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High-temperature macroscopic entanglement

Vlatko Vedral
Optics Section, Blackett Laboratory, Imperial College London,
Prince Consort Road, London SW7 2BZ, UK
E-mail: v.vedral@imperial.ac.uk

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Abstract. In this paper, we intend to show that macroscopic entanglement is possible at high temperatures. We have analysed multipartite entanglement produced by the η-pairing mechanism, which features strongly in the fermionic lattice models of high T_c superconductivity. This problem is shown to be equivalent to calculating multipartite entanglement in totally symmetric states of qubits. It is demonstrated that we can conclusively calculate the relative entropy of entanglement within any subset of qubits in the overall symmetric state. Three main results are then presented. First, the condition for superconductivity, namely existence of the off-diagonal long-range order (ODLRO), is dependent not on two-site entanglement but just classical correlations as the sites become more and more distant. Secondly, the entanglement that does survive in the thermodynamical limit is the entanglement of the total lattice and, at half-filling, it scales with the log of the number of sites. It is this entanglement that will exist at temperatures below the superconducting critical temperature, which can currently be as high as 160 K. Finally, it is proved that a complete mixture of symmetric states does not contain any entanglement in the macroscopic limit. On the other hand, a mixture of symmetric states possesses the same two qubit entanglement features as the pure states involved, in the sense that the mixing does not destroy entanglement for a finite number of qubits, albeit it does decrease it. Furthermore, maximal mixing of symmetric states does not destroy ODLRO and classical correlations. We discuss generalizations to the subsystems of any dimensionality (i.e. higher than spin-half).

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Contents

1. Introduction

Entanglement is, currently, one of the most studied phenomena in physics. Often shrouded in mystery, its basic premise is quite simple—entanglement is a correlation between distant particles that exists outside of any description offered by classical physics. Although this may, at first glance, appear to be an innocuous statement, in reality it is not so. Predictions from the theory of entanglement have confounded some of the greatest minds in science. Einstein famously dubbed it *spukhafte Fernwirkungen*: 'spooky action at a distance'. As we look deeper into the fabric of nature, this 'spooky' connection between particles is appearing everywhere, and its consequences are affecting the very (macroscopic) world that we experience. At the implementational level, using entanglement researchers have succeeded in teleporting information between two parties, designing cryptographic systems that cannot be broken and speeding up computations that would classically take a much longer time to execute [1]. Even though these applications have generated significant interest, we believe we have only scratched the 'tip of the iceberg' in terms of what entanglement is and indeed what we can do with it.

Although entanglement is experimentally present, pretty much beyond dispute, in microscopic systems, such as two photons or two atoms, many people find it difficult to accept that this phenomenon can exist and even have effects macroscopically. Based on our everyday intuition, we would, for example, find it very hard to believe that two cats or two human beings can be quantum-entangled. Yet, quantum physics does not tell us that there is a limitation to the existence of entanglement. It can, in principle and as far as we understand, be present in systems of any size and under many different external conditions.

The usual argument against seeing macroscopic entanglement is that large systems have a large number of degrees of freedom interacting with the rest of the universe and it is this interaction that is responsible for destroying entanglement. If we can exactly tell the state that a system is in, then this system cannot be entangled with any other system. In everyday life, objects exist at room (or comparable) temperature, so their overall state is quantum mechanically described by a very mixed state (this mixing due to temperature is, of course, also due to the interaction with a large 'hot' environment). Mixing of states that are entangled, in general, reduces entanglement and ultimately all the entanglement will vanish if the temperature is high enough. The question then is: how high is the highest temperature before we no longer see any entanglement? And how large can the body be so that entanglement is still present? Can we, for example, have macroscopic entanglement at room temperature?

Entanglement has recently been shown to affect macroscopic properties of solids, such as its magnetic susceptibility and heat capacity, but at a very low (critical) temperature [2]. This extraordinary result demonstrates that entanglement can have a significant effect on the macroscopic world. The basic reason for this dependence is simple. Magnetic susceptibility is proportional to the correlation between nuclear spins in the solid. As we mentioned before, entanglement offers a higher degree of correlation than anything allowed by classical physics, and the corresponding quantum susceptibility, which fully agrees with experimental results [2], is higher than that predicted by using just classical correlations (for further theoretical support for this, cf [3]). It is now very important to go beyond this low-temperature regime and experimentally test entanglement at increasingly higher temperatures.

Thinking that high-temperature entanglement is linked with (perhaps even responsible for) some other high-temperature quantum phenomena, such as high-temperature superconductivity, is tempting. After all, superconductivity is a manifestation of the existence of the off-diagonal long-range order (ODLRO) [4], which is a form of correlation that still persists in the thermodynamical (macroscopic) limit.[1] However, it is not immediately clear whether this correlation contains any quantum entanglement. Our main intention in this paper is to show that it does. This correlation contains multipartite entanglement between all electron pairs in the superconductor. To calculate this, we need to be able to quantify entanglement exactly and be able to discriminate entanglement from any form of classical correlation.

A great deal of effort has gone into theoretically understanding and quantifying entanglement [6]. There are a large number of different proposed measures; these measures capture different aspects of entanglement. In this paper, we will be interested in a measure based on the (asymptotic) distinguishability of entangled states from separable (disentangled) states, known as the relative entropy of entanglement [7, 8]. The main advantage of this measure is that it is easily defined for any number of systems of any dimensionality, which is not the case for entanglement of formation or distillation [6]. We have argued that a number of results in quantum information and computation follow from the relative entropy function [6].

There is, unfortunately, no closed form for the relative entropy of entanglement, but this measure can still be computed for a large class of relevant states such as the pure bipartite states, Werner states and many others [8]. Most recently, Wei *et al* [9] have succeeded in obtaining a formula for the relative entropy of entanglement for any number of totally symmetric pure states of n qubits using a very simple and elegant argument (some partial results have been obtained previously in this direction by Plenio and Vedral [10] using different methods, but only for three qubit symmetric states). We will use and extend these results further with the idea of applying them to a specific model of a superconductor.

