Semi-robust static Hedging of Barrier Options

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To Werner, Renate and Kim

Who supported me throughout the time I was writing this thesis without questioning the important role I allowed this work to take in my life.

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ABSTRACT

We explore how to put the theory on static hedges of barrier options into use. We discuss a polynomial expansion of the exact static hedge in a stationary diffusion model provided by [9] and we develop an explicit expression of an asymptotic static hedge, which is constructed to perform well for short maturities. We derive a semi-robust static hedge in a sense, that it is model independent and depends upon one's beliefs about the future values of implied volatility only.

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Introduction

Barrier options, also known as knock-in or knock-out options, are vanilla options, which come into existence or go out of existence, when certain prices has been reached. For example an up-and-out put (UOP) with strike K and barrier U is a vanilla put struck at K, which expires worthless, if the barrier U is hit during the lifetime of the derivative. That is we are considering a derivative with the following payoff at the time of maturity T:

$$\mathbf{1}_{\{\sup_{t\in[0,T]} S_t<U\}} \cdot (K - S_T)^+$$

Barrier options were first traded in the late sixties over the OTC market around the time vanilla options started to trade. Today they remain one of the most liquidly traded exotic options, especially in the foreign exchange market. One of the main reasons for this is, that barrier options trade at lower prices than the corresponding vanilla options. This attracts investors with a strong view on the underlying asset.

Therefore it is not surprising, that the literature on pricing and hedging barrier options is vast. In the classical Black-Scholes setting established by [21] analytic formulae for the prices of barrier options are available. This result developed by E. Reiner and M. Rubinstein in [23] heavily relies on the symmetry property of the underlying Brownian motion, which is well known as the *reflection principle*. Its counterpart in the PDE language is the famous *method of images*. However, this cannot be generalized to an arbitrary diffusion model in a straight-forward way. Nevertheless the theory for static hedging in stationary diffusion models established in [9] essentially uses this type of symmetry in a more general sense. In our project we shall closely follow this lead.

Classical Black-Scholes theory also provides a dynamic hedging strategy in a straightforward way. The great drawback thereof is that it requires to hold large positions in the underlying as we approach the barrier. In fact, the strategy is to hold $\Delta(S_t)$ units of the underlying asset at time t, where $\Delta(S_t)$ is the sensitivity of the value of the derivative with respect to changes in the underlying asset provided that the current level is S_t. It is intuitive, that $\Delta(S)$ is in fact exploding as $S \nearrow U$. In practice this can be "solved" by hedging with a slightly higher barrier (again using the example of an up-and-out put). However, this

is still expensive as barrier options exhibit a high gamma, especially for short maturities – a fact, that makes trading of large amounts in the underlying necessary and therefore the bid-ask becomes spread important.

An alternative approach is the use of a so called *static hedge*. The fundamental idea appears to be in some sense similar to the one of Heath, Jarrow and Morton [17], which changed the approach to Fixed Income modelling. We do no longer consider prices of vanilla European options as derivatives on the underlying asset, but rather as exogenously given by the market. From this standpoint it is natural to use plain vanilla options for the purpose of hedging exotic options.

Following [9] we understand a static hedge[1] of a UOP with strike K and barrier U as a portfolio consisting of one vanilla put struck at K and a European derivative with payoff function $g : [0, \infty) \to \mathbb{R}$ where g has support in $[U, \infty)$ and is such that the European derivative with payoff $g(S_T)$ has the same price as the vanilla put struck at K, whenever the underlying hits the barrier.

To understand why this is a static hedge in the intuitive sense consider the following trading strategy: Suppose at time zero we sell an UOP with strike K and barrier U, buy a vanilla put struck at K and sell a European derivative with payoff $g(S_T)$, where T is the time of maturity of all the instruments under consideration. In case the underlying hits the barrier at any time within the lifetime of our trade, the UOP is worthless and we can liquidate our hedge at no cost. In case it does not, the payoff at T of the UOP we sold and the vanilla put we bought coincide and $g(S_T) = 0$ — the net payoff is zero.

J. Bowie and P. Carr show in [5], that in case of the Black (zero-drift Black-Scholes) model the function g becomes

$$g(x) = -\frac{K}{U}\left(x - \frac{U^2}{K}\right)^+$$

Note that this means the European derivative, which is shorted in the static hedge portfolio at initiation is $\frac{K}{U}$ calls struck at $\frac{U^2}{K}$. This motivates our choice to consider asymptotic static hedges g of this form later on.

As a next step people worked on ways to relax the drift condition assumed in [5]. E. Derman, D. Ergener and I. Kani [14] develop an algorithm for computing the static hedge of a barrier option in an arbitrary binomial model using options with a single strike, but multiple maturities. A couple of years later P. Carr and A. Chou [7] provide explicit formulae

[1] A rigorous definition of this notion can be found in Definition 2.1.1.

for static hedges in the Black-Scholes model with drift using options with the same expiry, but multiple strikes. To this end they use static replication with "down-and-in Arrows" by showing that the value of such a path-dependent security with strike K and barrier H matches the value of a suitably weighted path-independent Arrow struck at the geometric reflection of K at H. The static hedge of an arbitrary down-and-in security then is simply a static portfolio of down-and-in Arrows struck below the barrier.

Note that in the classical setting addressed above the problem of pricing essentially corresponds to the problem of static hedging. In more general diffusion models however, these two tasks become quite different.

A first generalization of the classical Black-Scholes setting is presented by P. Carr, K. Ellis and V. Gupta in [8]. The authors make use of the so called "put-call-symmetry" (PCS), which is a modification of the well-known put-call parity (PCP). However, in contrary to the latter, the PCS is not model-independent. It assumes the drift of the underlying price process to be zero and more importantly the volatility structure is assumed to be *symmetric* in the sense that the volatility is given as $\sigma(F_t, t) = \sigma(\frac{F^2}{F_t}, t)$ for some constant F. Then the PCS states that

$$C(K)K^{-\frac{1}{2}} = P(H)H^{-\frac{1}{2}} \qquad \text{with } (KH)^{\frac{1}{2}} = F$$

However, a model in which the above relation holds true, cannot account for the skew of the implied volatility. This is a significant disadvantage – especially in the light of the fact, that the static hedge is sensitive to the skewness of implied volatility.

A significant generalization is the main result in [9], which provides a formula for g in an arbitrary time-homogeneous diffusion model. In general (including the well known case of the CEV model) the derivative with payoff g is not the multiple of a traded derivative and it is difficult to approximate numerically. This is the motivation for S. Nadtochiy and P. Carr to analyse the short-maturity behaviour and an *asymptotic* hedge in section 3.3 of [9] for the CEV model. The central idea is to consider hedging portfolios, which consist of a vanilla put and a multiple of vanilla calls and see how good one can possibly do. In this thesis we further develop this idea and provide an asymptotic single-call hedge for an arbitrary local volatility, which satisfies certain smoothness and boundedness assumptions. We compare our result to an asymptotic expansion in the time-to-maturity variable of the exact static hedge in [9].

Interestingly the comparative statics analysis in [12] shows that the dynamic hedging strategy under the CEV model is significantly different to the one in the lognormal case – the hedges can even have different signs. This indicates, that hedges for barrier options

are sensitive with respect to changes in the implied volatility surface and intuitively this gives rise to the hope, that what we are doing in the case of static hedging might be relevant.

If we consider the more general case of an arbitrary, possibly time-inhomogeneous diffusion model, the result [2] by C. Bardos, D. Douady and A. Fursikov from theory of parabolic PDE's tells us, that we do not get an exact static hedge in the sense above any more.

As we mentioned already the literature on static hedging of barrier options is vast and a significant part of it focuses on the numerical approximation of the hedge using optimization based methods. Here we briefly want to mention two examples, which make use of this approach and exhibit some similarities to our project. First, M. Avelaneda and A. Parás [1] propose an algorithm for hedging portfolios using options to manage volatility risk. This is in some sense consistent with the industry practice of delta and vega hedging derivative books. Second, J. Maruhn and E. Sachs [20] robustify hedges in a stochastic volatility setup. To this end they calculate numerically, what they call a "cost optimal static super-replicating strategy" and – as they claim – achieve robust hedge portfolios, which are only marginally more expensive than the respective barrier options.

What both of these papers have in common with our project is the objective of deriving robust hedges considering vanilla derivatives. However the approaches are quite different – the papers above are solving a related optimization problem numerically, whereas our approach is based on analytic considerations.

Another well established way to think about pricing exotic derivatives is to derive model independent sub- and super-replicating strategies corresponding to price bounds of the derivative under consideration. Here the central idea is to infer information about prices of exotic claims from market prices of vanilla options by no arbitrage arguments. Price bounds derived in this way have the appealing property, that they are robust to model mis-specification.

The central question is of course, how good the bounds are, which can be achieved using this technique. In [6] the authors conclude that in the case of barrier options it is often sufficient to consider vanilla calls and puts of all strikes as possible hedging instruments to place tight bounds on the possible prices. The more recent works [10] and [11] of A.M.G. Cox and J. Obloj focus on the robust pricing and hedging of double touch barrier options and double no-touch options respectively.

In chapter 4 we discuss how we can use the previously developed model dependent re-

sults as building blocks to obtain semi-robust static hedges, which depend on our believes in the future values of the implied volatility only.

To this end we can leverage the following monotonicity property: If the implied volatility increases for strikes below K and decreases for strikes above U, this only increases the value of our hedge portfolio (as we are long a put struck at K and short a call struck at $K^* > U$). This allows us to provide a super-replicating strategy, if we give ourselves a lower bound for the implied vol for strikes below K and an upper bound for the vol above U — essentially this is making use of the fact that convex payoffs are monotone with respect to implied volatility, which allows us to provide an "extreme model". The next step is to use a model dependent result to connect the obtained "extreme model" to a static hedge.

In the final section of 4 we illustrate this idea in detail by applying it to our asymptotic static hedge. That means, that we are losing generality, but at the same type greatly gain tractability. We want this to be seen as a first step, there is more to be done in further research.

Our contribution is to develop a middle ground between the entirely model dependent approach and completely model-independent consideration. The advantage of the robust hedge developed in our thesis is its neat and natural interpretation. It is connecting a given belief on the future dynamics of the implied volatility surface to a semi-robust static hedge. This allows us to take a bet on implied volatility, while having a good understanding of the risk we are taking.

Our thesis is organized as follows. Section 1.1 defines the class of diffusion models we are considering and introduces our global assumptions. In section 1.2 we derive the mathematical results, which form the base of the subsequent theory. Subsection 1.2.1 transforms our setting to a canonical Sturm-Liouville problem, subsection 1.2.2 summarizes the classical results linked to thereof and subsection 1.2.3 provides asymptotic expansions – most importantly the one of the Laplace transform of put and call prices. Section 2.1 discusses the exact static hedge from [9], section 2.2 provides an approach to compute the latter and section 2.3 contains a numerical illustration. In section 3.1 we introduce the asymptotic hedge rigorously and derive an explicit formula for it. Section 3.2 illustrates the latter by considering the cases of Black and CEV model. Chapter 4 robustifies the asymptotic hedge making use of the building blocks developed in the previous chapters. Section 4.1 discusses a method to obtain semi-robust static hedges, section 4.2 studies the case of Gatheral-type implied volatility and section 4.3 presents our semi-robust asymptotic hedge.

Chapter 1

Model setup and mathematical toolbox

1.1 Model setup

We assume that the financial market under consideration prices all contingent claims by (discounted) expectations of the respective payoffs at time of maturity T under some risk neutral measure $\mathbb{Q}$. The riskless rate is assumed to be zero for simplicity.
In particular we consider an asset S_t following a time-homogeneous local volatility model with zero drift, that is the $\mathbb{Q}$-dynamics of S_t are given by

$$\mathrm{d}S_t = \sigma(S_t)\mathrm{d}W_t \text{ for } t \in [0,T] \text{ and } S_0 = s \in (0,\infty), \tag{1.1}$$

where $(W_t)_{t\in[0,T]}$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t))$.

Note that the assumption on the drift of the underlying is naturally true for the largest market of interest as the most liquid instruments in the Foreign Exchange market are futures and forwards. It is a well-known fact, that both of them are $\mathbb{Q}$-martingales, if we assume the riskless rate to be deterministic.

In addition we intend to consider asymptotic hedges for short maturities, where the influence of the drift and in a similar way the riskless interest rate is of lower order. E.g. in the case of the Constant Elasticity of Variance (CEV) model it is shown in [19], that the introduction of a drift has no impact on the strike K^* of the single-call hedge. It changes the number of calls, say η, held only. In principle our derivation can be repeated in the case of a local volatility model with drift. However, this complicates our computations and impacts the integrability assumptions we have to make during our derivation.

The following assumption on the local volatility function is central.

(A1) *We assume the local volatility function $\sigma : (0,\infty) \to (0,\infty)$ to be three times continuously differentiable. Furthermore we suppose that $\frac{\sigma(x)}{x}$ is bounded from above and away from zero.*

Note that this implies that 0 and ∞ are no *exit* boundary points in the sense of Feller's classification of boundary points of a diffusion (compare Chapter 2 in [4]). Since S starts at $s \in (0, \infty)$ it neither hits 0 nor ∞. This assumption is rather strong as it rules out the case of the underlying S being modelled as a CEV process, since we have $\mathbb{P}(S_t = 0) > 0$ in this case. Nevertheless it turns out to be the appropriate assumption for our purpose as our main application is to consider a local volatility function, which comes from a parametric model of the implied volatility surface. This is illustrated in Chapter 4, when we consider the case of the implied volatility coming from the Gatheral-family.

Throughout our thesis we denote the time t price of a European-type contingent claim with payoff g and time-to-maturity $\tau = T - t$ by

$$u^g(S_t, \tau) = \mathbb{E}^{\mathbb{Q}} \left[g(S_{t+\tau}) | \, S_t \right], \tag{1.2}$$

in case the above expectation is well defined. It is a well-known fact, that this is a local $\mathbb{Q}$-martingale – in case it is a true $\mathbb{Q}$-martingale one shows easily by an application of Itô's Lemma, that u^g satisfies the Black-Scholes partial differential equation (PDE), which in our setup becomes

$$\frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}u(x,\tau) - ru(x,\tau) - \frac{\partial}{\partial \tau}u(x,\tau) = 0, \tag{1.3}$$

subject to $u^g(x, 0) = g(x)$. Going forward we call payoffs of this type *admissible*. However, it is well known that the payoffs of puts and calls, which are the main focus of our project, belong to this class.

Throughout this project, we denote the payoff of a European call and put struck at K, when the level of the underlying is x, by $C(x, K) = C(x)$ and $P(x, K) = P(x)$ respectively. Their respective prices for time-to-maturity τ are denoted by $c(x, \tau, K) = c(x, \tau)$ and $p(x, \tau, K) = p(x, \tau)$.

The main instrument of interest is an up-and-out put (UOP) with strike K and barrier U. As indicated in the introduction, this is a contingent claim with the following payoff at the time of maturity T:

$$\mathbf{1}_{\left\{\sup_{t\in[0,T]} S_t < U\right\}} \cdot (K - S_T)^+$$

1.2 Mathematical toolbox

In this section we derive the mathematical tools, which are essential for our later considerations. We decided to present them in one section for two reasons: First to avoid repetitions and second to better illustrate how they are linked together.

1.2.1 Pricing in Laplace space

In this subsection we transform the pricing problem introduced above to Laplace-space. The main motivation for doing this, is that the corresponding pricing PDE 1.3 becomes an ODE, which can easily be transformed into a canonical Sturm-Liouville problem. This of course has the significant advantage, that we have all the well-known results of classical Sturm-Liouville theory at our disposal.

This approach used in [9] and in here is not entirely new, but has already been introduced by D. Davidov and V. Linetsky in [12] and [13]. However, both of these papers do not apply the above technique to the problem of static hedging, but only to the problem of pricing of barrier options.

