

LAGRANGIANS OF HYPERGRAPHS: THE FRANKL-FÜREDI CONJECTURE HOLDS ALMOST EVERYWHERE

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ABSTRACT. Frankl and Füredi conjectured in 1989 that the maximum Lagrangian of all r -uniform hypergraphs of fixed size m is realised by the initial segment of the colexicographic order. In particular, in the principal case $m = \binom{t}{r}$ their conjecture states that every $H \subseteq \mathbb{N}^{(r)}$ of size $\binom{t}{r}$ satisfies

$$\max\left\{\sum_{A \in H} \prod_{i \in A} y_i : y_1, y_2, \dots \geq 0; \sum_{i \in \mathbb{N}} y_i = 1\right\} \leq \frac{1}{t^r} \binom{t}{r}.$$

We prove the above statement for all $r \geq 4$ and large values of t (the case $r = 3$ was settled by Talbot in 2002). More generally, we show for any $r \geq 4$ that the Frankl-Füredi conjecture holds whenever $\binom{t-1}{r} \leq m \leq \binom{t}{r} - \gamma_r t^{r-2}$ for a constant $\gamma_r > 0$, thereby verifying it for ‘most’ $m \in \mathbb{N}$.

Furthermore, for $r = 3$ we make an improvement on the results of Talbot [8] and Tang, Peng, Zhang and Zhao [9].

1. INTRODUCTION

Multilinear polynomials are of central interest in most branches of modern mathematics, and extremal combinatorics is by no means an exception. In particular, a large number of hypergraph Turán problems reduce to calculating or estimating the Lagrangian of a hypergraph, which is a constrained maximum of the multilinear function naturally associated with the hypergraph.

To set the scene, we need a few definitions. We follow standard notation of extremal combinatorics (see e.g. [1]). In particular, for $n, r \in \mathbb{N}$, we write $[n]$ for the set $\{1, \dots, n\}$ and, given a set X , by $X^{(r)}$ we denote the set family $\{A \subseteq X : |A| = r\}$. Dealing with finite families of finite sets we will be freely switching between the set system and the hypergraph points of view: with no loss of generality, we can assume our hypergraphs to be defined on \mathbb{N} , yet we write $e(H)$ for the number of sets (‘edges’) in H .

For a finite r -uniform hypergraph $H \subseteq [n]^{(r)}$ and a vector of real numbers (referred as a *weighting*) $\vec{y} := (y_1, \dots, y_n)$ consider a multilinear polynomial function

$$L(H, \vec{y}) := \sum_{A \in H} \prod_{i \in A} y_i.$$

The *Lagrangian* of H is defined as its maximum on the standard simplex

$$\lambda(H) := \max\{L(H, \vec{y}) : y_1, \dots, y_n \geq 0; \sum_{i=1}^n y_i = 1\};$$

note that, by compactness, the maximum does always exist (but need not be unique).

The above notion was introduced in 1965 by Motzkin and Strauss [7] for $r = 2$, that is for graphs, in order to give a new proof of Turán’s theorem. Later it was extended to uniform hypergraphs, where the Lagrangian plays an important role in governing densities of blow-ups. In particular, using Lagrangians of r -graphs, Frankl and Rödl [4] disproved a conjecture of Erdős [2] by exhibiting infinitely many non-jumps for hypergraph Turán densities. In the following years the Lagrangian has found numerous applications in hypergraph Turán problems; for more details we refer to a survey by Keevash [6] and the references therein. Further results, which appeared after the publication of [6], include [5] and [9].

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In this paper we address the problem of maximising the Lagrangian itself over all r -graphs with a fixed number of edges. Let $H^{m,r}$ be the subgraph of $\mathbb{N}^{(r)}$ consisting of the first m sets in the colexicographic order (recall that this is the ordering on $\mathbb{N}^{(r)}$ in which $A < B$ if $\max(A \triangle B) \in B$). In 1989 Frankl and Füredi [3] conjectured that the maximum Lagrangian of an r -graph on m edges is realised by $H^{m,r}$.

Conjecture 1.1 ([3]). $\lambda(H^{m,r}) = \max\{\lambda(H) : H \subseteq \mathbb{N}^{(r)}, e(H) = m\}$.

In an important special case, which we refer to as the *principal case*, Conjecture 1.1 states that for $m = \binom{t}{r}$ the maximum Lagrangian is attained on $H^{m,r} = [t]^{(r)}$, where we have $\lambda(H^{m,r}) = \lambda([t]^{(r)}) = \frac{1}{t^r} \binom{t}{r}$. While initially the Frankl-Füredi conjecture was motivated by applications to hypergraph Turán problems, we think it also interesting in its own right, as it makes a natural and general statement about maxima of multilinear functions.