The purpose of this paper is to investigate possible links between high-temperature entanglement and high-temperature superconductivity with the intention of showing that entanglement can persist at higher temperatures. We analyse a particular mechanism, namely the η-pairing of electrons based on Yang [11], which was originally proposed to explain high-temperature superconductivity. The main difference between this pairing mechanism and the usual Bardeen, Cooper and Schrieffer (BCS) electron pairing [12] for (low-temperature)

[1] The paper introducing the off-diagonal long-range order as a criterion for condensation was originally proposed in [5].

superconductivity is that, in the former, electrons that are positioned at the same site are paired, whereas, in the latter, electrons forming Cooper pairs are separated by a certain finite average distance (the so-called coherence length, typically of the order of hundreds of nanometres). The physical reason behind electron pairing is also believed to be different in a high-temperature superconductor, but we do not wish to enter into a discussion of these details here (see e.g. [13]). We will, however, look at the η model in a different way, using totally symmetric states, and this will make calculation of entanglement easier. Wei *et al* [9] have recently made significant progress in calculating the relative entropy of entanglement for the symmetric state. We extend their approach to the calculation of the relative entropy of entanglement for mixed symmetric state arising from tracing over some qubits in pure states, and apply it to understanding the various relations between entanglements of a subset of qubits and their relation to the total entanglement. We show that, although two-site entanglement disappears as the distance between sites diverges (a conclusion also reached by Zanardi and Wang [14] in a different way), the total entanglement still persists in the thermodynamical limit. Furthermore, it scales logarithmically with the number of qubits. Therefore, it is this total entanglement that should be compared with ODLRO and not the two-site entanglement. Whereas the two-site entanglement vanishes thermodynamically, two-site classical correlations are still present and so is the entanglement between two clusters of qubits (two-cluster entanglement in η states has also been analysed by Fan [15]). We show that all aspects of our analysis can easily be generalized to higher than half-spin systems. My hope is that the present work, which is just a first step in exploring high-temperature entanglement, will be extended to different models with states other than symmetric and that this will allow us to have a much more complete understanding of entanglement and the role it plays in the macroscopic world.

2. η-pairing in superconductivity

The model that we describe now consists of a number of lattice sites, each of which can be occupied by fermions having spin-up or spin-down internal states. Let us introduce fermion creation and annihilation operators, $c_{i,s}^{\dagger}$ and $c_{i,s}$ respectively, where the subscript i refers to the ith lattice site and s refers for the value of the spin, $\uparrow$ or $\downarrow$. Since fermions obey the Pauli exclusion principle, we can have at most two fermions attached to one and the same site. The c operators therefore satisfy the anticommutation relations:

$$\{c_{i,s}, c_{j,t}^{\dagger}\} = \delta_{ij}\delta_{s,t} \tag{1}$$

and cs and $c^{\dagger}$s anticommute as usual. (Some general features of fermionic entanglement—arising mainly from the Pauli exclusion principle—have been analysed in [14], [16]–[18].)

We only need assume that our model has the interaction which favours formation of Cooper pairs of fermions of opposite spin at each site [11]. The actual Hamiltonian is not relevant for my present purposes. It suffices to say that Yang originally considered the Hubbard model for which the η states are eigenstates (but none of them is a ground state [11]). A generalization of the Hubbard model was presented in [19] and in a specific regime of this new model the η states do become lowest energy eigenstates (this is a fact that will become relevant when we talk about high-temperature entanglement). Both these models have been used to simulate high-temperature superconductivity, since in high superconducting materials, the coherent length of each Cooper pair is, on average, much smaller than for a normal superconductor.

Suppose, now, that there are n sites and suppose, further, that we introduce an operator $\eta^\dagger$ that creates a coherent superposition of a Cooper pair in each of the lattice sites,

$$\eta^\dagger = \sum_{i=1}^{n} c_{i,\uparrow}^\dagger c_{i,\downarrow}^\dagger. \tag{2}$$

The $\eta^\dagger$ operator can be applied to the vacuum a number of times, each time creating a new coherent superposition. However, the number of applications, k, cannot exceed the number of sites, n, since we cannot have more than one pair per site due to the exclusion principle. We now introduce the following basis:

$$|k, n-k\rangle := \frac{1}{\sqrt{\binom{n}{k}}} (\eta^\dagger)^k |0\rangle, \tag{3}$$

where the factor in front is just the necessary normalization. Here, the vacuum state $|0\rangle$ is annihilated by all c operators, $c_{i,s}|0\rangle = 0$. We note in passing that the originally defined η operators can also have phase factors dependent on the location of the site on the lattice. We can have a set of operators such as

$$\eta_k = \sum_{n} \mathrm{e}^{\mathrm{i}kn} c_{n,\uparrow}^\dagger c_{n,\downarrow}^\dagger. \tag{4}$$

All the states generated with any η_k from the vacuum will be shown to have the same amount of entanglement so that the extra phases will be ignored in the rest of the paper (i.e. we will only consider the $k = 0$ states).

We can think of the η states in the following way. Suppose that $k = 2$. Then this means that we will be creating two η-pairs in total, but they cannot be created in the same lattice site. The state $|2, n-2\rangle$ is therefore a symmetric superposition of all combinations of creating two pairs at two different sites. Let us, for the moment, use the label 0 when the site is unoccupied and 1 when it is occupied. Then the state $|2, n-2\rangle$ is

$$|2, n-2\rangle = \frac{1}{\sqrt{\binom{n}{2}}} (|\underbrace{000}_{n-2}\ldots\underbrace{11}_{2}\rangle + \cdots |\underbrace{11}_{2}\ldots\underbrace{000}_{n-2}\rangle), \tag{5}$$

i.e. it is an equal superposition of states containing 2 states $|1\rangle$ and $n-2$ states $|0\rangle$. These states, due to their high degree of symmetry, are much easier to handle than general arbitrary superpositions and we can compute entanglement for them between any number of sites. Note that in this description each site effectively holds one quantum bit, whose 0 signifies that the site is empty and 1 signifies that the site is full.

The main characteristic of η states is the existence of the long-range off-diagonal order (ODLRO), which implies its various superconducting features, such as the Meissner effect and flux quantization [20]. The ODLRO is defined by the off-diagonal matrix elements of the two-site reduced density matrix being finite in the limit when the distance between the sites diverges. Namely,

$$\lim_{|i-j|\to\infty} \langle c_{j,\uparrow}^\dagger c_{j,\downarrow}^\dagger c_{i,\downarrow} c_{i,\uparrow}\rangle \longrightarrow \alpha, \tag{6}$$

where α is a constant (independent of n). We will show that although the existence of off-diagonal matrix elements does not guarantee the existence of entanglement between the two sites, it does guarantee the existence of multi-site entanglement between all the sites. Note that here, by 'correlations', we mean correlations between the number of electrons positioned at different sites i and j. Namely, we are looking at the probability of one site being occupied (empty) given that the other site is occupied (empty). This is different from spin–spin correlations, which would look at the occurrences of both electron spins being up or down, or one being up and the other being down [16].