First we recall the definition of the Laplace transform of a function f

$$\mathcal{L}(f)(\lambda) := \int_0^\infty e^{-\lambda y} f(y) \mathrm{d}y$$

for all $\lambda \in \mathbb{C}$ and $f : [0, \infty) \to \mathbb{C}$, such that the above integral is absolutely convergent. Denote

$$\hat{p}(x, \lambda, K) = \mathcal{L}(p(x, \cdot, K))(\lambda), \qquad \hat{c}(x, \lambda, K) = \mathcal{L}(c(x, \cdot, K))(\lambda), \tag{1.4}$$

and, more generally

$$\hat{u}(x, \lambda) = \mathcal{L}(u^g(x, \cdot))(\lambda)$$

Next we show how the pricing PDE 1.3 simplifies, when we switch to Laplace space: Taking Laplace transform of equation 1.3 and making use of the property

$$\mathcal{L}\left(\frac{\partial}{\partial \tau} u(x, \cdot)\right)(\lambda) = \lambda \hat{u}(x, \lambda) - u(x, 0)$$

and the initial condition $u^g(x, 0) = g(x)$ we get

$$\frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}\hat{u}^g(x, \lambda) - (\lambda + r)\hat{u}^g(x, \lambda) = -g(x) \tag{1.5}$$

To reduce the Laplace transform of the pricing PDE for European derivatives 1.5 to the canonical form of the Sturm-Liouville problem (SL), we perform (following [9]) the change of variables

$$Z(x) = \sqrt{2}\int_U^x \frac{1}{\sigma(y)}\mathrm{d}y \quad \text{for } x \in (0, \infty). \tag{1.6}$$

Observe that the transformation Z is chosen such that it maps the upper barrier U to zero. In addition we have by assumption (A1) that $\lim_{x\to 0} Z(x) = -\infty$ and $\lim_{x\to\infty} Z(x) = \infty$. Further define $X(z) = Z^{-1}(z)$ for $z \in (-\infty, \infty)$ and the so-called *potential* q by

$$\begin{aligned} q(z) &= \frac{1}{8}\sigma'\left(X(z)\right)^2 - \frac{1}{2\sqrt{2}}\sigma''\left(X(z)\right)X'(z) \\ &= \frac{1}{8}\sigma'\left(X(z)\right)^2 - \frac{1}{4}\sigma''(X(z))\sigma(X(z)) \quad \text{for } z \in (-\infty, \infty) \end{aligned} \tag{1.7}$$

In addition we are now considering *normalized payoff functions* in the following sense:

$$\tilde{g}(z) = g(X(z))\sqrt{\frac{\sigma(U)}{\sigma(X(z))}} \tag{1.8}$$

If we further define $\xi := -\lambda$, the normalized Laplace transform of a European derivative with payoff $g(z)$ satisfies a Sturm-Liouville problem (SL) in its canonical form

$$\frac{\partial^2}{\partial z^2}\tilde{u}^g(z,\xi) + (\xi - q(z))\tilde{u}^g(z,\xi) = -\tilde{g}(z) \tag{1.9}$$

with $z \in (-\infty, \infty)$.

Integrability of the potential q is a convenient assumption to apply classical Sturm-Liouville theory as it is presented for example by Titchmarsh in [24]. However, we want to illustrate by the following example, that this is not an appropriate assumption for our purpose.

Consider a Black or zero-drift Black-Scholes model, i.e. assume $\sigma(s) = \sigma s$ for some constant $\sigma > 0$. Then we have $q(z) := \frac{1}{8}\sigma^2$. Clearly, this is not integrable on $(-\infty, \infty)$. Therefore it is crucial to make a slightly weaker assumption in order to not rule out most of the classical models, which are of interest in finance.

(A2) We assume, that the limits $l_1 := \lim_{z\to-\infty} q(z)$, $l_2 := \lim_{z\to\infty} q(z)$ *exist and*

$$\int_{-\infty}^{0} |q(z) - l_1| \mathrm{d}z + \int_{0}^{\infty} |q(z) - l_2| \mathrm{d}z < \infty, \tag{1.10}$$

where the potential q is defined in equation 1.7. Furthermore we assume q' to be absolutely integrable and bounded on $(-\infty, \infty)$.

Note that in case of the Black model, we have $l_1 = l_2 = \frac{1}{8}\sigma^2$ and assumption (A2) is satisfied.

The following lemma provides a sufficient condition for assumption (A2) to hold true. Although it might not be as intuitive as the latter assumption, it is easier to check as it includes conditions on the local volatility only and in fact, this is what we will use going forward.

Lemma 1.2.1. *Let assumption (A1) be satisfied. Further assume the following condition to be true.*

(A3) Let the local volatility σ satisfy

$$\sup_{x\geq 0}\left\{x^2\cdot|\sigma'''(x)|\right\}<\infty \quad \textit{and} \quad \int_0^\infty \left(x^2\vee 1\right)|\sigma'''(x)|\mathrm{d}x<\infty \tag{1.11}$$

Then assumption (A2) is satisfied.

Proof. Assumption (A3) implies, that σ'' is absolutely bounded on $[0,\infty)$. Using partial integration one easily shows, that σ' is absolutely bounded as well. Knowing this, one shows that the following equations hold true:

$$\int_U^\infty \left|\sigma''(x)\right|\left(\int_U^x \frac{|\sigma(y)|+1}{\sigma(y)}\mathrm{d}y\right)\mathrm{d}x<\infty \tag{1.12}$$

$$\int_0^U \left|\sigma''(x)\right|\left(\int_x^U \frac{|\sigma(y)|+1}{\sigma(y)}\mathrm{d}y\right)\mathrm{d}x<\infty \tag{1.13}$$

$$\int_0^\infty \left|\sigma'''(x)\right|\sigma(x)\mathrm{d}x<\infty \tag{1.14}$$

This allows us to make use of the proof of Lemma 1 in [9] (see page 33, Appendix A), which implies that assumption (A2) is satisfied.

□

1.2.2 Results from Sturm-Liouville theory

In this section we repeat the summary of results from Sturm-Liouville (SL) theory in section 2.1 and 2.2 of [24] as they are presented in [9]. For a rigorous treatment in the setup time-homogeneous diffusion models we refer the interested reader to the latter reference.

We are mainly interested in a representation of the solution to the canonical SL problem 1.9 in terms of solutions to the homogeneous SL problem

$$\tilde{u}_{zz}(z,\xi)+(\xi-q(z))\tilde{u}(z,\xi)=0 \tag{1.15}$$

The central result is the following:

$$\tilde{u}^g(z,\xi)=\frac{\psi_2(z,\xi)\int_{-\infty}^z\psi_1(y,\xi)\tilde{g}(y)\mathrm{d}y+\psi_1(z,\xi)\int_z^\infty\psi_2(y,\xi)\tilde{g}(y)\mathrm{d}y}{m_2(\xi)-m_1(\xi)} \tag{1.16}$$

is a solution to the SL problem 1.9, where $\psi_1(\cdot,\xi)\in\mathcal{L}^2(-\infty,0)$ and $\psi_2(\cdot,\xi)\in\mathcal{L}^2(0,\infty)$ are constructed as a $\mathcal{L}^2$-limit of solutions to the homogeneous SL problem 1.15 with zero boundary conditions on $(l,0)$ as $l\searrow-\infty$ and on $(0,m)$ as $m\nearrow\infty$ respectively. $m_1(\xi)$ and $m_2(\xi)$ are unique constants such that

$$\begin{aligned}\psi_1(z,\xi)&=\theta(z,\xi)+m_1(\xi)\phi(z,\xi)\\ \psi_2(z,\xi)&=\theta(z,\xi)+m_2(\xi)\phi(z,\xi)\end{aligned} \tag{1.17}$$

with $\phi(\cdot,\xi)$ and $\theta(\cdot,\xi)$ being solutions to the homogeneous SL problem 1.15 satisfying $\phi(0,\xi)=0$, $\partial_z\phi(0,\xi)=-1$ and $\theta(0,\xi)=1$, $\partial_z\theta(0,\xi)$ respectively.

1.2.3 Asymptotic expansions

In both of the following chapters it is central to have an asymptotic expansion of the Laplace transform of put and call prices at the barrier U available. This is provided as a main result in this section under some additional assumptions.

Of course we want to make use of Sturm-Liouville theory and apply the change of variables stated in section 3.1, which converts our pricing ODE to a canonical Sturm-Liouville problem (compare equations 1.6 - 1.8).

By $\tilde{p}(z,\xi)$ and $\tilde{c}(z,\xi)$ we denote the normalized Laplace transforms of the put and call price respectively. Both of them are solutions to the Pricing ODE in the form of a canonical Sturm-Liouville problem 1.9 with $g(x)=P(x,K)=(K-x)^+$ and $g(x)=C(x,K^*)=(x-K^*)^+$ respectively. Therefore we can apply equation 1.16 to represent them as integrals over ψ_1 and ψ_2. Using $\psi_1(0,\xi)=1=\psi_2(0,\xi)$ we get (recall that $\tilde{P}$ and $\tilde{C}$ denote the normalized put and call payoffs):

$$\begin{aligned}\tilde{p}(0,\xi)&=\frac{\int_{-\infty}^0\psi_1(y,\xi)\tilde{P}(y,K)\mathrm{d}y+\int_0^\infty\psi_2(y,\xi)\tilde{P}(y,K)\mathrm{d}y}{m_2(\xi)-m_1(\xi)}\\ \tilde{c}(0,\xi)&=\frac{\int_{-\infty}^0\psi_1(y,\xi)\tilde{C}(y,K^*)\mathrm{d}y+\int_0^\infty\psi_2(y,\xi)\tilde{C}(y,K^*)\mathrm{d}y}{m_2(\xi)-m_1(\xi)}\end{aligned}\tag{1.18}$$

Assuming $K<U<K^*$ this simplifies to

$$\begin{aligned}\tilde{p}(0,\xi)&=\frac{\int_{-\infty}^0\psi_1(y,\xi)\tilde{P}(y,K)\mathrm{d}y}{m_2(\xi)-m_1(\xi)}\\ \tilde{c}(0,\xi)&=\frac{\int_0^\infty\psi_2(y,\xi)\tilde{C}(y,K^*)\mathrm{d}y}{m_2(\xi)-m_1(\xi)}\end{aligned}\tag{1.19}$$

Our next step is to establish an asymptotic expansion for ψ_1 and ψ_2. To achieve this we make use of a result from [24] (Lemma 6.2, p.119) on how one can construct them iteratively. We define

$$\begin{aligned}\chi_1(z,\xi)&=e^{iz\sqrt{\xi}}\\ \chi_{n+1}(z,\xi)&=e^{iz\sqrt{\xi}}+\frac{1}{2i\sqrt{\xi}}\int_0^z e^{i(z-x)\sqrt{\xi}}q(x)\chi_n(x,\xi)\mathrm{d}x+\frac{1}{2i\sqrt{\xi}}\int_z^\infty e^{i(x-z)\sqrt{\xi}}q(x)\chi_n(x,\xi)\mathrm{d}x,\end{aligned}$$

where we use the square root that maps $\mathbb{C}\backslash[0,\infty)$ continuously to the upper half plane. This version of the square root is used throughout the remainder of this thesis.

Next we present a classical Lemma from Sturm-Liouville theory, which is taken from [24] and [9] for the most part.

Lemma 1.2.2. *If the potential q is absolutely integrable on $[0,\infty)$, the above iteration converges uniformly for ξ changing in $H := \mathbb{C}\ \left(\{z : |z| \leq 16J^2\} \cup [0,\infty)\right)$ with $J := \int_0^\infty |q(z)|\mathrm{d}z$ to a solution of*

$$\chi(z,\xi) = e^{iz\sqrt{\xi}} + \frac{1}{2i\sqrt{\xi}} \int_0^z e^{i(z-x)\sqrt{\xi}} q(x)\chi(x,\xi)\mathrm{d}x + \frac{1}{2i\sqrt{\xi}} \int_z^\infty e^{i(x-z)\sqrt{\xi}} q(x)\chi(x,\xi)\mathrm{d}x \quad (1.20)$$

Let in addition q' be absolutely integrable and bounded on $[0,\infty)$. For short maturities we get the asymptotic expansion

$$\chi(z,\xi) = e^{iz\sqrt{\xi}} \left\{ 1 + \frac{1}{2i\sqrt{\xi}} \int_0^z q(x)\mathrm{d}x + \mathcal{O}\left(|\xi|^{-1}\right) \right\} \quad \text{as } |\xi| \to \infty, \quad (1.21)$$

where the residual and its first two derivatives in z can be estimated uniformly in z on $[0,\infty)$.

Proof. If χ is a solution to the above integral equation, it is shown by straight-forward calculations, that χ is two times continuously differentiable and solves the homogeneous SL-problem 1.15.

To proof that the iteration converges uniformly in ξ we follow section 6.2 in [24] (p. 119). By construction we have:

$$\chi_2(z,\xi) - \chi_1(z,\xi) = \frac{e^{iz\sqrt{\xi}}}{2i\sqrt{\xi}} \int_0^z q(x)\mathrm{d}x + \int_z^\infty e^{2i(x-z)\sqrt{\xi}} q(x)\mathrm{d}y \quad (1.22)$$

Taking absolute value gives:

$$\left| e^{-iz\sqrt{\xi}} \right| |\chi_2(z,\xi) - \chi_1(z,\xi)| \leq \frac{J}{|2\sqrt{\xi}|} \quad (1.23)$$

By induction one shows:

$$\left| e^{-iz\sqrt{\xi}} \right| |\chi_{n+1}(z,\xi) - \chi_n(z,\xi)| \leq \left(\frac{J}{|2\sqrt{\xi}|} \right)^n \quad (1.24)$$

Therefore, χ_n is convergent, if $\sqrt{\xi} > \frac{1}{2}J$, i.e. in particular for ξ changing uniform in any compact on H. Moreover we have:

$$\begin{aligned} |\chi_n(z,\xi) - \chi_2(z,\xi)| &\leq |\chi_3(z,\xi) - \chi_2(z,\xi)| + ... + |\chi_n(z,\xi) - \chi_{n-1}(z,\xi)| \\ &\leq \sum_{n=2}^{\infty} \left(\frac{J}{2|\sqrt{\xi}|} \right)^n = \frac{1}{4} \left| e^{iz\sqrt{\xi}} \right| \frac{\frac{J^2}{|\sqrt{\xi}|^2}}{1 - \frac{1}{2}\frac{J}{|\sqrt{\xi}|}} \end{aligned}$$

As we know that $\chi_n(x) \to \chi(x)$ as $n \to \infty$ we have:

$$\left|e^{-iz\sqrt{\xi}}\right| \cdot |\chi(z,\xi)) - \chi_2(z,\xi)| \leq \frac{\left(\frac{J}{2|\sqrt{\xi}|}\right)^2}{1 - \frac{1}{2}\frac{J}{|\sqrt{\xi}|}} \tag{1.25}$$

Clearly, this estimate is uniform in z. Similarly we can derive an iterative scheme for the first and second derivative of $e^{-iz\sqrt{\xi}}\chi(z,\xi)$. Making use of the assumption, that q' is bounded and integrable we obtain similar estimates. Now, only the asymptotic expansion remains to be shown.

$$\begin{aligned}
\chi_2(z,\xi) &= e^{iz\sqrt{\xi}} + \frac{1}{2i\sqrt{\xi}} \int_0^z e^{i(z-x)\sqrt{\xi}} q(x) e^{ix\sqrt{\xi}} \mathrm{d}x + \frac{1}{2i\sqrt{\xi}} \int_z^\infty e^{i(x-u)\sqrt{\xi}} q(x) e^{ix\sqrt{\xi}} \mathrm{d}x \\
&= e^{iz\sqrt{\xi}} \left(1 + \frac{1}{2i\sqrt{\xi}} \int_0^z q(x)\mathrm{d}x + \frac{1}{2i\sqrt{\xi}} \int_z^\infty e^{2i(x-z)\sqrt{\xi}} q(x)\mathrm{d}x\right) \\
&= e^{iz\sqrt{\xi}} \left\{1 + \frac{1}{2i\sqrt{\xi}} \int_0^z q(x)\mathrm{d}x + \frac{1}{\xi} \int_z^\infty \left(e^{2i(x-z)\sqrt{\xi}}\right)' q(x)\mathrm{d}x\right\} \\
&= e^{iz\sqrt{\xi}} \left\{1 + \frac{1}{2i\sqrt{\xi}} \int_0^z q(x)\mathrm{d}x + \mathcal{O}\left(|\xi|^{-1}\right)\right\},
\end{aligned} \tag{1.26}$$

where we applied partial integration in the last step and made use of the integrability assumption on both, q and q'. This finishes the proof.

□

Note, that the assumption made in the above Lemma is too strong to apply it to our setup. Even in case of a zero-drift Black-Scholes model, the corresponding potential will not be integrable, but a non-zero constant. Recall, that exactly for this reason we made assumption (A2) in section 3.1 instead.

The following observation is key to understand how one can apply Lemma 1.2.2 to our problem. Introduce the change of variables $\bar{\xi} := \xi - l_2$ and $\bar{q}(z) := q(z) - l_2$. Then $\bar{\xi} - \bar{q}(z) = \xi - q(z)$ and $\tilde{u}(z,\xi)$ satisfies the homogeneous SL-equation 1.15 if and only if $\tilde{u}(z,\bar{\xi})$ satisfies

$$\tilde{u}_{zz}\left(z,\bar{\xi}\right) + \left(\bar{\xi} - \bar{q}(z)\right)\tilde{u}\left(z,\bar{\xi}\right) = 0 \tag{1.27}$$

By assumption (A2) $\bar{q}$ is integrable and therefore we can apply Lemma 1.2.2 on the homogeneous SL-problem 1.27.

