A Probabilistic proof of some integral formulas involving the Meijer G-function

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Abstract

New integral formulas involving the Meijer G-function are derived using recent results concerning distributional characterisations and distributional transformations in probability theory.

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1 Introduction and main results

The Meijer G-function is a very general function which includes many simpler special functions as special cases. The Meijer G-function is defined by the contour integral:

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s}\frac{\prod_{j=1}^m \Gamma(s+b_j)\prod_{j=1}^n \Gamma(1-a_j-s)}{\prod_{j=n+1}^p \Gamma(s+a_j)\prod_{j=m+1}^q \Gamma(1-b_j-s)}\,\mathrm{d}s,$$

where c is a real constant defining a Bromwich path separating the poles of $F(s+b_j)$ from those of $F(1-a_j-s)$ and where we use the convention that the empty product is 1. A more detailed discussion of the Meijer G-function and examples are given in [3], pp. 206–222; see also [9] and references therein.

In this paper, we derive new integral formulas involving the Meijer G-function. We prove these results using a probabilistic approach, using recent results from the theory of distributional characterisations and distributional transformations in probability theory that are given [6]. Our main result is as follows.

Theorem 1.1. *Let n be a positive integer and suppose that $a_1,\ldots,a_n > -1$. Then, for all $x>0$,*

$$G^{n,0}_{0,n}(x \,|\, a_1,\ldots,a_n) = \int_x^\infty G^{n,0}_{n,n}\left(\frac{x}{t} \,\middle|\, \begin{matrix} a_1+1,\ldots,a_n+1 \\ a_1,\ldots,a_n \end{matrix}\right) G^{n,0}_{0,n}(t \,|\, a_1,\ldots,a_n)\,\mathrm{d}t. \qquad (1.1)$$

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If $a_1, \ldots, a_n$ are distinct, then (1.1) simplifies to

$$G^{n,0}_{0,n}(x \,|\, a_1, \ldots, a_n) = \sum_{k=1}^{n} \left(\prod_{j=k}^{n} \frac{1}{a_j - a_k} \right) \int_x^\infty \left(\frac{x}{t} \right)^{a_k} G^{n,0}_{0,n}(t \,|\, a_1, \ldots, a_n) \, \mathrm{d}t,$$

and if $a_1 = \cdots = a_n = a$, then (1.1) simplifies to

$$G^{n,0}_{0,n}(x \,|\, a, \ldots, a) = \frac{1}{(n-1)!} \int_x^\infty \left(\frac{x}{t} \right)^{a} \left[\log\left(\frac{t}{x} \right) \right]^{n-1} G^{n,0}_{0,n}(t \,|\, a, \ldots, a) \, \mathrm{d}t. \tag{1.2}$$

The rest of this article is organised as follows. In Section 2, we present some preliminary results from probability theory that we will use to prove Theorem 1.1. In particular, we state a useful characterisation of the distribution of the product of n independent gamma random variables and introduce an associated distributional transformation. In Section 3, we establish some properties of this distributional transformation. In Section 4, we use these properties to prove Theorem 1.1. We conclude by noting that the approach used in this paper to prove Theorem 1.1 could in principle be used to prove other integral formulas involving special functions.

2 Preliminary results from probability theory

In this section, we introduce the results from probability theory that are required in our proof of Theorem 1.1.

2.1 Products of random variables

One of the ways the Meijer G-function enters probability theory is through the study of products of independent random variables. It was shown by [11] that probability density functions of products of independent beta, gamma and central normal random variables are Meijer G-functions. The density function of the product of n independent standard normal random variables with density $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}$, $x \in \mathbb{R}$, is given by

$$p(x) = \frac{1}{(2\pi)^{n/2}} G^{n,0}_{0,n}\left(\frac{x^2}{2^n} \,\middle|\, 0, \ldots, 0 \right), \quad x \in \mathbb{R}. \tag{2.3}$$

