \pi(\mathbf{C}) + \min_{a \in \pi(\mathbf{C})} \dim(\pi^{-1}(a) \cap \mathbf{C})$$

5. For any irreducible closed $\mathbf{C}$ in $\mathbf{X}^n$ and projection map $\pi : \mathbf{X}^n \to \mathbf{X}^m$, there is a subset $\mathbf{V}$ relatively open in $\pi(\mathbf{C})$ such that

$$\min_{a \in \pi(\mathbf{C})} \dim(\pi^{-1}(a) \cap \mathbf{C}) = \dim(\pi^{-1}(v) \cap \mathbf{C})$$

for every $v \in \mathbf{V} \cap \pi(\mathbf{C})$.

The conditions on dimensions of points and dimensions of unions are trivial. Let $\exists e\hat{C}$ define an irreducible closed subset. By Lemma 6.3, $\hat{C} = \bigvee D^\delta$ for some closed irreducible $D(k) \subseteq \hat{C}(k)$. If some $\exists e\hat{C}_1$ defines a proper closed subset of $\exists e\hat{C}(L,k)$ then it has the same enumeration of variables because both are canonically presented. Thus $\hat{C}_1(k) \subset \hat{C}(k)$ by Lemmas 5.1 and 6.1. But then $\dim(\hat{C}_1 \wedge D^\delta)(k) < \dim D^\delta(k) = \dim \hat{C}(k)$ for some δ, hence $\dim \hat{C}_1(k) < \dim \hat{C}(k)$, verifying condition 3.

Now suppose that we have a projection $\pi : L^{n_1+n_2} \times k^{m_1+m_2} \to L^{n_1} \times k^{m_1}$. Then $\pi(\exists e\hat{C}(L,k))$ is defined by $\exists e\hat{C}'$ where $\hat{C}' = \exists z\hat{C}$ for some appropriate $z \in k$. Thus it remains to prove that

$$\dim \hat{C}(k) = \dim \exists z\hat{C}(k) + \min \dim(\hat{C}(a,k))$$

where $\hat{C}(a,k) = \pi^{-1}(a) \cap \hat{C}(k)$. But this is known for algebraic varieties, thus giving us 4. Condition 5 is proved similarly. □

Theorem 6.1. *(L,k) is a Zariski structure.*

Proof. By Lemma 6.4, it remains to establish semi-properness of projection maps, but this is immediate by constructibility. □

Chapter 7

Equivariant Zariski Structures and Beyond

We conclude our investigation of equivariant algebras and their associated structures by constructing a functor nSpec : $\mathsf{Equiv}(k)_{\Gamma}^{op} \to \mathsf{Zar}$, where $\mathsf{Equiv}(k)_{\Gamma}$ is defined to be the full subcategory of $\mathsf{Equiv}(k)$ consisting of those equivariant k-algebras whose associated theories are Γ-rigid. Some additional remarks on equivariant algebras are made. We conclude with a demonstration that both of the nSpec functors obtained in this thesis can be 'glued' together on an appropriate class of k-algebras.

7.1 Functorial Correspondence

As in Chapter 2, given an object A of $\mathsf{Equiv}(k)_{\Gamma}$, we choose a candidate $\mathrm{nSpec}\, A = (L, \mathbb{K})$ where $\mathbb{K}$ is a large algebraically closed field. We demonstrate that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{B}^{op} \arrow[r] \arrow[d, hook] & \mathsf{Zar}^{c} \arrow[d, hook] \\
\mathsf{Equiv}(k)_{\Gamma}^{op} \arrow[r] & \mathsf{Zar}
\end{tikzcd}$$

where B is taken to be the category of commutative affine Hopf k-algebras, because it is anti-equivalent to the category of affine algebraic groups (Appendix C). The reader may wish to review the contents of Appendix C (in particular Definition C.4 and Proposition C.1) before embarking on the following results.

Lemma 7.1. *Let H be a Hopf algebra, A a H-module algebra. Suppose that $\mathbf{A}_1, \mathbf{A}_2 \in A$ are eigenvectors of the action of H on A, i.e. there are characters $\chi_i : H \to k$ such that*

$$h \cdot \mathbf{A}_i = \chi_i(h)\mathbf{A}_i \qquad h \in H \qquad i = 1, 2$$

Then $\mathbf{A}_1\mathbf{A}_2$ is a H-eigenvector with character $\chi(h) = \sum_{(h)} \chi_1(h')\chi_2(h'')$ for every $h \in H$.

Proof. Given $h \in H$ we have

$$h \cdot (\mathbf{A}_1\mathbf{A}_2) = \sum_{(h)} (h' \cdot \mathbf{A}_1)(h'' \cdot \mathbf{A}_2) = \sum_{(h)} \chi_1(h')\chi_2(h'')\mathbf{A}_1\mathbf{A}_2$$

as required. □

Proposition 7.1. *Given a morphism* $\varphi : A \to B$ *in* $\mathsf{Equiv}(k)_\Gamma$*, there is a morphism of Zariski structures* $\mathrm{nSpec}\,\varphi : \mathrm{nSpec}\,B \to \mathrm{nSpec}\,A$.

Proof. Let $\mathrm{nSpec}\,A = (L_A, V_A, \mathbb{K})$, $\mathrm{nSpec}\,B = (L_B, V_B, \mathbb{K})$. Suppose that A is equivariant with respect to the Hopf subalgebra H and elements $\mathbf{U}_{11}, \ldots \mathbf{U}_{1n_1}$ of A. Then there is a Hopf subalgebra H' of B such that

1. B is equivariant with respect to H' and $\mathbf{U}_{21}, \ldots, \mathbf{U}_{2n_2}$.
2. $\varphi(\mathbf{U}_{1i})$ is a monomial in $\mathbf{U}_{21}, \ldots \mathbf{U}_{2n_2}$ for each i.
3. $\varphi|_H : H \to H'$.