This is the central idea we use to construct the solutions $\psi_1(z,\xi)$, $\psi_2(z,\xi)$ in the sense of section 1.2.2. This is done rigorously in the following Lemma, which is a modified version of Lemmata 2 and 5 in [9].

Lemma 1.2.3. *Let assumptions (A1) and (A2) hold true. Then* $J_1 := \int_{-\infty}^{0} |q(z) - l_1| \mathrm{d}z < \infty$ *and* $J_2 := \int_0^{\infty} |q(z) - l_2| \mathrm{d}z < \infty$. *Denote by* ψ_1 *and* ψ_2 *the respective solutions of the homogeneous SL-problem 1.15 in the sense they are defined in equation 1.17. Then the following asymptotic expansions hold true:*

(i) As $|\xi| \to \infty$ *satisfying* $\Im(\sqrt{\xi}) \geq \sqrt{16J_2^2 - l_2 \wedge 0}$ *we have*

$$\psi_2(z,\xi) = e^{iz\sqrt{\xi}} \left(1 + \frac{1}{2i\sqrt{\xi}} \int_0^z q(y)\mathrm{d}y + \mathcal{O}\left(|\xi|^{-1}\right)\right) \tag{1.28}$$

and the residual as well as its first and second derivative in z can be estimated uniformly on $[0,\infty)$ *and on any compact in* $(-\infty, 0]$.

(ii) As $|\xi| \to \infty$ *satisfying* $\Im(\sqrt{\xi}) \geq \sqrt{16J_1^2 - l_1 \wedge 0}$ *we have*

$$\psi_1(z,\xi) = e^{-iz\sqrt{\xi}} \left(1 - \frac{1}{2i\sqrt{\xi}} \int_0^z q(y)\mathrm{d}y + \mathcal{O}\left(|\xi|^{-1}\right)\right) \tag{1.29}$$

Again, the residual and its first and second derivative in z can be estimated uniformly on $(-\infty, 0]$ *and on any compact in* $[0,\infty)$.

Proof. We only proof part (i) — part (ii) follows along the same lines replacing "z" by "$-z$". First we apply the asymptotic expansion from Lemma 1.2.2 (i.e. equation 1.21) to the homogeneous SL-problem in equation 1.27. We get

$$\chi(z,\xi) = e^{iz\sqrt{\xi - l_2}} \left(1 + \frac{1}{2i\sqrt{\xi - l_2}} \int_0^z (q(x) - l_2)\mathrm{d}x + \mathcal{O}\left(|\xi|^{-1}\right)\right) \tag{1.30}$$

Next, observe that we have for trivial reasons

$$\sqrt{\xi} - \sqrt{\xi - l_2} = \frac{l_2}{\sqrt{\xi} + \sqrt{\xi - l_2}} \tag{1.31}$$

Therefore as $|\xi| \to \infty$ it holds:

$$\begin{aligned} \frac{1}{\sqrt{\xi - l_2}} &= \frac{1}{\sqrt{\xi} - \frac{l_2}{\sqrt{\xi}+\sqrt{\xi - l_2}}} = \frac{1}{\sqrt{\xi}} + \mathcal{O}\left(|\xi|^{-\frac{3}{2}}\right), \\ e^{iz\sqrt{\xi - l_2}} &= e^{iz\sqrt{\xi}} \cdot e^{-iz\frac{l_2}{\sqrt{\xi}+\sqrt{\xi - l_2}}} = e^{iz\sqrt{\xi}} \left(1 - \frac{1}{2} l_2 iz \frac{1}{\sqrt{\xi}} + \mathcal{O}\left(|\xi|^{-1}\right)\right) \end{aligned} \tag{1.32}$$

As a result equation 1.30 becomes:

$$\begin{aligned} \chi(z,\xi) &= e^{iz\sqrt{\xi}} \left(1 + \frac{1}{2i\sqrt{\xi}} \int_0^z (q(y) - l_2)\mathrm{d}y + l_2 z + \mathcal{O}\left(|\xi|^{-1}\right)\right) \\ &= e^{iz\sqrt{\xi}} \left(1 + \frac{1}{2i\sqrt{\xi}} \int_0^z q(y)\mathrm{d}y + \mathcal{O}\left(|\xi|^{-1}\right)\right) \end{aligned} \tag{1.33}$$

Using equation 1.33 we have that $\chi(0,\xi) = 1 + \mathcal{O}\left(|\xi|^{-1}\right)$ as $|\xi| \to \infty$. In addition we have

$$\frac{1}{1 + \mathcal{O}\left(|\xi|^{-1}\right)} = 1 + \mathcal{O}\left(|\xi|^{-1}\right) \tag{1.34}$$

Now, we use equation (34) from [9], that is

$$\psi_2(z,\xi) = \frac{\chi(z,\xi)}{\chi(0,\xi)} \tag{1.35}$$

and therefore we get the desired expansion 1.28 uniformly for $z \in [0,\infty)$.

Next, we extend the result to all compacts in $(-\infty, 0]$, where we follow the proof of Lemma 5 in [9] closely. The idea used for this result can be described as follows: For a given $l \in (-\infty, 0]$ we shift the potential function q by l units to the right, get a SL problem of the form we considered above and apply the same reasoning. More precisely:

Let $l \in (-\infty, 0]$ and define $\tilde{q}(z) := q(z-l)$. Denote by $\tilde{\chi}(\cdot,\xi)$ the solution to the SL-problem with potential $\tilde{q}$, which is $\mathcal{L}^2(0,\infty)$. The corresponding asymptotic expansion 1.33 tells us, that $\tilde{\chi}(-l,\xi) \neq 0$ for ξ with $|\xi|$ large enough. As $\tilde{\chi}(z-l,\xi)/\tilde{\chi}(-l,\xi) = \psi_2(z,\xi)$ we conclude as above, that the expansion 1.28 holds uniformly over $z \in [l,\infty)$. We conclude knowing, that l was arbitrary.

□

Having in mind, that we want to achieve an asymptotic expansion of equation 1.19 we now derive an asymptotic expansion for m_1 and m_2.

Lemma 1.2.4. *Again, we use the notation from section 1.2.2 and assume that (A1) and (A2) hold true.*

(i) As $|\xi| \to \infty$ *satisfying* $\Im(\sqrt{\xi}) \geq \sqrt{16J_2^2 - l_2 \wedge 0}$ *we have*

$$m_2(\xi) = -i\sqrt{\xi - l_2}\left(1 + \mathcal{O}\left(|\xi|^{-1}\right)\right) \tag{1.36}$$

(ii) Similarly we have as $|\xi| \to \infty$ *satisfying* $\Im(\sqrt{\xi}) \geq \sqrt{16J_1^2 - l_1 \wedge 0}$

$$m_1(\xi) = i\sqrt{\xi - l_1}\left(1 + \mathcal{O}\left(|\xi|^{-1}\right)\right) \tag{1.37}$$

Proof. We proof part (i) only. Part (ii) follows along similar arguments. For the first part we follow the proof of Lemma 2 in [9] closely.

Notice, that if we switch to the SL problem with potential $\bar{q} := q - l_2$ presented in equation 1.27, then m_2 in the problem under consideration corresponds to the new problem, up to a shift of variable $\xi \mapsto \xi - l_2$. This allows us to focus on the case $l_2 = 0$ only.

Again, we make use of equation (34) in [9]:

$$\frac{\chi(z,\xi)}{\chi(0,\xi)} = \psi_2(z,\xi) = \theta(z,\xi) + m_2(\xi)\phi(z,\xi) \tag{1.38}$$

with $\phi(\cdot,\xi)$ and $\theta(\cdot,\xi)$ being solutions to the homogeneous SL problem 1.15 satisfying $\phi(0,\xi) = 0$, $\partial_z\phi(0,\xi) = -1$ and $\theta(0,\xi) = 1$, $\partial_z\theta(0,\xi)$ respectively (according to the definition in 1.2.2). Differentiating the above with respect to z and plugging in zero gives:

$$m_2(\xi) = -\frac{\partial\chi(0,\xi)}{\xi(0,\xi)} \tag{1.39}$$

Differentiating the integral equation 1.20 with respect to z directly gives the first equation.

To show the stated asymptotic expansion we first recall the expansion for χ from Lemma 1.2.2 stated in equation 1.21.

$$\chi(z,\xi) = e^{iz\sqrt{\xi}}\left(1 + \frac{1}{2i\sqrt{\xi}}\int_0^z q(x)\mathrm{d}x + \mathcal{O}\left(|\xi|^{-1}\right)\right) \tag{1.40}$$

Therefore we have

$$\begin{aligned}
\int_0^\infty e^{iy\sqrt{\xi}}q(y)\chi(y,\xi)\mathrm{d}y &= \int_0^\infty e^{2iy\sqrt{\xi}}q(y)\left(1 + \frac{1}{2i\sqrt{\xi}}\int_0^z q(x)\mathrm{d}x + \mathcal{O}\left(|\xi|^{-1}\right)\right)\mathrm{d}y \\
&= \frac{1}{2i\sqrt{\xi}}\int_0^\infty \left(e^{2iy\sqrt{\xi}}\right)' q(y)\mathrm{d}y + \mathcal{O}\left(|\xi|^{-\frac{1}{2}}\right) \\
&= \frac{1}{2i\sqrt{\xi}}\left(-q(0) - \int_0^\infty e^{2iy\sqrt{\xi}}q'(y)\mathrm{d}y + \mathcal{O}\left(|\xi|^{-\frac{1}{2}}\right)\right) \\
&= \mathcal{O}\left(|\xi|^{-1}\right),
\end{aligned} \tag{1.41}$$

where we made assumptions on the integrability of q and its derivative. As a result we have:

$$\begin{aligned}
m_2(\xi) &= -i\sqrt{\xi}\frac{1 - \frac{1}{2i\sqrt{\xi}}\int_0^\infty e^{iy\sqrt{\xi}}q(y)\chi(y,\xi)\mathrm{d}y}{1 + \frac{1}{2i\sqrt{\xi}}\int_0^\infty e^{iy\sqrt{\xi}}q(y)\chi(y,\xi)\mathrm{d}y} \\
&= -i\sqrt{\xi}\frac{1 + \mathcal{O}\left(|\xi|^{-1}\right)}{1 + \mathcal{O}\left(|\xi|^{-1}\right)} \\
&= -i\sqrt{\xi}\left(1 + \mathcal{O}\left(|\xi|^{-1}\right)\right)
\end{aligned} \tag{1.42}$$

This finishes the proof.

□

We are now in the position to provide an asymptotic expansion of the normalized Laplace transforms of call and put prices in ξ. The main idea is to plug the results from Lemma 1.2.3 and Lemma 1.2.4 into equation 1.19 and expand the resulting integrals. This can be done using the widely known *stationary phase method*. However, in our case it turns out, that it is more convenient to make this precise using *partial integration*.

Proposition 1.2.1. *Let assumption (A3) hold true. Then the asymptotic expansion of the Laplace transform of the price of a put with strike K and current level of the underlying $U > K$ is given by*

$$\hat{p}(U,\xi,K) = \frac{\sqrt{\sigma(U)\sigma(K)}}{2\sqrt{2}i\xi^{\frac{3}{2}}} e^{-iZ(K)\sqrt{\xi}} \left(1 - \frac{1}{2i}\int_U^K \widetilde{\sigma}(x)\mathrm{d}x \frac{1}{\sqrt{\xi}} + \mathcal{O}\left(|\xi|^{-1}\right)\right) \tag{1.43}$$

for $|\xi| \to \infty$ satisfying $\Im(\sqrt{\xi}) \geq C$ for some constant C, where $\widetilde{\sigma}$ is defined by

$$\tilde{\sigma}(x) := \frac{1}{4\sqrt{2}} \frac{(\sigma'(x))^2}{\sigma(x)} - \frac{1}{2\sqrt{2}}\sigma''(x) \tag{1.44}$$

Similarly the one of the price of a call with strike $K^* > U$ is given by

$$\hat{c}(U,\xi,K^*) = \frac{\sqrt{\sigma(U)\sigma(K^*)}}{2\sqrt{2}i\xi^{\frac{3}{2}}} e^{iZ(K^*)\sqrt{\xi}} \left(1 + \frac{1}{2i}\int_U^{K^*} \widetilde{\sigma}(x)\mathrm{d}x \frac{1}{\sqrt{\xi}} + \mathcal{O}\left(|\xi|^{-1}\right)\right) \tag{1.45}$$

again for $|\xi| \to \infty$ satisfying $\Im(\sqrt{\xi}) \geq C$ for some constant C.

Proof. We show the expansion for the Laplace transform of the put price only. For the analogous result for the put price one can argue along the same lines. First, we combine both parts of Lemma 1.2.4 to get

$$\begin{aligned} \frac{1}{m_2(\xi) - m_1(\xi)} &= -\frac{1}{2i\sqrt{\xi}} \frac{1}{1 + \mathcal{O}\left(|\xi|^{-1}\right)} \\ &= -\frac{1}{2i\sqrt{\xi}} \left(1 + \mathcal{O}\left(|\xi|^{-1}\right)\right) \end{aligned} \tag{1.46}$$

Next we plug the achieved asymptotic expansions into 1.19 and use Lemma 1.2.3 (ii) to get get:

$$\tilde{p}(0,\xi) = -\frac{1}{2i\sqrt{\xi}} \int_{-\infty}^0 e^{-iy\sqrt{\xi}} \left(1 - \frac{1}{2i\sqrt{\xi}}\int_0^y q(x)\mathrm{d}x + \mathcal{O}\left(|\xi|^{-1}\right)\right) \tilde{P}(y,K)\mathrm{d}y \tag{1.47}$$

After simple rearranging we get:

$$\begin{aligned} \tilde{p}(0,\xi) = &- \frac{\sqrt{\sigma(U)}}{2i\sqrt{\xi}} \int_{-\infty}^{Z(K)} e^{-iy\sqrt{\xi}} \frac{K - X(y)}{\sqrt{\sigma(X(y)}}\mathrm{d}y \\ &- \frac{\sqrt{\sigma(U)}}{4\xi} \int_{-\infty}^{Z(K)} e^{-iy\sqrt{\xi}} \frac{K - X(y)}{\sqrt{\sigma(X(y))}} \left(\int_0^y q(x)\mathrm{d}x\right) \mathrm{d}y \\ &- \frac{\sqrt{\sigma(U)}}{\xi^{\frac{3}{2}}} \int_{-\infty}^{Z(K)} e^{-iy\sqrt{\xi}} \frac{K - X(y)}{\sqrt{\sigma(X(y))}} \mathcal{O}\left(1\right) \mathrm{d}y \end{aligned} \tag{1.48}$$

We consider the two terms above separately. For the first term, we have by partial integration (for fixed $k \geq 2$):

$$\begin{aligned} &\int_{-\infty}^{Z(K)} e^{-iy\sqrt{\xi}} \frac{K - X(y)}{\sqrt{\sigma(X(y))}}\mathrm{d}y \\ &\qquad = e^{-iZ(K)\sqrt{\xi}} \left(\sum_{l=2}^{k-1} (-1)^{(l-1)} \left(\frac{1}{-i\sqrt{\xi}}\right)^l f^{(l-1)}(Z(K)) + \mathcal{O}(|\xi|^{-\frac{k}{2}})\right), \end{aligned} \tag{1.49}$$

where f is defined by $f(y) := (K - X(y))/\sqrt{\sigma(X(y))}$, if $f \in C^{(k-1)}$ and we have for all $l = 0, 1, ..., k-1$:

$$\int_{-\infty}^{Z(K)} e^{-iy\sqrt{\xi}} f^{(l)}(y)\mathrm{d}y < \infty \tag{1.50}$$

We want the first two coefficients of this expansion, i.e. we need this condition to be satisfied with $k = 4$.