A random variable with density (2.3) has *product normal* distribution with variance 1, denoted by $\mathrm{PN}(n, 1)$. The density of the product of n independent gamma random variables with density $\frac{\lambda^{r_i}}{\Gamma(r_i)} x^{r_i - 1} \mathrm{e}^{-\lambda x}$, $x > 0$, $\lambda > 0$, $r_i > 0$, $i = 1, \ldots, n$, (denoted by $\mathrm{Gamma}(r_i, \lambda)$) is given by

$$p(x) = \frac{\lambda^n}{\prod_{j=1}^n \Gamma(r_j)} G^{n,0}_{0,n}(\lambda^n x \,|\, r_1 - 1, \ldots, r_n - 1), \quad x > 0, \tag{2.4}$$

and a random variable with density (2.4) is said to have a *product gamma* distribution, which we denote by $\mathrm{PG}(r_1, \ldots, r_n, \lambda)$. In this paper, for simplicity, we take $\lambda = 1$. Finally, the density of the product of n independent beta $\mathrm{Beta}(a_i, b_i)$ random variables with density $\frac{\Gamma(a_i+b_i)}{\Gamma(a_i)\Gamma(b_i)}x^{a_i-1}(1-x)^{b_i-1}$, $0<x<1$, $a_i,b_i>0$, $i=1,\ldots,n$, (denoted by $\mathrm{Beta}(a_i,b_i)$) is given by

$$p(x)=\bigg(\prod_{i=1}^m\frac{\Gamma(a_i+b_i)}{\Gamma(a_i)}\bigg)G^{n,0}_{n,n}\bigg(x\;\bigg|\;\begin{matrix}a_1+b_1-1,\ldots,a_n+b_n-1\\ a_1-1,\ldots,a_n-1\end{matrix}\bigg),\quad 0<x<1.$$

2.2 Stein characterisations

Recently, the products of independent beta, gamma and central normal random variables have received attention [4, 6] in the context of the probabilistic technique Stein's method, introduced in 1972 by Stein [12]. In the works [4] and [6], so-called Stein characterisations were obtained for products of independent beta, gamma and central normal random variables, and the characterisations of products of gammas and normals will be of particular interest to us in this paper.

Before presenting these characterisations, we introduce some notation. For $r\in\mathbb{R}$, we define the operator T_r by $T_rf(x)=xf'(x)+rf(x)$ and we let D denote the usual differential operator. Also, let $B_{r_1,\ldots,r_n}$ denote the iterated operator $T_{r_1}\cdots T_{r_n}$. Then the following characterisations of the product normal (see [4], Proposition 2.3) and product gamma distributions (see [6], Proposition 2.8) hold:

Proposition 2.1. *Suppose $Z\sim\mathrm{PN}(n,\sigma^2)$. Let $f\in C^n(\mathbb{R})$ be such that $\mathbb{E}|Zf(Z)|<\infty$ and $\mathbb{E}|Z^{k-1}f^{(k)}(Z)|<\infty$, $k=1,\ldots,n$. Then*

$$\mathbb{E}[\mathcal{A}_Zf(Z)]=0, \tag{2.5}$$

where $\mathcal{A}_Zf(x)=\sigma^2A_nf(x)-xf(x)$. Conversely, if $\mathbb{E}[\mathcal{A}_Zf(W)]=0$ for all $f\in C^n(\mathbb{R})$ such $\mathbb{E}|\mathcal{A}_Zf(Z)|<\infty$, then $\mathcal{L}(W)=\mathrm{PN}(n,\sigma^2)$.

Suppose now that $Y\sim\mathrm{PG}(r_1,\ldots,r_n,1)$. Let $f\in C^n(\mathbb{R}^+)$ be such that $\mathbb{E}|Yf(Y)|<\infty$ and $\mathbb{E}|Y^kf^{(k)}(Y)|<\infty$, $k=0,\ldots,n$, where $f^{(0)}\equiv f$. Then

$$\mathbb{E}[\mathcal{A}_Yf(Y)]=0, \tag{2.6}$$

where $\mathcal{A}_Yf(x)=B_{r_1,\ldots,r_n}f(x)-xf(x)$. Conversely, if $\mathbb{E}[\mathcal{A}_Yf(W)]=0$ for all $f\in C^n(\mathbb{R}^+)$ such that $\mathbb{E}|\mathcal{A}_Yf(Y)|<\infty$, then $\mathcal{L}(W)=\mathrm{PG}(r_1,\ldots,r_n,1)$.