By Lemma 7.1, without loss of generality, we can assume that the $\varphi(\mathbf{U}_{1i})$ occur amongst the $\mathbf{U}_{21}, \ldots, \mathbf{U}_{2n_2}$. Indeed, suppose that some $\varphi(\mathbf{U}_{1i})$ does not occur amongst the $\mathbf{U}_{2j}$ for $1 \le j \le n_2$. By Proposition C.1, B is a H'-module algebra under the adjoint representation. By repeated application of Lemma 7.1, $\varphi(\mathbf{U}_{1i})$ is a H'-eigenvector because it is generated by the $\mathbf{U}_{2j}$. Thus we can add it to the set of generators for B. Recall that by the definition of an equivariant algebra (Definition 4.5), if $\Pi' = \langle \eta'_1, \ldots, \eta'_{n_2} \rangle$ is the group associated to B, then for each $1 \le j \le n_2$ there must be a $k \le n_2$ such that $\eta_j^{-1} = \eta_k$. Thus by adding $\varphi(\mathbf{U}_{1i})$ as a generator, we must also add another generator so that this property remains satisfied. This can be done because $\varphi(\mathbf{U}_{1i})$ is a monomial in the $\mathbf{U}_{2j}$.

Now we put $B' = \{\phi(\mathbf{U}_{1i}) : 1 \le i \le n_1\}$. If Π is the group associated with A, we put $\phi(\Pi)$ as the subgroup of Π' generated by those η'_j for which $\mathbf{U}_{2j} \in B'$. Because $\varphi_H : H \to H'$, we have a corresponding morphism of varieties $f : V_B \to V_A$. Partition V_B into orbits of $\phi(\Pi)$:

$$V_B = \bigcup_{x \in \Lambda'} \phi(\Pi)x$$

for some set of representatives Λ'. The map $\mathrm{nSpec}\,\varphi : L_B \to L_A$ is then constructed fiberwise on orbits, using an inductive procedure analogous to the proof of Theorem 5.1. Let $\phi(v, w, x)$ define a basic closed subset of $\operatorname{nSpec} A$; thus ϕ is of the form

$$\exists\lambda\exists\mu\exists e\exists y\exists\gamma\exists b\left(\bigwedge_{i=1}^{s}\bigwedge_{j=1}^{s_i}\pi(v_{ij})=y_i\wedge\lambda_{ij}e_i=v_{ij}\wedge\mathbf{E}(e_i,y_i)\wedge\phi\wedge\bigwedge_{(i,j)\in\Theta}\phi_{ij}\wedge C(\lambda,\mu,y,z,\gamma,b,x)\right)$$

where C is Zariski closed and Γ-invariant.

Claim: Let $\pi_A : L_A \to V_A$ (and $\pi_B : L_B \to V_B$). The preimage of $\phi(\operatorname{nSpec} A)$ is defined by

$$\exists\lambda\exists\mu\exists e\exists e''\exists y\exists\gamma\exists b\left(\begin{array}{c}\bigwedge_{i=1}^{s}\bigwedge_{j=1}^{s_i}\pi_B(v_{ij})=y_i\wedge\lambda_{ij}e_i=v_{ij}\wedge\mathbf{E}(e_i,y_i)\wedge\\ \bigwedge_{i=1}^{p}\bigwedge_{j=1}^{p_i}\pi_B(w_{ij})=z_i\wedge\mu_{ij}e''_i=w_{ij}\wedge\mathbf{E}(e''_i,z'_i)\wedge\\ \bigwedge_{(i,j)\in\Theta'}\phi'_{ij}\wedge C'(\lambda,\mu,y,\gamma,b,x)\end{array}\right)$$

where

1. $\Theta' \subseteq \{1 \leq i,j \leq p+s\}$
2. ϕ'_{ij} is ϕ_{ij} with the $\mathbf{U}_{1i}$ replaced with their images under φ.
3. $C'(\lambda,\mu,y,z,\gamma,b,x)$ holds if and only if
    - $C(\lambda,\mu,f(y),\gamma,b,x)$ holds in $\operatorname{nSpec} A$.
    - $z'=(z_i)\in f^{-1}(z)$ where $z=(\pi_A(e'_i))$.

Proof. Immediate by construction of $\operatorname{nSpec}\varphi$. ☐

Because C' is closed, the formula in the claim defines a closed set in $\operatorname{nSpec} B$, as required. ☐

Whereas the proof of Proposition 2.5 (stating the analogous result for Azumaya structures) involved subtleties regarding projections of variables, the proof of Proposition 7.1 does not. This is because the general core formulas in the equivariant case are ‘finer’; namely one has a lot more choice about what information is and is not expressed linking the various fibers.

Corollary 7.1. *There is a functor* $\operatorname{nSpec} : \mathsf{Equiv}(k)_\Gamma^{op} \to \mathsf{Zar}$ *extending* $\mathsf{B}^{op} \to \mathsf{Zar}^c$.

Proof. Immediate by Proposition 7.1. We note that if H is a commutative affine Hopf k-algebra then the structure corresponding to H is a line bundle $\pi : L \to G$ where G is an affine algebraic group. Each fiber $L_g = \pi^{-1}(g)$ is the G-module given by the character $\chi_g : G \to k$ associated to $g \in G$. It is clear that this structure is definably interpretable in k. ☐

As in the Azumaya case, a certain amount of algebraic structure is recoverable from an abstract theory T which looks like T_A. If T is formulated in the language

$$\mathcal{L} = (L, V, k, \pi, \mathbf{E}, \mathbf{U}_i, h_j, C : 1 \leq j \leq m, 1 \leq i \leq n)$$

and satisfies the axioms of Definition 4.7, let F be the free algebra over k on the generators $h_j, \mathbf{U}_i$. A model $(L, k) \models T$ will be a representation of $A = F/I$ where I is the annihilator of (L, k). By Theorem 5.1, the algebra A is determined up to the cardinality of the uncountable algebraically closed field k. It need not be the case that $T = T_A$ for some equivariant k-algebra A.

Example 7.1. *Let A be the k-algebra with generators $\mathbf{U}, \mathbf{V}^{\pm 1}$ subject to the relation*

$$\mathbf{U}\mathbf{V} = q\mathbf{V}\mathbf{U}$$

where $\mathbf{V}$ is invertible and q is a generic parameter. Then A is not equivariant (it is not even semi-equivariant). Yet there is an abstract theory T satisfying the axioms of Definition 4.7 from which A can be recovered. T is formulated in the language $\mathcal{L} = (L, V, k, \pi, \mathbf{U}, \mathbf{V}^{\pm 1}, q)$, V is the affine line,

$$\eta_{\mathbf{V}}(x) = qx \qquad \eta_{\mathbf{V}^{-1}}(x) = q^{-1}x$$

$\Gamma = \{1\}$ and $\lambda_{\mathbf{V}}(y, x) = \lambda_{\mathbf{V}^{-1}}(y, x) = y - x$. Clearly T is Γ-rigid.