First, we calculate f' and f'' explicitly. Using the fact that $X'(y) = \sigma(X(y))/\sqrt{2}$ we get:

$$f'(y) = -\frac{1}{\sqrt{2}}\sqrt{\sigma(X(y))} - (K - X(y))\,\frac{\sigma'(X(y))}{2\sqrt{2}\sqrt{\sigma(X(y))}} \tag{1.51}$$

and

$$f''(y) = -\,(K - X(y))\left(\frac{\sigma''(X(y))\sqrt{\sigma(X(y))}}{4\sqrt{2}} - \frac{\sigma'(X(y))^2\sqrt{\sigma(X(y))}}{8}\right) \tag{1.52}$$

Making use of assumption (A3) one can check, that for $l = 0, 1, 2, 3$ we have for some constants b_l and c_l:

$$f'(y) \leq b_l e^{c_l|y|} \tag{1.53}$$

Now, we choose ξ such that $\Im(\sqrt{\xi}) > \max\{l_0, ..., l_3\}$ to show equation 1.50 for $k = 4$. Therefore we have:

$$\int_{-\infty}^{Z(K)} e^{-iy\sqrt{\xi}}\frac{K - X(y)}{\sqrt{\sigma(X(y))}}\mathrm{d}y = -e^{-iZ(K)\sqrt{\xi}}\left(\frac{\sqrt{\sigma(K)}}{\sqrt{2}\xi} + \mathcal{O}\left(|\xi|^{-2}\right)\right) \tag{1.54}$$

Similarly we get for the second term:

$$\begin{aligned}\int_{-\infty}^{Z(K)} & e^{-iy\sqrt{\xi}} g(y)\mathrm{d}y \\ &= e^{-iZ(K)\sqrt{\xi}}\left(\sum_{l=2}^{k-1}(-1)^{(l-1)}\left(\frac{1}{-i\sqrt{\xi}}\right)^l g^{(l-1)}(Z(K)) + \mathcal{O}\left(|\xi|^{-\frac{k}{2}}\right)\right),\end{aligned} \tag{1.55}$$

where g is defined by

$$g(y) = \frac{\int_0^y q(x)\mathrm{d}x\,(K - X(y))}{\sqrt{\sigma(X(y))}} = f(y)\int_0^y q(x)\mathrm{d}x, \tag{1.56}$$

if $g \in C^{(k-1)}$ and we have for all $l = 0, 1, ..., k-1$:

$$\int_{-\infty}^{Z(K)} e^{-iy\sqrt{\xi}} g^{(l)}(y)\mathrm{d}y < \infty \tag{1.57}$$

Here we just need the first order coefficient of the expansion, i.e. the latter condition has to be satisfied for $k = 3$. Furthermore we have:

$$g'(y) = q(y)f(y) + f'(y)\int_0^y q(x)\mathrm{d}x \tag{1.58}$$

Again, we show using assumption (A3), that equation 1.57 is satisfied for $k = 3$ and we get for the second integral:

$$\int_{-\infty}^{Z(K)} e^{-iy\sqrt{\xi}} g(y)\mathrm{d}y = e^{-iZ(K)\sqrt{\xi}} \left(-\frac{\sigma(K)}{\sqrt{2}\xi} \int_0^{Z(K)} q(x)\mathrm{d}x + \mathcal{O}\left(|\xi|^{-\frac{3}{2}}\right) \right) \tag{1.59}$$

For the third term we apply partial integration two times and make use of the fact, that the first and second derivative of the residual can be estimated uniformly in y. This gives:

$$\int_{-\infty}^{Z(K)} e^{-iy\sqrt{\xi}} \frac{K - X(y)}{\sqrt{\sigma(X(y))}} \mathcal{O}(1)\,\mathrm{d}y = e^{-iZ(K)\sqrt{\xi}} \mathcal{O}\left(|\xi|^{-1}\right) \tag{1.60}$$

Combining the three terms from above we get:

$$\tilde{p}(0,\xi) = \frac{\sqrt{\sigma(U)}\sqrt{\sigma(K)}}{2\sqrt{2}i\xi^{\frac{3}{2}}} e^{-iZ(K)\sqrt{\xi}} \left(1 - \frac{1}{2i\sqrt{\xi}} \int_0^{Z(K)} q(x)\mathrm{d}x + \mathcal{O}\left(|\xi|^{-1}\right) \right) \tag{1.61}$$

Moreover we have:

$$\begin{aligned} \int_0^{Z(y)} q(z)\mathrm{d}z &= \int_U^y \left(\frac{1}{4 \cdot 2}(\sigma'(x))^2 - \frac{\sigma''(x)}{2\sqrt{2}} \frac{1}{Z'(x)} \right) Z'(x)\mathrm{d}x \\ &= \int_U^y \left(\frac{1}{4\sqrt{2}} \frac{(\sigma'(x))^2}{\sigma(x)} - \frac{1}{2\sqrt{2}}\sigma''(x) \right) \mathrm{d}x \end{aligned} \tag{1.62}$$

We conclude recalling the definition of $\widetilde{\sigma}$ in equation 1.44 and making the following observation (use 1.6 and 1.8):

$$\hat{p}(U,\xi,K) = \tilde{p}(Z(U),\xi) \frac{\sigma(U)}{\sigma(X(Z(U))} = \tilde{p}(0,\lambda) \tag{1.63}$$

This finishes the proof. □

Chapter 2

Exact static hedge and its approximation

Before we rigorously define the concept of an *exact static hedging* and discuss the result achieved by [9] we briefly want to outline how all this fits in the overall structure of our thesis. Our main goal is to develop a robustness result for the static hedge of our up-and-out put. To this end we use the exact static hedge and its approximations in the setting of time-homogeneous local volatility models as a building block to connect implied volatility to static hedges. This final step is done in chapter 4, but for this and the subsequent chapter let us focus on entirely model dependent considerations.

2.1 The exact static hedge

Following [9] we consider an up-and-out put (UOP) option with strike K, barrier U and maturity T, which is assumed to be fixed and is the maturity of all instruments under consideration. To motivate our formal definition of an exact static hedge, we repeat our intuitive explanation from the introduction using a PDE standpoint. From this point of view both, the price of a vanilla put and a UOP as prices of traded derivatives satisfy the same pricing PDE – however, the price of the UOP has to satisfy an additional zero boundary condition at $x = U$. Therefore, if we can find a payoff G such that, to the left of the barrier G coincides with the corresponding put payoff and the price of the European-type option with payoff G is zero along $x = U$, then its price also has to coincide with the price of the UOP provided that the barrier has not yet been hit.

This intuitive argument can be made precise considering the following trading strategy. At initiation of a short position of one UOP contract we open a long position in a European derivative with payoff G. In case the underlying hits the barrier U during its lifetime, we liquidate our portfolio at no cost as the prices of the two derivatives coincide (indeed they are both equal to zero). In case the underlying does not hit the barrier U, the payoffs

of the two derivatives coincide. We conclude that their prices must be the same at all times.

Therefore we will look for G in the form $G(x) = (K - x)^+ - g(x)$, where g has support in $[U, \infty)$ and therefore $G(S_T) = (K - S_T)^+$, if S_T has not hit U. The problem boils down to choosing $g(x)$ such that the price of a European derivative with payoff $G(S_T)$ conditional on the current level of the underlying being U is equal to zero for all times $t \in [0, T]$. Equivalently, we want the European-type derivative with payoff $g(S_T)$ to be equal to the price of a vanilla put, in case the current level of the underlying is U (again for all times $t \in [0, T]$). In the notation introduced above, this is summarized in the following definition.

Definition 2.1.1 (Exact static hedge)**.** *A European derivative with payoff g is called an exact static hedge, if*

$$p(U, t, K) = u^g(U, t) \ \ \forall t \in [0, T], \quad \textit{and } g(x) = 0 \ \ \forall x < U \tag{2.1}$$

P. Carr and S. Nadtochiy derive in Theorem 1, page 19 of [9] an explicit formula for the static hedge g under some regularity assumptions. More precisely the normalized payoff of the exact static hedge is given by

$$\tilde{g}(z) = \frac{1}{2\pi i} \int_\Gamma \psi_1(z, \xi) \frac{\Upsilon(\xi)}{m_2(\xi) - m_1(\xi)} \mathrm{d}\xi, \tag{2.2}$$

where ψ_1, m_1 and m_2 are as in section 1.2.2, Υ is given by

$$\Upsilon(\xi) = \int_{-\infty}^0 \psi_1(y, \xi)(K - X(y))^+ \frac{\sqrt{\sigma(U)}}{\sqrt{\sigma(X(y))}} \mathrm{d}y \tag{2.3}$$

and the integration contour Γ is constructed in equations (38)-(40) of [9]. It is the image of an interval of the form $[-\infty + iJ, \infty + iJ]$ for a constant $J > 0$ specified in equation (38) of [9] under the mapping $\xi \mapsto \xi^2$, where the direction of the "interval-contour" is "from left to right".

This is a satisfactory result from a theoretical point of view – however, some issues remain to be addressed. In most cases the contour integral cannot be calculated explicitly and numerical approximation is not straight-forward. In fact, there is no known programmable scheme for this problem. In the subsequent section we illustrate a method, which can in principle be used to obtain a polynomial expansion. However, in practice it is a tedious job to calculate the coefficients – again a programmable iteration is yet to be found. This forces us to use an approximation of a (rather small) order.

2.2 Polynomial expansion

The following discussion outlines a method, which in principle allows us to obtain a polynomial expansion of the static hedge in equation 2.2.

Our starting point is the latter equation. We make the transformation $s := -i\sqrt{\xi}$, which changes the contour of integration to $\Gamma' = -i\sqrt{\Gamma} = [\delta + i\infty, \delta - i\infty]$ and therefore we get:

$$\tilde{g}(z) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \frac{2s\psi_1(z,s)\Upsilon(s)}{m_2(s) - m_1(s)} \mathrm{d}s \tag{2.4}$$

Now, we assume an expansion of the above integrand in terms of s^{-k}, $k \in \{2, 3, 4, ...\}$ to be given, that is we assume to have $c_k(z)$, $k = 2, 3, 4, ...$ such that

$$2s\psi_1(z,s)\frac{\Upsilon(s)}{m_2(s) - m_1(s)} = \sum_{k=2}^{\infty} c_k(z) \frac{e^{s(z+Z(K))}}{s^k} \tag{2.5}$$

This might seem rather unnatural at this point, but in fact we can prove, that this is indeed the case. However, we decided to only provide a very informal argument as we are not going to build on this result later in our thesis: One can make use of the iteration for χ given in chapter 1 and it is easy to show, that it converges for all large enough ξ. This means its limit is analytic and we obtain a series expansion

$$2s\psi_1(z,s)e^{-s(z+Z(K))}\frac{\Upsilon(s)}{m_2(s) - m_1(s)} = \sum_{k=2}^{\infty} c_k(z) s^{-k} \tag{2.6}$$

The fact, that the series expansion starts at $k = 2$ becomes clear later on in this chapter, when we calculate the first coefficients explicitly.

Next, we explain why we are allowed to interchange series and integration. To this end it is sufficient to show, that

$$\sum_{k=2}^{\infty} c_k(z) \int_{\delta - i\infty}^{\delta + i\infty} \frac{e^{s(z+Z(K))}}{s^k} \mathrm{d}s \tag{2.7}$$

is absolutely convergent. We have:

$$\begin{aligned}
\int_{\delta - i\infty}^{\delta + i\infty} \left| \frac{e^{s(z+Z(K))}}{s^k} \right| \mathrm{d}s &= e^{\delta(z+Z(K))} \int_{-\infty}^{\infty} \frac{1}{|iu + \delta|^k} \mathrm{d}u \\
&= \frac{e^{\delta(z+Z(K))}}{\delta^k} \int_{-\infty}^{\infty} \frac{1}{|iu/\delta + 1|^k} \mathrm{d}u \\
&= \frac{e^{\delta(z+Z(K))}}{\delta^{(k-1)}} \int_{-\infty}^{\infty} \frac{1}{|iu + 1|^k} \mathrm{d}u \\
&\leq \frac{1}{\delta^{(k-1)}} \pi e^{\delta(z+Z(K))},
\end{aligned} \tag{2.8}$$

since $\int_{-\infty}^{\infty} 1/|iu+1|^2 \mathrm{d}u = \pi$ and $|iu+1|^{(k+1)} \geq |iu+1|^k$ for $k = 2, 3, 4, ...$ Therefore we choose $\delta > 1$ and get

$$\tilde{g}(z) = \sum_{k=2}^{\infty} c_k(z) \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \frac{e^{s(z+Z(K))}}{s^k} \mathrm{d}s \tag{2.9}$$

Integrals of this type can be calculated explicitly using the well-known theorem of residues and Jordan's Lemma. More precisely we consider two case:

First, assume $z < -Z(K)$. In this case we close the integration contour with a "half circle to the right". Jordan's Lemma essentially tells us that the integral over the "half-circle" is vanishing as its radius goes to infinity. We have no poles to the right of Γ' and therefore by the theorem of residues, the integral over the concatenated contour is zero. As a result the integral under consideration must be zero as well.

If, however, $z \geq -Z(K)$ we close the contour with a "half-circle to the left" and proceed by calculating the residuals. We observe that for $k = 2, 3, 4, ...$ we have

$$\int_{\delta - i\infty}^{\delta + i\infty} \frac{e^{s(z-z_0)}}{s^k} \mathrm{d}s = \sum_{l=0}^{\infty} \int_{\delta - i\infty}^{\delta + i\infty} s^{l-k} \frac{(z-z_0)^l}{l!} = 2\pi i \frac{(z-z_0)^{k-1}}{(k-1)!} \tag{2.10}$$

Therefore we have for $z \geq -Z(K)$

$$\tilde{g}(z) = \sum_{k=2}^{\infty} \frac{c_k(z)}{(k-1)!} \left(z + Z(K)\right)^{k-1} \tag{2.11}$$

That is for arbitrary $z \in (-\infty, \infty)$ we have for fixed $N > 2$

$$\tilde{g}(z) = \sum_{k=2}^{\infty} \frac{c_k(z)}{(k-1)!} \left(\left(z + Z(K)\right)^+\right)^{k-1} \tag{2.12}$$

We change back variables to the original domain and get a representation of our static hedge:

$$g(x) = \tilde{g}(Z(x)) \sqrt{\frac{\sigma(x)}{\sigma(U)}} = \sqrt{\frac{\sigma(x)}{\sigma(U)}} \sum_{k=2}^{\infty} \frac{c_k(Z(x))}{(k-1)!} \left(\left(Z(x) + Z(K)\right)^+\right)^{k-1} \tag{2.13}$$

We can truncate the above expansion to get an asymptotic approximation of g of arbitrary order. This is summarized in the following definition.

Definition 2.2.1. *We define the asymptotic approximation of order $n \in \mathbb{N}$ for the exact static hedge g from equation 2.2 by*

$$g_n(x) = \sqrt{\frac{\sigma(x)}{\sigma(U)}} \sum_{k=2}^{n+1} \frac{c_k(Z(x))}{(k-1)!} \left(\left(Z(x) + Z(K)\right)^+\right)^{k-1} \tag{2.14}$$

To illustrate this result we calculate the first two terms in this expansion explicitly, that is we calculate c_2 and c_3. Making use of the results from Sturm-Liouville theory in chapter 1 this is an easy task.