Similar characterisations have been obtained for many standard probability distributions (see [8] for an overview of the current literature), and lie at the heat of Stein's method, by characterising distributions in a very convenient manner for the purpose of deriving approximation theorems in probability theory. For a detailed account of Stein's method for normal approximation see [2], and for a simple, general introduction see [10]. Whilst Stein characterisations are typically used as part of Stein's method, they have utility in other areas, such as obtaining formulas for moments of probability distributions [5] and deriving formulas for probability density functions and characteristic functions [4, 6]. In this paper, we shall see consider a rather curious application of the product gamma Stein equation to establishing new integral formulas for the Meijer G-function.

2.3 Distributional transformations

The charactering equation (2.5) motivates a distributional transformation ([4], Definition 1.2) which generalises the zero bias transformation (see [7]). For W a mean zero random variable with variance 1, the random variable $W^{*(n)}$ is said to have the *W-zero biased distribution of order n* if, for all $f \in C^n(\mathbb{R})$ such that the relevant expectations exist,

$$\mathbb{E}[Wf(W)] = \mathbb{E}[DT_1^{n-1}f(W^{*(n)})]. \tag{2.7}$$

This distributional transformation was introduced in [4], and a number of interesting properties were obtained, which, in conjugation with the characterisation (2.5), allows one to prove product normal approximation theorems (see [4], Section 4). An analogous distributional transformation is motivated by the characterising equation (2.6):

Definition 2.2. *Let W be a non-negative random variable with $0 < \mathbb{E}W < \infty$. We say that $W^{G(n)}$ has the W*-gamma biased distribution of order n *with shape parameters $r_1, \ldots, r_n > 0$ if, for all $f \in C^n(\mathbb{R}^+)$ such that the relevant expectations exist,*

$$\mathbb{E}[Wf(W)] = \mathbb{E}[B_{r_1,\ldots,r_n}f(W^{G(n)})], \tag{2.8}$$

where $r_1, \ldots, r_n$ are such that $\prod_{k=1}^n r_k = \mathbb{E}W$.

It was established in lemma 2.6 of [6] that for any such W there exists a unique random variable $W^{G(n)}$ that has the W-gamma biased distribution of order n. Combining this fact with the characterising equation (2.6) for the $\mathrm{PG}(r_1, \ldots, r_n, 1)$ distribution immediately gives the following lemma.

Lemma 2.3. *Let W be a non-negative random variable with $0 < \mathbb{E}W < \infty$, and let $W^{G(n)}$ have the W-gamma biased distribution of order n with shape parameters $r_1, \ldots, r_n$, in accordance with Definition 2.2. Then the $\mathrm{PG}(r_1, \ldots, r_n, 1)$ distribution is the unique fixed point of the W-gamma biased distribution of order n with shape parameters $r_1, \ldots, r_n$.*

In the next section, we shall collect further properties of this distributional transformation. These properties may, in future works, prove useful in deriving product gamma approximation theorems, although in this paper we will exploit its properties to prove Theorem 1.1.

3 Properties of the gamma bias transformation

In this section, we establish some properties of the gamma bias transformation of order n from which we shall deduce Theorem 1.1. Firstly, we present a lemma, that was given in [6], which gives an inverse operator for the iterated operator $B_{r_1,\ldots,r_n} = T_{r_1} \cdots T_{r_n}$.