3. General description of symmetric states

The states we will analyse here will always be of the form

$$|\Psi(n,k)\rangle \equiv |k, n-k\rangle := \frac{1}{\sqrt{\binom{n}{k}}}(\hat{S}|\underbrace{000}_{k}\ldots\underbrace{11}_{n-k}\rangle), \tag{7}$$

where $\hat{S}$ is the total symmetrization operator. We will also consider mixtures of these states, which become relevant when we talk about systems at finite temperatures. Symmetric states arise, for example, in the Dicke model, where n atoms simultaneously interact with a single mode of the electromagnetic field [21]. They are, furthermore, very important as they happen to be eigenstates of many models in solid state physics and, in particular, they are eigenstates of the Hubbard and related models supporting the η pairing mechanism. The analysis presented in this study will be applicable to any of these systems and not just the η model. The η mechanism will be significant here because of its potential to support high-temperature entanglement.

We will start computing the entanglement between every pair of qubits (sites) in the above state $|\Psi(n,k)\rangle$. A simpler task would be first to mention if and when every pair of qubits in a totally symmetric state is entangled. For this, we need to only compute the reduced two-qubit density matrix which can be written as

$$\sigma_{12}(k) = a|00\rangle\langle 00| + b|11\rangle\langle 11| + 2c|\psi^+\rangle\langle\psi^+|, \tag{8}$$

where $|\psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ and

$$a = \frac{\binom{n-2}{k-2}}{\binom{n}{k}} = \frac{k(k-1)}{n(n-1)}, \tag{9}$$

$$b = \frac{\binom{n-2}{k}}{\binom{n}{k}} = \frac{(n-k)(n-k-1)}{n(n-1)}, \tag{10}$$

$$c = \frac{\binom{n-2}{k-1}}{\binom{n}{k}} = \frac{k(n-k)}{n(n-1)}. \tag{11}$$

We can easily check that $a + b + 2c = 1$ and so the state is normalized. This density matrix is the same no matter how far the two sites are from each other, since the state is symmetric, and must therefore be identical for all qubits. We can easily test the Peres–Horodecki (partial transposition) condition [22] for separability of this state. Two qubits are entangled if and only if they are inseparable (meaning that they are not of the form $\sum_i p_i\rho_1^i \otimes \rho_2^i$, where p_i are probabilities and

ρ_1^i and ρ_2^i any states of the first and second qubits, respectively) which leads to states $\sigma_{12}(k)$ being entangled if and only if

$$a + b - \sqrt{(a-b)^2 + 4c^2} < 0, \tag{12}$$

which leads to

$$(k-1)(n-k-1) < k(n-k). \tag{13}$$

This equation is satisfied for all $n \geqslant 2$ (two qubits or more) and $1 \leqslant k \leqslant n-1$. So, apart from the case when the total state is of the form $|000..0\rangle$ or $|111..1\rangle$, there is always two-qubit entanglement present in symmetric states. Note, however, that in the limit of n and k becoming large—no matter what their ratio may be—the value of the left-hand side approaches the value of the right-hand side and entanglement thus disappears. This is a very interesting property of symmetric states and we will be able to quantify it exactly in the next section.

An important point to make is that the two-point correlation function used in the calculation of the ODLRO in equation (6) is, in fact, just one of the 16 numbers we need for the full two-site density matrix (the independent number of real parameters is actually 15, because of normalization). In our simplified case of symmetric states in the η-pairing model, this off-diagonal element is equal to c. However, for the density matrix, we still need to know a and b, and these numbers clearly affect the amount of entanglement. Imagine, for example, the situation where $a = b$. Then the condition for entanglement is that $a - c < 0$, which does not hold if $a \geqslant c$ and such a density matrix is certainly possible. So, the first lesson is that two-site entanglement is not the same as the existence of ODLRO and, therefore, two-site entanglement is not relevant for superconductivity. This does not mean, of course, that there is no entanglement in the whole of the lattice. In the next section, we will calculate exactly this. We will determine the relative entropy of entanglement for all symmetric states and all their substates. We will be able to extend the method of Wei *et al* [9] and analyse many relationships between various subsets of symmetric states, including the amount of entanglement in any subset of qubits (or sites).

4. Relative entropy of entanglement for symmetric states

The symmetric states are very convenient for studying various features of multipartite entanglement simply because, as already indicated, one can compute exactly the relative entropy of entanglement for any reduced state including the total symmetric state for any n and k. It is expected that, because they possess a high degree of symmetry, they will also display a high degree of entanglement. It is precisely for this reason that they are suitable to allow the existence of entanglement at high temperatures. This will now be analysed in detail.

We will first introduce the relative entropy of entanglement. This measure is very suitable because it is completely general: it can be defined for any number of subsystems of any dimensionality [6]. It also has close connections with macroscopic thermodynamical quantities as explained in [3]. The relative entropy of entanglement measures the distance between a state and the nearest disentangled (separable) state. If $\mathcal{D}$ is the set of all disentangled states (i.e. states of the form $\sum_i p_i \rho_1^i \otimes \rho_2^i \cdots \otimes \rho_n^i$, where p_i is any probability distribution), the measure of entanglement for a state σ is then defined as

$$E(\sigma) := \min_{\rho \in \mathcal{D}} S(\sigma || \rho), \tag{14}$$

where $S(\sigma||\rho) = \mathrm{tr}(\sigma\log\sigma - \sigma\log\rho)$ is the relative entropy between the two density matrices ρ and σ. To compute this measure for any state σ, we need to find its closest disentangled state ρ. Finding this closest state is, in general, still an open problem; however, it has recently been solved for pure symmetric states by Wei *et al* [9].