First we note, that $\Upsilon(\xi)$ is the Laplace transform of the price of a put, when the current level of the underlying is U, and therefore we have by 1.2.1 part (i) (in this informal derivation we just assume, that assumption (A3) is satisfied):

$$\Upsilon(\xi) = \hat{p}(U,\xi) = \frac{\sqrt{\sigma(U)\sigma(K)}}{2\sqrt{2}i\xi^{\frac{3}{2}}} e^{-iZ(K)\sqrt{\xi}} \left(1 - \frac{1}{2i\sqrt{\xi}} \int_U^K \widetilde{\sigma}(x)\mathrm{d}x + \mathcal{O}\left(|\xi|^{-1}\right)\right) \tag{2.15}$$

Combining this with the asymptotic expansion of ψ_1 in Lemma 1.2.3 (ii), we have the following asymptotic expansion of the integrand in equation 2.2.

$$\frac{\sqrt{\sigma(U)\sigma(K)}}{2\sqrt{2}i\xi^{\frac{3}{2}}} e^{-i(z+Z(K))\sqrt{\xi}} \left\{1 - \frac{1}{2i\sqrt{\xi}} \left(\int_U^K \widetilde{\sigma}(x)\mathrm{d}x + \int_0^z q(x)\mathrm{d}x\right) + \mathcal{O}\left(|\xi|^{-\frac{1}{2}}\right)\right\} \tag{2.16}$$

After changing variables the integrand is

$$\frac{\sqrt{\sigma(U)\sigma(K)}}{\sqrt{2}s^2} e^{s(z+Z(K))} \left\{1 + \frac{1}{2s} \left(\int_U^K \widetilde{\sigma}(x)\mathrm{d}x + \int_0^z q(x)\mathrm{d}x\right) + \mathcal{O}\left(|\xi|^{-\frac{1}{2}}\right)\right\} \tag{2.17}$$

and therefore we have

$$\begin{aligned} c_2(z) &= \frac{\sqrt{\sigma(U\sigma(K)}}{\sqrt{2}} \\ c_3(z) &= \frac{\sqrt{\sigma(U)\sigma(K)}}{2\sqrt{2}} \left(\int_U^K \widetilde{\sigma}(x)\mathrm{d}x + \int_0^z q(x)\mathrm{d}x\right) \end{aligned} \tag{2.18}$$

Therefore a second order approximation of the exact static hedge is given by

$$\begin{aligned} g_2(x) = &\frac{\sqrt{\sigma(x)\sigma(K)}}{\sqrt{2}} \left(Z(x) + Z(K)\right)^+ \\ &+ \frac{\sqrt{\sigma(x)\sigma(K)}}{4\sqrt{2}} \left(\int_0^{Z(x)} q(y)\mathrm{d}y - \int_K^U \widetilde{\sigma}(y)\mathrm{d}y\right) \left(\left(Z(x) + Z(K)\right)^+\right)^2 \end{aligned} \tag{2.19}$$

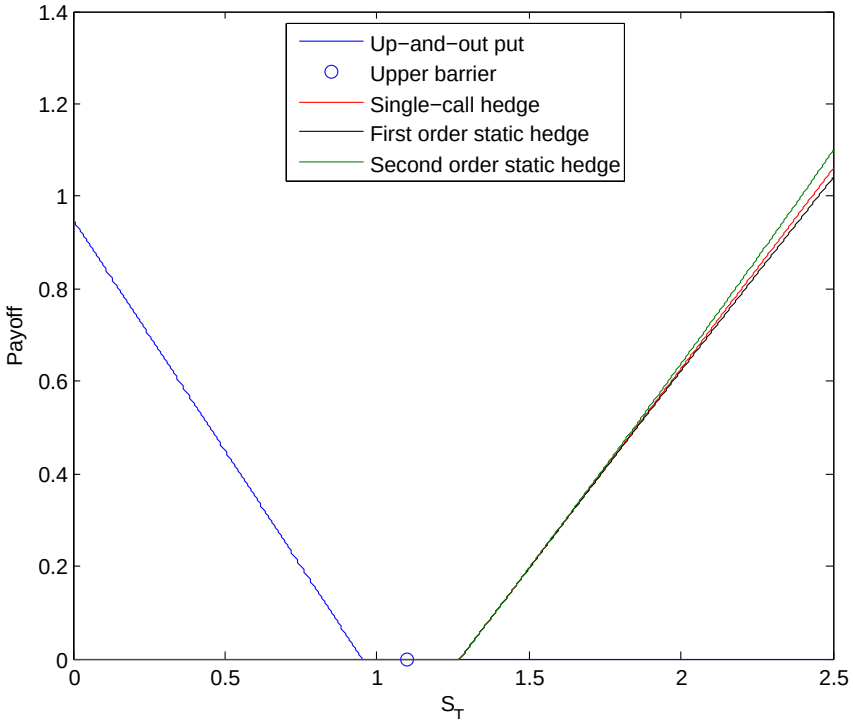
To simplify this we make use of the definition of $Z(x)$ and apply equation 1.62. Then we have the first and second order approximation of the exact static hedge given by:

$$g_1(x) = \sqrt{\sigma(x)\sigma(K)} \left(\int_U^x \frac{1}{\sigma(y)}\mathrm{d}y - \int_K^U \frac{1}{\sigma(y)}\mathrm{d}y\right)^+ \tag{2.20}$$

and

$$\begin{aligned} g_2(x) = &\sqrt{\sigma(x)\sigma(K)} \left(\int_U^x \frac{1}{\sigma(y)}\mathrm{d}y - \int_K^U \frac{1}{\sigma(y)}\mathrm{d}y\right)^+ \\ &+ \frac{\sqrt{\sigma(x)\sigma(K)}}{2\sqrt{2}} \left(\int_U^x \widetilde{\sigma}(y)\mathrm{d}y - \int_K^U \widetilde{\sigma}(y)\mathrm{d}y\right) \left(\left(\int_U^x \frac{1}{\sigma(y)}\mathrm{d}y - \int_K^U \frac{1}{\sigma(y)}\mathrm{d}y\right)^+\right)^2 \end{aligned} \tag{2.21}$$

Figure 2.1: First and second order static hedge vs. exact static hedge



2.3 Numerical illustration

In this section we briefly illustrate the second order approximation of the exact static hedge numerically. Note that $\widetilde{\sigma}$ can easily be expressed using σ only. We use its definition in equation 1.44 to get

$$\int_K^x \widetilde{\sigma}(y)\mathrm{d}y = \frac{1}{4\sqrt{2}} \int_K^x \frac{\sigma'(y)^2}{\sigma(y)}\mathrm{d}y - \frac{1}{2\sqrt{2}} \left(\sigma'(x) - \sigma'(K)\right) \tag{2.22}$$

This can be used to calculate our approximation numerically in a straight-forward way.

In general this involves numerical integration of expressions including σ and its derivatives, which are explicitly known. This is not a difficult task, but in the case of the Black-model (a zero-drift Black-Scholes model) things are even easier. We can calculate the second order approximation of the static hedge explicitly and compare it to the well-known explicit formula of the latter.

Figure 2.1 illustrates the result. More precisely it presents the payoff of an up-and-out put with strike $K = 0.95$ assuming that the underlying hasn't yet hit the upper barrier

$U = 1.1$ and the corresponding static hedges. The line in red shows the well-known exact static hedge – in this case a call with strike $U^2/K = 1.34$ and slope $K/U = 0.81$. The black line represents the first order approximation of the exact static hedge, the line in green corresponds to the second order approximation of the latter.

Note, that our first order approximation is against the widespread intuition not linear. This is a result of the fact, that we are expanding in the "z-variable", which essentially is a log transformation of the original domain.

Apparently this first order approximation is already doing a good job in this case. Up to 2 (twice the current level of the underlying) there is no significant difference to the exact static hedge (more precisely the difference is 0.0053, i.e. about 0.5% of the current level of the underlying).

Both, the first and the second order approximation of the exact static hedge behave asymptotically like a logarithm. Since the exact static hedge is linear in case of the black model, they will both underestimate it for very large values of S. As the second order approximation contains a $\log^2$-term, it will perform better for extreme values of the underlying.

The approximation of the static hedge used here is explicitly given by:

$$\begin{aligned} g_2(x) &= \sqrt{\sigma^2 xK}\left(\int_U^x \frac{1}{\sigma y}\mathrm{d}y - \int_K^U \frac{1}{\sigma y}\mathrm{d}y\right)^+ \\ &\qquad + \frac{\sqrt{\sigma^2 xK}}{16}\left\{\left(\int_U^x \frac{1}{\sigma y}\mathrm{d}y \int_K^U \frac{1}{\sigma y}\mathrm{d}y\right)^+\right\}^2 \left(\int_U^x \frac{\sigma^2}{\sigma y}\mathrm{d}y - \int_K^U \frac{\sigma^2}{\sigma y}\mathrm{d}y\right) \\ &= \sqrt{xK}\left(\log(x) - 2\log(U) + \log(K)\right)^+ \\ &\qquad + \frac{\sqrt{xK}}{16}\left\{\left(\log(x) - 2\log(U) + \log(K)\right)^+\right\}^2 \left(\log(x) - 2\log(U) + \log(K)\right) \end{aligned}$$

Observe, that the approximation of the exact static hedge does not depend on the total level of the volatility σ.

A satisfactory answer to the question of how to approximate the exact static hedge from equation 2.2 needs a bound on the residual term of the discussed expansion in equation 2.14. This would allow us to introduce global super- and sub-replicating strategies and an estimate on their precision. Unfortunately it turns out, that the classical estimates on χ provided in [24] are not sharp enough to lead to useful error bounds.

Furthermore we acknowledge, that the calculation of the coefficients in our expansion is a very tedious task. The calculation of the first two coefficients appeared to be relatively

easy in this chapter as we were able to leverage the results from our mathematical toolbox 1.2. Therefore it is desirable to find a programmable formulation of this task.

Both of these problems are well beyond the scope of this project and are left for further research. Now we focus on a more intuitive and at the same time more tractable approximation of the exact static hedge. Inspired by the well-understood case of a Black model considered by [5] we now want to study an asymptotic hedge for short maturities, which consists of a put and a multiple of vanilla calls only.

Chapter 3

Asymptotic hedge

3.1 The asymptotic hedge

Recall the definition of the exact static hedge in Definition 2.1.1: We call a European derivative with payoff $g(S_T)$ at maturity T an exact static hedge of an up-and-out put with strike K and barrier U, if the function g has support on (U, ∞) and the price of the European derivative u^g as a function of the underlying evaluated at U equals the one of a vanilla put for all intermediate times t.

The idea of this chapter is to consider payoff functions g, which represent a portfolio of η calls struck at $K^* > U$ only and see how good we can do with them, if we restrict our focus to short maturities. That is we consider $g(x) = \eta\,(x - K^*)^+$ for $K^* > U$. Note, that functions of this type clearly have support in $[U, \infty)$.

Naturally the next question is which condition we should use to determine the parameters K^* and η. A first idea may be to postulate that the *absolute* hedging error at the barrier is going to zero as time-to-maturity is going to zero. However, since $K < U < K^*$ this limit is zero as a result of the underlying following a continuous process – independent of the choice of K^* and η. It turns out, that the *relative* hedging error is the appropriate criteria to use. This is made precise in the following definition.

Definition 3.1.1 (Asymptotic hedge)**.** *The pair (K^*, η) is an asymptotic (or according to [9] a single-call) hedge for an up-and-out put with strike K and barrier U, if*

$$\frac{p(U, \lambda, K) - \eta c(U, \lambda, K^*)}{p(U, \lambda, K)} \to 0, \qquad \tau \to 0 \tag{3.1}$$

The aim of this chapter is to find an explicit expression for K^* and η. To this end we apply Laplace transform and solve the corresponding problem in Laplace-space making use of the tools developed in chapter 1.

Keeping in mind, that considering the limit $\tau \to 0$ in the original space is "equivalent" to $\lambda \to \infty$ in Laplace space, the Laplace transformed quantities from Definition 3.1.1 should intuitively satisfy

$$\frac{\hat{p}(U,\lambda,K) - \eta\hat{c}(U,\lambda,K^*)}{\hat{p}(U,\lambda,K)} \to 0, \qquad |\lambda| \to \infty \tag{3.2}$$

In the following theorem we find a candidate asymptotic hedge (K^*, η) by applying the asymptotic expansion results for the Laplace transform of a vanilla put and call from chapter 1 to the above equation. Then we proof, that (K^*, η) indeed is the asymptotic hedge.

Theorem 3.1.1. *Let the assumption (A3) hold true. Then the parameter combination* (K^*, η) *with* K^* *solving*

$$\int_U^{K^*} \frac{1}{\sigma(y)}\mathrm{d}y = \int_K^U \frac{1}{\sigma(y)}\mathrm{d}y \tag{3.3}$$

and η *defined by*

$$\eta = \sqrt{\frac{\sigma(K)}{\sigma(K^*)}} \tag{3.4}$$

is the asymptotic hedge in the sense of Definition 3.1.1.

Proof. To show the second statement we generalize the idea from the proof of proposition 2 in [9] (see appendix B, page 34 of this reference) and introduce the constants c_1, c_2 and c_3 given by

$$\begin{aligned} c_1 &:= \frac{\sqrt{\sigma(U)\sigma(K)}}{4\sqrt{2}} \left(\int_U^{K^*} \widetilde{\sigma}(x)\mathrm{d}x - \int_K^U \widetilde{\sigma}(x)\mathrm{d}x \right) \in (-\infty, \infty) \\ c_2 &:= \sqrt{2} \int_K^U \frac{1}{\sigma(y)} \geq 0 \\ c_3 &:= \frac{\sigma(U)\sigma(K)}{2\sqrt{2}} \geq 0 \end{aligned} \tag{3.5}$$

Choosing K^* and η as specified in equations 3.3 and 3.4 respectively, we have:

$$\sqrt{\sigma(U)\sigma(K)}e^{-iZ(K)\sqrt{-\lambda}} = \eta\sqrt{\sigma(U)\sigma(K^*)}e^{iZ(K^*)\sqrt{-\lambda}} \tag{3.6}$$

Note, that we can intuitively identify $-i\sqrt{-\lambda} \to \sqrt{\lambda}$, but to be consistent with our definition of the square root in chapter 1, we keep our notation the way it is. By equations 1.43 and 1.45 from chapter 1, we have for $|\lambda| \to \infty$

$$\begin{aligned} &\hat{p}(U,\lambda,K) - \eta\hat{c}(U,\lambda,K^*) \\ &= \frac{\sqrt{\sigma(U)\sigma(K)}}{2\sqrt{2}(-i\sqrt{-\lambda})^3} e^{-iZ(K)\sqrt{-\lambda}} \left(\left(\int_U^K \widetilde{\sigma}(x)\mathrm{d}x + \int_U^{K^*} \widetilde{\sigma}(x)\mathrm{d}x \right) \frac{1}{-i\sqrt{-\lambda}} + \mathcal{O}\left(|\lambda|^{-1}\right) \right) \\ &= c_1 \frac{e^{ic_2\sqrt{-\lambda}}}{\lambda^2} + \frac{e^{ic_2\sqrt{-\lambda}}}{(-i\sqrt{-\lambda})^5}\mathcal{O}(1) \end{aligned} \tag{3.7}$$

Similarly we have for the Laplace transform of the put price:

$$\begin{aligned}\hat{p}(U,\lambda,K) &= \frac{\sqrt{\sigma(U)\sigma(K)}}{2\sqrt{2}(-i\sqrt{-\lambda})^3}e^{-iZ(K)\sqrt{-\lambda}}\left(1+\mathcal{O}\left(|\lambda|^{-\frac{1}{2}}\right)\right)\\ &= c_3\frac{e^{ic_2\sqrt{-\lambda}}}{(-i\sqrt{-\lambda})^3}+\frac{e^{ic_2\sqrt{-\lambda}}}{\lambda^2}\mathcal{O}(1)\end{aligned} \tag{3.8}$$

First, we consider the inverse Laplace transform of the first integral in equation 3.7. We have:

$$\frac{\partial^2}{\partial\tau^2}\left(\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda\tau+ic_2\sqrt{-\lambda}}}{\lambda^2}\mathrm{d}\lambda\right)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda\tau+ic_2\sqrt{-\lambda}}\mathrm{d}\lambda=\frac{c_2}{2\sqrt{\pi\tau^3}}e^{-\frac{c_2^2}{4\tau}}, \tag{3.9}$$

where we made use of the fact, that the function $f(\lambda)=e^{ic_2\sqrt{-\lambda}}$ is the Laplace transform of the first hitting time of a Brownian motion – more precisely $T_{c_2}:=\inf\{t\geq 0: B_t=c_2\}$ – and the density of this random variable is well-known. Therefore:

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda\tau+ic_2\sqrt{-\lambda}}}{\lambda^2}\mathrm{d}\lambda=\int_0^\tau\left(\int_0^s\frac{c_2}{2\sqrt{\pi t^3}}e^{-\frac{c_2^2}{4t}}\mathrm{d}t\right)\mathrm{d}s=\int_0^\tau\left(2\int_{\frac{c_2}{\sqrt{2s}}}^\infty\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\mathrm{d}x\right)\mathrm{d}s,$$

where we changed variables to $x=c_2/\sqrt{2t}$ in the last step. Now we make use of the well known asymptotic relation

$$\int_x^\infty e^{-\frac{y^2}{2}}\mathrm{d}y=\frac{1}{x}e^{-\frac{x^2}{2}}\left(1+o(1)\right) \tag{3.10}$$

and get as $\tau\to 0$

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda\tau+ic_2\sqrt{-\lambda}}}{\lambda^2}\mathrm{d}\lambda=\frac{2}{\sqrt{\pi}c_2}\int_0^\tau\sqrt{s}e^{-\frac{c_2^2}{4s}}\mathrm{d}s\left(1+o(1)\right) \tag{3.11}$$

Next we apply analogous arguments to the inverse Laplace transform of the first term in equation 3.8. In this case things are easier and we get by differentiating once with respect to τ and once with respect to c_2:

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda\tau+ic_2\sqrt{-\lambda}}}{\lambda^{\frac{3}{2}}}\mathrm{d}\lambda &= \int_0^\tau\left(\int_{c_2}^\infty\frac{u}{2\sqrt{\pi s^3}}e^{-\frac{u^2}{4s}}\mathrm{d}u\right)\mathrm{d}s\\ &= \int_0^\tau\left(\int_{\frac{c_2}{\sqrt{2s}}}^\infty\frac{u}{\sqrt{\pi s}}e^{-\frac{u^2}{2}}\mathrm{d}u\right)\mathrm{d}s=\frac{1}{\sqrt{\pi}}\int_0^\tau\frac{1}{\sqrt{s}}e^{-\frac{c_2^2}{4s}}\mathrm{d}s\end{aligned} \tag{3.12}$$

Next we consider the second term from equation 3.7. Here our approach has to be different as we need to derive an estimate on its absolute value because of the higher order term involved. First observe, that

$$\tau\lambda+ic_2\sqrt{-\lambda}=\tau\left(-i\sqrt{-\lambda}-\frac{c_2}{2\tau}\right)^2-\frac{c_2^2}{4\tau} \tag{3.13}$$

Therefore we have by changing variables to $s := (-i\sqrt{-\lambda} - c/(2\tau))^2$:

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda\tau+ic_2\sqrt{-\lambda}}}{(-i\sqrt{-\lambda})^5}\mathrm{d}\lambda = e^{-\frac{c_2^2}{4\tau}} \int_{\Gamma'} \frac{e^{\tau s}}{-i\sqrt{-s}\left(-i\sqrt{-s}+\frac{c_2}{2\tau}\right)^4}\mathrm{d}s, \tag{3.14}$$

where Γ' is the image of $[\gamma - i\infty, \gamma + i\infty]$ under the mapping $\lambda \mapsto s(\lambda)$. For $s = (-i\sqrt{-(\gamma+iu)} - c_2/(2\tau))^2 \in \Gamma'$ for some $u \in \mathbb{R}$ it holds

$$\Re(s) = \gamma - \frac{c_2}{2\tau}\sqrt{2\left(\sqrt{\gamma^2+u^2}+\gamma\right)} + \frac{c_2^2}{4\tau^2} \leq \gamma - \frac{c_2}{\tau}\sqrt{\gamma} + \frac{c_2^2}{4\tau^2}, \tag{3.15}$$

since $\Re(s)$ is maximal at $u = 0$ for obvious reasons. Now, fix $\tau > 0$ and choose $\gamma = \gamma(\tau) = c_2^2/(4\tau^2) > 0$. This gives:

$$\Re(s) \leq \frac{c_2^2}{4\tau^2} - \frac{c_2}{\tau}\frac{c_2}{2\tau} + \frac{c_2^2}{4\tau^2} = 0 \tag{3.16}$$

Therefore it holds $|e^{\tau s}| = e^{\tau\Re(s)} = 1$ and we have:

$$\begin{aligned}
\left|\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda\tau+ic_2\sqrt{-\lambda}}}{(-i\sqrt{-\lambda})^5}\mathrm{d}\lambda\right| &\leq e^{-\frac{c_2^2}{4\tau}} \int_{\Gamma'} \frac{1}{\left|\sqrt{-s}(-i\sqrt{-s}+\frac{c_2}{2\tau})^4\right|}|\mathrm{d}s| = e^{-\frac{c_2^2}{4\tau}} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{|\lambda|^{\frac{5}{2}}}|\mathrm{d}\lambda| \\
&= e^{-\frac{c_2^2}{4\tau}} \int_{-\infty}^{\infty} \frac{1}{(\gamma(\tau)^2+u^2)^{\frac{5}{4}}}\mathrm{d}u = e^{-\frac{c_2^2}{4\tau}} \frac{B\left(\frac{1}{2},\frac{3}{4}\right)}{\gamma(\tau)^{\frac{3}{2}}} \\
&= e^{-\frac{c_2^2}{4\tau}} \frac{8B\left(\frac{1}{2},\frac{3}{4}\right)}{c_2^3}\tau^3,
\end{aligned} \tag{3.17}$$

where $B(p,q)$ is the Beta-function. Similarly we get for the second term in equation 3.8:

$$\left|\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda\tau+ic_2\sqrt{-\lambda}}}{\lambda^2}\mathrm{d}\lambda\right| \leq e^{-\frac{c_2^2}{4\tau}} \frac{4B\left(\frac{1}{4},\frac{1}{2}\right)}{c_2^2}\tau^2 \tag{3.18}$$

Now it is time to combine the results we derived above. For small enough τ (that is large enough γ) we can bound the $\mathcal{O}(1)$ term in equation 3.7 by a constant, say M. We have:

$$\begin{aligned}
|p(U,\tau,K) - \eta c(U,\tau,K^*)| &\leq \frac{2c_1}{\sqrt{\pi}c_2}\int_0^\tau \sqrt{s}e^{-\frac{c_2^2}{4s}}\mathrm{d}s + \frac{c_1}{2\pi}e^{-\frac{c_2^2}{4\tau}}\left|\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda\tau+ic_2\sqrt{-\lambda}}}{(-i\sqrt{-\lambda})^5}\mathcal{O}(1)\mathrm{d}\lambda\right| \\
&\leq \frac{2c_1}{\sqrt{\pi}c_2}\int_0^\tau \sqrt{s}e^{-\frac{c_2^2}{4s}}\mathrm{d}s + \frac{4c_1MB\left(\frac{1}{2},\frac{3}{4}\right)}{\pi c_2^2}e^{-\frac{c_2^2}{4s}}\tau^3
\end{aligned}$$

Similarly we have for τ small enough and N being the bound for the higher order term in equation 3.8:

$$\begin{aligned}
|p(U,\tau,K)| &\geq \frac{c_3}{\sqrt{\pi}}\int_0^\tau \frac{1}{\sqrt{s}}e^{-\frac{c_2^2}{4s}}\mathrm{d}s - \frac{c_3}{2\pi}e^{-\frac{c_2^2}{4\tau}}\left|\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda\tau+ic_2\sqrt{-\lambda}}}{(-i\sqrt{-\lambda})^3}\mathcal{O}(1)\mathrm{d}\lambda\right| \\
&\geq \frac{c_3}{\sqrt{\pi}}\int_0^\tau \frac{1}{\sqrt{s}}e^{-\frac{c_2^2}{4s}}\mathrm{d}s - \frac{2c_3NB\left(\frac{1}{4},\frac{1}{2}\right)}{\pi c_2^2}e^{-\frac{c_2^2}{4\tau}}\tau^2 > 0
\end{aligned}$$

for τ small enough since we have by partial integration:

$$\int_0^\tau \frac{1}{\sqrt{s}} e^{-\frac{c_2^2}{4s}} \mathrm{d}s = \frac{\tau^{\frac{3}{2}}}{c_2^2} e^{-\frac{c_2^2}{4\tau}} + (1 + \mathcal{O}(\tau)) \tag{3.19}$$

Finally, we combine the above estimates and conclude with an application of L'Hôpital's rule.

□

The following remark shows how the first order approximation of the exact static hedge in equation 2.21, which was presented in chapter 2, is linked to the asymptotic hedge we obtained in the above theorem. As both of them are in their sense "first order approximations" to the exact static hedge we find the expected similarities.

Remark 3.1.1. *We assume the first order approximation of the static hedge to be given. That is a European-type derivative with the payoff function*

$$g_1(x) = \sqrt{\sigma(x)\sigma(K)} \left(\int_U^x \frac{1}{\sigma(y)} \mathrm{d}y - \int_K^U \frac{1}{\sigma(y)} \mathrm{d}y \right)^+ \tag{3.20}$$

(i) The strike of the asymptotic hedge K^* can be characterized as the largest root of g_1. We have:

$$g_1(K^*) = 0 \Leftrightarrow \left(\int_U^{K^*} \frac{1}{\sigma(y)} \mathrm{d}y - \int_K^U \frac{1}{\sigma(y)} \mathrm{d}y \right)^+ = 0 \tag{3.21}$$

As the left hand side is increasing in K^*, the largest root is characterized by

$$\int_U^{K^*} \frac{1}{\sigma(y)} \mathrm{d}y = \int_K^U \frac{1}{\sigma(y)} \mathrm{d}y \tag{3.22}$$

This is equivalent to equation 3.3.

(ii) At K^* characterized above, the slope of the asymptotic hedge and the first order static hedge coincide. This can be seen easily:

$$g_1'(x) = \frac{\sigma'(x)\sqrt{\sigma(K)}}{2\sqrt{\sigma(x)}} \left(\int_U^x \frac{1}{\sigma(y)} \mathrm{d}y - \int_K^U \frac{1}{\sigma(y)} \mathrm{d}y \right) + \sqrt{\frac{\sigma(K)}{\sigma(x)}} \tag{3.23}$$

And therefore by equation 3.4:

$$g_n\prime(K^*) = g_1'(K^*) = \sqrt{\frac{\sigma(K)}{\sigma(K^*)}} = \eta \tag{3.24}$$

3.2 Examples

The aim of this section is to illustrate the main result, Theorem 3.1.1, using two well known examples, where we naturally get back the classical results. Moreover we illustrate the performance of our approximations in case of the CEV model.

3.2.1 Black (Zero drift Black-Scholes) model

We consider the well known Black model, that is a Black-Scholes model with zero drift, where the dynamics of the underlying asset are given by

$$\mathrm{d}S_t = \sigma S_t \mathrm{d}W_t, \quad S_0 = s > 0, \tag{3.25}$$

with constant volatility $\sigma > 0$. In our notation this means we are considering $\sigma(s) = \sigma s$. Therefore equation 3.3 from Theorem 3.1.1 becomes

$$\int_U^{K^*} \frac{1}{\sigma y}\mathrm{d}y = \int_K^U \frac{1}{\sigma y}\mathrm{d}y \tag{3.26}$$

Simply calculating the above integrals, this is equivalent to

$$\log(K^*) = 2\log(U) - \log(K), \quad \text{i.e. } K^* = \frac{U^2}{K} \tag{3.27}$$

Using formula 3.4 we now compute easily

$$\eta = \sqrt{\frac{\sigma(K)}{\sigma(K^*)}} = \sqrt{\frac{K}{K^*}} = \sqrt{\frac{K^2}{U^2}} = \frac{K}{U} \tag{3.28}$$

This tells us, that in a Black-Scholes setting with zero drift, the asymptotic static hedge for an up-and-out put with strike K and barrier U is long one vanilla put struck at K and short $\frac{K}{U}$ vanilla calls struck at $\frac{U^2}{K}$. Of course this coincides with the classical exact static hedge established in [23].

Note that the static hedge is in particular independent of the volatility σ. Since the implied volatility surface in this setup is constant and equal to σ for all strikes, this observation indicates, that our static hedge should not depend on the overall level of the volatility surface.

3.2.2 Constant Elasticity of Variance (CEV) model

Under the Constant Elasticity of Variance (CEV) model (again assuming zero drift) the dynamics of the underlying are given by

$$\mathrm{d}S_t = S_t^{1+\beta}\mathrm{d}W_t, \quad S_0 = s > 0, \tag{3.29}$$

with a constant $\beta < 0$. That is in our notation assuming $\sigma(s) = s^{1+\beta}$. Clearly $\frac{\sigma(s)}{s} = s^{\beta}$ is not bounded as $s \to 0$ – this contradicts assumption (A2). In fact this tells us, that the underlying S_t does hit zero with positive probability.

Strictly speaking this means, that we cannot apply the results from Theorem 3.1.1. However, we illustrate that the static hedge obtained by solving the equations 3.3 and 3.4 does still coincide with the result derived in Chapter 3.3 of [9].

Using 3.29 equation 3.3 becomes

$$\int_U^{K^*} \frac{1}{y^{1+\beta}}\mathrm{d}y = \int_K^U \frac{1}{y^{1+\beta}}\mathrm{d}y \tag{3.30}$$

This is equivalent to

$$\frac{U^{-\beta} - (K^*)^{-\beta}}{\beta} = \frac{K^{-\beta} - U^{-\beta}}{\beta}, \quad \text{i.e. } K^* = \left(2U^{-\beta} - K^{-\beta}\right)^{-\frac{1}{\beta}} \tag{3.31}$$

By equation 3.4

$$\eta = \sqrt{\frac{\sigma(K)}{\sigma(K^*)}} = \sqrt{\frac{K^{1+\beta}}{(K^*)^{1+\beta}}} = \left(\frac{K}{K^*}\right)^{\frac{1+\beta}{2}} \tag{3.32}$$

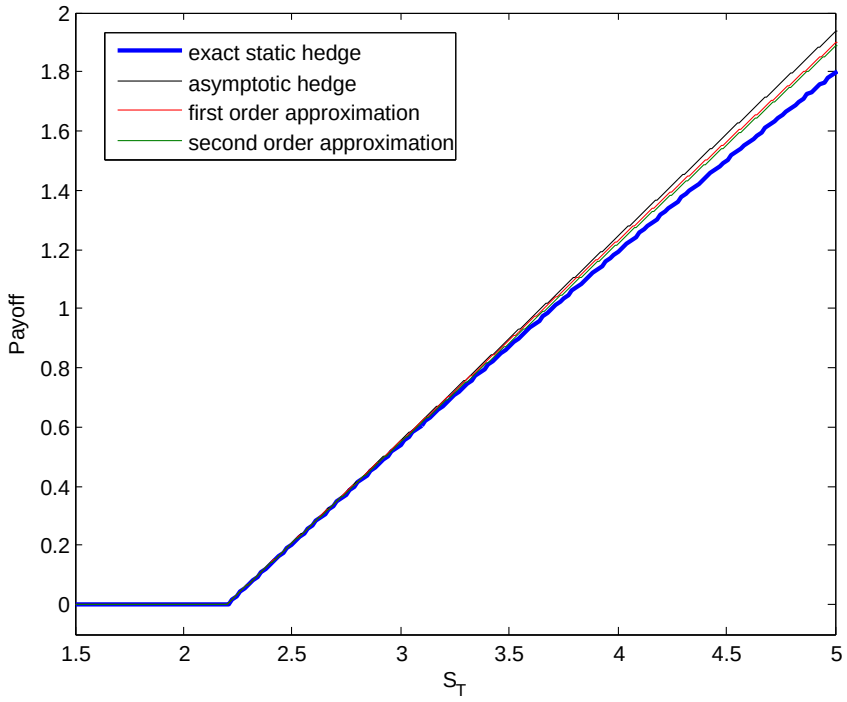
Both results are in line with the ones obtained in Section 3.3 of [9].

3.2.3 Exact static hedge and our approximations

Now we want to illustrate how the exact static hedge compares to both types of approximations under consideration in case of the CEV model. The main reason for us to choose the CEV model is that an explicit formula for the exact static hedge is known (compare section 3.2 of [9]).

We follow the latter reference in choosing the model parameters. We consider a zero-drift CEV model with parameter $\beta = -0.5$ and an up-and-out put option with strike $K = 0.5$ and barrier $U = 1.2$, the current level of the underlying asset is assumed to be $S = 1$. The calculation of the asymptotic hedge is straight-forward making use of the previous example

Figure 3.1: Exact static hedge in CEV model and its approximations

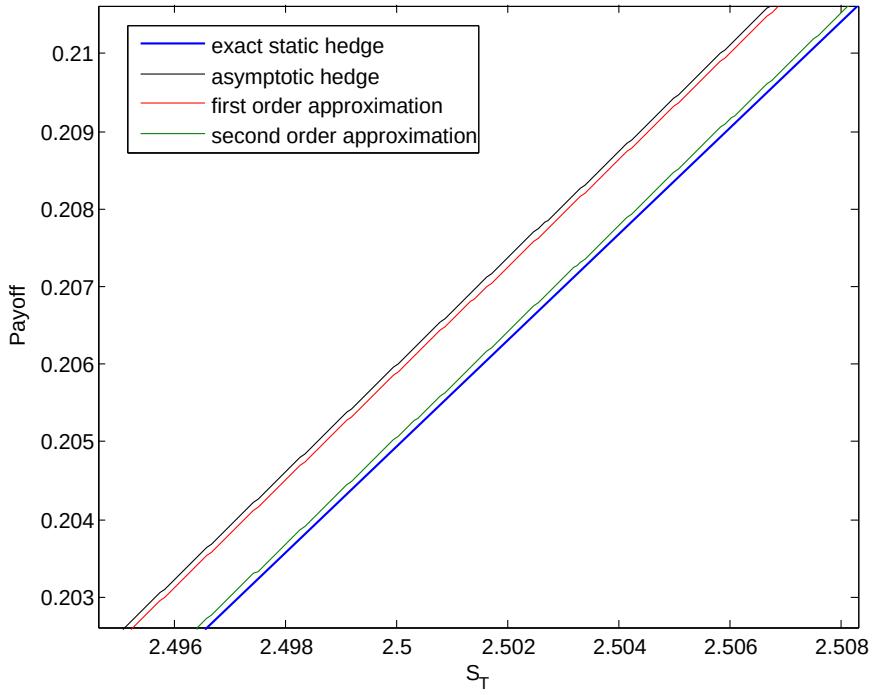


and the first and second order approximation of the exact static hedge are obtained by plugging $\sigma(x) = x^{1+\beta}$ into equation 2.21. The result is shown in figure 3.1.

Again our approximations seem to do a good job for values, which are relevant when considering reasonably short maturities. This has already been the case for the Black model illustrated in figure 2.1. In fact, it is hard to distinguish between the exact static hedge shown as a blue line and all three approximations for values of the underlying asset below three. For larger values it seems to be the case, that the asymptotic hedge overestimates the payoff of the exact static hedge due to the fact, that it is linear and does not capture the curvature of the exact hedge in this particular case.

However, there is no significant difference between the first an the second order approximation, which are plotted as red and green lines respectively. To allow us to distinguish between the two we provide a "zoom-in" in figure 3.2 for values of the underlying around 2.5. Although the difference is very small in absolute value, we can clearly see, that the second order approximation is performing better than the one of first order, which appears to be very close to the asymptotic static hedge in this case.