Lemma 3.1. *Let $U_1, \ldots, U_n$ be independent $\mathrm{Beta}(r_j, 1)$ random variables with $r_j > 0$, and define $V_n = \prod_{j=1}^n U_j$. Define the operator $H_{r_1,\ldots,r_n}$ by $H_{r_1,\ldots,r_n}f(x) = (\prod_{k=1}^n r_k)^{-1}\mathbb{E}f(xV_n)$. Then*

(i) $H_{r_1,\ldots,r_n}$ is the right-inverse of the operator $B_{r_1,\ldots,r_n}$ in the sense that

$$B_{r_1,\ldots,r_n}H_{r_1,\ldots,r_n}f(x) = f(x).$$

(ii) Suppose now that $f \in C^n(\mathbb{R})$. Then, for any $n \geq 1$,

$$H_{r_1,\ldots,r_n} B_{r_1,\ldots,r_n} f(x) = f(x).$$

Therefore, $H_{r_1,\ldots,r_n}$ is the inverse of $B_{r_1,\ldots,r_n}$ when the domain of $B_{r_1,\ldots,r_n}$ is $C^n(\mathbb{R})$.

With Lemma 3.1 stated, we can now obtain a useful relationship between the gamma bias distribution of order n in terms of the size bias distribution, which is analogous to the relationship (see [4]) between the zero bias distribution of order n in terms of the square bias distribution (defined in [2]). If $W \geq 0$ has mean $\mu > 0$, we say W^s has the W-size biased distribution if, for all f such that $\mathbb{E}Wf(W)$ exists,

$$\mathbb{E}Wf(W) = \mu \mathbb{E}f(W^s).$$

The size bias coupling is commonly used in Stein's method; for an application of this coupling to normal approximation see Baldi, Rinott and Stein [1].

Proposition 3.2. *Let W be a non-negative random variable with $0 < \mathbb{E}W < \infty$, and let W^s have the W-size bias distribution. Let W^s and $\{U_k\}_{1\leq k\leq n}$ be mutually independent, with $U_k \sim \mathrm{Beta}(r_k, 1)$, where $r_1, \ldots, r_n > 0$ are such that $\prod_{k=1}^n r_k = \mathbb{E}W$. Define $V_n = \prod_{k=1}^n U_k$. Then, the random variable*

$$W^{G(n)} \stackrel{\mathcal{D}}{=} V_n W^s \tag{3.9}$$

has the W-gamma bias distribution of order n with shape parameters $r_1, \ldots, r_n$.

Proof. Let $f \in C_c$, the set of continuous functions on $\mathbb{R}^+$ with compact support. Recall from Lemma 3.1 that $B_{r_1,\ldots,r_n} H_{r_1,\ldots,r_n} g(x) = g(x)$ for any g. Thus,

$$\begin{aligned}\mathbb{E}f(W^{G(n)}) &= \mathbb{E}B_{r_1,\ldots,r_n} H_{r_1,\ldots,r_n} f(W^{G(n)}) = \mathbb{E}W H_{r_1,\ldots,r_n} f(W) \\ &= \prod_{k=1}^n (1/r_k) \mathbb{E}Wf(V_n W) = \prod_{k=1}^n (1/r_k) \mathbb{E}W \mathbb{E}f(V_n W^s) = \mathbb{E}f(V_n W^s).\end{aligned}$$

Since the expectation of $f(W^{G(n)})$ and $f(V_n W^s)$ are equal for all $f \in C_c$, the random variables $W^{G(n)}$ and $V_n W^s$ must be equal in distribution. $\square$

We now note some formulas for the probability density function of the product of n independent beta random variables, that we will use in the proof of Proposition 3.4.

Lemma 3.3. *Let $\{U_k\}_{1\leq k\leq n}$ be mutually independent $\mathrm{Beta}(r_k, 1)$ random variables with $r_1, \ldots, r_n > 0$. Then, the density function of $V_n = \prod_{k=1}^n U_k$ is given by*

$$p_{V_n}(x) = \bigg(\prod_{i=1}^n r_i\bigg) G^{n,0}_{n,n}\bigg(x \,\bigg|\, \begin{matrix} r_1, \ldots, r_n \\ r_1 - 1, \ldots, r_n - 1 \end{matrix}\bigg), \quad 0 < x < 1. \tag{3.10}$$