Wei *et al* showed that a convenient and intuitive way of writing the closest disentangled state to the symmetric state $|k, n-k\rangle$ is [9]

$$\rho = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\,\phi|\phi^{\otimes n}\rangle\langle\phi^{\otimes n}|, \tag{15}$$

where

$$|\phi^{\otimes n}\rangle = (\sqrt{k/n}|0\rangle + \sqrt{(n-k)/n}\mathrm{e}^{\mathrm{i}\phi}|1\rangle)^{\otimes n} \tag{16}$$

is the tensor product of n states each of which is a superposition of states $|0\rangle$ and $|1\rangle$ with probabilities k/n and $1-k/n$, respectively. This ρ was proved to achieve the minimum of the relative entropy by showing that it saturates an independently obtained lower bound. The relative entropy of entanglement of the total state is now easily computed. Since $\sigma = |k, n-k\rangle\langle k, n-k|$ is a pure state, $\mathrm{tr}\,\sigma\log\sigma = 0$ and we only need to compute $-\langle k, n-k|\log\rho|k, n-k\rangle$, which is equal to

$$E(|k, n-k\rangle) = -\log\binom{n}{k} + k\log\frac{n}{k} + (n-k)\log\frac{n}{n-k}. \tag{17}$$

Note that entanglement is largest when $n = 2k$ as is intuitively expected (i.e. the largest number of terms is then present in the expansion of the state in terms of the computational basis states). Then, for large n, it can be seen that the amount of entanglement grows as

$$E(|n/2, n/2\rangle) \approx \tfrac{1}{2}(\log n + 2) \tag{18}$$

and so (in the leading order) entanglement grows logarithmically with the number of qubits in the state. To obtain this formula we have used Sterling's approximation for the factorial

$$n! \approx 2.507 n^{n+1/2}\mathrm{e}^{-n}. \tag{19}$$

Most of the results in this paper will asymptotically have the form $\alpha\log n + \beta$, where $\alpha > 0$ and β are constants that will usually be omitted as we are only concerned about the general form of the behaviour.

We now return to the question of different phases introduced between different elements of the superposition in the symmetric states. Let us consider states of the form

$$|1, n-1, \theta\rangle = |00..1\rangle + \mathrm{e}^{\mathrm{i}\theta}|00..10\rangle + \mathrm{e}^{(n-1)\mathrm{i}\theta}|10..0\rangle, \tag{20}$$

where we have $k = 1$ ones and $n-1$ zeros and θ is any phase. The simplest way to see that entanglement does not depend on the phase θ is to define a new basis at the mth site as $|\tilde{0}\rangle = |0\rangle$, $|\tilde{1}\rangle = \exp\{(m-1)\theta\}|1\rangle$. This way the phases have been absorbed by the basis states and the resulting state is, in the tilde basis,

$$|1, n-1, \theta\rangle = |\tilde{0}\tilde{0}..\tilde{1}\rangle + |\tilde{0}\tilde{0}..\tilde{1}\tilde{0}\rangle + |\tilde{1}\tilde{0}..\tilde{0}\rangle. \tag{21}$$

The amount of entanglement must therefore be independent of any phase difference of the above type and this is, of course, true for symmetric states with any number of zeros and ones. All considerations from this point onwards will therefore immediately apply to all these states will different phases.

We can also compute the two-site relative entropy of entanglement exactly. The closest disentangled state is in this case the same as in equation (15) with $n = 2$. In the computational basis we have

$$\rho = \left(\frac{k}{n}\right)^2 |00\rangle\langle 00| + \left(\frac{n-k}{n}\right)^2 |11\rangle\langle 11| + \left(\frac{2k(n-k)}{n^2}\right) |\psi^+\rangle\langle\psi^+|. \tag{22}$$

That this is a minimum can be seen from the fact that the relative entropy of the state of two qubits is

$$S(\sigma||\rho) = -S(\sigma) - \langle\psi^+|\log\rho|\psi^+\rangle - \langle 00|\log\rho|00\rangle - \langle 11|\log\rho|11\rangle \tag{23}$$

$$\geqslant -S(\sigma) - \log\langle\psi^+|\rho|\psi^+\rangle - \log\langle 00|\rho|00\rangle - \log\langle 11|\rho|11\rangle, \tag{24}$$

the inequality following from concavity of the log function. Suppose now that ρs only non-zero elements are $\rho_{00} = \langle 00|\rho|00\rangle$, $\rho_{11} = \langle 11|\rho|11\rangle$ and $\rho_{++} = \langle\psi^+|\rho|\psi^+\rangle$. Given that it has to be separable, i.e. that $2\sqrt{\rho_{00}\rho_{11}} \geqslant \rho_{++}$ (which follows from the Peres–Horodecki criterion), and that, at the same time, it has to be closest to σ, we can conclude that $\rho_{00} = k/n$. The other entries of ρ then follow.

To prove that ρ is the minimum in a rigorous fashion, we need to show that any variation of the type $(1-x)\rho + x\omega$, where ω is any separable state, leads to a higher relative entropy (a method similar to [8]). Since relative entropy is a convex function, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} S(\sigma||(1-x)\rho + x\omega) \geqslant 0. \tag{25}$$

In fact, since relative entropy is convex in the second argument it is enough to assume that ω is just a product state.

For $a > 0$, $\log a = \int_0^\infty (at-1)/(a+t)\,\mathrm{d}t/(1+t^2)$ and, thus, for any positive operator A, $\log A = \int_0^\infty (At-1)/(A+t)\,\mathrm{d}t/(1+t^2)$. Let $f(x,\omega) = S(\sigma||(1-x)\rho + x\omega)$. Then

$$\begin{aligned}\frac{\partial f}{\partial x}(0,\omega) &= -\lim_{x\to 0}\mathrm{Tr}\left\{\frac{\sigma(\log((1-x)\rho + x\omega) - \log\rho)}{x}\right\}\\ &= \mathrm{Tr}\left\{\left(\sigma\int_0^\infty (\rho+t)^{-1}(\rho-\omega)(\rho+t)^{-1}\,\mathrm{d}t\right)\right\}\\ &= 1 - \int_0^\infty \mathrm{Tr}(\sigma(\rho+t)^{-1}\omega(\rho+t)^{-1})\,\mathrm{d}t\\ &= 1 - \int_0^\infty \mathrm{Tr}((\rho+t)^{-1}\sigma(\rho+t)^{-1}\omega)\,\mathrm{d}t.\end{aligned} \tag{26}$$