Thus the full subcategory of Zar consisting of large saturated models of theories which satisfy Definition 4.7 will contain Zariski structures which do not lie in the image of nSpec. An equivalence of categories via nSpec does not therefore seem possible. The following conjecture remains open.

Conjecture 7.1. *If A is a non-commutative equivariant algebra then* nSpec A *is a non-classical Zariski structure.*

7.2 Quantized Universal Enveloping Algebras

Despite some important examples falling under the umbrella of equivariant algebras, there are many important algebras which are not equivariant. There is one such collection of algebras that the author firmly had in mind when formulating the mathematics of the previous three chapters; namely the quantized universal enveloping algebras at generic parameter.

Definition 7.1. *Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra. Let n be the rank of $\mathfrak{g}$, $C = (a_{ij})$ the Cartan matrix of $\mathfrak{g}$ with respect to a choice of Cartan subalgebra $\mathfrak{h}$ and simple roots $\alpha_1, \ldots, \alpha_n$. Let $q \in k$ be generic, $q_i = q^{d_i}$. The **quantized enveloping algebra of $\mathfrak{g}$ over k** (denoted $U_q(\mathfrak{g})$) is the k-algebra with generators $E_i, F_i, K_i^{\pm 1}$, $1 \leq i \leq n$ subject to the relations*

1. $K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j \qquad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j$.

2. $K_i K_j = K_j K_i$.

3. $E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{1}}$.

4. *The quantized Serre relations:*

$$\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} E_i^{1-a_{ij}-l} E_j E_i^l = 0 \qquad (i \neq j)$$

$$\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} F_i^{1-a_{ij}-l} F_j F_i^l = 0 \qquad (i \neq j)$$

We refer the reader to Appendix B for appropriate facts and notation concerning semisimple Lie algebras. The presence of the quantized Serre relations (if non-trivial) prevent $U_q(\mathfrak{g})$ from being semi-equivariant, let alone equivariant. Nevertheless, if we discard the quantized Serre relations, we do obtain a semi-equivariant algebra $\tilde{U}_q(\mathfrak{g})$: take $H = k[K_i^{\pm 1} : 1 \leq i \leq n]$ and equip it with the group Hopf algebra structure, namely that given by

$$\Delta(K_i) = K_i \otimes K_i \qquad \epsilon(K_i) = 1 \qquad S(K_i) = K_i^{-1}$$

for each i. Then E_i and F_i are eigenvectors of the adjoint action of H on $\tilde{U}_q(\mathfrak{g})$ by 1 and the remaining relations satisfied by E_i, F_i in 3 are of the required form. It transpires that there is an equivariant algebra $U_q'(\mathfrak{g})$ of which $\tilde{U}_q(\mathfrak{g})$ is an epimorphic image, namely the k-algebra subject to the relations $1, 2$ and

$$[E_i, F_i] = \frac{K_i - K_i^{-1}}{q - q^{-1}} \qquad 1 \leq i \leq n \tag{7.1}$$

Proposition 7.2. *$U_q'(\mathfrak{g})$ is equivariant and its theory is Γ-rigid.*

Proof. This is a straightforward generalization of the corresponding result for $U_q(\mathfrak{sl}_2(k))$. We shall assume for simplicity that $\mathfrak{g}$ is simply laced (i.e. that $d_i = 1$ for all i). Define the vectors $\mathbf{a}_j = (a_{1j}, \ldots, a_{nj})$ for $1 \leq j \leq n$ and put $\mathbf{A} = \mathbb{Z}\langle \mathbf{a}_1, \ldots \mathbf{a}_n \rangle$. Then $\Pi = q^{\mathbf{A}}$ acts on H (by multiplication) and the E_i, F_i move between fibers according to this action on the base. Define

$$\lambda_{E_i}(y) = -\lambda_{F_i}(y) = \frac{y_i + y_i^{-1}}{q - q^{-1}} \qquad y = (y_1, \ldots y_n) \qquad 1 \leq i \leq n$$

$$P_{E_i}(x, y) = P_{F_i}(x, y) = y^2 - x$$

Then these maps and polynomials satisfy the required conditions. As with the $U_q(\mathfrak{sl}_2(k))$ case, we have to be careful about picking the roots and for this purpose, we partition $(k^*)^n$ into cosets of $q^{\mathbf{A}}$:

$$(k^*)^n = \bigcup_{x \in \Lambda} q^{\mathbf{A}} x$$

where Λ is a set of representatives. For $x \in \Lambda$, choose any y such that $y^2 = x$ (i.e. $y = (y_1, \ldots, y_n)$ with $y_i^2 = x_i$ for every i). If $z \in q^{\mathbf{A}}x$ then there is $\mathbf{a} \in \mathbf{A}$ such that $z = q^{\mathbf{a}}x$ and we choose the square root $q^{\mathbf{a}/2}y$ of z. The associated theory is trivially Γ-rigid (see the proof of Proposition 5.3). $\square$

It is unlikely that $\tilde{U}_q(\mathfrak{g})$ itself is equivariant, although the author has been unable to prove this. A calculation using the above λ_{E_i}, λ_{F_i}, P_{E_i} demonstrates that they do not satisfy $E_iF_j - F_jE_i = 0$ for $i \neq j$. It also does not seem possible that a more sophisticated selection of functions and polynomials can rectify this without violating the corresponding relations for (7.1). Nevertheless, that there is an epimorphism $U_q'(\mathfrak{g}) \to \tilde{U}_q(\mathfrak{g})$ suggests (by a possible generalization of Corollary 7.1) that a putative geometric structure corresponding to $\tilde{U}(\mathfrak{g})$ could map to the Zariski structure nSpec $U'(\mathfrak{g})$.

7.3 Total Equivariance

We isolate a particularly nice class of equivariant algebras with the following definition.