When the r_k are distinct the density of V_n can be written as

$$p_{V_n}(x) = \bigg(\prod_{i=1}^n r_i\bigg) \sum_{k=1}^n \frac{x^{r_k - 1}}{\prod_{j\neq k}^n (r_j - r_k)}, \quad 0 < x < 1, \tag{3.11}$$

and the distribution function of V_n is given by

$$F_{V_n}(x) = \sum_{k=1}^{n} \left(\prod_{j\neq k}^{n} \frac{r_j}{r_j - r_k} \right) x^{r_k}, \quad 0 < x < 1. \tag{3.12}$$

Proof. Formula (3.10) follows immediately from (2.1). We prove that formula (3.11) holds by induction on n. The result holds for $n = 1$, so suppose that for some $n \geq 1$,

$$p_{V_n}(v) = \left(\prod_{i=1}^{n} r_i \right) \sum_{k=1}^{n} \frac{v^{r_k - 1}}{\prod_{j\neq k}^{n}(r_j - r_k)}, \quad 0 < v < 1.$$

By the inductive hypothesis, the joint density of V_n and an independent Beta$(r_{n+1}, 1)$ random variable U_{n+1} is given by

$$p_{U_{n+1},V_n}(u, v) = \left(\prod_{i=1}^{n+1} r_i \right) u^{r_{n+1}-1} \sum_{k=1}^{n} \frac{v^{r_k - 1}}{\prod_{j\neq k}^{n}(r_j - r_k)}, \quad 0 < u, v < 1.$$

Making the change of variables $X = U_{n+1}V_n$, we have

$$p_{X,V_n}(x, v) = \left(\prod_{i=1}^{n+1} r_i \right) x^{r_{n+1}-1} \sum_{k=1}^{n} \frac{v^{r_k - r_{n+1} - 1}}{\prod_{j\neq k}^{n}(r_j - r_k)}, \quad 0 < x < v < 1,$$

and the marginal distribution of X is given by

$$\begin{aligned}
p_X(x) &= \left(\prod_{i=1}^{n+1} r_i \right) x^{r_{n+1}-1} \sum_{k=1}^{n} \int_x^1 \frac{v^{r_k - r_{n+1} - 1}}{\prod_{j\neq k}^{n}(r_j - r_k)} \, \mathrm{d}v \\
&= \left(\prod_{i=1}^{n+1} r_i \right) \sum_{k=1}^{n} \left(\frac{x^{r_k - 1}}{\prod_{j\neq k}^{n+1}(r_j - r_k)} - \frac{x^{r_{n+1} - 1}}{\prod_{j\neq k}^{n+1}(r_j - r_k)} \right) \\
&= \left(\prod_{i=1}^{n+1} r_i \right) \left[\sum_{k=1}^{n} \frac{x^{r_k - 1}}{\prod_{j\neq k}^{n+1}(r_j - r_k)} + \frac{x^{r_{n+1} - 1}}{\prod_{j=1}^{n}(r_j - r_{n+1})} \right] \\
&= \left(\prod_{i=1}^{n+1} r_i \right) \sum_{k=1}^{n+1} \frac{x^{r_k - 1}}{\prod_{j\neq k}^{n+1}(r_j - r_k)},
\end{aligned}$$

which completes the inductive proof. Formula (3.12) for the distribution function of V_n now follows immediately on integrating the formula for the density function of V_n over the interval $(0, x)$. $\square$

We are now in a position to prove the main result of this section: a formula for the distribution function and the density of the gamma bias transformation of order n. The formula simplifies for certain values of the shape parameters $r_1, \ldots, r_n$.