For our minimal guess ρ in equation (22) we can then write

$$\frac{\partial f}{\partial x}(0,\omega)-1 = -\operatorname{Tr}\left\{\omega\int_0^\infty(\rho+t)^{-1}\sigma(\rho+t)^{-1}\,\mathrm{d}t\right\}$$
$$= \frac{n}{n-1}\frac{k-1}{k}\langle 00|\omega|00\rangle+\frac{n}{n-1}\frac{n-k-1}{n-k}\langle 11|\omega|11\rangle$$
$$+\frac{n}{n-1}\langle\psi^+|\omega|\psi^+\rangle, \tag{27}$$

where we have used the fact that $\int_0^\infty (p+t)^{-2}\,\mathrm{d}t = p^{-1}$. Since the expression in the previous equation is always less than or equal to a unity if $\omega = |\alpha\beta\rangle\langle\alpha\beta|$ (i.e. a product state), it follows that

$$\left|\frac{\partial f}{\partial x}(0,\omega)-1\right| \leqslant 1. \tag{28}$$

Thus it also follows that $(\partial f/\partial x)(0, |\alpha\beta\rangle\langle\alpha\beta|) \geqslant 0$. Since any separable state can be written in the form $\rho = \sum_i r_i|\alpha^i\beta^i\rangle\langle\alpha^i\beta^i|$, therefore

$$\frac{\partial f}{\partial x}(0,\rho) = \sum_i r_i\frac{\partial f}{\partial x}(0, |\alpha^i\beta^i\rangle\langle\alpha^i\beta^i|) \geqslant 0. \tag{29}$$

This confirms that ρ is the minimum since the gradient is positive.

Therefore the relative entropy of entanglement between any two sites is

$$E_{12} = a\log a - b\log b - 2c\log 2c$$
$$-a\log\left(\frac{k}{n}\right)^2 - b\log\left(\frac{n-k}{n}\right)^2 - 2c\log\left(\frac{2k(n-k)}{n^2}\right)$$
$$= \log\left(\frac{n}{n-1}\right)+\frac{k(k-1)}{n(n-1)}\log\left(\frac{k-1}{k}\right)+\frac{(n-k)(n-k-1)}{n(n-1)}\log\left(\frac{n-k-1}{n-k}\right). \tag{30}$$

We see that when $n, k, n-k \to \infty$, then $E_{12}\to 0$ as it should be from our discussion of the separability criterion. This can be thought of as one way of recovering the 'quantum to classical' correspondence in the limit of large number of systems present in the state: locally, between any two sites, entanglement does vanish, although globally, and as will be seen in more detail, entanglement still persists.

Entanglement between any number of qubits, $l \leqslant k$, can also be calculated using the same method . The state after we trace out all but l qubits is given by

$$\sigma_l = \sum_{i=0}^{l}\binom{l}{l-i}\frac{\binom{n-l}{k-i}}{\binom{n}{k}}|i, l-i\rangle\langle i, l-i|. \tag{31}$$

The closest disentangled state is given by

$$\rho_l = \sum_{i=0}^{l} \binom{l}{i} \left(\frac{k}{n}\right)^{l-i} \left(\frac{n-k}{n}\right)^{i} |i, l-i\rangle\langle i, l-i|, \tag{32}$$

as can be shown by the above method. The relative entropy of entanglement is now given by

$$E_l = \sum_{i=0}^{l} \binom{l}{l-i} \frac{\binom{n-l}{k-i}}{\binom{n}{k}} \log \left\{ \binom{l}{l-i} \frac{\binom{n-l}{k-i}}{\binom{n}{k}} \left(\frac{n}{k}\right)^{l-i} \left(\frac{n}{n-k}\right)^{i} \binom{l}{i}^{-1} \right\}. \tag{33}$$

This is a very interesting quantity as it allows us to speak about entanglement involving any number of qubits. What do we expect from it? We expect that entanglement grows exponentially with l, for a fixed total number of qubits, n. This can be confirmed using the Sterling formula. Note that entanglement grows at this rate even though the states we are talking about are mixed, since $n - l$ qubits have been traced out. Another way of seeing why entanglement grows exponentially with the number of qubits included for a total fixed number of qubits is to look at the opposite regime. For any finite fixed l, we should have that in the large $n, k, n-k$ limit the amount of entanglement between l tends to zero. This decrease with larger and larger n occurs at an exponential rate.

Finally, we calculate the entanglement between l qubits and the remaining $n - l$ qubits. Since the whole state that we are now examining is pure, the relative entropy of entanglement is given by the entropy of the l qubits:

$$S_{12\ldots l} = -\sum_{i=0}^{l} \binom{l}{l-i} \frac{\binom{n-l}{k-i}}{\binom{n}{k}} \log \left\{ \binom{l}{l-i} \frac{\binom{n-l}{k-i}}{\binom{n}{k}} \right\}. \tag{34}$$

What are the properties of this expression when we take the various asymptotic limits? How is this quantity related to other entanglements calculated here? We expect that for the half-filling, $n/k = 2$, and $n, l \to \infty$, the entropy becomes $\log l$, since we basically have a maximal mixture in the symmetric subspace of l qubits. This can be confirmed by a simple application of the Sterling approximation formula used before. The result is in agreement with the fact that total entanglement grows at the rate of the log of the number of qubits, since two-cluster entanglement is a lower bound for the total entanglement in the state between all the qubits.

5. Classical versus quantum correlations

In this section we would like to investigate the relationship between classical and quantum correlations for symmetric states, and both in relation to the already introduced concept of ODLRO. First of all, it is clear that in the limit of $n \to \infty$ all bipartite (or two-site) entanglement disappears (this was seen both from the Peres–Horodecki criterion and from the direct computation of the relative entropy). In spite of this, the ODLRO still exists and the two quantities are therefore not related. In other words, two-site entanglement is not relevant for superconductivity. However, the main point of this section is that the two-site classical correlations still survive in the limit of $n \to \infty$. To show this, let us first of all define bipartite classical correlations.

A quantum state can have zero amount of entanglement, but still have non-zero classical correlations. An example is the state $|00\rangle\langle 00| + |11\rangle\langle 11|$. Classical correlations between systems A and B in the state σ_{AB} can be defined as [23]

$$C_A(\sigma_{AB}) := \max_{A_i^\dagger A_i} S(\sigma_B) - \sum_i p_i S(\sigma_B^i) = \max_{A_i^\dagger A_i} \sum_i p_i S(\sigma_B^i || \sigma_B), \tag{35}$$

where $\sigma_B^i = \mathrm{tr}_A \sigma_{AB}^i$, $\sigma_{AB}^i = A_i \sigma_{AB} A_i^\dagger$, and $\sum_i A_i^\dagger A_i = 1$ is the most general measurement on system A. The same can be defined with the most general measurement performed on B, so that we obtain

$$C_B(\sigma_{AB}) := \max_{B_i^\dagger B_i} S(\sigma_A) - \sum_i p_i S(\sigma_A^i) = \max_{B_i^\dagger B_i} \sum_i p_i S(\sigma_A^i || \sigma_A). \tag{36}$$

The physical motivation behind the above definition is the following: classical correlations between A and B tell us how much information we can obtain about A (B) by performing measurements in B (A). It is the (maximum) difference between the entropy of A (B) before and after the measurement on B (A) is performed. There is some evidence that $C_A = C_B$ [23], but this equality will not be relevant here.