For $r = 2$ the validity of Conjecture 1.1 is easy to see and follows from the arguments of Motzkin and Strauss [7]. In fact, the Lagrangian of a graph H is attained by equi-distributing the weights between the vertices of the largest clique of H , resulting in $\lambda(H) = \frac{\omega(H)-1}{2\omega(H)}$. Since $H^{m,r}$ has the largest clique size over all graphs on m edges, Conjecture 1.1 holds.

On the other hand, the situation for hypergraphs is far more complex, since for $r \geq 3$, unlike in the graph case, no direct way of inferring $\lambda(H)$ from the structure of H is known. Hence one is confined to estimating the Lagrangians of different r -graphs against each other without calculating them directly.

For $r = 3$ Talbot [8] proved that Conjecture 1.1 holds whenever $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} - (t-1) = \binom{t}{3} - (2t-3)$ for some $t \in \mathbb{N}$. Note that this range covers an asymptotic density 1 subset of \mathbb{N} , and also includes the principal case $m = \binom{t-1}{3}$. Recently Tang, Peng, Zhang and Zhao [9] extended the above range to $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} - \frac{1}{2}(t-1)$. Furthermore, Conjecture 1.1 is known to hold when $\binom{t}{3} - m$ is a small constant, but for the remaining values of m it is still open.

In contrast to this, for $r \geq 4$ much less has been known so far, as Talbot's proof method for $r = 3$, perhaps surprisingly, does not immediately transfer. Talbot showed in the same paper [8] that for every $r \geq 4$ there is a constant $\gamma_r > 0$ such that if $\binom{t-1}{r} \leq m \leq \binom{t}{r} - \gamma_r t^{r-2}$ and H is supported on t vertices (that is, ignoring isolated vertices, H is a subgraph of $[t]^{(r)}$), then indeed $\lambda(H) \leq \lambda(H^{m,r})$. Still, for no value of m , apart from some trivial ones, Conjecture 1.1 has been known to hold. Our main goal in this article is to close this gap by confirming the Frankl-Füredi Conjecture for 'most' values of m for any given $r \geq 4$, including the principal case for large m .

Theorem 1.2. *For every $r \geq 4$ there exists $\gamma_r > 0$ such that for all $\binom{t-1}{r} \leq m \leq \binom{t}{r} - \gamma_r t^{r-2}$ we have*

$$\lambda(H^{m,r}) = \max\{\lambda(H) : H \subseteq \mathbb{N}^{(r)}, e(H) = m\}.$$

Corollary 1.3. *For every $r \geq 4$ there exists $t_r \in \mathbb{N}$ such that for all $t \in \mathbb{N}$ with $t \geq t_r$ we have*

$$\max\left\{\lambda(H) : H \subseteq \mathbb{N}^{(r)}, e(H) = \binom{t}{r}\right\} = \lambda([t]^{(r)}) = \frac{1}{t^r} \binom{t}{r}.$$

By monotonicity, we obtain another immediate corollary, which can be viewed as a strong approximate version of Conjecture 1.1.

Corollary 1.4. *For every $r \geq 4$ there exists $t_r \in \mathbb{N}$ such that for all $t \geq t_r$ the following holds. Suppose that $\binom{t-1}{r} < m \leq \binom{t}{r}$ and that H is an r -graph with $e(H) = m$. Then*

$$\lambda(H) \leq \frac{1}{t^r} \binom{t}{r}.$$

When H is supported on $[t]$ we give a proof of a stronger statement, namely that in this case we can take $\gamma_r = (1 + o(1))/(r-2)!$ in Theorem 1.2. More precisely, we claim the following.

Theorem 1.5. *For every $r \geq 3$ there exists a constant $\delta_r > 0$ such that for all $\binom{t-1}{r} \leq m \leq \binom{t}{r} - \binom{t-2}{r-2} - \delta_r t^{r-9/4}$ we have*

$$\lambda(H^{m,r}) = \max\{\lambda(H) : H \subseteq [t]^{(r)}, e(H) = m\}.$$

For $r = 3$ it was implicitly shown by Talbot in [8] that for any $\binom{t-1}{3} < m \leq \binom{t}{3}$, that is for all $m \in \mathbb{N}$, the 3-graph maximising the Lagrangian amongst all m -edge 3-graphs can be assumed to be supported on $[t]$. Combined with Theorem 1.5, this yields, for large m , an improvement of the bounds in [8] and [9].

Corollary 1.6. *There exists a constant $\delta_3 > 0$ such that for all $\binom{t-1}{3} \leq m \leq \binom{t}{3} - (t-2) - \delta_3 t^{3/4}$ we have*

$$\lambda(H^{m,r}) = \max\{\lambda(H) : H \subseteq \mathbb{N}^{(r)}, e(H) = m\}.$$

Our proofs use a number of previously known properties of the Lagrangian, as well as induction on r and some facts about uniform set systems such as the Kruskal-Katona theorem.

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