Definition 7.2. *An equivariant k-algebra A is* ***totally equivariant*** *if any maximal commutative subalgebra has the structure of a Hopf algebra with respect to which A is equivariant.*

Example 7.2. *Let $\mathcal{O} = \mathcal{O}_q((k^\times)^2)$ be the quantum 2-torus, i.e. the k-algebra with generators $\mathbf{U}^{\pm 1}, \mathbf{V}^{\pm 1}$ subject to the relation*

$$\mathbf{U}\mathbf{V} = q\mathbf{V}\mathbf{U}$$

where q is generic. Then $\mathcal{O}_q$ is totally equivariant.

Proof. The algebra $\mathcal{O}_q$ is equivariant because $\mathcal{O}_{\mathbf{q}}((k^\times)^n)$ is (see Subsection 4.2.2). Now any maximal commutative subalgebra H must be generated by some $c\mathbf{U}^p\mathbf{V}^q$ and its inverse, where $p, q \in \mathbb{Z}$ and $c \in k^\times$. Thus taking the additional generators $\mathbf{V}^{-q}\mathbf{U}^{1-p}$, $\mathbf{V}^{1-q}\mathbf{U}^{-p}$ and their inverses gives the whole of $\mathcal{O}_q$. It is easy to see that these generators are eigenvectors for H under the adjoint action, either directly or by use of Lemma 7.1. $\square$

A totally equivariant algebra A has many associated Zariski structures, each depending on the particular Hopf subalgebra chosen. For those maximal commutative subalgebras which are conjugated by an automorphism of A, there is a corresponding isomorphism of the associated Zariski structures by Corollary 7.1. In Example 7.2, one could consider the maximal commutative subalgebras $k[\mathbf{U}^{\pm 1}]$ and $k[\mathbf{V}^{\pm 1}]$. By total equivariance, there are two corresponding Zariski structures nSpec $\mathcal{O}_q$ and nSpec $\mathcal{O}_q'$ respectively. Let φ be the k-algebra automorphism of $\mathcal{O}_q$ given by

$$\varphi(\mathbf{U}) = \mathbf{V} \qquad \varphi(\mathbf{V}) = \mathbf{U}^{-1}$$

Then φ is an equivariant automorphism and it follows that there is a corresponding Zariski isomorphism $\operatorname{nSpec} \mathcal{O}_\epsilon \simeq \operatorname{nSpec} \mathcal{O}'_\epsilon$.

7.4 Towards Descent

Having obtained two functors $\operatorname{nSpec} : \mathsf{Azum}(k)^{op}_{def} \to \mathsf{Zar}$ and $\operatorname{nSpec} : \mathsf{Equiv}(k)^{op}_{\Gamma} \to \mathsf{Zar}$, we now demonstrate that they can be pasted together on a suitable overlap. Firstly, given an Azumaya k-algebra A and $\mathbf{A} \in A$, it is known that one can define the characteristic polynomial $\chi_{\mathbf{A}}(x)$ of $\mathbf{A}$ ([LB07]). If A is an algebra of PI degree N, $\chi_{\mathbf{A}}$ is of degree N. This notion agrees with the usual notion of characteristic polynomial when A is a matrix algebra. For reasons to do with integral closure, the coefficients of $\chi_{\mathbf{A}}$ all lie in $Z(A)$.

Proposition 7.3. *Let A be a prime k-algebra Azumaya over its center. Suppose that A is generated by $\mathbf{U}_1, \dots, \mathbf{U}_d$ as a k-algebra where*

1. *$\chi_{\mathbf{U}_1}(k)$ is a Galois covering of $V = \operatorname{Spec} Z(A)$ with group Γ.*
2. *There is a maximal commutative subalgebra H of A containing $Z(A)$ and $\mathbf{U}_1$.*
3. *A is equivariant with respect to H and those $\mathbf{U}_i \notin H$.*
4. *$\Pi = \Gamma$.*

Then A is definably irreducibly representable.

Proof. There exist regular functions $a_1, \dots, a_N$ on V such that

$$\chi_{\mathbf{U}_1}(t) = t^N + a_1 t^{N-1} + \cdots + a_N$$

For each $x \in V$, let $\chi_{\mathbf{U}_1}(t, x) = t^N + a_1(x) t^{N-1} + \cdots + a_N(x)$. By 1, $\chi_{\mathbf{U}_1}(t, x)$ is separable in k. Hence there is basis $\{e_1, \dots, e_N\}$ of the unique irreducible module M_x corresponding to $x \in V$ consisting of $\mathbf{U}_1$-eigenvectors, with eigenvalues given by solutions to $\chi_{\mathbf{U}_1}(t, x) = 0$. By re-indexing if necessary, we assume that $\mathbf{U}_i \notin H$ for all $i \geq m > 1$. Those $\mathbf{U}_i$ for which $i < m$ fix each eigenvector of $\mathbf{U}_1$, because they commute with $\mathbf{U}_1$. By conditions 3 and 4, given $\mathbf{U}_i$ with $i \geq m$ we have

$$\mathbf{U}_i e_j = \lambda_i(y_i) e_k \qquad P_i(y_i, x) = 0$$

where x is the eigenvalue of e_j and e_k is the eigenvector with eigenvalue $\eta_i x$. But $P_i(y, x) = y^{n_i} - \mu_i(x)$ for some $\mu_i \in \Pi$, thus each P_i gives a Galois covering of $\operatorname{Spec} H$. Composing these with our original covering $\chi_{\mathbf{U}_1}(k)$ of V gives the required Galois covering of V in terms of which all actions of $\mathbf{U}_i$ are definable. Stipulating that each central element acts by the appropriate scalar in M_x completes the proof. $\square$

For a number of the aforementioned equivariant algebras defined in terms of a generic parameter, if this parameter is replaced by an appropriate primitive l-th root of unity, the resulting algebra will satisfy the assumptions of Proposition 7.3, e.g. $\mathcal{O}_\epsilon((k^\times)^2)$, $U'_\epsilon(\mathfrak{g})$.