Now, applying this measure of classical correlations to the two-site reduced density matrix from the overall symmetric state, ρ_{12}, we obtain

$$\begin{aligned} C &= -a \log a - b \log b - c \log c + \tfrac{1}{2}((a + c/2) \log(a + c/2) + (b + c/2) \log(b + c/2)) \\ &= (r - 2r^2) \log r + ((1 - r) - 2(1 - r)^2) \log(1 - r) - 2r(1 - r) \log 2r(1 - r), \end{aligned} \tag{37}$$

where $r = k/n$ is the fraction of ones in the state (the so-called filling factor in any 'Cooper pair' lattice model, including the η model). We now see that at half-filling—when ODLRO is maximal—the classical two-site correlations also survive asymptotically since $C_A = C_B = 0.5$. Therefore, all the correlations between any two sites are here due to classical correlations.

Note, incidentally, that we cannot have the situation in which entanglement exists between two parties, while at the same time classical correlations vanish. Quantum correlations presuppose the existence of classical correlations. This, of course, relies on the fact that entanglement is defined in a reasonable way, namely that when we talk about two-site entanglement we must trace the other sites out. We are not allowed to perform measurements on other sites and condition the remaining entanglement on them. Measurements that generate entanglement are, first of all, unrealistic for a macroscopic object which thermalizes very quickly. Even if we were to allow such measurements, then the state after them will still have classical correlations of at least the same magnitude as entanglement. So, it cannot be that entanglement is important for the issues of superconductivity, phase transitions, condensation, etc, and that classical correlations are not.

As an example, let us take the 'maxmum singlet fraction' in the two-site density matrix σ_{12} as our definition of entanglement. This is the maximum fraction of a maximally entangled state in the state σ_{12}, which is in this case equal to c, and this is the same as ODLRO. So, if the maximum singlet fraction is used to measure entanglement, then entanglement also persists in the thermodynamical limit. In fact, as will be shown later, this measure also survives when we mix symmetric states, because it is a linear measure. The maximum singlet fraction, however, is not a realistic measure of entanglement as it is not easily accessible experimentally, hence we do not use it in this paper.

To make our analysis more complete we also show how to calculate mutual information [6] for symmetric states. This quantity tells us about the total (quantum plus classical) correlations in a given state. Mutual information is equal to the relative entropy between the state itself and the product of individual qubit density matrices, obtained by tracing out all the other qubits. This product state is easily written as

$$\rho_{prod} = \left(\frac{k}{n}|0\rangle\langle 0| + \frac{n-k}{n}|1\rangle\langle 1|\right)^{\otimes n}. \tag{38}$$

The mutual information is now given by

$$I(|k, n-k\rangle) = n\left(-\frac{k}{n}\log\frac{k}{n} - \frac{n-k}{n}\log\frac{n-k}{n}\right), \tag{39}$$

and this is basically just the sum of individual qubit entropies. Since the qubit entropy (the quantity in parenthesis in the above equation) is a finite quantity for a given ratio $r = k/n$, the total mutual information grows linearly with the number of qubits n. Furthermore, since entanglement grows as $\log n$, we conclude that classical correlations grow roughly as $n - \log n$ (for this conclusion to be exact, classical and quantum correlations as defined here would have to add up to mutual information; while this is true for some states [23], it is certainly not true in general).

The fact that classical correlations and mutual information survive the thermodynamical limit does not imply that there in no meaning left for entanglement when it comes to superconductivity and ODLRO. Only now, we must talk either about the bipartite entanglement between two clusters of sites (computed in the previous section) or the multipartite entanglement between all sites. Since the overall state across all sites is pure in our considerations so far, this means that two-site non-vanishing classical correlations (or equivalently ODLRO) must imply entanglement between two clusters, each of which contains one of the sites and such that the union of the two clusters is the whole lattice. This simply must be the case, since, otherwise, if the clusters were not entangled, the total state would be a product of the states of individual clusters, and this means that even classical correlations would be zero, which is a contradiction. Furthermore, the fact that any two such clusters are entangled, implies that the multipartite entanglement also exists, since this entanglement is by definition larger than any bipartite entanglement (as, for multipartite entanglement, we are looking for the closest separable state over all sites, rather than just over the two clusters).

6. Thermal entanglement and superconductivity

There is a critical temperature beyond which any superconductor becomes a normal conductor. The basic idea behind computing this temperature according to BCS is the following. At a very low temperature, only the ground state of the system is populated and for a superconductor this state involves a collection of Cooper pairs with different momenta values around the Fermi surface. This state can be, somewhat loosely, thought of as a Cooper pair condensate, and it is this condensation that is the key to superconductivity. It took initially a long time to understand how the pairs are formed, since electrons repel each other and therefore should not be bound together. The attraction is provided by electrons interacting with the positive ions left in the lattice. We can think of one electron moving and dragging along the lattice, which then pulls other electrons thereby providing the necessary attraction [12]. When the temperature starts to increase, the Cooper pairs start to break up, leading to the transition to the normal conductor. What this 'breaking up' means is that higher than ground states start to get populated by electrons, and these are states where an electron is created with say momentum k and spin-up, but no electron is created in the $-k$ momentum state. From the BCS analysis this critical temperature can be calculated to be of the form [12]

$$T_c \approx \frac{\hbar\omega}{k} \mathrm{e}^{-1/\lambda}, \tag{40}$$

where $\hbar\omega$ is the energy shell around the Fermi surface which is engaged in formation of Cooper pairs, k the Boltzmann constant and λ a parameter equal to the product of the electron density at the Fermi surface $N(0)$ and the effective electronic attractive coupling, V. The critical temperature formula is valid in the weak coupling regime where $\lambda = N(0)V \ll 1$.