Figure 3.2: Exact static hedge in CEV model and its approximations - zoom in



The main conclusion we draw from this illustration is that all the proposed approximations of the exact static hedge seem to perform well. For reasonably short maturities each of them appears to be equally good as we cannot observe significant differences between the alternatives under consideration.

In the final chapter we discuss ways to robustify the obtained static hedges. Encouraged by the above results, we decide to perform an explicit calculation for the case of the asymptotic static hedge, which is the most tractable one of the alternatives under consideration and the loss in performance does not appear to be substantial.

Chapter 4

Semi-robust hedge

The central idea of the following discussion is to consider a local volatility model as a building block to develop a semi-robust asymptotic static hedge. Levering the results from the previous chapter we illustrate a method, which allow us to obtain a semi-robust static hedge in a sense, that it is reflecting one's believes on the future values of implied volatility. This idea is discussed in detail in the following section.

4.1 From implied to local volatility

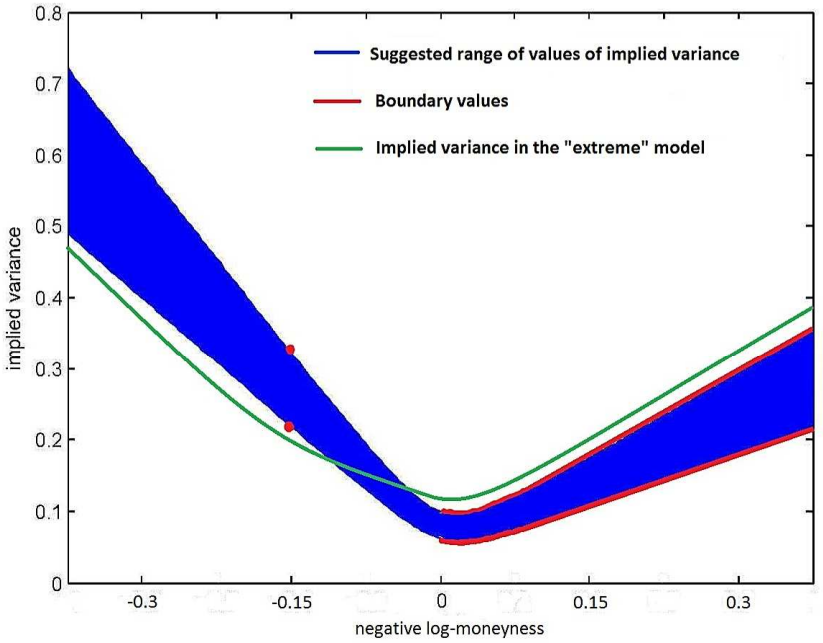
The method we suggest to construct model independent hedges consists of two steps. First we construct a range of future implied volatilities, which we believe will not be exceeded. To this end we could for example look at historic dynamics of the implied volatility surface. This input is thought to incorporate a trader's view on the future values of the implied volatility.

A generic example of such a range is given in figure 4.1. Here the implied volatility is plotted as a function of the negative log-moneyness $k = \log(K/S)$. The displayed blue strip could be the range of implied volatilities we believe is not exceeded in the lifetime of our derivative contract. We use this example throughout the chapter and we provide the corresponding range of asymptotic hedges in figure 4.5, which is discussed in section 4.3.

The second step goes back to the monotonicity argument, which was given in the introduction. Since our asymptotic hedge is a combination of a put long struck at $K < U$ and a short position of η calls struck at $K^* > U$, it clearly can be decomposed in a concave and a convex payoff. Therefore their respective prices are monotone with respect to implied volatility.

The method we describe works for both, super- and sub-replicating strategies – however, for the sake of simplicity, we will focus on the super-replicating strategy for now. The cen-

Figure 4.1: Range of Gatheral-type implied variance



tral idea is to choose an implied volatility, which is identified with an "extreme model". We construct this as follows: First we choose a lower bound for the future values of the implied volatility at the point $k = \log(K/S)$ corresponding to the strike of the put K and an upper bound for the implied volatility above $u = \log(U/S)$ – that is the range of the strike of the static hedge in terms of negative log-moneyness. These "boundary points" are highlighted in red in figure 4.1 and an implied volatility corresponding to a generic "extreme model" is shown in green.

Now, we make use of the well known fact, that a local volatility model can be calibrated to any implied volatility surface. To avoid possible confusion arising from notation we remind ourselves, that so far we have been considering a local volatility model

$$\mathrm{d}S_t = \sigma(S_t)\mathrm{d}W_t, \quad t \in [0, T], \tag{4.1}$$

where we denoted the coefficient in the SDE describing the dynamics of the underlying by σ. As this turns out to be more convenient here, we introduce the local volatility as a function of the negative log-moneyness $k = \log(K/S)$, that is a function $\sigma_{loc}(k)$, such that

$$\sigma_{loc}(k)) := \frac{\sigma(Se^k)}{Se^k} \tag{4.2}$$

Using this notation we can use the well-known result (see e.g. [3]), which allows us to

connect the local volatility $\sigma_{loc}(k)$ to the implied volatility $\Sigma(k)$ for short maturities by

$$\frac{1}{\Sigma(k)} = \frac{1}{k}\int_0^k \frac{1}{\sigma^{loc}(x)}\mathrm{d}x \tag{4.3}$$

Keeping in mind that we want to consider asymptotic hedges for short maturities, this is very much the right result for our purpose.

To deal with this explicitly we now introduce a specific parametric form of implied volatility.

4.2 The Gatheral family

Going forward we are working with implied volatility of the Gatheral-type, which is defined rigorously in Definition 4.2.1 below. However, this assumption is quite reasonable as it is common practice in the industry to obtain implied volatility surfaces by fitting a Gatheral parametric family to the observed implied volatilities from traded vanilla options. Arguably we therefore do "observe" volatility surfaces of this type.

Definition 4.2.1. *An implied volatility surface is from the Gatheral family*[1]*, if for each fixed expiration T the implied variance $\Sigma^2(k)$ as a function of the log-strike or more precisely the "negative* log-*moneyness" $k = \log(K/S_t)$ is of the form*

$$\Sigma^2(k) = a + b\left(\rho(k-m) + \sqrt{(k-m)^2+\sigma^2}\right), \tag{4.4}$$

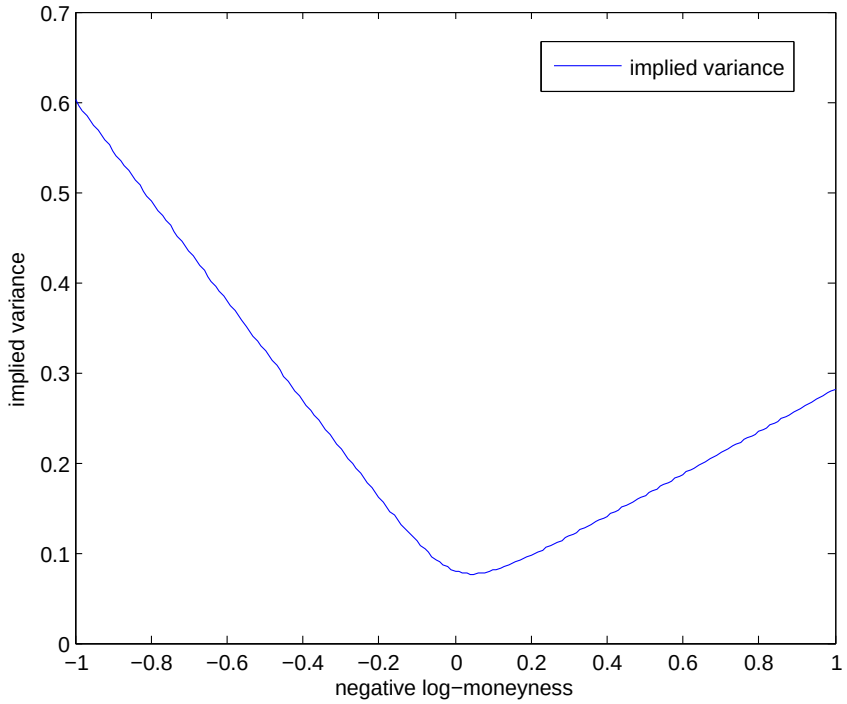
where the coefficients a, b, ρ, σ and m depend on the expiration T.

From our point of view there are two main reasons why this family is so popular. First, it is relatively easy to eliminate calendar spread arbitrage, when fitting it to observed market data and interpolating between "time-slices". Second the parameters used in 4.4 have intuitive meaning and we can easily assign them a "natural range".

- m: The translation parameter – m determines the location of the minimum of the implied volatility as a function of k. Choosing $m = 0$ means the minimum of the implied volatility function is assumed to be at zero. This is a reasonable assumption as we are looking at the implied volatility as a function of $k = \log(K/S_t)$, i.e. $k = 0$ corresponds to $K = S_t$ – the at-the-money option. We choose to focus on this type of Gatheral-implied volatility going forward.

[1] This is the *stochastic volatility inspired* of implied volatility surfaces proposed in [15], pp 37-42.

Figure 4.2: Implied variance for the Gatheral family in our base case



- ρ: Skewness – this is typically negative. For our base case[2] we assume $\rho = -0.4$.
- a: Overall level of variance. For our base scenario we let $a = 0.04$, that is a volatility of 20%.
- b: Convexity – is expected to be positive. For illustration purposes we assume $b = 0.4$.
- σ: Determines how smooth the volatility smile is at the money. For the moment we take $\sigma = 0.1$.

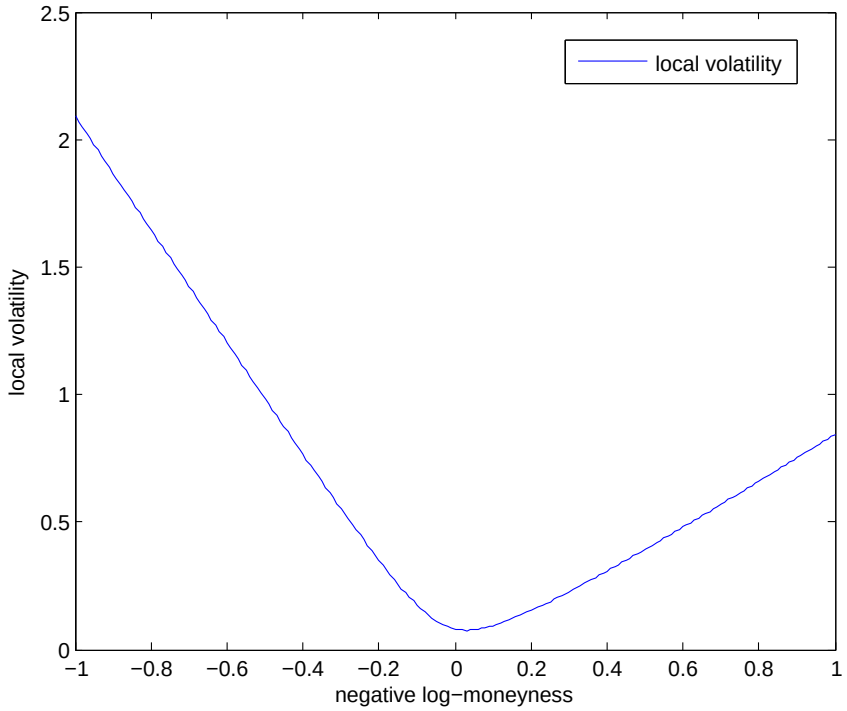
Figure 4.2 illustrates the form of the Gatheral-type implied variance using the parameter combination of the "base case" described above.

As a next step, we plug the functional form of the implied volatility $\Sigma(k)$ defined in equation 4.4 into equation 4.3:

$$\frac{1}{\sqrt{a + b\left(\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}\right)}} = \frac{1}{k}\int_0^k \frac{1}{\sigma_{loc}(x)}\mathrm{d}x \tag{4.5}$$

[2] All values of this scenario are taken from [16].

Figure 4.3: Local volatility function corresponding to the implied variance in Figure 4.2



Rearranging and differentiating with respect to k gives

$$\frac{1}{\sigma_{loc}(k)} = \frac{2\Sigma^2(k) - kb\left(\rho + \frac{k-m}{\sqrt{(k-m)^2+\sigma^2}}\right)}{2\Sigma^3(k)}$$

Rearranging gives an explicit expression for the local volatility $\sigma_{loc}(k)$

$$\sigma_{loc}(k) = \frac{2\sqrt{(k-m)^2+\sigma^2}\Sigma^3(k)}{(2\Sigma^2(k) - bk\rho)\sqrt{(k-m)^2+\sigma^2} - bk(k-m)} \tag{4.6}$$

Figure 4.3 illustrates the local volatility function corresponding to the implied variance plotted in figure 4.2 in the sense of equation 4.6.

Our idea is to apply the results from the previous chapters to obtain a semi-robust static hedge. However the local volatility σ_{loc} obtained from Gatheral family in equation 4.6 does not meet the assumptions made in section 1.1. Having in mind that in this discussion the local volatility is only a building block to connect a Gatheral-type implied volatility to a static hedge we explain in the following remark, which adjustments allow our earlier analysis to succeed nonetheless.

Remark 4.2.1. *Clearly σ_{loc} and therefore $\sigma(s)$ are smooth functions, i.e. the first part of assumption (A1) is satisfied. However, the second part implies that σ_{loc} is bounded as $k \to \infty$ and as $k \to -\infty$. However, one sees from equation 4.6 directly, that σ_{loc} behaves like a square root asymptotically. This is clearly contradicting the second part of assumption (A1). In fact the underlying asset S_t, in the local volatility model introduced by σ_{loc} is only a local martingale. This is not only a mathematical subtleness, but causes the classical theory for pricing derivatives to break down. For example we do not get the classical pricing PDE 1.3 any more.*

However, there is an easy way around that. The Gatheral-type implied volatility we are starting with is fitted to observed market data and implied volatility will be quoted for a certain range of strikes only. For our purpose however, it is sufficient to model the implied volatility according to the Gatheral-type paramatric family within this range only. Outside of this range we choose σ_{loc} in such a way, that it is $C^3(\mathbb{R})$ and constant outside some finite interval. Using this modified local volatility, we can easily argue, that both, assumptions (A1) and (A3) are satisfied.

Note, that for example our asymptotic static hedge derived in chapter 3 does not depend on the values of $\sigma_{loc}(k)$ for very large or very small k: One sees this directly using Theorem 3.1.1. Similarly, the coefficients of the asymptotic expansion from chapter 2 do not depend on the extreme values of σ. Ideally, we would like to obtain a uniform bound on the residuals. However, this is not straight-forward using classical estimates and is therefore left for further research.

We will now consider the concept of asymptotic hedging introduced in chapter 3, which is much more tractable.

4.3 Semi-robust asymptotic hedge

In our final section we want to apply the discussed technique to explicitly obtain a *semi-robust asymptotic hedge*. The main result in chapter 2 is designed to serve as a building block for this case.