Proposition 3.4. *Let W be a random variable with $\mathbb{E}W = \prod_{k=1}^{n} r_k$, and let $W^{G(n)}$ have the W-gamma biased distribution of order n with shape parameters $r_1, \ldots, r_n > 0$.*

(i) Let V_n be distributed as the product of the mutually independent random variables $U_k \sim \mathrm{Beta}(r_k, 1)$, $k = 1, \ldots, n$. Then, the distribution function of $W^{G(n)}$ is given by

$$F_{W^{G(n)}}(w) = 1 - \prod_{k=1}^{n}(1/r_k)\mathbb{E}\Big[W\Big(1 - F_{V_n}\Big(\frac{w}{W}\Big)\Big)\mathbf{1}(W \geq w)\Big]. \tag{3.13}$$

In particular, if the r_k are all distinct, we have

$$F_{W^{G(n)}}(w) = \mathbb{E}\Bigg[W\Bigg[1 - \sum_{k=1}^{n}\frac{1}{r_k}\Bigg(\prod_{j\neq k}^{n}\frac{1}{r_j - r_k}\Bigg)\Big(\frac{w}{W}\Big)^{r_k}\Bigg]\mathbf{1}(W \geq w)\Bigg]. \tag{3.14}$$

If $r_1 = \cdots = r_n = r$ the distribution function of $W^{G(n)}$ can be written as

$$F_{W^{G(n)}}(w) = 1 - \frac{1}{(n-1)!r^n}\mathbb{E}\Big[W\gamma\Big(n, r\log\Big(\frac{W}{w}\Big)\Big)\mathbf{1}(W \geq w)\Big], \tag{3.15}$$

where $\gamma(a, x) = \int_0^x t^{a-1}\mathrm{e}^{-t}\,\mathrm{d}t$.

(ii) The density function of $W^{G(n)}$ is given by

$$p_{W^{G(n)}}(w) = \mathbb{E}\Big[G^{n,0}_{n,n}\Big(\frac{w}{W}\,\Big|\,\begin{matrix} r_1, \ldots, r_n \\ r_1 - 1, \ldots, r_n - 1\end{matrix}\Big)\mathbf{1}(W \geq w)\Big]. \tag{3.16}$$

If the r_k are distinct, we have

$$p_{W^{G(n)}}(w) = \mathbb{E}\Bigg[\sum_{k=1}^{n}\Bigg(\prod_{j\neq k}^{n}\frac{1}{r_j - r_k}\Bigg)\Big(\frac{w}{W}\Big)^{r_k - 1}\mathbf{1}(W \geq w)\Bigg]. \tag{3.17}$$

If $r_1 = \cdots = r_n = r$ the density function of $W^{G(n)}$ is given by

$$p_{W^{G(n)}}(w) = \frac{1}{(n-1)!}\mathbb{E}\Bigg[\Big(\frac{w}{W}\Big)^{r-1}\Big(\log\Big(\frac{W}{w}\Big)\Big)^{n-1}\mathbf{1}(W \geq w)\Bigg]. \tag{3.18}$$

Proof. (i) In the proof of Proposition 3.2 we showed that $\mathbb{E}f(W^{G(n)}) = \prod_{k=1}^{n}(1/r_k)\mathbb{E}Wf(V_nW)$ for all bounded functions f. By taking $f(x) = \mathbf{1}(x \leq w)$ we have

$$F_{W^{G(n)}}(w) = \prod_{k=1}^{n}(1/r_k)\mathbb{E}[W\mathbf{1}(V_nW \leq w)] = 1 - \prod_{k=1}^{n}(1/r_k)\mathbb{E}[W\mathbf{1}(V_nW \geq w)], \tag{3.19}$$

as $\mathbb{E}W = \prod_{k=1}^{n} r_k$. Formula (3.13) now follows. If the r_k are distinct, then, from formula (3.12) for the distribution function of V_n and (3.13), we deduce formula (3.14).

Suppose now that $r_1 = \cdots r_n = r$. It is straightforward to verify that $-\log(U_k)$ follows the $\mathrm{Exp}(r)$ distribution. Hence, $-\log(V_r)$ follows the $\mathrm{Gamma}(n, r)$ distribution, and thus

$$F_{W^{G(n)}}(w) = \frac{1}{r^n}\mathbb{E}\Big[W\int_0^{-\log(\frac{w}{W})}\frac{r^n}{(n-1)!}t^{n-1}\mathrm{e}^{-rt}\,\mathrm{d}t\mathbf{1}(W \geq w)\Big].$$

Making the change of variables $u = rt$ gives

$$\int_0^{-\log(\frac{w}{W})} t^{n-1}\mathrm{e}^{-rt}\,\mathrm{d}t = \frac{1}{r^n}\int_0^{-r\log(\frac{w}{W})} u^{n-1}\mathrm{e}^{-u}\,\mathrm{d}u = \frac{1}{r^n}\gamma\bigg(n, r\log\bigg(\frac{W}{w}\bigg)\bigg),$$

and formula (3.15) now follows.