Proposition 7.4. *Let A be a k-algebra satisfying the assumptions of Proposition 7.3. Then the images of A under both* nSpec *functors are Zariski isomorphic.*

Proof. Let $\operatorname{nSpec} A = (E, V, \mathbb{K})$ be the associated Azumaya structure, $\operatorname{nSpec} A' = (L, W, \mathbb{K})$ the associated equivariant structure. Suppose that

1. $Z(A)$ is generated by $h_1, \dots, h_m$ over k.
2. H is generated by the h_i and $\mathbf{U}_1, \dots, \mathbf{U}_{m'}$.
3. $\mathbf{V}_1, \dots, \mathbf{V}_n$ are the remaining generators of A, which all are eigenvectors of the adjoint action of H.

Because each character on H restricts to a character on $Z(A)$, we have a projection map $p : W \to V$. Partition W up into orbits of Π:

$$W = \bigcup_{y \in \Lambda} \Pi y$$

where Λ is a set of representatives. Because $\Pi = \Gamma$, it follows that the orbits of Π coincide with the fibers of p. Thus a slight modification of the inductive fiberwise construction in the proof of Theorem 5.1 will suffice to extend p to a map $L \to E$ (just treat each irreducible module in E as a collection of N one-dimensional fibers) which is also seen to be bijective. That we have a Zariski isomorphism is now routine, argued along the lines of Propositions 2.5 and 7.1. $\square$

The proof of Proposition 7.4 indicates, in a sense, that the Azumaya structure of A is a geometric quotient of the associated equivariant structure where both of these exist. The content of Proposition 7.4 is that the two functors nSpec can be patched together on the algebras satisfying the conditions of Proposition 7.3. The construction of additional functors from categories of noncommutative k-algebras to Zar could, perhaps, lead to a theory of descent governing how these functors are glued on intersections.

Chapter 8

Epilogue

After the foregoing investigations, we have arrived at two nSpec functors which afford us something of a limited glimpse at the rich and varied world of Zariski structures. Limited is the operative adjective. Azumaya algebras are about as close to commutative algebras as one can get. On the other hand, equivariant algebras demand a level of symmetry which is hugely restrictive. Despite these criticisms, we hope that what has been achieved counts as progress. Perhaps the strongest vindication of our methods can be found in the final section of Chapter 7; namely that what seemed like two different constructions do find common ground.

Apart from the resolution of outstanding conjectures (2.1, 7.1), any further investigation should have the proof of Conjecture 1.1 as its primary (or close to primary) aim. And perhaps the end result will be a patchwork quilt of different functors to Zar for which an appropriate descent theory exists. The material in Chapters 2 and 3 is somewhat rudimentary, and the author had hoped that Conjecture 2.1 would have been solved by now. Ultimately, what is required is a more sophisticated understanding of what elimination of imaginaries means in an arbitrary abelian category in order that a finer result may be proved using a particular abelian subcategory of ${}_A\mathsf{Mod}$. The remarks at the end of Chapter 3 also air some important issues, notably what should count as a continuously varying family of topological structures and what consequences this could have for intersection theory.

At a more fundamental level, 'categorification' of the theory of Zariski structures would extend the scope of its applicability and bring it more into line with the functor-theoretic methods of [Pres09]. The combined methods of both approaches could find some application in the analysis of Gabriel-Rosenberg spectra of abelian categories ([Ros08]). As far as geometric representation theory is concerned, a natural (albeit challenging) stepping stone would be to attempt a model-theoretic understanding of the results of [BK04]. For a classical flag variety, the equivariant line bundles alluded to in the introduction do have a first-order axiomatization. Let Λ be the set of characters of G, B a fixed Borel subgroup. We consider the language

$$\mathcal{L}_G = (k, V, L_\lambda, \pi_\lambda, \mathbf{U}_g, \sigma_\lambda : \lambda \in \Lambda, g \in G)$$

where $\pi_\lambda : L_\lambda \to V$ and $\mathbf{U}_g : L_\lambda \to L_\lambda$ for each element $g \in G$. The theory T_G says the following:

1. k is an algebraically closed field of characteristic 0.
2. V is the projective variety G/B.
3. For each $\lambda \in \Lambda$, $\pi_\lambda : L_\lambda \to V$ is a surjective map and for each $x \in V$, $L_{\lambda,x} = \pi_\lambda^{-1}(x)$ is a one-dimensional k-vector space.
4. For each λ, σ_λ is a set-theoretic section of L_λ, namely $\pi_\lambda \circ \sigma_\lambda = 1_V$. For each $x \in V$, we put $\mathbf{e}_\lambda(x) = \sigma_\lambda(x)$.
5. The maps $\mathbf{U}_g$ are linear and act G-equivariantly on each L_λ, namely $\mathbf{U}_g(\mathbf{e}_\lambda(x)) = \mathbf{e}_\lambda(gx)$ for each $g \in G$, $x \in V$ and $\lambda \in \Lambda$.

By contrast, for quantized flag varieties adopting a category-theoretic framework seems indispensable, and perhaps is only amenable to that model theory which is suitably categorified.

Appendix A

Some Noncommutative Algebra

All rings are assumed to be associative with unit. The center of a ring R will always be denoted $Z(R)$ unless stated otherwise. If R is a commutative domain, then $Q(R)$ denotes its field of fractions.

A.1 Polynomial identity rings and central simple algebras

We denote the free algebra on generators $x_1, \ldots, x_n$ over $\mathbb{Z}$ by $\mathbb{Z}\langle x_1, \ldots, x_n \rangle$.

Definition A.1. *Let R be a ring. A (noncommutative) polynomial $f \in \mathbb{Z}\langle x_1, \ldots, x_n \rangle$ is a* ***polynomial identity*** *of R if $f(r_1, \ldots, r_n) = 0$ for all $r_i \in R$. A* ***polynomial identity ring*** *(PI ring) is a ring which satisfies some monic polynomial in $\mathbb{Z}\langle x_1, \ldots, x_n \rangle$.*

The **minimal degree** of a PI ring R is the least degree of a monic polynomial identity for R. Trivially all commutative rings are PI rings of minimal degree 2. The following remark gives some elementary properties of PI rings.

Remark A.1.