The formula for the critical temperature is usually used for other mechanisms of electron pairing, and not just coupling via the phonon lattice modes as in the BCS model [12]. Importantly for us, the formula also features in models for explaining and designing high T_c superconducting materials. If the attraction, say between an electron and a hole, is of the order of Coulomb forces, $\hbar\omega \approx 1$ eV m, and for the weak coupling of, say, $\lambda = 0.2$, the critical temperature we obtain is 100 K. So if the material is below this temperature, it is then superconducting. Anything above 70–90 K is considered to be high-temperature superconductivity, since it can be achieved by cooling with liquid nitrogen (which is a standard and easy method of cooling). What seems to be the mechanism behind high-temperature superconductivity is the fact that the energy gap between the ground superconducting, electron-pair state and the excited states is large enough not to be easily excited as the temperature increases well beyond zero temperature. The exact way in which this is achieved is still an open question. In the models mentioned here the ground state is one of the symmetric states from the previous sections. Therefore, we can conclude that as long as we have high-temperature superconductivity, the total state should also be macroscopically entangled. Superconduction and hence entanglement can currently exist at temperatures of about 160 K.

We would now like to explicitly calculate and show how entanglement disappears as the temperature increases for any model having the η pairing state as the ground state. For this, we need to describe other states that would be mixed in with the symmetric η states as the temperature increases. They, of course, depend on the actual Hamiltonian. For instance, in the Hubbard model in [11], states of the type

$$\xi_a^\dagger|0\rangle = \sum_i c_{i,\downarrow}^\dagger c_{i+a,\uparrow}^\dagger |0\rangle \tag{41}$$

are important; here we create a spin singlet state but at sites separated by the distance $a \neq 0$. If we have $2k$ electrons in total, then $2k-2$ would be paired in the lowest energy state, and the remaining two electrons would not be. This would give us the state of the form

$$|\xi\rangle := \eta^{k-1}\xi_a^\dagger|0\rangle. \tag{42}$$

Note that this state is a symmetric combination of states which have $k-1$ electron pairs distributed among n sites and the last electron pair is in two different sites separated by the distance a. These two sites are different from the other $k-2$ sites due to Pauli's exclusion principle. Even higher states are obtained by having two electron pairs existing outside of the symmetric state and so on. The exact form of these, as noted before, depends on the exact form of the Hamiltonian. Even simple Hamiltonians are frequently very difficult to diagonalize and their eigenstates are still by and large unknown. Given this, it may be difficult to calculate the exact amount of entanglement when, at finite temperature, the ground state is mixed with higher energy states. We will, therefore, make a simplifying assumption that, if the ground state is $|k, n-k\rangle$, the higher energy states can be written as $|k-1, n-k+1\rangle$, $|k-2, n-k+2\rangle$ and so on. All these will in fact be assumed to be symmetric and we will ignore the extra unpaired electrons as far as entanglement is concerned (they will only contribute to the eigenvalue of energy as it were).

This assumption leads us to consider mixtures of symmetric states. The symmetric states will be mixed with probabilities in accordance to Boltzmann's exponential law or the Fermi–Dirac law if we talk about η pairs. The distribution we use will be immaterial for our argument. The total state, σ_T, is

$$\sigma_T = \sum_{k=0}^{n} p_k |\Psi(k,n)\rangle\langle\Psi(k,n)|, \tag{43}$$

where, in the case of η pairs, the probabilities are

$$p_i = \frac{1}{\mathrm{e}^{E_i/kT}+1}, \tag{44}$$

where p_i is the probability of occupying the ith energy level. The reduced two-site state can be calculated to be

$$\sigma_{12} = \sum_{k=0}^{n} p_k \sigma_{12}(k). \tag{45}$$

The condition for inseparability now becomes

$$\sum_{k,l} p_k p_l k(n-l)\{(n-k)l-(k-1)(n-l-1)\} > 0. \tag{46}$$

We see that the thermal averaging is in a sense inconsequential for the existence of entanglement as the factors $p_k p_l$ are probabilities and are always non-negative. For inequality to hold (i.e. to have non-zero bipartite entanglement present) we need that $1 \leqslant k, l \leqslant n-1$. This is the same condition as before when the total state was pure. Thus, surprisingly, the condition for inseparability is completely independent of temperature (although two-site states do become separable in the macroscopic limit even at zero temperature, as noted before).

We now look at the entanglement of the symmetric mixed state as a whole. Can we still calculate the relative entropy of entanglement? This is in general very difficult to do for multiparty mixed states, and some partial methods for upper bounds have only been presented recently [3]. We conjecture that the closest disentangled state is now presumably the thermal average of the closest disentangled states for individual k's (this, we believe, is the same as the conjecture in [9], for which Wei *et al* have offered a great deal of 'circumstantial evidence'; for example, closest separable states have to possess the same symmetry as the entangled states for which they minimize the relative entropy [24]). We believe that this bound is exact and that this can be proven using methods for calculating two-site entanglement, but we have not been able to show this yet. Even if this is not true, our method at least gives us a very good upper bound which is sufficient to show how total entanglement vanishes as T becomes high. The relative entropy of entanglement between these two states is given by (the right-hand side of the inequality)

$$E(\sigma_T) \leqslant \sum_k p_k \log p_k - \sum_k p_k \langle \Psi(k,n)| \log \left(\sum_l p_l \rho_l \right) |\Psi(k,n)\rangle, \tag{47}$$

where ρ_l is the closest disentangled state to the pure symmetric state containing l ones and $n-l$ zeros. We have already seen that

$$\rho_l = \sum_{i=1}^{l} \binom{l}{i} \left(\frac{k}{n}\right)^{l-i} \left(\frac{n-k}{n}\right)^{i} |\Psi(l,n)\rangle\langle\Psi(l,n)|, \tag{48}$$

so that

$$\begin{aligned} E(\sigma_T) &\leqslant \sum_k p_k \log p_k \\ &\quad - \sum_k p_k \langle \Psi(k,n)| \log \left\{ \sum_l p_l \sum_{i=1}^{l} \binom{l}{i} \left(\frac{k}{n}\right)^{l-i} \left(\frac{n-k}{n}\right)^{i} \right\} |\Psi(l,n)\rangle\langle\Psi(l,n)||\Psi(k,n)\rangle \\ &= -\sum_k p_k \log \sum_{i=1}^{k} \binom{k}{i} \left(\frac{k}{n}\right)^{k-i} \left(\frac{n-k}{n}\right)^{i}. \end{aligned} \tag{49}$$

The interesting conclusion here is the following. Suppose that we are at a high temperature and that all symmetric states are equally likely, i.e. $p_k = 1/(n+1)$ for all values k (basically, our state is an equal mixture of all symmetric states). This, of course, in an approximation to the true density matrix, but it becomes more and more accurate with the increase in temperature and it ceases to be so when states other than symmetric become mixed in. The (upper bound to) entanglement is then given by

$$E(\sigma_{T\to\infty}) \leqslant \frac{1}{n+1} \log \sum_{i=1}^{k} \binom{k}{i} \left(\frac{k}{n}\right)^{k-i} \left(\frac{n-k}{n}\right)^{i}. \tag{50}$$

The fraction inside the log tends to n^2 as n becomes large, so that entanglement scales as $\log n/n$. This is to be expected since entanglement grows as $\log n$ with n, but the mixedness grows linearly with the number of state involved, $n+1$. Therefore, in the thermodynamical limit, the overall mixed state entanglement also disappears. This has to eventually occur, of course, if we believe that entanglement is intimately linked with superconductivity and superconductivity also vanishes at sufficiently high temperatures.