(ii) The general formula follows from differentiating the right-hand side of (3.13) with respect to w, and then applying formula (3.10) for the density of V_n. Formula (3.17) follows from substituting the formula (3.11) for the density of V_n into (3.16). Finally, we consider the case $r_1 = \cdots = r_n = r$. For $a > 0$, the function $\gamma(n, r\log(a/w))$ is differentiable on $(0, a)$, with derivative

$$\frac{\mathrm{d}}{\mathrm{d}w}\bigg[\gamma\bigg(n, r\log\bigg(\frac{a}{w}\bigg)\bigg)\bigg] = -\frac{r^n}{w}\bigg(\log\bigg(\frac{a}{w}\bigg)\bigg)^{n-1}\bigg(\frac{w}{a}\bigg)^r. \tag{3.20}$$

Using (3.20) and dominated convergence now yields formula (3.18). □

4 Proof of Theorem 1.1 and concluding remarks

4.1 Proof of Theorem 1.1

Let us first consider the general case $a_1, \ldots, a_n > -1$. For ease of notation, let $r_j = a_j + 1$ for $j = 1, \ldots, n$. Let $W \sim \mathrm{PG}(r_1, \ldots, r_n, 1)$, which has density

$$p(x) = KG^{n,0}_{0,n}(x \,|\, r_1 - 1, \ldots, r_n - 1), \quad x > 0, \tag{4.21}$$

where $K = \prod_{k=1}^n (1/r_k)$. From formula (3.16), we have that the density of W-gamma biased distribution of order n with shape parameters $r_1, \ldots, r_n$ is given by

$$p_{W^{G(n)}}(x) = K\int_x^\infty G^{n,0}_{n,n}\bigg(\frac{x}{t}\,\bigg|\,\begin{matrix} r_1, \ldots, r_n \\ r_1 - 1, \ldots, r_n - 1\end{matrix}\bigg)G^{n,0}_{0,n}(x \,|\, r_1 - 1, \ldots, r_n - 1)\,\mathrm{d}t, \quad x > 0. \tag{4.22}$$

But, by Lemma 2.3, the $\mathrm{PG}(r_1, \ldots, r_n, 1)$ distribution is the unique fixed point of the W-gamma biased distribution of order n with shape parameters $r_1, \ldots, r_n$. Thus, (4.21) and (4.22) are equal for all $x > 0$, from which we deduce formula (1.1). The formulas for the special cases of distinct $a_1, \ldots, a_n$ and $a_1 = \cdots = a_n = a$ following similarly, with the difference being that we apply formulas (3.17) and (3.18) instead of (3.16). □

4.2 Discussion

The approach used in this paper to obtain the integral formulas of Theorem 1.1 could also be used to arrive at integral formulas for other special functions. The first step would be to obtain an appropriate Stein characterisation of a probability distribution P, whose probability density function is given in terms of special functions. An associated distributional transformation would then have to be obtained that contains P as a fixed point. Finally, a formula for the density of the distributional transformation of P would then need to be obtained, from which we would deduce an integral formula involving special functions.

For example, the PN$(n, 1)$ characterisation (2.5) and the zero bias transformation of order n could be used together to obtain integral formulas involving the Meijer G-function. However, doing this just leads to a formula that is equivalent to (1.2) with $a = 0$, and reduces to it after a simple change of variables. This is essentially due to the fact that the $\Gamma(\frac{1}{2}, \frac{1}{2})$ distribution, the chi-square distribution with one degree of freedom, has the same distribution as the square of a standard normal random variable.

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