1. *If R is a PI ring of minimal degree d then the subrings and homomorphic images of R are PI rings of minimal degree at most d.*
2. *If S is finitely generated as a left module over a commutative subring R then S is a PI ring.*

Proof. 1 is immediate by definition. For 2, see [MR01] Corollary 13.4.9. □

Definition A.2. *A* ***central simple algebra*** *over a field k is a finite simple k-algebra A such that $Z(A) = k$.*

By Wedderburn's theorem, $A \simeq M_n(D)$ for a central simple division algebra D. Suppose that A is central simple with $Z(A) = k$ and let l be a field extension of k. Then $A \otimes_k l$ is central simple over l. If $l = \overline{k}$, the algebraic closure of k, then $D \otimes l = \overline{k}$ and in this case $A \otimes l$ is in fact a matrix algebra over l.

Remark A.2. *If A is a central simple algebra then $\dim_{Z(A)}(A)$ is a square.*

Proof. Put $k = Z(A)$. Then $\dim_k(A) = \dim_{\overline{k}}(A \otimes \overline{k}) = \dim_{\overline{k}}(M_n(\overline{k})) = n^2$ for some n. ☐

The following two important theorems link PI rings with central simple algebras.

Theorem A.1 (Kaplansky)**.** *Let R be a primitive PI ring of minimal degree d. Then R is a central simple algebra of dimension $(d/2)^2$ over its center.*

Proof. [MR01] Theorem 13.3.8. ☐

Theorem A.2 (Posner)**.** *Let R be a prime PI ring of minimal degree d with center Z. Put $\Sigma = Z \setminus \{0\}$ and $Q(R) = \Sigma^{-1}R$. Then $Q(R)$ is a central simple algebra over $Q(Z)$ and $\dim_{Q(Z)} Q(R) = (d/2)^2$.*

Proof. [MR01] Theorem 13.6.5. ☐

Posner's theorem can be used to define the **PI degree** of any prime PI ring. Indeed, if R is a prime PI ring of minimal degree d then the PI degree of R, $\deg_{PI}(R)$, is defined to be $d/2$. It is easy to see that if R is a prime PI ring and $\mathfrak{P}$ is a prime ideal of R, then $\deg_{PI}(R/\mathfrak{P}) \leq \deg_{PI}(R)$. There are prime ideals for which equality does hold and such primes are said to be **regular** ([BG02] Lemma III.1.2).

A.2 Affine PI triples

All fields in this section will be assumed to be algebraically closed unless stated otherwise. We shall be concerned with those k-algebras which have large central subalgebras. Such algebras are characterized by the following definition.

Definition A.3. *A tuple (k, Z_0, A) is an* ***affine PI triple*** *if A is an affine (i.e. finitely generated) k-algebra and A is a finitely generated module over some central k-algebra Z_0.*

Lemma A.1 (Artin-Tate)**.** *Let $T \subseteq S \subseteq R$ be rings. Suppose that T is Noetherian, that S is central in R and that R is an affine T-algebra and a finitely generated S-module. Then S is an affine T-algebra.*

Proof. [MR01] Lemma 13.9.10. ☐

The terminology of Definition A.3 is justified as follows. If (k, Z_0, A) is an affine PI triple then Lemma A.1 gives that Z_0 is an affine k-algebra. By the Hilbert basis theorem, Z_0 is therefore

Noetherian and is a PI ring because it is commutative. A is also Noetherian and a PI ring (by Lemma A.1 and Remark A.1).

Proposition A.1. *Let* (k, Z_0, A) *be an affine PI triple. If* M *is an irreducible* A*-module then* M *is finite-dimensional over* k *and* $\operatorname{ann}_{Z_0}(M)$ *is a maximal ideal of* Z_0*.*

Proof. Put $\overline{A} = A/\operatorname{ann}_A(M)$. Then $\overline{A}$ is a primitive ring and is a PI ring by Remark A.1. By Kaplansky's theorem (Theorem A.1), $\overline{A}$ is a central simple algebra and $Z(\overline{A})$ is therefore an affine k-algebra by Lemma A.1. Thus $Z(\overline{A})$ is a finite field extension of k. But $Z_0/\operatorname{ann}_{Z_0}(M)$ embeds into $Z(\overline{A})$, so $\operatorname{ann}_{Z_0}(M)$ is a maximal ideal of Z_0 and $Z_0/\operatorname{ann}_{Z_0}(M)$ is finite-dimensional over k. Thus M is finite-dimensional over k. □

Let $\mathsf{Irred}(A)$ denote the set of isomorphism classes of irreducible A-modules. If M is an irreducible A-module, we denote its isomorphism class by $[M]$. Proposition A.1 tells us that we can define a map

$$\chi : \mathsf{Irred}(A) \to \operatorname{Spec} Z_0 \qquad [M] \mapsto \operatorname{ann}_{Z_0}(M)$$

where $\operatorname{Spec} Z_0$ actually denotes the set of **characters** of Z_0, namely k-algebra homomorphisms $Z_0 \to k$. We shall call χ the **central character map** associated to the affine PI triple (k, Z_0, A).

Proposition A.2. *Let* (k, Z_0, A) *be an affine PI triple where* A *is generated as a* Z_0*-module by* n *elements. Then the central character map* χ *is surjective with finite fibers consisting of* $\leq n$ *elements.*

Proof. The proof supplied here is slightly more general. Let $\mathfrak{p}$ be a prime ideal of Z_0 and localize A at $\mathfrak{p}$ to obtain $A_{\mathfrak{p}}$. Let $\mathfrak{m}$ be the unique maximal ideal of $(Z_0)_{\mathfrak{p}}$. Because $A_{\mathfrak{p}} \neq 0$, by Nakayama's lemma $\mathfrak{m}A_{\mathfrak{p}} \neq A_{\mathfrak{p}}$. Then $A_{\mathfrak{p}}/\mathfrak{m}A_{\mathfrak{p}}$ is a $Q(Z_0/\mathfrak{p})$-vector space of dimension between 1 and n (by Nakayama again). Now $\operatorname{Spec}(A_{\mathfrak{p}}/\mathfrak{m}A_{\mathfrak{p}})$ is the fiber of the (scheme-theoretic) map $\operatorname{Spec} A \to \operatorname{Spec} Z_0$ over $\mathfrak{p}$ and it therefore has between 1 and n elements. In particular if $\mathfrak{m}$ is a maximal ideal of Z_0 then there is a prime ideal $\mathfrak{P}$ of A such that $\mathfrak{P} \cap Z_0 = \mathfrak{m}$. Now $\mathfrak{P}$ is contained in a maximal left ideal and the quotient of A by this ideal is a simple A-module whose Z_0-annihilator is $\mathfrak{m}$, as required. □

A.3 Azumaya algebras

We now isolate a class of affine PI triples for which the central character map is bijective. The **enveloping algebra** of R is defined to be $R^e = R \otimes_Z R^{op}$. There is a canonical homomorphism $R^e \to \operatorname{End}_Z(R)$ given by $a \otimes b \mapsto (r \mapsto arb)$.