One kind of entanglement that we can say survives the thermodynamical high-temperature limit is the average of entanglements of individual symmetric states. This average entanglement is given by

$$E_{avr} = \sum_k p_k E(|k, n-k\rangle) = \frac{1}{2}\sum_k p_k \log \frac{k(n-k)}{n}. \tag{51}$$

Note that if all probabilities behave as $1/n$, i.e. the symmetric state is maximally mixed, then the entanglement scales as $\log n$ (the same as pure state at half-filling). This is expected, as there are $n+1$ states and each one has entanglement proportional to $\log n$ and, so, on average, entanglement also behaves as $\log n$. However, this average entanglement, as we argued before is not a good measure as it requires to address the symmetric states individually and discriminate them from each other. This is not just difficult in practice, but is in fact frequently even impossible in principle.

It is interesting to note that the ODLRO does survive the mixing of symmetric states. Even when we have an equal mixture of all symmetric states the average ODLRO is given by

$$\frac{1}{n+1}\sum_{k=0}^{n} \frac{k}{n}\frac{(n-k)}{n} = \frac{1}{2} - \frac{1}{6}\frac{2n+1}{n} \rightarrow \frac{1}{6}, \tag{52}$$

where the arrow indicates the convergence when n is large. Of course, at sufficiently high temperatures, the system will leave the subspace of symmetric states and other states will also start to contribute. This eventually does lead to vanishing of ODLRO, but the total entanglement and ODLRO may still disappear at different temperatures. To calculate this exactly, we would need a much more detailed model and a more extensive and careful calculation which lie outside the scope of the present paper. (Note that the same conclusions hold for the maximum singlet fraction in the two-site density matrix which also survives the mixing in the thermodynamical limit; this is unfortunately and as pointed out before, not a suitable measure of entanglement in our setting.)

We conclude by showing that total correlations—quantum and classical—as quantified by the mutual information [6] can also easily be calculated for thermal mixtures of symmetric states. Let us assume again that the symmetric states are maximally mixed and each appears with the probability $1/(n+1)$. Then the mutual information is given by

$$I = \frac{n}{n+1}\sum_k \left(-\frac{k}{n}\log\frac{k}{n} - \frac{n-k}{n}\log\frac{n-k}{n}\right) - \log(n+1). \tag{53}$$

For large n this expression reduces to

$$I \rightarrow n - \log n. \tag{54}$$

Since we know that thermal entanglement disappears in this limit, it is natural that the mutual information is equal to the classical correlations and this then coincides with the conclusion following equation (39).

7. *D*-dimensional symmetric states

Extensions of all the considerations in this study to D-dimensions are seen to be very straightforward (similar generalizations to higher dimension symmetric states were also considered in [9]). We should actually be able to reproduce all the above results in the generalized form, such that instead of qubits we have qutrits, and so on. The generic symmetric state would now be written as

$$|n_1, n_2, \ldots, n_d\rangle \tag{55}$$

and it would be a totally symmetrized state of n_1 states $|1\rangle$, n_2 states $|2\rangle$ and so on (it is also realistic to assume that the total number of particles is conserved). This could, for example, represent higher spin fermions which can occupy different lattice sites as in the rest of the paper. The closest state to the one in equation (55) in terms of the relative entropy is a mixture of the states of the type

$$(\sqrt{n_1/N}|1\rangle + \mathrm{e}^{\mathrm{i}\phi}\sqrt{n_2/N}|2\rangle + \cdots \mathrm{e}^{\mathrm{i}(d-1)\phi}\sqrt{n_d/N}|d\rangle)^{\otimes n}, \tag{56}$$

with the phase ϕ completely randomized as before. Knowledge of this closest state allows us to compute the relative entropy of entanglement of any number of subsystems of this system. All other results follow in exactly the same way. Nothing fundamental is changed in higher dimensions and hence we will not say anything more on this topic.

8. Discussion and conclusions

In this paper we have analysed the η-pairing mechanism which leads to eigenstates of the Hubbard and similar models used in explaining high-temperature superconductivity. We have shown that they correspond to multi-qubit symmetric states, where the qubit is made up of an empty and a full site (two-electron spin singlet state). We have also shown how to calculate entanglement and classical correlations for such states. For pure states, entanglement of the total state increases at the rate $\log n$ with the number of qubits n, whereas two-site entanglement vanishes at the rate $1/n$. The two-site classical correlations, on the other hand, persist in the thermodynamical limit. So, the ODLRO can be associated for pure states with total entanglement or two-site classical correlations, but not with the two-site entanglement. We have also demonstrated that the total entanglement for maximally mixed symmetric states disappears at the rate $(\log n)/n$. Various mutual information measures, which quantify the total amount of correlations in a given state, are also computed and shown to be consistent with the calculations of classical and quantum correlations.

There are many interesting issues raised by this work. Even if a consensus is reached on the correct model for high T_c superconductivity, and this is shown to contain multipartite electron entanglement—which we have argued for in this paper—we are still left with the question of being able to extract and use this entanglement. At present there are no methods of extraction. Perhaps we can somehow extract electrons from the superconductor and then use them for quantum teleportation or other forms of quantum information processing.

It is presently believed that to perform a reliable and scalable quantum computation we may need to be at very low temperatures, but the existence of high-temperature macroscopic entanglement may just challenge this dogma. Be that as it may, we believe that the argument in favour of the existence of high-temperature entanglement does show that entanglement may be much more ubiquitous than is presently thought. This may force us to push the boundary between the classical and the quantum world towards taking seriously the concept that quantum mechanics is indeed universal and should be applied at all levels of complexity, independent of the number or, indeed, nature of particles involved.

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