As a first step we make use of Theorem 3.1.1 to connect a local volatility, which is derived from a Gatheral-type implied volatility and was modified in the sense of remark 4.2.1, to an asymptotic hedge. For simplicity we assume the current level of the underlying to be $S = 1$ – this saves us from normalizing the strikes and therefore simplifies our formulae. We get

by equation 3.3 from theorem 3.1.1:

$$\int_U^{K^*} \frac{1}{\sigma(y)}\mathrm{d}y = \int_K^U \frac{1}{\sigma(y)}\mathrm{d}y \tag{4.7}$$

Transforming to log-coordinates, i.e. making a simple substitution $k = \log(y)$, gives

$$\int_{\log(U)}^{\log(K^*)} \frac{e^k}{\sigma(e^k)}\mathrm{d}k = \int_{\log(K)}^{\log(U)} \frac{e^k}{\sigma(e^k)}\mathrm{d}k \tag{4.8}$$

Now, we express σ in terms of σ_{loc} by making use of equation 4.2. We get

$$\int_{\log(U)}^{\log(K^*)} \frac{1}{\sigma_{loc}(k)}\mathrm{d}k = \int_{\log(K)}^{\log(U)} \frac{1}{\sigma_{loc}(k)}\mathrm{d}k \tag{4.9}$$

Next we show how this equation can be expressed as a simple equation of the implied volatility Σ only. This greatly simplifies the problem of calculating K^*. First we rewrite the above equation in the following form:

$$\int_0^{\log(K^*)} \frac{1}{\sigma_{loc}(k)}\mathrm{d}k = 2\int_0^{\log(U)} \frac{1}{\sigma_{loc}(k)}\mathrm{d}k - \int_0^{\log(K)} \frac{1}{\sigma_{loc}(k)}\mathrm{d}k \tag{4.10}$$

By equation 4.3 this is equivalent to

$$\frac{\log(K^*)}{\Sigma(\log(K^*))} = 2\frac{\log(U)}{\Sigma(\log(U))} - \frac{\log(K)}{\Sigma(\log(K))} \tag{4.11}$$

In the problem of calculating K^* for a given Gatheral-type implied volatility the right-hand-side is a given constant, say

$$C := 2\frac{\log(U)}{\Sigma(\log(U))} - \frac{\log(K)}{\Sigma(\log(K))} \tag{4.12}$$

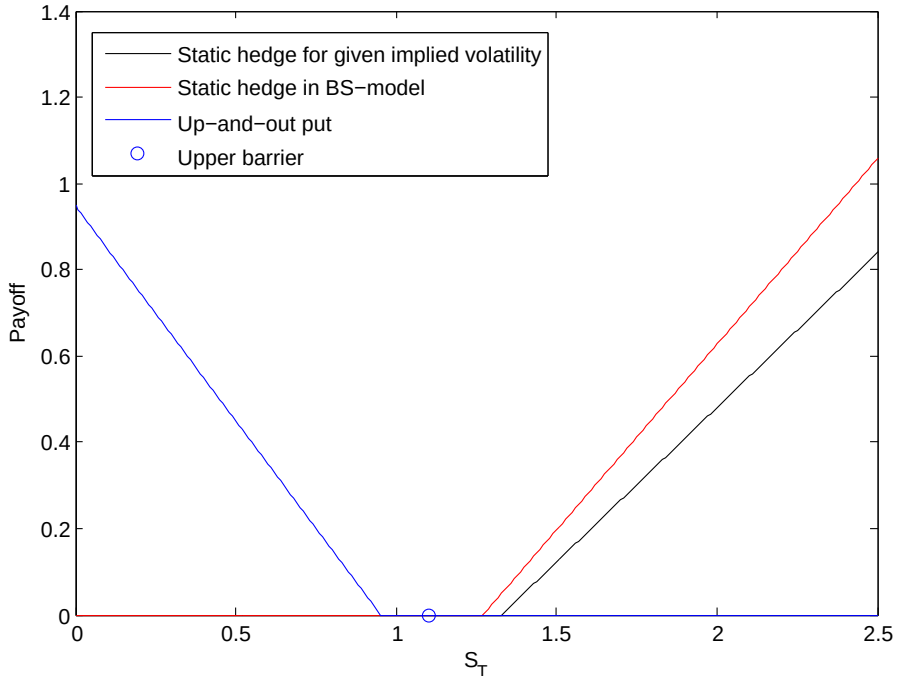
Using the functional form of $\widetilde{\Sigma}$ given in equation 4.4, we see that $\log(K^*)$ is a solution to the following equation in k

$$\frac{k}{\sqrt{a + b\left(\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}\right)}} = C \tag{4.13}$$

As a result k is essentially a root of a polynomial of degree four. In principle this can be solved explicitly using Ferrari's well known formula. However, this is somewhat tedious and we prefer to use a simple Newton-Raphson scheme for our numerical experiments.

The latter is easy to implement and has a good speed of convergence in our case as the left-hand-side of equation 4.13 exhibits a well behaved derivative, namely $\frac{1}{\sigma_{loc}(k)}$ by equation 4.3. As a first guess we take $k = \log(U^2/K)$ – the strike of the static hedge in case of a flat implied volatility. The exact implementation in MATLAB can be found in the appendix

Figure 4.4: Asymptotic static hedge for the implied volatility in figure 4.2



together with the code for all the other numerical experiments performed.

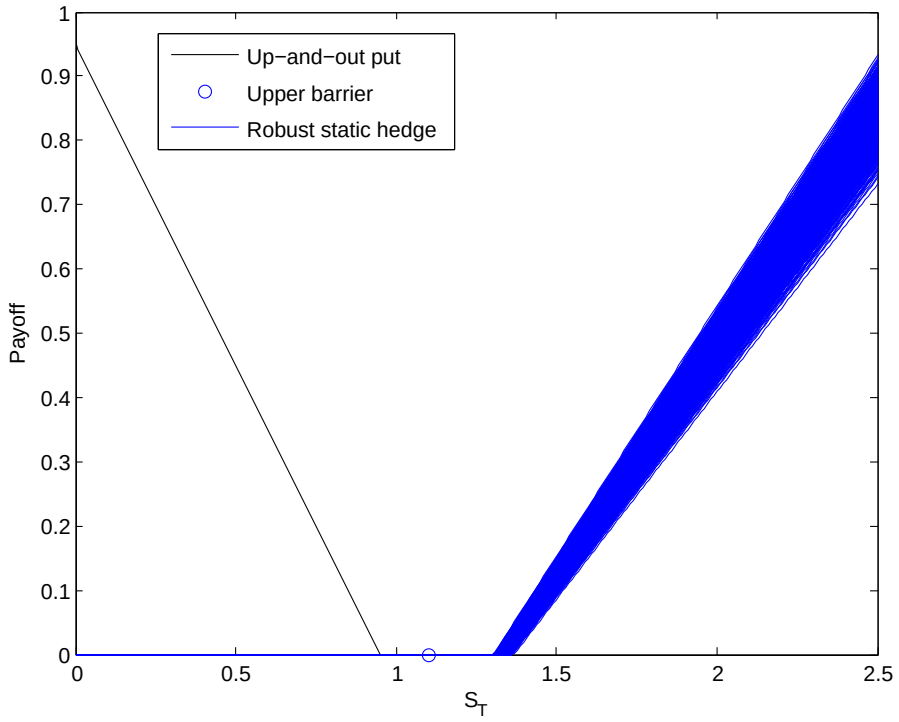
To obtain our static hedge the slope η remains to be calculated. Given the results of Theorem 3.1.1 this is an easy task. Recalling formula 3.4 and the relation of σ_{loc} and σ provided in equation 4.2 we simply get

$$\eta = \sqrt{\frac{\sigma(K)}{\sigma(K^*)}} = \sqrt{\frac{\sigma_{loc}(\log(K))K}{\sigma_{loc}(\log(K^*))K^*}} \tag{4.14}$$

The result we get using the above method is illustrated in figure 4.4. Here we again used the Gatheral-type implied volatility with the parameter combination $(a, b, \rho, \sigma, m) = (0.04, 0.4, -0.4, 0.1, 0)$ from figures 4.2 and 4.3. Here we see the asymptotic static hedge for an up-and-out put with strike $K = 0.95$ and barrier $U = 1.1$. The exact values of the static hedge are $(K^*, \eta) = (1.3312, 0.7190)$ compared to $(\frac{U^2}{K}, \frac{K}{U}) = (1.2737, 0.8636)$ in a Black-Scholes model.

As a second step, we apply the above procedure to a whole range of implied volatilities and obtain a range of corresponding asymptotic hedges. E.g. we can use the family of implied volatilities in figure 4.1, which was already discussed at the beginning of this chapter. More precisely we show the range of implied volatility surfaces of Gatheral-type, which are obtained by taking the base scenario $(a, b, \rho, \sigma) = (0.04, 0.4, -0.4, 0.1)$ and vary

Figure 4.5: Robust static hedge for range of implied volatility in figure 4.1



each parameter between 85% and 115% of its base value.

The corresponding range of static hedges is shown in figure 4.5, where we again hedged the up-and-out put option with strike $K = 0.95$ and upper barrier $U = 1.1$ (as we did in case of the asymptotic hedge in figure 4.4).

From this result we can directly construct a semi-robust super-replicating strategy. To this end we take the call with the smallest tradable strike, that is above the range of hedges shown in blue. Denote this by K^*. Next, we infer the slope η of the "right border" of the range of asymptotic hedges. Then we take a long position of one put struck at K and a short position of η calls struck at K^*. This strategy is super-replicating, if the future values of implied volatility stay within the strip of implied volatilities illustrated in figure 4.1.

Of course, we can obtain a semi-robust asymptotic hedge using our "extreme" model from figure 4.1 as well. To this end we fit a Gatheral-type implied volatility to the corresponding implied volatility, e.g. the green line in 4.1 and use the above technique to obtain the corresponding asymptotic hedge.

Observe that this method gives us the intuition, that our semi-robust static hedge will always be cheaper than the "conservative" super-replicating strategy obtained by model-independent considerations: Consider constructing our static hedge from any given "extreme" model. No matter how extreme, i.e. essentially how skewed, we choose our model to be, there will always be infinitely many, which are "more extreme", i.e. exhibit more skewness, and all of them will have been accounted for by the model-independent super-replicating strategy.

As a matter of fact this discount in comparison to the model-independent super-hedge does not come for free. It means that we are taking on some risk by hedging the up-and-out put using our static hedging strategy. This would even be the case, if we had a way to calculate the exact static hedge corresponding to our belief in the future values of the implied volatility without using any approximation. However, the contribution of our work is, that we have a good understanding of the risk we are taking. This provides a tool, which allows us to scale this risk according to our appetite.

Conclusion

This thesis discusses two approaches to calculate and approximate the static hedge of an up-and-out put option in a local volatility model under some regularity assumptions. We discuss a polynomial expansion of the exact static hedge provided by [9] and we develop an explicit expression of an asymptotic hedge for short maturities. This work is done with the objective to obtain a semi-robust static hedge.

Both of the above results have the common property, that they perform well for short maturities. For the single-call hedge this is obvious – by definition we ensure that the relative hedging error is going to zero as time-to-maturity goes to zero. As for our expansion of the exact static hedge, this is a result of the fact, that our expansion is accurate at the "kink" point of the exact static hedge. Keeping in mind, that we assume the underlying asset to move continuously in time, this approximate hedge should do well for short maturities. We have indication, that even for reasonably long maturities our approximations appear to perform well.

However we leave the question of finding bounds on the approximation error, which arises when truncating our expansion for the exact static hedge, for further research. This estimate is of interest as it would allow us to give global super- and sub-replicating strategies. These hedges have the advantage, that they are ensured to do well for all maturities.

We develop a method, which allows us to obtain semi-robust sub- and super-replicating strategies. This way we provide a middle ground between the conservative sub- and super-replicating strategies obtained by entirely model-independent considerations and the classical model-dependent hedges. The appealing property of our result is that it allows us to observe the "state of the world" in terms of implied volatility, which we have intuition for.

Appendix

Here we present the MATLAB code, which is used for our numerical experiments throughout the thesis.

Second order approximation of exact static hedge

The following code is used in chapter 2 to plot the first and second order approximation of the exact static hedge in the Black model case.

```
% calculates second order approx. of exact static hedge in BS model

% Input parameters (spot is assumed to be 1)
K = 0.95;    % stike
U = 1.1;     % barrier
k = log(K);
u = log(U);

% define grid for S
a=2.5;       % range
S = linspace(0.00001,a,2500);
s = log(S);

% calculate first and second order approximation
g = zeros(1,length(s));
g2 = zeros(1,length(s));
for is = 1:length(S)
      if(S(is) ≥ U^2/K)
          h = (log(S(is))-2*log(U)+log(K));
          g(is) = sqrt(S(is)*K) * h;
          g2(is) = g(is) + sqrt(S(is)*K)/16 *  h^3;
      end
end

% Plots
plot(S,max(K-S,0))
hold on
```

```
plot(U,0,'o')
plot(S,K/U*max(S-U^2/K,0),'r')
plot(S,g,'k')
plot(S,g2,'c')
hold off
xlabel('S_T'); ylabel('Payoff');
leg=legend('Up-and-out put','Upper barrier','Exact static hedge','First ...
    order static hedge','Second order static hedge');
set(leg,'Location','North');
```

The remainder of the code is used for our numerical illustrations in chapter 4.

The single-call hedge

First we present the function we use to get the asymptotic hedge (K^*, η) for an up-and-out put with strike K and barrier U given an arbitrary Gatheral-type implied volatility with parameters (a, b, ρ, σ) (the translation parameter m is assumed to be zero).

```
function sh = static_hedge(K,U,a,b,rho,sigma)
% Calculates asymptotic static hedge (K*,η) of an UOP @(K,U) for a
% Gatheral implied vol with parameter (a,b,ρ,σ,0) using Newton method


% Defining parameters
k = log(K);
u = log(U);
C = 2*u/Sigma(u,a,b,rho,sigma,0) - k/ Sigma(k,a,b,rho,sigma,0);

% Starting value
kstar = log(U^2/K);
temp = -1;

while(abs(kstar-temp) > 1e-12)
    temp = kstar;
    y = kstar/ Sigma(kstar,a,b,rho,sigma,0)-C;
    dy = 1/sigmatilde(kstar,a,b,rho,sigma);
    kstar = kstar - y/dy;
end
Kstar = exp(kstar);

% calculate η
eta = sigmatilde(k,a,b,rho,sigma)/sigmatilde(kstar,a,b,rho,sigma);
eta = sqrt(eta*K/Kstar);

```

```
27 % Return static hedge
28 sh = [Kstar,eta];
```

The function used to calculate the Gatheral-type implied volatility $\Sigma(k)$ with a set of parameters (a, b, ρ, σ, m) is straight-forward to implement using equation 4.4:

```
1 function Sigma = Sigma(k,a,b,rho,sigma,m)
2 % evaluates a implied volatility of the Gatheral type at the level of ...
      -log-moneyness k
3 Sigma = a + b * (rho*(k-m)+ sqrt((k-m).^2 + sigma^2));
4 Sigma = sqrt(Sigma);
```

So is the function to compute the local volatility function $\widetilde{\sigma}(k)$ as a function of the minus-log-moneyness k by equation 4.6:

```
1 function sigmatilde = sigmatilde(k,a,b,rho,sigma)
2 % calculates local vol as a fct of -log-moneyness corresponding to a Gatheral
3 % type implied vol with parameter (a,b,rho,sigma,0)
4 t1 = sqrt(k.^2+sigma^2);
5 t2 = Sigma(k,a,b,rho,sigma,0);
6 sigmatilde = 2*t1.*t2.^3 ./ (t1.*(2*t2.^2-b*rho*k)-b*k.^2);
```

The robust single-call hedge

Here we present the code used to obtain the robust static hedge, which is illustrated in figure 4.5.

```
1  % Calculates robust static hedge
2  clear all
3  close all
4  clc
5
6  tic
7
8  % Input parameters (spot is assumed to be 1)
9  K = 0.95;
10 U = 1.1;
11
12 % base case
13 a = 0.04;        % overall level
14 b = 0.4;         % convexity
15 rho = -0.4;      % skewness
16 sigma = 0.1;     % smoothness
```

```
17
18 % Set up ranges for parameters
19 s = 0.15;       % (relative)shock
20 d = 5;         %  number of steps
21 Ra = linspace((1-s)*a,(1+s)*a,d);
22 Rb = linspace((1-s)*b,(1+s)*b,d);
23 Rrho = linspace((1-s)*rho,(1+s)*rho,d);
24 Rsig = linspace((1-s)*sigma,(1+s)*sigma,d);
25
26 % set up 4 dimensional cubes for Kstar and eta
27 Kstar = zeros(d,d,d,d);
28 eta = Kstar;
29 h = 1:d;
30
31 % Calculate cube of (Kstar,eta)
32 for ia = h
33 for ib = h
34 for irho = h
35 for isig = h
36     sh = static_hedge(K,U,Ra(ia),Rb(ib),Rrho(irho),Rsig(isig));
37     Kstar(ia,ib,irho,isig)=sh(1);
38     eta(ia,ib,irho,isig)=sh(2);
39 end
40 end
41 end
42 end
43
44 % plots
45 x = 0:0.01:2.5; % grid
46
47 % implied vol surfaces under consideration
48 % implied vol surface
49 figure(1)
50 plot(-1:0.01:1,(Sigma(-1:0.01:1,a,b,rho,sigma,0)).^2);
51 hold on
52 for ia = h
53 for ib = h
54 for irho = h
55 for isig = h
56 plot(-1:0.01:1,(Sigma(-1:0.01:1,Ra(ia),Rb(ib),Rrho(irho),Rsig(isig),0)).^2);
57 end
58 end
59 end
60 end
61 hold off
62 title('Gatheral type implied variance surfaces under consideration')
63 xlabel('log-strike'); ylabel('implied variance');
```

```
64 legend('Family of implied variances considered')
65
66 % Robust hedge
67 figure(2)
68 plot(x,max(K-x,0),'k')
69 hold on
70 plot(U,0,'o')
71 for ia = h
72 for ib = h
73 for irho = h
74 for isig = h
75     plot(x,eta(ia,ib,irho,isig)*max(x-Kstar(ia,ib,irho,isig),0))
76 end
77 end
78 end
79 end
80 hold off
81 title('Robust static hedge');
82 xlabel('S_T'); ylabel('Payoff');
83 leg=legend('Up-and-out put','Upper barrier','Robust static hedge');
84 set(leg,'Location','North');
85
86 toc
```

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