Definition A.4. *A ring R is an* ***Azumaya algebra*** *over its center Z if R is a finitely generated projective Z-module and the canonical homomorphism $R^e \to \mathrm{End}_Z(R)$ is an isomorphism.*

Because R is finitely generated as a Z-module, it follows from Remark A.1 that R is a PI ring.

Proposition A.3. *Let R be an Azumaya algebra with center Z and let $\mathfrak{m}$ be a maximal ideal of Z. Then $R/\mathfrak{m}R$ is a central simple algebra with center $Z/\mathfrak{m}$.*

Proof. [MR01], Proposition 13.7.11. ☐

Recall that if M is a projective module over a commutative ring R, then the localization $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for every prime ideal $\mathfrak{p}$ of R ([Lang02], X Theorem 4.4). Thus we can define the **rank of M at $\mathfrak{p}$**, denoted $\mathrm{rk}_\mathfrak{p}(M)$, to be the rank of the $R_\mathfrak{p}$-module $M_\mathfrak{p}$. By continuity of rank, it follows that if M is a projective module over a domain then M has constant rank, and we can define the **rank** of M to be $\mathrm{rk}(M) = \mathrm{rk}_{(0)}(M)$. Now let R be a prime Azumaya algebra over its center Z, $\mathfrak{m}$ a maximal ideal of Z. By Nakayama's lemma, we know that the minimum number of generators of $R_\mathfrak{m}$ as a $Z_\mathfrak{m}$-module ($= \mathrm{rk}\, R$) is equal to $\dim_k(R/\mathfrak{m}R)$ where $k = Z/\mathfrak{m}$. But by Proposition A.3, $R/\mathfrak{m}R$ is central simple over k, hence its k-dimension is n^2 for some n. We say that R is an **Azumaya algebra of rank n^2**. Note that by Posner's theorem (Theorem A.2) and related remarks, $\deg_{PI}(R) = n$ (because $\dim_{Q(Z)}(Q(Z) \otimes R) = \mathrm{rk}_{(0)} R = n$).

Theorem A.3 (Artin-Procesi)**.** *Let R be a prime ring with center Z. The following are equivalent:*

1. *R is an Azumaya algebra of rank n^2 over Z.*
2. *R is a PI ring with $\deg_{PI}(R) = n$ whose prime ideals are all regular.*
3. *R is a PI ring with $\deg_{PI}(R) = n$ whose maximal ideals are all regular.*

Proof. [MR01], Theorem 13.7.14. ☐

Proposition A.4. *Let (k, Z_0, A) be a prime affine PI triple with k algebraically closed. Let $\deg_{PI}(A) = n$. Let Z be the center of A, $\mathfrak{M}$ a maximal ideal of A and put $\mathfrak{m} = \mathfrak{M} \cap Z$. The following are equivalent:*

1. *$\mathfrak{M}$ is a regular maximal ideal of A.*
2. *$A_\mathfrak{m}$ is Azumaya over $Z_\mathfrak{m}$.*
3. *$\mathfrak{M} = \mathfrak{m}A$.*

4. *The unique irreducible left* $(A/\mathfrak{M})$*-module has dimension* n*.*

5. $A/\mathfrak{M} \simeq M_n(k)$*.*

Proof. [BG02] Theorem III.1.6. ☐

Corollary A.1. *Let* $(k, Z(A), A)$ *be an Azumaya affine PI triple. Then the central character map is a bijection.*

Proof. By Theorem A.3, all prime ideals of A are regular, hence so are all maximal ideals. The result now follows by Proposition A.4 and the fact that being Azumaya is a local property. ☐

Appendix B

Semisimple Lie Algebras

We include some fundamental results on semisimple Lie algebras in this appendix primarily for the purpose of setting up some notation. We assume that the reader is familiar with the definitions of a Lie algebra and Lie algebra representation (in particular the adjoint representation). More details can be found in many sources, e.g. [Hum73].

A Lie algebra $\mathfrak{g}$ is said to be **semisimple** if it has no non-zero abelian ideals. If $\mathfrak{g}$ is semisimple it possesses a **Cartan subalgebra** $\mathfrak{h}$, namely a maximal abelian subalgebra consisting of ad-semisimple elements. We have an eigenspace decomposition $\mathfrak{g}$ under the action of $\operatorname{ad}\mathfrak{h}$ called the **Cartan decomposition**:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

where

1. $\mathfrak{h} = \mathfrak{g}_0 = \{x \in \mathfrak{g} : [x, \mathfrak{h}] = 0\}$

2. Φ consists of **roots**, namely those $\alpha : \mathfrak{h} \to k$ such that

   $$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [x, y] = \alpha(y)y \text{ for all } y \in \mathfrak{h}\}$$

   is non-zero.

Φ forms a **reduced root system** in $\mathfrak{h}^*$. By the properties of root systems, Φ contains a subset Φ^+ of **positive roots** and Φ is the disjoint union of Φ^+ and $\Phi^- = -\Phi^+$. There is a basis Δ of $\mathfrak{h}^*$ consisting of **simple roots**:

$$\Delta = \{\alpha_1, \dots \alpha_n\} \subseteq \Phi^+$$

Here $n = \dim_k \mathfrak{h}$ is called the **rank** of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ splits as a direct sum

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \qquad \mathfrak{n}^\pm = \sum_{\alpha \in \Phi^\pm} \mathfrak{g}_\alpha$$

called the **triangular decomposition** of $\mathfrak{g}$.

The **Killing form** $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to k$ is the symmetric bilinear form defined by $(x, y) = \mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y)$. It is $\mathfrak{g}$-invariant and non-degenerate. Moreover, the Killing form is non-degenerate when restricted to $\mathfrak{h}$. The **Cartan matrix** of $\mathfrak{g}$ is the $n \times n$ matrix $C = (a_{ij})$ with

$$a_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

Any reduced root system is the sum of irreducible root systems. The latter correspond to **simple** Lie algebras. If $\mathfrak{g}$ is simple, the simple roots of $\mathfrak{g}$ fall under the following two cases:

1. They are all of the same length (i.e. (α, α) is the same for every simple α), and we are said to be in the **simply laced case**.
2. They have two lengths: **long** and **short**.

The form is normalized so that $(\alpha, \alpha) = 2$ for all short roots. With this normalization, the integers $d_i = (\alpha_i, \alpha_i)/2$ for $1 \leq i \leq n$ belong to $\{1, 2, 3\}$. Putting D as the diagonal matrix with entries $d_1, \ldots, d_n$, the matrix DC is symmetric.

Appendix C

Hopf Algebras

More details concerning Hopf algebras can be found in [Kas94].

C.1 Coalgebras

Let k be a field.

Definition C.1. *A* ***coalgebra*** *over k is a triple (C, Δ, ϵ) where C is a vector space, $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$ are linear maps satisfying the following axioms:*

1. *The diagram*

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "1_C \otimes \Delta"] \\
C \otimes C \arrow[r, "\Delta \otimes 1_C"'] & C \otimes C \otimes C
\end{tikzcd}$$

   commutes.

2. *The diagram*

$$\begin{tikzcd}
k \otimes C & C \otimes C \arrow[l, "\epsilon \otimes 1_C"'] \arrow[r, "1_C \otimes \epsilon"] & C \otimes k \\
& C \arrow[ul, "\simeq"] \arrow[u, "\Delta"'] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

   commutes.

The maps Δ and ϵ are called the ***coproduct*** *and* ***counit*** *of C respectively.*

One obtains the Definition C.1 by writing out the definition of a k-algebra diagrammatically and reversing all of the arrows. Let (C, Δ, ϵ) be a coalgebra. If $x \in C$ then $\Delta(x) \in C \otimes C$, hence

$$\Delta(x) = \sum_i x'_i \otimes x''_i$$

for some $x_i', x_i'' \in C$. We adopt the **Sweedler notation**

$$\Delta(x) = \sum_{(x)} x' \otimes x''$$

to get rid of the subscripts.

C.2 Bialgebras and Hopf Algebras

It is possible to have vector spaces which come equipped with both an algebra and a coalgebra structure. Naturally these two structures should interact in some way, and this leads to the definition of a bialgebra.

Definition C.2. *A **bialgebra** over k is a quintuple $(H, \mu, \eta, \Delta, \epsilon)$ where*

1. *(H, μ, η) is an algebra.*
2. *(H, Δ, ϵ) is a coalgebra.*
3. *The maps Δ and ϵ are algebra homomorphisms.*

Let $(H, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. For $f, g \in \mathrm{End}(H)$, define $f * g$ to be the composition of the maps

$$H \overset{\Delta}{\to} H \otimes H \overset{f \otimes g}{\to} H \otimes H \overset{\mu}{\to} H$$

The resulting map on $\mathrm{End}(H)$ is bilinear and is called the **convolution**.

Definition C.3. *Let $(h, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. An endomorphism S of H is called an **antipode** for H if*

$$S * 1_H = 1_H * S = \eta \circ \epsilon$$

*A **Hopf algebra** is a bialgebra with an antipode.*

Coordinate rings of linear algebraic groups are important examples of commutative Hopf algebras. If $\mathfrak{g}$ is a Lie algebra, its universal enveloping algebra $U(\mathfrak{g})$ has Hopf algebra structure given by

$$\Delta(x) = 1 \otimes x + x \otimes 1 \qquad \epsilon(x) = 0 \qquad s(x) = -x$$

for every $x \in U(\mathfrak{g})$. This is a **cocommutative** Hopf algebra; namely $\Delta \circ \tau = \Delta$ where $\tau(x \otimes y) = y \otimes x$ for all $x, y \in U(\mathfrak{g})$. Quantum groups, like the quantized universal enveloping algebras $U_q(\mathfrak{g})$, are neither commutative nor cocommutative.

C.3 Adjoint representation

Let H be a Hopf algebra. If a, x and elements of H, define

$$a \cdot x = \sum_{(a)} a' x S(a'') \qquad x^a = \sum_{(a)} S(a') x a''$$

These endow H with the structure of a left- (respectively right-) module over itself and are called the **left-** (respectively **right-**) **adjoint representation** of H. These definitions generalize the adjoint action of a Lie algebra on itself and the action of a group on itself by conjugation.

Definition C.4. *Let H be a k-bialgebra. An algebra A is a H-**module algebra** if*

1. *A is a H-module.*
2. *The multiplication $\mu : A \otimes A \to A$ and unit $\eta : k \to A$ of A are morphisms of H-modules.*

By a H-module, one means a module over the algebra (H, μ, η). In Definition C.4, the field k is equipped with H-module structure given by $h \cdot c = \epsilon(h)c$ for every $h \in H$ and $c \in k$. The tensor product $A \otimes A$ is equipped with the following H-module structure:

$$h \cdot (a \otimes b) = \Delta(h)(a \otimes b) = \sum_{(h)} (h' \cdot a) \otimes (h'' \cdot b)$$

Proposition C.1. *The left (respectively right) adjoint representation of H turns H into a left (respectively right) H-module algebra.*

Proof. We prove this for the left adjoint representation; the proof for the right adjoint representation being analogous. Firstly, for $a, x \in H$ we have

$$a \cdot \eta(1) = \sum_{(a)} a' S(a'') = \epsilon(a) 1_H$$

where 1_H is the identity element of H. For $a, x, y \in H$,

$$\begin{aligned} \sum_{(a)} (a' \cdot x)(a'' \cdot y) &= \sum_{(a)} a' x S(a'') a''' y S(a'''') \\ &= \sum_{(a)} a' x \epsilon(a'') y S(a''') \\ &= \sum_{(a)} a' x y S(a'') \\ &= a \cdot (xy) \end{aligned}$$

as required. □

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